UJMA

# Universal Journal of Mathematics and Applications

VOLUME II
ISSUE I

ISSN 2619-9653

http://dergipark.gov.tr/ujma

VOLUME II ISSUE I
ISSN 2619-9653

March 2019
http://dergipark.gov.tr/ujma

# UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

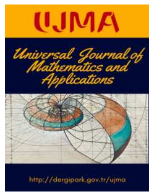
# Contents

*Universal Journal of Mathematics and Applications*, 2 (1) (2019) 1-10
***Research paper***

UJMA

**Universal Journal of Mathematics and Applications**

*Journal Homepage: www.dergipark.gov.tr/ujma*
*ISSN 2619-9653*
*DOI: http://dx.doi.org/10.32323/ujma.520682*




# Special Graceful Labelings of Irregular Fences and Lobsters

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**Article Info**

***Keywords:*** *Fence, Graceful labelling, Lobster.*
***2010 AMS:*** *05C30, 05C78.*
***Received:*** *2 February 2019*
***Accepted:*** *6 March 2019*
***Available online:*** *20 March 2019*

**Abstract**

Irregular fences are subgraphs of $P_m \times P_n$ formed with $m$ copies of $P_n$ in such a way that two consecutive copies of $P_n$ are connected with one or two edges; if two edges are used, then they are located in levels separated an odd number of units. We prove here that any of these fences admits a special kind of graceful labeling, called $\alpha$-labeling. We show that there is a huge variety of this type of fences presenting a closed formula to determine the number of them that can be built on the grid $[1,m] \times [1,n]$. If only one edge is used to connect any pair of consecutive copies of $P_n$, the resulting graph is a tree. We use the $\alpha$-labelings of this type of fences to construct and label a subfamily of lobsters, partially answering the long standing conjecture of Bermond that states that all lobsters are graceful. The final labeling of the lobsters presented here is not only graceful, it is an $\alpha$-labeling, therefore they can be used to produce new graceful trees.

## 1. Introduction

Suppose $G$ is a graph of order $n$ and size $m$. An injective function $f : V(G) \to \{0,1,\dots,m\}$ is called a *graceful labeling* of $G$ if every edge $uv$ of $G$ has assigned a *weight*, defined by $|f(u) - f(v)|$, and the set of all weights induced by $f$ on the edges of $G$ is $\{1,2,\dots,m\}$. A graph that admits a graceful labeling is called *graceful*. This labeling, together with three other labelings, was introduced by Rosa [1] as a mean to study a problem in combinatorial design associated with the decomposition of the complete graph $K_{2m+1}$ into copies of any tree of size $m$. Rosa proved that if there is a graceful labeling of a tree $T$ of size $m$, then there exists a (cyclic) decomposition of $K_{2m+1}$ into copies of $T$. Several applications of gracefully labeled graphs are known, we can mentioned here the ones presented by Bloom and Golomb [2] and [3], and the ones given by Brankovic and Wanless [4].

An $\alpha$*-labeling* of $G$ is a graceful labeling $f$ for which there exists an integer $\lambda$, called the *boundary value* of $f$, such that for each edge $uv$ of $G$, either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$. If $G$ admits an $\alpha$-labeling, then it is called an $\alpha$*-graph*. This definition of an $\alpha$-graph implies that $G$ is bipartite and $\lambda$ is the smaller of the two vertex labels that yield the weight 1. This type of labeling is the most restrictive one among the four labelings introduced by Rosa [1]. The existence of an $\alpha$-labeling implies the existence of several other types of labelings; so, they are located at the center of this research area. Not all graphs are graceful or $\alpha$, this fact motivates the search of new families of graphs admitting these types of labelings.

Let $G$ be a graph of order $n$ and size $m$. Suppose that $f$ is a graceful labeling of $G$. The labeling $\bar{f} : V(G) \to \{0,1,\dots,m\}$, defined as $\bar{f} = m - f(v)$ for every $v \in V(G)$, is called the *complementary* labeling of $f$; this is also a graceful labeling; thus, its existence can be used to prove that the number of graceful labelings of any graph is always even. Let $g$ be a labeling of $G$ defined as $g(v) = c + f(v)$ for every $v \in V(G)$; we say that $g$ is a *c units shifting* of $f$. It is not difficult to see that both, $f$ and $g$, induce the same weights. Suppose now that $f$ is an $\alpha$-labeling of $G$ with boundary value $\lambda$; the labeling $h$, defined for every $v \in V(G)$ as

$$h(v) = \begin{cases} f(v) & \text{if } f(v) \leq \lambda, \\ d - 1 + f(v) & \text{if } f(v) > \lambda, \end{cases}$$

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is called a *d-graceful labeling* of $G$. This type of labeling was introduced in 1982, independently, by Maheo and Thuillier [5] and Slater [6].

Suppose that $f(v)-f(u)=w>0$, then $h(v)-h(u)=d-1+f(v)-f(u)=d-1+w$. Since $1\leq w\leq m$, we get that $d\leq d-1+w\leq d-1+m$. In other terms, the weights induced by $h$ on the edges of $G$ are $d,d+1,\dots,d-1+m$. This property of the $\alpha$-labelings has been widely used to construct new graceful and $\alpha$-graphs starting with smaller $\alpha$-graphs. The *reverse* of $f$, denoted by $f_r$, is another $\alpha$-labeling of $G$, it is defined as

$$f_r(v)=\begin{cases}\lambda-f(v) & \text{if } f(v)\leq\lambda,\\ m+\lambda+1-f(v) & \text{if } f(v)>\lambda.\end{cases}$$

Note that $f$ and $f_r$ have the same boundary value; in addition, if $f(v)-f(u)=w$, for any weight $w\in\{1,2,\dots,m\}$, then $f_r(v)-f_r(u)=m+\lambda+1-f(v)-\lambda+f(u)=m+1-(f(v)-f(u))=m+1-w$.

In Section 2 we present an $\alpha$-labeling for a large family of connected subgraphs of the grid $P_m\times P_n$. This family, denoted by $\mathscr{F}$, is formed by all the graphs built in the following way:

For every $i\in\{1,2,\dots,m\}$, let $P_n^i$ be the path of order $n$ with vertex set $V(P_n^i)=\{v_{i,0},v_{i,1},\dots,v_{i,n-1}\}$ and edge set $E(P_n^i)=\{v_{i,0}v_{i,1}, v_{i,1}v_{i,2},\dots,v_{i,n-2}v_{i,n-1}\}$. Now, for every $i\in\{1,2,\dots,m-1\}$, decide whether $P_n^i$ is connected to $P_n^{i+1}$ with one or two edges (also called *links*). If only one edge connects them, then choose any $j\in\{0,1,\dots,n-1\}$ and connect with an edge the vertices $v_{i,j}$ and $v_{i+1,j}$. If two edges connect them, then choose $j_1,j_2\in\{0,1,\dots,n-1\}$, where $|j_2-j_1|$ is odd, and introduce the edges $v_{i,j_1}v_{i+1,j_1}$ and $v_{i,j_2}v_{i+1,j_2}$. Given that the number of edges connecting two copies of $P_n$ may vary, we refer to this type of graph as an *irregular fence*. In Figure 1.1 we show all the nonisomorphic fences in $\mathscr{F}$ built on $[1,3]\times[1,4]$. We claim that all the irregular fences are $\alpha$-graphs.

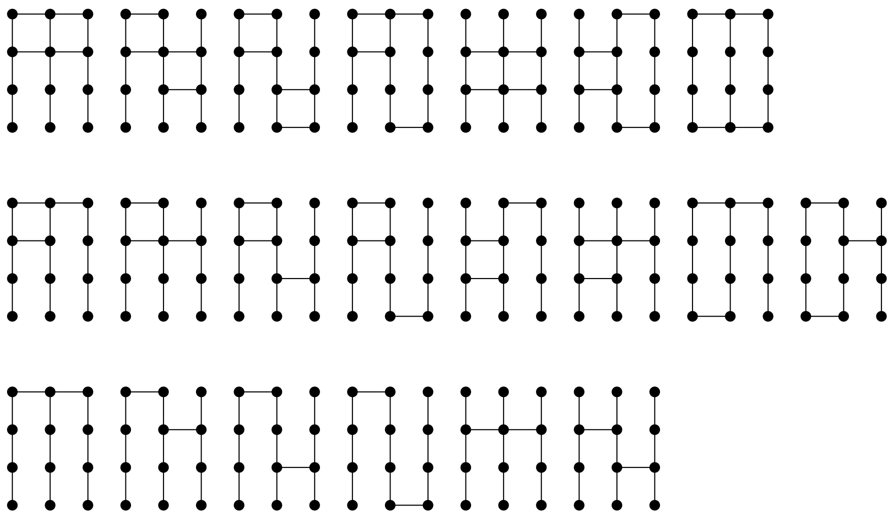


**Figure 1.1:** All nonisomorphic irregular fences built on $[1,3]\times[1,4]$

In Section 3 we study this type of irregular fences from an enumerative perspective. We present a closed formula for the number of nonisomorphic irregular fences built on $[1,m]\times[1,n]$. When every pair of consecutive copies of $P_n$ is connected with only one edge, the resulting fence corresponds to a type of tree called *path-like tree*; it is known that they are $\alpha$-trees [7]. In Section 4 we consider a subfamily of the path-like trees built on $[1,m]\times[1,5]$, with the extra property that they are lobsters. We characterize the lobsters that are irregular fences, therefore, $\alpha$-trees; in addition we show that some other $\alpha$-lobsters can be obtained from them by adding pendant vertices to some or all the vertices at distance one from the central path.

All graphs considered in this work are simple, i.e., no loops nor multiple edges are allowed. We mainly follow the notation and terminology used in [8] and [9].

## 2. $\alpha$-labelings of irregular fences

As we mentioned before, $\alpha$-labelings were introduced by Rosa [1]; he presented a labeling scheme for caterpillars that can be easily adapted for the case of paths. For the sake of completeness, we present here Rosa's $\alpha$-labeling of the path $P_n$; we use this labeling in the construction of the $\alpha$-labeled irregular fences.

**Lemma 2.1.** *For every $n\geq 1$, the path $P_n$ is an $\alpha$-graph.*

Assuming that $V(P_n)=\{v_0,v_1,\dots,v_{n-1}\}$ and $E(P_n)=\{v_0v_1,v_1v_2,\dots,v_{n-2}v_{n-1}\}$, the $\alpha$-labeling $f:V(P_n)\to\{0,1,\dots,n-1\}$ is defined as:

$$f(v_i)=\begin{cases}\frac{i}{2} & \text{if } i \text{ is even},\\ n-\frac{i+1}{2} & \text{if } i \text{ is odd}.\end{cases}$$

The labeling $f$ has boundary value $\lambda = \frac{n-2}{2}$ when $n$ is even and $\lambda = \frac{n-1}{2}$ when $n$ is odd. Moreover, $f(v_0) = 0$ regardless the parity of $n$ but $f(v_{n-1}) = \frac{n}{2} = \lambda + 1$ when $n$ is even and $f(v_{n-1}) = \frac{n-1}{2} = \lambda$ when $n$ is odd. We say that $v \in V(P_n)$ is a *black* vertex if $f(v) \le \lambda$, otherwise $v$ is a *white* vertex. In Figure 2.1 we show two examples of this labeling on $P_{12}$ and $P_{17}$. Just for the examples, the boundary value is on a red vertex.

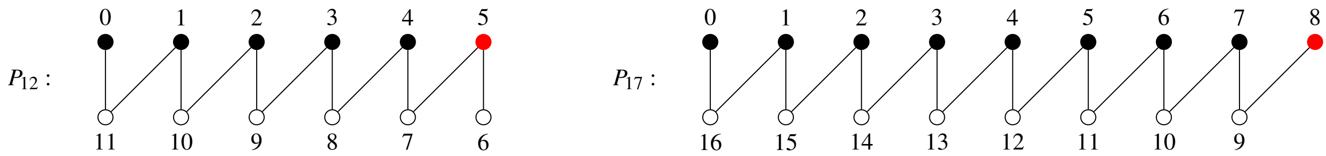



**Figure 2.1:** $\alpha$-labelings of $P_{12}$ and $P_{17}$

The construction of the $\alpha$-labeled irregular fences, built on $P_m \times P_n$, is based on an embedding of the path $P_{mn}$ on the grid $[1,m] \times [1,n]$. The division algorithm tell us that for each $i \in \{1,2,\ldots,mn\}$, there exist unique $q$ and $r$ such that $i = qn + r$, where $0 \le r < n$. Using this fact we can define the embedding of $P_{mn}$ on the grid $[1,m] \times [1,n]$ to be the bijective function $\phi : \{v_0, v_1, \ldots, v_{mn-1}\} \to [1,m] \times [1,n]$, where

$$\phi(v_i) = \begin{cases} (q+1, r+1) & \text{if } q \text{ is even,} \\ (q+1, n-r) & \text{if } q \text{ is odd.} \end{cases}$$

Once the embedding is done, we proceed to label the vertices of $P_{mn}$ using the function $f$ given in Lemma 2.1. In the first part of Figure 2.2 we show an embedding of $P_{15}$ on the grid $[1,5] \times [1,3]$, on the second part we exhibit the $\alpha$-labeling of this path at this embedding.

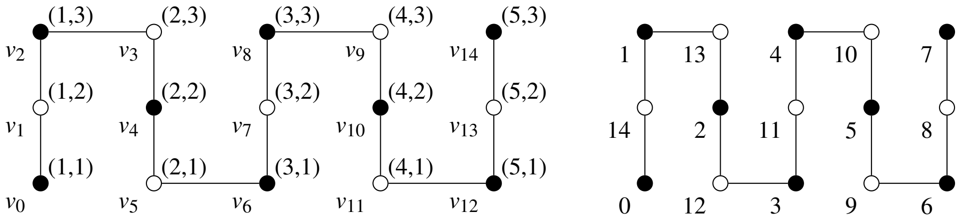
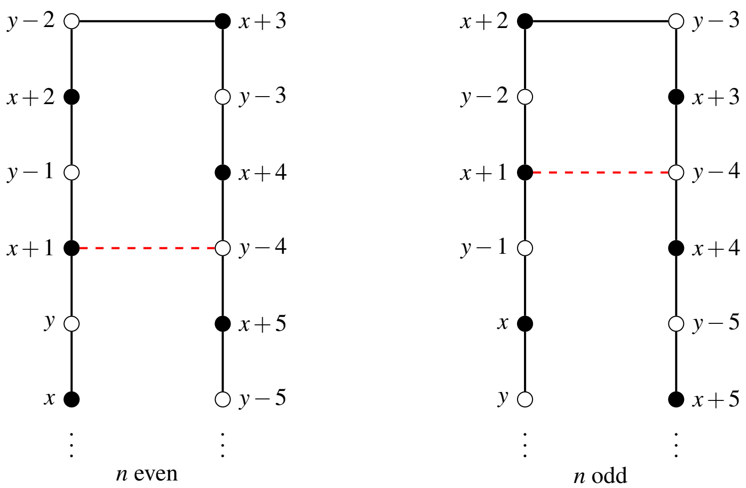



**Figure 2.2:** Embedding of $P_{15}$ on $[1,5] \times [1,3]$ and its $\alpha$-labeling

In the following lemmas we present the essential results that will allow us to prove that any irregular fence in $\mathscr{F}$ is an $\alpha$-graph.

**Lemma 2.2.** *Any fence built on $[1,2] \times [1,n]$, with only one edge connecting the two copies of $P_n$, is an $\alpha$-graph.*

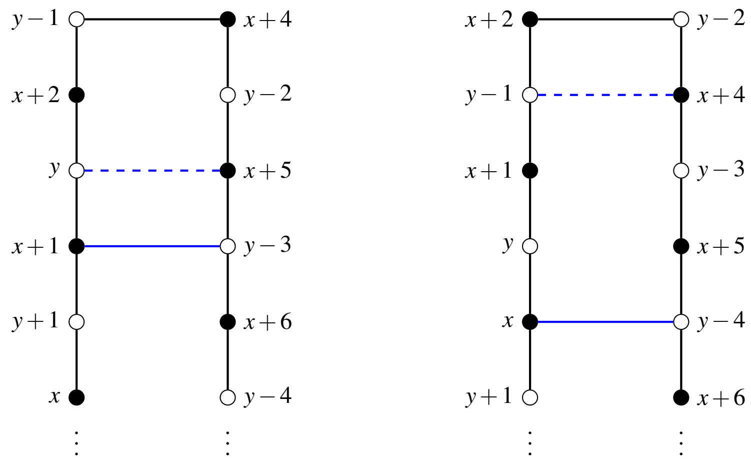
*Proof.* Suppose that $P_{2n}$ has been embedded in the grid $[1,2] \times [1,n]$ in the way described before. In addition, assume that $P_{2n}$ has been labeled using the function $f$ given in Lemma 2.1. In the following diagram we show this labeling where the labels on the black vertices are at most $\lambda$, the boundary value of $f$, while the labels on the white vertices are at least $\lambda + 1$. Note that the edge connecting the vertices on $(1,n)$ and $(2,n)$ has weight $y - x - 5$, independently of the parity of $n$.




If for any feasible value of $t$, the vertices on $(1, n-t)$ and $(2, n-t)$ are connected, the new edge also has weight $y - x - 5$. This implies that all the horizontal "edges" on this embedding of $P_{2n}$ have the same weight and any of them can be used to connect the two copies of $P_n$, being the final fence an $\alpha$-graph. $\square$

**Lemma 2.3.** *Any fence built on $[1,2]\times[1,n]$, with two edges connecting the two copies of $P_n$, is an $\alpha$-graph.*

*Proof.* As we did in Lemma 2.2, suppose that $P_{2n}$ has been embedded in the grid $[1,2]\times[1,n]$, in the way described before, and that it has been labeled using the $\alpha$-labeling $f$ in Lemma 2.1. In the following diagram, we show new labelings for the two copies of $P_n$.




These labelings are obtained from $f$ by fixing the labels on the black vertices of the first copy of $P_n$ and adding one unit to all other vertices. In this way, the edges on the first copy of $P_n$ have the weights $n+2, n+3, \ldots, 2n$; the weights on the edges of the second copy of $P_n$ are $1, 2, \ldots, n-1$. We use all the labels in $\{0, 1, \ldots, 2n\}$ except $\lceil \frac{n}{2} \rceil$. Since the white vertices on the second copy of $P_n$ were augmented one unit while the black vertices on the first copy were fixed, any line connecting a black vertex with a white vertex will be an edge of weight $y-x-4$. Similarly, any line connecting a white vertex with a black vertex will be an edge of weight $y-x-5$ because the labels of both endvertices were augmented one unit. Hence, by connecting both copies of $P_n$ with two edges, one of each kind, that is, one black-white and one white-black, we obtain an $\alpha$-labeled irregular fence. This fence is in $\mathscr{F}$ because these types of edges are in alternated levels. This concludes the proof. □

In Figure 2.3 we show four examples of these labeled irregular fences, two for each lemma.

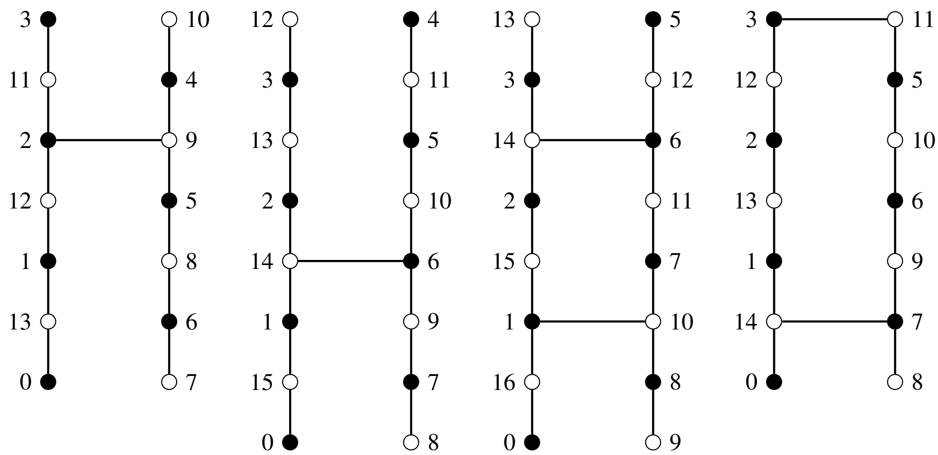



**Figure 2.3:** $\alpha$-labelings of four irregular fences

**Theorem 2.4.** *If $G$ is an irregular fence in $\mathscr{F}$, then $G$ is an $\alpha$-graph.*

*Proof.* Suppose that $G$ is an irregular fence built on $P_m \times P_n$ such that it contains $1 \le k \le m-1$ pairs of consecutive copies of $P_n$ connected by two edges. Thus, $G$ has size $m(n-1)+(m-1)+k = mn-1+k$. Assume that the path $P_{mn}$ has been labeled using the function $f$ in Lemma 2.1 and is embedded in the grid $[1,m]\times[1,n]$. Thus, the weights induced on the edges of every copy of $P_n$ are consecutive integers, and the horizontal edges, of this embedding of $P_{mn}$, have weights $(m-1)n, (m-2)n, \ldots, 2n, n$.
Now we delete all the horizontal edges connecting consecutive copies of $P_n$ in $P_{mn}$. Once this is done, we draw new horizontal edges following the pattern in $G$, In this way, we have a labeling of $G$; based on Lemma 2.2, this is an $\alpha$-labeling when $G$ is a tree, that is, when only one edge connects any pair of consecutive copies of $P_n$. If this is not the case, i.e., when there are $k>0$ pairs of consecutive copies of $P_n$ connected with two edges, these two horizontal edges have the same weight. To eliminate this duplicity, we apply the procedure used in the proof of Lemma 2.3.
Suppose that $i_1, i_2, \ldots, i_k$ are the indices for which there are two horizontal edges connecting $P_n^{i_j}$ and $P_n^{i_j+1}$. For every $i \le i_j$, the labels of the black vertices of all $P_n^i$ are fixed and all the other labels are augmented in one unit. In this way, these horizontal edges have different

weights that are consecutive integers. Once this process has been applied to every pair of consecutive copies of $P_n$ connected by two edges, the resulting labeling is indeed an $\alpha$-labeling of $G$. In fact, since there are exactly $k$ pairs of consecutive copies of $P_n$ connected by two edges, the original labels of the white vertices have been shifted $k$ units, avoiding the duplicity of vertex labels; the weights on each copy of $P_n$ are consecutive integers, and the weights on the horizontal edges complement the ones on the vertical edges. Therefore, the final labeling of $G$ is an $\alpha$-labeling and $G$ is an $\alpha$-graph. $\square$

In Figure 2.4 we show an example of this labeling where $G$ is built on $P_{10} \times P_{10}$ and $k = 7$.

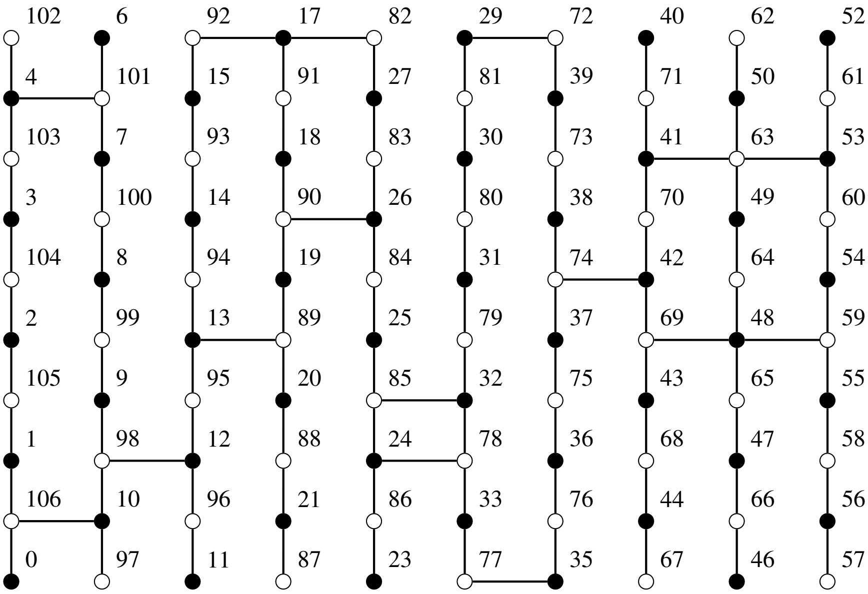



**Figure 2.4:** $\alpha$-labelings of a fence of size 106 built on $[1,10] \times [1,10]$

## 3. Enumerating irregular fences

Motivated by the result in the previous section, we want to determine the number of this type of fences. In [10], we found the number of fences that can be built on the grid $[1,m] \times [1,n]$. Using that result, we present here a closed formula for the number of nonisomorphic irregular fences built on the grid.

We start by counting the number of irregular fences that can be built on $[1,2] \times [1,n]$. Since the grid $[1,m] \times [1,n]$ can be seen as a linear amalgamation of $m-1$ copies of $[1,2] \times [1,n]$ we refer to the fences on $[1,2] \times [1,n]$ as *building blocks*, or just *blocks*, of $[1,m] \times [1,n]$. Thus, a block in an irregular fence consists of two copies of $P_n$ and 1 or 2 (horizontal) links (edges), It is not difficult to see that the number of blocks with only one link is $C(n,1) = n$, i.e., the number of ways of selecting one element from $\{1,2,\ldots,n\}$. To determine the number of blocks with two links we may count the 2-element subsets of $\{1,2,\ldots,n\}$, such that the difference between the two elements is odd. Thus, for any subset $\{i,j\}$, with $i < j$, the possible values for $j$ are determined by the value of $i$. When $i$ is odd, there are $\lfloor \frac{n}{2} \rfloor - \frac{i-1}{2}$ possible values for $j$. When $i$ is even, there are $\lceil \frac{n}{2} \rceil - \frac{i}{2}$ possible values for $j$.

Hence, when $n$ is even, the number of 2-element subsets satisfying the conditions is given by

$$\sum_{i=1}^{\frac{n}{2}} i + \sum_{i=1}^{\frac{n}{2}-1} i = 2\sum_{i=1}^{\frac{n}{2}-1} i + \frac{n}{2} = \frac{2(\frac{n}{2}-1)\frac{n}{2}}{2} + \frac{n}{2} = \frac{n}{2}\left(\frac{n}{2} - 1 + 1\right) = \frac{n^2}{4}.$$

When $n$ is odd, this number is

$$2\sum_{i=1}^{\frac{n-1}{2}} i = \frac{2(\frac{n-1}{2})(\frac{n+1}{2})}{2} = \frac{n^2-1}{4}.$$

Therefore, the number of blocks is $n + \frac{n^2}{4} = \frac{n^2+4n}{4}$ when $n$ is even and $n + \frac{n^2-1}{4} = \frac{n^2+4n-1}{4}$ when $n$ is odd. For $n \geq 1$, the sequence $a(n)$ formed by these values corresponds to the sequence A002620 in OEIS [11].

Another number needed in our counting process is the number of symmetric blocks. Once again, we start analyzing the case where the block has exactly one link. If $n$ is even, there are no symmetric blocks. If $n$ is odd, there is only one symmetric block. We have a similar situation when the block has two links. When $n$ is odd there are no two numbers $i < j$ in $\{1,2,\ldots,n\}$ such that $j-i$ is odd and $i-1 = n-j$. When $n$ is even, for every $1 \leq i \leq \frac{n}{2}$, the number $j = n+1-i$ belongs to $\{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, n\}$, $j-i = n+1-i-i = n+1-2i$ is odd and $i-1 = n-j = n-(n+1-i) = i-1$. Then, if $s(n)$ denotes the number of symmetric blocks, we get

$$s(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

For $n \geq 1$, the sequence $s(n)$ is the sequence A152271 in OEIS [12].
Now we turn our attention to the general case. Given any irregular fence $F$ built on $[1,m] \times [1,n]$, there are other three fences that are isomorphic to $F$: when $F$ is rotated $180^\circ$ around a central vertical axis, when $F$ is rotated $180^\circ$ around a central horizontal axis, and when $F$ is rotated $180^\circ$ around a central axis perpendicular to the plane containing $F$. Thus, there are three possible situations: $F$ has four different representations, $F$ has two different representations, or $F$ has one representation. Let $T$ be the set of all irregular fences on $[1,m] \times [1,n]$; we define $V$ to be the subset of $T$ containing the fences with a vertical symmetry, $H$ to be the subset of $T$ containing the fences with a horizontal symmetry, $C$ to be the subset of $T$ containing the fences with a central symmetry, and $A$ to be the subset of $T$ containing the fences with all these symmetries. In Figure 3.1 we show four examples, one for each of these subsets.

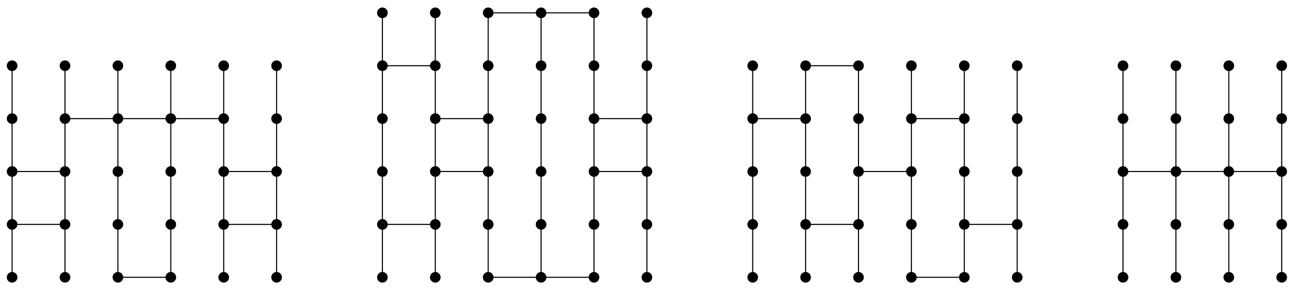



**Figure 3.1:** Different types of symmetric fences

Since the fences in $A$ have all the described symmetries, each of them appears only once in the list of all possible fences built on $[1,m] \times [1,n]$. Every element of $V - A$, $H - A$, or $C - A$ appears twice in this list. Every nonsymmetric fence appears four times in the list. Thus, if we take the addition of cardinalities

$$|T| + |V| + |H| + |C|$$

every fence is counted four times. Therefore, the number of nonisomorphic irregular fences built on $[1,m] \times [1,n]$ is given by

$$f(m,n) = \frac{1}{4}(|T| + |V| + |H| + |C|).$$

In order to find a closed formula for $f(m,n)$ we just need to determine explicitely these four cardinalities.

Based on the number of blocks and symmetric blocks, found above, and the fact that $[1,m] \times [1,n]$ can be formed with $m-1$ copies of $[1,2] \times [1,n]$, we can say that

$$|T| = \begin{cases} (\frac{n^2+4n}{4})^{m-1} & \text{if } n \text{ is even,} \\ (\frac{n^2+4n-1}{4})^{m-1} & \text{if } n \text{ is odd.} \end{cases}$$

If $F$ is a fence in $V$, then its $i$th block is identical to its $(m-i)$th block. This implies that we need to determine the number of posibilities for the first $\lfloor \frac{m}{2} \rfloor$ blocks. Thus,

$$|V| = \begin{cases} (\frac{n^2+4n}{4})^{\lfloor \frac{m}{2} \rfloor} & \text{if } n \text{ is even,} \\ (\frac{n^2+4n-1}{4})^{\lfloor \frac{m}{2} \rfloor} & \text{if } n \text{ is odd.} \end{cases}$$

If $F \in H$, each block in $F$ must be symmetric. So,

$$|H| = \begin{cases} (\frac{n}{2})^{m-1} & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

When $F \in C$, there are two cases that we need to analyze that depend on the parity of $m$. Recall that in this case the $i$th block of $F$ is represented up side down in the $(m-i)$th block.

If $m$ is even and $i = \frac{m}{2}$, then $i = m - i$. This implies that the $i$th block of $F$ must be symmetric. So,

$$|C| = \begin{cases} (\frac{n^2+4n}{4})^{\frac{m-2}{2}} \cdot \frac{n}{2} & \text{if } n \text{ is even,} \\ (\frac{n^2+4n-1}{4})^{\frac{m-2}{2}} \cdot 1 & \text{if } n \text{ is odd.} \end{cases}$$

If $m$ is odd

$$|C| = \begin{cases} \left(\frac{n^2+4n}{4}\right)^{\frac{m-1}{2}} & \text{if } n \text{ is even,} \\ \left(\frac{n^2+4n-1}{4}\right)^{\frac{m-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Thus, we have found a closed formula for $F(m,n)$. We summarize these results in the following theorem.

**Theorem 3.1.** *The number $f(m,n)$ of nonisomorphic irregular fences built on $[1,m]\times[1,n]$ is:*

- *When both m and n are even.*
  $f(m,n) = \frac{1}{4}\left(\left(\frac{n^2+4n}{4}\right)^{m-1} + \left(\frac{n^2+4n}{4}\right)^{\frac{m}{2}} + \left(\frac{n}{2}\right)^{m-1} + \left(\frac{n^2+4n}{4}\right)^{\frac{m-2}{2}} \cdot \frac{n}{2}\right)$
- *When m is even and n is odd.*
  $f(m,n) = \frac{1}{4}\left(\left(\frac{n^2+4n-1}{4}\right)^{m-1} + \left(\frac{n^2+4n-1}{4}\right)^{\frac{m}{2}} + 1 + \left(\frac{n^2+4n-1}{4}\right)^{\frac{m-2}{2}}\right)$
- *When m is odd and n is even.*
  $f(m,n) = \frac{1}{4}\left(\left(\frac{n^2+4n}{4}\right)^{m-1} + \left(\frac{n^2+4n}{4}\right)^{\frac{m-1}{2}} + \left(\frac{n}{2}\right)^{m-1} + \left(\frac{n^2+4n}{4}\right)^{\frac{m-1}{2}}\right)$
- *When both m and n are odd*
  $f(m,n) = \frac{1}{4}\left(\left(\frac{n^2+4n-1}{4}\right)^{m-1} + \left(\frac{n^2+4n-1}{4}\right)^{\frac{m-1}{2}} + 1 + \left(\frac{n^2+4n-1}{4}\right)^{\frac{m-1}{2}}\right)$

In Table 1, read by rows, we show the first values of $f(m,n)$ for $2 \le m,n \le 10$. We have omitted the cases where $m=1$ or $n=1$ because $f(m,n)=1$.

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 3 | 5 | 6 | 9 | 10 | 14 | 15 | 20 |
| 3 | 4 | 9 | 21 | 36 | 66 | 100 | 160 | 225 | 330 |
| 4 | 10 | 39 | 150 | 366 | 918 | 1810 | 3640 | 6315 | 11100 |
| 5 | 25 | 169 | 1060 | 3721 | 12789 | 32761 | 83296 | 177241 | 375925 |
| 6 | 70 | 819 | 8360 | 40626 | 190917 | 620830 | 1994944 | 5134095 | 13143500 |
| 7 | 196 | 3969 | 65808 | 443556 | 2849526 | 11764900 | 47783680 | 148718025 | 459591750 |
| 8 | 574 | 19719 | 525600 | 4875786 | 42730578 | 223502230 | 1146718720 | 4312651995 | 16085261781 |
| 9 | 1681 | 97969 | 4196416 | 53597041 | 640749609 | 4245955921 | 27519010816 | 125061956881 | 562969695625 |
| 10 | 5002 | 489219 | 33564800 | 589530846 | 9611072577 | 80672576050 | 660454273024 | 3626791798575 | 19703925162500 |

**Table 1:** Initial values for the numebr $f(m,n)$ of nonisomorphic irregular fences builon $[1,m]\times[1,n]$

## 4. Lobsters with an $\alpha$-labeling

A *lobster* is a tree with the property that the removal of all its leaves results in a caterpillar, and a *caterpillar* is a tree with the property that the removal of all its leaves results in a path. We refer to this path as the *central path* of the lobster. An alternative definition was given in [13]. Let $P$ be any of the longest paths in a tree $T$; $T$ is called a *k-distance* tree if every vertex is at distance at most $k$ from $P$. Thus, paths are 0-distance trees, caterpillars are 1-distance trees, and lobsters are 2-distance trees.

It was conjectured by Bermond [14] that all lobsters are graceful. Several families of graceful lobsters are known. Using the construction of Stanton and Zarnke [15] it is possible to obtain a graceful labeling of any lobster constructed by attaching, to every vertex of a path, a leaf of the star $K_{1,n}$. Burzio and Ferrarese [16] proved that any tree obtained from a graceful tree by replacing each edge with a path of fixed length is graceful. Thus, if the starting tree is a caterpillar and every edge is replaced with a path of length 2, the resulting graph is a lobster. This is one of the strongest results in this area, the weakest part is that the distance between any two leaves, at distance two, is always even. This problem is solved in the work of Wang et al. [17], as well as in the series of articles of Mishra and Panagrahi [18], [19], [20], and [21]. In all these papers, the lobsters considered share the property that all the vertices in the central path have degree larger than two and the subtrees attached to them must satisfy some structural conditions. Morgan [13] proved that all lobsters with a perfect matching are graceful. In a similar line, Krop [22] showed the same for lobsters with an almost perfect matching.

In this section we explore lobsters that are path-like trees and how to use the $\alpha$-labeling, given in Section 2, to produce new $\alpha$-labeled lobsters.

Suppose that the path $P_{5m}$ has been labeled using the labeling in Lemma 2.1, and embedded in the grid $[1,m]\times[1,5]$, as we did in Section 2. Thus, every column in this embedding is a copy of $P_5$; moreover, the labeling of the $i$th copy of $P_5$ is a $d_i$-graceful labeling shifted $c_i$ units, where $d_i = n(m-i)+1$ and

$$c_i = \begin{cases} \frac{n(i-1)}{2} & \text{if } i \text{ is odd,} \\ \frac{n(i-1)+1}{2} & \text{if } i \text{ is even.} \end{cases}$$

We claim that when every copy of $P_5$ is replaced by a copy of any caterpillar of diameter four, the result still holds; that is, we can concatenate the central vertices of these caterpillars to obtain a lobster with an $\alpha$-labeling. In Figure 4.1 we show the labeling scheme given by Rosa [1] to get an $\alpha$-labeling of a caterpillar of size $n-1$.

Let $G$ be a caterpillar of diameter 4 and order $n$. If all the leaves of $G$ are deleted, we get the path $P_3$; thus, we can use the notation $C(n_1,n_2,n_3)$ to denote the caterpillar of order $n = n_1+n_2+n_3+3$, obtained from $P_3$ by attaching $n_i$ pendant vertices to the vertex $v_i$ of $P_3$. In Figure 4.2 we show an $\alpha$-labeling $f$ of $C(n_1,n_2,n_3)$ together with the reverse of its complementary labeling.

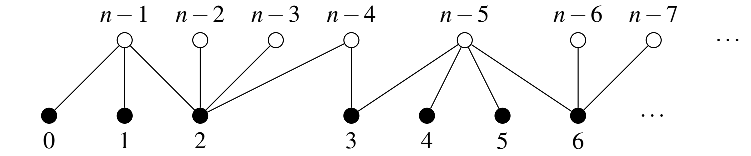



**Figure 4.1:** $\alpha$-labeling scheme of a caterpillar of size $n-1$

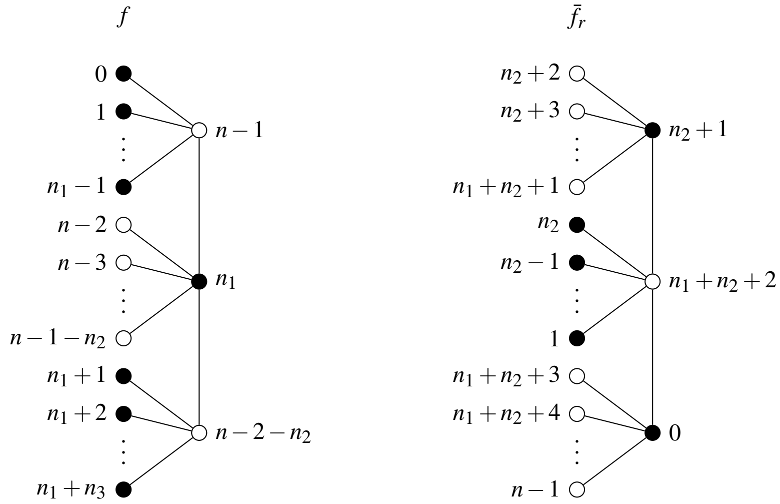



**Figure 4.2:** $\alpha$-labelings of $C(n_1,n_2,n_3)$

**Lemma 4.1.** *The lobster $L$, obtained by connecting with an edge the central vertices of two copies of the caterpillar $C(n_1,n_2,n_3)$, is an $\alpha$-tree.*

*Proof.* The caterpillar $C(n_1,n_2,n_3)$ has size $n_1,n_2,n_3+2$; the $\alpha$-labeling $f$ of it has boundary value $\lambda = n_1+n_3$. Then, we label the first copy of this caterpillar using the labeling $f$, which is transformed into a $(n+1)$-graceful labeling. In this way, its central vertex has label $n_1$. The second copy of the caterpillar is originally labeled using $\overline{f}_r$, this labeling is shifted $n_1+n_3+1$ units, thus there is no repetition of labels between both copies. The new label of the central vertex of the second copy is $(n_1+n_2+2)+(n_1+n_3+1) = n+n_1$. Hence, if we connect with an edge the central vertices, this edge will have weight $n$. Therefore, the lobster $L$ is an $\alpha$-tree. $\square$

This process can be applied to any number of copies of $C(n_1,n_2,n_3)$, in the same way that it was applied to any number of copies of $P_n$ in Section 2. Thus, we get the following theorem.

**Theorem 4.2.** *For each $1 \le i \le k$, let $G_i$ be a copy of the caterpillar $C(n_1,n_2,n_3)$. If for every $1 \le i \le k-1$, the central vertex of $G_i$ is connected with an edge to the central vertex of $G_{i+1}$, then the resulting graph is a lobster that admits an $\alpha$-labeling.*

In Figure 4.3 we show an example of this construction using the caterpillar $C(2,4,3)$ four times. We must observe that the lobsters obtained using these caterpillars do not have a perfect (or almost perfect) matching.

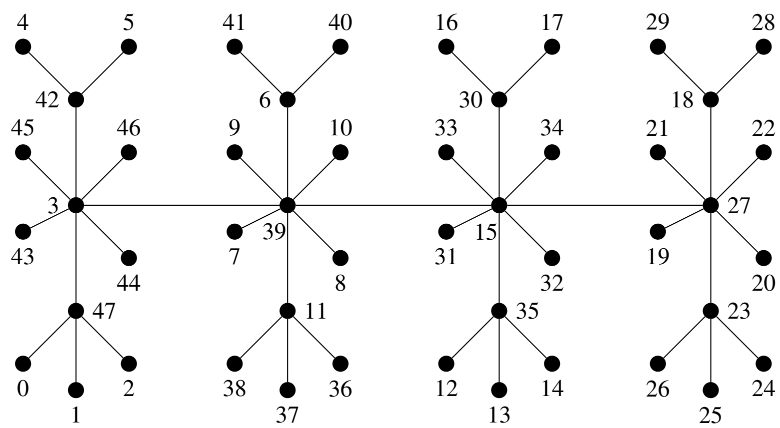



**Figure 4.3:** $\alpha$-labeling of a lobster in $\mathscr{G}$

For each $1 \leq i \leq k$, let $G_i$ be a copy of the caterpillar $C(n_1, n_2, n_3)$. The family $\mathscr{G}_k$ consists of all lobsters formed connecting with an edge the central vertices of $G_i$ and $G_{i+1}$ where $1 \leq i \leq k-1$. Thus we can say that all members of $\mathscr{G}_k$ are $\alpha$-trees. Furthermore, for any $G \in \mathscr{G}_k$, the $\alpha$-labeling of $G$, obtained using Theorem 4.2, assigns the label 0 to a leaf of $G$ and the label $\lambda$ (when $k$ is odd) or $\lambda+1$ (when $k$ is even) to another leaf, and the distance between these leaves is $k+3$, that is, the diameter of $G$. In [23] we proved that if $B_1, B_2, \ldots, B_k$ is a collection of $\alpha$-labeled blocks, with boundary value $\lambda_i$, then the graph obtained amalgamating the vertex labeled 0 in $B_i$ with the vertex labeled $\lambda_{i-1}$ in $B_{i-1}$, for every $2 \leq i \leq k$, is an $\alpha$-graph. We refer to this process as the $(0,\lambda)$-*amalgamation*. As we showed before, if $G$ is a caterpillar, there exists an $\alpha$-labeling of $G$ that assigns the labels 0 and $\lambda$ (when the diameter is even) or 0 and $\lambda+1$ (when the diameter is odd) on the leaves of a path of maximum length in $G$. These two properties allow us to prove the following theorem.

**Theorem 4.3.** *Let $G_1, G_2, \ldots, G_t$ be a collection of $\alpha$-graphs, such that $G_i \in \mathscr{G}_{k_i}$ or $G_i$ is a caterpillar. Then, the lobster $L$, obtained via $(0-\lambda)$-amalgamation of these graphs, is an $\alpha$-tree.*

*Proof.* Suppose that $f_i$ is an $\alpha$-labeling of $G_i$ with boundary value $\lambda_i$. If $G_i$ is a caterpillar, we assume that $f_i$ is the labeling $f$ in Figure 4.1. If $G_i$ is a lobster in $\mathscr{G}_{k_i}$, we assume that $f_i$ is the labeling obtained in Theorem 4.2. In both cases, the vertex of $G_i$ labeled 0 belongs to a path of maximum length in $G_i$. If the vertex of $G_i$ labeled $\lambda_i$ is on a leaf, then we can identify the vertex labeled 0 in $G_{i+1}$ with the vertex labeled $\lambda_i$ in $G_i$. The $\alpha$-labeling of the new graph, denoted by $\Gamma_{i+1}$, is obtained by shifting $\lambda_i$ units the labeling $f_{i+1}$ and transforming $f_i$ into a $d_i$-graceful labeling where $d_i - 1$ is the size of $G_{i+1}$. If the boundary value of this labeling of $\Gamma_{i+1}$ is on a leaf, we concatenate $\Gamma_{i+1}$ with $G_{i+2}$, to obtain an $\alpha$-graph $\Gamma_{i+2}$, and so on until all the amalgamations are done. If the boundary value of this labeling of $\Gamma_{i+1}$ is not on a leaf, then we use the complementary labeling, which puts its boundary value on a leaf, and connect $\Gamma_{i+1}$ with $G_{i+2}$, and continue in this way until all the amalgamations are done. Given the position of the vertices labeled 0 and $\lambda_i$, the final graph is a lobster with an $\alpha$-labeling. $\square$

In Figure 4.4 we show an example of this construction where $G_1 \in \mathscr{G}_2$, $G_2$ is a caterpillar of size 10, and $G_3 \in \mathscr{G}_3$.

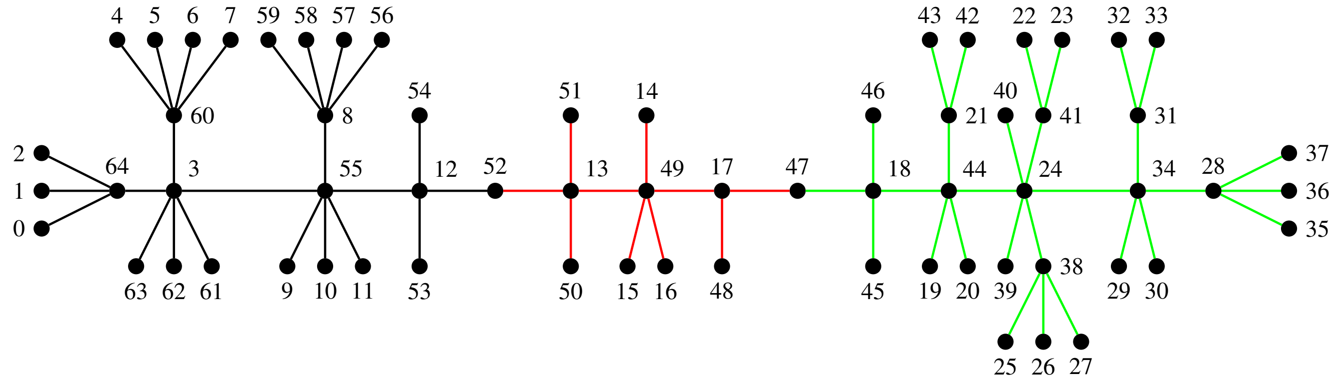



**Figure 4.4:** $\alpha$-labeling of a lobster

## 5. Conclusions

There is a wide variety of fences, we explored here one of these varieties where two consecutive copies of $P_n$ are connected by one or two links, if two links are used, the distance between them is odd. These constraints can be modified to explore the existence of $\alpha$-labelings of general fences, where the number of links is not restricted to 1 or 2. We think that all fences admit an $\alpha$-labeling, except when the fence is isomorphic to the cycle $C_n$ with $n \equiv 2 (\mathrm{mod}\ 4)$, that is not a graceful graph.

The construction of $\alpha$-lobsters presented in Theorem 4.3 can be use in a more general case, where a lobster could be decomposed into sublobsters, each of them with an $\alpha$-labeling that assigns the labels 0 and $\lambda$ to leaves $u$ and $v$ such that the distance between them equals the diameter of the sublobster. We think that this technique should be explored with more details in future works.

## 6. Acknowledgment

We want to thank the referees for their comments, suggestions, and fine job, that certainly helped us to improve the quality of this work.

## References

[1] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris, (1967), 349-355.
[2] G. S. Bloom, S. W. Golomb, *Applications of numbered undirected graphs*, Proc. IEEE, **65** (1977), 562-570.
[3] G. S. Bloom and S. W. Golomb, *Numbered complete graphs, unusual rulers, and assorted applications*, In Theory and Applications of Graphs, Lecture Notes in Math., 642, Springer-Verlag,New York, 1978, 53-65.
[4] L. Brankovic, I. M. Wanless, *Graceful labelling: State of the art, applications and future directions*, Math Comput. Sci., **5** (2011), 11-20.
[5] M. Maheo, H. Thuillier, *On d-graceful graphs*, Ars Combin., **13** (1982), 181-192.
[6] P. J. Slater, *On k-graceful graphs*, Proc. of the 13th S.E. Conf. on Combinatorics, Graph Theory and Computing, (1982), 53-57.
[7] C. Barrientos, Difference Vertex Labelings, Ph.D. Thesis, Universitat Politecnica de Catalunya, Barcelona, 2004.
[8] G. Chartrand, L. Lesniak, *Graphs & Digraphs*, 2nd ed. Wadsworth & Brooks/Cole, Monterey, 1986.
[9] J. A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin., 21(#DS6), 2018.
[10] C. Barrientos, S. Minion, *Counting and labeling grid related graphs*, preprint.

[11] V. Jovovic, https://oeis.org/A002620, 2000.
[12] C. Barrientos, https://oeis.org/A152271, 2019.
[13] D. Morgan, *All lobsters with perfect matchings are graceful*, Electron. Notes Discrete Math., **11** (2002), 6 pages.
[14] J. C. Bermond, *Graceful graphs, radio antennae and French windmills*, *Graph Theory and Combinatorics*, Pitman, London, 1979, 18-37.
[15] R. Stanton, C. Zarnke, *Labeling of balanced trees*, Proc. 4th Southeast Conf. Combin., *Graph Theory and Combinatorics*, (1973), 479-495.
[16] M. Burzio, G. Ferrarese, *The subdivision graph of a graceful tree is a graceful tree*, Discrete Math., **181** (1998) 275-281.
[17] J. G. Wang, D. J. Jin, X.-G Lu, D. Zhang, *The gracefulness of a class of lobster trees*, Math. Comput. Modelling, **20** (1994), 105-110.
[18] D. Mishra, A. C. Panda, *Some new family of graceful lobsters*, Adv. Appl. Discrete Math., **14**(1) (2014), 1-24.
[19] D. Mishra, P. Panigrahi, *Graceful lobsters obtained by component moving of diameter four trees*, Comput. Math. Appl., **50** (2005), 367-380.
[20] D. Mishra, P. Panigrahi, *Some graceful lobsters with all three types of branches incident on the vertices of the central path*, Comput. Math. Appl., **56** (2008), 1382-1394.
[21] D. Mishra, P. Panigrahi, *Some new classes of graceful lobsters obtained from diameter four trees*, Math. Bohem., **135** (2010), 257-278.
[22] E. Krop, *Lobsters with an almost perfect matching are graceful*, Bull. Inst. Combin. Appl., **74** (2015), 21-24.
[23] C. Barrientos, *Graceful labelings of chain and corona graphs*, Bull. Inst. Combin. Appl., **34** (2002), 17-26.

*Universal Journal of Mathematics and Applications*, 2 (1) (2019) 11-13
**Research paper**

UJMA

# Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: DOI: http://dx.doi.org/10.32323/ujma.469745

# A Characterization of Left Regularity

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**Article Info**

***Keywords:*** *Anshel-Clay near-ring, Integral near-ring, Left regular, Regular*
***2010 AMS:*** *16E50, 16Y30, 20M17*
***Received:*** *12 October 2018*
***Accepted:*** *6 December 2018*
***Available online:*** *20 March 2019*

**Abstract**

We show that a zero-symmetric near-ring $N$ is left regular if and only if $N$ is regular and isomorphic to a subdirect product of integral near-rings, where each component is either an Anshel-Clay near-ring or a trivial integral near-ring. We also show that a zero-symmetric near-ring is regular without nonzero nilpotent elements if and only if the multiplicative semigroup of N is a union of disjoint groups.

## 1. Introduction

A (right) near-ring is an algebraic system $(N,+,\cdot)$ such that $(1)$ $(N,+)$ is a (not necessarily abelian) group, $(2)$ $(N,\cdot)$ is a semigroup and $(3)$ the multiplication $\cdot$ is right distributive over the addition $+$ . From $(3)$ we obtain that $0x=0$ for all $x\in N$. The near-ring of constant functions on a group $(G,+)$ shows that in general $x0\neq 0$ in a near-ring. $N$ is called zero-symmetric, if $x0=0$ for all $x\in N$. A near-ring $N$ is called regular, if for all $x\in N$ there
exists an element $y\in N$ such that $x=xyx$. $N$ is called left (right) regular, if for all $x\in N$ there exists $y\in N$ such that $x=yx^2$ $(x=x^2y)$. $N$ is called integral, if $N$ has no nonzero divisors of zero. A zero-symmetric integral near-ring $N$ is called trivial, if $xy=x$ for all $x,y\in N, y\neq 0$. The set $N-\{0\}$ shall be denoted by $N^*$. For this and other terminology we refer to [1]. In the next section we define Anshel-Clay near-rings and characterize them in the class of nontrivial integral near-rings. Then we show that a zero-symmetric near-ring $N$ is left regular, if and only if $N$ is regular and isomorphic to a subdirect product of near-rings, which are either trivial integral near-rings or Anshel-Clay near-rings. We also prove for an arbitrary zero-symmetric near-ring $N$ that the multiplicative semigroup $(N,\cdot)$ is a union of disjoint groups, if and only $N$ is regular without nonzero nilpotent elements.

## 2. Left regular near-rings

**Definition 2.1.** [2] *A near-ring $N$ is called Anshel-Clay near-ring (ACN), if $N*$ is a disjoint union of subsets $A_i, i\in I$, where $I$ is an index set, such that the following conditions hold:*

1. $|A_i|\geq 2$ *for all* $i\in I$.
2. $(A_i,\cdot)$ *is a group with neutral element* $1_i$ *for all* $i\in I$.
3. *For all* $i,j\in I$, *the mapping* $x\mapsto 1_jx$ *for* $x\in A_i$ *is a group isomorphism from* $(A_i,\cdot)$ *onto* $(A_j,\cdot)$.
4. *Each* $1_i, i\in I$, *is a right identity of* $N$.

As we shall see in the next result, condition 3 follows from the other conditions, so when we say that $N$ is an ACN, we mean that $N$ satisfies conditions $1,2,4$. Anshel-Clay near-rings have been defined in [2], but they occured implicitely in previous papers on planar and strongly uniform near-rings, see for example [3], [4], [5] and [6]. In [2] and in [7] these near-rings have been used to coordinatise certain noncommutative spaces.

**Theorem 2.2.** *Let $N$ be an ACN. Then*

1. $A_i=\{n\in N^* \mid 1_in=n\}$
2. $A_i=1_iN^*$
3. *For all* $i,j\in I, h_{ij}:A_i\to A_j, h_{ij}(x):=1_jx$ *for* $x\in A_i$ *is a group isomorphism.*

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*Proof.* 1. Since $1_i$ is the identity of the group $A_i, A_i \subseteq \{n \in N^* | 1_i n = n\}$. Conversely, let $n \in N^*$ such that $1_i n = n$. Since $N^* = \bigcup_{j \in I} A_j$, $n \in A_j$ for some $j \in I$. Suppose that $i \neq j$ and let $n^{-1}$ denote the inverse of $n$ in $A_j$. Then $1_j = nn^{-1} = (1_i n)n^{-1} = 1_i(nn^{-1}) = 1_i 1_j = 1_i$, since $1_j$ is a right identity of $N$. Thus $1_i = 1_j$, a contradiction, since $i \neq j$ implies $A_i \cap A_j = \varnothing$. It follows that $\{n \in N^* | 1_i n = n\} \subseteq A_i$.
2. From 1. we have that $A_i \subseteq 1_i N^*$. Conversely, if $n = 1_i m \in 1_i N^*$, then $1_i n = 1_i(1_i m) = 1_i^2 m = 1_i m = n$, hence from 1. $1_i N^* \subseteq A_i$.
3. Let $x, y \in A_i$. Since $1_j$ is a right identity of $N, h_{ij}(xy) = 1_j xy = 1_j x 1_j y = h(x)h(y)$, so $h_{ij}$ is a group homomorphism. Now suppose that $1_j x = 1_j y$, for some elements $x, y \in A_i$. Then $1_i(1_j x) = 1_i(1_j y)$. Since $1_i 1_j = 1_i$, we obtain $1_i x = 1_i y$. By 1. it follows that $x = y$, so $h_{ij}$ is injective. If $x$ is an arbitrary element of $A_j$, then $1_i x \in A_i$ by 2., hence $h_{ij}(1_i x) = 1_j(1_i x) = (1_j 1_i)x = 1_j x = x$, which shows that $h_{ij}$ is an isomorphism. □

A near-field is a near-ring with identity, where every nonzero element is invertible.

**Theorem 2.3.** *1. Every ACN is a zero-symmetric, nontrivial integral near-ring.*
*2. Let N be an ACN. Then N is a near-field, if and only if N has an identity, if and only if I is a one element set.*

*Proof.* 1. If $x0 \neq 0$ for some $x \in N$, then $x0 \in A_i$ for some $i \in I$, since $N^* = \bigcup_{i \in I} A_i$. From $(x0)^2 = x(0x)0 = x0$ we obtain $x0 = 1_i$. Thus $1_i n = (x0)n = x(0n) = x0 = 1_i$ for all $n \in N^*$. By 2. of Theorem 2.2, it follows that $A_i = 1_i N^* = \{1_i\}$, which contradicts condition 1 in the definition of an ACN. It follows that $N$ is zero-symmetric. Now suppose $xy = 0$ for some elements $x, y \in N$. If $y \neq 0$, then $y \in A_i$ for some $i \in I$. If $y^{-1}$ is the inverse of $y$ in $A_i$, then $0 = 0y^{-1} = (xy)y^{-1} = x(yy^{-1}) = x1_i = x$, thus $N$ is integral. If $N$ is a trivial integral near-ring, then $xy = x$ for all $y \neq 0$, hence $A_i = 1_i N^* = \{1_i\}$ for all $i \in I$, a contradiction.
2. If $N$ is an ACN with identity 1, then $1 = 1_i$ for all $i \in I$, since each $1_i$ is a right identity of N. Thus $N$ is a near-field. □

Next we characterize which nontrivial integral near-rings are Anshel-Clay near-rings.

**Theorem 2.4.** *For a zero-symmetric, nontrivial integral near-ring $N$, the following are equivalent:*

1. *$N$ is an ACN*
2. *$\forall n \in N^* : Nn = N$*
3. *$N$ is left regular*
4. *$N$ is regular*

*Proof.* Let $N$ be an ACN and let $n \in N^*$. Then $n \in A_i$ for some $i \in I$. Since $1_i$ is a right identity of $N$, $N = N1_i = Nn^{-1}n \subseteq Nn$, hence $Nn = N$. Next, suppose $Nx = N$ for all $x \in N, x \neq 0$. Then $Nx^2 = (Nx)x = Nx = N$, for all $x \neq 0$, hence there exists an element $y \in N$, such that $x = yx^2$, thus $N$ is left regular. That 3. implies 4. has been shown in [8], Proposition 1. Finally we show that 4. implies 1. If $0 \neq e \in N$ is idempotent, then for all $n \in N$ we have $(ne - n)e = ne^2 - ne = ne - ne = 0$. Since $N$ is integral it follows that $ne = n$, hence each idempotent is a right identity of $N$. Now suppose that $N$ is regular and let $n \in N$. Then there exists an element $x \in N$ such that $n = nxn$. Then $nx$ is idempotent, hence $n = n(nx) = n^2 x$. It follows that $N$ is regular and right regular. By [9], Theorem 4.3, $N^* = \bigcup_{i \in I} A_i$, where $A_i$ is a group with identity $1_i$ for $i \in I$ and $A_i \cap A_j = \varnothing$ if $i \neq j$. As we have seen before, each $1_i$ is a right identity of $N$. Now we can show like in Theorem 2.2, No. 3, that $h_{ij} : A_i \to A_j$, $h_{ij}(x) = 1_j x$ for $x \in A_i$ is a group isomorphism. If $A_i = \{1_i\}$ for all $i \in I$, then $(N^*, \cdot)$ is a band since each $1_i, i \in I$, is a right identity of $N$. Since this contradicts our assumption that $N$ is a nontrivial integral near-ring, it follows that $|A_i| \geq 2$ for all $i \in I$, hence $N$ is an ACN. □

An idempotent $e$ of a near-ring $N$ is called right semi-central in $N$, if $eN = eNe$. It is easy to show that $e$ is right semi-central in $N$ if and only if $en = ene$ for all $n \in N$. $N$ is called right semi-central, if every idempotent $e$ of $N$ is right semi-central in $N$ (see [10]). Let $N$ be an integral near-ring and $i, n \in N$. $i$ is called a left identity of $n$, if $in = n$. Note that if $n \neq 0$ has a left identity $i$, then $i$ is uniquely determined, since $i_1 n = n = i_2 n$ implies $(i_1 - i_2)n = 0$, hence $i_1 = i_2$.

**Theorem 2.5.** *For a zero-symmetric regular near-ring N the following are equivalent:*

1. *N has no nonzero nilpotent elements.*
2. *N is right semi-central.*
3. *N is isomorphic to a subdirect product of Anshel-Clay near-rings and trivial integral near-rings.*
4. *N is left regular.*

*Proof.* That 1. implies 2. has been shown in [10], Cor. 7. Conversely, suppose that there exists an element $n \in N, n \neq 0, n^2 = 0$. Since $N$ is regular, $n = nxn$ for some $x \in N$. Then $e := nx$ is idempotent and $n = en = ene$, since $e$ is semi-central by assumption, so $n = ne = n^2 x = 0$, a contradiction, which shows the equivalence of 1. and 2. Next we show that 1. implies 3. By [11], $N$ is isomorphic to a subdirect product of integral near-rings $N_i, i \in I$. Since $N$ is regular, each $N_i$ is also regular. Therefore, if $N_i$ is a nontrivial integral near-ring, then $N_i$ is an ACN by Theorem 2.4. Since each ACN is integral by Theorem 2.3 it follows that 3. implies 1. Since 4. implies 1. is clear, it remains to show that 3. implies 4. Suppose that $N$ is isomorphic to a subdirect product of near-rings $N_i, i \in I$, where each $N_i$ is an ACN or a trivial integral near-ring. Let $n \in N$. We have to show that there exists an element $x \in N$ such that $n = xn^2$. Since $N$ is regular, there exists $y \in N$ such that $n = nyn$. $N$ is isomorphic to a subdirect product of the near-rings $N_i$, so $n = (n_i)_{i \in I}$, $y = (y_i)_{i \in I}$, for some $n_i, y_i \in N_i$, $i \in I$. Then $n_i = n_i y_i n_i$ and $e_i := n_i y_i$ is an idempotent for all $i \in I$. Since $(n_i - n_i e_i)e_i = 0_i$ and each $N_i$ is integral, we obtain $n_i = n_i e_i = n_i^2 y_i$. Let $x := ny^2$. Then for all $i \in I$, $n_i^2 x_i = n_i(n_i^2 y_i)y_i = n_i^2 y_i = n_i$ and $n_i x_i n_i = (n_i^2 y_i)(y_i n_i) = n_i y_i n_i = n_i$, hence $n^2 x = n = nxn$. Now fix an element $i \in I$ and suppose that $N_i$ is an ACN. Then there exists an index set $J_i$ such that $N_i = \{0_i\} \cup \bigcup_{j \in J_i} A_j$, using the terminology of Definition 2.1 Suppose $n_i \neq 0$. Since $n_i = n_i x_i n_i$, $n_i x_i$ is a left identity for $n_i$. There exists an element $j \in J_i$, such that $n_i \in A_j$. But then $1_j$ is also a left identity for $n_i$, hence by the uniqueness of the left identity, $n_i x_i = 1_j$. Note that $x_i$ is also an element of $A_j$. This follows from Theorem 2.2, since $n_i \in A_j$ and $1_j x_i = 1_j n_i y_i^2 = (1_j n_i) y_i^2 = n_i y_i^2 = x_i$. Therefore we obtain that $x_i n_i = 1_j = n_i x_i$, hence $n_i^2 x_i = n_i = n_i x_i n_i = x_i n_i^2$. This equation also holds if $n_i = 0$, so it holds for all $i \in I$, where $N_i$ is an ACN. Since the previous equation is obviously true for all those $i \in I$, where $N_i$ is a trivial integral near-ring, we conclude that $n^2 x = n = xn^2$. Thus $N$ is left regular. □

In [12], the equivalence of 1. and 4. has been shown with a different proof. From Theorem 2.5 and [9], Theorem 4.3 we also obtain

**Theorem 2.6.** *For a zero-symmetric near-ring $N$, the following are equivalent:*

1. *$N$ is regular without nonzero nilpotent elements.*
2. *The multiplicative semigroup of $N$ is a union of disjoint groups.*

## References

[1] G. Pilz, *Near-rings, The theory and its applications*, 2 ed., North-Holland Mathematics Studies 23, North-Holland Publishing Co., Amsterdam, 1983, MR0721171.
[2] J. Andre, H. H. Ney, *On Anshel-Clay-nearrings, Near-rings and near-fields* (Oberwolfach, 1989), 15-20, Math. Forschungsinst. Oberwolfach, Schwarzwald, 1995. MR1725312
[3] M. Anshel, J. R. Clay, *Planar algebraic systems: Some geometric interpretations*, J. Algebra 10 (1968), 166-173, MR0241473.
[4] G. Ferrero, *Stems planari e BIB-disegni*, Riv. Mat. Univ. Parma, **2**(11) (1970), 79-96, MR031332
[5] G. Szeto, *The Sub-Semi-Groups excluding zero of a near-ring*, Monatsh. Math., **77** (1973), 357-362, MR0330237.
[6] G. Szeto, *Planar and strongly uniform near-rings*, Proc. Amer. Math. Soc., **44** (1974), 269-274, MR0340351.
[7] H. H. Ney, *Anshel-Clay near-rings and semiaffine parallelogramspaces*, Near-rings and near-fields (Fredericton, NB, 1993), 203-207, Math. Appl., 336, Kluwer Acad. Publ., Dordrecht, 1995, MR1366480.
[8] G. Mason, *Strongly regular near-rings*, Proc. Edinburgh Math. Soc., **23**(1) (1980), 27-35, MR0582019.
[9] A. H. Clifford, G. B. Preston, *The algebraic theory of semigroups*, **1**(7), Mathematical Surveys, Amer. Math. Soc., Providence, R.I., (1977), MR0132791 7.
[10] Y. U. Cho, *A study on near-rings with semi-central idempotents*, ar East J. Math. Sci., **98**(6), (2015), 759-762.
[11] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc., **2** (1970), 363-368, MR0263877.
[12] V. G. Marin, *Some properties of regular near-algebras, Ring-theoretical constructions* (Russian), Mat. Issled, **49** (1979), 105-114, 162-163, MR0544982.

*Universal Journal of Mathematics and Applications, 2 (1) (2019) 14-23*
***Research paper***

UJMA

**Universal Journal of Mathematics and Applications**

*Journal Homepage: www.dergipark.gov.tr/ujma*
*ISSN 2619-9653*
*DOI: DOI: http://dx.doi.org/10.32323/ujma.454890*

# *D*-Stratification and Hierarchy Graphs of the Space of Order 2 and 3 Matrix Pencils

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**Article Info**

***Keywords:*** *Equivalence relation, Kronecker normal form, Linear systems, Matrix pencils, Stratification.*
***2010 AMS:*** *15A21, 15A22, 93B10, 93B35,*
***Received:*** *22 August 2018*
***Accepted:*** *2 February 2019*
***Available online:*** *20 March 2019*

**Abstract**

Small changes in the entries of a matrix pencil may lead to important changes in its Kronecker normal form. Studies about the effect of small perturbations have been made when considering the stratification associated with the strict equivalence between matrix pencils. In this work, we consider a partition in the space of pairs of matrices associated to regular matrix pencils, which will be proved to be a finite stratification of the space of such matrix pencils, called D-stratification. Matrix pencils in the same strata are those having some prescribed Segre indices. We study the effect of perturbations which lead to changes in the Kronecker canonical form, preserving the order of the nilpotent part. Our goal is to determine which *D*-strata can be reached. In the cases where the order of the matrix pencils is 2 or 3, we obtain the corresponding hierarchy graphs, illustrating the *D*-strata that can be reached when applying some small perturbations.

## 1. Introduction

Jordan normal form of a square matrix $A$ is not stable under small perturbations, small changes in its entries may change the Jordan normal form of the matrix. In [1, 2], V. I. Arnold identified nearby canonical structures using miniversal deformations. H. den Boer and G. Ph. A. Thijsse, A. S. Markus and E. E. Parilis (see [3, 4]) found Jordan normal form of matrices which could be obtained from a given Jordan matrix by arbitrary small perturbations. The changes in the normal form of a matrix when only elements in some concrete positions can be changed are studied by different authors for example, (see [5, 6]).
An informal introduction to perturbations of matrices up to different equivalence relations is given in [7]. Changes of the canonical form for order two and three matrices under congruence were given in [8].
V. I. Arnold introduced the sets of matrices having the same Segre characteristics (and differing only in the continuous invariants) as bundles of matrices. Gibson proved in [9] that this partition is actually a Whitney stratification, the closure of each stratum being the union of strata (of strictly lower dimension).
In [10], [11] a different stratification of the space of square matrices is considered, being the matrices in the same stratum those having the same Drazin inverse (see [12] for this relation between matrices).
The stratification of the space of pairs of matrices related to linear control systems can be found in [13], where a proof of this stratification being Whitney regular in a particular case is included (the general case is an open problem). Bifurcation diagrams were obtained in [14]. Later, the same author proved that the partition of the space of quadruples of matrices according to the set of discrete structural invariants is a stratification (see [15]). All possible Kronecker canonical forms of matrix pencils in a neighbourhood of any given pencil were described by Pokrzywa in [16].
Stratifications can be represented by hierarchy graphs, the nodes being the strata and the edges the covering relations; that is to say, the possible paths from one bundle to another one.
The closure of a stratum consists of all those strata which can be reached applying a small perturbation. Closure relations for matrices under conjugation and matrix pencils under strict equivalence were studied in [17, 18]. In [19] Hasse diagrams for the closure ordening for order two matrices under *congruece were constructed. E. Elmroth, P. Johansson and B. Kågström presented in [18] Stratigraph, a Java-based tool for the computation and visualization of canonical information and stratification hierarchies for matrices and matrix pencils.

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Closure relations for matrix pencils under strict equivalence were studied in [20] and [14]. All possible Kronecker canonical forms of matrix pencils in a neighbourhood of any given matrix pencil were described by Pokrzywa in [16].
As usual, we will denote by $M_n(\mathbb{C})$ the set of square matrices of order $n$ with coefficients in $\mathbb{C}$ and by $Gl_n(\mathbb{C})$ the set of all invertible matrices of order $n$.

## 2. Motivation

Let us consider a linear control dynamical system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

with $E, A$ square matrices of order n.
In the case where $E$ is an invertible matrix, we can pre-multiply the equation above by $E^{-1}$, thus obtaining:

$$\dot{x}(t) = A_1 x(t) + B_1 u(t)$$

If $E$ is not invertible, and assuming that the matrix pencil $\lambda E + A$ is regular (to ensure the system has a unique solution) the system splits into a slow and fast subsystems, according to the response's speed to the changes in the control input. These subsystems can be obtained using the Kronecker normal form of the matrix pencil (or its Weierstraß normal form, since we only consider regular matrix pencils). It can be found, for example, in [21]. Let us recall it, we have assumed the matrix pencil $\lambda E + A$ to be regular, there exist invertible matrices $P$, $Q$ such that

$$Q(\lambda E + A)P = \lambda \begin{pmatrix} I_{n_0} & 0 \\ 0 & \mathscr{N}_E \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & I_{n-n_0} \end{pmatrix}$$

with $\mathscr{N}_E$ a nilpotent matrix, $G \in M_{n_0}(\mathbb{C})$ a matrix in Jordan reduced form,

$$G = \begin{pmatrix} J & \\ & N_G \end{pmatrix}$$

where $\det(J) \neq 0$ and $N_G$ is a nilpotent matrix.
The Kronecker reduced form of the matrix pencil $\lambda E + A$, or Weierstraß form, is:

$$\begin{aligned} \lambda E_c + A_c &= \lambda \begin{pmatrix} I_{n_0} & 0 \\ 0 & \mathscr{N}_E \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & I_{n-n_0} \end{pmatrix} \\ &= \lambda \begin{pmatrix} I_\nu & 0 & 0 \\ 0 & I_{n_0-\nu} & 0 \\ 0 & 0 & \mathscr{N}_E \end{pmatrix} + \begin{pmatrix} J & 0 & 0 \\ 0 & N_G & 0 \\ 0 & 0 & In-n_0 \end{pmatrix} \end{aligned}$$

Applying suitable basis change and pre-multiplication to the systems' equation (2.1)

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

the system splits into two subsystems:

$$\left.\begin{aligned} \begin{pmatrix} I_\nu & 0 \\ 0 & I_{n_0-\nu} \end{pmatrix} \dot{x}_1(t) &= \begin{pmatrix} J & 0 \\ 0 & N_G \end{pmatrix} x_1(t) + \begin{pmatrix} B_1^1 \\ B_1^2 \end{pmatrix} u(t) \\ \mathscr{N}_E \dot{x}_2(t) &= x_2(t) + B_2 u(t) \end{aligned}\right\}$$

The first system is referred to as the slow subsystem and the second one as the fast subsystem. We will denote them by $\Sigma_S$ and $\Sigma_F$ , respectively. The solution of the fast subsystem is well-known (see [22], 1989). Obviously, if E is an invertible matrix, the fast subsystem does not appear ($n_0 = n$).
In turn, the slow subsystem $\Sigma_S$ splits into two subsystems:

$$\left.\begin{aligned} \Sigma_J \quad \dot{y}_(t) &= Jy(t) + B_1^1 u(t) \\ \Sigma_{N_G} \quad \dot{z}_(t) &= N_G z(t) + B_1^2 u(t) \end{aligned}\right\}$$

That is to say, the initial system can be divided into three independent subsystems:

$$\left.\begin{aligned} \Sigma_J \qquad \dot{y}_(t) &= Jy(t) + B_1^1 u(t) \\ \Sigma_{N_G} \qquad \dot{z}_(t) &= N_G z(t) + B_1^2 u(t) \\ \Sigma_F \quad \mathscr{N}_E \dot{x}_2(t) &= x_2(t) + B_2 u(t) \end{aligned}\right\}$$

The solutions to the subsystems above are:

$$
\begin{aligned}
y(t) &= e^{Jt}y_0 + \int_0^t e^{J(t-\tau)}B_1^1 u(\tau)d\tau, \\
z(t) &= e^{N_G t}z_0 + \int_0^t e^{N(t-\tau)}B_1^2 u(\tau)d\tau \\
&= \sum_{i=0}^{n_0-\nu-1}\frac{1}{i!}N_G^i t^i z_0 + \sum_{i=0}^{n_0-\nu-1}\frac{1}{i!}\int_0^t N_G^i t^i B_1^2 u(\tau)d\tau, \\
x_2 &= -\sum_{i=1}^{n-n_0-1}\mathscr{N}^i B u^{(i)}(t)
\end{aligned}
$$

and while the solution of the first subsystem, $\Sigma_J$ is a matrix series, the solution of the second and third ones, $\Sigma_{N_G}$ and $\Sigma_F$ are polynomial matrices. The first one depends on the integral of control function and the second one on the derivatives of this function. This supposes an important difference when computing solutions (recall that the different methods and algorithms to compute the exponential of a matrix $J$ are not absolutely satisfactory). On the other hand, the formula to compute the exponential of the matrix is the same independently of the exact value of the eigenvalues and relies only on the Segre characteristics of the matrix.
This suggests to considering a partition in the space of regular matrix pencils, which will be called $D$-stratification, where strata will consist of matrix pencils having the same Segre characteristics in matrix $J$ (and not taking into account the Segre characteristics of matrices $N_G$ and $\mathscr{N}_E$).

## 3. $D$-Partition of the space of pairs of matrices associated to regular matrix pencils

We will denote by $\mathscr{X}$ the set of regular matrix pencils $\lambda E + A$, with $E, A \in M_n(\mathbb{C})$ (which is an open subset of the space of pencils of matrices identified with the space of pair of matrices $M_n(\mathbb{C}) \times M_n(\mathbb{C})$, thus a differentiable manifold).
We define an equivalence relation in $\mathscr{X}$ according to the Segre characteristic of matrix $J$ in the Kronecker reduced form of the matrix pencil $\lambda E + A$.

**Definition 3.1.** *Given two regular matrix pencils $\lambda E + A \in \mathscr{X}$, $\lambda E' + A' \in \mathscr{X}$ with Kronecker reduced forms*

$$
\begin{aligned}
\lambda E_c + A_c &= \lambda \begin{pmatrix} I_\nu & 0 & 0 \\ 0 & I_{n_0-\nu} & 0 \\ 0 & 0 & \mathscr{N}_E \end{pmatrix} + \begin{pmatrix} J & 0 & 0 \\ 0 & N_G & 0 \\ 0 & 0 & I_{n-n_0} \end{pmatrix} \\
\lambda E'_c + A'_c &= \lambda \begin{pmatrix} I_{\nu'} & 0 & 0 \\ 0 & I_{n'_0-\nu'} & 0 \\ 0 & 0 & \mathscr{N}'_E \end{pmatrix} + \begin{pmatrix} J' & 0 & 0 \\ 0 & N'_{G'} & 0 \\ 0 & 0 & I_{n-n'_0} \end{pmatrix}
\end{aligned}
$$

*they will be said to be D-equivalent if $\nu = \nu'$, $n_0 = n'_0$ and $J$ and $J'$ have the same Segre characteristics.*

Note that equivalent pencils in the same orbit under classical equivalent relation of matrix pencils (see [23]) are equivalent under $D$-equivalence relation considered. Therefore equivalent classes are the union of orbits. But matrix pencils in the same stratum of the stratification induced for classical equivalence relation are not necessarily in the same orbit under $D$-equivalence.

**Example 3.2.** *The following matrix pencils*

$$
\lambda E_1 + A_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 2 & & \\ & 3 & \\ & & 1 \end{pmatrix}
$$

$$
\lambda E_2 + A_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 2 & & \\ & 0 & \\ & & 1 \end{pmatrix}
$$

*are in the same stratum when considering the stratification induced for classical equivalence (strict equivalence) relation but they are not in the same orbit under D-equivalence.*

Equivalent classes under $D$-equivalence relation can be obtained dividing classical strata.

**Example 3.3.** *Let $S$ be the classical stratum consisting of matrix pencils with canonical reduced form*

$$
\lambda \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \mathscr{N}_E \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_{n_0} & \\ & & & I_{n-n_0} \end{pmatrix}
$$

*where $\lambda_i \neq \lambda_j$ if $i \neq j$, $1 \leq i, j \leq n$.*
*This stratum splits into the following D-equivalence classes:*

$$\lambda \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \mathcal{N}_E \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_{n_0} & \\ & & & I_{n-n_0} \end{pmatrix}, \qquad \lambda_i \neq 0, \;\; \lambda_i \neq \lambda_j \;\text{ if }\; i \neq j$$

$$\lambda \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \mathcal{N}_E \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_{n_0-1} & & \\ & & & 0 & \\ & & & & I_{n-n_0} \end{pmatrix}, \quad \lambda_i \neq 0, \;\; \lambda_i \neq \lambda_j \;\text{ if }\; i \neq j$$

In order to obtain all the strata, we can proceed as follows. First of all we divide each stratum on the finite classical stratification into a finite number of equivalent classes, separating the orbits in the stratum having some zero-eigenvalue Jordan block and then we joint the sets having the same Segre characteristic correspoding to non-singular part of the Jordan matrix.

**Example 3.4.** *Let us consider the classical strata corresponding to*

$$\lambda E_1 + A_1 = \lambda \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & 1 & & \\ & & \lambda_2 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \; \lambda_1 \neq \lambda_2$$

$$\lambda E_2 + A_2 = \lambda \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & 1 & & \\ & & \lambda_2 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \; \lambda_1 \neq \lambda_2$$

*The stratum corresponding to $\lambda E_1 + A_1$ may be divided into*

$$\lambda E_1^1 + A_1^1 = \lambda \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \; \lambda_1 \neq 0$$

$$\lambda E_1^2 + A_1^2 = \lambda \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \; \lambda_1 \neq 0$$

*The stratum corresponding to $\lambda E_2 + A_2$ may be divided into*

$$\lambda E_2^1 + A_2^1 = \lambda \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \; \lambda_1 \neq 0$$

$$\lambda E_2^2 + A_2^2 = \lambda \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \; \lambda_1 \neq 0$$

*Then, we joint*

$$(\lambda E_1^1 + A_1^1) \cup (\lambda E_2^1 + A_2^1), \quad (\lambda E_1^2 + A_1^2) \cup (\lambda E_2^2 + A_2^2)$$

*That is to say, $\lambda E_1^1 + A_1^1$ and $\lambda E_2^1 + A_2^1$ are in the same stratum (and all D-equivalent to both pencils) and $\lambda E_1^2 + A_1^2$ and $\lambda E_2^2 + A_2^2$ are in the same stratum (and all D-equivalent to both pencils).*
*Observe that the equivalent classes of $\lambda E_2^i + A_2^i$ $(i = 1, 2)$ is in the frontier (or boundary) of the equivalent classes of $\lambda E_1^i + A_1^i$ $(i = 1, 2)$:*

$$\lim_{\varepsilon\to 0} \quad \lambda\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \varepsilon \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} =$$

$$\lambda\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$$\lim_{\varepsilon\to 0} \quad \lambda\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \varepsilon \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} =$$

$$\lambda\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

**Proposition 3.5.** *The partition of $\mathscr{X}$ into D-equivalence classes is a finite partition.*

We will denote $D$-equivalence classes as $D(\nu, n_0, \sigma)$, referring to the orders of matrices $J$, $G$ and the Segre characteristics of $J$. A deeper study of these sets is made below.

**Proposition 3.6.** *Let $J(\sigma)$ the stratum in $Gl_\nu(\mathbb{C})$ under similarity and let $\mathscr{N}il(n-n_0)$ and let $\mathscr{N}il(n_0-\nu)$ be the smooth manifolds of nilpotent matrices of size $n-n_0$ and $n_0-\nu$ respectively. Then, there is a smooth monomorphism from the set $J(\sigma)\times\mathscr{N}il(n-n_0)\times\mathscr{N}il(n_0-\nu)$ to $M_n(\mathbb{C})\times M_n(\mathbb{C})$,*

$$\begin{array}{rl} \varphi : J(\sigma)\times\mathscr{N}il(n-n_0)\times\mathscr{N}il(n_0-\nu) & \longrightarrow M_n(\mathbb{C})\times M_n(\mathbb{C}) \\ (A,\mathscr{N},N) & \longrightarrow \left(\begin{pmatrix} I_\nu & & \\ & I_{n_0} & \\ & & \mathscr{N} \end{pmatrix}, \begin{pmatrix} A & & \\ & N & \\ & & I_{n_{n_0}} \end{pmatrix}\right) \end{array}$$

*Proof.* It is straightforward that $\varphi$ is injective and differentiable. □

**Corollary 3.7.** *For each $\mathscr{N}\in\mathscr{N}il(n-n_0)$, the map*

$$\begin{array}{rl} \varphi_{\mathscr{N}} : J(\sigma)\times\mathscr{N}il(n_0-\nu) & \longrightarrow M_n(\mathbb{C})\times M_n(\mathbb{C}) \\ (A,\mathscr{N},N) & \longrightarrow \left(\begin{pmatrix} I_\nu & & \\ & I_{n_0-\nu} & \\ & & \mathscr{N} \end{pmatrix}, \begin{pmatrix} A & & \\ & N & \\ & & I_{n-n_0} \end{pmatrix}\right) \end{array}$$

*is a smooth monomorphism.*

**Remark 3.8.** *The D-equivalent class $D(\nu, n_0, \sigma)$ is the set of equivalent pairs to $\varphi(J(\sigma)\times\mathscr{N}il_{n-n_0}\times\mathscr{N}il_{n_0-\nu})$*

$$\begin{array}{l} D(\nu,n_0,\sigma) = \\ \{(A,B) = Q\varphi(A,\mathscr{N},N)P \mid (A,\mathscr{N},N)\in J(\sigma)\times\mathscr{N}il_{n-n_0}\times\mathscr{N}il_{n_0-\nu}, P,Q\in Gl_n(\mathbb{C})\} \end{array}$$

*and each D-equivalence class $D(\nu,n_0,\sigma)$ is a disjoint union of the sets of equivalent pairs to $\varphi_{\mathscr{N}}(J(\sigma)\times\mathscr{N}\times\mathscr{N}il_{n_0-\nu})$. Therefore, as a consequence of [10], $D(\nu,n_0,\sigma)$ is a disjoint union of differentiable manifolds.*

In the following section we will show that $D(\nu,n_0,\sigma)$ are, actually, differentiable manifolds.

## 4. Regularity of strata

First of all we reasoning that the orbits configuring the strata are complex differentiable submanifolds of the set of matrix pencils $M_n(\mathbb{C})\times M_n(\mathbb{C})$.
Since orbits under classical equivalent relation of matrix pencils are orbits under the action of the Lie group $\mathscr{G}=\{(P,Q)\in Gl(n;\mathbb{C})\times Gl(n;\mathbb{C})\}$ under the $\alpha_{\lambda E+A}$ action:

$$\begin{array}{rl} \alpha_{\lambda E+A} : \mathscr{G}\times M_n(\mathbb{C}) & \longrightarrow M_n(\mathbb{C})\times M_n(\mathbb{C}) \\ (P,Q) & \longrightarrow \lambda PEQ + PAQ \end{array}$$

**Proposition 4.1.** *The orbits of matrix pencils under classical equivalent relation are complex differentiable submanifolds of $M_n(\mathbb{C})\times M_n(\mathbb{C})$*

*Proof.* Taking into account that $\alpha_{\lambda E+A}$ is a rational map and $\mathscr{G}$ is obviously a constructible set, Chevalley's theorem (see for example [24]) states that $\alpha_{\lambda E+A}(\mathscr{G}=\mathscr{O}(\lambda E+A)$ is also constructible. Then $\mathscr{O}(\lambda E+A)$ has a nonsingular point. Taking into account that given any two points on the orbit there is a diffeomorphism of $M_n(\mathbb{C})\times M_n(\mathbb{C})$ preserving the orbit and mapping one onto the other, it follows that every point on the orbit is nonsingular. Hence $\mathscr{O}(\lambda E+A)$ is a complex differentiable manifold. $\square$

The orbits can be parameterized by

$$\begin{array}{ll}\alpha_1 : V & \longrightarrow \mathscr{O}(\lambda E+A)\\ (P,Q) & \longrightarrow \alpha_1(P,Q)=\alpha_{\lambda E+A}(P,Q)\end{array}$$

where $V$ is a submanifold of $\mathscr{G}$ minitransversal to the stabilizer defined as $Stab(\lambda E+A)=\{(P,Q)\mid \alpha_{\lambda E+A}(P,Q)=\lambda E+A\}$.
From $\alpha_1$ we can construct a local diffeomorphism at $(\lambda E+A,\lambda E+A\in\Gamma\times\mathscr{O}(\lambda E+A)$ which preserves the orbits as follows

$$\begin{array}{rl}\beta:\Gamma\times\mathscr{O}(\lambda E+A) & \longrightarrow M_n(\mathbb{C})\times M_n(\mathbb{C})\\ ((\lambda E+A)+(\lambda X+Y),(\lambda E'+A') & \longrightarrow \alpha_{(\lambda E+A)+(\lambda X+Y)}(\alpha_1^{-1}(\lambda E'+A'))\end{array}$$

where $\Gamma$ is a variety transversal to the orbit under strict equivalence of matrix pencils.
To study the regularity of strata, we first reduce the problem to the intersection with a variety $\Gamma$ transversal to the orbits of any element of the $D$-equivalence class. The variety considered in this paper is the miniversal deformation obtained in [23]:
$\Gamma=(\lambda E+A)+\{\lambda X+Y\}$, that has the following form for regular matrix pencils in canonical reduced form:

$$\left(\lambda\begin{pmatrix}I & \\ & N\end{pmatrix}+\begin{pmatrix}J & \\ & I\end{pmatrix}\right)+\left(\lambda\begin{pmatrix}0 & \\ & X_N\end{pmatrix}+\begin{pmatrix}Y_J & \\ & 0\end{pmatrix}\right)$$

where $N+X_N$ and $J+Y_J$ are miniversal deformations of square matrices under similarity (for instance, given in [1]).

**Remark 4.2.** $J+Y_J=\begin{pmatrix}J_1 & \\ & J(0)\end{pmatrix}+\begin{pmatrix}Y_{J_1} & \\ & Y_{J(0)}\end{pmatrix}$.

**Proposition 4.3.** *For this particular variety $\Gamma$ considered above,*

*a) If $\lambda X+Y\neq\lambda 0+0$ then $(\lambda E+A)+(\lambda X+Y)\notin\mathscr{O}(\lambda E+A)$.*
*b) $(\lambda E+A)+(\lambda X+Y)\in D(\nu,n_0,\sigma)$ if and only if $J_1+Y_{J_1}$ has the same Segre symbol than $J_1$, and $J(0)+Y_{J(0)}$ and $N+Y_N$ are nilpotent.*

**Lemma 4.4.** *Let $\lambda E+A$ be a matrix pencil in $M_n(\mathbb{C})\times M_n(\mathbb{C})$, $\mathscr{O}(\lambda E+A)$ its orbit, $D(\nu,n_0,\sigma)$ its stratum and $\Gamma$ the variety transversal to the orbit defined in [23]. Then, in a neighborhood of $\lambda E+A$, $D(\nu,n_0,\sigma)$ is a complex differentiable submanifold of $\mathscr{X}$ if and only if $D(\nu,n_0,\sigma)\cap\Gamma$ is*

*Proof.* Assume that $D(\nu,n_0,\sigma)$ is regular at $\lambda E+A$. Taking into account that $\Gamma$ is transversal to the orbit it is also transversal to $D(\nu,n_0,\sigma)$. Hence $D(\nu,n_0,\sigma)\cap\Gamma$ is regular at $\lambda E+A$.
Conversely, assume that $D(\nu,n_0,\sigma)\cap\Gamma$ is regular at $\lambda E+A$. Considering the local diffeormorphism $\beta$ we have

$$D(\nu,n_0,\sigma)=\beta(D(\nu,n_0,\sigma)\cup\Gamma)\times\mathscr{O}(\lambda E+A)$$

locally at $\lambda E+A$. Therefore $D(\nu,n_0,\sigma)$ is regular at $\lambda E+A$.

$\square$

Then we can conclude the following result.

**Theorem 4.5.** *The sets of the from $D(\nu,n_0,\sigma)$ are differentiable submanifolds of $\mathscr{X}$.*

*Proof.* Let $\lambda E+A$ be a regular pencil, $\mathscr{O}(\lambda E+A)$ its orbit and $D(\nu,n_0,\sigma)$ its stratum. We must prove that $D(\nu,n_0,\sigma)$ is regular at $\lambda E+A$. Taking into account (as we said before) that given any two points in the orbit there is a diffeomorphism of $M_n(\mathbb{C})\times M_n(\mathbb{C})$ preserving the orbit and mapping one onto the other. We consider the pencil in its reduced form.
By 4.4 it suffices to prove that $D(\nu,n_0,\sigma)\cap\Gamma$ is regular at $\lambda E+A$, for that we consider the following map

$$\begin{array}{rl}\phi: M_\nu(\mathbb{C})\times M_{n-n_0}(\mathbb{C})\times M_{n_0-\nu}(\mathbb{C}) & \longrightarrow M_n(\mathbb{C})\times M_n(\mathbb{C})\\ (A,B,C) & \longrightarrow \lambda\begin{pmatrix}I & & \\ & I & \\ & & B\end{pmatrix}+\begin{pmatrix}A & & \\ & B & \\ & & I\end{pmatrix}\end{array}$$

that is, clearly, a diffeomorphim such that

$$\phi(S_s(J(\sigma))\cap\Gamma_s(J)\times S_s(\mathscr{N}_E)\cap\Gamma_s(\mathscr{N}_E)\cap\mathscr{N}il_{n-n_0}\times S_s(N_G)\cap\Gamma_s(N_G)\cap\mathscr{N}il_{n_0-\nu})=D(\nu,n_0,\sigma)\cap\Gamma$$

(where $S_s(J(\sigma))$, $S_s(\mathscr{N}_E)$, $S_s(N_G)$ are the Segre strata of the square matrices $J(\sigma)$, $\mathscr{N}_E$ and $N_G$ under similarity and $\Gamma_s(J(\sigma))$, $\Gamma_s(\mathscr{N}_E)$ and $\Gamma_s(N_G)$ are linear varieties transversal to the Segre orbit of $J(\sigma)$, $\mathscr{N}_E$ an $N_G$ respectively and hence also transversal to $S_s(J(\sigma))$, $S_s(\mathscr{N}_E)$ and $S_s(N_G)$ at $J(\sigma)$, $\mathscr{N}_E$ and $N_G$ respectively)
Following [9], Segre strata are regular so, $S_s(J(\sigma))\cap\Gamma_s(J(\sigma))$, $S_s(\mathscr{N}_E)\cap\Gamma_s(\mathscr{N}_E)$ and $S_s(N_G)\cap\Gamma_s(N_G)$ are regular at $J(\sigma)$, $\mathscr{N}_E$ and $N_G$ respectively, and the proof is completed. $\square$

**Proposition 4.6.** *D-equivalence classes verify the frontier condition. That is to say, each frontier of strata consists of strata of strictly lower dimension.*

**Corollary 4.7.** *The partition of $\mathscr{X}$ into sets of the form $D(\nu, n_0, \sigma)$ constitutes a (finite) stratification of $\mathscr{X}$, which will be called D-stratification.*

## 5. Hierarchy diagrams

Given a non-standard regular pencil, not all small perturbations but some of them lead to a standard pencil with all the non-zero eigenvalues being different from each other.

**Example 5.1.** *Let us consider*

$$\lambda E + A = \lambda \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

*The small perturbation*

$$(\lambda E + A)(\varepsilon) = \lambda \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varepsilon_1 & 1 \\ & & & \varepsilon_2 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & \varepsilon_3 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \ \forall \varepsilon_i \neq 0, i = 1, 2, 3,$$

*is equivalent to*

$$(\lambda E_c + A_c)(\varepsilon) = \lambda \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & \varepsilon_3 & & \\ & & \frac{1}{\varepsilon_1} & \\ & & & \frac{1}{\varepsilon_2} \end{pmatrix}.$$

But between the initial pencil and the more generic one which was obtained, we can find other matrix pencils as, for example,

$$(\lambda E + A)(\varepsilon) = \lambda \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varepsilon & 1 \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \ \forall \varepsilon \neq 0$$

that is equivalent to

$$(\lambda E_c + A_c)(\varepsilon) = \lambda \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & \frac{1}{\varepsilon} & & \\ & & 0 & \\ & & & 1 \end{pmatrix}.$$

Therefore we are interested in finding all possible types of pencils that we can find in a neighbourhood of a given pencil and in what hierarchic position.

The construction of a hierarchy diagram is based upon two facts. First, the order of matrices $\mathscr{N}_E$ and $N_G$ in the reduced form can be the same or smaller than the original one when applying a small perturbation. The hierarchy diagram in the case the order of these matrices are the same can be deduced from the hierarchy diagrams in the case where square matrices under similarity are considered.

Taking into account the construction of each stratum we can deduce the hierarchic structure from the stratification induced by classical equivalence, by means of breaking joining equivalent strata and replacing them in the closure hierarchic.

We present the hierarchic closure for $n = 2$.

First of all, we show the list of all equivalent classes with a representant of each class.

| $D(\nu, n_0, \sigma)$ | $\lambda A + B$ |
|---|---|
| $D(0,0,-)$ | $\lambda \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ |
| $D(0,1,-)$ | $\lambda \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$ |
| $D(1,1,(1))$ | $\lambda \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & \\ & 1 \end{pmatrix}$ |
| $D(0,2,-)$ | $\lambda \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$ |
| $D(1,2,(1))$ | $\lambda \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & \\ & 0 \end{pmatrix}$ |
| $D(2,2,(2))$ | $\lambda \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & \\ 1 & \lambda_1 \end{pmatrix}$ |
| $D(2,2,(1,1))$ | $\lambda \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & \\ & \lambda_1 \end{pmatrix}$ |
| $D(2,2,((1);(1)))$ | $\lambda \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$ |

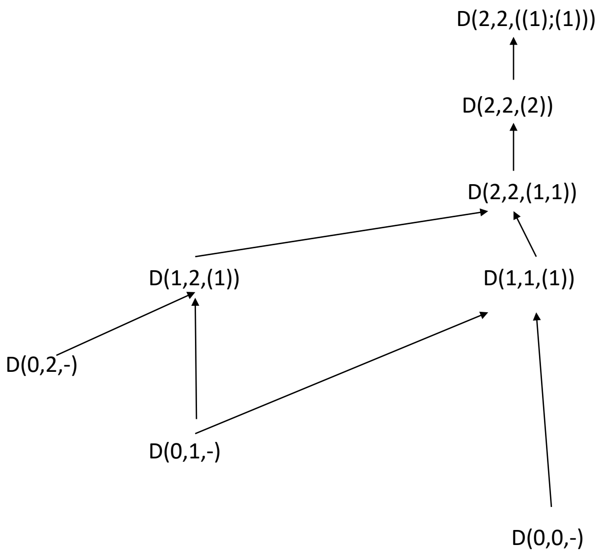


**Figure 5.1:** Hierarchic closure for $n = 2$

Then, the hierarchic closure is given as Figure 5.1.

Notation $D(\nu, n_0, \sigma) \to D(\nu', n'_0, \sigma')$ indicates that $D(\nu, n_0, \sigma) \subset \overline{D(\nu', n'_0, \sigma')}$ where $\overline{D(\nu', n'_0, \sigma')}$ is the closure of $D(\nu', n'_0, \sigma')$.

Now we present the case $n = 3$.

As in the case $n = 2$ we present the list of all equivalence classes

| $D(\nu, n_0, \sigma)$ | $\lambda A + B$ |
|---|---|
| $D(0,0,-)$ | $\lambda \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ |
| $D(0,1,-)$ | $\lambda \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ |
| $D(1,1,(1))$ | $\lambda \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ |
| $D(2,2,((1);(1)))$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & 1 \end{pmatrix}$ |
| $D(2,2,(2))$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} \lambda & 1 & \\ & \lambda & \\ & & 1 \end{pmatrix}$ |
| $D(2,2,(1,1))$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & 1 \end{pmatrix}$ |
| $D(1,2,(1))$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & \\ & 0 & \\ & & 1 \end{pmatrix}$ |
| $D(0,2,-)$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$ |
| $D(3,3,((1);(1);(1)))$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$ |
| $D(3,3,((2);(1)))$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$ |
| $D(3,3,(3))$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{pmatrix}$ |
| $D(3,3,(2,1))$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_1 \end{pmatrix}$ |
| $D(3,3,((1,1);(1)))$ | $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$ |

$$\begin{array}{cc}
D(3,3,((1);(1);(1)) & \lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix} \\
D(2,3,((1);(1)) & \lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & 0 \end{pmatrix} \\
D(2,3,(2)) & \lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & 0 \end{pmatrix} \\
D(2,3,(1,1)) & \lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & 0 \end{pmatrix} \\
D(1,3,(1)) & \lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \\
D(0,3,-) & \lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}
\end{array}$$

Then, the hierarchic closure is given in figure 5.2.

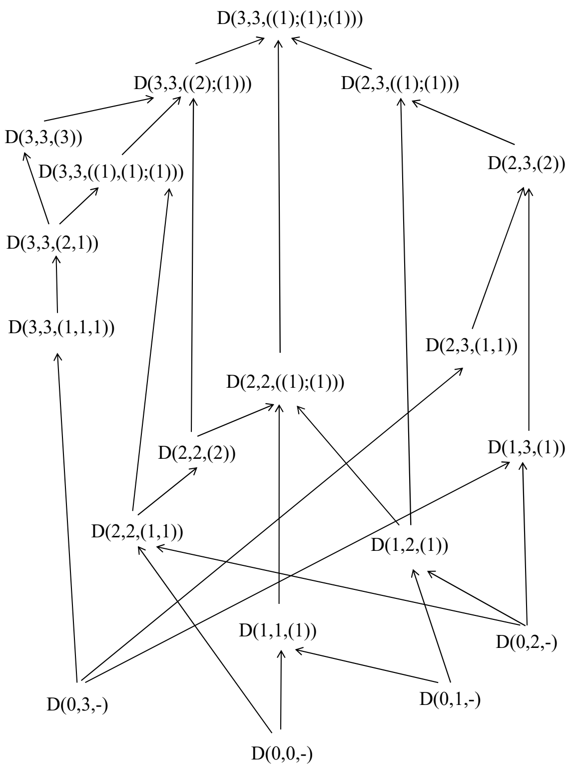


**Figure 5.2:** Hierarchic closure for $n = 3$

Notation $D(\nu, n_0, \sigma) \to D(\nu', n_0', \sigma')$ indicates that $D(\nu, n_0, \sigma) \subset \overline{D(\nu', n_0', \sigma')}$ where $\overline{D(\nu', n_0', \sigma')}$ is the closure of $D(\nu', n_0', \sigma')$.

## 6. Conclusion

In this work, a partition called D-stratification, in the space of pairs of matrices associated to regular matrix pencils preserving the order of the nilpotent parts has been considered and it was proved to be is a finite stratification of the space of such matrix pencils. This study shows the effect of perturbations over the Kronecker canonical form of a prescribed pencil. We present the $D$-strata that can be reached in the cases where the order of the matrix pencils is 2 or 3 and obtain the corresponding hierarchy graphs, thus illustrating the $D$-strata that is possible to reach when applying small perturbations.

## References

[1] V. I. Arnold, *On matrices depending on parameters*, Russian Math. Surveys, **26**(2), (1971), 29–43.
[2] V. I. Arnold, *Singularity Theory*, Cambridge University Press, 1981.
[3] H. den Boer, G. Ph. A. Thijsse, *Semi-stability of sums of partial multiplicities under additive perturbation*, Integral Equations Operator Theory, **3** (1980), 23–42.
[4] A. S. Markus, E. E. Parilis, *The change of the Jordan structure of a matrix under small perturbations*, Linear Algebra and its Applications, **54**, (1983), 139–152.
[5] M. A. Beitia, I. de Hoyos, I. Zaballa, *The change of similarity invariants under row perturbations: Generic cases*, Linear Algebra Appl., **429** (2008), 482–496.

[6] M. A. Beitia, I. de Hoyos, I. Zaballa, *The change of similarity invariants under row perturbations*, Linear Algebra Appl., **429**(5–6) (2008), 1302–1333.
[7] L. Klimenko, V.V. Sergeichuk, *An informal introduction to perturbations of matrices determined up to similarity or congruence*, Sao Paulo Journal of Mathematical Sciences, **8**(1), (2014), 1–22.
[8] A. Dmytryshyn, V. Futorny, B. Kågström, L. Klimenko, V.V. Sergeichuk, *Change of the congruence canonical form of 2-by-2 and 3-by-3 matrices under perturbations and bundles of matrices under congruence*, Linear Algebra Appl., **469** (2015), 305–334.
[9] C. G. Gibson, *Regularity of the Segre Stratification*, Math. Proc. Camb. Phil. Soc. **80**, (1976), 91–97.
[10] M. D. Magret, *Stratification of the space of matrices defining linear dynamical systems*, Int. J. Complex Systems in Science, **6**(1), (2016), 29–35.
[11] M. D. Magret, *Perturbation of matrices preserving the Segre characteristics of the non-nilpotent Jordan blocks*, CEDYA 2017 - Communications Book, (2017), 429–432.
[12] J. Clotet, M. D. Magret, *Familias diferenciables de inversas de Drazin*, UPC Commons, 2005.
[13] M. I. García-Planas, *Estudio geométrico de familias diferenciables de parejas de matrices: deformaciones versales, cambios de base globales, estratificación de Brunvsky*. Ph D. Thesis, UPC, 1994.
[14] M. I. García-Planas, V.V. Sergeichuk, *Generic families of matrix pencils and their bifurcation diagrams*, Linear Algebra and its Applications 332, (2001), 165–179.
[15] M. I. García-Planas, *Kronecker stratification of the space of quadruples of matrices*, SIAM Journal on Matrix Analysis and Applications, **19**(4), (1998), 872–885.
[16] A. Pokrzywa, *On perturbations and the equivalence orbit of a matrix pencil*, Linear Algebra and its Applications, **82**, (1986), 99–121.
[17] A. Edelman, E. Elmroth, B. Kågström, *A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm*, SIAM J. Matrix Anal. Appl., **31**(3) (1999), 667–699.
[18] E. Elmroth, P. Johansson, B. Kågström, *Computation and presentation of graphs displaying closure hierarchies of Jordan and Kronecker structures*, Numer. Linear Algebra Appl., **8**(6-7) (2001), 381–399.
[19] V. Futorny, L. Klimenko, V.V. Sergeichuk, *Change of the congruence canonical form of 2-by-2 matrices under perturbations*, Electron. J. Linear Algebra, **27**, (2014), 146–154.
[20] A. Edelman, E. Elmroth, B. Kågström, *A geometric approach to perturbation theory of matrices and matrix pencils, Part I: Versal deformations*, SIAM J. Matrix Anal. Appl., **18**(3) (1997), 653–692.
[21] F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York (1959).
[22] L. Dai. *Singular Control Systems*, Springer Verlag, New York, 1989.
[23] M. I. García-Planas, V.V. Sergeichuk, *Simplest miniversal deformations of matrices, matrx pencils, and contragredient matrix pencils*, Linear Algebra and its Applications, 302/303, (1999), 45–61.
[24] J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, (1981).

*Universal Journal of Mathematics and Applications*, 2 (1) (2019) 24-32
**Research paper**

UJMA

# Universal Journal of Mathematics and Applications

*Journal Homepage: www.dergipark.gov.tr/ujma*
*ISSN 2619-9653*
*DOI: DOI: http://dx.doi.org/10.32323/ujma.472929*

# Filtering of Multidimensional Stationary Processes with Missing Observations

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### Article Info

***Keywords:*** *Minimax-Robust estimate, Least favourable spectral density*
***2010 AMS:*** *60G10, 60G25, 60G35, 62M20, 93E10, 93E11*
***Received:*** *22 October 2018*
***Accepted:*** *4 March 2019*
***Available online:*** *20 March 2019*

### Abstract

The problem of the mean-square optimal linear estimation of linear functionals which depend on the unknown values of a multidimensional continuous time stationary stochastic process from observations of the process with a stationary noise is considered. Formulas for calculating the mean-square errors and the spectral characteristics of the optimal linear estimates of the functionals are derived under the condition of spectral certainty, where spectral densities of the signal and the noise processes are exactly known. The minimax (robust) method of estimation is applied in the case of spectral uncertainty, where spectral densities of the processes are not known exactly, while some sets of admissible spectral densities are given. Formulas that determine the least favorable spectral densities and minimax spectral characteristics of the optimal estimates are derived for some special sets of admissible spectral densities.

## 1. Introduction

The problem of estimation of the unknown values of stochastic processes is of constant interest in the theory and applications of stochastic processes. The formulation of the interpolation, extrapolation and filtering problems for stationary stochastic sequences with known spectral densities and reducing the estimation problems to the corresponding problems of the theory of functions belongs to Kolmogorov [1]. Effective methods of solution of the estimation problems for stationary stochastic processes were developed by Wiener [2] and Yaglom [3, 4]. Further results are presented in the books by Rozanov [5], Hannan [6], Box et. al [7], Brockwell and Davis [8].
The crucial assumption of most of the methods developed for estimating the unobserved values of stochastic processes is that the spectral densities of the involved stochastic processes are exactly known. However, in practice, complete information on the spectral densities is impossible in most cases. In this situation, one finds the parametric or nonparametric estimate of the unknown spectral density and then apply one of the traditional estimation methods provided that the selected density is the true one. This procedure can result in significant increasing of the value of the error of estimate as Vastola and Poor [9] have demonstrated with the help of some examples. To avoid this effect one can search estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximum value of the errors of estimates. The paper by Grenander [10] was the first one where this approach to extrapolation problem for stationary processes was proposed. Several models of spectral uncertainty and minimax-robust methods of data processing can be found in the survey paper by Kassam and Poor [11]. In the papers by Franke [12], [13] Franke and Poor [14] the minimax extrapolation and filtering problems for stationary sequences were investigated with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities. In the papers by Moklyachuk [15, 16] the extrapolation, interpolation and filtering problems for functionals which depend on the unknown values of stationary processes and sequences are investigated. The estimation problems for functionals which depend on the unknown values of multidimensional stationary stochastic processes is the aim of the investigation by Moklyachuk and Masyutka [17, 18]. In their book Moklyachuk and Golichenko [19] presented results of investigation of the interpolation, extrapolation and filtering problems for periodically correlated stochastic sequences. In the papers by Luz and Moklyachuk [20], Luz2016 results of an investigation of the estimation problems for functionals which depend on the unknown values of stochastic sequences with stationary increments are described. Prediction problem

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for stationary sequences with missing observations is investigated in papers by Bondon [21, 22], Cheng, Miamee and Pourahmadi [23], Cheng and Pourahmadi [24], Kasahara, Pourahmadi and Inoue [25], Pourahmadi, Inoue and Kasahara [26], Pelagatti [27]. In papers by Moklyachuk and Sidei [28] - [31] an approach is developed to an investigation of the interpolation, extrapolation and filtering problems for stationary stochastic processes with missing observations.
In this article, we deal with the problem of the mean-square optimal linear estimation of the functional

$$A\vec{\xi}=\int\limits_{R^s}\vec{a}(t)^{\top}\vec{\xi}(-t)dt,$$

which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t)$ from observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ at points $t\in\mathbb{R}^{-}\backslash S$, $S=\bigcup\limits_{l=1}^{s}[-M_l-N_l,-M_l]$, $R^s=[0,\infty)\backslash S^{+}$, $S^{+}=\bigcup\limits_{l=1}^{s}[M_l,\,M_l+N_l]$. The case of spectral certainty, as well as the case of spectral uncertainty, are considered. Formulas for calculating the spectral characteristic and the mean-square error of the optimal linear estimate of the functional are derived under the condition of spectral uncertainty, where the spectral densities of the processes are exactly known. In the case of spectral uncertainty, where the spectral densities are not exactly known while a set of admissible spectral densities is given, the minimax method is applied. Formulas for determination the least favorable spectral densities and the minimax-robust spectral characteristics of the optimal estimates of the functional are proposed for some specific classes of admissible spectral densities.

## 2. Hilbert space projection method of filtering

Let $\vec{\xi}(t)=\{\xi_k(t)\}_{k=1}^{T},t\in\mathbb{R}$, and $\vec{\eta}(t)=\{\eta_k(t)\}_{k=1}^{T},t\in\mathbb{R}$, be uncorrelated mean square continuous multidimensional stationary stochastic processes with zero first moments, $E\vec{\xi}(t)=\vec{0}$, $E\vec{\eta}(t)=\vec{0}$, absolutely continuous spectral functions and spectral density matrices which satisfy the minimality condition

$$\int_{-\infty}^{\infty}(b(\lambda))^{\top}(F(\lambda)+G(\lambda))^{-1}\overline{b(\lambda)}d\lambda<\infty, \tag{2.1}$$

where $b(\lambda)=\sum\limits_{l=1}^{s}\int\limits_{-M_l-N_l}^{-M_l}\vec{\alpha}(t)e^{it\lambda}dt$ is a nontrivial function of the exponential type. Under this condition the error-free estimate of the process $\vec{\xi}(t)+\vec{\eta}(t)$ is impossible (see, for example, Rozanov [5]).
Suppose that we have observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ at points $t\in\mathbb{R}^{-}\backslash S$, where

$$S=\bigcup_{l=1}^{s}[-M_l-N_l,-M_l],\; R^s=[0,\infty)\backslash S^{+},\; S^{+}=\bigcup_{l=1}^{s}[M_l,\,M_l+N_l]$$

.
The main purpose of this article is to find the mean-square optimal linear estimate of the functional

$$A\vec{\xi}=\int\limits_{R^s}\vec{a}(t)^{\top}\vec{\xi}(-t)dt,$$

which depends on the unknown values of the process $\vec{\xi}(t)$.
We will assume that the function $\vec{a}(t)$ satisfies the condition

$$\sum_{k=1}^{T}\int\limits_{R^s}|a_k(t)|dt<\infty. \tag{2.2}$$

This condition ensures that the functional $A_s\vec{\xi}$ has a finite second moment.
It follows from the spectral decompositions of the processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ (see Gikhman and Skorokhod [32])

$$\vec{\xi}(t)=\int\limits_{-\infty}^{\infty}e^{it\lambda}Z_{\xi}(d\lambda),\qquad \vec{\eta}(t)=\int\limits_{-\infty}^{\infty}e^{it\lambda}Z_{\eta}(d\lambda),$$

where $Z_{\xi}(d\lambda)$ and $Z_{\eta}(d\lambda)$ are vector valued orthogonal stochastic measures, that the functional $A\vec{\xi}$ can be represented in the form

$$A\vec{\xi}=\int\limits_{-\infty}^{\infty}(A(\lambda))^{\top}Z_{\xi}(d\lambda),\quad A(\lambda)=\int\limits_{R^s}\vec{a}(t)e^{-it\lambda}dt.$$

Consider the Hilbert space $H=L_2(\Omega,\mathscr{F},P)$ generated by random variables $\xi$ with zero mathematical expectations, $E\xi=0$, finite variations, $E|\xi|^2<\infty$, and inner product $(\xi,\eta)=E\xi\overline{\eta}$. Denote by $H^s(\xi+\eta)$ the closed linear subspace generated by elements $\{\xi_k(t)+\eta_k(t):t\in\mathbb{R}^{-}\backslash S,k=\overline{1,T}\}$ in the Hilbert space $H=L_2(\Omega,\mathscr{F},P)$.
Let $L_2(F+G)$ be the Hilbert space of complex-valued functions $\vec{a}(\lambda)=\{a_k(\lambda)\}_{k=1}^{T}$ such that

$$\int_{-\infty}^{\infty}\vec{a}(\lambda)^{\top}(F(\lambda)+G(\lambda))\overline{\vec{a}(\lambda)}d\lambda=\int_{-\infty}^{\infty}\sum_{k,l=1}^{T}a_k(\lambda)\overline{a_l(\lambda)}(f_{kl}(\lambda)+g_{kl}(\lambda))d\lambda<\infty.$$

Denote by $L_2^s(F+G)$ the subspace of $L_2(F+G)$ generated by functions

$$e^{it\lambda}\delta_k,\ \delta_k=\{\delta_{kl}\}_{l=1}^{T},\ k=\overline{1,T},\ t\in\mathbb{R}^-\backslash S.$$

Denote by $\hat{A}_s\vec{\xi}$ the optimal linear estimate of the functional $A_s\vec{\xi}$ from observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ and denote by $\Delta(F,G)=E\left|A_s\vec{\xi}-\hat{A}_s\vec{\xi}\right|^2$ the mean-square error of the estimate $\hat{A}_s\vec{\xi}$.

The mean-square optimal linear estimate $\hat{A}_s\vec{\xi}$ of the functional $A_s\vec{\xi}$ is of the form

$$\hat{A}\vec{\xi}=\int_{-\infty}^{\infty}(h(\lambda))^{\top}(Z_{\xi}(d\lambda)+Z_{\eta}(d\lambda)),$$

where $h(\lambda)=\{h_k(\lambda)\}_{k=1}^{T}\in L_2^s(F+G)$ is the spectral characteristic of the estimate, and the mean-square error $\Delta(h;F,G)$ of the estimate is determined by formula

$$\Delta(h;F,G)=E\left|A\vec{\xi}-\hat{A}\vec{\xi}\right|^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}(A(\lambda)-h(\lambda))^{\top}F(\lambda)\overline{(A(\lambda)-h(\lambda))}d\lambda+\frac{1}{2\pi}\int_{-\infty}^{\infty}(h(\lambda))^{\top}G(\lambda)\overline{h(\lambda)}d\lambda.$$

Since we suppose that the spectral densities of the stationary processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ are known, we can apply the method of orthogonal projections in the Hilbert spaces proposed by A. N. Kolmogorov [1] in order to find the optimal estimate. According to this method, the optimal linear estimation of the functional $A\vec{\xi}$ is a projection of the element $A\vec{\xi}$ of the space $H$ on the subspace $H^s(\xi+\eta)$. The estimate is determined by two conditions:

$$1)\hat{A}\vec{\xi}\in H^s(\xi+\eta),$$
$$2)A\vec{\xi}-\hat{A}\vec{\xi}\perp H^s(\xi+\eta).$$

Under the second condition, the spectral characteristic $h(\lambda)$ of the optimal linear estimate $\hat{A}\vec{\xi}$ satisfies the relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[(A(\lambda))^{\top}F(\lambda)-(h(\lambda))^{\top}(F(\lambda)+G(\lambda))\right]e^{-it\lambda}d\lambda=0,\quad t\in\mathbb{R}^-\backslash S. \tag{2.3}$$

Consider the function $(C(\lambda))^{\top}=(A(\lambda))^{\top}F(\lambda)-(h(\lambda))^{\top}(F(\lambda)+G(\lambda))$ and its Fourier transform

$$\vec{\mathbf{c}}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C(\lambda)e^{-it\lambda}d\lambda,\quad t\in\mathbb{R}.$$

It follows from relation (2.3), that the function $\mathbf{c}(t)$ can be nonzero only on the set $U=S\cup[0,\infty)$. Hence, the function $C(\lambda)$ is of the form

$$C(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}\vec{\mathbf{c}}(t)e^{it\lambda}dt+\int_{0}^{\infty}\vec{\mathbf{c}}(t)e^{it\lambda}dt,$$

and the spectral characteristic of the estimate $\hat{A}\vec{\xi}$ is of the form

$$(h(\lambda))^{\top}=(A(\lambda))^{\top}F(\lambda)(F(\lambda)+G(\lambda))^{-1}-(C(\lambda))^{\top}(F(\lambda)+G(\lambda))^{-1}.$$

It follows from the first condition, $\hat{A}\vec{\xi}\in H^s(\xi+\eta)$, which determines the estimate of the functional $A\vec{\xi}$, that for any $t\in U$ the following relation holds true

$$\int_{-\infty}^{\infty}\left((A(\lambda))^{\top}F(\lambda)(F(\lambda)+G(\lambda))^{-1}-(C(\lambda))^{\top}(F(\lambda)+G(\lambda))^{-1}\right)e^{-it\lambda}d\lambda=0. \tag{2.4}$$

Let us define the following operators in the space $L_2(U)$

$$(\mathbf{B}\mathbf{x})(t)=\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}(F(\lambda)+G(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du+\frac{1}{2\pi}\int_{0}^{\infty}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}(F(\lambda)+G(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du,$$

$$(\mathbf{R}\mathbf{x})(t)=\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{i\lambda(u+t)}d\lambda du+\frac{1}{2\pi}\int_{0}^{\infty}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du,$$

$$(\mathbf{Q}\mathbf{x})(t)=\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(F(\lambda)+G(\lambda))^{-1}G(\lambda)e^{i\lambda(u-t)}d\lambda du+$$
$$+\frac{1}{2\pi}\int_{0}^{\infty}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(F(\lambda)+G(\lambda))^{-1}G(\lambda)e^{i\lambda(u-t)}d\lambda du,$$

$$\vec{\mathbf{x}}(t) \in L_2(U), \quad t \in U.$$

The equality (2.4) can be represented in the form

$$\int_{-\infty}^{\infty}\int_{R^s} \vec{a}(u)^\top F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{i(u-t)}dud\lambda - \int_{-\infty}^{\infty}\left(\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} \vec{\mathbf{c}}(t)^\top (F(\lambda)+G(\lambda))^{-1}e^{i(u-t)\lambda}du\right)d\lambda$$
$$-\int_{-\infty}^{\infty}\int_{0}^{\infty} \vec{\mathbf{c}}(t)^\top (F(\lambda)+G(\lambda))^{-1}e^{i(u-t)\lambda}dud\lambda = 0, \quad t \in U. \quad (2.5)$$

Let $\vec{\mathbf{a}}(t)$ be a function such that

$$\vec{\mathbf{a}}(t) = \vec{0},\, t \in S, \quad \vec{\mathbf{a}}(t) = \vec{a}(t), t \in R^s \quad \vec{\mathbf{a}}(t) = \vec{0},\, t \in S^+.$$

Making use of the introduces above notations, we can represent equality (2.5) in terms of linear operators in the space $L_2(U)$

$$(\mathbf{Ra})(t) = (\mathbf{Bc})(t), \quad t \in U.$$

Assume that the operator $\mathbf{B}$ is invertible (see paper by Salehi [33] for more details). Then the function $\vec{\mathbf{c}}(t)$ can be found and it is calculated by the formula

$$\vec{\mathbf{c}}(t) = (\mathbf{B}^{-1}\mathbf{Ra})(t), \quad t \in U.$$

The spectral characteristic $h(\lambda)$ of the estimate $\hat{A}\vec{\xi}$ is calculated by the formula

$$(h(\lambda))^\top = (A(\lambda))^\top F(\lambda)(F(\lambda)+G(\lambda))^{-1} - (C(\lambda))^\top (F(\lambda)+G(\lambda))^{-1},$$
$$C(\lambda) = \sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} (\mathbf{B}^{-1}\mathbf{Ra})(t)e^{it\lambda}dt + \int_{0}^{\infty}(\mathbf{B}^{-1}\mathbf{Ra})(t)e^{it\lambda}dt. \quad (2.6)$$

The mean-square error of the estimate $\hat{A}\vec{\xi}$ is calculated by the formula

$$\Delta(h;F,G) = \frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top G(\lambda)+(C(\lambda))^\top)(F(\lambda)+G(\lambda))^{-1}F(\lambda)(F(\lambda)+G(\lambda))^{-1}((A(\lambda))^\top G(\lambda)+(C(\lambda))^\top)^* d\lambda +$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top F(\lambda)-(C(\lambda))^\top)(F(\lambda)+G(\lambda))^{-1}G(\lambda)(F(\lambda)+G(\lambda))^{-1}((A(\lambda))^\top G(\lambda)+(C(\lambda))^\top)^* d\lambda =$$
$$= \langle(\mathbf{Ra})(t),(\mathbf{B}^{-1}\mathbf{Ra})(t)\rangle + \langle(\mathbf{Qa})(t),\vec{\mathbf{a}}(t)\rangle, \quad (2.7)$$

where

$$\langle\vec{a}(t),\vec{b}(t)\rangle = \sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} a_k(t)\overline{b_k(t)}dt + \int_{0}^{\infty} a_k(t)\overline{b_k(t)}dt$$

is the inner product in the space $L_2(U)$.
The obtained results can be summarized in the form of theorem.

**Theorem 2.1.** *Let $\vec{\xi}(t)$ and $\vec{\eta}(t)$ be uncorrelated multidimensional stationary stochastic processes with the spectral densities $F(\lambda)$ and $G(\lambda)$ which satisfy the minimality condition (2.1). Let condition (2.2) be satisfied and let the operator* $\mathbf{B}$ *be invertible. The spectral characteristic $h(\lambda)$ and the mean-square error $\Delta(h;F,G)$ of the optimal linear estimate of the functional $A\vec{\xi}$ which depends on the unknown values of the process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$, $t \in \mathbb{R}^-\backslash S$ are calculated by formulas (2.6), (2.7).*

## 3. Minimax-robust method of filtering

In the previous sections, we deal with the filtering problem under the condition that we know spectral densities of the processes. In this case, we derived formulas for calculating the spectral characteristics and the mean-square errors of the optimal estimates of the introduced functionals. In the case of spectral uncertainty, where full information on spectral densities is impossible while it is known that spectral densities belong to some specified classes of admissible densities, the minimax method of filtering is reasonable. This method gives us a procedure of finding estimates which minimize the maximum values of the mean-square errors of the estimates for all spectral densities from the given class of admissible spectral densities. For the description of the minimax method, we propose the following definitions (see book by Moklyachuk and Masytka [18] for more details).

**Definition 3.1.** *For a given class of spectral densities $D = D_F \times D_G$ the spectral densities $F^0(\lambda) \in D_F$, $G^0(\lambda) \in D_G$ are called least favorable in class $D$ for the optimal linear filtering of the functional $A\vec{\xi}$ if the following relation holds true*

$$\Delta\left(F^0,G^0\right) = \Delta\left(h\left(F^0,G^0\right);F^0,G^0\right) = \max_{(F,G)\in D_F\times D_G} \Delta\left(h\left(F,G\right);F,G\right).$$

**Definition 3.2.** *For a given class of spectral densities $D=D_F\times D_G$ the spectral characteristic $h^0(\lambda)$ of the optimal linear filtering of the functional $A\vec{\xi}$ is called minimax-robust if there are satisfied conditions*

$$h^0(\lambda)\in H_D=\bigcap_{(F,G)\in D_F\times D_G} L_2^s(F+G),$$

$$\min_{h\in H_D}\max_{(F,G)\in D}\Delta(h;F,G)=\max_{(F,G)\in D}\Delta\left(h^0;F,G\right).$$

From the introduced definitions and formulas derived above, we can obtain the following statement.

**Lemma 3.3.** *Spectral densities $F^0(\lambda)\in D_F$, $G^0(\lambda)\in D_G$ satisfying the minimality condition (2.1) are the least favorable in the class $D=D_F\times D_G$ for the optimal linear filtering of the functional $A\vec{\xi}$, if the Fourier coefficients of the functions*

$$(F^0(\lambda)+G^0(\lambda))^{-1},\quad F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1},\quad F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}G^0(\lambda)$$

*determine operators $\mathbf{B}^0,\mathbf{R}^0,\mathbf{Q}^0$, which give a solution of the constrained optimization problem*

$$\max_{(F,G)\in D_F\times D_G}(\langle(\mathbf{Ra})(t),(\mathbf{B}^{-1}\mathbf{Ra})(t)\rangle+\langle(\mathbf{Qa})(t),\vec{\mathbf{a}}(t)\rangle)=\langle(\mathbf{R}^0\mathbf{a})(t),((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)\rangle+\langle(\mathbf{Q}^0\mathbf{a})(t),\vec{\mathbf{a}}(t)\rangle. \tag{3.1}$$

*The minimax spectral characteristic $h^0=h(F^0,G^0)$ is determined by formula (2.6) if $h(F^0,G^0)\in H_D$.*

For more detailed analysis of properties of the least favorable spectral densities and the minimax-robust spectral characteristics we observe that the least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ and the minimax spectral characteristic $h^0=h(F^0,G^0)$ form a saddle point of the function $\Delta(h;F,G)$ on the set $H_D\times D$. The saddle point inequalities

$$\Delta\left(h^0;F,G\right)\leq\Delta\left(h^0;F^0,G^0\right)\leq\Delta\left(h;F^0,G^0\right),\ \forall h\in H_D,\forall F\in D_F,\forall G\in D_G,$$

hold true if $h^0=h(F^0,G^0)$, $h(F^0,G^0)\in H_D$, where $(F^0,G^0)$ is a solution of the constrained optimization problem

$$\sup_{(F,G)\in D_F\times D_G}\Delta\left(h(F^0,G^0);F,G\right)=\Delta\left(h(F^0,G^0);F^0,G^0\right). \tag{3.2}$$

The linear functional $\Delta\left(h\left(F^0,G^0\right);F,G\right)$ is calculated by the formula

$$\Delta\left(h\left(F^0,G^0\right);F,G\right)=$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)(F^0(\lambda)+G^0(\lambda))^{-1}F(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*d\lambda+$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)(F^0(\lambda)+G^0(\lambda))^{-1}G(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}((A(\lambda))^\top G^0(\lambda)-(C^0(\lambda))^\top)^*d\lambda,$$
$$C^0(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\lambda}dt+\int_0^{\infty}((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\lambda}dt.$$

The constrained optimization problem (3.2) is equivalent to the unconstrained optimization problem (see book by Pshenichnyj [34])

$$\Delta_D(F,G)=-\Delta(h(F^0,G^0);F,G)+\delta((F,G)\left|D_F\times D_G\right.)\to\inf, \tag{3.3}$$

where $\delta((F,G)\left|D_F\times D_G\right.)$ is the indicator function of the set $D=D_F\times D_G$. Solution of the problem (3.3) is characterized by the condition $0\in\partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is the subdifferential of the convex functional $\Delta_D(F,G)$ at point $(F^0,G^0)$, namely, the set of all continuous linear functionals $\Lambda$ on $L_1\times L_1$ satisfying the inequality $\Delta_D(F,G)-\Delta_D(F^0,G^0)\geq\Lambda(F,G)-\Lambda(F^0,G^0)$. This condition makes it possible to find the least favourable spectral densities in some special classes of spectral densities $D$ (see books by Ioffe and Tihomirov [35], Pshenichnyj [34], Rockafellar [36]).
Note, that the form of the functional $\Delta(h(F^0,G^0);F,G)$ is convenient for application of the Lagrange method of indefinite multipliers for finding solution of the problem (3.2). Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine least favourable spectral densities in some special classes of spectral densities (see books by Moklyachuk [37, 15], Moklyachuk and Masyutka [18] for additional details).

## 4. Least favorable spectral densities in the class $D=D_0\times D_{1\delta}$

Consider the problem of minimax filtering of the functional $A\vec{\xi}$ in the case where spectral densities of the processes belong to the following classes of admissible spectral densities $D=D_0\times D_{1\delta}$,

$$D_0^1=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Tr}\,F(\lambda)d\lambda=p\right.\right\},$$

$$D_{1\delta}^1=\left\{G(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathrm{Tr}(G(\lambda)-G_1(\lambda))\right|d\lambda\leq\delta\right.\right\};$$

$$D_0^2=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}f_{kk}(\lambda)d\lambda=p_k, k=\overline{1,T}\right.\right\},$$
$$D_{1\delta}^2=\left\{G(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|g_{kk}(\lambda)-g_{kk}^1(\lambda)\right|d\lambda\leq\delta_k, k=\overline{1,T}\right.\right\};$$
$$D_0^3=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle B_1,F(\lambda)\rangle d\lambda=p\right.\right\},$$
$$D_{1\delta}^3=\left\{G(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}|\langle B_2,G(\lambda)-G_1(\lambda)\rangle|d\lambda\leq\delta\right.\right\},$$
$$D_0^4=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\lambda)d\lambda=P\right.\right\},$$
$$D_{1\delta}^4=\left\{G(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|g_{ij}(\lambda)-g_{ij}^1(\lambda)\right|d\lambda\leq\delta_i^j, i,j=\overline{1,T}\right.\right\},$$

where $G_1(\lambda)$ is a known and fixed spectral density matrix, $\delta,p,\delta_k,p_k,k=\overline{1,T}$, $\delta_i^j,i,j=\overline{1,T}$, are given numbers, $P,B_1,B_2$ are given positive definite Hermitian matrices.
The classes $D_{1\delta}$ describe the "$\delta$-neighborhood" models in the space $L_1$ of a given bounded spectral density matrix $G_1(\lambda)$.
From the condition $0\in\partial\Delta_D(F^0,G^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.
For the first pair $D_0^1\times D_{1\delta}^1$ we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=\alpha^2(F^0(\lambda)+G^0(\lambda))^2, \tag{4.1}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=\beta^2\gamma(\lambda)(F^0(\lambda)+G^0(\lambda))^2, \tag{4.2}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathrm{Tr}\,(G^0(\lambda)-G_1(\lambda))\right|d\lambda=\delta, \tag{4.3}$$

where $\alpha^2,\beta^2$ are Lagrange multipliers, $|\gamma(\lambda)|\leq1$ and

$$\gamma(\lambda)=\mathrm{sign}\,(\mathrm{Tr}\,(G^0(\lambda)-G_1(\lambda)))\quad\text{if}\quad\mathrm{Tr}\,(G^0(\lambda)-G_1(\lambda))\neq0.$$

For the second pair $D_0^2\times D_{1\delta}^2$, we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\left\{\alpha_k^2\delta_{kl}\right\}_{k,l=1}^T(F^0(\lambda)+G^0(\lambda)), \tag{4.4}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\left\{\beta_k^2\gamma_k(\lambda)\delta_{kl}\right\}_{k,l=1}^T(F^0(\lambda)+G^0(\lambda)), \tag{4.5}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|g_{kk}^0(\lambda)-g_{kk}^1(\lambda)\right|d\lambda=\delta_k,\ k=\overline{1,T}, \tag{4.6}$$

where $\alpha_k^2,\beta_k^2$ are Lagrange multipliers, $\delta_{kl}$ are Kronecker symbols, $|\gamma_k(\lambda)|\leq1$ and

$$\gamma_k(\lambda)=\mathrm{sign}\,(g_{kk}^0(\lambda)-g_{kk}^1(\lambda))\quad\text{if}\quad g_{kk}^0(\lambda)-g_{kk}^1(\lambda)\neq0,\ k=\overline{1,T}.$$

For the third pair $D_0^3\times D_{1\delta}^3$, we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=\alpha^2(F^0(\lambda)+G^0(\lambda))B_1^\top(F^0(\lambda)+G^0(\lambda)), \tag{4.7}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=\beta^2\gamma'(\lambda)(F^0(\lambda)+G^0(\lambda))B_2^\top(F^0(\lambda)+G^0(\lambda)), \tag{4.8}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\left\langle B_2,G^0(\lambda)-G_1(\lambda)\right\rangle\right|d\lambda=\delta, \tag{4.9}$$

where $\alpha^2,\beta^2$ are Lagrange multipliers, $|\gamma'(\lambda)|\leq1$ and

$$\gamma'(\lambda)=\mathrm{sign}\left\langle B_2,G^0(\lambda)-G_1(\lambda)\right\rangle\quad\text{if}\quad\left\langle B_2,G^0(\lambda)-G_1(\lambda)\right\rangle\neq0.$$

For the fourth pair $D_0^4\times D_{1\delta}^4$, we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)-(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\vec{\alpha}\cdot\vec{\alpha}^*(F^0(\lambda)+G^0(\lambda)), \tag{4.10}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)+(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\left\{\beta_{ij}\gamma_{ij}(\lambda)\right\}_{i,j=1}^T(F^0(\lambda)+G^0(\lambda)), \tag{4.11}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\right|d\lambda=\delta_i^j,\ i,j=\overline{1,T}, \tag{4.12}$$

where $\vec{\alpha},\beta_{ij}$ are Lagrange multipliers, $|\gamma_{ij}(\lambda)|\leq 1$ and

$$\gamma_{ij}(\lambda)=\frac{g_{ij}^0(\lambda)-g_{ij}^1(\lambda)}{\left|g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\right|}\quad \text{if}\quad g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\neq 0,\ i,j=\overline{1,T}.$$

Thus, the following statement holds true.

**Theorem 4.1.** *The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ in the classes $D_0\times D_{1\delta}$ for the optimal linear filtering of the functional $A\vec{\xi}$ are determined by relations (4.1) – (4.3) for the first pair $D_0^1\times D_{1\delta}^1$ of sets of admissible spectral densities; by relations (4.4) – (4.6) for the second pair $D_0^2\times D_{1\delta}^2$ of sets of admissible spectral densities; by relations (4.7) – (4.9) for the third pair $D_0^3\times D_{1\delta}^3$ of sets of admissible spectral densities; by relations (4.10) – (4.12) for the fourth pair $D_0^4\times D_{1\delta}^4$ of sets of admissible spectral densities; the minimality condition (2.1), the constrained optimization problem (3.1) and restrictions on densities from the corresponding classes $D_0\times D_{1\delta}$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (2.6).*

**Corollary 4.2.** *Assume that the spectral density $G(\lambda)$ is known. Let the function $F^0(\lambda)+G(\lambda)$ satisfy the minimality condition (2.1). The spectral density $F^0(\lambda)$ is the least favorable in the classes $D_0^k$, $k=\overline{1,4}$ for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (4.1), (4.4), (4.7), (4.10), respectively, and the pair $(F^0(\lambda),G(\lambda))$ is a solution of the optimization problem (3.1). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (2.6).*

**Corollary 4.3.** *Assume that the spectral density $F(\lambda)$ is known. Let the function $F(\lambda)+G^0(\lambda)$ satisfy the minimality condition (2.1). The spectral density $G^0(\lambda)$ is the least favorable in the classes $D_{1\delta}^k$, $k=\overline{1,4}$ for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (4.2) – (4.3), (4.5) – (4.6), (4.8) – (4.9), (4.11) – (4.12), respectively, and the pair $(F(\lambda),G^0(\lambda))$ is a solution of the optimization problem (3.1). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (2.6).*

## 5. Least favorable spectral densities in the class $D=D_{2\delta}\times D_\varepsilon$

Consider the problem of filtering of the functional $A\vec{\xi}$ in the case where spectral densities of the processes belong to the class of admissible spectral densities $D_{2\delta}\times D_\varepsilon$,

$$D_{2\delta}^1=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathrm{Tr}(F(\lambda)-F_1(\lambda))|^2d\lambda\leq\delta\right.\right\};$$

$$D_\varepsilon^1=\left\{G(\lambda)\left|\mathrm{Tr}\,G(\lambda)=(1-\varepsilon)\mathrm{Tr}\,G_1(\lambda)+\varepsilon\mathrm{Tr}\,W(\lambda),\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Tr}\,G(\lambda)d\lambda=q\right.\right\};$$

$$D_{2\delta}^2=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f_{kk}(\lambda)-f_{kk}^1(\lambda)\right|^2d\lambda\leq\delta_k,k=\overline{1,T}\right.\right\};$$

$$D_\varepsilon^2=\left\{G(\lambda)\left|g_{kk}(\lambda)=(1-\varepsilon)g_{kk}^1(\lambda)+\varepsilon w_{kk}(\lambda),\frac{1}{2\pi}\int_{-\infty}^{\infty}g_{kk}(\lambda)d\lambda=q_k,k=\overline{1,T}\right.\right\};$$

$$D_{2\delta}^3=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}|\langle B_1,F(\lambda)-F_1(\lambda)\rangle|^2d\lambda\leq\delta\right.\right\};$$

$$D_\varepsilon^3=\left\{G(\lambda)\left|\langle B_2,G(\lambda)\rangle=(1-\varepsilon)\langle B_2,G_1(\lambda)\rangle+\varepsilon\langle B_2,W(\lambda)\rangle,\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle B_2,G(\lambda)\rangle d\lambda=q\right.\right\};$$

$$D_{2\delta}^4=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f_{ij}(\lambda)-f_{ij}^1(\lambda)\right|^2d\lambda\leq\delta_i^j,i,j=\overline{1,T}\right.\right\},$$

$$D_\varepsilon^4=\left\{G(\lambda)\left|G(\lambda)=(1-\varepsilon)G_1(\lambda)+\varepsilon W(\lambda),\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\lambda)d\lambda=Q\right.\right\},$$

where $F_1(\lambda),G_1(\lambda)$ are known and fixed spectral densities, $W(\lambda)$ is unknown spectral density, $q,\delta,q_k,\delta_k,k=\overline{1,T}$, $\delta_i^j,i,j=\overline{1,T}$, are given numbers, $Q,B_1,B_2$ are given positive definite Hermitian matrices.
The classes $D_{2\delta}$ describe the "$\delta$-neighborhood" models in the space $L_2$ of the given bounded spectral density $F_1(\lambda)$, the classes $D_\varepsilon$ describe the "$\varepsilon$-contamination" models of spectral densities.
From the condition $0\in\partial\Delta_D(F^0,G^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.
For the first pair $D_{2\delta}\times D_\varepsilon^1$, we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=\alpha^2\mathrm{Tr}\,(F^0(\lambda)-F_1(\lambda))(F^0(\lambda)+G^0(\lambda))^2, \tag{5.1}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathrm{Tr}\,(F^0(\lambda)-F_1(\lambda))\right|^2 d\lambda=\delta, \tag{5.2}$$

$$((A(\lambda))^{\top}F^0(\lambda)-(C^0(\lambda))^{\top})^*((A(\lambda))^{\top}F^0(\lambda)-(C^0(\lambda))^{\top})=(\beta^2+\gamma(\lambda))(F^0(\lambda)+G^0(\lambda))^2, \tag{5.3}$$

where $\alpha^2,\beta^2$ are Lagrange multipliers, $\gamma(\lambda)\leq 0$ and $\gamma(\lambda)=0$ if $\mathrm{Tr}\,F^0(\lambda)>(1-\varepsilon)\mathrm{Tr}\,G_1(\lambda)$.
For the second pair $D^2_{2\delta}\times D^2_{\varepsilon}$, we have equations

$$((A(\lambda))^{\top}G^0(\lambda)+(C^0(\lambda))^{\top})^*((A(\lambda))^{\top}G^0(\lambda)+(C^0(\lambda))^{\top})=(F^0(\lambda)+G^0(\lambda))\left\{\alpha_k^2(f^0_{kk}(\lambda)-f^1_{kk}(\lambda))\delta_{kl}\right\}_{k,l=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{5.4}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f^0_{kk}(\lambda)-f^1_{kk}(\lambda)\right|^2 d\lambda=\delta_k,\ k=\overline{1,T}, \tag{5.5}$$

$$((A(\lambda))^{\top}F^0(\lambda)-(C^0(\lambda))^{\top})^*((A(\lambda))^{\top}F^0(\lambda)-(C^0(\lambda))^{\top})=(F^0(\lambda)+G^0(\lambda))\left\{(\beta_k^2+\gamma_k(\lambda))\delta_{kl}\right\}_{k,l=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{5.6}$$

where $\alpha_k^2,\beta_k^2$ are Lagrange multipliers, $\gamma_k(\lambda)\leq 0$ and $\gamma_k(\lambda)=0$ if $g^0_{kk}(\lambda)>(1-\varepsilon)g^1_{kk}(\lambda)$.
For the third pair $D^3_{2\delta}\times D^3_{\varepsilon}$, we have equations

$$((A(\lambda))^{\top}G^0(\lambda)+(C^0(\lambda))^{\top})^*((A(\lambda))^{\top}G^0(\lambda)+(C^0(\lambda))^{\top})=\alpha^2\left\langle B_1,F^0(\lambda)-F_1(\lambda)\right\rangle(F^0(\lambda)+G^0(\lambda))^2, \tag{5.7}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\left\langle B_1,F^0(\lambda)-F_1(\lambda)\right\rangle\right|^2 d\lambda=\delta, \tag{5.8}$$

$$((A(\lambda))^{\top}F^0(\lambda)-(C^0(\lambda))^{\top})^*((A(\lambda))^{\top}F^0(\lambda)-(C^0(\lambda))^{\top})=(\beta^2+\gamma'(\lambda))(F^0(\lambda)+G^0(\lambda))B_2^{\top}(F^0(\lambda)+G^0(\lambda)), \tag{5.9}$$

where $\alpha^2,\beta^2$ are Lagrange multipliers, $\gamma'(\lambda)\leq 0$ and $\gamma'(\lambda)=0$ if $\langle B_2,G^0(\lambda)\rangle>(1-\varepsilon)\langle B_2,G_1(\lambda)\rangle$.
For the fourth pair $D^4_{2\delta}\times D^4_{\varepsilon}$ we have equations

$$((A(\lambda))^{\top}G^0(\lambda)+(C^0(\lambda))^{\top})^*((A(\lambda))^{\top}G^0(\lambda)+(C^0(\lambda))^{\top})=(F^0(\lambda)+G^0(\lambda))\left\{\alpha_{ij}(f^0_{ij}(\lambda)-f^1_{ij}(\lambda))\right\}_{i,j=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{5.10}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f^0_{ij}(\lambda)-f^1_{ij}(\lambda)\right|^2 d\lambda=\delta_i^j,\ i,j=\overline{1,T}, \tag{5.11}$$

$$((A(\lambda))^{\top}F^0(\lambda)-(C^0(\lambda))^{\top})^*((A(\lambda))^{\top}F^0(\lambda)-(C^0(\lambda))^{\top})=(F^0(\lambda)+G^0(\lambda))(\vec{\beta}\cdot\vec{\beta}^*+\Gamma(\lambda))(F^0(\lambda)+G^0(\lambda)), \tag{5.12}$$

where $\vec{\beta},\alpha_{ij}$ are Lagrange multipliers, $\Gamma(\lambda)\leq 0$ and $\Gamma_3(\lambda)=0$ if $G^0(\lambda)>(1-\varepsilon)G_1(\lambda)$.
Thus, the following statement holds true.

**Theorem 5.1.** *The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ in the classes $D_{2\delta}\times D_{\varepsilon}$ for the optimal linear filtering of the functional $A_s\vec{\xi}$ are determined by relations (5.1) – (5.3) for the first pair $D^1_{2\delta}\times D^1_{\varepsilon}$ of sets of admissible spectral densities; by relations (5.4) – (5.6) for the second pair $D^2_{2\delta}\times D^2_{\varepsilon}$ of sets of admissible spectral densities; by relations (5.7) – (5.9) for the third pair $D^3_{2\delta}\times D^3_{\varepsilon}$ of sets of admissible spectral densities; by relations (5.10) – (5.12) for the fourth pair $D^4_{2\delta}\times D^4_{\varepsilon}$ of sets of admissible spectral densities; the minimality condition (2.1), the constrained optimization problem (3.1) and restrictions on densities from the corresponding classes $D_{2\delta}\times D_{\varepsilon}$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A_s\vec{\xi}$ is determined by the formula (2.6).*

**Corollary 5.2.** *Assume that the spectral density $G(\lambda)$ is known. Let the function $F^0(\lambda)+G(\lambda)$ satisfy the minimality condition (2.1). The spectral density $F^0(\lambda)$ is the least favorable in the classes $D^k_{2\delta}$, $k=\overline{1,4}$ for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (5.1) – (5.2), (5.4) – (5.5), (5.7) – (5.8), (5.10) – (5.11), respectively, and the pair $(F^0(\lambda),G(\lambda))$ is a solution of the optimization problem (3.1). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (2.6).*

**Corollary 5.3.** *Assume that the spectral density $F(\lambda)$ is known. Let the function $F(\lambda)+G^0(\lambda)$ satisfy the minimality condition (2.1). The spectral density $G^0(\lambda)$ is the least favorable in the classes $D^k_{\varepsilon}$, $k=\overline{1,4}$ for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (5.3), (5.6), (5.9), (5.12), respectively, and the pair $(F(\lambda),G^0(\lambda))$ is a solution of the optimization problem (3.1). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (2.6).*

## 6. Conclusion

In the article, we propose methods of the mean-square optimal linear filtering of functionals which depend on the unknown values of the multidimensional stationary stochastic process based on observations of the process with an additive stationary stochastic noise process. The case of spectral certainty, as well as the case of spectral uncertainty, are considered. In the case of spectral certainty, where the spectral density matrices of the stationary processes are exactly known, we apply a method based on orthogonal projections in a Hilbert space and derive formulas for calculating the spectral characteristics and the mean-square errors of the optimal estimates of the functionals. In the case of spectral uncertainty, where the spectral density matrices of the stationary processes are not exactly known while some sets of admissible spectral density matrices are given, we apply the minimax-robust method of estimation. This method allows us to find estimates that minimize the maximum values of the mean-square errors of estimates for all spectral density matrices from a given class of admissible spectral density matrices and derive relations which determine the least favourable spectral density matrices. These least favourable spectral density matrices are solutions of the optimization problem $\Delta_D(F,G) = -\Delta(h(F^0,G^0);F,G) + \delta((F,G)\,|D_F \times D_G\,) \to \inf$, which is characterized by the condition $0 \in \partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is the subdifferential of the convex functional $\Delta_D(F,G)$ at point $(F^0,G^0)$. The form of the functional $\Delta(h(F^0,G^0);F,G)$ is convenient for application of the Lagrange method of indefinite multipliers for finding a solution to the optimization problem. The complexity of the problem is determined by the complexity of calculation of the subdifferential of the convex functional $\Delta_D(F,G)$. Making use of the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine the least favourable spectral densities in some special classes of spectral densities. These are: classes $D_0$ of densities with the moment restrictions, classes $D_{1\delta}$ which describe the "$\delta$-neighborhood" models in the space $L_1$ of a given bounded spectral density, classes $D_{2\delta}$ which describe the "$\delta$-neighborhood" models in the space $L_2$ of a given bounded spectral density, classes $D_\varepsilon$ which describes the "$\varepsilon$-contamination" models of spectral densities.

## References


[1] A. N. Kolmogorov, *Selected works by A. N. Kolmogorov. Vol. II: Probability theory and mathematical statistics*, Ed. by A. N. Shiryayev. Math. Appl. Soviet Series., 26. Kluwer, 1992.
[2] N. Wiener, *Extrapolation, Interpolation and Smoothing of Stationary Time Series. With Engineering Applications*, The M. I. T. Press, Massachusetts Institute of Technology, Cambridge, Mass., 1966.
[3] A. M. Yaglom, *Correlation Theory of Stationary and Related Random Functions. Vol. 1: Basic results*, Springer Series in Statistics, Springer-Verlag, New York etc., 1987.
[4] A. M. Yaglom, *Correlation Theory of Stationary and Related Random Functions. Vol. 2: Supplementary notes and references*, Springer Series in Statistics, Springer-Verlag, New York etc., 1987.
[5] Yu. A. Rozanov, *Stationary Stochastic Processes*, San Francisco-Cambridge-London-Amsterdam: Holden-Day, 1967.
[6] E. J. Hannan, *Multiple time series*, Wiley, 1970.
[7] G. E. P. Box, G. M. Jenkins, G. C. Reinsel, G. M. Ljung, *Time series analysis. Forecasting and control*, 5th ed., Wiley, 2016.
[8] P. J. Brockwell, R. A. Davis, *Time series: Theory and methods*, 2nd ed., Springer, 1998.
[9] K. S. Vastola, H. V. Poor, *An analysis of the effects of spectral uncertainty on Wiener filtering*, Automatica, **28** (1983), 289–293.
[10] U. Grenander, *A prediction problem in game theory*, Arkiv för Matematik, **3** (1957), 371–379.
[11] S. A. Kassam, H. V. Poor, *Robust techniques for signal processing: A survey*, Proceedings of the IEEE, **73**(3) (1985), 433–481.
[12] J. Franke, *On the robust prediction and interpolation of time series in the presence of correlated noise*, J. Time Ser. Analysis, **5**(4) (1984), 227–244.
[13] J. Franke, *Minimax robust prediction of discrete time series*, Z. Wahrscheinlichkeitstheor. Verw. Gebiete, **68** (1985), 337–364.
[14] J. Franke, H. V. Poor, *Minimax-robust filtering and finite-length robust predictors*, Robust and Nonlinear Time Series Analysis. Lecture Notes in Statists., Springer-Verlag, **26** (1984), 87–126.
[15] M. P. Moklyachuk, *Robust Estimations of Functionals of Stochastic Processes*, Kyiv University, Kyiv, 2008.
[16] M. P. Moklyachuk, *Minimax-robust estimation problems for stationary stochastic sequences*, Statist. Optim. Inform. Comput., **3**(4) (2015), 348–419.
[17] M. P. Moklyachuk, O. Yu. Masyutka, *Robust filtering of stochastic processes*, Theory of Stochastic Processes, **13**(1-2) (2007), 166–181.
[18] M. P. Moklyachuk, O. Yu. Masyutka, *Minimax-Robust Estimation Technique for Stationary Stochastic Processes*, LAP Lambert Academic Publishing, 2012.
[19] M. P. Moklyachuk, I. I. Golichenko, *Periodically Correlated Processes Estimates*, LAP Lambert Academic Publishing, 2016.
[20] M. M. Luz, M. P. Moklyachuk, *Filtering problem for functionals of stationary sequences*, Statist. Optim. Inform. Comput., **4**(1) (2016), 68–83.
[21] P. Bondon, *Prediction with incomplete past of a stationary process*, Stochastic Process and their Applications. **98** (2002), 67–76.
[22] P. Bondon, *Influence of missing values on the prediction of a stationary time series*, Journal of Time Series Analysis, **26**(4) (2005), 519–525.
[23] R. Cheng, A. G. Miamee, M. Pourahmadi, *Some extremal problems in $L^p(w)$*, Proc. Amer. Math. Soc., **126** (1998), 2333–2340.
[24] R. Cheng, M. Pourahmadi, *Prediction with incomplete past and interpolation of missing values*, Statist. Probab. Lett., **33** (1996), 341–346.
[25] Y. Kasahara, M. Pourahmadi, A. Inoue, *Duals of random vectors and processes with applications to prediction problems with missing values*, Statist. Probab. Lett., **79**(14) (2009), 1637–1646.
[26] M. Pourahmadi, A. Inoue, Y. Kasahara, *A prediction problem in $L^2(w)$*, Proc. Amer. Math. Soc., **135**(4) (2007), 1233–1239.
[27] M. M. Pelagatti, *Time Series Modelling with Unobserved Components*, New York: CRC Press, 2015.
[28] M. P. Moklyachuk, M. I. Sidei, *Interpolation problem for stationary sequences with missing observations*, Statist. Optim. Inform. Comput., **3**(3) (2015), 259-275.
[29] M. P. Moklyachuk, M. I. Sidei, *Filtering problem for stationary sequences with missing observations*. Statist. Optim. Inform. Comput., **4**(4) (2016), 308–325.
[30] M. P. Moklyachuk, M. I. Sidei, *Filtering Problem for functionals of stationary processes with missing observations*, Commun Optim. Theory, (2016), 1-18, Article ID 21 .
[31] M. P. Moklyachuk, M. I. Sidei, *Extrapolation problem for stationary sequences with missing observations*, Statist. Optim. Inform. Comput., **5**(3) (2017), 212–233.
[32] I. I. Gikhman, A. V. Skorokhod, *The Theory of Stochastic Processes. I.*, Springer, 2004.
[33] H. Salehi, *Algorithms for linear interpolator and interpolation error for minimal stationary stochastic processes*, Ann. Probab., **7**(5) (1979), 840–846.
[34] B. N. Pshenichnyj, *Necessary Conditions of an Extremum*, Pure Appl. Math., 4. New York: Marcel Dekker, 1971.
[35] A. D. Ioffe, V. M. Tikhomirov, *Theory of extremal problems*, Studies in Mathematics and its Applications, Vol. 6. North-Holland Publishing Company. Amsterdam, New York, Oxford, 1979.
[36] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1997.
[37] M. P. Moklyachuk, *Nonsmooth Analysis and Optimization*, Kyiv University, Kyiv, 2008.
[38] M. M. Luz, M. P. Moklyachuk, *Minimax-robust filtering problem for stochastic sequences with stationary increments and cointegrated sequences*, Statist. Optim. Inform. Comput., **2**(3) (2014), 176–199.

*Universal Journal of Mathematics and Applications*, 2 (1) (2019) 33-35
**Research paper**

UJMA

# Universal Journal of Mathematics and Applications

*Journal Homepage:* www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: DOI: http://dx.doi.org/10.32323/ujma.456605

# Energy Conditions for Hamiltonian and Traceable Graphs

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**Article Info**

***Keywords:*** *Energy, Hamiltonian, Graph, Traceable*
***2010 AMS:*** *05C45, 05C50.*
***Received:*** *31 August 2018*
***Accepted:*** *29 November 2018*
***Available online:*** *20 March 2019*

**Abstract**

A graph is called Hamiltonian (resp. traceable) if the graph has a Hamiltonian cycle (resp. path), a cycle (resp. path) containing all the vertices of the graph. The energy of a graph is defined as the sum of the absolute values of the eigenvalues of the graph. In this note, we present new conditions based on energy for Hamiltonain and traceable graphs.

## 1. Introduction

All the graphs considered in this note are undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let $G$ be a graph of order $n$ with $e$ edges. We use $\delta(G)$ and $\chi(G)$ to denote the minimum degree and the chromatic number of $G$, respectively. The independence number, denoted $\alpha = \alpha(G)$, is defined as the size of the largest independent set in $G$. The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G)$ of the adjacency matrix $A(G)$ of $G$ are called the eigenvalues of $G$. We use $S^+(G)$ (resp. $S^-(G)$) to denote the sum of the squares of the positive (resp. negative) eigenvalues of $G$. Notice that $S^+(G) + S^-(G) = 2e(G)$ for a graph $G$. The energy, denoted $Eng(G)$, of $G$ is defined as $\sum_{i=1}^{n} |\mu_i(G)|$ (see [2]). A cycle $C$ in a graph $G$ is called a Hamiltonian cycle of $G$ if $C$ contains all the vertices of $G$. A graph $G$ is called Hamiltonian if $G$ has a Hamiltonian cycle. A path $P$ in a graph $G$ is called a Hamiltonian path of $G$ if $P$ contains all the vertices of $G$. A graph $G$ is called traceable if $G$ has a Hamiltonian path. In this note, we will present the energy conditions for Hamiltonian and traceable graphs. The results are as follows.

**Theorem 1.1.** *Let $G$ be a $k$-connected ($k \geq 2$) graph with $n \geq 3$ vertices and $e$ edges. If*

$$Eng(G) \geq 2\sqrt{\frac{2e(\chi - 1)(n-k-1)}{\chi}},$$

*then $G$ is Hamiltonian.*

**Theorem 1.2.** *Let $G$ be a $k$-connected graph with $n \geq 3$ vertices and $e$ edges. If*

$$Eng(G) \geq 2\sqrt{\frac{2e(\chi - 1)(n-k-2)}{\chi}},$$

*then $G$ is traceable or $K_{1,3}$.*

## 2. Lemmas

In order to prove Theorem 1.1, we need the following results as our lemmas. Lemma 2.1 below is Theorem 2.3 on Pages 484 in [3].

**Lemma 2.1.** *Let $G$ be a graph. Then*

$$\chi \geq 1 + \max\left\{\frac{S^+}{S^-}, \frac{S^-}{S^+}\right\}.$$

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Lemma 2.2 below is Theorem 3.14 on Pages 88 and 89 in [4].

**Lemma 2.2.** *Let G be a graph. If the number of eigenvalues of G which are greater than, less than, and equal to zero are p, q, and r, respectively, then*

$$\alpha \leq r + \min\{p, q\},$$

*where $\alpha$ is the independence number of G.*

## 3. Proofs

Next, we will present proofs for Theorems 1.1 and 1.2. Some ideas from [5] are used in our proofs.

**Proof of Theorem 1.1.** Let $G$ be a graph satisfying the conditions in Theorem 1.1. Suppose, to the contrary, that $G$ is not Hamiltonian. If $n = 3$, $G$ must be Hamiltonian since $G$ is $k$-connected ($k \geq 2$). From now on, we assume that $n \geq 4$. Since $G$ is $k$-connected ($k \geq 2$), $G$ has a cycle. Choose a longest cycle $C$ in $G$ and give an orientation on $C$. Since $G$ is not Hamiltonian, there exists a vertex $u_0 \in V(G) - V(C)$. By Menger's theorem, we can find $s$ ($s \geq k$) pairwise disjoint (except for $u_0$) paths $P_1$, $P_2$, ..., $P_s$ between $u_0$ and $V(C)$. Let $v_i$ be the end vertex of $P_i$ on $C$, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1$, $v_2$, ..., $v_s$ agrees with the orientation of $C$. We use $v_i^+$ to denote the successor of $v_i$ along the orientation of $C$, where $1 \leq i \leq s$. Since $C$ is a longest cycle in $G$, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s$ and the index $s+1$ is regarded as 1. Moreover, $S := \{u_0, v_1^+, v_2^+, ..., v_s^+\}$ is independent (otherwise $G$ would have cycles which are longer than $C$). Then $\alpha \geq s+1 \geq k+1$.

Let $\mu_1 \geq \mu_2 \geq ... \geq \mu_p$ be the $p$ positive eigenvalues of $G$ and let $\mu_{n-q+1} \geq \mu_{n-q+2} \geq ... \geq \mu_n$ be the $q$ negative eigenvalues of $G$. Then $n-(p+q)$ is the number of eigenvalues of $G$ which are equal to zero. Since $\sum_{i=1}^{p} \mu_i + \sum_{i=n-q+1}^{n} \mu_i = \text{trace of } A = 0$, $\sum_{i=1}^{p} |\mu_i| = \sum_{i=n-q+1}^{n} |\mu_i|$. Thus we have that

$$Eng(G) = 2\sum_{i=1}^{p} |\mu_i| = 2\sum_{i=n-q+1}^{n} |\mu_i|.$$

From Lemma 2.1, we have that

$$\chi \geq 1 + \frac{S^+}{S^-} = 1 + \frac{S^+}{2e - S^+} = \frac{2e}{2e - S^+}, \ \chi \geq 1 + \frac{S^-}{S^+} = 1 + \frac{S^-}{2e - S^-} = \frac{2e}{2e - S^-}.$$

Therefore we further have that

$$S^+ \leq \frac{2e(\chi - 1)}{\chi}, \ S^- \leq \frac{2e(\chi - 1)}{\chi}.$$

From Cauchy-Schwarz inequality, we have that

$$\frac{Eng(G)}{2} = \sum_{i=1}^{p} |\mu_i| \leq \sqrt{p\sum_{i=1}^{p} \mu_i^2} = \sqrt{pS^+} \leq \sqrt{\frac{2e(\chi-1)p}{\chi}}.$$

Similarly, we have that

$$\frac{Eng(G)}{2} = \sum_{i=n-q+1}^{n} |\mu_i| \leq \sqrt{q\sum_{i=n-q+1}^{n} \mu_i^2} = \sqrt{qS^-} \leq \sqrt{\frac{2e(\chi-1)q}{\chi}}.$$

Therefore we get that

$$Eng(G) = \frac{Eng(G)}{2} + \frac{Eng(G)}{2}$$

$$\leq \sqrt{\frac{2e(\chi-1)p}{\chi}} + \sqrt{\frac{2e(\chi-1)q}{\chi}} = \sqrt{\frac{2e(\chi-1)}{\chi}}(\sqrt{p} + \sqrt{q}).$$

From Lemma 2.2, we have that $\alpha \leq n-(p+q)+\min\{p,q\} \leq n-p-q+p = n-q$ and $\alpha \leq n-(p+q)+\min\{p,q\} \leq n-p-q+q = n-p$. Thus $p \leq n-\alpha$ and $q \leq n-\alpha$. Therefore we have that

$$2\sqrt{\frac{2e(\chi-1)(n-k-1)}{\chi}} \leq Eng(G) \leq 2\sqrt{\frac{2e(\chi-1)(n-\alpha)}{\chi}}$$

$$\leq 2\sqrt{\frac{2e(\chi-1)(n-s-1)}{\chi}} \leq 2\sqrt{\frac{2e(\chi-1)(n-k-1)}{\chi}}.$$

From the above proofs, we have that

$$S^+ = S^- = \frac{2e(\chi-1)}{\chi},$$

$$\mu_1 = \mu_2 = \cdots = \mu_p, \mu_{n-q+1} = \mu_{n-q+2} = \cdots = \mu_n,$$
$$p = q = n - \alpha, \alpha = s + 1 = k + 1.$$

Thus $p\mu_1^2 = S^+ = S^- = q\mu_n^2$. Since $p = q$, $\mu_1^2 = \mu_n^2$. Hence $\mu_1 = -\mu_n$. Since $G$ is connected and $\mu_1 = -\mu_n$, $G$ is a bipartite graph. From Perron-Frobenius theorem, we have that $\mu_1 > \mu_2$. Since $\mu_1 = \mu_2 = \cdots = \mu_p$, we must have $p = 1$. Now $\alpha = n - p = n - 1$, which implies that $G$ cannot be 2-connected, a contradiction.

Therefore the proof of Theorem 1 is complete. □

**Proof of Theorem 1.2.** Let $G$ be a graph satisfying the conditions in Theorem 1.2. Suppose, to the contrary, that $G$ is not traceable. If $n = 3$, $G$ must be traceable since $G$ is $k$-connected ($k \geq 1$). From now on, we assume that $n \geq 4$. Choose a longest path $P$ in $G$ and give an orientation on $P$. Let $x$ and $y$ be the two end vertices of $P$. Since $G$ is not traceable, there exists a vertex $u_0 \in V(G) - V(P)$. By Menger's theorem, we can find $s$ ($s \geq k$) pairwise disjoint (except for $u_0$) paths $P_1$, $P_2$, ..., $P_s$ between $u_0$ and $V(P)$. Let $v_i$ be the end vertex of $P_i$ on $P$, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1$, $v_2$, ..., $v_s$ agrees with the orientation of $P$. Since $P$ is a longest path in $G$, $x \neq v_i$ and $y \neq v_i$, for each $i$ with $1 \leq i \leq s$, otherwise $G$ would have paths which are longer than $P$. We use $v_i^+$ to denote the successor of $v_i$ along the orientation of $P$, where $1 \leq i \leq s$. Since $P$ is a longest path in $G$, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s-1$. Moreover, $S := \{u_0, v_1^+, v_2^+, ..., v_s^+, x\}$ is independent (otherwise $G$ would have paths which are longer than $P$). Then $\alpha \geq s + 2 \geq k + 2$.

Using the arguments similar to the ones in Proof of Theorem 1.1, we have that

$$2\sqrt{\frac{2e(\chi-1)(n-k-2)}{\chi}} \leq Eng(G) \leq 2\sqrt{\frac{2e(\chi-1)(n-\alpha)}{\chi}}$$
$$\leq 2\sqrt{\frac{2e(\chi-1)(n-s-2)}{\chi}} \leq 2\sqrt{\frac{2e(\chi-1)(n-k-2)}{\chi}}.$$

Therefore we have that

$$S^+ = S^- = \frac{2e(\chi-1)}{\chi},$$
$$\mu_1 = \mu_2 = \cdots = \mu_p, \mu_{n-q+1} = \mu_{n-q+2} = \cdots = \mu_n, p = q = n - \alpha, \alpha = s + 2 = k + 2.$$

Thus $p\mu_1^2 = S^+ = S^- = q\mu_n^2$. Since $p = q$, $\mu_1^2 = \mu_n^2$. Hence $\mu_1 = -\mu_n$. Since $G$ is connected and $\mu_1 = -\mu_n$, $G$ is a bipartite graph. From Perron-Frobenius theorem, we have that $\mu_1 > \mu_2$. Since $\mu_1 = \mu_2 = \cdots = \mu_p$, we must have $p = 1$. Now $\alpha = n - p = n - 1$. So $G$ is $K_{1,n-1}$ with $n \geq 4$. Since now $k = 1$ and $n - 1 = \alpha = k + 2$, we have that $G$ is $K_{1,3}$.

Therefore the proof of Theorem 1.2 is complete.

## References


[1] J. A. Bondy, U. S. R. Murty, *Graph Theory with Applications*, The Macmillan Press LTD, 1976.
[2] I. Gutman, *The energy of a graph*, Berichte der Mathematisch-Statistischen Sektion im Forschungszentrum Graz, **103** (1978), 1 – 12.
[3] T. Ando, M. Lin, *Proof of a conjectured lower bound on the chromatic number of a graph*, Linear Algebra Appl., **485** (2015), 480 – 484.
[4] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, 3rd Edition, Johann Ambrosius Barth, 1995.
[5] R. Li, *A sharp upper bound for the energy of a connected graph*, Manuscript, July 2018.

*Universal Journal of Mathematics and Applications*, 2 (1) (2019) 36-41
**Research paper**

UJMA

# Universal Journal of Mathematics and Applications

*Journal Homepage:* www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: DOI: http://dx.doi.org/10.32323/ujma.418626

# The Existence of a Global Attractor for one Fourth Order Hyperbolic Equation with Memory Operator

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**Article Info**

***Keywords:*** *Bounded absorbing set, Memory operator, Minimal global attractor, Semilinear hyperbolic equation*
***2010 AMS:*** *35L30, 35L76*
***Received:*** *26 April 2018*
***Accepted:*** *29 November 2018*
***Available online:*** *20 March 2019*

**Abstract**

In this work, the initial-boundary value problem for one fourth order semilinear hyperbolic equation with memory operator is considered (here the memory operator is under the operator of differentiation with respect to time variable). The asymptotic compactness of semigroup generated by this problem is proved. The existence of a minimal global attractor for this problem is also proved.

## 1. Introduction

Nonlinear equations with memory operator, especially the equations with hysteresis have great importance among the partial differential equations. Nonlinear relations of hysteresis type appear in ferromagnetism, ferroelectricity, superconductivity, plasticity, friction, etc. The research of solutions of partial differential equations with hysteresis nonlinearities is a nontrivial problem. Such equations, when hysteresis operator is under the operator of differentiation with respect to time variable, have special difficulties.
The research of asymptotic behaviour of a dynamic system, which is originated by the corresponding initial-boundary value problem, has a special significance. For the equations with hysteresis nonlinearities, these questions have not been almost investigated. In this field, only particular cases were considered. For instance, the asymptotic character of solutions of the initial-boundary value problem for one quasilinear parabolic equation, in which the hysteresis operator is under the operator of differentiation with respect to time variable, was investigated in [1]. The similar results were obtained in [2], [3]. In [4], [5], [6] the corresponding problems were researched by the application of the results of nonlinear semigroup theory.
In this work, the asymptotic result for solutions of the initial-boundary value problem for one semilinear hyperbolic equation with memory operator is obtained and the existence of a minimal global attractor for this problem is proved.

## 2. Problem statement and reliminaries

Here we use the concepts and notations which were introduced in [7].
Let $\Omega \subset R^N$ $(N \geq 1)$ be a bounded, connected set with a smooth boundary $\Gamma$. We consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\left[u + \mathrm{F}(u)\right] + \Delta^2 u + |u|^p u = h \text{ in } Q = \Omega \times (0,T), \tag{2.1}$$

$$u = 0,\ \Delta u = 0,\ (x,t) \in \Gamma \times [0,T], \tag{2.2}$$

$$\left[u + \mathrm{F}(u)\right]|_{t=0} = u^{(0)} + w^{(0)},\ \frac{\partial u}{\partial t}\Big|_{t=0} = u^{(1)} \text{ in}\,\Omega, \tag{2.3}$$

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where $p>0$ and F is a memory operator (at any instant t, $\mathrm{F}(u)$ may depend not only on $u(t)$, but also on the previous evolution of $u$), which acts from $\mathrm{M}\left(\Omega;C^0([0,T])\right)$ to $\mathrm{M}\left(\Omega;C^0([0,T])\right)$. Here $\mathrm{M}\left(\Omega;C^0([0,T])\right)$ is a space of strongly measurable functions $\Omega\to C^0([0,T])$. We assume that the operator F is applied at each point $x\in\Omega$ independently: the output $[\mathrm{F}(u(x,\cdot))](t)$ depends on $u(x,\cdot)|_{[0,t]}$, but not on $u(y,\cdot)|_{[0,t]}$ for any $y\neq x$ (for more details see [7]).
We assume that

$$\begin{cases} \text{for}\,\forall\, \upsilon_1,\ \upsilon_2\in \mathrm{M}\left(\Omega;\mathrm{C}^0([0,\mathrm{T}])\right) \text{ and for}\,\forall\, \mathrm{t}\in[0,\mathrm{T}],\text{ if } \upsilon_1=\upsilon_2 \\ \text{in } [0,\mathrm{t}],\text{a.e. in } \Omega,\text{ then } [\mathrm{F}(\upsilon_1)](\cdot,\mathrm{t})=[\mathrm{F}(\upsilon_2)](\cdot,\mathrm{t}) \text{ a.e. in}\,\Omega; \end{cases} \tag{2.4}$$

$$\begin{cases} \forall\left\{\upsilon_n\in \mathrm{M}\left(\Omega;C^0([0,T])\right)\right\}_{n\in\mathrm{N}},\text{ if } \upsilon_\mathrm{n}\to\upsilon \text{ uniformly in } [0,\mathrm{T}] \\ \text{a.e. in } \Omega,\text{ then } \mathrm{F}(\upsilon_\mathrm{n})\ \to \mathrm{F}(\upsilon) \text{ uniformly in } [0,\mathrm{T}] \text{ a.e. in } \Omega; \end{cases} \tag{2.5}$$

$$\begin{cases} \exists\, L>0,\ \exists g\in L^2(\Omega):\forall\,\upsilon\in M\left(\Omega;C^0([0,T])\right) \\ \|[\mathrm{F}(\upsilon)](x,\cdot)\|_{C^0([0,T])}\leq L\|\upsilon(x,\cdot)\|_{C^0([0,T])}+g(x),\ \text{a.e. in } \Omega; \end{cases} \tag{2.6}$$

$$\begin{cases} \forall\,\upsilon\in\mathrm{M}\left(\Omega;C^0([0,T])\right),\forall\,[t_1,t_2]\subset[0,T], \\ \text{if}\,\upsilon(\mathrm{x},\cdot)\ \text{is affine in } [\mathrm{t}_1,\mathrm{t}_2] \text{ a.e. in}\,\Omega,\text{then} \\ \{[\mathrm{F}(\upsilon)](x,t_2)-[\mathrm{F}(\upsilon)](x,t_1)\}\,[\upsilon(x,t_2)-\upsilon(x,t_1)]\geq 0,\ \text{a.e. in } \Omega; \end{cases} \tag{2.7}$$

$$\begin{cases} \exists\, 0<L_1<1,\forall\,\upsilon\in M\left(\Omega;C^0([0,T])\right),\forall\,[t_1,t_2]\subset[0,T], \\ \text{if } \upsilon(x,\cdot) \text{ is affine in } [t_1,t_2] \text{ a.e. in } \Omega,\text{ then} \\ |[\mathrm{F}(\upsilon)]\ (x,t_2)-[\mathrm{F}(\upsilon)]\ (x,t_1)|\leq L_1\,|\upsilon(x,t_2)-\upsilon(x,t_1)|\ \ \text{a.e. in } \Omega. \end{cases} \tag{2.8}$$

$$\begin{cases} \exists\, 0<L_2<1,\forall\,\upsilon\in M\left(\Omega;C^0([0,T])\right),\forall\,[t_1,t_2]\subset[0,T], \\ \text{if } \upsilon(x,\cdot) \text{ is affine in } [t_1,t_2] \text{ a.e. in } \Omega,\text{ then} \\ |[\mathrm{F}(u)]\,(x,t_2)-[\mathrm{F}(\upsilon)]\,(x,t_2)-([\mathrm{F}(u)]\ (x,t_1)-[\mathrm{F}(\upsilon)]\,(x,t_1))|\leq \\ \leq L_2\,|u(x,t_2)-\upsilon(x,t_2)-(u(x,t_1)-\upsilon(x,t_1))|. \end{cases} \tag{2.9}$$

Let $V=H_0^2(\Omega)\bigcap L^{p+2}(\Omega)$ and

$$u^{(0)}\in V,\ w^{(0)}\in L^2(\Omega),u^{(1)}\in L^2(\Omega),\ h\in L^2(\Omega). \tag{2.10}$$

**Definition 2.1.** *A function $u\in L^2(0,T;V)\bigcap H^1\left(0,T;L^2(\Omega)\right)$ is said to be a solution of problem (2.1)-(2.3) if $\mathrm{F}(u)\in L^2(Q)$, and*

$$\iint_Q\left\{-\frac{\partial u}{\partial t}\cdot\frac{\partial\upsilon}{\partial t}-[u+\mathrm{F}(u)]\frac{\partial\upsilon}{\partial t}+\Delta u\cdot\Delta\upsilon+|u|^p u\upsilon\right\}dxdt=$$

$$=\iint_Q h\upsilon dxdt+\int_\Omega\left[u^{(0)}(x)+w^{(0)}(x)+u^{(1)}(x)\right]\ \upsilon(x,0)\,dx,$$

*for every $\upsilon\in L^2(0,T;V)\bigcap H^1\left(0,T;L^2(\Omega)\right)$ $(\upsilon(\cdot,T)=0$ a.e. in $\Omega)$.*

Well posedness of problem (2.1)-(2.3) without F, was studied by different authors (see, for example [8]). The initial boundary problem for the parabolic equation without the nonlinear term $|u|^p u$ and with $\Delta u$ was studied in [7].
The following theorems about existence and uniqueness of solutions of problem (2.1)-(2.3) can be proved in the same way as the corresponding theorems from [9].

**Theorem 2.2.** *Assume that (2.4)-(2.8),(2.10) hold. Then problem (2.1)-(2.3) has at least one solution such that*

$$u\in W^{1,\infty}\left(0,T;L^2(\Omega)\right)\cap L^\infty(0,T;V),\ \ \mathrm{F}(u)\in H^1\left(0,T;L^2(\Omega)\right).$$

**Theorem 2.3.** *Assume that the hypotheses of Theorem 2.2 hold,*

$$p\leq\frac{2}{N-2},N\geq 3\,(p\text{ is arbitrary and finite when } N{=}2) \tag{2.11}$$

*and F fulfils the following condition*

$$\begin{cases} \forall r>0,\ \exists L(r)>0:\forall t\in(0,T],\forall\upsilon_1,\upsilon_2\in\{u\in L^2(Q_t):\|u\|_{L^2(Q_t)}\leq r\}: \\ \|F(\upsilon_1)-F(\upsilon_2)\|_{L^2(Q_t)}\leq L(r)\|\upsilon_1-\upsilon_2\|_{L^2(Q_t)}. \end{cases} \tag{2.12}$$

*Then problem (2.1)-(2.3) has only one solution.*

As an example of an operator which satisfies the mentioned conditions, we can present the Bouc operator (see, for example [10]):

$$[B(u)](t)=\alpha u(t)+\int_0^t f\left(\int_s^t |u'(\tau)|d\tau\right)\varphi(u(s))u'(s)ds,$$

here $\alpha$ is a positive constant, $f$ and $\varphi$ are continuous real functions, with $f$ positive and nondecreasing.
The following theorem is obtained from Theorem 2.2 and Theorem 2.3 by the common theory of solvability of linear hyperbolic equations.

**Theorem 2.4.** *Assume that the conditions of Theorem 2.3 hold. Then for arbitrary $T>0$ problem (2.1)-(2.3) has only one solution $u\in C^1\left([0,T];L^2(\Omega)\right)\cap C\left([0,T];H_0^2(\Omega)\right)$.*

By the condition (2.11): $V=H_0^2(\Omega)\bigcap L^{p+2}(\Omega)=H_0^2(\Omega)$. We set $E=H_0^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$. Then under the conditions of Theorem 2.4, Problems (2.1)-(2.3) generates the semigroup $\{S(t)\}_{t\geq 0}$ in $E$ by the formula:

$$S(t)\left(u^{(0)},u^{(1)},w^{(0)}\right)=(u,u_t,w),$$

where $u$ is a unique solution of this problem.

**Definition 2.5.** *(see [11]) A bounded set $B_0\subset E$ is said to be absorbing if for an arbitrary bounded set $B\subset E$ there exists $t_1(B)$ such that $S(t)B\subset B_0$ for all $t\geq t_1(B)$.*

**Definition 2.6.** *(see [11]) Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a metric space $(X,d)$. A smallest, nonempty, bounded, closed set $A\subset X$ that satisfies*

$$\lim_{t\to\infty}\sup_{\upsilon\in B}\inf_{u\in A} d(S(t)\upsilon,u)=0,$$

for each bounded set $B\subset X$, is called a minimal global attractor of $\{S(t)\}_{t\geq 0}$.
The following theorem about the existence of a bounded absorbing set for Problems (2.1)-(2.3) can be proved in the same way as the corresponding theorem in [12].

**Theorem 2.7.** *Under the conditions (2.4)-(2.12), Problems (2.1)-(2.3) has a bounded absorbing set $B_0\subset E$.*

In this work, we first prove the asymptotic compactness of a semigroup, generated by problem (2.1)-(2.3), and then the basic theorem about the existence of a minimal global attractor for this problem.
Note that, a semigroup $\{S(t)\}_{t\geq 0}$, defined on a metric space $(X,d)$, is called asymptotically compact, if for arbitrary bounded set $B\subset X$ such, that $\bigcup\limits_{t\geq 0}S(t)B$ is bounded in $(X,d)$, the sequence $\{S(t_k)\upsilon_k\}_{k=1}^{\infty}$, $t_k\to\infty$, $\upsilon_k\in B$ has a convergent subsequence.

## 3. Basic Results

**Theorem 3.1.** *Assume that (2.4)-(2.12) hold. Then the semigroup $\{S(t)\}_{t\geq 0}$, generated by problem (2.1)-(2.3), is asymptotically compact in $E$.*

**The proof of Theorem 3.1.** It suffices to prove that for any bounded set $B$ from $E$ and for arbitrary $\varepsilon>0$ there exists $T=T(\varepsilon,B)$ such that

$$\limsup_{i\to\infty}\sup_{q\in N}\left\|S(T)\theta_{i+q}-S(T)\theta_i\right\|_E\leq\varepsilon, \tag{3.1}$$

where $\{\theta_i\}$ is the sequence from $B$ and $\{S(t)\theta_i\}$ converges $*-$weekly in $L^{\infty}(0,\infty;E)$.
We prove this by the method of time discretization (see [7]).
For any $m\in N$, we set $k=\frac{T}{m}$ and:

$$u_m^0=u^{(0)},\ w_m^0=w^{(0)},\ u_m^1=u^{(0)}+ku^{(1)},$$
$$u_m^{-1}=u^{(0)}-ku^{(1)},\ u_m^n(x)=u(x,nk),\ n=2,\dots,m,$$
$$w_m^n(x)=[\mathrm{F}(u_m)](x,nk),\ n=1,\dots,m,\ \text{ a.e. in }\Omega,$$

$u_m(x,\cdot)=$ linear time interpolate of $u(x,nk)$ for $n=0,1,\dots,m$ a.e. in $\Omega$, $w_m(x,\cdot)=$linear time interpolate of $w(x,nk)$ for $n=1,\dots,m$ a.e. in $\Omega$.
We will use the techniques used in [13, 14, 15]. We set $\theta_i=\left\{u_i^{(0)},u_i^{(1)},w_i^{(0)}\right\}$, $u_i=S(t)\theta_i$ $(i=1,2,\dots)$.
We consider the following problems for $l=i,j$ $(i,j=1,2,\dots)$:

$$\frac{u_{lm}^n-2u_{lm}^{n-1}+u_{lm}^{n-2}}{k^2}+\frac{u_{lm}^n-u_{lm}^{n-1}}{k}+\frac{w_{lm}^n-w_{lm}^{n-1}}{k}+$$
$$+\Delta^2u_{lm}^n+|u_{lm}^n|^p u_{lm}^n=h\text{ in }V',\ n=1,2,\dots,m,$$
$$u_{lm}^0=u_l^{(0)},\ w_{lm}^0=w_l^{(0)},u_{lm}^1=u_l^{(0)}+ku_l^{(1)},u_{lm}^{-1}=u_l^{(0)}-ku_l^{(1)},$$

and obtain, that

$$\frac{u_{im}^n-u_{jm}^n-2\left(u_{im}^{n-1}-u_{jm}^{n-1}\right)+u_{im}^{n-2}-u_{jm}^{n-2}}{k^2}+\frac{u_{im}^n-u_{jm}^n-(u_{im}^{n-1}-u_{jm}^{n-1})}{k}+$$

$$+\frac{w_{im}^{n}-w_{jm}^{n}-(w_{im}^{n-1}-w_{jm}^{n-1})}{k}+$$

$$+\Delta^{2}\left(u_{im}^{n}-u_{jm}^{n}\right)+|u_{im}^{n}|^{p}u_{im}^{n}-\left|u_{jm}^{n}\right|^{p}u_{jm}^{n}=0,\ \text{in}\,V',\ n=1,2,...,m. \tag{3.2}$$

$$u_{lm}^{0}-u_{jm}^{0}=u_{l}^{(0)}-u_{j}^{(0)},\ w_{lm}^{0}-w_{jm}^{0}=w_{l}^{(0)}-w_{j}^{(0)},$$

$$u_{lm}^{1}-u_{jm}^{1}=u_{l}^{(0)}-u_{l}^{(0)}+k\left(u_{l}^{(1)}-u_{j}^{(1)}\right),u_{lm}^{-1}-u_{jm}^{-1}=u_{l}^{(0)}-u_{l}^{(0)}-k\left(u_{l}^{(1)}-u_{j}^{(1)}\right). \tag{3.3}$$

By multiplying both sides of equality (3.2) by $u_{im}^{n}-u_{jm}^{n}-(u_{im}^{n-1}-u_{jm}^{n-1})$, summing for $n=s,...,m$ for arbitrary $s\in\{1,2,...,m\}$ , integrating by $\Omega$ and using the condition (2.7), we can obtain the following relation:

$$E\left(u_{im}^{m}-u_{jm}^{m}\right)+\frac{k}{2}(1-L_{1})\sum_{n=s}^{m}\int_{\Omega}\left(\frac{u_{im}^{n}-u_{jm}^{n}-(u_{im}^{n-1}-u_{jm}^{n-1})}{k}\right)^{2}dx+$$

$$+\sum_{n=s}^{m}\int_{\Omega}\left(|u_{im}^{n}|^{p}u_{im}^{n}-\left|u_{jm}^{n}\right|^{p}u_{jm}^{n}\right)\left(u_{im}^{n}-u_{jm}^{n}-\left(u_{im}^{n-1}-u_{jm}^{n-1}\right)\right)dx\leq$$

$$\leq E\left(u_{im}^{s-1}-u_{jm}^{s-1}\right), \tag{3.4}$$

where

$$E\left(u_{im}^{n}-u_{jm}^{n}\right)=$$

$$=\frac{1}{2}\int_{\Omega}\left|\Delta\left(u_{im}^{n}-u_{jm}^{n}\right)\right|^{2}dx+\frac{1}{2}\int_{\Omega}\left(\frac{u_{im}^{n}-u_{jm}^{n}-(u_{im}^{n-1}-u_{jm}^{n-1})}{k}\right)^{2}dx.$$

It is evident that for arbitrary $\delta>0$ there exists $c_{2}(\delta)>1$ such, that

$$|u-\upsilon|^{2}\leq\delta+c_{2}(\delta)|u-\upsilon|^{2}\ ,\ \forall u,\upsilon\in R.$$

We can obtain the following inequality from (3.4), when $s=1$ :

$$k\sum_{n=1}^{m}\int_{\Omega}\left(\frac{u_{im}^{n}-u_{jm}^{n}-(u_{im}^{n-1}-u_{jm}^{n-1})}{k}\right)^{2}dx\leq\delta T mes\Omega+C_{1}\frac{C_{2}(\delta)}{1-L_{1}}\|B\|_{V}^{2}+$$

$$+\frac{C_{2}(\delta)}{1-L_{1}}\sum_{n=1}^{m}\int_{\Omega}\left(\left|u_{jm}^{n}\right|^{p}u_{jm}^{n}-|u_{im}^{n}|^{p}u_{im}^{n}\right)\left(u_{im}^{n}-u_{jm}^{n}-\left(u_{im}^{n-1}-u_{jm}^{n-1}\right)\right)dx,\ \forall\delta>0. \tag{3.5}$$

By multiplying both sides of the equality (3.2) by $u_{im}^{n}-u_{jm}^{n}$, summing for $n=1,...,m$ , integrating by $\Omega$ and using the condition (2.7), we have

$$-k\left(1+\frac{L_{1}^{2}}{2\nu}\right)\sum_{n=1}^{m}\int_{\Omega}\left(\frac{u_{im}^{n}-u_{jm}^{n}-(u_{im}^{n-1}-u_{jm}^{n-1})}{k}\right)^{2}dx+$$

$$+\frac{k}{2}\int_{\Omega}\left(\frac{u_{im}^{m}-u_{jm}^{m}-(u_{im}^{m-1}-u_{jm}^{m-1})}{k}\right)^{2}dx+$$

$$+\frac{k}{2}\int_{\Omega}\left(\frac{u_{im}^{0}-u_{jm}^{0}-(u_{im}^{-1}-u_{jm}^{-1})}{k}\right)^{2}dx+$$

$$+\int_{\Omega}\frac{u_{im}^{m}-u_{jm}^{m}-(u_{im}^{m-1}-u_{jm}^{m-1})}{k}\left(u_{im}^{m}-u_{jm}^{m}\right)dx-$$

$$-\int_{\Omega}\frac{u_{im}^{0}-u_{jm}^{0}-(u_{im}^{-1}-u_{jm}^{-1})}{k}\left(u_{im}^{0}-u_{jm}^{0}\right)dx+$$

$$+\int_{\Omega}\left(\frac{1}{2}\left(u_{im}^{m}-u_{jm}^{m}\right)^{2}-\frac{1}{2}\left(u_{im}^{0}-u_{jm}^{0}\right)^{2}\right)dx+$$

$$+k\left(1-\frac{\nu c_{\Omega}^{2}}{2}\right)\sum_{n=1}^{m}\int_{\Omega}\left|\Delta\left(u_{im}^{n}-u_{jm}^{n}\right)\right|^{2}dx\leq0, \tag{3.6}$$

By (3.3) and due to the existence of a bounded absorbing set, we have

$$E\left(u^{n}\left(T\right)\right)\leq c\left(\|B\|_{E}\right)\ \text{for}\,\forall T\geq 0,$$

where $\|B\|_E=\sup\limits_{\upsilon\in B}\|\upsilon\|_E$. Then from (3.6), we obtain the following

$$k\left(1-\frac{\nu c_{\Omega}^{2}}{2}\right)\sum_{n=1}^{m}\left\|\Delta\left(u_{im}^{n}-u_{jm}^{n}\right)\right\|^{2}\leq$$

$$\leq C_{3}\left(\|B\|_{V}\right)+k\left(1+\frac{L_{1}^{2}}{2\nu}\right)\sum_{n=1}^{m}\left\|\frac{u_{im}^{n}-u_{jm}^{n}-(u_{im}^{n-1}-u_{jm}^{n-1})}{k}\right\|^{2}. \tag{3.7}$$

Multiplying (3.5) by $\nu_1$ and summing it with (3.7), we have

$$k\sum_{n=1}^{m}E\left(u_{im}^{n}-u_{jm}^{n}\right)\leq\frac{\nu_{1}\delta T mes\Omega}{2\nu_{2}}+\frac{\tilde{C}\left(\|B\|_{V},\delta\right)}{2\nu_{2}}+$$

$$+k\frac{\nu_{1}C_{2}\left(\delta\right)}{2\nu_{2}\left(1-L_{1}\right)}\sum_{n=1}^{m}\int_{\Omega}\left(\left|u_{jm}^{n}\right|^{p}u_{jm}^{n}-\left|u_{im}^{n}\right|^{p}u_{im}^{n}\right)\frac{u_{im}^{n}-u_{jm}^{n}-\left(u_{im}^{n-1}-u_{jm}^{n-1}\right)}{k}dx,\ \forall\delta>0\,, \tag{3.8}$$

where

$$\nu_{2}=\min\left\{\nu_{1}-1-\frac{L_{1}^{2}}{2\nu},\,1-\frac{\nu c_{\Omega}^{2}}{2}\right\},$$

and $\nu$, $\nu_1$ are chosen such that

$$1-\frac{\nu c_{\Omega}^{2}}{2}>0\,,\nu_{1}-1-\frac{L_{2}^{2}}{2\nu}>0.$$

Summing (3.4) for $s=1,\ldots,m$, using the condition $L_1<1$, (3.8) and lemma 2.2 from [15], we can obtain that

$$\limsup_{i\to\infty}\limsup_{j\to\infty}E\left(u_{im}^{m}-u_{jm}^{m}\right)\leq\frac{\nu_{1}mes\Omega}{2\nu_{2}}\delta+\frac{C\left(\|B\|_{V},\delta\right)}{2\nu_{2}T},\ \forall\delta>0$$

and consequently,

$$\limsup_{i\to\infty}\ \sup_{q\in N}E\left(u_{(i+q)m}^{m}-u_{im}^{m}\right)\leq\frac{2\nu_{1}mes\Omega}{\nu_{2}}\delta+\frac{2C\left(\|B\|_{V},\delta\right)}{\nu_{2}T},\forall\delta>0\,,\forall T>0.$$

From the last relation we obtain, that for arbitrary $\varepsilon>0$ there exists $T=T\left(\varepsilon,B\right)$ and $\delta>0$ such that

$$\limsup_{i\to\infty}\ \sup_{q\in N}\left[\frac{1}{2}\left\|\Delta\left(u_{(i+q)m}^{m}-u_{im}^{m}\right)\right\|^{2}+\frac{1}{2}\left\|\frac{u_{(i+q)m}^{m}-u_{im}^{m}-\left(u_{(i+q)m}^{m-1}-u_{im}^{m-1}\right)}{k}\right\|^{2}\right]\leq\varepsilon,$$

whence defining

$$\tilde{u}_{m}\left(x,t\right)=u_{m}^{n}\left(x\right),\text{if}\left(\text{n}-1\right)\text{k}<\text{t}\leq\text{nk},\ \text{n}=1,2,\ldots,\text{m};\ \text{a.e. in}\ \Omega$$

and defining $\tilde{w}_m,\tilde{f}_m$ similarly, we obtain that

$$\limsup_{i\to\infty}\ \sup_{q\in N}\left[\frac{1}{2}\left\|\Delta\left(\tilde{u}_{(i+q)m}-\tilde{u}_{im}\right)(T)\right\|^{2}+\frac{1}{2}\left\|\left(u_{(i+q)m}-u_{im}\right)_{t}(T)\right\|^{2}\right]\leq\varepsilon. \tag{3.9}$$

Since (see, [9])
$u_m\to u$ weakly star in $H^1\left(0,T;L^2\left(\Omega\right)\right)\bigcap L^\infty\left(0,T;H_0^1\left(\Omega\right)\right)$,
$\tilde{u}_m\to u$ weakly star in $L^\infty\left(0,T;H_0^1\left(\Omega\right)\right)$,
as $m\to\infty$, then passing to the limit as $m\to\infty$ in the inequality (3.9), we obtain (3.1).
Theorem 3.1 is proved.

**Theorem 3.2.** *(the basic theorem) Assume that* (2.4)-(2.12) *hold. Then Problems* (2.1)-(2.3) *has a minimal global attractor which is invariant and compact.*

**The proof of Theorem 3.2.** According to Theorem 2.7, under the conditions (2.4)-(2.12), problem (2.1)-(2.3) has a bounded absorbing set and by Theorem 3.1, the semigroup $\{S(t)\}_{t\geq 0}$, generated by this problem is asymptotically compact. Therefore according to Theorem 3.2 from [11], the problem (2.1)-(2.3) has a minimal global attractor, which is invariant and compact.
Theorem 3.2 is proved.

# References


[1] N. Kenmochi, A. Visintin, *Asymptotic stability for nonlinear evolution problems with hysteresis*, Europ. J. Appl. Math. **5** (1994), 39–56.
[2] P. Krejci, *Hysterezis and periodic solutions of semilinear and quasilinear wave equations*, Math. Z. **193** (1986), 247–264.
[3] P. Krejci, *Asymptotic stability of periodic solutions to the wave equation with hysteresis*, In: Models of hysteresis (A.Visintin, ed.), Longman, Harlow, 77-90, 1993.
[4] M. Hilpert, *On uniqueness for evolution problems with hysteresis*, Mathematical Models for Phase Change Problems, (J.F.Rodrigues,ed.), Birkhauser, Basel, **88**, 1989, 377-388.
[5] A. Visintin, *Hysteresis and Semigroups, in "Models of Hysteresis"*, A.Visintin, ed. Longman, Harlow, (1993) 192-206.
[6] A. Visinton, *Quasilinear hyperbolic equations with hysteresis*, Rend. Mat. Acc. Lincei, **15(9)**(3-4) (2004), 235-247.
[7] A. Visintin, *Differential Models of Hysteresis*, Springer, 1993.
[8] J. L. Lions, *Some Solution Methods for Nonlinear Boundary Problems*, Moscow, Mir, 1972 (in Russian).
[9] S. E. Isayeva, *The mixed problem for one semilinear hyperbolic equation with memory*, Transactions of NAS of Azerbaijan **XXX**(1) (2010), 105–112.
[10] R. Bouc, *Modele mathematique d'hysteresis et application aux systemes a un degre de liberte*, Thése, Marseille, 1966.
[11] O. A. Ladyzhenskaya, *On the determination of minimal attractors for the Navier-Stokes equations and other partial differential equations*, Uspekhi Mat. Nauk **42**(6) (1987), 25–60. English translation: Russian Math. Surveys **42**(6) (1987), 27–73.
[12] S.E. Isayeva, *The existence of an absorbing set for one mixed problem with memory*, Transactions of NAS of Azerbaijan **XXXIII(1)** (2013), 27–35.
[13] I. Chueshov, I. Lasiecka, *Attractors for second-order evolution equations with a nonlinear damping*, J. Dynam. Differential Equations **16(2)** (2004), 469–512.
[14] I. Lasiecka, A. R. Ruzmaikina, *Finite dimensionality and regularity of attractors for 2-D semilinear wave equation with nonlinear dissipation*, J. Math. Anal. Appl. **270** (2002), 16–50.
[15] A. Kh. Khanmamedov, *Global attractors for wave equations with nonlinear interior damping and critical exponents*, J. Differential Equations **230** (2006), 702–719.

*Universal Journal of Mathematics and Applications*, 2 (1) (2019) 42-43
**Research paper**

UJMA

# Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: DOI: http://dx.doi.org/10.32323/ujma.473484

# $\alpha$-Supraposinormality of Operators in Dense Norm-Attainable Classes

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**Article Info**

***Keywords:*** *Hyponormal operator, Norm-attainable class, Posinormal operator, $\alpha$-supraposinormality*
***2010 AMS:*** *47B20*
***Received:*** *22 October 2018*
***Accepted:*** *29 November 2018*
***Available online:*** *20 March 2019*

**Abstract**

The notion of supraposinormality was introduced by Rhaly in a superclass of posinormal operators. In this paper, we give an extension of this notion of supraposinormality to $\alpha$-supraposinormality of operators in the dense norm-attainable class.

## 1. Introduction

Characterization of normality has been done in different aspects by many mathematicians. In [1, 2, 3] and the references therein, they showed characterizations of posinormality and gave some spectral properties of posinormal operators. The relationship between a hyponormal operator and a posinormal operator has also been considered [1]. The author in [2] further introduced a superclass of the posinormal operators and determined sufficient conditions for this superclass to be posinormal and hyponormal. The idea of norm-attainabilty has also been considered by quite a number of authors, for instance, [4, 5] considered conditions for norm-attainability for elementary operators. In this paper, we are interested in characterizing $\alpha-$supraposinormal operators in dense norm-attainable classes. At this point, we give some useful notations. From [1] it is known that an operator $A$ on a Hilbert space $H$ is posinormal if and only if $\gamma^2 A^*A \geq AA^*$ for some $\gamma \geq 0$. $A$ is hyponormal when $\gamma = 1$. The operator $A$ is dominant if $Ran(A-\lambda) \subset Ran(A-\lambda)^*$ for all $\lambda$ in the spectrum of $A$; $A$ is dominant if and only if $A-\lambda$ is posinormal for all complex numbers $\lambda$. Hyponormal operators are necessarily dominant. If $A$ is posinormal, then $KerA \subset KerA^*$. Moreover, $A$ is norm-attainable if there exists a unit vector $x \in H$ such that $\|Ax\| = \|A\|$, where $\|.\|$ is the usual operator norm [5]. The class of all norm-attainable operators is denoted by $NA(H)$. In this work, without loss of generality, $NA(H)$ is taken to be norm dense and separable unless otherwise stated and $NA(H) \subseteq B(H)$.

## 2. Preliminaries

In this section, we give some definitions and auxiliary results which are useful in the sequel.

**Definition 2.1.** *Let $A \in NA(H)$, we say that $A$ is supraposinormal if there exist positive operators $S$ and $T$ on $H$ such that $ASA^* = A^*TA$, where at least one of $S$, $T$ has dense range. The ordered pair $(T,S)$ is called an interrupter pair associated with $A$.*

**Definition 2.2.** *Let $A \in NA(H)$, then for some positive integer $\alpha$ we say that $A$ is $\alpha$-supraposinormal if there exist positive invertible operators $S$ and $T$ on $H$ such that $A^{\alpha}SA^* = A^{\alpha *}TA$, where at least one of $S$, $T$ has a separable range and $A$ is self-adjoint. For simplicity we denote an $\alpha$-supraposinormal operator by $A^{\alpha}$.*

**Definition 2.3.** *Let $A \in NA(H)$, we say that $A$ is totally supraposinormal if $A$-$\lambda$ is supraposinormal for all complex numbers $\lambda$.*

We know that the superclass of operators contains all operators which are posinormal, hyponormal, invertible, positive, coposinormal and norm-attainable [3]. If $A$ is posinormal, then $AA^* = A^*PA$ for some positive operator $P$, so $A$ is supraposinormal with interrupter pair $(I,P)$. If $A$ is coposinormal, then $A^*A = AQA^*$ for some positive operator $Q$, so $A$ is supraposinormal with interrupter pair $(Q,I)$.

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**Remark 2.4.** *Analogously from [3], the collection $\mathscr{S}$ of all supraposinormal operators on $H$ forms a cone in $NA(H)$, and $\mathscr{S}$ is involutive. Indeed, it is easy to see that $\mathscr{S}$ is closed under scalar multiplication, so $\mathscr{S}$ contains all $\alpha A$ for $A \in \mathscr{S}$ and $\alpha \geq 0$, and therefore $\mathscr{S}$ is a cone. Moreover, it is equally easy to see that $A$ is supraposinormal if and only if $A^*$ is supraposinormal, so $\mathscr{S}$ is closed under involution since $NA(H)$ is a $C^*$-algebra.*

## 3. Main Results

In this section, we give the main results in this paper. We begin with the following proposition.

**Lemma 3.1.** *Let $A \in NA(H)$ satisfy $A^\alpha QA^* = A^{\alpha *}PA$ for positive invertible operators $P,Q \in NA(H)$ and a positive integer $\alpha$. The following conditions hold:*

*(i). If $Q$ has separable and norm dense range, then $A$ is supraposinormal and $KerA^\alpha \subset KerA^{\alpha *}$.*
*(ii). If $P$ has separable dense range, then $A$ is supraposinormal and dominant. Moreover, $KerA^\alpha \subset KerA^{\alpha *}$.*
*(iii). If $Q$ is positive invertible and norm-attainable, then the $\alpha$-supraposinormal operator $A$ is $\alpha$-posinormal and hence $\alpha$-hyponormal.*
*(iv). If $P$ is positive invertible and norm-attainable, then the $\alpha$-supraposinormal operator $A$ is $\alpha$-coposinormal.*
*(v). If $P$ and $Q$ are both positive invertible and norm-attainable, then $A$ is both posinormal and coposinormal with $KerA^\alpha = KerA^{\alpha *}$ and $RanA^\alpha = RanA^{\alpha *}$.*
*(vi). If $P$ and $Q$ are both positive invertible, norm-attainable and either is dominant, then $A$ is both $\alpha$-coposinormal and norm-attainable with $KerA^\alpha \cap KerA^{\alpha *} = RanA^\alpha \cap RanA^{\alpha *}$.*

*Proof.* Proofs of $(i)-(v)$ follow analogously from [3]. For the proof of $(vi)$, We consider the orthogonal complements of $KerA^\alpha \cap KerA^{\alpha *}$ and $RanA^\alpha \cap RanA^{\alpha *}$. Since $NA(H)$ is a $C^*$-algebra, normality and norm-attainabilty of $P$ and $Q$ are necessary. Hence, Fugledge-Putman theorem for posinormal and norm attainable class suffices. This completes the proof. □

**Theorem 3.2.** *Let $A^\alpha - \lambda$ be supraposinormal for distinct real values $\lambda = r_1, r_2, ..., r_k$, and assume that the same interrupter pair $(Q,P)$ serves $A^\alpha - \lambda$ in each value of the sequence. Then $Q = P$ and $Ker(A^\alpha - \lambda) = Ker(A^\alpha - \lambda)^*$ when $\lambda = r_1, r_2, ..., r_k$*

*Proof.* We first consider three cases when $\lambda = 0, r_1$, and $r_2$, as in [3]. For any positive integer $\alpha$, $(A^\alpha - \lambda)Q(A^\alpha - \lambda)^* = (A^\alpha - \lambda)^* P(A^\alpha - \lambda)$ for we find that for $k = 1$ and 2, $(A - r_k)Q(A - r_k)^* = (A^\alpha - r_k)^* P(A^\alpha - r_k)$ reduces to $PA^\alpha + A^\alpha * P + r_k Q = QA^\alpha * + A^\alpha Q + r_k P$. Therefore, $(r_1 - r_2)Q = (r_1 - r_2)P$, so $Q = P$. The fact that $Ker(A^\alpha - \lambda) = Ker(A^\alpha - \lambda)^*$ for $\lambda = 0, r_1$, and $r_2$ follows from [2], Corollary 3.2. For the complete sequence upto $r_k$, we consider Caratheodory's extension theorem and by Proposition (**??**), the proof is complete. □

For a generalization consider the following corollary.

**Corollary 3.3.** *If $A^\alpha \in B(H)$ is totally supraposinormal and the same two positive operators $Q,P \in B(H)$ form an interrupter pair $(Q,P)$ for $A^\alpha - \lambda$ for all complex numbers $\lambda$, then $Q = P$; it also follows that $Ker(A^\alpha - \lambda) = Ker(A^\alpha - \lambda)^*$ for all $\lambda$ if and only if $A^\alpha = Ker(A^{\alpha *}$.*

*Proof.* The proof is analogous to that of [3], Corollary 3. □

## 4. Conclusion

We conclude with the following open question: Does there exist an operator $A^\alpha$ that is totally $\alpha$-supraposinormal but neither norm-attainable nor dominant/codominant in a non-separable space?

## 5. Acknowledgement

This work was partially supported by the DFG research Grant No. 1603991000

## References

[1] C. S. Kubrusly and B. P. Duggal, *On posinormal operators*, Adv. Math. Sci. Appl. **17** (1) (2007), 131-147.
[2] G. Leibowitz, *Rhaly matrices*, J. Math. Anal. Appl. **128** (1) (1987), 272-286.
[3] H. C. Rhaly Jr., *A superclass of the posinormal operators*, Newyork J. Math. **20** (2014), 497-506.
[4] N. B. Okelo, *The norm-attainability of some elementary operators*, Appl. Math. E-Notes **13** (2013), 1-7.
[5] N. B. Okelo, J. O. Agure and P. O. Oleche, *Certain conditions for norm-attainability of elementary operators and derivations,* Int. J. of Math. and Soft Comp. **3** (1) (2013), 1-7.

*Universal Journal of Mathematics and Applications, 2 (1) (2019) 44-47*
***Research paper***

**Universal Journal of Mathematics and Applications**

*Journal Homepage: www.dergipark.gov.tr/ujma*
*ISSN 2619-9653*
*DOI: DOI: http://dx.doi.org/10.32323/ujma.509358*

# On the Quotients of Regular Operators

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**Article Info**

***Keywords:*** *Banach lattice, L-weakly compact operator, M-weakly compact operator, Quotient space, Regular operators.*
***2010 AMS:*** *46B42, 47B60, 47B65*
***Received:*** *7 January 2019*
***Accepted:*** *26 February 2019*
***Available online:*** *20 March 2019*

**Abstract**

We give some results about quotients of regular operators on Banach lattices by the linear span of the positive M-weakly and positive L-weakly compact operators. We also present a representation of the quotient space created by the linear span of the positive L-weakly compact operators.

## 1. Introduction

A significant number of studies concerning $L$-weakly and $M$-weakly compact operators has been produced in the literature up to the present. The existing studies concern mostly the relationship between these operators and other operator classes or pathological properties of them. Recently, certain results regarding the order structure of these operator classes have been obtained [1]. Therefore, it is natural to consider other spaces with new ordered structure. The quotient spaces represent of the constructing new spaces from old ones. For this reason, we investigate certain order properties of quotients of the regular operators by our operators which are Banach lattice.

We refer to [2, 3, 4, 5] for unexplained concepts and properties about Banach lattices and positive operators. In the rest of this article, $E$ and $F$ are assumed as Banach lattices, $X$ and $Y$ are assumed Banach spaces unless otherwise stated, and neither of them is the zero space, as we will not indicate this fact in every result. $\mathscr{L}(E,F)$ $\left(\text{resp. } \mathscr{L}^{+}(E,F)\right)$ denotes all linear bounded (resp. positive) operators from $E$ to $F$. In general, the linear span of the positive operators $\mathscr{L}^{r}(E,F)$ which is called regular operators is neither a vector lattice (or Riesz space) nor a Banach space with respect to operator norm $\|.\|$. However, when another norm namely so-called regular norm $\|.\|_r$ is defined by

$$\|T\|_r = \inf\left\{\|S\| : S \in \mathscr{L}^{+}(E,F),\ |Tx| \leq S|x|,\ \forall x \in E\right\}$$

then $\mathscr{L}^{r}(E,F)$ turns into a Banach space. Also, the equality $\|T\|_r = \||T|\|$ is satisfied whenever $T \in \mathscr{L}^{r}(E,F)$ has a modulus. However, there are situations in which $\mathscr{L}^{r}(E,F)$ is a Riesz space. For example, $(\mathscr{L}^{r}(E,F), \|.\|_r)$ is a Banach lattice provided that $F$ is Dedekind complete or $E$ is atomic with an order continuous norm ([6], Theorem 3.3 and 3.4).

The operator $T \in \mathscr{L}(X,E)$ is called $L$-weakly compact if $T(B_X)$, where $B_X$ is the closed unit ball of $X$, is an $L$-weakly compact in the sense that every disjoint sequence $(y_n)_{n\in\mathbb{N}}$ in the $sol(T(B_X))$ is norm null. The operator $T \in \mathscr{L}(E,X)$ is called $M$-weakly compact whenever the sequence $(Tx_n)_{n\in\mathbb{N}}$ is norm null for every bounded disjoint sequence $(x_n)$ in E. From now on, we use the notations $\mathscr{W}_M(E,F)$ and $\mathscr{W}_L(E,F)$ for all operators which is $M$-weakly and $L$-weakly compact, respectively. There is very important duality property between our operators and is stated as follows: $T \in \mathscr{W}_M(E,F)$ (resp. $T \in \mathscr{W}_L(E,F)$) if and only if $T^* \in \mathscr{W}_L(F^*,E^*)$ (resp. $T^* \in \mathscr{W}_M(F^*,E^*)$) where $T^*$ is adjoint operator for $T$. $\mathscr{W}_M(E,F)$ and $\mathscr{W}_L(E,F)$ are subclasses of weakly compact operators and are closed in $\mathscr{L}(E,F)$ with the operator norm.

$E$ is said to have an order continuous norm whenever $\inf\{\|x_\alpha\|\} = 0$ for every downwards directed net $(x_\alpha)$ such that $\inf\{x_\alpha\} = 0$ in $E$. For example, $c_0$, $\ell_p$ and $L^p(\mu)$ $(1 \leq p < \infty)$ have order continuous norm whereas $\ell_\infty$ and $c$ do not have respect their usual norms. The order continuous part of $E$ is defined

$$E^a = \{x \in E : |x| \geqslant x_\alpha \downarrow 0 \Rightarrow \|x_\alpha\| \to 0\}$$

For example, for an atomless measure $\mu$, $(L^\infty(\mu))^a = \{0\}$ and $(\ell^\infty)^a = c_0$. From Proposition 2.4.10, Proposition 3.6.2 in [4], we see that $E^a$ is a closed order ideal and all $L$-weakly compact subsets is included in $E^a$. Therefore, $E^a = \{0\}$ $\left(\text{resp. } (E')^a = \{0\}\right)$ if and only if $\mathscr{W}_L(E,F) = \{0\}$ (resp. $\mathscr{W}_M(E,F) = \{0\}$). For this reason, we assume that $E^a \neq \{0\}$ $\left(\text{resp. } (E')^a \neq \{0\}\right)$.

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## 2. Quotients

For the Riesz space $E$, we recall that the linear subspace $A \subset E$ is an ideal whenever $|u| \leq |v|$ and $v \in A$ implies $u \in A$. On the other hand, the quotient space $E \diagup A$ consists of all equivalence classes modulo $A$. We use the notation $[u]$ for the equivalence classes containing the element $u \in E$. The quotient $E \diagup A$ is a Riesz space according to the partial ordering: $[u] \leq [v]$ whenever there exist elements $u_1 \in [u]$ and $v_1 \in [v]$ such that $u_1 \leq v_1$ ([7], Sec. 18, also for equivalent ordering). The natural quotient map $\pi : E \to E \diagup A$, $\pi(u) = [u]$ is a linear, surjective Riesz (lattice) homomorphism in that $|\pi(u)| = \pi(u)$ for all $u \in E$ and its null space is $A$. Since a null space of a Riesz homomorfizm should be ideal, we consider quotients of Riesz spaces by the ideals. Moreover, the positive cone of $E \diagup A$ is $\pi(E^+) = \{[u] : u \in E^+\}$. Besides, for the closed subspace $A$ of a normed space $X$, the function

$$\|.\| : X \diagup A \to \mathbb{R}, \|u\| = dist(u, A) = \inf\{\|u - y\| : y \in A\} = \inf\{\|u\| : u \in [x]\}$$

define the so-called quotient norm on the quotient space $X \diagup A$. Moreover, $X \diagup A$ with quotient norm is also a Banach space provided that $X$ is a Banach space. Therefore, $E \diagup A$ with the quotient norm is a Banach lattice whenever $E$ is a Banach lattice and $A$ is a closed ideal of $E$ ([2],Proposition II.5.4).

Theorem 2.2 of [8] show that a regular operator which is both $L$-weakly compact and $M$-weakly compact do not need to have a modulus. Also, Theorem 2.3 of [8] show that a modulus $|T|$ for the operator $T \in \mathscr{W}_L(E,F) \cap \mathscr{W}_M$ do not need to be $L$-weakly or $M$-weakly compact. These examples make it clear that $\mathscr{W}_M(E,F)$ (resp. $\mathscr{W}_L(E,F)$) and $\mathscr{W}_M(E,F) \cap \mathscr{L}^r(E,F)$ (resp. $\mathscr{W}_L(E,F) \cap \mathscr{L}^r(E,F)$) are not vector lattice generally. Nevertheless, considering smaller subclasses

$$\mathscr{W}_L^r(E,F) = \{T_1 - T_2 : T_1, T_2 \in \mathscr{W}_L^+(E,F)\}$$

and

$$\mathscr{W}_M^r(E,F) = \{T_1 - T_2 : T_1, T_2 \in \mathscr{W}_M^+(E,F)\}$$

we have nice order theoretic structures. Recently the following result have been proven:

**Theorem 2.1** ([1] Theorem 2.2 and 2.3)**.** *$\mathscr{W}_L^r(E,F)$, equipped with the regular norm, is a Dedekind complete Banach lattice. Similarly, if $F$ is a Dedekind complete Banach lattice, then $\mathscr{W}_M^r(E,F)$ equipped with the regular norm is a Dedekind complete Banach lattice.*

As we can also see in the proof of Theorems 2.2 and 2.3 in [1], $\mathscr{W}_L^r(E,F)$, equipped with the regular norm, is closed in $\mathscr{L}^r(E,F)$. Similarly, if $F$ is a Dedekind complete Banach lattice, then $\mathscr{W}_M^r(E,F)$ equipped with the regular norm is closed in $\mathscr{L}^r(E,F)$. Note that our operator classes have also domination property, in other words, the inequality $0 \leq S \leq T$ implies that $S$ is in the class of operators as $T$. Thus, the next two results are clear from Proposition II.5.4 of [2].

**Corollary 2.2.** *If $\mathscr{L}^r(E,F)$ with the regular norm is a Banach lattice, then $\mathscr{L}^r(E,F) \diagup \mathscr{W}_L^r(E,F)$ with the quotient norm is a Banach lattice.*

**Corollary 2.3.** *If $F$ is a Dedekind complete, then $\mathscr{L}^r(E,F) \diagup \mathscr{W}_M^r(E,F)$ with the quotient norm is a Banach lattice.*

Dedekind completeness of the quotients depends on the quotient map $\pi : E \to E \diagup A$ to be order continuous since $A$ is the kernel of the quotient map $\pi$. Note that our operator classes are not generally band in $\mathscr{L}^r(E,F)$. For example, this can be seen when $E = F = c_0$. The next proposition state a situation that our quotients are Dedekind complete. Recall that a Banach lattice $E$ is an $AM$-space if $\|\sup\{x,y\}\| = maks\{\|x\|, \|y\|\}$ for all $x, y \in E^+$ and a strong order unit in Riesz space $E$ is an element $e \in E^+$ whenever for every $x \in E$ there is $\lambda \in \mathbb{R}$ such that $-\lambda e \leq x \leq \lambda e$.

**Proposition 2.4.** *If $E$ is an $AM$-space with strong order unit and the norm on $F$ is order continuous, then $\mathscr{L}^r(E,F) \diagup \mathscr{W}_L^r(E,F)$ and $\mathscr{L}^r(E,F) \diagup \mathscr{W}_M^r(E,F)$ are Dedekind complete.*

*Proof.* It is easy to see that $\mathscr{W}_M^r(E,F)$ and $\mathscr{W}_L^r(E,F)$ are bands in $\mathscr{L}^r(E,F)$ under the assumptions. Therefore, the quotient map is order continuous ([3], Theorem 7.9). Since $F$ is Dedekind complete $\mathscr{L}^r(E,F)$ is Dedekind complete so does our quotients. □

From this point on, we assume that $\mathscr{L}^r(E,F)$ is a Banach lattice whenever we mention about quotients of it.

A Riesz space $E$ is said to be a lattice ordered algebra whenever $E$ is also an associative algebra such that the product of positive elements is positive. In addition, if $E$ is a Banach lattice, then $E$ is called a Banach lattice algebra provided that $\|xy\| \leq \|x\| \|y\|$ holds for all $x, y \in E^+$. It is well known that the composition of two positive operators is positive. Therefore, $\mathscr{L}^r(E,E)$ (briefly $\mathscr{L}^r(E)$) is closed under composition. This makes $\mathscr{L}^r(E)$ with the regular norm into a Banach lattice algebra whenever $E$ is Dedekind complete. In this case,the identity of $\mathscr{L}^r(E)$ has the norm one. Moreover, if a linear subset $\mathscr{U}$ of the space $\mathscr{L}^r(E)$ is two sided ideal, in the sense that for every $S \in \mathscr{U}$ and for every $T \in \mathscr{L}^r(E)$, the compositions $ST$ and $TS$ belong to $\mathscr{U}$, then $\mathscr{L}^r(E) \diagup \mathscr{U}$ is also Banach lattice algebra.

On the contrary, regarding the regular weakly compact and regular compact operators, Example 1.2 in [9] shows that regular $L$-weakly and regular $M$-weakly compact operators do not need to be two sided ideals in $\mathscr{L}^r(E)$. In the same paper, it is proven that $\mathscr{W}_M(E) \cap \mathscr{L}^r(E)$ (resp. $\mathscr{W}_L(E) \cap \mathscr{L}^r(E)$) is a two sided ideal in $\mathscr{L}^r(E)$ if and only if $E^*$ (resp. $E$)has an order continuous norm ([9], Theorem 3.3 and 3.4). Similar results can be given for $\mathscr{W}_M^r(E)$ and $\mathscr{W}_L^r(E)$.

**Theorem 2.5.** *$\mathscr{W}_M^r(E)$ (resp. $\mathscr{W}_L^r(E)$) is a two sided ideal in $\mathscr{L}^r(E)$ if and only if $E^*$ (resp. $E$) has an order continuous norm.*

*Proof.* Proof is the same as with Theorem 3.3 and 3.4 in [9]. □

Since $\mathscr{W}_L^r(E)$ and $\mathscr{W}_M^r(E)$ are norm closed subspaces of $\mathscr{L}^r(E)$, the following result is obvious.

**Corollary 2.6.** *If $E$ is Dedekind complete and $E^*$ (resp. $E$) has an order continuous norm then $\mathscr{L}^r(E) \diagup \mathscr{W}_M^r(E)$ (resp. $\mathscr{L}^r(E) \diagup \mathscr{W}_L^r(E)$) with quotient norm is Banach lattice algebra.*

Order continuity properties of our operator classes regarding regular norm is given in [1] as it follows.

**Theorem 2.7** ([1], Theorem 3.1 ve 3.2)**.** *The regular norm on $\mathscr{W}_L^r(E,F)$ (resp. $\mathscr{W}_M^r(E,F)$) is order continuous if and only if $E^*$ (resp. $F$) has an order continuous norm.*

Order continuity is a hereditary property for the quotients by the closed ideals ([5], Example 1.5). But the regular norm is not order continuous in general. In the context of the order continuity of regular norm, some results were given by Z.Chen et all in [10].

**Theorem 2.8** ([10], Proposition 1)**.** *If the regular norm on $\mathscr{L}^r(E,F)$ is order continuous, then the norms both on $E^*$ and $F$ are order continuous.*

**Theorem 2.9** ([10], Theorem 2)**.** *The following statements are equivalent.*

1. $\mathscr{L}^r(E,F)$ is a vector lattice and the regular norm on $\mathscr{L}^r(E,F)$ is order continuous.
2. Every positive operator $T: E \to F$ is $L$- and $M$-weakly compact.

As a consequence, we obtain the following:

**Corollary 2.10.** *The regular norm on $\mathscr{L}^r(E,F)$ is order continuous if and only if $\mathscr{L}^r(E,F)\diagup\mathscr{W}_M^r(E,F)=\{0\}$ and $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)=\{0\}$.*

Note that $E$ has order continuous norm if and only if $E$ is $\sigma$-Dedekind complete and there does not exist any sublattice of $E$ isomorphic to $\ell_\infty$ ([4], Corollary.2.4.3).

**Theorem 2.11** ([11], Theorem 2)**.** *Let $E=(E,\tau)$ be a Dedekind $\sigma-$complete Riesz space, let $\tau$ be locally convex-solid, and let $M$ be a $\tau$-closed ideal of $E$. If $E$ contains a copy of $\ell_\infty$, then $E\diagup M$ or $M$ contains a lattice copy of $\ell_\infty$.*

Combining Theorem 2.7 and Theorem 2.8 with Theorem 2.11, we obtain the following result:

**Corollary 2.12.** *If $F$ is Dedekind complete, the following statements hold.*

1. If $F$ has order continuous norm, but $E^*$ does not have then $\mathscr{L}^r(E,F)\diagup\mathscr{W}_M^r(E,F)$ also does not have order continuous norm.
2. If $E^*$ has order continuous norm, but $F$ does not have then $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)$ also does not have order continuous norm.

*Proof.* If $E^*$ (resp. $F$) does not have order continuous, then $\mathscr{L}^r(E,F)$ contains a lattice copy of $\ell_\infty$. Since $F$ (resp. $E^*$) has order continuous norm, then $\mathscr{W}_M^r(E,F)$ (resp. $\mathscr{W}_L^r(E,F)$) has order continuous norm. Therefore, $\mathscr{L}^r(E,F)\diagup\mathscr{W}_M^r(E,F)$ (resp. $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)$) does not have order continuous norm. □

## 3. A Representation of $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)$

In [12], a representation of the weak Calkin algebra $\mathscr{L}(E)\diagup\mathscr{W}(E)$ where $\mathscr{W}(E)$ denotes the class of weakly compact operators on $E$ is given. Similarly, in this section, we present a representation of the quotient $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)$.
We consider the operator $R(S): E^{**}\diagup E^a \to F^{**}\diagup F^a$ for every $S\in\mathscr{L}^r(E,F)$ as follows

$$R(S)([x^{**}])=[S^{**}x^{**}].$$

Thus, we can define the induced map

$$\begin{aligned} R:\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F) &\to \mathscr{L}^r(E^{**}\diagup E^a, F^{**}\diagup F^a)\\ S+\mathscr{W}_L^r(E,F) &\to R(S). \end{aligned}$$

In the following proposition, we present some properties of the above operator.

**Proposition 3.1.** *The following assumptions hold for the map $R$.*

1. *If $S\in\mathscr{W}_L^r(E,F)$ then $R(S)=0$.*
2. *$R$ is a positive linear map.*
3. *For $S\in\mathscr{L}^r(E,F)$ $\|R(S)\|_{F^{**}\diagup F^a}\le\|S^{**}\|_{E^{**}}$, so $\|R\|\le 1$.*
4. *If $E=F$ then $R([I_E])=I_{E^{**}\diagup E^a}$.*
5. *Whenever $ST$ is defined as $R(ST)=R(S)R(T)$.*

*Proof.* (1) Since $L$-weakly compact operators take values in $F^a$ and are a subclass of weakly compact operators, then $S^{**}(E^{**})\subseteq F^a$ hold for all $S\in\mathscr{W}_L^r(E,F)$.
(2) Let choose $0\le S\in\mathscr{L}^r(E,F)$ and $[x^{**}]\in(E^{**}\diagup E^a)^+$.Then $x^{**}\in(E^{**})^+$ so $S^{**}x^{**}\in(F^{**})^+$. It means that $R([S])[x^{**}]\in(F^{**}\diagup F^a)^+$, i.e. $R$ is positive. The linearity of $R$ is a routine verification.
(3) For $T,S\in\mathscr{L}^r(E,F)$, $x^{**}\in E^{**}$

$$\begin{aligned} \|R(S)[x^{**}]\|_{F^{**}\diagup F^a} &= dist(R(S)[x^{**}],F^a)=\inf\{\|R(S)[x^{**}]-y\|: y\in F^a\}\\ &= \inf\{\|S^{**}x^{**}+z-y\|: z,y\in F^a\}\\ &= \inf\{\|S^{**}x^{**}-(y-z)\|: z,y\in F^a\}\\ &= \inf\{\|S^{**}x^{**}-u\|: u\in F^a\}\\ &\le \inf\{\|S^{**}x^{**}-S^{**}w\|: w\in E^a\}\\ &\le \|S^{**}\|\inf\{\|x^{**}-w\|: w\in E^a\}=\|S^{**}\|\,\|[x^{**}]\| \end{aligned}$$

show that $\|R(S)\| \leq \|S^{**}\| = \|S\|$ so $\|R\| \leq 1$.
(4) It follows that for every $x^{**} \in E^{**}$

$$R([I_E])[x^{**}] = \left[(I_E)^{**} x^{**}\right] = [x^{**}] = I_{E^{**}/E^a}[x^{**}].$$

(5) Let $E$, $F$, $G$ be Banach lattices and $S \in \mathscr{L}^r(E,F)$, $T \in \mathscr{L}^r(G,E)$. For every $x^{**} \in E^{**}$ we obtain that

$$\begin{aligned} R([ST])[x^{**}] &= \left[(ST)^{**} x^{**}\right] = [S^{**}(T^{**}x^{**})] = [S^{**}(R([T])[x^{**}])] \\ &= R([S])R([T])[x^{**}]. \end{aligned}$$

□

**Definition 3.2.** *The pair of Banach lattices $(E,F)$ has the invariant modulus property if the equality $|T^*| = |T|^*$ holds for every $T \in \mathscr{L}^r(E,F)$ for which $|T|$ exists in $\mathscr{L}^r(E,F)$.*

**Theorem 3.3.** *If $(E,F)$ and $(F^*,E^*)$ have invariant modulus property, then $R$ is a Riesz homomorphism.*

*Proof.* For $S \in \mathscr{L}^r(E,F)$ and $x^{**} \in E^{**}$ we obtain

$$R(|[S]|)[x^{**}] = R([|S|])[x^{**}] = \left[|S|^{**} x^{**}\right] = [|S^{**}|x^{**}].$$

On the other hand, by the help Riesz-Kantorovich formulae we get

$$\begin{aligned} R(|[S]|)[x^{**}] &= [\sup\{|S^{**}y^{**}| : |y^{**}| \leq x^{**}\}] = [\sup\{|R([S])[y^{**}]| : |y^{**}| \leq x^{**}\}] \\ &= \sup\{|R([S])[y^{**}]| : [|y^{**}|] \leq [x^{**}]\} = \sup\{|R([S])[y^{**}]| : |[y^{**}]| \leq [x^{**}]\} \\ &= |R([S])|[x^{**}]. \end{aligned}$$

□

Analogously to Gantmacher's theorem (see [3], Theorem 17.2) can be modified for $L$-weakly compact operators as follows:

**Lemma 3.4.** *If $E^*$ has Schur property and $T \in \mathscr{L}(E,F)$, then the following statements are equivalent.*

1. $T \in \mathscr{W}_L(E,F)$,
2. $T^{**}(E^{**}) \subseteq F^a$,
3. $T^* : \left((F^a)^*, \sigma\left((F^a)^*, F^a\right)\right) \to (E^*, \sigma(E^*, E^{**}))$,
4. $T^* \in \mathscr{W}_M(E,F)$.

*Proof.* $(1 \Rightarrow 2 \Rightarrow 3)$ is clear from Theorem 17.2 in [3].
$(3 \Rightarrow 4)$ If $(f_n)_{n\in\mathbb{N}}$ is a norm bounded disjoint sequence in $F^*$ so in $(F^a)^*$, then we have $f_n \to 0$ in the topology $\sigma\left((F^a)^*, F^a\right)$ since $F^a$ has order continuous norm (see [4], Corollary 2.4.3). Thus, $(T^* f_n)$ is $\sigma(E^*, E^{**})$-null sequence by the hypothesis. Hence $\|T^* f_n\| \to 0$ since $E^*$ has Schur property. This show that $T^*$ is an $M$-weakly compact operator.
$(4 \Rightarrow 1)$ is clear from Theorem 18.13 in [3]. □

**Lemma 3.5.** *If $F^a$ has Schur property and $T \in \mathscr{L}(E,F)$ then the following statements are equivalent.*

1. $T \in \mathscr{W}_L(E,F)$,
2. $T^{**}(E^{**}) \subseteq F^a$,
3. $T^* \in \mathscr{W}_M(E,F)$.

*Proof.* $(1 \Rightarrow 2)$ and $(3 \Rightarrow 1)$ are obvious. $(2 \Rightarrow 3)$ If $T^{**}(E^{**}) \subseteq F^a$ hold then the operator $T$ is weakly compact (see [3], Theorem 17.2). Hence $T^{**}(B_{E^{**}})$, where $B_{E^{**}}$ is unit ball of $E^{**}$, is relatively weakly compact subset of $F^a$, so is $L$-weakly compact from Corollary 3.6.8 in [4] since $F^a$ has Schur property. □

**Theorem 3.6.** *Suppose that $E^*$ or $F$ has Schur property. Then, $S \in \mathscr{W}_L^r(E,F)$ if and only if $R(S) = 0$.*

*Proof.* Necessity has been proved in Proposition 3.1. Suppose that $R(S) = 0$. The equality $R(S) = 0$ implies $S^{**}(E^{**}) \subseteq F^a$. Hence, $S \in \mathscr{W}_L^r(E,F)$ from Lemma 3.4 and Lemma 3.5 which we are looking for. □

Hence, if $E^*$ and $F$ has Schur property with order continuous norm, then $S + \mathscr{W}_L^r(E,F) \to R(S)$ provides a representation of the quotient $\mathscr{L}^r(E,F) / \mathscr{W}_L^r(E,F)$, and its image $\{R(S) : S \in \mathscr{L}^r(E)\}$ whenever $E = F$ is a subalgebra of $\mathscr{L}^r(E^{**}/E^a)$ containing the identity.

# References


[1] E. Bayram, A. W. Wickstead, *Banach lattices of L-weakly and M-weakly compact operators*, Arch. Math. (Basel) **108**(2017), 293–299.
[2] H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
[3] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Pure and Applied Mathematics, vol. 119, Academic Press, Inc., Orlando, FL, 1985.
[4] P. Meyer-Nieberg, *Banach Lattices*, Universitext, Springer-Verlag, Berlin, 1991.
[5] W. Wnuk, *Banach Lattices with Order Continuous Norms*, Advenced Topics in Mathematics, Polish Scientific Publishers PWN, Warsaw, 1999.
[6] A. W. Wickstead, *Regular operators between Banach lattices*, Positivity, TrendsMath., Birkhäuser, Basel, 2007.
[7] W. A. J. Luxemburg, A.C. Zaanen, *Riesz Spaces I*, North-Holland Publ., Amsterdam, 1971.
[8] Z. L. Chen and A. W. Wickstead, *L-weakly and M-weakly compact operators*, Indag. Math. (N.S.), **10**(3) (1999), 321-336.
[9] E. Bayram , W. Wnuk, *Some Algebra Ideals Of Regular Operators*, Comment. Math. **532** (2013), 127-133.
[10] Z. L. Chen, Y. Feng, J.X. Chen, *The Order Continuity of the Regular Norm on Regular Operator Spaces*, Abstr. Appl. Anal., (2013), Article ID 183786, 7 pages.
[11] M. Wojtowicz, *Copies of $\ell_\infty$ in quotients of locally solid Riesz spaces*, Arch. Math. **80**(2003), 294–301.
[12] M. Gonzalez, E. Saksman, H.O. Tylli, *Representing non-weakly compact operators*, Studia Math., **113** (1995), 265-282.