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# Another Two-Parameter Sujatha Distribution with Properties and Applications

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**Abstract**

In this paper another two-parameter Sujatha distribution (ATPSD), which includes exponential distribution and Sujatha distribution as particular cases, has been proposed. Statistical properties including shapes for varying values of parameters, moments, coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability of ATPSD have been discussed. The method of moment estimation and the method of maximum likelihood estimation have been discussed for estimating its parameters. Finally, applications of ATPSD have been discussed with two real lifetime datasets.

## 1. Introduction

The statistical analysis and modeling of lifetime data are crucial in almost all areas of knowledge including medical science, engineering, behavioral science, insurance and finance among others. The two important one parameter lifetime distributions for modeling lifetime data were exponential distribution and Lindley distribution, introduced by Lindley [1]. The Lindley distribution is defined by its probability density function (pdf) and cumulative distribution function (cdf)

$$f_1(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};\quad x>0,\ \theta>0$$

$$F_1(x;\theta)=1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x};\quad x>0,\ \theta>0$$

where $\theta$ is a scale parameter. Ghitany et al [2] have discussed statistical properties, estimation of parameter and application of Lindley distribution. Shanker et al [3] have detailed critical study on applications of exponential and Lindley distributions for modeling lifetime data from biomedical science and engineering and observed that in majority of datasets these two distributions are not suitable. In search for a new lifetime distribution which gives a better fit than both exponential and Lindley distributions, Shanker [4] proposed Sujatha distribution defined by its pdf and cdf

$$f_2(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x};\ x>0,\ \theta>0 \tag{1.1}$$

$$F_2(x;\theta)=1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x};\ x>0,\ \theta>0$$

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where $\theta$is a scale parameter. It should be noted that Lindley distribution is a convex combination of exponential$(\theta)$ and a gamma$(2,\theta)$ distributions whereas Sujatha distribution is a convex combination of exponential$(\theta)$, a gamma$(2,\theta)$ and a gamma$(3,\theta)$ distributions.

The first four moments about origin and central moments of Sujatha distribution obtained by Shanker [4] are

$$\mu_1^{'}=\frac{\theta^2+2\theta+6}{\theta(\theta^2+\theta+2)} \quad \mu_2^{'}=\frac{2(\theta^2+3\theta+12)}{\theta^2(\theta^2+\theta+2)} \quad \mu_3^{'}=\frac{6(\theta^2+4\theta+20)}{\theta^3(\theta^2+\theta+2)} \quad \mu_4^{'}=\frac{24(\theta^2+5\theta+30)}{\theta^4(\theta^2+\theta+2)}$$

$$\mu_2=\frac{\theta^4+4\theta^3+18\theta^2+12\theta+12}{\theta^2(\theta^2+\theta+2)^2}$$

$$\mu_3=\frac{2(\theta^6+6\theta^5+36\theta^4+44\theta^3+54\theta^2+36\theta+24)}{\theta^3(\theta^2+\theta+2)^3}$$

$$\mu_4=\frac{3(3\theta^8+24\theta^5+172\theta^6+376\theta^5+736\theta^4+864\theta^3+912\theta^2+480\theta+240)}{\theta^4(\theta^2+\theta+2)^4}$$

Shanker [4] has discussed its important properties including shapes of density function for varying values of parameter, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability. Shanker [4] discussed the maximum likelihood estimation of parameter and showed the applications of Sujatha distribution to model lifetime data from biomedical science and engineering. Shanker [5] has introduced Poisson- Sujatha distribution (PSD), a Poisson mixture of Sujatha distribution, and studied its properties, estimation of parameter and applications to model count data. Shanker and Hagos [6] have discussed zero-truncated Poisson- Sujatha distribution (ZTPSD) and applications for modeling count data excluding zero counts. Shanker and Hagos [7] have also studied size-biased Poisson- Sujatha distribution and its applications for count data excluding zero counts.

Shanker [8] proposed a two-parameter quasi Sujatha distribution (QSD), which includes Sujatha distribution and size-biased Lindley distribution (SBLD) as particular cases for $\alpha=\theta$ and $\alpha=0$, respectively and defined by its pdf and cdf

$$f_3(x;\theta,\alpha)=\frac{\theta^2}{\alpha\theta+\theta+2}\left(\alpha+\theta x+\theta x^2\right)e^{-\theta x}; \quad x>0,\ \theta>0,\ \alpha\geq 0$$

$$F_3(x;\theta,\alpha)=1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\alpha\theta+\theta+2}\right]e^{-\theta x};\ x>0,\ \theta>0,\ \alpha\geq 0.$$

Statistical properties including moments based measures, hazard rate function, mean residual life function, stochastic ordering, mean deviation, Bonferroni and Lorenz curves, stress-strength reliability along with estimation of parameters using both the method of moments and the method of maximum likelihood and applications of QSD have been discussed by Shanker [8]. It has been established through a real lifetime data that QSD gives much closer fit over one parameter exponential, Lindley and Sujatha distributions and two-parameter lognormal, gamma and Weibull distributions.

Recently, Mussie and Shanker [9] proposed a two-parameter Sujatha distribution (TPSD) defined by its pdf and cdf

$$f_4(x;\theta,\alpha)=\frac{\theta^3}{\alpha\theta^2+\theta+2}\left(\alpha+x+x^2\right)e^{-\theta x}; \quad x>0,\ \theta>0,\ \alpha\geq 0$$

$$F_4(x;\theta,\alpha)=1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\alpha\theta^2+\theta+2}\right]e^{-\theta x}; \quad x>0,\theta>0,\ \alpha\geq 0.$$

Like QSD, TPSD also includes Sujatha distribution and SBLD as particular cases for $\alpha=1$ and $\alpha=0$, respectively.

Statistical properties including coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, stress-strength reliability along with estimation of parameters using both the method of moments and the method of maximum likelihood and applications of TPSD have been discussed in Mussie and Shanker [9]. In search for a new two-parameter Sujatha distribution (NTPSD), Mussie and Shanker [10] proposed the following NTPSD defined by its pdf and cdf

$$f_5(x;\theta,\alpha)=\frac{\theta^3}{\theta^2+\alpha\theta+2}\left(1+\alpha x+x^2\right)e^{-\theta x}; \quad x>0,\ \theta>0,\ \alpha\geq 0$$

$$F_5(x;\theta,\alpha)=1-\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x}, \quad x>0,\theta>0,\ \alpha\geq 0.$$

Mussie and Shanker [10] have discussed statistical properties including coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, stress-strength reliability, and estimation of the parameters using maximum likelihood estimation. Some numerical examples have also been presented by Mussie and Shanker [10] to test the goodness of fit of NTPSD over one parameter exponential, Lindley and Sujatha distributions and two-parameter Sujatha distribution (TPSD).

Note that NTPSD includes Sujatha distribution and Akash distribution proposed by Shanker [11] as particular cases for $\alpha = 1$ and $\alpha = 0$, respectively.

In this paper another two-parameter Sujatha distribution (ATPSD) which includes both exponential distribution and Sujatha distribution as particular cases has been suggested. Its important properties including hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, stress-strength reliability have been discussed. The estimation of the parameters has been discussed using maximum likelihood estimation. Two numerical examples have been given to test the goodness of fit of ATPSD and the fit has been compared with two-parameter QSD, TPSD and NTPSD and one parameter exponential, Lindley and Sujatha distributions.

## 2. Another two-parameter Sujatha distribution

Another two-parameter Sujatha distribution (ATPSD) having parameters $\theta$ and $\alpha$ is defined by its pdf.

$$f_6(x;\theta,\alpha) = \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left(1+\alpha x+\alpha x^2\right)e^{-\theta x}; \quad x>0,\ \theta>0,\ \alpha\geq 0, \tag{2.1}$$

where $\theta$ is a scale parameter and $\alpha$ is a shape parameter. It can easily be verified that (2.1) reduces to Sujatha distribution (1.1), introduced by Shanker [4] and exponential distribution for $\alpha = 1$ and $\alpha = 0$, respectively.

Like Sujatha distribution, ATPSD (2.1) is also a three-component mixture of exponential $(\theta)$, gamma $(2,\theta)$ and gamma $(3,\theta)$ distributions. We have

$$f_6(x;\theta,\alpha) = p_1 g_1(x;\theta) + p_2 g_2(x;\theta,2) + (1-p_1-p_2) g_3(x;\theta,3)$$

where

$$p_1 = \frac{\theta^2}{\theta^2+\alpha\theta+2\alpha}, \quad p_2 = \frac{\alpha\theta}{\theta^2+\alpha\theta+2\alpha}, \quad g_1(x;\theta) = \theta e^{-\theta x}; \quad x>0, \theta>0$$

$$g_2(x;\theta,2) = \frac{\theta^2}{\Gamma(2)}e^{-\theta x}x^{2-1}; \quad x>0, \theta>0, \quad g_3(x;\theta,3) = \frac{\theta^3}{\Gamma(3)}e^{-\theta x}x^{3-1}; \quad x>0, \theta>0.$$

The corresponding cdf of ATPSD (2.1) can be obtained as

$$F_6(x;\theta,\alpha) = 1-\left[1+\frac{\alpha\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2\alpha}\right]e^{-\theta x}; \quad x>0,\ \theta>0,\ \alpha\geq 0.$$

Behaviors of the pdf and the cdf of ATPSD for varying values $\theta$ of $\alpha$ the parameters and are shown in Figure 2.1 and Figure 2.2, respectively.

## 3. Moments and related measures

The moment generating function of ATPSD (2.1) can be obtained as

$$\begin{aligned}
M_X(t) &= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\int_0^\infty e^{-(\theta-t)x}(1+\alpha x+\alpha x^2)dx \\
&= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[\frac{1}{(\theta-t)}+\frac{\alpha}{(\theta-t)^2}+\frac{2\alpha}{(\theta-t)^3}\right] \\
&= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+\frac{\alpha}{\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k+\frac{2\alpha}{\theta^3}\sum_{k=0}^{\infty}\binom{k+2}{k}\left(\frac{t}{\theta}\right)^k\right] \\
&= \sum_{k=0}^{\infty}\frac{\theta^2+\alpha\theta(k+1)+\alpha(k+1)(k+2)}{\theta^2+\alpha\theta+2\alpha}\left(\frac{t}{\theta}\right)^k.
\end{aligned}$$

Thus, the $r^{th}$ moment about origin of ATPSD (2.1), obtained as the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$, is given by

$$\mu_r' = \frac{r!\left\{\theta^2+(r+1)\alpha\theta+(r+1)(r+2)\alpha\right\}}{\theta^r(\theta^2+\alpha\theta+2\alpha)}; \quad r=1,2,3,\ldots$$

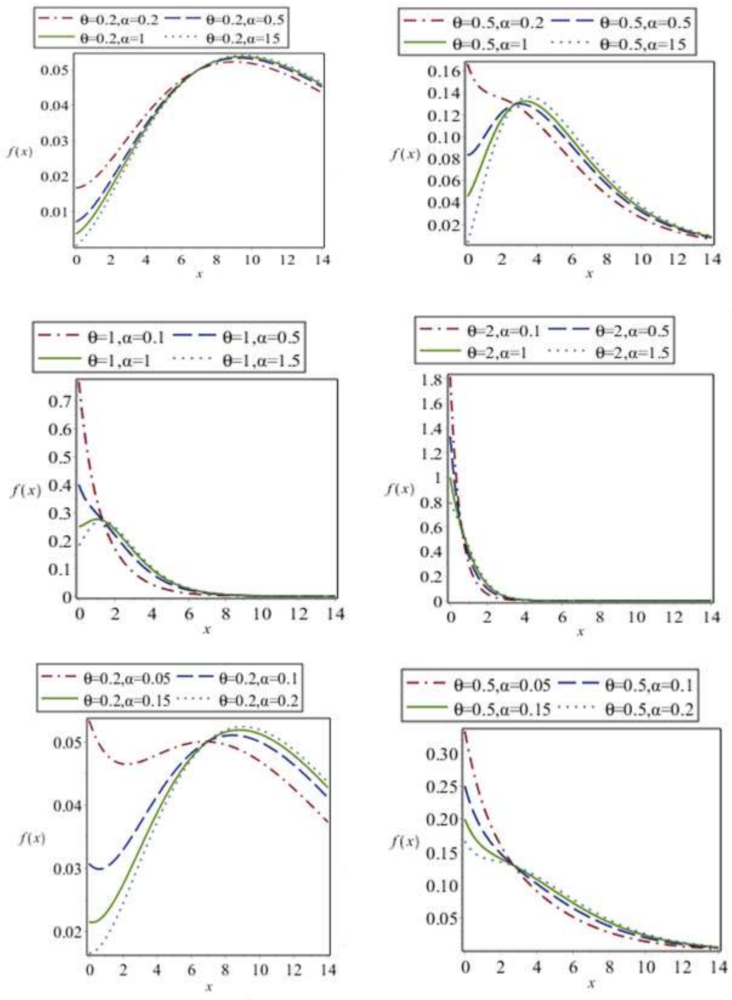


**Figure 2.1:** Behavior of the pdf of ATPSD for varying values of the parameters $\theta$ and $\alpha$

The first four moments about origin of ATPSD are obtained as

$$\mu_1' = \frac{\theta^2+2\alpha\theta+6\alpha}{\theta(\theta^2+\alpha\theta+2\alpha)}, \quad \mu_2' = \frac{2(\theta^2+3\alpha\theta+12\alpha)}{\theta^2(\theta^2+\alpha\theta+2\alpha)},$$

$$\mu_3' = \frac{6(\theta^2+4\alpha\theta+20\alpha)}{\theta^3(\theta^2+\alpha\theta+2\alpha)}, \quad \mu_4' = \frac{24(\theta^2+5\alpha\theta+30\alpha)}{\theta^4(\theta^2+\alpha\theta+2\alpha)}.$$

Using the relationship between moments about the mean and moments about the origin, the moments about mean of ATPSD are obtained as

$$\mu_2 = \frac{2\alpha^2\theta^2+4\alpha\theta^3+\theta^4+12\alpha^2\theta+16\alpha\theta^2+12\alpha^2}{\theta^2(\theta^2+\alpha\theta+2\alpha)^2}$$

$$\mu_3 = \frac{2(2\alpha^3\theta^3+6\alpha^2\theta^4+6\alpha\theta^5+\theta^6+18\alpha^3\theta^2+42\alpha^2\theta^3+30\alpha\theta^4+36\alpha^3\theta+36\alpha^2\theta^2+24\alpha^3)}{\theta^3(\theta^2+\alpha\theta+2\alpha)^3}$$

$$\mu_4 = \frac{3\left(\begin{array}{c}8\alpha^4\theta^4+32\alpha^3\theta^5+44\alpha^2\theta^6+24\alpha\theta^7+3\theta^8+96\alpha^4\theta^3+320\alpha^3\theta^4+344\alpha^2\theta^5\\ +128\alpha\theta^6+336\alpha^4\theta^2+768\alpha^3\theta^3+408\alpha^2\theta^4+480\alpha^4\theta+576\alpha^3\theta^2+240\alpha^4\end{array}\right)}{\theta^4(\theta^2+\alpha\theta+2\alpha)^4}$$

The coefficient of variation (C.V), coefficient of skewness $(\sqrt{\beta_1})$, coefficient of kurtosis $(\beta_2)$ and index of dispersion $(\gamma)$ of ATPSD are given by

$$CV = \frac{\sigma}{\mu_1} = \frac{\sqrt{2\alpha^2\theta^2+4\alpha\theta^3+\theta^4+12\alpha^2\theta}+16\alpha\theta^2+12\alpha^2}{\theta^2+2\alpha\theta+6\alpha}$$

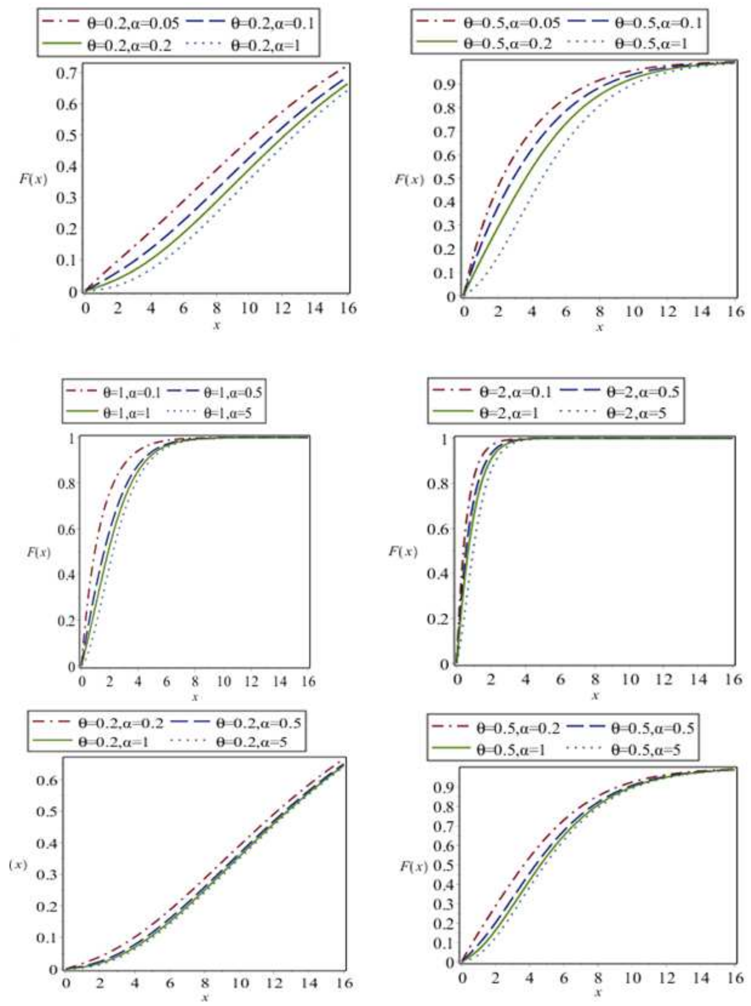

**Figure 2.2:** Behavior of the cdf of ATPSD for varying values of the parameters $\theta$ and $\alpha$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\frac{2(2\alpha^3\theta^3+6\alpha^2\theta^4+6\alpha\theta^5+\theta^6+18\alpha^3\theta^2+42\alpha^2\theta^3+30\alpha\theta^4+36\alpha^3\theta+36\alpha^2\theta^2+24\alpha^3)}{(2\alpha^2\theta^2+4\alpha\theta^3+\theta^4+12\alpha^2\theta+16\alpha\theta^2+12\alpha^2)^{3/2}}$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\mu_4=\frac{3\begin{pmatrix}8\alpha^4\theta^4+32\alpha^3\theta^5+44\alpha^2\theta^6+24\alpha\theta^7+3\theta^8+96\alpha^4\theta^3+320\alpha^3\theta^4+344\alpha^2\theta^5\\+128\alpha\theta^6+336\alpha^4\theta^2+768\alpha^3\theta^3+408\alpha^2\theta^4+480\alpha^4\theta+576\alpha^3\theta^2+240\alpha^4\end{pmatrix}}{(2\alpha^2\theta^2+4\alpha\theta^3+\theta^4+12\alpha^2\theta+16\alpha\theta^2+12\alpha^2)^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{2\alpha^2\theta^2+4\alpha\theta^3+\theta^4+12\alpha^2\theta+16\alpha\theta^2+12\alpha^2}{\theta(\theta^2+\alpha\theta+2\alpha)(\theta^2+2\alpha\theta+6\alpha)}$$

It can be easily verified that these statistical constants of ATPSD reduce to the corresponding statistical constants of Sujatha and exponential distributions at $\alpha=1$ and $\alpha=0$ respectively.

The behaviors of C.V., $\sqrt{\beta_1}$, $\beta_2$ and $\gamma$, for varying values of the parameters $\theta$ and $\alpha$ have been shown numerically in Tables 1,2,3 and 4. For a given value of $\alpha$, C.V. increases as the value of $\theta$ increases. But for values $1\leq\theta\leq5$, C.V. decreases as the value of $\alpha$ increases.

Since $\sqrt{\beta_1}>0$, ATPSD is always positively skewed, and this means that ATPSD is a suitable model for positively skewed lifetime data.

Since $\beta_2>3$, ATPSD is always leptokurtic, which means that ATPSD is more peaked than the normal curve, thus ATPSD is suitable for lifetime data which are leptokurtic.

Index of dispersion of ATPSD for varying values of parameters and .
As long as $0\leq\theta\leq1$ and $0\leq\alpha\leq5$, the nature of ATPSD is over dispersed $(\sigma^2>\mu_1')$ and for $1\leq\theta\leq5$ and $0\leq\alpha\leq5$, the nature of ATPSD is under dispersed $(\sigma^2<\mu_1')$.

**Table 1:** C.V. of ATPSD for varying values of parameters $\theta$ and $\alpha$.

| $\alpha/\theta$ | 0.2 | 0.5 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.640678 | 0.790787 | 0.945247 | 1.013246 | 1.015576 | 1.012149 | 1.009163 |
| 0.5 | 0.614004 | 0.702377 | 0.836660 | 0.962250 | 0.99661 | 1.005655 | 1.007547 |
| 1 | 0.604466 | 0.662392 | 0.761739 | 0.892143 | 0.951190 | 0.977525 | 0.989835 |
| 2 | 0.599565 | 0.639569 | 0.708329 | 0.816497 | 0.882958 | 0.922627 | 0.946881 |
| 3 | 0.597911 | 0631470 | 0.687023 | 0.778162 | 0.840635 | 0.882516 | 0.911036 |
| 4 | 0.597080 | 0.627321 | 0.675561 | 0.755148 | 0.812529 | 0.853461 | 0.883077 |
| 5 | 0.596580 | 0.624798 | 0.668399 | 0.739814 | 0.792609 | 0.831700 | 0.861102 |

**Table 2:** Coefficient skewness ($\sqrt{\beta_1}$) of ATPSD for varying values of parameters $\theta$ and $\alpha$

| $\alpha/\theta$ | 0.2 | 0.5 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 0.2 | 1.114210 | 1.248863 | 1.643745 | 1.984109 | 2.045477 | 2.048672 | 2.041683 |
| 0.5 | 1.133836 | 1.154381 | 1.365976 | 1.745907 | 1.916915 | 1.984875 | 2.011218 |
| 1 | 1.145006 | 1.145839 | 1.247611 | 1.535588 | 1.733747 | 1.848046 | 1.912879 |
| 2 | 1.151691 | 1.153577 | 1.201582 | 1.377838 | 1.541364 | 1.662624 | 1.748583 |
| 3 | 1.154099 | 1.158904 | 1.194050 | 1.320932 | 1.451434 | 1.558992 | 1.642899 |
| 4 | 1.155338 | 1.162202 | 1.192975 | 1.294584 | 1.402342 | 1.496069 | 1.573228 |
| 5 | 1.156092 | 1.164402 | 1.193449 | 1.280549 | 1.372667 | 1.455018 | 1.525066 |

**Table 3:** Coefficient of kurtosis ($\beta_2$) of ATPSD for varying values of parameters $\theta$ and $\alpha$

| $\alpha/\theta$ | 0.2 | 0.5 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 0.2 | 4.869916 | 5.109996 | 6.633262 | 8.690336 | 9.230612 | 9.313262 | 9.289003 |
| 0.5 | 4.941280 | 4.924032 | 5.510204 | 7.214400 | 8.270528 | 8.774988 | 9.001011 |
| 1 | 4.973635 | 4.944566 | 5.170213 | 6.214990 | 7.193906 | 7.868405 | 8.297711 |
| 2 | 4.991667 | 4.984856 | 5.082378 | 5.625000 | 6.285420 | 6.865586 | 7.326691 |
| 3 | 4.997968 | 5.004164 | 5.080437 | 5.451135 | 5.931292 | 6.392963 | 6.794445 |
| 4 | 5.001174 | 5.015076 | 5.087074 | 5.380592 | 5.758343 | 6.135306 | 6.478444 |
| 5 | 5.003116 | 5.022048 | 5.093943 | 5.346882 | 5.661781 | 5.979645 | 6.275987 |

**Table 4:** Index of dispersion ($\gamma$) of ATPSD for varying values of parameters $\theta$ and $\alpha$

| $\alpha/\theta$ | 0.2 | 0.5 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 0.2 | .5643939 | 2.751515 | 1.451923 | 0.641667 | 0.391930 | 0.279936 | 0.217569 |
| 0.5 | 5.357375 | 2.466667 | 1.400000 | 0.694444 | 0.431884 | 0.306064 | 0.235088 |
| 1 | 5.252329 | 2.313480 | 1.305556 | 0.696429 | 0.452381 | 0.325758 | 0.251067 |
| 2 | 5.197861 | 2.220551 | 1.218487 | 0.666667 | 0.451356 | 0.334416 | 0.262078 |
| 3 | 5.179408 | 2.186717 | 1.180000 | 0.643382 | 0.441667 | 0.332150 | 0.263431 |
| 4 | 5.170124 | 2.169221 | 1.158508 | 0.627273 | 0.432547 | 0.327778 | 0.261904 |
| 5 | 5.164536 | 2.158531 | 1.144817 | 0.615741 | 0.424979 | 0.323306 | 0.259524 |

# 4. Reliability properties

In this section, reliability properties of ATPSD including hazard rate function, mean residual life function, stochastic ordering and stress-strength reliability have been discussed.

## 4.1. Hazard rate function and mean residual life function

Let $X$ be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. The hazard rate function (also known as failure rate function), $h(x)$ and the mean residual function, $m(x)$ of $X$ are respectively defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{p(X < x + \Delta x | X > x)}{\Delta x} = \frac{f(x)}{1 - F(x)}$$

and

$$m(x) = E[X - x | X > x] = \frac{1}{1 - F(x)} \int_x^\infty [1 - F(t)]dt$$

The corresponding hazard rate function, $h(x)$ and the mean residual function $m(x)$ of ATPSD (2.1) are thus obtained as

$$h(x) = \frac{\theta^3(1 + \alpha x + \alpha x^2)}{\theta^2(1 + \alpha x + \alpha x^2) + \alpha(2\theta x + \theta + 2)}$$

and

$$m(x) = \frac{\theta^2 + \alpha\theta + 2\alpha}{[(\theta^2 + \alpha\theta + 2\alpha) + \alpha\theta x(\theta x + \theta + 2)]e^{-\theta x}} \int_x^\infty \left[1 + \frac{\alpha\theta t(\theta t + \theta + 2)}{\theta^2 + \alpha\theta + 2\alpha}\right] e^{-\theta t} dt = \frac{\alpha\theta^2 x^2 + \alpha\theta(\theta + 4)x + (\theta^2 + 2\alpha\theta + 6\alpha)}{\theta[\alpha\theta^2 x^2 + \alpha\theta(\theta + 2)x + (\theta^2 + \alpha\theta + 2\alpha)]}$$

It can be easily verified that $h(0) = \frac{\theta^3}{\theta^2 + \alpha\theta + 2\alpha} = f(0)$ and $m(0) = \frac{\theta^2 + 2\alpha\theta + 6\alpha}{\theta(\theta^2 + \alpha\theta + 2\alpha)} = \mu_1'$
It can also be easily verified that these expressions for $h(x)$ and $m(x)$ of ATPSD reduce to the corresponding $h(x)$ and $m(x)$ of Sujatha distribution ar $\alpha = 1$.

The behavior of $h(x)$ and $m(x)$ of ATPSD for different values of its parameters $\theta$ and $\alpha$ are shown in Figures 4.1 and 4.2, respectively. It is obvious from the graphs of $h(x)$ that it takes different shapes including monotonically increasing, upside bathtub and downside bathtub, etc., whereas $m(x)$ is monotonically decreasing function.

## 4.2. Stochastic ordering

Stochastic ordering of positive continuous random variable is an important tool for judging the comparative behavior of continuous distributions. A random variable $X$ is said to be smaller than a random variable $Y$ in the

i. stochastic order $(X \leq_{st} Y)$ if $F_x(x) \geq F_y(x)$ for all $x$,
ii. hazard rate order $(X \leq_{ht} Y)$ if $h_x(x) \geq h_y(x)$ for all $x$,
iii. mean residual life order $(X \leq_{mrl} Y)$ if $m_x(x) \leq m_y(x)$ for all $x$,
iv. likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_x(x)}{f_y(x)}$ decreases in $x$.

The following results due to Shaked and Shanthikumar [12] are well known for establishing stochastic ordering of distributions

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mlr} Y$$

$$\Downarrow_{x \leq_{st} y}$$

The ATPSD (2.1) is ordered with respect to the strongest "likelihood ratio" ordering as established in the following theorem:

**Theorem 4.1.** *Let $X \sim ATPSD(\theta_1, \alpha_1)$ and $Y \sim ATPSD(\theta_2, \alpha_2)$. If $\theta_1 > \theta_2$ and $\alpha_1 = \alpha_2$ or $\theta_1 = \theta_2$ and $\alpha_1 < \alpha_2$, then $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.*

*Proof.* We have

$$\frac{f_x(x;\theta_1,\alpha_1)}{f_y(x;\theta_2,\alpha_2)} = \frac{\theta_1^3(\theta_2^2 + \alpha_2\theta_2 + 2\alpha_2)}{\theta_2^3(\theta_1^2 + \alpha_1\theta_1 + 2\alpha_1)} \left(\frac{1 + \alpha_1 x + \alpha_1 x^2}{1 + \alpha_2 x + \alpha_2 x^2}\right) e^{-(\theta_1 - \theta_2)x}; \quad x > 0$$

$$\ln \frac{f_x(x;\theta_1,\alpha_1)}{f_y(x;\theta_2,\alpha_2)} = \ln \left[\frac{\theta_1^3(\theta_2^2 + \alpha_2\theta_2 + 2\alpha_2)}{\theta_2^3(\theta_1^2 + \alpha_1\theta_1 + 2\alpha_1)}\right] + \ln \left(\frac{1 + \alpha_1 x + \alpha_1 x^2}{1 + \alpha_2 x + \alpha_2 x^2}\right) - (\theta_1 - \theta_2)x; \quad x > 0$$

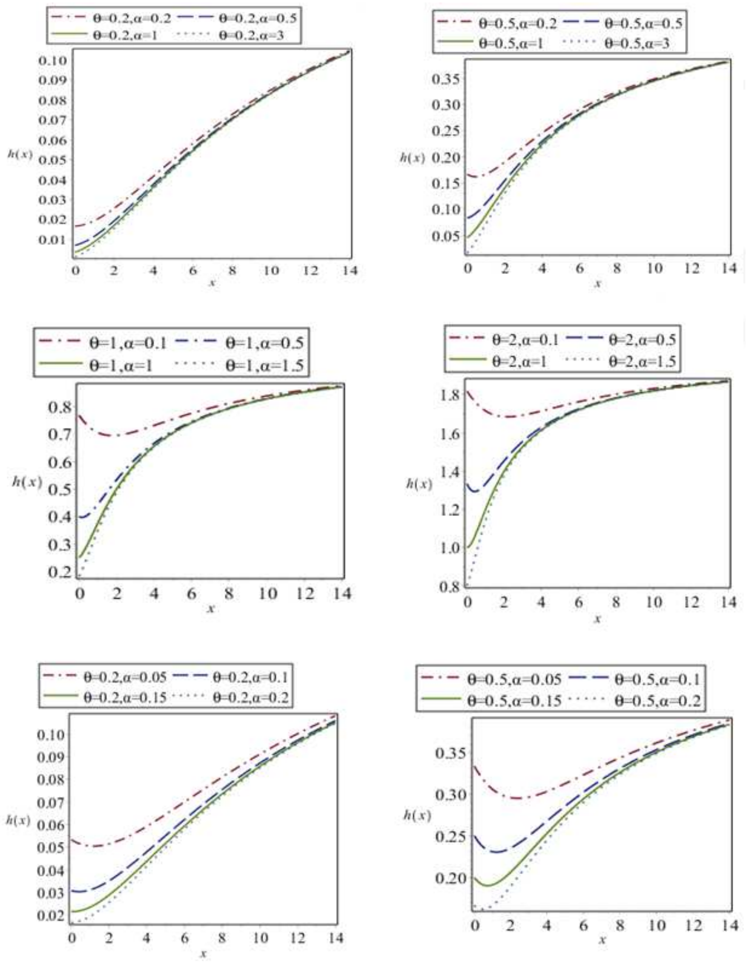


**Figure 4.1:** Behavior of the $h(x)$ of ATPSD for selected values of the parameters $\theta$ and $\alpha$

This gives

$$\frac{d}{dx}\ln\frac{f_x(x;\theta_1,\alpha_1)}{f_y(x;\theta_2,\alpha_2)}=\frac{(\alpha_1-\alpha_2)-(\alpha_1-\alpha_2)x^2}{(1+\alpha_1 x+x^2)(1+\alpha_2 x+x^2)}-(\theta_1-\theta_2).$$

Thus for $(\theta_1>\theta_2 and \alpha_1=\alpha_2)$ or $(\theta_1=\theta_2 and \alpha_1<\alpha_2)$, $\frac{d}{dx}\ln\frac{f_x(x;\theta_1,\alpha_1)}{f_y(x;\theta_2,\alpha_2)}<0$. This means that $X\leq_{lr}Y$ and hence $X\leq_{hr}Y$, $X\leq_{mrl}Y$ and $X\leq_{st}Y$. This shows flexibility of ATPSD over Sujatha distribution. □

### 4.3. Stress-strength reliability

The stress-strength reliability of a component illustrates the life of the component which has random strength $X$ that is subjected to random stress $Y$. When the stress of the component $Y$ applied to it exceeds the strength of the component $X$, the component fails instantly and the component will function satisfactorily till $X>Y$. Therefore, $R=P(Y<X)$ is a measure of the component reliability and is known as stress-strength reliability in statistical literature. It has extensive application in almost all areas of knowledge especially in engineering such as structure, deterioration of rocket motor, static fatigue of ceramic component, aging of concrete pressure vessels etc.

Let $X$ and $Y$ be independent strength and stress random variables having ATPSD (2.1) with parameter $(\theta_1,\alpha_1)$ and $(\theta_2,\alpha_2)$ respectively. Then the stress-strength reliability $R$ of ATPSD can be obtained as

$$R=P(Y<X)=\int_0^\infty P(Y<X|X=x)f_x(x)dx=\int_0^\infty f_6(x;\theta_1,\alpha_1)F_6(x;\theta_2,\alpha_2)dx$$

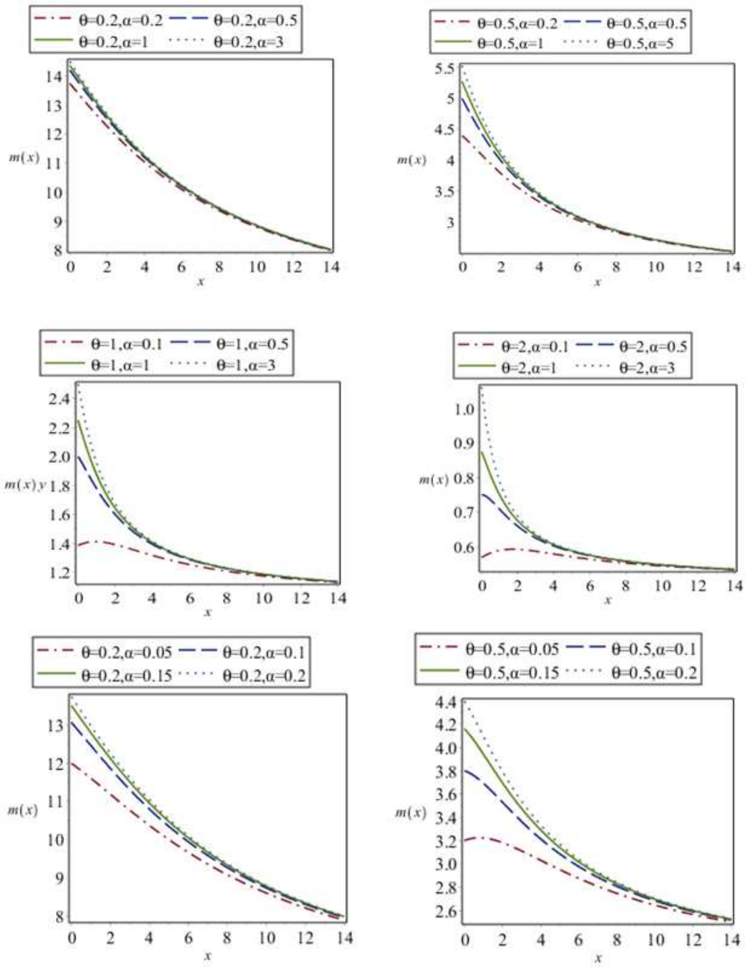


**Figure 4.2:** Behavior of the $m(x)$ of ATPSD for selected values of the parameters $\theta$ and $\alpha$

$$
\begin{aligned}
R &= 1-\frac{\theta_1^3[\theta_2^6+(\alpha_1+2\alpha_2+4\theta_1)\theta_2^5+(3\alpha_1\alpha_2+3\alpha_1\theta_1+7\alpha_1\alpha_2\theta_1+6\theta_1^2+2\alpha_1+6\alpha_2)\theta_2^4]}{(\theta_1^2+\alpha_1\theta_1+2\alpha_1)(\theta_2^2+\alpha_2\theta_2+2\alpha_2)(\theta_1+\theta_2)^5}\\
&+\frac{\theta_1^3(7\alpha_1\alpha_2\theta_1+3\alpha_1\theta_1^2+9\alpha_2\theta_1^2+4\theta_1^3+20\alpha_1\alpha_2+4\alpha_1\theta_1+18\alpha_2\theta_1)\theta_2^3}{(\theta_1^2+\alpha_1\theta_1+2\alpha_1)(\theta_2^2+\alpha_2\theta_2+2\alpha_2)(\theta_1+\theta_2)^5}\\
&+\frac{\theta_1^3(5\alpha_1\alpha_2\theta_1^2+\alpha_1\theta_1^3+5\alpha_1\alpha_2\theta_1^3+\theta_1^4+30\alpha_1\alpha_2\theta_1+2\alpha_1\theta_1^2+20\alpha_2\theta_1^2+40\alpha_1\alpha_2)\theta_2^2}{(\theta_1^2+\alpha_1\theta_1+2\alpha_1)(\theta_2^2+\alpha_2\theta_2+2\alpha_2)(\theta_1+\theta_2)^5}\\
&+\frac{\theta_1^3[(\alpha_1\alpha_2\theta_1^2+\alpha_2\theta_1^3+12\alpha_1\alpha_2\theta_1+10\alpha_2\theta_1^2+20\alpha_1\alpha_2)\theta_1\theta_2+2(\theta_1^2+\theta_1+2)\alpha_1\theta_1^2]}{(\theta_1^2+\alpha_1\theta_1+2\alpha_1)(\theta_2^2+\alpha_2\theta_2+2\alpha_2)(\theta_1+\theta_2)^5}
\end{aligned}
$$

It can easily be verified that the stress-strength reliability of Sujatha distribution and exponential distribution are particular cases of stress-strength reliability of ATPSD at $\alpha_1=\alpha_2=1$, and $\alpha_1=\alpha_2=0$, respectively.

# 5. Statistical properties

## 5.1. Mean deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and the median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined as

$$\delta_1(X)=\int_0^\infty |x-\mu|f(x)dx \quad \delta_2(X)=\int_0^\infty |x-M|f(x)dx$$

respectively., where $\mu=E(X)$ and $M=median(X)$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the following relationships

$$
\begin{aligned}
\delta_1(X) &= \int_0^\mu(\mu-x)f(x)dx+\int_\mu^\infty(x-\mu)f(x)dx \qquad (5.1)\\
&= \mu F(\mu)-\int_0^\mu f(x)dx-\mu[1-F(\mu)]+\int_\mu^\infty xf(x)dx=2\mu F(\mu)-2\mu+2\int_\mu^\infty xf(x)dx=2\mu F(\mu)-2\int_\mu^\infty xf(x)dx
\end{aligned}
$$

and

$$\begin{aligned}\delta_2(X) &= \int_0^M (M-x)f(x)dx + \int_M^\infty (x-M)f(x)dx \\ &= MF(M) - \int_0^M f(x)dx - M[1-F(M)] + \int_M^\infty xf(x)dx = \mu + 2\int_M^\infty xf(x)dx = \mu - 2\int_0^M xf(x)dx\end{aligned} \tag{5.2}$$

Using the pdf (2.1) and expression for the mean of ATPSD, we get

$$\int_0^\mu xf_6(x)(x;\theta,\alpha)dx = \mu - \frac{[\theta^3(\alpha\mu^3+\alpha\mu^2+\mu)+\theta^2(3\alpha\mu^2+2\alpha\mu+1)+2\alpha\theta(3\mu+1)+6\alpha]e^{-\theta\mu}}{\theta(\theta^2+\alpha\theta+2\alpha)} \tag{5.3}$$

$$\int_0^M xf_6(x)(x;\theta,\alpha)dx = \mu - \frac{[\theta^3(\alpha M^3+\alpha M^2+M)+\theta^2(3\alpha M^2+2\alpha M+1)+2\alpha\theta(3M+1)+6\alpha]e^{-\theta M}}{\theta(\theta^2+\alpha\theta+2\alpha)} \tag{5.4}$$

Using expressions from (5.1), (5.2), (5.3) and (5.4) and after some tedious algebraic simplifications, the mean deviation about the mean, $\delta_1(X)$ and the mean deviation about the median, $\delta_2(X)$ of ATPSD are obtained as

$$\delta_1(X) = \frac{2[\theta^3(\alpha\mu^2+\alpha\mu+1)+2\alpha\theta(2\mu+1)+6\alpha]e^{-\theta\mu}}{\theta(\theta^2+\alpha\theta+2\alpha)}$$

and

$$\delta_2(X) = \frac{2[\theta^3(\alpha M^2+\alpha M+M)+\theta^3(3\alpha M^2+2\alpha M+1)+2\alpha\theta(3M+1)+6\alpha]e^{-\theta M}}{\theta(\theta^2+\alpha\theta+2\alpha)} - \mu$$

### 5.2. Bonferroni and Lorenz curves and indicies

The Bonferroni and Lorenz curves, introduced by Bonferroni [13] and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography and medical science. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx = \frac{1}{p\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^\infty xf(x)dx\right] \tag{5.5}$$

and

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx = \frac{1}{\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right] = \frac{1}{\mu}\left[\mu - \int_q^\infty xf(x)dx\right] \tag{5.6}$$

respectively or equivalently

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(x)dx$$

and

$$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)dx$$

respectively, where $\mu = E(x)$ and $q = F^{-1}(p)$.
The Bonferroni and Gini indices are thus defined as

$$B = 1 - \int_0^1 B(p)dp \tag{5.7}$$

and

$$B = 1 - 2\int_0^1 L(p)dp \tag{5.8}$$

respectively.
Using pdf of ATPSD (2.1), we get

$$\int_q^\infty xf_6(x;\theta,\alpha)dx = \frac{\{\theta^3(\alpha q^3+\alpha q^2+q)+\theta^2(3\alpha q^2+2\alpha q+1)+2\alpha\theta(3q+1)+6\alpha\}e^{-\theta q}}{\theta(\theta^2+\alpha\theta+2\alpha)} \tag{5.9}$$

Now using equation (5.9), (5.5) and (5.6), we get

$$B(p) = \frac{1}{p}\left[1 - \frac{\{\theta^3(\alpha q^3+\alpha q^2+q)+\theta^2(3\alpha q^2+2\alpha q+1)+2\alpha\theta(3q+1)+6\alpha\}e^{-\theta q}}{\theta^2+2\alpha\theta+6\alpha}\right] \tag{5.10}$$

and

$$L(p)=1-\frac{\{\theta^3(\alpha q^3+\alpha q^2+q)+\theta^2(3\alpha q^2+2\alpha q+1)+2\alpha\theta(3q+1)+6\alpha\}e^{-\theta q}}{\theta^2+2\alpha\theta+6\alpha} \tag{5.11}$$

Now using the equations (5.10) and (5.11) in (5.7) and (5.8), the Bonferroni and Gini indices of ATPSD (2.1) are obtained as

$$B=1-\frac{\{\theta^3(\alpha q^3+\alpha q^2+q)+\theta^2(3\alpha q^2+2\alpha q+1)+2\alpha\theta(3q+1)+6\alpha\}e^{-\theta q}}{\theta^2+2\alpha\theta+6\alpha}$$

$$G=\frac{2\{\theta^3(\alpha q^3+\alpha q^2+q)+\theta^2(3\alpha q^2+2q+1)+2\alpha\theta(3q+1)+6\alpha\}e^{-\theta q}}{\theta^2+2\alpha\theta+6\alpha}-1$$

## 6. Estimation of parameters

In this section, the estimation of parameters of ATPSD using method of moments and method of maximum likelihood have been discussed.

### 6.1. Method of Moment Estimates (MOME)

Since ATPSD (2.1) has two parameters to be estimated, the first two moments about the origin are required to estimate its parameters using method of moments. Equating the population mean to the sample mean, we have

$$\bar{x}=\frac{\theta^2+2\alpha\theta+6\alpha}{\theta(\theta^2+\alpha\theta+2\alpha)}=\frac{\theta^2+\alpha\theta+2\alpha}{\theta(\theta^2+\alpha\theta+2\alpha)}+\frac{\alpha(\theta+4)}{\theta(\theta^2+\alpha\theta+2\alpha)}$$

$$\bar{x}=\frac{1}{\theta}+\frac{\alpha(\theta+4)}{\theta(\theta^2+\alpha\theta+2\alpha)}$$

$$(\theta^2+\alpha\theta+2\alpha)=\frac{\alpha(\theta+4)}{\theta\bar{x}-1} \tag{6.1}$$

Again equating the second population moment with the corresponding sample moment, we have

$$m_2^{'}=\frac{2(\theta^2+3\alpha\theta+12\alpha)}{\theta^2(\theta^2+\alpha\theta+2\alpha)}=\frac{2(\theta^2+\alpha\theta+2\alpha)}{\theta^2(\theta^2+\alpha\theta+2\alpha)}+\frac{4\alpha(\theta+5)}{\theta^2(\theta^2+\alpha\theta+2\alpha)}$$

$$m_2^{'}=\frac{2}{\theta^2}+\frac{4\alpha(\theta+5)}{\theta^2(\theta^2+\alpha\theta+2\alpha)}$$

$$\theta^2+\alpha\theta+2\alpha=\frac{4\alpha(\theta+5)}{m_2^{'}\theta^2-2} \tag{6.2}$$

Equations (6.1) and (6.2) give the following cubic equation in $\theta$

$$\alpha m_2^{'}\theta^3+4(m_2^{'}-\bar{x})\theta^2-2\alpha(10\bar{x}-1)\theta+12\alpha=0. \tag{6.3}$$

Solving equation (6.3) using any iterative method such as Newton-Raphson method, Regula-Falsi method or Bisection method, method of moment estimation(MOME)$\tilde{\theta}$ of $\theta$ can be obtained and substituting the value of $\tilde{\theta}$ in equation (6.1), MOME $\tilde{\alpha}$ of $\alpha$ can be obtained as

$$\tilde{\alpha}=\frac{(1-\tilde{\theta}\bar{x})(\tilde{\theta})^2}{(\tilde{\theta})^2\bar{x}-\tilde{\theta}+2(\bar{x}-1)(\tilde{\theta})-6}$$

### 6.2. Maximum likelihood estimates(MLE)

Let $(x_1, x_2, \cdots, x_n)$ be random sample from ATPSD (2.1). The likelihood function $L$ is given by

$$L = \left(\frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\right)\prod_{i=1}^{n}\left(1+\alpha x_i+\alpha x_i^2\right)e^{-n\theta\bar{x}}.$$

The natural log likelihood function is thus obtained as

$$\ln L = n\left[3\ln\theta - \ln(\theta^2+\alpha\theta+2\alpha)\right] + \sum_{i=1}^{n}\ln(1+\alpha x_i+\alpha x_i^2) - -n\theta\bar{x},$$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimate(MLE's) $(\hat{\theta}, \hat{\alpha})$ of $(\theta, \alpha)$ are then the solutions of the following non-linear equations

$$\frac{\partial \ln L}{\partial \theta} = \frac{3n}{\theta} - \frac{n(2\theta+\alpha)}{\theta^2+\alpha\theta+2\alpha} - n\bar{x} = 0$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n(\theta+2)}{\theta^2+\alpha\theta+2\alpha} + \sum_{i=1}^{n}\frac{\left(x_i+x_i^2\right)x_i}{1+\alpha x_i+\alpha x_i^2} = 0$$

These two natural log likelihood equations do not seem to be solved directly, because they cannot be expressed in closed forms. The (MLE's) $(\hat{\theta}, \hat{\alpha})$ of $(\theta, \alpha)$ can be computed directly by solving the natural log likelihood equation using Newton-Raphson iteration available in R-software till sufficiently close values $\hat{\theta}$ of $\hat{\alpha}$ are obtained. The initial values of parameters $\theta$ and $\alpha$ are the MOME $(\hat{\theta}, \hat{\alpha})$ of the parameters $(\theta, \alpha)$.

## 7. Applications

In this section the goodness of fit of ATPSD using maximum likelihood estimation has been discussed with the following two real lifetime datasets.

**Data set 1:** This data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark [14].

1.1 1.4 1.3 1.7 1.9 1.8 1.6 2.2 1.7 2.7 4.1 1.8 1.5 1.2 1.4 3 1.7 2.3 1.6 2

**Data Set 2:** The following data represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test, available in Lawless [15].

1.4 5.1 6.3 10.8 12.1 18.5 19.7 22.2 23.0 30.6 37.3 46.3 53.9 59.8 66.2

In order to compare lifetime distributions, values of $-2\ln L$ , AIC (Akaike Information Criterion) and K-S Statistic ( Kolmogorov-Smirnov Statistic) for the above data sets have been computed. The formulae for computing AIC and K-S Statistic are as follows:

$AIC = -2\ln L + 2k$ and $K-S = \sup_x |F_n(x) - F_0(x)|$, where $k$ is the number of parameters, $n$ is the sample size, and the $F_n(x)$ is empirical distribution function. The MLE $(\hat{\theta}, \hat{\alpha})$ along with their standard errors, S.E $(\hat{\theta}, \hat{\alpha})$, $-2\ln L + 2k$, AIC and K-S statistics and p-value of the fitted distributions are presented in the Table 5.
The best distribution is the distribution which corresponds to the lower values of $-2\ln L$, AIC, and K-S statistic. It is obvious from the goodness of fit test that ATPSD gives much closer fit over two parameters QSD, TPSD and NTPSD and one parameter Sujatha, Lindley and exponential distributions.

## 8. Concluding remarks

Another two-parameter Sujatha distribution (ATPSD) has been introduced which includes exponential distribution and Sujatha distribution as particular cases. Moments about origin and moments about mean of ATPSD have been obtained. The behaviors of coefficient of variation, coefficient of skewness, coefficient of kurtosis, Index of dispersion, hazard rate function and mean residual life function of ATPSD have been discussed with varying values of the parameters. The stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability of ATPSD have been discussed. Both method of moment and the method of maximum likelihood have been discussed for estimating parameters. Finally, two examples of observed real lifetime datasets have been presented to show the applications and goodness of fit of ATPSD over two parameters QSD, TPSD and NTPSD and one parameter Sujatha, Lindley and exponential distributions and it has been observed that ATPSD gives much closer fit.

**Table 5:** MLE's, S.E $(\hat{\theta}, \hat{\alpha})$, $-2\ln L$, AIC and K-S statistics and p-value of the fitted distributions of the given datasets

| Dataset | Distrubition | MLE's | S.E | $-2\log L$ | AIC | K-S | P-value |
|---|---|---|---|---|---|---|---|
| 1 | ATPSD | $\hat{\theta}$=1.35, $\hat{\alpha}$=25.79 | 0.18; 37.26 | 50.22 | 54.22 | 0.300 | 0.0551 |
| 1 | NTPSD | $\hat{\theta}$=1.06, $\hat{\alpha}$=42.47 | 0.165; 69.76 | 52.74 | 56.74 | 0.324 | 0.0298 |
| 1 | TPSD | $\hat{\theta}$=0.59, $\hat{\alpha}$=103.48 | 0.14; 171.61 | 66.50 | 70.50 | 0.448 | 0.0007 |
| 1 | QSD | $\hat{\theta}$=0.56, $\hat{\alpha}$=104.85 | 0.131; 170.04 | 66.21 | 70.21 | 0.442 | 0.0008 |
| 1 | Sujatha | $\hat{\theta}$=1.136 | 0.15 | 57.50 | 59.5 | 0.359 | 0.0116 |
| 1 | Lindley | $\hat{\theta}$=0.82 | 0.14 | 60.50 | 62.50 | 0.391 | 0.0044 |
| 1 | Exponential | $\hat{\theta}$=0.53 | 0.12 | 65.67 | 67.67 | 0.440 | 0.0009 |
| 2 | ATPSD | $\hat{\theta}$=0.084, $\hat{\alpha}$=0.007 | 0..19; 0.009 | 128.02 | 132.02 | 0.107 | 0.9890 |
| 2 | NTPSD | $\hat{\theta}$=0.088, $\hat{\alpha}$=30.18 | 0.018; 43.30 | 129.75 | 133.75 | 0.131 | 0.9300 |
| 2 | TPSD | $\hat{\theta}$=0.094, $\hat{\alpha}$=49.77 | 0.016; 52.97 | 128.59 | 132.59 | 0.199 | 0.5240 |
| 2 | QSD | $\hat{\theta}$=0.084, $\hat{\alpha}$=12.264 | 0.02; 15.334 | 128.02 | 132.02 | 0.107 | 0.9890 |
| 2 | Sujatha | $\hat{\theta}$=0.107 | 0.159 | 132.87 | 134.87 | 0.178 | 0.6661 |
| 2 | Lindley | $\hat{\theta}$=0.070 | 0.013 | 128.81 | 130.81 | 0.110 | 0.9836 |
| 2 | Exponential | $\hat{\theta}$=0.036 | 0.009 | 129.48 | 131.48 | 0.156 | 0.8068 |

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# Analysis of Einstein Field Equations of Static Plane Symmetric Space-Time in General Relativity via Lie Approach

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**Abstract**

In this study, Einstein field equations of static plane symmetric space-time is discussed by means of Lie approach. Lie point symmetries are computed for the considered equations. Moreover, one dimensional conjugacy classes for symmetry algebra are reported. Additionally, similarity variables are derived which are further utilized to find the solutions for parameters $A$ and $B$ of the considered space-time. Both linearly independent and dependent relationships between the unknown parameters $A$ and $B$ are observed for different cases. At the end, some well-known solutions are presented.

## 1. Introduction

It is now exactly 100 years since the birth of General Relativity (GR). Exact solutions of the Einstein field equations have played very important roles in the study of physical problems. Some well-known examples includes the Schwarzschild and Kerr solutions to discuss black holes, the Friedmann solutions in the context of cosmology and the plane wave solutions which ended the controversial discussion about the existence of gravitational radiation [1]. The Einstein field equations are highly non-linear second order partial differential equations (PDEs). On the other hand, Lie's theory gives a mathematical and systematic approach to investigate the solutions of differential equations. The application of Lie group theory for the solution of nonlinear ordinary differential equation is one of the most fascinating and significant area of research. It is well-established that from Lie's theory one can not only construct a class of exact solutions but can also find new solutions using different invariant transformations. It also gives most widely applicable technique to find the closed form solution of differential equations. Investigation of these solutions plays a vital role for the understanding of the physical aspects of these differential equations.
Lie symmetry analysis is considered to be a systematic way to derive an ansatz which can be further utilized to reduce the dimension of the differential equations (DEs). The Lie analysis also plays a important role in the algebraic analysis of DEs [2]. Thus it would be worthwhile to investigate the solutions of Einstein field equations using Lie symmetry approach.
The space-time admitting three parameter group of motions of the Euclidean plane is said to possess plane symmetry and is known as a plane symmetric space-time. Such space-time possesses many properties equivalent to those of spherical symmetry. The plane symmetric space-time has been extensively investigated by many researchers from various standpoints. Taub [3], Bondi [4], Bondi and Pirani-Robinson [5] defined and studied plane wave solutions. They considered the concept of group of motions of space-time which played a fundamental role in the plane gravitational waves. It has been established that the plane symmetric space-time admits the plane wave solutions of GR field equations [6].
Here the aim is to explore the similarity solutions of static plane symmetric vacuum solutions in GR. For this purpose, Lie point symmetry method is used. In particular, exact solutions of field equations are presented. The paper is organized as follow: Lie analysis of plane symmetric vacuum solutions is given in Section **2**. In the last section, results are summarized.

## 2. Lie analysis for plane symmetric vacuum solutions

Let us consider static plane symmetric space-time [7]:

$$ds^2 = A(x)dt^2 - dx^2 - B(x)(dy^2 + dz^2).$$

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The Ricci scalar for this space-time is given by

$$R=\frac{1}{2}\left[\frac{2A''}{A}-\left(\frac{A'}{A}\right)^2+\frac{2A'B'}{AB}+\frac{4B''}{B}-\left(\frac{B'}{B}\right)^2\right],$$

where prime represents derivative with respect to $x$.
The Einstein field equations are:

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu}. \tag{2.1}$$

Using Eq.(2.1), GR vacuum field equations becomes:

$$\frac{4B''}{B}-\left(\frac{B'}{B}\right)^2=0, \tag{2.2}$$

$$\left(\frac{B'}{B}\right)^2+\frac{2A'B'}{AB}=0, \tag{2.3}$$

$$\frac{2A''}{A}+\frac{2B''}{B}-\left(\frac{A'}{A}\right)^2-\left(\frac{B'}{B}\right)^2+\frac{A'B'}{AB}=0. \tag{2.4}$$

Manipulating Eqs.(2.2-2.4), one can get:

$$\frac{A'B'}{AB}+\left(\frac{B'}{B}\right)^2-\frac{2B''}{B}=0, \qquad \frac{2A''}{A}-\left(\frac{A'}{A}\right)^2+\frac{A'B'}{AB}-\frac{2B''}{B}=0. \tag{2.5}$$

Thus a system of second order non-linear differential equations with two unknowns namely $A$ and $B$ is obtained.
The Lie analysis of (2.5) requires one parameter Lie group of transformations:

$$\tilde{x} = x+\lambda\, \xi(x,\, A,\, B)+O(\lambda^2), \tag{2.6}$$

$$\tilde{A} = A+\lambda\, \phi(x,\, A,\, B)+O(\lambda^2), \tag{2.7}$$

$$\tilde{B} = B+\lambda\, \psi(x,\, A,\, B)+O(\lambda^2), \tag{2.8}$$

which leave the solutions of differential equation invariant.
Let

$$X\equiv\xi(x,\,A,\,B)\frac{\partial}{\partial x}+\phi(x,\,A,\,B)\frac{\partial}{\partial A}+\psi(x,\,A,\,B)\frac{\partial}{\partial B}$$

be a Lie point symmetry generator of (2.5) if it satisfies the invariance condition i.e.

$$X^{[2]}[Eq.(2.5)]\,|_{Eq.(2.5)}=0, \tag{2.9}$$

where $X^{[2]}$ is the second prolongation [2].
From Eqs. (2.6)-(2.8), one can write:

$$\frac{dx}{\xi(x,\,A,\,B)}=\frac{dA}{\phi(x,\,A,\,B)}=\frac{dB}{\psi(x,\,A,\,B)}=\lambda,$$

which is known as the characteristics equation [2]. Using Eq. (2.9) one can get an over determined system of linear partial differential equations which yields:

$$\xi=c_1x+c_2, \quad \phi=c_3A, \quad \psi=c_4B, \tag{2.10}$$

where $c_i$ $(i=1,2,3,4)$ are the constants of integration. Now Lie point symmetries for the system (2.5) are constructed by setting one constants equal to unity and remaining constants to zero in (2.10):

$$X_1=x\frac{\partial}{\partial x},\; X_2=\frac{\partial}{\partial x},\; X_3=A\frac{\partial}{\partial A},\; X_4=B\frac{\partial}{\partial B}.$$

### 2.1. Optimal system

The finite-dimensional Lie algebra of the system (2.5) is $L=\{X_1,X_2,X_3,X_4\}$. As we know there are infinitely many one-dimensional Lie algebras. The next aim is to obtain a class from these infinitely many one-dimensional Lie algebras which is their best representative. This class is called the optimal system of one-dimensional algebras.

In order to construct the conjugacy classes for one-dimensional algebra, following definitions from [2] are presented.

1: If $X_i$ and $X_j$ are Lie point generators then their Lie bracket or commutator $[X_i,\, X_j]$ is defined in a unique way, satisfying:

$$[X_i,\, X_j]=X_j\Big(X_i\Big)-X_i\Big(X_j\Big).$$

The nonzero commutation relationship for $L$ is:

$$[X_1,\, X_2] = X_2.$$

2: $\pounds_i$ and $\pounds_j$ are said to be equivalent conjugacy classes if

$$\pounds_i = Ad\, X_i\left(\pounds_j\right),$$

where $X_i \in L$.
The general element will be

$$\pounds = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4.$$

After applying adjoint transformation on $\pounds$ and renaming the coefficients, one gets [8]:

$$\pounds_1 =< X_2 >, \quad \pounds_2 =< aX_3 + bX_4 >, \quad \pounds_3 =< X_2 + \varepsilon(aX_3 + bX_4) >, \quad \pounds_4 =< X_1 + \alpha(aX_3 + bX_4) >,$$

where $a = \cos\theta$, $b = \sin\theta$ and $\theta \in [0,\pi)$. While $\alpha$ is a real number and $\varepsilon = \pm 1$.

## 2.2. Reduction by calculating similarity variables

This section is devoted for the computation of similarity variables and their corresponding reduced equations.

### 2.2.1. $\pounds_1 =< X_2 >$

Using Eq.(2.10), the characteristic system becomes:

$$\frac{dx}{1} = \frac{dA}{0} = \frac{dB}{0},$$

which can be separated into the following linear equations:

$$(i): \ \frac{dA}{0} = \frac{dx}{1} \quad (ii): \ \frac{dx}{1} = \frac{dB}{0}.$$

It leads towards a constant solution.

### 2.2.2. $\pounds_2 =< aX_3 + bX_4 >$

I. $a = 1,\ b = 0$.

It gives:

$$X_3 = A\frac{\partial}{\partial A}.$$

Using Eq.(2.10), this operator has the following characteristic system:

$$\frac{dA}{A} = \frac{dx}{0} = \frac{dB}{0},$$

which can be separated into the following linear equations:

$$(i): \ \frac{dA}{A} = \frac{dx}{0} \quad (ii): \ \frac{dA}{A} = \frac{dB}{0}.$$

Integrating $((i)-(ii))$, one by one we get the following results respectively:

$$x = \xi_1, \quad B = \xi_2,$$

where $\xi_1$ and $\xi_2$ are constants of integration. Designating second invariant as a function of first one gives:

$$B = g(\xi_1),$$

where $A$ is a constant function. Using $B = g(\xi_1)$ in the system (2.5), one can obtain $A = B =$ constant, leaving a Minkowskian space-time.

II. $a = 0, b = 1,$ **(Linearly independent relation)**

Similarly,

$$x = \xi_1, \quad A = \xi_2.$$

Designating second invariant as a function of first one it yields: $A = g(\xi_1)$ and $B$ is a constant function. System (2.5) with $A = g(\xi_1)$ gives the following solution

$$A = C_3(x - 2C_2)^2, \ B = C_1,$$

where $C_i$'s are real constants. By some suitable substitutions, it can be shown that this solution corresponds to conformally flat space-time.

## 3. Concluding remarks

This paper is devoted to investigate the plane symmetric solutions in GR. Here the Lie point symmetry approach has been used to investigate the plane symmetric solutions. When infinitesimal generator is applied to the system (2.5), an over-determined system of linear partial differential equations is obtained. Thus solution of this system gives four non-trivial Lie point generators, which are further utilized to compute the one-dimensional optimal system. Different cases are generated from these classes which in turn suggest both linearly independence and dependence relationships between the parameters $A$ and $B$ for plane symmetric space-time in GR.
It is expected that this approach may recover some new solutions. It would be worthwhile to extend the analysis to explore the solutions for other space-times. In particular, the solutions of field equations in modified theories of gravity can be interested to discuss the issue of dark energy and expansion of universe. Investigation of non-vacuum plane symmetric solutions in $f(R)$ gravity using Lie approach is under process.

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# The Heat Transfer Problem in a Rigid and Nonconvex Gray Body with Temperature Dependent Thermal Conductivity

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**Abstract**

This work studies coupled steady-state conduction-radiation heat transfer in a non-convex gray body when the thermal conductivity temperature-dependent. The gray-body assumption is an improvement with respect to the black body model, because in this model a portion of the incident radiant energy can be reflected from the body boundary. The problem is mathematically described by a nonlinear partial differential equation subjected to a nonlinear boundary condition involving a Fredholm operator which arises from the non-convexity of the body. In this problem the absolute temperature distribution is the unknown, as in the case of a black body. The Kirchhoff transformation is employed to linearize the partial differential equation, giving rise to new boundary conditions. The solution of the problem is constructed by a proposed iterative procedure, employing sequences that involve the temperature and the radiosity. The convergence is explicitly demonstrated. Besides, an error estimate, for each element, is presented. It is remarkable that the results obtained for black bodies are a particular case of this work. In other words, the results presented in reference [1] consists of a particular case of this paper, obtained when the emissivity is equal to one.

## 1. Introduction

Although many times the black body assumption may be a good approximation for engineering purposes, thermal radiant energy is emitted, absorbed and reflected by any real body. This article may be considered as a generalization of [1], aiming at a more realistic description, since real bodies boundaries actually reflect part of the incident radiant energy.
When the body is surrounded by a rarefied atmosphere or when the temperature levels are high, this thermal radiation exchange must be taken into account. As a first approximation, the body could be treated as a blackbody. Nevertheless, a blackbody [1] does not reflect thermal radiant energy and, many times, this fact does not allow a good mathematical description for bodies in which the reflection is not negligible. The simplest physical model for taking into account the reflection of thermal radiant energy is based on the gray body assumption. This assumption may be regarded as an improvement of the black body hypothesis, since it takes into account the reflection of thermal radiant energy. In other words, it is allowed that a part of the incident thermal radiant energy is not absorbed, but reflected.
When the body boundary is assumed to be opaque, a conduction heat transfer process takes place inside it, while a thermal radiation heat transfer takes place from/to the body boundary.
In this work the body boundary is assumed opaque and gray. In other words, it is assumed that, after reaching the boundary, the incident thermal radiant energy be divided into two parcels: one reflected and one absorbed (transferred to the inner part of the body by conduction heat transfer).
Non-convex bodies are characterized by a direct thermal radiant energy interchange among non-neighboring points on body boundary, since from all the energy coming from body boundary a portion of the emitted and a portion of the reflected thermal radiant energy reaches directly the same boundary. This effect causes more complex boundary conditions [2], [3].

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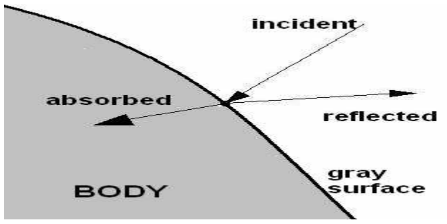


**Figure 1.1:** Scheme of a gray surface

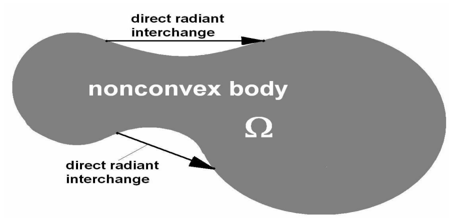


**Figure 1.2:** A nonconvex body. Radiant interchange among points on the boundary must be accounted for.

The mathematical modeling of the energy transfer process in a gray body is the subject of this work, in which it is assumed that the body may be non-convex and may have a temperature dependent thermal conductivity. The considered phenomenon can be found almost everywhere, since most bodies are not convex and possess conductivities that depend on the temperature.
Thermal radiation heat transfer and conduction heat transfer coupling occurs on the boundary of the body, since the normal heat flux must be continuous at any boundary point. So, an inherently nonlinear boundary condition must be satisfied.
In addition, conduction heat transfer is the energy transfer in the interior of the body. An elliptic partial differential equation [4] describes this steady-state conduction heat transfer. This partial differential equation becomes nonlinear in case of temperature-dependent thermal conductivity.
The thermal conductivity depends on the temperature [5], [6], [7], [8] and on the pressure [9] for all real materials. Under the rigid body assumption, in some cases, these dependences may be disregarded. However in some cases, it is important to consider a temperature-dependent thermal conductivity, since a body can experience large variations of conductivity [10],[11],[12],[13],[14].
Thermal radiant energy is emitted from all the points on the boundary of any real body because all the body points are at a temperature level above the absolute zero. In addition, there is a reflection of thermal radiant energy from all the points on the boundary which receive any thermal radiation.
This body may be represented by the bounded open set $\Omega$. When $\Omega$ is not convex, a part of the thermal radiant energy emitted and/or reflected from body boundary will reach, directly, the body, acting as an external temperature dependent heat source.
Assuming the body rigid, opaque and at rest, the energy transfer process inside $\Omega$ occurs solely by conduction heat transfer.
So, there is a coupling between a conduction heat transfer (inside $\Omega$) and a thermal radiant heat transfer (from/to $\partial\Omega$), concerning the energy transfer.
The main objective of this work is to construct the solution for the steady-state energy transfer process in a gray body with temperature-dependent thermal conductivity. Also existence and uniqueness of the solution of this solution are demonstrated.
Reference [1] may be considered as a particular case of the present work, which arises when the emissivity is equal to one (that means black body).
The mathematical description for a gray body is much more complex than the description for a black body, due to the boundary conditions. These boundary conditions must take into account that only a part of the incident thermal radiant energy is absorbed.

## 2. Mathematical description

The following partial differential equation [15], [16] describes the conduction heat transfer process inside $\Omega$, considering steady-state regimen

$$\nabla \cdot (k \nabla T) + \dot{q} = 0, \ \ in \ \ \Omega \ , \qquad k = \hat{k}(T) \tag{2.1}$$

in which $\dot{q}$ represents a known nonnegative field (an internal heat source) and $k$ is the thermal conductivity, a real positive-valued quantity depending on the temperature $T$ .
Since $k = \hat{k}(T)$, equation (2.1) is not linear.
Considering a gray body, the thermal radiant energy emitted from a point on the boundary $\partial\Omega$ (per unit time and per unit area) is given by $\varepsilon\sigma|T^3|T$. In this expression, the Stefan-Boltzmann constant is represented by $\sigma$, while the emissivity is is represented by $\varepsilon$, being such that $0 < \varepsilon \leq 1$ . When $\varepsilon = 1$ the black body hypothesis is assumed [15]. The incident thermal radiant energy at a given point $\mathbf{x} \in \partial\Omega$ (per unit time and per unit area) is given by [17],

$$H = \hat{H}(\mathbf{x}) = \int_{\mathbf{y}\in\partial\Omega} \hat{B}(\mathbf{y})\, K dS + \hat{s}(\mathbf{x}), \qquad K = \hat{K}(\mathbf{x},\mathbf{y}) \qquad \forall \mathbf{x} \in \partial\Omega \tag{2.2}$$

in which $B = \hat{B}(\mathbf{x})$ is called radiosity (the sum of both emitted and reflected energy at a given point on the boundary). In equation (2.2) two effects are considered: the existence of an external thermal radiant source, $s = \hat{s}(\mathbf{x})$, and also the thermal radiation effect which emerges from points on $\partial\Omega$ and reaches the point $\mathbf{x} \in \partial\Omega$. The quantity $T = \hat{T}(\mathbf{x})$ in (2.2) stands for the absolute temperature at the point $\mathbf{x} \in \partial\Omega$.

The kernel $K$ depends solely on the shape of $\Omega$ [17], being such that

$$0 \leq \int_{\mathbf{y}\in\partial\Omega} K dS = \hat{\eta}(\mathbf{x}) \leq \beta < 1, \qquad \forall \mathbf{x} \in \partial\Omega \tag{2.3}$$

in which $\beta$ is a constant.
The difference between the emitted and the absorbed thermal radiant energy is called thermal radiant heat flux from/to body boundary and is expressed in this work as $q_{RAD}$. This difference is exactly the difference between the radiosity and the incident thermal radiation. In other words,

$$q_{RAD} = B - H \ \ on \ \ \partial\Omega$$

$$B = \varepsilon\sigma|T^3|T + (1-\varepsilon)H$$

in which $(1-\varepsilon)$ is the reflectance, since the body is opaque [17]. In a more convenient form

$$B = \hat{B}(\mathbf{x})\varepsilon\sigma|\hat{T}(\mathbf{x})|^3\hat{T}(\mathbf{x}) + (1-\varepsilon)\int_{\mathbf{y}\in\partial\Omega}\hat{B}(\mathbf{y})KdS + \hat{s}(\mathbf{x}), \qquad \forall \mathbf{x} \in \partial\Omega \tag{2.4}$$

Once $T$ is known, equation (2.4) is a Fredholm integral equation of the second kind.
The continuity of the normal heat flux across the boundary is ensured when the normal conduction heat flux equals the thermal radiant heat flux on $\partial\Omega$.
In consequence,

$$\mathbf{q}\cdot\mathbf{n} = -k\nabla T\cdot\mathbf{n} = B - H, \ \ on \ \ \partial\Omega$$

$$\begin{aligned} \nabla\cdot(k\nabla T) + \dot{q} &= 0 \ \ in \ \ \Omega \\ -k\nabla T\cdot\mathbf{n} &= B - H \ \ on \ \ \partial\Omega \\ B &= \varepsilon\sigma|T^3|T + (1-\varepsilon)H \ \ on \ \ \partial\Omega \\ H &= \Im[B] + \ \ on \ \ \partial\Omega \end{aligned} \tag{2.5}$$

in which the unknown is the absolute temperature field $T$ and the linear operator $\Im[B]$ is defined by

$$\Im[B] = \int_{\mathbf{y}\in\partial\Omega}\hat{B}(\mathbf{y})KdS, \quad \mathbf{x}\in\partial\Omega$$

## 3. The Kirchhoff transformation

In order to employ the Kirchhoff transformation, the first step is the definition of an operator $\aleph$ such that [18]

$$\aleph[T] = \int_0^t \hat{T}(\xi)d\xi \tag{3.1}$$

At this point one can take advantage of the fact that $\nabla\cdot(k\nabla T) = \nabla\cdot(\nabla\omega)$. In this case, the following differential equation is satisfied by the function $\omega = \aleph[T]$

$$\nabla\cdot(\nabla\omega) + \dot{q} = 0, \ \ in \ \ \Omega \tag{3.2}$$

Once $\dot{q}$ is known, equation (3.2) is linear.
Considering any real material, the thermal conductivity $k$ is always a positive valued function. Actually, for any material, the thermal conductivity has a positive lower bound and the following inequality always holds

$$\hat{k}(T) \geq \delta > 0 \quad for\ any\ T, \quad \delta = costant$$

So, the existence of the inverse of the Kirchhoff transformation, denoted by $T = \aleph^{-1}[\omega]$ is automatically ensured.
At this point it is convenient to use the Kirchhoff transformation to express the considered problem (problem (2.5)) in terms of the variable $\omega$

$$\begin{aligned} \nabla\cdot(\nabla\omega) + \dot{q} &= 0 \ \ in \ \ \Omega \\ -\nabla\omega\cdot\mathbf{n} &= B - H, \ \ on \ \ \partial\Omega \\ B &= \varepsilon\sigma|T|^3T + (1-\varepsilon)H, \ \ on \ \ \partial\Omega \\ H &= \Im[B] + s, \ \ on \ \ \partial\Omega \\ T &= \aleph^{-1}[\omega] \end{aligned} \tag{3.3}$$

in which the unknown is $\omega$. It important to note that the determination of $\omega$ leads to the unique determination of the absolute temperature. Problem (3.3) may be also represented as

$$\begin{aligned} \nabla\cdot(\nabla\omega)+\dot{q}=0 \;\; in \;\; \Omega \\ -\nabla\omega\cdot\mathbf{n}=\varepsilon\sigma|T|^3T-\varepsilon\,(\Im[B]+s) \;\; on \;\; \partial\Omega \\ B=\varepsilon\sigma|T|^3T+(1-\varepsilon)\,(\Im[B]+s) \;\; on \;\; \partial\Omega \\ T=\aleph^{-1}[\omega] \\ \Im[B]=\int_{\mathbf{y}\in\partial\Omega}\hat{B}(\mathbf{y})\,KdS, \quad \mathbf{x}\in\partial\Omega \end{aligned} \tag{3.4}$$

## 4. Constructing the solution of problems with nonconvexity effects

To reach the solution of problem (2.5), the sequences $[\Phi^1,\Phi^2,\Phi^3,\dots]$ and $[\Psi^1,\Psi^2,\Psi^3,\dots]$ will be constructed. Their elements are obtained from

$$\begin{aligned} \nabla\cdot(\nabla\Psi^{i+1})+\dot{q}=0, \;\; in \;\; \Omega \\ -(\nabla\Psi^{i+1})\cdot\mathbf{n}=\varepsilon\sigma|\Phi^{i+1}|^3\Phi^{i+1}-\varepsilon\,(\Im[\Theta^{i+1}]+s), \;\; on \;\; \partial\Omega \\ \Theta^{i+1}=\varepsilon\sigma|\Phi^{i+1}|^3\Phi^{i+1}+(1-\varepsilon)\,(\Im[\Theta^{i+1}]+s), \;\; on \;\; \partial\Omega \\ \Phi^i=\aleph^{-1}[\Psi^i] \\ i=1,2,3,\dots \end{aligned} \tag{4.1}$$

in which

$$\Im[\Theta^i]=\int_{\mathbf{y}\in\partial\Omega}\hat{\Theta}^i(\mathbf{y})KdS\,, \quad \mathbf{x}\in\partial\Omega \quad , \quad i=1,2,3,\dots$$

and in such a way that

$$\omega\equiv\lim_{i\to\infty}\Psi^i \qquad and \qquad T\equiv\lim_{i\to\infty}\Phi^i$$

with $\Phi^0\equiv 0$ and $\Theta^0\equiv 0$. The existence of each element of the sequences is ensured from the coerciveness of the associated operator [19], [1].

**Proposition 4.1.** *The solution of the problem (3.3), denoted by $T$ and such that $\omega=\aleph[T]$, is the limit of the sequence $[\Phi^1,\Phi^2,\Phi^3,\dots]$ whose elements are obtained from (4.1), with $\Phi^0\equiv 0$ and $\Theta^0\equiv 0$.*

*Proof.* (Part one).
The fields $\Phi^1$ and $\Psi^1$ are non negative.
Since $\dot{q}\geq 0$, the following nonempty subset $\partial\Omega^-$ may be defined

$$\partial\Omega^-\equiv\{\mathbf{x}\in\partial\Omega \;\; such\ that \;\; (\nabla\Phi^1)\cdot\mathbf{n}\leq 0\}$$

$$\inf_{\Omega}\Psi^1=\inf_{\partial\Omega}\Psi^1=\inf_{\partial\Omega^-}\Psi^1 \tag{4.2}$$

and, from the boundary conditions of (4.1),

$$\varepsilon\sigma|\Phi^1|^3\Phi^1-\varepsilon\,(\Im[\Theta^0]+s)\geq 0 \;\; on \;\; \partial\Omega^- \quad\Rightarrow\quad \varepsilon\sigma|\Phi^1|^3\Phi^1-\varepsilon\, s\geq 0 \;\; on \;\; \partial\Omega^-$$

$$\Phi^1\geq 0 \;\; on \;\; \partial\Omega^- \quad\Rightarrow\quad \inf_{\Omega^-}\Phi^1\geq 0$$

Therefore, from (4.2),

$$\Psi^1=\aleph[\Phi^1]=\int_0^{\Phi^1}\hat{k}(\xi)d\xi\geq 0 \;\; on \;\; \partial\Omega^- \quad\Rightarrow\quad \Psi^1\geq 0 \;\; in \;\; \Omega$$

and, from (3.1),

$$\Phi^1\geq 0 \;\; in \;\; \Omega \tag{4.3}$$

□

*Proof.* (Part two).
The fields $\Phi^i$ and $\Psi^i$ are such that $\Phi^{i+1} \geq \Phi^i$ and $\Psi^{i+1} \geq \Psi^i$.
From (4.1) it comes that

$$\begin{aligned} \nabla \cdot (\nabla (\Psi^{i+1} - \Psi^i)) = 0, \;\; in \;\; \Omega \\ -(\nabla (\Psi^{i+1} - \Psi^i)) \cdot \mathbf{n} = \varepsilon\sigma|\Phi^{i+1}|^3\Phi^{i+1} - \varepsilon\sigma|\Phi^i|^3\Phi^i - \\ -(\varepsilon\Im[\Theta^i] + s) - (\varepsilon\Im[\Theta^{i-1}] + s), \;\; on \;\; \partial\Omega \end{aligned} \tag{4.4}$$

Now the nonempty subset $\partial\Omega^-_{i+1}$ may be defined as

$$\partial\Omega^-_{i+1} \equiv \{\mathbf{x} \in \partial\Omega \;\; such \; that \;\; (\nabla(\Phi^{i+1} - \Phi^i)) \cdot \mathbf{n} \leq 0\}$$

such that

$$\inf_{\Omega}(\Psi^{i+1} - \Psi^i) = \inf_{\partial\Omega}(\Psi^{i+1} - \Psi^i) = \inf_{\partial\Omega^-_{i+1}}(\Psi^{i+1} - \Psi^i) \tag{4.5}$$

$$\varepsilon\sigma|\Phi^{i+1}|^3\Phi^{i+1} - \varepsilon\sigma|\Phi^i|^3\Phi^i \geq \varepsilon(\Im[\Theta^i] - \Im[\Theta^{i-1}]) \;\; on \;\; \partial\Omega^-_{i+1}$$

Therefore, considering that (4.3) is verified and $\Phi^0 \equiv 0$,

$$\Phi^1 = \varepsilon\sigma|\Phi^1|^3\Phi^1 + (1-\varepsilon)s \geq 0 \;\; on \;\; \partial\Omega \;\;\; \Rightarrow \;\;\; \Im[\Theta^1] \geq \Im[\Theta^0] \;\; on \;\; \partial\Omega$$

Thus,

$$\Phi^2 - \Phi^1 \geq 0 \;\; on \;\; \partial\Omega^-_2$$

$$\Psi^2 - \Psi^1 \geq 0 \;\; on \;\; \partial\Omega^-_2$$

Hence, from (4.5)

$$\inf_{\Omega}(\Psi^2 - \Psi^1) = \inf_{\partial\Omega}(\Psi^2 - \Psi^1) = \inf_{\partial\Omega^-_2}(\Psi^2 - \Psi^1) \;\;\; \Rightarrow \;\;\; \Psi^2 \geq \Psi^1 \;\; in \;\; \Omega$$

and, consequently

$$\Phi^2 \geq \Phi^1 \;\; on \;\; \partial\Omega \;\; and \;\; \Theta^2 \geq \Theta^1 \;\; on \;\; \partial\Omega$$

The above procedure can be repeated in order to lead to the conclusion that $\Psi^3 \geq \Psi^2$ in $\Omega$, that $\Phi^3 \geq \Phi^2$ on $\partial\Omega$ and also that $\Theta^3 \geq \Theta^2$ on $\partial\Omega$. So, the following may be written,

$$\Psi^{i+1} \geq \Psi^i \;\; in \;\; \Omega \;\; , \;\; \Phi^{i+1} \geq \Phi^i \;\; on \;\; \partial\Omega \;\; and \;\; \Theta^{i+1} \geq \Theta^i \;\; on \;\; \partial\Omega \tag{4.6}$$

The above result (2.3) is very important because it allows ensuring that the sequences $[\Psi^1, \Psi^2, \Psi^3, \ldots]$, $[\Phi^1, \Phi^2, \Phi^3, \ldots]$ and $[\Theta^1, \Theta^2, \Theta^3, \ldots]$ are non decreasing.

□

*Proof.* (Part three). Convergence of the sequence $[\Theta^1, \Theta^2, \Theta^3, \ldots]$.
Now one should prove that the sequence $[\Theta^1, \Theta^2, \Theta^3, \ldots]$ converges in the norm of $L^1(\partial\Omega)$. To achieve this objective problem (4.4) is considered, allowing to conclude that

$$\int_{\partial\Omega}(\varepsilon\sigma|\Phi^{i+1}|^3\Phi^{i+1} - \varepsilon\sigma|\Phi^i|^3\Phi^i)dS = \int_{\mathbf{x}\in\partial\Omega}\int_{\mathbf{y}\in\partial\Omega}\varepsilon(\hat{\Theta}^i(\mathbf{y}) - \hat{\Theta}^{i-1}(\mathbf{y}))KdS$$

Considering the following property of the kernel $K(\mathbf{x}, \mathbf{y}) \equiv K(\mathbf{y}, \mathbf{x})$, equation (2.3) enables to write

$$\int_{\partial\Omega}\varepsilon\sigma(|\Phi^{i+1}|^3\Phi^{i+1} - |\Phi^i|^3\Phi^i)dS \leq \beta\int_{\partial\Omega}\varepsilon(\Theta^i - \Theta^{i-1})dS \tag{4.7}$$

In addition, from (4.1)

$$\Theta^i - \Theta^{i-1} = \varepsilon\sigma(|\Phi^i|^3\Phi^i - |\Phi^{i-1}|^3\Phi^{i-1}) + (1-\varepsilon)(\Im[\Theta^{i-1}] - \Im[\Theta^{i-2}]) \tag{4.8}$$

Equation (4.8) enables to write

$$\int_{\partial\Omega}(\Theta^i-\Theta^{i-1})dS=\int_{\partial\Omega}\varepsilon\sigma(|\Phi^i|^3\Phi^i-|\Phi^{i-1}|^3\Phi^{i-1})dS+\int_{\partial\Omega}(1-\varepsilon)(\Im[\Theta^{i-1}]-\Im[\Theta^{i-2}])dS$$

and, from (2.3), the following inequality comes

$$\int_{\partial\Omega}(\Theta^i-\Theta^{i-1})dS\leq\int_{\partial\Omega}\varepsilon\sigma(|\Phi^i|^3\Phi^i-|\Phi^{i-1}|^3\Phi^{i-1})dS+\beta\int_{\partial\Omega}(1-\varepsilon)(\Theta^{i-1}-\Theta^{i-2})dS$$

Now, taking equation (4.7) into account, the following holds

$$\int_{\partial\Omega}(\Theta^i-\Theta^{i-1})dS\leq\beta\int_{\partial\Omega}\varepsilon(\Theta^{i-1}-\Theta^{i-2})dS+\beta\int_{\partial\Omega}(1-\varepsilon)(\Theta^{i-1}-\Theta^{i-2})dS$$

and, hence,

$$\int_{\partial\Omega}(\Theta^i-\Theta^{i-1})dS\leq\beta\int_{\partial\Omega}(\Theta^{i-1}-\Theta^{i-2})dS$$

Since $\Theta^{i+1}\geq\Theta^i$ there is a contraction, ensuring the convergence of the sequence $[\Theta^1,\Theta^2,\Theta^3,\dots]$ in $L^1(\partial\Omega)$.
In addition, for any integer $i\geq 0$

$$\|\Theta^{i+1}-\Theta^i\|_{L^1(\partial\Omega)}=\int_{\partial\Omega}(\Theta^{i+1}-\Theta^i)dS\leq\beta^i\int_{\partial\Omega}\Theta^1dS=\bar{M}\beta^i \tag{4.9}$$

in which $\bar{M}$ is a positive constant. Therefore it follows that

$$\Theta^\infty\equiv\lim_{i\to\infty}\Theta^i \ \ on\ \ \partial\Omega\,,\quad with\ \ \Theta^\infty\in L^1(\partial\Omega)$$

□

*Proof.* (Part four). Convergence of the sequence $[\Phi^1,\Phi^2,\Phi^3,\dots]$ on the boundary.
At this point the convergence of the sequence $[\Phi^1,\Phi^2,\Phi^3,\dots]$ can be demonstrated . Aiming to this, one can consider again (4.4), to write

$$\int_{\partial\Omega}(\varepsilon\sigma|\Phi^{i+1}|^3\Phi^{i+1}-\varepsilon\sigma|\Phi^i|^3\Phi^i)dS=\int_{\mathbf{x}\in\partial\Omega}\int_{\mathbf{y}\in\partial\Omega}\varepsilon(\hat{\Theta}^i(\mathbf{y})-\hat{\Theta}^{i-1}(\mathbf{y}))KdS$$

Inequality (2.3) allows to write

$$\int_{\partial\Omega}(\varepsilon\sigma|\Phi^{i+1}|^3\Phi^{i+1}-\varepsilon\sigma|\Phi^i|^3\Phi^i)dS\leq\beta\int_{\partial\Omega}\varepsilon(\Theta^i-\Theta^{i-1})dS$$

giving rise to

$$\int_{\partial\Omega}\sigma(|\Phi^{i+1}|^3\Phi^{i+1}-|\Phi^i|^3\Phi^i)dS\leq\bar{M}\beta^i\left(\frac{1-\varepsilon}{1-\beta\varepsilon}\right)^{i-1}\quad,\quad i=1,2,3,\dots$$

In addition, considering that $\Phi^{i+1}\geq\Phi^i\geq 0$, it comes that

$$(\|\Phi^{i+1}-\Phi^i\|_{L^4(\partial\Omega)})^4=\int_{\partial\Omega}|\Phi^{i+1}-\Phi^i|^4dS\leq\bar{M}\beta^i\left(\frac{1-\varepsilon}{1-\beta\varepsilon}\right)^{i-1}\quad,\quad i=1,2,3,\dots$$

The convergence is quite obvious for $\beta=0$. According to reference [20] there is a contraction for $0<\beta<1$. So, since $\Phi^i$ must belong to the Banach space $L^4(\partial\Omega)$, the sequence $[\Phi^1,\Phi^2,\Phi^3,\dots]$ converges in this space. This allows concluding that

$$\Phi^\infty\equiv\lim_{i\to\infty}\Phi^i \ \ on\ \ \partial\Omega\,,\quad with\ \ \Phi^\infty\in L^4(\partial\Omega)$$

□

*Proof.* (Part five). The limit of the sequence $[\Psi^1, \Psi^2, \Psi^3, \dots]$ is a solution of (4.4).
From problem (4.1), the limit of the sequence $[\Psi^1, \Psi^2, \Psi^3, \dots]$, expressed as by $\Psi^\infty$, may be represented as the solution of

$$\begin{aligned} \nabla \cdot (\nabla \Psi^\infty) + \dot{q} = 0 \quad in \;\; \Omega \\ -(\nabla \Psi^\infty) \cdot \mathbf{n} = \varepsilon\sigma \left| \aleph^{-1}[\Psi^\infty] \right|^3 \left( \aleph^{-1}[\Psi^\infty] \right) - E \quad on \;\; \partial\Omega \end{aligned}$$

in which $E = \hat{E}(\mathbf{x})$ is a known function defined on the boundary as follows

$$E = \varepsilon \left( \Im[\Theta^\infty] + s \right) = \varepsilon \left( \int_{\mathbf{y}\in\partial\Omega} \hat{\Theta}^\infty K dS + s \right) \quad on \;\; \partial\Omega$$

Therefore it may be concluded that $\omega \equiv \Psi^\infty$, limit of the sequence $[\Psi^1, \Psi^2, \Psi^3, \dots]$, is a solution of problem (3.3), thus demonstating the solution existence. The uniqueness of the solution is demonstrated in the next step.

□

*Proof.* (Part six). The limit of the sequence corresponds to the unique solution of (3.3).
To demonstrate that the limit of the sequence $[\Psi^1, \Psi^2, \Psi^3, \dots]$ is the unique solution of (3.4), one can assume the existence of a field $u$, which is supposed different from $\Psi^\infty$, and satisfies

$$\begin{aligned} \nabla \cdot (\nabla u) + \dot{q} = 0 \;\; in \;\; \Omega \\ -\nabla u \cdot \mathbf{n} = \varepsilon\sigma |v|^3 v - \varepsilon \left( \Im[z] + s \right) \;\; on \;\; \partial\Omega \\ z = \varepsilon\sigma |v|^3 v + (1-\varepsilon) \left( \Im[z] + s \right) \;\; on \;\; \partial\Omega \\ v = \aleph^{-1}[u] \\ \Im[z] = \int_{\mathbf{y}\in\partial\Omega} \hat{z}(\mathbf{y}) K dS \,, \quad \mathbf{x} \in \partial\Omega \end{aligned} \tag{4.10}$$

Defining the nonempty subset $\partial\Omega_u^-$ as

$$\partial\Omega_u^- \equiv \left\{ \mathbf{x} \in \partial\Omega \quad such\ that \quad (\nabla\, u) \cdot \mathbf{n} \le 0 \right\}$$

And, reminding that $(\nabla\, v) \cdot \mathbf{n} \le 0$ on $\partial\Omega_u^-$, it comes that

$$\inf_{\Omega} u = \inf_{\partial\Omega} u = \inf_{\partial\Omega_u^-} u \qquad and \qquad \inf_{\Omega} v = \inf_{\partial\Omega} v = \inf_{\partial\Omega_u^-} v \tag{4.11}$$

$$\begin{aligned} \varepsilon\sigma |v|^3 v - \varepsilon \left( \Im[z] + s \right) \ge 0 \;\; on \;\; \partial\Omega_u^- \\ z = \varepsilon\sigma |v|^3 v + (1-\varepsilon) \left( \Im[z] + s \right) \;\; on \;\; \partial\Omega_u^- \\ v = \aleph^{-1}[u] \end{aligned} \tag{4.12}$$

Since the quantity $s$ is nonnegative and (4.11) is verified, (4.12) leads to

$$\begin{aligned} \varepsilon\sigma |v|^3 v - \varepsilon \left( \Im[z] + s \right) \ge 0 \;\; on \;\; \partial\Omega_u^- \qquad and \qquad z = \varepsilon\sigma |v|^3 v + (1-\varepsilon) \left( \Im[z] + s \right) \\ \Rightarrow \qquad \sigma |v|^3 v \ge \left( \Im[z] + s \right) \;\; on \;\; \partial\Omega_u^- \\ \Rightarrow \quad \sigma |v|^3 v \ge z \;\; on \;\; \partial\Omega_u^- \quad \Rightarrow \quad \sigma \left| \inf_{\partial\Omega_u^-} v \right|^3 \inf_{\partial\Omega_u^-} v \ge \inf_{\partial\Omega_u^-} z \quad \Rightarrow \quad \sigma \left| \inf_{\partial\Omega} v \right|^3 \inf_{\partial\Omega} v \ge \inf_{\partial\Omega} z \end{aligned} \tag{4.13}$$

In addition, from (4.12) and (4.13), it comes that

$$\inf_{\partial\Omega} z \ge \varepsilon \inf_{\partial\Omega} z + (1-\varepsilon) \ge \inf_{\partial\Omega} \left( \Im[z] \right) \tag{4.14}$$

Since,

$$\inf_{\partial\Omega} \left( \Im[z] \right) \ge \hat{\eta}(\mathbf{x}) \inf_{\partial\Omega} z \qquad for\ some \;\; \mathbf{x} \in \partial\Omega$$

from (4.14), one can state that

$$\inf_{\partial\Omega} z \ge \inf_{\partial\Omega} \left( \Im[z] \right) \ge \hat{\eta}(\mathbf{x}) \inf_{\partial\Omega} z \qquad for\ some \;\; \mathbf{x} \in \partial\Omega$$

in which $0 \le \hat{\eta}(\mathbf{x}) < \beta$. Therefore,

$$\inf_{\partial\Omega} z \geq 0 \quad \Rightarrow \quad \inf_{\partial\Omega} v \geq 0 \quad \Rightarrow \quad \inf_{\Omega} v \geq 0$$

Now, considering (3.1), the following holds

$$\inf_{\partial\Omega} v \geq 0 \quad \Rightarrow \quad \inf_{\partial\Omega} u \geq 0 \quad \Rightarrow \quad v \geq 0 \;\; in \;\; \Omega \quad \Rightarrow \quad u \geq 0 \;\; in \;\; \Omega$$

Combining (4.10) with (4.1), gives rise to

$$\begin{aligned} \nabla \cdot \left(\nabla\left(u-\Psi^{i+1}\right)\right) = 0 \quad in \;\; \Omega \\ -\left(\nabla\left(u-\Psi^{i+1}\right)\right)\cdot \mathbf{n} = \varepsilon\sigma\left(|v|^3 v - |\Phi^{i+1}|^3\Phi^{i+1}\right) - \varepsilon\left(\Im[z]-\Im[\Theta^i]\right) \quad on \;\; \partial\Omega \\ z-\Theta^{i+1} = \varepsilon\sigma\left(|v|^3 v - |\Phi^{i+1}|^3\Phi^{i+1}\right) + (1-\varepsilon)\left(\Im[z]-\Im[\Theta^i]\right) \quad on \;\; \partial\Omega \\ v = \aleph^{-1}[u]\;, \;\; \Phi^i = \aleph^{-1}[\Psi^i] \\ \Im[z] = \int_{\mathbf{y}\in\partial\Omega} \hat{z}(\mathbf{y})\, K dS\;, \quad \mathbf{x}\in\partial\Omega \\ \Im[\Theta^i] = \int_{\mathbf{y}\in\partial\Omega} \hat{\Theta}^i(\mathbf{y})\, K dS\;, \quad \mathbf{x}\in\partial\Omega \end{aligned} \tag{4.15}$$

Defining the nonempty subset $\partial\Omega_u^{i+1}$ as

$$\partial\Omega_u^{i+1} \equiv \left\{\mathbf{x}\in\partial\Omega \quad such\ that \quad \nabla\left(u-\Psi^{i+1}\right)\cdot\mathbf{n} \leq 0\right\}$$

it comes that

$$\inf_{\Omega}\left(u-\Psi^{i+1}\right) = \inf_{\partial\Omega}\left(u-\Psi^{i+1}\right) = \inf_{\partial\Omega_u^{i+1}}\left(u-\Psi^{i+1}\right)$$

So,

$$u \geq \Psi^{i+1} \quad on \;\; \partial\Omega_u^{i+1} \quad \Rightarrow \quad u \geq \Psi^{i+1} \quad in \;\; \Omega \;\; for\ any\ i$$

and, accounting for (3.1) it comes that

$$u \geq \Psi^{i+1} \quad in \;\; \Omega \quad \Longleftrightarrow \quad v \geq \Phi^{i+1} \quad in \;\; \Omega \;\; for\ any\ i \tag{4.16}$$

Since

$$\varepsilon\sigma\left(|v|^3 v - |\Phi^{i+1}|^3\Phi^{i+1}\right) - \varepsilon\left(\Im[z]-\Im[\Theta^i]\right) \geq 0 \quad on \;\; \partial\Omega_u^{i+1}$$

and (4.6) and (4.16) hold, the following must also hold

$$\begin{aligned} z \geq \Theta^i \quad on \;\; \partial\Omega \quad \Rightarrow \quad v \geq \Psi^{i+1} \quad on \;\; \partial\Omega_u^{i+1} \quad \Rightarrow \\ \Rightarrow \quad u \geq \Psi^{i+1} \quad on \;\; \partial\Omega_u^{i+1} \quad \Rightarrow \quad u \geq \Psi^{i+1} \quad in \;\; \Omega \quad \Rightarrow \\ \Rightarrow \quad v \geq \Phi^{i+1} \quad in \;\; \Omega \quad \Rightarrow \quad z \geq \Theta^{i+1} \quad on \;\; \partial\Omega \quad \Rightarrow \quad \ldots \end{aligned}$$

Therefore, the following result can be ensured

$$z \geq \Theta^i \;\; on \;\; \partial\Omega \quad \Rightarrow \quad u \geq \Psi^{i+1} \quad in \;\; \Omega \quad and \quad v \geq \Phi^{i+1} \quad in \;\; \Omega \quad , \quad for\ any\ i \tag{4.17}$$

Since $\Theta^0 \equiv 0$ on $\partial\Omega$, (4.17) yields

$$u \geq \Psi^\infty \quad and \quad v \geq \Phi^\infty \quad in \;\; \Omega \quad and \quad z \geq \Theta^\infty \quad on \;\; \partial\Omega \tag{4.18}$$

The existence of the limits $\Psi^\infty$, $\Phi^\infty$ and $\Theta^\infty$ allows to write (4.15) , for $i\to\infty$

$$
\begin{aligned}
\nabla\cdot\left(\nabla\left(u-\Psi^{\infty}\right)\right)=0 \quad in \;\; \Omega \\
-\left(\nabla\left(u-\Psi^{\infty}\right)\right)\cdot\mathbf{n}=\varepsilon\sigma\left(|v|^{3}v-|\Phi^{\infty}|^{3}\Phi^{\infty}\right)-\varepsilon\left(\Im[z]-\Im[\Theta^{\infty}]\right) \quad on \;\; \partial\Omega \\
z-\Theta^{\infty}=\varepsilon\sigma\left(|v|^{3}v-|\Phi^{\infty}|^{3}\Phi^{\infty}\right)+(1-\varepsilon)\left(\Im[z]-\Im[\Theta^{\infty}]\right) \quad on \;\; \partial\Omega \\
v=\aleph^{-1}[u]\,,\;\; \Phi^{i}=\aleph^{-1}[\Psi^{\infty}] \\
\Im[z]=\int_{\mathbf{y}\in\partial\Omega}\hat{z}(\mathbf{y})\,KdS\,,\quad \mathbf{x}\in\partial\Omega \\
\Im[\Theta^{\infty}]=\int_{\mathbf{y}\in\partial\Omega}\hat{\Theta}^{\infty}(\mathbf{y})\,KdS\,,\quad \mathbf{x}\in\partial\Omega
\end{aligned}
\tag{4.19}
$$

Problem (4.19) allows to write

$$
\begin{aligned}
\int_{\partial\Omega}\left\{\varepsilon\sigma\left(|v|^{3}v-|\Phi^{\infty}|^{3}\Phi^{\infty}\right)-\varepsilon\left(\Im[z]-\Im[\Theta^{\infty}]\right)\right\}dS=0 \\
\int_{\partial\Omega}\left(z-\Theta^{\infty}\right)dS=\int_{\partial\Omega}\left\{\varepsilon\sigma\left(|v|^{3}v-|\Phi^{\infty}|^{3}\Phi^{\infty}\right)-\varepsilon\left(\Im[z]-\Im[\Theta^{\infty}]\right)\right\}dS
\end{aligned}
\tag{4.20}
$$

in which

$$
\int_{\partial\Omega}\left(\Im[z]-\Im[\Theta^{\infty}]\right)dS=\int_{\mathbf{x}\in\partial\Omega}\left(\int_{\mathbf{y}\in\partial\Omega}\left(\hat{z}(\mathbf{y})\hat{\Theta}(\mathbf{y})K\right)dS\right)dS\leq\beta\int_{\partial\Omega}(z-\Theta)dS \tag{4.21}
$$

Therefore, combining (4.20) with (4.21)

$$
\int_{\partial\Omega}(z-\Theta^{\infty})dS\leq\beta\int_{\partial\Omega}(z-\Theta^{\infty})dS \tag{4.22}
$$

From (4.18) and (4.22), since $0\leq\beta<1$, it can be concluded that

$$
z=\Theta^{\infty} \quad on \;\; \partial\Omega
$$

Hence,

$$
v=\Phi^{\infty} \quad in \;\; \Omega
$$

and the difference $u-\Psi^{\infty}$ is zero everywhere (that means $u=\Psi^{\infty}$ in $\Omega$), since it is the unique solution of the well-known Dirichlet problem [4]

$$
\begin{aligned}
\nabla\cdot\left(\nabla\left(u-\Psi^{\infty}\right)\right)=0 \quad in \;\; \Omega \\
u-\Psi^{\infty}=0 \quad on \;\; \partial\Omega
\end{aligned}
$$

Therefore, the temperature $T$, which is the solution of (2.5), is such that

$$
T\equiv\lim_{i\to\infty}\Psi^{i}=\Psi^{\infty} \quad in \;\; \Omega
$$

while $\omega(\omega=\aleph[T])$ is given by $\Psi^{\infty}$ and $B$ is given by $\Theta^{\infty}$.

□

## 5. On the difference between the solution and the i-th element of the sequence

At this section the goal is present a tool that enables verifying whether the element $\Phi^{i}$ is sufficiently precise for simulation purposes. Aiming at this goal, one can consider problem (4.15), replacing $u$ by $\omega$, $v$ by $T$ and $z$ by $B$, as follows

$$
\begin{aligned}
\nabla\cdot\left(\nabla\left(\omega-\Psi^{i+1}\right)\right)=0 \quad in \;\; \Omega \\
-\left(\nabla\left(\omega-\Psi^{i+1}\right)\right)\cdot\mathbf{n}=\varepsilon\sigma\left(|T|^{3}T-|\Phi^{i+1}|^{3}\Phi^{i+1}\right)-\varepsilon\left(\Im[B]-\Im[\Theta^{i}]\right) \quad on \;\; \partial\Omega \\
B-\Theta^{i+1}=\varepsilon\sigma\left(|T|^{3}T-|\Phi^{i+1}|^{3}\Phi^{i+1}\right)+(1-\varepsilon)\left(\Im[B]-\Im[\Theta^{i}]\right) \quad on \;\; \partial\Omega \\
T=\aleph^{-1}[\omega]\,,\;\; \Phi^{i}=\aleph^{-1}[\Psi^{i}] \\
\Im[B]=\int_{\mathbf{y}\in\partial\Omega}\hat{B}(\mathbf{y})\,KdS\,,\quad \mathbf{x}\in\partial\Omega
\end{aligned}
$$

Since $\nabla\cdot\left(\nabla\left(\omega-\Psi^{i+1}\right)\right)$ is zero everywhere, the boundary condition yields

$$\int_{\partial\Omega}\varepsilon\left(\sigma|T|^3T-|\Phi^{i+1}|^3\Phi^{i+1}\right)dS=\int_{\mathbf{x}\in\partial\Omega}\left(\int_{\mathbf{y}\in\partial\Omega}\varepsilon\left(\hat{B}(\mathbf{y})\hat{\Theta}^i(\mathbf{y})K\right)dS\right)dS$$

and, therefore,

$$\int_{\partial\Omega}\varepsilon\left(\sigma|T|^3T-|\Phi^{i+1}|^3\Phi^{i+1}\right)dS\leq\beta\int_{\partial\Omega}\varepsilon\left(B-\Theta^i\right)dS \tag{5.1}$$

Since

$$\int_{\partial\Omega}\varepsilon\left(B-\Theta^i\right)dS=\int_{\partial\Omega}\varepsilon\sigma\left(|T|^3T-|\Phi^{i+1}|^3\Phi^{i+1}\right)dS+\int_{\partial\Omega}(1-\varepsilon)\left(\Im[B]-\Im[\Theta^i]\right)dS \tag{5.2}$$

Combining (2.3), (5.1) and (5.2), one may write

$$\begin{aligned}\int_{\partial\Omega}\varepsilon\left(\sigma|T|^3T-\sigma|\Phi^{i+1}|^3\Phi^{i+1}\right)dS\leq\beta\int_{\partial\Omega}\varepsilon\left(B-\Theta^i\right)dS+\beta\int_{\partial\Omega}\varepsilon\left(\Theta^{i+1}-\Theta^i\right)dS\\ \int_{\partial\Omega}\left(B-\Theta^{i+1}\right)dS\leq\int_{\partial\Omega}\varepsilon\sigma\left(|T|^3T-|\Phi^{i+1}|^3\Phi^{i+1}\right)dS+\\ +\beta\int_{\partial\Omega}(1-\varepsilon)\left(\Theta^{i+1}-\Theta^i\right)dS+\beta\int_{\partial\Omega}(1-\varepsilon)\left(B-\Theta^i+1\right)dS\end{aligned} \tag{5.3}$$

Inequalities (5.3) allow to write

$$\int_{\partial\Omega}\left(B-\Theta^{i+1}\right)dS\leq\frac{\beta}{1-\beta}\int_{\partial\Omega}\left(\Theta^{i+1}-\Theta^i\right)dS$$

then, from (4.9)

$$\int_{\partial\Omega}\left(B-\Theta^{i+1}\right)dS\leq\frac{\beta}{1-\beta}\int_{\partial\Omega}\left(\Theta^{i+1}-\Theta^i\right)dS\leq\frac{\beta^{i+1}}{1-\beta}\int_{\partial\Omega}\Theta^1dS$$

From (4.1), when $i=1$, it comes that

$$\int_{\Omega}\dot{q}dV=\int_{\partial\Omega}\left(\varepsilon\sigma|\Phi^1|^3\Phi^1-\varepsilon s\right)dS+\int_{\partial\Omega}\left(\Theta^1-s\right)dS$$

Therefore,

$$\int_{\partial\Omega}(B-\Theta^i)dS\leq\frac{\beta^i}{1-\beta}\left(\int_{\Omega}\dot{q}dV+\int_{\partial\Omega}sdS\right) \tag{5.4}$$

and, from (5.1),

$$\int_{\partial\Omega}(\sigma|T|^3T-\sigma|\Phi^i|^3\Phi^i)dS\leq\frac{\beta^i}{1-\beta}\left(\int_{\Omega}\dot{q}dV+\int_{\partial\Omega}sdS\right) \tag{5.5}$$

Since $T\geq\Phi^i$, inequality (5.5) yields

$$\left\|T-\Phi^i\right\|_{L^4(\partial\Omega)}\leq C\beta^{i/4} \tag{5.6}$$

in which $\|\ \|_{L^4(\partial\Omega)}$ represents the usual $L^4(\partial\Omega)$ norm and $C$ is a constant defined as

$$C=\left(\tfrac{1}{1-\beta}\left(\int_{\Omega}\dot{q}dV+\int_{\partial\Omega}sdS\right)\right)^{1/4}$$

If the kernel $K$ is bounded, estimate (5.6) may be improved. In this case, a nonempty subset $\partial\Omega_{++}^{i+1}$ can be defined as

$$\partial\Omega_{++}^{i+1}\equiv\left\{\mathbf{x}\in\partial\Omega\quad \textit{such that}\quad\nabla\left(\omega-\Psi^{i+1}\right)\cdot\mathbf{n}\leq0\right\}$$

Therefore,

$$\sup_{\Omega}\left(\omega-\Psi^{i+1}\right)=\sup_{\partial\Omega}\left(\omega-\Psi^{i+1}\right)=\sup_{\partial\Omega_{++}^{i+1}}\left(\omega-\Psi^{i+1}\right)$$

and

$$\varepsilon\left(\sigma|T|^3T-\sigma|\Phi^{i+1}|^3\Phi^{i+1}\right)\leq\int_{\mathbf{y}\in\partial\Omega}\left(\hat{B}(\mathbf{y})-\hat{\Theta}^i(\mathbf{y})K\right)dS \quad on\ \partial\Omega_{++}^{i+1}$$

Considering that the kernel is bounded and $\varepsilon>0$, one may conclude that

$$\sigma|T|^3T-\sigma|\Phi^{i+1}|^3\Phi^{i+1}\leq\sup_{\mathbf{x}\in\partial\Omega,\mathbf{y}\in\partial\Omega}K\int_{\mathbf{y}\in\partial\Omega}\left(\hat{B}(\mathbf{y})-\hat{\Theta}^i(\mathbf{y})\right)dS \quad on\ \partial\Omega_{++}^{i+1}$$

and, accounting for inequation (5.4)

$$|T|^3T-|\Phi^{i+1}|^3\Phi^{i+1}\leq\sup_{\mathbf{x}\in\partial\Omega,\mathbf{y}\in\partial\Omega}(K)\left(\frac{\beta^{i+1}}{\sigma(1-\beta)}\left(\int_{\Omega}\dot{q}dV+\int_{\partial\Omega}sdS\right)\right) \quad on\ \partial\Omega_{++}^{i+1}$$

Since $T\geq\Phi^{i+1}$, it comes that

$$T-\Phi^{i+1}\leq C\left(\sup_{\mathbf{x}\in\partial\Omega,\mathbf{y}\in\partial\Omega}(K)\right)^{1/4}\left(\frac{\beta^{i+1}}{\sigma}\right)^{1/4} \quad on\ \partial\Omega_{++}^{i+1}$$

and the supremum of $T-\Phi^{i+1}$ on $\partial\Omega_{++}^{i+1}$ is such that

$$\sup_{\partial\Omega_{++}^{i+1}}(T-\Phi^{i+1})\leq C\left(\sup_{\mathbf{x}\in\partial\Omega,\mathbf{y}\in\partial\Omega}(K)\right)^{1/4}\left(\frac{\beta^{i+1}}{\sigma}\right)^{1/4}$$

Also,

$$\sup_{\partial\Omega_{++}^{i+1}}\left(\omega-\Psi^{i+1}\right)=\sup_{\partial\Omega_{++}^{i+1}}\left(\int_0^T\hat{k}(\xi)d\xi-\int_0^{\Phi^{i+1}}\hat{k}(\xi)d\xi\right)=\sup_{\partial\Omega_{++}^{i+1}}\left(\int_0^T\hat{k}(\xi)d\xi\right)\leq$$
$$\leq\left(\sup_{\partial\Omega_{++}^{i+1}}\left(T-\Phi^{i+1}\right)\right)\left(\sup_{\Phi^{i+1}<\xi<T}\left(\hat{k}(\xi)\right)\right)\leq\left(\sup_{\partial\Omega_{++}^{i+1}}\left(T-\Phi^{i+1}\right)\right)\left(\sup_{0<\xi<\infty}\left(\hat{k}(\xi)\right)\right)$$

and

$$\sup_{\Omega}\left(\omega-\Psi^{i+1}\right)=\sup_{\Omega}\left(\int_0^T\hat{k}(\xi)d\xi-\int_0^{\Phi^{i+1}}\hat{k}(\xi)d\xi\right)=\sup_{\Omega}\left(\int_0^T\hat{k}(\xi)d\xi\right)$$

Now, considering that

$$\sup_{\Omega}\left(\omega-\Psi^{i+1}\right)\geq\left(\inf_{0<\xi<\infty}\hat{k}(\xi)\right)\sup_{\Omega}\left(T-\Phi^{i+1}\right)$$

It may be concluded that

$$\sup_{\Omega}\left(T-\Phi^{i+1}\right)\leq\frac{\left(\sup_{0<\xi<\infty}\hat{k}(\xi)\right)}{\left(\inf_{0<\xi<\infty}\hat{k}(\xi)\right)}\left\{\sup_{\mathbf{x}\in\partial\Omega,\mathbf{y}\in\partial\Omega}(K)\left(\frac{\beta^{i+1}}{\sigma}\left(\int_{\Omega}\dot{q}dV+\int_{\partial\Omega}sdS\right)\right)\right\}^{1/4}$$

This could also be expressed as

$$\left\|T-\Phi^i\right\|_{L^\infty(\Omega)}\leq\bar{C}\beta^{1/4} \tag{5.7}$$

In (5.7) $\bar{C}$ represents a nonnegative constant, which is given by

$$\bar{C}=\frac{\left(\sup_{0<\xi<\infty}\hat{k}(\xi)\right)}{\left(\inf_{0<\xi<\infty}\hat{k}(\xi)\right)}\left\{\sup_{\mathbf{x}\in\partial\Omega,\mathbf{y}\in\partial\Omega}\left(K\right)\left(\frac{\beta^{i+1}}{\sigma}\left(\int_{\Omega}\dot{q}dV+\int_{\partial\Omega}sdS\right)\right)\right\}^{1/4}$$

and $\| \ \|_{L^{\infty}(\Omega)}$ represents the norm of the supremum.
In order to emphasize the relevance of this result, it is important to notice that any real surface $K(\mathbf{x},\mathbf{y})$ is bounded.

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# Positive Solutions for Systems of Fourth Order Two-Point Boundary Value Problems with Parameter

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**Abstract**

This paper deals with the existence of positive solutions for a system of nonlinear singular fourth-order differential equations with a parameter $\lambda$ subject two-point boundary conditions. Our analysis relies on the Krasnoselskii fixed point theorem and under suitable conditions, we derive explicit eigenvalue intervals of $\lambda$ for the existence of at least one positive solution for the system.

## 1. Introduction

We are concerned with determining intervals of the parameter $\lambda$ for which there exist positive solutions for the following boundary value problem of nonlinear differential system BVPs

$$\begin{cases} u^{(4)}(t)+\lambda a(t) f(v(t))=0, & 0<t<1, \\ v^{(4)}(t)+\lambda b(t) g(u(t))=0, & 0<t<1, \end{cases} \tag{1.1}$$

subject to two-point boundary conditions.

$$\begin{cases} u(0)=0, u'(0)=0, u''(1)=0, u'''(1)=0, \\ v(0)=0, v'(0)=0, v''(1)=0, v'''(1)=0. \end{cases} \tag{1.2}$$

The theory of multi-point boundary value problems for ordinary differential equations arises in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities that can be set up as a multi-point boundary value problem. Many problems in the theory of elastic stability can be handled as multi-point boundary value problems too. Higher-order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydromagnetic stability and astronomy, be a mandlong wave theory, induction motors, engineering and applied physics. The boundary value problems of higher-order have been examined due to their mathematical importance and applications in different areas of applied sciences. In particular, third-order, fourth-order and nth order were considered, see [1] - [14] and the references therein.
Fourth-order ordinary differential equations are models for bending or deformation of elastic beams, and therefore have important applications in mechanics, engenieering and physical sciences; see [15] - [19]. Many authors have studied the beam equation under various boundary conditions and by different approaches. For example, Bouteraa et al. [20, 21], studied the existence and nonexistence of positive solutions of two types boundary value problem for a nonlinear fourth-order differential equation

$$\begin{cases} u^{(4)}(t)=\lambda f(t,u(t)), & t\in(0,1), \\ u(0)=u'(0)=u'(1)=u'''(1)+\psi(u(1))=0, \end{cases}$$

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and

$$\begin{cases} u^{(4)}(t)-\rho^4 u(t)=f(t,u(t)),\ t\in[0,\omega], \\ u^{(i)}(0)=u^{(i)}(\omega), \qquad i\in\{0,1,2,3\}, \end{cases}$$

where $\lambda>0,\ \rho\in\mathbb{R}^+,\ f\in C([0,\omega]\times\mathbb{R},\mathbb{R})$ and $\psi\in C([0,\infty),[0,\infty))$.
By applying iterative method, Djourdem et al. [22] obtained the existence of monotone positive solution of the following nonlinear BVP

$$u^{(4)}(t)=f(t,u(t)), \quad t\in[0,1],$$

$$u'(0)=u''(0)=u(1)=0,\ u'''(\eta)+\alpha u(0)=0,$$

where $f\in C([0,1]\times[0,+\infty),[0,+\infty))$ , $\alpha\in[0,6)$ and $\eta\in\left[\frac{2}{3},1\right)$.
However, there are few works that deal with multi-point boundary value problem for a coupled systems of nonlinear differential equations, see [23] - [25]. Motivated by the above montioned works and works of coupled systems of nonlinear differential equations, we discuss the existence of positive solutions of BVPs (1.1)-(1.2). Ours analysis relies on the Guo-Krasnoselskii's fixed point theorem for operators leaving a Banach space cone invariant [26]. A Green function play a fundamental role in defining an appropriate operator on a suitable cone. The aim of this paper is to etablish some simple criteria for the existence of single positive solutions of the BVPs (1.1)-(1.2) in explicit intervals for $\lambda$. This paper is organized as follows. In section 2, we present some preliminaries and lemmas that will be used to prove our main results. In section 3, we discuss the existence of single positive solution of BVPs (1.1)-(1.2), and an example explain our conditions are applicable.

## 2. Preliminaries

Let $B=C([0,1],\mathbb{R})$ be a Banach space endowed with usual supermum norm, and $B^+=C\left([0,1],\mathbb{R}^+\right)$.

**Lemma 2.1.** *Let $y(\cdot)\in C[0,1]$. If $u\in C^4[0,1]$, then the BVP*

$$\begin{cases} u^{(4)}(t)=y(t), & 0\le t\le 1 \\ u(0)=u'(0)=u''(1)=u'''(1)=0 \end{cases}$$

*has a unique solution*

$$u(t)=\int_0^1 H(t,s)y(s)\,ds,$$

*where*

$$H(t,s)=\begin{cases} \frac{1}{6}t^2(3s-t), & 0\le t\le s\le 1, \\ \frac{1}{6}s^2(3t-s), & 0\le s\le t\le 1. \end{cases}$$

*Proof.* The derivatives of the function $H$ with respect to $t$ is

$$\frac{\partial}{\partial t}H(t,s)=\begin{cases} \frac{1}{2}s^2-\frac{1}{2}(s-t)^2,\ 0\le t\le s\le 1 \\ \frac{1}{2}s^2, \qquad 0\le s\le t\le 1 \end{cases}$$

Since the derivative of the function $H$ with respect to $t$ is nonnegative for all $t\in[0,1]$, $H$ is nondecreasing function of $t$ that attaints its maximum when $t=1$. Then

$$\max_{0\le t\le 1} H(t,s)=H(1,s)=\frac{1}{2}s^2-\frac{1}{6}s^3=\psi(s) \tag{2.1}$$

□

**Lemma 2.2.** *Let $y(\cdot)\in B^+$. Then, the unique solution $u(t)$ of BVPs $(1.1)-(1.2)$ is nonnegative and satisfies*

$$\min_{t\in[\theta,1]} u(t)\ge\frac{2\theta^3}{3}\|u\|,$$

*where $\theta\in(0,1)$.*

*Proof.* Let $y(\cdot)\in B^+$, then from $H(t,s)\ge 0$, we know $u\in B^+$. Set $u(t_0)=\|u\|$, $t_0\in(0,1]$. we first prove that

$$\frac{H(t,s)}{H(t_0,s)}\ge\frac{2}{3}t^3, \quad t,t_0,s\in(0,1].$$

In fact, we can consider four cases:
(1) if $0<t,t_0\le s\le 1$, then

$$\frac{H(t,s)}{H(t_0,s)}=\frac{t^2(3s-t)}{t_0^2(3s-t_0)}\ge\frac{t^2(2s)}{3t_0-t_0}\ge\frac{t^2(2s)}{3}\ge\frac{t^2(2t)}{3}=\frac{2t^3}{3},$$

(2) if $0 \le t \le s \le t_0 \le 1$, then

$$\frac{H(t,s)}{H(t_0,s)} = \frac{t^2(3s-t)}{s^2(3t_0-s)} \ge \frac{t^2(3s-t)}{3t_0-s} \ge \frac{t^2(3s-s)}{3t_0} \ge \frac{t^2(2s)}{3} \ge \frac{t^2(2t)}{3} \ge \frac{2t^3}{3},$$

(3) if $0 < s \le t, t_0 \le 1$, then

$$\frac{H(t,s)}{H(t_0,s)} = \frac{s^2(3t-s)}{s^2(3t_0-s)} = \frac{3t-s}{3t_0-s} \ge \frac{3t-s}{3t_0} \ge \frac{3t-s}{3} \ge \frac{2t+t-s}{3} \ge \frac{2t}{3} \ge \frac{2t^3}{3},$$

(4) if $0 \le t_0 \le s \le t \le 1$, then

$$\frac{H(t,s)}{H(t_0,s)} = \frac{s^2(3t-s)}{t_0^2(3s-t_0)} \ge \frac{t_0^2(3t-s)}{t_0^2(3t-t_0)} \ge \frac{3t-s}{3t} \ge \frac{3t-t}{3t} \ge \frac{2t}{3} \ge \frac{2t^3}{3}$$

Therefore, for $t \in [\theta,1]$, we obtain

$$\begin{aligned} u(t) &= \int_0^1 H(t,s)y(s)\,ds \\ &= \int_0^1 \frac{H(t,s)}{H(t_0,s)} H(t_0,s)y(s)\,ds \\ &\ge \int_0^1 \frac{2t^3}{3} H(t_0,s)y(s)\,ds = \frac{2t^3}{3}u(t_0) \ge \frac{2\theta^3}{3}\|u\|. \end{aligned}$$

The proof is complete. □

If we let

$$K = \left\{ u \in B: \ u(t) \ge 0 \ on \ [0,1] \ and \ \min_{t\in[\theta,1]} u(t) \ge \frac{2\theta^3}{3}\|u\| \right\},$$

then it is easy to see that $K$ a cone in $B$.
We not that the BVPs $(1.1)-(1.2)$ has a solutin $(u(t),v(t))$ if, and only if

$$u(t) = \lambda \int_0^1 H(t,s)a(s) f\left(\lambda \int_0^1 H(s,\nu)b(\nu)g(u(\nu))\,d\nu\right) ds, \quad t \in [0,1],$$

and

$$v(t) = \lambda \int_0^1 H(t,s)b(s)g(u(s))\,ds, \quad t \in [0,1]. \tag{2.2}$$

Now, we define an integral operator $Q_\lambda : B^+ \to B$ by

$$(Q_\lambda u)(t) = \lambda \int_0^1 H(t,s)a(s) f\left(\lambda \int_0^1 H(s,\nu)b(\nu)g(u(\nu))\,d\nu\right) ds, \quad u \in K. \tag{2.3}$$

We adopt the following hypothesies:
$(H_1)$ $a,b \in C((0,1),[0,\infty))$ and each does not vanish identically on any subinterval.

$(H_2)$ $f,g \in C([0,\infty),[0,\infty))$.

$(H_3)$ All of $f_0 = \lim_{x\to 0^+} \frac{f(x)}{x}$, $g_0 = \lim_{x\to 0^+} \frac{g(x)}{x}$, $f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}$, $g_\infty = \lim_{x\to\infty} \frac{g(x)}{x}$ exist as real numbers.

$(H_4)$ $g(0) = 0$ and $f$ is increasing function.

**Lemma 2.3.** *Let $\lambda > 0$ and $K$ be the cone defined above.*

*(i) If $u \in B^+$ and $v : [0,1] \to [0,\infty)$ is defined by (2.2), then $v \in K$.*

*(ii) If $Q_\lambda$ is the integral operator defined by (2.3), then $Q_\lambda(K) \subset K$.*

*(iii) Suppose $(H_1)$ and $(H_2)$ hold. Then $Q_\lambda : K \to B$ is completely continuous.*

*Proof.* Let $u \in B^{+}$ and $v$ be defined by (2.2).
$(i)$ By the nonnegativity of $H$, $b$ and $g$ it follows that $v(t) \geq 0$, $t \in [0,1]$. In view of $(H_1), (H_2)$, we obtain

$$\int_0^1 H(s,\nu)\, b(\nu)\, g(u(\nu))\, d\nu \geq \int_0^1 \min_{s\in[\theta,1]} H(s,\nu)\, b(\nu)\, g(u(\nu))\, d\nu$$

from which, we take

$$\min_{s\in[\theta,1]} \int_0^1 H(s,\nu)\, b(\nu)\, g(u(\nu))\, d\nu \geq \int_0^1 \min_{s\in[\theta,1]} H(s,\nu)\, b(\nu)\, g(u(\nu))\, d\nu$$

Consequently, employing (2.2), we obtain

$$\begin{aligned}
\int_0^1 H(s,\nu)\, b(\nu)\, g(u(\nu))\, d\nu &\geq \int_0^1 \min_{s\in[\theta,1]} H(s,\nu)\, b(\nu)\, g(u(\nu))\, d\nu \\
&\geq \frac{2\theta^3}{3} \int_0^1 H(s_0,\nu)\, b(\nu)\, g(u(\nu))\, d\nu \\
&\geq \frac{2\theta^3}{3} v(s_0),\ s_0 \in (0,1] \\
&\geq \frac{2\theta^3}{3} \|v\|.
\end{aligned}$$

Therefore

$$\min_{0<s<1} v(s) \geq \frac{2\theta^3}{3} \|v\|.$$

Which give that $v \in K$.
$(ii)$ Obviously, for $v \in K$, $Q_\lambda(u) \in B^{+}$. For $t \in [0,1]$, we have

$$\begin{aligned}
\|Q_\lambda u(t)\| &= \max_{0\leq t\leq 1} \lambda \int_0^1 H(t,s)\, a(s)\, f(v(s))\, ds \\
&\leq \lambda \int_0^1 H(1,s)\, a(s)\, f(v(s))\, ds
\end{aligned}$$

and

$$\begin{aligned}
Q_\lambda u(t) &= \lambda \int_0^1 H(t,s)\, a(s)\, f(v(s))\, ds \\
&= \lambda \int_0^1 \frac{H(t,s)}{H(1,s)} H(1,s)\, a(s)\, f(v(s))\, ds \\
&\geq \frac{2\theta^3}{3} \lambda \int_0^1 H(1,s)\, a(s)\, f(v(s))\, ds \\
&\geq \frac{2}{3}\theta^3 \|Q_\lambda u(t)\|.
\end{aligned}$$

Which give that $Q_\lambda u \in K$. Therefore $Q_\lambda : K \to K$ .
$(iii)$ It is not difficult to show that the operator $Q_\lambda : K \to B$ is completely continuous. □

Our analysis relies on the following Krasnoselskii's fixed point theorem of cone expansion-compression type.

**Theorem 2.4.** *(See [26]) Let $E$ be a Banach space and $K \subset E$ be a cone in $E$. Assume $\Omega_1$ and $\Omega_2$ are open subset of $E$ with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$, $Q : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that*

*$(i)$ $\|Qu\| \leq \|u\|$, for all $u \in K \cap \partial\Omega_1$ and $\|Qu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_2$, or*
*$(ii)$ $\|Qu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Qu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_2$*
*Then $Q$ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.*

Throughout this paper, we shall use the following notations:

$$L_1 = max\left\{ \left[ \left(\frac{2\theta^3}{3}\right)^2 \int_\theta^1 \psi(v)\, a(v)\, f_\infty dv \right]^{-1}, \left[ \left(\frac{2\theta^3}{3}\right)^2 \int_\theta^1 \psi(v)\, a(v)\, g_\infty dv \right]^{-1} \right\},$$

$$L_2 = min\left\{ \left[ \int_0^1 \psi(v)\, a(v)\, f_0 dv \right]^{-1}, \left[ \int_0^1 \psi(v)\, b(v)\, g_0 dv \right]^{-1} \right\},$$

$$L_3 = max\left\{ \left[ \left(\frac{2\theta^3}{3}\right)^2 \int_\theta^1 \psi(v)\, a(v)\, f_0 dv \right]^{-1}, \left[ \left(\frac{2\theta^3}{3}\right)^2 \int_\theta^1 \psi(v)\, a(v)\, g_0 dv \right]^{-1} \right\}$$

and

$$L_4 = min\left\{ \left[ \int_0^1 \psi(v)\, a(v)\, f_\infty dv \right]^{-1}, \left[ \int_0^1 \psi(v)\, b(v)\, g_\infty dv \right]^{-1} \right\}.$$

## 3. Main results

This section deals with the existence of at least one positive solution for BVPs (1.1)-(1.2). Our analysis relies on the Krasnoselskii fixed point theorem Theorem 2.4.

**Theorem 3.1.** *Under assumptions $(H_1), (H_2)$ and $(H_3)$, the BVPs (1.1)-(1.2) has a non-negative solution $(u,v)$ for any $\lambda$ satisfying $L_1 < \lambda < L_2$.*

*Proof.* Let $L_1 < \lambda < L_2$ and choose $\varepsilon > 0$ such that

$$max\left\{ \left[ \left(\frac{2\theta^3}{3}\right)^2 \int_\theta^1 \psi(v)\, a(v)\, (f_\infty - \varepsilon)\, dv \right]^{-1}, \left[ \left(\frac{2\theta^3}{3}\right)^2 \int_\theta^1 \psi(v)\, a(v)\, (g_\infty - \varepsilon)\, dv \right]^{-1} \right\} \le \lambda,$$

and

$$\lambda \le min\left\{ \left[ \int_0^1 \psi(v)\, a(v)\, (f_0 + \varepsilon)\, dv \right]^{-1}, \left[ \int_0^1 \psi(v)\, b(v)\, (g_0 + \varepsilon)\, dv \right]^{-1} \right\}.$$

From the definitions of $f_0$ and $g_0$ there exists an $R_1 > 0$ such that

$$f(u) \le (f_0 + \varepsilon)\, u,\ 0 < u \le R_1,$$

and

$$g(u) \le (g_0 + \varepsilon)\, u,\ 0 < u \le R_1.$$

Let $u \in K$ with $\|u\| = R_1$. From (2.1) and choice of $\varepsilon$, we have

$$\lambda \int_0^1 H(t,s)\, b(v)\, g(u(v)) \le \lambda \int_0^1 \psi(v)\, b(v)\, g(u(v))\, dv$$

$$\le \lambda \int_0^1 \psi(v)\, b(v)\, (g_0 + \varepsilon)\, u(v)\, dv$$

$$\le \|u\|\, \lambda \int_0^1 \psi(v)\, b(v)\, (g_0 + \varepsilon)\, dv$$

$$\le R_1 = \|u\|.$$

Consequently, from (2.1) and choice of $\varepsilon$, we obtain

$$\begin{aligned} Q_\lambda u(t) &= \lambda \int_0^1 H(t,s)a(s) f\left(\lambda \int_0^1 H(s,\nu)b(\nu)g(u(\nu))d\nu\right) ds \\ &\leq \lambda \int_0^1 \psi(s)a(s) f\left(\lambda \int_0^1 H(s,\nu)b(\nu)g(u(\nu))d\nu\right) ds \\ &\leq \lambda \int_0^1 \psi(s)a(s)(f_0+\varepsilon)\left[\lambda \int_0^1 H(s,\nu)b(\nu)g(u(\nu))d\nu\right] ds \\ &\leq \lambda \int_0^1 \psi(s)a(s)(f_0+\varepsilon)R_1 ds \\ &\leq R_1 = \|u\|. \end{aligned}$$

So, $\|Tu\| \leq \|u\|$. If we set $\Omega_1 = \{u \in B: \ \|u\| < R_1\}$, then

$$\|Q_\lambda u\| \leq \|u\|, \ for\, u \in K \cap \partial\Omega_1.$$

Considering the definitions of $f_\infty$ and $g_\infty$ there exists an $\overline{R}_2 > 0$ such that

$$f(u) \geq (f_\infty - \varepsilon)u, \quad u \geq \overline{R}_2,$$

and

$$g(u) \geq (g_\infty - \varepsilon)u, \quad u \geq \overline{R}_2.$$

Let $u \in K$ and $R_2 = max\left\{2R_1, \frac{3\overline{R}_2}{2\theta^3}\right\}$ with $\|u\| = R_2$, then

$$\min_{s\in[\theta,1]} u(s) \geq \frac{2}{3}\theta^3 \|u\| \geq \bar{R}_2$$

Therefore, from (2.2) and choice of $\varepsilon$, we have

$$\begin{aligned} \lambda \int_0^1 H(t,s)b(\nu)g(u(\nu))d\nu &\geq \frac{2\theta^3}{3}\lambda \int_0^1 H(1,\nu)b(\nu)g(u(\nu))d\nu \\ &\geq \frac{2\theta^3}{3}\lambda \int_\theta^1 \psi(\nu)b(\nu)(g_\infty - \varepsilon)u(\nu)d\nu \\ &\geq \|u\| \left(\frac{2\theta^3}{3}\right)^2 \lambda \int_\theta^1 \psi(\nu)b(\nu)(g_\infty - \varepsilon)d\nu \\ &\geq R_2 = \|u\|. \end{aligned}$$

Consequently, from (2.2), we obtain

$$\begin{aligned} Q_\lambda u(t) &\geq \frac{2\theta}{3}\lambda \int_\theta^1 \psi(s)a(s) f\left(\lambda \int_\theta^1 H(s,\nu)b(\nu)g(u(\nu))d\nu\right) ds \\ &\geq \frac{2\theta^3}{3}\lambda \int_\theta^1 \psi(s)a(s)(f_\infty - \varepsilon)\left[\lambda \int_\theta^1 H(s,\nu)b(\nu)g(u(\nu))d\nu\right] ds \\ &\geq \lambda\gamma \int_\theta^1 \psi(s)a(s)(f_\infty - \varepsilon)R_2 ds \end{aligned}$$

$$\geq \lambda \gamma^2 \int_{\theta}^{1} \psi(s) a(s) (f_\infty - \varepsilon) R_2 ds$$

$$\geq R_2 = \|u\|.$$

So, $\|Q_\lambda u\| \geq \|u\|$. If we set $\Omega_2 = \{u \in B : \ \|u\| < R_2\}$, then

$$\|Q_\lambda u\| \geq \|u\|, \ for \ u \in K \cap \partial\Omega_2.$$

From of part $(ii)$ of Theorem 2.4 to (3.1)and (3.1), the operator $Q_\lambda$ has a fixed point $u^* \in K \cap \left(\bar{\Omega}_2/\Omega_1\right)$. As such and with $v$ defined by

$$v(t) = \lambda \int_0^1 H(t,s) b(s) g(u(s)) ds.$$

This means that the BVPs (1.1)-(1.2) has a nonnegative solution $(u,v)$ for the given $\lambda$. The proof is complete. □

**Theorem 3.2.** *Under assumptions* $(H_1), (H_2), (H_3)$ *and* $(H_4)$*, the BVPs (1.1)-(1.2) has a non-negative solution* $(u,v)$ *for any* $\lambda$ *satisfying* $L_3 < \lambda < L_4$.

*Proof.* Let $L_3 < \lambda < L_4$ and choose $\varepsilon > 0$ such that

$$max\left\{\left[\left(\frac{2\theta^3}{3}\right)^2 \int_{\theta}^{1} \psi(v) a(v) (f_0 - \varepsilon) dv\right]^{-1}, \left[\left(\frac{2\theta^3}{3}\right)^2 \int_{\theta}^{1} \psi(v) a(v) (g_0 - \varepsilon) dv\right]^{-1}\right\} \leq \lambda,$$

and

$$\lambda \leq min\left\{\left[\int_0^1 \psi(v) a(v) (f_\infty + \varepsilon) dv\right]^{-1}, \left[\int_0^1 \psi(v) b(v) (g_\infty + \varepsilon) dv\right]^{-1}\right\}.$$

From the definitions of $f_0$ and $g_0$ there exists an $R_1 > 0$ such that

$$f(u) \geq (f_0 - \varepsilon) u, \quad 0 < u \leq R_1,$$

and

$$g(u) \geq (g_0 - \varepsilon) u, \quad 0 < u \leq R_1.$$

Now $g(0) = 0$ and so there exists $0 < R_2 < R_1$ such that

$$\lambda g(u) \leq \frac{R_1}{\int_0^1 \psi(v) b(v) dv}, \quad 0 \leq u \leq R_2.$$

Let $u \in K$ with $\|u\| = R_2$. Then

$$\lambda \int_0^1 H(t,s) b(v) g(u(v)) \leq \lambda \int_0^1 \psi(v) b(v) g(u(v)) dv$$

$$\leq \frac{\int_0^1 \psi(v) b(v) R_1 dv}{\int_0^1 \psi(v) b(v) dv}$$

$$\leq R_1 = \|u\|.$$

Therefore, by (2.2), we obtain

$$Q_\lambda u(t) = \lambda \int_0^1 H(t,s) a(s) f\left(\lambda \int_0^1 H(s,v) b(v) g(u(v)) dv\right) ds$$

$$\geq \frac{2\theta^3}{3} \lambda \int_{\theta}^{1} \psi(s) a(s) f\left(\frac{2\theta^3}{3} \lambda \int_{\theta}^{1} \psi(v) b(v) g(u(v)) dv\right) ds$$

$$\geq \frac{2\theta^3}{3} \lambda \int_{\theta}^{1} \psi(s) a(s) (f_0 - \varepsilon) \left[\left(\frac{2\theta^3}{3}\right)^2 \lambda \int_{\theta}^{1} \psi(v) b(v) (g_0 - \varepsilon) \|u\| dv\right] ds$$

$$\geq \|u\| \frac{2\theta^3}{3} \lambda \int_{\theta}^{1} \psi(v) a(v) (f_0 - \varepsilon) dv$$

$$\geq \|u\| \left(\frac{2\theta^3}{3}\right)^2 \lambda \int_{\theta}^{1} \psi(v) a(v) (f_0 - \varepsilon) dv$$

$$\geq \|u\|.$$

So, $\|Q_\lambda u\| \geq \|u\|$. If we set $\Omega_1 = \{u \in B : \|u\| < R_2\}$, then

$$\|Q_\lambda u\| \geq \|u\|, \quad u \in K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right).$$

Considering the definitions of $f_\infty$ and $g_\infty$ there exists $\overline{R}_1 > 0$ such that

$$f(u) \leq (f_\infty + \varepsilon) u, \quad u \geq \overline{R}_1,$$

and

$$g(u) \leq (g_\infty + \varepsilon) u, \quad u \geq \overline{R}_1.$$

We consider two cases: $g$ is bounded or $g$ is unbounded.

Case$(i)$. Assume that $g$ is bounded, say $g(x) \leq N, \quad N > 0$ for all $0 < x < \infty$. Then, for $u \in K$

$$\lambda \int_0^1 H(t,s) b(v) g(u(v)) \leq \lambda \int_0^1 \psi(v) b(v) g(u(v)) dv.$$

Let

$$M = max\left\{ f(u) \ : \ 0 \leq u \leq N\lambda \int_0^1 \psi(v) b(v) dv \right\},$$

and let

$$R_3 > max\left\{ 2R_2, \, M\lambda \int_0^1 \psi(s) a(s) ds \right\}.$$

Then, for $u \in K$ with $\|u\| = R_3$, we obtain

$$Q_\lambda u(t) \leq \lambda \int_0^1 \psi(s) a(s) M ds$$

$$\leq R_3 = \|u\|,$$

so that $\|Q_\lambda u\| \leq \|u\|$. If we set $\Omega_2 = \{u \in B : \|u\| \leq R_3\}$, then, for $u \in K \cap \partial\Omega_2$,

$$\|Q_\lambda u\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2.$$

Case$(ii)$. $g$ is unbounded, there exists $R_3 > max\left\{2R_2, \overline{R}_1\right\}$ such that $g(u) \leq g(R_3)$, for $0 < u \leq R_3$ Similarly, there exists $R_4 > max\left\{R_3, \, \lambda \int_0^1 \psi(v) b(v) g(R_3) dv\right\}$ such that $f(u) \leq f(R_4)$, for $0 < u \leq R_4$.
Let $u \in K$ with $\|u\| = R_4$, from$(H_4)$, we have

$$Q_\lambda u(t) \leq \lambda \int_0^1 \psi(s) a(s) f\left(\lambda \int_0^1 \psi(v) b(v) g(R_3) dv\right) ds$$

$$\leq \lambda \int_0^1 \psi(v) a(v) f(R_4) dv$$

$$\leq \lambda \int_0^1 \psi(v) a(v) (f_\infty + \varepsilon) R_4 dv$$

$$\leq R_4 = \|u\|.$$

So, $\|Q_\lambda u\| \leq \|u\|$. If we set $\Omega_2 = \{u \in C[0,1] \mid \|u\| < R_4\}$, then

$$\|Q_\lambda u\| \leq \|u\|, \quad for \quad u \in K \cap \partial\Omega_2.$$

Thus, in either of cases, From of part $(ii)$ of Theorem 2.4, the operator $Q_\lambda$ has a fixed point $u^* \in K \cap \left(\bar{\Omega}_2/\Omega_1\right)$. This means that the BVPs (1.1)-(1.2) has a nonnegative solution $(u,v)$ for the chosen value of $\lambda$. The proof is complete. □

## 4. Examples

Consider the nonlinear differential equations with parameter $\lambda$,

$$\begin{cases} u^{(4)}(t)) = \lambda t v(t)\left(v(t)e^{-v(t)} + \frac{v(t)+K}{1+\eta v(t)}\right), & 0<t<1, \\ v^{(4)}(t) = \lambda t u(t)\left(u(t)e^{-u(t)} + \frac{u(t)+K}{1+\eta u(t)}\right), & 0<t<1, \end{cases} \tag{4.1}$$

subject to two-point boundary conditions

$$\begin{cases} u(0)=0, u'(0)=0, u''(1)=0, u'''(1)=0, \\ v(0)=0, v'(0)=0, v''(1)=0, v'''(1)=0. \end{cases} \tag{4.2}$$

Where $a(t)=b(t)=t$, $f(v)=v\left(ve^{-v}+\frac{v+K}{1+\eta v}\right)=$, $g(u)=u\left(1+\frac{u+K}{1+\eta u}\right)$. By simple calculations, we have $g(0)=0$, $f_\infty = g_\infty = \frac{1}{\eta}$, $f_0 = g_0 = K$. Choosing $\theta=\frac{1}{3}$, $\eta=100$, *and* $K=10^4$, we obtain $L_3 \cong 1,1817237$, $L_4 \cong 9,1666667$.
By Theorem 3.2 it follow that for every $\lambda$ such that $1,1817237 < \lambda < 9,1666667$, the BVPs (4.1)-(4.2) has a nonnegative solution $(u,v)$ for given $\lambda$.

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# Numerical Solution for Hybrid Fuzzy Differential Equation by Fifth Order Runge-Kutta Nystrom Method

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**Abstract**

This study discusses a numerical methods for hybrid fuzzy differential equations by fifth order RK Nystrom Method for fuzzy differential equations. We prove the convergence result and give numerical examples to illustrate the theory.

## 1. Introduction

The topic of fuzzy differential equations(FDEs) has been rapidly growing in recent years. The concept of fuzzy derivative was first introduced by Chang and Zadeh [1] , it was followed up by Dubois and Prade [2] by using the extension principal in their approach. Other methods have been discussed by Puri and Ralescu [3] and Goetschel and Voxman [4]. Kandel and Byatt [5] applied the concept of fuzzy differential equation (FDE) to the analysis of fuzzy dynamical problems. The FDE and the initial value problem(Cauchy problem) were rigorously treated by Kaleva [6, 7], Seikkala [8], He and Yi [9] , Kloeden [10] and by other researchers [11, 12]. Recently several authors has investigate hybrid FDEs [13, 14, 15, 16].

Hybrid systems are devoted to modeling, design, and validation of interactive systems of computer programs and continuous systems. These are, control systems that are capable of controlling complex systems which have discrete dynamics event as well as continuous time dynamics can be modeled by hybrid system. Hybrid system evolve in continuous time like differential systems but undergo fundamental changes in their governing equations at a sequence of discrete times. For analytical results on stability properties and comparison theorems we refer to [3, 8, 17, 18].

In this paper, we develop numerical methods for solving hybrid fuzzy differential equations by Runge-Kutta Nystrom method using the Seikkala derivative. In Section 2 we list some basic definitions for fuzzy valued functions. In Section 3 we review hybrid fuzzy differential systems. In Section 4 the Runge-Kutta Nystrom method of order five for solving hybrid fuzzy differential equations and a convergence theorem are discussed. Section 5 contains a some numerical examples to illustrate the theory.

## 2. Preliminaries

Denote by $E^1$ the set of all functions $u : R \to [0,1]$ such that (i) $v$ is normal, that is, there exist an $x_0 \in R$ such that $v(x_0) = 1$, (ii) $u$ is a fuzzy convex, that is, for $x, y \in R$ and $0 \le \lambda \le 1$, $v(\lambda x + (1-\lambda)y) \ge \min\{v(x), v(y)\}$, (iii) $v$ is upper semi continuous, and (iv) $[v]^0 \equiv$ the closure of $\{x \in R : v(x) > 0\}$ is compact. For $0 < r \le 1$, we define $[v]^r = \{x \in R : v(x) \ge r\}$. An example of a $v \in E^1$ is given by

$$v(x) = \begin{cases} 4x-3, & \text{if } x \in (0.75, 1], \\ -2x+3, & \text{if } x \in (1, 1.5), \\ 0, & \text{if } x \notin (0.75, 1.5). \end{cases} \tag{2.1}$$

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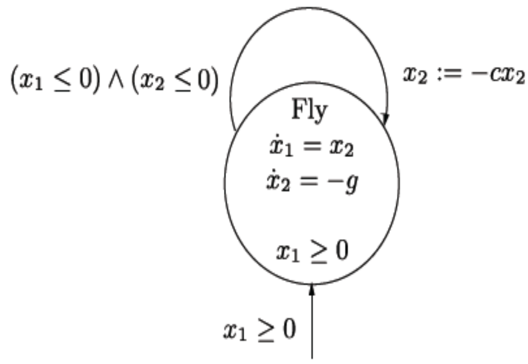


**Figure 3.1:** Bouncing ball.

The $r$-level sets of $u$ in (2.1) are given by

$$[v]^r = [0.75 + 0.25r,\ 1.5 - 0.5r]. \tag{2.2}$$

For later purpose, we define $\hat{0} \in E^1$ as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x \neq 0$.
Next we review the Seikkala derivative [8] of $x : I \to E^1$ where $I \subset R$ is an interval. If $[x(t)]^r = [\underline{x}^r(t), \overline{x}^r(t)]$ for all $t \in I$ and $r \in [0,1]$, then $[x'(t)]^r = [(\underline{x}^r)'(t), (\overline{x}^r)'(t)]$ if $x'(t) \in E^1$. Next consider the initial value problem(IVP)

$$x'(t) = g(t, x(t)),\ x(0) = x_0, \tag{2.3}$$

where $f : [0,\infty) \times R \to R$ is continuous. We would like to interpret (2.3) using the Seikkala derivative and $x_0 \in E^1$. Let $[x_0]^r = [\underline{x}_0^r, \overline{x}_0^r]$ and $[x(t)]^r = [\underline{x}^r(t), \overline{x}^r(t)]$. By the Zadeh extension principle we get $g : [0,\infty) \times E^1 \to E^1$ where

$$[g(t,x)]^r = [\min\{g(t,v) : v \in [\underline{x}^r(t), \overline{x}^r(t)]\}, \max\{g(t,v) : v \in [\underline{x}^r(t), \overline{x}^r(t)]\}].$$

Then $x : [0,\infty) \to E^1$ is a solution of (2.3) using the Seikkala derivative and $x_0 \in E^1$ if

$$(\underline{x}^r)'(t) = \min\{g(t,v) : v \in [\underline{x}^r(t), \overline{x}^r(t)]\}, \quad \underline{x}^r(0) = \underline{x}_0^r,$$
$$(\overline{x}^r)'(t) = \max\{g(t,v) : v \in [\underline{x}^r(t), \overline{x}^r(t)]\}, \quad \overline{x}^r(0) = \overline{x}_0^r,$$

for all $t \in [0,\infty)$ and $r \in [0,1]$. Lastly consider an $g : [0,\infty) \times R \times R \to R$ which is continuous and the IVP

$$\begin{cases} x'(t) = g(t, x(t), k), \\ x(0) = x_0. \end{cases} \tag{2.4}$$

As in [19], to interpret (2.4) using the Seikkala derivative and $x_0, k \in E^1$, by the Zadeh extension principle we use $g : [0,\infty) \times E^1 \times E^1 \to E^1$ where

$$[g(t,x,k)]^r = [\min\{g(t,v,v_k) : v \in [\underline{x}^r(t), \overline{x}^r(t)],\ \ v_k \in [\underline{k}^r, \overline{k}^r]\},$$
$$\max\{g(t,v,v_k) : v \in [\underline{x}^r(t), \overline{x}^r(t)],\ \ v_k \in [\underline{k}^r, \overline{k}^r]\}],$$

where $k^r = [\underline{k}^r, \overline{k}^r]$. Then $x : [0,\infty) \to E^1$ is a solution of (2.4) using the Seikkala derivative and $x_0, k \in E^1$ if

$$(\underline{x}^r)'(t) = \min\{g(t,v,v_k) : v \in [\underline{x}^r(t), \overline{x}^r(t)], v_k \in [\underline{k}^r, \overline{k}^r]\}, \quad \underline{x}^r(0) = \underline{x}_0^r,$$
$$(\overline{x}^r)'(t) = \max\{g(t,v,v_k) : v \in [\underline{x}^r(t), \overline{x}^r(t)], v_k \in [\underline{k}^r, \overline{k}^r]\}, \quad \overline{x}^r(0) = \overline{x}_0^r,$$

for all $t \in [0,\infty)$ and $r \in [0,1]$.

## 3. The hybrid fuzzy differential systems

Hybrid systems have been used to model several cyber-physical systems, including physical systems with impact, logic-dynamic controllers, and even Internet congestion.
A canonical example of a hybrid system is the bouncing ball, the physical system with impact. Here, the ball (thought of as a point-mass) is dropped from an initial height and bounces off the ground, dissipating its energy with each bounce. The ball exhibits continuous dynamics between each bounce; however, as the ball impacts the ground, its velocity undergoes a discrete change modeled after an inelastic collision.
A mathematical description of the bouncing ball follows. Let $x_1$ be the height of the ball and $x_2$ be the velocity of the ball. A hybrid system describing the ball is as follows:
When $x \in C = \{x_1 \ge 0\}$, flow is governed by $\dot{x}_1 = x_2, \dot{x}_2 = -g$, where g is the acceleration due to gravity. These equations state that when the ball is above ground, it is being drawn to the ground by gravity.
When $x \in D = \{x_1 = 0\}$, jumps are governed by $x_1^+ = x_1, x_2^+ = -\gamma x_2$, where $0 < \gamma < 1$ is a dissipation factor. This is saying that when the height of the ball is zero (it has impacted the ground), its velocity is reversed and decreased by a factor of $\gamma$. Effectively, this describes the nature of the inelastic collision.

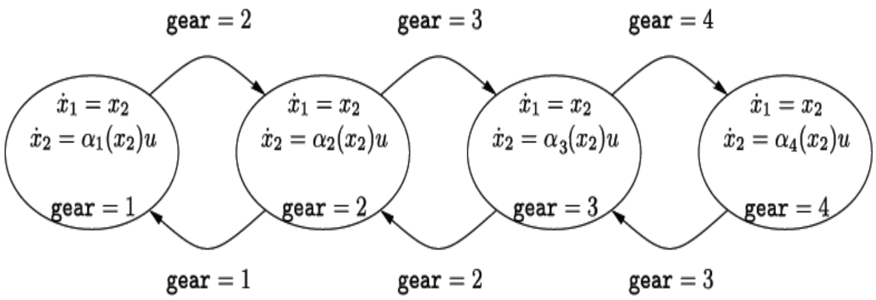


**Figure 3.2:** A hybrid system modeling a car with four gears.

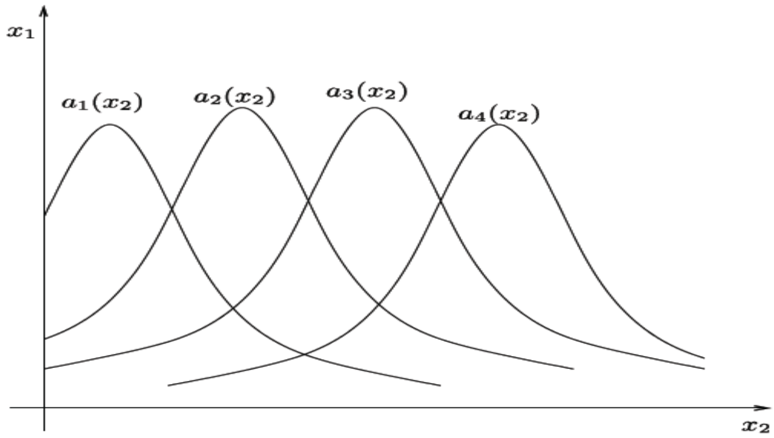


**Figure 3.3:** The efficiency functions of the different gears

## Car Gear shift:

The gear shift example describes a control design problem where both the continuous and the discrete controls need to be determined. Figure 3.2 shows a model of a car with a gear box having four gears.The longitudinal position of the car along the road is denoted by $x_1$ and its velocity by $x_2$ (lateral dynamics are ignored). The model has two control signals; the gear denoted $gear \in \{1,...,4\}$ and the throttle position denoted $u \in [u_{min}, u_{max}]$. Gear shifting is necessary because little power can be ignored by the engine at very low or very high engine speed. The function $\alpha_i$ represents the efficiency of the gear i. Typical shapes of the function $\alpha_i$ are shown in the Figure 3.3.
How many real valued continuous states does this model have? How many discrete states?
Several interesting control problems can be posed for this simple car model. For example, what is the optimal control strategy to drive from (a,0) to (b,0) in a minimum time? The problem is not trivial if we include the reasonable assumption that each gear shift takes a certain amount of time. The optimal control of hybrid system, may be derived using the theory of optimal control of hybrid systems.

Consider the hybrid fuzzy differential system

$$\begin{cases} x'(t) = g(t, x(t), \lambda_k(x_k)), & t \in [t_k, t_{k+1}], \\ x(t_k) = x_k, \end{cases} \tag{3.1}$$

where $'$ denotes Seikkala differentiation, $0 \le t_0 < t_1 < \cdots < t_k < \cdots, t_k \to \infty$,
$g \in C[R^+ \times E^1 \times E^1, E^1], \lambda_k \in C[E^1, E^1]$. To be specific the system look like

$$x'(t) = \begin{cases} x_0'(t) = g(t, x_0(t), \lambda_0(x_0)), & x_0(t_0) = x_0, \quad t_0 \le t \le t_1, \\ x_1'(t) = g(t, x_1(t), \lambda_1(x_1)), & x_1(t_1) = x_1, \quad t_1 \le t \le t_2, \\ \dots \\ x_k'(t) = g(t, x_k(t), \lambda_k(x_k)), & x_k(t_k) = x_k, \quad t_k \le t \le t_{k+1}, \\ \dots \end{cases}$$

Discuss the existence and uniqueness of solution of (3.1) hold for each $[t_k, t_{k+1}]$, by the solution of (2.3) we mean the following function:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t), & t_0 \le t \le t_1, \\ x_1(t), & t_1 \le t \le t_2, \\ \dots \\ x_k(t), & t_k \le t \le t_{k+1}, \\ \dots \end{cases}$$

We note that the solution of (3.1) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in E^1$ and $k = 0, 1, 2, \dots$

Using a representation of fuzzy numbers studied by Goestschel and Voxman [4] and Wu and Ma [18], we may represent $x \in E^1$ by a pair of fuctions $(\underline{x}(r), \overline{x}(r)), 0 \le r \le 1$, such that (i) $\underline{x}(r)$ is bounded, left continuous, and nondecreasing, (ii) $\overline{x}(r)$ is bounded, left continuous, and nonincreasing, and (iii) $\underline{x}(r) \le \overline{x}(r)$, $0 \le r \le 1$. For example , $v \in E^1$ given in ((2.1) is represented by $(\underline{v}(r), \overline{v}(r)) = (0.75 + 0.25r, 1.5 - 0.5r)$, $0 \le r \le 1$, which is similar to $[v]^r$ given by (2.2).

Therefore we may replace (3.1) by an equivalent system

$$\begin{cases} \underline{x}'(t) = \underline{g}(t, x, \lambda_k(x_k)) \equiv F_k(t, \underline{x}, \overline{x}), \quad \underline{x}(t_k) = \underline{x}_k, \\ \overline{x}'(t) = \overline{g}(t, x, \lambda_k(x_k)) \equiv G_k(t, \underline{x}, \overline{x}), \quad \overline{x}(t_k) = \overline{x}_k, \end{cases} \tag{3.2}$$

which possesses a unique solution $(\underline{x}, \overline{x})$ which is a fuzzy function. That is for each $t$, the pair $[\underline{x}(t;r), \overline{x}(t;r)]$ is a fuzzy number, where $\underline{x}(t;r), \overline{x}(t;r)$ are respectively the solutions of the parametric form given by

$$\begin{cases} \underline{x}'(t;r) = F_k[t, \underline{x}(t;r), \overline{x}(t;r)], \quad \underline{x}(t_k;r) = \underline{x}_k(r), \\ \overline{x}'(t;r) = G_k[t, \underline{x}(t;r), \overline{x}(t;r)], \quad \overline{x}(t_k;r) = \overline{x}_k(r), \end{cases} \tag{3.3}$$

for $r \in [0,1]$.

## 4. The Runge-Kutta Nystrom method

In this section, for a hybrid fuzzy differential equation (3.1) we develop the fifth order Runge-Kutta Nystrom method when $f$ and $\lambda_k$ in (2.3) can be obtained via the Zadeh extension principle from $f \in C[R^+ \times R \times R, R]$ and $\lambda_k \in C[R, R]$(since we are using the Seikkala derivative). We assume that the existence and uniqueness of solutions of (3.1) hold for each $[t_k, t_{k+1}]$.
For a fixed $r$, to integrate the system in (3.3) in $[t_0, t_1], [t_1, t_2], \ldots, [t_k, t_{k+1}], \ldots$, we replace each interval by a set of $N_k + 1$ discrete equally spaced grid points (including the end points) at which exact solution $x(t;r) = (\underline{x}(t;r), \overline{x}(t;r))$ is approximated by some $(\underline{y}_k(t;r), \overline{y}_k(t;r))$. For each the chosen grid points on $[t_k, t_{k+1}]$ at $t_{k,n} = t_k + nh_k, h_k = \frac{t_{k+1} - t_k}{N_k}$, $0 \le n \le N_k$, Let $(\underline{Y}_k(t;r), \overline{Y}_k(t;r)) \equiv (\underline{x}(t;r), \overline{x}(t;r))$, $(\underline{Y}_k(t;r), \overline{Y}_k(t;r))$ and $(\underline{y}_k(t;r), \overline{y}_k(t;r))$ may be denoted respectively by $(\underline{Y}_{k,n}(r), \overline{Y}_{k,n}(r))$ and $(\underline{y}_{k,n}(r), \overline{y}_{k,n}(r))$. We allow the $N_k$'s to vary over the $[t_k, t_{k+1}]$'s so that the $h_k$'s may be comparable.
The Runge-Kutta Nystrom method is a fifth order approximation of $\underline{Y}'_k(t;r)$ and $\overline{Y}'_k(t;r)$. To develop the Runge-Kutta Nystrom method for (2.3), and define

$$\begin{aligned} y_{k,n+1}(r) - \underline{y}_{k,n}(r) &= \sum_{i=1}^{6} w_i \underline{k}_i(t_{k,n}; y_{k,n}(r)), \\ \overline{y}_{k,n+1}(r) - \overline{y}_{k,n}(r) &= \sum_{i=1}^{6} w_i \overline{k}_i(t_{k,n}; y_{k,n}(r)), \end{aligned}$$

where $w_1, w_2, w_3, w_4, w_5$ and $w_6$ are constants and

$$\begin{aligned} \underline{k}_1(t_{k,n}; y_{k,n}(r)) &= \min\left\{ h_k g\left(t_{k,n}, v, \lambda_k(v_k)\right) \right| \\ &\qquad v \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], v_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \Big\}, \\ \overline{k}_1(t_{k,n}; y_{k,n}(r)) &= \max\left\{ h_k g\left(t_{k,n}, v, \lambda_k(v_k)\right) \right| \\ &\qquad v \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], v_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \Big\}, \\ \underline{k}_2(t_{k,n}; y_{k,n}(r)) &= \min\left\{ h_k g\left(t_{k,n} + \frac{1}{3}h_k, v, \lambda_k(v_k)\right) \right| \\ &\qquad v \in [\underline{z}_{k_1}(t_{k,n}, y_{k,n}(r)), \overline{z}_{k_1}(t_{k,n}, y_{k,n}(r))], v_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \Big\}, \\ \overline{k}_2(t_{k,n}; y_{k,n}(r)) &= \max\left\{ h_k g\left(t_{k,n} + \frac{1}{3}h_k, v, \lambda_k(v_k)\right) \right| \\ &\qquad v \in [\underline{z}_{k_1}(t_{k,n}, y_{k,n}(r)), \overline{z}_{k_1}(t_{k,n}, y_{k,n}(r))], v_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \Big\}, \\ \underline{k}_3(t_{k,n}; y_{k,n}(r)) &= \min\left\{ h_k g\left(t_{k,n} + \frac{2}{5}h_k, v, \lambda_k(v_k)\right) \right| \\ &\qquad v \in [\underline{z}_{k_2}(t_{k,n}, y_{k,n}(r)), \overline{z}_{k_2}(t_{k,n}, y_{k,n}(r))], v_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \Big\}, \end{aligned}$$

$$
\begin{aligned}
\overline{k}_3(t_{k,n};y_{k,n}(r)) &= \max\Big\{h_k g\Big(t_{k,n}+\frac{2}{5}h_k, v, \lambda_k(v_k)\Big)\Big| \\
&\qquad v\in[\underline{z}_{k_2}(t_{k,n},y_{k,n}(r)),\overline{z}_{k_2}(t_{k,n},y_{k,n}(r))], v_k\in[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\Big\},\\
\underline{k}_4(t_{k,n};y_{k,n}(r)) &= \min\Big\{h_k g\Big(t_{k,n}+h_k, v, \lambda_k(v_k)\Big)\Big| \\
&\qquad v\in[\underline{z}_{k_3}(t_{k,n},y_{k,n}(r)),\overline{z}_{k_3}(t_{k,n},y_{k,n}(r))], v_k\in[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\Big\},\\
\overline{k}_4(t_{k,n};y_{k,n}(r)) &= \max\Big\{h_k g\Big(t_{k,n}+h_k, v, \lambda_k(v_k)\Big)\Big| \\
&\qquad v\in[\underline{z}_{k_3}(t_{k,n},y_{k,n}(r)),\overline{z}_{k_3}(t_{k,n},y_{k,n}(r))], v_k\in[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\Big\},\\
\underline{k}_5(t_{k,n};y_{k,n}(r)) &= \min\Big\{h_k g\Big(t_{k,n}+\frac{2}{3}h_k, v, \lambda_k(v_k)\Big)\Big| \\
&\qquad v\in[\underline{z}_{k_4}(t_{k,n},y_{k,n}(r)),\overline{z}_{k_4}(t_{k,n},y_{k,n}(r))], v_k\in[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\Big\},\\
\overline{k}_5(t_{k,n};y_{k,n}(r)) &= \max\Big\{h_k g\Big(t_{k,n}+\frac{2}{3}h_k, v, \lambda_k(v_k)\Big)\Big| \\
&\qquad v\in[\underline{z}_{k_4}(t_{k,n},y_{k,n}(r)),\overline{z}_{k_4}(t_{k,n},y_{k,n}(r))], v_k\in[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\Big\},\\
\underline{k}_6(t_{k,n};y_{k,n}(r)) &= \min\Big\{h_k g\Big(t_{k,n}+\frac{4}{5}h_k, v, \lambda_k(v_k)\Big)\Big| \\
&\qquad v\in[\underline{z}_{k_5}(t_{k,n},y_{k,n}(r)),\overline{z}_{k_5}(t_{k,n},y_{k,n}(r))], v_k\in[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\overline{k}_6(t_{k,n};y_{k,n}(r)) &= \max\Big\{h_k g\Big(t_{k,n}+\frac{4}{5}h_k, v, \lambda_k(v_k)\Big)\Big| \\
&\qquad v\in[\underline{z}_{k_5}(t_{k,n},y_{k,n}(r)),\overline{z}_{k_5}(t_{k,n},y_{k,n}(r))], v_k\in[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\Big\},\\
\underline{z}_{k_1}(t_{k,n},y_{k,n}(r)) &= \underline{y}_{k,n}(r)+\frac{1}{3}\underline{k}_1(t_{k,n},y_{k,n}(r)),\\
\overline{z}_{k_1}(t_{k,n},y_{k,n}(r)) &= \overline{y}_{k,n}(r)+\frac{1}{3}\overline{k}_1(t_{k,n},y_{k,n}(r)),\\
\underline{z}_{k_2}(t_{k,n},y_{k,n}(r)) &= \underline{y}_{k,n}(r)+\frac{4}{25}\underline{k}_1(t_{k,n},y_{k,n}(r))+\frac{6}{25}\underline{k}_2(t_{k,n},y_{k,n}(r)),\\
\overline{z}_{k_2}(t_{k,n},y_{k,n}(r)) &= \overline{y}_{k,n}(r)+\frac{4}{25}\overline{k}_1(t_{k,n},y_{k,n}(r))+\frac{6}{25}\overline{k}_2(t_{k,n},y_{k,n}(r)),\\
\underline{z}_{k_3}(t_{k,n},y_{k,n}(r)) &= \underline{y}_{k,n}(r)+\frac{1}{4}\underline{k}_1(t_{k,n},y_{k,n}(r))\\
&\quad -\frac{12}{4}\underline{k}_2(t_{k,n},y_{k,n}(r))+\frac{15}{4}\underline{k}_3(t_{k,n},y_{k,n}(r)),\\
\overline{z}_{k_3}(t_{k,n},y_{k,n}(r)) &= \overline{y}_{k,n}(r)+\frac{1}{4}\overline{k}_1(t_{k,n},y_{k,n}(r))\\
&\quad -\frac{12}{4}\overline{k}_2(t_{k,n},y_{k,n}(r))+\frac{15}{4}\overline{k}_3(t_{k,n},y_{k,n}(r)),\\
\underline{z}_{k_4}(t_{k,n},y_{k,n}(r)) &= \underline{y}_{k,n}(r)+\frac{6}{81}\underline{k}_1(t_{k,n},y_{k,n}(r))\\
&\quad +\frac{90}{81}\underline{k}_2(t_{k,n},y_{k,n}(r))-\frac{50}{81}\underline{k}_3(t_{k,n},y_{k,n}(r))+\frac{8}{81}\underline{k}_4(t_{k,n};y_{k,n}(r)),\\
\overline{z}_{k_4}(t_{k,n},y_{k,n}(r)) &= \overline{y}_{k,n}(r)+\frac{6}{81}\overline{k}_1(t_{k,n},y_{k,n}(r))\\
&\quad +\frac{90}{81}\overline{k}_2(t_{k,n},y_{k,n}(r))-\frac{50}{81}\overline{k}_3(t_{k,n},y_{k,n}(r))+\frac{8}{81}\overline{k}_4(t_{k,n};y_{k,n}(r)),\\
\underline{z}_{k_5}(t_{k,n},y_{k,n}(r)) &= \underline{y}_{k,n}(r)+\frac{6}{75}\underline{k}_1(t_{k,n},y_{k,n}(r))\\
&\quad +\frac{36}{75}\underline{k}_2(t_{k,n},y_{k,n}(r))+\frac{10}{75}\underline{k}_3(t_{k,n},y_{k,n}(r))+\frac{8}{75}\underline{k}_4(t_{k,n};y_{k,n}(r)),\\
\overline{z}_{k_5}(t_{k,n},y_{k,n}(r)) &= \overline{y}_{k,n}(r)+\frac{6}{75}\overline{k}_1(t_{k,n},y_{k,n}(r))\\
&\quad +\frac{36}{75}\overline{k}_2(t_{k,n},y_{k,n}(r))+\frac{10}{75}\overline{k}_3(t_{k,n},y_{k,n}(r))+\frac{8}{75}\overline{k}_4(t_{k,n};y_{k,n}(r)),
\end{aligned}
$$

Next we define

$$\begin{aligned} S_k[t_{k,n},\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)] &= 23\underline{k}_1(t_{k,n},y_{k,n}(r)+125\underline{k}_3(t_{k,n},y_{k,n}(r)) \\ &\quad -81\underline{k}_5(t_{k,n},y_{k,n}(r))+125\underline{k}_6(t_{k,n},y_{k,n}(r)), \\ T_k[t_{k,n},\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)] &= 23\overline{k}_1(t_{k,n},y_{k,n}(r)+125\overline{k}_3(t_{k,n},y_{k,n}(r)) \\ &\quad -81\overline{k}_5(t_{k,n},y_{k,n}(r))+125\overline{k}_6(t_{k,n},y_{k,n}(r)) \end{aligned}$$

.

The exact solution at $t_{k,n+1}$ is given by

$$\begin{cases} \underline{Y}_{k,n+1}(r) \approx \underline{Y}_{k,n}(r) + \dfrac{1}{192} S_k[t_{k,n},\underline{Y}_{k,n}(r),\overline{Y}_{k,n}(r)], \\ \overline{Y}_{k,n+1}(r) \approx \overline{Y}_{k,n}(r) + \dfrac{1}{192} T_k[t_{k,n},\overline{Y}_{k,n}(r),\overline{Y}_{k,n}(r)]. \end{cases}$$

The approximate solution is given by

$$\begin{cases} \underline{y}_{k,n+1}(r) \approx \underline{y}_{k,n}(r) + \dfrac{1}{192} S_k[t_{k,n},\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)], \\ \overline{y}_{k,n+1}(r) \approx \overline{y}_{k,n}(r) + \dfrac{1}{192} T_k[t_{k,n},\overline{y}_{k,n}(r),\overline{y}_{k,n}(r)]. \end{cases} \tag{4.1}$$

**Lemma 4.1.** *Suppose $k \in Z^+$, $\varepsilon_k > 0$, $r \in [0,1]$, and $h_k < 1$ are fixed. Let $\{Z_{k,n}(r)\}_{n=0}^{N_k}$ be the fifth order R-K Nystrom method approximation with $N = N_k$ to the fuzzy IVP:*

$$\begin{cases} x'(t) = f(t,x(t),\lambda_k(x_k)), \quad t \in [t_k,t_{k+1}], \\ x(t_k) = x_k, \end{cases} \tag{4.2}$$

*If $\{y_{k,n}(r)\}_{n=0}^{N_k}$ denotes the result of (3.3) from some $y_{k,0}(r)$, then there exists $\delta_k > 0$ such that $|\underline{z}_{k,0}(r) - \underline{y}_{k,0}(r)| < \delta_k$, $|\overline{z}_{k,0}(r) - \overline{y}_{k,0}(r)| < \delta_k$ imply $|\underline{z}_{k,0}(r) - \underline{y}_{k,0}(r)| < \varepsilon_k$, $|\overline{z}_{k,0}(r) - \overline{y}_{k,0}(r)| < \varepsilon_k$.*

**Theorem 4.2.** *Consider the systems (3.2) and (4.1). For a fixed $k \in Z^+$ and $r \in [0,1]$,*

$$\lim_{h_0,\ldots,h_k \to 0} \underline{y}_{k,N_k}(r) = \underline{x}(t_{k+1};r), \tag{4.3}$$

$$\lim_{h_0,\ldots,h_k \to 0} \overline{y}_{k,N_k}(r) = \overline{x}(t_{k+1};r). \tag{4.4}$$

## 5. Numerical examples

Consider the fuzzy differential equation

$$x'(t) = x(t), \;\; x(0;r) = [0.75+0.25r, \; 1.125-0.125r], \;\; 0 \le r \le 1. \tag{5.1}$$

By the fifth order Runge Kutta Nystorm method with N=10

$$y(1.0;r) = (0.75+0.25r)(c_{0,1})^{10}, \;\; (1.125-0.125r)(c_{0,1})^{10}, \tag{5.2}$$

where $y(t;r)$ denotes an approximate solution of (5.1). Since the exact solution of (5.1) is $x(t;r) = [(0.75+0.25r)e^t, \; (1.125-0.125r)e^t]$, $0 \le r \le 1$, we see that
$x(1;r) = [(0.75+0.25r)e, \; (1.125-0.125r)e]$, $0 \le r \le 1$, which compares well with (5.2). By the fifth order Runge Kutta Nystorm method with $N = 10$,

$$y(1.0;r) = [(0.75+0.25r)(c_{0,1})^{10}_{,} \; (1.125-0.125r)(c_{0,1})^{10}], \;\; 0 \le r \le 1, \tag{5.3}$$

where $c_{0,1} = 1 + h + \dfrac{(h)^2}{2} + \dfrac{(h)^3}{6} + \dfrac{(h)^4}{24} + \dfrac{(h)^5}{120}$.

**Example 5.1.** *Next consider the following hybrid fuzzy IVP,*

$$\begin{cases} x^{'}(t) = x(t) + m(t)\lambda_k x(t_k), \;\; t \in [t_k,t_{k+1}], \;\; t_k = k, \;\; k = 0,1,2,3,..., \\ x(t;r) = [(0.75+0.25r)e^t, \; (1.125-0.125r)e^t], \;\; 0 \le r \le 1, \end{cases} \tag{5.4}$$

*where*

$$m(t) = \begin{cases} 2(t(mod 1)) & \text{if } t(mod 1) \le 0.5, \\ 2(1-t(mod 1) & \text{if } t(mod 1) > 0.5, \end{cases}$$

$$\lambda_k(\mu)=\begin{cases}\hat{0}, & \text{if } k=0\\ \mu, & \text{if } k\in\{1,2,\dots\}.\end{cases}$$

*The hybrid fuzzy IVP* (5.4) *is equivalent to the following systems of fuzzy IVPs:*

$$\begin{cases} x_0'(t)=x_0(t), \quad t\in[0,1],\\ x_0(0;r)=[(0.75+0.25r)e,(1.125-0.125r)e], \quad 0\le r\le 1,\\ x_i'(t)=x_i(t)+m(t)x_{i-1}(t), t\in[t_i,t_{i+1}], x_i(t)=x_{i-1}(t_i), i=1,2,..., \end{cases}$$

*In* (5.4)*,* $x(t)+m(t)\lambda_k(x(t_k))$ *is continous function of* $t,x$ *and* $\lambda_k(x(t_k))$*. Therefore by Example* 6.1 *of Kaleva* [6]*, for each* $k=0,1,2,\dots$*, the fuzzy IVP*

$$\begin{cases} x'(t)=x(t)+m(t)\lambda_k(x(t_k)), \quad t\in[t_k,t_{k+1}], t_k=k,\\ x(t_k)=x_{t_k}, \end{cases} \tag{5.5}$$

*has a unique solution on* $[t_k,t_{k+1}]$*. To numerically solve the hybrid fuzzy IVP* (5.4) *we will apply the Runge-Kutta method of order five for hybrid fuzzy differential equation with* $N=10$ *to obtain* $y_{1,2}(r)$ *approximating* $x(2.0;r)$*. Let* $f:[0,\infty)\times R\times R\to R$ *be given by*

$$f(t,x,\lambda_k(x(t_k)))=x(t)+m(t)\lambda_k(x(t_k)), \;\; t_k=k, \;\; k=0,1,2,\dots, \tag{5.6}$$

*where* $\lambda_k:R\to R$ *is given by*

$$\lambda_k(x)=\begin{cases}0, & \text{if } k=0\\ x, & \text{if } k\in\{1,2,\dots\}.\end{cases}$$

*By Example 1 of [19],* (5.1) *gives*

$$y_{1,0}(r)=[(0.75+0.25r)(c_{0,1})^{10},(1.125-0.125r)(c_{0,1})^{10}].$$

*Next suppose* $k=1$ *and* $n=0$*. Then*

$$\underline{y}_{1,1}(r)=\underline{y}_{1,0}(r)+\frac{1}{192}S_1[1.0,\underline{y}_{1,0}(r),\overline{y}_{1,0}(r)],$$

$$\overline{y}_{1,1}(r)=\overline{y}_{1,0}(r)+\frac{1}{192}T_1[1.0,\underline{y}_{1,0}(r),\overline{y}_{1,0}(r)].$$

*To obtain* $y_{1,1}(r)$*,* $i=1,2,3,4,5$

$$\underline{y}\left(1+\frac{i}{10};r\right)=\underline{y}\left(1+\frac{i-1}{10};r\right)c_{0,1}+\left[\frac{2i-1}{100}+\frac{3i-2}{3000}+\frac{4i-3}{120000}+\frac{5i-4}{6000000}+\frac{i-1}{60000000}\right]\underline{y}(1.0;r),$$

$$\overline{y}\left(1+\frac{i}{10};r\right)=\overline{y}\left(1+\frac{i-1}{10};r\right)c_{0,1}+\left[\frac{2i-1}{100}+\frac{3i-2}{3000}+\frac{4i-3}{120000}+\frac{5i-4}{6000000}+\frac{i-1}{60000000}\right]\overline{y}(1.0;r),$$

*Then for* $i=6,7,8,9,10$

$$\underline{y}\left(1+\frac{i}{10};r\right)=\underline{y}\left(1+\frac{i-1}{10};r\right)c_{0,1}+\left[\frac{1}{5}-\left(\frac{2i-2}{100}+\frac{i-1}{1000}+\frac{i-1}{30000}+\frac{i-1}{1200000}+\frac{i-1}{60000000}\right)\right]\underline{y}(1.0;r),$$

$$\overline{y}\left(1+\frac{i}{10};r\right)=\overline{y}\left(1+\frac{i-1}{10};r\right)c_{0,1}+\left[\frac{1}{5}-\left(\frac{2i-2}{100}+\frac{i-1}{1000}+\frac{i-1}{30000}+\frac{i-1}{1200000}+\frac{i-1}{60000000}\right)\right]\overline{y}(1.0;r).$$

*Let*

$$c_{2,0}=(c_{0,1})^{10}+\sum_{k=1}^{5}(c_{0,1})^{10-k}\left[\frac{2k-1}{100}+\frac{3k-2}{3000}+\frac{4k-3}{120000}+\frac{5k-4}{6000000}+\frac{k-1}{60000000}\right]+\sum_{k=6}^{10}(c_{0,1})^{10-k}\left[\frac{1}{5}-\left(\frac{2k-2}{100}+\frac{k-1}{1000}+\frac{k-1}{30000}+\frac{k-1}{1200000}+\frac{k-1}{60000000}\right)\right].$$

| $t$ | r | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| 1.1 | 2.274208 | 2.350015 | 2.425822 | 2.501629 | 2.577436 | 2.653243 | 2.729050 | 2.804857 | 2.880664 | 2.956471 | 3.032277 |
| 1.2 | 2.577355 | 2.663267 | 2.749179 | 2.835091 | 2.921003 | 3.006915 | 3.092826 | 3.178738 | 3.264650 | 3.350562 | 3.436474 |
| 1.3 | 2.955267 | 3.053776 | 3.152285 | 3.250794 | 3.349303 | 3.447812 | 3.546321 | 3.644830 | 3.743339 | 3.841848 | 3.940357 |
| 1.4 | 3.415807 | 3.529668 | 3.643528 | 3.757388 | 3.871249 | 3.985109 | 4.098969 | 4.212829 | 4.326690 | 4.440550 | 4.554410 |
| 1.5 | 3.967666 | 4.099921 | 4.232177 | 4.364432 | 4.496688 | 4.628943 | 4.761199 | 4.893454 | 5.025710 | 5.157966 | 5.290221 |
| 1.6 | 4.578278 | 4.730887 | 4.883496 | 5.036106 | 5.188715 | 5.341324 | 5.493934 | 5.646543 | 5.799152 | 5.951761 | 6.104371 |
| 1.7 | 5.210226 | 5.383900 | 5.557575 | 5.731249 | 5.904923 | 6.078597 | 6.252271 | 6.425946 | 6.599620 | 6.773294 | 6.946968 |
| 1.8 | 5.865754 | 6.061279 | 6.256805 | 6.452330 | 6.647855 | 6.843380 | 7.038905 | 7.234430 | 7.429956 | 7.625481 | 7.821006 |
| 1.9 | 6.547342 | 6.765587 | 6.983832 | 7.202077 | 7.420321 | 7.638566 | 7.856811 | 8.075056 | 8.293300 | 8.511545 | 8.729790 |
| 2.0 | 7.257731 | 7.499655 | 7.741580 | 7.983504 | 8.225429 | 8.467353 | 8.709277 | 8.951202 | 9.193126 | 9.435050 | 9.676975 |

**Table 1:** The approximation solution by RK Nystrom method to the IVP(15) - $\underline{x}(t;r)$

| $t$ | r | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| 1.1 | 3.411312 | 3.373409 | 3.335505 | 3.297602 | 3.259698 | 3.221795 | 3.183891 | 3.145988 | 3.108084 | 3.070181 | 3.032277 |
| 1.2 | 3.866033 | 3.823077 | 3.780121 | 3.737165 | 3.694209 | 3.651253 | 3.608298 | 3.565342 | 3.522386 | 3.479430 | 3.436474 |
| 1.3 | 4.432901 | 4.383647 | 4.334392 | 4.285138 | 4.235883 | 4.186629 | 4.137375 | 4.088120 | 4.038866 | 3.989611 | 3.940357 |
| 1.4 | 5.123711 | 5.066781 | 5.009851 | 4.952921 | 4.895991 | 4.839061 | 4.782131 | 4.725201 | 4.668270 | 4.611340 | 4.554410 |
| 1.5 | 5.951499 | 5.885371 | 5.819243 | 5.753115 | 5.686988 | 5.620860 | 5.554732 | 5.488604 | 5.422477 | 5.356349 | 5.290221 |
| 1.6 | 6.867417 | 6.791112 | 6.714808 | 6.638503 | 6.562199 | 6.485894 | 6.409589 | 6.333285 | 6.256980 | 6.180675 | 6.104371 |
| 1.7 | 7.815339 | 7.728502 | 7.641665 | 7.554828 | 7.467991 | 7.381154 | 7.294317 | 7.207480 | 7.120643 | 7.033805 | 6.946968 |
| 1.8 | 8.798632 | 8.700869 | 8.603107 | 8.505344 | 8.407581 | 8.309819 | 8.212056 | 8.114294 | 8.016531 | 7.918768 | 7.821006 |
| 1.9 | 9.821014 | 9.711891 | 9.602769 | 9.493647 | 9.384524 | 9.275402 | 9.166280 | 9.057157 | 8.948035 | 8.838912 | 8.729790 |
| 2.0 | 10.88659 | 10.76563 | 10.64467 | 10.52371 | 10.40274 | 10.28178 | 10.16082 | 10.03986 | 9.918899 | 9.797937 | 9.676975 |

**Table 2:** The approximation solution by RK Nystrom method to the IVP(15) - $\overline{x}(t;r)$

*Then*
$y_{2.0;r} = c_{2.0}y_1(1.0;r),$
$= [c_{2,0}(0.75+0.25r)(c_{1.0})^{10}, c_{2,0}(1.125-0.125r)(c_{1.0})^{10}],\ 0 \le r \le 1.$

*Since the exact solution of* (5.4) *for* $t \in [1,1.5]$ *is* $x(t;r) = x(1;r)(3e^{t-1}-2t), 0 \le r \le 1$, $x(1.5;r) = x(1;r)(3\sqrt{e}-3),\ 0 \le r \le 1$. *Then* $x(1.5;r)$ *is approximately 5.29022058 and* $y_{1,1}$ *is approximately 5.29022158. Since the exact solution of* (5.4) *for* $t \in [1.5,2]$ *is* $x(t;r) = x(1;r)(2t-2+e^{t-1.5}(3\sqrt{e}-4)),\ 0 \le r \le 1$.
*Therefore* $x(2.0;r) = x(1;r)(2+3e-4\sqrt{e})$. *Then* $x(2.0;r)$ *is approximately 9.676975672 and* $y_1(2.0;1)$ *is approximately 9.676975795. The approximate solution by fifth order Runge Kutta Nystrom method is plotted at* $t \in [0,2]$*(see Table 1-4 and Figure 3.1). The exact and approximate solution by fifth order Runge Kutta Nystrom method is plotted at* $t = 2$. *(see Table 1-4 and Figure 3.2).*

**Example 5.2.** *Next consider the following hybrid fuzzy IVP,*

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k x(t_k),\ t \in [t_k, t_{k+1}],\ t_k = k,\ k = 0,1,2,3,..., \\ x(t;r) = [(0.75+0.25r)e^t, (1.125-0.125r)e^t], \quad 0 \le r \le 1, \end{cases} \tag{5.7}$$

*where*

$$m(t) = |sin(\pi t)|, \quad k = 0,1,2,\ldots,$$

| $t$ | r | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| 1.1 | 2.274208 | 2.350015 | 2.425822 | 2.501629 | 2.577436 | 2.653243 | 2.729050 | 2.804857 | 2.880664 | 2.956471 | 3.032278 |
| 1.2 | 2.577355 | 2.663267 | 2.749179 | 2.835091 | 2.921003 | 3.006915 | 3.092826 | 3.178738 | 3.264650 | 3.350562 | 3.436474 |
| 1.3 | 2.955267 | 3.053776 | 3.152285 | 3.250794 | 3.349303 | 3.447812 | 3.546321 | 3.644830 | 3.743339 | 3.841848 | 3.940357 |
| 1.4 | 3.415808 | 3.529668 | 3.643528 | 3.757388 | 3.871249 | 3.985109 | 4.098969 | 4.212829 | 4.326690 | 4.440550 | 4.554410 |
| 1.5 | 3.967666 | 4.099921 | 4.232177 | 4.364432 | 4.496688 | 4.628944 | 4.761199 | 4.893455 | 5.025710 | 5.157966 | 5.290221 |
| 1.6 | 4.578278 | 4.730887 | 4.883497 | 5.036106 | 5.188715 | 5.341324 | 5.493934 | 5.646543 | 5.799152 | 5.951762 | 6.104371 |
| 1.7 | 5.210226 | 5.383901 | 5.557575 | 5.731249 | 5.904923 | 6.078597 | 6.252272 | 6.425946 | 6.599620 | 6.773294 | 6.946969 |
| 1.8 | 5.865754 | 6.061280 | 6.256805 | 6.452330 | 6.647855 | 6.843380 | 7.038905 | 7.234431 | 7.429956 | 7.625481 | 7.821006 |
| 1.9 | 6.547343 | 6.765587 | 6.983832 | 7.202077 | 7.420322 | 7.638566 | 7.856811 | 8.075056 | 8.293301 | 8.511545 | 8.729790 |
| 2.0 | 7.257731 | 7.499656 | 7.741580 | 7.983504 | 8.225429 | 8.467353 | 8.709278 | 8.951202 | 9.193126 | 9.435051 | 9.676975 |

**Table 3:** The Exact solution to the IVP(15) - $\underline{x}(t;r)$

| *t* | r | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| 1.1 | 3.411312 | 3.373409 | 3.335505 | 3.297602 | 3.259698 | 3.221795 | 3.183891 | 3.145988 | 3.108084 | 3.070181 | 3.032278 |
| 1.2 | 3.866033 | 3.823077 | 3.780121 | 3.737165 | 3.694209 | 3.651254 | 3.608298 | 3.565342 | 3.522386 | 3.479430 | 3.436474 |
| 1.3 | 4.432901 | 4.383647 | 4.334392 | 4.285138 | 4.235884 | 4.186629 | 4.137375 | 4.088120 | 4.038866 | 3.989611 | 3.940357 |
| 1.4 | 5.123712 | 5.066781 | 5.009851 | 4.952921 | 4.895991 | 4.839061 | 4.782131 | 4.725201 | 4.668271 | 4.611340 | 4.554410 |
| 1.5 | 5.951499 | 5.885371 | 5.819243 | 5.753116 | 5.686988 | 5.620860 | 5.554732 | 5.488605 | 5.422477 | 5.356349 | 5.290221 |
| 1.6 | 6.867417 | 6.791113 | 6.714808 | 6.638503 | 6.562199 | 6.485894 | 6.409589 | 6.333285 | 6.256980 | 6.180676 | 6.104371 |
| 1.7 | 7.815340 | 7.728503 | 7.641665 | 7.554828 | 7.467991 | 7.381154 | 7.294317 | 7.207480 | 7.120643 | 7.033806 | 6.946969 |
| 1.8 | 8.798632 | 8.700869 | 8.603107 | 8.505344 | 8.407582 | 8.309819 | 8.212056 | 8.114294 | 8.016531 | 7.918769 | 7.821006 |
| 1.9 | 9.821014 | 9.711892 | 9.602769 | 9.493647 | 9.384525 | 9.275402 | 9.166280 | 9.057157 | 8.948035 | 8.838913 | 8.729790 |
| 2.0 | 10.88659 | 10.76563 | 10.64467 | 10.52371 | 10.40274 | 10.28178 | 10.16082 | 10.03986 | 9.918900 | 9.797937 | 9.676975 |

**Table 4:** The Exact solution to the IVP(15) - $\overline{x}(t;r)$

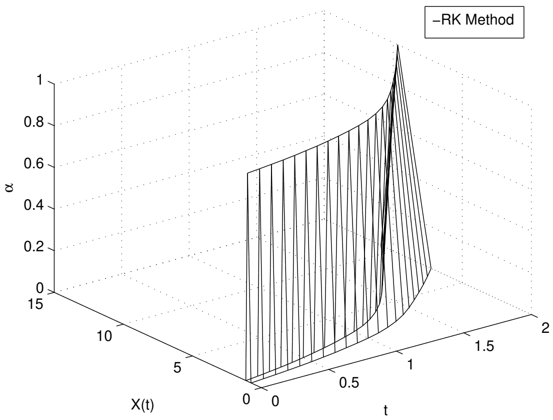


**Figure 5.1:** (for h=0.1)

$$\lambda_k(\mu) = \begin{cases} \hat{0}, & if\, k = 0 \\ \mu, & if\, k \in \{1,2,\ldots\}. \end{cases}$$

*Then* $x(t) + m(t)\lambda_k(x(t_k))$ *is continuous function of* $t, x$ *and* $\lambda_k(x(t_k))$*. Therefore by Example* 6.1 *of Kaleva* [6]*, for each* $k = 0,1,2,...$*, the fuzzy IVP*

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), & t \in [t_k, t_{k+1}], t_k = k, \\ x(t_k) = x_{t_k}, \end{cases} \tag{5.8}$$

*has a unique solution on* $[t_k, t_{k+1}]$*. To numerically solve the hybrid fuzzy IVP (36) we will apply the Runge-Kutta Method of order five for hybrid fuzzy differential equations with* $N = 10$*.*
*To obtain* $y_{1,1}(r)$*,*

$$C_1 = 125h + \frac{200}{3}h^2 + 20h^3,$$
$$C_2 = 23h + 24h^2 + 12h^3 + \frac{16}{5}h^4 + \frac{8}{5}h^5,$$

$$\underline{y}(1.1;r) = \underline{y}(1.0;r)c_{0,1} + \frac{1}{192}\Bigg[C_1 \sin\frac{\pi}{25} + 125h\sin\frac{2\pi}{25}$$

$$-81h\sin\frac{\pi}{15} + \frac{16}{3}h^2\sin\frac{\pi}{10} + \frac{24}{5}h^4\sin\frac{\pi}{30}\Bigg]\underline{y}(1.0;r),$$

$$\overline{y}(1.1;r) = \overline{y}(1.0;r)c_{0,1} + \frac{1}{192}\Bigg[C_1 \sin\frac{\pi}{25} + 125h\sin\frac{2\pi}{25}$$

$$-81h\sin\frac{\pi}{15} + \frac{16}{3}h^2\sin\frac{\pi}{10} + \frac{24}{5}h^4\sin\frac{\pi}{30}\Bigg]\overline{y}(1.0;r),$$

*Then for i=1,2,3,...,10.*

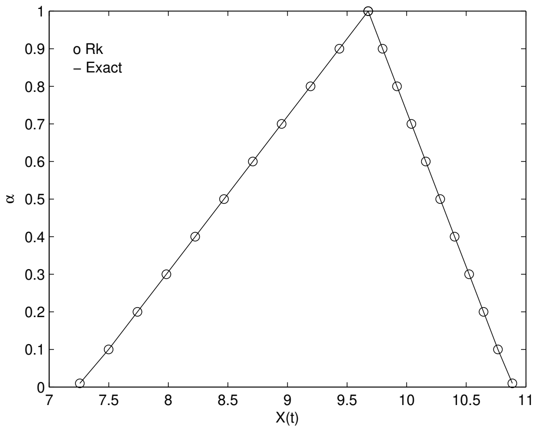


**Figure 5.2:** (for h=0.1)

$$\underline{y}\left(1+\frac{i}{10};r\right)=\underline{y}\left(1+\frac{i-1}{10};r\right)+\frac{1}{192}\left[C_1\sin\frac{(5i-3)\pi}{50}+C_2\sin\frac{(i-1)\pi}{10}\right.$$
$$-81h\sin\frac{(3i-1)\pi}{30}+125h\sin\frac{(5i-1)\pi}{50}+\frac{16}{3}h^2\sin\frac{\pi i}{10}$$
$$\left.+\frac{24}{5}h^4\sin\frac{(3i-2))\pi}{30}\right]\underline{y}(1.0;r),$$

$$\overline{y}\left(1+\frac{i}{10};r\right)=\overline{y}\left(1+\frac{i-1}{10};r\right)+\frac{1}{192}\left[C_1\sin\frac{(5i-3)\pi}{50}+C_2\sin\frac{(i-1)\pi}{10}\right.$$
$$-81h\sin\frac{(3i-1)\pi}{30}+125h\sin\frac{(5i-1)\pi}{50}+\frac{16}{3}h^2\sin\frac{\pi i}{10}$$
$$\left.+\frac{24}{5}h^4\sin\frac{(3i-2))\pi}{30}\right]\overline{y}(1.0;r).$$

*Let*

$$c_{2,0} = (c_{0,1})^{10}+\sum_{k=1}^{10}(c_{0,1})^{10-k}\frac{1}{192}\left[C_1\sin\frac{(5k-3)\pi}{50}+C_2\sin\frac{(k-1)\pi}{10}-81h\sin\frac{(3k-1)\pi}{30}\right.$$
$$\left.+125h\sin\frac{(5k-1)\pi}{50}+\frac{16}{3}h^2\sin\frac{\pi k}{10}+\frac{24}{5}h^4\sin\frac{(3k-2))\pi}{30}\right],$$

*Then*
$y_{2.0;r}=c_{2.0}y_1(1.0;r),$
$=[c_{2,0}(0.75+0.25r)(c_{1.0})^{10},c_{2,0}(1.125-0.125r)(c_{1.0})^{10}],\ 0\le r\le 1.$

*for* $t\in[1,2]$, *the exact solution of* (5.7) *satisfies*

$$\underline{x}(t;r)=\underline{x}(1;r)\frac{\pi\cos(\pi t)+\sin(\pi t)}{\pi^2+1}+\frac{e^t}{e}\underline{x}(1;r)\left(1+\frac{\pi}{\pi^2+1}\right),$$
$$\bar{x}(t;r)=\bar{x}(1;r)\frac{\pi\cos(\pi t)+\sin(\pi t)}{\pi^2+1}+\frac{e^t}{e}\bar{x}(1;r)\left(1+\frac{\pi}{\pi^2+1}\right).$$

*Therefore*

$$x(1;r)=\big[(0.75+0.25r)e,(1.125-0.125r)e\big],$$
$$x(2;r)=\Big(\frac{\pi}{\pi^2+1}+e\big(1+\frac{\pi}{\pi^2+1}\big)\Big)x(1;r).$$

*Then* $x(2.0;1)$ *is approximately 10.31033432 where as* $y_1(2.0;1)$ *is approximately 10.31033708. The approximate solution by fifth order Runge Kutta Nystrom method is plotted at* $t\in[0,2]$*(see Table 5-8 and Figure 3.3). The exact and approximate solution by fifth order Runge Kutta Nystrom method is plotted at* $t=2$. *(see Table 5-8 and Figure 5.4).*

## 6. Conclusion

In this paper we have discussed hybrid fuzzy differential systems and applied fifth order Runge-Kutta Nystorm method. In the proposed method is convergent to order $O(h^6)$.

| *t* | r | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| 1.1 | 2.285975 | 2.362174 | 2.438373 | 2.514572 | 2.590772 | 2.666971 | 2.743170 | 2.819369 | 2.895568 | 2.971767 | 3.047967 |
| 1.2 | 2.622836 | 2.710264 | 2.797691 | 2.885119 | 2.972547 | 3.059975 | 3.147403 | 3.234831 | 3.322259 | 3.409687 | 3.497114 |
| 1.3 | 3.049276 | 3.150919 | 3.252561 | 3.354204 | 3.455846 | 3.557489 | 3.659131 | 3.760774 | 3.862416 | 3.964059 | 4.065701 |
| 1.4 | 3.559976 | 3.678642 | 3.797307 | 3.915973 | 4.034639 | 4.153305 | 4.271971 | 4.390637 | 4.509303 | 4.627968 | 4.746634 |
| 1.5 | 4.145197 | 4.283371 | 4.421544 | 4.559717 | 4.697890 | 4.836064 | 4.974237 | 5.112410 | 5.250583 | 5.388757 | 5.526930 |
| 1.6 | 4.792142 | 4.951881 | 5.111619 | 5.271357 | 5.431095 | 5.590833 | 5.750571 | 5.910309 | 6.070047 | 6.229785 | 6.389523 |
| 1.7 | 5.486649 | 5.669538 | 5.852426 | 6.035314 | 6.218203 | 6.401091 | 6.583979 | 6.766868 | 6.949756 | 7.132644 | 7.315533 |
| 1.8 | 6.215071 | 6.422240 | 6.629409 | 6.836578 | 7.043747 | 7.250916 | 7.458086 | 7.665255 | 7.872424 | 8.079593 | 8.286762 |
| 1.9 | 6.966156 | 7.198362 | 7.430567 | 7.662772 | 7.894977 | 8.127183 | 8.359388 | 8.591593 | 8.823798 | 9.056004 | 9.288209 |
| 2.0 | 7.732750 | 7.990509 | 8.248267 | 8.506025 | 8.763784 | 9.021542 | 9.279300 | 9.53705 | 9.794817 | 10.05257 | 10.31033 |

**Table 5:** The approximation solution by RK Nystrom method to the IVP(18) - $\underline{x}(t;r)$

| *t* | r | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| 1.1 | 3.428962 | 3.390863 | 3.352763 | 3.314664 | 3.276564 | 3.238465 | 3.200365 | 3.162265 | 3.124166 | 3.086066 | 3.047967 |
| 1.2 | 3.934254 | 3.890540 | 3.846826 | 3.803112 | 3.759398 | 3.715684 | 3.671970 | 3.628256 | 3.584542 | 3.540828 | 3.497114 |
| 1.3 | 4.573914 | 4.523093 | 4.472272 | 4.421450 | 4.370629 | 4.319808 | 4.268987 | 4.218165 | 4.167344 | 4.116523 | 4.065701 |
| 1.4 | 5.339964 | 5.280631 | 5.221298 | 5.161965 | 5.102632 | 5.043299 | 4.983966 | 4.924633 | 4.865300 | 4.805967 | 4.746634 |
| 1.5 | 6.217796 | 6.148710 | 6.079623 | 6.010536 | 5.941450 | 5.872363 | 5.803276 | 5.734190 | 5.665103 | 5.596017 | 5.526930 |
| 1.6 | 7.188214 | 7.108345 | 7.028476 | 6.948607 | 6.868738 | 6.788869 | 6.709000 | 6.629131 | 6.549261 | 6.469392 | 6.389523 |
| 1.7 | 8.229974 | 8.138530 | 8.047086 | 7.955642 | 7.864197 | 7.772753 | 7.681309 | 7.589865 | 7.498421 | 7.406977 | 7.315533 |
| 1.8 | 9.322607 | 9.219023 | 9.115438 | 9.011853 | 8.908269 | 8.804684 | 8.701100 | 8.597515 | 8.493931 | 8.390346 | 8.286762 |
| 1.9 | 10.44923 | 10.33313 | 10.21703 | 10.10092 | 9.984824 | 9.868722 | 9.752619 | 9.636517 | 9.520414 | 9.404311 | 9.288209 |
| 2.0 | 11.59912 | 11.47024 | 11.34136 | 11.21248 | 11.08360 | 10.95473 | 10.82585 | 10.69697 | 10.56809 | 10.43921 | 10.31033 |

**Table 6:** The approximation solution by RK Nystrom method to the IVP(18)- $\overline{x}(t;r)$ for

| *t* | r | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| 1.1 | 2.285975 | 2.362174 | 2.438373 | 2.514572 | 2.590772 | 2.666971 | 2.743170 | 2.819369 | 2.895568 | 2.971767 | 3.047967 |
| 1.2 | 2.622836 | 2.710264 | 2.797691 | 2.885119 | 2.972547 | 3.059975 | 3.147403 | 3.234831 | 3.322259 | 3.409687 | 3.497114 |
| 1.3 | 3.049276 | 3.150919 | 3.252561 | 3.354204 | 3.455846 | 3.557489 | 3.659131 | 3.760774 | 3.862416 | 3.964059 | 4.065702 |
| 1.4 | 3.559976 | 3.678642 | 3.797307 | 3.915973 | 4.034639 | 4.153305 | 4.271971 | 4.390637 | 4.509303 | 4.627968 | 4.746634 |
| 1.5 | 4.145197 | 4.283371 | 4.421544 | 4.559717 | 4.697890 | 4.836064 | 4.974237 | 5.112410 | 5.250583 | 5.388757 | 5.526930 |
| 1.6 | 4.792142 | 4.951880 | 5.111619 | 5.271357 | 5.431095 | 5.590833 | 5.750571 | 5.910309 | 6.070047 | 6.229785 | 6.389523 |
| 1.7 | 5.486649 | 5.669538 | 5.852426 | 6.035314 | 6.218203 | 6.401091 | 6.583979 | 6.766867 | 6.949756 | 7.132644 | 7.315532 |
| 1.8 | 6.215071 | 6.422240 | 6.629409 | 6.836578 | 7.043747 | 7.250916 | 7.458085 | 7.665255 | 7.872424 | 8.079593 | 8.286762 |
| 1.9 | 6.966156 | 7.198362 | 7.430567 | 7.662772 | 7.894977 | 8.127183 | 8.359388 | 8.591593 | 8.823798 | 9.056003 | 9.288209 |
| 2.0 | 7.732750 | 7.990509 | 8.248267 | 8.506025 | 8.763784 | 9.021542 | 9.279300 | 9.537059 | 9.794817 | 10.05257 | 10.31033 |

**Table 7:** The exact solution to the IVP(18) - $\underline{x}(t;r)$

| *t* | r | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| 1.1 | 3.428963 | 3.390863 | 3.352763 | 3.314664 | 3.276564 | 3.238465 | 3.200365 | 3.162265 | 3.124166 | 3.086066 | 3.047967 |
| 1.2 | 3.934254 | 3.890540 | 3.846826 | 3.803112 | 3.759398 | 3.715684 | 3.671970 | 3.628256 | 3.584542 | 3.540828 | 3.497114 |
| 1.3 | 4.573914 | 4.523093 | 4.472272 | 4.421450 | 4.370629 | 4.319808 | 4.268987 | 4.218165 | 4.167344 | 4.116523 | 4.065702 |
| 1.4 | 5.339964 | 5.280631 | 5.221298 | 5.161965 | 5.102632 | 5.043299 | 4.983966 | 4.924633 | 4.865300 | 4.805967 | 4.746634 |
| 1.5 | 6.217796 | 6.148710 | 6.079623 | 6.010536 | 5.941450 | 5.872363 | 5.803276 | 5.734190 | 5.665103 | 5.596017 | 5.526930 |
| 1.6 | 7.188214 | 7.108345 | 7.028476 | 6.948607 | 6.868738 | 6.788869 | 6.709000 | 6.629131 | 6.549261 | 6.469392 | 6.389523 |
| 1.7 | 8.229974 | 8.138530 | 8.047086 | 7.955642 | 7.864197 | 7.772753 | 7.681309 | 7.589865 | 7.498421 | 7.406977 | 7.315532 |
| 1.8 | 9.322607 | 9.219022 | 9.115438 | 9.011853 | 8.908269 | 8.804684 | 8.701100 | 8.597515 | 8.493931 | 8.390346 | 8.286762 |
| 1.9 | 10.44923 | 10.33313 | 10.21703 | 10.10092 | 9.984824 | 9.868722 | 9.752619 | 9.636516 | 9.520414 | 9.404311 | 9.288209 |
| 2.0 | 11.59912 | 11.47024 | 11.34136 | 11.21248 | 11.08360 | 10.95473 | 10.82585 | 10.69697 | 10.56809 | 10.43921 | 10.31033 |

**Table 8:** The exact solution to the IVP(18) - $\overline{x}(t;r)$

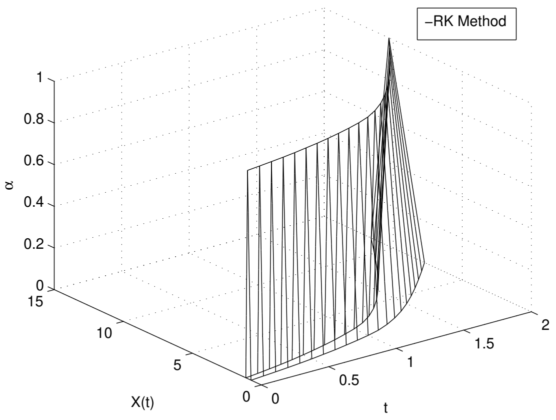


**Figure 5.3:** (for h=0.1)

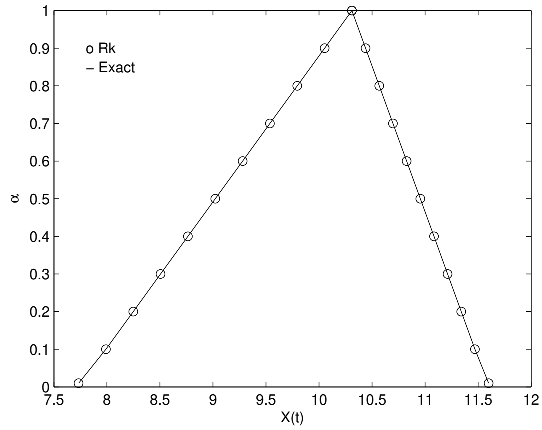


**Figure 5.4:** (for h=0.1)

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# Existence Criteria for Katugampola Fractional Type Impulsive Differential Equations with Inclusions

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**Abstract**

In this paper, we consider the existence and uniqueness of solutions to the impulsive differential equations with inclusions involving Katugampola fractional derivative. With the help of properties of Katugampola fractional calculus and fixed point methods, we derive existence and uniqueness results. Finally, an example is given to illustrate our theoretical results.

## 1. Introduction

Fractional calculus and its potential applications have gained a lot of importance, mainly because fractional calculus has become a powerful tool with more accurate and successful results in modelling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering. Many fields such as physics, fluid mechanics, viscoelasticity, heat conduction in materials with memory, chemistry and engineering can be described by fractional differential equations, see the basic books [1, 2, 3]. Recently, some basic theory for fractional differential equations and inclusions was discussed see the papers [4, 5, 6, 7, 8, 9] and the references therein. In recent years, attention has been paid to establish sufficient conditions for the existence results to differential systems involving Katugampola fractional derivatives see the papers [10, 11, 12]. In recent years, numerous contributions have been made in the theory and applications of (impulsive) fractional differential equations. The theory of impulsive differential equations and impulsive differential inclusions has been an object interest because of its wide applications in physics, biology, engineering, medical fields, industry and technology. The reason for this applicability arises from the fact that impulsive differential problems are an appropriate model for describing process which at certain moments change their state rapidly and which cannot be described using the classical differential problems. For some of these applications we refer to [13, 14, 15, 16, 17]. During the last ten years, impulsive differential inclusions with different condition have intensely studied by many mathematicians. At present, the foundations of the general theory of impulsive differential equations and inclusions are already laid, and many of them are investigated in details in the papers of Benchohra et al. [18, 19].
In this paper we are concerned with the existence of the following Katugampola fractional impulsive differential inclusions of the type,

$$\begin{cases} {}^{\rho}D^{\omega}_{0^+}u(x) \in H(x,u(x)), & x \in \mathfrak{J} = [0,T], x \neq x_m, m = 1,2,\dots k, 1 < \omega \leq 2, \\ \Delta u|_{x=x_m} = I_m(u(x_m^-)), & m = 1,2,\dots,k, \\ \Delta u'|_{x=x_m} = \bar{I}_m(u(x_m^-)), & m = 1,2,\dots,k, \\ u(0) = u_0,\ u'(0) = u_1, \end{cases} \tag{1.1}$$

where ${}^{\rho}D^{\omega}$ is the Katugampola fractional derivative in Caputo sense, $H : \mathfrak{J} \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map, $[P(\mathbb{R})$ is the family of all nonempty subset of $\mathbb{R}]$, $I_m$ and $\bar{I}_m : \mathbb{R} \to \mathbb{R}$, $m = 1,2,\cdots k$, and $u_0, u_1 \in \mathbb{R}$, $0 = x_0 < x_1 < \cdots < x_k < x_{k+1} = T$, $\Delta u|_{x=x_m} = u(x_m^+) - u(x_m^-)$, $\Delta u'|_{x=x_m} = u'(x_m^+) - u'(x_m^-)$, $u(x_m^+) = \lim_{l\to 0^+} u(x_m + l)$ and $u(x_m^-) = \lim_{l \to 0^-} u(x_m + l)$ denotes the right and left limits of $u(x)$ at $x = x_m$, $m = 1,2,\dots k$.

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## 2. Prerequisites

In this section, we introduce notations, definitions, lemmas and theorems that will be used for the main results.
Let $\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup\{|u(x)| : 0 \le x \le T\},$$

and let $\mathfrak{L}^1(\mathfrak{J},\mathbb{R})$ be the Banach space of functions $u:\mathfrak{J}\to\mathbb{R}$ that are Lebesgue integrable with the norm

$$\|u\|_{\mathfrak{L}^1} = \int_0^T |u(x)|\,\mathrm{d}x.$$

The space $\mathfrak{AC}^1(\mathfrak{J},\mathbb{R})$ consists of functions $u:\mathfrak{J}\to\mathbb{R}$, which are absolutely continuous, whose first derivative $u'$ is absolutely continuous. Let $(Y,\|\cdot\|)$ be a Banach space and let us assume that

$$Q_{cl}(Y) = \{X \in P(Y) : X \text{ closed}\},$$

$$Q_b(Y) = \{X \in P(Y) : X \text{ bounded}\},$$

$$Q_{cp}(Y) = \{X \in P(Y) : X \text{ compact}\},$$

and,

$$Q_{cp,c}(Y) = \{X \in P(Y) : X \text{ compact and convex}\}.$$

A multivalued map $F : Y \to Q(Y)$ is convex(closed) valued if $F(t)$ is convex(closed) for all $t \in Y$. $F$ is bounded on bounded sets if $F(B) = \cup_{t\in B}F(t)$ is bounded in $Y$ for all $B \in Q_b(Y)$(i.e., $\sup_{t\in B}\{\sup\{|u| : u \in F(t)\}\} < \infty$). $F$ is called upper semi-continuous on $Y$ if for each $t_0 \in Y$, the set $F(t_0)$ is a nonempty closed subset of $Y$, and if for each open set $M$ of $Y$ containing $F(t_0)$, there exists an open neighborhood $M_0$ of $t_0$ such that $F(M_0) \subseteq M$. $F$ is said to be completely continuous if $F(\mathscr{B})$ is relatively compact for every $\mathscr{B} \in Q_b(Y)$. If the multivalued map $F$ is completely continuous with nonempty compact values, then $F$ is upper semi-continuous if and only if $F$ has a closed graph (i.e., $t_n \to t_*$, $u_n \to u_*$, $u_n \in F(t_n) \Rightarrow u_* \in F(t_*)$ ). $F$ has a fixed point if there is $t \in Y$ such that $t \in F(t)$.
The fixed point set of the multivalued operator $F$ will be denoted by Fix $F$. A multivalued map $F : \mathfrak{J} \to Q_{cl}(\mathbb{R})$ is said to be measurable if for every $u \in \mathbb{R}$, the function

$$x \mapsto d(u,F(x)) = \inf\{|u-w| : w \in F(x)\},$$

is measurable. See the books of Aubin and Cellina [5], Deimling [20] and Hu and Papageorgiou [21] for more details.

**Definition 2.1.** *A multivalued map $H : \mathfrak{J}\times\mathbb{R} \to P(\mathbb{R})$ is said to be Carathéodory if*

*(i) $x \mapsto H(x,y)$ is measurable for each $y \in \mathbb{R}$;*
*(ii) $y \mapsto H(x,y)$ is upper semi-continuous for almost all $x \in \mathfrak{J}$.*

*For each $u \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$, define the set of selections of $H$ by*

$$S_{H,u} = \{g \in \mathfrak{L}^1(\mathfrak{J},\mathbb{R}) : g(x) \in H(x,u(x)) \ a.e.\ x \in \mathfrak{J}\}.$$

Let $(Y,d)$ be a metric space induced from the normed space $(Y,|\cdot|)$. Consider $G_d : P(Y)\times P(Y) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$F_d(A,B) = max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(Q_{b,cl}(Y),G_d)$ is a metric space and $(Q_{cl}(Y),G_d)$ is a generalized metric space. See the paper [22].

**Definition 2.2.** *A multivalued operator $M : Y \to Q_{cl}(Y)$ is called*

*(a) $\gamma$-Lipschitz if and only if there exists $\gamma > 0$ such that*

$$G_d(M(t),M(u)) \le \gamma d(t,u),\ for\ each\ t,u \in Y,$$

*(b) a contraction if and only if it is $\gamma$-Lipschitz with $\gamma < 1$.*

**Lemma 2.3.** *[23] Let $(Y,d)$ be a complete metric space. If $M : Y \to Q_{cl}(Y)$ is a contraction, then Fix $M \neq \emptyset$.*

**Definition 2.4.** *[2, 3] The fractional(arbitrary) order integral of the function $h \in \mathfrak{L}^1([a,b],\mathbb{R}_+)$ of order $\omega \in \mathbb{R}_+$ is defined by*

$$I_a^\omega h(x) = \frac{1}{\Gamma(\omega)}\int_a^x (x-s)^{\omega-1}h(s)\mathrm{d}s,$$

*where $\Gamma$ is the gamma function. When $a = 0$, we write $I^\omega h(x) = h(x) * \psi_\omega(x)$, where $\psi_\omega(x) = \frac{x^{\omega-1}}{\Gamma(\omega)}$ for $x > 0$, and $\psi_\omega(x) = 0$ for $x \le 0$, and $\psi_\omega \to \delta(x)$ as $\omega \to 0$, where $\delta$ is the delta function.*

**Definition 2.5.** *[2, 3] For a function $h$ given on the interval $[a,b]$, the Caputo fractional-order derivative of $h$, is defined by*

$$({}^cD^{\omega}_{a^+}h)(x) = \frac{1}{\Gamma(n-\omega)}\int_a^x (x-s)^{n-\omega-1}h^{(n)}(s)\mathrm{d}s,$$

*where $n=[\omega]+1$ and $[\omega]$ denotes the integer part of the real number $\omega$.*

Now, we consider the definitions of the generalized fractional operators introduced in [10, 11, 12].

**Definition 2.6.** *The generalized left-sided fractional integral ${}^{\rho}I^{\omega}_{a^+}h$ of order $\omega \in \mathbb{C}(Re(\omega)>0)$ is defined by*

$$({}^{\rho}I^{\omega}_{a^+}h)(x) \quad = \tfrac{\rho^{1-\omega}}{\Gamma(\omega)}\textstyle\int_a^x (x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}h(s)\mathrm{d}s, \tag{2.1}$$

*for $x>a$, if the integral exists.*

**Definition 2.7.** *The generalized fractional derivative, corresponding to the generalized fractional integral (2.1), is defined for $x>a$, by*

$$\begin{aligned}({}^{\rho}D^{\omega}_{a^+}h)(x) &= \left(x^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n({}^{\rho}I^{n-\omega}_{a^+}h)(x)\\ &= \frac{\rho^{\omega-n+1}}{\Gamma(n-\omega)}\left(x^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\int_a^x (x^{\rho}-s^{\rho})^{n-\omega-1}s^{\rho-1}h(s)\mathrm{d}s,\end{aligned} \tag{2.2}$$

*if the integral exists.*

**Definition 2.8.** *The Caputo-type generalized fractional derivative, ${}^{\rho}_{c}D^{\omega}_{a^+}$ is defined via the above generalized fractional derivative (2.2) as follows*

$${}^{\rho}_{c}D^{\omega}_{a^+}h(x) = \left({}^{\rho}D^{\omega}_{a^+}\left[h(s)-\sum_{m=0}^{n-1}\frac{h^{(m)}(a)}{m!}(s-a)^m\right]\right)(x),$$

*where $n=\lceil Re(\omega)\rceil$.*

Sufficient conditions are given in [2] for the fractional differential and integral to exist.

## 3. The convex case

In this section, we discuss about the existence of solutions for the problem (1.1) when the right hand side has the convex values. For this, we assume that $H$ is a compact, convex valued and multivalued map. Consider the Banach space,

$$\begin{aligned}\mathfrak{PC}(\mathfrak{J},\mathbb{R}) = \{u:\mathfrak{J}\to\mathbb{R}: u\in\mathfrak{C}((x_m,x_{m+1}],\mathbb{R}),\ m=0,1,\dots k+1\\ \text{and there exist } u(x_m^-),u(x_m^+),\ m=1,2,\dots k \text{ with } u(x_m^-)=u(x_m)\},\end{aligned}$$

with the norm

$$\|u\|_{\mathfrak{PC}} = \sup_{x\in\mathfrak{J}}|u(x)|.$$

Set $\mathfrak{J}' := [0,T]\backslash\{x_1,x_2,\dots x_k\}$.

**Definition 3.1.** *A function $u\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})\cap\cup_{m=0}^{k}\mathfrak{AC}^1((x_m,x_{m+1}),\mathbb{R})$ with its $\omega$-derivative exists on $\mathfrak{J}'$ is said to be a solution of (1.1), if there exists a function $g\in\mathfrak{L}^1([0,T],\mathbb{R})$ such that $g(x)\in H(x,u(x))$ a.e. $x\in\mathfrak{J}$ satisfies the differential equation ${}^{\rho}D^{\omega}_{0^+}u(x)=g(x)$ on $\mathfrak{J}'$, and conditions*

$$\Delta u|_{x=x_m} = I_m(u(x_m^-)),\ m=1,2,\dots k,$$

$$\Delta u'|_{x=x_m} = \bar{I}_m(u(x_m^-)),\ m=1,2,\dots k,$$

$$u(0)=u_0,\ u'(0)=u_1,$$

*are satisfied. Let $h:[a,b]\to\mathbb{R}$ be a continuous function. We need the following lemmas for the existence of solutions for the problem (1.1).*

**Lemma 3.2.** *[9] Let $\omega>0$, then the differential equation*

$${}^{\rho}D^{\omega}_{0^+}h(x)=0,$$

*has solutions $h(x)=b_0+b_1(\frac{x^{\rho}}{\rho})+b_2(\frac{x^{\rho}}{\rho})^2+\cdots+b_{n-1}(\frac{x^{\rho}}{\rho})^{(n-1)}$, $b_i\in\mathbb{R}$, $i=0,1,2,\dots,n-1$, $n=[\omega]+1$.*

**Lemma 3.3.** *[9] Let $\omega>0$, then*

$${}^{\rho}I^{\omega}_{0^+}\left({}^{\rho}D^{\omega}_{0^+}h(x)\right) = h(x)+b_0+b_1(\frac{x^{\rho}}{\rho})+b_2(\frac{x^{\rho}}{\rho})^2+\cdots+b_{n-1}(\frac{x^{\rho}}{\rho})^{(n-1)},$$

*for some $b_i\in\mathbb{R}$, $i=0,1,2,\dots,n-1$, $n=[\omega]+1$.*

From Lemma 3.2, and 3.3, we get the following results which is useful in the following sequel.

**Lemma 3.4.** *Let $1<\omega\leq 2$ and let $g\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$. A function $u$ is a solution of the fractional integral equation*

$$u(x)=\begin{cases} u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s, & \text{if } x\in[0,x_1],\\ u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^m\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s & \\ +\frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{i=1}^m(x^{\rho-1}-x_i^{\rho-1})\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-2}s^{\rho-1}g(s)\mathrm{d}s & \\ +\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s & \\ +\sum_{i=1}^m I_i(u(x_i^-))+\sum_{i=1}^m(x-x_i)\bar{I}_i(u(x_i^-)), & \text{if } x\in(x_m,x_{m+1}],\ m=1,2,\ldots k, \end{cases} \tag{3.1}$$

*if and only if $u$ is a solution of the fractional initial value problem*

$$\begin{cases} {}^\rho D_{0^+}^\omega u(x)=g(x), & \text{for each } x\in\mathfrak{J}',\\ \Delta u|_{x=x_m}=I_m(u(x_m^-)), & m=1,2,\ldots,k,\\ \Delta u'|_{x=x_m}=\bar{I}_m(u(x_m^-)), & m=1,2,\ldots,k,\\ u(0)=u_0,\ u'(0)=u_1. \end{cases} \tag{3.2}$$

*Proof.* Assume that $u$ satisfies (3.2). If $x\in[0,x_1]$ then ${}^\rho D_{0^+}^\omega u(x)=g(x)$. From Lemma 3.3, we get

$$u(x)=b_0+b_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s.$$

Hence $b_0=u_0$, $b_1=u_1$. Thus

$$u(x)=u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s.$$

If $x\in(x_1,x_2]$, then from Lemma 3.3, we arrive to

$$u(x)=b_0+b_1(\frac{x^\rho-x_1^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_1}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s. \tag{3.3}$$

$$\begin{aligned} \Delta u|_{x=x_1} &= u(x_1^+)-u(x_1^-)\\ &= b_0-\left(u_0+u_1(\frac{x_1^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^{x_1}(x_1^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\right)\\ &= I_1(u(x_1^-)). \end{aligned}$$

Hence,

$$b_0=u_0+u_1(\frac{x_1^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^{x_1}(x_1^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s+I_1(u(x_1^-)). \tag{3.4}$$

$$\begin{aligned} \Delta u'|_{x=x_1} &= u'(x_1^+)-u'(x_1^-)\\ &= b_1-\left(u_1+\frac{\rho^{2-\omega}(x^{\rho-1}-x_1^{\rho-1})}{\Gamma(\omega-1)}\int_0^{x_1}(x_1^\rho-s^\rho)^{\omega-2}s^{\rho-1}g(s)\mathrm{d}s\right)\\ &= \bar{I}_1(u(x_1^-)), \end{aligned}$$

and

$$b_1=u_1+\frac{\rho^{2-\omega}}{\Gamma(\omega-1)}(x^{\rho-1}-x_1^{\rho-1})\int_0^{x_1}(x_1^\rho-s^\rho)^{\omega-2}s^{\rho-1}g(s)\mathrm{d}s+\bar{I}_1(u(x_1^-)). \tag{3.5}$$

Then by (3.3)-(3.5), we get

$$\begin{aligned} u(x) &= u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^{x_1}(x_1^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}(x^{\rho-1}-x_1^{\rho-1})\int_0^{x_1}(x_1^\rho-s^\rho)^{\omega-2}s^{\rho-1}g(s)\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_1}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\ &+ I_1(u(x_1^-))+(x-x_1)\bar{I}_1(u(x_1^-)). \end{aligned}$$

If $x\in(x_m,x_{m+1}]$, then from Lemma 3.3, we get (3.1). Conversely, assume that $u$ satisfies the equation (3.1). If $x\in[0,x_1]$ then $u(0)=u_0$, $u'(0)=u_1$ and using the concept that ${}^\rho D_{0^+}^\omega$ is the left inverse of ${}^\rho I_{0^+}^\omega$, we get,

$${}^\rho D_{0^+}^\omega u(x)=g(x),\ \text{for each } x\in[0,x_1].$$

If $x \in [x_m, x_{m+1})$, $m = 1,2,\dots k$ and using the fact that ${}^{\rho}D^{\omega}_{0^+}L = 0$, where $L$ is a constant, we have

$$^{\rho}D^{\omega}_{0^+}u(x) = g(x), \text{ for each } x \in [x_m, x_{m+1}).$$

Also, we can easily prove that

$$\Delta u|_{x=x_m} = I_m(u(x_m^-)),\ m = 1,2,\dots,k,$$

$$\Delta u'|_{x=x_m} = \bar{I}_m(u(x_m^-)),\ m = 1,2,\dots,k.$$

□

By using the nonlinear alternative of Leray-Schauder type for multivalued maps [24], we can prove our first result. For this, we assume the following hypotheses:

(A1) $H : \mathfrak{J} \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map.
(A2) There exists $q \in \mathfrak{C}(\mathfrak{J}, \mathbb{R}^+)$ and $\Phi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\|H(x,y)\|_P = \sup\{|g| : g \in H(x,y)\} \le q(x)\Phi(|y|), \text{ for } x \in \mathfrak{J} \text{ and } y \in \mathbb{R}.$$

(A3) There exist $\Phi^*, \bar{\Phi}^* : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$|I_m(y)| \le \Phi^*(|y|), \text{ for } y \in \mathbb{R},$$
$$|\bar{I}_m(y)| \le \bar{\Phi}^*(|y|), \text{ for } y \in \mathbb{R}.$$

(A4) There exists a number $N > 0$ such that

$$\frac{N}{|u_0| + T^{\rho}\left|\frac{u_1}{\rho}\right| + a\Phi(N) + k\bar{\Phi}^*(N) + kT^{\rho}\bar{\Phi}^*(N)} > 1,$$

where $q^0 = \sup\{q(x) : x \in \mathfrak{J}\}$ and $a = \frac{kT^{\rho\omega}q^0}{\rho^{\omega}\Gamma(\omega+1)} + \frac{kT^{\rho\omega}q^0}{\rho^{\omega-1}\Gamma(\omega)} + \frac{T^{\rho\omega}q^0}{\rho^{\omega}\Gamma(\omega+1)}$.
(A5) There exists $\tilde{l} \in \mathfrak{L}^1(\mathfrak{J}, \mathbb{R}^+)$ such that

$$G_d(H(x,y), H(x,\bar{y})) \le \tilde{l}(x)\,|y - \bar{y}| \text{ for } a.e.\ x \in \mathfrak{J},\ y, \bar{y} \in \mathbb{R},$$

$d(0, H(x,0)) \le \tilde{l}(x),\ a.e.\ x \in \mathfrak{J}.$

**Theorem 3.5.** *Under assumptions* (A1)-(A5)*, the initial value problem* (1.1) *has at least one solution on* $\mathfrak{J}$.

*Proof.* We transform the problem (1.1) into a fixed point problem. Consider the multivalued operator

$$\begin{aligned} M(u) &= \{h \in \mathfrak{PC}(\mathfrak{J},\mathbb{R}) : h(x) = u_0 + u_1(\frac{x^{\rho}}{\rho}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} g(s)\mathrm{d}s \\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)} \sum_{0<x_m<x} (x^{\rho-1} - x_m^{\rho-1}) \int_{x_{m-1}}^{x_m} (x_m^{\rho} - s^{\rho})^{\omega-2} s^{\rho-1} g(s)\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} g(s)\mathrm{d}s \\ &+ \sum_{0<x_m<x} I_m(u(x_m^-)) + \sum_{0<x_m<x} (x - x_m)\bar{I}_m(u(x_m^-)),\ g \in S_{H,u}\}. \end{aligned}$$

Clearly from Lemma 3.4, fixed points of $M$ are solutions to (1.1). We shall prove that $M$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type [24]. The proof of the theorem contains five steps.
**Step 1:** *$M(u)$ is convex for each $u \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$.* If $h_1, h_2 \in M(u)$, then there exist $g_1, g_2 \in S_{H,u}$ such that for each $x \in \mathfrak{J}$, we obtain,

$$\begin{aligned} h_i(x) &= u_0 + u_1(\frac{x^{\rho}}{\rho}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} g_i(s)\mathrm{d}s \\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)} \sum_{0<x_m<x} (x^{\rho-1} - x_m^{\rho-1}) \int_{x_{m-1}}^{x_m} (x_m^{\rho} - s^{\rho})^{\omega-2} s^{\rho-1} g_i(s)\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} g_i(s)\mathrm{d}s \\ &+ \sum_{0<x_m<x} I_m(u(x_m^-)) + \sum_{0<x_m<x} (x - x_m)\bar{I}_m(u(x_m^-)),\ i = 1,2. \end{aligned}$$

Let $0 \leq d \leq 1$, then for each $x \in \mathfrak{J}$, we have

$$\begin{aligned}(dh_1+(1-d)h_2)(x) &= u_0+u_1(\frac{x^{\rho}}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}[dg_1(s)+(1-d)g_2(s)]\mathrm{d}s\\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1}-x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^{\rho}-s^{\rho})^{\omega-2}s^{\rho-1}[dg_1(s)+(1-d)g_2(s)]\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}[dg_1(s)+(1-d)g_2(s)]\mathrm{d}s\\ &+ \sum_{0<x_m<x}I_m(u(x_m^-))+\sum_{0<x_m<x}(x-x_m)\bar{I}_m(u(x_m^-)).\end{aligned}$$

Since $S_{H,u}$ is convex(because $H$ has convex values), we get

$$dh_1+(1-d)h_2 \in M(u).$$

**Step 2:** *M maps bounded sets into bounded sets in* $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$. Let $B_{\Omega^*}=\{u\in\mathfrak{PC}(\mathfrak{J},\mathbb{R}):\|u\|_\infty\leq\Omega^*\}$ be bounded set in $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ and $u\in B_{\Omega^*}$. Then for each $h\in M(u)$ and $x\in\mathfrak{J}$, we get $(A2)$-$(A3)$,

$$\begin{aligned}|h(x)| &\leq |u_0|+\left|\frac{u_1}{\rho}\right|T^{\rho}+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}|g(s)|\mathrm{d}s\\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1}-x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^{\rho}-s^{\rho})^{\omega-2}s^{\rho-1}|g(s)|\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}|g(s)|\mathrm{d}s\\ &+ \sum_{0<x_m<x}\left|I_m(u(x_m^-))\right|+\sum_{0<x_m<x}(x-x_m)\left|\bar{I}_m(u(x_m^-))\right|\\ &\leq |u_0|+\left|\frac{u_1}{\rho}\right|T^{\rho}+\frac{kT^{\rho\omega}q^0}{\rho^{\omega}\Gamma(\omega+1)}\Phi(\Omega^*)+\frac{T^{\rho\omega}q^0}{\rho^{\omega-1}\Gamma(\omega)}\Phi(\Omega^*)+\frac{T^{\rho\omega}q^0}{\rho^{\omega}\Gamma(\omega+1)}\Phi(\Omega^*)+k\Phi^*(\Omega^*)+k\bar{\Phi}^*(\Omega^*).\end{aligned}$$

Thus,

$$\|h\|_\infty \leq |u_0|+\left|\frac{u_1}{\rho}\right|T^{\rho}+\frac{kT^{\rho\omega}q^0}{\rho^{\omega}\Gamma(\omega+1)}\Phi(\Omega^*)+\frac{T^{\rho\omega}q^0}{\rho^{\omega-1}\Gamma(\omega)}\Phi(\Omega^*)+\frac{T^{\rho\omega}q^0}{\rho^{\omega}\Gamma(\omega+1)}\Phi(\Omega^*)+k\Phi^*(\Omega^*)+k\bar{\Phi}^*(\Omega^*):=\tilde{\ell}.$$

**Step 3:** *M maps bounded sets into equicontinuous sets of* $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$. Let $t_1,t_2\in\mathfrak{J}$, $t_1<t_2$, $B_{\Omega^*}$ be a bounded set of $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ as in Step 2, let $u\in B_{\Omega^*}$ and $h\in M(u)$, then

$$\begin{aligned}|h(t_2)-h(t_1)| &\leq \left|\frac{u_1}{\rho}\right|(t_2^{\rho}-t_1^{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<t_2-t_1}\int_{x_{m-1}}^{x_m}(x_m^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}|g(s)|\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{t_1}\left|(t_2^{\rho}-s^{\rho})^{\omega-1}-(t_1^{\rho}-s^{\rho})^{\omega-1}\right|\left|s^{\rho-1}\right||g(s)|\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{t_1}^{t_2}\left|(t_2^{\rho}-s^{\rho})^{\omega-1}\right|\left|s^{\rho-1}\right||g(s)|\mathrm{d}s\\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<t_2-t_1}(t_2^{\rho-1}-x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^{\rho}-s^{\rho})^{\omega-2}\left|s^{\rho-1}\right||g(s)|\mathrm{d}s\\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<t_1}(t_2^{\rho-1}-t_1^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^{\rho}-s^{\rho})^{\omega-2}\left|s^{\rho-1}\right||g(s)|\mathrm{d}s\\ &+ \sum_{0<x_m<t_2-t_1}\left|I_m(u(x_m^-))\right|+\sum_{0<x_m<t_2-t_1}(t_2-x_m)\left|\bar{I}_m(u(x_m^-))\right|\\ &+ \sum_{0<x_m<t_1}(t_2-t_1)\left|\bar{I}_m(u(x_m^-))\right|.\end{aligned}$$

From the hypotheses $(A2)$ and $(A3)$, we can easily show that the right hand side of the above inequality tends to zero independently of $u$ as $t_1\to t_2$. From Step 1 to Step 3 together with the Arzelá-Ascoli theorem, we can conclude that $M:\mathfrak{PC}(\mathfrak{J},\mathbb{R})\to P(\mathfrak{PC}(\mathfrak{J},\mathbb{R}))$ is completely continuous.

**Step 4:** *M has a closed graph.* Let $u_n\to u_*$, $h_n\in M(u_n)$ and $h_n\to h_*$. we want to prove that $h_*\in M(u_*)$. $h_n\in M(u_n)$ means that there exists $g_n \in S_{H,u_n}$ such that, for each $x \in \mathfrak{J}$,

$$\begin{aligned} h_n(x) &= u_0 + u_1(\frac{x^\rho}{\rho}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} g_n(s) \mathrm{d}s \\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)} \sum_{0<x_m<x} (x^{\rho-1} - x_m^{\rho-1}) \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-2} s^{\rho-1} g_n(s) \mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} g_n(s) \mathrm{d}s \\ &+ \sum_{0<x_m<x} I_m(u_n(x_m^-)) + \sum_{0<x_m<x} (x - x_m) \bar{I}_m(u_n(x_m^-)). \end{aligned}$$

we want to prove that, there exists $g_* \in S_{H,u_*}$ such that, for each $x \in \mathfrak{J}$,

$$\begin{aligned} h_*(x) &= u_0 + u_1(\frac{x^\rho}{\rho}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} g_*(s) \mathrm{d}s \\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)} \sum_{0<x_m<x} (x^{\rho-1} - x_m^{\rho-1}) \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-2} s^{\rho-1} g_*(s) \mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} g_*(s) \mathrm{d}s \\ &+ \sum_{0<x_m<x} I_m(u_*(x_m^-)) + \sum_{0<x_m<x} (x - x_m) \bar{I}_m(u_*(x_m^-)). \end{aligned} \tag{3.6}$$

Since $H(x,\cdot)$ is upper semi continuous, then for every $\varepsilon > 0$, there exist $m_0(\varepsilon) \geq 0$ such that for every $m \geq m_0$, we get

$$g_n(x) \in H(x, u_n(x)) \subset H(x, u_*(x)) + \varepsilon B(0,1), \ a.e.\ x \in \mathfrak{J}.$$

Since $H(\cdot,\cdot)$ has compact values, then there exists a subsequence $g_{m_n}(\cdot)$ such that

$$g_{m_n}(\cdot) \to g_*(\cdot), \text{ as } n \to \infty,$$

$$g_*(x) \in H(x, u_*(x)), \ a.e.\ x \in \mathfrak{J}.$$

Using the concept that the functions $I_m$ and $\bar{I}_m$, $m = 1,2,\ldots k$ are continuous, we can easily prove that $h_*$ and $g_*$ satisfy (3.6).

**Step 5:** *A priori bounds on solutions*. Let $u \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ be such that $u \in \mu M(u)$ for $\mu \in (0,1)$. Then there exists $g \in S_{H,u}$ such that, for each $x \in \mathfrak{J}$,

$$\begin{aligned} |u(x)| &\leq |u_0| + \left|\frac{u_1}{\rho}\right| T^\rho + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} q(s) \Phi(|u(s)|) \mathrm{d}s \\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)} \sum_{0<x_m<x} (x^{\rho-1} - x_m^{\rho-1}) \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-2} s^{\rho-1} q(s) \Phi(|u(s)|) \mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} q(s) \Phi(|u(s)|) \mathrm{d}s \\ &+ \sum_{0<x_m<x} \Phi^*(|u(s)|) + \sum_{0<x_m<x} \bar{\Phi}^*(|u(s)|) \\ &\leq |u_0| + \left|\frac{u_1}{\rho}\right| T^\rho + \frac{kT^{\rho\omega} q^0}{\rho^\omega \Gamma(\omega+1)} \Phi(\|u\|_\infty) + \frac{kT^{\rho\omega} q^0}{\rho^{\omega-1} \Gamma(\omega)} \Phi(\|u\|_\infty) + \frac{kT^{\rho\omega} q^0}{\rho^\omega \Gamma(\omega+1)} \Phi(\|u\|_\infty) \\ &+ k\Phi^*(\|u\|_\infty) + k\bar{\Phi}^*(\|u\|_\infty). \end{aligned}$$

Thus,

$$\frac{\|u\|_\infty}{|u_0| + \left|\frac{u_1}{\rho}\right| T^\rho + a\Phi(\|u\|_\infty) + k\Phi^*(\|u\|_\infty) + k\bar{\Phi}^*(\|u\|_\infty)} \leq 1.$$

Then by $(A4)$, there exists $N$ such that $\|u\|_\infty \neq N$. Let

$$U = \{u \in \mathfrak{PC}(\mathfrak{J},\mathbb{R}) : \|u\|_\infty < N\}.$$

The operator $M : \overline{U} \to P(\mathfrak{PC}(\mathfrak{J},\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of $U$, there is no $u \in \partial U$ such that $u \in \mu M(u)$ for some $\mu \in (0,1)$. From the concepts of the nonlinear alternative of Leray-Schauder type [24], we conclude that $M$ has a fixed point $u$ in $\overline{U}$ which is a solution of the problem (1.1). This completes the proof of the theorem. □

## 4. The nonconvex case

In this section, we discuss about the concepts for the existence of solutions for the problem (1.1), when the right hand side has a nonconvex value. Now, we adopt the concepts from Bressan and Colombo [25], Covitz and Nodler [23], and some existence results for nonconvex valued differential inclusion in [5, 21]. We consider the following hypotheses for the next theorem:

(A6) $H:\mathfrak{J}\times\mathbb{R}\to Q_{cp}(\mathbb{R})$ has the property that $H(\cdot,y):\mathfrak{J}\to Q_{cp}(\mathbb{R})$ is measurable, convex valued and integrable bounded for each $y\in\mathbb{R}$.
(A7) There exist constants $\ell^*,\bar{\ell}^*>0$ such that

$$|I_m(y)-I_m(\bar{y})|\le \ell^*|y-\bar{y}|,\ \text{for each } y,\bar{y}\in\mathbb{R} \text{ and } m=1,2,\dots k,$$

$$|\bar{I}_m(y)-\bar{I}_m(\bar{y})|\le \bar{\ell}^*|y-\bar{y}|,\ \text{for each } y,\bar{y}\in\mathbb{R} \text{ and } m=1,2,\dots k.$$

**Theorem 4.1.** *Assume that* $(A5)$-$(A7)$. *If*

$$\left[\frac{k\tilde{l}T^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)}+\frac{k\tilde{l}T^{\rho\omega}}{\rho^{\omega-1}\Gamma(\omega)}+\frac{\tilde{l}T^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)}+k(\ell^*+T^{\rho}\bar{\ell}^*)\right]<1, \tag{4.1}$$

*where* $\tilde{l}=\sup\{\tilde{l}(x):x\in\mathfrak{J}\}$, *then* (1.1) *has one solution on* $\mathfrak{J}$.

*Proof.* For each $u\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$, the set $S_{H,u}$ is nonempty. From $(A6)$ and (see [26, Theorem III.6], $H$ has a measurable selection. We shall prove that $M$ satisfies the assumptions of Lemma 2.3. The proof contains two steps.
**Step 1:** $M(u)\in Q_{cl}(\mathfrak{PC}(\mathfrak{J},\mathbb{R}))$ *for each* $u\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$. Let $(u_n)_{n\ge0}\in M(u)$ such that $u_n\to\tilde{u}$ in $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$. Then, $\tilde{u}\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ and there exists $g_n\in S_{H,u}$ such that for each $x\in\mathfrak{J}$,

$$\begin{aligned}
u_n(x) &= u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}g_n(s)\mathrm{d}s\\
&+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1}-x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-2}s^{\rho-1}g_n(s)\mathrm{d}s\\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g_n(s)\mathrm{d}s\\
&+ \sum_{0<x_m<x}I_m(u(x_m^-))+\sum_{0<x_m<x}(x-x_m)\bar{I}_m(u(x_m^-)).
\end{aligned}$$

From $(A5)$ and $H$ has compact values, we may pass to a subsequence if necessary to get that $g_n$ converges weakly to $g$ in $\mathfrak{L}^1_w(\mathfrak{J},\mathbb{R})$(the space endowed with the weak topology). A standard argument shows that $g_n$ converges strongly to $g\in S_{H,u}$. Then, for each $x\in\mathfrak{J}$,

$$\begin{aligned}
u_n(x)\to\tilde{u}(x) &= u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\
&+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1}-x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-2}s^{\rho-1}g(s)\mathrm{d}s\\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\
&+ \sum_{0<x_m<x}I_m(u(x_m^-))+\sum_{0<x_m<x}(x-x_m)\bar{I}_m(u(x_m^-)).
\end{aligned}$$

So, $\tilde{u}\in M(u)$.
**Step 2:** *There exists* $\gamma<1$ *such that* $G_d(M(u),M(\bar{u}))\le\gamma\|u-\bar{u}\|_\infty$ *for each* $u,\bar{u}\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$. Let $u,\bar{u}\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ and $h_1\in M(u)$. Then there exists $g_1(x)\in H(x,u(x))$ such that for each $x\in\mathfrak{J}$,

$$\begin{aligned}
h_1(x) &= u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}g_1(s)\mathrm{d}s\\
&+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1}-x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-2}s^{\rho-1}g_1(s)\mathrm{d}s\\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g_1(s)\mathrm{d}s\\
&+ \sum_{0<x_m<x}I_m(u(x_m^-))+\sum_{0<x_m<x}(x-x_m)\bar{I}_m(u(x_m^-)).
\end{aligned}$$

From $(A5)$, we get

$$G_d\left(H(x,u(x)),H(x,\bar{u}(x))\right)\le\tilde{l}(x)\,|u(x)-\bar{u}(x)|.$$

Therefore, there exists $z\in H(x,\bar{u}(x))$ such that

$$|g_1(x)-z|\le\tilde{l}(x)\,|u(x)-\bar{u}(x)|,\ x\in\mathfrak{J}.$$

Consider $U : \mathfrak{J} \to P(\mathbb{R})$ given by

$$U(x) = \{z \in \mathbb{R} : |g_1(x) - z| \leq \tilde{l}(x)\,|u(x) - \bar{u}(x)|\}.$$

Since the multivalued operator $V(x) = U(x) \cap H(x,\bar{u}(x))$ is measurable (See [He, Proposition III.4]), there exists a function $g_2(x)$ which is a measurable selection for $V$. So, $g_2(x) \in H(x,\bar{u}(x))$, and for each $x \in \mathfrak{J}$,

$$|g_1(x) - g_2(x)| \leq \tilde{l}(x)\,|u(x) - \bar{u}(x)|.$$

Now, we define for each $x \in \mathfrak{J}$ ,

$$\begin{aligned}
h_2(x) &= u_0 + u_1\left(\frac{x^\rho}{\rho}\right) + \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-1}s^{\rho-1}g_2(s)\mathrm{d}s\\
&+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1} - x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-2}s^{\rho-1}g_2(s)\mathrm{d}s\\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}g_2(s)\mathrm{d}s\\
&+ \sum_{0<x_m<x} I_m(u(x_m^-)) + \sum_{0<x_m<x}(x - x_m)\bar{I}_m(u(x_m^-)).
\end{aligned}$$

Then, for each $x \in \mathfrak{J}$,

$$\begin{aligned}
|h_1(x) - h_2(x)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-1}s^{\rho-1}|g_1(s) - g_2(s)|\,\mathrm{d}s\\
&+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1} - x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-2}s^{\rho-1}|g_1(s) - g_2(s)|\,\mathrm{d}s\\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}|g_1(s) - g_2(s)|\,\mathrm{d}s\\
&+ \sum_{0<x_m<x}\left|I_m(u(x_m^-)) - I_m(\bar{u}(x_m^-))\right| + \sum_{0<x_m<x}(x - x_m)\left|\bar{I}_m(u(x_m^-)) - \bar{I}_m(\bar{u}(x_m^-))\right|,\\
&\leq \frac{\tilde{l}\rho^{1-\omega}}{\Gamma(\omega)}\sum_{m=1}^{k}\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-1}s^{\rho-1}|u(s) - \bar{u}(s)|\,\mathrm{d}s\\
&+ \frac{\tilde{l}\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1} - x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-2}s^{\rho-1}|u(s) - \bar{u}(s)|\,\mathrm{d}s\\
&+ \frac{\tilde{l}\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}|u(s) - \bar{u}(s)|\,\mathrm{d}s\\
&+ \sum_{m=1}^{k}\ell^*\left|u(x_m^-) - \bar{u}(x_m^-)\right| + \sum_{m=1}^{k}\bar{\ell}^*(x - x_m)\left|u(x_m^-) - \bar{u}(x_m^-)\right|,\\
&\leq \frac{k\tilde{l}T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}\|u - \bar{u}\|_\infty + \frac{k\tilde{l}T^{\rho\omega}}{\rho^{\omega-1}\Gamma(\omega)}\|u - \bar{u}\|_\infty\\
&+ \frac{\tilde{l}T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}\|u - \bar{u}\|_\infty + k\ell^*\|u - \bar{u}\|_\infty + kT^\rho\bar{\ell}^*\|u - \bar{u}\|_\infty.
\end{aligned}$$

Thus,

$$\|h_1 - h_2\|_\infty \leq \left[\frac{k\tilde{l}T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + \frac{k\tilde{l}T^{\rho\omega}}{\rho^{\omega-1}\Gamma(\omega)} + \frac{\tilde{l}T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + k(\ell^* + T^\rho\bar{\ell}^*)\right]\|u - \bar{u}\|_\infty.$$

By an similar relation obtained by interchanging the roles of $u$ and $\bar{u}$, we get

$$G_d(M(u), M(\bar{u})) \leq \left[\frac{k\tilde{l}T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + \frac{k\tilde{l}T^{\rho\omega}}{\rho^{\omega-1}\Gamma(\omega)} + \frac{\tilde{l}T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + k(\ell^* + T^\rho\bar{\ell}^*)\right]\|u - \bar{u}\|_\infty.$$

From (4.1), $M$ is a contraction and by Lemma 2.3, $M$ has a fixed point $u$ which is a solution to (1.1). Hence the proof is complete. □

Now, we prove a result for problem (1.1) by using the concept of the nonlinear alternative of Leray-Schauder type [24] for single-valued maps with a selection theorem due to Bressan-Colombo for lower semi-continuous multivalued maps with decomposable values, also see [27] for multivalued maps with decomposable values and their properties.
Let $E$ be a subset of $[0,T] \times \mathbb{R}$. $E$ is $\mathfrak{L} \otimes \mathscr{B}$ measurable if $E$ belongs to the $\sigma$- algebra generated by all sets of the form $J \times \mathfrak{D}$, where $J$ is Lebesgue measurable in $[0,T]$ and $\mathfrak{D}$ is Borel measurable in $\mathbb{R}$. A subset $E$ of $\mathfrak{L}^1([0,T],\mathbb{R})$ is decomposable if for all $y,z \in E$ and $J \subset [0,T]$ measurable, $y\chi_J + z\chi_{[0,T]-J} \in E$, where $\chi$ stands for the characteristic function.
Let $F : Y \to P(Y)$ be a multivalued operator with nonempty closed values. $F$ is lower semi-continuous, if the set $\{t \in Y : F(t) \cap B \neq \emptyset\}$ is open for any open set $B$ in $Y$.

**Definition 4.2.** *Let $X$ be a separable metric space and let $M : X \to P(\mathfrak{L}^1([0,T],\mathbb{R}))$ be a multivalued operator, then $M$ has Bressan-Colombo property if*

*(1) $M$ is lower semi-continuous;*
*(2) $M$ has nonempty closed and decomposable values.*

*Let $H : [0,T]\times\mathbb{R} \to P(\mathbb{R})$ be a multivalued map with a nonempty compact values. Consider to $H$ the multivalued operator $\mathfrak{H} : \mathfrak{PC}([0,T],\mathbb{R}) \to P(\mathfrak{L}^1([0,T],\mathbb{R}))$ by letting*

$$\mathfrak{H}(u) = \{z \in \mathfrak{L}^1([0,T],\mathbb{R}) : z(x) \in H(x,u(x)) \ for\ a.e.\ x \in [0,T]\}.$$

*The operator $\mathfrak{H}$ is called the Niemytzki operator associated with $H$.*

**Definition 4.3.** *Let $H : [0,T]\times\mathbb{R} \to P(\mathbb{R})$ be a multivalued function with nonempty compact values. We say $H$ is of lower semi-continuous type if its associated Niemytzki operator $\mathfrak{H}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Now we present a selection theorem due to Bressan and Colombo [25].

**Theorem 4.4.** *[25] Let $X$ be a separable metric space and let the operator $M : X \to P(\mathfrak{L}^1([0,T],\mathbb{R}))$ be a multivalued satisfying property Bressan and Colombo. Then $M$ has a continuous selection, that is there exists a continuous function(single-valued) $\tilde{f} : X \to \mathfrak{L}^1([0,1],\mathbb{R})$ such that $\tilde{f}(u) \in M(u)$ for every $u \in X$.*

Next we introduce the following hypotheses:

(A8) $H : [0,T]\times\mathbb{R} \to P(\mathbb{R})$ is a nonempty compact valued multivalued map such that:
  (a) $(x,u) \mapsto H(x,u)$ is $\mathfrak{L}\otimes\mathscr{B}$ measurable;
  (b) $u \mapsto H(x,u)$ is lower semi-continuous for a.e. $x \in [0,T]$.
(A9) For each $p > 0$, there exists a function $h_p \in \mathfrak{L}^1([0,T],\mathbb{R}^+)$ such that

$$\|H(x,u)\|_P \le h_p(x) \ for\ a.e.\ x \in [0,T],$$

and for $u \in \mathbb{R}$ with $|u| \le p$.

The following lemma plays important role in our main result.

**Lemma 4.5.** *[28] Let $H : [0,T]\times\mathbb{R} \to P(\mathbb{R})$ be a multivalued map with nonempty, compact values. Assume that $(A8)$, $(A9)$ hold, then $H$ is of lower semi-continuous.*

**Theorem 4.6.** *Suppose that hypotheses $(A2)$-$(A4)$, $(A8)$, and $(A9)$ are satisfied. Then the problem (1.1) has at least one solution.*

*Proof.* From hypotheses $(A8)$, $(A9)$ and Lemma 4.5, $F$ is of lower semi-continuous type. Then from Theorem 4.4, there exists a continuous function $h \in \mathfrak{PC}([0,T],\mathbb{R}) \to \mathfrak{L}^1([0,T],\mathbb{R})$ such that $h(u) \in \mathfrak{H}(u)$ for all $u \in \mathfrak{PC}([0,T],\mathbb{R})$. Consider the problem

$$\begin{cases} {}^{\rho}D^{\omega}_{0^+}u(x) \in h(u)(x), & \text{for a.e. } x \in \mathfrak{J} = [0,T],\ x \ne x_k,\ m = 1,2,\ldots k,\ 1 < \omega \le 2, \\ \Delta u|_{x=x_m} = I_m(u(x_m^-)), & m = 1,2,\ldots,k, \\ \Delta u'|_{x=x_m} = \bar{I}_m(u(x_m^-)), & m = 1,2,\ldots,k, \\ u(0) = u_0,\ u'(0) = u_1. \end{cases} \tag{4.2}$$

If $u$ is a solution of (4.2), then $u$ is a solution of (1.1). Problem (4.2) can be reformulated as a fixed point problem for the operator $M_1 : \mathfrak{PC}([0,T],\mathbb{R}) \to \mathfrak{PC}([0,T],\mathbb{R})$ defined by

$$\begin{aligned} M_1(u)(x) &= u_0 + u_1(\frac{x^\rho}{\rho}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-1}s^{\rho-1}h(u)(s)\mathrm{d}s \\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1} - x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-2}s^{\rho-1}h(u)(s)\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}h(u)(s)\mathrm{d}s \\ &+ \sum_{0<x_m<x} I_m(u(x_m^-)) + \sum_{0<x_m<x}(x - x_m)\bar{I}_m(u(x_m^-)). \end{aligned}$$

Using $(A2)$-$(A4)$ and from similar argument as in Theorem 3.5, we can prove that the operator $M_1$ satisfies all conditions in the Leray-Schauder alternative. □

## 5. Topological structure of the solution set

In this section, we present a theorem on the topological structure of the set of solutions to (1.1).

**Theorem 5.1.** *Assume that $(A1)$, $(A5)$ and the following hypotheses hold:*

*(A10) There exists $q_1 \in \mathfrak{C}(\mathfrak{J},\mathbb{R}^+)$ such that $\|H(x,y)\|_P \le q_1(x)$ for $x \in \mathfrak{J}$ and $y \in \mathbb{R}$.*
*(A11) There exists $d_1, d_2 > 0$ such that*

$$|I_m(y)| \le d_1,\ for\ y \in \mathbb{R},$$
$$|\bar{I}_m(y)| \le d_2,\ for\ y \in \mathbb{R}.$$

*Then the solution set of* (1.1) *is not empty and is compact in* $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$.

*Proof.* Let

$$S=\{u\in\mathfrak{PC}(\mathfrak{J},\mathbb{R}): u \text{ is a solution of } (1.1)\}.$$

From Theorem 3.5, $S\neq\emptyset$. Now, we want to prove that $S$ is compact. Let $(u_n)_{n\in\mathbb{N}}\in S$, then there exists $g_n\in S_{H,u_n}$ and $x\in\mathfrak{J}$ such that

$$\begin{aligned} u_n(x) &= u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}g_n(s)\mathrm{d}s\\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1}-x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-2}s^{\rho-1}g_n(s)\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g_n(s)\mathrm{d}s\\ &+ \sum_{0<x_m<x}I_m(u_n(x_m^-))+\sum_{0<x_m<x}(x-x_m)\bar{I}_m(u_n(x_m^-)). \end{aligned}$$

From hypotheses $(A1)$, $(A10)$ and $(A11)$, we can show that there exists an $N_1>0$ such that $\|u_n\|_\infty\leq N_1$ for every $n\geq 1$. As in Step 3 in Theorem 3.5, we can prove that the set $\{u_n: n\geq 1\}$ is equicontinuous in $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$. By Arzelá-Ascoli theorem, we can say that, there exists a subsequence(denoted again by $\{u_n\}$) of $\{u_n\}$ such that $u_n$ converges to $u$ in $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$. We shall prove that there exist $g(\cdot)\in F(\cdot,u(\cdot))$ and $x\in\mathfrak{J}$ such that

$$\begin{aligned} u(x) &= u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1}-x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-2}s^{\rho-1}g(s)\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\ &+ \sum_{0<x_m<x}I_m(u(x_m^-))+\sum_{0<x_m<x}(x-x_m)\bar{I}_m(u(x_m^-)). \end{aligned}$$

Since $H(x,\cdot)$ is upper semi-continuous, for every $\varepsilon>0$, there exists $n_0(\varepsilon)\geq 0$ such that for every $n\geq n_0$, we get

$$g_n(x)\in H(x,u_n(x))\subset H(x,u(x))+\varepsilon B(0,1),\ a.e.\ x\in\mathfrak{J}.$$

Since $H(\cdot,\cdot)$ has compact values, there exists subsequence $g_{n_m}(\cdot)$ such that

$$g_{n_m}(\cdot)\to g(\cdot) \text{ as } m\to\infty,$$

$$g(x)\in H(x,u(x)),\ a.e.\ x\in\mathfrak{J}.$$

Therefore,

$$|g_{n_m}(x)|\leq q_1(x),\ a.e.\ x\in\mathfrak{J}.$$

By Lebesgue's dominated convergence theorem, we say that $g\in\mathfrak{L}^1(\mathfrak{J},\mathbb{R})$ which implies that $g\in S_{H,u}$. Thus, for $x\in\mathfrak{J}$, we get

$$\begin{aligned} u(x) &= u_0+u_1(\frac{x^\rho}{\rho})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\ &+ \frac{\rho^{2-\omega}}{\Gamma(\omega-1)}\sum_{0<x_m<x}(x^{\rho-1}-x_m^{\rho-1})\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-2}s^{\rho-1}g(s)\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\ &+ \sum_{0<x_m<x}I_m(u(x_m^-))+\sum_{0<x_m<x}(x-x_m)\bar{I}_m(u(x_m^-)). \end{aligned}$$

Then,

$$S\in P_{cp}(\mathfrak{PC}(\mathfrak{J},\mathbb{R})).$$

□

## 6. An example

**Example 6.1.** *We consider the Katugampola fractional impulsive differential inclusions of the type,*

$$\begin{cases} {}^{\rho}D^{\omega}_{0^+}u(x) \in H(x,u(x)), \quad a.\ e. \ \ x \in \mathfrak{J} = [0,T],\ x \neq \frac{1}{3},\ 1 < \omega \leq 2, \\ \Delta u|_{x=\frac{1}{3}} = \frac{1}{6+\left|u(\frac{1}{3}^-)\right|}, \\ \Delta u'|_{x=\frac{1}{3}} = \frac{1}{8+\left|u(\frac{1}{3}^-)\right|}, \\ u(0) = u_0,\ u'(0) = 0. \end{cases} \tag{6.1}$$

Let us assume the values $T = 1$, $k = 1$, $x_1 = \frac{1}{3}$, $\rho = 1$, and $x_0 = x_1 = 0$. Set

$$H(x,u) = \{g \in \mathbb{R} : f_1(x,u) \leq g \leq f_2(x,u)\},$$

where the functions $f_1, f_2 : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ are given.

$$I_1(u(x_1)) = \frac{1}{6+\left|u(\frac{1}{3}^-)\right|},\ \bar{I}_1(u(x_1)) = \frac{1}{8+\left|u(\frac{1}{3}^-)\right|}.$$

Then the eqaution (6.1) takes the form (1.1). We consider for each $x \in \mathfrak{J}$, the function $f_1(x,\cdot)$ is lower semi-continuous(i.e.,the set $\{u \in \mathbb{R} : f_1(x,u) > \lambda\}$ is open for each $\lambda \in \mathbb{R}$), and assume that for each $x \in \mathfrak{J}$, $f_2(x,\cdot)$ is upper semi-continuous(i.e.,the set $\{u \in \mathbb{R} : f_2(x,u) < \lambda\}$ is open for each $\lambda \in \mathbb{R}$). Assume that there are $q \in \mathfrak{C}(\mathfrak{J},\mathbb{R}^+)$ and $\Phi : [0,\infty) \to (0,\infty)$ continuous and non decreasing such that

$$\max(|f_1(x,u)|, |f_2(x,u)|) \leq q(x)\Phi(|u|),\ x \in \mathfrak{J} \text{ and } u \in \mathbb{R}.$$

Assume that there exists a constant $N > 0$ such that

$$\frac{N}{\left(\frac{2q^0}{\Gamma(\omega+1)} + \frac{q^o}{\Gamma(\omega)}\right)\Phi(N) + \frac{7}{24}} > 1.$$

From this, $H$ is compact and convex valued and it is upper semi-continuous [20]. Since all the conditions of Theorem 3.5 are satisfied, the problem (6.1) has at least one solution $u$ on $\mathfrak{J}$.

## 7. Conclusion

In this article, Leray-Schauder type, Bressan and Colombo, Covitz and Nodler concepts are used to prove the the Katugampola fractional type impulsive differential equations with inclusions. The obtained conditions ensure that the existence of at least one solution to the proposed problem. Further, an example is investigated for the problem.

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# Some Transmuted Software Reliability Models

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**Abstract**

The Hausdorff approximation of the shifted Heaviside function $h_{t_0}(t)$ by general transmuted family of cumulative distribution functions is studied and a value for the error of the best approximation is derived in this paper.
The outcomes of numerical examples confirm theoretical conclusions and they are derived by the help of CAS Mathematica.
Real data set which is proposed by Musa in [1] using general transmuted exponential software reliability model is examined.

## 1. Introduction

In this article we investigate the Hausdorff approximation of the shifted Heaviside function $h_{t_0}(t)$ by quadratic and cubic transmuted exponential cumulative distribution functions, based on Owoloko et al. model [2] and Rahman et al. [3] model.
Using CAS Mathematica we illustrate the results by given by us software modules.

### 1.1. Preliminaries

**Definition 1.1.** *[3] Let $T$ be a random variable with cumulative distribution function (c.d.f.) $C(t)$. Then a general transmuted family, called $k$–transmuted family is defined as:*

$$M(t) = C(t) + (1 - C(t)) \sum_{i=1}^{k} \lambda_i (C(t))^i \tag{1.1}$$

*with $\lambda_i \in [-1,1]$ for $i = 1,2,\ldots,k$ and $-k \le \sum_{i=1}^{k} \lambda_i < 1$.*

For the quadratic transmuted family, see Shaw et Buckley [4].

The exponential distribution is a widely used lifetime distribution.
The (c.d.f.) of exponential distribution is given by:

$$C(t) = 1 - e^{-\frac{t}{\theta}}, \; t \in [0,\infty).$$

**Definition 1.2.** *The quadratic transmuted exponential family is defined by (see, Owoloko et al. [2]):*

$$M_1(t) = \left(1 - e^{-\frac{t}{\theta}}\right)\left(1 + \lambda e^{-\frac{t}{\theta}}\right). \tag{1.2}$$

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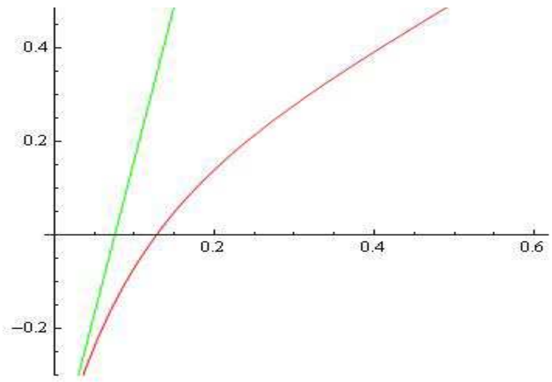


**Figure 2.1:** The functions $F(d)$ and $G(d)$.

**Remark.** From (1.1), we have

$$M(t) = C(t) + (1 - C(t))\left(\lambda_1 C(t) + \lambda_2 C^2(t)\right)$$

If $\lambda_2 = 0$ and $\lambda_1 = \lambda$ we have

$$\begin{aligned} M(t) &= C(t) + (1 - C(t))\,\lambda C(t) = \\ &= \left(1 - e^{-\frac{t}{\theta}}\right) + e^{-\frac{t}{\theta}}\lambda\left(1 - e^{-\frac{t}{\theta}}\right) = \\ &= \left(1 - e^{-\frac{t}{\theta}}\right)\left(1 + \lambda e^{-\frac{t}{\theta}}\right). \end{aligned}$$

**Definition 1.3.** *The (c.d.f.) of cubic transmutes exponential family is defined by:*

$$M_2(t) = (1+\lambda_1)\left(1 - e^{-\frac{t}{\theta}}\right) + (\lambda_2 - \lambda_1)\left(1 - e^{-\frac{t}{\theta}}\right)^2 - \lambda_2\left(1 - e^{-\frac{t}{\theta}}\right)^3. \tag{1.3}$$

We will note that the determination of compulsory in area of the Software Reliability Theory components, such as confidence intervals and confidence bounds, should also be accompanied by a serious analysis of the value of the best Hausdorff approximation [5] of the Heaviside function $h_{t_0}(t)$ by cumulative functions of type (1.1)–(1.2) - the subject of study in the present paper.

## 2. Main results

### 2.1. A note on the quadratic transmuted exponential family (1.2) [2]

Without loosing of generality we will look at the following (c.d.f.):

$$M_1^*(t) = \left(1 - e^{-\frac{t}{\theta}}\right)\left(1 + \lambda e^{-\frac{t}{\theta}}\right), \tag{2.1}$$

with

$$t_0 = -\theta \ln \frac{-1+\lambda+\sqrt{1+\lambda^2}}{2\lambda}; \quad M_1^*(t_0) = \frac{1}{2}.$$

The one–sided Hausdorff distance $d$ between the function $h_{t_0}(t)$ and the function (2.1) satisfies the relation

$$M_1^*(t_0 + d) = 1 - d. \tag{2.2}$$

The next theorem gives estimations for lower and upper bounds for $d$

**Theorem 2.1.** *Let*

$$p = -\frac{1}{2},$$

$$q = \frac{1}{2\lambda\theta}\left((1+\theta)2\lambda + (1-\lambda)^2 - (1-\lambda)\sqrt{1+\lambda^2}\right).$$

*For the one–sided Hausdorff distance $d = d(\lambda,\theta)$ between $h_{t_0}(t)$ and the function (2.1) the following inequalities hold for:*

$$2.1q > e^{1.05}$$

$$d_l = \frac{1}{2.1q} < d < \frac{\ln(2.1q)}{2.1q} = d_r. \tag{2.3}$$

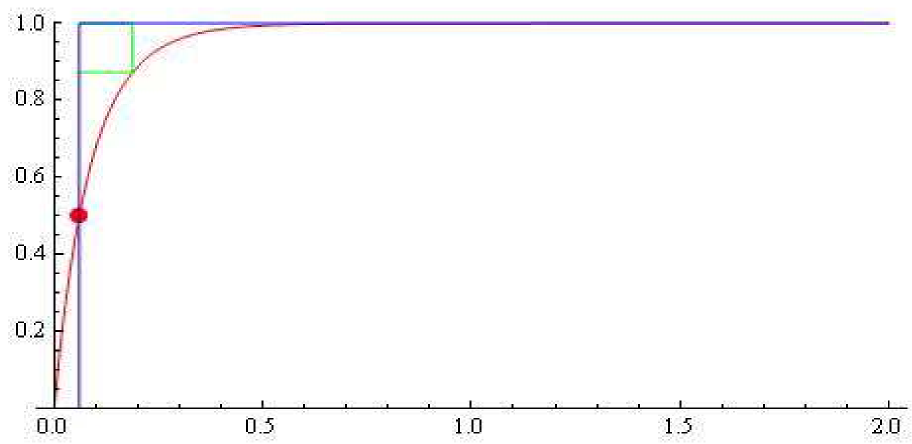


**Figure 2.2:** The model (2.1) for $\lambda = 0.2$, $\theta = 0.1$, $t_0 = 0.0598729$ ; H–distance $d = 0.127524$, $d_l = 0.0721072$, $d_r = 0.189613$.

*Proof.* We consider the function:

$$F(d) = M^*(t_0 + d) - 1 + d.$$

The function $F$ is increasing because $F'(d) > 0$.
Consider the function

$$G(d) = p + qd.$$

We obtain $G(d) - F(d) = O(d^2)$ by the help of Taylor expansion.
Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 2.1).
Evidently, $G'(d) > 0$.
Further, for $2.1q > e^{1.05}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

The proof of the theorem is completed. □

The model (2.1) for $\lambda = 0.2$, $\theta = 0.1$, $t_0 = 0.0598729$ is visualized on Fig. 2.2.

From the nonlinear equation (2.2) and inequalities (2.3) we have: $d = 0.127524$, $d_l = 0.0721072$, $d_r = 0.189613$.

## 2.2. A note on cubic transmuted exponential (c.d.f.) (1.3)

We consider the following family:

$$M_2^*(t) = (1+\lambda_1)\left(1 - e^{-\frac{t}{\theta}}\right) + (\lambda_2 - \lambda_1)\left(1 - e^{-\frac{t}{\theta}}\right)^2 - \lambda_2\left(1 - e^{-\frac{t}{\theta}}\right)^3. \tag{2.4}$$

Let $t_0$ is the positive root of the nonlinear equation

$$M_2^*(t_0) - \frac{1}{2} = 0.$$

The one–sided Hausdorff distance $d_1$ between the function $h_{t_0}(t)$ and the function (2.4) satisfies the relation

$$M_2^*(t_0 + d_1) = 1 - d_1.$$

Let

$$p_1 = e^{-\frac{3t_0}{\theta}}\left(\lambda_2 - (\lambda_1 + 2\lambda_2)e^{\frac{t_0}{\theta}} + (\lambda_1 + \lambda_2 - 1)e^{\frac{2t_0}{\theta}}\right),$$

$$q_1 = \frac{e^{-\frac{3t_0}{\theta}}}{\theta}\left(-3\lambda_2 + \theta e^{\frac{3t_0}{\theta}} + 2(\lambda_1 + 2\lambda_2)e^{\frac{t_0}{\theta}} + (1 - \lambda_1 - \lambda_2)e^{\frac{2t_0}{\theta}}\right).$$

In the next theorem lower and upper bounds for $d_1$ are given.

**Theorem 2.** For the one–sided Hausdorff distance $d_1$ between $h_{t_0}(t)$ and the function (2.4) the following inequalities are satisfied for:

$$2.1q_1 > e^{1.05}$$

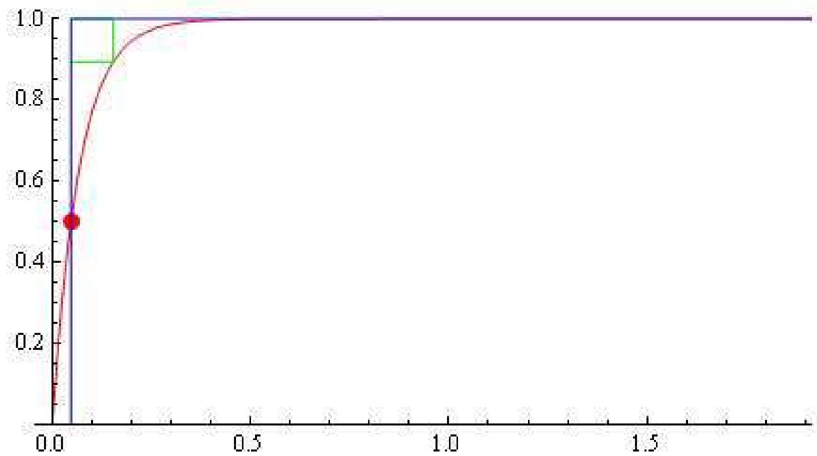


**Figure 2.3:** The model (2.4) for $\lambda_1 = 0.01$, $\lambda_2 = 0.05$, $\theta = 0.07$, $t_0 = 0.0473211$, $t_0 = 0.191515$; H–distance $d_1 = 0.106188$, $d_{l_1} = 0.0569506$, $d_{r_1} = 0.163197$.

$$d_{l_1} = \frac{1}{2.1q_1} < d_1 < \frac{\ln(2.1q_1)}{2.1q_1} = d_{r_1}.$$

The proof uses the ideas given here and will be skipped.

The model (2.4) for $\lambda_1 = 0.01$, $\lambda_2 = 0.05$, $\theta = 0.07$, $t_0 = 0.0473211$ is visualized on Fig. 2.3.

## 3. Numerical examples. Concluding remarks

Dataset, was proposed by Musa in [1]. The testing period is during the first 12 hours. The number of failures in each hour is given in Table 1.

| *Hour* | *Number of Failures* | *Cumulative failures* |
|---|---|---|
| 1 | 27 | 27 |
| 2 | 16 | 43 |
| 3 | 11 | 54 |
| 4 | 10 | 64 |
| 5 | 11 | 75 |
| 6 | 7 | 82 |
| 7 | 2 | 84 |
| 8 | 5 | 89 |
| 9 | 3 | 92 |
| 10 | 1 | 93 |
| 11 | 4 | 97 |
| 12 | 7 | 104 |

**Table 1:** Dataset [1]

The fitted model

$$M_2^*(t) = 104\left((1+\lambda_1)\left(1-e^{-\frac{t}{\theta}}\right) + (\lambda_2-\lambda_1)\left(1-e^{-\frac{t}{\theta}}\right)^2 - \lambda_2\left(1-e^{-\frac{t}{\theta}}\right)^3\right).$$

uses the data of Table 1 for the estimated parameters:

$$\lambda_1 = 0.207896;\ \ \lambda_2 = -0.733145;\ \ \theta = 3.44044$$

is plotted on Fig. 3.1.

In many cases it is appropriate to use the following deterministic software reliability model [1]:

$$M_3(t) = a^{b^{\frac{k_1}{t}}}.$$

The fitted model $M_3(t)$ based on the data of Table 1 for the estimated parameters:

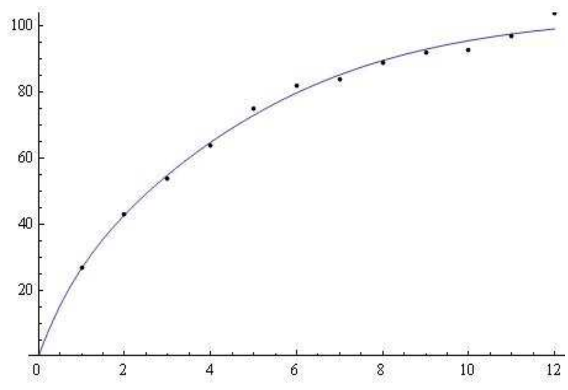


**Figure 3.1:** Approximation solution.

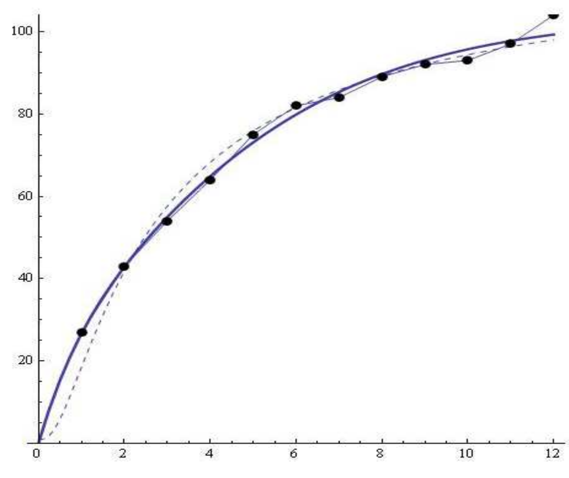


**Figure 3.2:** Comparison between the models: $M_2^*(t)$ – (thick) and $M_3(t)$ – (dashed).

$$a = 118.71;\ b = 0.667769;\ k_1 = 1.21843$$

is plotted on Fig. 3.2.

A good fit by the presented model $M_2^*(t)$ using for an example real data set is shown.

Obviously, studying of phenomenon ”super saturation” is mandatory element along with other important components - ”confidence bounds” and ”confidence intervals” when dealing with questions from Software Reliability Models domain.

For some software reliability models, see [6]–[47].

We hope that the results will be useful for specialists in this scientific area.

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# Erratum to ”On Convolution surfaces in Euclidean spaces”

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**Abstract**

In the present study we give some corrections for our paper which published in the first volume of this journal.

## 1. Erratum to ”On Convolution surfaces in Euclidean spaces”

**Page 89.**

**Theorem 3.2.** Let $M\star N$ be a convolution surface of a paraboloid $M$ and a translation surface $N$ given with the parametrization (3.4). Then the Gaussian curvature of the convolution surface is

$$K_{M\star N}=\frac{4cf''g''}{(f''+2)\left(g''+2c\right)\left((f')^2+(g')^2+1\right)^2}.$$

**Proof.** Let $M\star N$ be a convolution surface of a paraboloid $M$ and a translation surface $N$ given with the parametrization (3.4) For simplicity we define $z=x+y$. Then the tangent space of $M\star N$ is spanned by

$$\begin{aligned} z_s &= \frac{1}{2}\left(f''+2,0,f'\left(f''+2\right)\right), \\ z_t &= \frac{1}{2c}\left(0,g''+2c,g'\left(g''+2c\right)\right). \end{aligned}$$

Hence the coefficients of first and second fundamental forms of the convolution surface $M\star N$ are

$$E=\langle z_s,z_s\rangle=\frac{1}{4}\left(\left(f'\right)^2+1\right)\left(f''+2\right)^2,$$

$$\begin{aligned} F=\langle z_s,z_t\rangle &= \frac{f'g'}{4c}\left(f''+2\right)\left(g''+2c\right), \\ G=\langle z_t,z_t\rangle &= \frac{1}{4c^2}\left(\left(g'\right)^2+1\right)\left(g''+2c\right)^2, \end{aligned} \tag{3.5}$$

and

$$e=\frac{\langle z_{ss},z_s\times z_t\rangle}{\sqrt{EG-F^2}}=\frac{f''(g''+2c)\left(f''+2\right)^2}{8c\sqrt{EG-F^2}},$$

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$$f=\frac{\langle z_{st},z_s\times z_t\rangle}{\sqrt{EG-F^2}}=0, \tag{3.6}$$

$$e=\frac{\langle z_{tt},z_s\times z_t\rangle}{\sqrt{EG-F^2}}=\frac{g''(f''+2)\,(g''+2c)^2}{8c^2\sqrt{EG-F^2}},$$

respectively. By definition the Gaussian curvature of the convolution surface $M\star N$ is given by

$$K_{M\star N}=\frac{eg-f^2}{EG-F^2}. \tag{3.7}$$

So, substituting (3.5) and (3.6) into (3.7) after some calculation we get the result.□

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As a consequence of previous theorem one can get the following results.

**Corollary 3.3**. Let $M\star N$ be a convolution surface of a paraboloid $M$ and a translation surface (3.4). If the convolution $M\star N$ is a flat surface, then at least one of the following cases occur;

$$f(s)=a_1s+a_2,\text{ or } g(t)=b_1t+b_2,$$

where $a_i$ and $b_j$ are real constants.

**Corollary 3.4.** The convolution surface $M\star N$ given with the parametrization $f(s)=a_1s+a_2$ and $g(t)=b_1t+b_2$ is a part of a plane.

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Finally,convolution surface $M\star N$ has the parametrization

$$(x+y)(s,t)=(\frac{h'+2ff'}{2f'}\cos t,\frac{h'+2ff'}{2cf'}\sin t,\frac{(h')^2}{4c\,(f')^2}(c\cos^2 t+\sin^2 t)+h(s)). \tag{3.10}$$

**Theorem 3.5.** Let $M\star N$ be a convolution surface of a paraboloid $M$ and a surface of revolution given with the parametrization (3.8). If $c=1$ then the convolution surface $M\star N$ also a surface of revolution with Gaussian curvature

$$K_{M\star N}=\frac{(\varphi^2+h)'\left\{(\varphi^2+h)''(\varphi+h)'-(\varphi^2+h)'(\varphi+h)''\right\}}{(\varphi+f)\left\{((\varphi^2+h)')^2+((\varphi+h)')^2\right\}^2};\ \ f^{'}\neq 0, \tag{3.11}$$

where $\varphi(s)=\frac{h'(s)}{2f'(s)}$ is a real valued differentiable function different from 1.

**Proof.** Similar to the proof of Theorem 3.2 we get the result. □

**Corollary 3.6.** Let $M\star N$ be a convolution surface of a paraboloid $M$ with $c=1$ and a surface of revolution (3.8). If the convolution surface $M\star N$ is a flat surface, then it is either a plane or a surface of revolution satisfying

$$(\varphi^2+h)''(\varphi+h)'-(\varphi^2+h)'(\varphi+h)''=0.$$

**Proof.** If $M\star N$ is a flat surface, then

$$(\varphi^2+h)'\left\{(\varphi^2+h)''(\varphi+h)'-(\varphi^2+h)'(\varphi+h)''\right\}=0 \tag{3.12}$$

holds. So, we have two possible cases; $\varphi^2+h=const.$ , or $(\varphi^2+h)''(\varphi+h)'-(\varphi^2+h)'(\varphi+h)''=0$. For the first case $M\star N$ is a part of a plane □.

**Page 90.**

Omit the Equation 3.14.
Finally, the sum $M\star N$ has the parametrization

$$(x+y)(s,t)=\begin{pmatrix}\left(\frac{2tp(s)-z'(s)}{2t}\right)\sin s+(p'(s)+t)\cos s\\ \left(\frac{z'(s)-2ctp(s)}{2ct}\right)\cos s+(p'(s)+t)\sin s\\ z(s)+\left(\frac{z'(s)^2}{4ct^2}\right)\left(c\sin^2 s+\cos^2 s\right)\end{pmatrix},\ t\neq 0. \tag{3.15}$$

**Page 91.**

**Theorem 3.7.** Let $M \star N$ be a convolution surface of a paraboloid $M$ with $c = 1$ and a right helicoid $N$ given with the parametrization (3.17). Then the Gaussian curvature of the convolution surface is

$$K_{M\star N} = -\frac{\psi'' \left\{(\psi'(t-k)t + \psi(\psi\psi' + k)\right\}(k-t) - \left\{\psi\psi' + (\psi')^2(k-t) + k\right\}^2}{\left\{(\psi')^2(k-t)^2 + (\psi\psi' + k)^2 + (\psi\psi' + t)^2\right\}^2};\ t \neq 0, \tag{3.18}$$

where

$$\psi(t) = \frac{-k}{2t},$$

is a real valued function.

**Proof.** Similar to the proof of Theorem 3.2 we get the result. □