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A Unified Family of Generalized q-Hermite Apostol Type Polynomials and its Applications

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Abstract
The intended objective of this paper is to introduce a new class of generalized q-Hermite based Apostol type polynomials by combining the q-Hermite polynomials and a unified family of q-Apostol-type polynomials. The generating function, series definition and several explicit representations for these polynomials are established. The q-Hermite-Apostol Bernoulli, q-Hermite-Apostol Euler and q-Hermite-Apostol Genocchi polynomials are studied as special members of this family and corresponding relations for these polynomials are obtained.

Keywords: q-Hermite polynomials, Generalized q-Apostol type polynomials, Generalized q-Hermite Apostol type polynomials, Explicit representation
2010 AMS: 11B73, 11B83, 11B68

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1. Introduction and preliminaries

The q-calculus has been extensively studied for a long time by many mathematicians, physicists and engineers. The q-calculus is a generalization of many subjects, like the hypergeometric series, complex analysis and particle physics. The q-analogues of many orthogonal polynomials and functions assume a very pleasant form reminding directly of their classical counterparts. The q-calculus is mostly being used by physicists at a high level. In short, q-calculus is quite a popular subject today.

Throughout the present paper, $\mathbb{C}$ indicates the set of complex numbers, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0$ indicates the set of non-negative integers. Further, the variable $q \in \mathbb{C}$ such that $|q| < 1$. The following q-standard notations and definitions are taken from [1].

The q-analogue of the shifted factorial $(a)_n$ is defined by

$$(a;q)_0 = 1, (a;q)_n = \prod_{m=0}^{n-1}(1-q^m a),\ n \in \mathbb{N}.$$

The q-analogues of a complex number a and of the factorial function are defined by

$$[a]_q = \frac{1-q^a}{1-q},\ \ q \in \mathbb{C}-\{1\};\quad a \in \mathbb{C},$$

$$[n]_q! = \prod_{m=1}^{n}[m]_q = \frac{(q;q)_n}{(1-q)^n},\ q \neq 1;\ n \in \mathbb{N},\ \ [0]_q! = 1,\ q \in \mathbb{C}.$$

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad k = 0, 1, \ldots, n.$$

The q-exponential function is defined as:

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, \quad |x| < |1-q|^{-1}. \tag{1.1}$$

The q-Hermite polynomials are special or limiting cases of the orthogonal polynomials as they contain no parameter other than q and appears to be at the bottom of a hierarchy of the classical q-orthogonal polynomials [2]. The q-Hermite polynomials constitute a 1-parameter family of orthogonal polynomials, which for $q = 1$ reduce to the well known Hermite polynomials. We recall that the q-Hermite polynomials $H_{n,q}(x)$ is defined by the following generating function [3]:

$$F_q(x,t) := F_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{[n]_q!}, \tag{1.2}$$

$$F_q(t) := \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} \frac{t^{2n}}{[2n]_q!!}, \quad [2n]_q!! = [2n]_q[2n-2]_q....[2]_q.$$

$$D_{q,x}H_{n,q}(x) = [n]_q H_{n-1,q}(x).$$

Recently, many mathematicians studied the unification of the Bernoulli and Euler polynomials. Luo and Srivastava [4, 5] introduced the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α. Further, the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x)$ of order α and the generalized Apostol-Genochhi polynomials $G_n^{(\alpha)}(x)$ of order α are investigated by Luo [6, 7]. Thereafter, in 2014 Ernst [8] defined the q-analogues of the generalized Apostol type polynomials.

The generalized q-Apostol-Bernoulli polynomials $B_{n,q,\lambda}^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function [8]:

$$\left(\frac{t}{\lambda e_q(t) - 1}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} B_{n,q,\lambda}^{(\alpha)}(x) \frac{t^n}{[n]_q!}. \tag{1.3}$$

The generalized q-Apostol-Euler polynomials $E_{n,q,\lambda}^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function [8]:

$$\left(\frac{2}{\lambda e_q(t) + 1}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} E_{n,q,\lambda}^{(\alpha)}(x) \frac{t^n}{[n]_q!}. \tag{1.4}$$

The generalized q-Apostol-Genocchi polynomials $G_{n,q,\lambda}^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function [8]:

$$\left(\frac{2t}{\lambda e_q(t) + 1}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} G_{n,q,\lambda}^{(\alpha)}(x) \frac{t^n}{[n]_q!}. \tag{1.5}$$

In view of equations (1.3)-(1.5), the generalized q-Apostol type polynomials $\mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b)$ $(\alpha \in \mathbb{N}_0, \lambda, a, b \in \mathbb{C})$ of order α are defined by the following generating function:

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t) - a^b}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} \mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b) \frac{t^n}{[n]_q!}, \tag{1.6}$$

where $\mathscr{P}_{n,q,\beta}^{(\alpha)}(k,a,b) = \mathscr{P}_{n,q,\beta}^{(\alpha)}(0;k,a,b)$ are the q-Apostol type numbers of order α.

If we take the limit $q \to 1$, the generalized q-Apostol type polynomials defined by equation (1.6) reduces to the unified Apostol type polynomials [9]. In fact, the following special cases hold:

$$\lim_{q \to 1} \mathscr{P}_{n,q,\lambda}^{(\alpha)}(x;1,1,1) = B_{n,\lambda}^{(\alpha)}(x),$$

$$\lim_{q\to 1} \mathscr{P}^{(\alpha)}_{n,q,\lambda}(x;0,-1,1) = E^{(\alpha)}_{n,\lambda}(x),$$

$$\lim_{q\to 1} \mathscr{P}^{(\alpha)}_{n,q,\frac{\lambda}{2}}(x;1,-1/2,1) = G^{(\alpha)}_{n,\lambda}(x),$$

where $B^{(\alpha)}_{n,\lambda}(x), E^{(\alpha)}_{n,\lambda}(x)$ and $G^{(\alpha)}_{n,\lambda}(x)$ are the generalized forms of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.

In the current article, the q-Hermite-Apostol type polynomials are introduced and their explicit relations are proved. The corresponding results for the q-Hermite-Apostol Bernoulli, q-Hermite-Apostol Euler and q-Hermite-Apostol Genocchi polynomials are established.

2. Generalized q-Hermite Apostol type polynomials

In this section, a new hybrid class of the generalized q-Hermite-Apostol type polynomials (GqHATyP), denoted by ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ is introduced by convoluting the q-Hermite polynomials and generalized q-Apostol type polynomials. In order to establish the generating function for the these polynomials, the following result is proved:

Theorem 2.1. *The following generating function for the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)(\alpha \in \mathbb{N}_0, \lambda, a, b \in \mathbb{C})$ *holds true:*

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t) - a^b}\right)^{\alpha} F_q(t) e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!}, \tag{2.1}$$

Proof. Expanding the exponential function $e_q(xt)$ and then replacing the powers of x, *i.e.* $x^0;x^1;x^2;\cdots;x^n$ by the correlating q-Hermite polynomials $H_{0,q}(x);H_{1,q}(x);\cdots;H_{n,q}(x)$ in the l.h.s. of equation (1.6) and after summing up the terms of the resultant equation and denoting the resultant GqHATyP in the r.h.s. by ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$, assertion (2.1) is proved. □

Taking $x = 0$ in equation (2.1), we get

$${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(k,a,b) = {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(0;k,a,b),$$

where ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(k,a,b)$ are the q-Hermite Apostol type numbers of order α.

Next, the series expansions for the GqHATyP ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ is obtained by proving the following result:

Theorem 2.2. *The following series expansions for the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ *hold true:*

$${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q \mathscr{P}^{(\alpha)}_{r,q,\beta}(k,a,b) H_{n-r,q}(x), \tag{2.2}$$

$${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q {}_H\mathscr{P}^{(\alpha)}_{r,q,\beta}(k,a,b) x^{n-r}. \tag{2.3}$$

Proof. Utilizing equations (1.2) and (1.6) in the l.h.s. of generating function (2.1) and then using Cauchy-product rule in the l.h.s. of the resultant equation, it follows that

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q \mathscr{P}^{(\alpha)}_{r,q,\beta}(k,a,b) H_{n-r,q}(x)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!}. \tag{2.4}$$

Equating the coefficients of identical powers of t in both sides of equation (2.4), assertion (2.2) follows.

Utilizing equation (1.1) in the l.h.s. of generating function (2.1), it follows that

$$\sum_{n=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} \sum_{r=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{r,q,\beta}(k,a,b)\frac{t^r}{[n]_r!},$$

which on applying the Cauchy product rule in the r.h.s. and then comparing the coefficients of same powers of t in both sides of resultant equation yields assertion (2.3).

□

S. No.	k; a; b; β	Generating function	Name of the polynomials
I.	$k=1$; $a=1$; $b=1$; $\beta=\lambda$	$\left(\frac{t}{\lambda e_q(t)-1}\right)^{(\alpha)} F_q(t)\, e_q(xt) = \sum_{n=0}^{\infty} {}_HB_{n,q,\lambda}^{(\alpha)}(x)\frac{t^n}{[n]_q!}$	The generalized q-Hermite-Apostol Bernoulli polynomials (GqHABP)
II.	$k=0$; $a=-1$; $b=1$; $\beta=\lambda$	$\left(\frac{2}{\lambda e_q(t)+1}\right)^{(\alpha)} F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_HE_{n,q,\lambda}^{(\alpha)}(x)\frac{t^n}{[n]_q!}$	The generalized q-Hermite-Apostol Euler polynomials (GqHAEP)
III.	$k=1$; $a=-1/2$; $b=1$; $\beta=\lambda/2$	$\left(\frac{2t}{\lambda e_q(t)+1}\right)^{(\alpha)} F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_HG_{n,q,\lambda}^{(\alpha)}(x)\frac{t^n}{[n]_q!}$	The generalized q-Hermite-Apostol Genocchi polynomials (GqHAGP)

Table 1. Certain members belonging to the generalized q-Hermite-Apostol family

Different members of the generalized q-Hermite-Apostol family can be obtained by making suitable selections of the parameters k,a,b and β in generating relation (2.1). Some of these members are listed in Table 1.

Proposition 2.3. *The following relations for the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b)$ *holds true:*

$$D_{q,t}e_q(xt) = x\, e_q(xt),$$
$$D_{q,x}\left({}_H\mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b)\right) = [n]_q\, {}_H\mathscr{P}_{n-1,q,\beta}^{(\alpha)}(x;k,a,b).$$

Theorem 2.4. *For each $n\in\mathbb{N}$ and for the q-commuting variables x and u such that $xu = qux$, the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}_{k,q,\beta}^{(\alpha)}(x;k,a,b)$ *satisfy the following relations:*

$${}_H\mathscr{P}_{n,q,\beta}^{(\alpha+\gamma)}(x;k,a,b) = \sum_{r=0}^{n}\begin{bmatrix} n \\ r\end{bmatrix}_q {}_H\mathscr{P}_{r,q,\beta}^{(\alpha)}(x;k,a,b)\mathscr{P}_{n-r,q,\beta}^{(\gamma)}(k,a,b). \tag{2.5}$$

$${}_H\mathscr{P}_{n,q,\beta}^{(\alpha+\gamma)}(x+u;k,a,b) = \sum_{r=0}^{n}\begin{bmatrix} n \\ r\end{bmatrix}_q {}_H\mathscr{P}_{r,q,\beta}^{(\alpha)}(x;k,a,b)\mathscr{P}_{n-r,q,\beta}^{(\gamma)}(u;k,a,b). \tag{2.6}$$

Proof. Replacing α by $\alpha+\gamma$ in definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\mathscr{P}_{n,q,\beta}^{(\alpha+\gamma)}(x;k,a,b)\frac{t^n}{[n]_q!} = \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha+\gamma} F_q(t)e_q(xt)$$
$$= \left(\sum_{r=0}^{\infty} {}_H\mathscr{P}_{r,q,\beta}^{(\alpha)}(x;k,a,b)\frac{t^r}{[r]_q!}\right)\left(\sum_{n=0}^{\infty} \mathscr{P}_{n,q,\beta}^{(\gamma)}(k,a,b)\frac{t^n}{[n]_q!}\right).$$

Using Cauchy-product rule in the r.h.s. of above equation, it follows that

$$\sum_{n=0}^{\infty} {}_H\mathscr{P}_{n,q,\beta}^{(\alpha+\gamma)}(x;k,a,b)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\begin{bmatrix} n \\ r\end{bmatrix}_q {}_H\mathscr{P}_{r,q,\beta}^{(\alpha)}(x;k,a,b)\mathscr{P}_{n-r,q,\beta}^{(\gamma)}(k,a,b)\frac{t^n}{[n]_q!} \tag{2.7}$$

Equating the coefficients of identical powers of t in both sides of equation (2.7), assertion (2.5) follows. Further, replacing α by $\alpha+\gamma$ and x by $x+u$ in Definition 2.1 and proceeding on the same lines of proof as above, assertion (2.6) follows. □

Theorem 2.5. *For each $n\in\mathbb{N}$ and for the q-commuting variables x and u such that $xu = qux$, the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}_{k,q,\beta}^{(\alpha)}(x;k,a,b)$ *satisfy the following relation:*

$$\beta^b\, {}_H\mathscr{P}_{n,q,\beta}^{(\alpha)}(x+1;k,a,b) - a^b\, {}_H\mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b) = \frac{2^{1-k}[n]_q!}{[n-k]_q!}\, {}_H\mathscr{P}_{n-k,q,\beta}^{(\alpha-1)}(x;k,a,b). \tag{2.8}$$

S. No.	Special polynomials	Results
I.	**GqHABP** ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$	${}_HB^{(\alpha)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q B^{(\alpha)}_{r,q,\lambda}\ H_{n-r,q}(x)$ ${}_HB_{n,q,\lambda}(\alpha)(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\ {}_HB^{(\alpha)}_{r,q,\lambda}\ x^{n-r}$ ${}_HB^{(\alpha+\gamma)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\ {}_HB^{(\alpha)}_{r,q,\lambda}(x)\ B^{(\gamma)}_{n-r,q,\lambda}$ ${}_HB^{(\alpha+\gamma)}_{n,q,\lambda}(x+u)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\ {}_HB^{(\alpha)}_{r,q,\lambda}(x)\ B^{(\gamma)}_{n-r,q,\lambda}(u)$ $\beta^b\,{}_HB^{(\alpha)}_{n,q,\lambda}(x+1)-a^b\,{}_HB^{(\alpha)}_{n,q,\lambda}(x)=\frac{2^{1-k}[n]_q!}{[n-k]_q!}\ {}_HB^{(\alpha-1)}_{n-k,q,\lambda}(x)$
II.	**GqHAEP** ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$	${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q E^{(\alpha)}_{r,q,\lambda}\ H_{n-r,q}(x)$ ${}_HE_{n,q,\lambda}(\alpha)(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\ {}_HE^{(\alpha)}_{r,q,\lambda}\ x^{n-r}$ ${}_HE^{(\alpha+\gamma)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\ {}_HE^{(\alpha)}_{r,q,\lambda}(x)\ E^{(\gamma)}_{n-r,q,\lambda}$ ${}_HE^{(\alpha+\gamma)}_{n,q,\lambda}(x+u)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\ {}_HE^{(\alpha)}_{r,q,\lambda}(x)\ E^{(\gamma)}_{n-r,q,\lambda}(u)$ $\beta^b\,{}_HE^{(\alpha)}_{n,q,\lambda}(x+1)-a^b\,{}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{2^{1-k}[n]_q!}{[n-k]_q!}\ {}_HE^{(\alpha-1)}_{n-k,q,\lambda}(x)$
II.	**GqHAGP** ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$	${}_HG^{(\alpha)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q G^{(\alpha)}_{r,q,\lambda}\ H_{n-r,q}(x)$ ${}_HG_{n,q,\lambda}(\alpha)(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\ {}_HG^{(\alpha)}_{r,q,\lambda}\ x^{n-r}$ ${}_HG^{(\alpha+\gamma)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\ {}_HG^{(\alpha)}_{r,q,\lambda}(x)\ G^{(\gamma)}_{n-r,q,\lambda}$ ${}_HG^{(\alpha+\gamma)}_{n,q,\lambda}(x+u)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\ {}_HG^{(\alpha)}_{r,q,\lambda}(x)\ G^{(\gamma)}_{n-r,q,\lambda}(u)$ $\beta^b\,{}_HG^{(\alpha)}_{n,q,\lambda}(x+1)-a^b\,{}_HG^{(\alpha)}_{n,q,\lambda}(x)=\frac{2^{1-k}[n]_q!}{[n-k]_q!}\ {}_HG^{(\alpha-1)}_{n-k,q,\lambda}(x)$

Table 2. Certain results for the GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$

Proof. From generating relation (2.1), we have

$$\sum_{n=0}^{\infty}\beta^b\,{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x+1;k,a,b)\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}a^b\,{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!}$$

$$=\beta^b\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}F_q(t)e_q((x+1)t)-a^b\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}F_q(t)e_q(xt)$$

$$=\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}F_q(t)e_q(xt)\left(\beta^b e_q(t)-a^b\right)$$

$$\sum_{n=0}^{\infty}\left(\beta^b\,{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x+1;k,a,b)-a^b\,{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\right)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}2^{1-k}\,{}_H\mathscr{P}^{(\alpha-1)}_{n,q,\beta}(x;k,a,b)\frac{t^{n+k}}{[n]_q!}.$$

Equating the coefficients of same powers of t in both sides of the above equation, assertion (2.8) follows. □

In view of Table 1, ceratin results for the GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$ are established and are given in Table 2.

In the next section, certain explicit representations for the GqHATyP ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ are established.

3. Explicit representations

In order to derive the explicit representations for the GqHATyP ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$, we recall the following definition:

Definition 3.1. *The generalized q-Stirling numbers $S_q(n,\nu,a,b,\beta)$ of the second kind of order ν is defined as [10]:*

$$\sum_{n=0}^{\infty} S_q(n,\nu,a,b,\beta)\frac{t^n}{[n]_q!} = \frac{(\beta^b e_q(t)-a^b)^\nu}{[\nu]_q!}.$$

Theorem 3.2. *The following explicit formula for the generalized q-Hermite based Apostol type polynomials ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ in terms of the generalized q-Stirling numbers of the second kind $S_q(n,\nu,a,b,\beta)$ holds true:*

$$ {}_H\mathscr{P}^{(\alpha)}_{n-\nu k,q,\beta}(x;k,a,b) = 2^{\nu(k-1)}\frac{[\nu]_q![n-\nu k]_q!}{[n]_q!}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q {}_H\mathscr{P}^{(\nu-\alpha)}_{l,q,\beta}(x;k,a,b)S_q(n-l,\nu,a,b,\beta). \tag{3.1}$$

Proof. From generating relation (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!} &= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^\alpha F_q(t)e_q(xt)\frac{(\beta^b e_q(t)-a^b)^\nu}{[\nu]_q!}\left(\frac{[\nu]_q!}{(\beta^b e_q(t)-a^b)^\nu}\right)\\ &= \frac{[\nu]_q!}{(2^{1-k}t^k)^\nu}\sum_{l=0}^{\infty} {}_H\mathscr{P}^{(\alpha-\nu)}_{l,q,\beta}(x;k,a,b)\frac{t^l}{[l]_q!}\left(\sum_{n=0}^{\infty} S_q(n,\nu,a,b,\beta)\frac{t^n}{[n]_q!}\right).\end{aligned}$$

Applying the Cauchy-product rule on the r.h.s. of the above equation, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^{n+\nu k}}{[n]_q!} =& [\nu]_q!2^{(k-1)\nu}\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q {}_H\mathscr{P}^{(\alpha-\nu)}_{l,q,\beta}(x;k,a,b)\right.\\ &\left.\times S_q(n-l,\nu,a,b,\beta)\right\}\frac{t^n}{[n]_q!}.\end{aligned} \tag{3.2}$$

Equating the coefficients of identical powers of t in both sides of equation (3.2) yields assertion (3.1). □

Theorem 3.3. *The following explicit relation for the generalized q-Hermite based Apostol type polynomials ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ in terms of the generalized q-Apostol Bernoulli polynomials $B_{n,q,\lambda}(x)$ holds true:*

$$\begin{aligned}{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b) =& \frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix} n+1 \\ r \end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix} r \\ m \end{bmatrix}_q B_{n+1-r,q,\lambda}(x)\right.\\ &\left.-\sum_{m=0}^{n+1}\begin{bmatrix} n+1 \\ m \end{bmatrix}_q B_{n+1-m,q,\lambda}(x)\right\}{}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b).\end{aligned} \tag{3.3}$$

Proof. Consider generating function (2.1) in the following form:

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^\alpha F_q(t)e_q(xt) = \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^\alpha F_q(t)\left(\frac{t}{\lambda e_q(t)-1}\right)\frac{\lambda e_q(t)-1}{t}e_q(xt),$$

which on simplifying and rearranging the terms becomes

$$\begin{aligned}\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^\alpha F_q(t)e_q(xt) =& \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^\alpha F_q(t)\left(\frac{t}{\lambda e_q(t)-1}\,e_q(xt)\right)\frac{\lambda}{t}e_q(t)\\ &-\frac{1}{t}\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^\alpha F_q(t)\left(\frac{t}{\lambda e_q(t)-1}\,e_q(xt)\right).\end{aligned} \tag{3.4}$$

Using equations (1.3) and (2.1) in equation (3.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!} =& \frac{1}{t}\left(\lambda\sum_{m=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b)\frac{t^m}{[m]_q!}\sum_{n=0}^{\infty} B_{n,q}(x;\lambda)\frac{t^n}{[n]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\right.\\ &\left.-\sum_{m=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b)\frac{t^m}{[m]_q!}\sum_{n=0}^{\infty} B_{n,q}(x;\lambda)\frac{t^n}{[n]_q!}\right).\end{aligned} \tag{3.5}$$

S. No.	Special polynomials	Explicit representations
I.	GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$	${}_HB^{(\alpha)}_{n-\nu,q,\lambda}(x)=\frac{[\nu]_q![n-\nu]_q!}{[n]_q!}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q {}_HB^{\nu-\alpha}_{l,q,\lambda}(x)\ S(n-l,\nu,1,1,\lambda)$ ${}_HB^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_q B_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_q B_{n+1-m,q,\lambda}(x)\right\}{}_HB^{(\alpha)}_{m,q,\lambda}$ ${}_HB^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2}\sum_{m=0}^{n}\left\{\lambda\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q E_{n-r,q,\lambda}(x)+E_{n-m,q,\lambda}(x)\right\}{}_HB^{(\alpha)}_{m,q,\lambda}$ ${}_HB^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_q G_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_q G_{n+1-m,q,\lambda}(x)\right\}{}_HB^{(\alpha)}_{m,q,\lambda}$
II.	GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$	${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{[\nu]_q!}{2^\nu}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q {}_HE^{\nu-\alpha}_{l,q,\lambda}(x)\ S(n-l,\nu,-1,1,\lambda)$ ${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_q B_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_q B_{n+1-m,q,\lambda}(x)\right\}{}_HE^{(\alpha)}_{m,q,\lambda}$ ${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2}\sum_{m=0}^{n}\left\{\lambda\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q E_{n-r,q,\lambda}(x)+E_{n-m,q,\lambda}(x)\right\}{}_HE^{(\alpha)}_{m,q,\lambda}$ ${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_q G_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_q G_{n+1-m,q,\lambda}(x)\right\}{}_HE^{(\alpha)}_{m,q,\lambda}$
III.	GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$	${}_HG^{(\alpha)}_{n-\nu,q,\lambda}(x)=\frac{[\nu]_q![n-\nu]_q!}{[n]_q!}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q {}_HG^{\nu-\alpha}_{l,q,\lambda}(x)\ S(n-l,\nu,-1/2,1,\lambda/2)$ ${}_HG^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_q B_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_q B_{n+1-m,q,\lambda}(x)\right\}{}_HG^{(\alpha)}_{m,q,\lambda}$ ${}_HG^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2}\sum_{m=0}^{n}\left\{\lambda\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q E_{n-r,q,\lambda}(x)+E_{n-m,q,\lambda}(x)\right\}{}_HG^{(\alpha)}_{m,q,\lambda}$ ${}_HG^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_q G_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_q G_{n+1-m,q,\lambda}(x)\right\}{}_HG^{(\alpha)}_{m,q,\lambda}$

Table 3. Explicit representations for the GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$

Comparing the coefficients of identical powers of t in both sides of equation (3.5) yields assertion (3.3). □

Similarly, we can prove the following results:

Corollary 3.4. *The following explicit relation for the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ *in terms of the the generalized q-Apostol Euler polynomials* $E_{n,q,\lambda}(x)$ *holds true:*

$${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)=\frac{1}{2}\sum_{m=0}^{n}\left\{\lambda\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q E_{n-r,q,\lambda}(x)+E_{n-m,q,\lambda}(x)\right\}{}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b).$$

Corollary 3.5. *The following explicit relation for the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ *in terms of the generalized q-Apostol Genocchi polynomials* $G_{n,q,\lambda}(x)$ *holds true:*

$$\begin{aligned}{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)=&\frac{1}{2[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_q G_{n+1-r,q,\lambda}(x)\right.\\&\left.-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_q G_{n+1-m,q,\lambda}(x)\right\}{}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b).\end{aligned}$$

In view of Table 1, ceratin explicit representations for the GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$ are established and are given in Table 3.

Note. It is to be observed that for $\lambda = 1$, the results derived above for the generalized q-Hermite-Apostol Bernoulli polynomials ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, the generalized q-Hermite-Apostol Euler polynomials ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and generalized q-Hermite-Apostol Genocchi polynomials ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$ gives the analogous results for the generalized q-Hermite Bernoulli polynomials ${}_HB^{(\alpha)}_{n,q}(x)$, the generalized q-Hermite Euler polynomials ${}_HE^{(\alpha)}_{n,q}(x)$ and generalized q-Hermite Genocchi polynomials ${}_HG^{(\alpha)}_{n,q}(x)$.

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Analytical and Solutions of Fourth Order Difference Equations

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Abstract
In this article, we presented the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n(\pm 1 \pm x_{n-2}x_{n-3})},$$

where the initial conditions x_{-3} ,x_{-2} ,x_{-1} and x_0 are arbitrary real numbers. Also, we studied some dynamic behavior of these equations.

Keywords: Difference equation, Stability, Global attractor, Linearized stability, Periodicity
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1. Introduction

Recently, there has been an increasing interest in the study of global behavior of rational difference equations. The reason behind that is because difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, etc. See [1]. Rational difference equations is an important class of difference equations where they have many applications in real life, for example, the difference equation $x_{n+1} = \frac{a+bx_n}{c+x_n}$, which is known by Riccati Difference Equation has an application in optics and mathematical biology. For more results of the investigation of the rational difference equation see ([2]-[36]) and the references therein.

Karatas [37] examined the global behavior of higher order difference equation

$$x_{n+1} = \frac{ax_{n-(2k+1)}}{b + cx_{n-2k}x_{n-(2k+1)}}.$$

In [38] Gumus et al. studied behavior of a third order difference equation

$$x_{n+1} = \frac{\alpha x_n}{\beta + \gamma x_{n-1}^p x_{n-2}^q}.$$

Elsayed [39] investigated the global of a higher order rational difference equation

$$x_{n+1} = a + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}.$$

In [40], Kulenovic has got the global stability, periodic nature and gave the solution of non-linear difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{A + Bx_{n-1}}.$$

Elsayed [41] obtained periodic solution of period two and three of the difference equation

$$x_{n+1} = \alpha + \frac{\beta x_n}{x_{n-1}} + \frac{\gamma x_{n-1}}{x_n}.$$

Al-Shabi and Abo-Zeid [42] studied the global stability, periodic and boundedness of the positive solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2r-1}}{B + Cx_{n-2l}x_{n-2k}}.$$

Amleh and Drymonis [43] investigated the global character of solution of a certain rational difference equation

$$x_{n+1} = \frac{(\alpha x_n + \beta x_n x_{n-1} + \gamma x_{n-1})\, x_n}{Ax_n + Bx_n x_{n-1} + Cx_{n-1}}.$$

Nirmaladevi and Karthikeyan [44] studied periodicity solution and the global stability of nonlinear difference equation

$$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}.$$

Elsayed and El-Dessoky [45] investigated behavior of the rational difference equation of the fourth order

$$x_{n+1} = ax_n + \frac{bx_n x_{n-2}}{cx_{n-2} + dx_{n-3}}.$$

In this paper we investigate the global asymptotic behavior and the form of the solutions of the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n(\pm 1 \pm x_{n-2}x_{n-3})},$$

where the initial conditions $x_{-3},\ x_{-2}\ , x_{-1}$ and x_0 are arbitrary real numbers.

Here, we will review some of the definitions and theorems used in solving special cases of difference equations:

Definition 1.1. *Let I be some interval of real numbers and let*

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial condition $x_{-k},\ x_{-k+1},\ \ldots,\ x_0 \in I$, *the difference equation*

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.2. *A point $x^* \in I$ is called an equilibrium point of (1.1) if*

$$x^* = F(x^*),$$

that is,

$$x_n = x^* \ for\ all \quad n \geq -k.$$

*is a solution of (1.1), or equivalently, x^* is **a fixed point** of F.*

Definition 1.3. *(Stability)*

Let x^* be an equilibrium point of (1.1).

*(i) The equilibrium point x^* of (1.1) is called **locally stable** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\{x_n\}_{n=-k}^{\infty}$ is a solution of (1.1) and*

$$|x_{-k} - x^*| + |x_{1-k} - x^*| + ... + |x_0 - x^*| < \delta,$$

then

$$|x_n - x^*| < \varepsilon \quad for\ all \qquad n \geq 0.$$

*(ii) The equilibrium point x^* of (1.1) is called **locally asymptotically stable** if it is locally stable, and if there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of (1.1) and*

$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + ... + |x_0 - x^*| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = x^*.$$

*(iii) The equilibrium point x^* of (1.1) is called **a global attractor** if for every solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) we have*

$$\lim_{n\to\infty} x_n = x^*$$

*(iv) The equilibrium point x^* of (1.1) is called **globally asymptotically stable** if it is locally stable and global attractor of (1.1).*

*(v) The equilibrium point x^* of (1.1) is called **unstable** if x^* is not **locally stable**.*

2. Linearized stability analysis

Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point x^*. Let

$$p_i = \frac{\partial F}{\partial u_i}(x^*, x^*, ..., x^*) \quad for \quad i = 0, 1, ..., k,$$

denote the partial derivatives of $F(u_0, u_1, ...u_k)$ evaluated at the equilibrium x^* of (1.1).

Then the equation

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + ... + p_k y_{n-k}\ , \quad n = 0, 1, ..., \tag{2.1}$$

is called **the linearized equation associated** of (1.1) about the equilibrium point x^* and the equation

$$\lambda^{k+1} - p_0 \lambda^k - ... - p_{k-1}\lambda - p_k = 0, \tag{2.2}$$

is called the characteristic equation of (2.1) about x^*.

The following result known as the Linear Stability Theorem is very useful in determining the local stability character of the equilibrium point x^* of (1.1).

Theorem 2.1. *[46] Assume that $p_0, p_2, ..., p_k$ are real numbers such that*

$$|p_0| + |p_1| + ... + |p_k| < 1,$$

or

$$\sum_{i=1}^{k} |p_i| < 1.$$

Then all roots of (2.2) lie inside the unit disk.

3. Qualitative behavior of solutions of $x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(1+x_{n-2}x_{n-3})}$

In this part, we study the some qualitative properties for the recursive equation in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n(1+x_{n-2}x_{n-3})}, \tag{3.1}$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0 are arbitrary positive real numbers.

Theorem 3.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of difference equation* (3.1). *Then for* $n = 0, 1, ...$

$$\begin{aligned}
x_{6n-3} &= d\prod_{i=0}^{n-1}\frac{(1+2icd)}{(1+(2i+1)cd)}\frac{(1+2iab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+2ibc)},\\
x_{6n-2} &= c\prod_{i=0}^{n-1}\frac{(1+(2i+1)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+2iab)}\frac{(1+2ibc)}{(1+(2i+1)bc)},\\
x_{6n-1} &= b\prod_{i=0}^{n-1}\frac{(1+(2i+2)cd)}{(1+(2i+1)cd)}\frac{(1+2iab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+(2i+2)bc)},\\
x_{6n} &= a\prod_{i=0}^{n-1}\frac{(1+(2i+1)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+(2i+2)ab)}\frac{(1+(2i+2)bc)}{(1+(2i+1)bc)},\\
x_{6n+1} &= \frac{cd}{a(1+cd)}\prod_{i=0}^{n-1}\frac{(1+(2i+2)cd)}{(1+(2i+3)cd)}\frac{(1+(2i+2)ab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+(2i+2)bc)},\\
x_{6n+2} &= \frac{ab(1+cd)}{d(1+bc)}\prod_{i=0}^{n-1}\frac{(1+(2i+3)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+(2i+2)ab)}\frac{(1+(2i+2)bc)}{(1+(2i+3)bc)},
\end{aligned}$$

where $x_{-3} = d, x_{-2} = c, x_{-1} = b, x_0 = a.$

Proof. For $n = 0$, the result holds. Now, assume that $n > 0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{6n-9} &= d\prod_{i=0}^{n-2}\frac{(1+2icd)}{(1+(2i+1)cd)}\frac{(1+2iab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+2ibc)},\\
x_{6n-8} &= c\prod_{i=0}^{n-2}\frac{(1+(2i+1)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+2iab)}\frac{(1+2ibc)}{(1+(2i+1)bc)},\\
x_{6n-7} &= b\prod_{i=0}^{n-2}\frac{(1+(2i+2)cd)}{(1+(2i+1)cd)}\frac{(1+2iab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+(2i+2)bc)},\\
x_{6n-6} &= a\prod_{i=0}^{n-2}\frac{(1+(2i+1)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+(2i+2)ab)}\frac{(1+(2i+2)bc)}{(1+(2i+1)bc)},\\
x_{6n-5} &= \frac{cd}{a(1+cd)}\prod_{i=0}^{n-2}\frac{(1+(2i+2)cd)}{(1+(2i+3)cd)}\frac{(1+(2i+2)ab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+(2i+2)bc)},\\
x_{6n-4} &= \frac{ab(1+cd)}{d(1+bc)}\prod_{i=0}^{n-2}\frac{(1+(2i+3)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+(2i+2)ab)}\frac{(1+(2i+2)bc)}{(1+(2i+3)bc)}.
\end{aligned}$$

From (3.1) that

$$
\begin{aligned}
x_{6n-3} &= \frac{x_{6n-6}x_{6n-7}}{x_{6n-4}\left(1+x_{6n-6}x_{6n-7}\right)}\\
&= \frac{ab\prod_{i=0}^{n-2}\left(\frac{1+2iab}{1+(2i+2)ab}\right)}{\frac{ab(1+cd)}{d(1+bc)}\prod_{i=0}^{n-2}\left(\frac{1+(2i+3)cd}{1+(2i+2)cd}\right)\left(\frac{1+(2i+1)ab}{1+(2i+2)ab}\right)\left(\frac{1+(2i+2)bc}{1+(2i+3)bc}\right)\left(1+ab\prod_{i=0}^{n-2}\left(\frac{1+2iab}{1+(2i+2)ab}\right)\right)}\\
&= \frac{d(1+bc)}{(1+cd)}\frac{\left[\frac{(1+2ab)(1+4ab)...(1+(2n-6)ab)(1+2(n-2)ab)}{(1+2ab)(1+4ab)...(1+(2n-4)ab(1+(2n-2)ab)}\right]}{\prod_{i=0}^{n-2}\left(\frac{1+(2i+3)cd}{1+(2i+2)cd}\right)\left(\frac{1+(2i+1)ab}{1+(2i+2)ab}\right)\left(\frac{1+(2i+2)bc}{1+(2i+3)bc}\right)\left(1+\frac{ab}{1+(2n-2)ab}\right)}\\
&= d\frac{\left(\frac{1}{1+(2n-2)ab}\right)}{\prod_{i=0}^{n-1}\left(\frac{1+(2i+1)cd}{1+2icd}\right)\left(\frac{1+2ibc}{1+(2i+1)bc}\right)\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)ab}{1+(2i+2)ab}\right)\left(\frac{1+(2n-1)ab}{1+(2n-2)ab}\right)}\\
&= \frac{d}{(1+(2n-1)ab)\prod_{i=0}^{n-1}\left(\frac{1+(2i+1)cd}{1+2icd}\right)\left(\frac{1+2ibc}{1+(2i+1)bc}\right)\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)ab}{1+(2i+2)ab}\right)}\\
&= \frac{d}{\prod_{i=0}^{n-1}\left(\frac{1+(2i+1)cd}{1+2icd}\right)\left(\frac{1+2ibc}{1+(2i+1)bc}\right)\left(\frac{1+(2i+1)ab}{1+2iab}\right)}.
\end{aligned}
$$

Consequently, we have

$$
x_{6n-3} = d\prod_{i=0}^{n-1}\left(\frac{1+2icd}{1+(2i+1)cd}\right)\left(\frac{1+2iab}{1+(2i+1)ab}\right)\left(\frac{1+(2i+1)bc}{1+2ibc}\right).
$$

Similarly, other relations can be obtained and thus, the proof has been proved. □

Theorem 3.2. *The difference equation* (3.1) *has a unique equilibrium point which is* 0 *and it is not locally asymptotically stable.*

Proof. For the equilibrium points of (3.1), is given by

$$
x^* = \frac{x^{*2}}{x^*\left(1+x^{*2}\right)},
$$

$$
1+x^{*2} = 1,
$$

or

$$
x^{*2} = 0.
$$

Then the unique equilibrium point is $x^* = 0$.
Let F be function defined by

$$
F(u,v,w) = \frac{vw}{u(1+vw)}.
$$

Therefore it follows that

$$
F_u(u,v,w) = \frac{-vw}{u^2(1+vw)},\ F_v(u,v,w) = \frac{w}{u(1+vw)^2},\ F_w(u,v,w) = \frac{v}{u(1+vw)^2},
$$

we see that

$$
F_u(x^*,x^*,x^*) = -1,\ F_v(x^*,x^*,x^*) = 1,\ F_w(x^*,x^*,x^*) = 1.
$$

This completes the proof by using Theorem 2.1. □

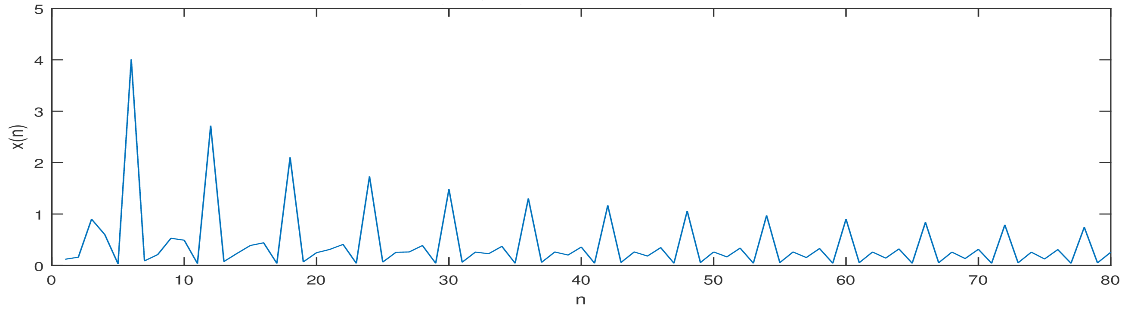

Figure 3.1. Plot of $x(n+1) = \frac{x(n-2)x(n-3)}{x(n)(1+x(n-2)x(n-3))}$

Numerical Examples

For statement the results of this part, we take into account numerical examples which illustrate different types of solutions to (3.1).

Example 3.3. *See Figure 3.1, since* $x_{-3} = 0.12,\ x_{-2} = 0.16,\ x_{-1} = 0.9$ *and* $x_0 = 0.6$.

4. Qualitative behavior of solutions of $x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n(-1+x_{n-2}x_{n-3})}$

Here, we obtain the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n(-1+x_{n-2}x_{n-3})}, \quad n = 0,1,..., \tag{4.1}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0.are arbitrary real numbers with $x_{-2}x_{-3} \neq 1,\ x_{-1}x_{-2} \neq 1$ and $x_0 x_{-1} \neq 1$.

Theorem 4.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of difference equation of* (4.1). *Then the equation* (4.1) *has unboundedness solutions and for* $n = 0,1,...$

$$\begin{aligned}
x_{6n-3} &= \frac{d(-1+bc)^n}{(-1+cd)^n(-1+ab)^n}, \quad x_{6n-2} = \frac{c(-1+cd)^n(-1+ab)^n}{(-1+bc)^n}, \\
x_{6n-1} &= \frac{b(-1+bc)^n}{(-1+cd)^n(-1+ab)^n}, \quad x_{6n} = \frac{a(-1+cd)^n(-1+ab)^n}{(-1+bc)^n}, \\
x_{6n+1} &= \frac{cd(-1+bc)^n}{a(-1+ab)^n(-1+cd)^{n+1}}, \quad x_{6n+2} = \frac{ab(-1+cd)^{n+1}(-1+ab)^n}{d(-1+bc)^{n+1}},
\end{aligned} \tag{4.2}$$

where $x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b$ *and* $x_0 = a$.

Proof. For $n = 0$ the conclusion holds. Now, assume that $n > 0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{6n-9} &= \frac{d(-1+bc)^{n-1}}{(-1+cd)^{n-1}(-1+ab)^{n-1}}, \quad x_{6n-8} = \frac{c(-1+cd)^{n-1}(-1+ab)^{n-1}}{(-1+bc)^{n-1}}, \\
x_{6n-7} &= \frac{b(-1+bc)^{n-1}}{(-1+cd)^{n-1}(-1+ab)^{n-1}}, \quad x_{6n-6} = \frac{a(-1+cd)^{n-1}(-1+ab)^{n-1}}{(-1+bc)^{n-1}}, \\
x_{6n-5} &= \frac{cd(-1+bc)^{n-1}}{a(-1+ab)^{n-1}(-1+cd)^n}, \quad x_{6n-4} = \frac{ab(-1+cd)^n(-1+ab)^{n-1}}{d(-1+bc)^n}.
\end{aligned}$$

Now we proof some of relations of (4.2).

$$x_{6n-2} = \frac{x_{6n-5}x_{6n-6}}{x_{6n-3}\left(-1+x_{6n-5}x_{6n-6}\right)} = \frac{(cd)\left(-1+cd\right)^{-1}}{\frac{d(-1+bc)^n}{(-1+cd)^n(-1+ab)^n}\left(-1+cd\left(-1+cd\right)^{-1}\right)}$$
$$= \frac{c\left(-1+cd\right)^{n-1}\left(-1+ab\right)^n}{\left(-1+bc\right)^n\left(-1+\frac{cd}{(-1+cd)}\right)} = \frac{c(-1+cd)^n\left(-1+ab\right)^n}{\left(-1+bc\right)^n}.$$

Also,

$$x_{6n-1} = \frac{x_{6n-4}x_{6n-5}}{x_{6n-2}\left(-1+x_{6n-4}x_{6n-5}\right)}$$
$$= \frac{\left(bc\left(-1+bc\right)^{n-1}\right)l\left(-1+bc\right)^n}{(c(-1+cd)^n(-1+ab)^n/(-1+bc)^n)\left(-1+\frac{bc}{-1+bc}\right)}$$
$$= \frac{b\left(-1+bc\right)^{n-1}}{\left(-1+cd\right)^n\left(-1+ab\right)^n\left(\frac{1}{-1+bc}\right)} = \frac{b\left(-1+bc\right)^n}{\left(-1+cd\right)^n\left(-1+ab\right)^n}.$$

Similarly, other relations can be obtained and thus, the proof has been proved. □

Theorem 4.2. *The difference equation (4.1) has a periodic solution of periodic six iff $ab=2$ and $b=d$ and we will take the form:*

$$\left\{d,c,b,a,\frac{cd}{a\left(-1+cd\right)},\frac{ab}{d},d,c,b,a,\frac{cd}{a\left(-1+cd\right)},\frac{ab}{d},\ldots\right\}.$$

Proof. Assume that there exists a prime period six solution of (4.1):

$$d,c,b,a,\frac{cd}{a\left(-1+cd\right)},\frac{ab}{d},d,c,b,a,\frac{cd}{a\left(-1+cd\right)},\frac{ab}{d},\ldots.$$

From (4.2), we get

$$x_{6n-3} = d = \frac{d\left(-1+bc\right)^n}{\left(-1+cd\right)^n\left(-1+ab\right)^n}, \quad x_{6n-2} = c = \frac{c\left(-1+cd\right)^n\left(-1+ab\right)^n}{\left(-1+bc\right)^n},$$
$$x_{6n-1} = b = \frac{b\left(-1+bc\right)^n}{\left(-1+cd\right)^n\left(-1+ab\right)^n}, \quad x_{6n} = a = \frac{a\left(-1+cd\right)^n\left(-1+ab\right)^n}{\left(-1+bc\right)^n},$$
$$x_{6n+1} = \frac{cd}{a\left(-1+cd\right)} = \frac{cd\left(-1+bc\right)^n}{a\left(-1+ab\right)^n\left(-1+cd\right)^{n+1}},$$
$$x_{6n+2} = \frac{ab}{d} = \frac{ab\left(-1+cd\right)^{n+1}\left(-1+ab\right)^n}{d\left(-1+bc\right)^{n+1}}.$$

Then we can see that

$$ab=2 \text{ and } b=d.$$

Conversely, suppose that $ab = 2$ *and* $b = d$. Then we see that

$$\begin{aligned}
x_{6n-3} &= \frac{d(-1+bc)^n}{(-1+cd)^n(-1+ab)^n} = \frac{d(-1+cd)^n}{(-1+cd)^n(-1+2)^n} = d,\\
x_{6n-2} &= \frac{c(-1+cd)^n(-1+ab)^n}{(-1+bc)^n} = \frac{c(-1+cd)^n(-1+2)^n}{(-1+cd)^n} = c,\\
x_{6n-1} &= \frac{b(-1+bc)^n}{(-1+cd)^n(-1+ab)^n} = \frac{b(-1+cd)^n}{(-1+cd)^n(-1+2)^n} = b,\\
x_{6n} &= \frac{a(-1+cd)^n(-1+ab)^n}{(-1+bc)^n} = a,\\
x_{6n+1} &= \frac{cd(-1+bc)^n}{a(-1+ab)^n(-1+cd)^{n+1}} = \frac{cd}{a(-1+cd)},\\
x_{6n+2} &= \frac{ab(-1+cd)^{n+1}(-1+ab)^n}{d(-1+bc)^{n+1}} = \frac{ab}{d}.
\end{aligned}$$

Thus we obtained a periodic solution of period six. □

Theorem 4.3. *Equation* (4.1) *has a periodic solution of period two iff* $ab = bc = cd = 2$ *(It also means* $a = c,\ b = d$*) and we will take the form:*

$$\{d, c, d, c, ...\}.$$

Proof. First assume that there exists a prime period two solution of (4.1):

$$d, c, d, c, ...$$

We see from the form of the solutions of (4.1) that

$$\begin{aligned}
x_{6n-3} = d &= \frac{d(-1+bc)^n}{(-1+cd)^n(-1+ab)^n}, \quad x_{6n-2} = c = \frac{c(-1+cd)^n(-1+ab)^n}{(-1+bc)^n},\\
x_{6n-1} = d &= \frac{b(-1+bc)^n}{(-1+cd)^n(-1+ab)^n}, \quad x_{6n} = c = \frac{a(-1+cd)^n(-1+ab)^n}{(-1+bc)^n},\\
x_{6n+1} = d &= \frac{cd(-1+bc)^n}{a(-1+ab)^n(-1+cd)^{n+1}}, \quad x_{6n+2} = c = \frac{ab(-1+cd)^{n+1}(-1+ab)^n}{d(-1+bc)^{n+1}}.
\end{aligned}$$

Thus we see that $ab = bc = cd = 2$.
Second suppose that $ab = bc = cd = 2$. Then we obtain

$$\begin{aligned}
x_{6n-3} &= \frac{d(-1+bc)^n}{(-1+cd)^n(-1+ab)^n} = \frac{d(-1+2)^n}{(-1+2)^n(-1+2)^n} = d,\\
x_{6n-2} &= \frac{c(-1+cd)^n(-1+ab)^n}{(-1+bc)^n} = \frac{c(-1+2)^n(-1+2)^n}{(-1+2)^n} = c,\\
x_{6n-1} &= \frac{b(-1+bc)^n}{(-1+cd)^n(-1+ab)^n} = \frac{b(-1+2)^n}{(-1+2)^n(-1+2)^n} = d,\\
x_{6n} &= \frac{a(-1+cd)^n(-1+ab)^n}{(-1+bc)^n} = c,\\
x_{6n+1} &= \frac{cd(-1+bc)^n}{a(-1+ab)^n(-1+cd)^{n+1}} = d,\\
x_{6n+2} &= \frac{ab(-1+cd)^{n+1}(-1+ab)^n}{d(-1+bc)^{n+1}} = c.
\end{aligned}$$

Thus we obtained a periodic solution of period two. □

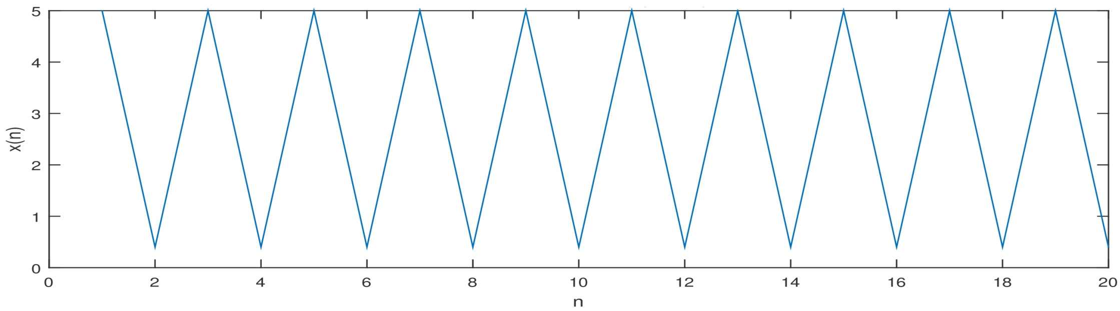

Figure 4.1. Plot of $x(n+1) = \frac{x(n-2)x(n-3)}{x(n)(-1+x(n-2)x(n-3))}$

Theorem 4.4. *Difference equation* (4.1) *has equilibrium points which are* $0, \pm\sqrt{2}$ *such that they are not locally asymptotically stable.*

Proof. We can write

$$x^* = \frac{x^{*2}}{x^*\left(-1+x^{*2}\right)}, \quad or \quad x^{*2}\left(x^{*2}-2\right) = 0,$$

consequently $0, \pm\sqrt{2}$ are the equilibrium points.

Suppose that $F : (0,\infty)^3 \to (0,\infty)$ be function defined by

$$F(u,v,w) = \frac{vw}{u(-1+vw)},$$

then

$$F_u(u,v,w) = \frac{-vw}{u^2(-1+vw)}, \; F_v(u,v,w) = \frac{-w}{u(-1+vw)^2}, \; F_w(u,v,w) = \frac{-v}{u(-1+vw)^2},$$

we see that,

$$F_u(x^*,x^*,x^*) = -1, \; F_v(x^*,x^*,x^*) = -1, \; F_w(x^*,x^*,x^*) = -1.$$

This completes the proof by using Theorem 2.1. □

Numerical Examples

We put some numerical examples which illustrate different types of solutions of (4.1).

Example 4.5. *When we put* $x_{-3} = 5, \; x_{-2} = 2/5, \; x_{-1} = 5$ *and* $x_0 = 2/5$. *See Figure 4.1.*

5. Qualitative behavior of solutions of $x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(1-x_{n-2}x_{n-3})}$

In this section, we get the expressions of the solution of the difference equation in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n(1-x_{n-2}x_{n-3})}, \quad n = 0,1,..., \tag{5.1}$$

where the initial conditions $x_{-3}, \; x_{-2}, \; x_{-1}$ *and* x_0 are arbitrary real numbers.

Theorem 5.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of the difference equation of* (5.1). *Then for* $n = 0,1,...$

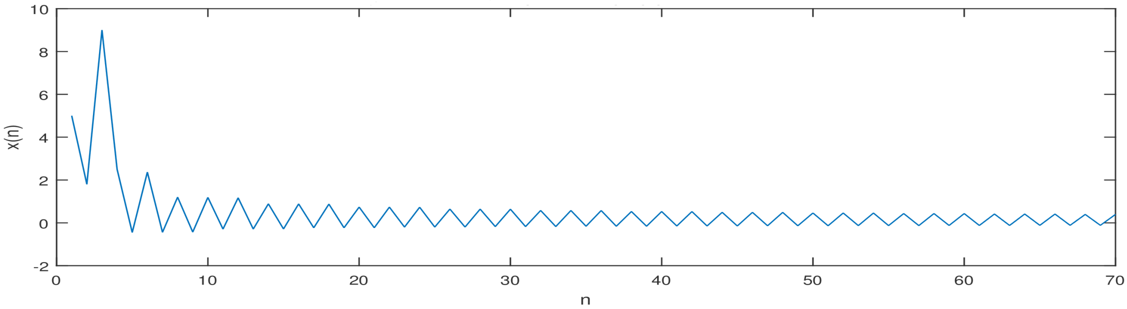

Figure 5.1. Plot of $x(n+1)=\frac{x(n-2)x(n-3)}{x(n)(1-x(n-2)x(n-3))}$

$$x_{6n-3}=d\prod_{i=0}^{n-1}\frac{(1-2icd)}{(1-(2i+1)cd)}\frac{(1-2iab)}{(1-(2i+1)ab)}\frac{(1-(2i+1)bc)}{(1-2ibc)},$$
$$x_{6n-2}=c\prod_{i=0}^{n-1}\frac{(1-(2i+1)cd)}{(1-(2i+2)cd)}\frac{(1-(2i+1)ab)}{(1-2iab)}\frac{(1-2ibc)}{(1-(2i+1)bc)},$$
$$x_{6n-1}=b\prod_{i=0}^{n-1}\frac{(1-(2i+2)cd)}{(1-(2i+1)cd)}\frac{(1-2iab)}{(1-(2i+1)ab)}\frac{(1-(2i+1)bc)}{(1-(2i+2)bc)},$$
$$x_{6n}=a\prod_{i=0}^{n-1}\frac{(1-(2i+1)cd)}{(1-(2i+2)cd)}\frac{(1-(2i+1)ab)}{(1-(2i+2)ab)}\frac{(1-(2i+2)bc)}{(1-(2i+1)bc)},$$
$$x_{6n+1}=\frac{cd}{a(1-cd)}\prod_{i=0}^{n-1}\frac{(1-(2i+2)cd)}{(1+(2i+3)cd)}\frac{(1-(2i+2)ab)}{(1+(2i+1)ab)}\frac{(1-(2i+1)bc)}{(1+(2i+2)bc)},$$
$$x_{6n+2}=\frac{ab(1-cd)}{d(1-bc)}\prod_{i=0}^{n-1}\frac{(1-(2i+3)cd)}{(1-(2i+2)cd)}\frac{(1-(2i+1)ab)}{(1-(2i+2)ab)}\frac{(1-(2i+2)bc)}{(1-(2i+3)bc)}.$$

Theorem 5.2. *Equation (5.1) has a unique equilibrium point which is 0 and it is not locally asymptotically stable.*

Example 5.3. *See Figure 5.1, we let* $x_{-3}=5,\ x_{-2}=1.8,\ x_{-1}=9$ *and* $x_{-0}=2.5$.

6. Qualitative behavior of solutions of $x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(-1-x_{n-2}x_{n-3})}$

In this part, we obtain the form of solution of the following difference equation

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(-1-x_{n-2}x_{n-3})},\quad n=0,1,..., \tag{6.1}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0 are arbitrary real numbers with $x_{-2}x_{-3}\neq-1,\ x_{-1}x_{-2}\neq-1$ and $x_0x_{-1}\neq-1$.

Theorem 6.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (6.1)*. Then* (6.1) *has the following solution for* $n=0,1,...$

$$x_{6n-3}=\frac{d(-1-bc)^n}{(-1-cd)^n(-1-ab)^n},\quad x_{6n-2}=\frac{c(-1-cd)^n(-1-ab)^n}{(-1-bc)^n},$$
$$x_{6n-1}=\frac{b(-1-bc)^n}{(-1-cd)^n(-1-ab)^n},\quad x_{6n}=\frac{a(-1-cd)^n(-1-ab)^n}{(-1-bc)^n},$$
$$x_{6n+1}=\frac{cd(-1-bc)^n}{a(-1-ab)^n(-1-cd)^{n+1}},\quad x_{6n+2}=\frac{ab(-1-cd)^{n+1}(-1-ab)^n}{d(-1-bc)^{n+1}}.$$

Theorem 6.2. *Difference equation* (6.1) *has a periodic solution of period six iff* $ab=-2$ *and* $b=d$ *and we will take the form:*

$$\left\{d,c,b,a,\frac{cd}{a(-1-cd)},\frac{ab}{d},d,c,b,a,\frac{cd}{a(-1-cd)},\frac{ab}{d},...\right\}.$$

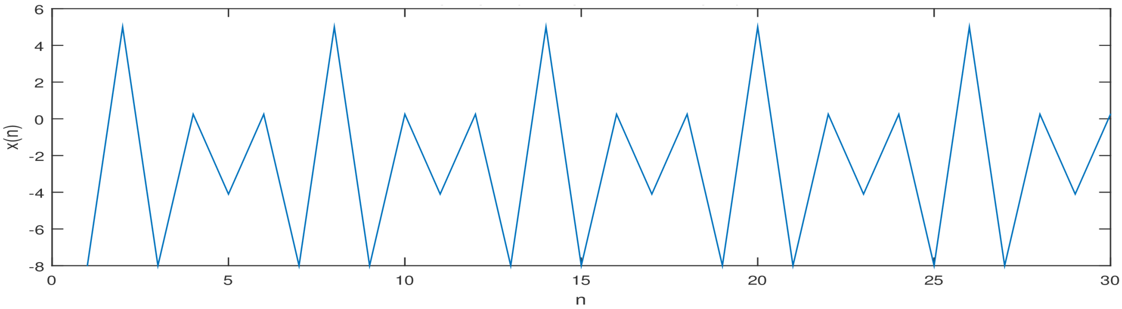

Figure 6.1. Plot of $x(n+1) = \frac{x(n-2)x(n-3)}{x(n)(-1-x(n-2)x(n-3))}$

Theorem 6.3. *Equation* (6.1) *has a periodic solution of period two iff* $ab = bc = cd = -2$ *and takes the form:* $\{d, c, d, c, ...\}$.

Theorem 6.4. *Difference equation* (6.1) *has equilibrium point which is* 0 *and it is not locally asymptotically stable.*

Example 6.5. *Figure 6.1 shows the period six solutions of* (6.1) *since* $x_{-3} = -8$, $x_{-2} = 5$, $x_{-1} = -8$ *and* $x_0 = 1/4$.

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A New Theorem on The Existence of Positive Solutions of Singular Initial-Value Problem for Second Order Differential Equations

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Abstract
We proved a new theorem on the existence of positive solutions to initial-value problems for second-order nonlinear singular differential equations. The existence of solutions is proven under considerably weaker than previously known conditions.

Keywords: Second order equations, Existence, Emden-Fowler equation
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1. Introduction

We consider the problem

$$\begin{aligned} (py')' + pqg(y) &= 0, \ \ t \in [0,T] \\ y(0) &= a > 0, \\ \lim_{t\to 0+} p(t)y'(t) &= 0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} (py')' + pqg(y) &= 0, \ \ t \in [0,T] \\ y(0) &= a > 0, \\ y'(0) &= 0, \end{aligned} \tag{1.2}$$

where $0 < T < \infty$, $p \geq 0$, $q \geq 0$ and $g : [0,\infty) \to [0,\infty)$.

Agarwal and O'Regan [1] established the existence theorems for the positive solution of the problem (1.1) and (1.2):

Theorem 1.1. *[1] Suppose the following conditions are satisfied*

$$p \in C[0,T) \cap C^1(0,T) \ \ \textit{with } p > 0 \textit{ on } (0,T) \tag{1.3}$$

$$q \in L^1_p[0,t^*] \text{ for any } t^* \in (0,T) \text{ with } q > 0 \text{ on } (0,T), \tag{1.4}$$

where $L^1_r[0,a]$ *is the space of functions* $u(t)$ *with* $\int_0^a |u(t)|\, r(t)dt < \infty$,

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)dxds < \infty \text{ for any } t^* \in (0,T) \tag{1.5}$$

and

$$g : [0,\infty) \to [0,\infty) \text{ is continuous, nondecreasing on } [0,\infty) \text{ and } g(u) > 0 \text{ for } u > 0. \tag{1.6}$$

Let

$$H(z) = \int_z^a \frac{dx}{g(x)} \text{ for } 0 < z \leq a$$

and assume

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)\tau(x)dxds < a \text{ for any } t^* \in (0,T), \tag{1.7}$$

here

$$\tau(x) = g\left(H^{-1}\left(\int_0^x \frac{1}{p(w)} \int_0^w p(z)q(z)dzdw\right)\right).$$

Then (1.1) *has a solution* $y \in C[0,T)$ *with* $py' \in C[0,T)$, $(py')' \in L^1_{pq}(0,T)$ *and* $0 < y(t) \leq a$ *for* $t \in [0,T)$. *In addition if either*

$$p(0) \neq 0$$

or

$$p(0) = 0 \text{ and } \lim_{t \to 0+} \frac{p(t)q(t)}{p'(t)} = 0$$

holds, then y is a solution of (1.2).

The condition (1.7) in connection with the definition of the function $\tau(x)$, makes this theorem difficult for an application. In [2] we proved more easy and applicable theorem:

Theorem 1.2. *Suppose* (1.3)-(1.5) *hold. In addition, we assume*

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)g(a)dxds < a$$

for any $t^* \in (0,T_0)$. *Then*

a) (1.1) *has a solution* $y \in C[0,T_0)$ *with* $py' \in C[0,T_0)$, $(py')' \in L^1_{pq}(0,T_0)$ *and* $0 < y(t) \leq a$ *for* $t \in [0,T_0)$.

b) If $\int_{T_0}^{T_1} \frac{1}{p(s)} \int_0^s p(x)q(x)g(T_0)ds < y(T_0)$, *and conditions* (1.3)-(1.6) *satisfied then solution can be extended into the interval* $[0,T_1)$.

In this paper we generalized the Theorem 1.2.

2. Main result

Theorem 2.1. *Suppose the following conditions are satisfied*

$$
\begin{aligned}
p &\in C[0,T)\cap C^1(0,T) \text{ with } p>0 \text{ on } (0,T],\\
q &\geq 0,\\
&\int_0^{t^*}\frac{1}{p(s)}\int_0^s p(x)q(x)dxds<\infty \text{ for any } t^*\in(0,T],
\end{aligned}
$$

$$g:[0,\infty)\to[0,\infty) \text{ is continuous, nondecreasing on } [0,\infty),$$

and assume

$$
\begin{aligned}
\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(a-\varphi(x))dxds &\leq a-\varphi(t),\\
\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(\varphi(x))dxds &\geq \varphi(t)
\end{aligned}
$$

for some $\varphi(t)\in C[0,T]$, with $0\leq\varphi(t)\leq a$. Then (1.1) has a solution $y\in C[0,T]$ with $py'\in C[0,T]$, $(py')'\in L^1_{pq}(0,T)$ and $0<y(t)\leq a$ for $t\in[0,T]$. In addition if either

$$p(0)\neq 0$$

or

$$p(0)=0 \text{ and } \lim_{t\to0+}\frac{p(t)q(t)}{p'(t)}=0$$

holds, then y is a solution of (1.2).

Remark 2.2. *The case $\varphi(t)\equiv0$, corresponds to the case of inequality*

$$\int_0^{t^*}\frac{1}{p(s)}\int_0^s p(x)q(x)g(a)dxds<a \text{ for any } t^*\in(0,T].$$

Proof of Theorem 2.1. Consider the sequence $\{y_n(t)\}, n=0,1,2,...$ with $y_0(t)\equiv a-\varphi(t)$,

$$y_n(t)=a-\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_{n-1}(x))dxds,\ n=1,2,...,\ t\leq T.$$

We have

$$
\begin{aligned}
y_0(t) &\equiv a-\varphi(t),\\
y_1(t) &= a-\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(a-\varphi(x))dxds\geq\varphi(t),
\end{aligned}
$$

$$
\begin{aligned}
y_2(t) &= a-\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_1(x))dxds\\
&\leq a-\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(\varphi(x))dxds\\
&\leq a-\varphi(t),
\end{aligned}
$$

$$\begin{aligned}
y_3(t) &= a-\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)g(y_2(x))dxds \geq \varphi(t)\\
y_4(t) &= a-\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)g(y_3(x))dxds \leq a-\varphi(t),\dots\\
y_{2n-1}(t) &\geq \varphi(t),\\
y_{2n}(t) &\leq a-\varphi(t),\dots
\end{aligned}$$

The sequences $\{y_{2n}(t)\}$ and $\{y_{2n+1}(t)\}$ are equicontinuous. Indeed we have

$$|y_n(t)-y_n(r)| = \int_r^t \frac{1}{p(s)}\int_0^s p(x)q(x)g(y_{n-1}(x))dxds \leq M\int_r^t \frac{1}{p(s)}\int_0^s p(x)q(x)dxds, \tag{2.1}$$

where

$$M = \max\{g(u) : 0 \leq u \leq a\}$$

and the right hand side of (2.1) can be taken $< \varepsilon$ for $|t-r| < \delta$, regardless of the choice of t and r: the function $\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)dxds$ is (uniformly) continuous on $[0,T]$. It follows from Ascoli Arzela Theorem that the sequence $\{y_{2n}(t)\}$ has the (uniformly) convergent subsequence, $y_{2n_k}(t) \to u(t)$. The Lebesgue dominated theorem guarantees that

$$\begin{aligned}
y_{2n_k+1}(t) &= a-\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)g(y_{2n_k}(x))dxds \to v(t),\\
v(t) &= a-\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)g(u(x))dxds\\
\text{and } u(t) &= a-\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)g(v(x))dxds.
\end{aligned}$$

If $u(t) = v(t)$ we have that the function $u(t)$ is the solution of the problem (1.1), indeed it follows from

$$u(t) = a-\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)g(u(x))dxds$$

that

$$\begin{aligned}
u'(t) &= -\frac{1}{p(t)}\int_0^t p(x)q(x)g(u(x))dx,\\
pu' &= -\int_0^t p(x)q(x)g(u(x))dx,\\
(pu')' &= -pqg(u).
\end{aligned}$$

So we suppose $u(t) \neq v(t)$. We have $u(0) = v(0) = a$ and if for example, $u(t) > v(t)$ on the interval $(0,b)$, then we obtain

$$u(b)-v(b) = \int_0^b \frac{1}{p(s)}\int_0^s p(x)q(x)\left[g(u(x))-g(v(x)\right]dxds > 0$$

and therefore $u(t) > v(t)$ on the whole interval $(0,T]$. The same holds for all points of intersections $t_0 : u(t_0) = v(t_0)$. That is if $u(t_0) = v(t_0)$, then for any $\varepsilon > 0$ there are infinitely many points $t_n \in [t_0, t_0+\varepsilon)$ such that $u(t_n) = v(t_n)$. Therefore, $u(t) > v(t)$ (or $<$) on $(t_0,T]$. Without loss of generality let us suppose $u(t) > v(t)$ on $(0,T]$

and consider the operator $N : C[0,T] \to C[0,T]$ defined by

$$Ny(t) = a-\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)g(y(x))dxds.$$

Next let

$$K = \{y \in C[0,T] : v(t) \leq y(t) \leq u(t) \text{ for } t \in [0,T]\}.$$

Clearly K is closed, convex, bounded subset of $C[0,T]$ and $N : K \to K$. Let us show that $N : K \to K$ is continuous and compact operator. Continuity follows from Lebesgue dominated convergence theorem: if $y_n(t) \to y(t)$, then $Ny_n(t) \to Ny(t)$. To show that N is completely continuous let $y(t) \in K$, then

$$|Ny(t) - Ny(r)| \leq M \left| \int_r^t \frac{1}{p(x)} \int_0^x p(z)q(z)dzds \right| \text{ for } t, r \in [0,T],$$

that is N completely continuous on $[0,T]$.

The Schauder-Tychonoff theorem guarantees that N has a fixed point $w \in K$, i.e. w is a solution of (1.1). It follows from

$$w'(t) = -\frac{1}{p(t)} \int_0^t p(x)q(x)g(w(x))dx,$$

that if $p(0) \neq 0$ then $w'(0) = 0$. Now if $p(0) = 0$ but $\lim_{t\to 0+} \frac{p(t)q(t)}{p'(t)} = 0$ we have from (**??**) that

$$w'(0+) = -\lim_{t\to 0+} \int_0^t \frac{p(x)q(x)}{p(t)} g(w(x))dx = 0,$$

that is w is a solution of (1.2).

The proof is completed.

Conflict of interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Analysis of the Dynamical System $\dot{x}(t) = Ax(t) + h(t,x(t)),\ x(t_0) = x_0$ in a Special Time-Dependent Norm

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Abstract
As the main new result, we show that one can construct a time-dependent positive definite matrix $R(t,t_0)$ such that the solution $x(t)$ of the initial value problem $\dot{x}(t) = Ax(t) + h(t,x(t)),\ x(t_0) = x_0$, under certain conditions satisfies the equation $\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R$ where $x_A(t)$ is the solution of the above IVP when $h \equiv 0$ and R is a constant positive definite matrix constructed from the eigenvectors and principal vectors of A and A^* and where $\|\cdot\|_{R(t,t_0)}$ and $\|\cdot\|_R$ are weighted norms. Applications are made to dynamical systems, and numerical examples underpin the theoretical findings.

Keywords: Nonlinear initial value problem with linear principal part, Vibration suppression, Monotonicity behavior, Two-sided bounds, Weighted norm
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1. Introduction

In this paper, the solution of the nonlinear initial value problem (for short: IVP) with linear principal part $\dot{x}(t) = Ax(t) + h(t,x(t)),\ x(t_0) = x_0$ is investigated in a special time-dependent weighted norm $\|\cdot\|_{R(t,t_0)}$ with positive definite matrix $R(t,t_0)$. It will be shown that under certain conditions, $R(t,t_0)$ can be constructed such that $\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R$ where $x_A(t)$ is the solution of the initial value problem $\dot{x}_A(t) = Ax_A(t),\ x_A(t_0) = x_0$ and R is a constant positive definite matrix constructed from the eigenvectors and principal vectors of A and A^*. In other words, the solution $x(t)$ in the time-dependent weighted norm $\|\cdot\|_{R(t,t_0)}$ is equal to the solution $x_A(t)$ of the pertinent linear IVP in the weighted norm $\|\cdot\|_R$. As a consequence, since $x_A(t)$ shows vibration suppression and monotonicity behavior under certain conditions, the same holds for $\|x(t)\|_{R(t,t_0)}$. This is the main new result.

The paper is structured as follows. In Section 2, the weighted norm $\|\cdot\|_R$ and, in Section 3, the biorthogonality of eigenvectors and principal vectors of the matrices A and A^* are recapitulated. Section 4 contains two fundamental matrices, namely one for the nonlinear IVP and one for the associated linear IVP. In Section 5, the matrix $R(t,t_0)$ is constructed, and the equation $\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R$ is derived. Section 6 contains an expression for $\|x(t)\|_{R(t,t_0)}$ in the norm $\|\cdot\|_2$, and Section 7 two-sided bounds on $\|x(t)\|$ in any vector norm $\|\cdot\|$. In Section 8, Applications to free nonlinear dynamical systems with linear principal part are given including numerical examples. Section 9 is the conclusion section.

2. The weighted norm $\|\cdot\|_R$ revisited

In this section, we revisit the results of [1] concerning the weighted norm $\|\cdot\|_R$, where R is a special positive definite matrix constructed from the eigenvectors and principal vectors of the adjoint A^* of a given system matrix A.

2.1 The case of a diagonalizable matrix A

We first turn to diagonalizable matrices A.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable. Let $\alpha_j = \lambda_j(A)$ be the eigenvalues and u_j be the associated left eigenvectors of A for $j = 1, \cdots, n$; further, let $A^* \in \mathbb{C}^{n\times n}$ be the adjoint matrix of A so that u_j^* are the right eigenvectors of A^* corresponding to the eigenvalues $\overline{\alpha_j}$ of A^* for $j = 1, \cdots, n$, i.e.,*

$$u_j A = \alpha_j u_j, \; j = 1, \cdots, n$$

and

$$A^* u_j^* = \overline{\alpha_j} u_j^*, \; j = 1, \cdots, n.$$

Let

$$\rho_j = \overline{\alpha_j} + \alpha_j = 2Re\,\alpha_j = 2Re\,\overline{\alpha_j}, \; j = 1, \cdots, n$$

and

$$R_j = u_j^* u_j, \; j = 1, \cdots, n. \tag{2.1}$$

Then,

$$A^* R_j + R_j A = \rho_j R_j, \; j = 1, \cdots, n.$$

In other words: The matrix eigenvalue problem

$$A^* V + VA = \mu V$$

has the n solution pairs

$$(\mu, V) = (\rho_j, R_j)$$

with real ρ_j and positive semi-definite matrix $R_j \in \mathbb{C}^{n\times n}$ for $j = 1, \cdots, n$. Further,

$$R := \sum_{j=1}^{n} R_j \tag{2.2}$$

is positive definite.

Proof. See [1, Theorems 4 - 6]. □

Remark 2.2. *Since R in (2.2) is positive definite, by*

$$\|u\|_R := (Ru, u)^{\frac{1}{2}}, \; u \in \mathbb{C}^n,$$

a weighted norm $\|\cdot\|_R$ is defined.

2.2 The case of a general square matrix A

In this subsection, we consider general square matrices A.

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$ have a canonical Jordan form consisting of r Jordan blocks. Let $\alpha_j = \lambda_j(A)$ be the eigenvalues and $u_1^{(j)}, \cdots, u_{m_j}^{(j)}$ be a chain of associated left principal vectors for $j = 1, \cdots, r$. Further, let $A^* \in \mathbb{C}^{n\times n}$ be the adjoint matrix of A so that $u_1^{(j)*}, \cdots, u_{m_j}^{(j)*}$ is a chain of right principal vectors of A^* corresponding to the eigenvalues $\overline{\alpha_j} = \lambda_j(A^*)$ for $j = 1, \cdots, r$, i.e.*

$$u_k^{(j)} A = \alpha_j u_k^{(j)} + u_{k-1}^{(j)}$$

with $u_0^{(j)} = 0,\ k = 1,\cdots,m_j,\ j = 1,\cdots,r$ *and*

$$A^* u_k^{(j)^*} = \overline{\alpha_j} u_k^{(j)^*} + u_{k-1}^{(j)\ *}$$

with $u_0^{(j)^*} = 0,\ k = 1,\cdots,m_j,\ j = 1,\cdots,r.$

Let

$$\rho_j = \overline{\alpha_j} + \alpha_j = 2Re\,\alpha_j = 2Re\,\overline{\alpha_j},\ j = 1,\cdots,r$$

and

$$R_j^{(k,k)} := u_k^{(j)^*} u_k^{(j)},\ k = 1,\cdots,m_j,\ j = 1,\cdots,r. \tag{2.3}$$

Then,

$$A^* R_j^{(1,1)} + AR_j^{(1,1)} = \rho_j R_j^{(1,1)},\ j = 1,\cdots,r.$$

In other words: The matrix eigenvalue problem

$$A^* V + VA = \mu V$$

has r solution pairs

$$(\mu, V) = (\rho_j, R_j^{(1,1)})$$

with real ρ_j. *Moreover, the matrices* $R_j^{(k,k)}$ *are positive semi-definite for* $k = 1,\cdots,m_j,\ j = 1,\cdots,r$. *Further,*

$$R_j := \sum_{k=1}^{m_j} R_j^{(k,k)},\ j = 1,\cdots,r \quad \text{and} \quad R := \sum_{j=1}^{r}\sum_{k=1}^{m_j} R_j^{(k,k)} \tag{2.4}$$

is positive definite.

Proof. See [1, Theorems 7 - 8]. □

Remark 2.4. *With* (2.4)*, also a weighted norm* $\|\cdot\|_R$ *can be defined.*

3. Biorthogonality system of principal vectors of A and A^* revisited

First, we investigate the case of a diagonalizable matrix A and then the case of a general square matrix. Even though the result for a diagonalizable matrix will be included in that for the case of a general square matrix, it seems nevertheless be worthwhile to study this case separately. This is also a review section.

3.1 Diagonalizable matrix A

In this subsection, we summarize a known result on the biorthogonality of the eigenvectors of matrices A and A^*. It can be shown that − for diagonalizable matrices A − the eigenvectors of A and A^* are biorthogonal (so that there is nothing to construct in this case).

For the sequel, we formulate the following *conditions* :

(C1) $A \in \mathbb{C}^{n\times n}$.

(C2) A is diagonalizable, and $\lambda_i,\ i = 1,\cdots,n$ are the eigenvalues of A as well as $p_i,\ i = 1,\cdots,n$ the associated eigenvectors.

(C3) $u_i^*,\ i = 1,\cdots,n$ are the eigenvectors of A^* corresponding to the eigenvalues $\overline{\lambda_i},\ i = 1,\cdots,n$ of A^*.

(C4) $\lambda_i \neq \lambda_j,\ i \neq j,\ i,j = 1,\cdots,n.$

Then, we have the following theorem.

Theorem 3.1. *(Biorthogonality relations of eigenvectors)*

Let the conditions (C1) - (C4) be fulfilled. Then, after appropriate normalization of the eigenvectors $p_i,\ i = 1,\cdots,n$ and $u_i^*, i = 1,\cdots,n$, one has the biorthogonality relations

$$(p_i, u_j^*) = \delta_{ij},\ i,j = 1,\cdots,n, \tag{3.1}$$

where $(\cdot,\cdot)$ is the usual scalar product on $\mathbb{C}^n \times \mathbb{C}^n$.

Proof. See [2, Theorem 1]. □

Remark 3.2. *The condition (C4) is not essential so that it can be omitted. For this, see [3, Theorem 3]. But, we keep it because it is fulfilled in our Numerical Example 1 in Section 8.*

3.2 General square matrix A

In this subsection (more precisely, in Theorem 3.3), we exploit the fact that a principal vector of stage k of matrix A resp. A^* remains a principal vector of stage k if one adds a linear combination of principal vectors of stages 1 to $k-1$ of A resp. A^*, as the case may be. Hereby, we can construct a biorthogonal set of principal vectors of A resp. A^* (provided that they are not already biorthogonal, in which case there is nothing to construct).

Like in Subsection 3.1, we formulate the following *conditions* :

$(C1')$ $A \in \mathbb{C}^{n\times n}$.

$(C2')$ $\lambda_i,\ i = 1,\cdots,r$ are the eigenvalues of A corresponding to the Jordan blocks $J_i(\lambda_i) \in \mathbb{C}^{m_i\times m_i},\ i = 1,\cdots,r$ with the chains of principal vectors $p_1^{(i)},\cdots,p_{m_i}^{(i)},\ i = 1,\cdots,r$.

$(C3')$ ${u_1^{(i)}}^*,\cdots,{u_{m_i}^{(i)}}^*,\ i = 1,\cdots,r$ are the principal vectors of A^* corresponding to the eigenvalues $\overline{\lambda_i},\ i = 1,\cdots,r$ of the Jordan blocks $J_i(\overline{\lambda_i}) \in \mathbb{C}^{m_i\times m_i},\ i = 1,\cdots,r$.

$(C4')$ $\lambda_i \neq \lambda_j,\ i \neq j,\ i,j = 1,\cdots,r$.

Then, we have

Theorem 3.3. *(Biorthogonality relations for principal vectors)*

Let the conditions $(C1')-(C4')$ be fulfilled. Then, the systems $\{p_1^{(l)},\cdots,p_{m_1}^{(l)};\cdots;p_1^{(r)},\cdots,p_{m_r}^{(r)}\}$ and $\{{u_1^{(1)}}^*,\cdots,{u_{m_1}^{(1)}}^*;\cdots;{u_1^{(r)}}^*,\cdots,{u_{m_r}^{(r)}}^*\}$ can be constructed such that the following biorthogonality relations hold:

$$(p_k^{(i)}, {u_l^{(i)}}^*) = \begin{cases} 1, & l = m_i - k + 1 \\ 0, & l \neq m_i - k + 1 \end{cases}$$

$k = 1,\cdots,m_i,\ i = 1,\cdots,r$ and

$$(p_k^{(i)}, {u_l^{(j)}}^*) = 0,\ i \neq j,$$

$k = 1,\cdots,m_i,\ l = 1,\cdots,m_j,\ i,j = 1,\cdots,r$.

So, with

$${v_l^{(i)}}^* := {u_{m_i-l+1}^{(i)}}^*, \tag{3.2}$$

$l = 1,\cdots,m_i,\ i = 1,\cdots,r$ one has the biorthogonality relations

$$(p_k^{(i)}, {v_l^{(i)}}^*) = \delta_{kl}, \tag{3.3}$$

$k,l = 1,\cdots,m_i,\ i = 1,\cdots,r$, and

$$(p_k^{(i)}, {v_l^{(j)}}^*) = 0,\ i \neq j, \tag{3.4}$$

$k = 1,\cdots,m_i,\ l = 1,\cdots,m_j,\ i,j = 1,\cdots,r$.

Proof. See [2, Theorem 2]. □

Remark 3.4. *The properties (3.3) and (3.4) can also be written as*

$$(p_k^{(i)}, {v_l^{(j)}}^*) = \delta_{ij}\delta_{kl}, \tag{3.5}$$

$k = 1,\cdots,m_i,\ i = 1,\cdots,r;\ l = 1,\cdots,m_j,\ j = 1,\cdots,r$.

4. Representations of the solution $x(t)$ of the IVP $\dot{x}(t) = Ax(t) + h(t,x(t)),\ t \geq t_0,\ x(t_0) = x_0$ and of $x_A(t)$ when $h \equiv 0$ by fundamental matrices

In the following, we discuss the existence, uniqueness, and boundedness of the solution of the initial value problem $\dot{x}(t) = Ax(t) + h(t,x(t)),\ t \geq t_0,\ x(t_0) = x_0$ as well as pertinent representations of $x(t)$ and $x_A(t)$ by use of fundamental matrices.

Let $t_0 \in \mathbb{R}_0^+$, let $\mathbb{F}$ be the field of real or complex numbers and $\mathbb{F}^n$ be the set of n-tuples with elements in $\mathbb{F}$. Further, let $\|\cdot\|$ be a norm on $\mathbb{F}^n$, let $\gamma > 0$ and $\mathbb{F}^n_\gamma = \{u \in \mathbb{F}^n \,|\, \|u\| \leq \gamma\}$. Finally, let $A \in \mathbb{F}^{n\times n}$ and $h(t,u) \in \mathbb{F}^n$, $t \geq t_0$, $u \in \mathbb{F}^n_\gamma$, and continuous. We investigate the initial value problem

$$\dot{x}(t) = Ax(t) + h(t,x(t)),\ t \geq t_0,\ x(t_0) = x_0. \tag{4.1}$$

Let $\lambda_j(A)$, $j = 1, \cdots, n$ be the eigenvalues of matrix A. The *spectral abscissa* $\nu[A]$ is defined as the maximum of the real parts of the eigenvalues, i.e.,

$$\nu[A] = \max_{j=1,\cdots,n} Re\,\lambda_j(A).$$

We suppose that $\nu[A] < 0$. Further, if the eigenvalues of matrix A play a role, we implicitly assume that $\mathbb{F} = \mathbb{C}$. For $\nu_{x_0}[A]$ and the index $\iota(\lambda(A))$ of an eigenvalue $\lambda(A)$, we refer the reader, e.g., to [4].

Let $\Phi_A(t,t_0)$ be the *fundamental matrix* (or *evolution*) pertinent to the problem $\dot{x}_A(t) = Ax_A(t)$ with the property $\Phi_A(t_0,t_0) = E$, where $E \in \mathbb{F}^{n\times n}$ is the identity matrix. Then, the initial value problem is equivalent to the integral equation

$$x(t) = \Phi_A(t,t_0)\,x_0 + \int_{t_0}^{t} \Phi_A(t,s)\,h(s,x(s))\,ds,\ t \geq t_0. \tag{4.2}$$

This is a common implicit representation of $x(t)$ using $\Phi_A(t,t_0) = \exp(A\,(t-t_0)$. But, we shall employ a different explicit one, below.

For the sequel, we state the following conditions:

(C_0) The function $h(\cdot,\cdot)$ is continuous on $D_\gamma := \{(t,u)\,|\, t \geq t_0,\ u \in \mathbb{F}^n_\gamma\} = \{(t,u)\,|\,t \geq t_0,\ u \in \mathbb{F}^n,\ \|u\| \leq \gamma\}$.

(C_1) The function $h(\cdot,\cdot)$ satisfies the (uniform) Lipschitz condition

$$\|h(t,u) - h(t,u')\| \leq L_h\,\|u-u'\|,\ t \geq t_0,\ u,u' \in \mathbb{F}^n_\gamma$$

with a positive constant L_h.

(C_2) For every $u \in \mathbb{F}^n_\gamma$,

$$\lim_{u\to 0} \frac{\|h(t,u)\|}{\|u\|} = 0 \qquad \text{uniformly with respect to t} \geq \text{t}_0.$$

Herewith, we have the following theorem:

Theorem 4.1. *(Existence, uniqueness, and boundedness of the solution)*

Let the conditions (C_0),(C_1), and (C_2) be fulfilled. Further, let the spectral abscissa $\nu[A] < 0$, and $x_0 \neq 0$, $\|x_0\|$ as well as L_h be sufficiently small.

Then, integral equation (4.2) and thus initial value problem (4.1) has a unique bounded solution for all $t \geq t_0$.

Proof. See [5, Theorem 1]. □

Remark 4.2. *Sufficient for (C_2) is the following condition:*

(C_2') There exists a constant $c_h > 0$ such that

$$\|h(t,u)\| \leq c_h\,\|u\|^\kappa, ;t \geq t_0,\ u \in \mathbb{F}^n,\ \|u\| \leq \gamma,$$

with $\kappa > 1$.

We need this stronger condition for the derivation of a lower bound on the solution $x(t)$ of (4.1).

Now, let $\Phi_{A,h}(t,t_0)$ be the fundamental matrix with $\Phi_{A,h}(t_0,t_0) = E$ pertinent to the IVP (4.1). Then, the representation of $x(t)$ using $\Phi_{A,h}(t,t_0)$ is given by

$$x(t) = \Phi_{A,h}(t,t_0)\,x_0. \tag{4.3}$$

The representation (4.3) for the solution $x(t)$ of the IVP (4.1) plays a major role in the subsequent sections.

5. Representation of the solution vector $x(t)$ of $\dot{x}(t) = Ax(t) + h(t, x(t)),\ x(t_0) = x_0$ in the weighted time-dependent norm $\|\cdot\|_{R(t,t_0}$

Let the conditions (C_0), (C_1), and (C_2') from Section 4 be fulfilled. We remind that the solution of

$$\dot{x}_A(t) = A\,x_A(t),\ t \geq t_0,\ x_A(t_0) = x_0 \tag{5.1}$$

can be written as

$$x_A(t) = \Phi_A(t, t_0)\,x_0 = e^{A(t-t_0)}\,x_0. \tag{5.2}$$

This is the representation of $x_A(t)$ by the fundamental matrix that plays a role, in the sequel. From (4.3), it follows

$$x(t) = \Phi_{A,h}(t,t_0)\,e^{-A(t-t_0)}\,e^{A(t-t_0)}\,x_0 = \Psi(t,t_0)\,e^{A(t-t_0)}\,x_0 \tag{5.3}$$

with

$$\Psi(t,t_0) := \Phi_{A,h}(t,t_0)\,e^{-A(t-t_0)}. \tag{5.4}$$

Thus,

$$x(t) = \Psi(t,t_0)x_A(t). \tag{5.5}$$

5.1 The case of a diagonalizable matrix A

Let the conditions $(C1)-(C4)$ be fulfilled. Then, according to [2, Theorem 5], one has the representation

$$x_A(t) = \sum_{k=1}^{n}(x_0, u_k^*)\,p_k\,e^{\lambda_k(t-t_0)},\ t \geq t_0, \tag{5.6}$$

with

$$A\,p_k = \lambda_k\,p_k,\ k = 1,\cdots,n$$

where $p_k, k = 1,\cdots,n$ and $u_k^*,\ k = 1,\cdots,n$ are biorthogonal, that is, where (3.1) is satisfied. Inserting (5.6) into (5.5) gives

$$x(t) = \sum_{k=1}^{n}(x_0,u_k^*)\,\Psi(t,t_0)\,p_k\,e^{\lambda_k(t-t_0)} = \sum_{k=1}^{n}(x_0,u_k^*)\ p_k(t,t_0)\,e^{\lambda_k(t-t_0)},\ t \geq t_0$$

with

$$p_k(t,t_0) := \Psi(t,t_0)\,p_k. \tag{5.7}$$

Define

$$P(t,t_0) := [p_1(t,t_0),\cdots,p_n(t,t_0))]. \tag{5.8}$$

Then,

$$P^{-1}(t,t_0)P(t,t_0) = E, \tag{5.9}$$

where E is the identity matrix. Set

$$P^{-1}(t,t_0) =: U(t,t_0) =: \begin{bmatrix} u_1(t,t_0) \\ u_2 t,t_0) \\ \cdots \\ u_n t,t_0) \end{bmatrix} \tag{5.10}$$

where $u_j(t,t_0),\ j = 1,\cdots,n$ are row vectors of length n.

From (5.8), (5.9), and (5.10), we have

$$u_j(t,t_0)\,p_k(t,t_0) = \delta_{jk}$$

or

$$(p_k(t,t_0), u_j^*(t,t_0)) = \delta_{jk}.$$

With (5.7), this leads to

$$(\Psi(t,t_0)\, p_k, u_j^*(t,t_0)) = \delta_{jk}$$

or

$$(p_k, \Psi^*(t,t_0)\, u_j^*(t,t_0)) = \delta_{jk}.$$

On the other hand, also

$$(p_k, u_j^*) = \delta_{jk}.$$

Subtracting both relations implies

$$(p_k, u_j^* - \Psi^*(t,t_0)\, u_j^*(t,t_0)) = 0,\ j,k = 1,\cdots,n$$

and thus

$$u_j^* - \Psi^*(t,t_0)\, u_j^*(t,t_0) = 0,\ j = 1,\cdots,n$$

or

$$u_j^*(t,t_0) = [\Psi^*(t,t_0)]^{-1}\, u_j^* = [\Psi^{-1}(t,t_0)]^*\, u_j^*,\ j = 1,\cdots,n.$$

This leads to

$$u_j(t,t_0) = u_j\, \Psi^{-1}(t,t_0),\ j = 1,\cdots,n.$$

Now, define

$$R_j(t,t_0) = u_j^*(t,t_0)\, u_j(t,t_0),\ j = 1,\cdots,n \tag{5.11}$$

and

$$R(t,t_0) = \sum_{j=1}^{n} R_j(t,t_0). \tag{5.12}$$

It is left to the reader to show that $R_j(t,t_0),\ j = 1,\cdots,n$ are positive semi-definite and that $R(t,t_0)$ is positive definite.

With (5.11) and (5.12), we obtain

$$\begin{aligned}
\|x(t)\|^2_{R(t,t_0)} &= (R(t,t_0)\, x(t), x(t)) = \sum_{j=1}^{n} (R_j(t,t_0)\, x(t), x(t)) \\
&= \sum_{j=1}^{n} ([\Psi^{-1}(t,t_0)]^*\, u_j^*\, [u_j\, \Psi^{-1}(t,t_0)]\, x(t), x(t)) \\
&= \sum_{j=1}^{n} (u_j^*\, u_j\, [\Psi^{-1}(t,t_0)\, x(t)], [\Psi^{-1}(t,t_0)\, x(t)]) \\
&= \sum_{j=1}^{n} (R_j\, x_A(t), x_A(t)) = (R\, x_A(t), x_A(t)) \\
&= \|x_A(t)\|^2_R
\end{aligned}$$

so that we have

Theorem 5.1. *Let the conditions (C_0), (C_1), and (C_2') be fulfilled. Further, let the spectral abscissa $\nu[A] < 0$ and $x_0 \neq 0$, $\|x_0\|$ as well as L_h be sufficiently small. Moreover, let the conditions (C1) - C4) be satisfied.*

Then, with (2.2) and (5.12),

$$\|x(t)\|^2_{R(t,t_0)} = \|x_A(t)\|^2_R$$

where, according to [1, (47)],

$$\|x_A(t)\|^2_R = \sum_{i=1}^{n} \|x_0\|^2_{R_i} e^{2\lambda_i(A)(t-t_0)},\ t \geq t_0.$$

In other words: The solution $x(t)$ to the nonlinear problem (4.1) in the time-dependent weighted norm $\|\cdot\|_{R(t,t_0)}$ is equal to the solution $x_A(t)$ of the pertinent linear problem (5.1) in the norm $\|\cdot\|_R$.

Remark 5.2. *We mention that the vectors $p_i, i = 1,\cdots,n$ and $u_i^*, i = 1,\cdots,n$ themselves need not be normed. For the representation (5.6), we only have to demand that relation (3.1) be satisfied.*

5.2 The case of a general square matrix A

Let the conditions $(C1') - (C4')$ from Section 3 be fulfilled. The relations (5.1)-(5.5) remain valid. But, here, instead of (5.6), according to [2, Theorem 6], we have

$$x_A(t) = \sum_{i=1}^{r}\sum_{k=1}^{m_i} (x_0, u^{(i)}_{m_i-k+1}{}^*) x_k^{(i)}(t),\ t \geq t_0 \tag{5.13}$$

with

$$x_k^{(i)}(t) = [p_1^{(i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p_{k-1}^{(i)}(t-t_0) + p_k^{(i)}]\, e^{\lambda_i(t-t_0)},\ t \geq t_0, \tag{5.14}$$

$k = 1,\cdots,m_i,\ i = 1,\cdots,r$ where $p_k^{(i)},\ k = 1,\cdots,m_i,\ i = 1,\cdots,r$ and $v_l^{(j)*} := u^{(j)}_{m_j-l+1}{}^*,\ l = 1,\cdots,m_j,\ j = 1,\cdots,r$ from (3.2) satisfy (3.5).

In the semi-norm $\|\cdot\|_{R_i^{(k,k)}}$, $x_A(t)$ has, according to [1, 4.2,(56), (57)], the form

$$\|x(t)\|^2_{R_i^{(k,k)}} = |p^{(i)}_{x_0,k-1}(t-t_0)|^2\, e^{2\lambda_i(A)(t-t_0)}\ , t \geq t_0$$

with $p^{(i)}_{x_0,k-1}(t-t_0)$ in (5.21) below. Next, we proceed as in Section 5.1. From (5.5) and (5.13), (5.14) we conclude that

$$\begin{aligned} x(t) &= \Psi(t,t_0)x_A(t) = \sum_{i=1}^{r}\sum_{k=1}^{m_i} (x_0, u^{(i)}_{m_i-k+1}{}^*) \times \\ &\quad [\Psi(t,t_0)\, p_1^{(i)} \tfrac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + \Psi(t,t_0)\, p_{k-1}^{(i)}(t-t_0) + \Psi(t,t_0)\, p_k^{(i)}]\, e^{\lambda_i(t-t_0)} \end{aligned}$$

so that

$$\begin{aligned} x(t) &= \sum_{i=1}^{r}\sum_{k=1}^{m_i} (x_0, u^{(i)}_{m_i-k+1}{}^*) \times \\ &\quad [p_1^{(i)}(t,t_0) \tfrac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p_{k-1}^{(i)}(t,t_0)\,(t-t_0) + p_k^{(i)}(t,t_0)]\, e^{\lambda_i(t-t_0)} \end{aligned}$$

with

$$p_j^{(i)}(t,t_0) = \Psi(t,t_0)\, p_j^{(i)},\ j = 1,\cdots,m_i, i = 1,\cdots,r. \tag{5.15}$$

Next, define

$$P(t,t_0) := [p_1^{(1)}(t,t_0),\cdots,p_{m_1}^{(1)}(t,t_0);\cdots;p_1^{(r)}(t,t_0),\cdots,p_{m_r}^{(r)}(t,t_0)]. \tag{5.16}$$

Then,

$$P^{-1}(t,t_0)\,P(t,t_0) = E. \tag{5.17}$$

Set

$$P^{-1}(t,t_0) =: V(t,t_0) =: \left[\begin{array}{c} v_1^{(1)}(t,t_0) \\ \vdots \\ v_{m_1}^{(1)}(t,t_0) \\ \hline \vdots \\ \hline v_1^{(r)}(t,t_0) \\ \vdots \\ v_{m_r}^{(r)}(t,t_0) \end{array}\right]$$

where $v_k^{(j)}(t,t_0)$ are row vectors of length n. From (5.15), (5.16), and (5.17), we have

$$v_k^{(i)}(t,t_0)\, p_s^{(j)}(t,t_0) = \delta_{ij}\,\delta_{ks},$$

$k = 1,\cdots,m_i$, $0 = 1,\cdots,r$; $s = 1,\cdots,m_j$, $j = 1,\cdots,r$ or

$$(p_s^{(j)}(t,t_0), v_k^{(i)^*}(t,t_0)) = \delta_{ij}\,\delta_{ks}.$$

With (5.15), this leads to

$$(\Psi(t,t_0)\, p_s^{(j)}, v_k^{(i)^*}(t,t_0)) = \delta_{ij}\,\delta_{ks}$$

or

$$(p_s^{(j)}, \Psi^*(t,t_0)\, v_k^{(i)^*}(t,t_0)) = \delta_{ij}\,\delta_{ks}.$$

On the other hand, also

$$(p_s^{(j)}, v_k^{(i)^*}) = \delta_{ij}\,\delta_{ks}.$$

Subtracting both relations implies

$$(p_s^{(j)}, v_k^{(i)^*} - \Psi^*(t,t_0)\, v_k^{(i)^*}(t,t_0)) = 0,\ j,k = 1,\cdots,n$$

and thus

$$v_k^{(i)^*} - \Psi^*(t,t_0)\, v_k^{(i)^*}(t,t_0) = 0,\ k = 1,\cdots,m_i,\ i = 1,\cdots,r$$

or

$$v_k^{(i)^*}(t,t_0) = [\Psi^*(t,t_0))]^{-1}\, v_k^{(i)^*} = [\Psi^{-1}(t,t_0)]^*\, v_k^{(i)^*},$$

$k = 1,\cdots,m_i$, $i = 1,\cdots,r$. This leads to

$$v_k^{(i)}(t,t_0) = v_k^{(i)}\,\Psi^{-1}(t,t_0)$$

$k = 1,\cdots,m_i$, $i = 1,\cdots,r$. Similarly to (3.2), define

$$v_l^{(i)^*}(t,t_0) := u_{m_i-l+1}^{(i)}{}^*(t,t_0),$$

as well as

$$R_j^{(k,k)}(t,t_0) = u_j^*(t,t_0)\, u_k^{(i)}(t,t_0) \tag{5.18}$$

$k = 1,\cdots,m_i$, $i = 1,\cdots,r$,

$$R_j(t,t_0) = \sum_{k=1}^{m_i} R_j^{(k,k)}(t,t_0),\ j = 1,\cdots,n \tag{5.19}$$

and

$$R(t,t_0) = \sum_{j=1}^{r} R_j(t,t_0). \tag{5.20}$$

Again, it is left to the reader to show that $R_j^{(k,k)}(t,t_0)$, $R_j(t,t_0)$ are positive semi-definite and that $R(t,t_0)$ is positive definite.

Herewith,

$$\begin{aligned}
\|x(t)\|^2_{R(t,t_0)} &= (R(t,t_0)x(t),x(t)) = \sum_{j=1}^{r}\sum_{k=1}^{m_i}(R_j^{(k,k)}(t,t_0)x(t),x(t)) \\
&= \sum_{j=1}^{r}\sum_{k=1}^{m_i}([\Psi^{-1}(t,t_0)]^* u_l^{(i)^*}[u_l^{(i)}\Psi^{-1}(t,t_0)]x(t),x(t)) \\
&= \sum_{j=1}^{r}\sum_{k=1}^{m_i}(u_l^{(i)^*}u_l^{(i)}[\Psi^{-1}(t,t_0)x(t)],[\Psi^{-1}(t,t_0)x(t)]) \\
&= \sum_{j=1}^{r}\sum_{k=1}^{m_i}(R_j^{(k,k)}x_A(t),x_A(t)) \\
&= \sum_{j=1}^{r}(R_j(t,t_0)x_A(t),x_A(t)) \\
&= (Rx_A(t),x_A(t)) \\
&= \|x_A(t)\|^2_R
\end{aligned}$$

so that we have

Theorem 5.3. *Let the conditions (C_0), (C_1), and (C_2') be fulfilled. Further, let the spectral abscissa $\nu[A] < 0$ and $x_0 \neq 0$, $\|x_0\|$ as well as L_h be sufficiently small. Moreover, let the conditions $(C1') - (C4')$ be satisfied.*

Then, with (2.4) and (5.20),

$$\|x(t)\|^2_{R(t,t_0)} = \|x_A(t)\|^2_R$$

where, according to [1, (57), (56)],

$$\|x_A(t)\|^2_R = \sum_{j=1}^{r}\sum_{k=1}^{m_i}|p^{(i)}_{x_0,k-1}(t-t_0)|^2 e^{2\lambda_i(A)(t-t_0)},\ t \geq t_0,$$

with

$$p^{(i)}_{x_0,k-1}(t-t_0) := (x_0, u_1^{(i)^*}\frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + u_{k-1}^{(i)^*}(t-t_0) + u_k^{(i)^*}). \tag{5.21}$$

In other words: The solution $x(t)$ to the nonlinear problem (4.1) in the time-dependent weighted norm $\|\cdot\|_{R(t,t_0)}$ is equal to the solution $x_A(t)$ of the pertinent linear problem (5.1) in the norm $\|\cdot\|_R$.

Remark 5.4. *We mention that the vectors $p_k^{(i)}$, $k = 1,\cdots,m_i$, $i = 1,\cdots,r$ and $v_k^{(i)^*}$, $i = 1,\cdots,r$ themselves need not be normed. For the representation (5.13), (5.14), we only have to demand that relation (3.5) be satisfied.*

6. An expression for $\|x(t)\|_{R(t,t_0)}$ in the norm $\|\cdot\|_2$

According to Section 5, under the respective conditions, one has

$$\|x(t)\|^2_{R(t,t_0)} = \|x_A(t)\|^2_R.$$

As a consequence of this, one obtains a series of corollaries. The first one follows from [6, Section 3].

6.1 The case of a diagonalizable matrix A

We first turn to diagonalizable matrices A.

Corollary 6.1. *Let the conditions (C_0), (C_1), and (C_2') as well as conditions (C1) - C4) be satisfied. Let R_j be given by (2.1) and R by (2.2) as well as $R_j(t,t_0)$ by (5.11) and $R(t,t_0)$ by (5.12). Further, let*

$$\psi_j(t) := (x_0, u_j^*)\, e^{Re\,\lambda_j(A)(t-t_0)},\ t \geq t_0, \tag{6.1}$$

$j = 1, \cdots, n$, as well as

$$\psi(t) := [\psi_1(t), \psi_2(t), \cdots, \psi_n(t)]^T. \tag{6.2}$$

Then,

$$|(x_0, u_j^*)| = \|x_0\|_{R_j},$$

$j = 1, \cdots, n$, and

$$\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R = \|\psi(t)\|_2,\ t \geq t_0.$$

Proof. See proof of [6, Lemma 3]. □

6.2 The case of a general square matrix A

In this subsection, we consider the general square matrices A.

Corollary 6.2. *Let the conditions (C_0), (C_1), and (C_2') as well as conditions $(C1') - (C4')$ be fulfilled. Let $R_j^{(k,k)}$, R_j, and R be given by (2.3) and (2.4), respectively, as well as $R_j^{(k,k)}(t,t_0)$, $R_j(t,t_0)$, and $R(t,t_0)$ by (5.18), (5.19), and (5.20), as the case may be. Moreover, let $p_{x_0,k-1}^{(j)}(t-t_0)$ be given by (5.21), and let*

$$\psi_k^{(j)}(t) := p_{x_0,k-1}^{(j)}(t-t_0)\, e^{Re\,\lambda_j(A)\,(t-t_0)}, \tag{6.3}$$

$k = 1, \cdots, m_j$, $j = 1, \cdots, r$, as well as

$$\psi^{(j)}(t) := [\psi_1^{(j)}(t), \cdots, \psi_{m_j}^{(j)}(t)]^T, \tag{6.4}$$

$i = 1, \cdots, r$ and

$$\psi(t) := \begin{bmatrix} \psi^{(1)}(t) \\ \psi^{(2)}(t) \\ \vdots \\ \psi^{(r)}(t) \end{bmatrix}. \tag{6.5}$$

Then,

$$|(x_0, u_k^{(j)^*})| = \|x_0\|_{R_j^{(k,k)}},$$

$k = 1, \cdots, m_j$, $j = 1, \cdots, r$, and

$$\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R = \|\psi(t)\|_2,\ t \geq t_0.$$

Proof. See [6, Lemma 4]. □

7. Two-sided bounds on $x(t)$ in any norm $\|\cdot\|$ based on $\psi(t)$

In [6, Sections 4.1 and 4.2], under certain conditions, we have established the two-sided bounds

$$X_0\,\|\psi(t)\| \le \|x_A(t)\| \le X_1\,\|\psi(t)\|,\ t \ge t_0, \tag{7.1}$$

with $\psi(t)$ from (6.1), (6.2) for the case of diagonalizable matrices A and with $\psi(t)$ from (6.3)-(6.5) for general square matrices A.

The same two-sided bounds will be derived for $x(t)$ instead of $x_A(t)$. As a preparation for this, we prove the following lemma.

Lemma 7.1. *(Two-sided bound on $\|x(t)\|$ by $\|x_A(t)\|$)*
Let the conditions (C_0), (C_1), and (C_2') be satisfied; further, let $\nu_{x_0}[A] = \nu[A]$ and let the spectral abscissa $\nu[A]$ of matrix A be negative and for every eigenvalue $\lambda(A)$ with $Re\lambda(A) = \nu[A]$ let the index $\iota(\lambda(A))$ be $\iota(\lambda(A)) = 1$, let $x_0 \ne 0$, and $\|x_0\|$ as well as c_h and L_h be sufficiently small.
Then, there exist positive constants X_0 and X_1 such that

$$X_0\,\|x_A(t)\| \le \|x(t)\| \le X_1\,\|x_A(t)\|,\ t \ge t_0. \tag{7.2}$$

Proof. From [5, Corollary 4], we obtain

$$X_{A,h,0}\,e^{\nu[A]\,(t-t_0)} \le \|x(t)\| \le X_{A,h,1}\,e^{\nu[A]\,(t-t_0)},\ t \ge t_0 \tag{7.3}$$

with $x(t) = \Phi_{A,h}(t,t_0)\,x_0$ and positive constants $X_{A,h,0}$ and $X_{A,h,1}$. For the special case $h \equiv 0$, this leads to

$$X_{A,0}\,e^{\nu[A]\,(t-t_0)} \le \|x_A(t)\| \le X_{A,1}\,e^{\nu[A]\,(t-t_0)},\ t \ge t_0$$

with $x_A(t) = \Phi_A(t,t_0)\,x_0$ and positive constants $X_{A,0}$ and $X_{A,1}$ or

$$\frac{1}{X_{A,1}}\,\|x_A(t)\| \le e^{\nu[A]\,(t-t_0)} \le \frac{1}{X_{A,0}}\,\|x_A(t)\|,\ t \ge t_0. \tag{7.4}$$

From (7.3) and (7.4), we conclude that ((7.2) is valid with

$$X_0 = \frac{X_{A,h,0}}{X_{A,1}}$$

and

$$X_1 = \frac{X_{A,h,1}}{X_{A,0}}.$$

□

7.1 The case of a diagonalizable matrix A

In order to obtain the two-sided bounds in (7.1) with $x(t)$ instead of $x_A(t)$, we first turn to diagonalizable matrices A. Here, we have

Corollary 7.2. *Let the conditions (C_0), (C_1), and (C_2') as well as conditions (C1) - (C4) be satisfied and let $\|\cdot\|$ be any vector norm. Let $\psi(t)$ be defined by (6.1) and (6.2). Let $x(t)$ be the solution of the initial value problem (4.1). Then, there exist positive constants X_0 and X_1 such that*

$$X_0\|\psi(t)\| \le \|x(t)\| \le X_1\|\psi(t)\|,\ t \ge t_0\,. \tag{7.5}$$

Proof. The proof of (7.5) follows from Lemma 7.1 and relation (7.1) which, in turn, is stated in [6, Section 4.1, Theorem 5]. □

Remark 7.3. *The two-sided bound (7.5) turns out to be much better than (7.3).*

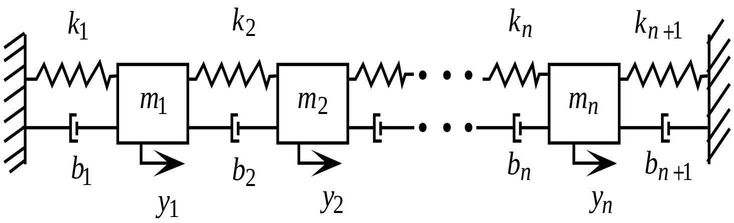

Figure 8.1. Multi-mass vibration model.

7.2 The case of a general square matrix A

In this subsection, we consider the general square matrices A.

Corollary 7.4. *Let the conditions* (C_0), (C_1), *and* (C_2') *as well as conditions* $(C1') - (C4')$ *be fulfilled, and let* $\|\cdot\|$ *be any vector norm. Let* $p^{(i)}_{x_0,k-1}(t-t_0)$ *be given by* (5.21) *and* $\psi(t)$ *by* (6.3)-(6.5)*. Let* $x(t)$ *be the solution of the initial value problem* (4.1)*. Then, there exist positive constants* X_0 *and* X_1 *such that*

$$X_0 \|\psi(t)\| \leq \|x(t)\| \leq X_1 \|\psi(t)\|,\ t \geq t_0\,. \tag{7.6}$$

Proof. The proof of (7.6) follows from Lemma 7.1 and relation (7.1) which, in turn, is stated in [6, Section 4.2, Theorem 6]. □

Remark 7.5. *The two-sided bound* (7.6) *turns out to be much better than* (7.3)*.*

8. Applications to free nonlinear dynamical systems with linear principal part

In this section, we consider applications to free nonlinear dynamical systems with linear principal part represented by a mechanical multi-mass vibratory system. Both the case of a diagonalizable and the case of a non-diagonalizable system matrix A is considered. For both cases, numerical examples illustrate the obtained results.

8.1 The multi-mass vibration model with nonlinear stiffness functions

We consider the multi-mass vibration model in Figure 8.1.

Here, k_i means the nonlinear stiffness function

$$k_i(v) = k_i^{(0)}\,(v + \eta\, v^3),\ v \in \mathbb{R}$$

with positive constants $k_i^{(0)}$, $i = 1, \cdots, n+1$ and with some parameter $\eta \geq 0$. For $\eta = 0$, we obtain a linear model, and otherwise a nonlinear model.

The equation of motion in vector form is given by

$$M\,\ddot{y} + B\,\dot{y} + q(y) = 0, \quad y(0) = y_0, \dot{y}(0) = \dot{y}_0$$

with

$$q(y) = K^{(0)}\,y + \eta\,q^{(3)}(y)$$

and

$$q^{(3)}(y) = \begin{bmatrix} k_1^{(0)}(y_1 - y_0)^3 - k_2^{(0)}(y_2 - y_1)^3 \\ k_2^{(0)}(y_2 - y_1)^3 - k_3^{(0)}(y_3 - y_2)^3 \\ \vdots \\ k_{n-1}^{(0)}(y_{n-1} - y_{n-2})^3 - k_n^{(0)}(y_n - y_{n-1})^3 \\ k_n^{(0)}(y_n - y_{n-1})^3 - k_{n+1}^{(0)}(y_{n+1} - y_n)^3 \end{bmatrix},$$

where $y_0 = y_{n+1} = 0$; the matrices M, B and $K^{(0)}$ are given by

$$M = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & m_3 & & \\ & & & \ddots & \\ & & & & m_n \end{bmatrix},$$

$$B = \begin{bmatrix} b_1+b_2 & -b_2 & & & & \\ -b_2 & b_2+b_3 & -b_3 & & & \\ & -b_3 & b_3+b_4 & -b_4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -b_{n-1} & b_{n-1}+b_n & -b_n \\ & & & & -b_n & b_n+b_{n+1} \end{bmatrix},$$

$$K^{(0)} = \begin{bmatrix} k_1^{(0)}+k_2^{(0)} & -k_2^{(0)} & & & & \\ -k_2^{(0)} & k_2^{(0)}+k_3^{(0)} & -k_3^{(0)} & & & \\ & -k_3^{(0)} & k_3^{(0)}+k_4^{(0)} & -k_4^{(0)} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -k_{n-1}^{(0)} & k_{n-1}^{(0)}+k_n^{(0)} & -k_n^{(0)} \\ & & & & -k_n^{(0)} & k_n+k_{n+1}^{(0)} \end{bmatrix}$$

with the *mass, damping,* and *stiffness matrices* M, B, and $K^{(0)}$, as the case may be, and the *displacement vector* y. In *state-space description*, this problem takes the form

$$\dot{x}(t) = Ax(t) + h(t,x(t)),\ t \geq 0,\ x(0) = x_0$$

with $x = [y^T, z^T]^T$, $z = \dot{y}$, and where the *system matrix* A is given by

$$A = \left[\begin{array}{c|c} 0 & E \\ \hline -M^{-1}K^{(0)} & -M^{-1}B \end{array}\right]$$

and h by

$$h(t,u) = \eta \begin{bmatrix} 0 \\ \\ -M^{-1}q^{(3)}(v) \end{bmatrix} =: h_0(u)$$

with $t \geq 0$, $u = [v^T, w^T]^T$, $v, w \in \mathbb{F}^n$. We mention that $x, u \in \mathbb{F}^m$ and $A \in \mathbb{F}^{m\times m}$ with $m = 2n$. From [5], it follows that

$$\|h(t,u)\| = \|h_0(u)\| \leq c_h \|u\|^3, t \geq 0, \quad u \in \mathbb{F}^m,$$

where $c_h = \eta\, \tilde{c}_h$ with a constant $\tilde{c}_h$ independent of η as well as

$$\|h(t,u) - h(t,u')\| \leq L_h \|u - u'\|, t \geq 0,\ u, u' \in \mathbb{F}^m_\gamma,$$

where $L_h = \eta\, \tilde{L}_h$, with a constant $\tilde{L}_h$ independent of η.

8.2 Numerical examples

Numerical Example 1: Matrix A diagonalizable

(i) *Data:*

The values $m_j,\ j = 1,\cdots,n$ and $b_j, k_j^{(0)},\ j = 1,\cdots,n+1$ are also specified as in earlier papers, namely as

$$\begin{array}{rcl} m_j &=& 1,\ j = 1,\cdots,n \\ k_j^{(0)} &=& 1,\ j = 1,\cdots,n+1 \end{array}$$

and

$$b_j = \begin{cases} 1/2,\ j\ even \\ 1/4,\ j\ odd. \end{cases}$$

Then,

$$M = E,$$

$$B = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & & & & \\ -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4} & & & \\ & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ & & & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

(if n is even), and

$$K^{(0)} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

We add the details in order to make the paper more readable on its own. Further, we choose $n = 5$ in this paper so that the state-space vector has dimension $m = 2n = 10$. For the initial time, we take

$$t_0 = 0$$

and

$$\eta \in \{0, 0.5, 1.0\}.$$

Finally, the initial conditions for $y(t)$ and $\dot{y}(t)$ are chosen as

$$y_0 = [-1,1,-1,1,-1]^T$$

as well as

$$\dot{y}_0 = [-1,-1,-1,-1,-1]^T.$$

(ii) *Computation of important quantities:*
Using the Matlab routine *eig.m*, one obtains

$$\begin{array}{rcl} \lambda_1(A^*) &=& -0.699760638780536 + 1.795981478159753i, \\ \lambda_2(A^*) &=& -0.699760638780536 - 1.795981478159753i, \\ \lambda_3(A^*) &=& -0.562668374040742 + 1.616358701643860i, \\ \lambda_4(A^*) &=& -0.562668374040742 - 1.616358701643860i, \\ \lambda_5(A^*) &=& -0.375000000000000 + 1.363589014329465i, \\ \lambda_6(A^*) &=& -0.375000000000000 - 1.363589014329465i, \\ \lambda_7(A^*) &=& -0.050239361219464 + 0.516371450711010i, \\ \lambda_8(A^*) &=& -0.050239361219464 - 0.516371450711010i, \\ \lambda_9(A^*) &=& -0.187331625959257 + 0.994521686465592i, \\ \lambda_{10}(A^*) &=& -0.187331625959257 - 0.994521686465592i. \end{array}$$

Therefore, $\lambda_j(A^*),\ j = 1, \cdots, m = 2n = 10$ and also $\lambda_j(A) = \overline{\lambda_j(A^*)},\ j = 1, \cdots, m = 2n = 10$ are distinct. Thus, matrix A is diagonalizable. Further, we obtain

$$U^* := [u_1^*, \cdots, u_{10}^*]$$

where

$$[u_1^*, \cdots, u_5^*] = \begin{bmatrix} 0.2680+0.0309i & 0.2680-0.0309i & -0.4533 & -0.4533 & 0.3314-0.2662i \\ -0.4491-0.0157i & -0.4491+0.0157i & 0.4039-0.0764i & 0.4039+0.0764i & 0.0539+0.0558i \\ 0.5119 & 0.5119 & 0.0563+0.1119i & 0.0563-0.1119i & -0.4393+0.1546i \\ -0.4370+0.0153i & -0.4370-0.0153i & -0.4321+0.0205i & -0.4321-0.0205i & -0.0354-0.0773i \\ 0.2439-0.0309i & 0.2439+0.0309i & 0.3970-0.1119i & 0.3970+0.1119i & 0.5101 \\ 0.0798-0.1237i & 0.0798+0.1237i & -0.1000+0.2519i & -0.1000-0.2519i & -0.1194-0.2759i \\ -0.0836+0.2122i & -0.0836-0.2122i & 0.0550-0.2290i & 0.0550+0.2290i & -0.0000+0.0000i \\ 0.0955-0.2474i & 0.0955+0.2474i & 0.1063-0.0140i & 0.1063+0.0140i & 0.0230+0.3285i \\ -0.0836+0.2122i & -0.0836-0.2122i & -0.0550+0.2290i & -0.0550-0.2290i & -0.0000-0.0000i \\ 0.0158-0.1237i & 0.0158+0.1237i & -0.0063-0.2379i & -0.0063+0.2379i & 0.0957-0.3478i \end{bmatrix},$$

and

$$[u_6^*, \cdots, u_{10}^*] = \begin{bmatrix} 0.3314+0.2662i & 0.0497-0.1441i & 0.0497+0.1441i & 0.3779 & 0.3779 \\ 0.0539-0.0558i & -0.0531-0.2199i & -0.0531+0.2199i & 0.3400-0.0254i & 0.3400+0.0254i \\ -0.4393-0.1546i & -0.0259-0.2635i & -0.0259+0.2635i & -0.0700-0.1592i & -0.0700+0.1592i \\ -0.0354+0.0773i & 0.0095-0.2322i & 0.0095+0.2322i & -0.3050+0.1050i & -0.3050-0.1050i \\ 0.5101 & -0.0755-0.1194i & -0.0755+0.1194i & -0.3079+0.1592i & -0.3079-0.1592i \\ -0.1194+0.2759i & -0.2568-0.0165i & -0.2568+0.0165i & 0.0314-0.3582i & 0.0314+0.3582i \\ -0.0000-0.0000i & -0.4417-0.0001i & -0.4417+0.0001i & -0.0079-0.3406i & -0.0079+0.3406i \\ 0.0230-0.3285i & -0.5104 & -0.5104 & -0.0857+0.0199i & -0.0857-0.0199i \\ -0.0000+0.0000i & -0.4417-0.0001i & -0.4417+0.0001i & 0.0079+0.3406i & 0.0079-0.3406i \\ 0.0957+0.3478i & -0.2536+0.0165i & -0.2536-0.0165i & 0.0543+0.3383i & 0.0543-0.3383i \end{bmatrix};$$

here the output results are given with only four decimal places for space reasons. The weighted matrix R is computed as

$$R = \begin{bmatrix} 1.2501 & -0.2868 & -0.1297 & 0.0135 & -0.0988 & 0.1966 & -0.1314 & -0.3356 & -0.0196 & 0.2662 \\ -0.2868 & 1.0887 & -0.3824 & -0.0842 & 0.0179 & -0.1567 & 0.2066 & 0.0440 & 0.0239 & 0.0319 \\ -0.1297 & -0.3824 & 1.1899 & -0.3531 & -0.1196 & 0.2780 & 0.0018 & 0.2200 & -0.1271 & -0.3402 \\ 0.0135 & -0.0842 & -0.3531 & 1.0873 & -0.3227 & -0.0173 & -0.0525 & -0.1892 & 0.1947 & 0.0505 \\ -0.0988 & 0.0179 & -0.1196 & -0.3227 & 1.2619 & -0.3017 & 0.0042 & 0.3092 & 0.0215 & 0.2699 \\ 0.1966 & -0.1567 & 0.2780 & -0.0173 & -0.3017 & 0.7620 & 0.2782 & 0.1039 & 0.0439 & -0.0256 \\ -0.1314 & 0.2066 & 0.0018 & -0.0525 & 0.0042 & 0.2782 & 0.8373 & 0.3358 & 0.1512 & 0.0458 \\ -0.3356 & 0.0440 & 0.2200 & -0.1892 & 0.3092 & 0.1039 & 0.3358 & 0.9171 & 0.3240 & 0.1085 \\ -0.0196 & 0.0239 & -0.1271 & 0.1947 & 0.0215 & 0.0439 & 0.1512 & 0.3240 & 0.8373 & 0.2919 \\ 0.2662 & 0.0319 & -0.3402 & 0.0505 & 0.2699 & -0.0256 & 0.0458 & 0.1085 & 0.2919 & 0.7685 \end{bmatrix},$$

We mention that the items in (i) and (ii) are already given in [6]. We have added them for the sake of completeness.

(iii) *Graph of* $y = \|x(t)\|_{R(t,t_0)}$ *for* $\eta \in \{0, 0.5, 1.0\}$*:*

In Figure 8.2, the curve $y = \|x(t)\|_{R(t,t_0)}$ for $\eta \in \{0, 0.5, 1.0\}$ is plotted.

The result in all three cases $\eta \in \{0, 0.5, 1.0\}$ is numerically identical with that in [6, Fig.4], i.e. for $y = \|x_A(t)\|_R$. But, the method of computing Figure 8.2 is different from that used for [6, Fig.4]; for this, see Section 8.3 below. The result of Figure 8.2 underpins the theoretical findings in Corollary 6.1.

The curve $y = \|x(t)\|_{R(t,t_0)}$ for $\eta \in \{0, 0.5, 1.0\}$ behaves essentially like $y = e^{-t}$ and clearly shows vibration suppression. Thus the curve in Figure 8.2 may serve as a measure of the damping property of the system.

The vibration behavior is due to the fact that the eigenvalues are pairwise conjugate complex. Since $Re\,\lambda_j(A^*) = Re\,\lambda_j(A) < 0,\ j = 1, \cdots, 10$, the system is asymptotically stable so that $\|x(t)\|_{R(t,t_0)} \to 0\ (t \to \infty)$.

(iv) *Two-sided bounds on* $y = \|x(t)\|_2$ *for* $\eta = 0$*:*

Now, we apply Corollary 7.2, first for $\eta = 0$ in order to check corresponding results from [6].

In Figure 8.3, the optimal upper and lower bounds $y = X_{1,2}\,\|\psi(t)\|_2$ and $y = X_{0,2}\,\|\psi(t)\|_2$ on $y = \|x(t)\|_2$ for $\eta = 0$ are shown where the optimal constants $X_{1,2}$ and $X_{0,2}$ are determined by the differential calculus of norms. Let $t_{s,u,2}$ and $t_{s,l,2}$ be the pertinent places of contact. For the initial guesses $t_{s,u,2_0} = 15.0$ and $t_{s,l,2_0} = 12.0$, the following results are obtained:

$$\begin{aligned} t_{s,u,2} &\doteq 15.204749, \\ X_{1,2} &\doteq 1.560408, \end{aligned}$$

and

$$\begin{aligned} t_{s,l,2} &\doteq 12.162025, \\ X_{0,2} &\doteq 0.803475. \end{aligned}$$

These results are numerically identical with those in [6]. However, the computational methods are different, see Section 8.3.

(v) *Two-sided bounds on* $y = \|x(t)\|_2$ *for* $\eta = 0.5$*:*

Now, we apply Corollary 7.2, for $\eta = 0.5$. In Figure 8.4, the pertinent optimal upper and lower bounds $y = X_{1,2}\,\|\psi(t)\|_2$ and $y = X_{0,2}\,\|\psi(t)\|_2$ on $y = \|x(t)\|_2$ are shown where the optimal constants $X_{1,2}$ and $X_{0,2}$ are determined by the differential calculus of norms. Let $t_{s,u,2}$ and $t_{s,l,2}$ be the pertinent places of contact. For the initial guesses $t_{s,u,2_0} = 19.0$ and $t_{s,l,2_0} = 16.0$, the following results are obtained:

$$\begin{aligned} t_{s,u,2} &\doteq 19.697311, \\ X_{1,2} &\doteq 1.444709, \end{aligned}$$

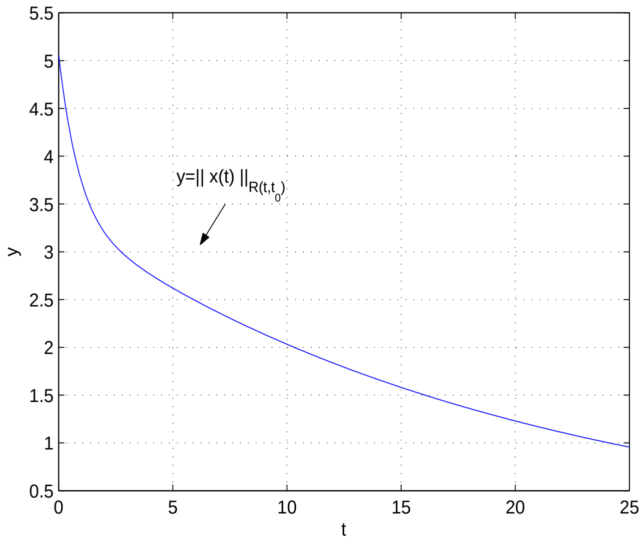

Figure 8.2. $y = \|x(t)\|_{R(t,t_0)}$ for $\eta \in \{0, 0.5, 1.0\}$ for diagonalizable system matrix A.

and

$$\begin{aligned} t_{s,l,2} &\doteq 16.737657, \\ X_{0,2} &\doteq 0.759622. \end{aligned}$$

These results are new. Computational details are given in Section 8.3.

Numerical Example 2: Matrix A non-diagonalizable

(i) *Construction of a non-diagonalizable matrix A:*

In the case $n = 2$ in Figure 8.1, we have

$$M = \left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right],$$

$$B = \left[\begin{array}{c|c} b_1 + b_2 & -b_2 \\ \hline -b_2 & b_2 + b_3 \end{array}\right],$$

and

$$K^{(0)} = \left[\begin{array}{c|c} k_1^{(0)} + k_2^{(0)} & -k_2^{(0)} \\ \hline -k_2^{(0)} & k_2^{(0)} + k_3^{(0)} \end{array}\right],$$

so that the pertinent characteristic equation reads

$$|\lambda^2 M + \lambda B + K^{(0)}| = \left|\begin{array}{c|c} \lambda^2 m_1 + \lambda(b_1 + b_2) + (k_1^{(0)} + k_2^{(0)}) & \lambda(-b_2) - k_2^{(0)} \\ \hline \lambda(-b_2) - k_2^{(0)} & \lambda^2 m_2 + \lambda(b_2 + b_3) + (k_2^{(0)} + k_3^{(0)}) \end{array}\right| = 0.$$

For the construction of a case with non-diagonalizable matrix A, we choose

$$b_2 = 0,\ m_2 = m_1 = 1,\ b_3 = b_1,\ k_3^{(0)} = k_1^{(0)}.$$

Then,

$$\lambda^2 m_1 + \lambda b_1 + (k_1^{(0)} + k_2^{(0)}) = s k_2^{(0)} \quad \text{with} \quad s \in \{+1, -1\}.$$

Hence, with $m_1 = 1$,

$$\lambda = -\frac{b_1}{2} \pm \sqrt{(\frac{b_1}{2})^2 - k_1^{(0)} - k_2^{(0)} + s k_2^{(0)}}.$$

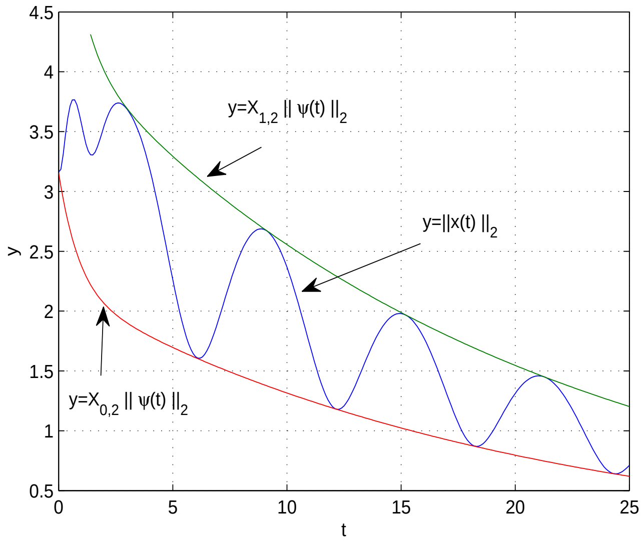

Figure 8.3. $y = \|x(t)\|_2$ for diagonalizable system matrix A and $\eta = 0$ as well as optimal upper and lower bounds.

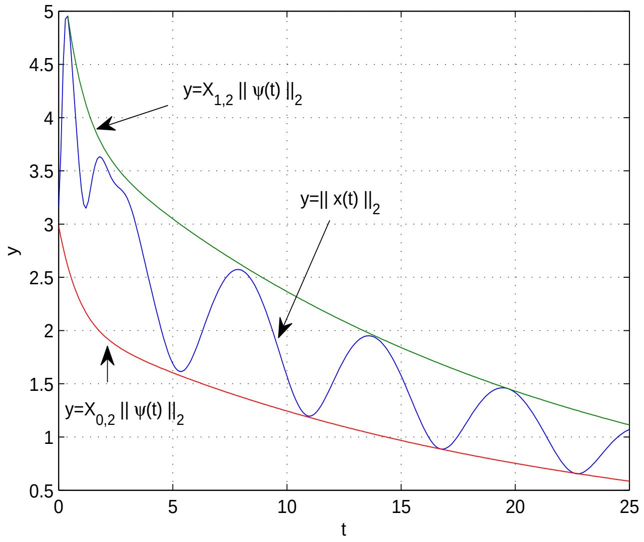

Figure 8.4. $y = \|x(t)\|_2$ for diagonalizable system matrix A and $\eta = 0.5$ as well as optimal upper and lower bounds.

Now, in order to get one real solution, we set

$$k_1^{(0)} := (\frac{b_1}{2})^2.$$

This implies

$$\lambda = \begin{cases} -\frac{b_1}{2}, & s = +1, \\ -\frac{b_1}{2} \pm i\sqrt{2k_2^{(0)}}, & s = -1. \end{cases}$$

(ii) *Data:*

As numerical values for the quantities not yet specified, we choose $b_1 = 1/4$, $k_2^{(0)} = 2^3 = 8$. On the whole, this delivers the following data:

$$m_1 = m_2 = 1;\ b_1 = 1/4, b_2 = 0, b_3 = 1/4;\ k_1^{(0)} = 1/64 = 1/2^4, k_2^{(0)} = 8, k_3^{(0)} = 1/64 = 1/2^4,$$

which leads to

$$M = \left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right],$$

$$B = \left[\begin{array}{c|c} b_1+b_2 & -b_2 \\ \hline -b_2 & b_2+b_3 \end{array}\right] = \left[\begin{array}{c|c} 0.25 & 0 \\ \hline 0 & 0.25 \end{array}\right],$$

and

$$K^{(0)} = \left[\begin{array}{c|c} k_1^{(0)}+k_2^{(0)} & -k_2^{(0)} \\ \hline -k_2^{(0)} & k_2^{(0)}+k_3^{(0)} \end{array}\right] = \left[\begin{array}{c|c} 1/64+8 & -8 \\ \hline -8 & 8+1/64 \end{array}\right] = \left[\begin{array}{c|c} 8.015625 & -8 \\ \hline -8 & 8.015625 \end{array}\right].$$

Further, we choose

$$t_0 = 0$$

as well as

$$y_0 = [-1, 1]^T$$

and

$$\dot{y}_0 = [-1, -1]^T.$$

(iii) *Computation of important quantities:*

Using the Matlab routine *jordan*, one obtains

$$\begin{aligned} \lambda_1(A^*) &= -0.1250 + 4.0000i, \\ \lambda_2(A^*) &= -0.1250 - 4.0000i, \\ \lambda_3(A^*) &= -0.1250, \\ \lambda_4(A^*) &= \lambda_3(A^*). \end{aligned}$$

Here, $m_1 = m_2 = 1$, and $m_3 = 2$. Thus, matrix A^* and therefore also A is not diagonalizable, and the computation of

$$U^* := [u_1^*, u_2^*, u_1^{(3)^*}, u_2^{(3)^*}]$$

gives

$$U^* = \begin{bmatrix} 0.25 + 0.0078125i & 0.25 - 0.0078125i & 0.0625 & 0.5 \\ -0.25 - 0.0078125i & -0.25 + 0.0078125i & 0.0625 & 0.5 \\ 0.0625i & 0.0625i & 0.5 & 0 \\ 0.0625i & 0.0625i & 0.5 & 0 \end{bmatrix}.$$

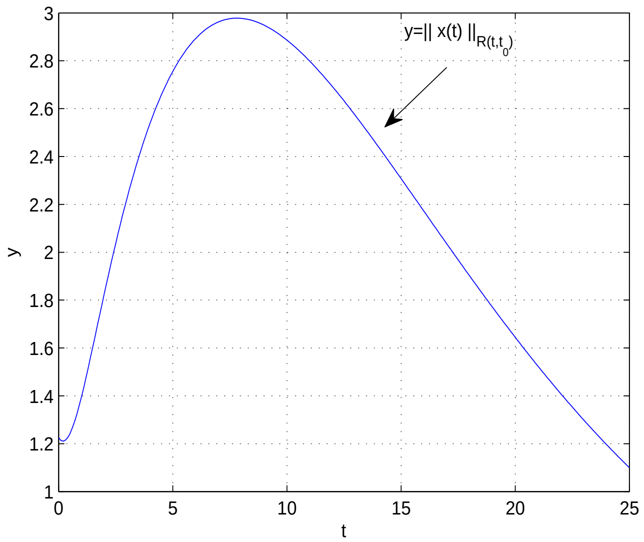

Figure 8.5. $y = \|x(t)\|_{R(t,t_0)}$ for non-diagonalizable system matrix A and $\eta \in \{0, 0.5, 1.0\}$.

The weighted matrix R is computed as

$$R = \begin{bmatrix} 0.379028320312500 & 0.128784179687500 & 0.032226562500000 & 0.030273437500000 \\ 0.128784179687500 & 0.379028320312500 & 0.030273437500000 & 0.032226562500000 \\ 0.032226562500000 & 0.030273437500000 & 0.257812500000000 & 0.242187500000000 \\ 0.030273437500000 & 0.032226562500000 & 0.242187500000000 & 0.257812500000000 \end{bmatrix}.$$

We mention that the items (i) - (iii) are already given in [6]. We have added them for the sake of completeness.

(iv) *Graph of* $y = \|x(t)\|_{R(t,t_0)}$ *for* $\eta \in \{0, 0.5, 1.0\}$*:*

In the norm $\|\cdot\|_{R(t,t_0)}$, we obtain Figure 8.5.

The result in all three cases is numerically identical with that in [6, Fig.9], i.e., for $y = \|x_A(t)\|_R$. But, again, the method of computing Figure 8.5 is different from that used for [6, Fig.8]; for this, see the computational aspects discussed in Section 8.3.

Remark 8.1. *The computation of the optimal two-sided bounds* $y = X_{1,2}\,\|\psi(t)\|_2$ *and* $y = X_{0,2}\,\|\psi(t)\|_2$ *on* $y = \|x(t)\|_2$ *for* $\eta = 0.5$ *corresponding to Figure 8.4 is left to the reader.*

8.3 Computational aspects

In this subsection, we say something about the used computer equipment, used Matlab programs, and the computation time.

(i) As to the *computer equipment*, the following hardware was available: an Intel Core2 Duo Processor at 3166 GHz, a 500 GB mass storage facility, and two 2048 MB high-speed memories. As software package for the computations, we used MATLAB, Version 7.11.

(ii) Whereas in [6], the computations were based on the representation $x_A(t) = e^{A(t-t_0)}x_0$ of the solution of the initial value problem $\dot{x}_A(t) = Ax_A(t),\ t \geq t_0,\ x_A(t_0) = x_0$, here for the solution of the nonlinear IVP $\dot{x}(t) = Ax(t) + h(t,x(t)),\ t \geq t_0,\ x(t_0) = x_0$, the Matlab program ODE45 is applied. In the case of $\eta = 0$, we obtain the same numerical result as in the linear case. The computation time is larger, however.

(iii) The *computation time t* of an operation was determined by the command sequence *t1=clock; operation; t=etime(clock,t1)*. It is put out in seconds, rounded to four decimal places. For the computation of the eigenvalues of matrix A, we used the command [XA,DA]=eig(A); the pertinent computation time for *Example 1* is less than 0.0001 s. For the computation of the 251 values $t, y(t)$ with $y(t) = \|x(t)\|_{R(t,t_0)},\ t = 0(0.1)25$ for, say, Figure 8.5, it took $t(tableforFigure8.5) = 7.8930s$. The computation times for the other figure are of a similar order.

9. Conclusion

In this paper, it is shown that one can construct a time-dependent positive definite matrix $R(t,t_0)$ such that the solution $x(t)$ of the nonlinear initial value problem with linear principal part $\dot{x}(t) = Ax(t) + h(t,x(t)),\ t \geq t_0,\ x(t_0) = x_0$ in the weighted norm $\|\cdot\|_{R(t,t_0)}$ is equal to the solution $x_A(t)$ of $\dot{x}_A(t) = Ax_A(t),\ t \geq t_0,\ x_A(t_0) = x_0$ in the weighted norm $\|\cdot\|_R$ where R is a constant positive definite matrix. As a consequence, if $\|x_A(t)\|_R$ shows vibration suppression or monotonicity behavior, so does $\|x(t)\|_{R(t,t_0)}$. Further, since $\|x_A(t)\|_R$ can be used to assess the damping behavior of the underlying dynamical problem, by the equation $\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R$ also the damping behavior of the nonlinear IVP can be assessed. The results are applied to dynamical systems, and examples underpin the theoretical findings.

One might object that incase of matrix A is not diagonalizable, the Jordan canonical form has to be calculated. But, the determination of the Jordan canonical form can be done by the *jordan routine* of MATLAB. Further, engineers usually reduce an originally large matrix A by a process called *condensation*. For these reduced matrices, it is usually no numerical problem to determine the canonical Jordan form, and it is then not costly to compute $\|x(t)\|_{R(t,t_0)}$. In addition, in engineering practice, often models with small matrices A are applied. For these models, the new method is likewise of major interest. Moreover, the matrices used in practice are in most cases diagonalizable. In these cases, no numerical problem at all exists.

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An Agile Optimal Orthogonal Additive Randomized Response Model

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Abstract
In this paper, a new additive randomized response model has been proposed. The properties of the proposed model have been studied. It has been shown theoretically that the suggested additive model is better than the one envisaged by [1] under very realistic conditions. Numerical illustrations are also given in support of the present study.

Keywords: Estimation of mean, Randomized response sampling, Respondents protection, Sensitive quantitative variable.
2010 AMS: Primary 62D05, Secondary 62D99

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1. Introduction

One problem with research on high – risk behavior is that respondents may consciously or unconsciously provide incorrect information. In psychological surveys, a social desirability bias has been observed as a major cause of distortion in standardized personality measures.. Survey researchers have similar concerns about the truth of survey results findings of such topics as drunk driving, use of marijuana, tax evasion, illicit drug use, induced abortion, shop lifting, child abuse, family disturbances, cheating in exams, HIV/AIDS, and sexual behavior. The most serious problem in studying certain social problems that are sensitive in nature (e.g. induced abortion, drug usage, tax evasion, etc.) is the lack of reliable measure of their incidence or prevalence. Thus to obtain trustworthy data on such confidential matters, especially the sensitive ones, instead of open surveys alternative procedures are required. Such an alternative procedure known as "randomized response technique" (RRT) was first introduced by [2]. It provides the opportunity of reducing response biases due to dishonest answers to sensitive questions. As a result, the technique assures a considerable degree of privacy protection in many contexts. Following the pioneering work of [2], many modifications are proposed in the literature. A good exposition of developments on randomized response techniques could refer to [3]-[18]. We below give the description of the model due to [1]

1.1 Additive model[1]:

Let there be k scrambling variables denoted by $S_j, j=1,2,\ldots,k$ whose mean $\theta_j (i.e. E(S_j)=\theta_j)$ and variance $\gamma_j^2 (i.e. V(S_j)=\gamma_j^2)$ are known. In [1] proposed optimal new orthogonal additive model named as (POONAM), each respondent selected in the sample is requested to rotate a spinner, as shown in Fig. 1.1, in which the proportion of the k shaded areas, say $P_1, P_2, \ldots P_k$ are

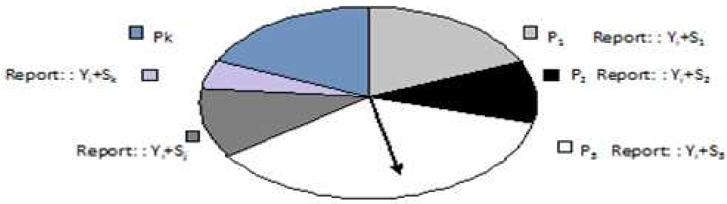

Figure 1.1. Spinner of POONAM[1]

orthogonal to the means of the k scrambling variables, say $\theta_1, \theta_2, ...\theta_k$ such that:

$$\sum_{i=1}^{k} P_j \theta_j = 0$$

and

$$\sum_{i=1}^{k} P_j = 1$$

Now if the pointer stops in the j^{th} shaded area, then the i^{th} respondent with the value of the sensitive variable, say Y_i, is requested report the scrambled response Z_i as:

$$Z_i = Y_i + S_i$$

Assuming that the sample of size n is drawn from the population using simple random sampling with replacement (SRSWR). [1] suggested an unbiased estimator of the population mean μ_Y as

$$\hat{\mu}_Y = \frac{1}{n} \sum_{i=1}^{n} Z_i$$

with variance

$$V(\hat{\mu}_Y) = \frac{1}{n}[\sigma_y^2 + \sum_{i=1}^{n} P_j(\theta_j^2 + \gamma_j^2)],$$

2. The proposed procedure:

Let $S_j, j = 1, 2, \ldots, k$ be k scrambling variables such that their distribution are known. In brief, let $E(S_j) = \theta_j)$ and variance $V(S_j) = \gamma_j^2)$ are known. Then, in the proposed additive model, each respondent selected in the sample is requested to rotate a spinner, as depicted in Fig. 2.1, in which the proportion of the $k$shaded areas, say $P_1, P_2, \ldots P_k$ are orthogonal to the means of the k scrambling variables, say $\theta_1, \theta_2, ...\theta_k$ such that:

$$\sum_{i=1}^{k} P_j \theta_j = 0$$

and

$$\sum_{i=1}^{k} P_j = 1$$

If the pointer stops in the j^{th} shaded area, then the i^{th} respondent with the value of the sensitive variable, say Y_i, is requested report the scrambled response Z_i^* as:

$$Z_i^* = Y_i + S_j^*$$

where $S_j^* = \frac{(a_j S_j + b_j \theta_j)}{(a_j + b_j)}$ and (a_j, b_j) being suitably chosen constants which may take real values and the functions of known parameters of scrambling variable S_j such as $\gamma_j, \theta_j, C_j(= \gamma_j/\theta_j), \beta_2(S_j) = \frac{\mu_4(S_j)}{\gamma_j^4}$ (coefficient of kurtosis), $G_1(S_j) = \frac{\mu_3(S_j)}{\gamma_j^3}$ is the Fisher's measure of skewness, $\mu_3(S_j)$ and $\mu_4(S_j)$ are third and fourth central moments of the scrambling variable S_j etc. Let a sample of size n be drawn from the population using the simple random sampling with replacement (SRSWR). Then we prove the following theorems.

Theorem 2.1. *An unbiased estimator of the population mean μ_Y is given by*

$$\hat{\mu}_{ST} = \frac{1}{n}\sum_{i=1}^{n} Z_i^*$$

Proof. Let E_1 and E_2 denote the expectation over the sampling design and the randomization device respectively, we have

$$E(\hat{\mu}_{ST}) = E_1 E_2[\frac{1}{n}\sum_{i=1}^{n} Z_i^*] = E_1[\frac{1}{n}\sum_{i=1}^{n} E_2(Z_i^*)] = E_1[\frac{1}{n}\sum_{i=1}^{n}(Y_i \sum_{i=1}^{n} P_j + \sum_{i=1}^{n} P_j\theta_j)]$$

$$E_1[\frac{1}{n}\sum_{i=1}^{n} Y_i] = \mu_Y,$$

since

$$\sum_{i=1}^{k} P_j\theta_j = 0$$

and

$$\sum_{i=1}^{k} P_j = 1$$

which completes the theorem. The variance of the proposed estimator $\hat{\mu}_{ST}$ is given in the following theorem. □

Theorem 2.2. *The variance of the proposed estimator $\hat{\mu}_{ST}$ is given by*

$$V(\hat{\mu}_{ST}) = \frac{1}{n}[\sigma_y^2 + \sum_{j=1}^{k} P_j\theta_j^2(1+\eta_j^2 C_j^2)],$$

where $\eta_j = a_j/(a_j + b_j)$ and $C_j = \gamma_j/\theta_j; j = 1, 2, ...k$.

Proof. Let V_1 and V_2 denote the variance over the sampling design and over the proposed randomization device, respectively, then we have

$$\begin{aligned} V(\hat{\mu}_Y) &= E_1 V_2(\hat{\mu}_Y) + V_1 E_2(\hat{\mu}_Y) = E_1[V_2[\frac{1}{n}\sum_{i=1}^{n}(Z_i^*)] + V_1[E_2(\frac{1}{n}\sum_{i=1}^{n}(Z_i^*))] \\ &= E_1[\frac{1}{n^2}\sum_{i=1}^{n} V_2(Z_i^*)] + V_1[(\frac{1}{n}\sum_{i=1}^{n} E_2(Z_i^*))] = [\frac{\sigma_y^2}{n} + E_1[\frac{1}{n^2}\sum_{i=1}^{n} V_2(Z_i^*)]. \end{aligned}$$

Note that

$$V_2(Z_i^*) = \sum_{j=1}^{k} P_j E_2(Y_i + S_j^*)^2 - Y_i^2 = Y_i^2 + \sum_{j=1}^{k} P_j\theta_j^2(1+\eta_j^2 C_j^2) - Y_i^2;$$

since

$$\sum_{i=1}^{k} P_j\theta_j = 0$$

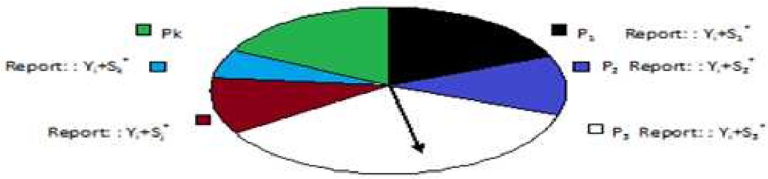

Figure 2.1. Spinner of Proposed Procedure

and

$$\sum_{i=1}^{k} P_j = 1$$

$$V_2(Z_i^*) = \sum_{j=1}^{k} P_j \theta_j^2 (1 + \eta_j^2 C_j^2)$$

where $\eta_j = a_j/(a_j + b_j)$ and $C_j = \gamma_j/\theta_j; j = 1, 2, ...k$.
Solving the above equations , we get

$$V(\hat{\mu}_{ST}) = \frac{1}{n}[\sigma_y^2 + \sum_{j=1}^{k} P_j \theta_j^2 (1 + \eta_j^2 C_j^2)],$$

This completes the proof of the theorem. □

3. Efficiency comparison

The proposed estimator $\hat{\mu}_{(ST)}$ will be more efficient than the estimator $\hat{\mu}_{(Y)}$ if

$$V(\hat{\mu}_{ST}) < V(\hat{\mu}_Y), if$$

i.e. if

$$\frac{1}{n}[\sigma_y^2 + \sum_{j=1}^{k} P_j \theta_j^2 (1 + \eta_j^2 C_j^2)] < \frac{1}{n}[\sigma_y^2 + \sum_{j=1}^{k} P_j \theta_j^2 (1 + C_j^2)]$$

i.e. if

$$[\sum_{j=1}^{k} P_j \theta_j^2 (1 + \eta_j^2 C_j^2)] < [\sum_{j=1}^{k} P_j \theta_j^2 (1 + C_j^2)]$$

i.e. if

$$[\sum_{j=1}^{k} P_j \theta_j^2 (\eta_j^2 - 1)] C_j^2 < 0$$

i.e. if

$$|\eta_j^2| < 1 \forall j = 1, 2, ...k$$

It follows from the above equation that we should choose the value of (a_j, b_j) such a way that

$$|\frac{a_j}{a_j + b_j}| < 1$$

σ_Y^2	θ_1	θ_2	θ_3	θ_4	$a_j = \frac{b_j}{4} and \eta_j = \frac{1}{5}$
25	1	5	10	-10.25	538.02
25	5	10	15	-17.50	310.22
25	10	15	20	-25.00	215.74
25	15	20	25	-32.50	171.97
125	1	5	10	-10.25	343.28
125	5	10	15	-17.50	251.87
125	10	15	20	-25.00	195.53
125	15	20	25	-32.50	163.61

Table 1. Percent relative efficiencies of the proposed estimator $\hat{\mu}_{(ST)}$ over the Singh (2010) estimator $\hat{\mu}_{(Y)}$.

σ_Y^2	$a_j = \frac{b_j}{4} and \eta_j = \frac{1}{5}$
25	1244.77
125	470.23
225	320.82
325	257.33
425	222.20
525	199.89
625	184.47
725	173.17
825	164.54

Table 2. Percent relative efficiencies of the proposed estimator $\hat{\mu}_{(ST)}$ over the Singh (2010) estimator $\hat{\mu}_{(Y)}$ for $\theta_j, j = 0, 1, 2...k$.

We have computed the percent relative efficiency (PRE) of the proposed estimator $\hat{\mu}_{(ST)}$ with respect to Singh's estimator $\hat{\mu}_Y$ by using the formula:

$$PRE(\hat{\mu}_{(ST)}, \hat{\mu}_{(Y)}) = \frac{[\sigma_y^2 + \sum_{j=1}^{k} P_j(\theta_j^2 + \gamma_j^2)]}{[\sigma_y^2 + \sum_{j=1}^{k} P_j(\theta_j^2 + \eta_j^2 \gamma_j^2)]} \times 100$$

By keeping the respondents cooperation in mind, we decided to choose $\gamma = 40, \gamma_1 = 30, \gamma_2 = 40, \gamma_3 = 20, \gamma_4 = 10, P_1 = 0.01, P_2 = 0.02, P_3 = 0.03, P_4 = 0.04$ with $k = 4$. In addition we choose different values $\sigma_y^2, \theta_1, \theta_2, \theta_3, \theta_4$.

It is observed that the values of $PRE(\hat{\mu}_{(ST)}, \hat{\mu}_{(Y)})$ are greater than 100. It follows that the proposed estimator $\hat{\mu}_{(ST)}$ is more efficient than the estimator $\hat{\mu}_{(Y)}$ due to [1] with a substantial gain in efficiency. Thus, based on our simulation results, the use of the proposed estimator $\hat{\mu}_{(ST)}$ over [1] estimator $\hat{\mu}_{(Y)}$ is recommended for all situations. We also consider a situation where $\theta_j = 0 for j = 1, 2, 3, 4$, and rest of the parameters are kept the same.The percent relative efficiency of the proposed estimator $\hat{\mu}_{(ST)}$ over [1] estimators $\hat{\mu}_{(Y)}$ has been shown. Numerical illustration clearly show that the percent relative efficiencies remain higher if the value of σ_y^2 is small. We have further considered the case $k = 2$ and computed the $PRE(\hat{\mu}_{(ST)}, \hat{\mu}_{(Y)})$ for different choices of parameters. Thus, based on our numerical findings, the proposed estimator $\hat{\mu}_{(ST)}$ is to be preferred over [1] estimator $\hat{\mu}_{(Y)}$ is recommended for all situations in real practice. It should be noted here that the experience is must in real surveys while making a choice of randomization device to be used in practice.

Discussion

In this article, we have suggested a new additive randomized response model and its properties are studied. We have proved the superiority of the proposed randomized response model over [1] randomized response models both theoretically and empirically.

σ_Y^2	θ_1	θ_2	θ_3	θ_4	$a_j = \frac{b_j}{4} and \eta_j = \frac{1}{5}$
25	1	1	2	-2.25	728.03
25	1	2	2	-2.75	505.45
25	2	1	3	-3.25	340.55
25	2	2	3	-3.75	239.00
125	1	1	2	-2.25	383.36
125	1	2	2	-2.75	320.09
125	2	1	3	-3.25	253.68
125	2	2	3	-3.75	199.62

Table 3. Percent relative efficiencies of the proposed estimator $\hat{\mu}_{(ST)}$ over the Singh (2010) estimator $\hat{\mu}_{(Y)}$.

σ_Y^2	$PRE's$
25	2380.00
125	556.00
225	353.33
325	275.38
425	234.12
525	208.57
625	191.20
725	178.62
825	169.09

Table 4. Percent relative efficiencies of the proposed estimator $\hat{\mu}_{(ST)}$ over the Singh (2010) estimator $\hat{\mu}_{(Y)}$ for $\theta_j, j = 0, 1, 2...k$.

σ_Y^2	θ_1	θ_2	θ_3	θ_4	$a_j = \frac{b_j}{4} and \eta_j = \frac{1}{5}$
25	1	5	10	-10.25	946.45
25	5	10	15	-17.50	715.96
25	10	15	20	-25.00	508.30
25	15	20	25	-32.50	373.52
125	1	5	10	-10.25	368.67
125	5	10	15	-17.50	340.15
125	10	15	20	-25.00	300.41
125	15	20	25	-32.50	261.38

Table 5. Percent relative efficiencies of the proposed estimator $\hat{\mu}_{(ST)}$ over the Singh (2010) estimator $\hat{\mu}_{(Y)}$ with $k = 2$.

σ_Y^2	θ_1	θ_2	θ_3	θ_4	$a_j = \theta_j and b_j = \gamma_j$ with $k = 2$
25	1	1	2	-2.25	1676.13
25	1	2	2	-2.75	1599.26
25	2	1	3	-3.25	1587.15
25	2	2	3	-3.75	1518.21
125	1	1	2	-2.25	424.94
125	1	2	2	-2.75	421.12
125	2	1	3	-3.25	420.26
125	2	2	3	-3.75	416.53

Table 6. Percent relative efficiencies of the proposed estimator $\hat{\mu}_{(ST)}$ over the Singh (2010) estimator $\hat{\mu}_{(Y)}$.

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On Signomial Constrained Optimal Control Problems

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Abstract
In this paper, using the notions of *variational differential system*, *adjoint differential system* and *modified Legendrian duality*, we formulate and prove necessary optimality conditions in signomial constrained optimal control problems.

Keywords: Optimal control, Maximum principle, Variational differential system, Adjoint differential system, Modified Legendrian duality.
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1. Introduction and problem formulation

Optimal control theory (see Lee and Markus [1], Pontriaguine *et al.* [2], Evans [3]), due to important applications in various branches of pure and applied science, has attracted many researchers over the years. Wagner [4] established a Pontryagin-type maximum principle associated with some Dieudonné-Rashevsky type problems governed by Lipcshitz functions. Later, Udrişte [5], using the *multi-time* concept, formulated and proved, under the simplified hypothesis, a maximum principle based on multiple/curvilinear integral cost functional and m-flow type PDE constraints. Treanţă and Vârsan [6] derived that solutions associated with an extended affine control system can be obtained as a limit process using solutions for a parameterized affine control system and weak small controls.

In this paper, taking into account Treanţă and Udrişte [7] and Treanţă [8], we introduce necessary conditions of optimality for a new class of optimal control problems involving signomial type constraints. For other different but connected points of view regarding this subject, the reader is directed to Mititelu and Treanţă [9] and Treanţă [10, 11].

In the following, for $x=(x^1,...,x^n)\in R^n$ we shall write $x>0$ if $x^i>0$, $i=\overline{1,n}$, and $x\geq 0$ if $x^i\geq 0$, $i=\overline{1,n}$. The set $R^n_+=\{x\in R^n : x\geq 0\}$ is said to be the *positive orthant* and, most of the times, we shall use the *open positive orthant* $\mathbf{P}^n=\{x\in R^n : x>0\}$. On the set $\mathbf{P}^n$, we consider the distinct monomials of the form $v^k=v^k(x)=(x^1)^{\alpha_{1k}}\cdots(x^n)^{\alpha_{nk}}$, $k=\overline{1,m}$, where α_{ik} are real numbers. If a^i_k are real numbers, then the functions $a^i_k v^k$, with summation upon k, are called *signomials*. The *controlled signomial dynamical systems* are defined as follows:

$$\dot{x}^i(t)=a^i_k v^k(x(t),u(t)), \quad i=\overline{1,n},$$

where $v^k(x,u):=(x^1)^{\alpha_{1k}}\cdots(x^n)^{\alpha_{nk}}(u^1)^{\gamma_{1k}}\cdots(u^r)^{\gamma_{rk}}$, $\alpha_{ik},\gamma_{\beta k}\in R$, $k=\overline{1,m}$, $i=\overline{1,n}$, $\beta=\overline{1,r}$, $t\in I\subseteq R$, and $\mathbf{P}^r\ni u=(u^\beta)$, $\beta=\overline{1,r}$, is a *control*.

Further, let us consider an optimal control problem based on a simple integral cost functional, constrained by a controlled signomial dynamical system:

$$\max_{u(\cdot),x_{t_0}} I(u(\cdot)) = \int_0^{t_0} X(x(t),u(t))\,dt \tag{1.1}$$

subject to

$$\dot{x}^i(t) = a^i_k v^k(x(t),u(t)), \quad i=\overline{1,n},\ k=\overline{1,m} \tag{1.2}$$

$$u(t) \in U,\ \forall t \in [0,t_0]; \quad x(0)=x_0,\ x(t_0)=x_{t_0}. \tag{1.3}$$

In the aforementioned optimal control problem we used the following terminology and notations: $t \in [0,t_0]$ is *parameter of evolution*, or the *time*; $[0,t_0] \subset R_+$ is the *time interval*; $x:[0,t_0] \to \mathbf{P}^n$, $x(t) = (x^i(t)),\ i=\overline{1,n}$, is a C^2-class function, called *state vector*; $u:[0,t_0] \to \mathbf{P}^r$, $u(t) = (u^\beta(t)),\ \beta = \overline{1,r}$, is a continuous *control vector*; U is the set of all admissible controls; the *running cost* $X(x(t),u(t))$ is a C^1-class function, called *autonomous Lagrangian*.

Through this work, the summation over the repeated indices is assumed. Further, we introduce the *Lagrange multiplier* $p(t) = (p_i(t))$, also called *co-state variable (vector)*, and a new Lagrange function

$$L(x(t),u(t),p(t)) = X(x(t),u(t)) + p_i(t)\left[a^i_k v^k(x(t),u(t)) - \dot{x}^i(t)\right].$$

In this way, we change the initial optimal control problem into a free optimization problem

$$\max_{u(\cdot),x_{t_0}} \int_0^{t_0} L(x(t),u(t),p(t))\,dt$$

subject to

$$u(t) \in U,\ p(t) \in P,\ \forall t \in [0,t_0]$$

$$x(0)=x_0,\ x(t_0)=x_{t_0},$$

where P is the set of co-state variables, which will be defined later. The *control Hamiltonian*

$$H(x(t),u(t),p(t)) = X(x(t),u(t)) + a^i_k p_i(t) v^k(x(t),u(t)),$$

or, equivalently,

$$H = L + p_i \dot{x}^i \quad (modified\ Legendrian\ duality)$$

permits us to rewrite the previous optimal control problem as follows

$$\max_{u(\cdot),x_{t_0}} \int_0^{t_0} \left[H(x(t),u(t),p(t)) - p_i(t)\dot{x}^i(t)\right] dt$$

subject to

$$u(t) \in U,\ p(t) \in P,\ \forall t \in [0,t_0]$$

$$x(0)=x_0,\ x(t_0)=x_{t_0}.$$

1.1 Variational and adjoint differential systems

Let us suppose that (1.2) is satisfied. Fix the control $u(t)$ and a corresponding solution $x(t)$ of (1.2). Let $x(t,\varepsilon)$ be a differentiable variation of the state variable $x(t)$, fulfilling

$$\dot{x}^i(t,\varepsilon) = a^i_k v^k\left(x(t,\varepsilon),u(t)\right)$$

$$x(t,0) = x(t), \quad i = \overline{1,n}.$$

Denote by $y^i(t) := x^i_\varepsilon(t,0)$. Taking the partial derivative with respect to ε, evaluating at $\varepsilon = 0$, we obtain the following system

$$\dot{y}^i(t) = a^i_k v^k_{x^j}\left(x(t),u(t)\right)\cdot y^j(t),$$

called *variational differential system*. The differential system

$$\dot{p}_j(t) = -a^i_k p_i(t) v^k_{x^j}\left(x(t),u(t)\right)$$

is called the *adjoint differential system* of the previous variational differential system since the scalar product $p_i(t)\cdot y^i(t)$ is a first integral of the two systems. Indeed, we have

$$\frac{d}{dt}\left[p_i(t)\cdot y^i(t)\right] = 0.$$

2. Main result

Let $\hat{u}(t) = \left(\hat{u}^\beta(t)\right)$, $\beta = \overline{1,r}$, be a continuous control vector defined on the closed interval $[0,t_0]$, with $\hat{u}(t) \in Int\,U$, which is an optimal point for the aforementioned control problem. Consider $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t)$ a variation of the optimal control vector $\hat{u}(t)$, where h is an arbitrary continuous vector function. We have $\hat{u}(t) \in Int\,U$ and, since a continuous function on a compact interval $[0,t_0]$ is bounded, there exists $\varepsilon_h > 0$ such that $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t) \in Int\,U$, $\forall\ |\varepsilon| < \varepsilon_h$. This ε is a "small" parameter used in our variational arguments.

Define $x(t,\varepsilon)$ as the state variable corresponding to the control variable $u(t,\varepsilon)$, i.e.,

$$\dot{x}^i(t,\varepsilon) = a^i_k v^k\left(x(t,\varepsilon),u(t,\varepsilon)\right), \quad i = \overline{1,n},\ \forall t \in [0,t_0]$$

and $x(0,\varepsilon) = x_0$. As well, consider (for $|\varepsilon| < \varepsilon_h$) the function (integral with parameter)

$$I(\varepsilon) := \int_0^{t_0} X\left(x(t,\varepsilon),u(t,\varepsilon)\right)dt.$$

Since $\hat{u}(t)$ is an optimal control variable we get $I(0) \geq I(\varepsilon)$, $\forall\ |\varepsilon| < \varepsilon_h$. Also, for any continuous vector function $p(t) = (p_i)(t) : [0,t_0] \to R^n$, we have

$$\int_0^{t_0} p_i(t)\left[a^i_k v^k\left(x(t,\varepsilon),u(t,\varepsilon)\right) - \dot{x}^i(t,\varepsilon)\right]dt = 0.$$

The variations involve

$$L\left(x(t,\varepsilon),u(t,\varepsilon),p(t)\right) = X\left(x(t,\varepsilon),u(t,\varepsilon)\right)$$

$$+p_i(t)\left[a^i_k v^k\left(x(t,\varepsilon),u(t,\varepsilon)\right) - \dot{x}^i(t,\varepsilon)\right]$$

and the associated function (integral with parameter)

$$I(\varepsilon) = \int_0^{t_0} L\left(x(t,\varepsilon),u(t,\varepsilon),p(t)\right)dt.$$

Now, assume that the co-state variable $p(t) = (p_i(t))$ is of C^1-class. The control Hamiltonian with variations

$$H\left(x(t,\varepsilon),u(t,\varepsilon),p(t)\right) = X\left(x(t,\varepsilon),u(t,\varepsilon)\right) + a^i_k p_i(t) v^k\left(x(t,\varepsilon),u(t,\varepsilon)\right)$$

changes the above integral with parameter as follows

$$I(\varepsilon) = \int_0^{t_0}\left[H\left(x(t,\varepsilon),u(t,\varepsilon),p(t)\right) - p_i(t)\dot{x}^i(t,\varepsilon)\right]dt.$$

Differentiating with respect to ε, evaluating at $\varepsilon = 0$, and using the formula of integration by parts, it follows

$$I'(0) = \int_0^{t_0} \left[H_{x^j}\left(x(t),\hat{u}(t),p(t)\right) + \dot{p}_j(t)\right] \cdot x^j_\varepsilon(t,0)dt$$

$$+ \int_0^{t_0} H_{u^\beta}\left(x(t),\hat{u}(t),p(t)\right) \cdot h^\beta(t)dt - \left(p_i(t) \cdot x^i_\varepsilon(t,0)\right)|_0^{t_0},$$

where $x(t)$ is the state variable corresponding to the optimal control variable $\hat{u}(t)$. We must have $I'(0) = 0$, for any continuous vector function $h(t) = \left(h^\beta(t)\right)$, $\beta = \overline{1,r}$. Also, the functions $x^i_\varepsilon(t,0)$ solve the following Cauchy problem

$$\nabla_t x^i_\varepsilon(t,0) = a^i_k v^k_x\left(x(t,0),u(t)\right) \cdot x_\varepsilon(t,0) + a^i_k v^k_u\left(x(t,0),u(t)\right) \cdot h(t)$$

$$t \in [0,t_0], \quad x_\varepsilon(0,0) = 0.$$

Consequently, we obtain

$$\frac{\partial H}{\partial u^\beta}\left(x(t),\hat{u}(t),p(t)\right) = 0, \quad \forall t \in [0,t_0]. \tag{2.1}$$

Using the adjoint differential system introduced in *Sect.* 1.1, we define P as the set of solutions for the following problem

$$\dot{p}_j(t) = -\frac{\partial H}{\partial x_j}\left(x(t),\hat{u}(t),p(t)\right), \quad p_j(t_0) = 0, \ \forall t \in [0,t_0]. \tag{2.2}$$

Moreover, we get

$$\dot{x}^j(t) = \frac{\partial H}{\partial p_j}\left(x(t),\hat{u}(t),p(t)\right), \quad x(0) = x_0, \ \forall t \in [0,t_0]. \tag{2.3}$$

Remark 2.1. *The algebraic system (2.1) describes the critical points of the control Hamiltonian H with respect to the control vector $u = (u^\beta)$.*

Now, taking into account the previous computations, we are able to formulate the main result of this paper.

Theorem 2.2. *(**Simplified maximum principle**) Let assume that the problem of maximizing the functional (1.1), subject to the signomial constraints (1.2) and to the conditions (1.3), with X, v^k of C^1-class, has an interior solution $\hat{u}(t) \in Int U$ determining the optimal state variable $x(t) = \left(x^i(t)\right)$. Then there exists the C^1-class co-state variable $p = (p_i)$, defined on the closed interval $[0,t_0]$, such that the relations (1.2), (2.1), (2.2) and (2.3) hold.*

Further, by using the new Lagrange function L and the above mentioned theorem, the following result is obvious.

Corollary 2.3. *Consider the problem of maximizing the functional (1.1), subject to the signomial constraints (1.2) and to the conditions (1.3), with X, v^k of C^1-class, has an interior solution $\hat{u}(t) \in Int U$ determining the optimal state variable $x(t) = \left(x^i(t)\right)$. Then there exists the C^1-class co-state variable $p = (p_i)$, defined on the closed interval $[0,t_0]$, such that*

$$\dot{x}^i(t) = a^i_k v^k\left(x(t),u(t)\right), \quad i = \overline{1,n}, \ k = \overline{1,m}$$

and the following Euler-Lagrange ODEs associated with the Lagrangian L

$$\frac{\partial L}{\partial u^\beta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}^\beta} = 0, \quad \beta = \overline{1,r}$$

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0, \quad \frac{\partial L}{\partial p_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{p}_i} = 0, \quad i = \overline{1,n}$$

are satisfied.

3. Conclusion and further development

In this paper, using the concepts of *variational differential system*, *adjoint differential system* and *modified Legendrian duality*, we have formulated and proved a simplified maximum principle associated with a signomial constrained optimal control problem. An immediate perspective of the present paper is to obtain the Euler-Lagrange and Hamilton ODEs, with many applications in Optimization Theory and Mechanics.

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Two Positive Solutions for a Fourth-Order Three-Point BVP with Sign-Changing Green's Function

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Abstract
This paper concerns the fourth-order three-point boundary value problem (BVP)

$$u^{(4)}(t) = f(t,u(t)), \quad t \in [0,1],$$

$$u'(0) = u''(0) = u(1) = 0, \ \alpha u''(1) - u'''(\eta) = 0,$$

where $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $\alpha \in [0,1)$ and $\eta \in \left[\frac{2\alpha+10}{15-2\alpha}, 1\right)$. Although the corresponding Green's function is sign-changing, we still obtain the existence of at least two positive and decreasing solutions under some suitable conditions on f by applying the two-fixed-point theorem due to Avery and Henderson. An example is also given to illustrate the main results.

Keywords: Completely continuous, fourth-order boundary value problem, Green's function, two positive solutions.
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1. Introduction

Fourth-order differential equations arise from a variety of different areas of applied mathematics, physics, engineering, material mechanics, fluid mechanics and so on [1, 2]. Many authors studied the existence of positive solutions for fourth-order m-point boundary value problems using different methods see [3]-[6] and the references therein.

In recent years, the existence and multiplicity of positive solutions of the boundary value problems with sign-changing Green's function has received much attention from many authors; see [7, 8, 9, 10, 11, 12, 13, 14].

In [15] Li, Sun and Kong considered the following BVP with an indefinitely signed Green's function

$$\begin{cases} u'''(t) = a(t) f(t,u(t)) = 0, \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad u''(\eta) = 0, \end{cases}$$

where $\alpha \in [0,2)$, $\eta \in \left[\frac{\sqrt{121+24\alpha}}{3(4+\alpha)}, 1\right)$. By means of the Guo-Krasnoselskii's fixed point theorem, existence results of positive solutions were obtained.

In [16] Xie et al. discuss the existence of triple positive solutions for the BVP

$$\begin{cases} u'''(t) = f(t, u(t)) = 0, & t \in (0,1), \\ u'(0) = 0,\ u(1) = \alpha u(\eta),\ u''(\eta) = 0, \end{cases}$$

where $0 < \alpha < 1$, $\max\left\{\frac{1+2\alpha}{1+4\alpha}, \frac{1}{2-\alpha}\right\} < \eta < 1$. The main tool used is the fixed point theorem due to Avery and Peterson.

It is to be observed that there are other types of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases; see the papers [17]-[21].

Inspired and motivated by the works mentioned above, in this paper we will study the following nonlinear fourth-order three-point BVP with sign-changing Green's function

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), & t \in [0,1] \\ u'(0) = u''(0) = u(1) = 0,\ \alpha u''(1) - u'''(\eta) = 0, \end{cases} \tag{1.1}$$

where $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $\alpha \in [0,1)$ and $\eta \in \left[\frac{2\alpha+10}{15-2\alpha}, 1\right)$. By imposing suitable conditions on f, we obtain the existence of at least two positive and decreasing solutions for the BVP (1.1).

To end this section, we state some fundamental definitions and the two-fixed-point theorem due to Avery and Henderson [22].

Let K be a cone in a real Banach space E.

Definition 1.1. *A functional $\psi : K \to \mathbb{R}$ is said to be increasing on K provided $\psi(x) \le \psi(y)$ for all $x, y \in K$ with $x \le y$, where $x \le y$ if and only if $y - x \in K$.*

Definition 1.2. *Let $\gamma : K \to [0,+\infty)$ be continuous. For each $d > 0$, one defines the set*

$$K(\gamma, d) = \{u \in K : \gamma(u) < d\}.$$

Theorem 1.3. *[22] Let ψ and γ be increasing, nonnegative, and continuous functionals on K, and let ω be a nonnegative continuous functional on K with $\omega(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(u) \le \omega(u) \le \psi(u), \qquad \|u\| \le M\gamma(u)$$

for all $u \in \overline{K(\gamma, c)}$. Suppose there exist a completely continuous operator $T : \overline{K(\gamma, c)} \to K$ and $0 < a < b < c$ such that

$$\omega(\lambda u) \le \lambda \omega(u) \quad for\ 0 \le \lambda \le 1,\ u \in \partial K(\omega, b),$$

and

(1) $\gamma(Tu) > c$ *for all* $u \in \partial K(\gamma, c)$;

(2) $\omega(Tu) < bc$ *for all* $u \in \partial K(\omega, b)$;

(3) $K(\psi, a) \ne \emptyset$ *and* $\psi(Tu) > a$ *for all* $u \in \partial K(\psi, a)$.

Then T has at least two fixed points u_1 and u_2 in $\overline{K(\gamma, c)}$ such that

$$a < \psi(u_1) \text{ with } \omega(u_1) < b,$$
$$b < \omega(u_2) \text{ with } \gamma(u_2) < c.$$

2. Preliminaries and lemmas

Let Banach space $E = C[0,1]$ be equipped with the norm $\|u\| = \max_{t \in [0,1]} |u(t)|$.

For the BVP

$$\begin{cases} u^{(4)}(t) = 0, & t \in (0,1), \\ u'(0) = u''(0) = u(1) = 0,\ \alpha u''(1) - u'''(\eta) = 0, \end{cases} \tag{2.1}$$

we have the following lemma.

Lemma 2.1. *The BVP (2.1) has only a trivial solution.*

Proof. It is simple to check. □

Now, for any $y \in E$, we consider the BVP

$$\begin{cases} u^{(4)}(t) = y(t) \quad t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, \ \alpha u''(1) - u'''(\eta) = 0, \end{cases} \tag{2.2}$$

After a direct computation, one may obtain the expression of Green's function of the BVP (2.2) as follows:
For $s \geq \eta$

$$G(t,s) = \begin{cases} -\frac{\alpha(1-t^3)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}, & 0 \leq t \leq s \leq 1, \\ \frac{(t-s)^3}{6} - \frac{\alpha(1-t^3)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}, & 0 \leq s \leq t \leq 1, \end{cases}$$

and for $s < \eta$

$$G(t,s) = \begin{cases} \frac{(1-t^3)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}, & 0 \leq t \leq s \leq 1, \\ \frac{(t-s)^3}{6} + \frac{(1-t^3)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Remark 2.2. *$G(t,s)$ has the following properties:*

$$G(t,s) \geq 0 \quad for \ 0 \leq s < \eta, \qquad G(t,s) \leq 0 \quad for \ \eta \leq s \leq 1.$$

Moreover, for $s \geq \eta$,

$$\max\{G(t,s) : t \in [0,1]\} = G(1,s) = 0,$$

$$\min\{G(t,s) : t \in [0,1]\} = G(0,s) = -\frac{\alpha(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}$$

and for $s < \eta$,

$$\max\{G(t,s) : t \in [0,1]\} = G(0,s) = \frac{1-\alpha(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6},$$

$$\min\{G(t,s) : t \in [0,1]\} = G(1,s) = 0.$$

Let

$$K_0 = \{y \in E : y(t) \ is\, nonnegative\, and\, decreasing\, on\ [0,1]\}.$$

Then K_0 is a cone in E.

Lemma 2.3. *Let $y \in K_0$ and $u(t) = \int_0^1 G(t,s)y(s)\,ds$, $t \in [0,1]$. Then u is the unique solution of the BVP (2.2) and $u \in K_0$. Moreover, $u(t)$ is concave on $[0,\eta]$.*

Proof. For $t \in [0,\eta]$, we have

$$\begin{aligned} u(t) = &\int_0^t \left[\frac{(t-s)^3}{6} + \frac{(1-t^3)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds \\ &+ \int_t^\eta \left[\frac{(1-t^3)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds \\ &+ \int_\eta^1 \left[-\frac{\alpha(1-t^3)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds. \end{aligned}$$

Since $y \in K_0$ and $\eta \geq \frac{2\alpha+10}{15-2\alpha}$ implies that $\eta > \frac{3\alpha}{4+2\alpha}$, we get

$$\begin{aligned}
u'(t) &= -\frac{\alpha t^2}{2(1-\alpha)}\int_0^\eta sy(s)\,ds - \frac{t^2}{2}\int_t^\eta y(s)\,ds \\
&\quad + \int_0^t \left[\frac{s^2-2ts}{2}\right] y(s)\,ds + \frac{\alpha t^2}{2(1-\alpha)}\int_\eta^1 (1-s)\,y(s)\,ds \\
&\leq y(\eta)\,[-\frac{\alpha t^2}{2(1-\alpha)}\int_0^\eta s ds - \frac{t^2}{2}\int_t^\eta y(s)\,ds \\
&\quad + \int_0^t \left[\frac{s^2-2ts}{2}\right] ds + \frac{\alpha t^2}{2(1-\alpha)}\int_\eta^1 \left(1-s(1-s)^3\right) ds] \\
&= \frac{t^2}{2} y(\eta)\,[\frac{\alpha(1-2\eta)}{2(1-\alpha)} + \frac{t}{3} - \eta] \\
&\leq \frac{t^2}{2} y(\eta)\,[\frac{\alpha(1-2\eta)}{2(1-\alpha)} - \frac{2\eta}{3}] \\
&\leq 0, \quad t \in [0,\eta].
\end{aligned}$$

At the same time, $y \in K_0$ and $\eta \geq \frac{2\alpha+10}{15-2\alpha} > \frac{1}{2}$ shows that

$$\begin{aligned}
u''(t) &= -\frac{\alpha t}{1-\alpha}\int_0^\eta sy(s)\,ds - t\int_t^\eta y(s)\,ds \\
&\quad - \int_0^t sy(s)\,ds + \frac{\alpha t}{1-\alpha}\int_\eta^1 (1-s)\,y(s)\,ds \\
&\leq y(\eta)\,[-\frac{\alpha t}{1-\alpha}\int_0^\eta s ds - t\int_t^\eta ds \\
&\quad - \int_0^t s ds + \frac{\alpha t}{1-\alpha}\int_\eta^1 (1-s)\,ds] \\
&= t y(\eta)\left[\frac{\alpha(1-2\eta)}{2(1-\alpha)} + \frac{t}{2} - \eta\right] \\
&\leq t y(\eta)\left[\frac{\alpha(1-2\eta)}{2(1-\alpha)}\right] \\
&\leq 0, \quad t \in [0,\eta].
\end{aligned}$$

For $t \in [\eta, 1)$, we have

$$\begin{aligned}
u(t) &= \int_0^\eta \left[\frac{(t-s)^3}{6} - \frac{\left(1-t^3\right)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds \\
&\quad + \int_\eta^t \left[\frac{(t-s)^3}{6} - \frac{\alpha\left(1-t^3\right)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds \\
&\quad + \int_t^1 \left[-\frac{\alpha\left(1-t^3\right)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds.
\end{aligned}$$

In view of $y \in K_0$ and $\eta > \frac{1}{2}$, we get

$$\begin{aligned}
u'(t) &= -\frac{\alpha t^2}{2(1-\alpha)}\int_0^\eta sy(s)\,ds + \int_0^\eta \left[\frac{s^2-2ts}{2}\right] y(s)\,ds \\
&\quad + \int_\eta^t \left[\frac{(t-s)^2}{2}\right] y(s)\,ds + \frac{\alpha t^2}{2(1-\alpha)}\int_\eta^1 (1-s)\,y(s)\,ds \\
&\le y(\eta)\,[-\frac{\alpha t^2}{2(1-\alpha)}\int_0^\eta s ds + \int_0^\eta \left[\frac{s^2-2ts}{2}\right] ds \\
&\quad + \int_\eta^t \left[\frac{(t-s)^2}{2}\right] ds + \frac{\alpha t^2}{2(1-\alpha)}\int_\eta^1 (1-s)\,ds] \\
&= \frac{t^2}{2} y(\eta) \left[\frac{\alpha(1-2\eta)}{2(1-\alpha)} + \frac{t-3\eta}{3}\right] \\
&\le \frac{t^2}{2} y(\eta)\,[\frac{\alpha(1-2\eta)}{2(1-\alpha)} + \frac{t-2\eta}{2}] \\
&= \frac{t^2}{2} y(\eta)\,[\frac{(1-2\eta)}{2(1-\alpha)}] \\
&\le 0, \quad t \in (\eta, 1].
\end{aligned}$$

Obviously, $u^{(4)}(t) = y(t)$ for $t \in [0,1]$, $u'(0) = u''(0) = u(1) = 0$, $\alpha u''(1) - u'''(\eta) = 0$. This shows that u is a solution of the BVP (2.2). The uniquess follows immediately from Lemma 2.1. Since $u'(t) \le 0$ for $t \in [0,1]$ and $u(1) = 0$, we have $u(t) \ge 0$ for $t \in [0,1]$. So, $u \in K_0$. In view of $u''(t) \le 0$ for $t \in [0,\eta]$, we know that $u(t)$ is concave on $[0,\eta]$. □

Lemma 2.4. *Let $y \in K_0$. Then the unique solution u of the BVP (2.2) satisfies*

$$\min_{t\in[0,\tau]} u(t) \ge \tau^* \|u\|,$$

where $\tau \in \left(0, \frac{1}{2}\right]$ and $\tau^* = \frac{\eta-\tau}{\eta}$.

Proof. From Lemma 2.3, we know that $u(t)$ is concave on $[0,\eta]$; thus for $t \in [0,\eta]$,

$$u(t) \ge \frac{\eta - t}{\eta} u(0) + \frac{t}{\eta} u(\eta).$$

At the same time, it follows from $u \in K_0$ that $\|u\| = u(0)$ which

$$u(t) \ge \frac{\eta - t}{\eta} \|u\|.$$

Therefore,

$$\min_{t\in[0,\tau]} u(t) = u(\tau) \ge \frac{\eta-\tau}{\eta} \|u\| = \tau^* \|u\|.$$

□

3. Main results

In what follows, we assume that f satisfies the following two conditions:

$(C1)$ For each $u \in [0,+\infty)$, the mapping $t \longmapsto f(t,u)$ is decreasing;

$(C2)$ For each $t \in [0,1]$, the mapping $u \longmapsto f(t,u)$ is increasing.

Let

$$K = \left\{ u \in K_0 : \min_{t\in[0,\tau]} u(t) \ge \tau^* \|u\| \right\}.$$

Then it is easy to see that K is a cone in E.

Now, we define an operator T as follows:

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))\, ds, \quad u \in K,\ t \in [0,1].$$

First, it is obvious that if u is a fixed point of T in K, then u is a nonnegative and decreasing solution of the BVP (1.1). Next, by Lemmas 2.3 and 2.4, we know that $T: K \to K$. Furthermore, although $G(t,s)$ is not continuous, it follows from known textbook results, for example, see [23], that $T: K \to K$ is completely continuous.

For convenience, we denote

$$A = \int_0^\tau G(\eta,s)\, ds, \qquad B = \int_0^\eta G(\tau,s)\, ds.$$

Theorem 3.1. *Assume that $(C1)$ and $(C2)$ hold. Moreover, suppose that there exist numbers a, b and c with $0 < a < b < \tau^* c$ such that*

$$f(\tau,c) > \frac{c}{A}, \tag{3.1}$$

$$f\left(0, \frac{b}{\tau^*}\right) < \frac{b}{B}, \tag{3.2}$$

$$f(\tau,\tau^* a) > \frac{a}{A}. \tag{3.3}$$

Then the BVP (1.1) has at least two positive and decreasing solutions.

Proof. First, we define the increasing, nonnegative, and continuous functionals γ, ω and ψ on K as follows:

$$\gamma(u) = \min_{t\in[0,\tau]} u(t) = u(\tau),$$
$$\omega(u) = \max_{t\in[\tau,1]} u(t) = u(\tau),$$
$$\psi(u) = \max_{t\in[0,1]} u(t) = u(0).$$

Obviously, for any $u \in K$, $\gamma(u) = \omega(u) \le \psi(u)$. At the same time, for each $u \in K$, in view of $\gamma(u) = \min_{t\in[0,\tau]} u(t) \ge \tau^* \|u\|$, we have

$$\|u\| \le \frac{1}{\tau^*}\gamma(u) \quad for\, u \in K.$$

Furthermore, we also note that

$$\omega(\lambda u) = \lambda\omega(u) \quad for \quad 0 \le \lambda \le 1,\ u \in K.$$

Next, for any $u \in K$, we claim that

$$\int_\tau^1 G(\eta,s) f(s,u(s))\, ds \ge 0. \tag{3.4}$$

In fact, it follows from $(C1)$, $(C2)$, and $\eta \ge \frac{2\alpha+10}{15-2\alpha}$ that

$$\begin{aligned}
&\int_\tau^1 G(\eta,s) f(s,u(s))\,ds\\
&= \int_\tau^\eta G(\eta,s) f(s,u(s))\,ds + \int_\eta^1 G(\eta,s) f(s,u(s))\,ds\\
&\geq f(\eta,u(\eta))\left[\int_\tau^\eta G(\eta,s)\,ds + \int_\eta^1 G(\eta,s)\,ds\right]\\
&= f(\eta,u(\eta))\\
&\quad \times[\int_\tau^\eta \left(\tfrac{(\eta-s)^3}{6} + \tfrac{(1-n^3)(1-\alpha(1-s))}{6(1-\alpha)} - \tfrac{(1-s)^3}{6}\right) ds + \int_\eta^1 \left(-\tfrac{\alpha(1-s)(1-\eta^3)}{6(1-\alpha)} - \tfrac{(1-s)^3}{6}\right) ds]\\
&= \tfrac{(1-\eta)}{24(1-\alpha)} f(\eta,u(\eta))\\
&\quad \times[(3+\alpha)\eta^3 + (3-\alpha)\eta^2 + (3-\alpha)\eta - \alpha - 1 - 2\left(3\eta - 2\alpha\eta + \alpha\eta^2 - 2\alpha + 3\right)\tau^2 + 4(1-\alpha)\tau^3]\\
&= \tfrac{(1-\eta)}{24(1-\alpha)} f(\eta,u(\eta))\\
&\quad \times[(3+\alpha)\eta^3 + (3-\alpha)\eta^2 + (3-\alpha)\eta - \alpha - 1 - 2\left(3\eta - 2\alpha\eta + \alpha\eta^2 - 2\alpha + 3\right)\tau^2 - 4\alpha\tau^3]\\
&\geq \tfrac{(1-\eta)}{24(1-\alpha)} f(\eta,u(\eta)) \times [(3+\alpha)\eta^3 + \left(3 - \tfrac{3}{2}\alpha\right)\eta^2 + \tfrac{3}{2}\eta - \tfrac{\alpha}{2} - \tfrac{5}{2}]\\
&\geq \tfrac{(1-\eta)}{24(1-\alpha)} f(\eta,u(\eta)) \times [\left(\tfrac{15}{4} - \tfrac{1}{2}\alpha\right)\eta - \tfrac{\alpha}{2} - \tfrac{5}{2}]\\
&\geq 0.
\end{aligned}$$

Now, we assert that $\gamma(Tu) > c$ for all $u \in \partial K(\gamma,c)$.
To prove this, let $u \in \partial K(\gamma,c)$; that is, $u \in K$ and $\gamma(u) = u(\tau) = c$. Then

$$u(t) \geq u(\tau) = c, \quad t \in [0,\tau]. \tag{3.5}$$

Since $(Tu)(t)$ is decreasing on $[0,1]$, it follows from (3.1), (3.4), (3.5), $(C1)$ and $(C2)$ that

$$\begin{aligned}
\gamma(Tu) &= (Tu)(\tau)\\
&\geq (Tu)(\eta)\\
&= \int_0^1 G(\eta,s) f(s,u(s))\,ds\\
&\geq \int_0^\tau G(\eta,s) f(s,u(s))\,ds\\
&\geq \int_0^\tau G(\eta,s) f(\tau,c)\,ds\\
&> \frac{c}{A}\int_0^\tau G(\eta,s)\,ds = c.
\end{aligned}$$

Then, we assert that $\omega(Tu) < b$ for all $u \in \partial K(\omega,b)$.
To see this, suppose that $u \in \partial K(\omega,b)$; that is, $u \in K$ and $\omega(u) = b$. Since $\|u\| \leq \frac{1}{\tau^*}\gamma(u) = \frac{1}{\tau^*}\omega(u)$, we have

$$0 \leq u(t) \leq \|u\| \leq \frac{b}{\tau^*}, \qquad t \in [0,\eta]. \tag{3.6}$$

In view of Remark 2.2, (3.2), (3.6), $(C1)$ and $(C2)$, we get

$$\begin{aligned}
\omega(Tu) &= (Tu)(\tau)\\
&= \int_0^1 G(\tau,s) f(s,u(s))\,ds\\
&\leq \int_0^\eta G(\tau,s) f(s,u(s))\,ds\\
&\leq \int_0^\eta G(\tau,s) f\left(0,\frac{b}{\tau^*}\right) ds\\
&< \frac{b}{B}\int_0^\eta G(\tau,s)\,ds\\
&= b.
\end{aligned}$$

Finally, we assert that $K(\psi,a) \neq \emptyset$ and $\psi(Tu) > a$ for all $u \in \partial K(\psi,a)$.
In fact, the constant function $\frac{a}{2} \in K(\psi,a)$. Moreover, for $u \in \partial K(\psi,a)$, that is $u \in K$ and $\psi(u) = u(0) = a$. Then

$$u(t) \geq \tau^* \|u\| = \tau^* u(0) = \tau^* a, \qquad t \in [0,\tau]. \tag{3.7}$$

Since $(Tu)(t)$ is decreasing on $[0,1]$, it follows from (3.3), (3.4), (3.7), $(C1)$ and $(C2)$ that

$$\begin{aligned}\psi(Tu) &= (Tu)(0)\\ &\geq (Tu)(\eta)\\ &= \int_0^1 G(\eta,s) f(s,u(s))\,ds\\ &\geq \int_0^\tau G(\eta,s) f(\tau,\tau * a)\,ds\\ &> \frac{a}{A}\int_0^\tau G(\eta,s)\,ds = a.\end{aligned}$$

To sum up, all the hypotheses of Theorem 1.3 are satisfied. Hence, T has at least two fixed points u_1 and u_2; that is, the BVP (1.1) has at least two positive and decreasing solutions u_1 and u_2 satisfying

$$\begin{aligned} a &< \max_{t\in[0,1]} u_1(t) \quad \textit{with} \quad \max_{t\in[\tau,1]} u_1(t) < b\\ b &< \max_{t\in[\tau,1]} u_2(t) \quad \textit{with} \quad \min_{t\in[0,\tau]} u_2(t) < c.\end{aligned}$$

□

4. An example

Consider the BVP

$$\begin{cases} u^{(4)}(t) = f(t,u(t)), \quad t\in[0,1],\\ u'(0) = u''(0) = u(1) = 0,\ \frac{1}{5}u''(1) - u'''\left(\frac{4}{5}\right) = 0, \end{cases} \tag{4.1}$$

where

$$f(t,u) = \begin{cases} \sqrt{u} + 9752 + \frac{1-t^3}{6}, & (t,u)\in[0,1]\times[0,169],\\ 340u - 47695 + \frac{1-t^3}{6}, & (t,u)\in[0,1]\times[169,170],\\ \frac{2021u^2}{5780} + \frac{1-t^3}{6}, & (t,u)\in[0,1]\times[170,+\infty]. \end{cases}$$

Since $\alpha = \frac{1}{5}$ and $\eta = \frac{4}{5}$, if we choose $\tau = \frac{1}{3}$, then a simple calculation shows that

$$A = \frac{1603}{162000}, \quad B = \frac{994}{10125}, \quad \tau^* = \frac{7}{12}.$$

Thus, if we let $a = 80$, $b = 98.7$ and $c = 300$, then it is easy to verify that all the conditions of Theorem 3.1. are satisfied. So, it follows from Theorem 3.1 that the BVP (4.1) has at least two positive and decreasing solutions.

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Solution of Singular Integral Equations of the First Kind with Cauchy Kernel

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Abstract
In this paper an analytic method is developed for solving Cauchy type singular integral equations of the first kind, over a finite interval. Chebyshev polynomials of the first kind, $T_n(x)$, second kind, $U_n(x)$, third kind, $V_n(x)$, and fourth kind, $W_n(x)$, corresponding to respective weight functions $W^{(1)}(x) = \frac{1}{\sqrt{1-x^2}}, W^{(2)}(x) = \sqrt{1-x^2}, W^{(3)}(x) = \sqrt{\frac{1+x}{1-x}}$, and $W^{(3)}(x) = \sqrt{\frac{1-x}{1+x}}$, have been used to obtain the complete analytical solutions for four different cases.

Keywords: Singular integral equation, Cauchy Kernel, Chebyshev polynomials, Weight function
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1. Introduction

Consider the Cauchy type singular integral equations (CSIEs) of the form

$$\int_{-1}^{1} \frac{\phi(t)}{(t-x)}\, dt = f(x) \quad , \quad -1 < x < 1 \tag{1.1}$$

where $f(x)$ is given real valued function and $\phi(t)$ is unknown function to be determined. Integral equation of the form (1.1) occurs in a variety of mixed boundary value problems of mathematical physics (cf. [1], [2], [3]).

Kim [4] solved CSIEs by using Gaussian quadrature and choose the zeros of Chebyshev polynomials of first and second kinds as the collocation and abscissa points. Chakrabarti and Berge [5] have proposed an approximate method for solving CSIEs using polynomial approximation of degree n and collocation points chosen to be the zeros of the Chebyshev polynomial of the first kind for all cases. Mandal and Panja [6] obtained the numerical solution of the second kind CSIEs using Daubechies scale function. Abdulkawi [7] uses differential transform method for the numerical solution of CSIEs of the first kind. Mondal and Mandal [8] used Chebyshev polynomials of first and second kind to obtain the exact solution of a simple hypersingular integral equation.

In this paper, we present the analytical solution of CSIEs (1.1). There are four basically important and interesting cases of equation (1.1), as given by the following :

Case I. $\Phi(x)$ is unbounded at both the end-point $x = \pm 1$.

Case II. $\Phi(x)$ is unbounded at the end $x=-1$, but bounded at the end $x=+1$.

Case III. $\Phi(x)$ is bounded at the end $x=-1$, but unbounded at the end $x=+1$.

Case IV. $\Phi(x)$ is bounded at both the end-point $x=\pm 1$.

It is well known that the complete analytical solutions of equation (1.1) for the above four cases, are given by the following expressions:

Case I :

$$\phi(x)=\frac{A_0}{\sqrt{1-x^2}}-\frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}f(t)}{t-x}dt \tag{1.2}$$

where A_0 is an arbitrary constant,

Case II :

$$\phi(x)=-\frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{f(t)}{t-x}dt, \tag{1.3}$$

Case III :

$$\phi(x)=-\frac{1}{\pi^2}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{f(t)}{t-x}dt, \tag{1.4}$$

Case IV : In this case the solution exists iff $\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt=0$ and the solution is given by

$$\phi(x)=-\frac{\sqrt{1-x^2}}{\pi^2}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}(t-x)}dt. \tag{1.5}$$

2. Chebyshev polynomials and their integral transforms

The Chebyshev polynomials of the first kind $T_n(x)$, second kind $U_n(x)$, third kind $V_n(x)$, and fourth kind $W_n(x)$, $n=0,1,2,.....$ are defined (cf. [9]) as follows :

$$T_n(x)=cosn\theta\ , \quad U_n(x)=\frac{sin(n+1)\theta}{sin\theta}\ ,$$

$$V_n(x)=\frac{cos(n+\frac{1}{2})\theta}{cos\frac{1}{2}\theta}\ , \quad W_n(x)=\frac{sin(n+\frac{1}{2})\theta}{sin\frac{1}{2}\theta}\ ,$$

where $x=cos\theta, 0\le\theta\le\pi$.

It is well known that the Chebyshev polynomials of first and second kinds are integral transforms of each other with respect to weighted Hilbert kernels, as given by

$$\int_{-1}^{1}\frac{T_n(t)}{\sqrt{1-t^2}(t-x)}dt=\pi U_{n-1}(x)\ ,\ \ -1<x<1\ , \tag{2.1}$$

and

$$\int_{-1}^{1}\frac{\sqrt{1-t^2}U_n(t)}{(t-x)}dt=-\pi T_{n+1}(x)\ ,\ \ -1<x<1\ . \tag{2.2}$$

Here the integral is to be interpreted as a Cauchy principal value integral.

It is further known that the third and fourth kind polynomials are similarly related :

$$\int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{V_n(t)}{(t-x)} dt = \pi W_n(x) \ , \ \ -1 < x < 1 \ , \tag{2.3}$$

and

$$\int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{W_n(t)}{(t-x)} dt = \pi V_n(x) \ , \ \ -1 < x < 1 \ . \tag{2.4}$$

Further, we will also use the result

$$\int_{-1}^{1} \frac{T_n(t)}{\sqrt{1-t^2}} dt = \begin{cases} \pi & , n = 0 \\ 0 & \text{otherwise} \ . \end{cases} \tag{2.5}$$

3. Method of the solution

Case I.

To obtain the solution of the integral equation (1.1) satisfying the end conditions given in Case I, we assume an expansion of $\phi(x)$ $(-1 \le x \le 1)$ in terms of Chebyshev polynomials of first kind $T_n(x)$ as

$$\phi(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} a_n T_n(x) \ , \ -1 < x < 1 \tag{3.1}$$

where $a_n (n = 0, 1, 2, ...)$ are unknown constants. Substituting (3.1) in the left side of (1.1) and assuming the validity of interchange of the order of integration and summation we obtain

$$\sum_{n=1}^{\infty} a_n [\pi U_{n-1}(t)] = f(t) \ , \ -1 < t < 1 \tag{3.2}$$

wherein the results

$$\int_{-1}^{1} \frac{T_0(t)}{\sqrt{1-t^2}(t-x)} dt = 0$$

and (2.1) have been utilized.

Multiplying both sides of equation (3.2) by $\frac{\sqrt{1-t^2}}{t-x}$ and then integrating with respect to t from -1 to 1 we obtain

$$\pi \sum_{n=1}^{\infty} a_n \int_{-1}^{1} \frac{\sqrt{1-t^2} U_{n-1}(t)}{(t-x)} dt = \int_{-1}^{1} \frac{\sqrt{1-t^2} f(t)}{(t-x)} dt \ , \ -1 < t < 1. \tag{3.3}$$

Now using the result (2.2) in (3.3), we obtain

$$\sum_{n=0}^{\infty} a_n T_n(x) = a_0 - \frac{1}{\pi^2} \int_{-1}^{1} \frac{\sqrt{1-t^2} f(t)}{(t-x)} dt \ , \ -1 < t < 1. \tag{3.4}$$

Using equation (3.4) in equation (3.1), we obtain the solution of the integral equation (1.1) as given by (1.2).

Case II.

In this case $\phi(x)$ can be written in terms of Chebyshev polynomials of fourth kind $W_n(x)$ as

$$\phi(x) = \sqrt{\frac{1-x}{1+x}} \sum_{n=0}^{\infty} b_n W_n(x)\ ,\ -1 < x < 1 \tag{3.5}$$

where $b_n(n = 0, 1, 2, ...)$ are unknown constants.

In a similar manner we obtain

$$\sum_{n=0}^{\infty} b_n[-\pi V_n(t)] = f(t)\ ,\ -1 < t < 1 \tag{3.6}$$

wherein the result (2.4) has been utilized.

Multiplying both sides of equation (3.6) by $\sqrt{\frac{1+t}{1-t}}\frac{1}{t-x}$ and then integrating with respect to t from -1 to 1 we obtain

$$-\pi \sum_{n=0}^{\infty} b_n \int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{V_n(t)}{t-x} dt = \int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{f(t)}{t-x} dt\ ,\ -1 < t < 1\ . \tag{3.7}$$

Now, using the result (2.3) in equation (3.7), we obtain

$$\sum_{n=0}^{\infty} b_n W_n(x) = -\frac{1}{\pi^2} \int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{f(t)}{t-x} dt\ ,\ -1 < t < 1\ . \tag{3.8}$$

Using equation (3.8) in equation (3.5), we obtain the solution of the integral equation (1.1) as given by (1.3).

Case III.

Here $\phi(x)$ can be written in terms of Chebyshev polynomials of third kind $V_n(x)$ as

$$\phi(x) = \sqrt{\frac{1+x}{1-x}} \sum_{n=0}^{\infty} c_n V_n(x)\ ,\ -1 < x < 1 \tag{3.9}$$

where $c_n(n = 0, 1, 2, ...)$ are unknown constants.

In a similar way we obtain

$$\sum_{n=0}^{\infty} c_n[\pi W_n(t)] = f(t)\ ,\ -1 < t < 1 \tag{3.10}$$

wherein the result (2.3) has been used.

Multiplying both sides of equation (3.10) by $\sqrt{\frac{1-t}{1+t}}\frac{1}{t-x}$ and then integrating with respect to t from -1 to 1 we obtain

$$\pi \sum_{n=0}^{\infty} c_n \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{W_n(t)}{t-x} dt = \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{f(t)}{t-x} dt\ ,\ -1 < t < 1. \tag{3.11}$$

Now, using the result (2.4) in equation (3.11), we obtain

$$\sum_{n=0}^{\infty} c_n V_n(x) = -\frac{1}{\pi^2} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{f(t)}{t-x} dt\ ,\ -1 < t < 1. \tag{3.12}$$

Using equation (3.12) in equation (3.9), we obtain the solution of the integral equation (1.1) as given by (1.4).

Case IV.

To obtain the bounded solution of the integral equation (1.1), we can expand $\phi(x)$ $(-1 \leq x \leq 1)$ in terms of Chebyshev polynomials of second kind $U_n(x)$ as

$$\phi(x) = \sqrt{1-x^2} \sum_{n=0}^{\infty} d_n U_n(x) \, , \; -1 < x < 1 \tag{3.13}$$

where $d_n (n = 0, 1, 2, ...)$ are unknown coefficients.

Proceeding in the same manner we obtain

$$\sum_{n=0}^{\infty} d_n [-\pi T_{n+1}(t)] = f(t) \, , \tag{3.14}$$

wherein the result (2.2) has been utilized.

By using the result (2.5) in (3.14) we obtain

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = 0$$

which is the solvability criterion for the existence of the bounded solution.

Multiplying both sides of equation (3.14) by $\frac{1}{\sqrt{1-t^2}(t-x)}$ and then integrating with respect to t from -1 to 1 we obtain

$$-\pi \sum_{n=0}^{\infty} d_n \int_{-1}^{1} \frac{T_{n+1}(t)}{\sqrt{1-t^2}(t-x)} dt = \int_{-1}^{1} \frac{\sqrt{1-t^2} f(t)}{\sqrt{1-t^2}(t-x)} dt \, , \; -1 < t < 1. \tag{3.15}$$

Now using the result (2.1) in equation (3.15), we obtain

$$\sum_{n=0}^{\infty} d_n U_n(x) = -\frac{1}{\pi^2} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}(t-x)} dt \, , \; -1 < t < 1. \tag{3.16}$$

Using equation (3.16) in equation (3.13), we obtain the solution of the integral equation (1.1) as given by (1.5).

4. Illustrative examples

Consider the singular integral equation

$$\int_{-1}^{1} \frac{\phi(t)}{(t-x)} dt = x^4 + 5x^3 + 2x^2 + x - (\frac{11}{8}) \; , \quad -1 < x < 1 \tag{4.1}$$

with the exact solutions for different cases as given by (cf. [10])

Case (I) : $\phi(t) = \frac{1}{\pi\sqrt{1-t^2}} \left[t^5 + 5t^4 + \frac{3}{2}(t^3 - t^2) - \frac{5}{2}t - \frac{9}{8}\right]$.

Case (II) : $\phi(t) = -\frac{1}{\pi}\sqrt{\frac{1-t}{1+t}} \left[t^4 + 6t^3 + \frac{15}{2}t^2 + 6t + \frac{7}{2}\right]$.

Case (III) : $\phi(t) = \frac{1}{\pi}\sqrt{\frac{1+t}{1-t}} \left[t^4 + 4t^3 - \frac{5}{2}t^2 + t - \frac{7}{2}\right]$.

Case (IV) : $\phi(t) = -\frac{1}{\pi}\sqrt{1-t^2} \left[t^3 + 5t^2 + \frac{5}{2}t + \frac{7}{2}\right]$.

Here $f(x) = x^4 + 5x^3 + 2x^2 + x - (\frac{11}{8})$. For the solution of the integral equation (4.1) satisfying the end conditions given in Case I, we can write $\phi(x)$ in the form given by (3.1). In a similar manner, we can obtain the expression given by (3.4).

Using Eq. (3.4) in Eq. (3.1) and then using the result

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}t^k}{(t-x)} dt = -\pi x^{k+1} + \sum_{i=0}^{k-1} \frac{1+(-1)^i}{4} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{i+1}{2})}{\Gamma(\frac{i+4}{2})} x^{k-i-1}, j = 1,2,..,$$

we obtain the solution of the IE (4.1) given in Case I.

In a similar way we can obtain the solutions of the integral equation (4.1) for all the other cases by using the methods described separately.

Remark 4.1. *The method of solution for obtaining the solution of (1.1) for the Case I is well known in the literature but is given here for the sake of completeness. The methods adopted for other cases, although simple, appear to be not known, and that is why these are given here.*

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