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Positive Linear Operators Preserving τ and τ^2

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ABSTRACT. In the paper we introduce a general class of linear positive approximation processes defined on bounded and unbounded intervals designed using an appropriate function. Voronovskaya type theorems are given for these new constructions. Some examples including well known operators are presented.

Keywords: Generalized operators, Voronovskaya theorem.

2010 Mathematics Subject Classification: 41A25, 41A36.

1. INTRODUCTION

In the theory of approximation by linear positive operators (l.p.o) Korovkin famous theorem has a crucial role to determine whether the corresponding sequence of l.p.o converges to the identity operator. However, Korovkin theorem for a sequence of l.p.o requires uniform convergence on an extended complete Tchebychev system, in special, the set of test functions $e_i(t) = t^i$, $i = 0, 1, 2$. In [6], to obtain better error estimation, J. P. King introduced and studied a generalization of the classical Bernstein operators. These operators preserve the test functions e_0 and e_2, while the classical Bernstein operators preserve the test functions e_0 and e_1. Starting from this approximation process King's idea has been successfully applied to several well known sequences of operators. In [5], the authors introduced the sequence of operators B_n^τ by

$$B_n^\tau(f;x) = \sum_{k=0}^{n} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right)\binom{n}{k}\tau^k(x)\left(1-\tau(x)\right)^{n-k},\ x \in [0,1],\ n \in \mathbb{N},$$

which is a new form of well-known Bernstein operators, where $\tau \in C[0,1]$ is a strictly increasing function , $\tau(0) = 0$, $\tau(1) = 1$. Shape preserving and convergence properties as well as the asymptotic behavior and saturation for the sequence (B_n^τ) were deeply studied using the test functions $\{1, \tau, \tau^2\}$. Durrmeyer version of the operators B_n^τ was introduced and studied in [1]. A similar idea was used for the operators defined on unbounded intervals given in [2].
In this short note, we introduce linear positive operators defined on bounded and unbounded intervals that preserve the functions τ and τ^2 such that $\tau \in C[0,1]$ is strictly increasing, $\tau(0) = 0$, $\tau(1) = 1$ (for the operators defined on the unbounded interval, we consider the function $\rho \in C[0,\infty)$ such that $\rho(0) = 0$ and $\rho'(x) > 0$ for $x \in [0,\infty)$). Then, we give a Voronovskaya type theorem for our general operators. Some examples including very well known operators are also obtained.

2. GENERALIZED OPERATORS

Let $L_n : C[0,1] \to C[0,1]$ be a sequence of l.p.o such that $L_n e_0 = e_0$ and $L_n e_1 = e_1$.

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Let $\tau : [0,1] \to [m, M]$ be continuous such that $0 < m < M$, $\tau^{'}(x) > 0$ for $x \in [0,1]$, $\tau(0) = m$ and $\tau(1) = M$. For any $f \in C[0,1]$ consider the function $\frac{f \circ \tau^{-1}}{e_1}$ such that

$$\frac{f \circ \tau^{-1}}{e_1}(m + (M-m)t) = \frac{f\left(\tau^{-1}(m + (M-m)t)\right)}{m + (M-m)t}, \ \ t \in [0,1].$$

For $x \in [0,1]$ and $f \in C[0,1]$ consider the operators

$$\left(V_n^L f\right)(x) = \tau(x) L_n \left(\frac{f \circ \tau^{-1}}{e_1}(m + (M-m)t); \frac{\tau(x) - m}{M - m}\right).$$

It is obvious that

$$V_n^L \tau(x) = \tau(x) \quad \text{and} \quad V_n^L \tau^2(x) = \tau^2(x).$$

2.1. **Examples.**

(1) Let $\tau(x) = x + 1$ and $L_n = B_n$, where (B_n) is the sequence of Bernstein operators. For $m = 1$ and $M = 2$,

$$V_n^B f(x) = (x+1) B_n \left(\frac{f(t)}{1+t}, x\right).$$

(2) Let $\tau(x) = e^{\mu x}$, $\mu > 0$ and $L_n = B_n$, where (B_n) is the sequence of Bernstein operators. For $m = 1$ and $M = e^{\mu}$,

$$V_n^B f(x) = e^{\mu x} B_n \left(\frac{f\left(\frac{1}{\mu}\log(1 + (e^{\mu} - 1)t)\right)}{1 + (e^{\mu} - 1)t}; \frac{e^{\mu x} - 1}{e^{\mu} - 1}\right).$$

Let $K_n : C[0,\infty) \to C[0,\infty)$ be a sequence of l.p.o such that $K_n e_0 = e_0$ and $K_n e_1 = e_1$.
Let $\rho : [0,\infty) \to [m,\infty)$ be continuous such that $m > 0$, $\rho^{'}(x) > 0$ for $x \in [0,\infty)$ and $\rho(0) = m$.
For $f \in C[0,\infty)$, consider the function $\frac{f \circ \rho^{-1}}{e_1}$ such that

$$\frac{f \circ \rho^{-1}}{e_1}(t + m) = \frac{f\left(\rho^{-1}(t+m)\right)}{t + m}, \ \ t \in [0,\infty).$$

For $x \in [0,\infty)$, consider the operators

$$\left(U_n^K f\right)(x) = \rho(x) K_n \left(\frac{f\left(\rho^{-1}(m+t)\right)}{m+t}; \rho(x) - m\right).$$

It is obvious that

$$\left(U_n^K \rho\right)(x) = \rho(x) \quad \text{and} \quad \left(U_n^K \rho^2\right)(x) = \rho^2(x).$$

2.2. **Examples.**

(1) Let $\rho(x) = e^{\mu x} + 1$, $x \geq 0$, $\mu > 0$ and $K_n = S_n$, where (S_n) is the sequence of Szász-Mirakyan operators. For $m = 2$, $\rho^{-1} : [2,\infty) \to [0,\infty)$, $\rho^{-1}(x) = \frac{1}{\mu}\log(x-1)$ and $x \in [2,\infty)$,

$$\left(U_n^S f\right)(x) = (e^{\mu x} + 1) S_n \left(\frac{f\left(\frac{1}{\mu}\log(1+t)\right)}{2+t}, e^{\mu x} - 1\right).$$

(2) Let $\rho(x) = e^{\mu x}$ and $K_n = T_n$, where (T_n) is the sequence of Baskakov operators. For $x \geq 0$, $m = 1$ and $\mu > 0$,

$$\left(U_n^B f\right)(x) = e^{\mu x} T_n \left(\frac{f\left(\frac{1}{\mu}\log(1+t)\right)}{1+t}; e^{\mu x} - 1 \right).$$

3. Transferring the Voronovskaya Result

Theorem 3.1. *Let $f \in C[0,1]$ with $f''(t)$ finite at any $t \in [0,1]$. Suppose that $L_n e_0 = e_0$, $L_n e_1 = e_1$ and*

$$V_n f(x) = \tau(x) L_n \left(\frac{f \circ \tau^{-1}}{e_1} (m + (M-m)t); \frac{\tau(x) - m}{M-m} \right).$$

If there exists $\alpha \in C[0,1]$ such that

$$\lim_{n\to\infty} n\left(L_n f(t) - f(t)\right) = \alpha(t) f''(t),$$

then we have

$$\begin{aligned} &\lim_{n\to\infty} n\left(V_n f(x) - f(x)\right) \\ = &\frac{(M-m)^2 \alpha\left(\frac{\tau(x)-m}{M-m}\right)}{\tau^2(x)\left(\tau'(x)\right)^3} \left[\tau'(x)\tau^2(x) f''(x) - \tau(x)\left(\tau(x)\tau''(x) + 2\left(\tau'(x)\right)^2\right) f'(x) \right. \\ + &\left. 2\left(\tau'(x)\right)^3 f(x) \right]. \end{aligned}$$

Proof. We have

$$\begin{aligned} &n\left(V_n f(x) - f(x)\right) \\ = &n\tau(x) \left[L_n \left(\frac{f \circ \tau^{-1}}{e_1}(m + (M-m)t); \frac{\tau(x)-m}{M-m} \right) - \frac{f(x)}{\tau(x)} \right], \\ = &n\tau(x) \left[L_n \left(\frac{f \circ \tau^{-1}}{e_1}(m + (M-m)t); \frac{\tau(x)-m}{M-m} \right) - \frac{f \circ \tau^{-1}}{e_1}(m + (M-m)t)\Big|_{\left(\frac{\tau(x)-m}{M-m}\right)} \right]. \end{aligned}$$

Thus we have from the hypothesis that

$$\lim_{n\to\infty} n\left(V_n f(x) - f(x)\right) = \tau(x)\alpha\left(\frac{\tau(x)-m}{M-m}\right) \frac{d^2}{du^2}\left(\frac{f\left(\tau^{-1}(m + (M-m)u)\right)}{m + (M-m)u} \right)\Bigg|_{u=\frac{\tau(x)-m}{M-m}},$$

with $u = \frac{\tau(x)-m}{M-m}$ and $\frac{dx}{du} = \frac{M-m}{\tau'(x)}$.

It is obvious that

$$\begin{aligned} \frac{d}{du}\left(\frac{f\left(\tau^{-1}(m + (M-m)u)\right)}{m + (M-m)u} \right)\Bigg|_{u=\frac{\tau(x)-m}{M-m}} &= \frac{d}{du}\left(\frac{f \circ \tau^{-1}}{e_1}(m + (M-m))(u) \right)\Bigg|_{u=\frac{\tau(x)-m}{M-m}}, \\ &= \frac{dx}{du}\frac{d}{dx}\left(\frac{f(x)}{\tau(x)}\right), \\ &= \frac{M-m}{\tau'(x)}\frac{d}{dx}\left(\frac{f(x)}{\tau(x)}\right), \\ &= (M-m)\frac{f'(x)\tau(x) - \tau'(x)f(x)}{\tau'(x)\tau^2(x)}, \end{aligned}$$

and

$$\begin{aligned}
&\frac{d^2}{du^2}\left(\frac{f\left(\tau^{-1}\left(m+(M-m)\,u\right)\right)}{m+(M-m)\,u}\right)\Bigg|_{u=\frac{\tau(x)-m}{M-m}} \\
= &\ \frac{M-m}{\tau'(x)}\frac{d}{dx}\left((M-m)\,\frac{f'(x)\,\tau(x)-f(x)\,\tau'(x)}{\tau'(x)\,\tau^2(x)}\right), \\
= &\ \frac{(M-m)^2}{\left(\tau(x)\,\tau'(x)\right)^3}\left[\tau'(x)\,\tau^2(x)\,f''(x)-\tau(x)\left(\tau(x)\,\tau''(x)+2\left(\tau'(x)\right)^2\right)f'(x)\right. \\
+ &\ \left.2\left(\tau'(x)\right)^3 f(x)\right].
\end{aligned}$$

Hence we have the desired result. □

Corollary 3.1. *Let $\tau(x)=e^{\mu x}$ and $L_n=B_n$, where (B_n) is the sequence of Bernstein operators. For $m=1$ and $M=e^{\mu}$, we get*

$$\lim_{n\to\infty} n\left(V_n f(x)-f(x)\right)=\frac{\left(e^{\mu x}-1\right)\left(e^{\mu}-e^{\mu x}\right)}{2\mu^2 e^{2\mu x}}\left(f''(x)-3\mu f'(x)+2\mu^2 f(x)\right).$$

Corollary 3.2. *Let $\tau(x)=x+1$ and $L_n=B_n$, where (B_n) is the sequence of Bernstein operators. For $m=1$ and $M=2$, we obtain*

$$\lim_{n\to\infty} n\left(V_n f(x)-f(x)\right)=\frac{x(1-x)}{2}\left(f''(x)-\frac{2}{x+1}f'(x)+\frac{2}{(x+1)^2}f(x)\right).$$

Theorem 3.2. *Let $f\in C[0,\infty)$ with $f''(t)$ finite, $t\in[0,\infty)$. Suppose that $K_n e_0=e_0$, $K_n e_1=e_1$ and*

$$U_n f(x)=\rho(x)\,K_n\left(\frac{f\circ\rho^{-1}}{e_1}(m+t)\,;\rho(x)-m\right).$$

If there exists $\gamma\in C[0,\infty)$ such that

$$\lim_{n\to\infty} n\left(K_n f(t)-f(t)\right)=\gamma(t)\,f''(t),$$

then we have

$$\begin{aligned}
&\lim_{n\to\infty} n\left(U_n f(x)-f(x)\right) \\
= &\ \frac{\gamma(\rho(x)-m)}{\rho^2(x)\left(\rho'(x)\right)^3}\left[\rho'(x)\,\rho^2(x)\,f''(x)-\rho(x)\left(\rho(x)\,\rho''(x)+2\left(\rho'(x)\right)^2\right)f'(x)\right. \\
+ &\ \left.2\left(\rho'(x)\right)^3 f(x)\right].
\end{aligned}$$

Proof. The proof of this theorem is similar to that of Theorem 1. □

Corollary 3.3. *Let $\rho(x)=e^{\mu x}+1$, $x\geq 0$, $\mu>0$ and $K_n=S_n$, where (S_n) is the sequence of Szász-Mirakyan operators. For $m=2$, we have*

$$\lim_{n\to\infty} n\left(U_n f(x)-f(x)\right)=\frac{e^{\mu x}-1}{2\mu^2 e^{2\mu x}}\left(f''(x)-\mu\frac{3e^{\mu x}+1}{e^{\mu x}+1}f'(x)+2\mu^2\frac{e^{2\mu x}}{\left(e^{\mu x}+1\right)^2}f(x)\right).$$

Corollary 3.4. *Let $\rho(x) = e^{\mu x}$, and $K_n = T_n$, where (T_n) is the sequence of Baskakov operators. For $x \geq 0$, $\mu > 0$ and $m = 1$, we get*

$$\lim_{n\to\infty} n\left(U_n f(x) - f(x)\right) = \frac{e^{\mu x}-1}{2\mu^2 e^{\mu x}} \left(f''(x) - 3\mu f'(x) + 2\mu^2 f(x) \right).$$

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Shift λ-Invariant Operators

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ABSTRACT. The present note is devoted to a generalization of the notion of shift invariant operators that we call it λ-invariant operators ($\lambda \geq 0$). Some properties of this new class are presented. By using probabilistic methods, three examples are delivered.

Keywords: Modulus of continuity, integral operator, convolution type operator, probabilistic distribution function.

2010 Mathematics Subject Classification: 41A35, 47B38.

1. INTRODUCTION

This research is mainly motivated by the work of G. A. Anastassiou and H. H. Gonska [6]. The authors have introduced a general family of integral type operators. Sufficient conditions were given for shift invariance and also the property of global smoothness preservation was studied. Let (X, d) be a metric space of real valued functions defined on D, where $D = \mathbb{R}$ or $D = \mathbb{R}_+$. An operator L which maps X into itself is called a shift invariant operator if and only if

$$Lf_\alpha = (Lf)_\alpha \text{ for any } f \in X \text{ and } \alpha > 0,$$

where $f_\alpha(\cdot) = f(\cdot + \alpha)$.
In this note we give a generalization of the notion of shift invariant operator. Some properties of this class are presented and a general family of such operators in the space of integrable functions $L^1(\mathbb{R})$ is introduced by using the convolution product of another operators with a scaling type function. By resorting to probabilistic methods, we indicate some classical operators as shift λ-invariant, where λ is calculated in each case.
We refer to the following operators: Szász-Mirakjan, Baskakov and Weierstrass. It is honest to mention that the value of λ does not target the whole sequence, it depends on the rank of the considered term.
The general results are concentrated in Section 2 and the applications are detailed in Section 3. It is known that the shift invariant operators are useful in wavelet analysis. Along with the paper [6], the subject was developed in other papers, among which we quote [3], [4], [5]. Until now, we have built a generalization of the shift invariant operators and we proved that the new class is consistent. The applications presented reinforce the significance of the construction. The use of this class of operators could lead for generating wavelet bases type. In this direction, the conditions for multiresolution analysis can be relaxed by using shift λ-invariant operators. Thus, we can talk about quasi-wavelet functions that can serve to reconstruct certain signals. We admit that this research direction is at an early stage.

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2. RESULTS

Firstly, we present the following informal definition.

Definition 2.1. *Let λ be a non negative number and X be a metric space of real valued functions defined on $\mathbb{R}$ or $\mathbb{R}_+$. An operator L acting on X is called a shift λ-invariant operator if*

$$|(Lf)_\alpha - Lf_\alpha| \le \lambda, \text{ for any } f \in X \text{ and } \alpha > 0. \tag{2.1}$$

Clearly, for $\lambda = 0$ we reobtain the notion of shift invariant operator.

Theorem 2.1. *Let A, B operators acting on a compact metric space X of real valued functions defined on $\mathbb{R}$ or $\mathbb{R}_+$.*
i) If A is a shift λ-invariant and B is a shift invariant, then AB is a shift λ-invariant operator.
ii) If A is a shift invariant, linear and positive operator, and B is a shift λ-invariant, then AB is a shift $\lambda\mu$-invariant operator, where $\mu = \|A\|$.
Call $\|A\| = \sup\{\|Ag\|_X : g \in X \text{ and } \|g\|_X \le 1\}$.

Proof. i) We take $g = Bf$ and, in concordance with the hypothesis, we can write successively

$$|(ABf)_\alpha - ABf_\alpha| = |(Ag)_\alpha - A(Bf_\alpha)| = |(Ag)_\alpha - Ag_\alpha| \le \lambda,$$

which implies the first statement of the theorem.
ii) Since B is a shift λ-invariant operator, we get

$$-\lambda \le (Bf)_\alpha - Bf_\alpha \le \lambda. \tag{2.2}$$

The operator A is linear and positive, consequently it is monotone, i.e., $Au \le Av$ for any u, v belong to X with the property $u \le v$.
A being a shift invariant operator too, relation (2.2) implies

$$|(ABf)_\alpha - ABf_\alpha| \le \lambda Ae_0,$$

where $e_0(x) = 1$, $x \in \mathbb{R}$ or $x \in \mathbb{R}_+$. Because of $0 < Ae_0 \le \|A\|$, the result follows. □

Remark. Assuming $Ae_0 = e_0$, relation usually verified by linear and positive operators (so called Markov type operators), we deduce that $\mu = 1$ and Theorem 2.1 (ii) guarantees that AB becomes a shift λ-invariant operator.
In what follows, starting from a sequence of shift λ-invariant operators and using a scaling type function, we construct a sequence of integral type operators.
For each $n \in \mathbb{N}$, let l_n be a shift λ_n-operator which maps the space $L^1(\mathbb{R})$ into itself. Also, we are fixing a function $\psi \in L^1(\mathbb{R})$ such that

$$\|\psi\|_1 = \int_{\mathbb{R}} |\psi(x)| dx \ne 0.$$

For any $f \in L^1(\mathbb{R})$, the convolution of $l_n f$ with ψ is a function named $L_n f$ which belongs to $L^1(\mathbb{R})$ and is defined by

$$(L_n f)(x) = (l_n f * \psi)(x) = \int_{\mathbb{R}} (l_n f)(y)\psi(x-y)dy. \tag{2.3}$$

It is known that the convolution product $*$ enjoys the commutativity property. Let $n \in \mathbb{N}$ arbitrarily be set. On the other hand, we have the following relations

$$(L_n f)_\alpha(x) = \int_{\mathbb{R}} (l_n f)(x + \alpha - u)\psi(u)du,$$

$$(L_n f_\alpha)(x) = \int_{\mathbb{R}} (l_n f_\alpha)(x - u)\psi(u)du,$$

which allow us to write

$$
\begin{aligned}
|(L_n f)_\alpha(x) - (L_n f_\alpha)(x)| &\leq \int_{\mathbb{R}} |(l_n f)(x+\alpha-u) - (l_n f_\alpha)(x-u)||\psi(u)|du \\
&= \int_{\mathbb{R}} |((l_n f)_\alpha - l_n f_\alpha)(x-u)||\psi(u)|du \\
&\leq \lambda_n \|\psi\|_1.
\end{aligned}
$$

We just ended the proof of the following result.

Theorem 2.2. *Let $L_n : L^1(\mathbb{R}) \to L^1(\mathbb{R})$, $n \geq 1$, be operators defined by (2.3). Then, for each $n \in \mathbb{N}$, L_n is a shift $\lambda_n\|\psi\|_1$-invariant operator.*

We notice that if we substitute in (2.3) the function ψ by $\|\psi\|_1^{-1}\psi$, then the operator L_n becomes shift λ_n-invariant, $n \in \mathbb{N}$.
As usual, we denote by $C_B(D)$ the Banach lattice of all bounded and continuous real functions on D endowed with the sup-norm $\|\cdot\|$. Also $C_B^1(D)$ denotes the subspace of $C_B(D)$ consisting of all functions which are continuously differentiable and bounded on D. We recall the definition of the first modulus of smoothness $\omega(f;\cdot)$ associated to the bounded function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$,

$$
\omega(f;\delta) = \sup_{\substack{x,y\in I \\ |x-y|\leq\delta}} |f(x) - f(y)|, \ \delta \geq 0. \tag{2.4}
$$

At this moment we need the following result.

Theorem 2.3. ([2, *Theorem 7.3.4*]) *Let the random variable Y have distribution μ, $E(Y) := x_0$ and $Var(Y) := \sigma^2$. Consider $f \in C_B^1(\mathbb{R})$. Then*

$$
|Ef(Y) - f(x_0)| = \left|\int_{\mathbb{R}} f d\mu - f(x_0)\right| \leq (1.5625)\omega\left(f'; \frac{\sigma}{2}\right)\sigma. \tag{2.5}
$$

In the above $E(Y)$, $Var(Y)$ represent the expected value and variance of Y, respectively.
We consider the random variables X_j, $j \geq 1$, independent and identically distributed and we introduce

$$
X_{j,\alpha} = X_j + \alpha, \ S_{n,\alpha} = \frac{1}{n}\sum_{j=1}^{n} X_{j,\alpha}, \ n \geq 1. \tag{2.6}
$$

Clearly, $S_{n,0}+\alpha = S_{n,\alpha}$. If we use the notations $E(X_{j,\alpha}) := x_{0,\alpha}$ and $Var(X_{j,\alpha}) := \sigma_\alpha^2$, by using the properties of the expectation respectively the variance, we obtain

$$
E(S_{n,\alpha}) = x_{0,0} + \alpha = x_{0,\alpha}, \ Var(S_{n,\alpha}) = \frac{\sigma_\alpha^2}{n} = \frac{\sigma_0^2}{n}.
$$

From (2.5) we deduce

$$
\begin{aligned}
|E[f(S_{n,\alpha})] - f(x_{0,\alpha})| &= \left|\int_{\mathbb{R}} f\left(\frac{t}{n}\right) dF_{n,\alpha}(t) - f(x_{0,\alpha})\right| \\
&\leq 1.5625\omega\left(f'; \frac{\sigma_0}{2\sqrt{n}}\right)\frac{\sigma_0}{\sqrt{n}},
\end{aligned} \tag{2.7}
$$

where $F_{n,\alpha}$ is the distribution function of the random variable $S_{n,\alpha}$.
It is known that by using probabilistic methods several classical positive and linear operators have been obtained. Pioneers in this research field can be mentioned here W. Feller [7] and

D.D. Stancu [9]. A recent and up-to-date approach to this study direction concerning Markov semigroups and approximation processes can be found in [1].
As in [9], for each $n \geq 1$, we choose

$$(L_n f)(x) = E[f \circ S_{n,0}(x)] = \int_{-\infty}^{\infty} f dF_{S_{n,0}(x)}, \tag{2.8}$$

where $F_{S_{n,0}}$ is the probability distribution of the variable $S_{n,0}$. Note that $L_n f$ is a bounded function and clearly satisfies $\|L_n f\| \leq \|f\|$.
Taking into account (2.6) and (2.7) we can write successively

$$\begin{aligned} |(L_n f)_\alpha(x) - (L_n f_\alpha)(x)| &\leq |E[f(S_{n,\alpha})] - f(x_{0,\alpha})| + |E[f_\alpha(S_{n,0})] - f_\alpha(x_{0,0})| \\ &\leq \mu\left(\omega\left(f'; \frac{\sigma_0}{2\sqrt{n}}\right) + \omega\left(f'_\alpha; \frac{\sigma_0}{2\sqrt{n}}\right)\right)\frac{\sigma_0}{\sqrt{n}} \\ &= 2\mu\omega\left(f'; \frac{\sigma_0}{2\sqrt{n}}\right)\frac{\sigma_0}{\sqrt{n}}, \end{aligned}$$

where $\mu = 1.5625$. Also, based on the definition (2.4), we used the identity $\omega(f_\alpha; \cdot) = \omega(f; \cdot)$ for each $\alpha \geq 0$.
Finally, using that $\omega(f'; \cdot)$ is a non-decreasing function, the above relations lead us to the following result.

Theorem 2.4. *Let S_n and L_n be defined by (2.6) and (2.7) respectively, where $f \in C_B^1(\mathbb{R})$. Let I be an interval such that $\sup_{x\in I} \sigma_0(x) = \gamma < \infty$. The following identity*

$$|(L_n f)_\alpha(x) - (L_n f_\alpha)(x)| \leq 3.125\omega\left(f'; \frac{\gamma}{2\sqrt{n}}\right)\frac{\gamma}{\sqrt{n}}, \; x \in I, \tag{2.9}$$

holds.

In view of relation (2.1), the above theorem says that L_n operator, subject of certain conditions, is a λ_n-invariant operator, where

$$\lambda_n = 3.125\omega\left(f'; \frac{\gamma}{2\sqrt{n}}\right)\frac{\gamma}{\sqrt{n}}.$$

Here λ_n's expression is complicated, consequently it is practically unattractive. With the desire to simplify it, we add an additional condition to function f. We require that f' satisfies a Lipschitz condition with a constant M and exponent β, $f' \in Lip_M \beta$, ($M \geq 0, 0 < \beta \leq 1$), that is

$$|f'(x_1) - f'(x_2)| \leq M|x_1 - x_2|^\beta, \; (x_1, x_2) \in I \times I.$$

The new requirement implies the continuity of f'. On the other hand, equivalent to this property is the inequality

$$\omega(f'; h) \leq Mh^\beta, \; h \geq 0, \tag{2.10}$$

see, e.g., [8, *page 49*].
Considering (2.9) and (2.10), the main result of this note will be read as follows.

Theorem 2.5. *Let S_n and L_n be defined by (2.6) and (2.8) respectively, where $f \in C_B(\mathbb{R})$ is differentiable on the domain such that $f' \in Lip_M \beta$. Let I be an interval and $\sup_{x\in I} \sigma_0(x) = \gamma < \infty$. Then, for each $n \in \mathbb{N}$, L_n is a λ_n-shift invariant operator, where*

$$\lambda_n = \frac{3.125}{2^\alpha} M \left(\frac{\gamma}{\sqrt{n}}\right)^{\beta+1}. \tag{2.11}$$

3. Applications

In this section we present three examples of classical operators, both of discrete and continuous type, which verify Theorem 2.4. We are able to indicate explicitly λ_n such that L_n may become a shift λ_n-invariant operator. In the following $\mathbb{N}_0$ stands for $\{0\} \cup \mathbb{N}$.
Set

$$E_2(\mathbb{R}_+) = \left\{ f \in C(\mathbb{R}_+) : \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\},$$

representing a Banach lattice endowed with the norm

$$\|f\|_* = \sup_{x \geq 0} (1+x^2)^{-1} |f(x)|.$$

Example 3.1. *Let X_j, $j \geq 1$, be i.i. random variables having Poisson distribution, i.e., for each $k \in \mathbb{N}_0$*

$$P(X_j = k) = e^{-x} \frac{x^k}{k!}, \ x \geq 0,$$

which implies $E(X_j) = x$ and $Var(X_j) = x$. Formula (2.8) leads us to Szász-Mirakjan operators defined for $f \in E_2(\mathbb{R}_+)$ as follows

$$(L_n f)(x) \equiv (M_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \ n \geq 1. \tag{3.12}$$

Further on, we consider $f \in C_B^1(\mathbb{R}_+)$ and $I = [0, a]$, $a > 0$ fixed.
Consequently we get $\gamma = \sqrt{a}$. Relation (2.9) yields

$$|(M_n f)_\alpha(x) - (M_n f_\alpha)(x)| \leq 3.125 \omega \left(f'; \frac{1}{2}\sqrt{\frac{a}{n}} \right) \sqrt{\frac{a}{n}}, \ x \in [0, a].$$

Example 3.2. *Let X_j, $j \geq 1$, be i.i. random variables following Pascal distribution, i.e., for each $k \in \mathbb{N}_0$*

$$P(X_j = k) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \ x \geq 0,$$

which implies $E(X_j) = x$ and $Var(X_j) = x + x^2$. Applying formula (2.8) we get Baskakov operators defined for $f \in E_2(\mathbb{R}_+)$ as follows

$$(L_n f)(x) \equiv (V_n f)(x) = \frac{1}{(1+x)^n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right), \ n \geq 1. \tag{3.13}$$

We take $f \in C_B^1(\mathbb{R}_+)$ and $I = [0, a]$, $a > 0$ fixed. This time we have $\gamma = \sqrt{a(a+1)}$ and (2.9) yields

$$|(V_n f)_\alpha(x) - (V_n f_\alpha)(x)| \leq 3.125 \omega \left(f'; \frac{1}{2}\sqrt{\frac{a^2+a}{n}} \right) \sqrt{\frac{a^2+a}{n}}, \ x \in [0, a].$$

Example 3.3. *Assume X_j, $j \geq 1$, are i.i. continuous Gaussian random variables having the normal distribution $N(x, \sigma)$. This means the probability density function is given by*

$$\mu(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-(t-x)^2/(2\sigma^2)), \ t \in \mathbb{R}.$$

It is known that $S_{n,0}$ has a normal distribution too, with $E(S_{n,0}) = x$ and $Var(S_{n,0}) = \sigma^2/n$. In this case, (2.8) yields the operator

$$(L_n f)(x) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} f(t) \exp(-n(t-x)^2/(2\sigma^2)) dt, \ f \in C_B(\mathbb{R}).$$

For $\sigma^2 = 0.5$ *it reduces to genuine Weierstrass operator* W_n.
For any $f \in C_B^1(\mathbb{R})$ *and* $I \subseteq \mathbb{R}$ *we have* $\gamma = 2^{-1/2}$ *and, in view of (2.9), we get*

$$|(W_n f)_\alpha(x) - (W_n f_\alpha)(x)| \le 3.125\omega\left(f'; \frac{1}{2\sqrt{2n}}\right)\frac{1}{\sqrt{2n}}, \; x \in I. \tag{3.14}$$

Remark. Taking into account the results (3.12), (3.13), (3.14), under the hypotheses of Theorem 2.5, we can state that the operators Szász-Mirakjan, Baskakov and Weierstrass of rank n are shift $C(\tau/n)^{(\beta+1)/2}$-invariant operators, where $C = 3.125M2^{-\beta}$ and τ is defined as follows: $\tau = a$ for the first operator, $\tau = a^2 + a$ for the second operator and $\tau = 0.5$ for the last operator.

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Inequalities for Synchronous Functions and Applications

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ABSTRACT. Some inequalities for synchronous functions that are a mixture between Čebyšev's and Jensen's inequality are provided. Applications for f-divergence measure and some particular instances including Kullback-Leibler divergence, Jeffreys divergence and χ^2-divergence are also given.

Keywords: Synchronous Functions, Lipschitzian functions, Čebyšev inequality, Jensen's inequality, f-divergence measure, Kullback-Leibler divergence, Jeffreys divergence measure, χ^2-divergence.

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1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure ν on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a ν-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for ν -a.e. (almost every) $x \in \Omega$, consider the *Lebesgue space*

$$L_w(\Omega, \nu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w(x)|f(x)|\,d\nu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\nu$ instead of $\int_\Omega w(x)\,d\nu(x)$.
Assume also that $\int_\Omega w d\nu = 1$. We have *Jensen's inequality*

$$\int_\Omega w(\Phi \circ f)\,d\nu \geq \Phi\left(\int_\Omega wf d\nu\right), \tag{1.1}$$

where $\Phi : [m, M] \to \mathbb{R}$ is a continuous convex function on the closed interval of real numbers $[m, M]$, $f : \Omega \to [m, M]$ is ν-measurable and such that $f, \Phi \circ f \in L_w(\Omega, \nu)$.
We say that the pair of measurable functions (f, g) are *synchronous* on Ω if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \tag{1.2}$$

for ν-a.e. $x, y \in \Omega$. If the inequality reverses in (1.2), the functions are called *asynchronous* on Ω.
If (f, g) are synchronous on Ω and $f, g, fg \in L_w(\Omega, \nu)$ then the following inequality, that is known in the literature as *Čebyšev's Inequality*, holds

$$\int_\Omega wfg d\nu \geq \int_\Omega wf d\nu \int_\Omega wg d\nu, \tag{1.3}$$

where $w(x) \geq 0$ for ν-a.e. (almost every) $x \in \Omega$ and $\int_\Omega w d\nu = 1$.
In this paper we establish some inequalities for synchronous functions that are a mixture between Čebyšev's and Jensen's inequality. Applications for f-divergence measure and some particular instances including Kullback-Leibler divergence, Jeffreys divergence and χ^2-divergence are also given.

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2. Inequalities for Synchronous Functions

We have the following inequality for synchronous functions:

Theorem 2.1. *Let* $\Phi, \Psi : [m, M] \to \mathbb{R}$ *be two synchronous functions on* $[m, M]$ *and* $w \geq 0$ *a.e. on* Ω *with* $\int_\Omega w d\nu = 1$. *If* $g : \Omega \to [m, M]$ *is* ν*-measurable and such that* $g, \Phi \circ g, \Psi \circ g, (\Phi \circ g)(\Psi \circ g) \in L_w(\Omega, \nu)$, *then*

$$\int_\Omega w(\Phi \circ g)(\Psi \circ g) d\nu + \Phi\left(\int_\Omega wgd\nu\right)\Psi\left(\int_\Omega wgd\nu\right) \geq \Phi\left(\int_\Omega wgd\nu\right)\int_\Omega w(\Psi \circ g) d\nu + \Psi\left(\int_\Omega wgd\nu\right)\int_\Omega w(\Phi \circ g) d\nu. \tag{2.4}$$

If the functions (Φ, Ψ) *are asynchronous, then the inequality in (2.4) reverses.*

Proof. Since Φ, Ψ are synchronous on $[m, M]$ and $\int_\Omega wgd\nu \in [m, M]$, then we have

$$\left[\Phi(g(x)) - \Phi\left(\int_\Omega wgd\nu\right)\right]\left[\Psi(g(x)) - \Psi\left(\int_\Omega wgd\nu\right)\right] \geq 0$$

for ν-a.e. $x \in \Omega$.
This is equivalent to

$$\Phi(g(x))\Psi(g(x)) + \Phi\left(\int_\Omega wgd\nu\right)\Psi\left(\int_\Omega wgd\nu\right) \geq \Phi\left(\int_\Omega wgd\nu\right)\Psi + \Psi\left(\int_\Omega wgd\nu\right)\Phi(g(x)) \tag{2.5}$$

for ν-a.e. $x \in \Omega$.
Now, if we multiply (2.5) by $w \geq 0$ a.e. on Ω and integrate, we deduce the desired result (2.4). □

Remark 2.1. *If the functions* $\Phi, \Psi : [m, M] \to \mathbb{R}$ *have the same monotonicity (opposite monotonicity) on* $[m, M]$, *then they are synchronous (asynchronous) and the inequality (2.4) holds for any* $g \in L_w(\Omega, \nu)$.

If $\Phi, \Psi : [m, M] \to \mathbb{R}$ are two synchronous functions on $[m, M]$, $x_i \in [m, M]$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n w_i = 1$, then by applying the inequality (2.4) for the discrete counting measure, we have

$$\sum_{i=1}^n w_i \Phi(x_i)\Psi(x_i) + \Phi\left(\sum_{i=1}^n w_i x_i\right)\Psi\left(\sum_{i=1}^n w_i x_i\right) \geq \Phi\left(\sum_{i=1}^n w_i x_i\right)\sum_{i=1}^n w_i \Psi(x_i) + \Psi\left(\sum_{i=1}^n w_i x_i\right)\sum_{i=1}^n w_i \Phi(x_i). \tag{2.6}$$

Example 2.1. *Let* $w \geq 0$ *a.e. on* Ω *with* $\int_\Omega w d\nu = 1$.
a). If $p, q > 0$ (< 0) *and* $g : \Omega \to [0, \infty)$ *is* ν*-measurable and such that* $g, g^p, g^q, g^{p+q} \in L_w(\Omega, \nu)$, *then*

$$\int_\Omega wg^{p+q} d\nu + \left(\int_\Omega wgd\nu\right)^p\left(\int_\Omega wgd\nu\right)^q \geq \left(\int_\Omega wgd\nu\right)^p \int_\Omega wg^q d\nu + \left(\int_\Omega wgd\nu\right)^q \int_\Omega wg^p d\nu. \tag{2.7}$$

If $p > 0(< 0)$, *and* $q < (> 0)$ *then the inequality (2.7) reverses.*

b). If $\alpha,\ \beta > 0\,(<0)$ *and* $g:\Omega\to\mathbb{R}$ *is* ν*-measurable and such that* $g,\ \exp(\alpha g),\ \exp(\beta g),\ \exp((\alpha+\beta)g)\in L_w(\Omega,\nu)$, *then*

$$\int_\Omega w\exp((\alpha+\beta)g)\,d\nu+\exp\left((\alpha+\beta)\int_\Omega wgd\nu\right)\\ \geq\exp\left(\alpha\int_\Omega wgd\nu\right)\int_\Omega w\exp(\beta g)\,d\nu+\exp\left(\beta\int_\Omega wgd\nu\right)\int_\Omega w\exp(\alpha g)\,d\nu. \tag{2.8}$$

If $\alpha>0(<0)$, *and* $\beta<(>0)$ *then the inequality (2.8) reverses.*
c). If $p>0$ *and* $g:\Omega\to(0,\infty)$ *is* ν*-measurable and such that* $g,\ g^p,\ \ln g,\ g^p\ln g\in L_w(\Omega,\nu)$, *then*

$$\int_\Omega wg^p\ln gd\nu+\left(\int_\Omega wgd\nu\right)^p\ln\left(\int_\Omega wgd\nu\right)\\ \geq\left(\int_\Omega wgd\nu\right)^p\int_\Omega w\ln gd\nu+\ln\left(\int_\Omega wgd\nu\right)\int_\Omega wg^pd\nu. \tag{2.9}$$

If $p<0$, *then the inequality (2.9) reverses.*

Corollary 2.1. *Let* $\Phi:[m,M]\to\mathbb{R}$ *be a measurable function on* $[m,M]$ *and* $w\geq0$ *a.e. on* Ω *and* $\int_\Omega wd\nu=1$. *If* $g:\Omega\to[m,M]$ *is* ν*-measurable and such that* $g,\ \Phi\circ g,\ (\Phi\circ g)^2\in L_w(\Omega,\nu)$, *then*

$$\frac{1}{2}\left[\int_\Omega w(\Phi\circ g)^2\,d\nu+\Phi^2\left(\int_\Omega wgd\nu\right)\right]\geq\Phi\left(\int_\Omega wgd\nu\right)\int_\Omega w(\Phi\circ g)\,d\nu. \tag{2.10}$$

We observe that the inequality (2.10) is of interest only if $\Phi\left(\int_\Omega wgd\nu\right)\neq0$. In this case, by dividing with $\Phi^2\left(\int_\Omega wgd\nu\right)>0$, we get

$$\frac{1}{2}\left[\frac{\int_\Omega w(\Phi\circ g)^2\,d\nu}{\Phi^2\left(\int_\Omega wgd\nu\right)}+1\right]\geq\frac{\int_\Omega w(\Phi\circ g)\,d\nu}{\Phi\left(\int_\Omega wgd\nu\right)}. \tag{2.11}$$

Remark 2.2. *Let* $\Phi:[m,M]\to\mathbb{R}$ *be a convex function on* $[m,M]$ *and* $w\geq0$ *a.e. on* Ω *with* $\int_\Omega wd\nu=1$. *If* $g:\Omega\to[m,M]$ *is* ν*-measurable and such that* $g,\ \Phi\circ g,\ (\Phi\circ g)^2\in L_w(\Omega,\nu)$ *and* $\Phi\left(\int_\Omega wgd\nu\right)>0$, *then by (2.11) we have*

$$\frac{1}{2}\left[\frac{\int_\Omega w(\Phi\circ g)^2\,d\nu}{\Phi^2\left(\int_\Omega wgd\nu\right)}+1\right]\geq\frac{\int_\Omega w(\Phi\circ g)\,d\nu}{\Phi\left(\int_\Omega wgd\nu\right)}\geq1. \tag{2.12}$$

This implies that

$$\frac{\int_\Omega w(\Phi\circ g)^2\,d\nu}{\Phi^2\left(\int_\Omega wgd\nu\right)}\geq1. \tag{2.13}$$

This inequality obviously holds for functions $\Phi:[m,M]\to\mathbb{R}$ *that are square convex, namely* Φ^2 *is convex. There are examples of convex functions* $\Phi:[m,M]\to\mathbb{R}$ *for which* Φ^2 *is not convex and* $\Phi\left(\int_\Omega wgd\nu\right)>0$ *holds. Indeed, if we consider* $\Phi:[-k,k]\to\mathbb{R},\ \Phi(t)=t^2-1$ *for* $k>1$ *then* $\Phi^2(t)=\left(t^2-1\right)^2$ *is convex on* $\left[-k,-\frac{\sqrt{3}}{3}\right]\cup\left[\frac{\sqrt{3}}{3},k\right]$ *and concave on* $\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)$. *Now, observe that for* $g(t)=t$, $\Omega=[0,k]$, $w(t)=\frac{1}{k}$ *we have*

$$\int_\Omega wgd\nu=\frac{1}{k}\int_0^k tdt=\frac{k}{2}$$

and

$$\Phi\left(\int_\Omega wgd\nu\right)=\Phi\left(\frac{k}{2}\right)=\frac{k^2}{4}-1$$

which is positive for $k > 2$.
This shows that the Jensen's type inequality (2.13) holds for larger classes than the square convex functions, namely for convex functions Φ *for which we have* $\Phi\left(\int_\Omega wgd\nu\right) > 0$.

Corollary 2.2. *Let* $\Phi : [m, M] \to \mathbb{R}$ *be a monotonic nondecreasing function on* $[m, M]$ *and* $w \geq 0$ *a.e. on* Ω *and* $\int_\Omega wd\nu = 1$. *If* $g : \Omega \to [m, M]$ *is* ν*-measurable and such that* $g, \Phi\circ g, g(\Phi \circ g) \in L_w(\Omega, \nu)$, *then*

$$\int_\Omega wg(\Phi \circ g)\, d\nu \geq \int_\Omega wgd\nu \int_\Omega w(\Phi \circ g)\, d\nu. \tag{2.14}$$

Remark 2.3. *We observe that, under the assumptions of Corollary 2.2 and if* $g : \Omega \to [m, M]$ *is convex and* $\int_\Omega wgd\nu > 0$, *then we get from (2.14) that*

$$\frac{\int_\Omega wg(\Phi \circ g)\, d\nu}{\int_\Omega wgd\nu} \geq \int_\Omega w(\Phi \circ g)\, d\nu \geq \Phi\left(\int_\Omega wgd\nu\right). \tag{2.15}$$

Example 2.2. *Let* $w \geq 0$ *a.e. on* Ω *with* $\int_\Omega wd\nu = 1$.
a). If $p \geq 1$ *and* $g : \Omega \to [m, M]$ *is* ν*-measurable and such that* $g, g^p, g^{p+1} \in L_w(\Omega, \nu)$, *then*

$$\frac{\int_\Omega wg^{p+1}d\nu}{\int_\Omega wgd\nu} \geq \int_\Omega wg^p d\nu \geq \left(\int_\Omega wgd\nu\right)^p. \tag{2.16}$$

b). If $\alpha > 0$ *and* $g : \Omega \to [m, M]$ *is* ν*-measurable and such that* $g, \exp(\alpha g), g\exp(\alpha g) \in L_w(\Omega, \nu)$, *then*

$$\frac{\int_\Omega wg\exp(\alpha g)\, d\nu}{\int_\Omega wgd\nu} \geq \int_\Omega w\exp(\alpha g)\, d\nu \geq \exp\left(\alpha \int_\Omega wgd\nu\right). \tag{2.17}$$

Corollary 2.3. *Let* $\Phi, \Psi : [m, M] \to \mathbb{R}$ *be two synchronous functions on* $[m, M]$, Ψ *also convex on* $[m, M]$ *and* $w \geq 0$ *a.e. on* Ω *with* $\int_\Omega wd\nu = 1$. *If* $g : \Omega \to [m, M]$ *is* ν*-measurable and such that* g, $\Phi \circ g$, $\Psi \circ g$, $(\Phi \circ g)(\Psi \circ g) \in L_w(\Omega, \nu)$ *and* $\Phi\left(\int_\Omega wgd\nu\right) > 0$, *then*

$$\int_\Omega w(\Phi \circ g)(\Psi \circ g)\, d\nu \geq \Psi\left(\int_\Omega wgd\nu\right) \int_\Omega w(\Phi \circ g). \tag{2.18}$$

Proof. From (2.4) and Jensen's inequality for Ψ we have

$$\begin{aligned}
&\int_\Omega w(\Phi \circ g)(\Psi \circ g)\, d\nu + \Phi\left(\int_\Omega wgd\nu\right)\Psi\left(\int_\Omega wgd\nu\right) \\
&\geq \Phi\left(\int_\Omega wgd\nu\right)\int_\Omega w(\Psi \circ g)\, d\nu + \Psi\left(\int_\Omega wgd\nu\right)\int_\Omega w(\Phi \circ g) \\
&\geq \Phi\left(\int_\Omega wgd\nu\right)\Psi\left(\int_\Omega wgd\nu\right) + \Psi\left(\int_\Omega wgd\nu\right)\int_\Omega w(\Phi \circ g)
\end{aligned}$$

and the inequality (2.18) is obtained. □

Let $\Phi, \Psi : [m, M] \to \mathbb{R}$ be two synchronous functions on $[m, M]$, Ψ also convex on $[m, M]$. If $x_i \in [m, M]$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n w_i = 1$, then by applying the inequality (2.18) for the discrete counting measure, we have

$$\sum_{i=1}^n w_i\Phi(x_i)\Psi(x_i) \geq \Psi\left(\sum_{i=1}^n w_i x_i\right)\sum_{i=1}^n w_i\Phi(x_i). \tag{2.19}$$

Example 2.3. *Let $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$.*
a). If $p > 0$, $q \geq 1$ and $g : \Omega \to [0, \infty)$ is ν-measurable and such that $g, g^p, g^q, g^{p+q} \in L_w(\Omega, \nu)$, then by (2.18) we have

$$\frac{\int_\Omega w g^{p+q} d\nu}{\int_\Omega w g^p} \geq \left(\int_\Omega w g d\nu\right)^q. \tag{2.20}$$

b). If $\alpha, \beta > 0$ and $g : \Omega \to \mathbb{R}$ is ν-measurable and such that $g, \exp(\beta g), \exp((\alpha+\beta) g) \in L_w(\Omega, \nu)$, then by (2.18) we have

$$\frac{\int_\Omega w \exp((\alpha+\beta) g) d\nu}{\int_\Omega w \exp(\beta g)} \geq \exp\left(\alpha \int_\Omega w g d\nu\right). \tag{2.21}$$

c). If $p \geq 1$ and $g : \Omega \to (0, \infty)$ is ν-measurable and such that $g, \ln g, g^p \ln g \in L_w(\Omega, \nu)$, then by (2.18) we have

$$\int_\Omega w g^p \ln g d\nu \geq \left(\int_\Omega w g d\nu\right)^p \int_\Omega w \ln g d\nu. \tag{2.22}$$

3. An Associated Functional

Let $\Phi, \Psi : I \to \mathbb{R}$ be two measurable functions on the interval I and $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$. If $g : \Omega \to I$ is ν-measurable and such that $g, \Phi \circ g, \Psi \circ g, (\Phi \circ g)(\Psi \circ g) \in L_w(\Omega, \nu)$, then we can consider the following functional

$$\begin{aligned} &\mathcal{F}(\Phi, \Psi; g, w) \\ &:= \int_\Omega w (\Phi \circ g)(\Psi \circ g) d\nu + \Phi\left(\int_\Omega w g d\nu\right) \Psi\left(\int_\Omega w g d\nu\right) \\ &- \Phi\left(\int_\Omega w g d\nu\right) \int_\Omega w (\Psi \circ g) d\nu - \Psi\left(\int_\Omega w g d\nu\right) \int_\Omega w (\Phi \circ g) d\nu. \end{aligned} \tag{3.23}$$

In particular, if $g, \Phi \circ g, \Psi \circ g, (\Phi \circ g)^2 \in L_w(\Omega, \nu)$, we have

$$\begin{aligned} &\mathcal{F}(\Phi; g, w) \\ &:= \int_\Omega w (\Phi \circ g)^2 d\nu + \Phi^2\left(\int_\Omega w g d\nu\right) - 2\Phi\left(\int_\Omega w g d\nu\right) \int_\Omega w (\Phi \circ g) d\nu \\ &\geq 0. \end{aligned} \tag{3.24}$$

Theorem 3.2. *Let $\Phi, \Psi : I \to \mathbb{R}$ be two measurable functions on I and $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$. If $g : \Omega \to I$ is ν-measurable and such that $g, \Phi \circ g, \Psi \circ g, (\Phi \circ g)^2, (\Psi \circ g)^2 \in L_w(\Omega, \nu)$, then*

$$\mathcal{F}^2(\Phi, \Psi; g, w) \leq \mathcal{F}(\Phi; g, w) \mathcal{F}(\Psi; g, w). \tag{3.25}$$

Proof. Observe that the following identity holds true

$$\begin{aligned} &\mathcal{F}(\Phi, \Psi; g, w) \\ &= \int_\Omega w(x) \left[\Phi(g(x)) - \Phi\left(\int_\Omega w g d\nu\right)\right] \left[\Psi(g(x)) - \Psi\left(\int_\Omega w g d\nu\right)\right] d\nu(x). \end{aligned} \tag{3.26}$$

Using the Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\begin{aligned} (3.27)\quad & \left|\int_{\Omega} w(x)\left[\Phi(g(x))-\Phi\left(\int_{\Omega} w g d\nu\right)\right]\left[\Psi(g(x))-\Psi\left(\int_{\Omega} w g d\nu\right)\right] d\nu(x)\right| \\ & \leq\left(\int_{\Omega} w(x)\left[\Phi(g(x))-\Phi\left(\int_{\Omega} w g d\nu\right)\right]^{2} d\nu(x)\right)^{1/2} \\ & \times\left(\int_{\Omega} w(x)\left[\Psi(g(x))-\Psi\left(\int_{\Omega} w g d\nu\right)\right]^{2} d\nu(x)\right)^{1/2} \\ & =\mathcal{F}^{1/2}(\Phi; g, w) \mathcal{F}^{1/2}(\Psi; g, w). \end{aligned}$$

On utilizing (3.26) and (3.27) we deduce the desired result (3.25). □

For the functions Φ, $\Psi: I \rightarrow \mathbb{R}$, the n-tuples of real numbers $x=(x_1,\ldots,x_n)\in I^n$ and the probability distribution $w=(w_1,\ldots,w_n)$ define the functionals

$$\begin{aligned} (3.28)\quad \mathcal{F}(\Phi, \Psi; x, w):= & \sum_{i=1}^{n} w_{i} \Phi(x_{i}) \Psi(x_{i})+\Phi\left(\sum_{i=1}^{n} w_{i} x_{i}\right) \Psi\left(\sum_{i=1}^{n} w_{i} x_{i}\right) \\ & -\Phi\left(\sum_{i=1}^{n} w_{i} x_{i}\right) \sum_{i=1}^{n} w_{i} \Psi(x_{i})-\Psi\left(\sum_{i=1}^{n} w_{i} x_{i}\right) \sum_{i=1}^{n} w_{i} \Phi(x_{i}) \end{aligned}$$

and

$$(3.29)\quad \mathcal{F}(\Phi; x, w):=\sum_{i=1}^{n} w_{i} \Phi^{2}(x_{i})+\Phi^{2}\left(\sum_{i=1}^{n} w_{i} x_{i}\right)-2 \Phi\left(\sum_{i=1}^{n} w_{i} x_{i}\right) \sum_{i=1}^{n} w_{i} \Phi(x_{i}).$$

From the inequality (3.25) we have

$$\mathcal{F}^{2}(\Phi, \Psi; x, w) \leq \mathcal{F}(\Phi; x, w) \mathcal{F}(\Psi; x, w).$$

Theorem 3.3. *Let $\Phi: I \rightarrow \mathbb{R}$ be an L-Lipschitzian function on I, with $L>0$, namely it satisfies the condition*

$$|\Phi(t)-\Phi(s)| \leq L|t-s| \text{ for any } t, s \in I,$$

and $w \geq 0$ a.e. on Ω with $\int_{\Omega} w d\nu=1$. If $g: \Omega \rightarrow I$ is ν-measurable and such that g, g^2, $\Phi\circ g$, $(\Phi\circ g)^2 \in L_w(\Omega,\nu)$, then

$$(3.30)\quad (0 \leq) \mathcal{F}^{1/2}(\Phi; g, w) \leq L \mathcal{D}(g, w),$$

where the dispersion $\mathcal{D}(g,w)$ is defined by

$$(3.31)\quad \mathcal{D}(g, w):=\left(\int_{\Omega} w g^{2} d\nu-\left(\int_{\Omega} w g d\nu\right)^{2}\right)^{1/2}.$$

Proof. By Lipschitz condition we have

$$\begin{aligned}
\mathcal{F}(\Phi; g, w) &= \int_\Omega w(x)\left[\Phi(g(x)) - \Phi\left(\int_\Omega wgd\nu\right)\right]^2 d\nu(x) \\
&\le L^2 \int_\Omega w(x)\left(g(x) - \int_\Omega wgd\nu\right)^2 d\nu(x) \\
&= L^2 \int_\Omega w(x)\left(g^2(x) - 2\left(\int_\Omega wgd\nu\right) g(x) + \left(\int_\Omega wgd\nu\right)^2\right) d\nu(x) \\
&= L^2 \left[\int_\Omega w(x) g^2(x) d\nu(x) - \left(\int_\Omega wgd\nu\right)^2\right] \\
&= L^2 \mathcal{D}^2(g, w).
\end{aligned}$$

□

Corollary 3.4. *Let $\Phi : [m, M] \to \mathbb{R}$ be an absolutely continuous function on $[m, M]$ with*

(3.32) $$\|\Phi'\|_{[m,M],\infty} := essup_{t\in[m,M]} |\Phi'(t)| < \infty$$

and $w \ge 0$ a.e. on Ω with $\int_\Omega wd\nu = 1$. If $g : \Omega \to [m, M]$ is ν-measurable and such that g, g^2, $\Phi \circ g$, $(\Phi \circ g)^2 \in L_w(\Omega, \nu)$, then

(3.33) $$(0 \le) \mathcal{F}^{1/2}(\Phi; g, w) \le \|\Phi'\|_{[m,M],\infty} \mathcal{D}(g, w).$$

The proof follows by Theorem 3.3 on observing that for and t, $s \in [m, M]$ we have

$$|\Phi(t) - \Phi(s)| = \left|\int_s^t \Phi'(u) du\right| \le |t - s| \|\Phi'\|_{[m,M],\infty}.$$

Corollary 3.5. *Let $\Phi : I \to \mathbb{R}$ be an L-Lipschitzian function on I, with $L > 0$, and $w \ge 0$ a.e. on Ω with $\int_\Omega wd\nu = 1$. If $g : \Omega \to I$ is ν-measurable and there exists the constant m, $M \in I$ such that*

(3.34) $$m \le g(x) \le M \text{ for } \nu\text{-a.e. } x \in \Omega,$$

then g, g^2, $\Phi \circ g$, $(\Phi \circ g)^2 \in L_w(\Omega, \nu)$ and

(3.35) $$(0 \le) \mathcal{F}^{1/2}(\Phi; g, w) \le \frac{1}{2}(M - m) L.$$

The proof follows by (3.30) and the *Grüss inequality* that states that

(3.36) $$\mathcal{D}(g, w) \le \frac{1}{2}(M - m)$$

provided that g satisfies the condition (3.34).

Corollary 3.6. *Let $\Phi : I \to \mathbb{R}$ be Lipschitzian with constant $L > 0$, $\Psi : I \to \mathbb{R}$ be Lipschitzian with constant $K > 0$ and $w \ge 0$ a.e. on Ω with $\int_\Omega wd\nu = 1$. If $g : \Omega \to I$ is ν-measurable and such that g, $\Phi \circ g$, $\Psi \circ g$, $(\Phi \circ g)^2$, $(\Psi \circ g)^2 \in L_w(\Omega, \nu)$, then*

(3.37) $$|\mathcal{F}(\Phi, \Psi; g, w)| \le LK\mathcal{D}^2(g, w).$$

Moreover, if $g : \Omega \to I$ is ν-measurable and there exists the constant m, $M \in I$ such that the condition (3.34) is satisfied, then

(3.38) $$|\mathcal{F}(\Phi, \Psi; g, w)| \le \frac{1}{4}(M - m)^2 LK.$$

The proof follows by (3.25), (3.30) and (3.35).
If $\Phi : I \to \mathbb{R}$ is Lipschitzian with constant $L > 0$, $\Psi : I \to \mathbb{R}$ is Lipschitzian with constant $K > 0$, the n-tuples of real numbers $x = (x_1, ..., x_n) \in I^n$ then for any probability distribution $w = (w_1, ..., w_n)$ we have by (3.37) that

$$|\mathcal{F}(\Phi, \Psi; x, w)| \leq LK \left(\sum_{i=1}^{n} w_i x_i^2 - \left(\sum_{i=1}^{n} w_i x_i \right)^2 \right). \tag{3.39}$$

If the interval I is closed, namely $I = [m, M]$ and $x = (x_1, ..., x_n) \in [m, M]^n$ then by (3.38) we get the simpler upper bound:

$$|\mathcal{F}(\Phi, \Psi; x, w)| \leq \frac{1}{4} (M - m)^2 LK. \tag{3.40}$$

Consider the functional

$$\begin{aligned} \mathcal{F}_{p,q}(g, w) := & \int_\Omega w g^{p+q} d\nu + \left(\int_\Omega w g d\nu \right)^p \left(\int_\Omega w g d\nu \right)^q \\ & - \left(\int_\Omega w g d\nu \right)^p \int_\Omega w g^q d\nu - \left(\int_\Omega w g d\nu \right)^q \int_\Omega w g^p d\nu, \end{aligned} \tag{3.41}$$

provided that $g > 0$, $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$, g, g^p, g^q, $g^{p+q} \in L_w(\Omega, \nu)$ and p, $q \in \mathbb{R} \setminus \{0\}$.
Assume that $g : \Omega \to [m, M] \subset (0, \infty)$ and for $p \neq 0$ define the constants

$$\Delta_p(m, M) := |p| \times \begin{cases} M^{p-1} & \text{if } p \geq 1, \\ m^{p-1} & \text{if } p < 1. \end{cases} \tag{3.42}$$

If we consider the function $\Phi : [m, M] \subset (0, \infty) \to (0, \infty)$, $\Phi(t) = t^p$ then $\Phi'(t) = pt^{p-1}$ and

$$\sup_{t \in [m, M]} |\Phi'(t)| = \Delta_p(m, M)$$

as defined by (3.42).

Proposition 3.1. *Let $g : \Omega \to [m, M] \subset (0, \infty)$ be ν-measurable and p, $q \in \mathbb{R} \setminus \{0\}$. Then we have the inequality*

$$|\mathcal{F}_{p,q}(g, w)| \leq \frac{1}{4} (M - m)^2 \Delta_p(m, M) \Delta_q(m, M). \tag{3.43}$$

The proof follows by Corollary 3.6 for the functions $\Phi(t) = t^p$ and $\Psi(t) = t^q$ for $p, q \in \mathbb{R} \setminus \{0\}$.
Consider now the functional

$$\begin{aligned} \mathcal{F}_{p,\ln}(g, w) := & \int_\Omega w g^p \ln g d\nu + \left(\int_\Omega w g d\nu \right)^p \ln \left(\int_\Omega w g d\nu \right) \\ & - \left(\int_\Omega w g d\nu \right)^p \int_\Omega w \ln g d\nu - \ln \left(\int_\Omega w g d\nu \right) \int_\Omega w g^p d\nu, \end{aligned} \tag{3.44}$$

provided that $p > 0$ and $g : \Omega \to (0, \infty)$ is ν-measurable and such that g, g^p, $\ln g$, $g^p \ln g \in L_w(\Omega, \nu)$.
If we take the function $\Psi(t) = \ln t$, $t \in [m, M] \subset (0, \infty)$, then $\sup_{t \in [m, M]} |\Psi'(t)| = \frac{1}{m}$.
Using Corollary 3.6 for the functions $\Phi(t) = t^p$ and $\Psi(t) = \ln t$ for $p \in \mathbb{R} \setminus \{0\}$ we can state the following result as well:

Proposition 3.2. *Let $g:\Omega\to[m,M]\subset(0,\infty)$ be ν-measurable and $p\in\mathbb{R}\setminus\{0\}$. Then we have the inequality*

$$|\mathcal{F}_{p,\ln}(g,w)|\leq\frac{1}{4m}(M-m)^2\Delta_p(m,M). \tag{3.45}$$

We have the following result:

Theorem 3.4. *Let $\Phi,\Psi:I\to\mathbb{R}$ be two measurable functions such that there exists the real constants γ,Γ with*

$$\gamma\leq\frac{\Phi(t)-\Phi(s)}{\Psi(t)-\Psi(s)}\leq\Lambda \tag{3.46}$$

for a.e. $t,s\in I$ with $t\neq s$. If $g:\Omega\to I$ is ν-measurable and such that $g,\Phi\circ g,\Psi\circ g,(\Phi\circ g)^2,(\Psi\circ g)^2\in L_w(\Omega,\nu)$, then we have the inequalities

$$\gamma\mathcal{F}(\Psi;g,w)\leq\mathcal{F}(\Phi,\Psi;g,w)\leq\Lambda\mathcal{F}(\Psi;g,w). \tag{3.47}$$

Proof. My multiplying (3.46) with $(\Psi(t)-\Psi(s))^2\geq0$ we get

$$\gamma(\Psi(t)-\Psi(s))^2\leq[\Phi(t)-\Phi(s)][\Psi(t)-\Psi(s)]\leq\Lambda(\Psi(t)-\Psi(s))^2$$

for a.e. $t,s\in I$.
This implies

$$\begin{aligned}\gamma w(x)&\left(\Psi(g(x))-\Psi\left(\int_\Omega wgd\nu\right)\right)^2\\&\leq w(x)\left[\Phi(g(x))-\Phi\left(\int_\Omega wgd\nu\right)\right]\left[\Psi(g(x))-\Psi\left(\int_\Omega wgd\nu\right)\right]\\&\leq\Lambda w(x)\left(\Psi(g(x))-\Psi\left(\int_\Omega wgd\nu\right)\right)^2\end{aligned} \tag{3.48}$$

for ν-a.e. $x\in\Omega$.
Integrating the inequality (3.48) on Ω and making use of the equality (3.26) we deduce the desired result (3.47). □

Corollary 3.7. *Let $\Phi,\Psi:[m,M]\to\mathbb{R}$ be continuous on $[m,M]$ and differentiable on (m,M). Assume that $\Psi'(t)\neq0$ for any $t\in(m,M)$ and*

$$\inf_{t\in(m,M)}\left(\frac{\Phi'(t)}{\Psi'(t)}\right)>-\infty,\quad\sup_{t\in(m,M)}\left(\frac{\Phi'(t)}{\Psi'(t)}\right)<\infty.$$

If $g:\Omega\to I$ is ν-measurable and such that $g,\Phi\circ g,\Psi\circ g,(\Phi\circ g)^2,(\Psi\circ g)^2\in L_w(\Omega,\nu)$, then we have the inequalities

$$\begin{aligned}\inf_{t\in(m,M)}\left(\frac{\Phi'(t)}{\Psi'(t)}\right)&\mathcal{F}(\Psi;g,w)\leq\mathcal{F}(\Phi,\Psi;g,w)\\&\leq\sup_{t\in(m,M)}\left(\frac{\Phi'(t)}{\Psi'(t)}\right)\mathcal{F}(\Psi;g,w).\end{aligned} \tag{3.49}$$

Proof. By *Cauchy's mean value theorem*, for any $t,s\in[m,M]$ with $t\neq s$ there exists a c between t and s such that

$$\frac{\Phi(t)-\Phi(s)}{\Psi(t)-\Psi(s)}=\frac{\Phi'(c)}{\Psi'(c)}.$$

Therefore, for any $t,s\in[m,M]$ with $t\neq s$ we have

$$\inf_{t\in(m,M)}\left(\frac{\Phi'(t)}{\Psi'(t)}\right)\leq\frac{\Phi(t)-\Phi(s)}{\Psi(t)-\Psi(s)}\leq\sup_{t\in(m,M)}\left(\frac{\Phi'(t)}{\Psi'(t)}\right).$$

By applying Theorem 3.4 for $\gamma=\inf_{t\in(m,M)}\left(\frac{\Phi'(t)}{\Psi'(t)}\right)$ and $\Gamma=\sup_{t\in(m,M)}\left(\frac{\Phi'(t)}{\Psi'(t)}\right)$ we get the desired result (3.49). □

Remark 3.4. *We observe that if $\Phi,\Psi:I\to\mathbb{R}$ are two measurable functions such that there exists the positive constant Θ with*

$$\left|\frac{\Phi(t)-\Phi(s)}{\Psi(t)-\Psi(s)}\right|\leq\Theta \tag{3.50}$$

for a.e. $t,s\in I$ with $t\neq s$ and $g:\Omega\to I$ is ν-measurable and such that g, $\Phi\circ g$, $\Psi\circ g$, $(\Phi\circ g)^2,(\Psi\circ g)^2\in L_w(\Omega,\nu)$, then we have the inequalities

$$|\mathcal{F}(\Phi,\Psi;g,w)|\leq\Theta\mathcal{F}(\Psi;g,w). \tag{3.51}$$

Moreover, if Φ,Ψ are as in Corollary 3.7, then we have

$$|\mathcal{F}(\Phi,\Psi;g,w)|\leq\sup_{t\in(m,M)}\left|\frac{\Phi'(t)}{\Psi'(t)}\right|\mathcal{F}(\Psi;g,w).$$

In the case of synchronous functions we can prove the following result as well:

Theorem 3.5. *Let $\Phi,\Psi:[m,M]\to\mathbb{R}$ be two synchronous functions on $[m,M]$ and $w\geq0$ a.e. on Ω with $\int_\Omega wd\nu=1$. If $g:\Omega\to[m,M]$ is ν-measurable and such that g, $\Phi\circ g$, $\Psi\circ g$, $(\Phi\circ g)(\Psi\circ g)$, $|\Phi|\circ g$, $|\Psi|\circ g$, $(|\Phi|\circ g)(|\Psi|\circ g)\in L_w(\Omega,\nu)$, then*

$$\begin{aligned}\mathcal{F}&(\Phi,\Psi;g,w)\\&\geq\max\{|\mathcal{F}(|\Phi|,\Psi;g,w)|,|\mathcal{F}(\Phi,|\Psi|;g,w)|,|\mathcal{F}(|\Phi|,|\Psi|;g,w)|\}\geq0.\end{aligned} \tag{3.52}$$

Proof. We use the continuity property of the modulus, namely

$$|a-b|\geq||a|-|b||,\ a,b\in\mathbb{R}.$$

Since Φ,Ψ are synchronous, then

$$\begin{aligned}&\left[\Phi(g(x))-\Phi\left(\int_\Omega wgd\nu\right)\right]\left[\Psi(g(x))-\Psi\left(\int_\Omega wgd\nu\right)\right]\\&=\left|\Phi(g(x))-\Phi\left(\int_\Omega wgd\nu\right)\right|\left|\Psi(g(x))-\Psi\left(\int_\Omega wgd\nu\right)\right|\\&\geq\begin{cases}\left||\Phi(g(x))|-\left|\Phi\left(\int_\Omega wgd\nu\right)\right|\right|\left|\Psi(g(x))-\Psi\left(\int_\Omega wgd\nu\right)\right|\\ \left|\Phi(g(x))-\Phi\left(\int_\Omega wgd\nu\right)\right|\left||\Psi(g(x))|-\left|\Psi\left(\int_\Omega wgd\nu\right)\right|\right|\\ \left||\Phi(g(x))|-\left|\Phi\left(\int_\Omega wgd\nu\right)\right|\right|\left||\Psi(g(x))|-\left|\Psi\left(\int_\Omega wgd\nu\right)\right|\right|\end{cases}\\&=\begin{cases}\left|\left(|\Phi(g(x))|-\left|\Phi\left(\int_\Omega wgd\nu\right)\right|\right)\left(\Psi(g(x))-\Psi\left(\int_\Omega wgd\nu\right)\right)\right|\\ \left|\left(\Phi(g(x))-\Phi\left(\int_\Omega wgd\nu\right)\right)\left(|\Psi(g(x))|-\left|\Psi\left(\int_\Omega wgd\nu\right)\right|\right)\right|\\ \left|\left(|\Phi(g(x))|-\left|\Phi\left(\int_\Omega wgd\nu\right)\right|\right)\left(|\Psi(g(x))|-\left|\Psi\left(\int_\Omega wgd\nu\right)\right|\right)\right|\end{cases}\end{aligned} \tag{3.53}$$

for any $x\in\Omega$.

By using the identity (3.26) and the first branch in (3.53) we have

$$
\begin{aligned}
&\mathcal{F}(\Phi,\Psi;g,w)\\
&=\int_{\Omega}w(x)\left[\Phi(g(x))-\Phi\left(\int_{\Omega}wgd\nu\right)\right]\left[\Psi(g(x))-\Psi\left(\int_{\Omega}wgd\nu\right)\right]d\nu(x)\\
&\geq\int_{\Omega}w(x)\left|\left(|\Phi(g(x))|-\left|\Phi\left(\int_{\Omega}wgd\nu\right)\right|\right)\left(\Psi(g(x))-\Psi\left(\int_{\Omega}wgd\nu\right)\right)\right|d\nu(x)\\
&\geq\left|\int_{\Omega}w(x)\left(|\Phi(g(x))|-\left|\Phi\left(\int_{\Omega}wgd\nu\right)\right|\right)\left(\Psi(g(x))-\Psi\left(\int_{\Omega}wgd\nu\right)\right)d\nu(x)\right|\\
&=|\mathcal{F}(|\Phi|,\Psi;g,w)|,
\end{aligned}
$$

which proves the first part of (3.52).
The second and third part of (3.52) can be proved in a similar way and the details are omitted. □

For the natural numbers $n, m \geq 1$ we consider the functions $\Phi(t) = t^{2n+1}$ and $\Psi(t) = t^{2m+1}$ for real numbers $t \in \mathbb{R}$. These functions are monotonic increasing on $\mathbb{R}$. If $g : \Omega \to \mathbb{R}$ is ν-measurable and such that g, g^{2n+1}, g^{2m+1}, $g^{2m+2n+2} \in L_w(\Omega,\nu)$, then by (3.52) we have the inequality

$$
\begin{aligned}
&\mathcal{F}\left((\cdot)^{2n+1},(\cdot)^{2m+1};g,w\right)\\
&\geq\max\left\{\left|\mathcal{F}\left(|\cdot|^{2n+1},(\cdot)^{2m+1};g,w\right)\right|,\right.\\
&\left.\left|\mathcal{F}\left((\cdot)^{2n+1},|\cdot|^{2m+1};g,w\right)\right|,\left|\mathcal{F}\left(|\cdot|^{2n+1},|\cdot|^{2m+1};g,w\right)\right|\right\}(\geq 0.)
\end{aligned}
\tag{3.54}
$$

4. Applications for f-Divergences

Let $(X,\mathcal{A})$ be a measurable space satisfying $|\mathcal{A}| > 2$ and μ be a σ-finite measure on $(X,\mathcal{A})$. Let $\mathcal{P}$ be the set of all probability measures on $(X,\mathcal{A})$ which are absolutely continuous with respect to μ. For $P, Q \in \mathcal{P}$, let $p = \frac{dP}{d\mu}$ and $q = \frac{dQ}{d\mu}$ denote the *Radon-Nikodym* derivatives of P and Q with respect to μ.
Two probability measures $P, Q \in \mathcal{P}$ are said to be *orthogonal* and we denote this by $Q \perp P$ if

$$P(\{q=0\}) = Q(\{p=0\}) = 1.$$

Let $f : [0,\infty) \to (-\infty,\infty]$ be a convex function that is continuous at 0, i.e., $f(0) = \lim_{u\downarrow 0} f(u)$.
In 1963, I. Csiszár [3] introduced the concept of f-divergence as follows.

Definition 4.1. *Let $P, Q \in \mathcal{P}$. Then*

$$I_f(Q,P) = \int_X p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \tag{4.55}$$

is called the f-divergence of the probability distributions Q and P.

Remark 4.5. *Observe that, the integrand in the formula (4.55) is undefined when $p(x) = 0$. The way to overcome this problem is to postulate for f as above that*

$$0f\left[\frac{q(x)}{0}\right] = q(x)\lim_{u\downarrow 0}\left[uf\left(\frac{1}{u}\right)\right],\ x \in X. \tag{4.56}$$

We now give some examples of f-divergences that are well-known and often used in the literature (see also [2]).
For f continuous convex on $[0,\infty)$ we obtain the $*$-*conjugate* function of f by

$$f^*(u) = uf\left(\frac{1}{u}\right), \quad u \in (0,\infty)$$

and

$$f^*(0) = \lim_{u\downarrow 0} f^*(u).$$

It is also known that if f is continuous convex on $[0,\infty)$ then so is f^*.
The following two theorems contain the most basic properties of f-divergences. For their proofs we refer the reader to Chapter 1 of [17] (see also [2]).

Theorem 4.6 (Uniqueness and Symmetry Theorem)**.** *Let f, f_1 be continuous convex on $[0,\infty)$. We have*

$$I_{f_1}(Q,P) = I_f(Q,P),$$

for all $P, Q \in \mathcal{P}$ if and only if there exists a constant $c \in \mathbb{R}$ such that

$$f_1(u) = f(u) + c(u-1),$$

for any $u \in [0,\infty)$.

Theorem 4.7 (Range of Values Theorem)**.** *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous convex function on $[0,\infty)$.*
For any $P, Q \in \mathcal{P}$, we have the double inequality

$$f(1) \le I_f(Q,P) \le f(0) + f^*(0). \tag{4.57}$$

(i) If $P = Q$, then the equality holds in the first part of (4.57).
If f is strictly convex at 1, then the equality holds in the first part of (4.57) if and only if $P = Q$;
(ii) If $Q \perp P$, then the equality holds in the second part of (4.57).
If $f(0) + f^*(0) < \infty$, then equality holds in the second part of (4.57) if and only if $Q \perp P$.

The following result is a refinement of the second inequality in Theorem 4.7 (see [2, Theorem 3]).

Theorem 4.8. *Let f be a continuous convex function on $[0,\infty)$ with $f(1) = 0$ (f is normalised) and $f(0) + f^*(0) < \infty$. Then*

$$0 \le I_f(Q,P) \le \frac{1}{2}[f(0) + f^*(0)]V(Q,P) \tag{4.58}$$

for any $Q, P \in \mathcal{P}$.

For other inequalities for f-divergence see [1], [4]-[15].
The concept of f-divergence can be extended in a similar way for non-convex functions.

Theorem 4.9. *Let $f, h : [0,\infty) \to \mathbb{R}$ be synchronous and measurable on $[0,\infty)$. For any $P, Q \in \mathcal{P}$ we have*

$$I_{fh}(Q,P) \ge f(1) I_h(Q,P) + h(1) I_f(Q,P) - f(1)h(1). \tag{4.59}$$

Moreover, if f is normalised, then

$$I_{fh}(Q,P) \ge h(1) I_f(Q,P). \tag{4.60}$$

If both f and h are normalised, then

$$I_{fh}(Q,P) \ge 0. \tag{4.61}$$

Proof. If we write the inequality (2.4) for the synchronous functions $(\Phi, \Psi) = (f, h)$, $w = p$, $g = \frac{q}{p}$, $\Omega = X$ and $\nu = \mu$ we have

$$\int_X pf\left(\frac{q}{p}\right) h\left(\frac{q}{p}\right) d\mu + f\left(\int_X q d\mu\right) h\left(\int_X q d\mu\right)$$
$$\geq f\left(\int_X q d\mu\right) \int_X ph\left(\frac{q}{p}\right) d\mu + h\left(\int_X q d\mu\right) \int_X pf\left(\frac{q}{p}\right) d\mu$$

that is equivalent to the desired result (4.59).
The rest is obvious. □

An important divergence in Information Theory is the *Kullback-Leibler divergence* obtained for the decreasing convex function $f(t) = -\ln t, t > 0$ and defined by

$$KL(P, Q) = \int_X p \ln\left(\frac{p}{q}\right) d\mu,$$

for any $P, Q \in \mathcal{P}$.
If $h : [0, \infty) \to \mathbb{R}$ is a decreasing function with $h(1) \geq 0$, then by (4.60) we have the inequality

$$I_{-h \ln(\cdot)}(Q, P) \geq h(1) KL(P, Q) \geq 0 \tag{4.62}$$

for any $P, Q \in \mathcal{P}$.
In particular, we have the following inequalities

$$I_{-(\cdot)^p \ln(\cdot)}(Q, P) \geq KL(P, Q) \geq 0 \tag{4.63}$$

and

$$I_{-\exp(-\alpha \cdot) \ln(\cdot)}(Q, P) \geq KL(P, Q) \exp(-\alpha) \geq 0 \tag{4.64}$$

for $p, \alpha > 0$.

Theorem 4.10. *Let $f, h : [0, \infty) \to \mathbb{R}$ be Lipschitzian on $[0, \infty)$ with the constants L and K, respectively. For any $P, Q \in \mathcal{P}$ we then have*

$$|I_{fh}(Q, P) - f(1) I_h(Q, P) - h(1) I_f(Q, P) + f(1) h(1)| \leq KL\chi^2(Q, P) \tag{4.65}$$

where

$$\chi^2(Q, P) = \frac{1}{2} \int_X p\left(\frac{q}{p} - 1\right)^2 d\mu = \int_X \frac{q^2}{p} d\mu - 1$$

is Karl Pearson's χ^2-divergence.
Moreover, if f is normalised, then

$$|I_{fh}(Q, P) - h(1) I_f(Q, P)| \leq KL\chi^2(Q, P). \tag{4.66}$$

If both f and h are normalised, then

$$|I_{fh}(Q, P)| \leq KL\chi^2(Q, P). \tag{4.67}$$

Proof. If we write the inequality (3.25) for the functions $(\Phi, \Psi) = (f, h)$, $w = p$, $g = \frac{q}{p}$, $\Omega = X$ and $\nu = \mu$ we have

$$\begin{aligned} (4.68)\qquad & \left| \int_X pf\left(\frac{q}{p}\right) h\left(\frac{q}{p}\right) d\mu + f\left(\int_X q d\mu\right) h\left(\int_X q d\mu\right) \right. \\ & \left. -f\left(\int_X q d\mu\right) \int_X ph\left(\frac{q}{p}\right) d\mu - h\left(\int_X q d\mu\right) \int_X pf\left(\frac{q}{p}\right) d\mu \right| \\ & \leq LK\left(\int_X \frac{q^2}{p} d\mu - 1\right), \end{aligned}$$

that is equivalent to the desired result (4.65).
The rest is obvious. □

If some bounds for the likelihood ratio are known, then we can state the following results as well.

Theorem 4.11. *Let $P, Q \in \mathcal{P}$ such that for $0 < r < 1 < R$ we have*

$$r \leq \frac{q}{p} \leq R \ \mu\text{-a.e. on } X. \tag{4.69}$$

If $f, h : [r, R] \to \mathbb{R}$ are Lipschitzian on $[r, R]$ with the constants L and K, then we have

$$\begin{aligned} (4.70)\qquad & \left| I_{fh}(Q, P) - f(1) I_h(Q, P) - h(1) I_f(Q, P) + f(1) h(1) \right| \\ & \leq \frac{1}{4}(R - r)^2 KL. \end{aligned}$$

Moreover, if f is normalised, then

$$\left| I_{fh}(Q, P) - h(1) I_f(Q, P) \right| \leq \frac{1}{4}(R - r)^2 KL. \tag{4.71}$$

If both f and h are normalised, then

$$\left| I_{fh}(Q, P) \right| \leq \frac{1}{4}(R - r)^2 KL. \tag{4.72}$$

If we consider the convex function $g(t) = (t - 1)\ln t$, then this function generates the *Jeffreys divergence measure*

$$J(P, Q) := \int_X (p - q)(\ln p - \ln q)\, d\mu$$

where $P, Q \in \mathcal{P}$.
If we take $f(t) = t - 1$, $h(t) = \ln t$ then f is Lipschitzian with the constant 1 and h is Lipschitzian with the constant $\frac{1}{r}$ on $[r, R]$ and by (4.72) we have

$$0 \leq J(P, Q) \leq \frac{1}{4r}(R - r)^2 \tag{4.73}$$

provided that $P, Q \in \mathcal{P}$ satisfy the condition (4.69).
The *Neyman Chi-square distance* is defined by

$$\chi_N^2(Q, P) := \frac{1}{2}\int_X \frac{(p - q)^2}{q} d\mu = \int_X \frac{p^2}{q} d\mu - 1 = \chi^2(P, Q)$$

and generated by the convex function $g(t) = \frac{(t-1)^2}{2t}$, $t > 0$.

Now, consider the functions $f(t) = \frac{1}{2}(t-1)^2$ and $h(t) = \frac{1}{t}$ defined on the interval $[r, R]$. Then $f'(t) = t - 1$ and

$$\max_{t\in[r,R]} |f'(t)| = \max\{1-r, R-1\} = \frac{R-r}{2} + \left|\frac{r+R}{2} - 1\right|.$$

Also $h'(t) = -\frac{1}{t^2}$ and

$$\max_{t\in[r,R]} |h'(t)| = \frac{1}{r^2}.$$

Then from (4.71) we have

$$\left|\chi_N^2(Q,P) - \chi^2(Q,P)\right| \leq \frac{1}{4}\left(\frac{R}{r} - 1\right)^2 \left(\frac{R-r}{2} + \left|\frac{r+R}{2} - 1\right|\right) \tag{4.74}$$

provided that $P, Q \in \mathcal{P}$ satisfy the condition (4.69).

Similar results may be obtained by utilizing (3.49), however the details are not presented here.

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A Quantitative Variant of Voronovskaja's Theorem for King-Type Operators

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ABSTRACT. In this note we establish a quantitative Voronovskaja theorem for modified Bernstein polynomials using the first order Ditzian-Totik modulus of smoothness.

Keywords: Bernstein operators, Voronovskaja theorem, King operators, First order Ditzian-Totik modulus of smoothness

2010 Mathematics Subject Classification: 41A25, 41A36.

1. INTRODUCTION

The Bernstein polynomials are defined by

$$(1.1) \qquad (B_n f)(x) \equiv B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$, $f \in C[0,1]$ and $n \geq 1$. Among the properties of Bernstein polynomials we mention the following asymptotic formula, called Voronovskaja's theorem: *if f is bounded on $[0,1]$, differentiable in some neighborhood of $x \in [0,1]$, and has second derivative $f''(x)$, then*

$$(1.2) \qquad \lim_{n\to\infty} n\left((B_n f)(x) - f(x)\right) = \frac{1}{2} x(1-x) f''(x).$$

Further properties:

$$(B_n e_0)(x) = 1, \quad (B_n e_1)(x) = x \quad \text{and} \quad (B_n e_2)(x) = x^2 + \frac{1}{n} x(1-x),$$

where $e_i(x) = x^i$, $x \in [0,1]$ and $i \in \{0,1,2,\ldots\}$. In [8] King constructed a Bernstein-type operator, which preserves the functions e_0 and e_2. By modification of $f\left(\frac{k}{n}\right)$ in (1.1), Aldaz et al. [1] defined Bernstein-King-type operators possessing e_0 and e_j as fixed points, where $j \in \{2,3,\ldots\}$ is arbitrary. These operators are given by

$$(1.3) \qquad (U_{n,j} f)(x) \equiv U_{n,j}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f(a_{n,k})$$

(see [1, Proposition 11]), where $x \in [0,1]$, $f \in C[0,1]$ and

$$(1.4) \qquad a_{n,k} = \sqrt[j]{\frac{k(k-1)\ldots(k-j+1)}{n(n-1)\ldots(n-j+1)}}, \quad n \geq j \geq 2.$$

The operators $U_{n,j}$ are linear and positive, $U_{n,j} e_0 = e_0$ and $U_{n,j} e_j = e_j$, respectively.

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The goal of the paper is to obtain a quantitative Voronovskaja-type theorem for $U_{n,j}$ with the aid of the first order Ditzian-Totik modulus of smoothness defined by

$$\omega_{\varphi}^{1}(f;\delta)=\sup_{0<h\leq\delta}\sup_{x\pm\frac{1}{2}h\varphi(x)}\left|f\left(x+\frac{1}{2}h\varphi(x)\right)-f\left(x-\frac{1}{2}h\varphi(x)\right)\right|, \tag{1.5}$$

where $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$. It is known [2, Theorem 2.1.1] that (1.5) is equivalent with the $K-$functional

$$K_{1,\varphi}(f;\delta)=\inf_{g\in W(\varphi)}\{\|f-g\|+\delta\|\varphi g'\|\},\quad \delta>0,$$

where $W(\varphi)=\{g\,|\,g\in AC_{loc}[0,1],\|\varphi g'\|<\infty\}$ and $g\in AC_{loc}[0,1]$ means that g is absolutely continuous in every closed interval $[a,b]\subseteq[0,1]$, i.e. there exists $C_1>0$ such that

$$C_1^{-1}\omega_{\varphi}^{1}(f;\delta)\leq K_{1,\varphi}(f;\delta)\leq C_1\omega_{\varphi}^{1}(f;\delta). \tag{1.6}$$

It is worth mentioning that Floater obtained a generalization of (1.2) in [4], dealing with the asymptotic behavior of differentiated Bernstein polynomials. Different quantitative versions of Floater's theorem were established in [5], [6], [7] and [3].

2. Main result

In the sequel we need some lemmas.

Lemma 2.1. *The inequalities $0\leq 1-x^n-(1-x)^n\leq nx(1-x)$ hold true for all $x\in[0,1]$ and $n=1,2,\ldots$*

Proof. For $x\in[0,1]$, we have

$$x^n+(1-x)^n\leq x+(1-x)=1. \tag{2.1}$$

For the second inequality, we have

$$\begin{aligned}1-x^n-(1-x)^n&=(1-x)(1+x+\ldots+x^{n-1})-(1-x)^n\\&=(1-x)[1+x+\ldots+x^{n-1}-(1-x)^{n-1}]\leq nx(1-x)\end{aligned}$$

iff $1+x+\ldots+x^{n-1}\leq nx+(1-x)^{n-1}$. We prove the former inequality by induction on n. If $n=1$, then $1\leq x+1$; we suppose that $1+x+\ldots+x^{n-1}\leq nx+(1-x)^{n-1}$. Then, by (2.1),

$$\begin{aligned}&1+x+\ldots+x^{n-1}+x^n\\&\leq nx+(1-x)^{n-1}+x^n=(n+1)x+(1-x)^{n-1}-x+x^n\\&=(n+1)x+(1-x)^n+(1-x)^{n-1}-(1-x)^n-x+x^n\\&=(n+1)x+(1-x)^n+x(1-x)^{n-1}-x+x^n\\&=(n+1)x+(1-x)^n-x(1-x^{n-1}-(1-x)^{n-1})\leq(n+1)x+(1-x)^n,\end{aligned}$$

which was to be proved. □

Lemma 2.2. *For the operator $U_{n,j}$ defined by (1.3)-(1.4) and $x\in[0,1]$, we have*

a) $0\leq U_{n,j}(xe_0-e_1;x)\leq\frac{1}{n}(j-1)$;
b) $U_{n,j}((e_1-xe_0)^2;x)\leq\frac{2}{n}((j-1)^2+1)\varphi^2(x)$;
c) $U_{n,j}((e_1-xe_0)^4;x)\leq\frac{8}{n^2}((j-1)^2+1)$.

Proof. Because $U_{n,j}$ is linear and preserves the functions e_0 and e_j, we obtain

$$\begin{aligned} U_{n,j}(xe_0 - e_1; x) &= x - U_{n,j}(e_1; x) = \sum_{k=0}^{n} p_{n,k}(x)\frac{k}{n} - \sum_{k=0}^{n} p_{n,k}(x) a_{n,k} \\ &= \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n} - a_{n,k}\right). \end{aligned} \tag{2.2}$$

For $k \in \{j, j+1, \ldots, n\}$, we have $\frac{k-j+1}{n-j+1} \leq \ldots \leq \frac{k-1}{n-1} \leq \frac{k}{n}$. Hence

$$\begin{aligned} 0 \leq \frac{k}{n} - a_{n,k} &\leq \frac{k}{n} - \frac{k-j+1}{n-j+1} = (j-1)\frac{n-k}{n(n-j+1)} \\ &\leq (j-1)\frac{n-j}{n(n-j+1)} \leq \frac{j-1}{n}. \end{aligned} \tag{2.3}$$

Therefore, in view of (2.2) and (2.3), we get

$$\begin{aligned} 0 \leq U_{n,j}(xe_0 - e_1; x) &= \sum_{k=1}^{j-1} p_{n,k}(x)\frac{k}{n} + \sum_{k=j}^{n-1} p_{n,k}(x)\left(\frac{k}{n} - a_{n,k}\right) \\ \leq \sum_{k=1}^{j-1} p_{n,k}(x)\frac{j-1}{n} &+ \sum_{k=j}^{n-1} p_{n,k}(x)\frac{j-1}{n} \leq \frac{j-1}{n}\sum_{k=0}^{n} p_{n,k}(x) = \frac{j-1}{n}. \end{aligned}$$

$b)$ Taking into account the inequality $(a+b)^2 \leq 2(a^2+b^2)$, (2.3) and Lemma 2.1, we find that

$$\begin{aligned} &U_{n,j}((e_1 - xe_0)^2; x) = \sum_{k=0}^{n} p_{n,k}(x)(a_{n,k} - x)^2 \\ \leq\ & 2\sum_{k=0}^{n} p_{n,k}(x)\left(a_{n,k} - \frac{k}{n}\right)^2 + 2\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n} - x\right)^2 \\ =\ & 2\sum_{k=1}^{j-1} p_{n,k}(x)\left(\frac{k}{n}\right)^2 + 2\sum_{k=j}^{n-1} p_{n,k}(x)\left(a_{n,k} - \frac{k}{n}\right)^2 + \frac{2}{n}x(1-x) \\ \leq\ & 2\left(\frac{j-1}{n}\right)^2 \sum_{k=1}^{j-1} p_{n,k}(x) + 2\left(\frac{j-1}{n}\right)^2 \sum_{k=j}^{n-1} p_{n,k}(x) + \frac{2}{n}x(1-x) \\ =\ & 2\left(\frac{j-1}{n}\right)^2 (1 - (1-x)^n - x^n) + \frac{2}{n}x(1-x) \leq 2\left(\frac{j-1}{n}\right)^2 nx(1-x) + \frac{2}{n}x(1-x) \\ =\ & \frac{2}{n}((j-1)^2 + 1)\varphi^2(x). \end{aligned}$$

$c)$ In view of $(a+b)^4 \leq 8(a^4+b^4)$, (2.3) and $\sum_{k=0}^{n}(k-nx)^4 p_{n,k}(x) = 3n^2\varphi^4(x)+n(\varphi^2(x)-6\varphi^4(x))$, we obtain

$$
\begin{aligned}
U_{n,j}((e_1-xe_0)^4;x) &= \sum_{k=0}^{n} p_{n,k}(x)(a_{n,k}-x)^4 \\
&\leq 8\sum_{k=0}^{n} p_{n,k}(x)\left(a_{n,k}-\frac{k}{n}\right)^4 + 8\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n}-x\right)^4 \\
&= 8\sum_{k=1}^{j-1} p_{n,k}(x)\left(\frac{k}{n}\right)^4 + 8\sum_{k=j}^{n-1} p_{n,k}(x)\left(a_{n,k}-\frac{k}{n}\right)^4 + \frac{8}{n^4}\sum_{k=0}^{n} p_{n,k}(x)\,(k-nx)^4 \\
&\leq 8\left(\frac{j-1}{n}\right)^4 \sum_{k=1}^{j-1} p_{n,k}(x) + 8\left(\frac{j-1}{n}\right)^4 \sum_{k=j}^{n-1} p_{n,k}(x) \\
&\quad + \frac{8}{n^4}(3n^2\varphi^4(x)+n(\varphi^2(x)-6\varphi^4(x))) \\
&\leq 8\left(\frac{j-1}{n}\right)^4 + \frac{8}{n^4}4n^2\varphi^2(x) \leq 8\left(\frac{j-1}{n}\right)^4 + \frac{8}{n^2} \leq \frac{8}{n^2}((j-1)^4+1).
\end{aligned}
$$

This completes the proof of the lemma. □

The main result is the following theorem.

Theorem 2.1. *Let $U_{n,j}$ be given by (1.3)-(1.4). Then there exists $C_2>0$ depending only on j such that*

$$
\begin{aligned}
\left| n(U_{n,j}(f;x)-f(x)) \right. &+ \left. f'(x)nU_{n,j}(xe_0-e_1;x) - \frac{1}{2}f''(x)nU_{n,j}((e_1-xe_0)^2;x) \right| \\
&\leq C_2\omega_\varphi^1\left(f'';\frac{1}{\sqrt{n}}\right)
\end{aligned}
\tag{2.4}
$$

for all $x\in[0,1]$, $f\in C^2[0,1]$ and $n\geq j\geq 2$. Furthermore

$$
0 \leq \liminf_{n\to\infty} nU_{n,j}(xe_0-e_1;x) \leq \limsup_{n\to\infty} nU_{n,j}(xe_0-e_1;x) \leq j-1 \tag{2.5}
$$

and

$$
\begin{aligned}
0 &\leq \liminf_{n\to\infty} nU_{n,j}((e_1-xe_0)^2;x) \\
&\leq \limsup_{n\to\infty} nU_{n,j}((e_1-xe_0)^2;x) \leq \frac{1}{2}((j-1)^2+1).
\end{aligned}
\tag{2.6}
$$

Proof. Because $U_{n,j}(f;0)=f(0)$ and $U_{n,j}(f;1)=f(1)$, the estimate (2.4) is satisfied for $x\in\{0,1\}$. Now let $x\in(0,1)$ and $t\in[0,1]$. By Taylor's formula, we have

$$
f(t) = f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\int_x^t (f''(u)-f''(x))(t-u)\,du.
$$

Hence

$$\left| U_{n,j}(f;x) - f(x) + f'(x)U_{n,j}(xe_0 - e_1;x) - \frac{1}{2}f''(x)U_{n,j}((e_1 - xe_0)^2;x) \right|$$
$$= \left| U_{n,j}\left(\int_x^t (f''(u) - f''(x))(t-u)\,du; x \right) \right|$$
$$\leq U_{n,j}\left(\left| \int_x^t |f''(u) - f''(x)||t-u|\,du \right|; x \right). \tag{2.7}$$

On the other hand

$$\left| \int_x^u \frac{dv}{\varphi(v)} \right| \leq \varphi^{-1}(x)|u-x|^{1/2} \left| \int_x^u \frac{dv}{|u-v|^{1/2}} \right| \leq 2\varphi^{-1}(x)|u-x|, \quad x \in (0,1), u \in [0,1]$$

(cf. [2, Lemma 9.6.1]). Hence, for all $g \in W(\varphi)$, we have

$$\begin{aligned}
&\left| \int_x^t |f''(u) - f''(x)||t-u|\,du \right| \\
&\leq \left| \int_x^t |f''(u) - g(u)||t-u|\,du \right| + \left| \int_x^t |g(u) - g(x)||t-u|\,du \right| \\
&\quad + \left| \int_x^t |g(x) - f''(x)||t-u|\,du \right| \\
&\leq \frac{1}{2}(t-x)^2\|f''-g\| + \left| \int_x^t \left| \int_x^u g'(v)\,dv \right| |t-u|\,du \right| + \frac{1}{2}(t-x)^2\|f''-g\| \\
&\leq (t-x)^2\|f''-g\| + \|\varphi g'\| \left| \int_x^t \left| \int_x^u \frac{dv}{\varphi(v)} \right| |t-u|\,du \right| \\
&\leq (t-x)^2\|f''-g\| + 2\varphi^{-1}(x)\|\varphi g'\| \left| \int_x^t |u-x||t-u|\,du \right| \\
&\leq (t-x)^2\|f''-g\| + 2\varphi^{-1}(x)|t-x|^3\|\varphi g'\|. \qquad (2.8)
\end{aligned}$$

Combining (2.7), (2.8), Hölder's inequality and Lemma 2.2, we get

$$\begin{aligned}
&\left| U_{n,j}(f;x) - f(x) + f'(x)U_{n,j}(xe_0 - e_1;x) - \frac{1}{2}f''(x)U_{n,j}((e_1 - xe_0)^2;x) \right| \\
&\leq \|f''-g\|U_{n,j}((e_1 - xe_0)^2;x) + 2\varphi^{-1}(x)\|\varphi g'\|U_{n,j}(|e_1 - xe_0|^3;x) \\
&\leq \|f''-g\|U_{n,j}((e_1 - xe_0)^2;x) \\
&\quad + 2\varphi^{-1}(x)\|\varphi g'\| \left(U_{n,j}((e_1 - xe_0)^2;x) \right)^{1/2} \left(U_{n,j}((e_1 - xe_0)^4;x) \right)^{1/2} \\
&\leq \frac{2}{n}((j-1)^2+1)\varphi^2(x)\|f''-g\| \\
&\quad + 2\varphi^{-1}(x)\|\varphi g'\| \sqrt{\frac{2}{n}((j-1)^2+1)}\varphi(x)\frac{2\sqrt{2}}{n}\sqrt{(j-1)^4+1} \\
&\leq \frac{8}{n}\sqrt{(j-1)^2+1}\sqrt{(j-1)^4+1}\left(\|f''-g\| + \frac{1}{\sqrt{n}}\|\varphi g'\| \right).
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in W(\varphi)$, we find

$$\left| n(U_{n,j}(f;x) - f(x)) + f'(x) n U_{n,j}(xe_0 - e_1;x) - \frac{1}{2} f''(x) n U_{n,j}((e_1 - xe_0)^2;x) \right| \leq 8\sqrt{(j-1)^2+1}\sqrt{(j-1)^4+1} K_{1,\varphi}\left(f; \frac{1}{\sqrt{n}}\right).$$

Hence, by (1.6), we obtain the estimation (2.4).
Finally, the estimations (2.5) follow from Lemma 2.2, a). Again, due to Lemma 2.2, b), we obtain

$$nU_{n,j}((e_1 - xe_0)^2;x) \leq 2((j-1)^2+1)\varphi^2(x) \leq \frac{1}{2}((j-1)^2+1).$$

Hence we find the estimations (2.6), which completes the proof of the theorem. □

Corollary 2.1. *There exists $C_3 > 0$ such that*

$$\left| n(U_{n,2}(f;x) - f(x)) + (f'(x) - x f''(x)) n U_{n,2}(xe_0 - e_1;x) \right| \leq C_3 \omega_\varphi^1\left(f''; \frac{1}{\sqrt{n}}\right)$$

for all $x \in [0,1]$, $f \in C^2[0,1]$ and $n \geq 2$. Furthermore

$$0 \leq \liminf_{n\to\infty} nU_{n,2}(xe_0 - e_1;x) \leq \limsup_{n\to\infty} nU_{n,2}(xe_0 - e_1;x) \leq 1.$$

Proof. It follows immediately from Theorem 2.1, taking into account that $U_{n,2}(e_0;x) = 1$, $U_{n,2}(e_2;x) = x^2$ and

$$\begin{aligned} U_{n,2}((e_1 - xe_0)^2;x) &= U_{n,2}(e_2;x) - 2xU_{n,2}(e_1;x) + x^2 U_{n,2}(e_0;x) \\ &= 2x(x - U_{n,2}(e_1;x)) = 2xU_{n,2}(xe_0 - e_1;x). \end{aligned}$$

□

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A Sequence of Kantorovich-Type Operators on Mobile Intervals

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ABSTRACT. In this paper, we introduce and study a new sequence of positive linear operators, acting on both spaces of continuous functions as well as spaces of integrable functions on $[0,1]$. We state some qualitative properties of this sequence and we prove that it is an approximation process both in $C([0,1])$ and in $L^p([0,1])$, also providing some estimates of the rate of convergence. Moreover, we determine an asymptotic formula and, as an application, we prove that certain iterates of the operators converge, both in $C([0,1])$ and, in some cases, in $L^p([0,1])$, to a limit semigroup. Finally, we show that our operators, under suitable hypotheses, perform better than other existing ones in the literature.

Keywords: Kantorovich-type operators, Positive approximation processes, Rate of convergence, Asymptotic formula, Generalized convexity.

2010 Mathematics Subject Classification: 41A36, 41A25, 47D06.

1. INTRODUCTION

In [13], the author proposed a modification of the classical Bernstein operators B_n on $[0,1]$ that, instead of fixing constants and the function x, fixes the constants and x^2, obtaining, in such a way, an order of approximation at least as good as the order of approximation of the operators B_n in the interval $[0, 1/3[$. More precisely, those operators are defined by setting, for every continuous function on $[0,1]$, $\widetilde{B}_n(f) = B_n(f) \circ r_n$, where, for every $x \in [0,1]$,

$$r_n(x) = \begin{cases} x^2 & \text{if } n = 1\,, \\ -\dfrac{1}{2(n-1)} + \sqrt{\dfrac{nx^2}{n-1} + \dfrac{1}{4(n-1)^2}} & \text{if } n \geq 2\,. \end{cases}$$

Subsequently, other modifications of the classical Bernstein operators, as well as of many other well-known operators, that fix suitable functions were introduced (see [2] and the references quoted therein). Here we limit ourselves to mention that, for example, in [9], the authors considered a family of sequences of operators $(B_{n,\alpha})_{n\geq 1}$, $\alpha \geq 0$, that preserve the constants and the function $x^2 + \alpha x$. A further extension was presented in [12]; in that paper, Gonska, Raşa and Piţul considered the operators $V_n^\tau(f) = B_n(f) \circ \tau_n$ $(f \in C([0,1]))$, where $\tau_n = (B_n(\tau))^{-1} \circ \tau$ and τ is a strictly increasing function on $[0,1]$ such that $\tau(0) = 0$ and $\tau(1) = 1$. In particular, the operators V_n^τ preserve the constants and the function τ.

In [10], instead, the authors introduced a modification of Bernstein operators fixing constants and a strictly increasing function τ in the following way: considering a strictly increasing function τ which is infinitely many times continuously differentiable on [0, 1] and such that $\tau(0) = 0$

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and $\tau(1) = 1$, they introduced the operators

$$B_n^\tau(f) = B_n(f \circ \tau^{-1}) \circ \tau \qquad (n \geq 1, f \in C([0,1])).$$

The authors studied shape preserving and approximation properties of the operators B_n^τ, and compared them, under suitable assumptions, with the B_n's and the $V_n^t au's$. General sequences of positive linear operators fixing τ and τ^2 have been recently studied in [1].
In this paper, motivated by works [7], [4] and [5], we present a Kantorovich-type modification of the operators B_n^τ. In particular, in [7], among other things, the authors introduced a sequence of positive linear operators $(C_n)_{n\geq 1}$ that generalize the classical Kantorovich operators on $[0,1]$ and present the advantage to reconstruct any integrable function on $[0,1]$ by means of its mean value on a finite numbers of subintervals of $[0,1]$ that do not need to be a partition of $[0,1]$.
Accordingly, in this work, for any integrable function f on $[0,1]$ we shall study the operators

$$C_n^\tau(f) = C_n(f \circ \tau^{-1}) \circ \tau \qquad (n \geq 1),$$

where τ is a strictly increasing function that is infinitely many times continuously differentiable on [0, 1] and such that $\tau(0) = 0$ and $\tau(1) = 1$.
The paper is organized as follows; after giving some preliminaries, we discuss some qualitative properties of the operators C_n^τ; in particular, we prove that they preserve some generalized convexity. We also prove that the sequence $(C_n^\tau)_{n\geq 1}$ is an approximation process for spaces of continuous as well as integrable functions and we evaluate the rate of convergence in both cases by means of suitable moduli of smoothness. As a byproduct, we obtain a simultaneous approximation result for the operators B_n^τ.
By using some results of [5], we prove that the operators C_n^τ satisfy an asymptotic formula with respect to a second order elliptic differential operator and, as an application, that suitable iterates of the C_n^τ's can be employed in order to constructively approximate strongly continuous semigroups in the function spaces considered in the paper.
Finally, as a further consequence of the above mentioned asymptotic formula, we compare the sequence $(C_n^\tau)_{n\geq 1}$ and the sequence $(C_n)_{n\geq 1}$, showing that, under suitable conditions, the former perform better.

2. Preliminaries

From now on, we denote by $C([0,1])$ the space of all real-valued continuous functions on the interval $[0,1]$. As usual, $C([0,1])$ will be equipped with the uniform norm $\|\cdot\|_\infty$.
For every $i \geq 1$, the symbol e_i stands for the functions $e_i(x) := x^i$ for all $x \in [0,1]$; moreover $\mathbf{1}$ will indicate the constant function on $[0,1]$ of constant value 1. If $X \subset \mathbb{R}$, we denote by $\mathbf{1}_X$ the characteristic function of X, defined by setting, for every $x \in \mathbb{R}$,

$$\mathbf{1}_X(x) := \begin{cases} 1 & \text{if } x \in X; \\ 0 & \text{if } x \notin X. \end{cases}$$

Moreover, for every $k \in \mathbb{N}$, we denote by $C^k([0,1])$ the space consisting of all real-valued functions which are continuously differentiable up to order k on $[0,1]$. In particular, if $f \in C^k([0,1])$, for every $i = 0,\ldots,k$, $D^{(i)}(f)$ is the derivative of order i of f. For simplicity, if $i = 1,2$, we might also use the usual symbols f' and f''. Further, $C^\infty([0,1])$ is the space of all real-valued functions which are infinitely many times continuously differentiable on $[0,1]$.
Finally, for every $p \in [1,+\infty[$, we denote by $L^p([0,1])$ the space of all (the equivalence classes of) Borel measurable real-valued functions on $[0,1]$ whose p^{th} power is integrable with respect

to the Borel-Lebesgue measure λ_1 on [0, 1]. The space $L^p([0,1])$ is endowed with the norm

$$\|f\|_p := \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p} \qquad (f \in L^p([0,1])).$$

In what follows we recall the definition of certain operators acting on the space $L^1([0,1])$ which represent a generalization of the classical Kantorovich operators on $[0,1]$. They were studied in [7, Examples 1.2, 1] and subsequently extended to the multidimensional setting in [4, 5].
Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers such that, for every $n \geq 1$, $0 \leq a_n < b_n \leq 1$. Then, consider the positive linear operator $C_n : L^1([0,1]) \longrightarrow C([0,1])$ defined by setting, for any $f \in L^1([0,1])$, $n \geq 1$ and $x \in [0,1]$,

$$C_n(f)(x) = \sum_{k=0}^{n} \left(\frac{n+1}{b_n - a_n}\int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(t)\,dt\right)\binom{n}{k} x^k(1-x)^{n-k}. \tag{2.1}$$

Since $C_n(\mathbf{1}) = \mathbf{1}$, the restriction to $C([0,1])$ of each C_n is continuous and we have $||C_n|| = 1$ for any $n \geq 1$, where $\|\cdot\|$ denotes the usual operator norm on $C([0,1])$.
We notice that if, in particular, $a_n = 0$ and $b_n = 1$ for any $n \geq 1$, the operators C_n turn into the classical Kantorovich operators on $[0,1]$.
For every $n \geq 1$,

$$C_n(e_1) = \frac{n}{n+1}\,e_1 + \frac{a_n + b_n}{2(n+1)}\,\mathbf{1}, \tag{2.2}$$

$$C_n(e_2) = \frac{1}{(n+1)^2}\left(n^2 e_2 + n e_1(1-e_1) + n(a_n+b_n)e_1 + \frac{b_n^2 + a_n b_n + a_n^2}{3}\,\mathbf{1}\right). \tag{2.3}$$

We also point out that (see [7, Formula (4.2)]), the operators C_n are closely related to the classical Bernstein operators on $[0,1]$.
In fact, if one denotes by B_n the n-th Bernstein operator on $C([0,1])$, for every $f \in L^1([0,1])$, considering the function

$$F_n(f)(x) := \frac{n+1}{b_n - a_n}\int_{\frac{nx+a_n}{n+1}}^{\frac{nx+b_n}{n+1}} f(t)\,dt = \int_0^1 f\left(\frac{(b_n-a_n)t + a_n + nx}{n+1}\right)dt \tag{2.4}$$

$(x \in [0,1], n \geq 1)$, it turns out that

$$C_n(f) = B_n(F_n(f)) \tag{2.5}$$

$(f \in L^1([0,1]), n \geq 1)$.
As quoted in the Introduction, in [10] the authors introduced a modification of Bernstein operators that fixes suitable functions.
More precisely, consider a function $\tau \in C^\infty([0,1])$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for every $x \in [0,1]$.
The operators introduced in [10] are defined by

$$B_n^\tau(f) := B_n(f \circ \tau^{-1}) \circ \tau \qquad (n \geq 1, f \in C([0,1])). \tag{2.6}$$

Namely, for every $f \in C([0,1])$, $n \geq 1$ and $x \in [0,1]$,

$$B_n^\tau(f)(x) := \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right). \tag{2.7}$$

After the above preliminaries, we pass to introduce a new sequence of positive linear operators acting on integrable functions on [0, 1], which is a combination of (2.1) and (2.6). More precisely, for any $f \in L^1([0,1])$ and $n \geq 1$, we set

$$C_n^\tau(f) := C_n(f \circ \tau^{-1}) \circ \tau; \tag{2.8}$$

hence, for every $f \in L^1([0,1])$, $n \geq 1$ and $x \in [0,1]$,

$$C_n^\tau(f)(x) = \sum_{k=0}^{n} \left(\frac{n+1}{b_n - a_n} \int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} \left(f \circ \tau^{-1}\right)(t)\, dt \right) \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k},$$

where we have used the fact that, thanks to the change of variable theorem, $f \circ \tau^{-1} \in L^1([0,1])$ provided $f \in L^1([0,1])$.
Note that, if $\tau = e_1$, the operators C_n^τ turn into the operators C_n defined by (2.1), and hence in the classical Kantorovich operators whenever $a_n = 0$ and $b_n = 1$ for every $n \geq 1$.
The operators C_n^τ can be viewed as integral modification of Kantorovich-type of the operators B_n^τ with mobile intervals.

3. Shape preserving properties of the C_n^τ's

This section is devoted to show some qualitative properties of the operators C_n^τ. To this end, we first remark that, taking (2.4), (2.5) and (2.8) into account, the following formula holds true:

$$C_n^\tau(f) = B_n(F_n(f \circ \tau^{-1})) \circ \tau \tag{3.9}$$

$(f \in L^1([0,1]), n \geq 1)$.
Hence, one can recover some properties of the operators C_n^τ by means of the relevant ones held by the B_n's.
First off, as $F_n(f)$ is increasing whenever f is (continuous and) increasing, the B_n's map (continuous) increasing functions into increasing functions (see, e.g., [3, Remark p. 461]), and τ is increasing, we have that the operators C_n^τ map (continuous) increasing functions into increasing functions.
The C_n^τ's preserve also a particular form of convexity.
We recall (see [17]) that a function $f \in C([0,1])$ is said to be convex with respect to τ if, for every $0 \leq x_0 < x_1 < x_2 \leq 1$, one has

$$\begin{vmatrix} 1 & 1 & 1 \\ \tau(x_0) & \tau(x_1) & \tau(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \geq 0.$$

In particular, it can be proven that a function f is convex with respect to τ if and only if $f \circ \tau^{-1}$ is convex.
In [7, Proof of Th. 4.3]) it has been shown that the operators C_n map (continuous) convex functions into (continuous) convex functions; hence, thanks to (2.8), the operators C_n^τ map (continuous) convex functions with respect to τ into (continuous) convex functions with respect to τ.
Moreover, we investigate the monotonicity of the sequence $(C_n^\tau)_{n\geq 1}$ on convex functions with respect to τ.

Proposition 3.1. *If $f \in C([0,1])$ is convex with respect to τ and increasing (resp., decreasing), then, for every $n \geq 1$,*

$$f \leq C_n^\tau(f) \quad \textit{on} \quad \left[0, \tau^{-1}\left(\frac{a_n + b_n}{2}\right)\right], \tag{3.10}$$

(resp.,

$$f \le C_n^\tau(f) \quad \textit{on} \quad \left[\tau^{-1}\left(\frac{a_n+b_n}{2}\right), 1\right]). \tag{3.11}$$

Moreover, if $(a_n)_{n\ge1}$ *and* $(b_n)_{n\ge1}$ *are constant sequences and* $f \in C([0,1])$ *is convex with respect to* τ*, then*

$$C_{n+1}^\tau(f) \le \frac{n+1}{n+2} C_n^\tau(f) + \frac{1}{n+2} B_{n+1}^\tau(f), \tag{3.12}$$

B_n^τ *being defined by* (2.7).

Proof. In [7, Proposiotion 4.5] it has been proven that, if g is convex and increasing, then $g \le C_n(g)$ on $\left[0, \frac{a_n+b_n}{2}\right]$. Hence because f is convex with respect to τ and increasing, $f \circ \tau^{-1}$ is convex and increasing, so that

$$f \circ \tau^{-1} \le C_n(f \circ \tau^{-1}) \quad \text{on} \quad \left[0, \frac{a_n+b_n}{2}\right],$$

and from this we get (3.10). Reasoning in the same way, one can establish (3.11).
Moreover, fix $f \in C([0,1])$ convex function with respect to τ. In [7, Theorem 4.4] it was established that, if $g \in C([0,1])$ is convex, then, for all $n \ge 1$, $C_{n+1}(g) \le \frac{n+1}{n+2} C_n(g) + \frac{1}{n+2} B_{n+1}(g)$, so that, by applying this result to $f \circ \tau^{-1}$, we get (3.12). □

Besides the convexity with respect to τ, the operators C_n^τ preserve another type of convexity. More precisely, given $\varphi \in C^\infty([0,1])$ such that $\varphi'(x) \neq 0$ for all $x \in [0,1]$ and $\varphi(0) = 0$, and $k \in \mathbb{N}$, a function $f \in C^k([0,1])$ is said to be φ-convex of order k if, for every $x \in [0,1]$,

$$D_\varphi^{(k)}(f)(x) := D^{(k)}(f \circ \varphi^{-1})(\varphi(x)) \ge 0.$$

For more details about φ-convex functions of order k see [14].
Since in our case $\tau : [0,1] \to [0,1]$ is a bijection and a positive function, it is easy to show that a function $f \in C^k([0,1])$ is τ-convex of order k if and only if

$$D_\tau^{(k)}(f) := D^{(k)}(f \circ \tau^{-1}) \ge 0.$$

In other words, f is τ-convex of order k iff $f \circ \tau^{-1}$ is k-convex. Here we recall that a function $g \in C^k([0,1])$ is said to be k-convex if $D^{(k)}(g) \ge 0$.
By using the fundamental theorem of calculus, F_n maps k-convex functions into k-convex functions and the same happens for the B_n's (see, for example, [6, Prop. A.2.5]). Then, thanks to (3.9) we have that the C_n^τ's map τ-convex functions of order k into τ-convex functions of order k.
We point out that the operators C_n^τ do not preserve the convexity. In order to construct an example, we use the following alternative representation for the operators C_n^τ: for every $n \ge 1$ and $f \in L^1([0,1])$,

$$C_n^\tau(f) = B_n^\tau(G_n^\tau(f \circ \tau^{-1})),$$

where

$$G_n^\tau(f)(x) := \frac{n+1}{b_n - a_n} \int_{\frac{n\tau(x)+a_n}{n+1}}^{\frac{n\tau(x)+b_n}{n+1}} f(t)\,dt\,.$$

Then, choosing $a_n = 0, b_n = 1$ for all $n \ge 1$, $\tau = (e_1+e_2)/2$ and $f = e_1$,

$$C_n^\tau(e_1) = \frac{n}{n+1} B_n^\tau(e_1) + \frac{1}{2(n+1)}.$$

Recalling that in this case $B_n^\tau(e_1)$ is not convex for lower n (see [10]), we get that the same happens for $C_n^\tau(e_1)$.
Now we pass to show that each C_n^τ preserves the class of Hölder continuous functions. Given $M > 0$ and $0 \le \alpha \le 1$, we shall write $f \in \mathrm{Lip}_M\alpha$ if

$$|f(x) - f(y)| \le M|x-y|^\alpha \quad \text{for every } 0 \le x, y \le 1.$$

In particular, if $\alpha = 1$, we get the space of all Lipschitz functions of Lipschitz constant M.
First observe that, from hypotheses on τ, both τ and τ^{-1} are Lipschitz functions. Precisely, $\tau \in \mathrm{Lip}_L 1$ with $L := \|\tau'\|_\infty$ and $\tau^{-1} \in \mathrm{Lip}_N 1$ with $N := (\min_{[0,1]} \tau')^{-1}$. Therefore, by recalling that $C_n(\mathrm{Lip}_M 1) \subset \mathrm{Lip}_{CM} 1$ with $C := \max\{1, |f(0)| + |f(1)|\}$ (see [7, Th. 4.1 and Example n. 1]), from (2.8) it follows that

$$C_n^\tau(\mathrm{Lip}_M 1) \subset \mathrm{Lip}_{CLMN} 1 \quad \text{for every } n \ge 1\,. \tag{3.13}$$

On account of [3, Cor. 6.1.20], since $\|C_n^\tau\| = 1$ and property (3.13) holds, for every $n \ge 1$, $f \in C([0,1])$, $\delta > 0$, $M > 0$ and $0 < \alpha \le 1$,

$$\omega(C_n^\tau(f), \delta) \le (1 + C)\omega(f, \delta) \quad \text{and} \quad C_n^\tau(\mathrm{Lip}_M\alpha) \subset \mathrm{Lip}_{(CLN)^\alpha M}\alpha\,.$$

Finally, for every $k \in \mathbb{N}$, denote by $\mathbb{P}_{\tau,k}$ the linear subspace generated by the set $\{\tau^i \,:\, i = 0, \dots, k\}$. $\mathbb{P}_{\tau,k}$ is said to be the space of the τ-polynomials of degree k. Since both the B_n's and the F_n's map polynomials of degree k into polynomials of degree k, taking (3.9) into account, we have that

$$C_n^\tau(\mathbb{P}_{\tau,k}) \subset \mathbb{P}_{\tau,k} \qquad (k \in \mathbb{N}, n \ge 1).$$

4. Approximation properties of the C_n^τ's

In this section, we prove that $(C_n^\tau)_{n\ge1}$ is a positive approximation process both in $C([0,1])$ and in $L^p([0,1])$, $1 \le p < +\infty$, and we provide some estimates of the rate of convergence, by means of suitable moduli of smoothness. As a byproduct of the uniform convergence, we obtain a property of the operators B_n^τ introduced in [10], which seems to be new.
We begin by stating the following result.

Theorem 4.1. *For every $f \in C([0,1])$, we have that*

$$\lim_{n\to\infty} C_n^\tau(f) = f \tag{4.14}$$

uniformly on $[0,1]$.

Proof. From (2.2) and (2.3) it easily follows that

$$C_n^\tau(\tau) = \frac{n}{n+1}\tau + \frac{a_n + b_n}{2(n+1)}\mathbf{1}, \tag{4.15}$$

$$C_n^\tau(\tau^2) = \frac{1}{(n+1)^2}\left(n^2\tau^2 + n\tau(1-\tau) + n(a_n + b_n)\tau + \frac{b_n^2 + a_n b_n + a_n^2}{3}\mathbf{1}\right); \tag{4.16}$$

since $C_n^\tau(\mathbf{1}) = \mathbf{1}$ and $\{\mathbf{1}, \tau, \tau^2\}$ is an extended Tchebychev system on [0,1], (4.14) comes directly by an application of Korovkin Theorem (see [3, Example 5, p. 246]). □

In order to get a quantitative version of the above uniform convergence, we use a result due to Paltanea (see [15]) which involves the usual modulus of continuity of the first and second order, denoted, respectively, by $\omega(f, \delta)$ and $\omega_2(f, \delta)$. To this end, we need some further preliminaries.
For $x \in [0,1]$, we denote by $e_{\tau,i}^x$ the function

$$e_{\tau,i}^x(t) = (\tau(t) - \tau(x))^i \quad (i = 0, 1, 2, \dots)\,.$$

When $\tau = e_1$ we shall simply write $\psi_x^i(t) = (t-x)^i$.
In particular, for any $n \geq 1$ and $x \in [0,1]$) (see (4.15) and (4.16)),

$$C_n^\tau(e_{\tau,2}^x)(x) = \frac{1-n}{(n+1)^2}\tau^2(x) + \frac{n-a_n-b_n}{(n+1)^2}\tau(x) + \frac{b_n^2+a_nb_n+a_n^2}{3(n+1)^2}. \tag{4.17}$$

Moreover, we recall the following result (see [11, Formula (8)]): there exists a constant $K > 0$ such that

$$K\psi_x^2(t) \leq \tau'(x)e_{\tau,2}^x(t) \quad \text{for every } x,t \in [0,1]\,. \tag{4.18}$$

Obviously, $K = 1$ if $\tau = e_1$.

Proposition 4.2. *Consider $n \geq 1$, $f \in C([0,1])$, $0 \leq x \leq 1$, and $\delta > 0$. Then*

$$|C_n^\tau(f)(x) - f(x)| \quad \leq \omega(f, \delta_n^\tau(x)) + \frac{3}{2}\omega_2(f, \delta_n^\tau(x))\,, \tag{4.19}$$

where

$$\delta_n^\tau(x) = \frac{\sqrt{\tau'(x)}}{(n+1)\sqrt{K}}\sqrt{(n-1)\tau(x)(1-\tau(x)) + (1-a_n-b_n)\tau(x) + \frac{b_n^2+a_nb_n+a_n^2}{3}}.$$

Moreover,

$$\|C_n^\tau(f) - f\|_\infty \leq \omega\left(f, \frac{\|\tau'\|_\infty^{1/2}}{\sqrt{K}\sqrt{n+1}}\right) + \frac{3}{2}\omega_2\left(f, \frac{\|\tau'\|_\infty^{1/2}}{\sqrt{K}\sqrt{n+1}}\right)\,. \tag{4.20}$$

Proof. Let $n \geq 1$, $f \in C([0,1])$, $0 \leq x \leq 1$ and $\delta > 0$. Paltanea's estimate ([15, Theorem 2.2.1]; see, also, [6, Theorem 1.6.2]) runs as follows:

$$\begin{aligned}|C_n^\tau(f)(x) - f(x)| &\leq |f(x)||C_n^\tau(\mathbf{1})(x) - 1| \\ &\quad + \delta^{-1}|C_n^\tau(\psi_x)(x)|\omega(f,\delta) + \left(C_n^\tau(\mathbf{1})(x) + (2\delta^2)^{-1}C_n^\tau(\psi_x^2)(x)\right)\omega_2(f,\delta) \\ &= \delta^{-1}|C_n^\tau(\psi_x)(x)|\omega(f,\delta) + (1 + (2\delta^2)^{-1}C_n^\tau(\psi_x^2)(x))\omega_2(f,\delta)\,.\end{aligned}$$

Cauchy-Schwarz inequality yields

$$|C_n^\tau(\psi_x)| \leq \sqrt{C_n^\tau(\psi_x^2)}\,,$$

so that

$$|C_n^\tau(f)(x) - f(x)| \leq \delta^{-1}\sqrt{C_n^\tau(\psi_x^2)(x)}\omega(f,\delta) + (1 + (2\delta^2)^{-1}C_n^\tau(\psi_x^2)(x))\omega_2(f,\delta)\,.$$

From (4.18), (4.17) and the positivity of C_n^τ's, we have

$$\begin{aligned}&KC_n^\tau(\psi_x^2)(x) \leq \tau'(x)C_n^\tau(e_{\tau,2}^x) \\ &= \frac{\tau'(x)}{(n+1)^2}\left\{(n-1)\tau(x)(1-\tau(x)) + (1-a_n-b_n)\tau(x) + \frac{b_n^2+a_nb_n+a_n^2}{3}\right\}.\end{aligned}$$

Therefore,

$$C_n^\tau(\psi_x^2) \leq \frac{\tau'(x)}{K(n+1)^2}\left\{(n-1)\tau(x)(1-\tau(x)) + (1-a_n-b_n)\tau(x) + \frac{b_n^2+a_nb_n+a_n^2}{3}\right\} \tag{4.21}$$

and, for $\delta = \delta_n^\tau(x)$, we get (4.19). Estimate (4.20) follows by noting that

$$\delta_n^\tau(x) \leq \frac{\|\tau'\|_\infty^{1/2}}{\sqrt{K}\sqrt{n+1}}$$

since $0 \leq \tau(x) \leq 1$. □

As a byproduct of Theorem 4.1, we present a simultaneous approximation result for the operators B_n^τ given by (2.7). As far as we know, this property is new.

Theorem 4.2. *Suppose that $a_n = 0$ and $b_n = 1$ for every $n \geq 1$. Then, for every $f \in C^1([0,1])$,*

$$B_{n+1}^\tau(f)' = \tau' C_n^\tau\left(f'/\tau'\right). \tag{4.22}$$

Moreover,

$$\lim_{n\to\infty} B_n^\tau(f)' = f' \quad \textit{uniformly on } [0,1]. \tag{4.23}$$

Proof. Let $x \in [0,1]$, $f \in C^1([0,1])$, and $n \geq 1$. From (2.7) if follows that

$$\begin{aligned}
B_{n+1}^\tau(f)'(x) &= \tau'(x) \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} \\
&\times (n+1)\left(\left(f\circ\tau^{-1}\right)\left(\frac{k+1}{n+1}\right) - \left(f\circ\tau^{-1}\right)\left(\frac{k}{n+1}\right)\right) \\
&= \tau'(x) \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (f\circ\tau^{-1})'(t)\, dt\right) \\
&= \tau'(x) C_n^\tau\left(\frac{f'}{\tau'}\right)(x),
\end{aligned}$$

and this completes the proof of (4.22). Formula (4.23) immediately follows from (4.22) and Theorem 4.1, because τ' is bounded. □

Now we prove that the sequence $(C_n^\tau)_{n\geq 1}$ is a positive approximation process also in $L^p([0,1])$ for any $p \in [1, +\infty[$.

Theorem 4.3. *Assume that*

$$\sup_{n\geq 1} \frac{1}{b_n - a_n} = M \in \mathbb{R}.$$

Then, for every $p \in [1, +\infty[$ and $f \in L^p([0,1])$,

$$\lim_{n\to\infty} C_n^\tau(f) = f \quad \textit{in} \quad L^p([0,1]). \tag{4.24}$$

Proof. By Theorem 4.1, for every $f \in C([0,1])$, $\lim_{n\to\infty} C_n(f) = f$ in L^p-norm, as well. Since $C([0,1])$ is dense in $L^p([0,1])$, in order to prove the statement it is sufficient to show, thanks to Banach-Steinhaus theorem, that the sequence of operators $C_n^\tau : L^p([0,1]) \to L^p([0,1])$ $(n \geq 1)$ is equicontinuous, i.e.,

$$\sup_{n\geq 1} \|C_n^\tau\|_{L^p, L^p} < +\infty.$$

To this end, for every $n \geq 1$, $f \in L^p([0,1])$ and $x \in [0,1]$, we preliminary notice that, since the function $|t|^p$ $(t \in \mathbb{R})$ is convex,

$$|C_n^\tau(f)(x)|^p \leq \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} \left[\frac{(n+1)}{(b_n - a_n)} \int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} \left(f\circ\tau^{-1}\right)(t)\, dt\right]^p.$$

By applying Jensen's inequality (see, e.g., [8, Theorem 3.9]) to the probability measure $\dfrac{n+1}{b_n-a_n}\,\mathbf{1}_{[\frac{k+a_n}{n+1},\frac{k+b_n}{n+1}]}\lambda_1$ on $[0,1]$, we get

$$\begin{aligned}|C_n^\tau(f)(x)|^p &\le \sum_{k=0}^{n}\binom{n}{k}\tau(x)^k(1-\tau(x))^{n-k}\frac{(n+1)}{(b_n-a_n)}\int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}}\left|\left(f\circ\tau^{-1}\right)(t)\right|^p\,dt\\ &= \sum_{k=0}^{n}\binom{n}{k}\tau(x)^k(1-\tau(x))^{n-k}\frac{(n+1)}{(b_n-a_n)}\int_{\tau^{-1}\left(\frac{k+a_n}{n+1}\right)}^{\tau^{-1}\left(\frac{k+b_n}{n+1}\right)}|f(y)\tau'(y)|^p\,dy\\ &\le \|\tau'\|_\infty^p\frac{(n+1)}{(b_n-a_n)}\sum_{k=0}^{n}\binom{n}{k}\tau(x)^k(1-\tau(x))^{n-k}\int_{\tau^{-1}\left(\frac{k+a_n}{n+1}\right)}^{\tau^{-1}\left(\frac{k+b_n}{n+1}\right)}|f(y)|^p\,dy.\end{aligned}$$

We point out that

$$\int_0^1\tau(x)^k(1-\tau(x))^{n-k}\,dx=\int_0^1\frac{t^k(1-t)^{n-k}}{\tau'(\tau^{-1}(t))}\,dt\le\frac{1}{\min_{y\in[0,1]}\tau'(y)}\frac{1}{\binom{n}{k}(n+1)}.$$

Hence, by integrating with respect to x, we obtain

$$\|C_n^\tau(f)\|_p^p\le MN\|f\|_p^p,$$

where

$$N:=\frac{\|\tau'\|_\infty^p}{\min_{y\in[0,1]}\tau'(y)};$$

hence $\|C_n^\tau\|_{L^p,L^p}\le(MN)^{1/p}<+\infty$. □

An estimate of the convergence in (4.24) can be obtained by using a result due to Swetits and Wood [16, Theorem 1] which involves the second-order integral modulus of smoothness defined, for $f\in L^p([0,1])$, $1\le p<+\infty$, as

$$\omega_{2,p}(f,\delta):=\sup_{0<t\le\delta}\|f(\cdot+t)-2f(\cdot)+f(\cdot-t)\|_p\quad(\delta>0).$$

We define

(4.25)
$$\beta_{n,p,\tau}:=\frac{1}{(n+1)\sqrt{K}}\times\left\|\sqrt{\tau'}\left\{(n-1)\tau(1-\tau)+(1-a_n-b_n)\tau+\frac{b_n^2+a_nb_n+a_n^2}{3}\mathbf{1}\right\}^{1/2}\right\|_p^{1/2}$$

and

$$\begin{aligned}\gamma_{n,p,\tau}&:=\frac{1}{(n+1)^{2p/(2p+1)}K^{p/(2p+1)}}\\ &\times\left\|\tau'\left\{(n-1)\tau(1-\tau)+(1-a_n-b_n)\tau+\frac{b_n^2+a_nb_n+a_n^2}{3}\mathbf{1}\right\}\right\|_p^{p/(2p+1)},\end{aligned}\tag{4.26}$$

where K is the strictly positive constant in (4.18).
Then we can state the following result.

Proposition 4.3. *Under the hypotheses of Theorem 4.3, for every $p\in[1,+\infty[$ there exists $C_p>0$ such that, for every $f\in L^p([0,1])$ and for n sufficiently large,*

$$\|C_n^\tau(f)-f\|_p\le C_p(\alpha_{n,p,\tau}^2\|f\|_p+\omega_{2,p}(f,\alpha_{n,p,\tau}))$$

where $\alpha_{n,p,\tau}=\max\{\beta_{n,p,\tau},\gamma_{n,p,\tau}\}$.

Proof. First we introduce the following auxiliary functions:

$$F_n^\tau(x) := C_n^\tau(\psi_x)(x), \; G_n^\tau(x) := C_n^\tau(\psi_x^2)(x) \quad (x \in [0,1], n \geq 1).$$

Hence, the result in [16] applied to the uniformly bounded sequence $(C_n^\tau)_{n\geq 1}$ yields that there exists a constant $C_p > 0$ such that

$$\|C_n^\tau(f) - f\|_p \leq C_p(\mu_{n,p}^2\|f\|_p + \omega_{2,p}(f, \mu_{n,p})),$$

where the sequence $\mu_{n,p} \to 0$ as $n \to \infty$ and it is defined as follows:

$$\begin{aligned}\mu_{n,p} &:= \max\left\{\|C_n^\tau(\mathbf{1}) - \mathbf{1}\|_p^{1/2}, \|F_n^\tau\|_p^{1/2}, \|G_n^\tau\|_p^{p/(2p+1)}\right\} \\ &= \max\left\{\|F_n^\tau\|_p^{1/2}, \|G_n^\tau\|_p^{p/(2p+1)}\right\}.\end{aligned}$$

By Cauchy-Schwarz inequality we have

$$|F_n^\tau|^p \leq (\sqrt{G_n^\tau})^p,$$

so

$$\mu_{n,p} \leq \max\left\{\|\sqrt{G_n^\tau}\|_p^{1/2}, \|G_n^\tau\|_p^{p/(2p+1)}\right\}.$$

From (4.21) it follows that $\|\sqrt{G_n^\tau}\|_p^{1/2} \leq \beta_{n,p,\tau}$ and $\|G_n^\tau\|_p^{p/(2p+1)} \leq \gamma_{n,p,\tau}$ (see (4.25) and (4.26)). Moreover,

$$\begin{aligned}\gamma_{n,p,\tau} &\leq \frac{\|\tau'\|_\infty^{p/(2p+1)}}{(n+1)^{2p/(2p+1)}K^{p/(2p+1)}}(n+1)^{p/(2p+1)} \\ &= \frac{\|\tau'\|_\infty^{p/(2p+1)}}{(n+1)^{p/(2p+1)}K^{p/(2p+1)}} \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Similarly,

$$\beta_{n,p,\tau} \leq \frac{\|\sqrt{\tau'}\|_\infty^{1/2}}{(n+1)\sqrt{K}}(n+1)^{1/4} = \frac{\|\sqrt{\tau'}\|_\infty^{1/2}}{(n+1)^{3/4}\sqrt{K}} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, setting $\alpha_{n,p,\tau} = \max\{\beta_{n,p,\tau}, \gamma_{n,p,\tau}\}$, we have that $\alpha_{n,p,\tau} \to 0$ as $n \to \infty$ and this completes the proof. □

5. Asymptotic formula for the C_n^τ's

In this section we establish an asymptotic formula for the operators C_n^τ, which, in addition, allows us to derive other properties of them. To this end, from now assume that

(5.27) $$\text{there exists } l := \lim_{n\to\infty}(a_n + b_n) \in \mathbb{R}$$

and consider the differential operator $(V_l, C^2([0,1]))$ defined by setting

$$V_l(u)(x) := \frac{1}{2}x(1-x)u''(x) + \left(\frac{l}{2} - x\right)u'(x),$$

$(u \in C^2([0,1]), x \in [0,1])$.

Theorem 5.4. *Assume that (5.27) holds true. Then, for each $f \in C([0,1])$, twice differentiable at a certain $x \in]0,1[$,*

$$\begin{aligned}\lim_{n\to\infty} n(C_n^\tau(f)(x) - f(x)) &= \frac{\tau(x)(1-\tau(x))}{2} D_\tau^2(f)(x) + \left(\frac{l}{2} - \tau(x)\right) D_\tau(f)(x) \\ &= \frac{\tau(x)(1-\tau(x))}{2\tau'(x)^2} f''(x) + \frac{1}{\tau'(x)}\left(\frac{l}{2} - \tau(x) - \frac{\tau(x)(1-\tau(x))}{2\tau'(x)^2}\tau''(x)\right) f'(x).\end{aligned} \tag{5.28}$$

Moreover, for every $u \in C^2([0,1])$

$$\lim_{n\to\infty} n(C_n^\tau(u) - u) = V_l(u \circ \tau^{-1}) \circ \tau \tag{5.29}$$

uniformly in $[0,1]$.

Proof. In [5, Theorem 3.1] it was proven that

$$\lim_{n\to\infty} n(C_n(u) - u) = V_l(u),$$

for every $u \in C^2([0,1])$ uniformly on $[0,1]$, but it is easy to prove that the same limit relationship holds pointwise for each $f \in C([0,1])$, twice differentiable at a certain $x \in]0,1[$. From this, formulas (5.28) and (5.29) easily follow. □

5.1. **An application to iterates of the operators C_n^τ.** In this subsection we show how iterates of operators C_n^τ can be employed in order to approximate constructively certain semigroups of operators. For unexplained terminology concerning Semigroup Theory and its connection with Approximation Theory, we refer, e.g., to [6, Chapter 2].

We begin by recalling that, as shown in [5, Theorem 3.2] the operator $(V_l, C^2([0,1]))$ is closable and its closure generates a Markov semigroup $(T_l(t))_{t\geq 0}$ on $C([0,1])$ such that, if $t \geq 0$ and if $(\rho_n)_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} \rho_n/n = t$, then

$$\lim_{n\to\infty} C_n^{\rho_n}(f) = T_l(t)(f) \qquad \text{uniformly on } [0,1]$$

for every $f \in C([0,1])$, where $C_n^{\rho_n}$ denotes the iterate of C_n of order ρ_n.

Moreover (see [5, Theorem 3.4, Remark 3.5,1]), if either $a_n = 0$ and $b_n = 1$ for every $n \geq 1$, or the following properties hold true

(i) $0 < b_n - a_n < 1$ for every $n \geq 1$;
(ii) there exist $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 1$;
(iii) $M_1 := \sup_{n\geq 1} n(1-(b_n - a_n)) < +\infty$,

for every $p \geq 1$, $(T_l(t))_{t\geq 0}$ extends to a positive C_0-semigroup $(\widetilde{T}(t))_{t\geq 0}$ on $L^p([0,1])$ such that, if $(\rho_n)_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} \rho_n/n = t$, then for every $f \in L^p([0,1])$,

$$\lim_{n\to\infty} C_n^{\rho_n}(f) = \widetilde{T}(t)(f) \qquad \text{in } L^p([0,1]).$$

We remark that, for every $f \in C([0,1])$ and $k \geq 1$,

$$(C_n^\tau)^k(f) = C_n^k(f \circ \tau^{-1}) \circ \tau.$$

From this we get the following result.

Theorem 5.5. *Under assumption (5.27), for every $f \in C([0,1])$, $t \geq 0$ and for every sequence $(\rho_n)_{n\geq 1}$ of positive integers such that $\lim_{n\to\infty} \rho_n/n = t$,*

$$\lim_{n\to\infty} (C_n^\tau)^{\rho_n}(f) = T_l(t)(f \circ \tau^{-1}) \circ \tau \qquad \textit{uniformly on } [0,1].$$

Moreover, assume that either $a_n = 0$ *and* $b_n = 1$ *for every* $n \geq 1$*, or the following properties hold true*

(i) $0 < b_n - a_n < 1$ *for every* $n \geq 1$*;*
(ii) there exist $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} b_n = 1$*;*
(iii) $M_1 := \sup_{n\geq 1} n(1 - (b_n - a_n)) < +\infty$.

Then, if $t \geq 0$ *and if* $(\rho_n)_{n\geq 1}$ *is a sequence of positive integers such that* $\lim_{n\to\infty} \rho_n/n = t$*, then for every* $f \in L^p([0,1])$*,*

$$\lim_{n\to\infty} (C_n^\tau)^{\rho_n}(f) = \widetilde{T}(t)(f \circ \tau^{-1}) \circ \tau \qquad \text{in } L^p([0,1]).$$

5.2. **Comparing the operators C_n^τ and C_n.** The asymptotic formula (5.28) can be also used to prove that, under suitable conditions, the operators C_n^τ perform better than the operators C_n in approximating certain functions. In fact, arguing as in the proof of [10, Theorem 9], we are able to show the following result.

Theorem 5.6. *Let* $f \in C^2([0,1])$ *and assume that there exists* $n_0 \in \mathbb{N}$ *such that, for every* $n \geq n_0$ *and* $x \in]0,1[$*,*

$$f(x) \leq C_n^\tau(f)(x) \leq C_n(f)(x)\,.$$

Then, for $x \in]0,1[$*,*

$$\begin{aligned} f''(x) &\geq \frac{\tau''(x)}{\tau'(x)} f'(x) + \frac{\tau'(x)(2\tau(x) - l)}{\tau(x)(1-\tau(x))} f'(x) \\ &\geq \left(1 - \frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))}\right) f''(x) + \frac{\tau'(x)^2(2x - l)}{\tau(x)(1-\tau(x))} f'(x)\,. \end{aligned} \tag{5.30}$$

In particular, $f'' \geq 0$ *in* $]0, l/2[$ *(resp., in* $]l/2, 1[$*) whenever* f *is decreasing in* $]0, l/2[$ *(resp.,* f *is increasing in* $]l/2, 1[$*).*

Conversely, assume that at a given point $x_0 \in]0,1[$*, (5.30) holds with strict inequalities. Then there exists* $n_0 \in \mathbb{N}$ *such that, for every* $n \geq n_0$*,*

$$f(x_0) < C_n^\tau(f)(x_0) < C_n(f)(x_0)\,.$$

Example 5.1. *Take*

$$\tau = \frac{e_2 + \alpha e_1}{1+\alpha} \qquad (\alpha > 0)$$

and suppose that $f \in C^2([0,1])$ *is increasing and strictly convex.*

Moreover, assume that the sequences $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 1}$ *are such that* $l = \lim_{n\to\infty}(a_n + b_n) = 2$*. We show that there exist* $x_\alpha \in]0,1[$ *and* $n_0 \in \mathbb{N}$ *such that, for each* $x \in]x_\alpha, 1]$ *and* $n \geq n_0$*,*

$$f(x) < C_n^\tau(f)(x) < C_n(f)(x)\,.$$

On account of Theorem 5.6, it is sufficient to prove that there exists $x_\alpha \in]0,1[$ *such that, for* $x \in]x_\alpha, 1[$*,*

$$\begin{aligned} f''(x) &> \frac{\tau''(x)}{\tau'(x)} f'(x) + \frac{\tau'(x)(2\tau(x) - 2)}{\tau(x)(1-\tau(x))} f'(x) \\ &> \left(1 - \frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))}\right) f''(x) + \frac{\tau'(x)^2(2x - 2)}{\tau(x)(1-\tau(x))} f'(x)\,. \end{aligned} \tag{5.31}$$

The first inequality in (5.31) is satisfied for $\alpha > 2f'(1)/M$*, where* $M = \min_{[0,1]} f''(x)$*. Indeed, for this choice,*

$$f''(x) > \frac{2}{2x+\alpha} f'(x) > \frac{2}{2x+\alpha} f'(x) - 2\frac{2x+\alpha}{x^2+\alpha x} f'(x), \quad x \in]0,1[\,.$$

The second inequality in (5.31) is obviously fulfilled for those x for which

$$\frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))} \geq 1 \tag{5.32}$$

and

$$\frac{\tau''(x)}{\tau'(x)} > 2\frac{\tau'(x)^2}{\tau(x)(1-\tau(x))}\left(x-1-\frac{\tau(x)-1}{\tau'(x)}\right). \tag{5.33}$$

From one hand (5.32) is verified for $x \in]y_\alpha, 1]$ where

$$y_\alpha := \frac{1-2\alpha+\sqrt{4\alpha^2+8\alpha+1}}{6}$$

(see [10, Corollary 11, (iii)]). On the other hand (5.33) is equivalent to solve (with respect to x) the following inequality:

$$g_\alpha(x) := (x^2+\alpha x)(1+x+\alpha)-(2x+\alpha)^2(1-x) > 0\,.$$

By observing that $g_\alpha(0) < 0$, $g_\alpha(1) > 0$, and evaluating the critical points of g_α and their position within the interval [0, 1] depending on $\alpha > 0$, we can conclude that, for every $\alpha > 0$, there exists $z_\alpha \in]0,1[$ such that $g_\alpha(z_\alpha) = 0$ and $g_\alpha(x) > 0$ for every $z_\alpha < x \leq 1$. By setting $x_\alpha = \max\{y_\alpha, z_\alpha\}$ $(\alpha > 2f'(1)/M)$, we get the claim.
We point out that, in the case $\alpha = 0$, $\tau = e_2$ and the corresponding operators C_n^τ are a Kantorovich-type modification on mobile intervals of the operators in [10, p. 159]. On the other hand, $\tau_\infty = \lim_{\alpha\to+\infty} \tau = e_1$ uniformly w.r.t. $x \in [0,1]$, so that $C_n^{\tau_\infty} = C_n$ for any $n \geq 1$.

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