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On groups with relatively small normalizers of nonprimary subgroups

Vladimir Antonov *

Abstract

We consider the structure of a finite nonsolvable group G in which for any nonprimary subgroup A the index $|N_G(A) : A{\cdot}C_G(A)|$ is equal unit or a prime number.

Keywords: finite group, subgroup, normalizer, centralizer.

If A is an arbitrary subgroup of a group G, then $N(A) \geq A{\cdot}C(A)$, and the index $|N(A) : A{\cdot}C(A)|$ equals to the order of a subgroup of $Out(A)$, which is induced by elements of G. In this paper we consider the structure of finite groups G in which for any nonprimary subgroup A the index $|N(A) : A{\cdot}C(A)|$ is a divisor of a certain prime number, i.e., it is equal to 1 or a prime number. We'll call these groups NP-groups.

Note that any subgroup and factor-group of a NP-group is also a NP-group. The aim of this article is to describe the structure of nonsolvable NP-groups.

1.1. Lemma. *If a nonsolvable NP-group G is a central product of two subgroups G_1 and G_2, then one of the factors is abelian.*

Proof. Suppose that G_1 is nonabelian. Then ([1], Corollary of Lemma 2) there exists a subgroup A of G_1 such that $|N_{G_1}(A) : A \cdot C_{G_1}(A)|{=}p$ for a prime p. If A is nonprimary and B is an arbitrary subgroup of G_2, then from the fact that $|N(AB) : AB \cdot C(AB)|$ divides a prime number, it follows that $N_{G_2}(B) = B \cdot C_{G_2}(B)$. Then G_2 is abelian (see [1]). If A is primary and $|A| = q^n$ for a prime q, then the equality $N_{G_2}(B) = B{\cdot}C_{G_2}(B)$ holds for any q'-subgroup B of G_2. By Lemma 4 from [1], $G_2 = Q \leftthreetimes H$, where H is an abelian Hall q'-subgroup of G_2. i.e. G_2 is solvable. If G_2 is nonabelian, then for any q'-subgroup A of G_1, the equality $|N_{G_1}(A) : A{\cdot}C_{G_1}(A)|$ holds too. But then the group G_1 is also solvable, which is impossible. □

1.2. Lemma. *If Q is a Sylow q-subgroup of a NP-group G, $C(Q) \leq Q$ and $N(Q) = (Q \leftthreetimes \langle a\rangle) \leftthreetimes \langle b\rangle$, where $a \neq 1 \neq b$, then a and b are elements of prime orders, and if $N(Q) = Q \leftthreetimes \langle x\rangle$, then $|x|$ is the product of no more than two prime factors.*

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Proof. In the first case, if we let $A = Q \leftthreetimes \langle a \rangle$, we get that $|b| = |N(A) : A{\cdot}C(A)|$ is a prime. And supposing $A = Q \leftthreetimes \langle c \rangle$, where c is an element of prime order r from $\langle a \rangle$, then from the equality $|N(A) : A{\cdot}C(A)| = \frac{|a|}{r}|b|$ we get that $|a| = r$. In second case, it's sufficient to choose a subgroup $A = Q \leftthreetimes \langle y \rangle$, where y is an element of prime order from $\langle x \rangle$. □

Later on we will repeatedly use Frattini's argument ([7], theorem 1.3.7): if $H \lhd G$ and P is a Sylow p-subgroup of H, then $G = H \cdot N(P)$. In a solvable group all Hall π-subgroups are conjugate. Therefore a similar proposition is true in a case where P is a Hall π-subgroup of a solvable group H. We will call this Frattini's argument as well.

1.3. Theorem. *A finite nonabelian simple group G is a NP-group if and only if G satisfies one of the following conditions:*

1) $G \cong PSL(2, q^n)$, $\frac{q^n-1}{(2,q^n-1)}$ *is either a prime or a product of two primes;*

2) $G \cong PSU(3, 2^{2n})$, *and either $n = 2$ or each of the numbers $(2^n - 1)$ and $\frac{2^n+1}{3}$ are primes;*

3) $G \cong Sz(2^n)$, $n \in \{3, 5\}$.

Proof. Necessity. Let G be a finite nonabelian simple NP-group. It is known that any nonabelian simple group is either an alternating group, a Lie type group, or a sporadic simple group.

First, assume that $G \cong A_n$. If $n = 5$, then $G \cong PSL(2, 4)$, and if $n = 6$, then $G \cong PSL(2, 9)$. If, however, $n > 6$ then G contains a subgroup which is isomorphic to A_7. Let $G = A_7$, $a = (1\ 2)(3\ 4)$, $b = (1\ 3)(2\ 4)$, $c = (5\ 6\ 7)$, $x = (1\ 2)(5\ 6)$, $y = (1\ 2\ 3)$ and $A = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$. Then $C(A) = A$ and $N(A) = A \leftthreetimes (\langle y \rangle \leftthreetimes \langle x \rangle)$, i.e. $|N(A) : A{\cdot}C(A)| = 6$, which is impossible.

Now let G be a simple Lie type group over the Galois field $GF(q^n)$, where q is a prime. Suppose that the Lie rank l of G is more than 2. If J is a parabolic subgroup of G, corresponding to two nonadjoit nodes of the Dynkin diagram of G, then ([4], Proposition 2.17) $\overline{J} = J/O_q(J) = (\overline{Y}_1 \times \overline{Y}_2) \cdot \overline{H}$, where $\overline{Y}_1$ and $\overline{Y}_2$ are Lie type groups of Lie rank 1 over $GF(q^n)$ and H is a Cartan subgroup of G. By Lemma 1.1 each of $\overline{Y_i}$ is a solvable group. Since ([4], Theorem 2.13) solvable Lie type groups are either $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$ or ${}^2B_2(2)$, so $q^n \in \{2, 3\}$. Let $p_i \in \pi(\overline{Y}_i) \setminus \{q\}$, $\overline{A_1}$ and $\overline{A}_2$ be Sylow p_1- and p_2-subgroup from $\overline{Y}_1$ and $\overline{Y}_2$, respectively, then for the nonprimary subgroup $A = A_1 \cdot A_2$ the index $|N(A) : A{\cdot}C(A)|$ is divisible by q^2, which is impossible.

Therefore $l \leq 2$. Let $l = 2$, i.e., G is isomorphic to one of the groups $A_2(q^n)$, $B_2(q^n)$, ${}^2A_3(q^n)$, ${}^2A_4(q^n)$, ${}^3D_4(q^n)$, ${}^2F_4(2^{2n+1})$, $n > 0$, $({}^2F_4(2))'$.

First suppose that the Cartan subgroup H of the group G is trivial. The group $({}^2F_4(2))'$ contains a subgroup K isomorphic to $PSL(2, 25)$, which is not NP-group, because it has a Cartan subgroup of order 12, which contradicts Lemma 1.2. Because of this, G is a group of classical type over the field $GF(2)$, i.e., either $G \cong A_2(2) = PSL(3, 2)$, or $G \cong B_2(2) = PSp(4, 2)$. It's left to be noticed that $PSL(3, 2) \cong PSL(2, 7)$, and that the group $PSp(4, 2) \cong S_6$ is not simple.

Therefore $H \neq 1$. Let J be a proper parabolic subgroup of G. Then $\overline{J} = J/O_q(J) = \overline{Y} \cdot \overline{H}$, where $\overline{Y}$ is a Lie type group of Lie rank 1. If $G \cong {}^2F_4(2^{2n+1})$, $n > 0$, then subgroup J can be chosen so that $\overline{Y} \cong {}^2B_2(2^{2n+1})$, and if $\overline{A}$ is a subgroup of the order $2^{2n+1}+2^{n+1}+1$ from $\overline{Y}$, then A is nonprimary and $|N_Y(A):A\cdot C_Y(A)|=4$, which is imposible. If $G\cong {}^2A_4(q^n)$, then $\overline{Y}\cong PSL(2,q^{2n})$, and if $\overline{H}_1$ is a Cartan subgroup of $\overline{Y}$, then the index $|N(H_1):H_1\cdot C(H_1)|=2\cdot|\overline{H}/\overline{H_1}|$ is not a prime.

In all the other cases, subgroup $\overline{J}$ may choosen in such a way that $\overline{Y}\cong A_1(q^n)$. If $q^n=2$ and $\overline{A}$ is a subgroup of order 3 of $\overline{Y}$, then by Frattini's argument we assume that $\overline{H}\le N(\overline{A})$ which also leads to a contradiction. However, if $q^n\neq 2$, then as $\overline{A}$ we can take a Cartan subgroup of $\overline{Y}$.

Therefore $l=1$. If Q is a Sylow q-subgroup of G, then $C(Q)\le Q$ and $N(Q)=Q\leftthreetimes H$, where H is a Cartan subgroup of G. From the definition of an NP-group and the fact that H is abelian, one of the following is true: $|H|=1$, $|H|$ is a prime number or $|H|=pr$ where p and r are primes. Since group $A_1(2)$ is solvable, then the first case is impossible.

First, suppose that G is a twisted group. Let $G\cong\ {}^2A_2(q^n)=PSU(3,q^{2n})$. Then $|H|=\frac{q^{2n}-1}{(3,q^n+1)}=(q^n-1)\cdot\frac{q^n+1}{(3,q^n+1)}$. If $q>2$ then $|H|$ is divisible by 8, which is impossible. Therefore $q=2$ and all of the numbers (2^n-1) and $\frac{2^n+1}{(3,2^n+1)}$ are primes. The primarity of (2^n-1) implies that either $n=2$ or n is an odd prime and then $(2^n+1,3)$=3, i.e., G is a group of type 2) from this Theorem.

The group ${}^2B_2(2^{2n+1})$ contains, as subgroups, the Frobenius groups of orders $(2^{2n+1}\pm 2^{n+1}+1)\cdot 4$. Therefore each of the numbers $2^{2n+1}+2^{n+1}+1$ and $2^{2n+1}-2^{n+1}+1$ must be powers of the primes. Because their product is equal $(2^2)^{2n+1}+1$ it is divisible by 5. But then either $2^{2n+1}+2^{n+1}+1=5^m$, or $2^{2n+1}-2^{n+1}+1=5^m$ for some number m.

Consider the first case. If $2^{2n+1}+2^{n+1}+1=5^m$, then either $n=4t$ or $n=4t-1$ for some $t>0$. Since $2^7+2^4+1=145\neq 5^m$, then $n\ge 4$ in any case. Let $m=2^kr$, where r is an odd number. Then from

$$2^{n+1}(2^n+1)=5^m-1=2^{k+2}\cdot\frac{5^r-1}{4}\cdot\prod_{i=0}^{k-1}\frac{5^{2^ir}+1}{2}$$

it follows that $k=n-1\ge 3$. But the inequality

$$\prod_{i=0}^{k-1}\frac{5^{2^ir}+1}{2}>2^{k+1}+1=2^n+1$$

is true for $k\ge 3$, which is impossible.

If, however, $2^{2n+1}-2^{n+1}+1=5^m$, then either $n=4t+1$ or $n=4t+2$ for some $t\ge 0$. The equality

$$2^{n+1}(2^n-1)=5^m-1=2^{k+2}\cdot\frac{5^r-1}{4}\cdot\prod_{i=0}^{k-1}\frac{5^{2^ir}+1}{2}$$

implies $k=n-1$. If $k>1$ then from $k\in\{4t,4t+1\}$ it follows that $k\ge 4$ and we have the contradiction again. Therefore, $k\in\{0,1\}$ and, consequently, $n\in\{1,2\}$, i.e., G is a group of the type 3) from this Theorem.

Let $G\cong {}^2G_2(3^{2n+1})$. Since the group ${}^2G_2(3)$ is nonsimple, then $n>0$. In this case (see [8]) G has a subgroup H such that $H=(V_4\times D)\leftthreetimes\langle b\rangle$, where

$|b| = 3$, $V_4 = \langle a_1 \rangle \times \langle a_2 \rangle$, $|a_i| = 2$, and D is isomorphic to the dihedral group of order $\frac{3^{2n+1}+1}{2}$. If a is an element of order $\frac{3^{2n+1}+1}{4}$ from D, then the subgroup $A = V_4 \times \langle a \rangle$ is nonprimary and $|N_H(A) : A \cdot C_H(A)| = 6$, which is impossible.

Now suppose that G is a classical nontwisted group of Lie type rank 1, i.e., $G \cong A_1(q^n) \cong PSL(2, q^n)$. In this case $|H| = \frac{q^n-1}{(2,q^n-1)}$. Because of this $\frac{q^n-1}{(2,q^n-1)}$ is either be a prime, or a product of two primes, i.e., G is a group of the type 1) from this Theorem.

Now using the survey [10] we can show that G cannot be a sporadic simple group. To demonstrate this, it's sufficient to show that any sporadic simple group contains a subgroup, which is not NP-group. Let G_p denote a Sylow p-subgroup of G for a prime p.

1) In the group M_{11} the subgroup G_3 is self-centralizing and its normaliser has a form $N(G_3) = G_3 \leftthreetimes K$, where K is isomorphic to the semi-dihedral group of order 16, again contrary with Lemma 1.2.

2) M_{12}, M_{23}, M_{24}, Co_3, Suz and McL contain M_{11}.

3) M_{22} and M_{24} contain A_7, F_{22} contains S_{10}, and F_{23} and F'_{24} contain S_{12}.

4) The group $O'N$ contains J_1, and in the group J_1 the subgroup $N(G_3)$ is a direct product of two dihedral groups of orders 6 and 10. If A is a subgroup from $N(G_3)$ of order 15, then $|N(A) : A \cdot C(A)|$ is divisible by 4.

5) In the group J_2 we have $N(G_3) = G_3 \leftthreetimes \langle a \rangle$, where $C(G_3) = G_3$ and $|a| = 8$.

6) In the groups J_3 and He the subgroup $N(G_{17})$ is a Frobenius group of order $17 \cdot 8$; in J_4 and Co_2 the subgroup $N(G_{29})$ is a Frobenius group of order $29 \cdot 28$, again contrary to Lemma 1.2 and Co_1 and F_2 contain Co_2.

7) The group F_1 contain an involution τ such that $C(\tau)/O_2(C(\tau)) \cong Co_2$.

8) In the groups Ly and F_3 the subgroups $N(G_{37})$ and $N(G_{19})$ are Frobenius groups of orders $37 \cdot 18$ and $19 \cdot 18$, respectively.

9) The group F_5 contains HS, and in the group HS the subgroup $N(G_3)$ is isomorphic to $S_3 \times S_5$, and if $A_3 \times A_5 \cong A \leq N(G_3)$, then $|N(A) : A \cdot C(A)|$ is divisible by 4.

10) The group Ru contains an involution τ such that $C(\tau) \cong V_4 \times Sz(8)$, and if $A \cong V_4 \times H$, where H is a subgroup of order 5 from $Sz(8)$, then $|N(A) : A \cdot C(A)|$ is divisible by 4.

Sufficiency. If A is a proper nonprimary subgroup of G, then $N(A) < G$. Therefore, it is sufficient to prove, that any maximal subgroup of G is a NP-group.

Suppose first that $G \cong PSL(2, q^n)$, where q is a prime. Since $\frac{q^n-1}{(2,q^n-1)}$ is either a prime or a product of two primes, then, it is not difficult to see, that either $n = 1$ or $q \in \{2, 3\}$ and n is either a prime or the square of a prime (odd, if $q = 3$).From Dickson's Theorem ([6], Theorem 2.8.27) it follows that the maximal subgroups of G are the groups from the following list: $N(Q) = Q \leftthreetimes \langle a \rangle$, where Q is a Sylow q-subgroup of G, $|a| = \frac{q^n-1}{(2,q^n-1)}$; the dihedral groups of the orders $2 \cdot \frac{q^n \pm 1}{(2,q^n-1)}$; S_4 for $q^n \equiv \pm 1(8)$, A_4 for $q^n \equiv \pm 3(8)$, A_5 for $q^n \equiv \pm 1(10)$; $PSL(2, q^p)$ for $n = p^2$. It's not difficult to check that all these groups are NP-groups.

If $G \cong PSU(3, 2^{2n})$, then since $(2^n - 1)$ is a prime, n is a prime too. From [5] it follows that the maximal subgroups of G are the groups of the following types: $N(Q) = Q \leftthreetimes \langle a\rangle$, where Q is a Sylow 2-subgroup of G, $|a| = \frac{2^{2n}-1}{(3,2^n+1)}$; $C(b) = \langle b\rangle \times B$, where $|b| = \frac{2^n+1}{(3,2^n+1)}$, $B \cong PSL(2,2^n)$; the Frobenius group $\langle a\rangle \leftthreetimes \langle b\rangle$, $|a| = \frac{2^{2n}-2^n+1}{(3,2^n+1)}$, $|b| = 3$; the Frobenius groups $(\langle a\rangle \times \langle b\rangle) \leftthreetimes C$, $|a| = 2^n+1$, $|b| = \frac{2^n+1}{(3,2^n+1)}$, $C \cong S_3$.

In the groups $Sz(2,2^{2n+1})$ for a prime n, the maximal subgroups are the groups of the following types (see [9]): $N(Q) = Q \leftthreetimes \langle a\rangle$, Q is a Sylow 2-subgroup, $|a| = 2^n-1$; the dihedral group of order $2\cdot(2^n-1)$; the Frobenius groups $\langle a\rangle \leftthreetimes \langle b\rangle$, $|a| = 2^n \pm 2^{\frac{n+1}{2}}+1$, $|b| = 4$. $\square$

Below F and F^* denote the Fitting subgroup and the generalized Fitting subgroup of G, respectively.

1.4. Theorem. *Let G be a nonsolvable nonsimple NP-group. Then one of the following holds:*

1) *subgroup $F = F^*$ is a nontrivial p-group for some prime p, and $G/F \cong PSL(2,4)$;*

2) *$G \cong \mathrm{Aut}(PSL(2,2^n))$, $n \in \{2,3\}$;*

3) *$G = Z(G)\cdot L$, $L \cong PSL(2,q^n)$ or $SL(2,q^n)$, the number $\frac{q^n-1}{(2,q^n-1)}$ is a prime, and if $n=1$ then either $q \not\equiv \pm1(8)$ or $Z(G)$ is a 2-group;*

4) *$G = Z(G)\times L$ and either $L \cong PSL(2,q^n)$, $\frac{q^n-1}{(2,q^n-1)}$ is a product of the two prime numbers and $Z(G)$ is a q-group, or $Z(G)$ is a 2-group and $L \cong PSU(2,2^{2n})$ is a group from Theorem 1.3;*

5) *$G = Z(G)\cdot L$, $Z(G)$ is a 3-group and L is isomorphic to the covering group for $PSL(2,9)$ with $|Z(L)| = 3$.*

Proof. Let G be a group satisfiy conditions of this Theorem. Let's assume first that $F = F^*$. Then $C(F) \le F$. If F is a nonprimary group, then $|G:F|$ is a prime and G is a solvable group. Therefore, F is a p-group for some prime p. Moreover, if A/F is a p'-subgroup of G/F, then $|N(A):A|$ divides a prime number.

Let G_1/F is a minimal normal subgroup of G/F. Then G_1 is a non-nilpotent group, and consequently, is nonprimary. Therefore $|G:G_1|$ is a divisor of a prime. Assume that $G = G_1$. Then G/F is a simple NP-group. i.e., a group from Theorem 1.3.

Let $G/F \cong PSU(3,2^{2n})$. If $p \ne 2$ and A/F is a Sylow 2-subgroup of G/F, then A is nonprimary, and $|N(A):A\cdot C(A)| = \frac{2^{2n}-1}{(3,2^n+1)}$ is not a prime. Therefore $p=2$. Then ([4], p.166), for subgroup H/F of order $\frac{2^{2n}-1}{(3,2^n+1)}$ from $N_{G/F}(A/F)$ the equality $C_{G/F}(H/F) = H/F \times L/F$, where $L/F \cong PSL(2,2^n)$, is true. Therefore, for the nonprimary subgroup H, the index $|N(H):H\cdot C(H)|$ divides by $|L/F|$, which is impossible.

In the case $G/F \cong Sz(8)$, a Sylow 2-subgroup of G/F has the order 2^6. Hence $p=2$. If A/F is a subgroup of order 5 from G/F, then $|N(A):A| = 4$, which is impossible. If $G/F \cong Sz(2^5)$, then by analogy $p=2$ and if A/F is a subgroup of order 25, then $|N(A):A| = 4$.

Therefore, $G/F \cong PSL(2,q^n)$. If $q \ne p$ and Q/F is a Sylow q-subgroup of G/F, then Q is nonprimary and the primarity of the number $|N_{G/F}(Q/F):Q/F|$

implies that $\frac{q^n-1}{(2,q^n-1)}$ is a prime. If aF is an element of order q from Q/F then the index $|N(\langle a,F\rangle):\langle a,F\rangle|$ divides a prime number and, therefore, $n\le 2$.

If $n=2$ then from the primarity of $\frac{q^2-1}{(2,q^2-1)}$ we get that $q=2$, i.e. $G/F\cong PSL(2,4)$. Let $n=1$. Since the groups $PSL(2,2)$ and $PSL(2,3)$ are solvable, and $PSL(2,5)\cong PSL(2,4)$ then we can suppose that $q>5$. Let A/F is a subgroup of the prime order r, where r divides $\frac{q+1}{2}$. If $r\ne p$ then the primarity of $|N(A):A|=2\cdot\frac{q+1}{2r}$ implies $r=\frac{q+1}{2}$. But the numbers $\frac{q-1}{2}$ and $\frac{q+1}{2}$ are primes at the same time only when $q=5$. Suppose now that $r=p$. Then by the arbitrariness of r, the equation $\frac{q+1}{2}=p^k$ is solvable. Since $q>5$ then the prime number $\frac{q-1}{2}$ is odd. But then $q+1$ is divisible by 4. i.e. $p=2$. Since one of the numbers, either k or $k+1$, is even, then the numbers $q=2^{k+1}-1$ and $\frac{q-1}{2}=2^k-1$ cannot both be prime at the same time.

Assume now that $q=p$ and aF is an element of prime order from a subgroup of order $\frac{q^n\pm1}{(2,q^n-1)}$ from G/F. Because $N_{G/F}(\langle aF\rangle)$ is isomorphic to the dihedral group of order $\frac{q^n\pm1}{(2,q^n-1)}\cdot 2$, and $|N(\langle a,F\rangle):\langle a,F\rangle|$ is a prime, then the numbers $\frac{q^n\pm1}{(2,q^n-1)}$ are primes. If q is odd, then $q^n=5$. But $PSL(2,5)\cong PSL(2,4)$. If $q=2$, then because (2^n-1) is a prime it follows that n is a prime. But then in the case $n>2$ the number 2^n+1 is not prime. Therefore, $G/F\cong PSL(2,4)$.

Suppose now that $G_1<G$. Then, by using what's already been proved, $G_1/F\cong PSL(2,4)$ and $G/F=(G_1/F)\leftthreetimes\langle aF\rangle$, where aF is an automorphism of the group G_1/F. Let A/F be a subgroup of order 5 from G_1/F. By Frattini's argument we can assume that $aF\in N_{G/F}(A/F)$. But then $|N(A):A\cdot C(A)|$ is divisible by 4.

Therefor, if $F=F^*$, then by the theorem conditions, G is of type 1). Because of this, we'll further assume that $F<F^*$. Then $F^*=F\cdot L$, when L is the layer of the group G. By Lemma 1.1, the subgroup F is abelian and F^*/F is a simple group, i.e., a group from Theorem 1.3. Moreover, one of the following holds: $F=1$, $G=F^*$ or $1<F<F^*<G$.

In the first case F^* is a group from Theorem 1.3 and $F^*<G\le \mathrm{Aut}(F^*)$. From the definition of the NP-group it follows that $|G/F^*|$ is a prime. The structure of the automorphism groups of Lie type groups (e.g. [4], theorem 4.238) implies that in our case $G=F^*\leftthreetimes\langle a\rangle$, a is a prime order automorphism of group F^*. Set $|a|=p$.

First assume that $F^*\cong PSL(2,q^n)$. Let Q be a Sylow q-subgroup of $F^*$and $B=Q\leftthreetimes H$ be a Borel subgroup of group F^*. By Frattini's argument we can assume that $a\in N(Q)$. But then $a\in N(N_{F^*}(Q))=N(B)$. Since $C(Q)\le Q$ and $|N(Q):Q|=|H|\cdot p$, then, by Lemma 1.2, the number $|H|=\frac{q^n-1}{(2,q^n-1)}$ must be a prime number. But then, as it was noted in the proof of Theorem 1.3, either $q\in\{2,3\}$, or $n=1$. By analogy, for a subgroup A of order $\frac{q^n+1}{(2,q^n-1)}$ from F^* the equality $|N(A):A\cdot C(A)|=2p$ implies that subgroup A must be a primary group.

Let $q=2$. The primarity of the number (2^n-1) implies that n is a prime. If $n>2$, then 2^n+1 is divisible by 3 and, consequently, $2^n+1=3^k$ for a number k. Let $k>2$. If $k=2r$ is even, then $2^n=3^k-1=(3^r-1)(3^r+1)$, which is impossible. However, if $k=2r+1$, then $3^k-1=2(1+3+3^2+\cdots+3^{2r})\ne 2^n$

where the second factor is odd. Therefor, if $q = 2$, then the group F^* is isomorphic to one of the groups $PSL(2,4)$ or $PSL(2,8)$.

If $q = 3$ then the primarity of the number $\frac{3^n-1}{2}$ implies that n is an odd prime. However, from that fact that $\frac{3^n+1}{2}$ is even and prime it follows that $\frac{3^n+1}{2} = 2^k$, i.e., $3^n = 2^{k+1} - 1$ for a number k. Since the number $\frac{3^n-1}{2} = 2^k - 1$ is prime, then k is an odd prime. But then $k + 1 = 2r$ and $3^n = (2^r - 1)(2^r + 1)$, which is impossible for $r > 1$. However if $r = 1$, then $k = 1$. But then $n = 1$ as well, which contradicts the primarity of the group F^*.

Finally, let q and $\frac{q-1}{2}$ be primes. If $q = 5$, then $F^* \cong PSL(2,4)$. However if $q > 5$, then $\frac{q-1}{2}$ is odd. Because $\frac{q+1}{2}$ is primary, we obtain that $\frac{q+1}{2} = 2^k$, i.e. $q = 2^{k+1} - 1$. But then $\frac{q-1}{2} = 2^k - 1$. Since one of the numbers k, $k+1$ is even, and $k > 2$, then the numbers $(2^k - 1)$ and $(2^{k+1} - 1)$ can't both be prime simultaneously.

Suppose now that $F^* \cong PSU(3, 2^{2n})$. If $p \neq 2$ and A is a Sylow 2-subgroup of F^*, then $|N(A) : A \cdot C(A)| = p \cdot (2^n - 1) \cdot \frac{2^n+1}{(3,2^n+1)}$, which is impossible. However, if $p = 2$ and H is a Cartan subgroup of F^*, then H is nonprimary and $|N(H) : H \cdot C(H)| = 4$.

If $F^* \cong Sz(2^3)$ or $Sz(2^5)$ and A is a subgroup of order 5 or 25 of F^*, respectively, then $|N(A) : A| = 4p$, which contradicts Lemma 1.2.

Therefore, if $F = 1$, then G is of a type 2) from this Theorem.

Consider the case when $G = F^*$, i.e., $G = F \cdot L$, where L is the layer of the group G. By Lemma 1.1, the subgroup F is abelian, i.e., $F = Z(G)$, and L is a quasi simple group. Since the group G isn't simple, then $F \neq 1$. If F is nonprimary, then the index $|N_L(A) : A \cdot C_L(A)|$ divides a prime for any subgroup $A \leq L$. By theorem 4 from [2] $L \cong PSL(2, q^n)$ or $SL(2, q^n)$, the number $\frac{q^n-1}{(2,q^n-1)}$ is a prime and if $n = 1$, then $q \not\equiv \pm 1(8)$, i.e., G is of type 3) from this Theorem.

Now suppose that F is a p-group for a prime p. Since the Schur multiplier of group $Sz(2^5)$ is trivial then either L is a group from Theorem 1.3 or L is isomorphic to a covering of group $PSL(2, q^n)$, $Sz(8)$ or $PSU(3, 2^{2n})$.

Let $L/Z(L) \cong Sz(8)$. Then $L/Z(L)$ contains the subgroups $A_1/Z(L)$ and $A_2/Z(L)$ of order 5 and 13, respectively, such that $|N_L(A_i) : A_i \cdot C(A_i)| = 4$. Since p isn't at least one of the numbers 5 or 13, then supposing $A = F \cdot A_i$, we get a contradiction with the definition of NP-group. If $L \cong Sz(2^5)$ then subgroups of order 25 and 41 should be taken as subgroups A_1 and A_2 in the group G.

Therefore, we can assume that $L/Z(L) \cong PSL(2, q^n)$ or $PSU(3, 2^{2n})$.

First, assume that $Z(L) = 1$, i.e., $G = Z(G) \times L$. If $L \cong PSL(2, q^n)$ and $p \neq q$, then the number $\frac{q^n-1}{(2,q^n-1)}$ should be prime. Moreover, if $n = 1$ and $q \equiv \pm 1(8)$, then $L/Z(L)$ contains a subgroup $H/Z(L) \cong S_4$. If $V/Z(L)$ is a four-group from $H/Z(L)$, then the equality $|N_{H/Z(L)}(V/Z(L)) : V/Z(L)| = 6$ implies that in this case subgroup V is primary, i.e., $p = 2$. However, if $p = q$, then the number $\frac{q^n-1}{(2,q^n-1)}$ could be the product of two primes. But, if $q^n \equiv \pm 1(8)$, then when checking a four-group $V/Z(L)$ again, we get that $p = 2$. But then $q^n = 2^n \not\equiv \pm 1(8)$. If, however $L/Z(L) \cong PSU(3, 2^{2n})$ and $p \neq 2$, then for a Sylow 2-subgroup A of L, the subgroup $A \cdot Z(L))$ is nonprimary and again we get a contradiction with the definition of NP-group.

Now suppose that $Z(L) \neq 1$. Since the Schur multiplier is trivial for groups $PSL(2, 2^n)$ when $n > 2$, we can assume that in the case of $L/Z(L) \cong PSL(2, q^n)$ the number q is odd. Then the order of the Schur multiplier is equal to 2 (i.e. $L \cong SL(2, q^n)$) for $q^n \neq 9$ and 6 for $q^n = 9$. Consider the second case. If $|Z(L)|$ is divisible by 2 and $Q/Z(L)$ is a Sylow 3-subgroup of the group $L/Z(L)$, then the subgroup Q is nonprimary and $|N(Q) : Q \cdot C(Q)| = 4$, which is impossible. Hence, when $q^n = 9$ the order of $Z(L)$ is equal to 3. In the case of $L/Z(L) \cong PSU(3, 2^{2n})$ the Schur multiplier order is equal to 3, and if $A/Z(L)$ is a Sylow 2-subgroup of $L/Z(L)$, then subgroup A is nonprimary and $|N_L(A) : A \cdot C_L(A)|$ is not a prime.

Therefore, if $G = F^*$ then G is a group of type 3) or 5) from this Theorem. Finally, consider the case when $1 < F < F^* < G$. Then, by using what's already been proved, F^* is a group of type 3) or 4), while G/F is a group of type 2) from this Theorem. Let $G = F^* \cdot \langle a \rangle$, $a^p \in F^*$. If A/F is a Sylow q-subgroup from F^*/F, then the fact that $|N(A) : A \cdot C(A)|$ is divisible by $p \cdot |H/F|$, where H/F is a Cartan subgroup of group F^*/F, implies that subgroup F is a q-group for a prime q. But then, for the nonprimary subgroup H, the index $|N(H) : H \cdot C(H)|$ is divisible by $2p$, which is impossible. $\square$

1.5. Note. It isn't difficult to see that the groups type 2) and 5) of Theorem 1.4 are NP-groups. For type 1) groups, the proof of the sufficiency requires the fulfillment of a number of additional restrictions. Let's note some of them.

Let t be a p'-element from G, A be a t-invariant subgroup from F and $H = F \leftthreetimes \langle t \rangle$. Then the index $|N_H(A \leftthreetimes \langle t \rangle) : (A \leftthreetimes \langle t \rangle) \cdot C_H(A \leftthreetimes \langle t \rangle)|$ divides p. Looking at the intersections of these subgroups with F and taking into account that $N_F(A \leftthreetimes \langle t \rangle) = A \cdot (N_F(A) \cap C(t))$, we get that

$$|A \cdot (N_F(A) \cap C(t)) : A \cdot (C_H(A) \cap C(t))| = |N_F(A) \cap C(t) : (C_H(A) \cap C(t)) \cdot (A \cap C(t))|,$$

i.e., $|C_{N_F(A)}(t) : C_A(t) \cdot C_{C_F(A)}(t)|$ divides p.

Let $N_{G/F}(\langle tF \rangle) = \langle tF \rangle \leftthreetimes \langle hF \rangle$ and A be a $\langle t, h \rangle$-invariant subgroup from F. Since $h \in N(A \leftthreetimes \langle t \rangle)$, then in the same notation $N_H(A \leftthreetimes \langle t \rangle) = (A \leftthreetimes \langle t \rangle) \cdot C_H(t)$. But then $C_{N_F(A)}(t) = C_A(t) \cdot C_{C_F(A)}(t)$. Since the subgroup $N_F(A)$ is also $\langle t, h \rangle$-invariant, then

$$C_{N_F(N_F(A))}(t) = (N_F(A) \cap C(t)) \cdot C_{C_F(N_F(A))}(t) = C_A(t) \cdot C_{C_F(A)}(t).$$

Continuing this process and taking into account that F satisfies the normaliser conditions, we get the equality $C_F(t) = C_A(t) \cdot C_{C_F(A)}(t)$.

Supposing that in this equation $A = [F, a]$ and taking into account that $F = [F, a] \cdot C_F(a)$, we get that $C_F(a) = C_{[F,a]}(a) \cdot C_{C_F([F,a])}(a)$, i.e., $F = [F, a] \cdot C_F([F, a])$.

By analogy we can prove, that if $p \neq 2$ and $(\langle aF \rangle \times \langle bF \rangle) \leftthreetimes \langle cF \rangle$ is a subgroup of order 12 from G/F and subgroup $A \leq F$ is $\langle a, b, c \rangle$-invariant, then $C_F(\langle a, b \rangle) = C_A(\langle a, b \rangle) \cdot C_{C_F(A)}(\langle a, b \rangle)$ and $F = [F, \langle a, b \rangle] \cdot C_F([F, \langle a, b \rangle])$.

Note that all these properties hold if subgroup F is abelian, i.e., in this case G is a NP-group.

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E-Bochner curvature tensor on $N(k)$-contact metric manifolds

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Abstract

The object of the present paper is to study E-Bochner curvature tensor B^e satisfying $R.B^e = 0$, $B^e.R = 0$, $B^e.B^e = 0$ and $B^e.S = 0$ in an n-dimensional $N(k)$-contact metric manifold.

Keywords: $N(k)$-contact metric manifold, Sasakian manifold, extended Bochner curvature tensor.

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1. Introduction

In modern mathematics the study of contact geometry has become a matter of growing interest due to its role in explaining physical phenomena in the context of mathematical physics. An important class of contact manifold is Sasakian manifold introduced by S. Sasaki [17]. Among the geometric properties of manifolds symmetry is an important one. A Riemannian manifold M is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. As a generalization of locally symmetric spaces, many geometers have considered semisymmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be semisymmetric if its curvature tensor R satisfies

$$R(X, Y).R = 0, \qquad X, Y \in T(M),$$

where $R(X, Y)$ acts on R as a derivation. In contact geometry, S. Tanno [18] showed that a semisymmetric K-contact manifold M is locally isometric to the unit sphere $S^n(1)$.

On the other hand, S. Bochner [8] introduced a Kähler analogue of the Weyl conformal curvatur tensor by purely formal considerations, which is now well known as the Bochner curvature tensor. A geometric meaning of the Bochner curvature

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tensor is given by D. E. Blair [6]. By using the Boothby-Wang's fibration [10], M. Matsumoto and G. Chuman [14] constructed C-Bochner curvature tensor from the Bochner curvature tensor. The C-Bochner curvature tensor is given by

$$
\begin{aligned}
B(X,Y)Z \quad = \quad & R(X,Y)Z + \frac{1}{n+3}[S(X,Z)Y - S(Y,Z)X \\
& + g(X,Z)QY - g(Y,Z)QX + S(\phi X,Z)\phi Y \\
& - S(\phi Y,Z)\phi X + g(\phi X,Z)Q\phi Y \\
& - g(\phi Y,Z)Q\phi X + 2S(\phi X,Y)\phi Z \\
& + 2g(\phi X,Y)Q\phi Z - S(X,Z)\eta(Y)\xi \\
& + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY \\
& + \eta(Y)\eta(Z)QX] - \frac{p+n-1}{n+3}[g(\phi X,Z)\phi Y \\
& - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] \\
& - \frac{p-4}{n+3}[g(X,Z)Y - g(Y,Z)X] \\
& + \frac{p}{n+3}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\
& + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X],
\end{aligned}
\tag{1.1}
$$

where S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and $p = \frac{n+r-1}{n+1}$, r being the scalar curvature of the manifold.

In [11], H. Endo extended the concept of C-Bochner curvature tensor to E-Bochner curvature tensor as follows:

$$
\begin{aligned}
B^e(X,Y)Z \quad = \quad & B(X,Y)Z - \eta(X)B(\xi,Y)Z \\
& - \eta(Y)B(X,\xi)Z - \eta(Z)B(X,Y)\xi.
\end{aligned}
\tag{1.2}
$$

Then, he showed that a K-contact manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold. In a similar way Kim, Choi, Özgür and Tripathi [13] extended the contact conformal curvature tensor in a $N(k)$-contact metric manifold. Again, Özgür and Sular studied $N(k)$-contact metric manifold with extended contact conformal curvature tensor in [15]. In [16], C-Bochner semisymmetric $N(k)$-contact metric manifolds are studied. Again, $N(k)$-contact metric manifolds satisfying $B(\xi,X).R = 0$ and $B(\xi,X).B = 0$ are studied in [12]. Beside these, Sasakian manifolds satisfying $B.S = 0$ has been studied in [1]. Motivated by these studies, we consider E-Bochner semisymmetry in a $N(k)$-contact metric manifold which is defined as follows:

1.1. Definition. An n-dimensional $N(k)$-contact metric manifold is said to be E-Bochner semisymmetric if

$$R(X,Y).B^e = 0, \tag{1.3}$$

where B^e is the E-Bochner curvature tensor.

Beside this, we also study $N(k)$-contact metric manifolds satifying $B^e(\xi, X).R = 0$, $B^e(\xi, X).B^e = 0$ and $B^e(\xi, X).S = 0$.

The present paper is organized as follows:
After preliminaries in Section 3, we study E-Bochner semisymmetry in a $N(k)$-contact metric manifold and prove that the manifold is E-Bochner semisymmetric if and only if it is either a Sasakian manifold or it is locally isometric to the product $E^{n+1}(0) \times S^n(4)$. Beside this, in this section we prove that a non-Sasakian $N(k)$-contact metric manifold M^n, $(n \geq 5)$, satisfies $R(\xi, U).B^e = 0$ if and only if it is locally isometric to the product $E^{n+1}(0) \times S^n(4)$. Also, some important corollaries are given here. Section 4 deals with the $N(k)$-contact metric manifold satisfying $B^e(\xi, X).R = 0$ and we prove that a $N(k)$-contact metric manifold M^n, $(n \geq 5)$, with $k \neq 0$, satisfies $B^e(\xi, X).R = 0$ if and only if it is a Sasakian manifold. In the next Section, we prove that a $N(k)$-contact metric manifold M^n, $(n \geq 5)$, satisfies $B^e(\xi, X).B^e = 0$ if and only if it is a Sasakian manifold. Finally, in Section 6, we prove that a $N(k)$-contact metric manifold M^n, $(n \geq 5)$, satisfies $B^e(\xi, X).S = 0$ if and only if it is either a Sasakian manifold or the Ricci tensor S satisfies the relation $S(X, Y) = k(n-1)\eta(X)\eta(Y)$.

2. Preliminaries

An n-dimensional manifold M^n, $(n \geq 5)$, is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([2], [3])

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0 \quad \text{and} \quad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be the compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

Then M^n becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.1), it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

for any vector fields $X, Y \in TM$. An almost contact metric structure becomes a contact metric structure if $g(X, \phi Y) = d\eta(X, Y)$, for all vector fields $X, Y \in TM$.

It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [4]. Again, on a Sasakian manifold [17] we have

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case : D. E. Blair, Th. Koufogiorgos and B. J. Papantoniou [5] introduced the (k,μ)-nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ [5] of a contact metric manifold M is defined by

$$\begin{aligned} N(k,\mu) \quad : \quad & p \longrightarrow N_p(k,\mu) \\ & = \{W \in T_pM : R(X,Y)W = (kI + \mu h)(g(Y,W)X - g(X,W)Y)\}, \end{aligned}$$

for all $X, Y \in TM$, where $(k,\mu) \in \mathbb{R}^2$. A contact metric manifold M^n with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold. If $\mu = 0$, the (k,μ)-nullity distribution reduces to k-nullity distribution [19]. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [19]

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$-contact metric manifold [7]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [3].

Given a non-Sasakian (k,μ)-contact manifold M, E. Boeckx [9] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-k}}$$

and showed that for two non-Sasakian (k,μ)-manifolds M_1 and M_2, we have $I_{M_1} = I_{M_2}$ if and only if up to a D-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds.

Thus we see that from all non-Sasakian (k,μ)-manifolds of dimension $(2n+1)$ and for every possible value of the invariant I, one (k,μ)-manifold M can be obtained with $I_M = 1$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I = \frac{1+c}{|1-c|}$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I < -1$.

Using this invariant, D. E. Blair, J-S. Kim and M. M. Tripathi [7] constructed an example of a $(2n+1)$-dimensional $N(1 - \frac{1}{n})$-contact metric manifold, $n > 1$. The example is given in the following:

Since the Boeckx invariant for a $(1-\frac{1}{n}, 0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting D-homothetic deformation will be a $(1 - \frac{1}{n}, 0)$-manifold. That is, for $k = c(2-c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2}, \qquad 0 = \frac{\mu + 2a - 2}{a}$$

for a and c. The result is

$$c = \frac{\sqrt{n} \pm 1}{n-1}, \qquad a = 1 + c$$

and taking c and a to be these values we obtain $N(1-\frac{1}{n})$-contact metric manifold.

However, for a $N(k)$-contact metric manifold M^n of dimension n, we have ([3], [5])

$$(2.4)\quad (\nabla_X\phi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX),$$

where $h=\frac{1}{2}\pounds_\xi\phi$, $\pounds$ denotes the Lie differentiation.

$$(2.5)\quad R(X,Y)\xi = k[\eta(Y)X-\eta(X)Y],$$

$$(2.6)\quad R(\xi,X)Y = k[g(X,Y)\xi-\eta(Y)X],$$

$$(2.7)\quad R(X,\xi)Y = k[\eta(Y)X-g(X,Y)\xi],$$

$$(2.8)\quad S(X,Y) = (n-3)g(X,Y)+(n-3)g(hX,Y)+[(n-1)k-(n-3)]\eta(X)\eta(Y),$$

$$(2.9)\quad S(\phi X,\phi Y) = S(X,Y)-(n-1)k\eta(X)\eta(Y)-2(n-3)g(hX,Y),$$

$$(2.10)\quad S(Y,\xi) = k(n-1)\eta(X).$$

Beside these, it can be easily verified that in a $N(k)$-contact metric manifold M^n, $n\geq 5$, the E-Bochner curvature tensor satisfies the following conditions:

$$(2.11)\quad B^e(X,Y)\xi = \frac{4(k-1)}{n+3}[\eta(X)Y-\eta(Y)X],$$

$$(2.12)\quad B^e(\xi,X)Y = \frac{4(k-1)}{n+3}[\eta(Y)X-\eta(X)\eta(Y)\xi],$$

$$(2.13)\quad B^e(X,\xi)Y = \frac{4(k-1)}{n+3}[\eta(X)\eta(Y)\xi-\eta(Y)X],$$

$$(2.14)\quad B^e(\xi,X)\xi = \frac{4(k-1)}{n+3}[X-\eta(X)\xi],$$

and

$$(2.15)\quad B^e(\xi,\xi)\xi = 0.$$

3. E-Bochner semisymmetric $N(k)$-contact metric manifolds

In this section we study E-Bochner semisymmetry in an n-dimensional $N(k)$-contact metric manifold. Therefore we have

$$(3.1)\quad (R(X,Y).B^e)(U,V)W = 0.$$

From (3.1) we have

$$(3.2)\quad \begin{aligned} &R(X,Y)B^e(U,V)W - B^e(R(X,Y)U,V)W \\ &\qquad - B^e(U,R(X,Y)V)W - B^e(U,V)R(X,Y)W = 0. \end{aligned}$$

Putting $X=\xi$ in (3.2) and using (2.6), we obtain

$$(3.3)\quad \begin{aligned} &k[g(Y,B^e(U,V)W)\xi - \eta(B^e(U,V)W)Y - g(Y,U)B^e(\xi,V)W \\ &+\eta(U)B^e(Y,V)W - g(Y,V)B^e(U,\xi)W + \eta(V)B^e(U,Y)W \\ &-g(Y,W)B^e(U,V)\xi + \eta(W)B^e(U,V)Y] = 0. \end{aligned}$$

From (3.3) we have either $k = 0$, or

$$\begin{aligned}(3.4)\qquad &[g(Y, B^e(U,V)W)\xi - \eta(B^e(U,V)W)Y - g(Y,U)B^e(\xi,V)W \\ &+\eta(U)B^e(Y,V)W - g(Y,V)B^e(U,\xi)W + \eta(V)B^e(U,Y)W \\ &-g(Y,W)B^e(U,V)\xi + \eta(W)B^e(U,V)Y] = 0.\end{aligned}$$

For $k = 0$, we have $R(X,Y)\xi = 0$ and hence the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$.

Again, replacing U and W by ξ in (3.4) and using (2.11), (2.12), (2.13) and (2.14), we obtain

$$(3.5)\qquad \frac{4(k-1)}{n+3} g(\phi Y, \phi V)\xi = 0.$$

Since $g(\phi Y, \phi V)\xi \neq 0$, in general, therefore we obtain from (3.5), $\frac{4(k-1)}{n+3} = 0$, that is, $k = 1$. Therefore in this case the manifold is a Sasakian manifold.
In view of the above discussions we state the following:

3.1. Proposition. Let M^n, $(n \geq 5)$, be a E-Bochner semisymmetric $N(k)$-contact metric manifold. Then the manifold is either
i) locally isometric to the product $E^{n+1}(0) \times S^n(4)$, or
ii) a Sasakian manifold.

3.1. Corollary. Any E-Bochner semisymmetric $N(k)$-contact metric manifold with $k \neq 0$, is a Sasakian manifold.

Since $\nabla B^e = 0$ implies $R.B^e = 0$, therefore from Proposition 3.1 we obtain the following:

3.2. Corollary. Any E-Bochner symmetric $N(k)$-contact metric manifold is either
i) locally isometric to the product $E^{n+1}(0) \times S^n(4)$, or
ii) a Sasakian manifold.

From Proposition 3.1, it follows that a non-Sasakian $N(k)$-contact metric manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$. Conversely, suppose $k = 0$. Then from (2.6) we have $R(\xi, X)Y = 0$. Hence $R(\xi, X).B^e = 0$. Thus we can state the following:

3.1. Theorem. Let M^n, $(n \geq 5)$, be a non-Sasakian $N(k)$-contact metric manifold. Then the manifold satisfies $R(\xi, X).B^e = 0$ if and only if it is locally isometric to the product $E^{n+1}(0) \times S^n(4)$.

4. $N(k)$-contact metric manifolds with $B^e(\xi,U).R=0$

This section deals with a $N(k)$-contact metric manifold which satisfies

(4.1) $\quad (B^e(\xi,U).R)(X,Y)Z=0.$

From (4.1) we have

$$B^e(\xi,U)R(X,Y)Z-R(B^e(\xi,U)X,Y)Z \\ -R(X,B^e(\xi,U)Y)Z-R(X,Y)B^e(\xi,U)Z=0. \tag{4.2}$$

Using (2.12) in (4.2), we get

$$\frac{4(k-1)}{n+3}[\eta(R(X,Y)Z)U-\eta(U)\eta(R(X,Y)Z)\xi-\eta(X)R(U,Y)Z \\ +\eta(X)\eta(U)R(\xi,Y)Z-\eta(Y)R(X,U)Z+\eta(Y)\eta(U)R(X,\xi)Z \\ -\eta(Z)R(X,Y)U+\eta(U)\eta(Z)R(X,Y)\xi]=0. \tag{4.3}$$

From (4.3) we have either $k=1$, or

$$\eta(R(X,Y)Z)U-\eta(U)\eta(R(X,Y)Z)\xi-\eta(X)R(U,Y)Z \\ +\eta(X)\eta(U)R(\xi,Y)Z-\eta(Y)R(X,U)Z+\eta(Y)\eta(U)R(X,\xi)Z \\ -\eta(Z)R(X,Y)U+\eta(U)\eta(Z)R(X,Y)\xi=0. \tag{4.4}$$

For $k=1$, the manifold is a Sasakian manifold.
Now, putting $X=Z=\xi$ in (4.4) and using (2.5) and (2.6), we obtain

(4.5) $\quad kg(\phi Y,\phi U)\xi=0.$

The relation (4.5) yields $k=0$, since $g(\phi Y,\phi U)\xi\neq 0$, in general. Therefore the manifold is locally isometric to the product $E^{n+1}(0)\times S^n(4)$. In view of the above discussion we state the following:

4.1. Proposition. Let M^n be a $N(k)$-contact metric manifold with $B^e(\xi,U).R=0$. Then the manifold is either
i) locally isometric to the product $E^{n+1}(0)\times S^n(4)$, or
ii) a Sasakian manifold.

From Proposition 4.1 it follows that $N(k)$-contact metric manifolds with $k\neq 0$ satisfying the condition $B^e(\xi,U).R=0$ is Sasakian. Conversely, in a Sasakian manifold we see from (2.12) that $B^e(\xi,X)Y=0$. Thus we can state the following:

4.1. Theorem. Let M^n, $(n\geq 5)$, be a $N(k)$-contact metric manifold with $k\neq 0$. Then the manifold satisfies $B^e(\xi,U).R=0$ if and only if the manifold is Sasakian.

5. $N(k)$-contact metric manifolds satisfying $B^e(\xi,U).B^e=0$

Let M^n, $n\geq 5$, be a $N(k)$-contact metric manifold which satisfies

(5.1) $\quad (B^e(\xi,U).B^e)(X,Y)Z=0.$

From (5.1) we have

$$B^e(\xi,U)B^e(X,Y)Z - B^e(B^e(\xi,U)X,Y)Z \\ -B^e(X,B^e(\xi,U)Y)Z - B^e(X,Y)B^e(\xi,U)Z = 0. \tag{5.2}$$

Using (2.12) the relation (5.2) yields

$$\frac{4(k-1)}{n+3}[\eta(B^e(X,Y)Z)U - \eta(U)\eta(B^e(X,Y)Z)\xi - \eta(X)B^e(U,Y)Z \\ +\eta(X)\eta(U)B^e(\xi,Y)Z - \eta(Y)B^e(X,U)Z + \eta(Y)\eta(U)B^e(X,\xi)Z \\ -\eta(Z)B^e(X,Y)U + \eta(U)\eta(Z)B^e(X,Y)\xi] = 0. \tag{5.3}$$

The equation (5.3) yields either $k = 1$, or

$$\eta(B^e(X,Y)Z)U - \eta(U)\eta(B^e(X,Y)Z)\xi - \eta(X)B^e(U,Y)Z \\ +\eta(X)\eta(U)B^e(\xi,Y)Z - \eta(Y)B^e(X,U)Z + \eta(Y)\eta(U)B^e(X,\xi)Z \\ -\eta(Z)B^e(X,Y)U + \eta(U)\eta(Z)B^e(X,Y)\xi = 0. \tag{5.4}$$

Now, putting $Z = \xi$ in (5.4) and using (2.11), (2.12) and (2.14), we get

$$B^e(X,Y)U = \frac{4(k-1)}{n+3}\eta(U)[\eta(X)Y - \eta(Y)X]. \tag{5.5}$$

For $k = 1$, the manifold is Sasakian. Conversely, in the first case if the manifold is Sasakian then from (2.12) we obtain $B^e(\xi,X)Y = 0$. Hence $B^e(\xi,U).B^e = 0$ is satisfied.

In the second case, it follows from (5.5) that $B^e(X,Y)U = 0$ if and only if the manifold is Sasakian. Hence in this case we also obtain that $B^e(\xi,U).B^e = 0$ holds if and only if the manifold is Sasakian.

Thus we can state the following:

5.1. Theorem. A $N(k)$-contact metric manifold M^n, $(n \geq 5)$, satisfies $B^e(\xi,U).B^e = 0$ if and only if the manifold is Sasakian.

Therefore from (5.5) we can conclude the following:

5.2. Theorem. Let M^n, $n \geq 5$, be a $N(k)$-contact metric manifold satisfying $B^e(\xi,U).B^e = 0$. Then either
i) the manifold is Sasakian, or
ii) the E-Bochner curvature tensor vanishes if and only if the manifold is a Sasakian manifold.

6. $N(k)$-contact metric manifolds satisfying $B^e(\xi,X).S = 0$

We devote this section to study $N(k)$-contact metric manifolds satisfying $B^e(\xi,X).S = 0$. Therefore we have

$$S(B^e(\xi,X)U,V) + S(U,B^e(\xi,X)V) = 0. \tag{6.1}$$

Using (2.12) in (6.1), we get

$$(6.2)\qquad \frac{4(k-1)}{n+3}[\eta(U)S(X,V)-\eta(X)\eta(U)S(\xi,V) \\ +\eta(V)S(U,X)-\eta(X)\eta(V)S(X,\xi)]=0.$$

The relation (6.2) we have either $k=1$, or

$$(6.3)\qquad -\eta(X)\eta(U)S(\xi,V)+\eta(U)S(X,V)-\eta(X)\eta(V)S(X,\xi)+\eta(V)S(U,X)=0.$$

Putting $U=\xi$ and using (2.10) and $\eta(\xi)=1$ in (6.3) yields

$$(6.4)\qquad S(X,V)=(n-1)k\eta(X)\eta(V).$$

Again, if the manifold satisfies the relation (6.4), then in view of (2.12) we have

$$(6.5)\qquad \begin{aligned} B^e(\xi,X).S(U,V) &= -S(B^e(\xi,X)U,V)-S(U,B^e(\xi,X)V) \\ &= -(n-1)k[\eta(B^e(\xi,X)U)\eta(V)+\eta(U)\eta(B^e(\xi,X)V)] \\ &= 0. \end{aligned}$$

Again, if the manifold is Sasakian then we easily obtain from (2.12) that $B^e(\xi,X).S=0$. In view of above discussion we state the following:

6.1. Theorem. Let M^n, $n\geq 5$, is a $N(k)-$contact metric manifold. Then the relation $B^e(\xi,X).S=0$ if and only if the manifold is either Sasakian or the Ricci tensor satisfies the relation $S(X,Y=k(n-1)\eta(X)\eta(Y)$.

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Application of nonhomogenous Cauchy-Euler differential equation for certain class of analytic functions

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Abstract

In this paper, some new subclasses of analytic functions with complex order are introduced by means of a family of nonhomogenous Cauchy-Euler differential equations as well as some differential operators available in literature. The main object of the paper is to determine coefficient bounds for the classes already mentioned, and obtain the results relevant to well-known work.

Keywords: Analytic Functions, Differential operator, Nonhomogenous Cauchy-Euler differential equation, Coefficient bound.

2000 AMS Classification: 30C45

1. Introduction and preliminaries

Let A denote the class of analytic functions f in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Thus each $f \in A$ has a Taylor series representation

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i \cdot \tag{1.1}$$

A function $f \in A$ is said to belong to the class $S^*(\xi)$ if it satisfies

$$\Re\left(1 + \frac{1}{\xi}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0, \quad (z \in \mathbb{U}; \xi \in \mathbb{C} \setminus \{0\}) \cdot \tag{1.2}$$

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In 1936, Roberton [7] proved that if $f(z) = z + \sum_{i=2}^{\infty} a_i z^i$ is in $S^*(1-\beta)$ and $C(1-\beta)$, then

$$|a_i| \leq \frac{\prod_{k=0}^{i-2}[k+2(1-\beta)]}{(i-2)!} \text{ and } |a_i| \leq \frac{\prod_{k=0}^{i-2}[k+2(1-\beta)]}{i!} \ (i \in \mathbb{N}^*; 0 \leq \beta < 1)\cdot$$

In 1983, Nasr and Aouf [8] proved that if $f(z) = z + \sum_{i=2}^{\infty} a_i z^i$ is in $S^*(b)$, then

$$|a_i| \leq \frac{\prod_{k=0}^{i-2}[k+2|b|]}{i!} \ (i \in \mathbb{N}^*; 0 \leq \beta < 1)\cdot$$

A function $f \in A$ is said to be in the class $C^*(\xi_1)$ if it satisfies the following inequality

$$\Re\left(1 + \frac{1}{\xi_1}\frac{zf''(z)}{f'(z)}\right) > 0, \quad (z \in \mathbb{U}; \xi_1 \in \mathbb{C} \setminus \{0\})\cdot \tag{1.3}$$

A function $f \in A$ is said to be in the class $K^*(\lambda, \alpha, \xi_2)$ if it also satisfies the following inequality

$$\Re\left[1 + \frac{1}{\xi_2}\left(\frac{z[\lambda z f'(z) + (1-\lambda)f(z)]'}{\lambda z f'(z) + (1-\lambda)f(z)} - 1\right)\right] > \alpha, 0 \leq \alpha, \lambda \leq 1, z \in \mathbb{U}; \xi_2 \in \mathbb{C} \setminus \{0\}\cdot \tag{1.4}$$

To get more detailed information about the class of function $K^*(\lambda, \alpha, \xi_2)$, we will refer the reader to Altintas et al. (see for example [9]–[16]).

For a function $f \in A$, we define the following differential operator:

$$\begin{aligned}
D^0 f(z) &= f(z), \\
D_\lambda^1(\alpha, \beta, \mu) f(z) &= \left(\frac{\alpha - \mu + \beta - \lambda}{\alpha + \beta}\right) f(z) + \left(\frac{\mu + \lambda}{\alpha + \beta}\right) z f'(z), \\
D_\lambda^2(\alpha, \beta, \mu) f(z) &= D(D_\lambda^1(\alpha, \beta, \mu) f(z)), \\
&\vdots \\
D_\lambda^n(\alpha, \beta, \mu) f(z) &= D(D_\lambda^{n-1}(\alpha, \beta, \mu) f(z))\cdot
\end{aligned} \tag{1.5}$$

If f is given by (1.1) then from (1.5) we have

$$D_\lambda^n(\alpha, \beta, \mu) f(z) = z + \sum_{i=2}^{\infty} \left(\frac{\alpha + (\mu + \lambda)(i-1) + \beta}{\alpha + \beta}\right)^n a_i z^i \tag{1.6}$$

$$(f \in A, \alpha, \beta, \mu, \lambda \geq 0, \alpha + \beta \neq 0, n \in N_o)$$

By specializing the parameters of $D_\lambda^n(\alpha, \beta, \mu) f(z)$ we get the following differential operators. If we substitute

- $\beta = 1$, $\mu = 0$, we get $D_\lambda^n(\alpha, 1, 0) f(z) = D^n f(z) = z + \sum_{i=2}^{\infty} (\frac{\alpha + \lambda(i-1) + 1}{\alpha + 1})^n a_i z^i$ of differential operator given by Aouf, El-Ashwah and El-Deeb [1].
- $\alpha = 1$, $\beta = o$, and $\mu = 0$, we get $D_\lambda^n(1, 0, 0) f(z) = D^n f(z) = z + \sum_{i=2}^{\infty} (1 + \lambda(i-1))^n a_i z^i$ of differential operator given by Al-Oboudi [2].
- $\alpha = 1$, $\beta = o$, $\mu = 0$ and $\lambda = 1$, we get $D_1^n(1, 0, 0) f(z) = D^n f(z) = z + \sum_{i=2}^{\infty} (i)^n a_i z^i$ of Sălăgean's differential operator [3].

- $\alpha = 1$, $\beta = 1$, $\lambda = 1$ and $\mu = 0$, we get $D_1^n(1,1,0)f(z) = D^n f(z) = z + \sum_{i=2}^{\infty}(\frac{i+1}{2})^n a_i z^i$ of differential operator given by Uralegaddi and Somanatha [4].
- $\beta = 1$, $\lambda = 1$ and $\mu = 0$, we get $D_1^n(\alpha,1,0)f(z) = D^n f(z) = z + \sum_{i=2}^{\infty}(\frac{i+\alpha}{\alpha+1})^n a_i z^i$ of differential operator given by Cho and Srivastava, and Cho and Kim [5, 6].

By using the operator $D_\lambda^n(\alpha,\beta,\mu)f(z)$ given by (1.6), we now introduce a new subclass of analytic functions defined as follows:

A function $f \in A$ is said to belong to the class $F(n,\alpha,b)$ if it satisfies

$$\Re\left\{1 + \frac{1}{b}\left(\frac{D_\lambda^{n+1}(\alpha,\beta,\mu)f(z)}{D_\lambda^n(\alpha,\beta,\mu)f(z)} - 1\right)\right\} > \alpha, 0 \le \alpha < 1, b \in C^*\cdot$$

A function $f \in A$ is said to belong to the subclass of analytic functions of order γ in $\mathbb{U}$, denoted by $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$, and is defined by

$$\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi) =$$

$$\left\{f \in A : \Re\left\{1 + \frac{1}{\xi}\left[\frac{z[\zeta D_\lambda^{n+1}(\alpha,\beta,\mu)f(z) + (1-\zeta)D_\lambda^n(\alpha,\beta,\mu)f(z)]'}{\zeta D_\lambda^{n+1}(\alpha,\beta,\mu)f(z) + (1-\zeta)D_\lambda^n(\alpha,\beta,\mu)f(z)} - 1\right]\right\} > \gamma\right\}, \tag{1.7}$$

$$0 \le \gamma, \zeta \le 1, z \in \mathbb{U}; \xi \in \mathbb{C} \setminus \{0\}\cdot$$

Using the class $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$, we obtain the following subclasses studied by various authors.

$\Psi(n,1,0,0,1,\lambda,\alpha,b) = B(n,\lambda,\alpha,b)$,
$\Psi(0,1,0,0,1,0,0,b) = S^*(b)$,
$\Psi(0,1,0,0,1,1,0,b) = C(b)$,
$\Psi(0,1,0,0,1,0,0,1-\beta) = S^*(1-\beta)$,
$\Psi(0,1,0,0,1,1,0,1-\beta) = C(1-\beta)$,
$\Psi(0,1,0,0,1,\lambda,\alpha,\xi_2) = K(\lambda,\alpha,\xi_2)$,
$\Psi(n,1,0,0,1,0,\alpha,b) = F(n,\alpha,b)\cdot$

The main object of the present investigation is to derive some coefficient bounds for functions in the subclass $\Phi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi,\tau)$ of A satisfying the following nonhomogenous Cauchy-Euler differential equation

$$z^2\frac{d^2w}{dz^2} + 2(1+\tau)z\frac{dw}{dz} + \tau(1+\tau)w = (1+\tau)(2+\tau)g(z) \tag{1.8}$$
$$(w = f(z) \in A; g(z) \in \Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi); \tau \in \mathbb{R}\backslash] - \infty, -1])\cdot$$

Also note that

$$\Phi(n,1,0,0,1,\lambda,\alpha,b,\mu) = T(n,\lambda,\alpha,b,\mu),$$
$$\Phi(0,1,0,0,1,\lambda,\alpha,b,\mu) = SK(\lambda,\alpha,b,\mu),$$
$$\Phi(n,1,0,0,1,0,\alpha,b,\mu) = SD(n,\alpha,b,\mu)\cdot$$

To get more detailed information about the above said classes, we will refer the reader to [16] and [17].

2. Coefficient estimates for the function class $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$

Now we give our first result as follows:

2.1. Theorem. *Let the function $f \in A$ be defined by (1.1). If the function f is in the class $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$, then*

$$\left|a_i\right| \le \frac{\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^n}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

Proof. Let the function $f \in A$ be given by (1.1). Define a function

$$H(z) = (\zeta)D_\lambda^{n+1}(\alpha,\beta,\mu)f(z) + (1-\zeta)D_\lambda^{n}(\alpha,\beta,\mu)f(z), \tag{2.1}$$

where $D_\lambda^{n}(\alpha,\beta,\mu)f(z)$ is differential operator be given in (1.6). We note that the function H is of the form

$$H(z) = z + \sum_{i=2}^{\infty} \mathfrak{T}_i z^i, \mathfrak{T}_i \tag{2.2}$$

$$= \Big(\frac{[\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^n}{[\alpha+\beta]^{n+1}}\Big)a_i\cdot$$

Using (1.7) and (2.1), we get

$$\Re\left\{1+\frac{1}{\xi}\left(\frac{zH'(z)}{H(z)}-1\right)\right\} > \gamma,\ (z\in\mathbb{U})\cdot \tag{2.3}$$

Now we define a function $h(z)$ by

$$h(z) = \frac{1+\frac{1}{\xi}\left(\frac{zH'(z)}{H(z)}-1\right)-\gamma}{1-\gamma}\cdot \tag{2.4}$$

We also suppose

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots . \tag{2.5}$$

So we obtain

$$1+\frac{1}{\xi}\left(\frac{zH'(z)}{H(z)}-1\right)-\gamma = (1-\gamma)(1+c_1z+c_2z^2+\cdots), \tag{2.6}$$

or, equivalently,

$$zH'(z) - H(z) = H(z)\xi(1-\gamma)(c_1z+c_2z^2+\cdots)\cdot \tag{2.7}$$

Using (2.7), we conclude that

$$(2-1)\mathfrak{T}_2 = \xi(1-\gamma)c_1,$$

$$(3-1)\mathfrak{T}_3 = \xi(1-\gamma)[c_1\mathfrak{T}_2 + c_2],$$

$$(4-1)\mathfrak{T}_4 = \xi(1-\gamma)[c_1\mathfrak{T}_3 + c_2\mathfrak{T}_2 + c_3],$$

$$\vdots$$

$$(i-1)\mathfrak{T}_i = \xi(1-\gamma)[c_1\mathfrak{T}_{i-1} + c_2\mathfrak{T}_{i-2} + \cdots + c_{i-1}]\cdot \tag{2.8}$$

As $|c_i| \leq 2,\ i = \{1, 2, 3, \cdots\}$, so from (2.8) we have

$$\big|\mathfrak{T}_2\big| = \big|\xi(1-\gamma)c_1\big| \leq 2\big|\xi\big|(1-\gamma), \tag{2.9}$$

$$2\big|\mathfrak{T}_3\big| = \big|\xi(1-\gamma)[c_1\mathfrak{T}_2 + c_2]\big| \leq \big|\xi\big|(1-\gamma)[2\mathfrak{T}_2 + 2]$$

$$\leq 2\big|\xi\big|(1-\gamma)[1 + 2\big|\xi\big|(1-\gamma)]. \tag{2.10}$$

$$3\big|\mathfrak{T}_4\big| = \big|\xi(1-\gamma)[c_1\mathfrak{T}_3 + c_2\mathfrak{T}_2 + c_3]\big|, \tag{2.11}$$

or

$$6\big|\mathfrak{T}_4\big| \leq 2\big|\xi\big|(1-\gamma)[\mathfrak{T}_3 + \mathfrak{T}_2 + 1]\big|$$

$$\leq 2\big|\xi\big|(1-\gamma)[1 + 2\big|\xi\big|(1-\gamma)][2 + 2\big|\xi\big|(1-\gamma)]. \tag{2.12}$$

Using (2.9), (2.10) and (2.12), we get

$$\big|\mathfrak{T}_2\big| \leq \frac{\prod_j [j + 2\big|\xi\big|(1-\gamma)]}{(2-1)!}, j = o,$$

$$\big|\mathfrak{T}_3\big| \leq \frac{\prod_j [j + 2\big|\xi\big|(1-\gamma)]}{(3-1)!}, j = 0, 1,$$

similarly

$$\big|\mathfrak{T}_4\big| \leq \frac{\prod_j [j + 2\big|\xi\big|(1-\gamma)]}{(3-1)!}, j = 0, 1, 2$$

therefore

$$\big|\mathfrak{T}_i\big| \leq \frac{\prod_{j=o}^{j-2} [j + 2\big|\xi\big|(1-\gamma)]}{(i-1)!}, j \in \mathbb{N}^*\cdot$$

By using the relationship between the functions $f(z)$ and $H(z)$, we have

$$\mathfrak{T}_i = \Big(\frac{[\alpha + \zeta(\mu+\lambda)(i-1) + \beta][\alpha + (\mu+\lambda)(i-1) + \beta]^n}{[\alpha+\beta]^{n+1}}\Big)a_i,$$

implies

$$\big|a_i\big| \leq \frac{\prod_{j=o}^{j-2} [j + 2\big|\xi\big|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha + \zeta(\mu+\lambda)(i-1) + \beta][\alpha + (\mu+\lambda)(i-1) + \beta]^n}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

Now, by choosing different values of $\Psi(n, \alpha, \beta, \mu, \lambda, \zeta, \gamma, \xi)$, we have the following corollaries:

2.2. Corollary. *If a function $f \in A$ is in the class $\Psi(n, \alpha, \mu, \lambda, \zeta, \gamma, \xi)$, then*

$$\big|a_i\big| \leq \frac{\prod_{j=o}^{j-2} [j + 2\big|\xi\big|(1-\gamma)][\alpha]^{n+1}}{i![\alpha + \zeta(\mu+\lambda)(i-1)][\alpha + (\mu+\lambda)(i-1)]^n}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

2.3. Corollary. *If a function $f \in A$ is in the class $B(n,\lambda,\alpha,b)$, then*

$$\left|a_i\right| \le \frac{\prod_{j=o}^{j-2}[j+2\left|b\right|(1-\alpha)]}{(i-1)![1+\lambda(i-1)][i]^n}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

2.4. Corollary. *If a function $f \in A$ is in the class $S^*(b)$, then*

$$\left|a_i\right| \le \frac{\prod_{j=o}^{j-2}[j+2\left|b\right|]}{(i-1)!}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

2.5. Corollary. *If a function $f \in A$ is in the class $C(b)$, then*

$$\left|a_i\right| \le \frac{\prod_{j=o}^{j-2}[j+2\left|b\right|]}{i!}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

2.6. Corollary. *If a function $f \in A$ is in the class $S^*(1-\beta)$, then*

$$\left|a_i\right| \le \frac{\prod_{j=o}^{j-2}[j+2(1-\beta)]}{(i-1)!}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

2.7. Corollary. *If a function $f \in A$ is in the class $C(1-\beta)$, then*

$$\left|a_i\right| \le \frac{\prod_{j=o}^{j-2}[j+2(1-\beta)]}{i!}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

2.8. Corollary. *If a function $f \in A$ is in the class $K(\lambda,\alpha,\xi_2)$, then*

$$\left|a_i\right| \le \frac{\prod_{j=o}^{j-2}[j+2(1-\xi_2)(1-\alpha)]}{(i-1)![1+\lambda(i-1)]}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

2.9. Corollary. *If a function $f \in A$ is in the class $F(n,\alpha,b)$, then*

$$\left|a_i\right| \le \frac{\prod_{j=o}^{j-2}[j+2\left|b\right|(1-\alpha)]}{(i-1)![i]^n}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

2.10. Corollary. *If a function $f \in A$ is in the class $B(n,\lambda,b)$, then*

$$\left|a_i\right| \le \frac{\prod_{j=o}^{j-2}[j+2\left|b\right|]}{(i-1)![1+\lambda(i-1)][i]^n}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

3. Coefficient bound for the class $\Phi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi,\tau)$

3.1. Theorem. *Let the function $f \in A$ be defined by (1.1). If the function f is in the class $\Phi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi,\tau)$, then*

$$\left|a_i\right| \le \frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^n(i+\tau)(i+1+\tau)},$$

$j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$

Proof. Let the function $f \in A$ be given by (1.1). Also let

$$f(z) = z + \sum_{i=2}^{\infty} v_i z^i \in \Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi), \text{ implies} \tag{3.1}$$

$$\left|v_i\right| \le \tag{3.2}$$

$$\frac{\prod_{j=o}^{j-2}[j+2|\xi|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^{n}}, j\in\mathbb{N}^*, i\in\mathbb{N}\setminus\{1\}\cdot$$

Since

$$a_i=\frac{(1+\tau)(2+\tau)}{(i+\tau)(i+1+\tau)}v_i\cdot$$

Using (3.2) we get

$$|a_i|\leq\frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2|\xi|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^{n}(i+\tau)(i+1+\tau)},$$

$j\in\mathbb{N}^*, i\in\mathbb{N}\setminus\{1\}\cdot$

Next we have the following corollaries:

3.2. Corollary. *If a function $f\in A$ is in the class $\Phi(n,\alpha,\mu,\lambda,\zeta,\gamma,\xi,\tau)$, then*

$$|a_i|\leq\frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2|\xi|(1-\gamma)][\alpha]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)][\alpha+(\mu+\lambda)(i-1)]^{n}(i+\tau)(i+1+\tau)}, j\in\mathbb{N}^*, i\in\mathbb{N}\setminus\{1\}\cdot$$

3.3. Corollary. *If a function $f\in A$ is in the class $\Phi(n,\alpha,\lambda,\zeta,\gamma,\xi,\tau)$, then*

$$|a_i|\leq\frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2|\xi|(1-\gamma)][\alpha]^{n+1}}{i![\alpha+\zeta\lambda(i-1)][\alpha+\lambda(i-1)]^{n}(i+\tau)(i+1+\tau)}, j\in\mathbb{N}^*, i\in\mathbb{N}\setminus\{1\}\cdot$$

3.4. Corollary. *If a function $f\in A$ is in the class $\Phi(n,\alpha,\zeta,\gamma,\xi,\tau)$, then*

$$|a_i|\leq\frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2|\xi|(1-\gamma)][\alpha]^{n+1}}{i![\alpha+\zeta(i-1)][\alpha+(i-1)]^{n}(i+\tau)(i+1+\tau)}, j\in\mathbb{N}^*, i\in\mathbb{N}\setminus\{1\}\cdot$$

3.5. Corollary. *If a function $f\in A$ is in the class $SK(\lambda,\gamma,\xi,\tau)$, then*

$$|a_i|\leq\frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2|\xi|(1-\gamma)]}{(i-1)!(1-\lambda+\lambda i)(i+\tau)(i+1+\tau)}, j\in\mathbb{N}^*, i\in\mathbb{N}\setminus\{1\}\cdot$$

3.6. Corollary. *If a function $f\in A$ is in the class $SD(n,\gamma,\xi,\tau)$, then*

$$|a_i|\leq\frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2|\xi|(1-\gamma)]}{(i-1)!(i+\tau)(i+1+\tau)}, j\in\mathbb{N}^*, i\in\mathbb{N}\setminus\{1\}\cdot$$

4. Conclusions

There are many different types of operators can be reached in the literature, see for example: [18]- [23], and many more. Some similar results can also be found for different type of classes associated with the many different differential operators.

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Coefficient bounds for certain classes of bi-univalent functions

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Abstract

In this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disk. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

Keywords: Analytic and univalent functions, bi-univalent functions, starlike and convex functions, coefficients bounds.

2000 AMS Classification: 30C45

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike of order α if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some $\alpha(0 \le \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{S}$ is said to be convex of order α if it satisfies

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{1.3}$$

for some $\alpha(0 \le \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are convex of order α in $\mathcal{U}$.

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Gao and Zhou [5] showed some mapping properties of the following subclass of $\mathcal{A}$:

$$\mathcal{R}(\alpha,\beta) = \{f \in \mathcal{A} : \Re((f'(z) + \beta z f''(z)) > \alpha, \quad \beta > 0,\ 0 \le \alpha < 1;\ z \in \mathcal{U}\}.$$

Yang and Liu [12, Theorem 3.1, p.9], proved that the class $\mathcal{R}(\alpha,\beta) \subset \mathcal{S}$ iff $2(1-\alpha)\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \le 1.$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathcal{U})$$

and

$$f^{-1}(f(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \ge \frac{1}{4})$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). Example of functions in the class Σ are

$$\frac{z}{1-z}, \qquad \log\frac{1}{1-z}, \qquad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{U}$ such as

$$\frac{2z-z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ.

Lewin [6] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [1] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [7], on the other hand, showed that $\max_{f\in\Sigma} |a_2| = 4/3$.

The coefficient estimate problem for each of the Taylor–Maclaurin coefficients $|a_n|$ $(n \ge 3; n \in \mathbb{N})$ is presumably still an open problem.

Brannan and Taha [2] (see also [10]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$(see [3]). Thus, following Brannan and Taha [2] (see also [10]), a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma[\alpha]$ of strongly bi-starlike functions of order $\alpha(0 < \alpha \le 1)$ if each of the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \le 1,\ z \in \mathcal{U})$$

and

$$\left|\arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \le 1,\ w \in \mathcal{U}),$$

where g is the extension of f^{-1} to $\mathcal{U}$. The classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes defined by (1.2) and (1.3), were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [7,8]).

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Srivastava *et al.* [9] (see also, [4] and [11]).

In order to derive our main results, we have to recall here the following lemma [8].

1.1. Lemma. *If $h \in \mathcal{P}$ then $|c_k| \leq 2$ for each k,*

where $\mathcal{P}$ is the family of all functions h analytic in $\mathcal{U}$ for which $\Re h(z) > 0$
$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ for $z \in \mathcal{U}$.

2. Coefficient bounds for the function class $\mathcal{H}_\Sigma(\alpha, \beta)$

2.1. Definition. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma(\alpha, \beta)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } |\arg(f'(z) + \beta z f''(z))| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{U}) \tag{2.1}$$

and

$$|\arg(g'(w) + \beta w g''(w))| < \frac{\alpha\pi}{2} \qquad (w \in \mathcal{U}), \tag{2.2}$$

where $\beta > 0, 0 < \alpha < 1,\ 2(1-\alpha)\sum\limits_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1,$ *and the function g is given by*

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2.3}$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{H}_\Sigma(\alpha, \beta)$.

2.2. Theorem. *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\alpha, \beta)$ where $\beta > 0, 0 < \alpha < 1,$ and $2(1-\alpha)\sum\limits_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha+2) + 4\beta(\alpha+\beta+2-\alpha\beta)}} \tag{2.4}$$

and

$$|a_3| \leq \frac{\alpha^2}{(1+\beta)^2} + \frac{2\alpha}{3(1+2\beta)}. \tag{2.5}$$

Proof. It follows from (2.1) and (2.2) that

$$f'(z) + \beta z f''(z) = [p(z)]^\alpha \tag{2.6}$$

and

$$g'(w) + \beta w g''(w) = [q(w)]^\alpha \tag{2.7}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.8}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$2(1+\beta)a_2 = \alpha p_1, \tag{2.10}$$

$$3(1+2\beta)a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.11}$$

$$-2(1+\beta)a_2 = \alpha q_1 \tag{2.12}$$

and

$$3(1+2\beta)(2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get

$$p_1 = -q_1 \tag{2.14}$$

and

$$8(1+\beta)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.15}$$

Now from (2.11), (2.13) and (2.15), we obtain

$$\begin{aligned} 6(1+2\beta)a_2^2 &= \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2) \\ &= \alpha(p_2+q_2) + \frac{4(\alpha-1)(1+\beta)^2}{\alpha}a_2^2. \end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{2(\alpha+2)+4\beta(\alpha+\beta+2-\alpha\beta)}.$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \le \frac{2\alpha}{\sqrt{2(\alpha+2)+4\beta(\alpha+\beta+2-\alpha\beta)}}.$$

This gives the bound on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11), we get

$$6(1+2\beta)a_3 - 6(1+2\beta)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2 - \left(\alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2\right). \tag{2.16}$$

Upon substituting the value of a_2^2 from (2.15) and observing that $p_1^2 = q_1^2$, it follows that

$$a_3 = \frac{\alpha^2 p_1^2}{4(1+\beta)^2} + \frac{\alpha(p_2 - q_2)}{6(1+2\beta)}.$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{\alpha^2}{(1+\beta)^2} + \frac{2\alpha}{3(1+2\beta)}.$$

This completes the proof of Theorem 2.2. $\square$

Putting $\beta = 1$ in Theorem 2.2, we have

2.3. Corollary. *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\alpha, 1)$ where $0 < \alpha < 1$, and $2(1-\alpha)\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m+1} \leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha+2)+12}} \tag{2.17}$$

and

$$|a_3| \leq \frac{9\alpha^2 + 8\alpha}{36}. \tag{2.18}$$

3. Coefficient bounds for the function class $\mathcal{H}_\Sigma(\gamma, \beta)$

3.1. Definition. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma(\gamma, \beta)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \Re\left(f'(z) + \beta z f''(z)\right) > \gamma \qquad (z \in \mathcal{U}) \tag{3.1}$$

and

$$\Re\left(g'(w) + \beta w g''(w)\right) > \gamma \qquad (w \in \mathcal{U}), \tag{3.2}$$

where $\beta > 0, 0 \leq \gamma < 1$, $2(1-\gamma)\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$, and the function g is given by (2.3).

3.2. Theorem. *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\gamma, \beta)$, where $\beta > 0, 0 \leq \gamma < 1$, and $2(1-\gamma)\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{3(1+2\beta)}} \tag{3.3}$$

and

$$|a_3| \leq \frac{(1-\gamma)^2}{(1+\beta)^2} + \frac{2(1-\gamma)}{3(1+2\beta)}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exist p and $q \in \mathcal{P}$ such that

$$f'(z) + \beta z f''(z) = \gamma + (1-\gamma)p(z) \tag{3.5}$$

and

$$g'(w) + \beta w g''(w) = \gamma + (1-\gamma)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$2(1+\beta)a_2 = (1-\gamma)p_1, \tag{3.7}$$

$$3(1+2\beta)a_3 = (1-\gamma)p_2, \tag{3.8}$$

$$-2(1+\beta)a_2 = (1-\gamma)q_1 \tag{3.9}$$

and

$$3(1+2\beta)(2a_2^2 - a_3) = (1-\gamma)q_2 \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$8(1+\beta)^2 a_2^2 = (1-\gamma)^2(p_1^2 + q_1^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we find that

$$6(1+2\beta)a_2^2 = (1-\gamma)(p_2 + q_2).$$

Thus, we have

$$\left|a_2^2\right| \leq \frac{(1-\gamma)}{6(1+2\beta)}(|p_2| + |q_2|) = \frac{2(1-\gamma)}{3(1+2\beta)}$$

which is the bound on $\left|a_2^2\right|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$6(1+2\beta)a_3 - 6(1+2\beta)a_2^2 = (1-\gamma)(p_2 - q_2)$$

or, equivalently,

$$a_3 = a_2^2 + \frac{(1-\gamma)(p_2 - q_2)}{6(1+2\beta)}.$$

Upon substituting the value of a_2^2 from (3.12), we obtain

$$a_3 = \frac{(1-\gamma)^2(p_1^2 + q_1^2)}{8(1+\beta)^2} + \frac{(1-\gamma)(p_2 - q_2)}{6(1+2\beta)}.$$

Applying Lemma 1.1 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{(1-\gamma)^2}{(1+\beta)^2} + \frac{2(1-\gamma)}{3(1+2\beta)}$$

which is the bound on $|a_3|$ as asserted in (3.4). $\square$

Putting $\beta = 1$ in Theorem 3.2, we have

3.3. Corollary. *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\gamma, 1)$, where $0 \leq \gamma < 1$, and $2(1-\gamma)\sum\limits_{m=1}^{\infty} \frac{(-1)^{m-1}}{m+1} \leq 1$.*

$$|a_2| \leq \frac{1}{3}\sqrt{2(1-\gamma)} \tag{3.13}$$

and

$$|a_3| \leq \frac{(1-\gamma)(9(1-\gamma)+8)}{36}. \tag{3.14}$$

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Some results on a cross-section in the tensor bundle

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Abstract

The present paper is devoted to some results concerning with the complete lifts of an almost complex structure and a connection in a manifold to its $(0, q)$-tensor bundle along the corresponding cross-section.

Keywords: Almost complex structure, Almost analytic tensor, Complete lift, Connection, Tensor bundle.

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1. Introduction

The behaviour of the lifts of tensor fields and connections on a manifold to its different bundles along the corresponding cross-sections are studied by several authors. For the case tangent and cotangent bundles, see [13, 14, 15] and also tangent bundles of order 2 and order r, see [3, 11]. In [2], the first author and his collaborator studied the complete lift of an almost complex structure in a manifold on the so-called pure cross-section of its (p, q)-tensor bundle by means of the Tachibana operator (for diagonal lift to the (p, q)-tensor bundle see [1] and for the $(0, q)$-tensor bundle see [5]). Moreover they proved that if a manifold admits an almost complex structure, then so does on the pure cross-section of its (p, q)-tensor bundle provided that the almost complex structure is integrable. In [6], the authors give detailed description of geodesics of the (p, q)- tensor bundle with respect to the complete lift of an affine connection.

The purpose of the present paper is two-fold. Firstly, to show the complete lift of an almost complex structure in a manifold to its $(0, q)$-tensor bundle along the corresponding cross-section, when restricted to the cross-section determined by an almost analytic tensor field, is an almost complex structure. Finally, to study the behavior of the complete lift of a connection on the cross-section of the $(0, q)$-tensor bundle.

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Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^{∞}. Also, we denote by $\Im_q^p(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im_q^p(T_q^0(M))$ the corresponding set on the $(0,q)$ -tensor bundle $T_q^0(M)$. The Einstein summation convention is used, the range of the indices i,j,s being always $\{1,2,...,n\}$.

2. Preliminaries

Let M be a differentiable manifold of class C^{∞} and finite dimension n. Then the set $T_q^0(M) = \cup_{P\in M} T_q^0(P)$, $q>0$, is the tensor bundle of type $(0,q)$ over M, where $\cup$ denotes the disjoint union of the tensor spaces $T_q^0(P)$ for all $P \in M$. For any point $\tilde{P}$ of $T_q^0(M)$ such that $\tilde{P} \in T_q^0(M)$, the surjective correspondence $\tilde{P} \to P$ determines the natural projection $\pi : T_q^0(M) \to M$. The projection π defines the natural differentiable manifold structure of $T_q^0(M)$, that is, $T_q^0(M)$ is a C^{∞}-manifold of dimension $n+n^q$. If x^j are local coordinates in a neighborhood U of $P \in M$, then a tensor t at P which is an element of $T_q^0(M)$ is expressible in the form $(x^j, t_{j_1 \dots j_q})$, where $t_{j_1 \dots j_q}$ are components of t with respect to natural base. We may consider $(x^j, t_{j_1 \dots j_q}) = (x^j, x^{\overline{j}}) = x^J$, $j=1,...,n$, $\overline{j} = n+1,...,n+n^q$, $J = 1,...,n+n^{p+q}$ as local coordinates in a neighborhood $\pi^{-1}(U)$.

Let $V = V^i \frac{\partial}{\partial x^i}$ and $A = A_{j_1 \dots j_q} dx^{j_1} \otimes \dots \otimes dx^{j_q}$ be the local expressions in U of a vector field V and a $(0,q)-$tensor field A on M, respectively. Then the vertical lift ${}^{V}A$ of A and the complete lift ${}^{C}V$ of V are given, with respect to the induced coordinates, by

$$(2.1) \qquad {}^{V}A = \begin{pmatrix} 0 \\ A_{j_1 \dots j_q} \end{pmatrix}$$

and

$$(2.2) \qquad {}^{C}V = \begin{pmatrix} V^j \\ -\sum\limits_{\lambda=1}^{q} t_{j_1 \dots m \dots j_q} \partial_{j_\lambda} V^m \end{pmatrix}.$$

Suppose that there is given a tensor field $\xi \in \Im_q^0(M)$. Then the correspondence $x \mapsto \xi_x$, ξ_x being the value of ξ at $x \in M$, determines a mapping $\sigma_\xi : M \mapsto T_q^0(M)$, such that $\pi \circ \sigma_\xi = id_M$, and the n dimensional submanifold $\sigma_\xi(M)$ of $T_q^0(M)$ is called the cross-section determined by ξ. If the tensor field ξ has the local components $\xi_{k_1 \cdots k_q}(x^k)$, the cross-section $\sigma_\xi(M)$ is locally expressed by

$$(2.3) \qquad \begin{cases} x^k = x^k, \\ x^{\overline{k}} = \xi_{k_1 \cdots k_q}(x^k) \end{cases}$$

with respect to the coordinates $(x^k, x^{\overline{k}})$ in $T_q^0(M)$. Differentiating (2.3) by x^j, we see that n tangent vector fields B_j to $\sigma_\xi(M)$ have components

$$(2.4) \qquad (B_j^K) = (\frac{\partial x^K}{\partial x^j}) = \begin{pmatrix} \delta_j^k \\ \partial_j \xi_{k_1 \cdots k_q} \end{pmatrix}$$

with respect to the natural frame $\left\{\partial_k, \partial_{\overline{k}}\right\}$ in $T_q^0(M)$.

On the other hand, the fibre is locally expressed by

$$\begin{cases} x^k = const., \\ t_{k_1\cdots k_q} = t_{k_1\cdots k_q}, \end{cases}$$

$t_{k_1\cdots k_q}$ being considered as parameters. Thus, on differentiating with respect to $x^{\overline{j}} = t_{j_1\cdots j_q}$, we see that n^q tangent vector fields $C_{\overline{j}}$ to the fibre have components

$$(2.5)\qquad (C_{\overline{j}}^K) = (\frac{\partial x^K}{\partial x^{\overline{j}}}) = \begin{pmatrix} 0 \\ \delta_{k_1}^{j_1}\cdots\delta_{k_q}^{j_q} \end{pmatrix}$$

with respect to the natural frame $\{\partial_k, \partial_{\overline{k}}\}$ in $T_q^0(M)$.

We consider in $\pi^{-1}(U) \subset T_q^0(M)$, $n + n^q$ local vector fields B_j and $C_{\overline{j}}$ along $\sigma_\xi(M)$. They form a local family of frames $\left[B_j, C_{\overline{j}}\right]$ along $\sigma_\xi(M)$, which is called the adapted $(B, C)-$frame of $\sigma_\xi(M)$ in $\pi^{-1}(U)$. Taking account of (2.2) on the cross-section $\sigma_\xi(M)$, and also (2.4) and (2.5), we can easily prove that, the complete lift ${}^CV$ has along $\sigma_\xi(M)$ components of the form

$$(2.6)\qquad {}^CV = \begin{pmatrix} V^j \\ -L_V\xi_{j_1\cdots j_q} \end{pmatrix}$$

with respect to the adapted (B, C)-frame. From (2.1), (2.4) and (2.5), the vertical lift ${}^VA$ also has components of the form

$$(2.7)\qquad {}^VA = \begin{pmatrix} 0 \\ A_{j_1\ldots j_q} \end{pmatrix}$$

with respect to the adapted (B, C)- frame.

3. Almost complex structures on a pure cross-section in the $(0, q)$-tensor bundle

A tensor field $\xi \in \Im_q^0(M)$ is called pure with respect to $\varphi \in \Im_1^1(M)$, if [2, 4, 5, 7, 8, 9, 10, 12]:

$$(3.1)\qquad \varphi_{j_1}^r\xi_{r\cdots j_q} = \cdots = \varphi_{j_q}^r\xi_{j_1\cdots r} = \overset{*}{\xi}_{j_1\cdots j_q}.$$

In particular, vector and covector fields will be considered to be pure.

Let $\overset{*}{\Im}{}_q^0(M)$ denotes a module of all the tensor fields $\xi \in \Im_q^0(M)$ which are pure with respect to φ. Now, we consider a pure cross-section $\sigma_\xi^\varphi(M)$ determined by $\xi \in \overset{*}{\Im}{}_q^0(M)$. The complete lift ${}^C\varphi$ of φ along the pure cross-section $\sigma_\xi^\varphi(M)$ to $T_q^0(M)$ has local components of the form

$${}^C\varphi = \begin{pmatrix} \varphi_l^k & 0 \\ -(\Phi_\varphi\xi)_{lk_1\ldots k_q} & \varphi_{k_1}^{r_1}\delta_{k_2}^{r_2}\ldots\delta_{k_q}^{r_q} \end{pmatrix}$$

with respect to the adapted $(B, C)-$frame of $\sigma_\xi^\varphi(M)$, where $(\Phi_\varphi\xi)_{lk_1\cdots k_q} = \varphi_l^m\partial_m\xi_{k_1\cdots k_q} - \partial_l\overset{*}{\xi}_{k_1\cdots k_q} + \sum_{a=1}^{q}(\partial_{k_a}\varphi_l^m)\xi_{k_1\cdots m\cdots k_q}$ is the Tachibana operator.

We consider that the local vector fields

$$ {}^{C}X_{(i)} = {}^{C}(\frac{\partial}{\partial x^{i}}) = {}^{C}(\delta_{i}^{h}\frac{\partial}{\partial x^{h}}) = \begin{pmatrix} \delta_{i}^{h} \\ 0 \end{pmatrix} $$

and

$$ {}^{V}X^{(\bar{i})} = {}^{V}(dx^{i_1}\otimes\cdots\otimes dx^{i_q}) = {}^{V}(\delta_{h_1}^{i_1}\cdots\delta_{h_q}^{i_q}dx^{h_1}\otimes\cdots\otimes dx^{h_q}) = \begin{pmatrix} 0 \\ \delta_{h_1}^{i_1}\cdots\delta_{h_q}^{i_q} \end{pmatrix} $$

$i = 1, ..., n, \bar{i} = n+1, ..., n+n^{q}$ span the module of vector fields in $\pi^{-1}(U)$. Hence, any tensor fields is determined in $\pi^{-1}(U)$ by their actions on ${}^{C}V$ and ${}^{V}A$ for any $V \in \Im_{0}^{1}(M)$ and $A \in \Im_{q}^{0}(M)$. The complete lift ${}^{C}\varphi$ along the pure cross-section $\sigma_{\xi}^{\varphi}(M)$ has the properties

$$ (3.2) \qquad \begin{cases} {}^{C}\varphi({}^{C}V) = {}^{C}(\varphi(V)) + {}^{V}((L_V\varphi)\circ\xi), \forall V \in \Im_{0}^{1}(M), (i) \\ {}^{C}\varphi({}^{V}A) = {}^{V}(\varphi(A)), \forall A \in \Im_{q}^{0}(M), \quad (ii) \end{cases} $$

which characterize ${}^{C}\varphi$, where $\varphi(A) \in \Im_{q}^{0}(M)$. Remark that ${}^{V}((L_V\varphi)\circ\xi)$ is a vector field on $T_{q}^{0}(M)$ and locally expressed by

$$ {}^{V}((L_V\varphi)\circ\xi) = \begin{pmatrix} 0 \\ (L_V\varphi)_{i_1}^{j}\xi_{j i_2\cdots i_q} \end{pmatrix} $$

with respect to the adapted (B, C)-frame, where $\xi_{i_1\cdots i_q}$ are local components of ξ in M [5].

3.1. Theorem. *Let M be an almost complex manifold with an almost complex structure φ. Then, the complete lift ${}^{C}\varphi \in \Im_{1}^{1}(T_{q}^{0}(M))$, when restricted to the pure cross-section determined by an almost analytic tensor ξ on M, is an almost complex structure.*

Proof. If $V \in \Im_{0}^{1}(M)$ and $A \in \Im_{q}^{0}(M)$, in view of the equations (i) and (ii) of (3.2), we have

$$ (3.3) \qquad ({}^{C}\varphi)^{2}({}^{C}V) = {}^{C}(\varphi^{2})({}^{C}V) + {}^{V}(N_{\varphi}\circ\xi)({}^{C}V) $$

and

$$ (3.4) \qquad ({}^{C}\varphi)^{2}({}^{V}A) = {}^{C}(\varphi^{2})({}^{V}A), $$

where $N_{\varphi,X}(Y) = (L_{\varphi X}\varphi - \varphi(L_X\varphi))(Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + \varphi^{2}[X, Y] = N_{\varphi}(X, Y)$ is nothing but the Nijenhuis tensor constructed by φ.

Let $\varphi \in \Im_{1}^{1}(M)$ be an almost complex structure and $\xi \in \Im_{q}^{0}(M)$ be a pure tensor with respect to φ. If $(\Phi_{\varphi}\xi) = 0$, the pure tensor ξ is called an almost analytic $(0, q)-$tensor. In [4, 7, 9], it is proved that $\xi\circ\varphi \in \Im_{q}^{0}(M)$ is an almost analytic tensor if and only if $\xi \in \Im_{q}^{0}(M)$ is an almost analytic tensor. Moreover if $\xi \in \Im_{q}^{0}(M)$ is an almost analytic tensor, then $N_{\varphi}\circ\xi = 0$. When restricted to the pure cross-section determined by an almost analytic tensor ξ on M, from (3.3), (3.4) and linearity of the complete lift, we have

$$ ({}^{C}\varphi)^{2} = {}^{C}(\varphi^{2}) = {}^{C}(-I_M) = -I_{T_{q}^{0}(M)}. $$

This completes the proof. $\square$

4. Complete lift of a symmetric affine connection on a cross-section in the $(0,q)$-tensor bundle

We now assume that ∇ is an affine connection (with zero torsion) on M. Let Γ_{ij}^h be components of ∇. The complete lift ${}^C\nabla$ of ∇ to $T_q^0(M)$ has components ${}^C\Gamma_{MS}^I$ such that

$$
\begin{aligned}
(4.1)\quad {}^C\Gamma_{ms}^i &= \Gamma_{ms}^i,\ {}^C\Gamma_{\overline{m}s}^i ={}^C\Gamma_{m\overline{s}}^i ={}^C\Gamma_{\overline{ms}}^i ={}^C\Gamma_{\overline{ms}}^{\overline{i}} = 0,\\
{}^C\Gamma_{m\overline{s}}^{\overline{i}} &= -\sum_{c=1}^{q}\Gamma_{mi_c}^{s_c}\delta_{i_1}^{s_1}...\delta_{i_{c-1}}^{s_{c-1}}\delta_{i_{c+1}}^{s_{c+1}}...\delta_{i_q}^{s_q},\\
{}^C\Gamma_{\overline{m}s}^{\overline{i}} &= -\sum_{c=1}^{q}\Gamma_{si_c}^{m_c}\delta_{i_1}^{m_1}...\delta_{i_{c-1}}^{m_{c-1}}\delta_{i_{c+1}}^{m_{c+1}}...\delta_{i_q}^{m_q},\\
{}^C\Gamma_{ms}^{\overline{i}} &= \sum_{c=1}^{q}(-\partial_m\Gamma_{si_c}^a+\Gamma_{mi_c}^r\Gamma_{sr}^a+\Gamma_{ms}^r\Gamma_{ri_c}^a)t_{i_1...i_{c-1}ai_{c+1}...i_q}\\
&\quad+\frac{1}{2}\sum_{b=1}^{q}\sum_{c=1}^{q}(\Gamma_{mi_c}^l\Gamma_{si_b}^r+\Gamma_{mi_b}^l\Gamma_{si_c}^r)t_{i_1...i_{b-1}ri_{b+1}...i_{c-1}li_{c+1}...i_q}\\
&\quad+\sum_{d=1}^{q}t_{i_1...l...i_q}R_{i_dkm}{}^l
\end{aligned}
$$

with respect to the natural frame in $T_q^0(M)$, where δ_j^i-Kronecker delta and $R_{ikm}{}^l$ is components of the curvature tensor R of ∇ [6].

We now study the affine connection induced from ${}^C\nabla$ on the cross-section $\sigma_\xi(M)$ determined by the $(0,q)-$tensor field ξ in M with respect to the adapted (B,C)-frame of $\sigma_\xi(M)$. The vector fields $C_{\overline{j}}$ given by (2.5) are linearly independent and not tangent to $\sigma_\xi(M)$. We take the vector fields $C_{\overline{j}}$ as normals to the cross-section $\sigma_\xi(M)$ and define an affine connection $\widetilde{\nabla}$ induced on the cross-section. The affine connection $\widetilde{\nabla}$ induced $\sigma_\xi(M)$ from the complete lift ${}^C\nabla$ of a symmetric affine connection ∇ in M has components of the form

$$(4.2)\qquad \widetilde{\Gamma}_{ji}^h=(\partial_jB_i{}^A+{}^C\Gamma_{CB}^AB_j{}^CB_i{}^B)B^h{}_A,$$

where $B^h{}_A$ are defined by

$$(B^h{}_A,C^h{}_A)=(B_i{}^A,C_i{}^A)^{-1}$$

and thus

$$(4.3)\qquad B^h{}_A=(\delta_i^h,0),\quad C^h{}_A=(-\partial_j\xi_{k_1...k_q},\delta_{k_1}^{j_1}...\delta_{k_q}^{j_q}).$$

Substituting (4.1), (2.4), (2.5) and (4.3) in (4.2), we get

$$\widetilde{\Gamma}_{ji}^h=\Gamma_{ji}^h,$$

where Γ_{ji}^h are components of ∇ in M.

From (4.2), we see that the quantity

$$(4.4)\qquad \partial_jB_i{}^A+{}^C\Gamma_{CB}^AB_j{}^CB_i{}^B-\Gamma_{ji}^hB_h{}^A$$

is a linear combination of the vectors $C_{\bar{i}}{}^{A}$. To find the coefficients, we put $A = \bar{h}$ in (4.4) and find

$$\nabla_j \nabla_i \xi_{h_1 \ldots h_q} + \sum_{\lambda=1}^{q} \xi_{h_1 \ldots l \ldots h_q} R_{h_\lambda ij}{}^{l}.$$

Hence, representing (4.4) by $\widetilde{\nabla}_j B_i{}^{A}$, we obtain

$$\widetilde{\nabla}_j B_i{}^{A} = (\nabla_j \nabla_i \xi_{h_1 \ldots h_q} + \sum_{\lambda=1}^{q} \xi_{h_1 \ldots l \ldots h_q} R_{h_\lambda ij}{}^{l}) C_{\bar{h}}{}^{A}. \tag{4.5}$$

The last equation is nothing but the equation of Gauss for the cross-section $\sigma_\xi(M)$ determined by $\xi_{h_1 \ldots h_q}$. Hence, we have the following proposition.

4.1. Proposition. *The cross-section $\sigma_\xi(M)$ in $T_q^0(M)$ determined by a $(0,q)$ tensor ξ in M with symmetric affine connection ∇ is totally geodesic if and only if ξ satisfies*

$$\nabla_j \nabla_i \xi_{h_1 \ldots h_q} + \sum_{\lambda=1}^{q} \xi_{h_1 \ldots l \ldots h_q} R_{h_\lambda ij}{}^{l} = 0.$$

Now, let us apply the operator $\widetilde{\nabla}_k$ to (4.5), we have

$$\widetilde{\nabla}_k \widetilde{\nabla}_j B_i{}^{A} = \nabla_k (\nabla_j \nabla_i \xi_{h_1 \ldots h_q} + \sum_{\lambda=1}^{q} \xi_{h_1 \ldots l \ldots h_q} R_{h_\lambda ij}{}^{l}) C_{\bar{h}}{}^{A}. \tag{4.6}$$

Recalling that

$$\widetilde{\nabla}_k \widetilde{\nabla}_j B_i{}^{A} - \widetilde{\nabla}_j \widetilde{\nabla}_k B_i{}^{A} = \widetilde{R}_{DCB}{}^{A} B_k{}^{D} B_j{}^{C} B_i{}^{B} - R_{kji}{}^{h} B_h{}^{A},$$

and using the Ricci identity for a tensor field of type $(0,q)$, from (4.6) we get

$$\begin{aligned} & \widetilde{R}_{DCB}{}^{A} B_k{}^{D} B_j{}^{C} B_i{}^{B} - R_{kji}{}^{h} B_h{}^{A} \\ = \ & [\sum_{\lambda=1}^{q} (\nabla_k R_{h_\lambda ij}{}^{l} - \nabla_j R_{h_\lambda ik}{}^{l}) \xi_{h_1 \ldots l \ldots h_q} - R_{kji}{}^{l} \nabla_l \xi_{h_1 \ldots h_q} \\ & - \sum_{\lambda=1}^{q} R_{kjh_\lambda}{}^{l} \nabla_i \xi_{h_1 \ldots l \ldots h_q} + \sum_{\lambda=1}^{q} R_{h_\lambda ij}{}^{l} \nabla_k \xi_{h_1 \ldots l \ldots h_q} - \sum_{\lambda=1}^{q} R_{h_\lambda ik}{}^{l} \nabla_j \xi_{h_1 \ldots l \ldots h_q}] C_{\bar{h}}{}^{A}. \end{aligned}$$

Thus we have the result below.

4.2. Proposition. $\widetilde{R}_{DCB}{}^{A} B_k{}^{D} B_j{}^{C} B_i{}^{B}$ *is tangent to the cross-section $\sigma_\xi(M)$ if and only if*

$$\begin{aligned} & \sum_{\lambda=1}^{q} (\nabla_k R_{h_\lambda ij}{}^{l} - \nabla_j R_{h_\lambda ik}{}^{l}) \xi_{h_1 \ldots l \ldots h_q} \\ = \ & R_{kji}{}^{l} \nabla_l \xi_{h_1 \ldots h_q} + \sum_{\lambda=1}^{q} R_{kjh_\lambda}{}^{l} \nabla_i \xi_{h_1 \ldots l \ldots h_q} - \sum_{\lambda=1}^{q} R_{h_\lambda ij}{}^{l} \nabla_k \xi_{h_1 \ldots l \ldots h_q} \\ & + \sum_{\lambda=1}^{q} R_{h_\lambda ik}{}^{l} \nabla_j \xi_{h_1 \ldots l \ldots h_q}. \end{aligned}$$

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On convergence of an implicit iterative algorithm for non self asymptotically non expansive mappings

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Abstract

In this paper, we study an implicit iterative algorithm for two finite families of nonself asymptotically nonexpansive mappings. We prove some weak and strong convergence theorems for this iterative algorithm. Our results improve and extend the corresponding results of Soltuz [4], Xu and Ori [5], Khan et al. [6].

Keywords: Implicit iteration process, Nonself asymptotically nonexpansive mappings, Common fixed point, Strong convergence, Weak convergence.

2000 AMS Classification: 47H09, 47H10

1. Introduction

Let E be a real normed linear space and K be a nonempty subset of E. A mapping $T : K \to K$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ holds for all $x, y \in K$. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \ge 1$. A mapping $T : K \to K$ is called uniformly L-Lipschitzian if there exists constant $L > 0$ such that $\|T^n x - T^n y\| \le L \|x - y\|$ for all $x, y \in K$ and $n \ge 1$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] as a generalization of the class of nonexpansive mappings. They proved that if K is a nonempty closed bounded subset of a real uniformly convex

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Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Implicit iterative algorithms for asymptotically nonexpansive self-mapping in Banach spaces have been studied extensively by various authors; see for example [12, 13, 14, 15]. However, if the domain of T , $D(T)$, is a proper subset of E (and this is the case in several applications), and T maps $D(T)$ into E, then these iterative algorithms may fail to be well defined.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to K$ is said to be a retraction if $P^2 = P$. It follows that, if a map P is a retraction, then $Py = y$ for all y in the range of P.

In 2003, Chidume, Ofoedu and Zegeye [2] further generalized the concept of asymptotically nonexpansive self-mapping, and proposed the concept of nonself asymptotically nonexpansive mapping, which is defined as follows:

1.1. Definition. [2] Let K be a nonempty subset of a real normed space E and $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ *is called asymptotically nonexpansive if there exists a sequence* $\{k_n\} \subset [1, \infty)$ *with* $\lim_{n\to\infty} k_n = 1$ *such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \tag{1.2}$$

for all $x, y \in K$ *and* $n \geq 1$.

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$$

for all $x, y \in K$ *and* $n \geq 1$.

Note that if P is an identity mapping, then the above definitions reduce to those of a self-mapping T . By using the following iterative algorithm:

$$x_{n+1} = P((1 - \alpha_n)\, x_n + \alpha_n T\, (PT)^{n-1}\, x_n), \;\; x_1 \in K, \;\; n \geq 1,$$

Chidume et al. [2] gave some strong and weak convergence theorems for nonself asymptotically nonexpansive mappings in uniformly convex Banach spaces. They also established a demiclosedness principle. Convergence problems of an iterative algorithm to a common fixed point for nonself asymptotically nonexpansive mappings in uniformly convex Banach spaces have been considered by several authors (see, for example, [16, 17, 18, 19, 20, 21, 22, 23] and the references therein).

1.2. Remark. [3] If $T : K \to E$ is asymptotically nonexpansive and $P : E \to K$ is a nonexpansive retraction, then $PT : K \to K$ is asymptotically nonexpansive. Indeed, for all $x, y \in K$ and $n \geq 1$, we have

$$\begin{aligned} \|(PT)^n x - (PT)^n y\| &= \left\|PT(PT)^{n-1}x - PT(PT)^{n-1}y\right\| \\ &\leq \left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \\ &\leq k_n \|x - y\| . \end{aligned}$$

However, the converse may not be true.

Keeping in view the above fact, Zhou et al. [3] introduced the following generalized definition in 2007.

1.3. Definition. [3] Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E into K. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive with respect to P if there exists sequences $\{k_n\} \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$(1.3) \qquad \|(PT)^n x - (PT)^n y\| \le k_n \|x - y\|, \quad \forall x, y \in K, \ n \ge 1.$$

T is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L > 0$ such that

$$(1.4) \qquad \|(PT)^n x - (PT)^n y\| \le L \|x - y\|, \quad \forall x, y \in K, \ n \ge 1.$$

Throughout this paper, $J = \{1, 2, ..., N\}$ denotes the set of first N natural numbers. In what follows we fix $x_0 \in K$ as a starting point of an algorithm unless stated otherwise, and take $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ sequences in $(0, 1)$.

In 2001, Soltuz [4] introduced the Mann-type implicit algorithm for a nonexpansive mapping as follows:

$$(1.5) \qquad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T x_n, \ n \ge 1.$$

Xu and Ori [5] introduced the following implicit iterative algorithm for a finite family of nonexpansive mappings $\{T_i : i \in J\}$.

$$(1.6) \qquad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \ n \ge 1,$$

where $T_n = T_{n(modN)}$ and the $modN$ function takes values in J.

More recently, Khan, Yildirim and Ozdemir [6] introduced an implicit iterative algorithm for two finite families $\{S_i : i \in J\}$ and $\{T_i : i \in J\}$ of nonexpansive mappings as follows:

$$(1.7) \qquad x_n = \alpha_n x_{n-1} + \beta_n S_n x_n + \gamma_n T_n x_n$$

where $T_n = T_{n(modN)}$, $S_n = S_{n(modN)}$ and $\alpha_n + \beta_n + \gamma_n = 1$.

In this paper, we modify the implicit iterative algorithm of Khan, Yildirim and Ozdemir [6] for two finite families of nonself asymptotically nonexpansive mappings as follows:

Let E be a real Banach space, and K be a nonempty closed convex subset of E which is also a nonexpansive retract of E with a retraction P. Let $S_i, T_i : K \to E$ $(i = 1, 2, ..., N)$ be two families of nonself asymptotictally nonexpansive mappings with respect to P. For arbitrarily chosen $x_0 \in K$,

$$(1.8) \qquad x_n = \alpha_n x_{n-1} + \beta_n (PS_n)^n x_n + \gamma_n (PT_n)^n x_n$$

where $T_n = T_{n(modN)}$ and $S_n = S_{n(modN)}$.

In other words, if $n = (k-1)N + i$, $i = i(n) \in \{1, 2, ..., N\}$, $k = k(n) \ge 1$ is a positive integer and $k(n) \to \infty$, as $n \to \infty$, then we set $S_n = S_i$, $T_n = T_i$ and (1.8) can be expressed in the following form:

$$(1.9) \qquad x_n = \alpha_n x_{n-1} + \beta_n \left(PS_{i(n)}\right)^{k(n)} x_n + \gamma_n \left(PT_{i(n)}\right)^{k(n)} x_n, \ n \ge 1.$$

If we take $T_i = S_i$ for all $i \in J$, then (1.8) reduces to modified Mann-type implicit iteration as follows:

$$(1.10) \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) (PT_n)^n x_n, \ n \ge 1.$$

In addition, if we take $T_i = S_i = T$ for all $i \in J$ and T is a nonexpansive self-mapping, then (1.8) reduces to (1.5). Note that (1.8) reduces to (1.6) when $S_i =$

T_i for all $i \in J$ are nonexpansive self-mappings. Also (1.8) reduces to (1.7) when S_i and T_i are two nonexpansive self-mappings for all $i \in J$.

The purpose of this paper is to study the weak and strong convergence of the implicit iterative algorithm (1.8) for approximating common fixed points of the two finite families $\{S_i : i \in J\}$ and $\{T_i : i \in J\}$ of nonself asymptotically nonexpansive mappings with respect to P in Banach spaces. The results presented in this paper extend and improve the corresponding results of Soltuz [4], Xu and Ori [5], Khan et al. [6].

2. Preliminaries

In this section, we review definitions and lemmas used for the rest of the paper as follow:

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of E is the function $\delta_E(\varepsilon) : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1,\ \ \|y\| = 1,\ \varepsilon = \|x-y\|\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Let E be a Banach space and $S(E) = \{x \in E : \|x\| = 1\}$. The space E said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$.

A Banach space E is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x.

Let K be a nonempty subset of a Banach space E. For $x \in K$, the inward set of x, $I_K(x)$, is defined by $I_K(x) := \{x + \lambda(u - x) : u \in K, \lambda \geq 1\}$. A mapping $T : K \to E$ is called weakly inward if $Tx \in cl[I_K(x)]$ for all $x \in K$, where $cl[I_K(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

Let $K \subset E$ be a closed convex and P a mapping of E onto K. Then P is said to be sunny [7] if $P(Px + t(x - Px)) = Px$ for all $x \in E$ and $t \geq 0$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p, if for each sequence $\{x_n\}$ in $D(T)$, the conditions $x_n \to x_0$ weakly and $Tx_n \to p$ strongly imply $Tx_0 = p$.

A mapping $T : K \to K$ is said to be completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{Tx_{n_j}\}$ converges to some element of $R(T)$.

A mapping $T : K \to K$ is said to demi-compact if any sequence $\{x_n\}$ in K satisfying $x_n - Tx_n \to 0$ as $n \to \infty$ has a convergent subsequence.

Two mappings $T, S : K \to K$ are said to satisfy condition (A') [8] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all

$t \in (0, \infty)$ such that

$$\frac{1}{2}\left(\|x - T_1 x\| + \|x - T_2 x\|\right) \geq f\left(d\left(x, F\right)\right)$$

for all $x \in K$, where $d\left(x, F\right) = \inf\left\{\|x - p\| : p \in F := F(T) \cap F(S)\right\}$.

We modify this definition for two finite families of nonself asymptotically nonexpansive mappings as follows: Let E be a Banach space, K a nonempty closed convex subset of E with nonempty fixed point set F and P a sunny nonexpansive retraction. Let $\{T_i : i \in J\}$ and $\{S_i : i \in J\}$ be two finite families of nonself asymptotically nonexpansive mappings of K with respect to P . These families are said to satisfy condition (B') if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f\left(0\right) = 0$, $f\left(t\right) > 0$ for all $t \in (0, \infty)$ such that

$$\max_{i \in J}\left\{\frac{1}{2}\left(\|x - PT_i x\| + \|x - PS_i x\|\right)\right\} \geq f\left(d(x, F)\right)$$

for all $x \in K$ where $F := \left(\cap_{i=1}^{N} F\left(S_i\right)\right) \cap \left(\cap_{i=1}^{N} F\left(T_i\right)\right)$.

In what follows, we shall make use of the following lemmas.

2.1. Lemma. [9] *If $\{a_n\}, \{b_n\}$ are two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq \left(1 + b_n\right) a_n, \ \ n \geq 1$$

and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim\limits_{n \to \infty} a_n$ exists.

2.2. Lemma. [10] *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n \to \infty} \|x_n\| \leq d, \ \ \limsup_{n \to \infty} \|y_n\| \leq d \ \text{and} \ \lim_{n \to \infty} \|\left(1 - t_n\right) x_n + t_n y_n\| = d$$

hold for some $d \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

2.3. Lemma. [11] *Let E be real smooth Banach space, let K be nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

2.4. Lemma. [3] *Let E be a real smooth and uniformly convex Banach space, K a nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with the sequence $k_n \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero.*

3. Main results

Let E be a Banach space and K a nonempty closed convex subset of E. Let $S_i, T_i : K \to E$ $(i \in J)$ be two families of nonself asymptotically nonexpansive mappings with respect to P with common sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that $\lim_{n \to \infty} k_n = \lim_{n \to \infty} l_n = 1$. Let $\{x_n\}$ be defined by (1.8). Then $x_1 = \alpha_1 x_0 + \beta_1 \left(PS_1\right)^n x_1 + \gamma_1 \left(PT_1\right)^n x_1$. Define a mapping $W_1 : K \to K$ by $W_1 =$

$\alpha_1 x_0 + \beta_1 (PS_1)^n x + \gamma_1 (PT_1)^n x$ for all $x \in K$ where $\alpha_n + \beta_n + \gamma_n = 1$. Existence of x_1 is guaranteed if W_1 has a fixed point. Thus for any $x, y \in K$, we have

$$\begin{aligned} \|W_1 x - W_1 y\| &= \|\alpha_1 x_0 + \beta_1 (PS_1)^n x + \gamma_1 (PT_1)^n x \\ &\quad -\alpha_1 x_0 - \beta_1 (PS_1)^n y - \gamma_1 (PT_1)^n y\| \\ &\leq \beta_1 \|(PS_1)^n x - (PS_1)^n y\| + \gamma_1 \|(PT_1)^n x - (PT_1)^n y\| \\ &\leq \beta_1 k_1 \|x - y\| + \gamma_1 l_1 \|x - y\| \\ &\leq (\beta_1 k_1 + \gamma_1 l_1) \|x - y\| \end{aligned}$$

If $\beta_1 k_1 + \gamma_1 l_1 < 1$, then W_1 is a contraction. By Banach Contraction Principle, W_1 has a unique fixed point. Thus the existence of x_1 is established. Similarly, the existence of $x_2, x_{3,}...$ is established. Thus the implicit iterative algorithm (1.8) is well defined when $\beta_n k_n + \gamma_n l_n < 1$. Therefore, the implicit iterative algorithm (1.8) can be employed for the approximation of common fixed points of two families of nonself asymptotically nonexpansive mappings $\{S_i : j \in J\}$ and $\{T_i : i \in J\}$.

From now on, we denote the set of common fixed points of the two families $S_i, T_i : K \to E$ $(i \in J)$ by $F := \left(\cap_{i=1}^N F(S_i)\right) \cap \left(\cap_{i=1}^N F(T_i)\right)$.

3.1. Convergence Theorems in Real Banach Spaces. In this section, we prove the strong convergence of the iterative algorithm (1.8) to a common fixed point of two families of nonself asymptotically nonexpansive mappings with respect to P in real Banach spaces.

First, we prove the following lemma.

3.1. Lemma. *Let E be a real Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $S_i, T_i : K \to E$ $(i \in J)$ be two families of nonself asymptotically nonexpansive mappings with respect to P with common sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$. Suppose that $\{x_n\}$ defined by (1.8) satisfies the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n k_n + \gamma_n l_n < 1$ *for each integer* $n \geq 1$.
(ii) $s \leq \alpha_n, \beta_n, \gamma_n \leq 1 - s$ *for some* $s \in (0, 1)$.

If $F \neq \varnothing$, then

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists for each* $p \in F$.
(2) *there exists a constant $M > 0$ such that $\|x_{n+m} - p\| \leq M \|x_n - p\|$ for all $m, n \geq 1$ and $p \in F$.*

Proof. (1) Let $p \in F$. Set $k_n = 1 + u_n$, $l_n = 1 + v_n$. Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, so $\sum_{n=1}^{\infty} u_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$. Using (1.8), we have

$$\begin{aligned} \|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|(PS_n)^n x_n - p\| + \gamma_n \|(PT_n)^n x_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + \beta_n k_n \|x_n - p\| + \gamma_n l_n \|x_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (\beta_n k_n + \gamma_n l_n) \|x_n - p\| \qquad (3.1) \end{aligned}$$

which leads to

$$[1 - \beta_n k_n - \gamma_n l_n] \|x_n - p\| \leq \alpha_n \|x_{n-1} - p\|.$$

Since $\beta_n k_n + \gamma_n l_n < 1$, then $1 - \beta_n k_n - \gamma_n l_n > 0$, that is,

$$1 - \beta_n (1 + u_n) - \gamma_n (1 + v_n) > 0$$

for all $n \geq 1$. Thus, from (3.1) we have

$$\begin{aligned} \|x_n - p\| &\leq \frac{\alpha_n}{1 - \beta_n (1 + u_n) - \gamma_n (1 + v_n)} \|x_{n-1} - p\| \\ &= \frac{1 - \beta_n - \gamma_n}{1 - \beta_n (1 + u_n) - \gamma_n (1 + v_n)} \|x_{n-1} - p\| \\ &= \left[1 + \frac{\beta_n u_n + \gamma_n v_n}{1 - \beta_n (1 + u_n) - \gamma_n (1 + v_n)}\right] \|x_{n-1} - p\| \\ &= \left[1 + \frac{\beta_n u_n + \gamma_n v_n}{1 - \beta_n - \gamma_n - \beta_n u_n - \gamma_n v_n}\right] \|x_{n-1} - p\| \\ &= \left[1 + \frac{\beta_n u_n + \gamma_n v_n}{\alpha_n - \beta_n u_n - \gamma_n v_n}\right] \|x_{n-1} - p\| . \end{aligned} \tag{3.2}$$

Since $s \leq \beta_n, \gamma_n \leq 1 - s$ for some $s \in (0, 1)$, $\lim_{n\to\infty} \beta_n u_n = \lim_{n\to\infty} \gamma_n v_n = 0$. Thus for any given $\frac{\epsilon_o}{2}, \frac{\epsilon_1}{2} \in (0, \frac{s}{2})$, $\epsilon = \max\{\epsilon_0, \epsilon_1\}$, there exists positive integer n_0 such that

$$s - \epsilon < \alpha_n - \beta_n u_n - \gamma_n v_n, \tag{3.3}$$

as $n \geq n_0$. By (3.2) and (3.3), we get

$$\|x_n - p\| \leq \left[1 + \frac{u_n}{s - \epsilon} + \frac{v_n}{s - \epsilon}\right] \|x_{n-1} - p\| . \tag{3.4}$$

Since $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, we obtain $\sum_{n=1}^{\infty} \left(\frac{u_n}{s-\epsilon} + \frac{v_n}{s-\epsilon}\right) < \infty$. It now follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$.

(2) As $1 + t \leq e^t$ for all $t > 0$, from (3.4), we obtain

$$\begin{aligned} \|x_{n+m} - p\| &\leq e^{\left(\frac{u_{n+m}}{s-\epsilon} + \frac{v_{n+m}}{s-\epsilon}\right)} \|x_{n+m-1} - p\| \\ &\leq e^{\left(\frac{u_{n+m}}{s-\epsilon} + \frac{v_{n+m}}{s-\epsilon}\right)} \left[e^{\left(\frac{u_{n+m-1}}{s-\epsilon} + \frac{v_{n+m-1}}{s-\epsilon}\right)} \|x_{n+m-2} - p\|\right] \\ &\leq e^{\left(\frac{u_{n+m}}{s-\epsilon} + \frac{v_{n+m}}{s-\epsilon}\right) + \left(\frac{u_{n+m-1}}{s-\epsilon} + \frac{v_{n+m-1}}{s-\epsilon}\right)} \|x_{n+m-2} - p\| \\ &\quad \vdots \\ &\leq e^{\sum_{j=n+1}^{n+m} \left(\frac{u_j}{s-\epsilon} + \frac{v_j}{s-\epsilon}\right)} \|x_n - p\| \\ &\leq M \|x_n - p\| \end{aligned} \tag{3.5}$$

where $M = \sum_{j=n+1}^{n+m} \left(\frac{u_j}{s-\epsilon} + \frac{v_j}{s-\epsilon}\right)$. That is $\|x_{n+m} - p\| \leq M \ \|x_n - p\|$ for all $m, n \geq 1$ and $p \in F$. □

3.2. Theorem. *Let E be a real Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $S_i, T_i : K \to E$ $(i \in J)$ be two families of nonself asymptotically nonexpansive mappings with respect to P with common sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that* $\sum_{n=1}^{\infty} (k_n - 1) < \infty$,

$\sum_{n=1}^{\infty}(l_n - 1) < \infty$. *Suppose that* $F \neq \varnothing$ *and* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n k_n + \gamma_n l_n < 1$ *for each integer* $n \geq 1$.
(ii) $s \leq \alpha_n, \beta_n, \gamma_n \leq 1 - s$ *for some* $s \in (0, 1)$.

Then $\{x_n\}$ *converges strongly to some common fixed point of* $\{S_i\}$ *and* $\{T_i\}$ $(i \in J)$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The necessity of the conditions is obvious. Let us prove the sufficiency of the theorem.

Taking infimum over all $p \in F$ in (3.4), we get

$$d(x_n, F) \leq \left[1 + \frac{u_n}{s-\epsilon} + \frac{v_n}{s-\epsilon}\right] d(x_{n-1}, F).$$

Thus, we obtain from Lemma 2.1 that $\lim_{n\to\infty} d(x_n, F)$ exists. But by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we first show that $\{x_n\}$ is a Cauchy sequence in E. Since $\lim_{n\to\infty} d(x_n, F) = 0$, given $\epsilon > 0$, there exists a constant n_0 such that for all $n \geq n_0$, we have

$$d(x_n, F) < \frac{\epsilon}{1+M} \tag{3.6}$$

where $M > 0$ is the constant in Lemma 3.1(2). So we can find $p \in F$ such that

$$\|x_n - p\| < \frac{\epsilon}{1+M}. \tag{3.7}$$

From Lemma 3.1(2) we get for all $n \geq n_0$ and $m \geq 1$ that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &\leq M\|x_n - p\| + \|x_n - p\| \\ &= (1+M)\|x_n - p\| \\ &< \epsilon. \end{aligned} \tag{3.8}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E and so it must converge to a point q in K. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since F is closed, so we have $q \in F$. □

3.2. Convergence Theorems in Real Uniformly Convex Banach Spaces. In this section, we prove some strong and weak convergence of algorithm (1.8) to a common fixed point of two families of nonself asymptotically nonexpansive mappings with respect to P in real uniformly convex and smooth Banach space.

3.3. Lemma. *Let* E *be a real uniformly convex Banach space and* K *a nonempty closed convex subset of* E *which is also a nonexpansive retract of* E. *Let* $S_i, T_i : K \to E$ $(i \in J)$ *be two families of nonself asymptotically nonexpansive mappings with respect to* P *with common sequences* $\{k_n\}, \{l_n\} \subset [1, \infty)$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. *Suppose that* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n k_n + \gamma_n l_n < 1$ *for each integer* $n \geq 1$.

(ii) $s \leq \alpha_n, \beta_n, \gamma_n \leq 1-s$ *for some* $s \in (0,1)$.

If $F \neq \varnothing$, *then* $\lim_{n\to\infty} \|x_n - (PT_i)\, x_n\| = \lim_{n\to\infty} \|x_n - (PS_i)\, x_n\| = 0$ $(i \in J)$.

Proof. From Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$. We suppose that $\lim\limits_{n\to\infty} \|x_n - p\| = d$. Then

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p\| &= \lim_{n\to\infty} \|\alpha_n (x_{n-1} - p) + \beta_n ((PS_n)^n x_n - p) \\
&\quad + \gamma_n ((PT_n)^n x_n - p)\| \\
&= \lim_{n\to\infty} \left\| (1-\gamma_n) \left[\frac{\alpha_n}{1-\gamma_n} (x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n} ((PS_n)^n x_n - p) \right] \right. \\
&\quad \left. + \gamma_n ((PT_n)^n x_n - p) \right\| \\
&= d \qquad (3.9)
\end{aligned}$$

Since T_i $(i \in J)$ are asymptotically nonexpansive mappings and $F \neq \varnothing$, we have $\|(PT_n)^n x_n - p\| \leq l_n \|x_n - p\|$ for each $p \in F$. Taking lim sup on both sides, we obtain

$$\limsup_{n\to\infty} \|(PT_n)^n x_n - p\| \leq \limsup_{n\to\infty} l_n \|x_n - p\| = d. \tag{3.10}$$

Using (3.2) and $\sum\limits_{n=1}^{\infty} u_n < \infty$, $\sum\limits_{n=1}^{\infty} v_n < \infty$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left\| \frac{\alpha_n}{1-\gamma_n} (x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n} ((PS_n)^n x_n - p) \right\| \\
\leq\; &\limsup_{n\to\infty} \left[\frac{\alpha_n}{1-\gamma_n} \|x_{n-1} - p\| + \frac{\beta_n}{1-\gamma_n} k_n \|x_n - p\| \right] \\
\leq\; &\limsup_{n\to\infty} \left[\frac{\alpha_n}{1-\gamma_n} \|x_{n-1} - p\| \right. \\
&\left. + \frac{\beta_n}{1-\gamma_n} k_n \left[1 + \frac{\beta_n u_n + \gamma_n v_n}{\alpha_n - \beta_n u_n - \gamma_n v_n} \right] \|x_{n-1} - p\| \right] \\
\leq\; &\limsup_{n\to\infty} \left[\frac{k_n (\alpha_n + \beta_n)}{1-\gamma_n} \|x_{n-1} - p\| \right. \\
&\left. + \frac{\beta_n k_n}{1-\gamma_n} \left(\frac{\beta_n u_n + \gamma_n v_n}{\alpha_n - \beta_n u_n - \gamma_n v_n} \right) \|x_{n-1} - p\| \right] \\
\leq\; &\limsup_{n\to\infty} \left[\frac{k_n (1-\gamma_n)}{1-\gamma_n} \|x_{n-1} - p\| \right. \\
&\left. + \frac{\beta_n k_n}{1-\gamma_n} \left(\frac{\beta_n u_n + \gamma_n v_n}{\alpha_n - \beta_n u_n - \gamma_n v_n} \right) \|x_{n-1} - p\| \right] \\
=\; &d. \qquad (3.11)
\end{aligned}$$

Now considering (3.9), (3.10) and (3.11) and applying Lemma 2.2, we obtain

$$\lim_{n\to\infty} \left\| \frac{\alpha_n}{1-\gamma_n} (x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n} ((PS_n)^n x_n - p) - ((PT_n)^n x_n - p) \right\| = 0$$

which means that

$$\begin{aligned}
&\lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}x_{n-1}+\frac{\beta_n}{1-\gamma_n}\left(PS_n\right)^n x_n-\left(PT_n\right)^n x_n\right\| \\
= &\lim_{n\to\infty}\left(\frac{1}{1-\gamma_n}\right)\left\|\alpha_n x_{n-1}+\beta_n\left(PS_n\right)^n x_n-\left(1-\gamma_n\right)\left(PT_n\right)^n x_n\right\|=0.
\end{aligned}$$

Since $s\le\gamma_n\le 1-s$, we have $1/\left(1-s\right)\le 1/\left(1-\gamma_n\right)\le 1/s$. Therefore,

$$\lim_{n\to\infty}\|x_n-(PT_n)^n x_n\|=0. \tag{3.12}$$

In the same manner, we can prove that

$$\lim_{n\to\infty}\|x_n-(PS_n)^n x_n\|=0. \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}
\|x_n-x_{n-1}\| &= \left\|\alpha_n x_{n-1}+\beta_n\left(PS_n\right)^n x_n+\gamma_n\left(PT_n\right)^n x_n-x_{n-1}\right\| \\
&= \|\left(\beta_n+\gamma_n\right)\left(x_n-x_{n-1}\right)-\beta_n\left(x_n-\left(PS_n\right)^n x_n\right) \\
&\quad -\gamma_n\left(x_n-\left(PT_n\right)^n x_n\right)\| \\
&\le \left(\beta_n+\gamma_n\right)\|x_n-x_{n-1}\|+\beta_n\left\|x_n-\left(PS_n\right)^n x_n\right\| \\
&\quad +\gamma_n\left\|x_n-\left(PT_n\right)^n x_n\right\|.
\end{aligned}$$

And then from (3.12) and (3.13),

$$\begin{aligned}
\|x_n-x_{n-1}\| &\le \frac{\beta_n}{\left(1-\beta_n-\gamma_n\right)}\left\|x_n-\left(PS_n\right)^n x_n\right\| \\
&\quad +\frac{\gamma_n}{\left(1-\beta_n-\gamma_n\right)}\left\|x_n-\left(PT_n\right)^n x_n\right\| \\
&\to 0,\quad \left(n\to\infty\right).
\end{aligned} \tag{3.14}$$

Since an asymptotically nonexpansive mapping with respect to P must be uniformly L-Lipschitzian with respect to P , then we have

$$\begin{aligned}
\|x_n-\left(PT_n\right)x_n\| &\le \|x_n-(PT_n)^n x_n\|+\|(PT_n)^n x_n-\left(PT_n\right)x_n\| \\
&\le \|x_n-(PT_n)^n x_n\|+L\left\|x_n-(PT_n)^{n-1}x_n\right\| \\
&\le \|x_n-(PT_n)^n x_n\|+L\left\|(x_n-x_{n-1})+(x_{n-1}-(PT_n)^{n-1}x_{n-1})\right. \\
&\quad \left.+((PT_n)^{n-1}x_{n-1}-(PT_n)^{n-1}x_n)\right\| \\
&\le \|x_n-(PT_n)^n x_n\|+L\left\|x_{n-1}-(PT_n)^{n-1}x_{n-1}\right\| \\
&\quad +L(L+1)\left\|x_n-x_{n-1}\right\|.
\end{aligned}$$

This together with (3.12) and (3.14) implies that $\lim_{n\to\infty}\|x_n-\left(PT_n\right)x_n\|=0$. Similarly we can prove that $\lim_{n\to\infty}\|x_n-\left(PS_n\right)x_n\|=0$. This completes the proof. □

We now prove the following strong convergence theorem by making use of Lemma 2.3.

3.4. Theorem. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. Let $S_i,T_i:K\to E$ $(i\in J)$ be two families of weakly inward and nonself asymptotically nonexpansive mappings with respect to P with common sequences $\{k_n\}$, $\{l_n\}\subset$*

$[1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n-1)<\infty,\ \sum_{n=1}^{\infty}(l_n-1)<\infty$. *Suppose that* $F\neq\varnothing$ *and* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n=1,\ \beta_nk_n+\gamma_nl_n<1$ *for each integer* $n\geq 1$.
(ii) $s\leq\alpha_n,\beta_n,\gamma_n\leq 1-s$ *for some* $s\in(0,1)$.

If one of $\{S_i\}$ *and* $\{T_i\}$ $(i\in J)$ *is completely continuous, then* $\{x_n\}$ *converges strongly to some common fixed point of* $\{S_i\}$ *and* $\{T_i\}$ $(i\in J)$.

Proof. By Lemma 3.1, $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p\in F$. Assume that S_k is completely continuous for some $k\in J$. Since P is nonexpansive, then there exists subsequence $\{PS_kx_{n_j}\}\subset\{PS_kx_n\}$ such that $PS_kx_{n_j}\to q$ as $j\to\infty$. Now $\left\|x_{n_j}-q\right\|\leq\left\|x_{n_j}-PS_kx_{n_j}\right\|+\left\|PS_kx_{n_j}-q\right\|\to 0$ by Lemma 3.3. Also $\|q-PS_kq\|=\lim_{n\to\infty}\left\|x_{n_j}-PS_kx_{n_j}\right\|=0$, we get $q=PS_kq$. Similarly, $q=PT_kq$ and so $q\in F$. Since S_k is chosen arbitrarily, it follows from Lemma 2.3 that $\{x_n\}$ converges strongly to some common fixed point of $\{S_i\}$ and $\{T_i\}(i\in J)$. This completes the proof. □

Our next strong convergence theorem is an application of Theorem 3.2.

3.5. Theorem. *Let* K *be a nonempty closed convex subset of a real uniformly convex and smooth Banach space* E *with* P *as a sunny nonexpansive retraction. Let* $S_i,T_i:K\to E$ $(i\in J)$ *be two families of weakly inward and nonself asymptotically nonexpansive mappings with respect to* P *with common sequences* $\{k_n\}$, $\{l_n\}\subset[1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n-1)<\infty,\ \sum_{n=1}^{\infty}(l_n-1)<\infty$. *Suppose that* $F\neq\varnothing$ *and* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n=1,\ \beta_nk_n+\gamma_nl_n<1$ *for each integer* $n\geq 1$.
(ii) $s\leq\alpha_n,\beta_n,\gamma_n\leq 1-s$ *for some* $s\in(0,1)$.

If $\{S_i\}$ *and* $\{T_i\}$ $(i\in J)$ *satisfy Condition* $(B^{'})$, *then* $\{x_n\}$ *converges strongly to some common fixed point of* $\{S_i\}$ *and* $\{T_i\}$ $(i\in J)$.

Proof. By Lemma 3.1, $\lim_{n\to\infty}\|x_n-p\|$ exists for any $p\in F$, and so $\lim_{n\to\infty}d(x_n,F)$ exists for all $p\in F$. Because $\{S_i\}$ and $\{T_i\}$ satisfy Condition $(B^{'})$, we have

$$f\left(d(x_n,F)\right)\leq\max_{i\in J}\left\{\frac{1}{2}\left(\|x_n-PT_ix_n\|+\|x_n-PS_ix_n\|\right)\right\}.$$

Applying Lemma 3.3, we get $\lim_{n\to\infty}f\left(d\left(x_n,F\right)\right)=0$. Since $f:[0,\infty)\to[0,\infty)$ is a nondecreasing function satisfying $f(0)=0,\ f(t)>0$ for all $t\in(0,\infty)$, therefore

$$\lim_{n\to\infty}d\left(x_n,F\right)=0.$$

Now applying Theorem 3.2, we obtain the result. This completes the proof. □

Finally, we give our weak convergence theorem.

3.6. Theorem. *Let* K *be a nonempty closed convex subset of a real uniformly convex and smooth Banach space* E *satisfying Opial's condition with* P *as a sunny nonexpansive retraction. Let* $S_i,T_i:K\to E$ $(i\in J)$ *be two families of weakly inward and nonself asymptotically nonexpansive mappings with respect to* P *with* *common sequences* $\{k_n\}, \{l_n\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(l_n-1)<\infty$. *Suppose that* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n=1$, $\beta_n k_n+\gamma_n l_n<1$ *for each integer* $n\geq 1$.
(ii) $s\leq \alpha_n,\beta_n,\gamma_n\leq 1-s$ *for some* $s\in(0,1)$.

If $F\neq\varnothing$, *then* $\{x_n\}$ *converges weakly to some common fixed point of* $\{S_i\}$ *and* $\{T_i\}$ $(i\in J)$.

Proof. By Lemma 3.1, $\lim_{n\to\infty}\|x_n-p\|$ exists and so $\{x_n\}$ is bounded. Note that PT_i and PS_i are self-mappings from K into itself. We now prove that $\{x_n\}$ has a unique weak subsequential limit in F. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to p_1 and p_2, respectively. By Lemma 3.3, we have $\lim_{n\to\infty}\|x_{n_k}-(PS_i)\,x_{n_k}\|=0$ and $\lim_{n\to\infty}\|x_{n_k}-(PT_i)\,x_{n_k}\|$, $(i\in J)$. Lemma 2.4 guarantees that $(I-PS_i)\,p_1=0$, i.e., $(PS_i)\,p_1=p_1$. Similary, $(PT_i)\,p_1=p_1$. By Lemma 2.3, we get $p_1\in F$. Again in the same way, we can prove that $p_2\in F$. For uniqueness, assume $p_1\neq p_2$, then by Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-p_1\| &= \lim_{k\to\infty}\|x_{n_k}-p_1\|\\ &< \lim_{k\to\infty}\|x_{n_k}-p_2\|\\ &= \lim_{j\to\infty}\left\|x_{n_j}-p_2\right\|\\ &< \lim_{j\to\infty}\left\|x_{n_j}-p_1\right\|\\ &= \lim_{n\to\infty}\|x_n-p_1\|,\end{aligned}$$

which is a contradiction and hence $p_1=p_2$. Consequently, $\{x_n\}$ converges weakly to a common fixed point *of* $\{S_i\}$ and $\{T_i\}$ $(i\in J)$. □

3.7. Remark. (1) If the error terms are added in (1.8) and assumed to be bounded, then the results of this paper still hold.

(2) Since (1.5), (1.6), (1.7) and (1.10) are special cases of (1.8), the results proved using these algorithms follow as a special case to our above results.

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A note on the basic Lichnerowicz cohomology of transversally locally conformally Kählerian foliations

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Abstract

In this paper we generalize the basic Lichnerowicz cohomology on transversally locally conformally Kählerian foliations and we study its relation with basic Bott-Chern cohomology and 0–th basic Dolbeault cohomology with values in the associated foliated weight bundle.

Keywords: transversally locally conformally Kählerian foliation, Lichnerowicz cohomology, Bott-Chern cohomology.

2000 AMS Classification: 53C12, 53C55, 57R30, 32C35, 55N30.

1. Introduction and preliminaries

1.1. Introduction. A locally conformally Kähler (LCK) manifold M is a complex manifold whose universal cover $\widetilde{M}$ has a Kähler metric g such that $\pi_1(M)$ acts on $(\widetilde{M}, g)$ holomorphically and conformally. The fundamental properties of LCK manifolds were studied by Vaisman, Kashiwada, Dragomir, Ornea and Verbitsky.

The Lichnerowicz cohomology, also known in literature as Morse-Novikov cohomology, is a cohomology defined for a smooth manifold M and a closed 1–form θ. It is defined by twisting the usual differential of the de Rham complex $\Omega^\bullet(M)$ of M; namely, the Lichnerowicz cohomology is the cohomology of a complex $(\Omega^\bullet(M), d_\theta)$, where d_θ is defined by $d_\theta\varphi = d\varphi - \theta \wedge \varphi$. This cohomology was originally defined by Lichnerowicz and Novikov in the context of Poisson geometry and Hamiltonian mechanics, respectively. Lichnerowicz cohomology is naturally defined for a LCK manifold with its canonical closed 1–form called the Lee form, [18, 24]. Using the complex structures, variants of Lichnerowicz cohomology are defined for the Dolbeault cohomology and the Bott-Chern cohomology. These can be considered as fundamental invariants of LCK manifolds as established mainly by Vaisman, Ornea and Verbitsky.

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In a paper by Barletta and Dragomir [2] is introduced a new class of foliations called transversally locally conformally Kähler foliations (transversally LCK foliations), which is a foliated version of LCK manifolds roughly in the following sense: This class of foliations has a LCK structure on the direction transverse to the leaves. For instance, the simple foliation defined by a C^∞ submersion $f : \mathcal{M} \to M$ of $\mathcal{M}$ onto a LCK manifold M is transversally LCK. The case where the dimension of the leaves is zero corresponds to the original LCK manifolds.

The aim of this note is to extend some theories related to Lichnerowicz cohomology and its variants to basic forms on transversally LCK foliations. In this sense, in the preliminary subsection following [4, 5], we make a short review on the de Rham and Dolbeault theory for basic forms on transversally (holomorphic) foliations. The second section is dedicated to study of the basic Lichnerowicz cohomology of transversally LCK foliations. The relation of this cohomology with basic Bott-Chern cohomology and 0–th basic Dolbeault cohomology with values in the foliated weight bundle of a transversally LCK foliation is also studied, obtaining a basic version of some known results in the case of LCK manifolds due to Ornea and Verbitski [18].

1.2. Preliminaries. Let us consider $\mathcal{M}$ an $(n+m)$–dimensional manifold which will be assumed to be connected and orientable. Differential forms (and in particular functions) will take their values in the field of complex numbers $\mathbb{C}$. If φ is a form, then $\overline{\varphi}$ denote its complex conjugate and we say that φ is *real* if $\varphi = \overline{\varphi}$.

1.1. Definition. A codimension n foliation $\mathcal{F}$ on $\mathcal{M}$ is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$ such that:

(i) $\{U_i\}$, $i \in I$ is an open covering of $\mathcal{M}$;

(ii) For every $i \in I$, $\varphi_i : U_i \to M$ are submersions, where M is an n–dimensional manifold;

(iii) The maps $f_{i,j} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ satisfy

$$\varphi_j = f_{i,j} \circ \varphi_i \tag{1.1}$$

for every $(i,j) \in I \times I$ such that $U_i \cap U_j \neq \emptyset$.

We recall that a *plaque* of the foliation $\mathcal{F}$ is given by any fibre of φ_i and by (1.1) we have that, on the intersection $U_i \cap U_j$, the plaques defined respectively by φ_i and φ_j coincide. The manifold $\mathcal{M}$ is decomposed into a family of disjoint immersed connected submanifolds of dimension m called the *leafs* of $\mathcal{F}$.

The foliation $\mathcal{F}$ is said *transversally orientable* if on M can be given an orientation which is preserved by all $f_{i,j}$. We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and by $\Gamma(\mathcal{F})$ the space of its global sections i.e. vector fields tangent to $\mathcal{F}$. Also, a differential form φ is called *basic* if it satisfies $i_X\varphi = \mathcal{L}_X\varphi = 0$ for every $X \in \Gamma(\mathcal{F})$, where i_X and $\mathcal{L}_X$ denote the interior product and Lie derivative with respect to X, respectively. A *basic function* is a function constant on the leaves; such functions form an algebra denoted by $\mathcal{F}_b(\mathcal{M})$. The quotient $Q\mathcal{F} = T\mathcal{M}/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$. A vector field $Y \in \mathcal{X}(\mathcal{M})$ is said to be *foliated* if, for every $X \in \Gamma(\mathcal{F})$ we have $[X,Y] \in \Gamma(\mathcal{F})$ and we denote by $\mathcal{X}(\mathcal{M},\mathcal{F})$ the algebra of foliated vector fields on $\mathcal{M}$. The space $\mathcal{X}(\mathcal{M}/\mathcal{F}) = \mathcal{X}(\mathcal{M},\mathcal{F})/\Gamma(\mathcal{F})$ is called the algebra of *basic vector fields* on $\mathcal{M}$.

In this paper a system of local coordinates adapted to the foliation $\mathcal{F}$ means coordinates $(z^1, \dots, z^n, y^1, \dots, y^m)$ on an open subset U on which the foliation is defined by the equations $dz^i = 0$, $i = 1, \dots, n$. If $\mathcal{F}$ is transversally holomorphic (see the below discussion) $z^1, \dots, z^n$ will be complex coordinates.

We recall that a transverse structure to $\mathcal{F}$ is a geometric structure on M invariant by all the local diffeomorphisms $f_{i,j}$. Such a transverse structure can be considered as a geometric structure on the leaf space $\mathcal{M}/\mathcal{F}$ which generally is not a manifold.

If M is a complex manifold and all $f_{i,j}$ are biholomorphic maps then we say that $\mathcal{F}$ is *transversally holomorphic*. In this case, any transversal to $\mathcal{F}$ inherits a complex structure. If, moreover, M is endowed with a Hermitian structure which is preserved by all $f_{i,j}$ then we say that $\mathcal{F}$ is *Hermitian*. In this case $f_{i,j}$ are in particular biholomorphic maps and isometries. Also, the normal bundle $Q\mathcal{F}$ is equipped with a Hermitian metric g "invariant along the leaves" which can be writen in a transverse local system of coordinates $(z^1, \dots, z^n)$ in the form $g = g_{j\overline{k}}(z, \overline{z})dz^j \otimes d\overline{z}^k$. If the associated basic 2–form $\omega = \frac{i}{2} g_{j\overline{k}}(z, \overline{z})dz^j \wedge d\overline{z}^k$ is closed then $\mathcal{F}$ is called *transversally Kählerian.*

Throughout this paper we consider $\mathcal{F}$ to be transversally holomorphic with $2n$ codimension. Let $\Omega^r(\mathcal{M}/\mathcal{F})$ be the space of all basic forms of degree r. It is easy to see that the exterior derivative of a basic form is also a basic form. Indeed, if $\varphi \in \Omega^r(\mathcal{M}/\mathcal{F})$ then $i_X\varphi = \mathcal{L}_X\varphi = 0$ for any $X \in \Gamma(\mathcal{F})$ and, then by Cartan's formulas $\mathcal{L}_X = i_X d + d i_X$ and $d^2 = 0$ it follows that $i_X d\varphi = \mathcal{L}_X d\varphi = 0$ for any $X \in \Gamma(\mathcal{F})$. Let us denote by $d_b = d|_{\Omega^\bullet(\mathcal{M}/\mathcal{F})}$ the restriction of exterior derivative to basic forms. Then we have $d_b : \Omega^\bullet(\mathcal{M}/\mathcal{F}) \longrightarrow \Omega^{\bullet+1}(\mathcal{M}/\mathcal{F})$ and the differential complex

$$(1.2) \qquad 0 \longrightarrow \Omega^0(\mathcal{M}/\mathcal{F}) \xrightarrow{d_b} \Omega^1(\mathcal{M}/\mathcal{F}) \xrightarrow{d_b} \dots \xrightarrow{d_b} \Omega^{2n}(\mathcal{M}/\mathcal{F}) \longrightarrow 0$$

which is called the *basic de Rham complex* of $\mathcal{F}$; its cohomology is the basic de Rham cohomology $H^\bullet(\mathcal{M}/\mathcal{F})$. Now, we consider $Q_{\mathbb{C}}\mathcal{F} = Q\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$ the complexified normal bundle of $\mathcal{F}$. Let J be the automorphism of $Q_{\mathbb{C}}\mathcal{F}$ associated to the complex structure; J satisfies $J^2 = -\mathrm{Id}$ and then has two eigenvalues i and $-i$ with associated eigensubbundles respectively denoted by $Q^{1,0}\mathcal{F}$ and $Q^{0,1}\mathcal{F} = \overline{Q^{1,0}\mathcal{F}}$. We have a splitting $Q_{\mathbb{C}}\mathcal{F} = Q^{1,0}\mathcal{F} \oplus Q^{0,1}\mathcal{F}$ which gives rise to decomposition

$$\bigwedge^{r}(Q^*_{\mathbb{C}}\mathcal{F}) = \bigoplus_{p+q=r} \bigwedge^{p,q}(Q^*_{\mathbb{C}}\mathcal{F}),$$

where $\bigwedge^{p,q}(Q^*_{\mathbb{C}}\mathcal{F}) = \bigwedge^p(Q^{1,0*}\mathcal{F}) \otimes \bigwedge^q(Q^{0,1*}\mathcal{F})$. Basic sections of $\bigwedge^{p,q}(Q^*_{\mathbb{C}}\mathcal{F})$ are called *basic forms of type* (p, q) on $(\mathcal{M}, \mathcal{F})$. They form a vector space denoted by $\Omega^{p,q}(\mathcal{M}/\mathcal{F})$. We have

$$(1.3) \qquad \Omega^r(\mathcal{M}/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(\mathcal{M}/\mathcal{F}).$$

As in the classical case of a complex manifold, see [15], the basic exterior derivative is decomposed into two operators

$$\partial_b : \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \to \Omega^{p+1,q}(\mathcal{M}/\mathcal{F})\,,\; \overline{\partial}_b : \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \to \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}).$$

We have $\partial_b^2 = \overline{\partial}_b^2 = 0$ and $\partial_b\overline{\partial}_b + \overline{\partial}_b\partial_b = 0$. The differential complex

$$(1.4) \qquad 0 \longrightarrow \Omega^{p,0}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_b} \Omega^{p,1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_b} \dots \xrightarrow{\overline{\partial}_b} \Omega^{p,n}(\mathcal{M}/\mathcal{F}) \longrightarrow 0$$

is called the *basic Dolbeault complex* of $\mathcal{F}$; its cohomology $H^{p,\bullet}(\mathcal{M}/\mathcal{F})$ is the basic Dolbeault cohomology of foliation $\mathcal{F}$.

2. Basic Lichnerowicz cohomology of transversally locally conformally Kählerian foliations

2.1. Basic Lichnerowicz cohomology. Let $(\mathcal{M},\mathcal{F})$ be a transversally holomorphic foliation and $\theta \in \Omega^1(\mathcal{M}/\mathcal{F})$ be a closed basic 1-form. Denote by $d_{b,\theta}$: $\Omega^r(\mathcal{M}/\mathcal{F}) \to \Omega^{r+1}(\mathcal{M}/\mathcal{F})$ the map $d_{b,\theta} = d_b - \theta\wedge$.

Since $d_b\theta = 0$, we easily obtain that $d_{b,\theta}^2 = 0$. The differential complex

$$(2.1) \qquad 0 \longrightarrow \Omega^{0}(\mathcal{M}/\mathcal{F}) \xrightarrow{d_{b,\theta}} \Omega^{1}(\mathcal{M}/\mathcal{F}) \xrightarrow{d_{b,\theta}} \dots \xrightarrow{d_{b,\theta}} \Omega^{2n}(\mathcal{M}/\mathcal{F}) \longrightarrow 0$$

is called the *basic Lichnerowicz complex* of $(\mathcal{M},\mathcal{F})$ and its cohomology groups $H_\theta^\bullet(\mathcal{M}/\mathcal{F})$ are called the *basic Lichnerowicz cohomology groups* of $(\mathcal{M},\mathcal{F})$.

This is a basic version of the classical Lichnerowicz cohomology, motivated by Lichnerowicz's work [13] or Lichnerowicz-Jacobi cohomology on Jacobi and locally conformal symplectic manifolds, see [1, 12]. We also notice that Vaisman in [24] studied it under the name of "adapted cohomology" on locally conformal Kähler (LCK) manifolds. Some notions concerning to a such basic Lichnerowicz cohomology of real foliations may be found in [7].

We notice that, locally, the basic Lichnerowicz complex becames the basic de Rham complex after a change $\varphi \mapsto e^f\varphi$ with f a basic function which satisfies $d_bf = \theta$, namely $d_{b,\theta}$ is the unique differential in $\Omega^\bullet(\mathcal{M}/\mathcal{F})$ which makes the multiplication by the smooth basic function e^f an isomorphism of cochain basic complexes $e^f : (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,\theta}) \to (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_b)$.

2.1. Proposition. *The basic Lichnerowicz cohomology depends only on the basic class of θ. In fact, we have the isomorphism $H^r_{\theta - d_bf}(\mathcal{M}/\mathcal{F}) \approx H^r_\theta(\mathcal{M}/\mathcal{F})$.*

Proof. Since $d_{b,\theta}(e^f\varphi) = e^f d_{b,\theta-d_bf}\varphi$ it results that the map $[\varphi] \mapsto [e^f\varphi]$ is an isomorphism between $H^r_{\theta-d_bf}(\mathcal{M}/\mathcal{F})$ and $H^r_\theta(\mathcal{M}/\mathcal{F})$. □

For the basic Lichnerowicz cohomology, similar basic complexes of Dolbeault and Bott-Chern type can be defined. Taking into account the decomposition $\theta = \theta^{1,0} + \theta^{0,1}$, consider the Hodge components of the basic Lichnerowicz differential $d_{b,\theta} = d_b - \theta\wedge$ as

$$(2.2) \qquad d_{b,\theta} = \partial_{b,\theta} + \overline{\partial}_{b,\theta}\,,\ \partial_{b,\theta} = \partial_b - \theta^{1,0}\wedge\,,\ \overline{\partial}_{b,\theta} = \overline{\partial}_b - \theta^{0,1}\wedge\,.$$

The diferential complex

$$(2.3) \qquad \dots \xrightarrow{\overline{\partial}_{b,\theta}} \Omega^{p,q-1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_{b,\theta}} \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_{b,\theta}} \dots$$

is called the *basic Dolbeault-Lichnerowicz complex* of $(\mathcal{M},\mathcal{F})$; its cohomology groups denoted by $H_\theta^{p,\bullet}(\mathcal{M}/\mathcal{F})$ are called the *basic Dolbeault-Lichnerowicz cohomology groups* of $(\mathcal{M},\mathcal{F})$.

The differential complex

$$\Omega^{p-1,q-1}(\mathcal{M}/\mathcal{F}) \stackrel{\partial_{b,\theta}\overline{\partial}_{b,\theta}}{\longrightarrow} \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \stackrel{\partial_{b,\theta}\oplus\overline{\partial}_{b,\theta}}{\longrightarrow} \Omega^{p+1,q}(\mathcal{M}/\mathcal{F}) \oplus \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}) \tag{2.4}$$

is called the *basic Bott-Chern-Lichnerowicz complex* of $(\mathcal{M},\mathcal{F})$ and its cohomology groups

$$H^{\bullet,\bullet}_{BCL}(\mathcal{M}/\mathcal{F}) = \frac{\mathrm{Ker}\{\Omega^{\bullet,\bullet} \stackrel{\partial_{b,\theta}}{\longrightarrow} \Omega^{\bullet+1,\bullet}\} \cap \mathrm{Ker}\{\Omega^{\bullet,\bullet} \stackrel{\overline{\partial}_{b,\theta}}{\longrightarrow} \Omega^{\bullet,\bullet+1}\}}{\mathrm{Im}\{\Omega^{\bullet-1,\bullet-1} \stackrel{\partial_{b,\theta}\overline{\partial}_{b,\theta}}{\longrightarrow} \Omega^{\bullet,\bullet}\}}$$

are called the *basic Bott-Chern-Lichnerowicz cohomology groups* of $(\mathcal{M},\mathcal{F})$.

In the end of this subsection we apply some considerations from [24] for basic forms and we obtain a relation between a twisted basic cohomology associated to θ and basic real cohomology of $(\mathcal{M},\mathcal{F})$. For every θ as above, let us consider now the auxiliary basic operator $\widetilde{d_b} = d_b - \frac{r}{2}\theta\wedge$ where r is the degree of the basic form acted on. We notice that $\widetilde{d_b}$ is an antiderivation of basic differential forms and it is easy to see that $\widetilde{d}_b^2 = -\frac{1}{2}\theta \wedge d_b$. Then $\widetilde{d_b}$ defines a *twisted basic cohomology* of basic differential forms of $(\mathcal{M},\mathcal{F})$, which is given by

$$H^{\bullet}_{\widetilde{d_b}}(\mathcal{M}/\mathcal{F}) = \frac{\mathrm{Ker}\,\widetilde{d_b}}{\mathrm{Im}\,\widetilde{d_b} \cap \mathrm{Ker}\,\widetilde{d_b}} \tag{2.5}$$

and is isomorphic to the cohomology of the basic complex $(\widetilde{\Omega}^{\bullet}(\mathcal{M}/\mathcal{F}), \widetilde{d_b})$ consisting of the basic differential forms $\varphi \in \Omega^{\bullet}(\mathcal{M}/\mathcal{F})$ satisfying $\widetilde{d}_b^2\varphi = -\theta \wedge d_b\varphi = 0$.

The basic complex $\widetilde{\Omega}^{\bullet}(\mathcal{M}/\mathcal{F})$ admits a basic subcomplex $\Omega^{\bullet}_{\theta}(\mathcal{M}/\mathcal{F})$, namely, the ideal generated by θ. On this subcomplex, $\widetilde{d_b} = d_b$, which means that it is a basic subcomplex of the usual basic de Rham complex of $(\mathcal{M},\mathcal{F})$. Hence, one has the homomorphisms

$$a : H^r(\Omega^{\bullet}_{\theta}(\mathcal{M}/\mathcal{F})) \to H^r_{\widetilde{d_b}}(\mathcal{M}/\mathcal{F})\,,\; b : H^r(\Omega^{\bullet}_{\theta}(\mathcal{M}/\mathcal{F})) \to H^r(\mathcal{M}/\mathcal{F},\mathbb{R}). \tag{2.6}$$

Now, we can easily construct a homomorphism

$$c : H^r_{\widetilde{d_b}}(\mathcal{M}/\mathcal{F}) \to H^{r+1}(\mathcal{M}/\mathcal{F},\mathbb{R}). \tag{2.7}$$

Indeed, if $[\varphi] \in H^r_{\widetilde{d_b}}(\mathcal{M}/\mathcal{F})$, where φ is $\widetilde{d_b}$-closed basic form, then we put $c([\varphi]) = [\theta \wedge \varphi]$, and this produces the homomorphism from (2.7). We notice that the existence of c gives some relation between $\widetilde{d_b}$ and the basic real cohomology of $(\mathcal{M},\mathcal{F})$.

2.2. Remark. If we consider the decomposition $\widetilde{d_b} = \widetilde{\partial}_b + \widetilde{\overline{\partial}}_b$ we can construct analogous of homomorphisms a, b and c from (2.6) and (2.7), respectively, for corresponding basic Dolbeault cohomology.

2.2. Basic Lichnerowicz cohomology of transversally LCK foliations. In this subsection we consider the notion of transversally locally conformally Kählerian foliation that is a version of locally conformally Kähler manifold notion, see [22, 23, 24], for transversally Kählerian foliations, and we investigate some problems related to basic Lichnerowicz cohomology for such structures.

2.3. Definition. A locally conformally transversally Kählerian foliation, briefly transversally LCK foliation, is a transversally Hermitian foliation $(\mathcal{M}, \mathcal{F}, g)$ for which an open covering $\{U_i\}$ of $\mathcal{M}$ exists, and for each i a basic function $\sigma_i : U_i \to \mathbb{R}$ such that $\widetilde{g}_i = e^{-\sigma_i}(g|_{U_i})$ is a transverse Kähler metric on U_i.

It is easy to see that $\theta|_{U_i} = d_b\sigma_i$ defines a global d_b–closed 1–form, and $(\mathcal{M}, \mathcal{F}, \omega)$ has the characteristic property [2]:

$$d_b\omega = \theta \wedge \omega, \tag{2.8}$$

where ω is the basic Hermitian form on $(\mathcal{M}, \mathcal{F})$. If we take $U_i = \mathcal{M}$, then $(\mathcal{M}, \mathcal{F}, \omega)$ is called globally conformally Kählerian foliation. The basic form θ is called the basic Lee form of $(\mathcal{M}, \mathcal{F}, \omega)$. It is exact iff $(\mathcal{M}, \mathcal{F}, \omega)$ is globally conformal Kählerian foliation.

2.4. Example. A simple foliation defined by a C^∞ submersion $f : \mathcal{M} \to M$ of $\mathcal{M}$ onto a LCK manifold M is transversally LCK foliation. The case where the dimension of the leaves is zero corresponds to the original LCK manifolds.

2.5. Remark. If $(\mathcal{M}, \mathcal{F})$ is a complex analytic foliated manifold, [20], then similarly to Proposition 1.1 from [22], the foliation $\mathcal{F}$ is a transversally LCK foliation if and only if its transverse bundle $Q\mathcal{F}$ has a Kähler metric which is locally conformally with a foliated Hermitian metric, or equivalently $(\mathcal{M}, \mathcal{F})$ has a Hermitian metric which is locally conformally with a bundle-like metric.

Now, if $(\mathcal{M}, \mathcal{F}, \omega)$ is a transversally LCK foliation with basic Lee form θ, then due to (2.8) we have $d_{b,\theta}\omega = 0$. Therefore, ω represents a cohomology class $[\omega]_L$ in the basic Lichnerowicz complex $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,\theta})$.

2.6. Definition. The basic cohomology class $[\omega]_L \in H^2_\theta(\mathcal{M}/\mathcal{F})$ is called the *basic Lichnerowicz class* of the transversally LCK foliation $(\mathcal{M}, \mathcal{F}, \omega)$.

This invariant is a basic version of Morse-Novikov class of LCK manifolds, see [18].

Also, if we consider the decomposition $d_{b,\theta} = \partial_{b,\theta} + \overline{\partial}_{b,\theta}$ we have $\partial_{b,\theta}\omega = \overline{\partial}_{b,\theta}\omega = 0$ and so ω represents a cohomology class $[\omega]_{BCL}$ in the basic Bott-Chern-Lichnerowicz complex of $(\mathcal{M}, \mathcal{F}, \omega)$.

2.7. Definition. If $(\mathcal{M}, \mathcal{F}, \omega)$ is a transversally LCK foliation then the cohomology class $[\omega]_{BCL} \in H^{1,1}_{BCL}(\mathcal{M}/\mathcal{F})$ is called the *basic Bott-Chern-Lichnerowicz class* of $(\mathcal{M}, \mathcal{F}, \omega)$.

Thus, for any transversally LCK foliation we have three basic cohomological invariants:

- the basic Lee class $[\theta] \in H^1(\mathcal{M}/\mathcal{F})$;
- the basic Lichnerowicz class $[\omega]_L \in H^2_\theta(\mathcal{M}/\mathcal{F})$;
- the basic Bott-Chern-Lichnerowicz class $[\omega]_{BCL} \in H^{1,1}_{BCL}(\mathcal{M}/\mathcal{F})$.

Now, using an argument inspired from [11], we briefly present an another basic cohomology associated to transversally LCK foliations which is connected with the basic Lichnerowicz cohomology of transversally LCK foliations. Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with basic Lee form θ. We consider the basic closed 1-forms θ_0 and θ_1 defined by

(2.9) $\quad \theta_0 = m\theta$ and $\theta_1 = (m+1)\theta,\ m \in \mathbb{R}$.

Denote by $H^\bullet_{\theta_0}(\mathcal{M}/\mathcal{F})$ and $H^\bullet_{\theta_1}(\mathcal{M}/\mathcal{F})$ the basic Lichnerowicz cohomologies of the basic complexes $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{\theta_0})$ and $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{\theta_1})$, repectively.

Now, let $\widehat{\Omega}^k(\mathcal{M}/\mathcal{F}) = \Omega^k(\mathcal{M}/\mathcal{F}) \oplus \Omega^{k-1}(\mathcal{M}/\mathcal{F})$ and $\widehat{d}_b : \widehat{\Omega}^k(\mathcal{M}/\mathcal{F}) \to \widehat{\Omega}^{k+1}(\mathcal{M}/\mathcal{F})$ be the basic differential operator defined by

(2.10) $\quad \widehat{d}_b(\varphi, \psi) = (d_{b,\theta_1}\varphi - \omega \wedge \psi, -d_{b,\theta_0}\psi)$.

Using (2.8), by direct calculus it follows $\widehat{d}_b^2 = 0$. Thus, we can consider the basic complex $(\widehat{\Omega}^\bullet(\mathcal{M}/\mathcal{F}), \widehat{d}_b)$ and $\widehat{H}^\bullet(\mathcal{M}/\mathcal{F})$ the associated basic cohomology. We have the following result which relates $\widehat{H}^\bullet(\mathcal{M}/\mathcal{F})$ with basic Lichnerowicz cohomologies $H^\bullet_{\theta_0}(\mathcal{M}/\mathcal{F})$ and $H^\bullet_{\theta_1}(\mathcal{M}/\mathcal{F})$.

2.8. Proposition. *Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with basic Lee form θ. Suppose that $i^k : \Omega^k(\mathcal{M}/\mathcal{F}) \to \widehat{\Omega}^k(\mathcal{M}/\mathcal{F})$ and $\pi_2^k : \widehat{\Omega}^k(\mathcal{M}/\mathcal{F}) \to \Omega^{k-1}(\mathcal{M}/\mathcal{F})$ are homomorphisms of $\mathcal{F}_b(\mathcal{M}, \mathbb{R})$-modules defined by*

$$i^k(\varphi) = (\varphi, 0) \text{ and } \pi_2^k(\varphi, \psi) = \psi,$$

for $\varphi \in \Omega^k(\mathcal{M}/\mathcal{F})$ and $\psi \in \Omega^{k-1}(\mathcal{M}/\mathcal{F})$. Then:

i) *The mappings i^k and π_2^k induce an exact sequence of basic complexes*

$$0 \longrightarrow (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,\theta_1}) \xrightarrow{i^k} (\widehat{\Omega}^\bullet(\mathcal{M}/\mathcal{F}), \widehat{d}_b) \xrightarrow{\pi_2^k} (\Omega^{\bullet-1}(\mathcal{M}/\mathcal{F}), -d_{b,\theta_0}) \longrightarrow 0.$$

ii) *This exact sequence induces a long exact sequence*

$$\ldots\ H^k_{\theta_1}(\mathcal{M}/\mathcal{F}) \xrightarrow{(i^k)^*} \widehat{H}^k(\mathcal{M}/\mathcal{F}) \xrightarrow{(\pi_2^k)^*} H^{k-1}_{\theta_0}(\mathcal{M}/\mathcal{F}) \xrightarrow{-\delta^{k-1}} H^{k+1}_{\theta_1}(\mathcal{M}/\mathcal{F})\ \ldots,$$

where the connecting homomorphism $-\delta^{k-1}$ is defined by

(2.11) $\quad (-\delta^{k-1})[\varphi] = [\varphi \wedge \omega],\ \forall\, [\varphi] \in H^{k-1}_{\theta_0}(\mathcal{M}/\mathcal{F})$.

From the above proposition, we obtain

2.9. Corollary. *Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with basic Lee form θ and such that the basic Lichnerowicz cohomology groups $H^k_{\theta_0}(\mathcal{M}/\mathcal{F})$ and $H^k_{\theta_1}(\mathcal{M}/\mathcal{F})$ have finite dimension, for all k. Then, the basic cohomology group $\widehat{H}^k(\mathcal{M}/\mathcal{F})$ has also finite dimension, for all k, and*

(2.12) $$\widehat{H}^k(\mathcal{M}/\mathcal{F}) \cong \frac{H^k_{\theta_1}(\mathcal{M}/\mathcal{F})}{\operatorname{Im}\delta^{k-2}} \oplus \ker \delta^{k-1},$$

where $\delta^k : H^k_{\theta_0}(\mathcal{M}/\mathcal{F}) \to H^{k+2}_{\theta_1}(\mathcal{M}/\mathcal{F})$ is the homomorphism given by (2.11).

2.10. Corollary. *Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with basic Lee form θ such that the dimensions of the basic cohomology groups $H^k_{\theta_0}(\mathcal{M}/\mathcal{F})$ and $H^k_{\theta_1}(\mathcal{M}/\mathcal{F})$ are finite, for all k. If the basic Lichnerowicz class $[\omega]_L$ vanish then, for all k, we have*

(2.13) $\quad \widehat{H}^k(\mathcal{M}/\mathcal{F}) \cong H^k_{\theta_1}(\mathcal{M}/\mathcal{F}) \oplus H^{k-1}_{\theta_0}(\mathcal{M}/\mathcal{F})$.

2.3. Basic Bott-Chern cohomology of transversally LCK foliations. In this subsection we present a link between basic Bott-Chern cohomology, basic Lichnerowicz cohomology and 0-th basic Dolbeault cohomology with values in the associated foliated weight bundle of a transversally LCK foliation $(\mathcal{M},\mathcal{F},\omega)$. The notions are introduced by analogy with the corresponding notions in the case of LCK manifolds, [18].

Let us consider further $(\mathcal{M},\mathcal{F},\omega)$ to be a transversally Kählerian foliation. Then the basic Kähler form ω determines the basic Kähler class $[\omega]\in H^{1,1}(\mathcal{M}/\mathcal{F})$, and the difference of basic Kähler forms which have the same basic Kähler class is measured by a basic potential f:

$$\omega_1-\omega=\partial_b\overline{\partial}_b f$$

see Proposition 3.5.1. from [4]. (This also follows from $\partial_b\overline{\partial}_b$-Lemma for transversally Kähler foliations, [17]). Thus the space of all basic Kähler structures on a transversally holomorphic foliation $(\mathcal{M},\mathcal{F})$ is locally modeled on $H^{1,1}(\mathcal{M}/\mathcal{F},\mathbb{R})\times(\mathcal{F}_b(\mathcal{M})/\mathrm{const})$. A similar local description exists for the set of all basic LCK-structures on a given transversally holomorphic foliation, if we fix the basic cohomology class $[\theta]$ of a basic Lee form. Using the basic Bott-Chern-Lichnerowicz class $[\omega]_{BCL}\in H^{1,1}_{BCL}(\mathcal{M}/\mathcal{F})$ of a basic LCK form ω, similarly to [18], we can obtain that the difference of two basic LCK forms in the same basic Bott-Chern-Lichnerowicz class is expressed by a basic potential, just like in transversally Kähler case, and the set of all basic LCK structures on a given transversally holomorphic foliation $(\mathcal{M},\mathcal{F})$ is locally parametrized by

$$H^{1,1}_{BCL}(\mathcal{M}/\mathcal{F})\times(\mathcal{F}_b(\mathcal{M})/\mathrm{Ker}\, d_{b,\theta}d^c_{b,\theta}), \tag{2.14}$$

where $d_{b,\theta}d^c_{b,\theta}=-2\sqrt{-1}\partial_{b,\theta}\overline{\partial}_{b,\theta}$.

In order to find a connection between basic Bott-Chern cohomology, basic Lichnerowicz cohomology and 0–th basic Dolbeault cohomology with values in the associated foliated weight bundle of a transversally LCK foliation $(\mathcal{M},\mathcal{F},\omega)$, we briefly recall some definitions concerning to foliated bundles and basic connections, see [5, 9, 14].

Let $G\hookrightarrow P\to M$ be a principal bundle with structural group $G\subset\mathrm{GL}(n,\mathbb{C})$. The group G acts on P on the right and on its Lie algebra $\mathcal{G}$ by the adjoint representation Ad i.e., for $g\in G$ and $X\in\mathcal{G}$, $\mathrm{Ad}_g(X)=gXg^{-1}$. We say that a principal G-bundle $P\to(\mathcal{M},\mathcal{F})$ is a *foliated principal bundle* if it is equipped with a foliation $\mathcal{F}_P$ (*the lifted foliation*) such that the distribution $T\mathcal{F}_P$ is invariant under the right action of G, is transversal to the tangent space to the fiber, and projects to $T\mathcal{F}$. A connection ω on P is called *adapted* to $\mathcal{F}_P$ if the associated horizontal distribution contains $T\mathcal{F}_P$. An adapted connection γ is called a *basic connection* if it is basic as a $\mathcal{G}$-valued form on $(P,\mathcal{F}_P)$.

Let us consider now $E\to(\mathcal{M},\mathcal{F})$ be a complex vector bundle defined by a cocycle $\{U_i,g_{ij},G\}$ where $\{U_i\}$ is an open cover of $\mathcal{M}$ and $g_{ij}:U_i\cap U_j\to G\subset\mathrm{GL}(n,\mathbb{C})$ are the transition functions. To such a vector bundle we can always associate a principal G-bundle $P\to(\mathcal{M},\mathcal{F})$ whose fibre is the group G and the transition functions are g_{ij} (viewed as translations on G). The complex vector bundle $E\to(\mathcal{M},\mathcal{F})$ is *foliated* if E is associated to a foliated principal G-bundle $P\to(\mathcal{M},\mathcal{F})$. Let $\Omega^\bullet(\mathcal{M},E)$ denote the space of forms on $(\mathcal{M},\mathcal{F})$ with values in E.

If a connection form γ on P is adapted, then we say that an associated covariant derivative operator ∇ on $\Omega^\bullet(\mathcal{M}, E)$ is *adapted* to the foliated bundle. We say that ∇ is a *basic* connection on E if in addition the associated curvature operator ∇^2 satisfies $i_X\nabla^2 = 0$ for every $X \in \Gamma(\mathcal{F})$. Note that ∇ is basic if the principal connection γ associated to ∇ is basic. Let $\Gamma(E)$ denote the smooth sections of E, and let ∇ denote a basic connection on E. We say that a section $s : \mathcal{M} \to E$ is a *basic section* if and only if $\nabla_X s = 0$ for all $X \in \Gamma(\mathcal{F})$. Let ∇_b denote the basic connection and $\Gamma_b(E)$ denote the space of basic sections of E.

Now, let us consider E to be a foliated complex line bundle over the transversally holomorphic foliation $(\mathcal{M}, \mathcal{F})$ with a flat basic connection ∇_b. We denote by $\Omega^{p,q}(\mathcal{M}/\mathcal{F}, E)$ the set of all basic (p, q)–forms on $(\mathcal{M}, \mathcal{F})$ with values in E. Consider the basic complex

$$\ldots \longrightarrow \Omega^{p-1,q-1}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\partial_{b,E}\overline{\partial}_{b,E}} \Omega^{p,q}(\mathcal{M}/\mathcal{F}, E)$$

$$\xrightarrow{\partial_{b,E}\oplus\overline{\partial}_{b,E}} \Omega^{p+1,q}(\mathcal{M}/\mathcal{F}, E) \oplus \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}, E) \longrightarrow \ldots, \tag{2.15}$$

where $\partial_{b,E}$ and $\overline{\partial}_{b,E}$ denote the $(1, 0)$ and $(0, 1)$-parts of the basic connection operator $\nabla_b : \Omega^\bullet(\mathcal{M}/\mathcal{F}, E) \to \Omega^{\bullet+1}(\mathcal{M}/\mathcal{F}, E)$. The cohomology of (2.15) denoted by $H^{p,q}_{BC}(\mathcal{M}/\mathcal{F}, E)$ is called the *basic Bott-Chern cohomology of* $(\mathcal{M}, \mathcal{F})$ *with values in* E.

2.11. Definition. Let $(\mathcal{M}, \mathcal{F}, \omega, \theta)$ be a transversally LCK foliation, and E its foliated weight bundle, that is, a trivial complex foliated line bundle with the flat basic connection $d_b - \theta$. Consider ω as a closed basic $(1, 1)$-form on $(\mathcal{M}, \mathcal{F})$ with values in E. Its basic Bott-Chern class $[\omega]_{BC} \in H^{1,1}_{BC}(\mathcal{M}/\mathcal{F}, E)$ is called the *basic Bott-Chern class of the transversally LCK foliation* $(\mathcal{M}, \mathcal{F}, \omega, \theta)$.

Now, similarly to [18], we give a characterization of $H^{1,1}_{BC}(\mathcal{M}/\mathcal{F}, E)$ in terms of basic Lichnerowicz cohomology of $(\mathcal{M}, \mathcal{F}, \theta)$ and 0–th basic Dolbeault cohomology of the foliated weight bundle E.

The 0–th basic Dolbeault cohomology with values in a foliated bundle E can be realized as cohomology of the complex

$$\Gamma_b(E) = \Omega^{0,0}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\overline{\partial}_{b,E}} \Omega^{0,1}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\overline{\partial}_{b,E}} \Omega^{0,2}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\overline{\partial}_{b,E}} \ldots \tag{2.16}$$

If E has a flat basic connection, then $\partial_{b,E} : \Omega^{0,1}(\mathcal{M}/\mathcal{F}, E) \to \Omega^{1,1}(\mathcal{M}/\mathcal{F}, E)$ induces a map

$$H^{0,1}(\mathcal{M}/\mathcal{F}, \mathcal{E}) \xrightarrow{\partial^*_{b,E}} H^{1,1}_{BC}(\mathcal{M}/\mathcal{F}, E) \tag{2.17}$$

from the 0-th basic Dolbeault cohomology with values in the underlying holomorphic foliated bundle (denoted as $\mathcal{E}$) to the basic Bott-Chern cohomology with values in E. The basic complex

$$\Gamma_b(E) = \Omega^{0,0}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\nabla^{1,0}_b} \Omega^{1,0}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\nabla^{1,0}_b} \Omega^{2,0}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\nabla^{1,0}_b} \ldots \tag{2.18}$$

computes the 0-th basic Dolbeault cohomology with values in holomorphic foliated bundle $\mathcal{E}'$ with a holomorphic structure defined by the complex conjugate of the $\nabla^{1,0}_b$-part of the basic connection. When the foliated bundle E is real, we have $\mathcal{E} \approx \mathcal{E}'$. Then the cohomology of the basic complex (2.18) is naturally identified with $\overline{H^{0,\bullet}(\mathcal{M}/\mathcal{F},\mathcal{E})}$. The map $\overline{\partial}_{b,E} : \Omega^{1,0}(\mathcal{M}/\mathcal{F},E) \to \Omega^{1,1}(\mathcal{M}/\mathcal{F},E)$ defines a homomorphism

$$\overline{H^{0,1}(\mathcal{M}/\mathcal{F},\mathcal{E})} \xrightarrow{\overline{\partial}^*_{b,E}} H^{1,1}_{BC}(\mathcal{M}/\mathcal{F},E) \tag{2.19}$$

which is entirely similar to (2.17).

Following step by step the proof of Theorem 4.7. from [18], we obtain an analogous result for basic cohomologies, which allows to compute the basic Bott-Chern cohomology classes in terms of 0–th basic Dolbeault cohomology and basic Lichnerowicz cohomology.

2.12. Theorem. *Let $(\mathcal{M},\mathcal{F})$ be a transversally holomorphic foliation and $E_{\mathbb{R}}$ a trivial real foliated line bundle with flat basic connection $d_b-\theta$, where θ is a real closed basic 1-form. Denote by E its complexification, and let $\mathcal{E}$ be the underlying holomorphic bundle. Then there is an exact sequence*

$$H^{0,1}(\mathcal{M}/\mathcal{F},\mathcal{E}) \oplus \overline{H^{0,1}(\mathcal{M}/\mathcal{F},\mathcal{E})} \xrightarrow{\partial^*_{b,E}+\overline{\partial}^*_{b,E}} H^{1,1}_{BC}(\mathcal{M}/\mathcal{F},E) \xrightarrow{\nu} H^2_\theta(\mathcal{M}/\mathcal{F}), \tag{2.20}$$

where $H^2_\theta(\mathcal{M}/\mathcal{F})$ is the basic Lichnerowicz cohomology, ν a tautological map, and the first arrow is obtained as a direct sum of (2.17) and (2.19).

From the above theorem, we immediately obtain

2.13. Corollary. *Let $(\mathcal{M},\mathcal{F},\omega,\theta)$ be a transversally LCK foliation, E the corresponding flat foliated weight bundle, and $\mathcal{E}$ the underlying holomorphic foliated bundle. Asume that $H^{0,1}(\mathcal{M}/\mathcal{F},\mathcal{E})=0$ and $H^2_\theta(\mathcal{M}/\mathcal{F})=0$. Then $H^{1,1}_{BC}(\mathcal{M}/\mathcal{F},E)=0$.*

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The spectrum of the operator $D(r, 0, 0, s)$ over the sequence spaces ℓ_p and bv_p

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Abstract

In this paper we have examined the spectra of the operator $D(r, 0, 0, s)$ on sequence spaces ℓ_p and bv_p.

Keywords: Spectra; resolvent operator; point spectrum; continuous spectrum; residual spectrum.

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1. Introduction

Spectral theory is an important branch of mathematics due to its application in other branches of science. It has been proved to be a standard tool of mathematical sciences because of its usefulness and application oriented scope in different fields. In numerical analysis, the spectral values may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge. In aeronautics, the spectral values may determine whether the flow over a wing is laminar or turbulent. In electrical engineering, it may determine the frequency response of an amplifier or the reliability of a power system. In quantum mechanics, it may determine atomic energy levels and thus, the frequency of a laser or the spectral signature of a star. In structural mechanics, it may determine whether an automobile is too noisy or whether a building will collapse in an earthquake. In ecology, the spectral values may determine whether a food web will settle into a steady equilibrium. In probability theory, they may determine the rate of convergence of a Markov process.

In summability theory, different classes of matrices have been investigated. Characterizations of matrix classes are found in Tripathy and Sen [29], Tripathy [30], Rath and Tripathy [21] and many others. There are particular types of

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summability methods like Nörlund mean, Riesz mean, Euler mean, Abel transformation etc. Matrix methods have been studied from different aspects recently by Altin et.al [10], Tripathy and Baruah [31] and others.

Spectral theory is a thrust area of research in Functional analysis. The spectra of different operators have been studied. There are different types of matrix operators on sequence spaces. The spectra of only a few of the matrix operators have been studied so far that to on some particular type of sequence spaces. The works those exist are mainly on cesàro, Schur, Hausdorff and some difference matrix operators.

Wenger [33] examined the fine spectrum of the integer power of the Cesàro operator in c and Rhoades [26, 27] generalized this result to the weighted mean methods and proposed a conjecture for their fine spectra on $B(\ell_p)$ respectively. Reade [23] worked the spectrum of the Cesàro operator in the sequence space c_0 and Rhoades [24] extended it to the fine spectrum of the weighted mean operators. The fine spectrum of the Cesàro operator on the sequence space ℓ_p has been studied by Gonzalez [17], where $1 < p < \infty$. Okutyi [19, 20] computed the spectrum of the Cesàro operator on the sequence spaces bv and $bv_0 = bv \cap c_0$ and Rhoades [27] extended that result to weighted mean methods over the space bv_0. Akhmedov and Basar [4, 5] have recently determined, independently than that of Gonalez [17], the fine spectrum of the Cesàro operator in the sequence spaces c_0, ℓ_∞ and ℓ_p, by the different way respectively, where $1 < p < \infty$.

The spectrum and the fine spectrum of the Rhally operators on the sequence spaces c_0 and c, under assumption that $\underset{n\to\infty}{lim}(n+1)a_n = L \neq 0$, have been examined by Yildirim [32]. Furthermore, Coskun [12] has studied the spectrum and fine spectrum for p-Cesàro operator acting on the space c_0. More recently, Malafosse [18] and Altay and Basar [8] and Akhmedov and Basar [4] have respectively studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c and $\ell_p, p \geq 1$; where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $||x||_{s_r} = \underset{k\in N}{\sup} \frac{|x_k|}{r^k}, (r > 0)$.

Also, Akhmedov and Basar [3], and Altay and Basar [8] have determined the fine spectrum with respect to Goldberg's classification [16] of the difference operator $\triangle$ and the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_p, bv_p and c_0 and c; respectively where the sequence space bv_p is defined in [7] by $bv_p = \{x = (x_k) \in w : \sum |x_k - x_{k-1}|^p < \infty\}, (1 \leq p < \infty)$.

Furthermore, the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_1 and bv has been studied by Furkan, Bilgic and Kayaduman [13]. Recently the fine spectrum of the operator $B(r, s)$ over ℓ_p and bv_p has been studied by Bilgiç and Furkan [11]. More recently, the fine spectrum of $B(r, s, t)$ over the sequence spaces c_0 and c and ℓ_p and bv_p have been studied by Furkan et al. [14, 15]. Srivastava and Kumar [28] have determined the spectrum and fine spectrum of the operator $\triangle_a$ over the sequence space c_0, where

$\triangle_a : c_0 \to c_0$ is defined by

$$\triangle_a x = \triangle_a(x_n) = (a_n x_n - a_{n-1} x_{n-1})_{n=0}^{\infty} \text{ with } x_{-1} = 0,$$

where (a_k) is either constant or strictly decreasing sequence of positive real numbers satisfying $\underset{k\to\infty}{lim} a_k = a > 0$ and $a_0 \leq 2a$.

The same problem, in the case when the sequence (a_k) is assumed to be constant except for finitely many elements was investigated by Akhmedov [2]. Ahmadov and Shabrawy [1] have studied the spectrum of the operator $\Delta_{a,b}$ over the sequence space c. Spectra of some particular type of matrix operator have been investigated from different aspects by Rath and Tripathy [22].

2. Preliminaries and Definition

Let X be a linear space. By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, where X is a Banach space then the adjoint operator T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$.

Let $T : D(T) \to X$ be a linear operator, defined on $D(T) \subset X$, where $D(T)$ denote the domain of T and X is a complex normed linear space. For $T \in B(X)$ we associate a complex number α with the operator $(T - \alpha I)$ denoted by T_α defined on the same domain $D(T)$, where I is the identity operator. The inverse $(T - \alpha I)^{-1}$, denoted by T_α^{-1} is known as the resolvent operator of T.

A regular value is a complex number α of T such that
(R_1) T_α^{-1} exists,
(R_2) T_α^{-1} is bounded and
(R_3) T_α^{-1} is defined on a set which is dense in X.

The resolvent set of T is the set of all such regular values α of T, denoted by $\rho(T)$. Its complement is given by $C \setminus \rho(T)$ in the complex plane C is called the spectrum of T, denoted by $\sigma(T)$. Thus the spectrum $\sigma(T)$ consist of those values of $\alpha \in C$, for which T_α is not invertible.

Classification of spectrum:

The spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows:

(i) The point(discrete) spectrum $\sigma_{pt}(T)$ is the set such that T_α^{-1} does not exist. Further $\alpha \in \sigma_{pt}(T)$ is called the eigen value of T.

(ii) The continuous spectrum $\sigma_c(T)$ is the set such that T_α^{-1} exists and satisfies (R_3) but not (R_2) that is T_α^{-1} is unbounded.

(iii) The residual spectrum $\sigma_r(T)$ is the set such that T_α^{-1} exists (may be bounded or not) but not satisfy (R_3), that is, the domain of T_α^{-1} is not dense in X .

This is to note that in finite dimensional case, continuous spectrum coincides with the residual spectrum and equal to the empty set and the spectrum consists of only the point spectrum.

Let E and F be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N = \{0, 1, 2, ...\}$. Then, we say that A defines a matrix mapping from E into F, denote by $A : E \to F$, if for every sequence $x = (x_n) \in E$ the sequence $Ax = \{(Ax)_n\}$ is in F where $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$, provided the right hand side converges for every $n \in N$ and $x \in E$.

Our main focus in this paper is on the operator $D(r, 0, 0, s)$, where

$$D(r,0,0,s) = \begin{pmatrix} r & 0 & 0 & 0 & 0 & ... \\ 0 & r & 0 & 0 & 0 & ... \\ 0 & 0 & r & 0 & 0 & ... \\ s & 0 & 0 & r & 0 & ... \\ 0 & s & 0 & 0 & r & ... \\ . & . & . & . & ... & \end{pmatrix},$$

Here we assume that r and s are complex parameters and $s \neq 0$.
Remark: In particular if we consider $r = -1$ and $s = 1$ then $D(-1, 0, 0, 1) = \triangle_3$

2.1. Lemma. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

2.2. Lemma. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the rows of A is bounded.*

2.3. Lemma. *T has a dense range if and only if T^* is one to one, where T^* denote the adjoint operator of T.*

3. The spectrum of the operator $D(r, 0, 0, s)$ on the sequence space ℓ_p, $(1 < p < \infty)$.

3.1. Theorem. *$D(r, 0, 0, s) : \ell_p \to \ell_p$ is a bounded linear operator satisfying the inequalities $(|r|^p + |s|^p)^{\frac{1}{p}} \leq ||D(r, 0, 0, s)||_{\ell_p} \leq |r| + |s|$.*

Proof. The linearity of $D(r, 0, 0, s)$ is trivial and so is omitted. Let us consider $e = (1, 0, 0, ...) \in \ell_p$. Then $D(r, 0, 0, s)e = (r, 0, 0, s, 0, ...)$ and $\frac{||D(r,0,0,s)e||_{\ell_p}}{||e||_{\ell_p}} = (|r|^p + |s|^p)^{\frac{1}{p}}$ which give us $(|r|^p + |s|^p)^{\frac{1}{p}} \leq ||D(r, 0, 0, s)||_{\ell_p}$, for any $p > 1$.
Next let $x = (x_k) \in \ell_p$ then by using Minkowski's inequality and taking $x_{-3} = x_{-2} = x_{-1} = 0$, we have,

$$||D(r,0,0,s)x||_{\ell_p} = (\sum_{k=0}^{\infty} |sx_{k-3} + rx_k|^p)^{\frac{1}{p}}$$

$$\leq (\sum_{k=0}^{\infty} |sx_{k-3}|^p)^{\frac{1}{p}} + (\sum_{k=0}^{\infty} |rx_k|^p)^{\frac{1}{p}} = (|r|+|s|)||x||_{\ell_p}$$

This implies $||D(r,0,0,s)||_{\ell_p} \leq |r| + |s|$. This completes the proof. □

3.2. Lemma. *Let $1 < p < \infty$ and let $A \in (\ell_\infty, \ell_\infty) \cap (\ell_1, \ell_1)$ then $A \in (\ell_p, \ell_p)$.*

3.3. Theorem. $\sigma(D(r,0,0,s), \ell_p) = \{\lambda \in C : |r-\lambda| \leq |s|\}$.

Proof. First, we prove that $(D(r,0,0,s)-\alpha I)^{-1}$ exists and is in (ℓ_p, ℓ_p) for $|r-\alpha| > |s|$ and then we have to show that the operator $(D(r,0,0,s)-\alpha I)$ is not invertible for $|r-\alpha| \leq |s|$.
Let $\alpha \notin \{\lambda \in C : |r-\lambda| \leq |s|\}$. Since $s \neq 0$ we have $\alpha \neq r$ and so $(D(r,0,0,s) - \alpha I)$ is triangle, hence $(D(r,0,0,s) - \alpha I)^{-1}$ exists.
Let,

$$\begin{pmatrix} r-\alpha & 0 & 0 & 0 & 0 & ... \\ 0 & r-\alpha & 0 & 0 & 0 & ... \\ 0 & 0 & r-\alpha & 0 & 0 & ... \\ s & 0 & 0 & r-\alpha & 0 & ... \\ 0 & s & 0 & 0 & r-\alpha & ... \\ . & . & . & . & . & ... \end{pmatrix} \begin{pmatrix} p_0 & 0 & 0 & 0 & 0 & ... \\ p_1 & p_0 & 0 & 0 & 0 & ... \\ p_2 & p_1 & p_0 & 0 & 0 & ... \\ p_3 & p_2 & p_1 & p_0 & 0 & ... \\ p_4 & p_3 & p_2 & p_1 & p_0 & ... \\ . & . & . & . & . & ... \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & ... \\ 0 & 1 & 0 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & 0 & ... \\ 0 & 0 & 0 & 1 & 0 & ... \\ 0 & 0 & 0 & 0 & 1 & ... \\ . & . & . & . & . & ... \end{pmatrix}$$

Then we have

$p_0 = \frac{1}{r-\alpha}$
$p_1 = 0$
$p_2 = 0$
$p_3 = -\frac{s}{(r-\alpha)^2}$
$p_4 = 0$
$p_5 = 0$
$p_6 = \frac{s^2}{(r-\alpha)^3}$
- - -
we obtain
$p_{3k} = \frac{(-s)^k}{(r-\alpha)^{k+1}}$, $(k \geq 0)$
and
$p_{3k+1} = 0$, $(k \geq 0)$
and
$p_{3k+2} = 0$, $(k \geq 0)$.

Hence, we get

$$(D(r,0,0,s)-\alpha I)^{-1} = \begin{pmatrix} \frac{1}{r-\alpha} & 0 & 0 & 0 & 0 & ... \\ 0 & \frac{1}{r-\alpha} & 0 & 0 & 0 & ... \\ 0 & 0 & \frac{1}{r-\alpha} & 0 & 0 & ... \\ -\frac{s}{(r-\alpha)^2} & 0 & 0 & \frac{1}{r-\alpha} & 0 & ... \\ 0 & -\frac{s}{(r-\alpha)^2} & 0 & 0 & \frac{1}{r-\alpha} & ... \\ . & . & . & . & & ... \end{pmatrix}$$

Now, $||(D(r,0,0,s)-\alpha I)^{-1}||_{(\ell_1,\ell_1)} = \sup_k \sum_{n=k}^{\infty} \left|\frac{s}{r-\alpha}\right|^{n-k} \left|\frac{1}{r-\alpha}\right| = \left|\frac{1}{r-\alpha}\right| \sum_{k=0}^{\infty} \left|\frac{1}{r-\alpha}\right|^n < \infty$.
Similarly it can be verified that $||(D(r,0,0,s)-\alpha I)^{-1}||_{(\ell_\infty,\ell_\infty)} < \infty$.
This shows that $(D(r,0,s)-\alpha I)^{-1} \in (\ell_\infty,\ell_\infty) \cap (\ell_1,\ell_1)$ and hence by Lemma 3.2 $(D(r,0,0,s)-\alpha I)^{-1} \in (\ell_p,\ell_p)$ i.e. $\alpha \notin \sigma(D(r,0,0,s),\ell_p)$. This shows that $\sigma(D(r,0,0,s),\ell_p) \subseteq \{\lambda \in C : |r-\lambda| \leq |s|\}$.
Conversely, let $\alpha \in \{\lambda \in C : |r-\lambda| \leq |s|\}$
Case 1: Let $\alpha \neq r$.
Then $(D(r,0,0,s)-\alpha I)$ is triangle, and hence $(D(r,0,0,s)-\alpha I)^{-1}$ exists but for $y = (1,0,0,...) \in \ell_p$, $(D(r,0,0,s)-\alpha I)^{-1} y = (x_k)$ gives $x_{3k} = \frac{(-s)^k}{(r-\alpha)^{k+1}}$, for $(k \geq 0)$ and $x_{3k+1} = 0$, $x_{3k+2} = 0$ for $(k \geq 0)$ therefore $(x_k) \notin \ell_p$ since $|s| \geq |r-\alpha|$ i.e. $(D(r,0,0,s)-\alpha I)^{-1} \notin B(\ell_p)$ which implies $\alpha \in \sigma(D(r,0,0,s),\ell_p)$.
Therefore $\{\lambda \in C : |r-\lambda| \leq |s|\} \subseteq \sigma(D(r,0,0,s),\ell_p)$.
Case 2: Let $\alpha = r$.
Then the operator $(D(r,0,0,s)-\alpha I) = D(0,0,0,s)$ is represented by the matrix

$$(D(r,0,0,s)-\alpha I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & ... \\ 0 & 0 & 0 & 0 & 0 & ... \\ s & 0 & 0 & 0 & 0 & ... \\ 0 & s & 0 & 0 & 0 & ... \\ . & . & . & . & & ... \end{pmatrix} = D(0,0,0,s).$$

Since $D(0,0,0,s)x = \theta$ implies $x = \theta$, $D(0,0,0,s) : \ell_p \to \ell_p$ is injective but not onto. Hence $D(0,0,0,s)$ is not invertible and so $\alpha \in \sigma(D(r,0,0,s),\ell_p)$. Therefore in this case also $\{\lambda \in C : |r-\lambda| \leq |s|\} \subseteq \sigma(D(r,0,0,s),\ell_p)$. This completes the proof. □

3.4. Theorem. $\sigma_{pt}(D(r,0,0,s),\ell_p) = \emptyset$.

Proof. Suppose that $D(r,0,0,s)x = \alpha x$ for $x \neq \theta = (0,0,0,...)$ in ℓ_p. Then by solving the system of linear equations we have
$rx_0 = \alpha x_0$
$rx_1 = \alpha x_1$
$rx_2 = \alpha x_2$
$sx_0 + rx_3 = \alpha x_3$
$sx_1 + rx_4 = \alpha x_4$
- - -
$sx_k + rx_{k+3} = \alpha x_{k+3}$

If $x_{n_0} \neq 0$ is the first non-zero entry of the sequence $x = (x_n)$, then $\alpha = r$ and from the equation $sx_{n_0} + rx_{n_0+3} = \alpha x_{n_0+3}$ we get $sx_{n_0} = 0$. Since $s \neq 0$. we must have $x_{n_0} = 0$, contradicting the fact that $x_{n_0} \neq 0$. This complete the proof.

If $T : \ell_p \to \ell_p$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : \ell_p^* \to \ell_p^*$ is defined by the transpose of the matrix A. It is well-known that the dual space ℓ_p^* of ℓ_p is isomorphic to ℓ_q with $p^{-1} + q^{-1} = 1$. $\square$

3.5. Theorem. $\sigma_{pt}(D(r, 0, 0, s)^*, \ell_p^*) = \{\lambda \in C : |r - \lambda| < |s|\}$.

Proof. Suppose that $D(r, 0, 0, s)^* x = \alpha x$ for $x \neq \theta$ in $\ell_p^* \cong \ell_q$ with $p^{-1} + q^{-1} = 1$. Then by solving the system of linear equations we have
$rx_0 + sx_3 = \alpha x_0$
$rx_1 + sx_4 = \alpha x_1$
$rx_2 + sx_5 = \alpha x_2$
$---$
$rx_k + sx_{k+3} = \alpha x_k$
$---$
we obtain that
$x_{3n} = \left(\frac{\alpha - r}{s}\right)^n x_0$,$(n \geq 1)$
and
$x_{3n+1} = \left(\frac{\alpha - r}{s}\right)^n x_1$,$(n \geq 1)$
and
$x_{3n+2} = \left(\frac{\alpha - r}{s}\right)^n x_2$,$(n \geq 1)$

From the above system of equations we have,
$\sum_{n=1}^{\infty} |x_n|^q = (|x_0| + |x_1| + |x_2|) \sum_{n=0}^{\infty} |\frac{\alpha - r}{s}|^{qn}$ this shows that $(x_n) \in \ell_q$ if and only if $|\alpha - r| < |s|$. This completes the proof. $\square$

3.6. Theorem. $\sigma_r(D(r, 0, 0, s), \ell_p) = \{\lambda \in C : |r - \lambda| < |s|\}$.

Proof. We show that the operator $D(r, 0, 0, s) - \alpha I)$ has an inverse and $\overline{R(D(r, 0, 0, s) - \alpha I))} \neq \ell_p$ for $\alpha \in \{\lambda \in C : |r - \lambda| < |s|\}$. For $\alpha \neq r$ the operator $(D(r, 0, 0, s) - \alpha I)$ is triangle and has an inverse. For $\alpha = r$, the operator $(D(r, 0, 0, s) - \alpha I)$ is one to one and hence has an inverse. But by Theorem 3.5 implies that $(D(r, 0, 0, s)^* - \alpha I)$ is not one to one for $|r - \alpha| < |s|$. Now using the Lemma 2.3 we can conclude that the range of $(D(r, 0, 0, s) - \alpha I)$ is not dense in ℓ_p, i.e. $\overline{R(D(r, 0, 0, s) - \alpha I))} \neq \ell_p$. This completes the proof. $\square$

3.7. Theorem. $\sigma_c(D(r, 0, 0, s), \ell_p) = \{\lambda \in C : |r - \lambda| = |s|\}$.

Proof. The proof immediately follows from the fact that the set of spectrum is the disjoint union of the point spectrum, residual spectrum and continuous spectrum, that is
$\sigma(D(r, 0, 0, s), \ell_p) = \sigma_{pt}(D(r, 0, 0, s), \ell_p) \cup \sigma_r(D(r, 0, 0, s), \ell_p) \cup \sigma_c(D(r, 0, 0, s), \ell_p)$.
$\square$

4. The Spectrum of the operator $D(r,0,0,s)$ on the sequence space bv_p

4.1. Theorem. $D(r,0,0,s) \in B(bv_p)$.

Proof. The linearity of $D(r,0,0,s)$ is trivial and so is omitted. Let us take $x = (x_k) \in bv_p$ then by using Minkowski's inequality and taking the negative indices $x_{-k} = 0$, we have

$$||D(r,0,0,s)x||_{bv_p} = \left(\sum_{k=0}^{\infty} |(rx_k + sx_{k-3}) - (rx_{k-1} + sx_{k-4})|^p\right)^{\frac{1}{p}}$$

$$\leq \left(|r|^p \sum_{k=0}^{\infty} |x_k - x_{k-1}|^p\right)^{\frac{1}{p}} + \left(|s|^p \sum_{k=0}^{\infty} |x_{k-3} - x_{k-4}|^p\right)^{\frac{1}{p}} = (|r| + |s|)||x||_{bv_p}$$

This gives $||D(r,0,0,s)||_{bv_p} \leq |r| + |s|$. □

4.2. Theorem. $\sigma(D(r,0,0,s), bv_p) = \{\lambda \in C : |r - \lambda| \leq |s|\}$.

Proof. First, we prove that $(D(r,0,0,s) - \alpha I)^{-1}$ exists and is in (bv_p, bv_p) for $|r - \alpha| > |s|$ and then we have to show that the operator $(D(r,0,0,s) - \alpha I)$ is not invertible for $|r - \alpha| \leq |s|$.
Let $\alpha \notin \{\lambda \in C : |r - \lambda| \leq |s|\}$. Since $s \neq 0$ we have $\alpha \neq r$ and so $(D(r,0,0,s) - \alpha I)$ is triangle, hence $(D(r,0,0,s) - \alpha I)^{-1}$ exists.
Let $y = (y_k) \in bv_p$.This implies that $(y_k - y_{k-1}) \in \ell_p$. Solving the system of equations $(D(r,0,0,s) - \alpha I)x = y$ we have as in the proof of Theorem 3.3 that

$$x_k - x_{k-1} = \sum_{j=0}^{k} p_{k-j+1}(y_j - y_{j-1}); (k \in N), \text{ where } x_{-1} = y_{-1} = 0$$

i.e.$(x_k - x_{k-1}) = (D(r,0,0,s) - \alpha I)^{-1}(y_k - y_{k-1})$. Since $(D(r,0,0,s) - \alpha I)^{-1} \in (bv_p, bv_p)$ by Theorem 3.3, $(x_k - x_{k-1}) \in \ell_p$. This implies that $x = (x_k) \in bv_p$ and hence $(D(r,0,0,s)-\alpha I)^{-1} \in (bv_p, bv_p)$ this shows that $\alpha \notin \sigma(D(r,0,0,s), bv_p)$. Hence $\sigma(D(r,0,0,s), bv_p) \subseteq \{\lambda \in C : |r - \lambda| \leq |s|\}$.

Conversely, let $\alpha \in \{\lambda \in C : |r - \lambda| \leq |s|\}$. If $r \neq \alpha$, then $(D(r,0,0,s) - \alpha I)$ is triangle, hence $(D(r,0,0,s) - \alpha I)^{-1}$ exists, but $y = (1,0,0,...) \in bv_p$ gives $x = (x_k)$ with $x_{3k} = \frac{(-s)^k}{(r-\alpha)^{k+1}}$, for $(k \geq 0)$ and $x_{3k} = x_{3k+1} = 0$, for $(k \geq 0)$. Clearly, $(x_k) \notin bv_p$ for $|s| \geq |r - \alpha|$. This shows that $\alpha \in (D(r,0,0,s), bv_p)$.
Next let, $r = \alpha$, then similar arguments as in the proof of Theorem 3.3. shows that the operator $D(r,0,0,s) - \alpha I = D(0,0,0,s)$ is not invertible, therefore in this case also $\alpha \in \sigma(D(r,0,0,s), bv_p)$. Thus, $\{\lambda \in C : |r - \lambda| \leq |s|\} \subseteq \sigma(D(r,0,0,s), bv_p)$. This completes the proof. □

Since the spectrum and fine spectrum of the matrix $D(r,0,0,s)$ as an operator on the sequence space bv_p are similar to that of the space ℓ_p in Section 3, to avoid the repetition of the similar statements we give the results in the following theorem without proof.

4.3. Theorem. *(i)* $\sigma_{pt}(D(r,0,0,s), bv_p) = \emptyset$.
(ii) $\sigma_{pt}(D(r,0,0,s)^*, bv_p^*) = \{\lambda \in C : |r - \lambda| < |s|\}$.
(iii)$\sigma_r(D(r,0,0,s), bv_p) = \{\lambda \in C : |r - \lambda| < |s|\}$.
(iv)$\sigma_c(D(r,0,0,s), bv_p) = \{\lambda \in C : |r - \lambda| = |s|\}$.

5. Conclusion

We can generalize our operator

$$(D(r,0,0,..(n-1)\text{times},s) = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & r & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ s & 0 & . & . & r & 0 & . & . & . \\ 0 & s & 0 & . & . & r & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}$$

If we take $r = -1$ and $s = 1$, then the operator $(D(r,0,0,..(n-1)\text{times},s)$ will be the same as the generalized difference operator $\triangle_n$. Further on considering the operator $(D(r,0,0,..(n-1)\text{times},s)$ in place of $D(r,0,0,s)$, one can get parallel all our results obtained in this paper.

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Some subclasses of meromorphic multivalent functions involving a generalized differential operator

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Abstract

Making use of a generalized differential operator which is defined by means of the Hadamard product, we introduce some new subclasses of meromorphic p-valent functions and investigate their inclusion relationships, integral preserving and convolution properties. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

Keywords: Meromorphic functions, multivalent functions, Hadamard product(or convolution), subordination between analytic functions, generalized differential operator.

2000 AMS Classification: 30C45, 30C80

1. Introduction and Preliminaries

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : \ z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

Let $f, \ g \in \Sigma_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_{n-p} z^{n-p}.$$

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Then the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) := z^{-p} + \sum_{n=1}^{\infty} a_{n-p} b_{n-p} z^{n-p} =: (g * f)(z).$$

Let $\mathcal{P}$ denote the class of functions of the form:

$$p\,(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition:

$$\Re(p\,(z)) > 0 \qquad (z \in \mathbb{U}).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\,(\omega(z)) \qquad (z \in \mathbb{U}).$$

Indeed, it is known that (see [12] or [13])

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Analogous to the operator defined recently by Selvaraj and Selvakumaran [20] and Aouf *et al.* [2], we introduce the following integral operator:

$$\mathcal{M}_{\lambda,g}^{\delta} : \Sigma_p \longrightarrow \Sigma_p$$

defined by

$$\mathcal{M}_{\lambda,g}^{0} f(z) = (f * g)(z),$$

$$\mathcal{M}_{\lambda,g}^{1} f(z) = (1+\lambda)(f * g)(z) + \frac{\lambda}{p} z (f * g)'(z),$$

$$\mathcal{M}_{\lambda,g}^{\delta} f(z) = \mathcal{M}_{\lambda,g}^{1} \left(\mathcal{M}_{\lambda,g}^{\delta-1} f(z) \right) \qquad (\delta \in \mathbb{N};\ \lambda \geqq 0). \tag{1.2}$$

If $f \in \Sigma_p$, then we have

$$\mathcal{M}_{\lambda,g}^{\delta} f(z) = z^{-p} + \sum_{n=1}^{\infty} \left(1 + \frac{n\lambda}{p}\right)^{\delta} a_{n-p} b_{n-p} z^{n-p}. \tag{1.3}$$

It easily follows from (1.2) that

$$\frac{\lambda z}{p} \left(\mathcal{M}_{\lambda,g}^{\delta} f\right)'(z) = \mathcal{M}_{\lambda,g}^{\delta+1} f(z) - (1+\lambda) \mathcal{M}_{\lambda,g}^{\delta} f(z). \tag{1.4}$$

Throughout this paper, we assume that

$$p,\ k \in \mathbb{N}, \qquad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right),$$

and

$$(1.5) \qquad f_{p,k}^{\delta}(\lambda;g;z) = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{jp}\mathcal{M}_{\lambda,g}^{\delta}f(\varepsilon_k^j z) = z^{-p} + \cdots \qquad (f \in \Sigma_p).$$

Clearly, for $k = 1$, we have

$$f_{p,1}^{\delta}(\lambda;g;z) = \mathcal{M}_{\lambda,g}^{\delta}f(z).$$

Making use of the integral operator $\mathcal{M}_{\lambda,g}^{\delta}$ and the above-mentioned principle of subordination between analytic functions, we now introduce and investigate the following subclasses of the class Σ_p of mermorphically p-valent functions.

1.1. Definition. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$ if it satisfies the following subordination condition:

$$(1.6) \qquad -\frac{z\left(\mathcal{M}_{\lambda,g}^{\delta}f\right)'(z)}{pf_{p,k}^{\delta}(\lambda;g;z)} \prec h(z) \qquad (z \in \mathbb{U}),$$

where

$$h \in \mathcal{P} \quad \text{and} \quad f_{p,k}^{\delta}(\lambda;g;z) \neq 0 \quad (z \in \mathbb{U}^*).$$

1.2. Remark. In a recent paper, Srivastava *et al.* [21] introduced an investigated a subclass $\Sigma_{p,k}(a,c;h)$ of Σ_p consisting of functions which are satisfy the following subordination condition:

$$-\frac{z\left(\mathcal{L}_p(a,c)f\right)'(z)}{pf_{p,k}(a,c;z)} \prec h(z) \qquad (z \in \mathbb{U};\ c \neq 0,-1,-2,\ldots),$$

where $h \in \mathcal{P}$,

$$\mathcal{L}_p(a,c)f(z) = \varphi_p(a,c;z) * f(z) = \left(z^{-p} + \sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}z^{n-p}\right) * f(z) \qquad (z \in \mathbb{U}^*),$$

and

$$f_{p,k}(a,c;z) = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{jp}(\mathcal{L}_p(a,c)f)(\varepsilon_k^j z) \neq 0 \qquad (z \in \mathbb{U}^*).$$

The above $(\mu)_n$ is the Pochhammer symbol defined by

$$(\mu)_0 = 1 \quad \text{and} \quad (\mu)_n = \mu(\mu+1)\cdots(\mu+n-1) \qquad (n \in \mathbb{N}).$$

It is also easy to see that, if we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = \varphi_p(a,c;z)$$

in the class $\mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$, then it reduces to the class $\Sigma_{p,k}(a,c;h)$.

More recently, Wang *et al.* [22] studied a subclass $\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;h)$ of Σ_p consisting of functions which are satisfy the following subordination condition:

$$-\frac{z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(z) + \alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \alpha f_{p,k}^{q,s}(\alpha_1+1;z)\right]} \prec h(z) \qquad (z \in \mathbb{U}),$$

where $h \in \mathcal{P}$,

$$H_p^{q,s}(\alpha_1)f(z) = h_p^{q,s}(\alpha_1;z)*f(z) = \left(z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^{n-p}}{n!}\right)*f(z) \qquad (z \in \mathbb{U}^*),$$

and

$$f_{p,k}^{q,s}(\alpha_1;z) = \frac{1}{k}\sum_{n=1}^{k-1} \varepsilon_k^{jp}(H_p^{q,s}(\alpha_1)f)(\varepsilon_k^j z) \neq 0 \qquad (z \in \mathbb{U}^*).$$

It is also easy to see that, if we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = (1+\alpha)h_p^{q,s}(\alpha_1;z) + \alpha h_p^{q,s}(\alpha_1+1;z)$$

in the class $\mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$, then it reduces to the class $\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;h)$.

1.3. Definition. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$ if it satisfies the following subordination condition:

$$-\frac{z\left(\mathcal{M}_{\lambda,g}^{\delta}f\right)'(z)}{p\varphi_{p,k}^{\delta}(\lambda;g;z)} \prec h(z) \qquad (z \in \mathbb{U}) \tag{1.7}$$

for some $\varphi \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$, where

$$h \in \mathcal{P} \quad \text{and} \quad \varphi_{p,k}^{\delta}(\lambda;g;z) \neq 0 \quad (z \in \mathbb{U}^*).$$

1.4. Remark. If we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = \varphi_p(a,c;z)$$

in the class $\mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$, then it reduces to the class $\mathcal{K}_{p,k}(a,c;h)$, which was also introduced and studied recently by Srivastava *et al.* [21].

If we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = (1+\alpha)h_p^{q,s}(\alpha_1;z) + \alpha h_p^{q,s}(\alpha_1+1;z)$$

in the class $\mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$, then it reduces to the class $\mathcal{G}_{p,k}^{q,s}(\alpha;\alpha_1;h)$, which was also introduced and studied recently by Wang *et al.* [22].

1.5. Definition. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{H}_{p,k}^{\delta}(\alpha,\lambda;g;h)$ if it satisfies the following subordination condition:

$$-(1-\alpha)\frac{z\left(\mathcal{M}_{\lambda,g}^{\delta}f\right)'(z)}{p\varphi_{p,k}^{\delta}(\lambda;g;z)} - \alpha\frac{z\left(\mathcal{M}_{\lambda,g}^{\delta+1}f\right)'(z)}{p\varphi_{p,k}^{\delta+1}(\lambda;g;z)} \prec h(z) \qquad (z \in \mathbb{U}), \tag{1.8}$$

for some $\alpha \geqq 0$ and $\varphi \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$, where

$$h \in \mathcal{P} \quad \text{and} \quad \varphi_{p,k}^{\delta+1}(\lambda;g;z) \neq 0 \quad (z \in \mathbb{U}^*).$$

1.6. Remark. If we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = \varphi_p(a,c;z)$$

in the class $\mathcal{H}_{p,k}^{\delta}(\lambda;g;h)$, then it reduces to the class $\mathcal{K}_{p,k}(\alpha;a,c;h)$, which was also introduced and studied recently by Srivastava *et al.* [21].

1.7. Remark. By suitably specifying the values of $p,\ k,\ \delta,\ \lambda,\ \alpha,\ g$ and h, the classes

$$\mathcal{S}_{p,k}^{\delta}(\lambda;g;h),\ \mathcal{K}_{p,k}^{\delta}(\lambda;g;h) \quad \text{and} \quad \mathcal{H}_{p,k}^{\delta}(\alpha,\lambda;g;h)$$

reduce to the various subclasses introduced and studied in [9, 10, 26, 27]. For some recent investigations on meromorphic functions, see (for example) the earlier works [1, 3, 4, 5, 6, 7, 8, 11, 14, 15, 17, 18, 23, 24, 25] and the references cited therein.

In order to establish our main results, we shall also make use of the following lemmas.

1.8. Lemma. (See [12]) *Let $\vartheta,\ \gamma \in \mathbb{C}$ with $\vartheta \neq 0$. Suppose that φ is convex and univalent in $\mathbb{U}$ with*

$$\varphi(0) = 1 \quad \textit{and} \quad \Re(\vartheta\varphi(z) + \gamma) > 0 \quad (z \in \mathbb{U}).$$

If $\mathfrak{p}$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$, then the following subordination

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\vartheta\mathfrak{p}(z) + \gamma} \prec \varphi(z) \qquad (z \in \mathbb{U})$$

implies that

$$\mathfrak{p}(z) \prec \varphi(z) \qquad (z \in \mathbb{U}).$$

1.9. Lemma. (See [13]) *Let η be analytic and convex univalent in $\mathbb{U}$ and let ζ be analytic in $\mathbb{U}$ with*

$$\Re(\zeta(z)) \geqq 0 \quad (z \in \mathbb{U}).$$

If $\mathfrak{q}$ is analytic in $\mathbb{U}$ with $\mathfrak{q}(0) = \eta(0)$, then the following subordination

$$\mathfrak{q}(z) + \zeta(z)z\mathfrak{q}'(z) \prec \eta(z) \qquad (z \in \mathbb{U})$$

implies that

$$\mathfrak{q}(z) \prec \eta(z) \qquad (z \in \mathbb{U}).$$

1.10. Lemma. *Let $f \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$. Then*

$$-\frac{z\left(f_{p,k}^{\delta}(\lambda;g;z)\right)'}{pf_{p,k}^{\delta}(\lambda;g;z)} \prec h(z) \qquad (z \in \mathbb{U}). \tag{1.9}$$

Proof. Making use of (1.5), we have

$$\begin{aligned} f_{p,k}^{\delta}(\lambda;g;\varepsilon_k^j z) &= \frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{np}\mathcal{M}_{\lambda,g}^{\delta}f(\varepsilon_k^{n+j}z) \\ &= \varepsilon_k^{-jp}\cdot\frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{(n+j)p}\mathcal{M}_{\lambda,g}^{\delta}f(\varepsilon_k^{n+j}z) \\ &= \varepsilon_k^{-jp}f_{p,k}^{\delta}(\lambda;g;z) \qquad (j \in \{0,1,\dots,k-1\}), \end{aligned} \tag{1.10}$$

and

$$\left(f_{p,k}^{\delta}(\lambda;g;z)\right)' = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{j(p+1)}(\mathcal{M}_{\lambda,g}^{\delta}f)'(\varepsilon_k^j z).$$

Hence

$$(1.11)\qquad \begin{aligned} -\frac{z\left(f^{\delta}_{p,k}(\lambda;g;z)\right)'}{pf^{\delta}_{p,k}(\lambda;g;z)} &= \frac{1}{k}\sum_{j=0}^{k-1}\frac{\varepsilon_k^{j(p+1)}z(\mathcal{M}^{\delta}_{\lambda,g}f)'(\varepsilon_k^j z)}{-pf^{\delta}_{p,k}(\lambda;g;z)} \\ &= \frac{1}{k}\sum_{j=0}^{k-1}\frac{\varepsilon_k^{j}z(\mathcal{M}^{\delta}_{\lambda,g}f)'(\varepsilon_k^j z)}{-pf^{\delta}_{p,k}(\lambda;g;\varepsilon_k^j z)} \qquad (z\in\mathbb{U}). \end{aligned}$$

Moreover, since $f\in\mathcal{S}^{\delta}_{p,k}(\lambda;g;h)$, we have

$$(1.12)\qquad -\frac{\varepsilon_k^{j}z(\mathcal{M}^{\delta}_{\lambda,g}f)'(\varepsilon_k^j z)}{pf^{\delta}_{p,k}(\lambda;g;\varepsilon_k^j z)} \prec h(z) \qquad (z\in\mathbb{U};\ j\in\{0,1,\ldots,k-1\}).$$

Noting that h is convex and univalent in $\mathbb{U}$, from (1.11) and (1.12), we conclude that the assertion (1.9) of Lemma 1.10 holds true. $\square$

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \qquad (z\in\mathbb{U}).$$

A function $f\in\mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in\mathbb{U};\ \alpha<1).$$

We denote this class by $\mathcal{S}^*(\alpha)$. A function $f\in\mathcal{A}$ is said to be prestarlike of order α in $\mathbb{U}$ if

$$\frac{z}{(1-\alpha)^{2-2\alpha}}*f(z)\in\mathcal{S}^*(\alpha) \qquad (z\in\mathbb{U};\ \alpha<1).$$

We denote this class by $\mathcal{R}(\alpha)$. It is clear that a $f\in\mathcal{A}$ is in the class $\mathcal{R}(0)$ if and only if f is convex univalent in $\mathbb{U}$ and that

$$\mathcal{R}(\frac{1}{2})=\mathcal{S}^*(\frac{1}{2}).$$

1.11. Lemma. (See [13]) *Let $\alpha<1$, $f\in\mathcal{R}(\alpha)$ and $\phi\in\mathcal{S}^*(\alpha)$. Then, for any analytic function H in $\mathbb{U}$,*

$$\frac{f*(\phi H)}{f*\phi}(\mathbb{U})\subset\overline{co}(H(\mathbb{U})),$$

where $\overline{co}(H(\mathbb{U}))$ denotes the close convex hull of $H(\mathbb{U})$.

In the present paper, we aim at proving such results inclusion relationships, integral preserving and convolution properties for each of the function classes. The results presented here would provide extensions of those given in a number of earlier works. Several other new results are also obtained.

2. A Set of Inclusion Relationships

We first provide some inclusion relationships for the function classes

$$\mathcal{S}_{p,k}^{\delta}(\lambda;g;h),\ \mathcal{K}_{p,k}^{\delta}(\lambda;g;h) \quad \text{and} \quad \mathcal{H}_{p,k}^{\delta}(\alpha,\lambda;g;h)$$

which were defined in preceding section.

2.1. Theorem. *Let* $h \in \mathcal{P}$ *with*

(2.1) $\quad \Re(h(z)) < 1 + \dfrac{1}{\lambda} \qquad (\lambda > 0;\ z \in \mathbb{U}).$

Then

$$\mathcal{S}_{p,k}^{\delta+1}(\lambda;g;h) \subset \mathcal{S}_{p,k}^{\delta}(\lambda;g;h).$$

Proof. By using (1.4) and (1.5), we have

(2.2)

$$(1+\lambda)f_{p,k}^{\delta}(\lambda;g;z)+\frac{\lambda z}{p}\left(f_{p,k}^{\delta}(\lambda;g;z)\right)' = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{jp}\mathcal{M}_{\lambda,g}^{\delta+1}f(\varepsilon_k^{j}z) = f_{p,k}^{\delta+1}(\lambda;g;z).$$

Let $f \in \mathcal{S}_{p,k}^{\delta+1}(\lambda;g;h)$ and suppose that

(2.3) $\quad \varpi(z) = -\dfrac{z\left(f_{p,k}^{\delta}(\lambda;g;z)\right)'}{pf_{p,k}^{\delta}(\lambda;g;z)} \qquad (z \in \mathbb{U}),$

then ϖ is analytic in $\mathbb{U}$ with $\varpi(0) = 1$. It follows from (2.2) and (2.3) that

(2.4) $\quad (1-\lambda) - \lambda\varpi(z) = \dfrac{f_{p,k}^{\delta+1}(\lambda;g;z)}{f_{p,k}^{\delta}(\lambda;g;z)}.$

Differentiating both sides of (2.4) with respect to z and using (2.3), we have

(2.5) $\quad \varpi(z) + \dfrac{z\varpi'(z)}{p(1+\frac{1}{\lambda}) - p\varpi(z)} = -\dfrac{z\left(f_{p,k}^{\delta+1}(\lambda;g;z)\right)'}{pf_{p,k}^{\delta+1}(\lambda;g;z)}.$

From (2.5) and Lemma 1.10, we find that

(2.6) $\quad \varpi(z) + \dfrac{z\varpi'(z)}{p(1+\frac{1}{\lambda}) - p\varpi(z)} \prec h(z) \qquad (z \in \mathbb{U}).$

Now, in view of (2.1) and (2.6), an application of Lemma 1.8 yields

(2.7) $\quad \varpi(z) \prec h(z) \qquad (z \in \mathbb{U}).$

Set

(2.8) $\quad q(z) = -\dfrac{z(\mathcal{M}_{\lambda,g}^{\delta}f)'(z)}{pf_{p,k}^{\delta}(\lambda;g;z)},$

then q is analytic in $\mathbb{U}$ with $q(0) = 1$. We obtain from (1.4) that

(2.9) $\quad f_{p,k}^{\delta}(\lambda;g;z)q(z) = (1+\dfrac{1}{\lambda})\mathcal{M}_{\lambda,g}^{\delta}f(z) - \dfrac{1}{\lambda}\mathcal{M}_{\lambda,g}^{\delta+1}f(z).$

Differentiating both sides of (2.9) and using (2.8), we get

$$(2.10)\quad zq'(z)+\left(p(1+\frac{1}{\lambda})+\frac{z\left(f_{p,k}^{\delta}(\lambda;g;z)\right)'}{f_{p,k}^{\delta}(\lambda;g;z)}\right)q(z)=-\frac{z(\mathcal{M}_{\lambda,g}^{\delta+1}f)'(z)}{\lambda f_{p,k}^{\delta}(\lambda;g;z)}.$$

Since $f\in\mathcal{S}_{p,k}^{\delta+1}(\lambda;g;h)$, we find from (2.2), (2.3) and (2.10) that

$$(2.11)\quad q(z)+\frac{zq'(z)}{p(1+\frac{1}{\lambda})-p\varpi(z)}=-\frac{z(\mathcal{M}_{\lambda,g}^{\delta+1}f)'(z)}{pf_{p,k}^{\delta+1}(\lambda;g;z)}\prec h(z)\qquad(z\in\mathbb{U}).$$

From (2.1) and (2.7), we observe that

$$\Re\left(p(1+\frac{1}{\lambda})-p\varpi(z)\right)>0.$$

Therefore, from (2.11) and Lemma 1.9, we conclude that

$$q(z)\prec h(z)\qquad(z\in\mathbb{U}),$$

which implies $f\in\mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$. The proof of Theorem 2.1 is thus completed. □

2.2. Theorem. *Let $h\in\mathcal{P}$ with*

$$(2.12)\quad \Re(h(z))<1+\frac{1}{\lambda}\qquad(\lambda>0;\ z\in\mathbb{U}).$$

Then

$$\mathcal{K}_{p,k}^{\delta+1}(\lambda;g;h)\subset\mathcal{K}_{p,k}^{\delta}(\lambda;g;h).$$

Proof. Let $f\in\mathcal{K}_{p,k}^{\delta+1}(\lambda;g;h)$, then there exists a function $\varphi\in\mathcal{S}_{p,k}^{\delta+1}(\lambda;g;h)$ such that

$$(2.13)\quad -\frac{z(\mathcal{M}_{\lambda,g}^{\delta+1}f)'(z)}{p\varphi_{p,k}^{\delta+1}(\lambda;g;z)}\prec h(z)\qquad(z\in\mathbb{U}).$$

An application of Theorem 2.1 yields $\varphi\in\mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$ and Lemma 1.10 leads to

$$(2.14)\quad \psi(z)=-\frac{z\left(\varphi_{p,k}^{\delta}(\lambda;g;z)\right)'}{p\varphi_{p,k}^{\delta}(\lambda;g;z)}\prec h(z)\qquad(z\in\mathbb{U}).$$

Let

$$(2.15)\quad q(z)=-\frac{z(\mathcal{M}_{\lambda,g}^{\delta}f)'(z)}{p\varphi_{p,k}^{\delta}(\lambda;g;z)}\qquad(z\in\mathbb{U}).$$

By using (1.4), (2.15) can be written as follows:

$$(2.16)\quad \varphi_{p,k}^{\delta}(\lambda;g;z)q(z)=(1+\frac{1}{\lambda})\mathcal{M}_{\lambda,g}^{\delta}f(z)-\frac{1}{\lambda}\mathcal{M}_{\lambda,g}^{\delta+1}f(z).$$

Differentiating both sides of (2.16) and using (2.2)(with f replaced by φ), we find that

$$(2.17)\quad q(z)+\frac{zq'(z)}{p(1+\frac{1}{\lambda})-p\psi(z)}=-\frac{z(\mathcal{M}_{\lambda,g}^{\delta+1}f)'(z)}{p\varphi_{p,k}^{\delta+1}(\lambda;g;z)}\qquad(z\in\mathbb{U}).$$

Combining (2.13) and (2.17), we obtain

$$q(z) + \frac{zq'(z)}{p(1+\frac{1}{\lambda}) - p\psi(z)} \prec h(z) \qquad (z \in \mathbb{U}). \tag{2.18}$$

Combining (2.12), (2.14) and (2.18), we deduce from Lemma 1.9 that

$$q(z) \prec h(z) \qquad (z \in \mathbb{U}),$$

which shows that $f \in \mathcal{K}^{\delta}_{p,k}(\lambda; g; h)$. □

By carefully selecting the function h involved in Theorem 2.1 and Theorem 2.2, we can obtain a number of useful corollaries.

2.3. Corollary. *Let* $0 < \alpha \leqq 1$, $-1 \leqq B < A \leqq 1$ *and*

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \qquad (z \in \mathbb{U}). \tag{2.19}$$

If $\lambda > \left[\left(\frac{1+A}{1+B}\right)^{\alpha} - 1\right]^{-1}$, *then*

$$\mathcal{S}^{\delta+1}_{p,k}(\lambda; g; h) \subset \mathcal{S}^{\delta}_{p,k}(\lambda; g; h) \quad \text{and} \quad \mathcal{K}^{\delta+1}_{p,k}(\lambda; g; h) \subset \mathcal{K}^{\delta}_{p,k}(\lambda; g; h).$$

Proof. The analytic function h defined by (2.19) is convex univalent in $\mathbb{U}$ (see [21]), $h(0) = 1$ and $h(\mathbb{U})$ is symmetric with respect to real axis. Thus $h \in \mathcal{P}$ and

$$0 < \left(\frac{1-A}{1-B}\right)^{\alpha} < \Re(h(z)) < \left(\frac{1+A}{1+B}\right)^{\alpha} \qquad (z \in \mathbb{U};\ 0 < \alpha \leqq 1;\ -1 \leqq B < A \leqq 1).$$

Hence, by using Theorem 2.1 and 2.2, we have the corollaryllary. □

2.4. Corollary. *Let* $0 < \alpha < 1$ *and*

$$h(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{\alpha z}}{1-\sqrt{\alpha z}}\right)\right)^2 \qquad (z \in \mathbb{U}). \tag{2.20}$$

If $\lambda > \frac{\pi^2}{2}\left(\log\left(\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)\right)^{-2}$, *then*

$$\mathcal{S}^{\delta+1}_{p,k}(\lambda; g; h) \subset \mathcal{S}^{\delta}_{p,k}(\lambda; g; h) \quad \text{and} \quad \mathcal{K}^{\delta+1}_{p,k}(\lambda; g; h) \subset \mathcal{K}^{\delta}_{p,k}(\lambda; g; h).$$

Proof. The function h defined by (2.20) is in the class $\mathcal{P}$ (see [19]) and $h(\bar{z}) = \overline{h(z)}$. Therefore

$$\frac{1}{2} < h(-1) < \Re(h(z)) < h(1) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)\right)^2 \qquad (z \in \mathbb{U};\ 0 < \alpha < 1).$$

Hence, by using Theorem 2.1 and 2.2, we have the corollary. □

2.5. Theorem. *Let* $h \in \mathcal{P}$ *with*

$$\Re(h(z)) < 1 + \frac{1}{\lambda} \qquad (\lambda > 0;\ z \in \mathbb{U}). \tag{2.21}$$

Then

$$\mathcal{H}^{\delta}_{p,k}(\alpha_2, \lambda; g; h) \subset \mathcal{H}^{\delta}_{p,k}(\alpha_1, \lambda; g; h) \qquad (0 \leqq \alpha_1 < \alpha_2).$$

Proof. For $f \in \mathcal{H}^{\delta}_{p,k}(\alpha_2, \lambda; g; h)$, there exists a function $\varphi \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$ satisfying $\varphi^{\delta+1}_{p,k}(\lambda; g; z) \neq 0$ such that

$$-(1-\alpha_2)\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda;g;z)} - \alpha_2\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda;g;z)} \prec h(z) \qquad (z \in \mathbb{U}). \tag{2.22}$$

Put

$$q(z) = -\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda;g;z)} \qquad (z \in \mathbb{U}).$$

Since $\varphi \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$, it follows from (2.14) to (2.17)(using in the proof of Theorem 2.2) and (2.22) that

$$q(z)+\frac{\alpha_2 zq'(z)}{p(1+\frac{1}{\lambda})-p\psi(z)} = -(1-\alpha_2)\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda;g;z)}-\alpha_2\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda;g;z)} \prec h(z) \qquad (z \in \mathbb{U}). \tag{2.23}$$

In light of (2.14) and (2.21), we thus observe that

$$\frac{1}{\alpha_2}\Re\left(p(1+\frac{1}{\lambda}-p\psi(z)\right) > 0 \qquad (z \in \mathbb{U}).$$

Hence, by (2.23) and Lemma 1.9, we have

$$q(z) \prec h(z) \qquad (z \in \mathbb{U}). \tag{2.24}$$

Since $0 \leqq \frac{\alpha_1}{\alpha_2} < 1$ and h is convex univalent in $\mathbb{U}$, we deduce from (2.22) and (2.24) that

$$\begin{aligned} &-(1-\alpha_1)\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda;g;z)} - \alpha_1\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda;g;z)} = \left(1-\frac{\alpha_1}{\alpha_2}\right)q(z) \\ &+\frac{\alpha_1}{\alpha_2}\left(-(1-\alpha_2)\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda;g;z)} - \alpha_2\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda;g;z)}\right) \prec h(z) \qquad (z \in \mathbb{U}). \end{aligned} \tag{2.25}$$

Thus $f \in \mathcal{H}^{\delta}_{p,k}(\alpha_1, \lambda; g; h)$. The proof of Theorem 2.5 is evidently completed. □

3. Integral Preserving Properties

In this section, we prove some integral preserving properties of the subclasses

$$\mathcal{S}^{\delta}_{p,k}(\lambda; g; h) \quad \text{and} \quad \mathcal{K}^{\delta}_{p,k}(\lambda; g; h).$$

3.1. Theorem. *Let $h \in \mathcal{P}$ with*

$$\Re(h(z)) < \frac{\Re(c)}{p} \qquad (z \in \mathbb{U};\ \Re(c) > p). \tag{3.1}$$

If $f \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$, then the function defined by

$$F(z) = \frac{c-p}{z^c}\int_0^z t^{c-1}f(t)dt \tag{3.2}$$

is also in the class $\mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$, provided that

$$F^{\delta}_{p,k}(\lambda; g; z) \neq 0 \qquad (z \in \mathbb{U}^*).$$

Proof. Let $f \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h))$, we then find from (3.2) that

$$c\mathcal{M}_{\lambda,g}^{\delta} F(z) + z(\mathcal{M}_{\lambda,g}^{\delta} F)'(z) = (c-p)\mathcal{M}_{\lambda,g}^{\delta} f(z). \tag{3.3}$$

By using (3.3), we get

$$cF_{p,k}^{\delta}(\lambda; g; z) + z\left(F_{p,k}^{\delta}(\lambda; g; z)\right)' = (c-p)f_{p,k}^{\delta}(\lambda; g; z). \tag{3.4}$$

Let

$$\chi(z) = -\frac{z\left(F_{p,k}^{\delta}(\lambda; g; z)\right)'}{pF_{p,k}^{\delta}(\lambda; g; z)}.$$

Then χ is analytic in $\mathbb{U}$, with $\chi(0) = 1$, and from (3.4) we observe that

$$c - p\chi(z) = (c-p)\frac{f_{p,k}^{\delta}(\lambda; g; z)}{F_{p,k}^{\delta}(\lambda; g; z)}. \tag{3.5}$$

Differentiating both sides of (3.5) with respect to z and using Lemma 1.10, we obtain

$$\chi(z) + \frac{z\chi'(z)}{c - p\chi(z)} = -\frac{z\left(f_{p,k}^{\delta}(\lambda; g; z)\right)'}{pf_{p,k}^{\delta}(\lambda; g; z)} \prec h(z). \tag{3.6}$$

In view of (3.6), Lemma 1.9 leads to $\chi(z) \prec h(z)$. If we let

$$\kappa(z) = -\frac{z(\mathcal{M}_{\lambda,g}^{\delta} F)'(z)}{pF_{p,k}^{\delta}(\lambda; g; z)}, \tag{3.7}$$

then κ is analytic in $\mathbb{U}$ with $\kappa(0) = 1$. It follows from (3.3) that

$$F_{p,k}^{\delta}(\lambda; g; z)\kappa(z) = -\frac{c-p}{p}\mathcal{M}_{\lambda,g}^{\delta} f(z) + \frac{c}{p}\mathcal{M}_{\lambda,g}^{\delta} F(z). \tag{3.8}$$

Differentiating both sides of (3.8) and using (3.7), we get

$$z\kappa'(z) + (c - p\chi(z))\kappa(z) = (c-p)\frac{z(\mathcal{M}_{\lambda,g}^{\delta} f)'(z)}{-pF_{p,k}^{\delta}(\lambda; g; z)}. \tag{3.9}$$

Since $f \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h))$, from (3.5) and (3.9), we deduce that

$$\kappa(z) + \frac{z\kappa'(z)}{c - p\chi(z)} = \frac{c-p}{c - p\chi(z)}\frac{z(\mathcal{M}_{\lambda,g}^{\delta} f)'(z)}{-pF_{p,k}^{\delta}(\lambda; g; z)} = -\frac{z(\mathcal{M}_{\lambda,g}^{\delta} f)'(z)}{pf_{p,k}^{\delta}(\lambda; g; z)} \prec h(z). \tag{3.10}$$

Combining $\Re(h(z)) < \frac{\Re(c)}{p}$ and $\chi(z) \prec h(z)$, we find that

$$\Re(c - p\chi(z)) > 0.$$

Therefore, from (3.10) and Lemma 1.9, we have $\kappa(z) \prec h(z)$, which implies that $F \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h))$. □

By arguments similar to those used in the proofs of Theorems 2.2 and 3.1, the following result can be proved. We omit the details involved.

3.2. Corollary. *Let $h \in \mathcal{P}$ with*

$$\Re(h(z)) < \frac{\Re(c)}{p} \qquad (z \in \mathbb{U}).$$

If $f \in \mathcal{K}^{\delta}_{p,k}(\lambda; g; h)$ with $\varphi \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$, then the function

$$F(z) = \frac{c-p}{z^c} \int_0^z t^{c-1} f(t)dt$$

belongs to the class $\mathcal{K}^{\delta}_{p,k}(\lambda; g; h)$ with

$$G(z) = \frac{c-p}{z^c} \int_0^z t^{c-1} \varphi(t)dt$$

provided that $G^{\delta}_{p,k}(\lambda; g; z) \neq 0$ $(z \in \mathbb{U}^*)$.

4. Convolution Properties

At last, we prove the convolution properties associated with the function classes

$$\mathcal{S}^{\delta}_{p,k}(\lambda; g; h) \quad \text{and} \quad \mathcal{K}^{\delta}_{p,k}(\lambda; g; h).$$

4.1. Theorem. *Let $h \in \mathcal{P}$ with*

$$\Re(h(z)) < 1 + \frac{1-\alpha}{p} \qquad (z \in \mathbb{U};\ \alpha < 1). \tag{4.1}$$

If $f \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$,

$$\phi \in \Sigma_p \quad \text{and} \quad z^{p+1}\phi(z) \in \mathcal{R}(\alpha). \tag{4.2}$$

Then

$$f * \phi \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h).$$

Proof. Let $f \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h))$ and suppose that

$$\rho(z) = z^{p+1} f^{\delta}_{p,k}(\lambda; g; z) \qquad (z \in \mathbb{U}). \tag{4.3}$$

Then $\rho \in \mathcal{A}$ and

$$H(z) := -\frac{z(\mathcal{M}^{\delta}_{\lambda,g} f)'(z)}{p f^{\delta}_{p,k}(\lambda; g; z)} \prec h(z) \qquad (z \in \mathbb{U}). \tag{4.4}$$

By using Lemma 1.10, we find that

$$\frac{z\rho'(z)}{\rho(z)} = p + 1 + \frac{z\left(f^{\delta}_{p,k}(\lambda; g; z)\right)'}{f^{\delta}_{p,k}(\lambda; g; z)} \prec p + 1 - ph(z) \qquad (z \in \mathbb{U}). \tag{4.5}$$

In view of (4.1) and (4.5), we have

$$\Re\left(\frac{z\rho'(z)}{\rho(z)}\right) > \alpha, \tag{4.6}$$

that is, that

$$\rho \in \mathcal{S}^*(\alpha).$$

For $\phi \in \Sigma_p$, it is easy to verify that

(4.7)

$$z^{p+1}\left(\mathcal{M}^{\delta}_{\lambda,g}(f*\phi)(\varepsilon_k^j z)\right)=(z^{p+1}\phi(z))*\left(z^{p+1}\mathcal{M}^{\delta}_{\lambda,g}f(\varepsilon_k^j z)\right)\qquad (j\in\{0,1,\ldots,k-1\})$$

and

$$z^{p+2}\mathcal{M}^{\delta}_{\lambda,g}(f*\phi)'(z)=(z^{p+1}\phi(z))*\left(z^{p+2}(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)\right). \tag{4.8}$$

Making use of (4.3), (4.4), (4.7) and (4.8), we find that

$$\begin{aligned}\Phi(z):=\frac{z\mathcal{M}^{\delta}_{\lambda,g}(f*\phi)'(z)}{\frac{p}{k}\sum_{j=0}^{k-1}\varepsilon_k^{jp}\mathcal{M}^{\delta}_{\lambda,g}(f*\phi)(\varepsilon_k^j z)}&=-\frac{(z^{p+1}\phi(z))*\left(z^{p+2}(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)\right)}{p(z^{p+1}\phi(z))*\left(z^{p+1}f^{\delta}_{p,k}(\lambda;g;z)\right)}\\ &=\frac{(z^{p+1}\phi(z))*(\rho(z)H(z))}{(z^{p+1}\phi(z))*(\rho(z))}\qquad (z\in\mathbb{U}).\end{aligned} \tag{4.9}$$

Since h is convex univalent in $\mathbb{U}$, it follows from (4.2), (4.4), (4.6), (4.9) and Lemma 1.11 that

$$\Phi(z)\prec h(z)\qquad (z\in\mathbb{U}).$$

Hence

$$f*\phi\in\mathcal{S}^{\delta}_{p,k}(\lambda;g;h).$$

□

By similarly applying the method of proof of Theorem 4.1, we can get the following result.

4.2. Corollary. *Let $h\in\mathcal{P}$ with*

$$\Re(h(z))<1+\frac{1-\alpha}{p}\qquad (z\in\mathbb{U};\ \alpha<1).$$

If $f\in\mathcal{K}^{\delta}_{p,k}(\lambda;g;h)$ with $\varphi\in\mathcal{S}^{\delta}_{p,k}(\lambda;g;h)$,

$$\phi\in\Sigma_p\quad\text{and}\quad z^{p+1}\phi(z)\in\mathcal{R}(\alpha).$$

Then

$$f*\phi\in\mathcal{K}^{\delta}_{p,k}(\lambda;g;h).$$

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Remarks on generalized quantum gates

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Abstract

In this paper, we give a characterization of generalized quantum gates. We also show that many important operators are generalized quantum gates, moreover, some of these operators can be represented as the convex combination of only two unitary operators. Our results answer what kinds of operations a duality quantum computing admits. We point out that the set of all generalized quantum gates coincides with the set of all restricted allowable generalized quantum gates. Thus, our results are also valid for restricted allowable generalized quantum gates.

Keywords: Duality quantum computer, generalized quantum gate, isometry.

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1. Introduction

Quantum gate, which is represented by unitary operator mathematically, is a fundamental tool in designing quantum circuits and quantum algorithms in quantum computers. Quantum computations are just executed by a series of quantum gates.

In [1], Professor Long proposed a new type of computing machine, the duality quantum computer, which exploits the wave-particle duality of quantum systems. A quantum wave can be decomposed into parts using slits or beam splitters, for example. The subwaves can move along separate paths and then be combined at which point they interfere. Therefore, one can perform different gate operations at different paths, a property called the duality parallelism. This enables us to perform computation using not only products of unitary operations, but also linear combinations of unitary operations, which was called the duality gates or the generalized quantum gates ([1]). In this sense, the duality computer offers additional

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capability in information processing which is superior to quantum parallelism in ordinary quantum computer.

In duality quantum computer, the quantum wave divider operation and quantum wave combiner operation are allowed ([1]). In such a picture, the duality gate, or generalized quantum gate, can be written as $\sum_{i=0}^{d-1} p_i U_i$, where U_i are unitary operators, d is the number of slits that the duality quantum computer passes, and p_i is the probability that the duality quantum computer passes through the i-th slit, and $\sum_i p_i = 1$. The quantum wave divider operation and quantum wave combiner operation and the properties of generalized quantum gates were discussed in [2]-[6]. In [7], a necessary condition for a contraction to be a generalized quantum gate was given. Furthermore, using spectral resolution, a characterization that a contraction is not a generalized quantum gate was given in [8].

In [9], the allowable generalized quantum gates were introduced which is of the form $\sum_{i=0}^{d-1} c_i U_i$, where U_i are unitary operators, c_i are complex numbers with module less than or equal to 1 and constrained by $|\sum_{i=0}^{d-1} c_i| \leq 1$. In [10], the authors discussed the realization of allowable generalized quantum gates. In [11], the restricted allowable generalized quantum gates were considered satisfying $0 < \sum_{i=0}^{d-1} |c_i| \leq 1$. The connections with other problems and some further generalizations and applications are also investigated in [12]-[13]. For the review of the development of duality quantum computing, duality quantum information processing and duality quantum communication during the past few years and related work, please refer to [14]-[16].

In this paper, we give a characterization of generalized quantum gates. We also show that many important operators are generalized quantum gates, moreover, some of these operators can be represented as the convex combination of only two unitary operators. Our results answer what kinds of operations a duality computing admits. We point out that the set of all generalized quantum gates coincides with the set of all restricted allowable generalized quantum gates. Thus, our results are also valid for restricted allowable generalized quantum gates.

2. Preliminaries

Let H be a separable complex Hilbert space, $B(H)$ the set of all bounded linear operators on H, $G(H)$ the set of all generalized quantum gates on H, $A \in B(H)$. Denote the range and the null space of A by $R(A)$ and $N(A)$, respectively. If $\|A\| \leq 1$, then A is said to be contractive. The set of all contractions in $B(H)$ is denoted by $B(H)_1$. If p is a positive integer and $(A^*A)^p - (AA^*)^p$ is a positive operator, then $A \in B(H)$ is called a p-hyponormal operator. Specifically, when $p = 1$, A is called a hyponormal operator. The set of all normal operators, inverse operators, finite rank operators and compact operators on H is denoted by $N(H)$, $Inv(H)$, $F(H)$ and $K(H)$, respectively.

Let $A \in B(H)$, if for each $x \in N(A)^\perp$, we have $\|Ax\| = \|x\|$, then A is called a partial isometry, where $N(A)^\perp$ is called the initial space of A, and $R(A)$ the final space. If $N(A) = 0$, then A is called an isometry. A surjective isometry is a unitary operator.

The following lemma is a well-known result for partial isometry.

2.1. Lemma ([17]). *For $U \in B(H)$, the following statements are equivalent:*
(1) U is a partial isometry;
(2) U^* is a partial isometry;
(3) U^*U is the orthogonal projection on $N(U)^\perp$;
(4) UU^* is the orthogonal projection on $R(U)$.

If $A \in B(H)$, the ascent $asc(A)$ of A is defined to be the smallest nonnegative integer k (if it exists) which satisfies that $N(A^k) = N(A^{k+1})$. If such k does not exist, then the ascent of A is defined as infinity. Similarly, the descent $des(A)$ of A is defined as the smallest nonnegative integer k (if it exists) for which $R(A^k) = R(A^{k+1})$ holds. If such k does not exist, then $des(A)$ is defined as infinity, too. If the ascent and the descent of T are finite, then they are equal ([18]).

$A \in B(H)$ is said to be semi-Fredholm if the range $R(A)$ is closed and at least one of $\dim N(A)$ and $\dim N(A^*)$ is finite, and the Fredholm index ind (A) of A is defined by ind $(A) = \dim N(A) - \dim N(A^*)$. $A \in B(H)$ is said to be Fredholm if A is semi-Fredholm and $-\infty <$ ind $(A) < \infty$.

Let $A \in B(H)$. For each positive integer n, define A_n to be the restriction of A to $R(A^n)$. If there is a positive integer n_0 such that $R(A_{n_0})$ is closed and A_{n_0} is Fredholm, then A is called B-Fredholm. It follows from [19] that if A is B-Fredholm and n_0 satisfies the above properties, then A_m is Fredholm and ind (A_m) =ind (A_n) for all $m \geq n_0$. Thus, we can define the index of a B-Fredholm operator A as the index of the Fredholm operator A_n, where n is any positive integer such that $R(A_n)$ is closed and A_n is a Fredholm operator.

We first introduce the following important operator classes ([18], [20]-[23]).

2.2. Definition. (i) $T \in B(H)$ is called Browder, if T is Fredholm operator and $asc(T) = des(T) < \infty$.

(ii) $T \in B(H)$ is said to be Drazin invertible, if there exists $T^D \in B(H)$ such that $TT^D = T^DT, T^DTT^D = T^D, T^{k+1}T^D = T^k$ for some nonnegative integer k.

(iii) $T \in B(H)$ is said to be generalized Drazin invertible, if there exist $A, B \in B(H)$ such that B is quasi-nilpotent and $TA = AT, ATA = A, TAT = A + B$.

(iv) $T \in B(H)$ is called Weyl, if T is Fredholm with index 0.

(v) $T \in B(H)$ is called B-Weyl, if T is a B-Fredholm with index 0.

The above operators have the following characterizations ([18],[20]-[23]).

2.3. Lemma. *(i) $T \in B(H)$ is Browder iff $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent on some finite dimensional space.*

(ii) $T \in B(H)$ is Drazin invertible iff $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent.

(iii) $T \in B(H)$ is generalized Drazin invertible iff $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasi-nilpotent.

(iv) $T \in B(H)$ is B-Fredholm iff $T = T_1 \oplus T_2$, where T_1 is Fredholm and T_2 is nilpotent.

(v) $T \in B(H)$ is B-Weyl iff $T = T_1 \oplus T_2$, where T_1 is Fredholm with index 0 and T_2 is nilpotent.

In [2], Gudder essentially proved the following result.

2.4. Theorem ([2]). *If $\dim H < \infty$, then $G(H) = B(H)_1$.*

In [7], Wang, Du and Dou proved the following theorem.

2.5. Theorem ([7])**.** *If $A \in B(H)_1$ is a finite-rank perturbation of a semi-Fredholm partial isometry with ind $(A) \neq 0$, then $A \notin G(H)$.*

It follows from this result that Theorem 2.4 does not hold when $\dim H = \infty$. But we have the following result.

2.6. Lemma ([24])**.** *If $A \in B(H)$ and $\|A\| < 1 - \frac{2}{n}$ for some $n > 2$, then there exist unitary operators $U_1, U_2, \cdots, U_n$ such that*

$$A = \frac{1}{n}(U_1 + U_2 + \cdots + U_n).$$

Denote $B(H)_1^\circ = \{A \in B(H) : \|A\| < 1\}$. Then from the above lemma, the following result follows.

2.7. Theorem ([7])**.** *$B(H)_1^\circ \subseteq G(H)$.*

By using spectral resolution, Du and Dou in [8] established the necessary and sufficient conditions for $A \notin G(H)$ when $A \in B(H)_1$. As a corollary, they obtained the following result.

2.8. Theorem ([8])**.** *If $A \in B(H)_1$, then A is not a generalized quantum gate if and only if A is a semi-Fredholm with ind $(A) \neq 0$ and A is a compact perturbation of a partial isometry.*

3. Main Results

In this section, we give a characterization of generalized quantum gates and show that many important operators are generalized quantum gates.

First, we need the following important lemma.

3.1. Lemma ([8])**.** *Let $A \in B(H)_1$. Then there exist two unitary operators U_1 and U_2 in $B(H)$ such that*

$$A = \frac{1}{2}(U_1 + U_2)$$

if and only if $\dim N(A) = \dim N(A^*)$.

3.2. Remark. The above lemma shows that each contractive normal operator is a generalized quantum gate. However, the conclusion does not hold for contractive p-hyponormal operator. In fact, if A is the forward shift on H, then it is easy to check that A is a contractive p-hyponormal operator, but not a generalized quantum gate.

3.3. Proposition. *If $A, B \in G(H)$, then $AB \in G(H)$ and $A \oplus B \in G(H \oplus H)$.*

Proof. It follows from the fact $A, B \in G(H)$ that there exist unitary operators $U_1, \cdots, U_n, V_1, \cdots, V_m$ such that

$$A = \sum_{i=1}^{n} p_i U_i \ , B = \sum_{j=1}^{m} q_j V_j, \tag{3.1}$$

where $p_i > 0, i = 1, 2, \cdots, n,\ q_j > 0, j = 1, 2, \cdots, m,\ \sum_{i=1}^n p_i = \sum_{j=1}^m q_j = 1$. Thus,

$$AB = \sum_{i=1}^n p_i U_i \sum_{j=1}^m q_j V_j = \sum_{i=1}^n \sum_{j=1}^m p_i q_j U_i V_j,$$

$$A \oplus B = \sum_{i=1}^n p_i U_i \oplus \sum_{j=1}^m q_j V_j = \sum_{j=1}^m \sum_{i=1}^n p_i q_j (U_i \oplus V_j).$$

Noting that $\sum_{i=1}^n \sum_{j=1}^m p_i q_j = 1$ and the facts that the product as well as the direct sum of unitary operators are still unitary operators, we obtain $AB \in G(H)$, and $A \oplus B \in G(H \oplus H)$. □

3.4. Remark. The following two examples show that the converse of Proposition 3.3 does not hold.

3.5. Example. Let A, B be operators on H defined by

$$A : H \to H, \quad \{z_1, z_2, z_3, \cdots\} \mapsto \{z_1, \frac{1}{2}z_2, \frac{1}{2}z_3, \cdots\},$$

$$B : H \to H, \quad \{z_1, z_2, z_3, \cdots\} \mapsto \{0, z_1, z_2, \cdots\}.$$

Then

$$AB : H \to H, \quad \{z_1, z_2, z_3, \cdots\} \mapsto \{0, \frac{1}{2}z_1, \frac{1}{2}z_2, \frac{1}{2}z_3, \cdots\}.$$

Note that $A \in Inv(H)$, $\|A\| = 1$, $\|AB\| = \frac{1}{2}$. Then it follows from Theorem 2.7 and Lemma 3.1 that $A, AB \in G(H)$, but by Theorem 2.8 we obtain that $B \notin G(H)$.

3.6. Example. Let A be the forward shift on H, that is,

$$A : H \to H, \quad \{z_1, z_2, z_3, \cdots\} \mapsto \{0, z_1, z_2, \cdots\},$$

$B = A^*$, and $C = I - AB$. Then

$$A \oplus B = \frac{1}{2}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A & -C \\ 0 & B \end{pmatrix}.$$

Note that $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & -C \\ 0 & B \end{pmatrix}$ are both unitary operators. Then it follows from Proposition 3.3 that $A \oplus B \in G(H \oplus H)$. But it is easy to see that neither A nor B is in $G(H)$.

Now, we give the following characterization of generalized quantum gates.

3.7. Theorem. *Let $A \in B(H)_1$. Then $A \in G(H) \iff A^n \in G(H)$ for each positive integer n.*

Proof. The necessity follows from Proposition 3.3 immediately. Sufficiency. Firstly, we show that if $A \notin G(H)$ and ind $(A) < 0$, then A can be represented as the compact perturbation of an isometry.

In fact, it follows from Theorem 2.8 that $A = U + K$, where U is a partial isometry and K is compact. Noting that ind $(U) =$ ind $(U + K) =$ ind $(A) < 0$, we have dim $N(U) < \infty$.

If the operator U is written in the following form with respect to the space decomposition $H = N(U)^{\perp} \oplus N(U)$:

$$U = \begin{pmatrix} U_1 & 0 \\ U_2 & 0 \end{pmatrix},$$

then

$$U^*U = \begin{pmatrix} U_1^* & U_2^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ U_2 & 0 \end{pmatrix} = \begin{pmatrix} U_1^*U_1 + U_2^*U_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows from the fact that U is a partial isometry and Lemma 2.1 that U^*U is the orthogonal projection on $N(U)^{\perp}$. Therefore, $U_1^*U_1 + U_2^*U_2 = I_{N(U)^{\perp}}$. Moreover, it follows from $\|U^*U\| = 1$ that $\|U_2^*U_2\| \le 1$. Also, note that $\dim N(U) < \infty$. So $U_2^*U_2$ is a positive contractive finite rank operator. Denote

$$U_2^*U_2 = \sum_{i=1}^{k} a_i |\alpha_i\rangle\langle\alpha_i|,$$

where $0 < a_i \le 1$ and $\{|\alpha_i\rangle\}_{i=1}^{k}$ is the orthogonal set in $N(U)^{\perp}$. If we extend $\{|\alpha_i\rangle\}$ into an orthonormal basis $\{|\alpha_1\rangle, |\alpha_2\rangle, \cdots, |\alpha_k\rangle, |\beta_1\rangle, |\beta_2\rangle, \cdots\}$ of $N(U)^{\perp}$, then we have

$$\begin{aligned} U_1^*U_1 &= I_{N(U)^{\perp}} - U_2^*U_2 = I_{N(U)^{\perp}} - \sum_{i=1}^{k} a_i |\alpha_i\rangle\langle\alpha_i| \\ &= \sum_j |\beta_j\rangle\langle\beta_j| + \sum_{i=1}^{k} (1 - a_i)|\alpha_i\rangle\langle\alpha_i|. \end{aligned}$$

Moreover, it is easy to see that

$$\langle\beta_j|U_1^*U_1|\beta_j\rangle = \langle\beta_j|\beta_j\rangle = 1, j = 1, 2, \cdots,$$

$$\langle\alpha_i|U_1^*U_1|\alpha_i\rangle = 1 - a_i, i = 1, 2, \cdots, k.$$

If $0 < a_i < 1, i = 1, 2, \cdots, k$, take $b_i = \sqrt{\frac{1}{1-a_i}} - 1, i = 1, 2, \cdots, k$, and let

$$\tilde{U_1}|\alpha_i\rangle = b_i U_1|\alpha_i\rangle, i = 1, 2, \cdots, k,$$

$$\tilde{U_1}|\beta_j\rangle = 0, j = 1, 2, \cdots .$$

Then it is easy to see that

$$\begin{aligned} \langle\alpha_i|(U_1 + \tilde{U_1})^*(U_1 + \tilde{U_1})|\alpha_i\rangle &= (1 + b_i)^2 \langle\alpha_i|U_1^*U_1|\alpha_i\rangle \\ &= (1 + b_i)^2(1 - a_i) = 1, \qquad i = 1, 2, \cdots, k, \end{aligned} \tag{3.2}$$

and

$$\langle\beta_j|(U_1 + \tilde{U_1})^*(U_1 + \tilde{U_1})|\beta_j\rangle = \langle\beta_j|U_1^*U_1|\beta_j\rangle = 1, j = 1, 2, \cdots . \tag{3.3}$$

If there exists some i, such that a_i is 1, for example, assume that $a_m = 1, 0 < a_i < 1, i = 1, 2, \cdots, k, i \ne m$. Take $b_i = \sqrt{\frac{1}{1-a_i}} - 1, i = 1, 2, \cdots, k, i \ne m$, and let

$$\tilde{U_1}|\alpha_m\rangle = |\alpha_m\rangle, \ \ \tilde{U_1}|\alpha_i\rangle = b_i U_1|\alpha_i\rangle, i = 1, 2, \cdots, k, i \ne m,$$

$$\tilde{U_1}|\beta_j\rangle = 0, j = 1, 2, \cdots .$$

Then we can check that (3.2) and (3.3) still hold. Thus, $U_1+\tilde{U_1}$ is an isometry on $N(U)^{\perp}$ and $\tilde{U_1}$ is a finite rank operator. Hence, A can be written as

$$\begin{aligned} A &= U+K=\begin{pmatrix} U_1 & 0 \\ U_2 & 0 \end{pmatrix}+K \\ &= \begin{pmatrix} U_1+\tilde{U_1} & 0 \\ 0 & I \end{pmatrix}+\begin{pmatrix} -\tilde{U_1} & 0 \\ U_2 & -I \end{pmatrix}+K=V+F+K, \end{aligned}$$

where V is an isometry, F is a finite rank operator and K is a compact operator. So A can be written in the form $A=V+L$, where V is an isometry and L is compact.

Now, in order to prove the sufficiency, we consider two cases:

Case (I). ind $(A)<0$. It is obvious that the conclusion holds for $n=1$. If the conclusion holds for $n=k$, that is, $A^k\in G(H)\Rightarrow A\in G(H)$, we need to prove it also holds for $n=k+1$, that is, $A^{k+1}\in G(H)\Rightarrow A\in G(H)$. If $A\notin G(H)$, then $A=V+L$, where V is an isometry and L is compact, by induction assumption we know that $A^k\notin G(H)$, thus $A^k=V_1+L_1$, where V_1 is an isometry and L_1 is compact. Therefore, $A^{k+1}=A^kA=V_1V+V_1L+L_1V+L_1L$, where V_1V is still an isometry and $V_1L+L_1V+L_1L$ is still compact. On the other hand, it follows from the fact that A is a semi-Fredholm operator with ind $(A)\neq 0$ that A^{k+1} is a semi-Fredholm operator with ind $(A^{k+1})\neq 0$. Thus, it follows from Theorem 2.4 that $A^{k+1}\notin G(H)$.

Case (II). ind $(A)>0$. In Case (I) we have proved that if $A\notin G(H)$ and ind $(A)<0$, then $A^n\notin G(H)$. Note that $A\in G(H)\iff A^*\in G(H)$. Thus, if $A\notin G(H)$ and ind $(A)>0$, we have $A^*\notin G(H)$ and ind $(A^*)<0$. It follows from the proof of Case (I) that $(A^n)^*=(A^*)^n\notin G(H)$, thus $A^n\notin G(H)$. The sufficiency is proved. □

3.8. Remark. If we get rid of the assumption $A\in B(H)_1$, then the sufficiency does not hold. In fact, if $A=\begin{pmatrix} \frac{1}{3}I & 2I \\ 0 & 0 \end{pmatrix}$, then $A^2=\begin{pmatrix} \frac{1}{9}I & \frac{2}{3}I \\ 0 & 0 \end{pmatrix}$. It is easy to see that $\|A\|>2$, $\|A^2\|<1$. Thus, it follows from Theorem 2.3 that $A^2\in G(H)$. But it is obvious that $A\notin G(H)$.

3.9. Remark. The following example shows that $AB\in G(H)$ can not guarantee that $BA\in G(H)$. Let

$$A=\begin{pmatrix} I & 9I \\ 0 & 0 \end{pmatrix},\quad B=\begin{pmatrix} \frac{1}{3}I & \frac{1}{4}I \\ 0 & 0 \end{pmatrix}.$$

Then $AB=\begin{pmatrix} \frac{1}{3}I & \frac{1}{4}I \\ 0 & 0 \end{pmatrix}$, $BA=\begin{pmatrix} \frac{1}{3}I & 3I \\ 0 & 0 \end{pmatrix}$, $\|AB\|<1$, $\|BA\|>1$. Thus, it follows from Theorem 2.7 that $AB\in G(H)$, but $BA\notin G(H)$.

Moreover, the following example shows that $A,B\in B(H)_1$, $AB\in G(H)$ does not imply that $BA\in G(H)$.

3.10. Example. Let A,B be the operators on H defined as follows:

$$A:H\to H,\quad \{x_1,x_2,x_3,\cdots\}\mapsto\{0,0,0,x_1,0,0,0,x_2,\cdots\},$$

$$B : H \to H, \quad \{x_1, x_2, x_3, \cdots\} \mapsto \{x_2, x_4, x_6, \cdots\}.$$

Then we have

$$AB : H \to H, \quad \{x_1, x_2, x_3, \cdots\} \mapsto \{0, 0, 0, x_2, 0, 0, 0, x_4, \cdots\},$$
$$BA : H \to H, \quad \{x_1, x_2, x_3, \cdots\} \mapsto \{0, x_1, 0, x_2, 0, x_3, \cdots\},$$

and

$$(AB)^* : H \to H, \quad \{x_1, x_2, x_3, \cdots\} \mapsto \{0, x_4, 0, x_8, 0, x_{12}, \cdots\},$$
$$(BA)^* : H \to H, \quad \{x_1, x_2, x_3, \cdots\} \mapsto \{x_2, x_4, x_6, \cdots\}.$$

Note that $AB \in B(H)_1$ and dim $N(AB)$ =dim $N((AB)^*) = \infty$. Then it follows from Lemma 3.1 that $AB \in G(H)$. On the other hand, it is easy to see that dim $N(BA) = 0$ and dim $N((BA)^*) = \infty$, so BA is a semi-Fredholm operator with ind $(BA) \neq 0$. Also, it is easy to verify that BA is an isometry. Thus, it follows from Theorem 2.8 that $BA \notin G(H)$.

Finally, by using Proposition 3.3, we show that many important operators are generalized quantum gates.

First, note that the invertible operators and nilpotent operators are both Fredholm operators with index 0, while the quasi-nilpotent operators are not semi-Fredholm operators. Then it follows from Theorem 2.8 that the contractive invertible operators, contractive nilpotent operators, contractive quasi-nilpotent operators and contractive Weyl operators are all generalized quantum gates. Moreover, it follow from Lemma 2.3 and Proposition 3.3 that the contractive Browder operators, contractive Drazin invertible operators, contractive generalized Drazin operators and contractive B-Weyl operators are all generalized quantum gates, too. Note that the Weyl operators are Fredholm operators with index 0. So it follows from Theorem 2.8 that the contractive Weyl operators are generalized quantum gates. On the other hand, since the index of each B-Fredholm operator is not always 0, the contractive B-Fredholm operator is not always a generalized quantum gate.

Furthermore, we show that many operators are not only generalized quantum gates, but can also be represented as the convex combination of two unitary operators.

3.11. Theorem. *Let $A \in B(H)$ be a Drazin invertible operator (resp. Browder operator, Weyl operator, B-Weyl operator). Then the following statements are equivalent:*

(1) $A \in G(H)$.

(2) $A \in B(H)_1$.

(3) $A = \frac{1}{2}(U_1 + U_2)$, where U_1 and U_2 are unitary operators.

Proof. We only prove the case when A is a Drazin invertible operator. It is obvious that (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) hold. Now, we prove (2) $\Rightarrow$ (3). In fact, since A is Drazin invertible, by Lemma 2.3, we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \to H_1 \oplus H_2,$$

where A_1 is invertible and A_2 are nilpotent. It is obvious that both A_1 and A_2 are contractions. We assert that dim $N(A_2) =$ dim $N(A_2^*)$. Indeed, if dim $(H_2) < \infty$, then dim $N(A_2)$ = dim $N(A_2^*) < \infty$; if dim $(H_2) = \infty$, it follows from A_2 is nilpotent that dim $N(A_2)$ = dim $N(A_2^*) = \infty$, thus, dim $N(A_2)$ = dim $N(A_2^*)$, and so dim $N(A)$ = dim $N(A^*)$. Therefore, it follows from Lemma 3.1 that there exist unitary operators $U_1, U_2 \in B(H)$ such that $A = \frac{1}{2}(U_1 + U_2)$. $\square$

3.12. Remark. If $A \in B(H)$ is a quasi-nilpotent operator, then dim $N(A)$ = dim $N(A^*)$ does not always hold. In fact, if A is the operator on H defined as follows:

$$A : H \to H, \quad \{x_1, x_2, x_3, x_4, \cdots\} \mapsto \{0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots\},$$

then it is easy to see that dim $N(A) = 0 \neq 1$ = dim $N(A^*)$. This shows that the generalized Drazin invertible operators can not always be represented as the convex combination of two unitary operators.

3.13. Remark. It is obvious that a generalized quantum gate is a restricted allowable generalized quantum gate. Now, we show the converse is also true. In fact, if $A = \sum_{i=0}^{d-1} c_i U_i$, where U_i are unitary operators and $0 < \sum_{i=0}^{d-1} |c_i| \leq 1$, then $A = \sum_{i=0}^{d-1} |c_i| e^{i\theta_i} U_i$. Let $p_i = |c_i|$, $\tilde{U}_i = e^{i\theta_i} U_i$. Then $A = \sum_{i=0}^{d-1} p_i \tilde{U}_i$, where $\sum p_i \leq 1$, and $\tilde{U}_i$ are unitary operators. If $\sum_{i=0}^{d-1} |c_i| = 1$, then $\sum p_i = 1$ and so $A \in G(H)$. If $\sum_{i=0}^{d-1} |c_i| < 1$, then $\sum p_i < 1$, and we have $\|A\| < 1$, that is, $A \in B(H)_1^\circ$. Thus, it follows from Theorem 2.7 that $A \in G(H)$. So A is a generalized quantum gate.

It follows from the above fact that our results are not only valid for the generalized quantum gates, but also for the restricted allowable generalized quantum gates.

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Modified ratio estimators using stratified ranked set sampling

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Abstract

Stratified Ranked Set Sampling (SRSS) combines the advantages of stratification and Ranked set sampling (RSS) to obtain an unbiased estimator for the population mean, with potentially significant gains in efficiency. The present paper deals with modified ratio estimators of finite population mean using information on coefficient of variation and co-efficient of kurtosis of auxiliary variable in Stratified Ranked Set Sampling. It has been shown that these methods are highly beneficial to the estimation based on Stratified Simple Random Sampling (SSRS). The bias and mean squared error of the proposed estimators with large sample approximation are derived. Theoretically, it is shown that these suggested estimators are asymptotically more efficient than the estimators in stratified simple random sampling. The results have been illustrated by numerical example.

Keywords: Stratified ranked set sampling, Ratio-type estimators, Ranked set sampling, Auxiliary variables, Mean squared error, Population mean, Coefficient of variation, Coefficient of kurtosis, Efficiency.

1. Introduction

The literature on ranked set sampling describes a great variety of techniques for using auxiliary information to obtain more efficient estimators. Ranked set sampling (RSS) was first suggested by McIntyre (1952) and Stratified Ranked Set Sampling was introduced by Samawi (1996) to increase the efficiency of estimator of population mean. The performance of the combined and the separate ratio estimates using the stratified ranked set sample (SRSS) was given by Samawi and Siam (2003). Kadilar et al. (2009) used ranked set sampling to improve ratio estimator given by Prasad (1989). Here we use the idea of SRSS instead of SSRS to improve the precision of ratio estimators given by Kadilar and Cingi (2003).

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The usual ratio estimator given by Cochran (1977) for the population mean $\overline{Y}$ in stratified random sampling is defined by

$$\overline{y}_{SSRS} = \overline{y}_{st}(\frac{\overline{X}}{\overline{x}_{st}}) \tag{1.1}$$

where $\overline{y}_{st} = \sum_{h=1}^{L} W_h \overline{y}_h$ and $\overline{x}_{st} = \sum_{h=1}^{L} W_h \overline{x}_h$ are the unbiased estimators of population mean $\overline{Y}$ and $\overline{X}$ respectively. Here $W_h = \frac{N_h}{N}$ is the weight of h^{th} stratum, where N_h is the h^{th} stratum size and N is the total population size $(h = 1, 2, \ldots, L)$ and L is the total number of strata in the population.

When the population coefficient of variation for h^{th} stratum C_{x_h} is known and motivated by Sisodia and Dwivedi(1981),Kadilar and Cingi(2003) suggested a modified ratio estimator for $\overline{Y}$ in stratified random sampling as

$$\overline{y}_{stSD} = \overline{y}_{st} \frac{\sum_{h=1}^{L} W_h(\overline{X}_h + C_{x_h})}{\sum_{h=1}^{L} W_h(\overline{x}_h + C_{x_h})} \tag{1.2}$$

When coefficient of kurtosis for h^{th} stratum, $\beta_{2h}(x)$ is known and motivated by Singh and Kakran(1993), Kadilar and Cingi(2003) developed ratio-type estimator for $\overline{Y}$ as

$$\overline{y}_{stSK} = \overline{y}_{st} \frac{\sum_{h=1}^{L} W_h(\overline{X}_h + \beta_{2h}(x))}{\sum_{h=1}^{L} W_h(\overline{x}_h + \beta_{2h}(x))} \tag{1.3}$$

Kadilar and Cingi (2003) considered ratio type estimators based on Upadhyaya and Singh (1999), using both coefficient of variation and kurtosis in stratified random sampling as

$$\overline{y}_{stUS1} = \overline{y}_{st} \frac{\sum_{h=1}^{L} W_h(\overline{X}_h \beta_{2h}(x) + C_{x_h})}{\sum_{h=1}^{L} W_h(\overline{x}_h \beta_{2h}(x) + C_{x_h})} \tag{1.4}$$

$$\overline{y}_{stUS2} = \overline{y}_{st} \frac{\sum_{h=1}^{L} W_h(\overline{X}_h C_{x_h} + \beta_{2h}(x))}{\sum_{h=1}^{L} W_h(\overline{x}_h C_{x_h} + \beta_{2h}(x))} \tag{1.5}$$

To the first degree of approximation the mean squared error(MSE) of the estimators $\overline{y}_{SSRS}, \overline{y}_{stSD}, \overline{y}_{stSK}, \overline{y}_{stUS1}$ and $\overline{y}_{stUS2}$ respectively are

$$(1.6)\qquad MSE(\overline{y}_{SSRS})=\sum_{h=1}^{L}\frac{W_h^2}{n_h}(S_{y_h}^2+R^2S_{x_h}^2-2RS_{x_hy_h})$$

$$(1.7)\qquad MSE(\overline{y}_{stSD})=\sum_{h=1}^{L}\frac{W_h^2}{n_h}(S_{y_h}^2+R^2\lambda_1^2S_{x_h}^2-2R\lambda_1S_{x_hy_h})$$

$$(1.8)\qquad MSE(\overline{y}_{stSK})=\sum_{h=1}^{L}\frac{W_h^2}{n_h}(S_{y_h}^2+R^2\lambda_2^2S_{x_h}^2-2R\lambda_2S_{x_hy_h})$$

$$(1.9)\qquad MSE(\overline{y}_{stUS1})=\sum_{h=1}^{L}\frac{W_h^2}{n_h}(S_{y_h}^2+R^2\gamma_1^2S_{x_h}^2-2R\gamma_1S_{x_hy_h})$$

$$(1.10)\qquad MSE(\overline{y}_{stUS2})=\sum_{h=1}^{L}\frac{W_h^2}{n_h}(S_{y_h}^2+R^2\gamma_2^2S_{x_h}^2-2R\gamma_2S_{x_hy_h})$$

where $\lambda_1=\frac{\sum_{h=1}^{L}W_h\overline{X}_h}{\sum_{h=1}^{L}W_h(\overline{X}_h+C_{x_h})}$, $\lambda_2=\frac{\sum_{h=1}^{L}W_h\overline{X}_h}{\sum_{h=1}^{L}W_h(\overline{X}_h+\beta_{2h}(x))}$, $\gamma_1=\frac{\sum_{h=1}^{L}W_h\overline{X}_h\beta_{2h}(x)}{\sum_{h=1}^{L}W_h(\overline{X}_h\beta_{2h}(x)+C_{x_h})}$, $\gamma_2=\frac{\sum_{h=1}^{L}W_h\overline{X}_hC_{x_h}}{\sum_{h=1}^{L}W_h(\overline{X}_hC_{x_h}+\beta_{2h}(x))}$, $S_{y_h}^2=\frac{\sum_{i=1}^{N_h}(Y_{hi}-\overline{Y}_h)^2}{N_h-1}$, $S_{x_h}^2=\frac{\sum_{i=1}^{N_h}(X_{hi}-\overline{Y}_h)^2}{N_h-1}$ and $S_{y_hx_h}=\frac{\sum_{i=1}^{N_h}(Y_{hi}-\overline{Y}_h)(X_{hi}-\overline{X}_h)}{N_h-1}$.

2. Stratified ranked set sample

In ranked set sampling, r independent random sets, each of size r and each unit in the set being selected with equal probability and without replacement , are selected from the population. The members of each random set are ranked with respect to the characteristic of the study variable or auxiliary variable. Then, the smallest unit is selected from the first ordered set and the second smallest unit is selected from the second ordered set. By this way, this procedure is continued until the unit with the largest rank is chosen from the r^{th} set. This cycle may be repeated m times, so $mr(=n)$ units have been measured during this process.

In stratified ranked set sampling, for the h^{th} stratum of the population, first choose r_h independent samples each of size r_h , $h=1,2,\ldots,L$. Rank each sample, and use RSS scheme to obtain L independent RSS samples of size r_h , one from each stratum. Let $r_1+r_2+\ldots+r_L=r$. This complete one cycle of stratified ranked set sample. The cycle may be repeated m times until $n=mr$ elements have been obtained. A modification of the above procedure is suggested here to be used for the estimation of the ratio using stratified ranked set sample. For the h^{th} stratum, first choose r_h independent samples each of size r_h of independent bivariate elements from the h^{th} subpopulation (Stratum) , $h=1,2,\ldots,L$. Rank each sample with respect to one of the variables say Y or X. Then use the RSS sampling scheme to obtain L independent RSS samples of size r_h one from each stratum. This complete one cycle of stratified ranked set sample. The cycle may be repeated m times until $n = mr$ bivariate elements have been obtained. We will use the following notation for the stratified ranked set sample when the ranking is on the variable X. For the k^{th} cycle and the h^{th} stratum, the SRSS is denoted by $\{(Y_{h[1]k}, X_{h(1)k}), (Y_{h[2]k}, X_{h(2)k}), \ldots, (Y_{h[r_h]k}, X_{h(r_h)k}) : k = 1, 2, \ldots, m; h = 1, 2, \ldots, L\}$, where $Y_{h[i]k}$ is the i^{th} Judgment ordering in the i^{th} set for the study variable and $X_{h(i)k}$ is the i^{th} order statistic in the i^{th} set for the auxiliary variable.

The combined ratio estimator of population mean $\overline{Y}$ given by Samawi and Siam (2003), using stratified ranked set sampling is defined as

$$\overline{y}_{SSRS} = \overline{y}_{[SRSS]}(\frac{\overline{X}}{\overline{x}_{(SRSS)}}) \tag{2.1}$$

where $\overline{y}_{[SRSS]} = \sum_{h=1}^{L} W_h \overline{y}_{h[r_h]}$ and $\overline{x}_{(SRSS)} = \sum_{h=1}^{L} W_h \overline{x}_{h(r_h)}$.

The Bias and MSE of the estimator $\overline{y}_{SRSS}$ to the first degree of approximation are respectively given by

$$B(\overline{y}_{SRSS}) = \overline{Y}\Big[\sum_{h=1}^{L} \frac{W_h^2}{n_h}\{\frac{S_{x_h}^2}{\overline{X}^2} - \frac{S_{x_h y_h}}{\overline{X}\overline{Y}}\} - \sum_{h=1}^{L} \frac{W_h^2}{n_h}\{\frac{m}{n_h}(\sum_{i=1}^{r_h} D_{x_h(i)}^2 - \sum_{i=1}^{r_h} D_{x_h(i)y_h[i]})\}\Big] \tag{2.2}$$

and

$$MSE(\overline{y}_{SRSS}) = \sum_{h=1}^{L} \frac{W_h^2}{n_h}\big[\{S_{y_h}^2 + R^2 S_{x_h}^2 - 2RS_{x_h y_h}\} - \overline{Y}^2\{\frac{m}{n_h}\sum_{i=1}^{r_h}(D_{y_h[i]} - D_{x_h(i)})^2\}\big] \tag{2.3}$$

where $n_h = mr_h, D_{y_h[i]}^2 = \frac{\tau_{y_h[i]}^2}{\overline{Y}^2}, D_{x_h(i)}^2 = \frac{\tau_{x_h(i)}^2}{\overline{X}^2}$ and $D_{x_h(i)y_h[i]} = \frac{\tau_{x_h(i)y_h[i]}}{\overline{Y}\overline{X}}$. Here we would also like to remind that $\tau_{x_h(i)} = \mu_{x_h(i)} - \overline{X}_h, \tau_{y_h[i]} = \mu_{y_h[i]} - \overline{Y}_h$ and $\tau_{x_h(i)y_h[i]} = (\mu_{x_h(i)} - \overline{X}_h)(\mu_{y_h[i]} - \overline{Y}_h)$ where $\mu_{x_h(i)} = E[x_{h(i)}], \mu_{y_h(i)} = E[y_{h(i)}], \overline{X}_h$ and $\overline{Y}_h$ are the means of the h^{th} stratum for the variables X and Y, respectively.

3. Proposed estimators in stratified ranked set sampling

Motivated by estimators given from (1.2) to (1.5) that incorporation of more and more parameters on auxiliary variable help increase the efficiencies of estimators and motivated by Kadilar and Cingi (2003), we propose ratio-type estimator for $\overline{Y}$ using stratified ranked set sampling, when the population coefficient of variation C_{x_h} of auxiliary variable from stratum to stratum $(h = 1, 2, \ldots, L)$, is known, as

$$(3.1)\qquad \overline{y}_{strMM1} = \overline{y}_{[SRSS]} \frac{\sum_{h=1}^{L} W_h(\overline{X}_h + C_{x_h})}{\sum_{h=1}^{L} W_h(\overline{x}_{h(r_h)} + C_{x_h})}$$

where $\overline{y}_{[SRSS]} = \sum_{h=1}^{L} W_h \overline{y}_{h[r_h]}$ and $\overline{x}_{(SRSS)} = \sum_{h=1}^{L} W_h \overline{x}_{h(r_h)}$ are the stratified ranked set sample means for variables and respectively.

To obtain bias and MSE of $\overline{y}_{strMM1}$, we put $\overline{y}_{[SRSS]} = \overline{Y}(1+\delta_0)$ and $\overline{x}_{(SRSS)} = \overline{X}(1+\delta_1)$ so that $E(\delta_0) = E(\delta_1) = 0$.

Now $V(\delta_0) = E(\delta_0^2) = \frac{V(\overline{y}_{[SRSS]}}{\overline{Y}^2} = \sum_{h=1}^{L} W_h^2 \frac{1}{mr_h} \frac{1}{\overline{Y}^2}[S_{y_h}^2 - \frac{m}{n_h} \sum_{i=1}^{r_h} \tau_{y_h[i]}^2]$
$= \sum_{h=1}^{L} \frac{W_h^2}{n_h}[\frac{S_{y_h}^2}{\overline{Y}^2} - \frac{m}{n_h} \sum_{i=1}^{r_h} D_{y_{h[i]}}^2]$.

Similarly,$E(\delta_1^2) = \sum_{h=1}^{L} \frac{W_h^2}{n_h}[\frac{S_{x_h}^2}{\overline{X}^2} - \frac{m}{n_h} \sum_{i=1}^{r_h} D_{x_{h(i)}}^2]$

and $E(\delta_0, \delta_1) = \sum_{h=1}^{L} \frac{W_h^2}{n_h}[\frac{S_{x_h y_h}}{\overline{XY}} - \frac{m}{n_h} \sum_{i=1}^{r_h} D_{x_{h(i)} y_{h[i]}}]$

Further to validate first degree of approximation, we assume that the sample size is large enough to get $|\delta_0|$ and $|\delta_1|$ as small so that the terms involving δ_0 and or δ_1 with degree greater than two will be negligible.

The Bias and MSE of the estimator $\overline{y}_{strMM1}$ to the first degree of approximation are respectively, given by

$B(\overline{y}_{strMM1}) = E(\overline{y}_{strMM1}) - \overline{Y}$

Here $\overline{y}_{strMM1} = \overline{Y}(1+\delta_0)(1+\lambda_1\delta_1)^{-1}$, where $\lambda_1 = \frac{\sum_{h=1}^{L} W_h \overline{X}_h}{\sum_{h=1}^{L} W_h(\overline{X}_h + C_{x_h})}$.

Now $E(\overline{y}_{strMM1}) = \overline{Y}[1 + \lambda_1^2 E(\delta_1^2) - \lambda_1 E(\delta_0\delta_1)]$,because$E(\delta_0) = E(\delta_1) = 0$.

(using taylor series expansion, where $O(\delta_1)$are power terms of δ_1 with powers more than 2 are neglected)

$$(3.2)\qquad B(\overline{y}_{strMM1}) = \overline{Y}[\sum_{h=1}^{L} \frac{W_h^2}{n_h}\{\frac{\lambda_1^2 S_{x_h}^2}{\overline{X}^2} - \frac{\lambda_1 S_{x_h y_h}}{\overline{XY}}\} - \sum_{h=1}^{L} \frac{W_h^2}{n_h}\{\frac{m}{n_h}(\lambda_1^2 \sum_{i=1}^{r_h} D_{x_h(i)}^2 - \lambda_1 \sum_{i=1}^{r_h} D_{x_h(i) y_h[i]})\}]$$

Now $MSE(\overline{y}_{strMM1}) = E(\overline{y}_{strMM1} - \overline{Y})^2 = \overline{Y}^2 E[\delta_0^2 + \lambda_1^2\delta_1^2 - 2\lambda_1\delta_0\delta_1]$

$= \overline{Y}^2[\sum_{h=1}^{L} \frac{W_h^2}{n_h}(\frac{S_{y_h}^2}{\overline{Y}^2} - \frac{m}{n_h}\sum_{i=1}^{r_h} D_{y_{h[i]}}^2) + \lambda_1^2 \sum_{h=1}^{L} \frac{W_h^2}{n_h}(\frac{S_{x_h}^2}{\overline{X}^2} - \frac{m}{n_h}\sum_{i=1}^{r_h} D_{x_{h(i)}}^2)$
$-2\lambda_1 \sum_{h=1}^{L} \frac{W_h^2}{n_h}(\frac{S_{x_h y_h}}{\overline{XY}} - \frac{m}{n_h}\sum_{i=1}^{r_h} D_{x_{h(i)} y_{h[i]}})]$

$$(3.3) \qquad \implies MSE(\overline{y}_{strMM1}) = \sum_{h=1}^{L} \frac{{W_h}^2}{n_h}\big[(S_{y_h}{}^2 + R^2\lambda_1^2 S_{x_h}^2 - 2R\lambda_1 S_{x_h y_h}) - \overline{Y}^2 \frac{m}{n_h}\sum_{i=1}^{r_h}(D_{y_h[i]} - \lambda_1 D_{x_h(i)})^2\big]$$

Adapting the estimators in (1.3) given by Kadilar and Cingi (2003), we propose another new ratio type estimator in stratified ranked set sampling as follows

$$(3.4) \qquad \overline{y}_{strMM2} = \overline{y}_{[SRSS]} \frac{\sum_{h=1}^{L} W_h[\overline{X}_h + \beta_{2h}(x)]}{\sum_{h=1}^{L} W_h[\overline{x}_{h(r_h)} + \beta_{2h}(x)]}$$

The Bias and MSE of $\overline{y}_{strMM2}$ can be found as follows-

$B(\overline{y}_{strMM2}) = E(\overline{y}_{strMM2}) - \overline{Y}$

Here $\overline{y}_{strMM2} = \overline{Y}(1+\delta_0)(1+\lambda_2\delta_1)^{-1}$, where $\lambda_2 = \frac{\sum_{h=1}^{L} W_h \overline{X}_h}{\sum_{h=1}^{L} W_h(\overline{X}_h + \beta_{2h}(x))}$.

Now $E(\overline{y}_{strMM2}) = \overline{Y}[1 + \lambda_2^2 E(\delta_1^2) - \lambda_2 E(\delta_0\delta_1)]$,because$E(\delta_0) = E(\delta_1) = 0$.

$$(3.5) \qquad \implies B(\overline{y}_{strMM2}) = \overline{Y}\Big[\sum_{h=1}^{L} \frac{W_h^2}{n_h}\{\frac{\lambda_2^2 S_{x_h}^2}{\overline{X}^2} - \frac{\lambda_2 S_{x_h y_h}}{\overline{XY}}\} - \sum_{h=1}^{L} \frac{W_h^2}{n_h}\{\frac{m}{n_h}(\lambda_2^2 \sum_{i=1}^{r_h} D_{x_h(i)}^2 - \lambda_2 \sum_{i=1}^{r_h} D_{x_h(i)y_h[i]})\}\Big]$$

Now $MSE(\overline{y}_{strMM2}) = E(\overline{y}_{strMM2} - \overline{Y})^2 = \overline{Y}^2 E[\delta_0^2 + \lambda_2^2\delta_1^2 - 2\lambda_2\delta_0\delta_1]$

$= \overline{Y}^2[\sum_{h=1}^{L} \frac{W_h^2}{n_h}(\frac{S_{y_h}^2}{\overline{Y}^2} - \frac{m}{n_h}\sum_{i=1}^{r_h} D_{y_{h[i]}}^2) + \lambda_2^2 \sum_{h=1}^{L} \frac{W_h^2}{n_h}(\frac{S_{x_h}^2}{\overline{X}^2} - \frac{m}{n_h}\sum_{i=1}^{r_h} D_{x_{h(i)}}^2)$
$-2\lambda_2 \sum_{h=1}^{L} \frac{W_h^2}{n_h}(\frac{S_{x_h y_h}}{\overline{XY}} - \frac{m}{n_h}\sum_{i=1}^{r_h} D_{x_{h(i)} y_{h[i]}})]$

$$(3.6) \qquad \implies MSE(\overline{y}_{strMM2}) = \sum_{h=1}^{L} \frac{{W_h}^2}{n_h}\big[(S_{y_h}{}^2 + R^2\lambda_2^2 S_{x_h}^2 - 2R\lambda_2 S_{x_h y_h}) - \overline{Y}^2 \frac{m}{n_h}\sum_{i=1}^{r_h}(D_{y_h[i]} - \lambda_2 D_{x_h(i)})^2\big]$$

Motivated by estimators (1.4) and (1.5) by Kadilar and Cingi (2003), we now propose two more ratio ÔÇôtype estimators , considering both coefficients of variation and kurtosis in stratified ranked set sampling as follows

$$(3.7)\qquad \overline{y}_{strMM3} = \overline{y}_{[SRSS]} \frac{\sum_{h=1}^{L} W_h(\overline{X}_h \beta_{2h}(x) + C_{x_h})}{\sum_{h=1}^{L} W_h(\overline{x}_{h(r_h)} \beta_{2h}(x) + C_{x_h})}$$

$$(3.8)\qquad \overline{y}_{strMM4} = \overline{y}_{[SRSS]} \frac{\sum_{h=1}^{L} W_h[\overline{X}_h C_{x_h} + \beta_{2h}(x)]}{\sum_{h=1}^{L} W_h[\overline{x}_{h(r_h)} C_{x_h} + \beta_{2h}(x)]}$$

The Bias and MSE of $\overline{y}_{strMM3}$ can be found as follows-

$B(\overline{y}_{strMM3}) = E(\overline{y}_{strMM3}) - \overline{Y}$

Here $\overline{y}_{strMM2} = \overline{Y}(1+\delta_0)(1+\gamma_1\delta_1)^{-1}$, where$\gamma_1 = \frac{\sum_{h=1}^{L} W_h \overline{X}_h \beta_{2h}(x)}{\sum_{h=1}^{L} W_h(\overline{X}_h \beta_{2h}(x) + C_{x_h})}$

Now $E(\overline{y}_{strMM3}) = \overline{Y}[1 + \gamma_1^2 E(\delta_1^2) - \gamma_1 E(\delta_0\delta_1)]$, because$E(\delta_0) = E(\delta_1) = 0.$

$$(3.9)\qquad \implies B(\overline{y}_{strMM3}) = \overline{Y}[\sum_{h=1}^{L} \frac{W_h^2}{n_h}\{\frac{\gamma_1^2 S_{x_h}^2}{\overline{X}^2} - \frac{\gamma_1 S_{x_h y_h}}{\overline{XY}}\} - \sum_{h=1}^{L} \frac{W_h^2}{n_h}\{\frac{m}{n_h}(\gamma_1^2 \sum_{i=1}^{r_h} D_{x_h(i)}^2 - \gamma_1 \sum_{i=1}^{r_h} D_{x_h(i) y_h[i]})\}]$$

Now $MSE(\overline{y}_{strMM3}) = E(\overline{y}_{strMM3} - \overline{Y})^2 = \overline{Y}^2 E[\delta_0^2 + \gamma_1^2\delta_1^2 - 2\gamma_1\delta_0\delta_1]$

$= \overline{Y}^2[\sum_{h=1}^{L} \frac{W_h^2}{n_h}(\frac{S_{y_h}^2}{\overline{Y}^2} - \frac{m}{n_h}\sum_{i=1}^{r_h} D_{y_h[i]}^2) + \gamma_1^2 \sum_{h=1}^{L} \frac{W_h^2}{n_h}(\frac{S_{x_h}^2}{\overline{X}^2} - \frac{m}{n_h}\sum_{i=1}^{r_h} D_{x_{h(i)}}^2)$
$-2\gamma_1 \sum_{h=1}^{L} \frac{W_h^2}{n_h}(\frac{S_{x_h y_h}}{\overline{XY}} - \frac{m}{n_h}\sum_{i=1}^{r_h} D_{x_h(i) y_h[i]})]$

$$(3.10)\qquad \implies MSE(\overline{y}_{strMM3}) = \sum_{h=1}^{L} \frac{{W_h}^2}{n_h}\left[(S_{y_h}{}^2 + R^2\gamma_1^2 S_{x_h}^2 - 2R\gamma_1 S_{x_h y_h}) - \overline{Y}^2 \frac{m}{n_h}\sum_{i=1}^{r_h}(D_{y_h[i]} - \gamma_1 D_{x_h(i)})^2\right]$$

Similarly bias and mean squared error of the estimator $\overline{y}_{strMM4}$ can be obtained respectively by changing the place of coefficient of kurtosis and coefficient of variation, as

$$(3.11)\quad B(\overline{y}_{strMM4}) = \overline{Y}\Big[\sum_{h=1}^{L}\frac{W_h^2}{n_h}\{\frac{\gamma_2^2 S_{x_h}^2}{\overline{X}^2} - \frac{\gamma_2 S_{x_h y_h}}{\overline{XY}}\} - \sum_{h=1}^{L}\frac{W_h^2}{n_h}\{\frac{m}{n_h}(\gamma_2^2\sum_{i=1}^{r_h} D_{x_h(i)}^2 - \gamma_2\sum_{i=1}^{r_h} D_{x_h(i)y_h[i]})\}\Big]$$

and

$$(3.12)\quad MSE(\overline{y}_{strMM4}) = \sum_{h=1}^{L}\frac{{W_h}^2}{n_h}\big[(S_{y_h}{}^2 + R^2\gamma_2^2 S_{x_h}^2 - 2R\gamma_2 S_{x_h y_h}) - \overline{Y}^2\frac{m}{n_h}\sum_{i=1}^{r_h}(D_{y_h[i]} - \gamma_2 D_{x_h(i)})^2\big]$$

where $\gamma_2 = \frac{\sum_{h=1}^{L} W_h \overline{X}_h C_{x_h}}{\sum_{h=1}^{L} W_h(\overline{X}_h C_{x_h} + \beta_{2h}(x))}$.

4. Efficiency Comparison

On comparing $(1.7), (1.8), (1.9)$ and (1.10) with $(3.3), (3.6), (3.10)$ and (3.12) respectively, we obtain

1) $MSE(\overline{y}_{stSD}) - MSE(\overline{y}_{strMM1}) = A_1 \geq 0$,
where $A_1 = \overline{Y}^2 \sum_{h=1}^{L} \frac{W_h^2}{n_h}\frac{m}{n_h}\sum_{i=1}^{r_h}(D_{y_h[i]} - \lambda_1 D_{x_h(i)})^2$
$\implies MSE(\overline{y}_{stSD}) \geq MSE(\overline{y}_{strMM1})$

2) $MSE(\overline{y}_{stSK}) - MSE(\overline{y}_{strMM2}) = A_2 \geq 0$,
where $A_2 = \overline{Y}^2 \sum_{h=1}^{L} \frac{W_h^2}{n_h}\frac{m}{n_h}\sum_{i=1}^{r_h}(D_{y_h[i]} - \lambda_2 D_{x_h(i)})^2$
$\implies MSE(\overline{y}_{stSK}) \geq MSE(\overline{y}_{strMM2})$

3) $MSE(\overline{y}_{stUS1}) - MSE(\overline{y}_{strMM3}) = A_3 \geq 0$,
where $A_3 = \overline{Y}^2 \sum_{h=1}^{L} \frac{W_h^2}{n_h}\frac{m}{n_h}\sum_{i=1}^{r_h}(D_{y_h[i]} - \gamma_1 D_{x_h(i)})^2$
$\implies MSE(\overline{y}_{stUS1}) \geq MSE(\overline{y}_{strMM3})$

4) $MSE(\overline{y}_{stUS2}) - MSE(\overline{y}_{strMM4}) = A_4 \geq 0$,
where $A_4 = \overline{Y}^2 \sum_{h=1}^{L} \frac{W_h^2}{n_h}\frac{m}{n_h}\sum_{i=1}^{r_h}(D_{y_h[i]} - \gamma_2 D_{x_h(i)})^2$
$\implies MSE(\overline{y}_{stUS2}) \geq MSE(\overline{y}_{strMM4})$

It is easily seen that the MSE of the proposed estimators given in (3.1) ,(3.4), (3.7), and (3.8) are always smaller than the estimator given in (1.2) to (1.5) respectively, because A_1, A_2, A_3 and A_4 all are non-negative values and thus it is shown that the proposed ratio types estimators $\overline{y}_{strMM1}, \overline{y}_{strMM2}$, $\overline{y}_{strMM3}$ and $\overline{y}_{strMM4}$ for the population mean using stratified ranked set sampling are asymptotically more efficient than the ratio estimators $\overline{y}_{stSD}, \overline{y}_{stSK}$, $\overline{y}_{stUS1}$ and $\overline{y}_{stUS2}$ given by Kadilar and Cingi (2003).

Table 1. Population Statistics

Stratum 1	Stratum 2	Stratum 3
$N_1 = 12$	$N_2 = 30$	$N_3 = 17$
$n_1 = 9$	$n_2 = 15$	$n_3 = 12$
$W_1 = 0.2034$	$W_2 = 0.5085$	$W_3 = 0.2881$
$\overline{X}_1 = 5987.83$	$\overline{X}_2 = 11682.73$	$\overline{X}_3 = 68662.29$
$\overline{Y}_1 = 11788$	$\overline{Y}_2 = 16862.27$	$\overline{Y}_3 = 227371.53$
$S^2_{x_1} = 27842810.5$	$S^2_{x_2} = 760238523$	$S^2_{x_3} = 12187889050$
$S^2_{y_1} = 153854583$	$S^2_{y_2} = 2049296094$	$S^2_{y_3} = 372428238550$
$S_{y_1x_1} = 62846173.1$	$S_{y_2x_2} = 1190767859$	$S_{y_3x_3} = 27342963562$
$C_{x_1} = 0.8812$	$C_{x_2} = 2.3601$	$C_{x_3} = 1.6079$
$\beta_{21}(x) = 1.8733$	$\beta_{22}(x) = 10.7527$	$\beta_{23}(x) = 8.935$
$R_1 = 1.97$	$R_2 = 1.44$	$R_3 = 3.31$

5. Numerical Illustration

To compare efficiencies of various proposed estimators of our study, here, we take a stratified population with 3 strata with sizes 12,30 & 17, respectively on page 1119 (Appendix) given by Singh(2003). The example considers the data of Tobacco for Area and Production in specified countries during 1998, where y is production (study variable) in metric tons and x is area (auxiliary variable) in hectares.

For the above population, the parameters are summarized in Table 1. Note that total population size $N = 59, \overline{Y} = 76485.42, \overline{X} = 26942.29$

From this population we took 5 ranked set samples of sizes $r_1 = 3, r_2 = 5$ and $r_3 = 4$ from stratum 1st , 2nd and 3rd respectively. Further each ranked set sample from each stratum were repeated with number of cycles $m = 3$.Hence sample sizes of stratified ranked set samples equivalent to stratified simple random samples of sizes $n_h(= mr_h)$ on considering arbitrary allocation.

The relative efficiency of the estimator is given by $RE = \frac{\overline{y}_{SSRS}}{\overline{y}_{SRRS}} * 100$

The estimated relative efficiencies of various stratified ranked set estimators in comparison with different stratified SRS estimators are shown in Table 2.

From Table 2, we see that stratified ranked set estimators are more efficient than corresponding stratified SRS estimators. Thus, if coefficient of variation and coefficient of kurtosis are known for auxiliary variable x,then, these proposed estimators can be used in practice.

Table 2

Variances of various stratified SRS estimators	$\overline{y}_{SSRS}$ 2245922261	$\overline{y}_{stSD}$ 2245878377	$\overline{y}_{stSK}$ 2245739510	$\overline{y}_{stUS1}$ 2245913748	$\overline{y}_{stUS2}$ 2245816148
Variances of corresponding stratified ranked set sampling estimators & Relative Efficiencies in % for	$\overline{y}_{SRSS}$	$\overline{y}_{strMM1}$	$\overline{y}_{strMM2}$	$\overline{y}_{strMM3}$	$\overline{y}_{strMM4}$
Sample 1	1938096069 115.8229	1938091889 115.8809	1938094290 115.8736	1938092227 115.8827	1938095047 115.8775
Sample 2	22135875617 105.1523	2135831913 105.1524	2135693676 105.1527	2135870922 105.1523	2135769977 105.1525
Sample 3	1414957661 158.7272	1414914192 158.729	1414776801 158.7345	1414952994 158.727361	1414852647 158.7315
Sample 4	2017429690 111.3259	2017405990 111.3251	2017338937 111.3219	2017427349 111.32582	2017377001 111.3236
Sample 5	1523423535 147.426	1523375065 147.4278	1523219867 147.4337	1523418280 147.426191	1523305279 147.4305

6. Conclusion

We have proposed new ratio-type estimators for strati´$\frac{1}{4}$üed ranked set sampling on the lines of the estimators of Kadilar and Cingi (2003) and obtained their MSE equations. By these equations, the MSEÔÇÖs of proposed estimators have been compared with corresponding stratified simple random sampling estimators given by Kadilar and Cingi (2003) and found that the proposed estimators have smaller MSEÔÇÖs than the corresponding estimators. These theoretical results have been supported by the above numerical example and thus it is concluded that the proposed ratio type estimators $\overline{y}_{SRSS}$, $\overline{y}_{strMM1}$, $\overline{y}_{strMM2}$, $\overline{y}_{strMM3}$ and $\overline{y}_{strMM4}$, for the population mean using stratified ranked set sampling are more efficient asymptotically than the corresponding ratio estimators $\overline{y}_{SSRS}$, $\overline{y}_{stSD}$, $\overline{y}_{stSK}$, $\overline{y}_{stUS1}$ and $\overline{y}_{stUS2}$ using SRS. With this conclusion, we hope to develop new estimators in other sampling methods in the forthcoming studies.

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The Kummer beta Birnbaum-Saunders: An alternative fatigue life distribution

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Abstract

Birnbaum and Saunders [11] introduced a positive continuous distribution commonly used in reliability studies. Based on this distribution, we propose and study the called Kummer beta Birnbaum-Saunders distribution for modeling fatigue life data. Various properties of the new distribution including explicit expressions for the moments, generating function, mean deviations, density function of the order statistics and their moments are derived. We investigate maximum likelihood estimation of the parameters. The superiority of the new distribution is illustrated by means of two failure real data sets.

Keywords: Birnbaum-Saunders distribution, Fatigue life distribution, Kummer beta distribution, Lifetime data, Maximum likelihood estimation.

2000 AMS Classification: 60E05, 62N05

1. Introduction

Fatigue is a structural damage which occurs when a material is exposed to stress and tension fluctuations. When the effect of vibrations on material specimens and structures is studied, the first point to be considered is the mechanism that could cause fatigue of these materials. To understand the fatigue process and the genesis of the fatigue life and cumulative damage distributions, we recall concepts related to crack, cycle, fatigue, and load.

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In summary, the fatigue process (*fatigue life*) begins with an imperceptible fissure, the initiation, growth, and propagation of which produces a dominant crack in the specimen due to cyclic patterns of stress, whose ultimate extension causes the rupture or failure of this specimen. The failure occurs when the total extension of the crack exceeds a critical threshold for the first time. The partial extension of a crack produced by fatigue in each cycle is modeled by a random variable which depends on the type of material, the magnitude of the stress, and the number of previous cycles, among other factors. More details about the fatigue process can be found, for example, in Valluri [71], Birnbaum and Saunders [11], Murthy [53], Saunders [68], and Volodin [75].

The most popular model used to describe the lifetime process under fatigue is the Birnbaum-Saunders (BS) distribution. However, it allows for unimodal hazard rates only, hence cannot provide reasonable fits for modeling phenomenon with bathtub hazard rates, which are common in reliability studies. The distributions allowing for unimodal and bathtub hazard rates are sufficiently complex (Nelson, [55]) and usually require five or more parameters.

Motivated by problems of vibration in commercial aircraft that caused fatigue in the materials, Birnbaum and Saunders [11], [12] proposed the two-parameter BS distribution, also known as the fatigue life distribution, with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, say $\mathrm{BS}(\alpha, \beta)$. This distribution can be used to model lifetime data and it is widely applicable to represent failure times of fatiguing materials. If Z is a standard normal random variable, the random variable X defined by

$$X = \beta \left[\frac{\alpha Z}{2} + \left\{ \left(\frac{\alpha Z}{2} \right)^2 + 1 \right\}^{1/2} \right]^2$$

has a $\mathrm{BS}(\alpha, \beta)$ distribution whose cumulative distribution function (cdf) is given by

$$G(x) = \Phi(\nu) \tag{1.1}$$

for $x > 0$, where $\nu = (1/\alpha)\rho(x/\beta)$, $\rho(z) = z^{1/2} - z^{-1/2}$, and $\Phi(\cdot)$ is the standard cdf. The parameter β is the median of the distribution, i.e., $G(\beta) = \Phi(0) = 1/2$. For any $k > 0$, $kX \sim \mathrm{BS}(\alpha, k\beta)$. Kundu et al. [36] investigated the shape of the BS hazard rate function. Results on improved statistical inference for this distribution are discussed by Wu and Wong [78] and Lemonte et al. [46], [48]. Díaz-Garcia and Leiva [21] proposed a new family of generalized BS distributions based on contoured elliptical distributions, whereas Guiraud et al. [29] introduced a non-central version of the BS distribution. The probability density function (pdf) corresponding to (1.1) is

$$g(x) = r(\alpha, \beta) x^{-3/2} (x + \beta) \exp\left[-\frac{\tau(x/\beta)}{2\alpha^2} \right] \tag{1.2}$$

for $x > 0$, where $r(\alpha, \beta) = \exp\left(\alpha^{-2}\right) \left(2\alpha\sqrt{2\pi\beta}\right)^{-1}$ and $\tau(z) = z - z^{-1}$. The fractional moments of (1.2) (Rieck, [65]) are

$$E\left(X^p\right) = \beta^p I(p, \alpha),$$

where

$$I(p, \alpha) = \frac{K_{p+1/2}\left(\alpha^{-2}\right) + K_{p-1/2}\left(\alpha^{-2}\right)}{2K_{1/2}\left(\alpha^{-2}\right)} \tag{1.3}$$

and $K_p(z)$ denotes the modified Bessel function of the third kind with p representing its order and z the argument. Its integral representation is $K_p(z) = 0.5 \int_{-\infty}^{\infty} \exp\{-z\cosh(t) - pt\}\, dt$. A discussion of this function can be found in Watson [77].

The Kummer beta (KB) distribution may be characterized by the pdf (Ng and Kotz, [57])

$$F_{\mathrm{KB}}(x) = Kx^{a-1}(1-x)^{b-1}e^{-cx} \tag{1.4}$$

for $0 < x < 1$, $a > 0$, $b > 0$ and $-\infty < c < \infty$, where

$$K^{-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\, {}_1F_1(a; a+b; -c),$$

where

$${}_1F_1(a; a+b; -c) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{a-1}(1-t)^{b-1}e^{-ct}dt = \sum_{k=0}^{\infty} \frac{(a)_k(-c)^k}{(a+b)_k k!}$$

is the confluent hypergeometric function (Abramowitz and Stegun, [1]), $\Gamma(\cdot)$ is the gamma function and $(d)_k = d(d+1)\cdots(d+k-1)$ denotes the ascending factorial. An important special case of (1.4) for $c = 0$ is the beta pdf.

There has been much theoretical developments with respect to the BS distribution. The developments have covered many aspects of the distribution. Some of these are: acceptance sampling (Balakrishnan et al., [6]; Aslam et al., [3]), Bayes estimation (Xu and Tang, [80]), bivariate generalizations (Kundu et al., [35]), bootstrap estimation (Lemonte et al., [48]), censored estimation (Barreto et al., [8]), confidence intervals (Leiva et al., [40]), discrimination (Butler-Mccullough, [13]), EM estimation (Balakrishnan et al., [7]), graphical estimation (Chang and Tang, [15]), hazard rate (Kundu et al., [36]), influence diagnostics (Li et al., [49]), interval estimation (Wang, [76]), log linear models (Rieck and Nedelman, [66]), matrix-variate generalizations (Caro-Lopera et al., [14]), maximum likelihood estimation (Engelhardt et al., [23]), mixture models (Patriota, [61]), moment estimation (Ng et al., [56]), moment generating function (Rieck, [65]), percentiles estimation (Vilca et al., [74]), random number generation (Leiva et al., [39]), reference analysis (Xu and Tang, [79]), regression models (Lemonte and Cordeiro, [44]), reliability models (Upadhyay et al., [70]), robust estimation (Paula et al., [62]), shape and change point analyses (Azevedo et al., [5]), statistical software (Barros et al., [9]), testing hypotheses (Lemonte and Ferrari, [47]), time series models (Bhatti, [10]), truncated versions (Ahmed et al., [2]), and univariate generalizations (Owen, [60]; Vilca and Leiva, [72]; Gómes et al., [26]; Leiva et al., [38]; Leiva et al., [41]; Athayde et al., [4]; Ferreira et al., [25]; Santos-Neto et al., [67]; Lemonte, [42]).

The BS distribution has also received wide ranging applications. Some recent applications include: modeling of hourly SO_2 concentrations at ten monitoring stations located in different zones in Santiago (Leiva et al., [37]); modeling of diameter at breast height distributions of near-natural complex structure silver fir-European beech forests (Podlaski, [64]); modeling of hourly dissolved oxygen (DO) concentrations observed at four monitoring stations located at different areas of Santiago (Leiva et al., [38]; Vilca et al., [73]); statistical analysis of redundant systems with one warm stand-by unit (Nikulin and Tahir, [59]).

Because of the widespread study and applications of the BS distribution, there is a need for new generalizations. This aim of this paper is to introduce a new generalization of the BS distribution.

For an arbitrary baseline cdf $G(x)$ with parameter vector γ and pdf $g(x)$, the Kummer beta generalized (denoted by the prefix "KB-G" for short) cdf defined in Pescim et al. [63] is

$$F_{\mathrm{KBG}}(x) = K \int_0^{G(x)} t^{a-1}(1-t)^{b-1} e^{-c\,t} dt, \tag{1.5}$$

where $a > 0$ and $b > 0$ are shape parameters which induce skewness, and thereby promote weight variation of the tails, whereas the parameter $-\infty < c < \infty$ "squeezes" the pdf to the left or right, i.e., it gives weights to the extremes of the pdfs. For more details, see Pescim et al. [63].

The pdf corresponding to (1.5) can be expressed as:

$$f_{\mathrm{KBG}}(x) = K g(x) G^{a-1}(x) \left\{1 - G(x)\right\}^{b-1} \exp\left\{-c\, G(x)\right\}. \tag{1.6}$$

Clearly, the KB pdf (1.4) is a basic exemplar of equation (1.6) for $G(x) = x$, where $x \in (0,1)$. Additionally, we obtain the classical beta distribution for $c = 0$. Equation (1.6) will be most tractable when both $G(x)$ and $g(x)$ have simple analytic expressions. Its major benefit is to offer more flexibility to extremes (right and/or left) of the pdfs and therefore it becomes suitable for analyzing data with high degree of asymmetry.

The shape parameters a, b and c have the following effects on $f(x)$: increasing values of a make the lower and upper tails of f lighter; increasing values of b make the upper tails of f lighter but they do not change the lower tails of f; increasing values of c make the lower and upper tails of f lighter. So, each of the shape parameters adds more flexibility.

The class of distributions (1.6) includes two important special cases: the beta-generalized (BG) and exponentiated generalized (EG) distributions defined by Eugene et al. [24] and Mudholkar et al. [52] when $c = 0$ and $c = 0$ and $b = 1$, respectively. We can note that the BG distributions can be limited in one aspect. They have only two additional shape parameters and so they can add only a limited structure to the generated distribution. For instance, a BG distribution may have problems to capture the behavior of random variables with symmetric but highly leptokurtic distributions. While the beta parameters offer explicit control over skewness when the parent is symmetric, they have less control over higher moments such as kurtosis. Further, the EG distribution still introduces only one extra shape parameter, whereas three may be required to control both tail weights and the distribution of weight in the center. Hence, the generated distribution (1.6) is a more flexible since it has one more shape parameter than the classical beta or exponentiated generators.

In this paper, we introduce a new five-parameter distribution called the Kummer beta Birnbaum-Saunders (KBBS) distribution which contains as sub-models the BS and beta Birnbaum-Saunders (BBS) (Cordeiro and Lemonte, [18]) distributions. The main motivation for this extension is that the new distribution is a highly flexible life distribution which admits different degrees of kurtosis and asymmetry. Moreover, the new distribution due to its flexibility in accommodating bathtub shaped and unimodal forms of the hazard rate function could be an important distribution in a variety of problems in survival analysis and reliability studies. The KBBS distribution is not only convenient for modeling comfortable bathtub shaped and unimodal hazard rates but it is also suitable for testing goodness of fit of its sub-models.

The KBBS distribution comes from (1.6) by taking $G(x)$ and $g(x)$ as the cdf and the pdf of the BS(α, β) distribution, respectively. We also provide a comprehensive description of some of its mathematical properties with the hope that it will attract wider applications in reliability, engineering and in other areas of research.

The article is outlined as follows. In Section 2, we define the KBBS distribution and plot its pdfs and hazard rate functions. Section 3 provides useful expansions for the pdf and the cdf. We obtain explicit expressions for the moments and generating function (Section 4), incomplete moments (Section 5), mean deviations, Bonferroni and Lorenz curves and reliability (Section 6) and order statistics (Section 7). Several expressions in Sections 3 to 7 involve infinite series. The computational issues relating to these infinite series are discussed in Section 8. In Section 9, we discuss maximum likelihood estimation and statistical inference. Also discussed in Section 9 is a simulation study assessing the performance of the maximum likelihood estimators (MLEs). Two applications presented in Section 10 reveal the usefulness of the new distribution for fatigue life data. Concluding remarks are noted in Section 11.

2. The KBBS distribution

By taking the cdf (1.1) and the pdf (1.2) of the BS distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, the cdf and the pdf of the KBBS distribution are obtained from equations (1.5) and (1.6) as

$$F(x) = K \int_0^{\Phi(\nu)} t^{a-1}(1-t)^{b-1} e^{-ct} dt \tag{2.1}$$

and

$$\begin{aligned} f(x) &= K r(\alpha,\beta) x^{-3/2}(x+\beta)\Phi(\nu)\left[1-\Phi(\nu)\right]^{b-1} \\ &\quad \times \exp\left\{-\left[\frac{\tau(x/\beta)}{2\alpha^2} + c\,\Phi(\nu)\right]\right\} \end{aligned} \tag{2.2}$$

for $x > 0$. Hereafter, we denote by X the random variable following (2.2), say $X \sim \text{KBBS}(a, b, c, \alpha, \beta)$. This pdf has four shape parameters a, b, c and α, which allow for a high degree of flexibility. The parameter c controls tail weights to the extremes of the distribution. The associated hazard rate function becomes

$$h(x) = \frac{K r(\alpha,\beta) x^{-3/2}(x+\beta)\Phi(\nu)}{\left[1-F(x)\right]\left[1-\Phi(\nu)\right]^{1-b}} \exp\left\{-\left[\frac{\tau(x/\beta)}{2\alpha^2} + c\,\Phi(\nu)\right]\right\}.$$

The study of the new distribution is important since it extends some distributions previously considered in the literature. In fact, the BS distribution (with parameters α and β) is clearly a basic exemplar for $a = b = 1$ and $c = 0$, with a continuous crossover towards distributions with different shapes (e.g., a specified combination of skewness and kurtosis). The KBBS distribution contains as sub-models the beta-BS (BBS) and the exponentiated Birnbaum-Saunders (EBS) (Cordeiro et al., [19]) distributions when $c = 0$ and $b = 1$ in addition to $c = 0$, respectively. Plots of the KBBS pdf and hazard rate functions for selected parameter values are displayed in Figures 1 and 2. It is evident that the shapes of the new pdf are much more flexible than the BS distribution. Further, it allows four major hazard shapes: increasing, decreasing, bathtub and unimodal hazard rates.

3. Expansions for cdf and pdf

Expansions for equations (2.1) and (2.2) can be derived using the concept of exponentiated distributions. Cordeiro et al. [19] defined a random variable Y following the EBS distribution with parameters α, β and $\gamma > 0$, say $Y \sim \mathrm{EBS}(\alpha, \beta, \gamma)$. The cdf and the pdf of Y are denoted by $H(y; \alpha, \beta, \gamma) = \Phi^{\gamma}(\nu)$ and $h(y; \alpha, \beta, \gamma) = \gamma g_{\alpha,\beta}(y)\Phi^{\gamma-1}(\nu)$, respectively, where ν is defined in (1.1). The properties of some exponentiated distributions have been studied by several authors, see Mudholkar and Srivastava [51] and Mudholkar et al. [52] for the exponentiated Weibull distribution, Gupta et al. [30] for the exponentiated Pareto distribution, Gupta and Kundu [31] for the exponentiated exponential distribution, Nadarajah and Gupta [54] for the exponentiated gamma distribution, Cordeiro et al. [20] for the exponentiated generalized gamma distribution, Lemonte and Cordeiro [45] for the exponentiated generalized inverse Gaussian distribution, and Lemonte et al. [43] for the exponentiated Kumaraswamy distribution.

By expanding the term $\exp[-c\,\Phi(\nu)]$ and the binomial in equation (2.2), we obtain the linear combination (for $a > 0$ integer)

$$f(x) = \sum_{j,k=0}^{\infty} w_{j,k} h(x; \alpha, \beta, a + j + k), \tag{3.1}$$

where $h(x; \alpha, \beta, a + j + k)$ denotes the $\mathrm{EBS}(\alpha, \beta, a + j + k)$ pdf and the coefficient $w_{j,k}$ is given by

$$w_{j,k} = \frac{K(-1)^{j+k} c^j}{j!(a + j + k)} \binom{b-1}{k}.$$

By integrating (3.1), we obtain

$$F(x) = \sum_{j,k=0}^{\infty} w_{j,k} \Phi^{a+j+k}(\nu). \tag{3.2}$$

If a is a positive non-integer, we can expand $\Phi^{a+j+k}(\nu)$ as

$$\Phi^{a+j+k}(\nu) = \sum_{r=0}^{\infty} s_r(a + j + k)\Phi^r(\nu), \tag{3.3}$$

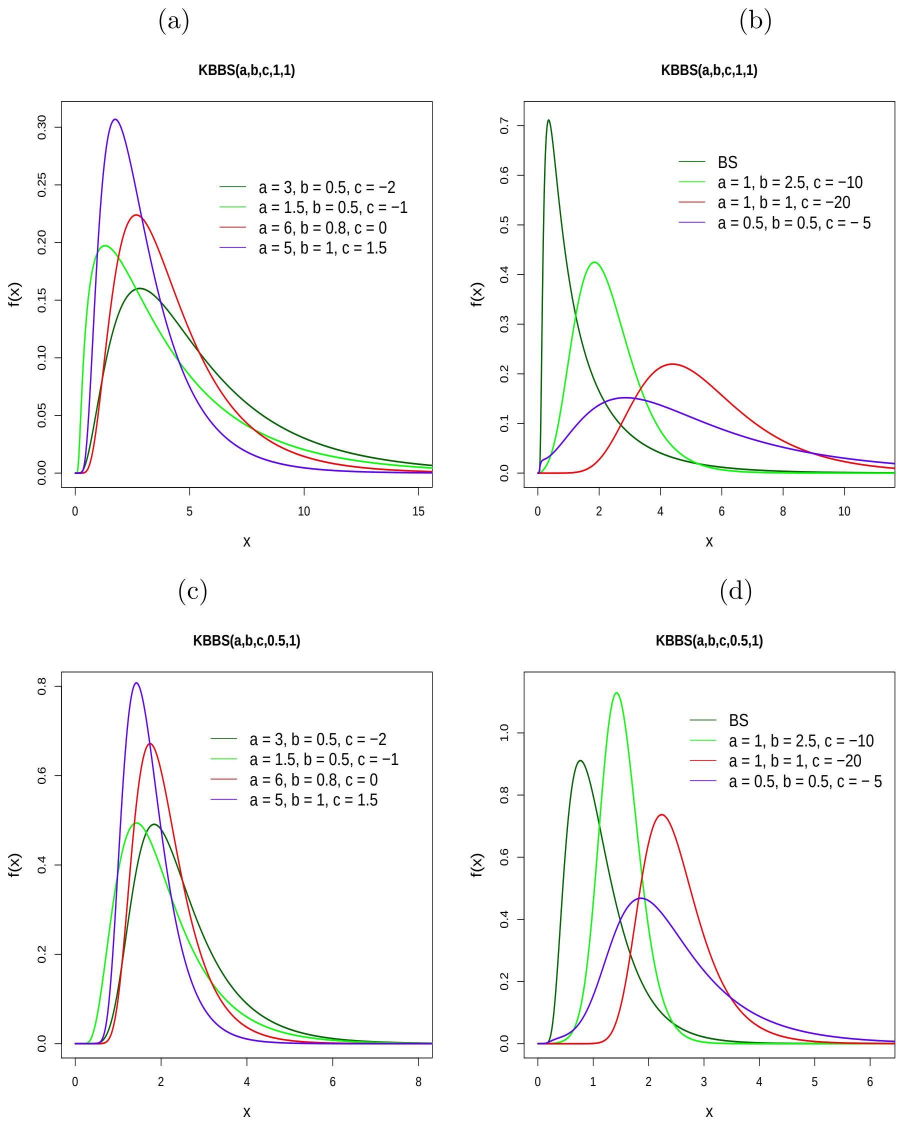


Figure 1. Plots of the pdf (2.2) for some parameter values.

where

$$s_r(m) = \sum_{k=r}^{\infty} (-1)^{k+r} \binom{m}{k} \binom{k}{r}.$$

Thus, from equations (1.2), (3.2) and (3.3), the KBBS cdf can be expressed as

$$F(x) = \sum_{r=0}^{\infty} b_r \Phi^r(\nu), \tag{3.4}$$

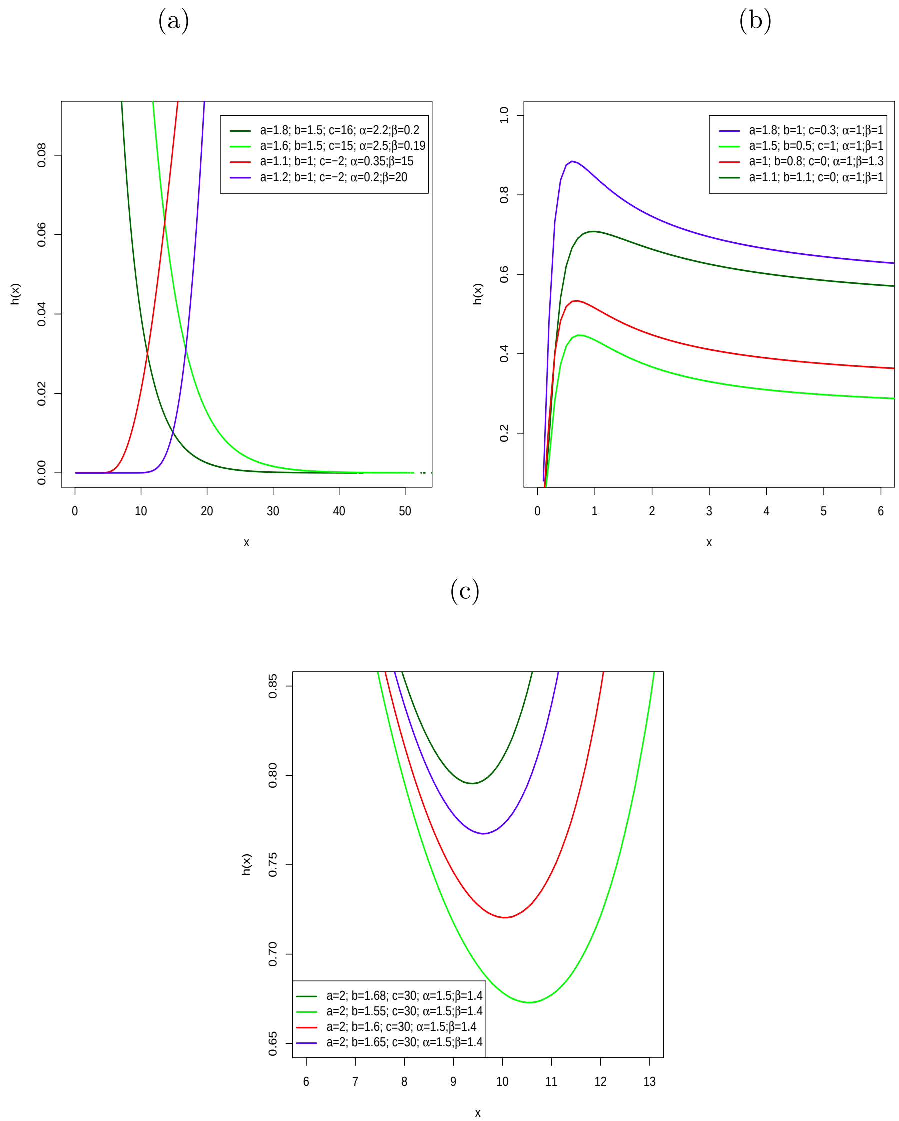


Figure 2. The KBBS hazard rate function. (a) Increasing and decreasing hazard rate function. (b) Unimodal hazard rate function. (c) Bathtub hazard rate function.

where

$$b_r = \sum_{j,k=0}^{\infty} w_{j,k} s_r(a+j+k).$$

For $a > 0$ real non-integer, the KBBS pdf expansion corresponding to (3.4) is obtained by simple differentiation

$$f(x) = \sum_{r=0}^{\infty} b_r h(x; \alpha, \beta, r). \tag{3.5}$$

Equation (3.5) reveals that the KBBS pdf is a linear combination of EBS pdfs. This result is important to derive some properties of the KBBS distribution from those of the EBS distribution.

4. Moments and generating function

4.1. Moments. The ordinary moments of X can be determined from the probability weighted moments (Greenwood et al., [28]) of the BS distribution formally defined for p and r non-negative integers by

$$\tau_{p,r-1} = \int_0^{\infty} x^p g(x) \Phi^{r-1}(\nu) dx. \tag{4.1}$$

The integral (4.1) can be easily computed numerically in software such as MAPLE, MATLAB, MATHEMATICA, Ox and R. Cordeiro and Lemonte [18] proposed an alternative representation to compute $\tau_{p,r-1}$ given by

$$\begin{aligned} \tau_{p,r-1} &= \frac{\beta^p}{2^{r-1}} \sum_{j=0}^{r-1} \binom{r-1}{j} \sum_{k_1,\ldots,k_j=0}^{\infty} A(k_1, \ldots, k_j) \\ &\quad \times \sum_{m=0}^{2s_j+j} (-1)^m \binom{2s_j + j}{m} I\left(p + s_j - m + \frac{j}{2}, \alpha\right), \end{aligned} \tag{4.2}$$

where $s_j = k_1 + \cdots + k_j$, $A(k_1, \ldots, k_j) = \alpha^{-2s_j - j} a_{k_1} \cdots a_{k_j}$, $a_k = (-1)^k 2^{(1-2k)/2} \left[\sqrt{\pi}(2k+1)\right]^{-1}$ and $I(p + (2s_j + j - 2m)/2, \alpha)$ is determined from (1.3).

The sth moment of X can be expressed from equation (3.5) as

$$\mu_s^{'} = \sum_{r=0}^{\infty} b_r \tau_{s,r-1}, \tag{4.3}$$

where $\tau_{s,r-1}$ is obtained from (4.1) and b_r is defined in (3.4).

he four first moments of the KBBS distribution were calculated by numerical integration and through infinite weighted sums in equation (4.3) using the statistical software package R. The values from both techniques are usually close when ∞ is replaced by a large number as 500 in (4.3). For selected values $a = 2$, $b = 1.5$, $c = 4$, $\alpha = 0.5$ and $\beta = 1$, Table (1) gives some numerical analysis for those moments and for variance, skewness and kurtosis.

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Plots of the skewness and kurtosis of the KBBS distribution as a function of c for selected values of a and b for $\alpha = 0.5$ and $\beta = 1.0$ are displayed in Figures 3 and 4. Figures 3a and 3b immediately indicate that the additional parameter c promotes high levels of asymmetry.

Table 1. Values of the four first moments, variance, skewness and kurtosis of the KBBS distribution for $a = 2$, $b = 1.5$, $c = 4$, $\alpha = 0.5$ and $\beta = 1$ obtained by numerical integration and through infinite weighted sums, where $j, k, r = 0, \ldots, p$.

Moments	Infinite weighted sums				Numerical integration
	p=50	p=100	p=250	p=500	
μ_1'	0.85967	0.85920	0.85898	0.85893	0.85890
μ_2'	0.83508	0.83355	0.83278	0.83258	0.83242
μ_3'	0.93435	0.92920	0.92633	0.92550	0.92479
μ_4'	1.23327	1.21506	1.20395	1.20042	1.19703
Variance	0.09604	0.0953	0.09492	0.09481	0.09471
Skewness	1.72439	1.6716	1.63790	1.62691	1.61629
Kurtosis	9.18582	8.6644	8.28549	8.14676	7.99257

(a) (b)

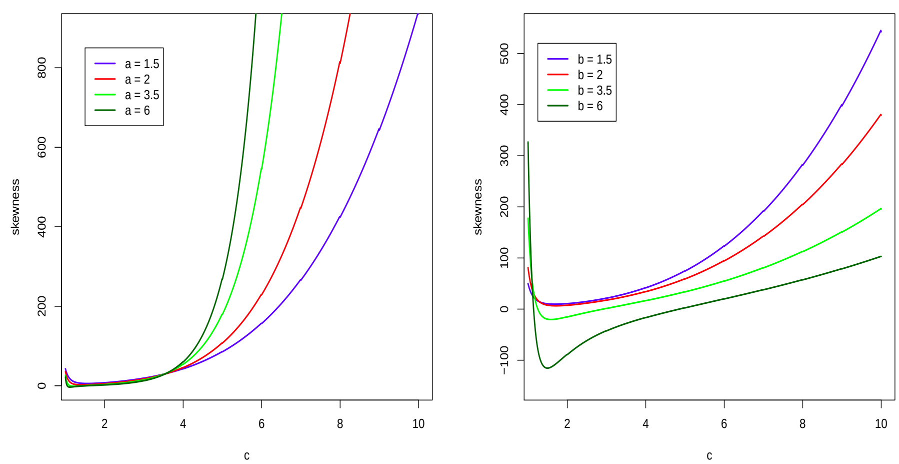


Figure 3. Skewness of the KBBS distribution as a function of c for some values of a and b for $\alpha = 0.5$ and $\beta = 1.0$. (a) $b = 1.5$ and (b) $a = 1.2$.

4.2. Generating function. Here, we provide a representation for the moment generating function (mgf) of X, say $M(t) = E\left[\exp(tX)\right]$, which is obtained as a linear combination of the mgf's of the EBS distributions. From expansion (3.5), we obtain

$$M(t) = \sum_{r=0}^{\infty} b_r M_r(t), \tag{4.4}$$

where $M_r(t)$ is the mgf of the EBS(α, β, r) distribution and b_r is defined by (3.4).

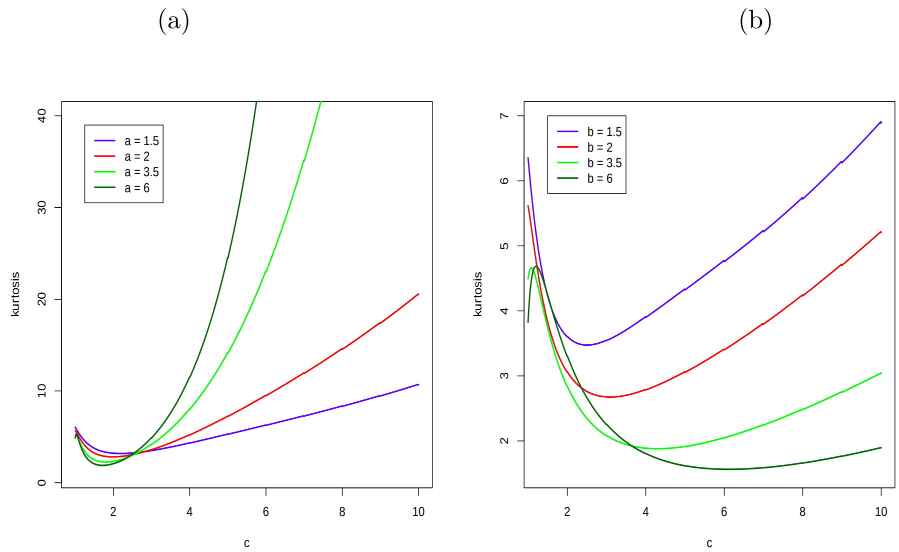


Figure 4. Kurtosis of the KBBS distribution as a function of c for some values of a and b for $\alpha = 0.5$ and $\beta = 1.0$. (a) $b = 1.5$ and (b) $a = 1.2$.

Thus, $M_r(t)$ can be expressed as

$$M_r(t) = r \int_0^\infty \exp(tx) g_{\alpha,\beta}(x) \Phi^{r-1}(\nu) dx, \tag{4.5}$$

where $g_{\alpha,\beta}(x)$ is the BS(α, β) pdf. Setting $u = \Phi(\nu)$ in (4.5), we have

$$M_r(t) = r \int_0^1 u^{r-1} \exp\left[tQ(u)\right] du, \tag{4.6}$$

where $x = Q(u)$ is the quantile function of the BS distribution and $u = \Phi(\nu)$ is given by (1.1).

Now, we derive a power series expansion for the quantile function of the EBS distribution that can be useful to calculate the mgf of the KBBS distribution. We use throughout an equation in Section 0.314 of Gradshteyn and Ryzhik [27] for a power series raised to a positive integer j given by

$$\left(\sum_{i=0}^{\infty} a_i x^i \right)^j = \sum_{i=0}^{\infty} c_{j,i} x^i, \tag{4.7}$$

where the coefficients $c_{j,i}$ (for $i = 1, 2, \ldots$) are easily computed from the recurrence equation

$$c_{j,i} = (i a_0)^{-1} \sum_{m=1}^{i} \left[m(j+1) - i\right] a_m c_{j,i-m} \tag{4.8}$$

and $c_{j,0} = a_0^j$. The coefficient $c_{j,i}$ can be determined from $c_{j,0}, \ldots, c_{j,i-1}$ and hence from the quantities $a_0, \ldots, a_i$. In fact, $c_{j,i}$ can be given explicitly in terms of the coefficients a_i, although it is not necessary for programming numerically our expansions in any algebraic or numerical software.

Following Cordeiro and Lemonte [18], we can invert $u = \Phi(\nu)$ if the condition $-2 < (x/\beta)^{1/2} - (\beta/x)^{1/2} < 2$ holds, to express x as a power series expansion of u

$$x = Q(u) = \sum_{i=0}^{\infty} \rho_i (u - 1/2)^i, \tag{4.9}$$

where the coefficients are $\rho_0 = \beta$, $\rho_{2q+1} = \beta\alpha^{2q+1}\binom{1/2}{q}4^{-q}$ for $q \geq 0$, $\rho_2 = \beta\alpha^2/2$ and $\rho_{2q} = 0$ for $q \geq 2$ and the quantities $e_{q,i}$ follow recursively from equations (4.7) and (4.8) by $e_{q,0} = d_0^q$ and

$$e_{q,i} = (i d_0)^{-1} \sum_{m=1}^{q} [m(q+1) - i]\, d_m e_{q,i-m}.$$

Here, the quantities d_m are defined by $d_m = 0$ (for $m = 0, 2, 4, \ldots$) and $d_m = j_{(m-1)/2}$ (for $m = 1, 3, 5, \ldots$), where the j_m's are calculated recursively from

$$j_{m+1} = \frac{1}{2(2m+3)} \sum_{v=0}^{m} \frac{(2v+1)(2m-2v+1) j_v j_{m-v}}{(v+1)(2v+1)}.$$

We have $j_0 = 1$, $j_1 = 1/6$, $j_2 = 7/120$, $j_3 = 127/7560$, and so on.

Substituting equation (4.9) into (4.6) and using the exponential expansion, we obtain

$$M_r(t) = r \sum_{p=0}^{\infty} \frac{t^p}{p!} \int_0^1 u^{r-1} \left(\sum_{i=0}^{\infty} \rho_i w^i \right)^p du, \tag{4.10}$$

where $w = u - 1/2$. From equations (4.7) and (4.8), we have

$$\left(\sum_{i=0}^{\infty} \rho_i w^i \right)^p = \sum_{i=0}^{\infty} \delta_{p,i} w^i = \sum_{i=0}^{\infty} \delta_{p,i} (u - 1/2)^i,$$

where $\delta_{p,0} = \rho_0^p$ and

$$\delta_{p,i} = (i \rho_0)^{-1} \sum_{m=1}^{i} [m(p+1) - i]\, \rho_m \delta_{p,i-m}.$$

Then, equation (4.10) becomes

$$M_r(t) = r \sum_{p,i=0}^{\infty} \frac{t^p}{p!} \delta_{p,i} \int_0^1 u^{r-1} (u - 1/2)^i \, du. \tag{4.11}$$

Using the binomial expansion in (4.11), the mgf of the EBS distribution can be expressed as

$$M_r(t) = \sum_{p=0}^{\infty} \delta_{p,i}^{*} t^p, \tag{4.12}$$

where

$$\delta_{p,i}^{*} = r\sum_{i=0}^{\infty}\sum_{q=0}^{i}\binom{i}{q}\frac{(-1)^{i-q}\delta_{p-i}}{p!(q+r)2^{i-q}}.$$

Finally, substituting (4.12) into (4.4), the mgf of the KBBS distribution reduces to

$$M(t) = \sum_{p=0}^{\infty}\eta_p t^p, \tag{4.13}$$

where

$$\eta_p = \sum_{r=0}^{\infty} b_r \delta_{p,r}^{*}.$$

5. Incomplete moments

Many important questions in econometrics require more than just knowing the mean of a distribution, but its shape as well. This is also obvious not only in the study of econometrics and income distributions but in many other areas of research. For empirical purposes, the shape of many distributions can be usefully described by what we call the incomplete moments. These types of moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments of a distribution. The nth incomplete moment of X is given by $T_n(y) = \int_0^y x^n f(x)dx$. By inserting (3.5) in $T_n(y)$, we obtain

$$T_n(y) = r(\alpha,\beta)\sum_{r=0}^{\infty} b_r \int_0^y x^{n-3/2}(x+\beta)\Phi^{r-1}(\nu)\exp\left\{-\frac{\tau(x/\beta)}{2\alpha^2}\right\}dx.$$

From Cordeiro and Lemonte [18], we have

$$\begin{aligned}\Phi^{r-1}(\nu) &= 2^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k_1,\ldots,k_j=0}^{\infty}\beta^{-(2s_j+j)/2}A(k_1,\ldots,k_j)\\ &\quad\times\sum_{m=0}^{2s_j+j}(-\beta)^m\binom{2s_j+j}{m}x^{(2s_j+j-2m)/2},\end{aligned}$$

where s_j and $A(k_1,\ldots,k_j)$ are defined in (4.2). Thus,

$$\begin{aligned}T_n(y) &= r(\alpha,\beta)\sum_{r=0}^{\infty} b_r 2^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k_1,\ldots,k_j=0}^{\infty}\beta^{-(2s_j+j)/2}A(k_1,\ldots,k_j)\\ &\quad\times\sum_{m=0}^{2s_j+j}(-\beta)^m\binom{2s_j+j}{m}\int_0^y x^{n+(2s_j+j-2m-3)/2}(x+\beta)\exp\left\{-\frac{\tau(x/\beta)}{2\alpha^2}\right\}dx.\end{aligned}$$

Let

$$D(p,q) = \int_0^q x^q\exp\left\{-\frac{x/\beta+\beta/x}{2\alpha^2}\right\}dx = \int_0^{q/\beta}u^q\exp\left\{-\frac{u+u^{-1}}{2\alpha^2}\right\}du.$$

From Terras [69], we can write

$$D(p,q)=\beta^{p+1}K_{p+1}\left(\alpha^{-2}\right)-q^{p+1}K_{p+1}\left(\frac{q}{2\alpha^2\beta},\frac{\beta}{2\alpha^2 q}\right),$$

where $K_p\left(x_1,x_2\right)$ denotes the incomplete Bessel function with arguments x_1 and x_2 and order p. For further details, see Jones [33], [34] and Harris [32].

Hence, the nth incomplete moment of X can be expressed as

$$\begin{aligned} T_n(y) &= r(\alpha,\beta)\sum_{r=0}^{\infty} b_r 2^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k_1,\ldots,k_j=0}^{\infty}\beta^{-(2s_j+j)/2}A\left(k_1,\ldots,k_j\right)\\ &\quad\times\sum_{m=0}^{2s_j+j}(-\beta)^m\binom{2s_j+j}{m}\\ &\quad\times\left\{D\left(n+\frac{2s_j+j-2m-1}{2},y\right)+\beta D\left(n+\frac{2s_j+j-2m-3}{2},y\right)\right\}. \end{aligned} \tag{5.1}$$

Equation (5.1) is the main result of this section.

6. Other measures

Here, we derive the means deviations, Lorenz and Bonferroni curves and the reliability of the KBBS distribution.

6.1. Mean deviations. We can derive the mean deviations about the mean $\mu_1^{'}\left(\delta_1\right)$ and about the median $M\left(\delta_2\right)$ in terms of the first incomplete moment. The median is obtained by inverting $F(M)=K\int_0^{\Phi(\nu)}t^{a-1}(1-t)^{b-1}e^{-ct}dt=1/2$ numerically. They can be expressed as

$$\delta_1=2\left[\mu_1^{'}F\left(\mu_1^{'}\right)-T_1\left(\mu_1^{'}\right)\right],\qquad \delta_2=\mu_1^{'}-2T_1\left(M\right),$$

where $T_1(\cdot)$ is the first incomplete moment of X given by (5.1) with $n=1$. We have

$$\begin{aligned} T_1(y) &= r(\alpha,\beta)\sum_{r=0}^{\infty} b_r 2^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k_1,\ldots,k_j=0}^{\infty}\beta^{-(2s_j+j)/2}A\left(k_1,\ldots,k_j\right)\\ &\quad\times\sum_{m=0}^{2s_j+j}(-\beta)^m\binom{2s_j+j}{m}\\ &\quad\times\left\{D\left(\frac{2s_j+j-2m+1}{2},y\right)+\beta D\left(\frac{2s_j+j-2m-1}{2},y\right)\right\}. \end{aligned} \tag{6.1}$$

The measures δ_1 and δ_2 are immediately calculated from (6.1) by setting $y=\mu_1^{'}$ and $y=M$, respectively.

An application of the mean deviations refer to the Lorenz and Bonferroni curves defined by $L(\pi)=T_1(q)/\mu_1^{'}$ and $B(\pi)=T_1(q)/\pi\mu_1^{'}$, respectively, where $q=F^{-1}(\pi)$ can be computed for a given probability π by inverting (2.1) numerically. These curves have applications in several fields. They measures are immediately calculated from equation (6.1).

6.2. Reliability. In the context of reliability, the stress-strength model describes the life of a component which has a random strength X_1 that is subjected to a random stress X_2. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = \Pr(X_1 < X_2)$ is a measure of component reliability which has many applications in engineering. We derive the reliability R when X_1 and X_2 have independent KBBS$(\alpha, \beta, a_1, b_1, c_1)$ and KBBS$(\alpha, \beta, a_2, b_2, c_2)$ distributions with the same shape parameters α and β.

The pdf of X_1 and the cdf of X_2 can be written from equations (3.1) and (3.2) as

$$f_1(x) = g(x) \sum_{i,j=0}^{\infty} w_{1,i,j}\,(a_1+i+j)\,\Phi^{a_1+i+j}(\nu), \qquad F_2(x) = \sum_{k,p=0}^{\infty} w_{2,k,p}\Phi^{a_2+k+p}(\nu),$$

respectively, where

$$w_{1,i,j} = \frac{K_1(-1)^{i+j}c_1^i}{i!\,(a_1+i+j)}\binom{b_1-1}{j}, \qquad w_{2,k,p} = \frac{K_2(-1)^{k+p}c_2^k}{k!\,(a_2+k+p)}\binom{b_2-1}{p}.$$

The reliability, R, is given by

$$R = \int_0^{\infty} f_1(x)F_2(x)dx \tag{6.2}$$

and then

$$R = \sum_{i,j,k,p=0}^{\infty} w_{1,i,j}w_{2,k,p}\int_0^{\infty} g(x)\Phi^{a_1+a_2+i+j+k+p-1}(\nu)dx.$$

From equation (3.3), we can write

$$\Phi^{a_1+a_2+i+j+k+p-1}(\nu) = \sum_{r=0}^{\infty} s_r\,(a_1+a_2+i+j+k+p-1)\,\Phi^r(\nu),$$

and then R reduces to

$$R = \sum_{i,j,k,p=0}^{\infty} w_{1,i,j}w_{2,k,p}\sum_{r=0}^{\infty} s_r\,(a_1+a_2+i+j+k+p-1)\,\tau_{0,r-1}, \tag{6.3}$$

where $\tau_{0,r-1}$ can be computed from (4.2).

7. Order statistics

Order statistics have been used in a wide range of problems, including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness-of-fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials.

Suppose $X_1, \ldots, X_n$ is a random sample from the KBBS distribution and let $X_{1:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. Using (3.4) and (3.5), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[g(x)\sum_{r=0}^{\infty} b_r\Phi^r(\nu)\right]\left[\sum_{r=0}^{\infty} b_r\Phi^r(\nu)\right]^{i+j-1}.$$

From equations (4.7) and (4.8), we obtain

$$\left[\sum_{r=0}^{\infty} b_r \Phi^r(\nu)\right]^{i+j-1} = \sum_{r=0}^{\infty} c_{i+j-1,r}\Phi^r(\nu),$$

where $c_{i+j-1,0} = b_0^{i+j-1}$ and

$$c_{i+j-1,r} = (rb_0)^{-1} \sum_{m=1}^{r} \left[m(i+j) - r\right] b_m c_{i+j-1,r-m}.$$

Hence, the pdf of the ith order statistic for the KBBS distribution can be expressed as

$$f_{i:n}(x) = \sum_{r=0}^{\infty} m_r\, h(x;\alpha,\beta,2r), \tag{7.1}$$

where

$$m_r = \frac{n!b_r}{(2r+1)(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} c_{i+j-1,r}.$$

Equation (7.1) is the main result of this section. It gives the pdf of the KBBS order statistics as a linear combination of EBS pdfs with parameters α, β and $2r$. So, several mathematical quantities of the KBBS order statistics such as ordinary and incomplete moments, generating function, mean deviations (and several others) can come immediately from those quantities of the EBS distribution.

8. Computational issues

Here, we show the practical values of (3.4), (3.5), (4.3), (4.13), (5.1), (6.3) and (7.1). These formulas with the infinite series truncated provide a simple way to compute the cdf, pdf, moments, mgf, incomplete moments, reliability and the pdf of order statistics. The question is: how large should the truncation limit be?

We now show evidence that each infinite summation can be truncated at twenty to yield sufficient accuracy. Let $D1$ denote the absolute difference between the integrated version, (2.1), and the truncated version of (3.4) averaged over $x = 0.01, 0.02, \ldots, 5$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D2$ denote the absolute difference between (1.6) and the truncated version of (3.5) averaged over $x = 0.01, 0.02, \ldots, 5$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D3$ denote the absolute difference between the truncated version of (4.3) and the integrated version,

$$\mu_s^{'} = Kr(\alpha,\beta) \int_0^{\infty} x^{s-3/2}(x+\beta)\Phi(\nu)\left[1-\Phi(\nu)\right]^{b-1} \exp\left\{-\left[\frac{\tau(x/\beta)}{2\alpha^2} + c\Phi(\nu)\right]\right\} dx,$$

averaged over $s = 1, 2, \ldots, 50$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D4$ denote the absolute difference between the truncated version of (4.13) and the integrated version,

$$\begin{aligned} M(t) &= Kr(\alpha, \beta) \int_0^\infty \exp(tx) x^{-3/2} (x + \beta) \Phi(\nu) \left[1 - \Phi(\nu)\right]^{b-1} \\ &\quad \times \exp\left\{ - \left[\frac{\tau(x/\beta)}{2\alpha^2} + c\Phi(\nu) \right] \right\} dx, \end{aligned}$$

averaged over $t = 0.01, 0.02, \ldots, 0.99$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D5$ denote the absolute difference between the truncated version of (5.1) and the integrated version,

$$T_n(y) = Kr(\alpha, \beta) \int_0^y x^{n-3/2} (x + \beta) \Phi(\nu) \left[1 - \Phi(\nu)\right]^{b-1} \exp\left\{ - \left[\frac{\tau(x/\beta)}{2\alpha^2} + c\Phi(\nu) \right] \right\} dx,$$

averaged over $n = 1, 2, \ldots, 50$, $y = 0.01, 0.02, \ldots, 5$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D6$ denote the absolute difference between the integrated version, (6.2), and the truncated version of (6.3) averaged over $a_1 = 0.01, 0.02, \ldots, 10$, $b_1 = 0.01, 0.02, \ldots, 10$, $c_1 = -10, -9.99, \ldots, 10$, $a_2 = 0.01, 0.02, \ldots, 10$, $b_2 = 0.01, 0.02, \ldots, 10$, $c_2 = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D7$ denote the absolute difference between (7.1) and the truncated version of (7.1) averaged over $i = 1, 2, \ldots, n$, $n = 1, 2, \ldots, 50$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$.

We obtained the following estimates after extensive computations: $D1 = 1.21 \times 10^{-21}$, $D2 = 9.43 \times 10^{-20}$, $D3 = 2.39 \times 10^{-33}$, $D4 = 3.54 \times 10^{-21}$, $D5 = 7.6 \times 10^{-25}$, $D6 = 1.68 \times 10^{-20}$ and $D7 = 2.78 \times 10^{-22}$. These estimates are small enough to suggest that the truncated versions of (3.4), (3.5), (4.3), (4.13), (5.1), (6.3) and (7.1) are reasonable for practical use.

It would ideal to show that each (untruncated) infinite series (like (3.4), (3.5), (4.3), (4.13), (5.1), (6.3) and (7.1)) is convergent and gives valid values for all values of its arguments. This will be a difficult mathematical problem and a possible future work.

9. Inference

Section 9.1 gives procedures for maximum likelihood estimation of the KBBS distribution. Section 9.2 assesses the performance of the MLEs in terms of biases, mean squared errors, coverage probabilities and coverage lengths by means of a simulation study.

9.1. Estimation. The estimation of the parameters of the KBBS distribution will be investigated by maximum likelihood. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample of this distribution with unknown parameter vector $\boldsymbol{\theta} = (\alpha, \beta, a, b, c)^T$.

The total log-likelihood function for $\boldsymbol{\theta}$ is

$$\begin{aligned}\ell(\boldsymbol{\theta}) \quad = \quad & n\log K + n\log r(\alpha,\beta) - \frac{3}{2}\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\left(x_i+\beta\right) - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\tau\left(x_i/\beta\right) \\ & -c\sum_{i=1}^{n}\Phi\left(\nu_i\right) + (a-1)\sum_{i=1}^{n}\log\Phi\left(\nu_i\right) + (b-1)\sum_{i=1}^{n}\log\left[1-\Phi\left(\nu_i\right)\right]. \qquad (9.1)\end{aligned}$$

The elements of score vector are given by

$$\begin{aligned}U_\alpha(\boldsymbol{\theta}) &= -\frac{n}{\alpha}\left(1+\frac{2}{\alpha^2}\right) + \frac{1}{\alpha^3}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}+\frac{\beta}{x_i}\right) \\ &\quad -\frac{1}{\alpha}\sum_{i=1}^{n}\nu_i\phi\left(\nu_i\right)\left\{\frac{a-1}{\Phi\left(\nu_i\right)} - \frac{b-1}{1-\Phi\left(\nu_i\right)} - 2c\right\}, \\ U_\beta(\boldsymbol{\theta}) &= -\frac{n}{2\beta} + \sum_{i=1}^{n}\frac{1}{x_i+\beta} + \frac{1}{2\alpha^2\beta}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}-\frac{\beta}{x_i}\right) \\ &\quad -\frac{1}{2\alpha\beta}\sum_{i=1}^{n}\tau\left(\sqrt{x_i/\beta}\right)\phi\left(\nu_i\right)\left\{\frac{a-1}{\Phi\left(\nu_i\right)} - \frac{b-1}{1-\Phi\left(\nu_i\right)} - c\right\}, \\ U_a(\boldsymbol{\theta}) &= \frac{n}{K}\frac{\partial K}{\partial a} + \sum_{i=1}^{n}\log\Phi\left(\nu_i\right), \\ U_b(\boldsymbol{\theta}) &= \frac{n}{K}\frac{\partial K}{\partial b} + \sum_{i=1}^{n}\log\left[1-\Phi\left(\nu_i\right)\right], \\ U_c(\boldsymbol{\theta}) &= \frac{n}{K}\frac{\partial K}{\partial c} + \sum_{i=1}^{n}\Phi\left(\nu_i\right),\end{aligned}$$

where $\phi(\cdot)$ is the standard normal pdf, $\nu_i = \alpha^{-1}\left\{\sqrt{x_i/\beta} - \sqrt{\beta/x_i}\right\}$ and $\tau\left(\sqrt{x_i/\beta}\right) = \sqrt{x_i/\beta} + \sqrt{\beta/x_i}$ for $i = 1,2,\ldots,n$. The partial derivatives of K with respect to a, b and c are

$$\begin{aligned}\frac{\partial K}{\partial a} &= -\frac{\left[\psi(a)-\psi(a+b)\right]{}_1F_1(a,a+b,-c) + \dfrac{\partial_1 F_1(a,a+b,-c)}{\partial a}}{B(a,b)\left[{}_1F_1(a,a+b,-c)\right]^2}, \\ \frac{\partial K}{\partial b} &= -\frac{\left[\psi(b)-\psi(a+b)\right]{}_1F_1(a,a+b,-c) + \dfrac{\partial_1 F_1(a,a+b,-c)}{\partial b}}{B(a,b)\left[{}_1F_1(a,a+b,-c)\right]^2}, \\ \frac{\partial K}{\partial c} &= \frac{a_1F_1(a+1,a+b+1,-c)}{(a+b)B(a,b)_1F_1(a,a+b,-c)},\end{aligned}$$

where

$$\begin{aligned}\frac{\partial_1 F_1(a,a+b,-c)}{\partial a} \quad = \quad & -\left[\psi(a)-\psi(a+b)\right]{}_1F_1(a,a+b,-c) \\ & -\sum_{k=0}^{\infty}\frac{(a)_k(-c)^k}{k!(a+b)_k}\left[\psi(a+b+k)-\psi(a+k)\right]\end{aligned}$$

and

$$\frac{\partial_1 F_1(a, a+b, -c)}{\partial c} = \psi(a+b)_1 F_1(a, a+b, -c) + \sum_{k=0}^{\infty} \frac{(a)_k (-c)^k}{k!(a+b)_k} \psi(a+b+k).$$

Maximization of (9.1) can be performed by using well established routines such as the `nlm` routine or optimize in the R statistical package. Setting these equations to zero, $\mathbf{U}(\boldsymbol{\theta}) = \mathbf{0}$, and solving them simultaneously yields the MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. These equations cannot be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm.

For interval estimation and hypothesis tests on the parameters in $\boldsymbol{\theta}$, we require the 5×5 total observed information matrix $\mathbf{J}(\boldsymbol{\theta}) = -\{U_{r,s}\}$, where the elements $U_{r,s}$ for $r, s = \alpha, \beta, a, b, c$ are given in the Appendix. The estimated asymptotic multivariate normal $N_5\left(\mathbf{0}, \mathbf{J}\left(\widehat{\boldsymbol{\theta}}\right)^{-1}\right)$ distribution of $\widehat{\boldsymbol{\theta}}$ can be used to construct approximate confidence regions for the parameters and for the hazard rate and survival functions. An asymptotic confidence interval with significance level γ for each parameter θ_r is given by

$$\mathrm{ACI}\left(\theta_r, 100(1-\gamma)\%\right) = \left(\widehat{\theta}_r - z_{1-\gamma/2}\sqrt{\widehat{\kappa}^{\theta_r,\theta_r}}, \widehat{\theta}_r + z_{1-\gamma/2}\sqrt{\widehat{\kappa}^{\theta_r,\theta_r}}\right),$$

where $\widehat{\kappa}^{\theta_r,\theta_r}$ is the rth diagonal element of $\mathbf{J}\left(\boldsymbol{\theta}\right)^{-1}$ estimated at $\widehat{\boldsymbol{\theta}}$ for $r = 1, \ldots, 4$, and $z_{1-\gamma/2}$ is the $100(1-\gamma/2)$ percentile of the standard normal distribution.

The likelihood ratio (LR) statistic is useful for comparing the new distribution with some of its sub-models. For example, we may adopt the LR statistic to check if the fit using the KBBS distribution is statistically "superior" to a fit using the BS distribution for a given data set. In any case, considering the partition $\boldsymbol{\theta} = \left(\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T\right)^T$, tests of hypotheses of the type $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus $H_A : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ can be performed using the LR statistic $w = 2\left\{\ell\left(\widehat{\boldsymbol{\theta}}\right) - \ell\left(\widetilde{\boldsymbol{\theta}}\right)\right\}$, where $\widehat{\boldsymbol{\theta}}$ and $\widetilde{\boldsymbol{\theta}}$ are the estimates of $\boldsymbol{\theta}$ under H_A and H_0, respectively. Under the null hypothesis H_0, w approaches χ_q^2 as $n \to \infty$, where q is the dimension of the vector $\boldsymbol{\theta}_1$ of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper 100γ percentile of the χ_q^2 distribution.

9.2. Simulation study. Here, we assess the performance of the MLEs with respect to sample size n. The assessment is based on a simulation study:

(1) generate ten thousand samples of size n from (2.1). The inversion method was used to generate samples, i.e., variates of the KBBS distribution were generated by solving

$$K \int_0^{\Phi(\rho(X/\beta)/\alpha)} t^{a-1}(1-t)^{b-1}\exp(-ct)dt = U,$$

where $U \sim U(0,1)$ is a uniform variate on the unit interval.

(2) compute the MLEs for the ten thousand samples, say $\left(\widehat{a}_i, \widehat{b}_i, \widehat{c}_i, \widehat{\alpha}_i, \widehat{\beta}_i\right)$ for $i = 1, 2, \ldots, 10000$.

(3) compute the standard errors of the MLEs for the ten thousand samples, say $\left(s_{\widehat{a}_i}, s_{\widehat{b}_i}, s_{\widehat{c}_i}, s_{\widehat{\alpha}_i}, s_{\widehat{\beta}_i}\right)$ for $i = 1, 2, \ldots, 10000$. The standard errors were computed by inverting the observed information matrices.

(4) compute the biases and mean squared errors given by

$$\mathrm{bias}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{h}_i - h\right),$$

$$\mathrm{MSE}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{h}_i - h\right)^2$$

for $h = a, b, c, \alpha, \beta$.

(5) compute the coverage probabilities and coverage lengths given by

$$\mathrm{CP}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} I\left\{\widehat{h}_i - 1.959964 s_{\widehat{h}_i} < h < \widehat{h}_i + 1.959964 s_{\widehat{h}_i}\right\},$$

$$\mathrm{CL}_h(n) = \frac{3.919928}{10000} \sum_{i=1}^{10000} s_{\widehat{h}_i}$$

for $h = a, b, c, \alpha, \beta$, where $I\{\cdot\}$ denotes the indicator function.

We repeated these steps for $n = 10, 11, \ldots, 100$ with $a = 1$, $b = 1$, $c = 1$, $\alpha = 1$ and $\beta = 1$, so computing $\mathrm{bias}_h(n)$, $\mathrm{MSE}_h(n)$, $\mathrm{CP}_h(n)$ and $\mathrm{CL}_h(n)$ for $h = a, b, c, \alpha, \beta$ and $n = 10, 11, \ldots, 100$.

Figure 5 shows how the five biases vary with respect to n. Figure 6 shows how the five mean squared errors vary with respect to n. Figure 7 shows how the five coverage probabilities vary with respect to n. Figure 8 shows how the five coverage lengths vary with respect to n. The broken line in Figure 5 corresponds to the biases being zero. The broken line in Figure 6 corresponds to the mean squared errors being zero. The broken line in Figure 7 corresponds to the nominal coverage probability of 0.95.

The following observations can be drawn from the figures: the biases for a, c and α are generally positive; the biases for b and β are generally negative; the biases appear smallest for the parameter, b; the biases appear largest for the parameter, c; the biases for each parameter either decrease or increase to zero as $n \to \infty$; the mean squared errors appear smallest for the parameters, b and β; the mean squared errors appear largest for the parameter, c; the mean squared errors for each parameter decrease to zero as $n \to \infty$; the coverage probabilities for each parameter are reasonably close enough to the nominal level for n greater than or equal to sixty; the coverage lengths appear smallest for the parameter, b; the coverage lengths appear largest for the parameters, a and c; the coverage lengths for each parameter decrease to zero as $n \to \infty$. These observations are for only one choice for (a, b, c, α, β), namely that $(a, b, c, \alpha, \beta) = (1, 1, 1, 1, 1)$. But the results were similar for other choices.

Section 10 presents two real data applications. The sample size for the first data set is sixty six. The sample size for the second data set is one hundred and one.

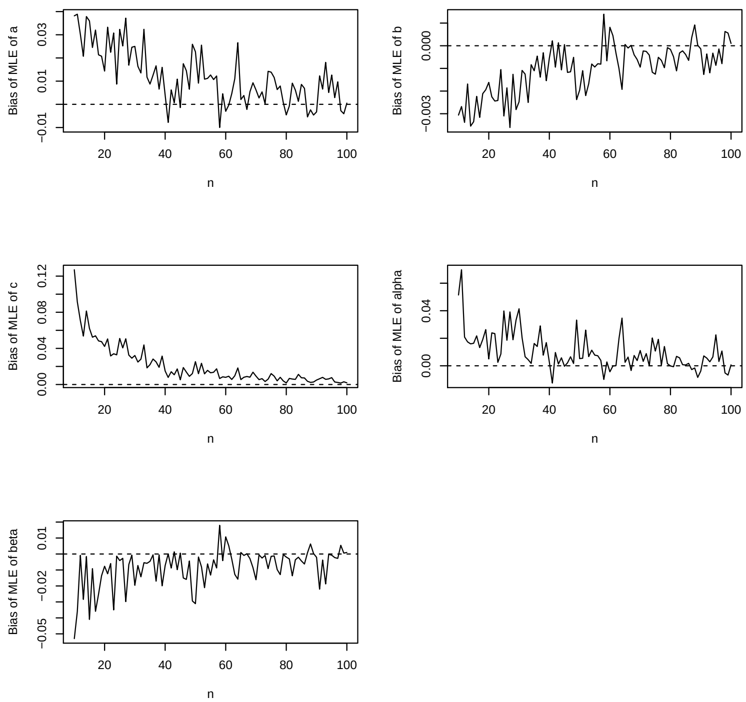


Figure 5. Biases of the MLEs of (a, b, c, α, β) versus $n = 10, 11, \ldots, 100$.

Hence, the biases for $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{\alpha}$ and $\widehat{\beta}$ can be expected to be less than 0.01, 0.001, 0.02, 0.02 and 0.01, respectively, for both data sets. The mean squared errors for $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{\alpha}$ and $\widehat{\beta}$ can be expected to be less than 0.005, 0.003, 0.02, 0.002 and 0.002, respectively, for both data sets. The coverage probabilities can be expected to be accurate for $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{\alpha}$ and $\widehat{\beta}$ for both data sets. The coverage lengths for $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{\alpha}$ and $\widehat{\beta}$ can be expected to be less than 0.3, 0.01, 0.2, 0.2 and 0.2, respectively, for both data sets. Hence, the point as well as interval estimates given in Section 10 can be considered accurate enough.

10. Applications

In this section, we use two data sets to compare the fits of the KBBS distribution with those of two sub-models (i.e., the beta-BS (BBS) and BS distributions) and also to the following non-nested models: the McDonald-Birnbaum-Saunders (McBS) (Cordeiro et al., [19]), the McDonald-gamma (McGa) (Marciano et al., [50]), the length-biased-Birnbaum-Saunders (LBS) (Leiva et al., [38]), the extended Birnbaum-Saunders (ExBS) (Leiva et al., [41]), the Marshall-Olkin extended Birnbaum-Saunders (MOEBS) (Lemonte, [42]) and the generalized Birnbaum-Saunders (GBS) (Owen, [60]) distributions. All the computations were performed using the R statistical software. Obviously, due to the genesis of the BS and

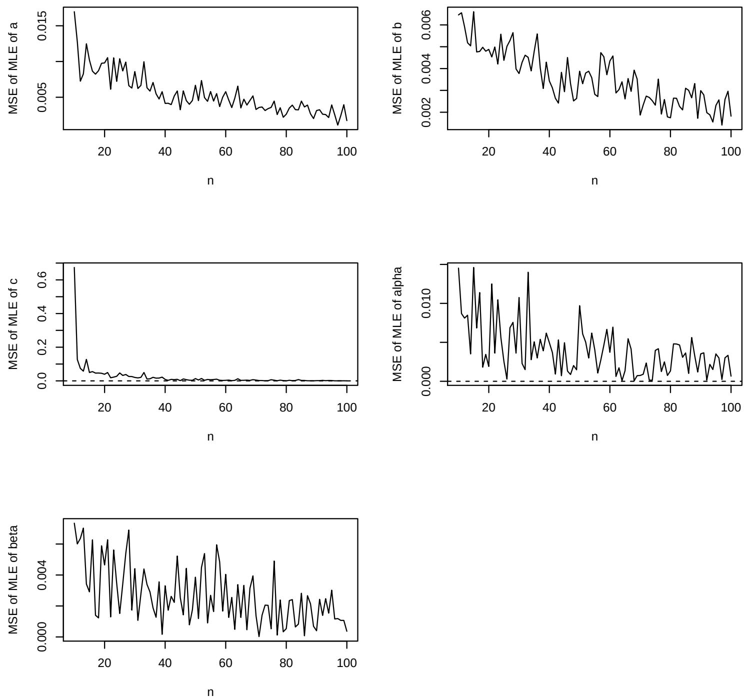


Figure 6. Mean squared errors of the MLEs of (a, b, c, α, β) versus $n = 10, 11, \ldots, 100$.

gamma distributions, the fatigue processes are ideally modeled by these distributions. Thus, the use of the KBBS distribution and its sub-models and also other lifetime distributions for fitting the data sets is justified.

10.1. Breaking stress of carbon fibres data. Here, we shall compare the fitted KBBS, BBS, BS, McBS, McGa, LBS, ExBS, MOEBS and GBS distributions to the data from Nichols and Padgett [58] on the breaking stress of carbon fibres (in Gba). Nichols and Padgett [58] described the data from a process which produces carbon fibers to be used in constructing fibrous composite materials. The carbon fiber fifty millimeters in length were sampled ($n = 66$) from the process, tested and their tensile strength were observed.

Firstly, in order to estimate the parameters, we consider the maximum likelihood estimation method discussed in Section 9. We take the estimates of α and β from the fitted BS distribution as starting values for the numerical iterative procedure. Table 2 lists the MLEs of the parameters and the values of the following statistics: Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). The results indicate that the KBBS distribution has the smallest values of these statistics, and so, it could be chosen as the more suitable distribution.

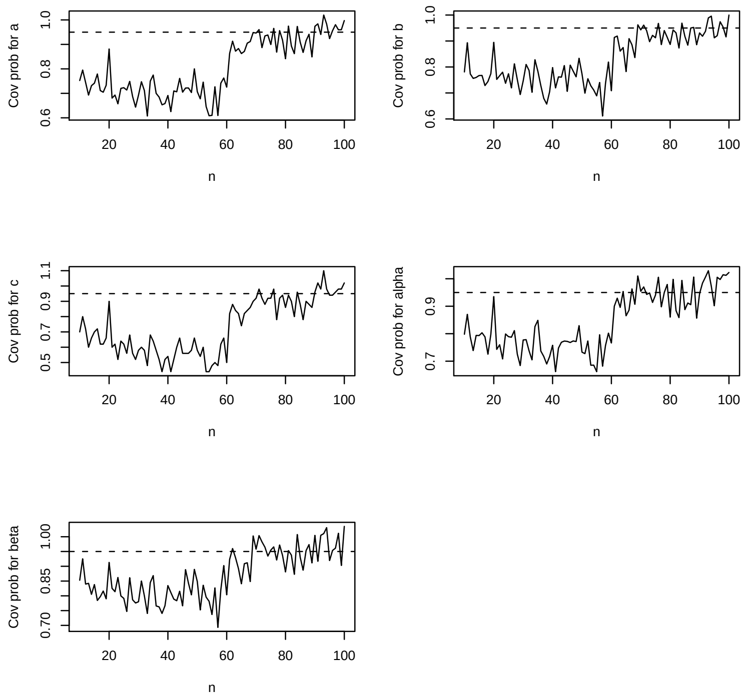


Figure 7. Coverage probabilities of the MLEs of (a, b, c, α, β) versus $n = 10, 11, \ldots, 100$.

Table 2. MLEs (standard errors in parentheses) and information criteria for breaking stress of carbon fibres data.

Model	α	β	a	b	c	AIC	BIC	CAIC
KBBS	0.6770	2.9944	0.3430	11.4176	-22.2353	179.6	190.6	181.6
	(0.5479)	(0.2732)	(0.2079)	(3.7988)	(5.4008)			
BBS	1.0452	57.5997	0.1990	1876.8935	0	191.6	200.4	193.0
	(0.0039)	(0.3413)	(0.0219)	(605.0512)				
BS	0.43712	2.51540	1	1	0	204.3	208.7	205.0
	(0.0380)	(0.1321)						

A comparison of the proposed distribution with some of its sub-models using LR statistics is given in Table 3. We reject the null hypotheses of the two LR tests in favor of the KBBS distribution. The rejection is extremely highly significant. This gives a clear evidence of the potential of the three skewness parameters when modeling real data.

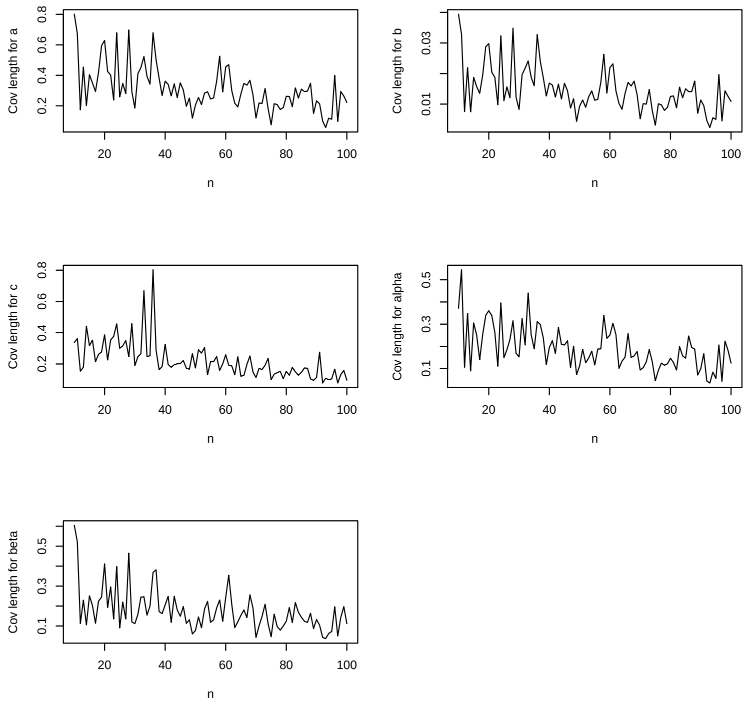


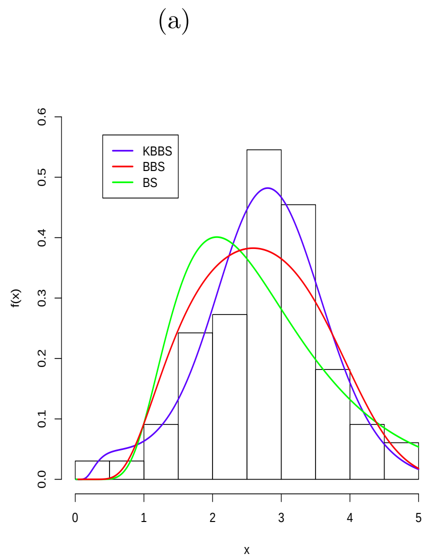
Figure 8. Coverage lengths of the MLEs of (a, b, c, α, β) versus $n = 10, 11, \ldots, 100$.

Table 3. LR statistics for breaking stress of carbon fibres data.

Model	Hypotheses	Statistic w	p-value
KBBS vs BBS	$H_0 : c = 0$ vs $H_1 : H_0$ is false	30.69	< 0.0001
KBBS vs BS	$H_0 : a = b = 1$ and $c = 0$ vs $H_1 : H_0$ is false	13.08	0.00029

In order to assess if the distribution is appropriate, Figures 9a and 9b display plots of the estimated pdfs and survival functions of the KBBS distribution and its sub-models. We can conclude that the KBBS distribution is a very suitable distribution to fit the data.

Secondly, we shall apply formal goodness-of-fit tests in order to verify which distribution fits the data better. We consider the Cramér-Von Mises (W^*) and Anderson-Darling (A^*) statistics. In general, the smaller the values of the statistics, W^* and A^*, the better the fit to the data. Let $F(x; \boldsymbol{\theta})$ be the cdf, where the form of F is known but $\boldsymbol{\theta}$ (a k-dimensional parameter vector, say) is unknown. To obtain the statistics, W^* and A^*, we proceed as follows:



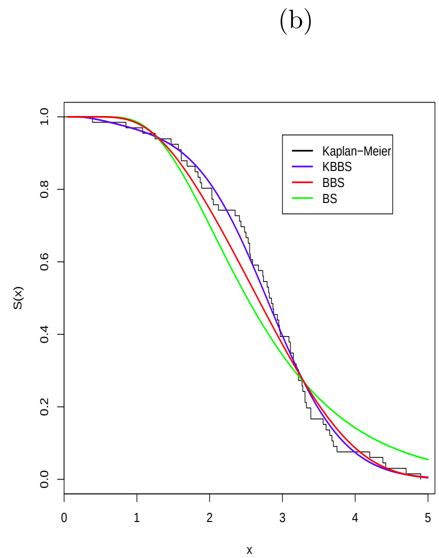


Figure 9. (a) Estimated pdfs of the KBBS distribution and its sub-models for breaking stress of carbon fibres data. (b) Empirical and estimated survival functions of the KBBS distribution and its sub-models for breaking stress of carbon fibres data.

(i) compute $v_i = F\left(x_i; \widehat{\boldsymbol{\theta}}\right)$, where the x_i's are in ascending order, $y_i = \Phi^{-1}(v_i)$ is the standard normal quantile function and $u_i = \Phi\left\{(y_i - \overline{y})/s_y\right\}$, where $\overline{y} = n^{-1}\sum_{i=1}^{n} y_i$ and $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \overline{y})^2$;

(ii) compute

$$W^2 = \sum_{i=1}^{n} \left\{u_i - (2i-1)/(2n)\right\}^2 + 1/(12n)$$

and

$$A^2 = -n - n^{-1}\sum_{i=1}^{n}\left\{(2i-1)\log(u_i) + (2n+1-2i)\log(1-u_i)\right\};$$

(iii) modify W^2 into $W^* = W^2(1{+}0.5/n)$ and A^* into $A^* = A^2\left(1 + 0.75/n + 2.25/n^2\right)$.

For further details, the reader is referred to Chen and Balakrishnan [16]. The values of the statistics, W^* and A^*, for the distributions are given in Table 4. Thus, according to these formal goodness-of-fit tests, the KBBS distribution fits the data better than its sub-models.

The MLEs (standard errors in parentheses) of the parameters of the LBS, MOEBS, GBS, ExBS, McBS and McGa distributions are listed in Table 5. On the basis of the statistics given in this table, the ExBS distribution yields a better fit than others. Overall, by comparing the measures in Tables 4 and 5, we conclude that the KBBS distribution outperforms all the distributions considered in Table 5. So, the proposed distribution can yield better fits than the LBS, MOEBS, GBS,

Table 4. Formal goodness-of-fit tests for breaking stress of carbon fibres data.

Model	Statistic	
	W^*	A^*
KBBS	0.0498	0.3123
BBS	0.2115	1.2216
BS	0.4603	2.5896

ExBS, McBS and McGa distributions and therefore may be an interesting alternative to these distributions for modeling fatigue lifetime data sets. These results illustrate the potentiality of the new distribution and the necessity for additional shape parameters.

Table 5. MLEs (standard errors in parentheses) and the measures, W^* and A^*, for breaking stress of carbon fibres data.

Model	Estimates					Statistic	
						W^*	A^*
ExBS	$\widehat{\alpha} = 3.3418$	$\widehat{\beta} = 0.5840$	$\widehat{\sigma} = 0.7586$	$\widehat{v} = -2.3019$	$\widehat{\lambda} = 0.0179$	0.0599	0.3825
	(1.8483)	(0.3675)	(0.1922)	(0.1145)	(0.0024)		
McBS	$\widehat{\alpha} = 3.8736$	$\widehat{\beta} = 0.1487$	$\widehat{a} = 18.8160$	$\widehat{\eta} = 35.5380$	$\widehat{c} = 29.00002$	0.0935	0.5223
	(0.1444)	(0.0923)	(0.4067)	(4.5916)	(1.2795)		
McGa	$\widehat{\alpha} = 28.5769$	$\widehat{\beta} = 2.3734$	$\widehat{a} = 0.1240$	$\widehat{b} = 48.0712$	$\widehat{c} = 0.2335$	0.0812	0.5173
	(4.0265)	(0.9942)	(0.5479)	(2.7540)	(0.1044)		
GBS	$\widehat{\alpha} = 0.5409$	$\widehat{\beta} = 2.6613$	$\widehat{\kappa} = 0.0009$			0.0742	0.4176
	(0.0249)	(0.1687)	(0.0001)				
MOEBS	$\widehat{\alpha} = 0.4358$	$\widehat{\beta} = 2.4723$	$\widehat{\eta} = 1.1187$			0.4453	2.5048
	(0.0379)	(0.1536)	(0.2439)				
LBS	$\widehat{\alpha} = 0.4410$	$\widehat{\beta} = 2.0919$				0.4192	2.3516
	(0.0396)	(0.1314)					

The QQ plots of the normalized quantile residuals was introduced by Dunn and Smyth [22] and more recently used by Cordeiro et al. [17]. Figures 10 and 11 show the improved fit achieved using the KBBS distribution over other distributions. We also emphasize the gain yielded by the KBBS distribution in relation to the BS, BBS, McBS, McGa, LBS, ExBS, MOEBS and GBS distributions.

10.2. Aluminum alloy fatigue data. The data refer to the fatigue life of 6061 - T6 aluminum coupons cut parallel to the direction of rolling and oscillated at eighteen cycles per second. It was reported and analyzed by Birnbaum and Saunders [12]. The KBBS distribution seems to be an appropriate distribution for fitting these data. Table 6 lists the MLEs (standard errors in parentheses) of the parameters. The results indicate that the KBBS distribution has the smallest values of the statistics (AIC and CAIC) in relation to its sub-models.

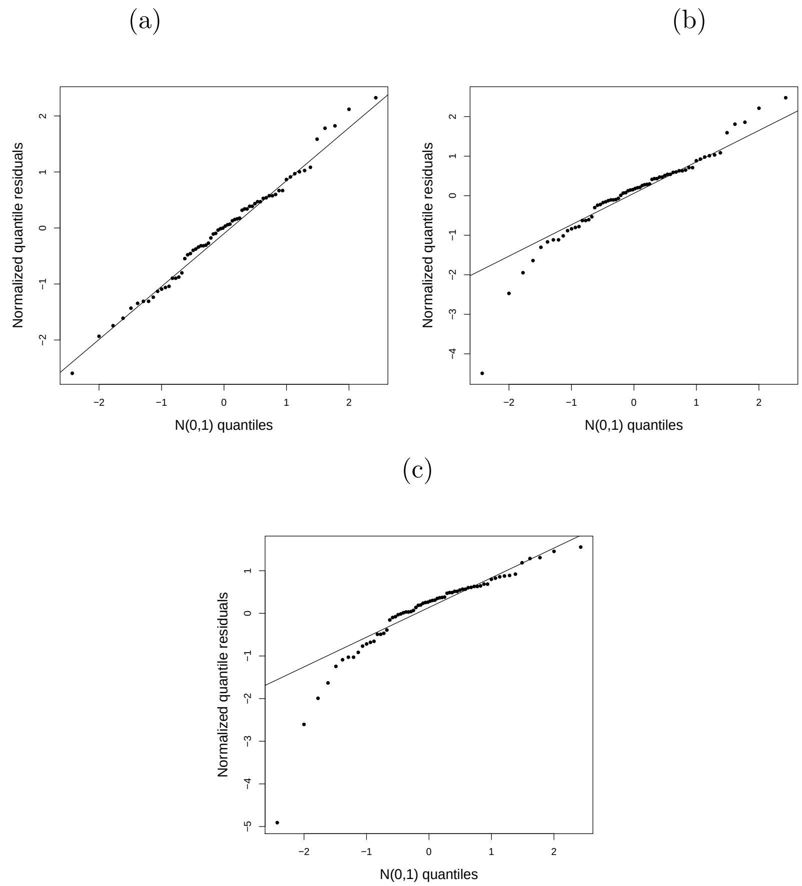


Figure 10. QQ plot of the normalized quantile residuals for the distributions: (a) KBBS, (b) BBS and (c) BS for breaking stress of carbon fibres data.

Table 6. MLEs (standard errors in parentheses) and information criteria for aluminum alloy fatigue data.

Model	α	β	a	b	c	AIC	BIC	CAIC
KBBS	0.9654	2065.821	0.9161	38.5452	-58.0575	1501.1	1514.2	1502.3
	(0.0809)	(24.1268)	(0.1069)	(1.4887)	(1.8296)			
BBS	0.2817	1600.382	0.6278	1.2967	0	1508.0	1518.5	1508.9
	(0.0114)	(29.1456)	(0.0457)	(0.0926)				
BS	0.3103	1336.377	1	1	0	1506.7	1512.0	1507.1
	(0.0218)	(40.7665)						

A test for the need for the third skewness parameter in the KBBS distribution can be based on the LR statistic described in Section 9. Applying the LR statistics to these data, the results are listed in Table 7. The p-values show that the proposed distribution yields the best fit to the data.

Table 7. LR statistics for aluminum alloy fatigue data.

Model	Hypotheses	Statistic w	p-value
KBBS vs BBS	$H_0 : c = 0$ vs $H_1 : H_0$ is false	8.92	0.0028
KBBS vs BS	$H_0 : a = b = 1$ and $c = 0$ vs $H_1 : H_0$ is false	11.65	0.0086

In Figure 12a, we provide the histogram of the data and the fitted KBBS, BBS and BS pdfs while in Figure 12b we display plots of the empirical and estimated survival functions of the KBBS distribution and some of its sub-models. We note that the KBBS distribution provides a satisfactory fit.

We can also perform formal goodness-of-fit tests in order to verify which distribution fits the data better. We apply the Cramér-Von Mises (W^*) and Anderson-Darling (A^*) statistics. The values of the statistics, W^* and A^*, for the KBBS distribution and its sub-models are given in Table 8. Thus, according to these formal tests, the KBBS distribution fits the data better than its sub-models.

Table 8. Formal goodness-of-fit test for aluminum alloy fatigue data.

Model	Statistic	
	W^*	A^*
KBBS	0.0249	0.1719
BBS	0.0758	0.5169
BS	0.1022	0.6806

The MLEs (standard errors in parentheses) of the parameters of the LBS, MOEBS, GBS, ExBS, McBS and McGa distributions are listed in Table 9. On the basis of the statistics given in this table, the ExBS distribution yields a better fit than others. Overall, by comparing the measures in Tables 8 and 9, we conclude that the KBBS distribution outperforms all the distributions considered in Table 9. So, the proposed distribution can yield a better fit than the LBS, MOEBS, GBS, ExBS, McBS and McGa distributions.

The QQ plots of the normalized quantile residuals in Figures 13 and 14 reveal the improvement in the fit achieved by the KBBS distribution over the others.

11. Concluding remarks

The Birnbaum-Saunders (BS) distribution is widely used to model times to failure for materials subject to fatigue. We proposed the Kummer beta generalized Birnbaum-Saunders (KBBS) distribution to extend the BS distribution introduced by Birnbaum and Saunders [11]. We provided a mathematical treatment of the new distribution including expansions for the cdfs and pdfs. We derived

Table 9. MLEs (standard errors in parentheses) and the measures, W^* and A^*, for aluminum alloy fatigue data.

Model	Estimates					Statistic	
						W^*	A^*
ExBS	$\widehat{\alpha} = 2.7389$	$\widehat{\beta} = 2.7518$	$\widehat{\sigma} = 2.0811$	$\widehat{v} = -7.2496$	$\widehat{\lambda} = -0.0586$	0.0292	0.2057
	(0.4777)	(0.9933)	(1.3579)	(0.4584)	(1.7623)		
McBS	$\widehat{\alpha} = 0.3624$	$\widehat{\beta} = 1600.3540$	$\widehat{a} = 0.8001$	$\widehat{\eta} = 18.4695$	$\widehat{c} = 16.6618$	0.0420	0.3119
	(0.0898)	(45.7822)	(0.4383)	(0.8048)	(0.3104)		
McGa	$\widehat{\alpha} = 21.2870$	$\widehat{\beta} = 0.0134$	$\widehat{a} = 0.4765$	$\widehat{b} = 0.9109$	$\widehat{c} = 0.0007$	0.0283	0.2038
	(3.2584)	(0.0021)	(0.0249)	(0.0397)	(0.0002)		
GBS	$\widehat{\alpha} = 10.2155$	$\widehat{\beta} = 1514.9490$	$\widehat{\kappa} = 0.00098$			0.0308	0.2074
	(2.4781)	(32.1458)	(0.00001)				
MOEBS	$\widehat{\alpha} = 0.3101$	$\widehat{\beta} = 1332.9960$	$\widehat{\eta} = 1.0379$			0.0999	0.6669
	(0.0218)	(48.3511)	(0.1850)				
LBS	$\widehat{\alpha} = 0.3109$	$\widehat{\beta} = 1216.4778$				0.4192	2.3516
	(0.0220)	(40.6364)					

expansions for the ordinary and incomplete moments, generating function, mean deviations and the moments of the order statistics. The estimation of parameters is approached by the method of maximum likelihood and the observed information matrix was derived. We considered likelihood ratio (LR) statistics and formal goodness-of-fit tests to compare the KBBS distribution with some of its sub-models and non-nested models. Applications of the KBBS distribution to two real data sets indicated that the new distribution provides consistently better fits than its sub-models and other lifetime models. We hope that this generalization may attract wider applications in the literature of the fatigue life distributions.

Appendix: Elements of the observed information matrix

The elements of the observed information matrix, $\mathbf{J}(\boldsymbol{\theta})$, for the parameters α, β, a, b and c are:

$$
\begin{aligned}
U_{\alpha\alpha} = {} & \frac{n}{\alpha^2} + \frac{6n}{\alpha^3} - \frac{3}{\alpha^4}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}+\frac{\beta}{x_i}\right) + \frac{2(a-1)}{\alpha^2}\sum_{i=1}^{n}\frac{\nu_i\phi(\nu_i)}{\Phi(\nu_i)} \\
& - \frac{2(b-1)}{\alpha^2}\sum_{i=1}^{n}\frac{\nu_i\phi(\nu_i)}{1-\Phi(\nu_i)} + \frac{a-1}{\alpha^3}\sum_{i=1}^{n}\left\{\frac{\nu_i^4\phi(\nu_i)}{\Phi(\nu_i)} - \frac{\alpha\nu_i^2\phi^2(\nu_i)}{\Phi(\nu_i)}\right\} \\
& - \frac{b-1}{\alpha^3}\sum_{i=1}^{n}\left\{\frac{\nu_i^4\phi(\nu_i)}{1-\Phi(\nu_i)} - \frac{\alpha\nu_i^2\phi^2(\nu_i)}{1-\Phi(\nu_i)}\right\} - \frac{4c}{\alpha}\sum_{i=1}^{n}\nu_i\phi(\nu_i)\left[1+\phi(\nu_i)\right],
\end{aligned}
$$

$$
\begin{aligned}
U_{\alpha\beta} = {} & -\frac{1}{\alpha^3\beta}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}-\frac{\beta}{x_i}\right) \\
& + \frac{a-1}{2\alpha^2\beta}\sum_{i=1}^{n}\left\{\frac{\alpha\nu_i\phi(\nu_i)}{\Phi(\nu_i)} + \frac{\nu_i^4\phi(\nu_i)}{\Phi(\nu_i)} - \frac{\alpha\nu_i^2\phi^2(\nu_i)}{\Phi^2(\nu_i)}\right\} \\
& - \frac{b-1}{2\alpha^2\beta}\sum_{i=1}^{n}\left\{\frac{\alpha\nu_i\phi(\nu_i)}{1-\Phi(\nu_i)} + \frac{\nu_i^4\phi(\nu_i)}{1-\Phi(\nu_i)} - \frac{\alpha\nu_i^2\phi^2(\nu_i)}{\left[1-\Phi(\nu_i)\right]^2}\right\} \\
& - \frac{c}{\alpha\beta}\sum_{i=1}^{n}\nu_i\phi(\nu_i)\,\tau\left(\sqrt{x_i/\beta}\right),
\end{aligned}
$$

$$
\begin{aligned}
U_{\beta\beta} = {} & \frac{n}{2\beta^2} - \sum_{i=1}^{n}(x_i+\beta)^{-2} - \frac{1}{\alpha^2\beta^3}\sum_{i=1}^{n}x_i \\
& + \frac{a-1}{2\alpha\beta^2}\sum_{i=1}^{n}\frac{\phi(\nu_i)\,\tau\left(\sqrt{x_i/\beta}\right)}{\Phi(\nu_i)} - \frac{b-1}{2\alpha\beta^2}\sum_{i=1}^{n}\frac{\phi(\nu_i)\,\tau\left(\sqrt{x_i/\beta}\right)}{1-\Phi(\nu_i)} \\
& - \frac{a-1}{4\alpha\beta^2}\sum_{i=1}^{n}\left\{\frac{-\alpha\nu_i\phi(\nu_i)}{\Phi(\nu_i)} + \frac{\nu_i\phi(\nu_i)\,\tau^2\left(\sqrt{x_i/\beta}\right)}{\alpha\Phi(\nu_i)} + \frac{\nu_i\phi^2(\nu_i)\,\tau^2\left(\sqrt{x_i/\beta}\right)}{\alpha\Phi^2(\nu_i)}\right\} \\
& - \frac{c}{\alpha\beta^2}\sum_{i=1}^{n}\left\{\alpha\nu_i\phi(\nu_i) + \alpha^{-1}\tau^2\left(\sqrt{x_i/\beta}\right)\phi^2(\nu_i)\right\},
\end{aligned}
$$

$$U_{aa} = \frac{n}{K}\left\{\frac{\partial^2 K}{\partial a^2} - \frac{1}{K}\left(\frac{\partial K}{\partial a}\right)^2\right\},$$

$$U_{bb} = \frac{n}{K}\left\{\frac{\partial^2 K}{\partial b^2} - \frac{1}{K}\left(\frac{\partial K}{\partial b}\right)^2\right\},$$

$$U_{cc} = \frac{n}{K}\left\{\frac{\partial^2 K}{\partial c^2} - \frac{1}{K}\left(\frac{\partial K}{\partial c}\right)^2\right\},$$

$$U_{ab} = \frac{n}{K}\left\{\frac{\partial^2 K}{\partial a\partial b} - \frac{1}{K}\frac{\partial K}{\partial a}\frac{\partial K}{\partial b}\right\},$$

$$U_{ac} = \frac{n}{K}\left\{\frac{\partial^2 K}{\partial a\partial c} - \frac{1}{K}\frac{\partial K}{\partial a}\frac{\partial K}{\partial c}\right\},$$

$$U_{bc} = \frac{n}{K}\left\{\frac{\partial^2 K}{\partial b\partial c} - \frac{1}{K}\frac{\partial K}{\partial b}\frac{\partial K}{\partial c}\right\},$$

where $\frac{\partial^2 K}{\partial a^2}$, $\frac{\partial^2 K}{\partial b^2}$, $\frac{\partial^2 K}{\partial c^2}$, $\frac{\partial^2 K}{\partial a \partial b}$, $\frac{\partial^2 K}{\partial a \partial c}$ and $\frac{\partial^2 K}{\partial b \partial c}$ are defined in Pescim et al. [63].

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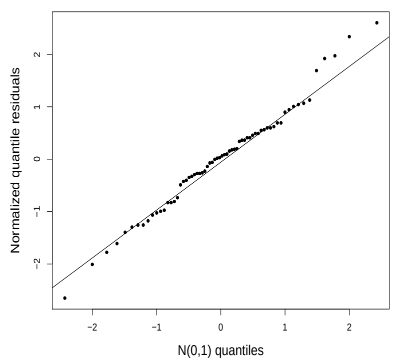
(a)



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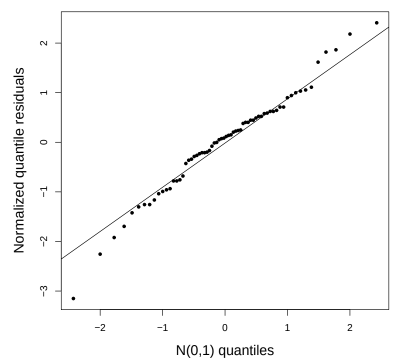

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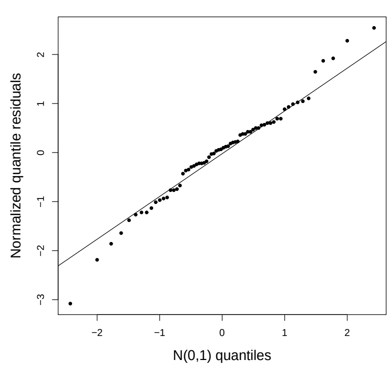

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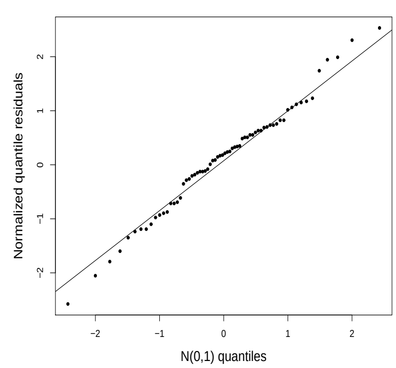

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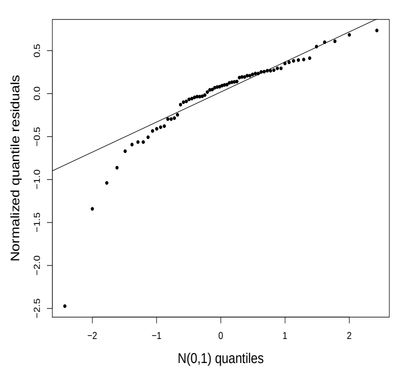

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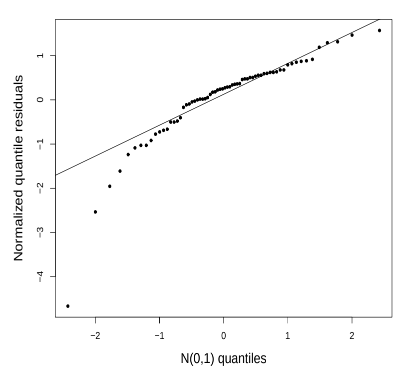

Figure 11. QQ plot of the normalized quantile residuals for the distributions: (a) ExBS, (b) McBS, (c) McGa, (d) GBS, (e) MOEBS and (f) LBS for breaking stress of carbon fibres data.

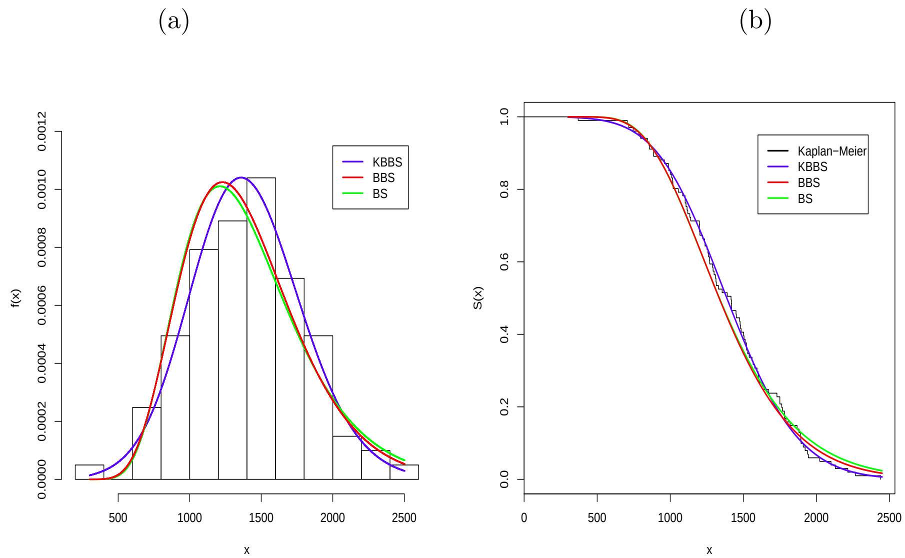

Figure 12. (a) Estimated pdfs of the KBBS distribution and its sub-models. (b) Empirical and estimated survival functions of the KBBS distribution and its sub-models.

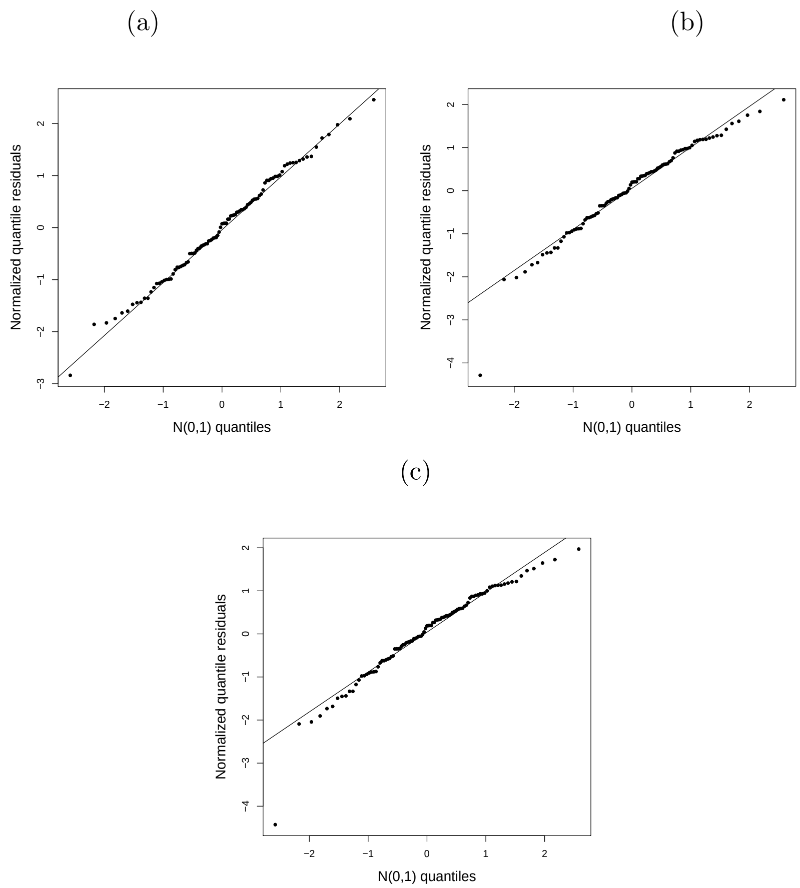

Figure 13. QQ plot of the normalized quantile residuals for the distributions: (a) KBBS, (b) BBS and (c) BS for aluminum alloy fatigue data.

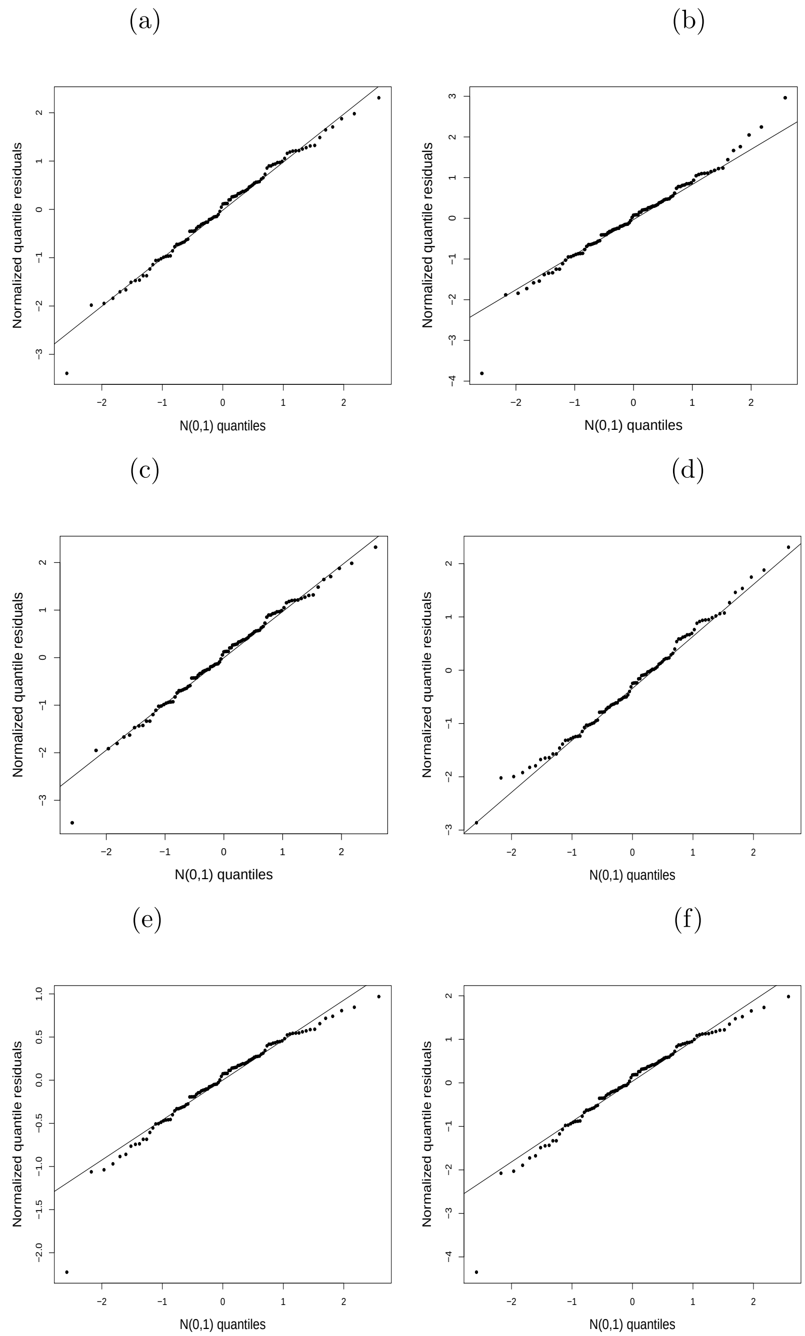

Figure 14. QQ plot of the normalized quantile residuals for the distributions: (a) ExBS, (b) McBS, (c) McGa, (d) GBS, (e) MOEBS and (f) LBS for aluminum alloy fatigue data.

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A general class of estimators for the population mean using multi-phase sampling with the non-respondents

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Abstract

This paper addresses the problem of non-response when estimating the population mean of the variable of interest. A general class of estimators is suggested for the unknown population mean of the study variable. Two vectors of the auxiliary information under multi-phase sampling scheme are used in presence of non response. The asymptotic variance of the proposed class is determined and compared with some other existing estimators theoretically and numerically. It is shown that the proposed class of estimators is more efficient than [13] and [17] classes of estimators.

Keywords: Auxiliary information, Multi-phase sampling, Non-response, Asymptotic variance, Efficiency.

2000 AMS Classification: 62D05

1. Introduction

While collecting information through the sample surveys there may arise several problems, one of the common problems is non-response. This happens especially in the surveys conducted through mails. When research participants can not be approached directly, they may refuse to acknowledge survey questionnaires sent to them through mails. The estimates obtained from such incomplete surveys are often biased. The data obtained from respondents group differs from that of non-respondents group and thus affects data reliability. [6] suggested non-respondents sub sampling scheme to handle this problem. In this scheme, initially the information is collected from the respondent's group through mail survey then non-respondents are re-contacted by using sub sampling process and complete

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information is retrieved through personal interviews. The main objective in sampling theory is to estimate the unknown population parameter of interest, and the information about population characteristic of the auxiliary variables may or may not be available. Of course the use of the auxiliary information increases the efficiency of the estimates of the population parameter. Many authors have proposed different estimators to estimate the population mean using the auxiliary information with or without considering non-response see [19], [7], [14], [12], [9], [13], [20], [8], [4], [16, 17], [10], [3] and references cited therein.

When information about the population characteristic of the auxiliary variable (X) is unavailable, in such situation it is estimated from a large first phase sample drawn from the population and then a smaller second phase sample is taken from the first phase sample to estimate the population parameter of the variable of interest (Y). [10] has considered regression-cum-ratio estimator using single auxiliary variable in two phase sampling scheme in presence of non-response. [17] have proposed regression-cum-ratio estimators considering different situations using two auxiliary variables in two-phase sampling scheme in presence of non-response.

Multi-phase sampling is very useful scheme in a situation when the variable of interest is very expensive and, connected with other cheaper auxiliary variables. Limited literature is available about this technique see for instance [18], [11], [1], [2] and [15].

This paper is inspired by the previous studies and the main objective is to propose a general class of estimators to estimate the unknown population mean $\bar{Y}$ under multi-phase sampling scheme using a vector $\underline{X} = (x_1, \ldots, x_k)^t$ with unknown population mean vector $\underline{\bar{X}} = (\bar{X}_1, \ldots, \bar{X}_k)^t$ and another vector $\underline{Z} = (z_1, \ldots, z_k)^t$ with known or unknown population mean vector $\underline{\bar{Z}} = (\bar{Z}_1, \ldots, \bar{Z}_k)^t$ in the presence of non-response. This research will also explore that the optimal estimators proposed in the generalized class are regression type estimators.

2. Notations and Background

Let us assume that P be a finite population of N distinct units. Let Y and X be the study and the auxiliary variables having values y_i and x_i, $i = (1, \ldots, N)$. Let X is correlated with Y and is used to estimate the unknown population mean $\bar{Y}$. When the mean of the auxiliary variable X is available, the ratio, product and regression estimators are used to increase the efficiency of the estimates of $\bar{Y}$. When $\bar{X}$ is unknown, two-phase sampling scheme is used, at first phase only $\bar{X}$ is estimated and the second phase is devoted for the estimation of $\bar{Y}$.

Let a large sample s' of size m_1 $(m_1 < N)$ is drawn by simple random sampling without replacement (SRSWOR) to collect information on the auxiliary variable X. It is assumed that all m_1 units provide complete information on X. In the second phase, a smaller sample s of size m_2 from m_1 units $(m_2 < m_1)$ is drawn by SRSWOR for obtaining information of the study variable Y. Suppose that non-response is present in second phase, in this situation, a subset s_1 of size m_2' supplies information on Y and the remaining $m_2^* = m_2 - m_2'$ units are non-respondents. Therefore, following the familiar technique of [6], a sub-sample s_{2r} of size $r = m_2^*/b$, $b > 1$ is selected from the m_2^* non-response units where r would be an integer otherwise must be rounded. Assuming that all r selected units show full response on second call. Consequently, the whole population is said to be stratified into two strata P_1 and P_2, where P_1 is the stratum of respondents of size N_1 that would give response on first call at second phase whereas P_2 is the stratum of non-respondents of size N_2 which would not respond on first call at second phase but will cooperate on the second call. Obviously N_1 and N_2 are not known in advance.

Now we can define a dummy variable $u = (y, x, z)$ in the following presentation

$$\bar{u}^* = d_1\bar{u}_1 + d_2\bar{u}_{2r},$$

where

$$\bar{u}' = \frac{\sum_{i=1}^{m_1} u_i}{m_1}, \quad \bar{u} = \frac{\sum_{i=1}^{m_2} u_i}{m_2}, \quad \bar{u}_1 = \frac{\sum_{i=1}^{m_2'} u_i}{m_2'}, \quad \bar{u}_{2r} = \frac{\sum_{i=1}^{r} u_i}{r}$$

and $u_i = (y_i, x_i, z_i)$ having i^{th} value. Also

$$d_1 = \frac{m_2'}{m_2} \qquad \text{and} \qquad d_2 = \frac{m_2^*}{m_2}.$$

Similarly we have

$$\bar{U} = D_1\bar{U}_1 + D_2\bar{U}_2,$$

where

$$\bar{U}_1 = \frac{\sum_{i=1}^{N_1} u_i}{N_1}, \quad \bar{U}_2 = \frac{\sum_{i=1}^{N_2} u_i}{N_2}, \quad D_1 = \frac{N_1}{N} \quad \text{and} \quad D_2 = \frac{N_2}{N}.$$

The variance of $\bar{u}^*$ is given by

$$\text{Var}(\bar{u}^*) = \theta_2 S_u^2 + \omega_2 S_{u(2)}^2 = \widetilde{S}_u^2, \tag{2.1}$$

where

$$S_u^2 = \frac{\sum_i^N (u_i - \bar{U})^2}{N-1}, \quad S_{u(2)}^2 = \frac{\sum_i^{N_2} (u_i - \bar{U}_2)^2}{N_2 - 1},$$

$$\theta_1 = \left(\frac{1}{m_1} - \frac{1}{N}\right), \quad \theta_2 = \left(\frac{1}{m_2} - \frac{1}{N}\right) \quad \text{and} \quad \omega_2 = \frac{N_2(b-1)}{m_2 N}.$$

One can define the covariance as

$$\text{Cov}(\bar{u}^*, \bar{v}^*) = \theta_2 S_{uv} + \omega_2 S_{uv(2)} = \widetilde{S}_{uv}, \tag{2.2}$$

where

$$S_{uv} = \frac{\sum_i^N (u_i - \bar{U})(v_i - \bar{V})}{N-1} \quad \text{and} \quad S_{uv(2)} = \frac{\sum_i^{N_2} (u_i - \bar{U}_2)(v_i - \bar{V}_2)}{N_2 - 1}, \quad (u = y, v = (x, z)).$$

From now on, we shall consider that the Asymptotic Variance (AV) of the considered estimators obtained by using a first order Taylor series [21].

[6] proposed an estimator $\bar{y}^*$ for the population mean $\bar{Y}$ when non response occurs

$$\bar{y}^* = d_1\bar{y}_1 + d_2\bar{y}_{2r}. \tag{2.3}$$

[13] proposed the following regression estimator using double sampling scheme when non-response occurs on Y and X both

$$t_{\text{OL}} = \bar{y}^* + \hat{\beta}_{yx}^* \left(\bar{x}' - \bar{x}^*\right), \tag{2.4}$$

where $\hat{\beta}^*_{yx} = \dfrac{s^*_{yx}}{s^{*2}_x}$ is a sample estimate of the population regression coefficient $\beta_{yx} = \dfrac{S_{yx}}{S^2_x}$ with $s^*_{yx} = \dfrac{\sum_{i=1}^{m_2'} y_i x_i + b\sum_{i=1}^{r} y_i x_i - m_2 \bar{y}^* \bar{x}^*}{m_2 - 1}$ and $s^{*2}_x = \dfrac{\sum_{i=1}^{m_2'} x_i^2 + b\sum_{i=1}^{r} x_i^2 - m_2 \bar{x}^{*2}}{m_2 - 1}$.
The asymptotic variance of t_{OL} is obtained by using the Taylor linearization and is given by

$$\mathrm{AV}(t_{\mathrm{OL}}) \cong \theta_1 S_y^2 + (\theta_2 - \theta_1) S_y^2 \left(1 - \rho^2_{yx}\right) + \omega_2 \left\{ S^2_{y(2)} + \beta^2_{yx} S^2_{x(2)} - 2\beta_{yx} S_{yx(2)} \right\},$$

where $\rho^2_{yx} = \dfrac{S^2_{yx}}{S^2_y S^2_x}$.

[10] proposed the following regression-cum-ratio estimator using [13] regression estimator

$$t_{\mathrm{K}} = t_{\mathrm{OL}} \left(\frac{a\bar{x}^* + b}{a\bar{x}' + b} \right)^{\alpha} \left(\frac{a\bar{x}' + b}{a\bar{x} + b} \right)^{\beta}, \tag{2.5}$$

where (a, b) are known constants and (α, β) are suitably chosen constants.

The minimum asymptotic variance of t_{K} is given by

$$\mathrm{minAV}(t_{\mathrm{K}}) \cong \theta_1 S_y^2 + (\theta_2 - \theta_1) S_y^2 \left(1 - \rho^2_{yx}\right) + \omega_2 S^2_{y(2)} \left(1 - \rho^2_{yx(2)}\right),$$

where $\rho^2_{yx(2)} = \dfrac{S^2_{yx(2)}}{S^2_{y(2)} S^2_{x(2)}}$.

[17] considered two auxiliary variables x and z in four different situations. We consider only two of those situations which are in our opinion the most interesting and related to our proposed class.

Situation I: – $\bar{X}$ unknown and $\bar{Z}$ known

When the population mean of the first auxiliary variable x is unknown and the population mean of the second auxiliary variable z is known. Also non-response occurs on both the study and the auxiliary variables. The proposed estimator is given by

$$t_{\mathrm{SK}(1)} = \left[\bar{y}^* + \hat{\beta}^*_{yx} (\bar{x}' - \bar{x}^*) \right] \frac{\bar{Z}}{\bar{Z} + \alpha(\bar{z}^* - \bar{Z})}, \tag{2.6}$$

where α is a suitably chosen constant.

The minimum asymptotic variance of the estimator $t_{\mathrm{SK}(1)}$ is given by

$$\mathrm{minAV}(t_{\mathrm{SK}(1)}) \cong \mathrm{AV}(t_{\mathrm{OL}}) - \frac{M^{*2}}{D^*},$$

where

$$M^* = \{\theta_2 S^2_z \beta_{yz} + \omega_2 S^2_{z(2)} \beta_{yz(2)}\} - \beta_{yx} \{(\theta_2 - \theta_1) S^2_z \beta_{xz} + \omega_2 S^2_{z(2)} \beta_{xz(2)}\},$$

$$D^* = \theta_2 S^2_z + \omega_2 S^2_{z(2)}.$$

Situation II: – $\bar{X}$ and $\bar{Z}$ both unknown

When the population means $\bar{X}$ and $\bar{Z}$ are unknown. Also non-response occurs on the study and the auxiliary variables. The estimator is given by

$$t_{\mathrm{SK}(2)} = \left[\bar{y}^* + \hat{\beta}^*_{yx}(\bar{x}' - \bar{x}^*)\right] \frac{\bar{z}'}{\bar{z}' + \gamma(\bar{z}^* - \bar{z}')}, \tag{2.7}$$

where γ is a suitably chosen constant.

The minimum asymptotic variance of $t_{\mathrm{SK}(2)}$ is given by

$$\min\mathrm{AV}(t_{\mathrm{SK}(2)}) \cong \mathrm{AV}(t_{\mathrm{OL}}) - \frac{N^{*2}}{D^*},$$

where

$$N^* = \left\{(\theta_2 - \theta_1)\, S_z^2 \beta_{yz} + \omega_2 S^2_{z(2)} \beta_{yz(2)}\right\} - \beta_{yx}\left\{(\theta_2 - \theta_1)\, S_z^2 \beta_{xz} + \omega_2 S^2_{z(2)} \beta_{xz(2)}\right\}.$$

It is important to note that our proposed general class is an extension of regression estimator. We consider specifically regression estimators because our consideration is to show that regression estimator(s) perform better than regression-cum-ratio estimator(s). Therefore, we expect that our proposed class of estimators perform better than [17] class.

3. Multi-Phase Scheme

By generalizing the multi-phase sampling scheme proposed by [2], we construct a sampling design with two vectors $\underline{X} = (x_1, \ldots, x_k)^t$ and $\underline{Z} = (z_1, \ldots, z_k)^t$ of auxiliary variables in such a way that at each i^{th} and $(i+1)^{th}$ phase, the samples s_i and s_{i+1} of sizes $m_i\,(m_i < N)$ and $m_{i+1}\,(m_{i+1} < m_i)$ are drawn by SRSWOR. At the i^{th} phase, the variables x_i and z_i are observed while at last phase, the auxiliary variables as well as y are measured, according to Table 1.

Table 1. Suggested multi-phase sampling design

1	2	3	$\ldots$	i	$\ldots$	k	$(k+1)$
m_1	m_2	m_3	$\ldots$	m_i	$\ldots$	m_k	m_{k+1}
(X_1, Z_1)	(X_1, Z_1)						
	(X_2, Z_2)	(X_2, Z_2)					
		(X_3, Z_3)	$\ddots$				
				(X_{i-1}, Z_{i-1})			
				(X_i, Z_i)	$\ddots$		
						(X_{k-1}, Z_{k-1})	
						(X_k, Z_k)	(X_k, Z_k)
							Y

4. General Class of Estimators

Using this multi phase sampling design, we propose a general class of estimators for the population mean $\bar{Y}$ when non response can occur on the study variable as well as on the auxiliary vectors.

$$t_{kk} = g\left(\bar{y}^{*}, \underline{\mathbf{w}}^{t}\right), \tag{4.1}$$

where

$$\underline{\mathbf{w}} = \left(\bar{x}_{1}^{(1)}, \bar{z}_{1}^{(1)}, \widetilde{x}_{1}^{(2)}, \widetilde{z}_{1}^{(2)}, \ldots, \bar{x}_{k}^{(k)}, \bar{z}_{k}^{(k)}, \widetilde{x}_{k}^{(k+1)}, \widetilde{z}_{k}^{(k+1)}\right)^{t}$$

and g is a function satisfying the following regularity conditions

- **C1:** $g : \mathcal{S} \to \mathbb{R}$ where $\mathcal{S} \subseteq \mathbb{R}^{4k+1}$ is a convex and bounded set containing the point $\left(\bar{Y}, \underline{\mathbf{W}}^{t}\right)$, with $\underline{\mathbf{W}} = \mathbb{E}\left(\underline{\mathbf{w}}\right) = \left(\bar{X}_{1}, \bar{X}_{1}, \bar{Z}_{1}, \bar{Z}_{1}, \ldots, \bar{X}_{k}, \bar{X}_{k}\bar{Z}_{k}, \bar{Z}_{k}\right)^{t}$;
- **C2:** it is continuous and bounded in $\mathcal{S}$;
- **C3:** its first and second partial derivatives exist and are continuous and bounded in $\mathcal{S}$;
- **C4:** $g\left(\bar{y}^{*}, \underline{\mathbf{W}}^{t}\right) = \bar{y}^{*}$.

In order to determine the minimum asymptotic variance (AV) of the class t_{kk}, let us indicate

$$g_{0} = \left.\frac{\partial g\left(\bar{y}^{*}, \underline{\mathbf{w}}^{t}\right)}{\partial \bar{y}^{*}}\right|_{\left(\bar{y}^{*}, \underline{\mathbf{w}}^{t}\right) = \left(\bar{Y}, \underline{\mathbf{W}}^{t}\right)}$$

and

$$g_{i} = \left.\frac{\partial g\left(\bar{y}^{*}, \underline{\mathbf{w}}^{t}\right)}{\partial w_{i}}\right|_{\left(\bar{y}^{*}, \underline{\mathbf{w}}^{t}\right) = \left(\bar{Y}, \underline{\mathbf{W}}^{t}\right)},$$

the first partial derivatives of g with respect to the component $\bar{y}^{*}$ and w_i, $i = 1, 2, \ldots, k$, of the vector $\underline{\mathbf{w}}$.

Expanding t_{kk} at the point $\left(\bar{Y}, \underline{\mathbf{W}}^{t}\right)$ in a first order Taylor's series, we get

$$\begin{aligned} t_{kk} \cong \bar{y}^{*} &+ \sum_{i=1}^{k} \eta_{i}\left(\bar{X}_{i} - \bar{x}_{i}^{(i)}\right) + \sum_{i=1}^{k} \xi_{i}\left(\bar{X}_{i} - \widetilde{x}_{i}^{(i+1)}\right) \\ &+ \sum_{i=1}^{k} \phi_{i}\left(\bar{Z}_{i} - \bar{z}_{i}^{(i)}\right) + \sum_{i=1}^{k} \varphi_{i}\left(\bar{Z}_{i} - \widetilde{z}_{i}^{(i+1)}\right), \end{aligned} \tag{4.2}$$

where first k indicates the general class and second k for having k auxiliary variables in the estimator t_{kk}.

Further, consider $\widetilde{u}_{i} = \delta_{u}\bar{u}_{i}^{*} + \bar{\delta}_{u}\bar{u}_{i}$ with $\bar{\delta_{u}} = 1 - \delta_{u}$ where δ_{u} is an indicator function taking value $\delta_{u} = 1$ when non-response occurs and 0 otherwise, and $u_{i} = \left(x_{i}, z_{i}\right), i = 1, \ldots, k$.

4.1. Situation 1: $\underline{\bar{X}}$ unknown and $\underline{\bar{Z}}$ known. In this situation it is assumed that the population mean vector $\underline{\bar{X}}$ is unknown but the mean vector $\underline{\bar{Z}}$ is known. Since $\underline{\bar{X}}$ is unknown so it is necessary to impose a constraint $\eta_i = -\xi_i$ and for computational reasons it may be useful to set $(\phi_i + \varphi_i) = \psi_i$. We can write Eq.(4.2) as

$$(4.3) \quad t_{1k} \cong \bar{y}^* + \sum_{i=1}^{k} \xi_i \left(\bar{x}_i^{(i)} - \widetilde{x}_i^{(i+1)}\right) + \sum_{i=1}^{k} \varphi_i \left(\bar{z}_i^{(i)} - \widetilde{z}_i^{(i+1)}\right) + \sum_{i=1}^{k} \psi_i \left(\bar{Z}_i - \bar{z}_i^{(i)}\right).$$

We can write (4.3) in a generalized vector form as

$$(4.4) \quad t_{1k} \cong \bar{y}^* + \left(\underline{\nu}' - \underline{\widetilde{\nu}}\right)^t \underline{\xi} + \left(\underline{\bar{Z}} - \underline{\bar{z}}'\right) \underline{\psi},$$

where

$$\underline{\nu}' = \left(\bar{x}_1^{(1)}, \bar{z}_1^{(1)}, \ldots, \bar{x}_k^{(k)}, \bar{z}_k^{(k)}\right)^t,$$

$$\underline{\widetilde{\nu}} = \left(\widetilde{x}_1^{(2)}, \widetilde{z}_1^{(2)}, \ldots, \widetilde{x}_k^{(k+1)}, \widetilde{z}_k^{(k+1)}\right)^t,$$

$$\underline{\xi} = \left(\xi_1, \varphi_1, \ldots, \xi_k, \varphi_k\right)^t,$$

$$\underline{\bar{z}}' = \left(\bar{z}_1^{(1)}, \ldots, \bar{z}_k^{(k)}\right)^t$$

and

$$\underline{\psi} = \left(\psi_1, \ldots, \psi_k\right)^t.$$

The asymptotic variance of the proposed class t_{1k}

$$(4.5) \quad \mathrm{AV}(t_{1k}) \cong \mathrm{Var}(\bar{y}^*) + \underline{\xi}^t \underline{\widetilde{\mathbf{S}}}_{xz} \underline{\xi} + \underline{\psi}^t \underline{\bar{S}}_{\bar{z}\bar{z}} \underline{\psi} - 2\left(\underline{\xi}^t \underline{\widetilde{\mathbf{S}}}_{yxz} + \underline{\psi}^t \underline{\bar{S}}_{y\bar{z}} - \underline{\xi}^t \underline{\bar{S}}_{xz\bar{z}} \underline{\psi}\right),$$

where

$$\mathrm{Var}(\bar{y}^*) = \widetilde{S}_y^2 = \theta_{k+1} S_y^2 + \omega_{k+1} S_{y(2)}^2,$$

$$\underline{\widetilde{\mathbf{S}}}_{xz} = E\left[(\underline{\nu}' - \underline{\widetilde{\nu}})(\underline{\nu}' - \underline{\widetilde{\nu}}^t\right] = \begin{bmatrix} \underline{\widetilde{S}}_{xx} & \underline{\widetilde{S}}_{xz} \\ \underline{\widetilde{S}}_{xz} & \underline{\widetilde{S}}_{zz} \end{bmatrix},$$

$$\underline{\widetilde{\mathbf{S}}}_{yxz} = E\left[(\bar{y}^* - \bar{Y})(\underline{\nu}' - \underline{\widetilde{\nu}})\right] = \begin{bmatrix} \underline{\widetilde{S}}_{yx} & \underline{\widetilde{S}}_{yz} \end{bmatrix}^t,$$

$$\underline{\bar{S}}_{\bar{z}\bar{z}} = E\left[(\underline{\bar{z}}' - \underline{\bar{Z}})(\underline{\bar{z}}' - \underline{\bar{Z}})^t\right],$$

$$\underline{\bar{S}}_{y\bar{z}} = E\left[(\bar{y}^* - \bar{Y})(\underline{\bar{z}}' - \underline{\bar{Z}})\right] = \begin{bmatrix} \theta_1 S_{yz_1} & \theta_2 S_{yz_2} & \ldots & \theta_k S_{yz_k} \end{bmatrix}^t,$$

and

$$\underline{\bar{S}}_{xz\bar{z}} = \begin{bmatrix} \underline{S}_{x\bar{z}} & \underline{S}_{z\bar{z}} \end{bmatrix}^t.$$

The details of above mentioned matrices are given in Appendix A.

Minimizing $\mathrm{AV}(t_{1k})$ w.r.t $\underline{\xi}$ and $\underline{\psi}$ leads to optimum vector

$$\underline{\psi} = \left\{\underline{\bar{S}}_{\bar{z}\bar{z}} - \underline{\bar{S}}_{xz\bar{z}}^t \underline{\widetilde{\mathbf{S}}}_{xz}^{-1} \underline{\bar{S}}_{xz\bar{z}}\right\}^{-1} \left\{\underline{\bar{S}}_{y\bar{z}} - \underline{\bar{S}}_{xz\bar{z}} \underline{\widetilde{\mathbf{S}}}_{xz}^{-1} \underline{\widetilde{\mathbf{S}}}_{yxz}\right\} = \underline{\psi}^{(o)}(\text{say}),$$

and

$$\underline{\xi} = \underline{\widetilde{\mathbf{S}}}_{xz}^{-1} \left(\underline{\widetilde{\mathbf{S}}}_{yxz} - \underline{\bar{S}}_{xz\bar{z}} \left\{\underline{\bar{S}}_{\bar{z}\bar{z}} - \underline{\bar{S}}_{xz\bar{z}}^t \underline{\widetilde{\mathbf{S}}}_{xz}^{-1} \underline{\bar{S}}_{xz\bar{z}}\right\}^{-1} \left\{\underline{\bar{S}}_{y\bar{z}} - \underline{\bar{S}}_{xz\bar{z}} \underline{\widetilde{\mathbf{S}}}_{xz}^{-1} \underline{\widetilde{\mathbf{S}}}_{yxz}\right\}\right) = \underline{\xi}^{(o)}(\text{say}).$$

Since it is not easy to express minimum AV (t_{1k}) in close form, in the following sub sections, we can express the class t_{1k} in two sub classes $t_{1k(1)}$ and $t_{1k(2)}$ with two and three phases.

4.1.1. *Two Phase.* Using Eq.(4.3), let consider (X_1, Z_1) auxiliary variables under two-phase sampling scheme assuming $\bar{X}_1$ unknown and $\bar{Z}_1$ known

$$t_{1k(1)} \cong \bar{y}^* + \xi_1\left(\bar{x}_1^{(1)} - \widetilde{x}_1^{(2)}\right) + \varphi_1\left(\bar{z}_1^{(1)} - \widetilde{z}_1^{(2)}\right) + \psi_1\left(\bar{Z}_1 - \bar{z}_1^{(1)}\right). \tag{4.6}$$

The asymptotic variance of $t_{1k(1)}$ can be written as

$$\text{AV}\ (t_{1k(1)}) \cong \widetilde{S}_y^2 + \xi_1^2\widetilde{S}_{x_1}^2 + \varphi_1^2\widetilde{S}_{z_1}^2 + \psi_1^2\bar{S}_{z_1}^2 - 2\xi_1\widetilde{S}_{yx_1} - 2\varphi_1\widetilde{S}_{yz_1} - 2\psi_1\bar{S}_{yz_1} + 2\xi_1\varphi_1\widetilde{S}_{x_1z_1}. \tag{4.7}$$

Minimizing Eq.(4.7) w.r.t $(\xi_1,\ \varphi_1,\ \psi)$, one can get

$$\xi_1 = \left(\widetilde{S}_{x_1}^2\widetilde{S}_{z_1}^2 - \widetilde{S}_{x_1z_1}^2\right)^{-1}\left(\widetilde{S}_{z_1}^2\widetilde{S}_{yx_1} - \widetilde{S}_{x_1z_1}\widetilde{S}_{yz_1}\right) = \xi_1^o(\text{say}),$$

$$\varphi_1 = \left(\widetilde{S}_{x_1}^2\widetilde{S}_{z_1}^2 - \widetilde{S}_{x_1z_1}^2\right)^{-1}\left(\widetilde{S}_{x_1}^2\widetilde{S}_{yz_1} - \widetilde{S}_{x_1z_1}\widetilde{S}_{yx_1}\right) = \varphi_1^o(\text{say})$$

and

$$\psi_1 = \left(\bar{S}_{z_1}^2\right)^{-1}\bar{S}_{yz_1} = \psi_1^o(\text{say}).$$

The minimum asymptotic variance of $t_{1k(1)}$ is given by

$$\text{minAV}(t_{1k(1)}) \cong \widetilde{S}_y^2 - \left[\frac{\widetilde{S}_{z_1}^2\widetilde{S}_{yx_1}^2 + \widetilde{S}_{x_1}^2\widetilde{S}_{yz_1}^2 - 2\widetilde{S}_{yx_1}\widetilde{S}_{yz_1}\widetilde{S}_{x_1z_1}}{\widetilde{S}_{x_1}^2\widetilde{S}_{z_1}^2 - \widetilde{S}_{x_1z_1}^2}\right] - \frac{\bar{S}_{yz_1}^2}{\bar{S}_{z_1}^2}, \tag{4.8}$$

$$\text{minAV}(t_{1k(1)}) \cong \widetilde{S}_y^2\left[1 - \left(\frac{\widetilde{\rho}_{yx_1}^2 + \widetilde{\rho}_{yz_1}^2 - 2\widetilde{\rho}_{yx_1}\widetilde{\rho}_{yz_1}\widetilde{\rho}_{x_1z_1}}{1 - \widetilde{\rho}_{x_1z_1}^2}\right) - \bar{\rho}_{yz_1}^2\right], \tag{4.9}$$

where $\widetilde{\rho}_{yu_1}^2 = \dfrac{\widetilde{S}_{yu_1}^2}{\widetilde{S}_y^2\widetilde{S}_{u_1}^2}$, $u_1 = (x_1,\ z_1)$ and $\widetilde{\rho}_{x_1z_1}^2 = \dfrac{\widetilde{S}_{x_1z_1}^2}{\widetilde{S}_{x_1}^2\widetilde{S}_{z_1}^2}$.

$$\text{minAV}(t_{1k(1)}) \cong \widetilde{S}_y^2\left[1 - \widetilde{R}_{y.x_1z_1}^2 - \bar{\rho}_{yz_1}^2\right] \tag{4.10}$$

where $\widetilde{R}_{y.x_1z_1}$ is a multiple correlation coefficient.

4.1.2. *Three Phase.* In this case, we take (X_1, X_2, Z_1, Z_2) auxiliary variables using three-phase sampling, assuming population means $(\bar{X}_1,\ \bar{X}_2)$ unknown and $(\bar{Z}_1,\ \bar{Z}_2)$ known

$$\begin{aligned} t_{1k(2)} \ \cong\ & \bar{y}^* + \xi_1\left(\bar{x}_1^{(1)} - \widetilde{x}_1^{(2)}\right) + \xi_2\left(\bar{x}_2^{(2)} - \widetilde{x}_2^{(3)}\right) + \varphi_1\left(\bar{z}_1^{(1)} - \widetilde{z}_1^{(2)}\right) \\ & + \varphi_2\left(\bar{z}_2^{(2)} - \widetilde{z}_2^{(3)}\right) + \psi_1\left(\bar{Z}_1 - \bar{z}_1^{(1)}\right) + \psi_2\left(\bar{Z}_2 - \bar{z}_2^{(2)}\right). \end{aligned} \tag{4.11}$$

The asymptotic variance of $t_{1k(2)}$

$$\mathrm{AV}(t_{1k(2)}) \cong \widetilde{S}_y^2 + \sum_{j=1}^{2} \xi_i^2 \widetilde{S}_{x_j}^2 + \sum_{j=1}^{2} \varphi_j^2 \widetilde{S}_{z_j}^2 - 2\left(\sum_{j=1}^{2} \xi_j \widetilde{S}_{yx_j} + \sum_{j=1}^{2} \varphi_j \widetilde{S}_{yz_j} + \sum_{j=1}^{2} \psi_j \bar{S}_{yz_j}\right)$$

$$+\sum_{j=1}^{2} \psi_j^2 \bar{S}_{z_j}^2 + 2\sum_{j=1}^{2} \xi_j \varphi_j \widetilde{S}_{x_j z_j} + 2\left(\psi_1 \psi_2 \bar{S}_{z_1 z_2} + \xi_1 \psi_2 \bar{\bar{S}}_{x_1 z_2} + \varphi_1 \psi_2 \bar{\bar{S}}_{z_1 z_2}\right). \tag{4.12}$$

The AV of $t_{1k(2)}$ is minimum when

$$\xi_2 = \left(\widetilde{S}_{x_2}^2 \widetilde{S}_{z_2}^2 - \widetilde{S}_{x_2 z_2}^2\right)^{-1}\left(\widetilde{S}_{z_2}^2 \widetilde{S}_{yx_2} - \widetilde{S}_{x_2 z_2} \widetilde{S}_{yz_2}\right) = \xi_2^o(\text{say}),$$

$$\varphi_2 = \left(\widetilde{S}_{x_2}^2 \widetilde{S}_{z_2}^2 - \widetilde{S}_{x_2 z_2}^2\right)^{-1}\left(\widetilde{S}_{x_2}^2 \widetilde{S}_{yz_2} - \widetilde{S}_{x_2 z_2} \widetilde{S}_{yx_2}\right) = \varphi_2^o(\text{say}),$$

$$\xi_1 = E^{-1}A = \xi_1^o(\text{say}),$$

$$\varphi_1 = E^{-1}B = \varphi_1^o(\text{say}),$$

$$\psi_1 = E^{-1}C = \psi_1^o(\text{say})$$

and

$$\psi_2 = E^{-1}D = \psi_2^o(\text{say}).$$

The details of (A, B, C, D, E) are given in Appendix B.

The resulting minimum asymptotic variance of $t_{1k(2)}$ is given by

$$\min\mathrm{AV}(t_{1k(2)}) = \widetilde{S}_y^2\left(1 - \bar{R}_{y.x_1 z_1 \bar{z}_1 \bar{z}_2}^2 - \widetilde{R}_{y.x_2 z_2}^2\right), \tag{4.13}$$

where $\bar{R}_{y.x_1 z_1 \bar{z}_1 \bar{z}_2}$ and $\widetilde{R}_{y.x_2 z_2}$ are the multiple correlation coefficients.

4.2. Situation 2: $\underline{\bar{X}}$ and $\underline{\bar{Z}}$ both unknown. In this situation it is assumed that the population mean vectors $\underline{\bar{X}}$ and $\underline{\bar{Z}}$ are unknown. From the expression (4.3), we have

$$t_{2k} \cong \bar{y}^* + \sum_{i=1}^{k} \xi_i\left(\bar{x}_i^{(i)} - \widetilde{x}_i^{(i+1)}\right) + \sum_{i+1}^{k} \varphi_i\left(\bar{z}_i^{(i)} - \widetilde{z}_i^{(i+1)}\right). \tag{4.14}$$

We can write Eq.(4.14) in a generalized vector form as

$$t_{2k} \cong \bar{y}^* + (\underline{\bar{\nu}}' - \underline{\widetilde{\nu}})\,\underline{\xi}. \tag{4.15}$$

The asymptotic variance of t_{2k} is

$$\mathrm{AV}(t_{2k}) \cong \mathrm{Var}(\bar{y}^*) + \underline{\xi}^t \underline{\widetilde{\mathbf{S}}}_{xz} \underline{\xi} - 2\underline{\xi}^t \underline{\widetilde{\mathbf{S}}}_{yxz}, \tag{4.16}$$

where $(\underline{\xi}, \underline{\widetilde{\mathbf{S}}}_{xz}, \underline{\widetilde{\mathbf{S}}}_{yxz})$ are defined earlier in Situation 1.

Minimization of $\mathrm{AV}(t_{2k})$ w.r.t g_{xz} leads to optimum vector

$$\underline{\xi}^o = \begin{bmatrix} \xi^o \\ \varphi^o \end{bmatrix} = \begin{bmatrix} (\underline{\widetilde{S}}_{xx}\underline{\widetilde{S}}_{zz} - \underline{\widetilde{S}}_{xz}^t \underline{\widetilde{S}}_{xz})^{-1}(\underline{\widetilde{S}}_{zz}\underline{\widetilde{S}}_{yx} - \underline{\widetilde{S}}_{xz}\underline{\widetilde{S}}_{yz}) \\ (\underline{\widetilde{S}}_{xx}\underline{\widetilde{S}}_{zz} - \underline{\widetilde{S}}_{xz}^t \underline{\widetilde{S}}_{xz})^{-1}(\underline{\widetilde{S}}_{xx}\underline{\widetilde{S}}_{yz} - \underline{\widetilde{S}}_{xz}\underline{\widetilde{S}}_{yx}) \end{bmatrix}.$$

Replacing $\underline{\xi}^o$ in Eq.(4.16), one can get the optimal estimator in the class which attains the minimum asymptotic variance bound given by

$$\text{(4.17)} \quad \min\text{AV}(t_{2k}) \cong \widetilde{S}_y^2 \left[1 - \underline{\widetilde{R}}^2_{y.xz}\right],$$

where $\underline{\widetilde{R}}^2_{y.xz}$ is a vector of the squared multiple correlation coefficients of Y on $\underline{X}$ and $\underline{Z}$ and is explained in detail in Appendix C.

4.2.1. *Two Phase.* Now suppose that both X_1 and Z_1 have unknown means, in this case expression (4.14) can be expressed as

$$\text{(4.18)} \quad t_{2k(1)} \cong \bar{y}^* + \xi_1 \left(\bar{x}_1^{(1)} - \widetilde{x}_1^{(2)}\right) + \varphi_1 \left(\bar{z}_1^{(1)} - \widetilde{z}_1^{(2)}\right).$$

The asymptotic variance of $t_{2k(1)}$ can be written as

$$\text{(4.19)} \quad \text{AV}(t_{2k(1)}) \cong \widetilde{S}_y^2 + \xi_1^2 \widetilde{S}_{x_1}^2 + \varphi_1^2 \widetilde{S}_{z_1}^2 - 2\xi_1 \widetilde{S}_{yx_1} - 2\varphi_1 \widetilde{S}_{yz_1} + 2\xi_1\varphi_1 \widetilde{S}_{x_1z_1}.$$

Now minimizing (4.19) w.r.t $(\xi_1,\ \varphi_1)$, we have

$$\xi_1 = \left(\widetilde{S}_{x_1}^2 \widetilde{S}_{z_1}^2 - \widetilde{S}_{x_1z_1}^2\right)^{-1} \left(\widetilde{S}_{z_1}^2 \widetilde{S}_{yx_1} - \widetilde{S}_{x_1z_1} \widetilde{S}_{yz_1}\right) = \xi_1^o(\text{say})$$

and

$$\varphi_1 = \left(\widetilde{S}_{x_1}^2 \widetilde{S}_{z_1}^2 - \widetilde{S}_{x_1z_1}^2\right)^{-1} \left(\widetilde{S}_{x_1}^2 \widetilde{S}_{yz_1} - \widetilde{S}_{x_1z_1} \widetilde{S}_{yx_1}\right) = \varphi_1^o(\text{say}).$$

One can write minimum AV of $t_{2k(1)}$ as

$$\text{(4.20)} \quad \min\text{AV}(t_{2k(1)}) \cong \text{Var}(\bar{y}^*) - \left[\frac{\widetilde{S}_{z_1}^2 \widetilde{S}_{yx_1}^2 + \widetilde{S}_{x_1}^2 \widetilde{S}_{yz_1}^2 - 2\widetilde{S}_{yx_1} \widetilde{S}_{yz_1} \widetilde{S}_{x_1z_1}}{\widetilde{S}_{x_1}^2 \widetilde{S}_{z_1}^2 - \widetilde{S}_{x_1z_1}^2}\right],$$

$$\text{(4.21)} \quad \min\text{AV } (t_{2k(1)}) = \widetilde{S}_y^2 \left(1 - \widetilde{R}^2_{y.x_1z_1}\right),$$

where $\widetilde{R}^2_{y.x_1z_1}$ is explained earlier in the Section (4.1.1).

4.2.2. *Three Phase.* Now consider the same auxiliary variables as taken earlier in Section (4.1.2) with unknown means

$$\text{(4.22)} \quad \begin{aligned} t_{2k(2)} &\cong \bar{y}^* + \xi_1 \left(\bar{x}_1^{(1)} - \widetilde{x}_1^{(2)}\right) + \xi_2 \left(\bar{x}_2^{(2)} - \widetilde{x}_2^{(3)}\right) \\ &\quad + \varphi_1 \left(\bar{z}_1^{(1)} - \widetilde{z}_1^{(2)}\right) + \varphi_2 \left(\bar{z}_2^{(2)} - \widetilde{z}_2^{(3)}\right). \end{aligned}$$

The asymptotic variance of $t_{2k(2)}$ can be expressed as

(4.23)

$$\text{AV}(t_{2k(2)}) \cong \widetilde{S}_y^2 + \sum_{j=1}^{2} \xi_i^2 \widetilde{S}_{x_j}^2 + \sum_{j=1}^{2} \varphi_j^2 \widetilde{S}_{z_j}^2 - 2\left(\sum_{j=1}^{2} \xi_j \widetilde{S}_{yx_j} + \sum_{j=1}^{2} \varphi_j \widetilde{S}_{yz_j}\right) + 2\sum_{j=1}^{2} \xi_j \varphi_j \widetilde{S}_{x_jz_j}.$$

Now minimizing (4.23) w.r.t $(\xi_1,\ \xi_2,\ \varphi_1,\ \varphi_2)$, we have

$$\underline{\xi}^o=\begin{bmatrix}\xi_1^o\\ \xi_2^o\\ \varphi_1^o\\ \varphi_2^o\end{bmatrix}=\begin{bmatrix}\left(\widetilde{S}_{x_1}^2\widetilde{S}_{z_1}^2-\widetilde{S}_{x_1z_1}^2\right)^{-1}\left(\widetilde{S}_{z_1}^2\widetilde{S}_{yx_1}-\widetilde{S}_{x_1z_1}\widetilde{S}_{yz_1}\right)\\ \left(\widetilde{S}_{x_2}^2\widetilde{S}_{z_2}^2-\widetilde{S}_{x_2z_2}^2\right)^{-1}\widetilde{S}_{z_2}^2\widetilde{S}_{yx_2}-\widetilde{S}_{x_2z_2}\widetilde{S}_{yz_2}\\ \left(\widetilde{S}_{x_1}^2\widetilde{S}_{z_1}^2-\widetilde{S}_{x_1z_1}^2\right)^{-1}\left(\widetilde{S}_{x_1}^2\widetilde{S}_{yz_1}-\widetilde{S}_{x_1z_1}\widetilde{S}_{yx_1}\right)\\ \left(\widetilde{S}_{x_2}^2\widetilde{S}_{z_2}^2-\widetilde{S}_{x_2z_2}^2\right)^{-1}\left(\widetilde{S}_{x_2}^2\widetilde{S}_{yz_2}-\widetilde{S}_{x_2z_2}\widetilde{S}_{yx_2}\right)\end{bmatrix}.$$

By replacing $(\xi_1^o,\xi_2^o,\varphi_1^o,\ \varphi_2^o)$ in (4.23), one can get the optimal estimator in the class which attains the minimum asymptotic variance bound given by:

$$\begin{aligned}\text{minAV}(t_{2k(2)})\cong\widetilde{S}_y^2&-\left(\frac{\widetilde{S}_{z_1}^2\widetilde{S}_{yx_1}^2+\widetilde{S}_{x_1}^2\widetilde{S}_{yz_1}^2-2\widetilde{S}_{yx_1}\widetilde{S}_{yz_1}\widetilde{S}_{x_1z_1}}{\widetilde{S}_{x_1}^2\widetilde{S}_{z_1}^2-\widetilde{S}_{x_1z_1}^2}\right)\\ &-\left(\frac{\widetilde{S}_{z_2}^2\widetilde{S}_{yx_2}^2+\widetilde{S}_{x_2}^2\widetilde{S}_{yz_2}^2-2\widetilde{S}_{yx_2}\widetilde{S}_{yz_2}\widetilde{S}_{x_2z_2}}{\widetilde{S}_{x_2}^2\widetilde{S}_{z_2}^2-\widetilde{S}_{x_2z_2}^2}\right),\end{aligned}\tag{4.24}$$

$$\text{minAV}(t_{2k(2)})\cong\widetilde{S}_y^2\left(1-\widetilde{R}_{y.x_1z_1}^2-\widetilde{R}_{y.x_2z_2}^2\right),\tag{4.25}$$

where $\widetilde{R}_{y.x_1z_1}^2$ and $\widetilde{R}_{y.x_2z_2}^2$ are the squared multiple correlation coefficients.
From Eq.(4.10), Eq.(4.13), Eq.(4.21) and Eq.(4.25), we can accomplish that $\left(t_{1k(1)},\ t_{1k(2)},\ t_{2k(1)}\text{ and }t_{2k(2)}\right)$ are regression type estimators in their optimal cases.

5. Numerical Comparison

In order to illustrate the gain in efficiency for the best estimator in the two situations discussed in previous section, we carried out a numerical study using the Population Census Report of Sialkot District (1998), Pakistan and this data is earlier used by [5]. Each variable is taken from rural locality. The description of variables is given below

Variable	**Description**
Y	Literacy Ratio
X_1	Population of primary but below matric
X_2	Population of matric and above
Z_1	Population of 18 years old and above
Z_2	Population of women 15-49 years old

We compute the Percent Relative Efficiency (PRE) of the considered estimators with respect to the [6] estimator $\bar{y}^*$, for different values of b, as

$$\text{PRE}(t_\bullet)=\frac{\text{Var}(\bar{y}^*)}{\text{AV}(\text{t}_\bullet)}\times 100,$$

where $t_\bullet=\left(t_{\text{OL}},t_{\text{K}},t_{\text{SK}(1)},t_{\text{SK}(2)},t_{1k(1)},t_{1k(2)},t_{2k(1)},t_{2k(2)}\right)$.

Table 2. PREs of the considered classes with respect to $\bar{y}^*$ for different values of b

			b			
m_1	m_2	Situation	2	3	4	5
160	100	t_{OL}	119.26	121.72	123.22	124.24
		t_{K}	120.52	123.69	125.65	126.98
		$t_{\mathrm{SK}(1)}$	119.56	123.05	125.48	127.24
		$t_{\mathrm{SK}(2)}$	121.34	124.98	127.31	128.91
		$t_{1k(1)}$	*128.95*	*131.98*	*133.91*	*135.22*
		$t_{2k(1)}$	124.78	128.71	131.20	132.92
	120	t_{OL}	116.96	120.11	122.00	123.26
		t_{K}	118.27	122.15	124.49	126.06
		$t_{\mathrm{SK}(1)}$	117.02	121.04	123.85	125.88
		$t_{\mathrm{SK}(2)}$	118.86	123.26	126.01	127.87
		$\mathrm{t}_{1k(1)}$	*127.18*	*130.87*	*133.11*	*134.62*
		$\mathrm{t}_{2k(1)}$	121.95	126.79	129.77	131.79
180	100	t_{OL}	121.50	123.45	124.64	125.44
		t_{K}	122.81	125.48	127.11	128.22
		$t_{\mathrm{SK}(1)}$	122.29	125.43	127.55	129.05
		$t_{\mathrm{SK}(2)}$	123.90	127.02	129.00	130.36
		$\mathrm{t}_{1k(1)}$	*130.82*	*133.42*	*135.07*	*136.20*
		$\mathrm{t}_{2k(1)}$	127.69	130.98	133.06	134.50
	120	t_{OL}	119.78	122.27	123.74	124.72
		t_{K}	121.16	124.38	126.31	127.59
		$t_{\mathrm{SK}(1)}$	120.24	123.90	126.35	128.08
		$t_{\mathrm{SK}(2)}$	122.07	125.80	128.10	129.66
		$\mathrm{t}_{1k(1)}$	*129.54*	*132.65*	*134.55*	*135.82*
		$\mathrm{t}_{2k(1)}$	125.57	129.59	132.06	133.72
200	100	t_{OL}	123.36	124.87	125.79	126.40
		t_{K}	124.71	126.95	128.31	129.24
		$t_{\mathrm{SK}(1)}$	124.72	127.49	129.31	130.58
		$t_{\mathrm{SK}(2)}$	126.03	128.71	130.39	131.54
		$\mathrm{t}_{1k(1)}$	*132.37*	*134.59*	*136.01*	*136.99*
		$\mathrm{t}_{2k(1)}$	130.12	132.85	134.59	135.80
	120	t_{OL}	122.14	124.05	125.18	125.92
		t_{K}	123.58	126.23	127.80	128.85
		$t_{\mathrm{SK}(1)}$	123.19	126.41	128.50	129.94
		$t_{\mathrm{SK}(2)}$	124.77	127.91	129.83	131.12
		$\mathrm{t}_{1k(1)}$	*131.51*	*134.12*	*135.72*	*136.80*
		$\mathrm{t}_{2k(1)}$	128.64	131.93	133.95	135.31

Table 3. PREs of the proposed class with respect to $\bar{y}^*$ for different values of b

				b			
m_1	m_2	m_3	Situation	2	3	4	5
200	140	100	$t_{1k(2)}$	136.32	141.78	145.44	148.04
			$t_{2k(2)}$	133.54	139.50	143.46	146.26
		110	$t_{1k(2)}$	136.50	142.56	146.57	149.40
			$t_{2k(2)}$	133.32	139.95	144.32	147.36
		120	$t_{1k(2)}$	136.72	143.41	147.79	150.86
			$t_{2k(2)}$	133.11	140.46	145.24	148.56
220	160	100	$t_{1k(2)}$	137.16	141.93	145.11	147.36
			$t_{2k(2)}$	135.34	140.44	143.81	146.17
		110	$t_{1k(2)}$	137.45	142.72	146.19	148.62
			$t_{2k(2)}$	135.36	141.02	144.70	147.27
		120	$t_{1k(2)}$	137.78	143.58	147.34	149.97
			$t_{2k(2)}$	135.40	141.64	145.66	148.43

In Table 2, we assume 40% non-response rate of the total population $N = 268$ (assuming last 107 units as non-respondents for second phase) and the numerical results of the estimators $t_{1k(1)}$ and $t_{2k(1)}$ with different combinations of sample sizes m_1 and m_2 are given. To compute PREs of the estimators $t_{1k(2)}$ and $t_{2k(2)}$, we consider 40% and 30% non-response rates of the total population at first and second phase (assuming last 80 units as non-respondents for third phase). The results are provided in Table 3 for different choices of sample sizes m_1, m_2 and m_3.

In Tables 2 and 3, it is seen that the PREs of all considered estimators are higher than $\bar{y}^*$ due to inclusion of the auxiliary information. Also, the PREs of all estimators increase with the increase of inverse sampling rate b. In Table 2, the estimators $(t_{\mathrm{OL}},\ t_{\mathrm{K}})$ with single auxiliary variable and $(t_{\mathrm{SK}(1)}, t_{\mathrm{SK}(2)})$ with two auxiliary variables perform almost similar but the proposed estimators $(\mathrm{t}_{1k(1)},\ \mathrm{t}_{2k(1)})$ show higher efficiency. It is also seen that the regression estimators $(\mathrm{t}_{1k(1)},\ \mathrm{t}_{2k(1)})$ perform better than the regression-cum-ratio estimators $(t_{\mathrm{SK}(1)},\ t_{\mathrm{SK}(2)})$.

When we consider more auxiliary variables and phases for t_{1k} and t_{2k} in Table 3, it is observed that they have considerable increase in efficiency. Moreover, it is observed that when we increase sample sizes, the PREs of all considered estimators also increase which confirms the large sample theory aspect.

It is also expected that the first proposed class t_{1k} shows always higher efficiency than the second class t_{2k}, because we are using more auxiliary information in Situation 1. Hence, we can conclude that the estimator t_{1k} is the best choice if $\underline{\bar{Z}}$ is known and, of course, t_{2k} is to use when $\underline{\bar{Z}}$ is unknown.

6. Conclusions

In this paper we propose a general class of estimators for the estimation of population mean $\bar{Y}$. For this we consider multi phase sampling scheme when auxiliary information is available. The effects of non-response on the study and on the auxiliary variables are discussed in detail. We determine the asymptotic variance for the proposed classes. To compare the efficiency of the suggested ones with other existing estimators in the literature, [6] estimator is used. It is noted that the performance of the proposed estimator t_{1k} is better than the other considered estimators. From the numerical analysis one can draw the conclusion that regression type estimators $(\mathrm{t}_{1k},\ \mathrm{t}_{2k})$ always perform better if compared to regression-cum-ratio estimators $\left(\mathrm{t}_{\mathrm{SK}(1)},\ \mathrm{t}_{\mathrm{SK}(2)}\right)$. However, our proposed generalized class of estimators is more efficient, in terms of asymptotic variance, if compared to all previous estimators available in the literature.

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Appendix A

The details of matrices in Eq.(4.5) are described here

$$\underline{\bar{S}}_{\bar{z}\bar{z}} = E\left[(\underline{\bar{z}}' - \underline{\bar{Z}})(\underline{\bar{z}}' - \underline{\bar{Z}})^t\right] = \begin{bmatrix} \theta_1 S_{z_1}^2 & \theta_1 S_{z_1 z_2} & \dots & \theta_1 S_{z_1 z_k} \\ \theta_1 S_{z_1 z_2} & \theta_2 S_{z_2}^2 & \dots & \theta_2 S_{z_2 z_k} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1 S_{z_1 z_k} & \theta_2 S_{z_2 z_k} & \dots & \theta_k S_{z_k}^2 \end{bmatrix},$$

$$\underline{\bar{S}}_{xz\bar{z}} = \left[\underline{S}_{x\bar{z}} \quad \underline{S}_{z\bar{z}}\right]^t,$$

where

$$\underline{S}_{x\bar{z}} = \begin{bmatrix} 0 & (\theta_2-\theta_1)S_{x_1 z_2} & 0 & \dots & 0 \\ 0 & 0 & (\theta_3-\theta_2)S_{x_2 z_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (\theta_k-\theta_{k-1})S_{x_{k-1} z_k} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$\underline{S}_{z\bar{z}} = \begin{bmatrix} 0 & (\theta_2-\theta_1)S_{z_1 z_2} & 0 & \dots & 0 \\ 0 & 0 & (\theta_3-\theta_2)S_{z_2 z_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (\theta_k-\theta_{k-1})S_{z_{k-1} z_k} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

$$\underline{\widetilde{\mathbf{S}}}_{xz} = E\left[(\underline{\bar{\nu}}' - \underline{\widetilde{\nu}})(\underline{\bar{\nu}}' - \underline{\widetilde{\nu}}^t\right] = \begin{bmatrix} \underline{\widetilde{S}}_{xx} & \underline{\widetilde{S}}_{xz} \\ \underline{\widetilde{S}}_{xz} & \underline{\widetilde{S}}_{zz} \end{bmatrix},$$

Now using the dummy variable u for the following representations

$$\underline{\widetilde{S}}_{uu} = \mathrm{diag}(\widetilde{S}^2_{u_i}),$$

where

$$\widetilde{S}^2_{u_i} = (\theta_{i+1} - \theta_i)\, S^2_{u_i} + \omega_{i+1} S^2_{u_i(2)},$$

and

$$\underline{\widetilde{S}}_{yu} = \begin{bmatrix} \widetilde{S}_{yu_1} & \widetilde{S}_{yu_2} & \ldots & \widetilde{S}_{yu_k} \end{bmatrix}^t,$$

where

$$\widetilde{S}_{yu_i} = (\theta_{i+1} - \theta_i) S_{yu_i} + \omega_{i+1} S_{yu_i(2)},$$

with

$$u_i = (x_i, z_i), \qquad i = 1, \ldots, k$$

$$\theta_i = \left(\frac{1}{m_i} - \frac{1}{N}\right), \quad \theta_{i+1} = \left(\frac{1}{m_{i+1}} - \frac{1}{N}\right), \quad \omega_{i+1} = \frac{N_{i+1}(b-1)}{m_{i+1}N}, \quad b > 1.$$

$$\underline{\widetilde{S}}_{uv} = \mathrm{diag}(\widetilde{S}_{u_i v_i}), \qquad (u_i = x_i, v_i = z_i).$$

where

$$\widetilde{S}_{u_i v_i} = (\theta_{i+1} - \theta_i)\, S_{u_i v_i} + \omega_{i+1} S_{u_i v_i(2)}.$$

Appendix B

Minimizing Eq.(4.12), the following terms are obtained

$$\begin{aligned} A = {} & \widetilde{S}^2_{z_1}\bar{S}^2_{z_1}\bar{S}^2_{z_2}\widetilde{S}_{yx_1} - \bar{S}^2_{z_1}\bar{S}^2_{z_2}\widetilde{S}_{yz_1}\widetilde{S}_{x_1z_1} - \widetilde{S}^2_{z_1}\bar{S}^2_{z_1}\widetilde{S}_{yz_2}\bar{\bar{S}}_{x_1z_2} - \widetilde{S}^2_{z_1}\widetilde{S}_{yx_1}\bar{S}^2_{z_1z_2} \\ & - \bar{S}^2_{z_1}\widetilde{S}_{yx_1}\bar{\bar{S}}^2_{z_1z_2} + \bar{S}^2_{z_1}\widetilde{S}_{yz_1}\bar{\bar{S}}_{x_1z_2}\bar{\bar{S}}_{z_1z_2} + \widetilde{S}^2_{z_1}\bar{S}_{yz_1}\bar{\bar{S}}_{x_1z_2}\bar{S}_{z_1z_2} \\ & + \bar{S}^2_{z_1}\bar{S}_{yz_2}\widetilde{S}_{x_1z_1}\bar{\bar{S}}_{z_1z_2} + \widetilde{S}_{yz_1}\widetilde{S}_{x_1z_1}\bar{S}^2_{z_1z_2} - \bar{S}_{yz_1}\widetilde{S}_{x_1z_1}\bar{S}_{z_1z_2}\bar{\bar{S}}_{z_1z_2}, \end{aligned}$$

$$\begin{aligned} B = {} & \widetilde{S}^2_{x_1}\bar{S}^2_{z_1}\bar{S}^2_{z_2}\widetilde{S}_{yz_1} - \bar{S}^2_{z_1}\bar{S}^2_{z_2}\widetilde{S}_{x_1z_1}\widetilde{S}_{yx_1} - \widetilde{S}^2_{x_1}\bar{S}^2_{z_1}\bar{\bar{S}}_{z_1z_2}\bar{S}_{yz_2} - \widetilde{S}^2_{x_1}\bar{S}^2_{z_1z_2}\widetilde{S}_{yz_1} \\ & - \bar{S}^2_{z_1}\bar{\bar{S}}^2_{x_1z_2}\widetilde{S}_{yz_1} + \bar{S}^2_{z_1}\bar{\bar{S}}_{x_1z_2}\bar{\bar{S}}_{z_1z_2}\widetilde{S}_{yx_1} + \widetilde{S}^2_{x_1}\bar{S}_{z_1z_2}\bar{\bar{S}}_{z_1z_2}\bar{S}_{yz_1} \\ & + \bar{S}^2_{z_1}\widetilde{S}_{x_1z_1}\bar{\bar{S}}_{x_1z_2}\bar{S}_{yz_2} + \widetilde{S}_{x_1z_1}\bar{S}^2_{z_1z_2}\widetilde{S}_{yx_1} - \widetilde{S}_{x_1z_1}\bar{S}_{z_1z_2}\bar{\bar{S}}_{x_1z_2}\bar{S}_{yz_1}, \end{aligned}$$

$$\begin{aligned} C = {} & \widetilde{S}^2_{x_1}\widetilde{S}^2_{z_1}\bar{S}^2_{z_2}\bar{S}_{yz_1} - \widetilde{S}^2_{x_1}\widetilde{S}^2_{z_1}\bar{S}_{z_1z_2}\bar{S}_{yz_2} - \widetilde{S}^2_{x_1}\bar{\bar{S}}^2_{z_1z_2}\bar{S}_{yz_1} - \widetilde{S}^2_{z_1}\bar{\bar{S}}^2_{x_1z_2}\bar{S}_{yz_1} \\ & - \bar{S}^2_{z_2}\widetilde{S}^2_{x_1z_1}\bar{S}_{yz_1} + \widetilde{S}^2_{x_1z_1}\bar{S}_{z_1z_2}\bar{S}_{yz_2} + \widetilde{S}^2_{z_1}\bar{\bar{S}}_{x_1z_2}\bar{S}_{z_1z_2}\widetilde{S}_{yx_1} + \widetilde{S}^2_{x_1}\bar{S}_{z_1z_2}\bar{\bar{S}}_{z_1z_2}\widetilde{S}_{yz_1} \\ & - \widetilde{S}^2_{x_1z_1}\bar{S}_{z_1z_2}\bar{\bar{S}}_{z_1z_2}\widetilde{S}_{yx_1} - \widetilde{S}^2_{x_1z_1}\bar{S}_{z_1z_2}\bar{\bar{S}}_{x_1z_2}\widetilde{S}_{yz_1} + 2\widetilde{S}^2_{x_1z_1}\bar{\bar{S}}_{z_1z_2}\bar{\bar{S}}_{x_1z_2}\bar{S}_{yz_1}, \end{aligned}$$

$$\begin{aligned} D = {} & \widetilde{S}^2_{x_1}\widetilde{S}^2_{z_1}\bar{S}^2_{z_1}\bar{S}_{yz_2} - \widetilde{S}^2_{x_1}\widetilde{S}^2_{z_1}\bar{S}_{z_1z_2}\bar{S}_{yz_1} - \widetilde{S}^2_{x_1}\bar{S}^2_{z_1}\bar{\bar{S}}_{z_1z_2}\widetilde{S}_{yz_1} - \widetilde{S}^2_{z_1}\bar{S}^2_{z_1}\bar{\bar{S}}_{x_1z_2}\widetilde{S}_{yx_1} \\ & - \bar{S}^2_{z_1}\widetilde{S}^2_{x_1z_1}\bar{S}_{yz_2} + \widetilde{S}^2_{x_1z_1}\bar{S}_{z_1z_2}\bar{S}_{yz_1} + \bar{S}^2_{z_1}\widetilde{S}_{x_1z_1}\bar{\bar{S}}_{z_1z_2}\widetilde{S}_{yx_1} + \bar{S}^2_{z_1}\widetilde{S}_{x_1z_1}\bar{\bar{S}}_{x_1z_2}\widetilde{S}_{yz_1}, \end{aligned}$$

$$\begin{aligned} E = {} & \widetilde{S}^2_{x_1}\widetilde{S}^2_{z_1}\bar{S}^2_{z_1}\bar{S}^2_{z_2} - (\widetilde{S}^2_{x_1}\widetilde{S}^2_{z_1}\bar{S}^2_{z_1z_2} + \widetilde{S}^2_{x_1}\bar{S}^2_{z_1}\bar{\bar{S}}^2_{z_1z_2} + \widetilde{S}^2_{z_1}\bar{S}^2_{z_1}\bar{\bar{S}}^2_{x_1z_2} \\ & + \bar{S}^2_{z_1}\bar{S}^2_{z_2}\widetilde{S}^2_{x_1z_1}) + \widetilde{S}^2_{x_1z_1}\bar{S}^2_{z_1z_2} + 2\bar{S}^2_{z_1}\widetilde{S}_{x_1z_1}\bar{\bar{S}}_{x_1z_2}\bar{\bar{S}}_{z_1z_2}. \end{aligned}$$

Appendix C

From (4.17), the vector of the squared multiple correlation coefficient $\underline{\widetilde{R}}^2_{y.xz}$ is explained as

$$\underline{\widetilde{R}}^2_{y.xz} = \widetilde{R}^2_{y.x_1z_1} + \widetilde{R}^2_{y.x_2z_2} + \cdots + \widetilde{R}^2_{y.x_\kappa z_\kappa},$$

where

$$\widetilde{R}^2_{y.x_iz_i} = \left(\frac{\widetilde{\rho}^2_{yx_i} + \widetilde{\rho}^2_{yz_i} - 2\widetilde{\rho}_{yx_i}\widetilde{\rho}_{yz_i}\widetilde{\rho}_{x_iz_i}}{1 - \widetilde{\rho}^2_{x_iz_i}} \right), \quad (i = 1, \ldots, \kappa),$$

$$\widetilde{\rho}^2_{yu_i} = \frac{\widetilde{S}^2_{yu_i}}{\widetilde{S}^2_y \widetilde{S}^2_{u_i}}, \qquad u_i = (x_i, z_i),$$

and

$$\widetilde{\rho}^2_{x_iz_i} = \frac{\widetilde{S}^2_{x_iz_i}}{\widetilde{S}^2_{x_i}\widetilde{S}^2_{z_i}}.$$

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Generalized modified linear systematic sampling scheme for finite populations

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Abstract

The present paper deals with a further modification on the selection of linear systematic sample, which leads to the introduction of a more generalized form of modified linear systematic sampling namely generalized modified linear systematic sampling (GMLSS) scheme, which is applicable for any sample size, irrespective of the population size whether it is a multiple of sample size or not. The performances of the proposed modified linear systematic sampling scheme are assessed with that of simple random sampling, circular systematic sampling for certain hypothetical populations as well as for some natural populations. As a result, it is observed that the proposed modified linear systematic sample means perform better than the simple random sample mean and circular systematic sample mean for estimating the population mean in the presence of linear trend among the population values. Further improvements on GMLSS are achieved by introducing Yates type end corrections.

Keywords: Linear Trend, Modified Systematic Sampling, Natural Population, Simple Random Sampling, Trend Free Sampling, Yates End Corrections, Yates Type End Corrections.

2000 AMS Classification: 62D05

1. Introduction

It is well known that in the presence of linear trend among the population values the systematic sampling performs better than the simple random sampling without replacement for estimating the population mean. Later several modifications have been made to improve the efficiency of the systematic sampling by introducing changes on the method of selection which includes centered systematic sampling by Madow [12], balanced systematic sampling by Sethi [15], modified systematic

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sampling by Singh, Jindal, and Garg [16] and changes on the estimators itself like Yates end corrections [26]. In all these sampling schemes mentioned above and also in the case of linear systematic sampling (LSS), it is assumed that the population size N is a multiple of sample size n and there is no restriction on the part of sample size. For a detailed discussion on estimation of finite population means, one may refer to Bellhouse and Rao [2], Chang and Huang [3], Cochran [4], Fountain and Pathak [5], Gupta and Kabe [6], Kadilar and Cingi [8], Khan et al. [9] [11], Murthy [14], Singh S. [17], Subramani [18] [19] [20] [21], Subramani and Singh [24] and the references cited therein. The circular systematic sampling (CSS) is an alternative to LSS whenever the population size is not a multiple of sample size with certain restrictions to get distinct units in the sample. For a discussion on the choices for the sampling interval for the case of CSS one may refer to Bellhouse [1], Sudakar [25] and Khan et al. [10]. Chang and Huang [3] have suggested a modification on linear systematic sampling and introduced the Remainder Linear systematic sampling (RLSS). The RLSS can be used for population size is not a multiple of sample size, where $N = n\widehat{k} + r$, $\widehat{k}$ is the sampling interval for RLSS and depends upon the remainder r. When the remainder is zero, the RLSS reduces to the usual linear systematic sampling.

Recently Subramani [22] [23] has introduced a modification on the selection of a systematic sample in the linear systematic sampling by choosing two random starts and is called as modified linear systematic sampling (MLSS) scheme. However the problem is that the MLSS is also applicable only for the cases where the population size is a multiple of sample size and is not valid when the population size is not a multiple of sample size. In this paper, a more general form of the MLSS is introduced which is applicable for any sample size whether it is even or odd and for any population size N, where $N = nk'$ or $N \neq nk'$, where k' is the sampling interval. For the sake of convenience, it is assumed that, without loss of generality, that $N = n_1k_1 + n_2k_2$ such that $n = n_1 + n_2$ and the value of $n_1 \geq n_2$. Further, it is shown that the MLSS method discussed by Subramani [22] [23] and the usual LSS are the particular cases of the proposed methods of generalized modified systematic sampling. The explicit expressions for the GMLSS sample means, the bias and the mean squared error are obtained for certain hypothetical populations with a perfect linear trend among the population values and are compared with that of simple random sampling. Further the relative performances of GMLSS are assessed with that of the simple random sampling, circular systematic sampling and remainder linear systematic sampling for certain natural populations. From the numerical comparison, it has been shown that the GMLSS perform better than the simple random sampling and circular systematic sampling for estimating the finite population means in the presence of linear trend. The entire above are explained with the help of hypothetical as well as natural populations.

2. Proposed Modified Systematic Sampling Scheme for any sample size

The proposed modified systematic sampling scheme is explained here, firstly with the help of examples and later the generalized case. For the sake of simplicity and for the benefit of the readers, selecting a modified linear systematic sample of size 5 from a population of size $N = 12$ and $N = 16$ is explained with the help of following examples.

2.1. Example. Let $N = 12$, $n = 5$, $n_1 = 3$, $k_1 = 2$, $n_2 = 2$, $k_2 = 3$

Step 1: Arrange the 12 population units as given below:

Arrangement of the Population units

i		j		
1	2	1	2	3
1	2	3	4	5
6	7	8	9	10
11	12			

Step 2: Select two random numbers $1 \leq i \leq 2$ and $1 \leq j \leq 3$; include all the elements in the columns corresponding to i and j. The selected samples are given below:

GMLSS Samples

Sample No.	i	j	Sampled Units
1	1	1	1,3,6,8,11
2	1	2	1,4,6,9,11
3	1	3	1,5,6,10,11
4	2	1	2,3,7,8,12
5	2	2	2,4,7,9,12
6	2	3	2,5,7,10,12

2.2. Example. Let $N = 12$, $n = 5$, $n_1 = 3$, $k_1 = 4$, $n_2 = 2$, $k_2 = 2$

Step 1: Arrange the 16 population units as given below:

Arrangement of the Population units

i				j	
1	2	3	4	1	2
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16		

Step 2: Select two random numbers $1 \leq i \leq 4$ and $1 \leq j \leq 2$; include all the elements in the columns corresponding to i and j. The selected samples are given below:

GMLSS Samples

Sample No.	i	j	Sampled Units
1	1	1	1,5,7,11,13
2	1	2	1,6,7,12,13
3	2	1	2,5,8,11,14
4	2	2	2,6,8,12,14
5	3	1	3,5,9,11,15
6	3	2	3,6,9,12,15
7	4	1	4,5,10,11,16
8	4	2	4,6,10,12,16

2.1. Generalized Modified Linear Systematic Sampling Scheme (GMLSS). The steps involved in selecting a generalized modified linear systematic sample (GMLSS) of size n from a population of size $N = n_1k_1 + n_2k_2$, where $n = n_1 + n_2$, k_1 and k_2 such that $k = k_1 + k_2$ are positive integers, are as follows:

1. Firstly arrange the N population units (labels) in a matrix with $k = k_1 + k_2$ columns as given in the Fig. 1. That is, the first n_2k population units are arranged row wise in the first n_2 rows with k elements each and the remaining $(n_1 - n_2)k_1$ population units are arranged row wise in the next $(n_1 - n_2)$ rows with k_1 elements each as in the arrangement given below in the Fig. 1.

2. The first k_1 columns are assumed as Set 1 and the next k_2 columns are assumed as Set 2.

3. Select two random numbers, i in between 1 and k_1 and j in between 1 and k_2, then select all the n_1 units in the i^{th} column of Set 1 and all the n_2 units in the j^{th} column of Set 2, which together give the sample of size n.

4. The step 3 leads to $k_1 \times k_2$ samples of size n.

Figure 1. Arrangement of the population units

$$
\begin{array}{cccccccccc}
1 & \cdots & i & \cdots & k_1 & k_1+1 & \cdots & k_1+j & \cdots & k \\
k+1 & \cdots & k+i & \cdots & k+k_1 & k+k_1+1 & \cdots & k+k_1+j & \cdots & 2k \\
2k+1 & \cdots & 2k+i & \cdots & 2k+k_1 & 2k+k_1+1 & \cdots & 2k+k_1+j & \cdots & 3k \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n_2-1)k+1 & \cdots & (n_2-1)k+i & \cdots & (n_2-1)k+k_1 & (n_2-1)k+k_1+1 & \cdots & (n_2-1)k+k_1+j & \cdots & n_2k \\
n_2k+1 & \cdots & n_2k+i & \cdots & n_2k+k_1 & & & & & \\
n_2k+k_1+1 & \cdots & n_2k+k_1+i & \cdots & n_2k+2k_1 & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & & \\
n_2k_2+(n_1-1)k_1+1 & \cdots & n_2k_2+(n_1-1)k_1+i & \cdots & n_2k_2+n_1k_1 & & & & &
\end{array}
$$

Since the generalized modified linear systematic sampling scheme has $k_1 \times k_2$ samples of size n and each unit in the Set-1 is included in k_1 samples and each unit in the Set-2 is included in k_2 samples, the first order and second order inclusion probabilities are obtained as given below:

$$\pi_i = \begin{cases} \frac{1}{k_1} & \text{if } i^{th} \text{ unit is from the Set-1,} \\ \frac{1}{k_2} & \text{if } i^{th} \text{ unit is from the Set-2.} \end{cases} \tag{2.1}$$

$$\pi_{ij} = \begin{cases} \frac{1}{k_1} & \text{if } i^{th} \text{ and } j^{th} \text{ units are from the same column of the Set-1,} \\ \frac{1}{k_2} & \text{if } i^{th} \text{ and } j^{th} \text{ units are from the same column of the Set-2,} \\ \frac{1}{k_1 \times k_2} & \text{if } i^{th} \text{ and } j^{th} \text{ units are from Set-1 and Set-2 respectively,} \\ 0 & \text{Otherwise.} \end{cases} \tag{2.2}$$

In general, for the given population size $N = n_1k_1 + n_2k_2$, the selected generalized modified linear systematic samples (labels of the population units) for the random starts i and j are given below:

$$S_{ij} = \begin{cases} i, i+k, i+2k, \cdots, i+(n_2-1)k, i+n_2k, i+n_2k+k_1, \cdots, i+n_2k+(n_1-n_2-1)k_1, \\ j+k_1, j+k_1+k, \cdots, j+k_1+(n_2-1)k \qquad (i = 1,2,3,\cdots,k_1 \text{ and } j = 1,2,3,\cdots,k_2) \end{cases} \tag{2.3}$$

The generalized modified linear systematic sample mean based on the random starts i and j is obtained as

$$\bar{y}_{gmlss} = \bar{y}_{ij} = \frac{1}{n}\left(\sum_{l=0}^{n_2-1} y_{i+kl} + \sum_{l=0}^{n_1-n_2-1} y_{i+n_2k+k_1l} + \sum_{l=0}^{n_2-1} y_{j+k_1+kl}\right) \tag{2.4}$$
$$(i = 1,2,3,\cdots,k_1 \text{ and } j = 1,2,3,\cdots,k_2)$$

Since the first order inclusion probabilities are not equal, the generalized modified linear systematic sample mean given above in Equation (2.4) is not an unbiased estimator. The mean squared error of the GMLSS mean can be obtained from the Equation (2.5) given below:

$$MSE\left(\bar{y}_{gmlss}\right) = \frac{1}{k_1k_2}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\left(\bar{y}_{ij} - \bar{Y}\right)^2 \tag{2.5}$$

3. Population with a Linear Trend

As stated earlier the linear systematic sampling (LSS) has less variance than the simple random sampling if the population consists solely of a linear trend among the population values. In this section, the relative efficiencies of the generalized modified systematic sampling schemes with that of simple random sampling for estimating the mean of finite populations with linear trend among the population values are assessed for certain hypothetical populations.

In this hypothetical population, the values of $N = n_1k_1 + n_2k_2$ population units are in arithmetic progression. That is,

$$Y_i = a + ib \qquad (i = 1,2,3,\cdots,N) \tag{3.1}$$

After a little algebra, one may obtain the generalized modified systematic sample means (GMLSS) with the random starts i and j and the population mean for the above hypothetical population are as given below:

$$\bar{y}_{gmlss} = \bar{y}_{ij} = a + \frac{1}{n}\left(n_1 i + n_2 j + n_2 k_1 + (k_1+k_2)n_2(n_1-1) + \frac{k_1(n_1-n_2)(n_1-n_2-1)}{2}\right) b \tag{3.2}$$
$$(i = 1,2,3,\cdots,k_1 \text{ and } j = 1,2,3,\cdots,k_2)$$

$$\bar{Y} = a + \left(\frac{k_1 n_1 + k_2 n_2 + 1}{2}\right) b \tag{3.3}$$

For the above population with a linear trend, variances of the simple random sample mean $V(\bar{y}_r)$ together with the Bias and Mean Squared Error of the generalized modified linear systematic sample means $Bias(\bar{y}_{gmlss})$ and $MSE(\bar{y}_{gmlss})$ are obtained as given below:

$$V(\bar{y}_r) = \frac{(N-n)(N+1)b^2}{12n} \tag{3.4}$$

$$Bias(\bar{y}_{gmlss}) = \frac{(k_1-k_2)n_2\{n_1-(n_2+1)\}}{2n} b \tag{3.5}$$

$$MSE(\bar{y}_{gmlss}) = \frac{1}{12n^2}\left\{n_1^2(k_1^2-1) - n_2^2(k_2^2-1) + 3n_2^2\{n_1-(n_2+1)\}^2(k_1-k_2)^2\right\} b^2 \tag{3.6}$$

Since the algebraic comparisons of the various expressions of the variance and the mean squared error given in Equations (3.4) and (3.6) are not possible due to the presence of several different parameters, we have compared them numerically and are presented in the Table 3.

3.1. Remark. If we put $k_1 = k_2$ in Equation (3.5), then $Bias(\bar{y}_{gmlss}) = 0$. That is, the Generalized modified linear systematic sample means become the modified linear systematic sample means of Subramani (2013a, b) and the resulting estimators are unbiased.

3.2. Remark. Even when $k_1 \neq k_2$ if we put $n_1 = n_2 + 1$ in Equation (3.5), the $Bias(\bar{y}_{gmlss}) = 0$. That is, the GMLSS estimators are unbiased estimators of the population mean.

3.3. Remark. When $n_2 = 0$ in Equation (3.5), the $Bias(\bar{y}_{gmlss}) = 0$. When $n_2 = 0$ which means $n_1 = n$, Equation (3.6) reduces to the variance of linear systematic sample mean. That is, the GMLSS estimators are unbiased estimators of the population mean and the resulting generalized modified linear systematic sample means become the linear systematic sample means.

3.4. Remark. The mean squared error of the Remainder Linear Systematic sample mean is given below:

$$MSE(\bar{y}_{rlss}) = \frac{1}{\widehat{k}(\widehat{k}+1)} \sum_{i=1}^{\widehat{k}} \sum_{j=1}^{\widehat{k}+1} (\bar{y}_{ij} - \bar{Y})^2 \tag{3.7}$$

where $\widehat{k}$ is the sampling inverval for RLSS.

3.5. Remark. Under the Remainder Linear systematic sampling procedure, the variance of Horvitz-Thompson estimator obtained by Chang and Huang [3] is given below:

$$V_{rlss}\left(\bar{y}_{HT}\right)=\frac{1}{N^2}\left\{(n-r)^2\widehat{k}^2\left\{\frac{1}{\widehat{k}}\sum_{i=1}^{\widehat{k}}\left(\bar{y}_{1i}-\bar{Y}_1\right)^2\right\}+r^2(\widehat{k}+1)^2\left\{\frac{1}{\widehat{k}+1}\sum_{i=1}^{\widehat{k}+1}\left(\bar{y}_{2i}-\bar{Y}_2\right)^2\right\}\right\} \tag{3.8}$$

where $\bar{y}_{1i}$ is the i^{th} sample mean of the Set-1

$\bar{y}_{2i}$ is the i^{th} sample mean of the Set-2

$\bar{Y}_1$ is the population mean of the first stratum

$\bar{Y}_2$ is the population mean of the second stratum

3.6. Remark. Under the Generalized modified linear systematic sampling, the variance of Horvitz-Thompson estimator is given below:

$$V_{gmlss}\left(\bar{y}_{HT}\right)=\frac{1}{N^2}\left\{n_1^2k_1^2\left\{\frac{1}{k_1}\sum_{i=1}^{k_1}\left(\bar{y}_{1i}-\bar{Y}_1\right)^2\right\}+n_2^2k_2^2\left\{\frac{1}{k_2}\sum_{i=1}^{k_2}\left(\bar{y}_{2i}-\bar{Y}_2\right)^2\right\}\right\} \tag{3.9}$$

where $\bar{y}_{1i}$ is the i^{th} sample mean of the Set-1

$\bar{y}_{2i}$ is the i^{th} sample mean of the Set-2

$\bar{Y}_1$ is the population mean of the Set-1

$\bar{Y}_2$ is the population mean of the Set-2

3.7. Remark. When $n_1 = (n-r)$, $k_1 = \widehat{k}$, $n_2 = r$, and $k_2 = \widehat{k}+1$, the generalized modified linear systematic sampling reduces to Remainder Linear systematic sampling for the population with a perfect linear trend. This shows that RLSS is the particular case of GMLSS for the population with a perfect linear trend. However the arrangement of the population units are different and hence the Set 1(2) and Stratum 1(2) are not one and the same.

4. Some Modifications on Generalized Modified Linear Systematic Sample Means

It has been shown in Section 3 that the estimators based on the generalized modified linear systematic sampling schemes (GMLSS) are, in general, not unbiased estimators. However, further improvements can be achieved by modifying the generalized modified linear systematic sample means as done by Subramani [21] in the case of modified linear systematic sample mean by introducing Yates type end corrections [26]. Consequently the proposed sampling scheme becomes completely trend free sampling (Mukerjee and Sengupta [13]).

4.1. Yates Type End Corrections for GMLSS Means. The modification involves the usual generalized modified linear systematic sampling, but the modified sample mean is defined as

$$\bar{y}^*_{gmlss} = \bar{y}_{gmlss} + \beta(y_1 - y_n) = \bar{Y} \tag{4.1}$$

That is, the units selected first and last are given the weights $\frac{1}{n}+\beta$ and $\frac{1}{n}-\beta$ respectively, whereas the remaining units will get the equal weight of $\frac{1}{n}$, so as to make the proposed estimator is equal to the population mean. That is, the value of β is obtained as

$$(4.2) \qquad \beta = \frac{\bar{Y} - \bar{y}_{gmlss}}{(y_1 - y_n)}$$

For the hypothetical population defined in Section 3, after a little algebra, we have obtained the value of β for the two random starts i and j as given below:

$$(4.3) \qquad \beta = \frac{(n_1 - n_2)\{k_1(n_2+1) - k_2n_2\} + 2\{k_2n_2 - n_1(i+1) - n_2(j+1)\}}{2n\{(i-j) - (k_1 + k(n_2-1))\}}$$

In the similar manner one can propose Yates type end corrections, alternate to Yates end corrections by giving different weights to two different units other than the first and last units. For example one may give different weights to two successive units; the first units of the two subsamples and so on.

If we give different weights to the two successive units between 1 and n_2 in Set 1 or successive units in Set 2 then the revised estimator $\bar{y}^{**}_{gmlss}$ for the case of generalized modified linear systematic sample mean $\bar{y}^{**}_{gmlss} = \bar{y}_{gmlss} + \beta_1(y_l - y_{l+1}) = \bar{Y}$, which yields the value of β_1 as is given below:

(4.4)

$$\beta_1 = \frac{\bar{Y} - \bar{y}_{gmlss}}{(y_l - y_{l+1})} = \frac{(n_2 - n_1)\{k_1(n_2+1) - k_2n_2\} - 2\{k_2n_2 - n_1(i+1) - n_2(j+1)\}}{2nk}$$

If we give different weights to the two successive units between $n_2 + 1$ and n_1 in Set 1 or the first units of the two sets then the revised estimator $\bar{y}^{***}_{gmlss}$ for the case of generalized modified linear systematic sample mean $\bar{y}^{***}_{gmlss} = \bar{y}_{gmlss} + \beta_2(y_{l'} - y_{l'+1}) = \bar{Y}$, which yields the value of β_2 as is given below:

(4.5)

$$\beta_2 = \frac{\bar{Y} - \bar{y}_{gmlss}}{(y_{l'} - y_{l'+1})} = \frac{(n_2 - n_1)\{k_1(n_2+1) - k_2n_2\} - 2\{k_2n_2 - n_1(i+1) - n_2(j+1)\}}{2nk_1}$$

If we give different weights to the first and the last units in the Set 1 then the revised estimator $\bar{y}^{****}_{gmlss}$ for the case of generalized modified linear systematic sample mean $\bar{y}^{****}_{gmlss} = \bar{y}_{gmlss} + \beta_3(y_1 - y_{n_1}) = \bar{Y}$, which yields the value of β_3 as is given below.

(4.6)

$$\beta_3 = \frac{\bar{Y} - \bar{y}_{gmlss}}{(y_1 - y_{n_1})} = \frac{(n_2 - n_1)\{k_1(n_2+1) - k_2n_2\} - 2\{k_2n_2 - n_1(i+1) - n_2(j+1)\}}{2n\{k_2n_2 + (n_1 - 1)k_1\}}$$

If we give different weights to the units $n_1 + 1$ and n in Set 2 (i.e. the first and the last units in Set 2), then the revised estimator $\bar{y}^{*****}_{gmlss}$ for the case of generalized modified linear systematic sample mean $\bar{y}^{*****}_{gmlss} = \bar{y}_{gmlss} + \beta_4(y_{l'} - y_{l'+1}) = \bar{Y}$, which yields the value of β_4 as is given below:

$$(4.7) \qquad \beta_4 = \frac{\bar{Y} - \bar{y}_{gmlss}}{(y_1 - y_{n_2})} = \frac{2(n_1i + n_2j) - N + n + n_1n_2(k_2 - k_1)}{2nk(n_2 - 1)}$$

4.1. Remark. In the presence of a perfect linear trend, the revised generalized modified linear systematic sampling estimators $\bar{y}^{*}_{gmlss} = \bar{y}^{**}_{gmlss} = \bar{y}^{***}_{gmlss} = \bar{y}^{****}_{gmlss} = \bar{y}^{*****}_{gmlss} = \bar{Y}$ and hence the variances are zero. That is, GMLSS becomes a trend free sampling (Mukerjee and Sengupta [13]).

5. Relative Performance of Modified Linear Systematic Sampling for Certain Natural Populations

The Generalized proposed modified linear systematic sampling schemes were introduced in Section 2, where as its means, bias and the mean squared error were derived in Section 3 for the hypothetical populations with a perfect linear trend among the population values. Further it has been shown (Remarks 3.1 to 3.3) that the usual linear systematic sampling (LSS) and the modified linear systematic sampling (MLSS) schemes are the particular cases of the proposed GMLSS scheme. Further it has been shown (Remark 3.7) that the RLSS is the particular case of the GMLSS for the population with a perfect linear trend. Since the expression of the mean squared error of GMLSS means is involved several parameters compared to simple random sampling without replacement, it is not feasible to make an algebraic comparison. Hence we have assessed the performances of GMLSS means with that of SRSWOR means, CSS means and RLSS means for a hypothetical population together with some natural populations considered by Murthy, p.228 [14]. The data were collected for estimating the output of 80 factories in a region. The data pertaining to the number of workers, fixed capital and the output are respectively denoted as population 2, population 3 and population 4, where as its labels (presuming a hypothetical population with a perfect linear trend, as considered as population 1. It is already established in Subramani [22] [23] that whenever the population size is a multiple of sample size the modified linear systematic sampling performs well compared to simple random sampling as well as linear systematic sampling. Hence we have obtained the variance of the simple random sample mean $V(\bar{y}_r)$, the variance of the circular systematic sample mean $V(\bar{y}_{css})$, the mean squared error of the remainder linear systematic sample mean, $MSE(\bar{y}_{rlss})$, the mean squared error of the generalized modified systematic sample mean, $MSE(\bar{y}_{gmlss})$, the variance of Horvitz Thompson estimator under RLSS, $V_{rlss}(\bar{y}_{HT})$ and the variance of Horvitz Thompson estimator under GMLSS, for the odd sample sizes from 7 to 25 so as the population size 80 is not a multiple of the sample size with various possible combinations of n_1 and n_2 such that $n_1 > n_2$ and are presented in Table 3. From the table values, it is seen that, out of the 40 cases considered the generalized modified linear systematic sample (GMLSS) means perform better than simple random sample mean and circular systematic sample mean in all the 40 cases. Further it is observed from Equation (3.6) that whenever the differences between k_1 and k_2 and n_1 and n_2 approaches zero simultaneously, then the efficiency of the GMLSS improves.

It is to be noted that for the fixed population size $N = 80$ there are 10 different choices for the sample of size $n = 11$. The mean squared error of the generalized modified linear systematic sample mean under the possible combinations of k_1, n_1, k_2 and n_2 for population 1 (population with a perfect linear trend) is given in the following Table 1:

Table 1. $MSE\left(\bar{y}_{gmlss}\right)$ under the possible combinations of k_1, n_1, k_2 and n_2 for population 1

N	n	k_1	n_1	k_2	n_2	$MSE\left(\bar{y}_{gmlss}\right)$
80	11	7	10	10	1	4.56
		4	9	22	2	98.56
		6	9	13	2	16.99
		8	9	4	2	8.32
		4	8	16	3	45.08
		7	8	8	3	2.80
		4	7	13	4	13.07
		8	7	6	4	3.04
		10	6	4	5	2.71
		5	6	10	5	2.30

From the possible combinations of k_1, n_1, k_2 and n_2, one can choose the optimum value of k_1, n_1, k_2 and n_2 such that the differences between k_1 and k_2, and n_1 and n_2 approaches zero simultaneously.

The different sample sizes with the population size $N = 80$ together with the sampling interval of Circular systematic sampling, Remainder linear systematic sampling and Generalized modified linear systematic sampling considered for the numerical comparisons of these sampling schemes are given below:

Table 2. Sample size and sampling interval of Circular systematic sampling (CSS), Remainder linear systematic sampling (RLSS) and Generalized modified linear systematic sampling (GMLSS)

N	n	CSS	RLSS				GMLSS			
		k^*	$\hat{k}$	$n-r$	r	$\hat{k}+1$	k_1	n_1	k_2	n_2
80	7	11	11	4	3	12	11	4	12	3
	9	9	8	1	8	9	8	5	10	4
	11	7	7	8	3	8	5	6	10	5
	13	6	6	11	2	7	8	7	4	6
	15	-	5	10	5	6	3	8	8	7
	17	5	4	5	12	5	4	11	6	6
	19	4	4	15	4	5	4	15	5	4
	21	-	3	4	17	4	4	17	3	4
	23	3	3	12	11	4	3	12	4	11
	25	3	3	20	5	4	3	20	4	5

where k^* is Sampling interval for Circular Systematic Sampling
and $\widehat{k}$ is Sampling interval for Remainder Linear Systematic Sampling

Table 3. Comparison of simple random sample, Circular systematic sample, Remainder Linear systematic sample, Horvitz-Thompson estimator based on RLSS, Generalized modified linear systematic sample means and Horvitz-Thompson estimator based on GMLSS for the four natural populations considered by Murthy (1967), page 228. The four populations are Population 1-Labels (presuming a hypothetical population with a perfect linear trend), Population 2-Number of workers, Population 3-Fixed capital and Population 4-The output

Population Number	n	$V(\bar{y}_r)$	$V(\bar{y}_{css})$	$MSE(\bar{y}_{rlss})$	$V_{rlss}(\bar{y}_{HT})$	$MSE(\bar{y}_{gmlss})$	$V_{gmlss}(\bar{y}_{HT})$
1	7	70.39	11.53	5.45	5.44	5.45	5.44
	9	53.25	6.59	5.33	5.45	3.25	3.38
	11	42.34	5.07	2.51	2.43	2.30	3.50
	13	34.79	3.41	2.18	2.11	1.79	2.69
	15	29.25	–	1.21	1.19	1.33	1.19
	17	25.01	3.84	1.10	1.20	0.89	0.97
	19	21.67	2.73	0.87	0.83	0.87	0.83
	21	18.96	–	0.84	0.92	0.84	0.92
	23	16.73	10.99	0.47	0.51	0.47	0.51
	25	14.85	2.85	0.48	0.45	0.48	0.45
2	7	9533.59	1908.46	1490.64	1638.59	1075.56	1024.33
	9	7211.86	1127.89	1089.76	1117.15	570.51	555.67
	11	5734.4	856.99	403.87	470.99	403.80	569.45
	13	4711.54	585.35	284.44	322.85	331.71	511.43
	15	3961.44	–	282.34	352.48	229.31	253.62
	17	3387.84	605.59	294.84	332.73	191.55	158.40
	19	2935.00	428.32	136.84	174.17	231.43	209.19
	21	2568.41	–	200.99	221.55	196.86	216.94
	23	2265.58	1458.65	150.12	197.91	95.33	95.01
	25	2011.20	420.57	75.82	105.19	138.26	121.73
3	7	93212.63	19244.93	12379.70	13481.46	10706.31	10267.41
	9	70512.45	11648.10	9916.21	10147.90	5716.93	5809.58
	11	56066.87	8139.36	2841.66	3074.49	6724.62	8492.78
	13	46066.09	7247.69	3141.28	3280.75	4227.09	6596.81
	15	38732.19	–	3759	4551.23	2479.96	3725.77
	17	33123.91	8457.46	4372.18	4901.98	1568.98	1529.61
	19	28696.32	4029.56	2373.67	2864.38	1934.37	1759.56
	21	25112.08	–	1336.18	1470.22	1534.75	1682.77
	23	22151.18	15277.04	820.73	1070.23	797.58	739.63
	25	19664.03	4967.74	326.32	411.64	2785.79	2459.51
4	7	439256.84	125737.19	71902.22	71792.05	80223.8	75178.72
	9	332284.09	67808.68	37887.33	37515.29	32907.60	30070.34
	11	264210.53	53569.57	20689.43	20220.72	25476.05	28633.15
	13	217082.67	37048.46	16479.84	16355.28	24579.86	39846.35
	15	182522.25	–	12708.17	13113.80	13021.36	22414.21
	17	156093.69	33489.79	10350.21	10625.27	8622.55	9824.10
	19	135229.03	23203.85	10372.55	10353.09	14168.90	13446.96
	21	118338.60	–	5886.70	6320.10	10217.66	11206.30
	23	104385.63	64036.67	3632.85	4219.70	6324.86	6336.19
	25	92665.14	19367.25	2507.05	2663.48	8598.95	7640.17

Table 4. Efficiency of the Generalized modified linear systematic sample means and Horvitz-Thompson estimator based on GMLSS for the 4 natural populations

Population Number	n	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9
1	7	12.92	2.12	1.00	1.00	12.94	2.12	1.00	1.00	1.00
	9	16.38	2.03	1.64	1.68	15.75	1.95	1.58	1.61	1.04
	11	18.41	2.20	1.09	1.06	12.10	1.45	0.72	0.69	1.52
	13	19.44	1.91	1.22	1.18	12.93	1.27	0.81	0.78	1.50
	15	21.99	–	0.91	0.89	24.58	–	1.02	1.00	0.89
	17	28.10	4.31	1.24	1.35	25.78	3.96	1.13	1.24	1.09
	19	24.91	3.14	1.00	0.95	26.11	3.29	1.05	1.00	0.95
	21	22.57	–	1.00	1.10	20.61	–	0.91	1.00	1.10
	23	35.60	23.38	1.00	1.09	32.80	21.55	0.92	1.00	1.09
	25	30.94	5.94	1.00	0.94	33.00	6.33	1.07	1.00	0.94
2	7	8.86	1.77	1.39	1.52	9.31	1.86	1.46	1.60	0.95
	9	12.64	1.98	1.91	1.96	12.98	2.03	1.96	2.01	0.97
	11	14.20	2.12	1.00	1.17	10.07	1.50	0.71	0.83	1.41
	13	14.20	1.76	0.86	0.97	9.21	1.14	0.56	0.63	1.54
	15	17.28	–	1.23	1.54	15.62	–	1.11	1.39	1.11
	17	17.69	3.16	1.54	1.74	21.39	3.82	1.86	2.10	0.83
	19	12.68	1.85	0.59	0.75	14.03	2.05	0.65	0.83	0.90
	21	13.05	–	1.02	1.13	11.84	–	0.93	1.02	1.10
	23	23.77	15.30	1.57	2.08	23.85	15.35	1.58	2.08	1.00
	25	14.55	3.04	0.55	0.76	16.52	3.45	0.62	0.86	0.88
3	7	8.71	1.80	1.16	1.26	9.08	1.87	1.21	1.31	0.96
	9	12.33	2.04	1.73	1.78	12.14	2.00	1.71	1.75	1.02
	11	8.34	1.21	0.42	0.46	6.60	0.96	0.33	0.36	1.26
	13	10.90	1.71	0.74	0.78	6.98	1.10	0.48	0.50	1.56
	15	15.62	–	1.52	1.84	10.40	–	1.01	1.22	1.50
	17	21.11	5.39	2.79	3.12	21.66	5.53	2.86	3.2	0.97
	19	14.83	2.08	1.23	1.48	16.31	2.29	1.35	1.63	0.91
	21	16.36	–	0.87	0.96	14.92	–	0.79	0.87	1.10
	23	27.77	19.15	1.03	1.34	29.95	20.65	1.11	1.45	0.93
	25	7.06	1.78	0.12	0.15	8.00	2.02	0.13	0.17	0.88
4	7	5.48	1.57	0.90	0.89	5.84	1.67	0.96	0.95	0.94
	9	10.10	2.06	1.15	1.14	11.05	2.26	1.26	1.25	0.91
	11	10.37	2.10	0.81	0.79	9.23	1.87	0.72	0.71	1.12
	13	8.83	1.51	0.67	0.67	5.45	0.93	0.41	0.41	1.62
	15	14.02	–	0.98	1.01	8.14	–	0.57	0.59	1.72
	17	18.10	3.88	1.20	1.23	15.89	3.41	1.05	1.08	1.14
	19	9.54	1.64	0.73	0.73	10.06	1.73	0.77	0.77	0.95
	21	11.58	–	0.58	0.62	10.56	–	0.53	0.56	1.10
	23	16.50	10.12	0.57	0.67	16.47	10.11	0.57	0.67	1.00
	25	10.78	2.25	0.29	0.31	12.13	2.53	0.33	0.35	0.89

where $R_1 = \frac{V(\bar{y}_r)}{MSE(\bar{y}_{gmlss})}$, $R_2 = \frac{V(\bar{y}_{css})}{MSE(\bar{y}_{gmlss})}$, $R_3 = \frac{MSE(\bar{y}_{rlss})}{MSE(\bar{y}_{gmlss})}$, $R_4 = \frac{V_{rlss}(\bar{y}_{HT})}{MSE(\bar{y}_{gmlss})}$, $R_5 = \frac{V(\bar{y}_r)}{V_{gmlss}(\bar{y}_{HT})}$, $R_6 = \frac{V(\bar{y}_{css})}{V_{gmlss}(\bar{y}_{HT})}$, $R_7 = \frac{MSE(\bar{y}_{rlss})}{V_{gmlss}(\bar{y}_{HT})}$, $R_8 = \frac{V_{rlss}(\bar{y}_{HT})}{V_{gmlss}(\bar{y}_{HT})}$, and $R_9 = \frac{V_{gmlss}(\bar{y}_{HT})}{MSE(\bar{y}_{gmlss})}$

6. Conclusion

In this paper a generalized version of modified linear systematic sampling (GMLSS) scheme is introduced irrespective of the sample size, whether it is odd or even and the population size is not a multiple of the sample size. The explicit expressions for the sample means and the mean squared error of the GMLSS estimators are derived for certain hypothetical populations with a perfect linear trend among the population values. The performances of the proposed sampling scheme are assessed with that of the simple random sampling without replacement, Circular systematic sampling and Remainder linear systematic sampling for a hypothetical population together with some natural populations considered by Murthy [14]. The comparative studies reveal that whenever there is a linear trend among the population values the generalized modified linear systematic sampling (GMLSS) performs well compared to simple random sampling. In the case of circular systematic sampling, the sample size and the population size must be a co-prime to get all distinct units in the sample. Hence the case with sample size 15 is not considered. Further it is observed that when $N = 80$ and $n = 21$ with $k = 4$ the first unit and the last unit are the same and hence this case is also not considered. From the above points, it is concluded that the proposed GMLSS is applicable for all possible combinations of population size and sample size even where the CSS fails. In comparing with the RLSS, the GMLSS performs better than the RLSS in majority of the cases.

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Cases of residual types in diagnostic checking for ARMA model

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Abstract

In this study, the residuals in time series analysis, which were classified in four different classes as "conditional residuals", "unconditional residuals", "innovation" and "normalized residuals", are calculated by a simulation study for the ARMA model under certain parameter values for different numbers of observation and their conditions in diagnostic checking are examined using the test statistic which belongs to Ljung Box.

Keywords: ARMA Models, Diagnostic Checking, Residual Types, Time Series.

1. Introduction

Box and Jenkins [2] worked on building and forecasting time series models and found out the method which is called Box-Jenkins Modelling Process in time series analysis. This approach has been improved by many studies for approximately forty years. Lutkepohl [8], Box et al. [3], Brockwell and Davis [4] and Wei [12] have popular text books about this subject. One of the most important points in the process of analyzing in time series is diagnostic checking. It can be determined which Box-Jenkins model is suitable for time series data in two ways. The first option is the examining of ACF and PACF plots of the time series. The second option tests the (H_0) null hypothesis where the autocorrelations of residuals are equal to zero in lag m. [1]. Thus, residual values play a significant role in diagnostic checking in time series models. Knowing the general structure of the residuals used in diagnostic checking is essential in order to apply this idea in practical applications effectively [9].

There are very little amount of study has been done about the residuals in literature. The residuals in time series analysis are classified in four different classes as, "conditional residuals", "unconditional residuals", "innovations" and "normalized residuals" in Mauricio [9]. Mauricio showed that when the considered model

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contains moving average roots near the unit circle and the number of observation is small, the unconditional residuals and normalized residuals give different decisions in diagnostic checking. In this study, the residuals in time series analysis, which were classified in 4 different classes as "conditional residuals", "unconditional residuals", "innovation" and "normalized residuals", are calculated by a simulation study for the $ARMA(1,1)$ model under certain parameter values for different numbers of observation and their conditions in diagnostic checking are examined using the test statistic which belongs to Ljung-Box.

The rest of this article is organized as follows. In Section 2, a description of the $ARMA(p,q)$ models used for modeling the time series and the Ljung Box test statistic used for diagnostic checking for the time series models are presented. Section 3 gives the details of the residual types adopted by Mauricio. A simulation study results are given for different lags, number of observations and parameter values for $ARMA(1,1)$ model and the different decisions in diagnostic checking are given for the resudial types in the realization study in Section 4. Finally, Section 5 concludes this research.

2. AUTOREGRESSIVE AND MOVING AVERAGE ($ARMA$) MODEL AND MODEL DIAGNOSTIG CHECKING

Autoregressive and moving average ($ARMA$) model is used in modeling stationary time series. The mentioned model is a combination of AR and MA models. This model is expressed with a certain number of values preceding time and residual series of data. If the model is made up of a combination of AR and MA model with p and q degrees respectively, $ARMA$ model is symbolized as $ARMA(p,q)$ and theoretically written as $\phi(B)\widetilde{W}_t = \theta(b)A_t$, where,

$$\phi(B) = 1 - \phi_1(B) - \phi_2(B)^2 - ... - \phi_p(B)^p$$

$$\theta(B) = 1 - \theta_1(B) - \theta_2(B)^2 - ... - \theta_p(B)^p$$

Invertibility and stationarity conditions for the model are, the roots of $\phi(B) = 0$ and $\theta(B) = 0$ be in the unit circle. $ARMA(1,1)$ model is written theoretically as,

$$\widetilde{W} = \phi_1\widetilde{W} + A_t - \theta_1 A_{t-1} \quad or \quad (1 - \phi_1 B)\widetilde{W}_t = (1 - \theta_1 B)A_t$$

Here, the invertibility and stationarity conditions are possible in $-1 < \phi_1 < 1$ and $-1 < \theta_1 < 1$ (Box et al., [3], Wei citeref:Wei). As mentioned above one of the approaches for diagnostic checking is using of test statistic which is based on residuals' sample autocorrelation functions. The null and alternative hypothesis for the test statistics as below:

$$\begin{cases} H_0: & Model\ is\ appropriate\ (\rho_1 = \rho_2 = ... = \rho_k = 0) \\ H_1: & Model\ is\ inappropriate \end{cases}$$

(ρ_i: Autocorrelation coefficients, $i = 1, 2, \ldots, k$).

In 1970, Box-Pierce derived the following Q_{BP} statistic having chi square distribution with approximate m degree of freedom for diagnostic checking citeref:BoxP.

$$Q_{BP} = n\sum_{k=1}^{m} r_k^2,\ k = 1, ..., m \tag{2.1}$$

Where n, number of observations, r_k, residuals' sample autocorrelation coefficients and m is number of lags. Ljung and Box [6] suggested to use the Q_{LB} statistic as below,

$$(2.2) \qquad Q_{LB} = n(n+2)\sum_{k=1}^{m} r_k^2/(n-k)$$

The autocorrelation coefficient of the residuals, which is also used for deriving the Ljung-Box test statistic expressed in the equation 2.2 above is used.

$$(2.3) \qquad r_k = \sum_{t=k+1}^{n} u_t u_{t-k} / \sum_{t=1}^{n} u_t^2 \quad , \ k = 1, \ldots, m, \ t = 1, \ldots, n$$

Here, $r = (r_1, r_2, \ldots, r_m)$ has multivariate normal distribution with zero average, $V(r_k) = (n-k)/n(n+2)$ and $Kov(r_k, r_1) = 0 \ \ (k \neq 1)$, so autocorrelation coefficient of the residuals has chi-square distribution with m degree of freedom for large n values [2]. Since the distribution of residual autocorrelations is $N(0, n^{-1}I_m)$, then distributions of both Ljung-Box's Q_{LB} and Box-Pierce's Q_{BP} are chi-square distributions with m degree of freedom and their expected values are m and their variances are $2m$. The expected value of Q_{LB} is m for finite numbers of n values, whereas,

$$(2.4) \qquad E(Q_{BP}) = \frac{nm}{n+2}\left(1 - \frac{m+1}{2n}\right)$$

As long as n be less than m, $E(Q_{BP})$ will be less than m. Moreover, for n values which are greater than m, variances of Q_{BP} and Q_{LB} are written as follows,

$$V(Q_{BP}) = 2m(1 + (m-10)/n), \quad V(Q_{LB}) = 2m(1 + (m-10)/n)$$

Here, the variance of Q_{LB} exceeds $2m$ value, however Monte Carlo simulation method has shown that its distribution is much closer than the distribution of Q_{BP} to chi-square with m degree of freedom [2], [6]. The test statistics, which are used for diagnostic checking, are called Portmanteau test statistics. There are a lot of Portmanteau test statistics in the literature and detailed information about structures and distributions of other Portmanteau test statistics could be found in many studies [2], [6], [7] [5], [1].

3. DEFINING AND CALCULATING OF RESIDUAL TYPES FOR $ARMA(p, q)$

Let see stationary time series process $\{W_t\}$ following the model and let $w = [w_1, w_2, ..., w_n]'$ generated by $\{W_t\}$.The theoretical representation of ARMA (Autoregressive - Moving Average) model is given below:

$$(3.1) \qquad \phi(B)\widetilde{W}_t = \theta(B)A_t$$

Here, $\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$ and $\theta(B) = 1 - \sum_{i=1}^{q} \theta_i B^i$ are polynomials with degrees of p and q, also B is a lag operator,$\widetilde{W}_t = W_t - E[W_t]$ and $\{A_t\}$ is a white noise process with $\sigma^2 > 0$. Regarding the model (3.1) , $\widetilde{W} =$

$[\widetilde{W_1},...,\widetilde{W_n}]'$, $A = [A_1,...,A_n]'$ and $U_* = [\widetilde{W}_{1-p},...,\widetilde{W}_0, A_{1-q},...,A_0]'$, observed time series $w = [w_1, w_2, ..., w_n]'$ can be seen as a particular realization of a random vector $W = [W_1, W_2, ..., W_n]'$ following the model,

$$D_\phi \widetilde{W} = D_\theta A + VU_* \tag{3.2}$$

Where D_ϕ and D_θ are $n \times n$ parameter matrices with ones as diagonal elements and $-\phi_j$ and $-\theta_j$ as elements that constitute the jth subdiagonal, respectively, and V is a $n \times (p+q)$ matrix with $V_{ij} = \phi_{p+i-j}(i = 1,...,p; j = 1,...,p)$ and $V_{ij} = -\theta_{q+i-j+p}(i = 1,...,q; j = p+i,...,p+q)$, where the remaining elements are zero [9].

Let us assume that the theoretical autocovariance matrix is $\sum_w = \sigma^{-2}E[\widetilde{W}\widetilde{W}']$, and $\hat{\sum}_w$ is an estimation of $\sum_w$. The autocovariance matrix can be given as follows from the equation (3.2) [9];

$$\sum_w = D_\phi^{-1}(D_\phi D_\theta' + V\Omega V')(D_\phi^{-1})' = K^{-1}(I + Z\Omega Z)(K^{-1})' \tag{3.3}$$

In equation (3.3) , $K = D_\theta^{-1}D_\phi, Z = -D_\theta^{-1}V$ and $\Omega = \sigma^{-2}E[U_*U_*']$ are parameter matrices of dimensions $n \times n$, $n \times (p+q)$ and $(p+q) \times (p+q)$, respectively, with Ω being readily expressible in terms of $\phi_1,...,\phi_p, \theta_1,...,\theta_q$, for example, in Ljung and Box [7]. Besides, here $\sum_0 = I + Z\Omega Z' = [I - Z(\Omega^{-1} + Z'Z)^{-1}Z']^{-1}$.

Using the relation (3.3), $\widetilde{w}' \hat{\sum}_w^{-1} \widetilde{w}$ can be written as,

$$\widetilde{w}' \hat{\sum_w}^{-1} \widetilde{w} = \widetilde{w}' \hat{K}' (I + \hat{Z}\hat{\Omega}\hat{Z})\hat{K}\widetilde{w} \tag{3.4}$$

$\hat{K}, \hat{Z}$ and $\hat{\Omega}$ symbolize the estimations of parameter matrices defined in equation (3.3). According to these theoretical information, residuals have been grouped by Mauricio [9] in 4 different classes as below:

3.1. Conditional Residuals. Conditional residuals are associated with relation (3.4) and defined as the elements of the $n \times 1$ vector $\hat{a}_0 = \hat{K}\tilde{w}$.

3.2. Unconditional Residuals. Unconditional residuals are associated with (3.4) and defined as the elements of the $n \times 1$ vector $\hat{a} = (I + \hat{Z}\hat{\Omega}\hat{Z}'^{-1}\hat{K}\tilde{w} = \hat{\sum}_0^{-1} \hat{a}_0$.

3.3. Innovations. Innovation residuals are associated with (3.4) and defined as the elements of the $n \times 1$ vector $\hat{e} = \hat{L}^{-1}\tilde{w} = (\hat{K}\hat{L})^{-1}\hat{a}_0$. Here, $\hat{L}$ is the estimation of the $n \times n$ unit lower- triangle matrix L in below factorization,

$\sum_w = LFL'$, or,

$$\sum_w = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ L_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_n \end{pmatrix} \begin{pmatrix} 1 & L_{12} & \cdots & L_{1n} \\ 0 & 1 & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

where $F_t > 0$, and $t = 1, 2, ..., n$.

From relation (3.4), the below equations will be result [9]:

$$\tilde{w}'\hat{\sum}_{w}^{-1}\tilde{w} = \hat{a}_0'\hat{\sum}_{0}^{-1}\hat{a}_0 = \hat{e}'\hat{F}^{-1}\hat{e}$$

where $\hat{F}$, is the estimation of matrix F.

3.4. Normalized Residuals. If we define lower-triangle matrix P as, $\sum_0 = I + Z\Omega Z' = PP'$, the definition of vector $\hat{v}$(normalized residuals) is,

$$\hat{v} = \hat{P}^{-1}\hat{a}_0 = \hat{P}'\hat{a} = \hat{F}^{-\frac{1}{2}}\hat{e}$$

where $\hat{P}$ is the estimation of matrix P [9], [10], [11].

4. SIMULATION, REALIZATION AND REAL WORLD DATA APPLICATION STUDIES

4.1. Simulation Study.

In this part, conditions of the residual types in diagnostic checking are compared by using Ljung-Box test statistic for $ARMA(1,1)$ model. Here, the main aim is not to test whether the model is appropriate or not, but it is to compare whether the types of residuals lead us to the same results for diagnostic checking or not. Data are generated under assumption of $ARMA(1,1)$ models which have autoregressive and moving average parameter values with $\phi = 0.5$ *and* $\theta = 0.6$, $\phi = 0.9$ *and* $\theta = 0.6$, $\phi = 0.1$ *and* $\theta = 0.9$, $\phi = 0.2$ *and* $\theta = 0.9$. By considering these four different parameter conditions, the simulation study has been done and types of residuals are calculated for all generated time series. In this simulation study, to see the rejection ratio of the null hypothesis "H_0: Model is appropriate", trials are repeated 100 times and total number of rejections are divided by 100. Furthermore, to see interaction between number of observation and rejection ratio, n=10, n=25, n=50, n=100, n=250 and n=500 observation numbers are taken and result are in Figure 1-4. Lags(m) are taken as 2,3,5,10.

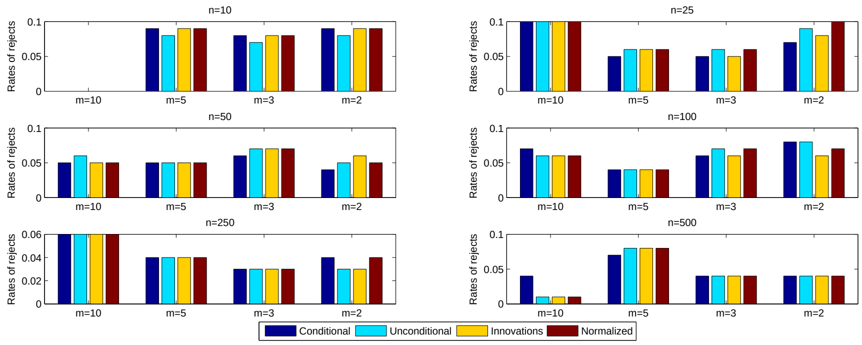

Figure 1. H_0 rejection ratio for $\phi = 0.5$ and $\theta = 0.6$,

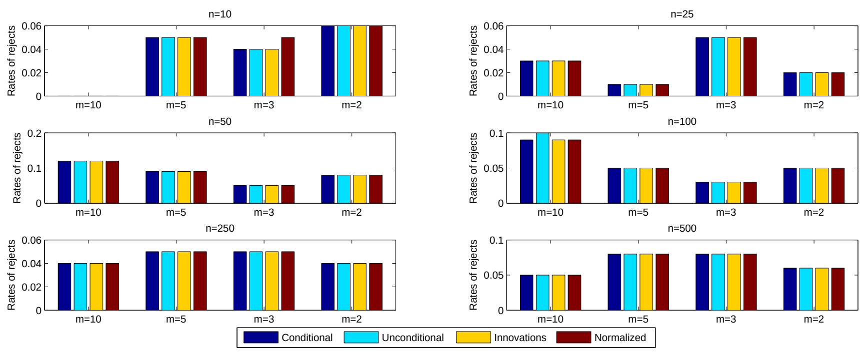

Figure 2. H_0 rejection ratio for $\phi = 0.9$ and $\theta = 0.6$

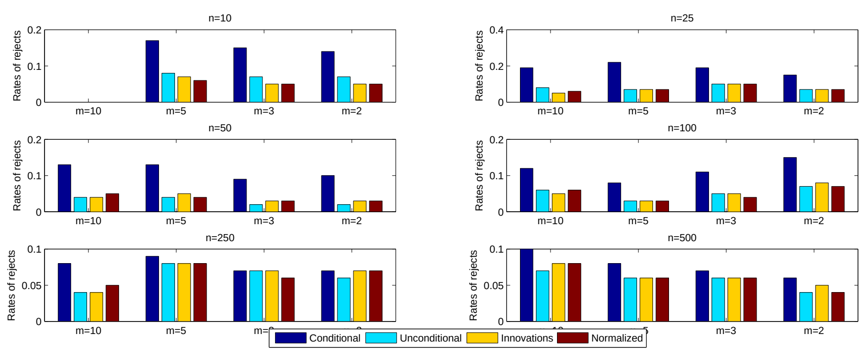

Figure 3. H_0 rejection ratio for $\phi = 0.1$ and $\theta = 0.9$

As seen in Figure 1, when autoregressive and moving average parameter values are $\phi = 0.5$ and $\theta = 0.6$, there is no difference between rejection ratio in four residual types with different number of observations and lags. Also in Figure 2, with $\phi = 0.9$ and $\theta = 0.6$, the same sitiation is valid. In Figure 3, $\phi = 0.1$ and

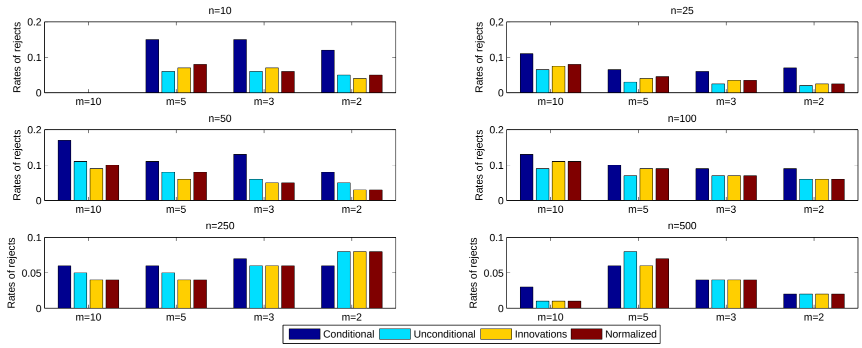

Figure 4. H_0 rejection ratio for $\phi = 0.2$ and $\theta = 0.9$

$\theta = 0.9$, the rejection ratio of conditional residuals is higher than the other types of residuals. And when the number of observation increases, the rejection ratios of four different types of residuals, are close to each other for all of the lags. In Figure 4, $\phi = 0.2$ and $\theta = 0.9$, the same situations as Figure 3 appear, rejection ratios of the conditional residuals have high values, rather to other residual types. Also the differences of the rejection ratios vanish when the number of observation increases.

Similar to Mauricio [9], as it can be seen from the bar charts (Figure 1-4), when the number of observations is small and $ARMA(1,1)$ model is close to non invertibility, conditional residuals have higher rejection ratios than the other three residual types. As mentioned above, these differences vanish when the number of observation increases. And when the number of observation values are large, the ratios of rejection, which are calculated from the different types of residual, are close to each other for all of the lags and parameter values. This situation is also lead us to use large number of observations in the modeling process.

For the negative values of and , the same situations are valid in diagnostic checking, because the differences of calculated residual types are related with non-invertible MA parameter value, so non-invertibility condition is provided not only MA parameter close to 1, but also close to -1. The invertibility condition is satistified when $-1 < \theta_1 < 1$.

Simulation results are consistent with the findings of Mauricio [9].

4.2. Realization Study.

The realization data is produced under the assumptation of suitable model, we consider the suitable model is $ARMA(1,1)$ model which have autoregressive and moving average parameter values with $\phi = 0.1$ and $\theta = 0.9$ in data production process. The number of observations is taken as 25. The realization data and calculated values of residual types are in Table 1.

Unit Number	Residual Types				
	Data	Conditional	Unconditional	Innovations	Normalized
1	2.6795	2.6795	0.9476	2.6795	2.0882
2	-1.3419	0.8017	-0.757	-0.1452	-0.1264
3	0.3349	1.1906	-0.2122	0.3699	0.3384
4	-0.3621	0.6768	-0.5866	-0.1171	-0.11
5	1.0322	1.6768	0.5405	0.9754	0.9322
6	1.0101	2.4159	1.3933	1.7087	1.6518
7	-2.0718	0.0014	-0.9189	-0.7357	-0.717
8	1.9296	2.1381	1.3098	1.5079	1.4781
9	-0.4807	1.2506	0.5051	0.6303	0.6205
10	0.4625	1.6361	0.9652	1.0604	1.0474
11	-0.6746	0.7517	0.1479	0.2103	0.2083
12	1.0333	1.7772	1.2338	1.2863	1.2764
13	-0.3375	1.1587	0.6696	0.699	0.6947
14	1.6726	2.7492	2.309	2.3277	2.3162
15	-2.673	-0.366	-0.7622	-0.766	-0.7629
16	0.6017	0.5396	0.1831	0.1851	0.1845
17	-2.2197	-1.7942	-2.1151	-2.1144	-2.1089
18	1.2676	-0.1253	-0.4141	-0.4036	-0.4027
19	0.7519	0.5124	0.2525	0.2634	0.2629
20	-0.1922	0.1938	-0.0401	-0.0311	-0.031
21	0.8995	1.0931	0.8825	0.8908	0.8898
22	0.9274	1.8212	1.6317	1.6373	1.6359
23	-1.8327	-0.2864	-0.4569	-0.4544	-0.4541
24	2.3883	2.3139	2.1604	2.1632	2.162
25	-1.4132	0.4305	0.2923	0.2926	0.2925

Table1. Realization Data and Calculated Residual Values for Residual Types

Lag	Residual Types			
	Conditional	Unconditional	Innovations	Normalized
m=10	43.42	9.55	8.88	9.87
m=5	25.89	3.63	4.2	4.47
m=3	18.33	3.38	3.71	4.17
m=2	14.31	3.33	3.66	4.11

Table2. Calculated Box-Jenkins Test Statistic Values for Different Residual Types and Lags

Lag	Residual Types			
	Conditional	Unconditional	Innovations	Normalized
m=10	Rejected	Not Rejected	Not Rejected	Not Rejected
m=5	Rejected	Not Rejected	Not Rejected	Not Rejected
m=3	Rejected	Not Rejected	Not Rejected	Not Rejected
m=2	Rejected	Not Rejected	Not Rejected	Not Rejected

Table 3. Decision Cases of H_0 Hypothesis For Different Residual Types and Lags

In Table 2, there are the calculated chi square values for different types of residual and different lags. The calculated chi square values from the conditional residuals give significant chi square test statistic value. This situation leads us to reject the "H_0: Model is appropriate" hypothesis when the conditional residuals are considered in diagnostic checking contrary to the other residuals as seen in Table 3.

4.3. Real World Data Application. In real world data application study, we used Gross Domestic Product data of Turkey (between 1998,Q1 and 2010,Q2), and considered that the estimated autocorrelation and moving average parameters are $\phi = 0.9$ and $\theta = 0.5$, respectively.

This identification yields the approximate model of order (1,1)

$$(1 - 0.9B)\widetilde{w}_t = (1 - 0.5B)a_t$$

The results of diagnostic checking (approximately values of test statistic and decision cases) could be found in Table 4-5.

Lag	Residual Types			
	Conditional	Unconditional	Innovations	Normalized
m=10	230	290	240	244
m=5	130	160	138	145
m=3	80	90	86	87
m=2	55	60	57	58

Table 4. Calculated Box-Jenkins Test Statistic Values for Different Residual Types and Lags

Lag	Residual Types			
	Conditional	Unconditional	Innovations	Normalized
m=10	Rejected	Rejected	Rejected	Rejected
m=5	Rejected	Rejected	Rejected	Rejected
m=3	Rejected	Rejected	Rejected	Rejected
m=2	Rejected	Rejected	Rejected	Rejected

Table 5. Decision Cases of H_0 Hypothesis For Different Residual Types and Lags

According the results in Table 4-5, it could be seen that the residuals lead us to have same decision in diagnostic checking as in the real word application study.

5. CONCLUSION

According to simulation study, especially when the number of observations are small and ARMA model's parameters are close to non invertibility, the differences between the calculated chi square test statistic values are becoming significant, and the ratios of rejection differ. The values of test statistics which are calculated from conditional residuals are bigger than the other three types of residuals. For ARMA(1,1) models which have parameter values close to non invertibility situations, using of conditional residuals give big ratio of the rejection than the other residuals types in diagnostic checking, but as the number of observation n increases, rates of rejections between the different types of residuals are in tendency to decrease. When the trials are done with the other parameter values which are not close to non invertible or stationary situation, four types of residuals give similar rates of rejection in the simulation studies. The moving average parameter in realization study is close to non-invertibility situation, thus "H_0: Model is appropriate" hypothesis is rejected, when the conditional residuals are considered in diagnostic checking contrary to the other residuals as seen in Table 3. On the contrary, there is no difference between four residual types according to diognostic checking results as in the real world data. As it can be seen from the bar charts (Figure 1-4), when the number of observations is small and $ARMA(1,1)$ model is close to non invertibility, conditional residuals have higher rejection ratios than the other three residual types.

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