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A Two-Parameter Weighted Rama Distribution with Properties and Application

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Abstract

In this paper, a two-parameter weighted Rama distribution which includes one parameter Rama distribution introduced by Shanker [1] has been proposed for modelling real lifetime data. Statistical properties of the distribution including shapes of a probability density function, moments and moment related measures, hazard rate function, mean residual life function, and stochastic orderings have been discussed. The estimation of its parameters has been discussed using the method of maximum likelihood. Application of the proposed distribution has been discussed.

1. Introduction

The concept of weighted distributions was firstly introduced by Fisher [2] to model ascertainment biases which were later formalized by Rao [3] in a unifying theory for problems where the observations fall in non-experimental, non-replicated and non-random. When an investigator records an observation in nature according to a certain stochastic model, the distribution of the recorded observation will not have the original distribution unless every observation is given an equal chance of being recorded. For example, suppose the original observation x_0 comes from a distribution having a probability density function (pdf.), $f_o(x,\theta_1)$, where θ_1 may be a parameter vector and observation x s recorded according to a probability re-weighted by a weight function $w(x,\theta_2) > 0$, θ_2 being a new parameter vector, then x comes from a distribution having pdf

$$f\left(x;\theta_1,\theta_2\right) = k\,w\left(x;\theta_2\right) f_o\left(x;\theta_2\right)$$

where k is a normalizing constant. Recall that such types of distributions are known as weighted distributions. The weighted distributions with a weight function $w\left(x,\theta_2\right) = x$ are called length-biased distribution. Patil and Rao [4]-[5] have examined some general probability models leading to weighted probability distributions, discussed their applications and showed the occurrence of $w(x,\theta_2) = x$ in a natural way in problems relating to sampling.
Shanker [1] has introduced Rama distribution for modelling behavioural Science data defined by its pdf and cumulative distribution function (cdf)

$$f_1\left(x;\theta\right) = \frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x} \;\; ;x>0,\ \theta>0 \tag{1.1}$$

$$F_1\left(x,\theta\right) = 1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\; ;x>0,\theta>0$$

Thus the r^{th} moment about origin μ'_r of Rama distribution (1.1) obtained by Shanker [1] is given by

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$$\mu_r{}' = \frac{r!\left[\theta^3+(r+1)(r+2)(r+3)\right]}{\theta^r\left(\theta^3+6\right)} \; ; r=1,2,3,...$$

The first four moments about origin of Rama distribution obtained by Shanker [1] are as follows

$$\mu_1{}' = \frac{\theta^3+24}{\theta\left(\theta^3+6\right)}$$

$$\mu_2{}' = \frac{2\left(\theta^3+60\right)}{\theta^2\left(\theta^3+6\right)}$$

$$\mu_3{}' = \frac{6\left(\theta^3+120\right)}{\theta^3\left(\theta^3+6\right)}$$

$$\mu_4{}' = \frac{24\left(\theta^3+210\right)}{\theta^4\left(\theta^3+6\right)}.$$

The moments about mean of Rama distribution are

$$\mu_2 = \frac{\theta^6+84\theta^3+144}{\theta^2\left(\theta^3+6\right)^2}$$

$$\mu_3 = \frac{2\left(\theta^9+198\theta^6+324\theta^3+864\right)}{\theta^3\left(\theta^3+6\right)^3}$$

$$\mu_4 = \frac{9\left(\theta^{12}+312\theta^9+2304\theta^6+10368\theta^3+10368\right)}{\theta^4\left(\theta^3+6\right)^4}.$$

Shanker [1] has discussed statistical properties including shapes of pdf for varying values of parameter, hazard rate function; mean residual life function, stochastic ordering, mean deviations, order statistic, Bonferroni and Lorenz curves, Renyi entropy measures, and stress-strength reliability of Rama distribution. Shanker [1] has also studied the estimation of the parameter of Rama distribution using both the method of maximum likelihood and the method of the moment along with an application.

In this paper, a two-parameter weighted Rama distribution which includes one parameter Rama distribution proposed by Shanker [1] has been introduced and studied. The statistical properties of the distribution including the coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, and stochastic ordering have been discussed. The method of maximum likelihood has been discussed for estimating parameters. The goodness of fit of the proposed distribution has been discussed with a real lifetime data and it shows a quite satisfactory fit over one parameter life time distributions including exponential, Lindley, Rama and two-parameter lifetime distributions including Gompertz, lognormal, Generalized exponential introduced by Gupta and Kundu [6], weighted Lindley introduced by Ghitany et al [7] and weighted Sujatha distribution introduced by Shanker et al [8].

2. Weighted Rama Distribution

The pdf of the weighted Rama distribution (WRD) can be expressed as

$$f(x;\theta,\alpha) = K x^{\alpha-1} f_o(x;\theta) \; ; \; x>0, \; \theta>0, \; \alpha>0$$

where, K is the normalizing constant and $f_o(x;\theta)$ is the pdf of Rama distribution given in (1.1). Thus the pdf of WRD can be obtained as

$$f_2(x;\theta,\alpha) = \frac{\theta^{\alpha+3}}{\theta^3+\alpha(\alpha+1)(\alpha+2)} \frac{x^{\alpha-1}}{\Gamma(\alpha)} \left(1+x^3\right) e^{-\theta x} \; ; x>0, \theta>0, \alpha>0 \tag{2.1}$$

where $\theta =$ scale parameter , $\alpha =$ shape parameter and

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy; y>0, \alpha>0$$

is the complete gamma function.

Further, pdf (2.1) can be expressed as a two-component mixture of gamma (θ,α) and gamma $(\theta,\alpha+3)$ distributions. We have

$$f_2(x;\theta,\alpha) = p\, g_1(x;\theta,\alpha) + (1-p)\, g_2(x;\theta,\alpha+3)$$

where

$$p = \frac{\theta^3}{\theta^3 + \alpha(\alpha+1)(\alpha+2)},$$

$$g_1(x;\theta,\alpha) = \frac{\theta^\alpha}{\Gamma(\alpha)} e^{-\theta x} x^{\alpha-1},$$

$$g_2(x;\theta,\alpha+3) = \frac{\theta^{\alpha+3}}{\Gamma(\alpha+3)} e^{-\theta x} x^{(\alpha+3)-1}.$$

The behaviour of the pdf of WRD for varying values of parameters θ and α are shown in Figure 2.1.

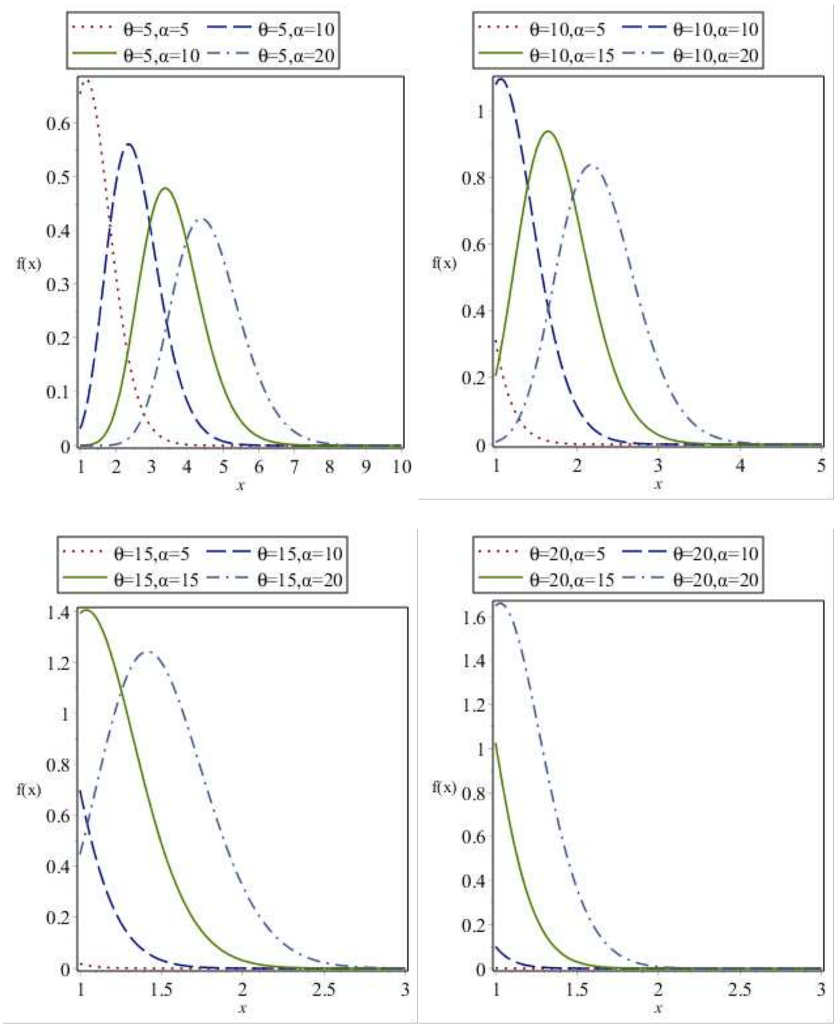

Figure 2.1: Behavior of the pdf of WRD for various values of the parameters θ and α

The cdf of WRD can be obtained as

$$F_2(x;\theta,\alpha) = 1 - \left[\frac{\begin{array}{c}(\theta x)^\alpha \left\{(\theta x)^2 + (\theta x)(\alpha+2) + (\alpha+1)(\alpha+2)\right\} e^{-\theta x} \\ + \left\{\theta^3 + \alpha(\alpha+1)(\alpha+2)\right\} \Gamma(\alpha,\theta x)\end{array}}{\left[\theta^3 + \alpha(\alpha+1)(\alpha+2)\right]\Gamma(\alpha)} \right] \quad ; x>0, \theta>0, \alpha>0$$

where $\Gamma(\alpha,\theta x)$ is the upper incomplete gamma function defined by

$$\Gamma(\alpha,z) = \int_z^\infty e^{-y} y^{\alpha-1} dy; \quad y \geq 0, \quad \alpha > 0.$$

The behaviour of the cdf of WRD for varying values of the parameters θ and α are shown in Figure 2.2.

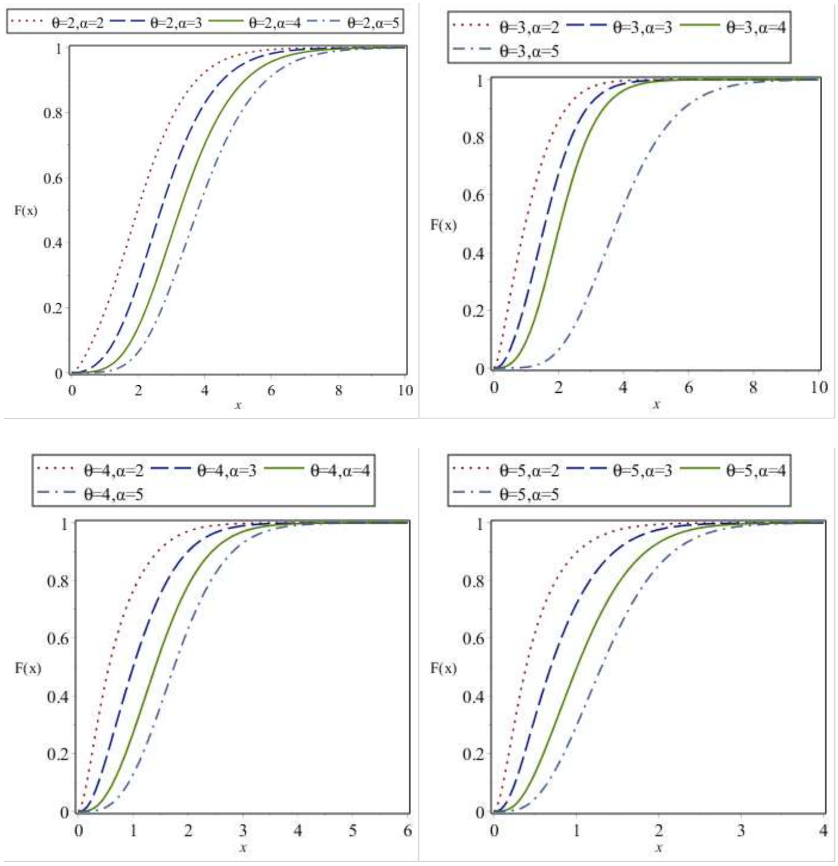

Figure 2.2: Behavior of the cdf of WRD for various values of the parameters θ and α

3. Statistical Constants

The r^{th} moment about origin, μ'_r of WRD (2.1) can be obtained as

$$\mu'_r = E\left(X^r\right) = \int_0^\infty x^r f_2\left(x;\theta,\alpha\right)dx = \int_0^\infty x^r \frac{\theta^{\alpha+3}x^{\alpha-1}\left(1+x^3\right)}{\Gamma\left(\alpha\right)\left[\theta^3+\alpha\left(\alpha+1\right)\left(\alpha+2\right)\right]}e^{-\theta x}dx$$

$$= \frac{\Gamma\left(\alpha+r\right)}{\Gamma\left(\alpha\right)}\;\frac{\theta^3+\left(\alpha+r\right)\left(\alpha+r+1\right)\left(\alpha+r+2\right)}{\theta^r\left[\theta^3+\alpha\left(\alpha+1\right)\left(\alpha+2\right)\right]};\, r=1,2,3,.... \tag{3.1}$$

The first four moments about origin of WRD, after substituting $r = 1$, 2, 3 and 4 in (3.1) are obtained as

$$\mu'_1 = \frac{\alpha\left\{\theta^3+\left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)\right\}}{\theta\left[\theta^3+\alpha\left(\alpha+1\right)\left(\alpha+2\right)\right]}$$

$$\mu'_2 = \frac{\alpha\left(\alpha+1\right)\left\{\theta^3+\left(\alpha+2\right)\left(\alpha+3\right)\left(\alpha+4\right)\right\}}{\theta^2\left[\theta^3+\alpha\left(\alpha+1\right)\left(\alpha+2\right)\right]}$$

$$\mu'_3 = \frac{\alpha\left(\alpha+1\right)\left(\alpha+2\right)\left\{\theta^3+\left(\alpha+3\right)\left(\alpha+4\right)\left(\alpha+5\right)\right\}}{\theta^3\left[\theta^3+\alpha\left(\alpha+1\right)\left(\alpha+2\right)\right]}$$

$$\mu'_4 = \frac{\alpha\left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)\left\{\theta^3+\left(\alpha+4\right)\left(\alpha+5\right)\left(\alpha+6\right)\right\}}{\theta^4\left[\theta^3+\alpha\left(\alpha+1\right)\left(\alpha+2\right)\right]}.$$

Now using the relationship $\mu_r = E(X-\mu'_1)^r = \sum_{k=0}^{r}\left(\begin{array}{c} r \\ k \end{array}\right)\mu'_k(-\mu'_1)^{r-k}$ between moments about mean and moments about origin, the moments about the mean of WRD are obtained as

$$\mu_2=\frac{\alpha\left\{\theta^6+2\left(\alpha^3+9\alpha^2+20\alpha+12\right)\theta^3+\left(\alpha^6+9\alpha^5+31\alpha^4+51\alpha^3+40\alpha^2+12\alpha\right)\right\}}{\theta^2\left(\theta^3+\alpha^3+3\alpha^2+2\alpha\right)^2}$$

$$\mu_3=\frac{2\alpha\left\{\begin{array}{l}\theta^9+3\left(\alpha^3+13\alpha^2+32\alpha+20\right)\theta^6+3\left(\alpha^6+8\alpha^5+25\alpha^4+38\alpha^3+28\alpha^2+8\alpha\right)\theta^3\\+\left(\alpha^9+12\alpha^8+60\alpha^7+162\alpha^6+255\alpha^5+234\alpha^4+116\alpha^3+24\alpha^2\right)\end{array}\right\}}{\theta^3\left(\theta^3+\alpha^3+3\alpha^2+2\alpha\right)^3}$$

$$\mu_4=\frac{3\alpha\left\{\begin{array}{l}(\alpha+2)\theta^{12}+4\left(\alpha^4+11\alpha^3+56\alpha^2+106\alpha+60\right)\theta^9\\+3\left(2\alpha^6+34\alpha^5+198\alpha^4+550\alpha^3+792\alpha^2+568\alpha+160\right)\alpha\theta^6\\+4\left(\alpha^8+20\alpha^7+156\alpha^6+642\alpha^5+1551\alpha^4+2274\alpha^3+1988\alpha^2+952\alpha+198\right)\alpha^2\theta^3\\+\left(\begin{array}{l}\alpha^{13}+20\alpha^{12}+173\alpha^{11}+856\alpha^{10}+2691\alpha^9+5628\alpha^8\\+7943\alpha^7+7480\alpha^6+4504\alpha^5+1568\alpha^4+240\alpha^3\end{array}\right)\end{array}\right\}}{\theta^4\left(\theta^3+\alpha^3+3\alpha^2+2\alpha\right)^4}.$$

It can be easily shown that at $\alpha=1$, the moments about the origin and the moments about mean of WRD reduces to the corresponding moments of Rama distribution.

The expressions for coefficient variation (C.V.) coefficient of skewness $\left(\sqrt{\beta_1}\right)$, the coefficient of kurtosis (β_2) and index of dispersion (γ) of WRD is thus given as

$$C.V=\frac{\sigma}{\mu_1{}'}=\frac{\sqrt{\alpha\left\{\theta^6+\left(2\alpha^3+18\alpha^2+40\alpha+24\right)\theta^3+\alpha^6+9\alpha^5+31\alpha^4+51\alpha^3+40\alpha^2+12\alpha\right\}}}{\alpha\left(\theta^3+\alpha^3+6\alpha^2+11\alpha+6\right)}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2{}^{3/2}}=\frac{2\alpha\left\{\begin{array}{l}\theta^9+\left(3\alpha^3+39\alpha^2+96\alpha+60\right)\theta^6+\left(3\alpha^6+24\alpha^5+75\alpha^4+114\alpha^3+84\alpha^2+24\alpha\right)\theta^3\\\alpha^9+12\alpha^8+60\alpha^7+162\alpha^6+255\alpha^5+234\alpha^4+116\alpha^3+24\alpha^2\end{array}\right\}}{\left\{\alpha\left(\theta^6+\left(2\alpha^3+18\alpha^2+40\alpha+24\right)\theta^3+\alpha^6+9\alpha^5+31\alpha^4+51\alpha^3+40\alpha^2+12\alpha\right)\right\}^{3/2}}$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{3(\alpha+2)\left\{\begin{array}{l}\theta^{12}+\left(4\alpha^3+36\alpha^2+152\alpha+120\right)\theta^9+\left(6\alpha^6+90\alpha^5+414\alpha^4+822\alpha^3+732\alpha^2+240\alpha\right)\theta^6\\+\left(4\alpha^9+72\alpha^8+480\alpha^7+1608\alpha^6+2988\alpha^5+3120\alpha^4+1712\alpha^3+384\alpha^2\right)\theta^3\\+\left(\alpha^{12}+18\alpha^{11}+137\alpha^{10}+582\alpha^9+1527\alpha^8+2574\alpha^7+2795\alpha^6+1890\alpha^5+724\alpha^4+120\alpha^3\right)\end{array}\right\}}{\left\{\alpha\left(\theta^6+\left(2\alpha^3+18\alpha^2+40\alpha+24\right)\theta^3+\alpha^6+9\alpha^5+31\alpha^4+51\alpha^3+40\alpha^2+12\alpha\right)^2\right\}}$$

$$\gamma=\frac{\sigma^2}{\mu'_1}=\frac{\left\{\theta^6+\left(2\alpha^3+18\alpha^2+40\alpha+24\right)\theta^3+\alpha^6+9\alpha^5+31\alpha^4+51\alpha^3+40\alpha^2+12\alpha\right\}}{\theta\left(\theta^3+\alpha^3+3\alpha^2+2\alpha\right)\left(\theta^3+\alpha^3+6\alpha^2+11\alpha+6\right)}.$$

The behaviour of the coefficient of variation (C.V.), the coefficient of skewness (C.S.), the coefficient of kurtosis (C.K.) and index of dispersion (I.D.) of WRD have been prepared for varying values of the parameters θ and α and presented in Figure 3.1.

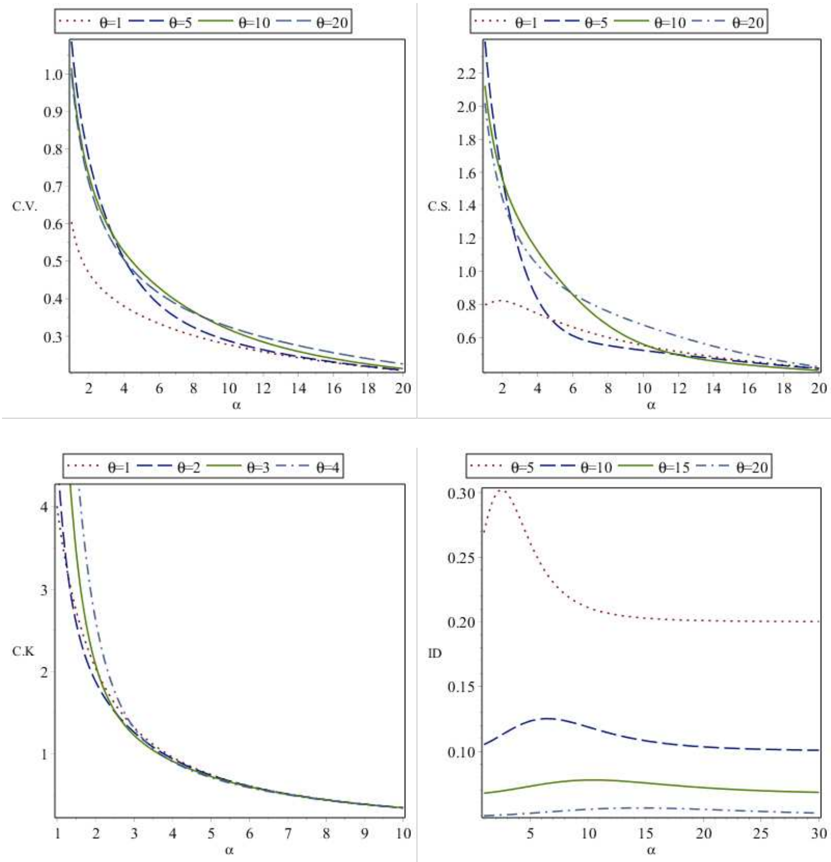

Figure 3.1: Behavior of C.V., C.S., C.K., and I.D of WRD for varying values of parameters θ and α

4. Survival Function ans Hazard Rate Function

The survival (reliability) function of WRD can be obtained as

$$S(x;\theta,\alpha) = 1 - F_2(x;\theta,\alpha) == \frac{\left\{ \begin{array}{c} (\theta x)^{\alpha}\left[(\theta x)^2+(\alpha+2)(\theta x)+(\alpha+1)(\alpha+2)\right]e^{-\theta x} \\ +\left[\theta^3+\alpha(\alpha+1)(\alpha+2)\right]\Gamma(\alpha,\theta x) \end{array} \right\}}{\left[\theta^3+\alpha(\alpha+1)(\alpha+2)\right]\Gamma(\alpha)}.$$

The hazard (or failure) rate function, $h(x)$ of WRD is thus obtained as

$$h(x) = \frac{f(x;\theta,\alpha)}{S(x;\theta,\alpha)} = \frac{\theta^{\alpha+3}x^{\alpha-1}\left(1+x^3\right)e^{-\theta x}}{\left\{ \begin{array}{c} \left[\theta^3+\alpha(\alpha+1)(\alpha+2)\right]\Gamma(\alpha,\theta x) \\ +(\theta x)^{\alpha}\left[(\theta x)^2+(\theta x)(\alpha+2)+(\alpha+1)(\alpha+2)\right]e^{-\theta x} \end{array} \right\}} \ ; \ x>0,\ \theta>0,\ \alpha>0.$$

The behaviour of $h(x)$ of WRD for varying values of parameters θ and α are shown in Figure 4.1.

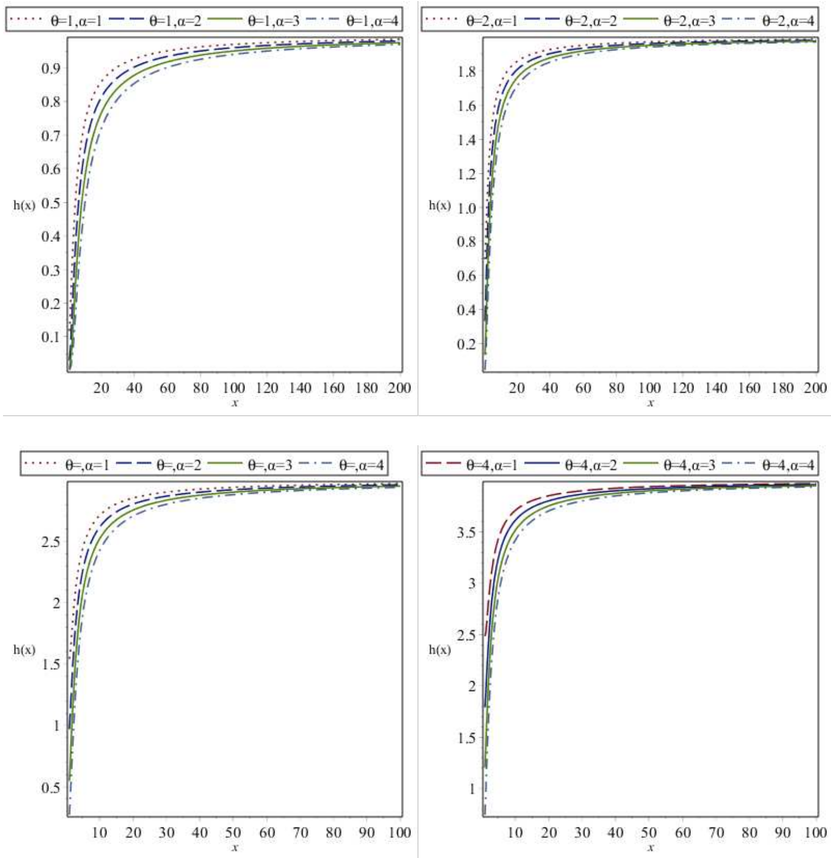

Figure 4.1: Behavior of C.V., C.S., C.K., and I.D of WRD for varying values of parameters θ and α

5. Mean Residual Life Function

The mean residual life function $\mu(x) = E(X - x|X > x)$ of the WRD can be obtained as

$$\begin{aligned}\mu(x) &= \frac{1}{S(x;\theta,\alpha)}\int_x^\infty y\, f_2(y;\theta,\alpha)\,dy \;-x\\ &= \frac{\left\{\begin{array}{l}(\theta x)^\alpha\left\{(\theta x)^2+2(\alpha+2)(\theta x)+\theta^3+(\alpha+1)(\alpha+2)(\alpha+3)\right\}e^{\theta x}\\ +\left[\left\{\alpha\theta^3+\alpha(\alpha+1)(\alpha+2)(\alpha+3)\right\}-\theta x\left\{\theta^3+\alpha(\alpha+1)(\alpha+2)\right\}\right]\Gamma(\alpha,\theta x)\end{array}\right\}}{\theta\left[(\theta x)^\alpha\left\{(\theta x)^2+(\alpha+2)(\theta x)+(\alpha+1)(\alpha+2)\right\}e^{-\theta x}+\left\{\theta^3+\alpha(\alpha+1)(\alpha+2)\right\}\Gamma(\alpha,\theta x)\right]}.\end{aligned}$$

It can be easily shown that

$$\mu(0) = \frac{\alpha\left[\theta^3+(\alpha+1)(\alpha+2)(\alpha+3)\right]}{\theta\left[\theta^3+\alpha(\alpha+1)(\alpha+2)\right]} = \mu_1'.$$

The behaviour of $\mu(x)$ of WRD for varying values of parametersθ and α are shown in Figure 5.1.

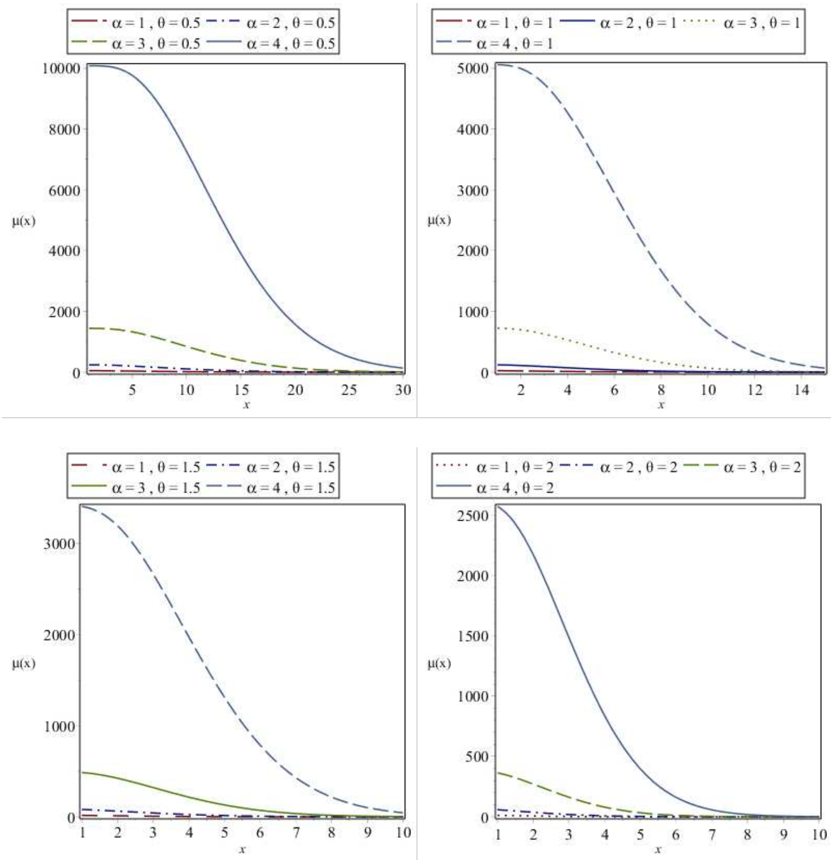

Figure 5.1: Behavior of $\mu(x)$ of the WRD for varying values of parameters θ and α

6. Stochastic Ordering

The stochastic ordering of positive continuous random variables is an important tool for examining their comparative behaviour. A random variable X is said to be smaller than a random variable Y in the

- stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x
- hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x
- mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)$ for all x
- likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following important interpretations due to Shaked and Shanthikumar [10] are well known for establishing the stochastic ordering of distributions.

$$\begin{array}{ccccc} X\leq_{lr}Y & \Rightarrow & X\leq_{hr}Y & \Rightarrow & X\leq_{mrl}Y \\ & & \Downarrow & & \\ & & X\leq_{st}Y. & & \end{array}$$

The WRD is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem.

Theorem 6.1. *Let X˜$WRD(\theta_1,\alpha_1)$ and Y˜$WRD(\theta_2,\alpha_2)$. If $\theta_1 > \theta_2$ and $\alpha_1 = \alpha_2$ (or $\alpha_1 < \alpha_2$ and $\theta_1 = \theta_2$), then $X\leq_{lr}Y$ and hence $X\leq_{hr}Y$, $X\leq_{mrl}Y$ and $X\leq_{st}Y$.*

Proof. We have

$$\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{\theta_1^{\alpha_1+3}\left(\theta_2^3+\alpha_2^3+3\alpha_2^2+2\alpha_2\right)\Gamma(\alpha_2)}{\theta_2^{\alpha_2+3}\left(\theta_1^3+\alpha_1^3+3\alpha_1^2+2\alpha_1\right)\Gamma(\alpha_1)}\, x^{\alpha_1-\alpha_2}\, e^{-(\theta_1-\theta_2)x}.$$

Now,

$$\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \ln\left(\frac{\theta_1^{\alpha_1+3}\left(\theta_2^3+\alpha_2^3+3\alpha_2^2+2\alpha_2\right)\Gamma(\alpha_2)}{\theta_2^{\alpha_2+3}\left(\theta_1^3+\alpha_1^3+3\alpha_1^2+2\alpha_1\right)\Gamma(\alpha_1)}\right)+(\alpha_1-\alpha_2)\ln x-(\theta_1-\theta_2)x.$$

This gives

$$\frac{d}{dx}\ln\left(\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)}\right) = \frac{\alpha_1-\alpha_2}{x}-(\theta_1-\theta_2).$$

Thus, if ($\alpha_1 = \alpha_2$ and $\theta_1 \geq \theta_2$) or ($\alpha_1 < \alpha_2$ *and* $\theta_1 \geq \theta_2$),then $\frac{d}{dx} \ln\left(\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)}\right) < 0$. This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$. This shows flexibility of WRD over Rama distribution. □

7. Maximum Likelihood Estimation of Parameters

Let $(x_1, x_2, x_3, \ldots, x_n)$ be a random sample of size n from WRD (2.1). The likelihood function, L of WRD is given by

$$L = \left[\frac{\theta^{\alpha+3}}{\Gamma(\alpha)\left\{\theta^3 + \alpha^3 + 3\alpha^2 + 2\alpha\right\}}\right]^n \prod_{i=1}^{n} x_i^{\alpha-1}\left(1 + x_i^3\right) e^{-n\,\theta\,\bar{x}}.$$

The natural log likelihood function is thus obtained as

$$\ln L = n\left[(\alpha+3)\ln\theta - \ln\left\{\theta^3 + \alpha(\alpha+1)(\alpha+2)\right\} - \ln\Gamma(\alpha)\right] + (\alpha-1)\sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln\left(1 + x_i^3\right) - n\,\theta\,\bar{x}.$$

The maximum likelihood estimates $(\widehat{\theta},)$ of (θ,α) is the solution of the following log likelihood equations.

$$\frac{\partial \ln L}{\partial \theta} = \frac{n(\alpha+3)}{\theta} - \frac{3n\theta^2}{\theta^3 + \alpha^3 + 3\alpha^2 + 2\alpha} - n\,\bar{x} = 0$$

$$\frac{\partial \ln L}{\partial \alpha} = n\ln\theta - \frac{n\left(3\alpha^2 + 6\alpha + 2\right)}{\theta^3 + \alpha^3 + 3\alpha^2 + 2\alpha} - n\,\psi(\alpha) + \sum_{i=1}^{n}\ln x_i = 0$$

where $\bar{x}$ is the sample mean and $\psi(\alpha)$ is the digamma function defined as

$$\psi(\alpha) = \frac{d}{d\alpha}\ln\Gamma(\alpha).$$

The MLE's $\left(\overset{\wedge}{\theta}, \overset{\wedge}{\alpha}\right)$ of parameters of WRD (θ,α) can be computed directly by solving the natural log likelihood equation using Newton-Raphson iteration available in R-software till sufficiently close estimates of $\overset{\wedge}{\theta}$ and $\overset{\wedge}{\alpha}$ are obtained. In this paper, initial values of θ and α are taken $\theta = 0.5$ and $\alpha = 1.5$, respectively.

8. A Numerical Example

A numerical example of real lifetime data has been presented to test the goodness of fit of WRD over other one parameter and two parameter life time distribution. The following data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm, available in Bader and Priest [9].

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966	1.997
2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490
2.511	2.514	2.535	2.554	2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684
2.697	2.726	2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585			

For this data set, WRD has been fitted along with one parameter exponential, Lindley , Rama distributions and two – parameter gamma distribution Gompertz distribution, generalized exponential distribution (GED) introduced by Gupta and Kundu [6], lognormal distribution ,weighted Sujatha distribution (WSD) introduced by Shanker and Shukla [8] and WLD. The pdf and cdf of gamma, Gompertz, lognormal, GED, WSD and WLD are presented in table 1. The ML estimates, values of $-2\ln L$, Akaike Information criteria (AIC), K-S statistics and p-value of the fitted distributions are presented in table 2. The AIC and K-S Statistics are computed using the following formulae: $AIC = -2\ln L + 2k$ and K-S $= \sup_x |F_n(x) - F_0(x)|$, where k = the number of parameters, n = the sample size , $F_n(x)$ is the empirical (sample) cumulative distribution function, and $F_0(x)$ is the theoretical cumulative distribution function. The best distribution is the distribution corresponding to lower values of $-2\ln L$, AIC, and K-S statistics.

Distributions	pdf	cdf
WSD	$f(x;\theta,\alpha)=\frac{\theta^{\alpha+2}}{\theta^2+\alpha\theta+\alpha(\alpha+1)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}\left(1+x+x^2\right)e^{-\theta x}$	$F(x;\theta,\alpha,\beta)=1-\frac{\left\{\theta^2+\alpha\theta+\alpha(\alpha+1)\right\}\Gamma(\alpha,\theta x)+(\theta x)^{\alpha}(\theta x+\theta+\alpha+1)e^{-\theta x}}{\{\theta^2+\alpha\theta+\alpha(\alpha+1)\}\Gamma(\alpha)}$
WLD	$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{(\theta+\alpha)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(1+x)e^{-\theta x}$	$F(x;\theta,\alpha)=1-\frac{(\theta+\alpha)\Gamma(\alpha,\theta x)+(\theta x)^{\alpha}e^{-\theta x}}{(\theta+\alpha)\Gamma(\alpha)}$
GED	$f(x;\theta,\alpha)=\theta\,\alpha\left(1-e^{-\theta x}\right)^{\alpha-1}e^{-\theta x}$	$F(x;\theta,\alpha)=\left(1-e^{-\theta x}\right)^{\alpha}$
Gamma	$f(x;\theta,\alpha)=\frac{\theta^{\alpha}}{\Gamma(\alpha)}e^{-\theta x}x^{\alpha-1}$	$F(x;\theta,\alpha)=1-\frac{\Gamma(\alpha,\theta x)}{\Gamma(\alpha)}$
Lognormal	$f(x;\theta,\alpha)=\frac{1}{\sqrt{2\pi}\alpha x}e^{-\frac{1}{2}\left(\frac{\log x-\theta}{\alpha}\right)^2}$	$F(x;\theta,\alpha)=\phi\left(\frac{\log x-\theta}{\alpha}\right)$
Lindley	$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$	$F(x;\theta)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}$
Gompertz	$f(x;\theta,\alpha)=\theta\,e^{\alpha x-\frac{\theta}{\alpha}(e^{\alpha x}-1)}$	$F(x;\theta,\alpha)=1-e^{-\frac{\theta}{\alpha}(e^{\alpha x}-1)}$

Table 1: The pdf and the cdf of fitted distributions

Distribution	Team sheet		$-2\ln L$	AIC	K-S	P-value
	$\hat{\theta}$	$\hat{\alpha}$				
WRD	9.5764	20.7494	98.76	102.76	0.055	0.983
WSD	9.7387	22.3612	99.99	103.99	0.057	0.975
WLD	9.6265	22.8938	101.95	105.95	0.059	0.973
Gamma	9.5380	23.3820	100.07	104.07	0.058	0.962
GED	2.0331	87.2847	109.24	113.24	0.087	0.613
Lognormal	0.8751	0.2124	102.72	106.73	0.103	0.713
Gompertz	0.0080	2.0420	107.25	111.250	0.085	0.673
Rama	0.130106	——-	211.49	213.49	0.324	0.000
Lindley	0.0702	——-	238.38	240.38	0.401	0.000
Exponential	0.4079	——-	261.73	263.73	0.447	0.000

Table 2: MLE's, - 2ln L, AIC, K-S Statistics and p-values of the fitted distributions

It is quite obvious from table 2 that WRD is competing well with two parameter lifetime distributions and gives a quite satisfactory fit the considered distributions. This means that, like other two-parameter lifetime distributions, WRD is also an important two-parameter lifetime distribution for modeling real lifetime data.

9. Concluding Remarks

A two-parameter weighted Rama distribution (WRD) which includes one parameter Rama distribution proposed by Shanker [1] has been suggested for modelling lifetime data from engineering. Its statistical properties including shapes of the probability density function for varying values of parameters, coefficients of variation, skewness, kurtosis, and index of dispersion have been studied. Its reliability measures including hazard rate function, mean residual life function, and the stochastic ordering have been discussed. The method of maximum likelihood has been discussed for estimating its parameters. The goodness of fit of the proposed distribution has been explained with a real lifetime data from engineering and the fit has been found quite satisfactory over one parameter exponential, Lindley and Rama distributions and two- parameter gamma, Gompertz, generalized exponential, lognormal, weighted Sujatha and weighted Lindley distributions.

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Hyperbolic Fibonacci Sequence

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Abstract

In this paper, we investigate the hyperbolic Fibonacci sequence and the hyperbolic Fibonacci numbers. Furthermore, we give recurrence relations, the golden ratio and Binet's formula for the hyperbolic Fibonacci sequence and Lorentzian inner product, cross product and mixed product for the hyperbolic Fibonacci vectors.

1. Introduction

For the Fibonacci sequence

$$1\,,\ 1\,,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ldots,F_n,\ldots$$

defined by the recurrence relation

$$F_n=F_{n-1}+F_{n-2}\,,\ \ (n\geq 3),$$

with $F_1=F_2=1$, it is well known that the n-th term of the Fibonacci sequence (F_n) [1]-[3]. Some recent generalizations have produced a variety of new and extended results,[4]-[8].
Hyperbolic numbers have applications in different areas of mathematics and theoretical physics. In particular, they are related to the Lorentz-Minkowski (Space-time) geometry in the plane as well as complex numbers are to Euclidean one [9] . The work on the function theory for hyperbolic numbers can be found in [10]-[15]. The set of hyperbolic numbers $\mathbb{H}$ can be described in the form as

$$\mathbb{H}=\{\,z=x+h\,y\,|\,h\notin\mathbb{R}\,,\ h^2=1\,,\ x,y\in\mathbb{R}\}. \tag{1.1}$$

Addition, substraction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$\begin{aligned} z_1\pm z_2 =&(x_1+h\,y_1)\pm(x_2+h\,y_2)=(x_1\pm x_2)+h\,(y_1\pm y_2),\\ z_1\times z_2 =&(x_1+h\,y_1)\times(x_2+h\,y_2)=x_1\,x_2+y_1\,y_2+h\,(x_1\,y_2+y_1\,x_2). \end{aligned} \tag{1.2}$$

On the other hand, the division of two hyperbolic numbers are given by

$$\begin{aligned} &\frac{z_1}{z_2}=\frac{x_1+h\,y_1}{x_2+h\,y_2}\\ &\frac{(x_1+h\,y_1)(x_2-h\,y_2)}{(x_2+h\,y_2)(x_2-h\,y_2)}=\frac{x_1\,x_2+y_1\,y_2}{x_2^2-y_2^2}+h\,\frac{(x_1\,y_2+y_1\,x_2)}{x_2^2-y_2^2}. \end{aligned} \tag{1.3}$$

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If $x_2^2 - y_2^2 \neq 0$, then the division $\frac{z_1}{z_2}$ is possible. Therefore, the hyperbolic number system is a non-division algebra.
The hyperbolic conjugation of $z = x + hy$ is defined by

$$\bar{z} = z^{\dagger} = x - hy\,,\ \bar{\bar{z}} = z.$$

For any z_1, z_2 hyperbolic numbers, can be written as follows:

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\ \overline{z_1 \times z_2} &= \overline{z_1} \times \overline{z_2}, \\ \|\overrightarrow{z}\|^2 &= z \times \bar{z} = x^2 - y^2. \end{aligned}$$

where z is time-like if $\|\overrightarrow{z}\|^2 > 0$, light-like if $\|\overrightarrow{z}\|^2 = 0$ and space-like if $\|\overrightarrow{z}\|^2 < 0$. The ring of hyperbolic numbers has zero-divisors. Moreover, these zero-divisors are also idempotent elements $\{e, e^{\dagger}\}$ for hyperbolic numbers, given by

$$e = \frac{1+h}{2},\ e^{\dagger} = \frac{1-h}{2}.$$

where $e\,e^{\dagger} = 0$, $e^2 = e$, $(e^{\dagger})^2 = e^{\dagger}$, $e + e^{\dagger} = 1$ and $e - e^{\dagger} = k$. Then, each hyperbolic number z can be written as follows:

$$z = x + hy = (x+y)\,e + (x-y)\,e^{\dagger} = z_1\,e + z_2\,e^{\dagger}.$$

These numbers are also called double, split, perplex, Lorentz and duplex numbers [12].

2. Hyperbolic Fibonacci sequence

The hyperbolic Fibonacci sequence defined by

$$\widetilde{F}_n = F_n + hF_{n+1}\,,\ \ (h^2 = 1) \tag{2.1}$$

with $\widetilde{F}_1 = 1 + h$, $\widetilde{F}_2 = 1 + 2h$ where $h^2 = 1$. That is, the hyperbolic Fibonacci sequence $\widetilde{F}_n$ is

$$h,\ 1+h,\ 1+2h,\ 2+3h,\ 3+5h, \ldots, (1+h)F_n + hF_{n-1}, \ldots \tag{2.2}$$

Using the equations (2.1) and (2.2) , it was obtained

$$\begin{aligned} &\widetilde{F}_{n+1} = (1+h)\,F_{n+1} + h\,F_n \\ &\widetilde{F}_{n+2} = (1+2h)\,F_{n+1} + (1+h)\,F_n \\ &\widetilde{F}_{n+3} = (2+3h)\,F_{n+1} + (1+2h)\,F_n \\ &\vdots \\ &\widetilde{F}_{n+r} = (F_n + hF_{n+1})\,F_{r+1} + (F_{n-1} + hF_n)\,F_r \\ &\vdots \end{aligned} \tag{2.3}$$

For the hyperbolic Fibonacci sequence, it was obtained the following properties:

$$\begin{aligned} &\widetilde{F}_{n+1}^2 + \widetilde{F}_n^2 = 2\,\widetilde{F}_{2n+1} + F_{2n+2}\,, \\ &\widetilde{F}_{n+1}^2 - \widetilde{F}_{n-1}^2 = 2\,\widetilde{F}_{2n} + F_{2n+1}\,, \\ &\widetilde{F}_{n+r} = \widetilde{F}_n F_{r+1} + \widetilde{F}_{n-1} F_r \quad (n \geq 3) \\ &\widetilde{F}_{n-1}\,\widetilde{F}_{n+1} - \widetilde{F}_n^2 = h\,(-1)^n\,, \\ &\widetilde{F}_{n-r}\,\widetilde{F}_{n+r} - \widetilde{F}_n^2 = h\,(-1)^{n-r+1}\,F_r^2\,, \\ &\widetilde{F}_n^2 + hF_{n+1}^2 = \widetilde{F}_{2n+1}\,, \\ &\widetilde{F}_n\,\widetilde{F}_m + \widetilde{F}_{n+1}\,\widetilde{F}_{m+1} = 2\,\widetilde{F}_{n+m+1} + F_{n+m+2}\,, \\ &\widetilde{F}_n\,\widetilde{F}_{m+1} - \widetilde{F}_{n+1}\,\widetilde{F}_m = h\,(-1)^m\,F_{n-m}\,, \\ &\tfrac{\widetilde{F}_{n+r} + (-1)^r\,\widetilde{F}_{n-r}}{\widetilde{F}_n} = L_r = F_{r+1} + (-1)^r F_{r-1}\,. \end{aligned}$$

Theorem 2.1. *If $\widetilde{F}_n$ is the hyperbolic Fibonacci number, then*

$$\lim_{n\to\infty} \tfrac{\widetilde{F}_{n+1}}{\widetilde{F}_n} = \tfrac{\alpha^2}{\alpha^2 - 1}$$

where $\alpha = (1+\sqrt{5})\big/2 = 1.618033..$ is the golden ratio.

Proof. We have for the Fibonacci number F_n,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$$

where $\alpha = (1+\sqrt{5})\big/2 = 1.618033..$ is the golden ratio [3].

Then, using this limit value for the hyperbolic Fibonacci number $\widetilde{F}_n$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{\widetilde{F}_{n+1}}{\widetilde{F}_n}=\lim_{n\to\infty}\frac{F_{n+1}+hF_{n+2}}{F_n+hF_{n+1}} &= \lim_{n\to\infty}\frac{(F_{n+1}+hF_{n+2})(F_n-hF_{n+1})}{F_n^2-F_{n+1}{}^2}\\ &\lim_{n\to\infty}\frac{F_{n+1}(F_n-F_{n+2})+h(F_nF_{n+2}-F_{n+1}^2)}{(F_n^2-F_{n+1}^2)}\\ &\lim_{n\to\infty}\frac{-F_{n+1}^2}{F_n^2-F_{n+1}^2}+h\lim_{n\to\infty}\frac{(-1)^{n+1}}{F_n^2-F_{n+1}^2}\\ &=\frac{-\alpha^2}{1-\alpha^2}+0\\ &=\frac{\alpha^2}{\alpha^2-1}\end{aligned}$$

where the identities $F_{n+2}=F_n+F_{n+1}$ and $F_{n+1}F_{n-1}-F_n{}^2=(-1)^n$ are used. □

Theorem 2.2. *The Binet formula* [1] *for the hyperbolic Fibonacci sequence is as follows;*

$$\widetilde{F}_n=\frac{1}{\alpha-\beta}\left(\bar{\alpha}\,\alpha^n-\bar{\beta}\,\beta^n\right).$$

Proof. If we use definition of the hyperbolic Fibonacci sequence and substitute first equation in footnote, then we get

$$\begin{aligned}\widetilde{F}_n &= F_n+hF_{n+1}\\ &=\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+h\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)\\ &=\frac{\alpha^n(1+h\alpha)-\beta^n(1+h\beta)}{\alpha-\beta}\\ &=\frac{\overline{\alpha}\,\alpha^n-\overline{\beta}\,\beta^n}{\alpha-\beta}\end{aligned}$$

where $\overline{\alpha}=1+h\alpha$ and $\overline{\beta}=1+h\beta$. □

3. Hyperbolic Fibonacci vectors

Let $\overrightarrow{z_1}=(x_1,x_2,x_3)$ and $\overrightarrow{z_2}=(y_1,y_2,y_3)$ be vectors in $\mathbb{R}^3$. The Lorentzian inner product of z_1 and z_2 is defined as [16]

$$z_1.z_2=\langle\overrightarrow{z_1},\overrightarrow{z_2}\rangle_L=x_1y_1+x_2y_2-x_3y_3.$$

This space denote by $\mathbb{L}^{2,1}$ or *Lorentz* $3-$*space* $\mathbb{L}^3$.
A hyperbolic Fibonacci vector is defined by

$$\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}=(\widetilde{\mathrm{F}}_n,\widetilde{\mathrm{F}}_{n+1},\widetilde{\mathrm{F}}_{n+2})$$

Also, from equations (2.1) and (2.3) it can be expressed as

$$\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}=\overrightarrow{\mathrm{F}}_{\mathrm{n}}+h\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}$$

where $\overrightarrow{\mathrm{F}}_{\mathrm{n}}=(F_n,F_{n+1},F_{n+2})$ and $\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}=(F_{n+1},F_{n+2},F_{n+3})$ are the hyperbolic Fibonacci vectors.
The product of the hyperbolic Fibonacci vector $\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}$ and the scalar $\lambda\in\mathbb{R}$ is given by

$$\lambda\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}=\lambda\overrightarrow{\mathrm{F}}_{\mathrm{n}}+h\lambda\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}$$

and $\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}$ and $\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{m}}$ are equal if and only if

$$\begin{aligned}F_n &= F_m\\ F_{n+1} &= F_{m+1}\\ F_{n+2} &= F_{m+2}.\end{aligned}$$

[1] Binet formula is the explicit formula to obtain the n-th Fibonacci and Lucas numbers. It is well known that for the Fibonacci and Lucas numbers, Binet formulas are

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

and

$$L_n=\alpha^n+\beta^n$$

respectively, where $\alpha+\beta=1$, $\alpha-\beta=\sqrt{5}$, $\alpha\beta=-1$ and $\alpha=(1+\sqrt{5})\big/2$, $\beta=(1-\sqrt{5})\big/2$,[7],[8].

Some examples of the hyperbolic Fibonacci vectors can be given easily as;

$$\begin{aligned}\overrightarrow{\widetilde{\mathrm{F}}_1} &= (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3)\\ &= (F_1, F_2, F_3) + h(F_2, F_3, F_4)\\ &= (1+h, 1+2h, 2+3h)\end{aligned}$$

$$\begin{aligned}\overrightarrow{\widetilde{\mathrm{F}}_2} &= (\widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_4)\\ &= (F_2, F_3, F_4) + h(F_3, F_4, F_5)\\ &= (1+2h, 2+3h, 3+5h)\end{aligned}$$

Theorem 3.1. *Let $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{m}}$ be two hyperbolic Fibonacci vectors. The Lorentzian inner product of $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{m}}$ is given by*

$$\left\langle \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}, \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{m}} \right\rangle_L = (F_{n+m+1} - F_{n+m+4}) + h(3F_{n+m+2} + 2F_{n+m+3} - F_{n+1}F_{m+1}). \tag{3.1}$$

Proof. The Lorentzian inner product of $\overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n} = (\widetilde{F}_n, \widetilde{F}_{n+1}, \widetilde{F}_{n+2})$ and $\overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{m} = (\widetilde{F}_m, \widetilde{F}_{m+1}, \widetilde{F}_{m+2})$ defined by

$$\begin{aligned}\left\langle \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}, \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{m}} \right\rangle_L &= \widetilde{F}_n\widetilde{F}_m + \widetilde{F}_{n+1}\widetilde{F}_{m+1} - \widetilde{F}_{n+2}\widetilde{F}_{m+2}\\ &= \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{m} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{m+1} \right\rangle\\ &\quad + h[\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{m+1} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{m} \right\rangle]\end{aligned}$$

where $\overrightarrow{\mathrm{F}_\mathrm{n}} = (F_n, F_{n+1}, F_{n+2})$ is the hyperbolic Fibonacci vector. Also, the equations (1.1), (1.2) and (1.3), we obtain

$$\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{m} \right\rangle = F_nF_m + F_{n+1}F_{m+1} - F_{n+2}F_{m+2} \tag{3.2}$$

$$\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{m+1} \right\rangle = F_{n+1}F_{m+1} + F_{n+2}F_{m+2} - F_{n+3}F_{m+3} \tag{3.3}$$

$$\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{m+1} \right\rangle = F_nF_{m+1} + F_{n+1}F_{m+2} - F_{n+2}F_{m+3} \tag{3.4}$$

and

$$\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{m} \right\rangle = F_{n+1}F_m + F_{n+2}F_{m+1} - F_{n+3}F_{m+2} \tag{3.5}$$

Then from equation (3.2), (3.3), (3.4) and (3.5), we have the equation (3.1). □

Special Case-1: For the Lorentzian inner product of the hyperbolic Fibonacci vectors $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}$ and $\overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n+1}$, we get

$$\begin{aligned}\left\langle \overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n}, \overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n+1} \right\rangle_L &= \widetilde{F}_n\widetilde{F}_{n+1} + \widetilde{F}_{n+1}\widetilde{F}_{n+2} - \widetilde{F}_{n+2}\widetilde{F}_{n+3}\\ &= \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{n+1} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{n+2} \right\rangle\\ &\quad + h[\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{n+2} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{n+1} \right\rangle]\\ &= (F_{2n+2} - F_{2n+5}) + h(2F_{2n+3} + F_{2n+5} - F_{n+2}F_{n+3})\end{aligned}$$

and

$$\begin{aligned}\left\langle \overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n}, \overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n} \right\rangle_L &= \widetilde{F}_n^2 + \widetilde{F}_{n+1}^2 - \widetilde{F}_{n+2}^2\\ &= \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{n} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{n+1} \right\rangle + h\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{n+1} \right\rangle\\ &= (F_{2n+1} - F_{2n+4}) + 2h(F_{2n+2} - F_{n+2}F_{n+3}).\end{aligned}$$

Then for the Lorentzian inner product of the hyperbolic vector [2], we have, using identities of the Fibonacci numbers

$$\begin{aligned}F_n^2 + F_{n+1}^2 &= F_{2n+1}\\ F_{n+3}^2 - F_{n+1}^2 &= F_{2n+2}\\ F_nF_m + F_{n+1}F_{m+1} &= F_{n+m+1}\end{aligned}$$

(see, [11]), we have

$$\begin{aligned}\left\|\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}\right\|^2 &= \left\langle \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}, \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}} \right\rangle_L = \widetilde{F}_n^2 + \widetilde{F}_{n+1}^2 - \widetilde{F}_{n+2}^2\\ &= (F_{2n+1} - F_{2n+2}) + 2h(F_{2n+2} - 2F_{n+2}F_{n+3}).\end{aligned}$$

[2] Lorentzian inner product of hyperbolic number as follows:

$$\langle \overrightarrow{z}, \overrightarrow{z} \rangle_L = {x_1}^2 + {x_2}^2 - {x_3}^2, \text{ [11]}.$$

The Lorentzian cross product [16],[17] of the vectors $\overrightarrow{z_1}$ and $\overrightarrow{z_2}$ in $\mathbb{L}^3$ is

$$\begin{aligned}\overrightarrow{z_1}\times_L\overrightarrow{z_2} &= \begin{vmatrix}-i & -j & k\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}\\ &= -i(x_2y_3-x_3y_2)+j(x_1y_3-x_3y_1)+k(x_1y_2-x_2y_1).\end{aligned}$$

Theorem 3.2. *Let $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}$ be two hyperbolic Fibonacci vectors. The Lorentzian cross product of $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}$ is given by*

$$\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}\times_L\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}=h(-1)^mF_{n-m}(i+j+k). \tag{3.6}$$

Proof. The Lorentzian cross product of $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}=\overrightarrow{\mathrm{F}}_{\mathrm{n}}+h\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}=\overrightarrow{\mathrm{F}}_{\mathrm{m}}+h\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}$ defined by

$$\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}\times\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}=(\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}})+(\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}})+h(\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}+\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}$$

where $\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}$ is the hyperbolic Fibonacci vector and $\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}$ is the Lorentzian cross product for the hyperbolic Fibonacci vectors $\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}$ and $\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{m}}$,

Now, we calculate the cross products $\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}$, $\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}$, $\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}$ and $\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}$: Using the property $F_mF_{n+1}-F_{m+1}F_n=(-1)^nF_{m-n}$, we get

$$\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}=(-1)^mF_{n-m}(i+j+k) \tag{3.7}$$

$$\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}=(-1)^{m+1}F_{n-m}(i+j+k) \tag{3.8}$$

$$\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}=(-1)^{m+1}F_{n-m-1}(i+j+k) \tag{3.9}$$

and

$$\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}=(-1)^mF_{n-m+1}(i+j+k). \tag{3.10}$$

Then from the equations (3.7), (3.8), (3.9) and (3.10), we obtain the equation (3.6). □

Theorem 3.3. *Let $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}$, $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_{\ell}}$ be the hyperbolic Fibonacci vectors. The Lorentzian mixed product of these vectors is*

$$\left\langle\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}\times_L\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}\,,\,\overrightarrow{\widetilde{\mathrm{F}}_{\ell}}\right\rangle=0. \tag{3.11}$$

Proof. Using the properties

$$\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}\times\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}=(\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}})+(\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}})+h(\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}+\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}})$$

and

$$\overrightarrow{\widetilde{\mathrm{F}}}_{\ell}=\overrightarrow{\mathrm{F}}_{\ell}+h\overrightarrow{\mathrm{F}}_{\ell+1}$$

we can write,

$$\begin{aligned}\left\langle\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}\times_L\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{m}},\overrightarrow{\widetilde{\mathrm{F}}}_{\ell}\right\rangle &=\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}},\overrightarrow{\mathrm{F}}_{\ell}\right\rangle+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}},\overrightarrow{\mathrm{F}}_{\ell}\right\rangle\\ &+h[\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}},\overrightarrow{\mathrm{F}}_{\ell+1}\right\rangle+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}},\overrightarrow{\mathrm{F}}_{\ell+1}\right\rangle]\\ &+h[\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}},\overrightarrow{\mathrm{F}}_{\ell}\right\rangle+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}},\overrightarrow{\mathrm{F}}_{\ell}\right\rangle]\\ &+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}},\overrightarrow{\mathrm{F}}_{\ell+1}\right\rangle+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}},\overrightarrow{\mathrm{F}}_{\ell+1}\right\rangle.\end{aligned}$$

Then from equations (3.7), (3.8), (3.9)and (3.10) and by using the Lorentzian inner product definition of the hyperbolic number, we obtain

$$\left\langle(i+j+k),\overrightarrow{\mathrm{F}_{\ell}}\right\rangle=F_{\ell}+F_{\ell+1}-F_{\ell+2}=0,$$

$$\left\langle(i+j+k),\overrightarrow{\mathrm{F}_{\ell+1}}\right\rangle=F_{\ell+1}+F_{\ell+2}-F_{\ell+3}=0.$$

Thus, we have the equation (3.11). □

4. Conclusion

The hyperbolic Fibonacci sequence defined by

$$\widetilde{F}_n = F_n + hF_{n+1}\ ,\ \ (h^2 = 1),$$

with $\widetilde{F}_1 = 1 + h\,,\ \widetilde{F}_2 = 1 + 2h$ where $h^2 = 1,$.

In addition, limit for the hyperbolic Fibonacci sequence and Binet's formula for the hyperbolic Fibonacci sequence is given. Furthermore, vectors and the Lorentzian inner product, cross product and mixed product of these vectors are given.

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Hyperspaces of Superparacompact Spaces and Continuous Maps

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Abstract

In the present paper we establish that the space $\exp_\beta X$ of compact subsets of a Tychonoff space X is superparacompact iff X is so. Further, we prove the Tychonoff map $\exp_\beta f : \exp_\beta X \to \exp_\beta Y$ is superparacompact iff a given map $f : X \to Y$ is superparacompact.

1. Introduction

In the present paper under space we mean a topological T_1-space, under compact a Hausdorff compact space and under map a continuous map.

A collection ω of subsets of a set X is said [1] to be *star-countable* (respectively, *star-finite*) if each element of ω intersects at most a countable (respectively, finite) set of elements of ω. A collection ω of subsets of a set X *refines* a collection Ω of subsets of X if for each element $A \in \omega$ there is an element $B \in \Omega$ such that $A \subset B$. They also say that ω is *a refinement* of Ω.

A finite sequence of subsets $M_0, \ldots, M_s$ of a set X is [2] a *chain* connecting sets M_0 and M_s, if $M_{i-1} \cap M_i \neq \varnothing$ for $i = 1, \ldots, s$. A collection ω of subsets of a set X is said to be *connected* if for any pair of sets $M, M' \subset X$ there exists a chain in ω connecting sets M and M'. The maximal connected subcollections of ω are called *components* of ω. A star-finite open cover of a space X is said to be *a finite-component cover* if the number of elements of each component is finite. A space X is said to be *superparacompact* if every open cover of X has a finite-component cover which refines it.

Note that any compact space is superparacompact, and any superparacompact space is strongly paracompact. Infinite discrete space is superparacompact, but it is not compact. Real line is strongly paracompact, but it is not superparacompact.

For a collection $\omega = \{O_\alpha : \alpha \in A\}$ of subsets of a space X we suppose $[\omega] = [\omega]_X = \{[O_\alpha]_X : \alpha \in A\}$. For a space X, its some subspace W and a set $B \subset X \setminus W$ they say [2] that an open cover λ of the space W picks out the set B in X if $B \cap (\cup[\lambda]_X) = \varnothing$.

The following criterion plays a key role in investigation the class of superparacompact spaces.

Theorem 1.1. *[3] A Tychonoff space X is superparacompact iff for every closed set F in βX lying in the growth $\beta X \setminus X$ there exists a finite-component cover λ of X pricking out F in βX (i. e. $F \cap (\cup[\lambda]_{\beta X}) = \varnothing$).*

D.Buhagiar and T.Miwa offered the following criterion of superparacompactness.

Theorem 1.2. *[4] A Tychonoff space X is superparacompact iff for every closed set F in perfect compactification bX lying in the growth $bX \setminus X$ there is a finite-component cover λ of X pricking out F in bX (i. e. $F \cap (\cup[\lambda]_{bX}) = \varnothing$).*

Let us recall a notion of the perfect compactification. For a topological space X and its subset A a set $Fr_X A = [A]_X \cap [X \setminus A]_X = [A]_X \setminus Int_X A$ is called [5] a boundary of A.

Let vX be a compact extension of a Tychonoff space X. If $H \subset X$ is an open set in X, then by $O(H)$ (or by $O_{vX}(H)$) we denote a maximal open set in vX satisfying $O_{vX}(H) \cap X = H$. It is easy to see that

$$O_{vX}(H) = \bigcup_{\substack{\Gamma \in \tau_{vX}, \\ \Gamma \cap X = H}} \Gamma,$$

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where τ_{vX} is the topology of the space vX.
A compactification vX of a Tychonoff space X is called *perfect with respect to an open set* H in X, if the equality $[Fr_X H]_{vX} = Fr_{vX} O_{vX}(X)$ holds. If vX is perfect with respect to every open set in X, then it is called *a perfect compactification* of the space X ([1], P. 232). A compactification vX of a space X is perfect iff for any two disjoint open sets U_1 and U_2 in X the equality $O(U_1 \bigcup U_2) = O(U_1) \bigcup O(U_2)$ is carried out. The Stone-Cěch compactification βX of X is perfect. The equality $O(U_1 \bigcup U_2) = O(U_1) \bigcup O(U_2)$ is satisfied for every pair of open sets U_1 and U_2 in X iff X is normal, and the compactification vX coincides with the Stone-Cěch compactification βX, i. e. $vX \cong \beta X$.
Let X be a space. By $\exp X$ we denote a set of all nonempty closed subsets of X. A family of sets of the view

$$O\langle U_1, ..., U_n\rangle = \{F \in \exp X :\ F \subset \bigcup_{i=1}^{n} U_n, F \cap U_1 \neq \varnothing, ..., F \cap U_n \neq \varnothing\}$$

forms a base of a topology on $\exp X$, where $U_1, \ldots, U_n$ are open nonempty sets in X. This topology is called *the Vietoris topology*. A space $\exp X$ equipped with Vietoris topology is called *hyperspace* of X. For a compact space X its hyperspace $\exp X$ is also a compact space (for details, see [6], [7], [8]).
Note for any space X it is well known that

$$[O\langle U_1, ..., U_n\rangle]_{\exp X} = O\left\langle [U_1]_X, ..., [U_n]_X \right\rangle.$$

Let $f : X \to Y$ be continuous map of compacts, $F \in \exp X$. We put

$$(\exp f)(F) = f(F).$$

This equality defines a map $\exp f : \exp X \to \exp Y$. For a continuous map f the map $\exp f$ is continuous. Really, it follows from the formula

$$(\exp f)^{-1} O\langle U_1, ..., U_m\rangle = O\langle f^{-1}(U_1), ..., f^{-1}(U_m)\rangle$$

what one can check directly. Note that if $f : X \to Y$ is an epimorphism, then $\exp f$ is also an epimorphism.
For a Tychonoff space X we put

$$\exp_\beta X = \{F \in \exp \beta X : F \subset X\}.$$

It is clear, that $\exp_\beta X \subset \exp X$. Consider the set $\exp_\beta X$ as a subspace of the space $\exp X$. For a Tychonoff spaces X the space $\exp_\beta X$ is also a Tychonoff space with respect to the induced topology.
For a continuous map $f : X \to Y$ of Tychonoff spaces we put

$$\exp_\beta f = (\exp \beta f)|_{\exp_\beta X},$$

where $\beta f : \beta X \to \beta Y$ is the Stone-Cěch compactification [5] of f (it is unique).
As it is well-known the action of functors on various categories of topological spaces and their continuous maps is one of the main problems of theory of covariant functors, in the present paper we investigate the action of the functor exp (the construction of taking of a hyperspace of a given space) on superparacompact spaces (section 2) and superparacompact maps (section 3).

2. Hyperspace of superparacompact spaces

It is well known that for a Tychonoff space X the set $\exp_\beta X$ is everywhere dense in $\exp \beta X$, i. e. $\exp \beta X$ is a compactification of the space $\exp_\beta X$. We claim $\exp \beta X$ is a perfect compactification of $\exp_\beta X$. At first we will prove the following technical statement.

Lemma 2.1. *Let γX be a compact extension of a space X and, V and W be disjoint open sets in γX. Let $V^X = X \cap V$ and $W^X = X \cap W$. Then the following equality is true:*

$$[X \setminus V^X]_{\gamma X} \cap [X \setminus W^X]_{\gamma X} = [X \setminus (V^X \cup W^X)]_{\gamma X}.$$

Proof. It is clear that $[X \setminus V^X]_{\gamma X} \cap [X \setminus W^X]_{\gamma X} \supset [X \setminus (V^X \cup W^X)]_{\gamma X}$. Let $x \in [X \setminus V^X]_{\gamma X} \cap [X \setminus W^X]_{\gamma X}$. Then each open neighbourhood Ox in γX of x intersects with the sets $X \setminus V^X$ and $X \setminus W^X$. Hence, $Ox \not\subset V^X$ and $Ox \not\subset W^X$. Therefore, since $V^X \cap W^X = \varnothing$, we have $Ox \not\subset V^X \cup W^X$, i. e. $Ox \cap X \setminus (V^X \cup W^X) \neq \varnothing$. By virtue of arbitrariness of the neighbourhood Ox we conclude that $x \in [X \setminus (V^X \cup W^X)]_{\gamma X}$. □

Theorem 2.2. *For a Tychonoff space X the space $\exp \beta X$ is a perfect compactification of the space $\exp_\beta X$.*

Proof. It is enough to consider basic open sets. Let U_1 and U_2 be disjoint open sets in X. Since βX is perfect compactification of X we have $O_{\beta X}(U_1 \cup U_2) = O_{\beta X}(U_1) \cup O_{\beta X}(U_2)$. Consider open sets

$$O\langle U_i\rangle = \{F : F \in \exp_\beta X, F \subset U_i\}, \qquad i = 1,\ 2$$

in $\exp_\beta X$. It is clear, that $O\langle U_1\rangle \cap O\langle U_2\rangle = \varnothing$. We will show that

$$O_{\exp \beta X}(O\langle U_1\rangle \cup O\langle U_2\rangle) = O_{\exp \beta X}(O\langle U_1\rangle) \cup O_{\exp \beta X}(O\langle U_2\rangle).$$

The inclusion $\supset$ follows from the definition of the set $O(H)$ (see [1], P. 234). That is why it is enough to show the inverse inclusion. Let $\Phi \subset \beta X$ be a closed set such that $\Phi \notin O_{\exp \beta X}(O\langle U_1\rangle) \cup O_{\exp \beta X}(O\langle U_2\rangle)$. Then $\Phi \in \exp \beta X \setminus O_{\exp \beta X}(O\langle U_i\rangle)$, $i = 1,\ 2$. From [1] (see, P. 234) we have

$$\exp \beta X \setminus O_{\exp \beta X}(O\langle U_i\rangle) = [\exp_\beta X \setminus O\langle U_i\rangle]_{\exp \beta X}, \qquad i = 1,\ 2.$$

Hence $\Phi \in [\exp_\beta X \setminus O\langle U_i\rangle]_{\exp\beta X}$, $i=1,\ 2$. Since $O\langle U_1\rangle \cap O\langle U_2\rangle = \varnothing$ by Lemma 2.1 we have

$$[\exp_\beta X \setminus O\langle U_1\rangle]_{\exp\beta X} \cap [\exp_\beta X \setminus O\langle U_2\rangle]_{\exp\beta X} = [\exp_\beta X \setminus O(\langle U_1\rangle \cup O\langle U_2\rangle)]_{\exp\beta X}.$$

Therefore, $\Phi \in [\exp_\beta X \setminus O_{\exp\beta X}(O\langle U_1\rangle \cup O\langle U_2\rangle)]_{\exp\beta X}$, what is equivalent $\Phi \in \exp\beta X \setminus O_{\exp\beta X}(\langle U_1\rangle \cup \langle U_2\rangle)$ (see [1], P. 234). In other words, $\Phi \notin O_{\exp\beta X}(\langle U_1\rangle \cup \langle U_2\rangle)$. Thus, we have established that inclusion $O_{\exp\beta X}(\langle U_1\rangle \cup \langle U_2\rangle) \subset O_{\exp\beta X}(O\langle U_1\rangle) \cup O_{\exp\beta X}(O\langle U_2\rangle)$ is also fair. □

Lemma 2.3. *Let $U_1,\dots,\ U_n;\ V_1,\dots,\ V_m$ be open subsets of a space X. Then $O\langle U_1,\dots,\ U_n\rangle \cap O\langle V_1,\dots,\ V_m\rangle \neq \varnothing$ iff for each $i\in\{1,\dots,n\}$ and for each $j\in\{1,\dots,m\}$ there exists, respectively $j(i)\in\{1,\dots,m\}$ and $i(j)\in\{1,\dots,n\}$, such that $U_i\cap V_{j(i)}\neq\varnothing$ and $U_{i(j)}\cap V_j\neq\varnothing$.*

Proof. Assume that for every $i\in\{1,\dots,n\}$ there exists $j(i)\in\{1,\dots,m\}$ such that $U_i\cap V_{j(i)}\neq\varnothing$ and for every $j\in\{1,\dots,m\}$ there exists $i(j)\in\{1,\dots,n\}$ such that $U_{i(j)}\cap V_j\neq\varnothing$. For any pair $(i,\ j)\in\{1,\dots,n\}\times\{1,\dots,m\}$ for which $U_i\cap V_j\neq\varnothing$, choose a point $x_{ij}\in U_i\cap V_j$ and make a closed set F consisting of these points. Then $F\subset\bigcup\limits_{i=1}^{n}U_i$ and $F\subset\bigcup\limits_{j=1}^{m}V_j$. Besides, $F\cap U_i\neq\varnothing$, $i=1,\dots,\ n$, and $F\cap V_j\neq\varnothing$, $j=1,\dots,\ m$. Therefore, $F\in O\langle U_1,\dots,\ U_n\rangle\cap O\langle V_1,\dots,\ V_m\rangle$.
Suppose there exists $i_0\in\{1,\dots,n\}$ such that $U_{i_0}\cap V_j=\varnothing$ for all $j\in\{1,\dots,m\}$. Then $U_{i_0}\cap\bigcup\limits_{j=1}^{m}V_j=\varnothing$ and for each $F\in O\langle U_1,\ \dots,\ U_n\rangle$ we have $F\not\subset\bigcup\limits_{j=1}^{m}V_j$. Hence, $F\notin O\langle V_1,\ \dots,\ V_m\rangle$. Similarly, every $\Gamma\in O\langle V_1,\ \dots,\ V_m\rangle$ lies in $\bigcup\limits_{j=1}^{m}V_j$ what implies $\Gamma\cap U_{i_0}=\varnothing$. From here $\Gamma\notin O\langle U_1,\ \dots,\ U_n\rangle$. Thus, $O\langle U_1,\ \dots,\ U_n\rangle\cap O\langle V_1,\ \dots,\ V_m\rangle=\varnothing$. □

Lemma 2.4. *Let υ be a finite-component cover of a Tychonoff space X. Then the family $\exp_\beta\upsilon=\{O\langle U_1,\dots,\ U_n\rangle : U_i\in\upsilon,\ i=1,\dots,\ n;\ n\in\mathbb{N}\}$ is a finite-component cover of the space $\exp_\beta X$.*

Proof. Let $O\langle G_1,\dots,\ G_k\rangle$ be an element of $\exp_\beta\upsilon$. Each $G_i\in\upsilon$ intersects with finite elements of υ. Let $|\{\alpha:\ G_i\cap U_\alpha\neq\varnothing,\ U_\alpha\in\upsilon\}|=n_i$, $i=1,\ 2,\dots,\ k$. Denote $\gamma=\{G_i\cap U_j:\ G_i\cap U_j\neq\varnothing,\ i=1,\ 2,\dots,\ k,\ U_j\in\upsilon\}$. Then $|\gamma|\leq n_1\cdot\ldots\cdot n_k$. Therefore, the set $O\langle G_1,\dots,\ G_k\rangle$ crosses not more then $\prod\limits_{i=1}^{k}n_i$ elements of $\exp_\beta\upsilon$. It means that the collection $\exp_\beta\upsilon$ is star-finite.
Let $F\in\exp_\beta X$. There is a subfamily $\upsilon_F\subset\upsilon$ such that $F\subset\bigcup\limits_{U\in\upsilon_F}U$. From a cover $\{F\cap U:\ U\in\upsilon_F,\ F\cap U\neq\varnothing\}$ of the compact F it is possible to allocate a finite subcover $\{F\cap U_i:\ i=1,\dots,m\}$. We have $F\in O\langle U_1,\dots,\ U_m\rangle$. So, the family $\exp_\beta\upsilon$ is a cover of $\exp_\beta X$. On the other hand by the definition of Vietoris topology the cover $\exp_\beta\upsilon$ is open. Thus, $\exp_\beta\upsilon$ is a star-finite open cover of $\exp_\beta X$.
We will show now that all components of the $\exp_\beta\upsilon$ are finite.
Let $M=O\langle G_1,\dots,G_s\rangle$ and $M^{'}=O\langle G_1^{'},\dots,\ G_t^{'}\rangle$ be arbitrary elements of $\exp_\beta\upsilon$. Further, let $\gamma_{G_iG_j^{'}}=\{U_l^{ij}:\ l=1,\ 2,\dots,\ n_{ij}\}$ be the maximal chain of υ connecting G_i and $G_j^{'}$, $i=1,\ 2,\dots,\ s;\ j=1,\ 2,\dots,\ t$. By definition these sets satisfy the following properties:

(1) $U_1^{ij}=G_i,\qquad i=1,\dots,s;\ j=1,\dots,t;$
(2) $U_{n_{ij}}^{ij}=G_j^{'},\qquad i=1,\dots,s;\ j=1,\dots,t;$
(3) $U_l^{ij}\cap U_{l+1}^{ij}\neq\varnothing,\qquad l=1,\dots,n_{ij}-1;\ i=1,\dots,s;\ j=1,\dots,t.$

If $s<t$ we have $O\langle G_1,\dots,G_s\rangle=O\langle U_1^{1j},\dots,U_1^{sj},\ U_1^{i_1(s+1)},\dots,U_1^{i_{t-s}t}\rangle$, where $j=1,\dots,t$ and $i_1,\dots,i_{t-s}\in\{1,\dots,s\}$. Further, $O\langle G_1^{'},\dots,G_t^{'}\rangle=O\langle U_{n_{i1}}^{i1},\dots,U_{n_{it}}^{it}\rangle$, $i=1,\dots,s$. Thus, the cover $\exp_\beta\upsilon$ has a chain connecting the given sets $M=O\langle G_1,\dots,\ G_s\rangle$ and $M^{'}=O\langle G_1^{'},\dots,\ G_t^{'}\rangle$. The case $s>t$ is analogously.
Now using Lemma 2.1 and calculating directly we find that each maximal chain of $\exp_\beta\upsilon$ connecting the sets $M=O\langle G_1,\dots,\ G_s\rangle$ and $M^{'}=O\langle G_1^{'},\dots,\ G_t^{'}\rangle$ has no more than $\prod\limits_{\substack{i=1,\\ j=1}}^{\substack{t\\ s}}n_{ij}$ elements. Thus, all components of $\exp_\beta\upsilon$ is finite. □

Theorem 2.5. *For a Tychonoff space X its hyperspace $\exp_\beta X$ is superparacompact iff X is superparacompact.*

Proof. As the superparacompactness is inherited to the closed subsets [2], the superparacompactness of $\exp_\beta X$ implies superparacompactness of the closed subset $X\subset\exp_\beta X$.
Let Ω be an open cover of $\exp_\beta X$. For each element $G\in\Omega$ there exists $O_G\langle U_1,\dots,U_n\rangle$ such that $O_G\langle U_1,\dots,U_n\rangle\subset G$, where $U_1,\dots,U_n$ are open sets in X. We can choose sets $G\in\Omega$ so that a collection of sets $O_G\langle U_1,\dots,\ U_n\rangle$ forms a cover of $\exp_\beta X$, what we denote by $\Omega^{'}$. It is easy to see that a collection $\omega^{'}=\bigcup\limits_{O_G\langle U_1,\dots,U_n\rangle\in\Omega^{'}}\{U_1,\dots,\ U_n\}$ is an open cover of X. There exists a finite-component cover ω of X which refines $\omega^{'}$. Then by Lemma 2.4 the collection

$$\exp_\beta\omega=\{O\langle V_1,\dots,\ V_k\rangle:\ V_i\in\omega,\ i=1,\dots,\ n;\ n\in\mathbb{N}\}$$

is a finite-component cover of $\exp_\beta X$ and it refines Ω. □

3. Superparacompactness of the map $\exp_\beta f$

For a continuous map $f:(X,\tau_X)\to(Y,\tau_Y)$ and $O\in\tau_Y$ a preimage $f^{-1}O$ is called *a tube* (above O). Remind, a continuous map $f:\ X\to Y$ is called [2] a T_0-*map*, if for each pair of distinct points $x,x^{'}\in X$, such that $f(x)=f(x^{'})$, at least one of these points has an open neighbourhood in X which does not contain another point. A continuous map $f:\ X\to Y$ is called *totally regular*, if for each point $x\in X$ and every closed set F in X not containing x there exists an open neighbourhood O of $f(x)$ such that in the tube $f^{-1}O$ the sets $\{x\}$ and F are functional separable. Totally regular T_0-map is said to be *a Tychonoff map*.
Obviously, each continuous map $f:\ X\to Y$ of a Tychonoff space X into a topological space Y is a Tychonoff map. In this case owing to the set $\exp_\beta X$ is a Tychonoff space concerning to Vietoris topology for every Tychonoff space X, the map $\exp_\beta f:\ \exp_\beta X\to\exp_\beta Y$ is a Tychonoff map.
A continuous, closed map $f:X\to Y$ is said to be *compact* if the preimage $f^{-1}y$ of each point $y\in Y$ is compact. A continuous map $f:X\to Y$ is compact iff for each point $y\in Y$ and every cover ω of the fibre $f^{-1}y$, consisting of open sets in X, there is an open neighbourhood O of y in Y such that the tube $f^{-1}O$ can be covered with a finite subfamily of ω.
A compact map $bf:\ b_fX\to Y$ is said to be *a compactification* of a continuous map $f:X\to Y$ if X is everywhere dense in b_fX and $bf|_X=f$. On the set of all compactifications of the map f it is possible to introduce a partial order: for the compactifications $b_1f:\ b_{1f}X\to Y$ and $b_2f:\ b_{2f}X\to Y$ of f we put $b_1f\le b_2f$ if there is a natural map of $b_{2f}X$ onto $b_{1f}X$. B. A. Pasynkov showed that for each Tychonoff map $f:X\to Y$ there exists its maximal compactification $g:Z\to Y$, which he denoted by βf, and the space Z where this maximal compactification defines by β_fX. To within homeomorphism for a given Tychonoff map f its maximal compactification βf is unique.

Remark 3.1. *Note that the maps b_1f, b_2f, βf are compactifications of the map f. The spaces $b_{1f}X$, b_{2f}, β_fX are some extensions of X but they are not obliged to be compactifications.*

A Tychonoff map $f:X\to Y$ is said to be *superparacompact*, if for every closed set F in β_fX lying in the growth $\beta_fX\setminus X$ there exists a finite-component cover λ of X pricking out F in β_fX (i. e. $F\cap(\cup[\lambda]_{\beta_fX})=\varnothing$) [3].
It is easy to see that one can define superparacompactness of a map as follows: a map $f:X\to Y$ is superparacompact if for each $y\in Y$ and every open cover Υ of $f^{-1}y$ in X there exists an open neighbourhood O of y in Y such that Υ has a finite-component cover υ of $f^{-1}O$ in X which refines Υ.

Definition 3.2. *A compactification $bf:b_fX\to Y$ of a Tychonoff map $f:X\to Y$ is said to be perfect compactification of f if for each point $y\in Y$ and for every disjoint open sets U_1 and U_2 in X there exists an open neighbourhood $O\subset Y$ of y such that the equality*

$$O_{b_fX}(U_1\cup U_2)\cap bf^{-1}O=\Big(O_{b_fX}(U_1)\cup O_{b_fX}(U_2)\Big)\cap bf^{-1}O$$

holds.

Let $f:\ X\to Y$ be a continuous map of a Tychonoff space X into a space Y. It is well known there exists a compactification vX of X such that f has a continuous extension $vf:\ vX\to Y$ on vX. It is clear, vf is a perfect compactification of f.
The following result is an analog of Theorem 1.2 for a case of maps.

Theorem 3.3. *Let $bf:\ b_fX\to Y$ be a perfect compactification of a Tychonoff map $f:X\ \to Y$. The map f is superparacompact iff for every closed set F in b_fX lying in the growth $b_fX\setminus X$ there exists a finite-component cover λ of X pricking out the set F in b_fX.*

Proof. The proof is carried out similar to the proof of Theorem 1.1 Π from [2]. □

Evidently a restriction $f|_\Phi:\ \Phi\to Y$ of a superparacompact map $f:\ X\to Y$ on the closed subset $\Phi\subset X$ is a superparacompact map.
The following result is a variant of Theorem 2.2 for a case of maps.

Theorem 3.4. *Let $f:\ X\to Y$ be a Tychonoff map. Then the map $\exp_\beta\beta f:\exp_\beta\beta_fX\to\exp_\beta Y$ is a perfect compactification of $\exp_\beta f:\ \exp_\beta X\ \to\exp_\beta Y$.*

Proof. The proof is similar to the proof of Theorem 2.2. Here the equality

$$(\exp_\beta\beta f)^{-1}O\langle U_1,...,U_m\rangle=O\langle\beta f^{-1}(U_1),...,\beta f^{-1}(U_m)\rangle$$

is used. □

The following statement is the main result of this section.

Theorem 3.5. *The Tychonoff map $\exp_\beta f:\ \exp_\beta X\to\exp_\beta Y$ is superparacompact iff a map $f:\ X\to Y$ is superparacompact.*

Proof. Let $\exp_\beta f:\ \exp_\beta X\to\exp_\beta Y$ be a superparacompact map. It implies that $f:\ X\to Y$ is a superparacompact map since $X\cong\exp_1X$ is closed set in $\exp_\beta X$.
Let now $f:\ X\to Y$ be a superparacompact map. Consider arbitrary $\Gamma\in\exp_\beta Y$ and an open cover Ω of $(\exp_\beta f)^{-1}(\Gamma)=\{F\in\exp_\beta X: f(F)=\Gamma\}$ in $\exp_\beta X$. For each element $G\in\Omega$ there exists $O_G\langle U_1,\dots,U_n\rangle$ such that $O_G\langle U_1,\dots,U_n\rangle\subset G$, where $U_1,\dots,U_n$ are open sets in X. We can choose sets $G\in\Omega$ so that a collection of sets $O_G\langle U_1,\dots,\ U_n\rangle$ forms a cover of $(\exp_\beta f)^{-1}(\Gamma)$, what we denote by $\Omega^{'}$. It is easy to see that a collection $\omega^{'}=\bigcup\limits_{O_G\langle U_1,\dots,U_n\rangle\in\Omega'}\{U_1,\dots,\ U_n\}$ is an open cover of $f^{-1}\Gamma$ in X. For each $y\in\Gamma$ there exists an open neighbourhood O_y of y in Y such that the collection $\omega_y=\{U\cap f^{-1}O_y:\ U\in\omega^{'}\}$ is an open cover of $f^{-1}y$ in X and ω_y has a finite-component cover $\omega_y^{'}$ of $f^{-1}O_y$ in X which refines ω_y. Gather such O_y and construct an open cover $\{O_y:y\in\Gamma\}$ of Γ in Y. Since $\Gamma\in\exp_\beta Y$ by construction of hyperspace, Γ is a compact subset of Y. Consequently, there exists a finite open subcover $\gamma=\{O_{y_1},\dots,O_{y_n}\}$ in Y, which covers Γ. Put

$\omega = \bigcup\limits_{O_{y_i}\in\gamma} \omega'_{y_i}$. Then ω is an open cover of $f^{-1}\left(\bigcup\limits_{U\in\omega} U\right)$ in X. By the construction ω is a finite-component cover, and it refines ω'. Hence, $\exp_\beta \omega$ is a finite-component cover of $(\exp_\beta f)^{-1}O\langle O_{y_1},\dots,O_{y_n}\rangle = \langle f^{-1}O_{y_1},\dots,f^{-1}O_{y_n}\rangle$ in $\exp_\beta X$ and it refines Ω.
So, for each $\Gamma \in \exp_\beta Y$ and every open cover Ω of $(\exp_\beta f)^{-1}\Gamma$ in $\exp_\beta X$ there exists an open neighbourhood $O\langle O_{y_1},\dots,O_{y_n}\rangle$ of Γ in $\exp_\beta Y$ such that Ω has a finite-component cover $\exp_\beta \omega$ of $(\exp_\beta f)^{-1}O\langle O_{y_1},\dots,O_{y_n}\rangle$ in $\exp_\beta X$ which refines Ω. Thus, the map $\exp_\beta f:\ \exp_\beta X \to \exp_\beta Y$ is superparacompact. $\square$

Corollary 3.6. *Let $f:\ X\to Y$ be a superparacompact map and Φ be a closed set in $\exp_\beta \beta_f X$ such that $\Phi\subset \exp_\beta \beta_f X\setminus \exp_\beta X$. Then there exists a finite-component cover Ω of $\exp_\beta X$ pricking out Φ in $\exp_\beta \beta_f X$ (i. e. $\Phi\cap(\cup[\Omega]_{\exp_\beta \beta_f X}) = \varnothing$).*

Corollary 3.7. *The functor $\exp_\beta$ lifts onto category of superparacompact spaces and their continuous maps.*

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Exponential Growth of Solutions for Nonlinear Coupled Viscoelastic Wave Equations

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Abstract

In this work, we consider an initial-boundary value problem related to the nonlinear coupled viscoelastic equations

$$\begin{cases} |u_t|^j u_{tt} - \Delta u_{tt} - div\left(|\nabla u|^{\alpha-2} \nabla u\right) - \Delta u + \int\limits_0^t g(t-s)\Delta u ds + |u_t|^{m-1} u_t = f_1(u,v), \\ |v_t|^j v_{tt} - \Delta v_{tt} - div\left(|\nabla v|^{\beta-2} \nabla v\right) - \Delta v + \int\limits_0^t h(t-s)\Delta v ds + |v_t|^{r-1} v_t = f_2(u,v). \end{cases}$$

We will show the exponential growth of solutions with positive initial energy.

1. Introduction

In this work we consider the following coupled system of viscoelastic wave equations:

$$\begin{cases} |u_t|^j u_{tt} - \Delta u_{tt} - div\left(|\nabla u|^{\alpha-2} \nabla u\right) - \Delta u + \int\limits_0^t g(t-s)\Delta u ds + |u_t|^{m-1} u_t = f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ |v_t|^j v_{tt} - \Delta v_{tt} - div\left(|\nabla v|^{\beta-2} \nabla v\right) - \Delta v + \int\limits_0^t h(t-s)\Delta v ds + |v_t|^{r-1} v_t = f_2(u,v), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n $(n = 1,2,3)$ with smooth boundary $\partial\Omega$, the constants $j > 0$, $\alpha \geq 2$, $\beta \geq 2$, $m \geq 1$, $r \geq 1$. Here, $f_1(u,v)$ and $f_2(u,v)$ are nonlinear functions defined as

$$\begin{cases} f_1(u,v) = a|u+v|^{2(p+1)}(u+v) + b|u|^p u|v|^{p+2}, \\ f_2(u,v) = a|u+v|^{2(p+1)}(u+v) + b|v|^p v|u|^{p+2} \end{cases} \tag{1.2}$$

in which the constants $a > 0$, $b > 0$, and p satisfies

$$\begin{cases} p > -1, & n = 1,2, \\ -1 < p \leq 1, & n = 3. \end{cases} \tag{1.3}$$

Let

$$f_1(u,v) = \frac{\partial F(u,v)}{\partial u} \text{ and } f_2(u,v) = \frac{\partial F(u,v)}{\partial v},$$

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where

$$F(u,v)=\frac{1}{2(p+2)}\left[a|u+v|^{2(p+2)}(u+v)+2b|uv|^{p+2}\right].$$

There are two positive constants $c_0,\ c_1$ such that

$$c_0\left(|u|^{2(r+2)}+|v|^{2(r+2)}\right)\leq 2(r+2)F(u,v)\leq c_1\left(|u|^{2(r+2)}+|v|^{2(r+2)}\right).$$

As a special case, for $\alpha=\beta=2$, the system (1.1) becomes the following system

$$\begin{cases} |u_t|^j u_{tt}-\Delta u_{tt}-\Delta u+\int\limits_0^t g(t-s)\Delta u ds+|u_t|^{m-1}u_t=f_1(u,v), \\ |v_t|^j v_{tt}-\Delta v_{tt}-\Delta v+\int\limits_0^t h(t-s)\Delta v ds+|v_t|^{r-1}v_t=f_2(u,v). \end{cases} \tag{1.4}$$

Liu [1] proved decay of the solutions for system (1.4) under some appropriate functions f_1 and f_2. Later, Said-Houari [2] studied exponential growth of the solutions for system (1.4). When $j=0$ and without the $\Delta u_{tt},\ \Delta v_{tt}$ terms, the system (1.4) has been investigated by some authors and results concerning local and global existence, blow up, decay of the solutions were obtained [3, 4, 5, 6, 7, 8]. Hao et al. [9] considered global nonexistence of the solution of (1.1), with negative initial energy.
Motivated by the above papers, in this work we prove the exponential growth of solutions for the problem (1.1), with positive initial energy. This work is organized as follows: In section 2, we present some lemmas and notations needed later of this paper. In section 3, exponential growth of the solution is proved.

2. Preliminaries

In this part, we give some assumptions and lemmas which will be used throughout this paper. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.
Now, we make the following assumptions on the C^1-nonnegative and nonincreasing relaxation functions g and h :

$$1-\int\limits_0^\infty g(s)ds=l>0,\ \ 1-\int\limits_0^\infty h(s)ds=k>0 \tag{2.1}$$

and $\forall s\geq 0$

$$g'(s)\leq 0,\ \ h'(s)\leq 0. \tag{2.2}$$

Let us define

$$\begin{aligned} I(t)=I(u,v)&=\left(1-\int\limits_0^t g(s)ds\right)\|\nabla u\|^2+\left(1-\int\limits_0^t h(s)ds\right)\|\nabla v\|^2 \\ &-2(p+2)\int\limits_\Omega F(u,v)dx+(g\circ\nabla u+h\circ\nabla v)+\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha+\frac{1}{\beta}\|\nabla v\|_\beta^\beta, \end{aligned} \tag{2.3}$$

$$\begin{aligned} J(t)=J(u,v)&=\frac{1}{2}\left(1-\int\limits_0^t g(s)ds\right)\|\nabla u\|^2+\frac{1}{2}\left(1-\int\limits_0^t h(s)ds\right)\|\nabla v\|^2 \\ &-\int\limits_\Omega F(u,v)dx+\frac{1}{2}(g\circ\nabla u+h\circ\nabla v)+\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha+\frac{1}{\beta}\|\nabla v\|_\beta^\beta \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} E(t)&=\frac{1}{j+2}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)+\frac{1}{2}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right) \\ &+\frac{1}{2}\left(1-\int\limits_0^t g(s)ds\right)\|\nabla u\|^2+\frac{1}{2}\left(1-\int\limits_0^t h(s)ds\right)\|\nabla v\|^2 \\ &-\int\limits_\Omega F(u,v)dx+\frac{1}{2}(g\circ\nabla u+h\circ\nabla v)+\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha+\frac{1}{\beta}\|\nabla v\|_\beta^\beta \end{aligned} \tag{2.5}$$

where

$$(\phi\circ\psi)(t)=\int\limits_0^t\phi(t-\tau)\int\limits_\Omega|\psi(t)-\psi(\tau)|^2dxd\tau.$$

Lemma 2.1. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$\begin{aligned} E'(t) &= -\left(\|u_t\|_{m+1}^{m+1} + \|v_t\|_{m+1}^{m+1}\right) + \frac{1}{2}\left(g' \circ \nabla u + h' \circ \nabla v\right) \\ &\quad - \frac{1}{2}\left(g(t)\|\nabla u\|^2 + h(t)\|\nabla v\|^2\right) \\ &\leq 0. \end{aligned} \tag{2.6}$$

Proof. Multiplying the first and second equation of (1.1) by u_t and v_t, respectively, integrating over $\Omega \times [0,t]$, then adding them together and integrating by parts, we obtain (2.6). □

3. Exponential growth of solutions

In this part, we are going to consider the exponential growth of the solution for the problem (1.1).
Firstly, we give following two lemmas.

Lemma 3.1. *[10, 11]. Suppose that (1.3) holds. Let (u,v) for $\eta > 0$*

$$\begin{aligned} \|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \leq \eta \Big[& \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha + \frac{1}{\beta}\|\nabla v\|_\beta^\beta \\ & + I_1\|\nabla u\|^2 + I_2\|\nabla v\|^2 \Big]^{p+2}, \end{aligned}$$

where

$$I_1 = \int_{\Omega_1} |u_t|\left(|u|^{2p+3} + |v|^{2p+3} + |u|^{p+1}|v|^{p+2}\right)dx,$$

$$I_2 = \int_{\Omega_2} |v_t|\left(|u|^{2p+3} + |v|^{2p+3} + |u|^{p+2}|v|^{p+1}\right)dx$$

and

$$\begin{aligned} \Omega_1 &= \{(x,t) : |u(x,t)| \leq 1,\ |v(x,t)| \leq 1\}, \\ \Omega_2 &= \{(x,t) : |u(x,t)| \leq 1,\ |v(x,t)| \geq 1\}. \end{aligned}$$

Lemma 3.2. *[10, 11]. Suppose that (1.3) holds. Let (u,v) be the solution of problem (1.1). Assume further that $E(0) < E_1$ and*

$$\left[\frac{1}{\alpha}\|\nabla u_0\|_\alpha^\alpha + \frac{1}{\beta}\|\nabla v_0\|_\beta^\beta + I(0)\right]^{\frac{1}{2}} > \alpha_1.$$

Then, there exists a constant $\alpha_2 > \alpha_1$ such that

$$\left[\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha + \frac{1}{\beta}\|\nabla v\|_\beta^\beta + I(t)\right]^{\frac{1}{2}} > \alpha_2,$$

$$\left(\|u+v\|_{2(p+2)}^{2(p+2)} + \|uv\|_{p+2}^{p+2}\right)^{\frac{1}{2(p+2)}} > B\alpha_2,$$

for all $t \in (0,T)$, where

$$B = \eta^{\frac{1}{2(p+2)}},\ \alpha_1 = B^{-\frac{p+2}{p+1}},\ E_1 = \left(\frac{1}{2} - \frac{1}{2(p+2)}\right)\alpha_1^2.$$

Theorem 3.3. *Suppose that (1.3) holds. Assume further that*

$$\max\{j+2, m+1, r+1\} < 2(p+2),$$

$$E(0) < E_1$$

and (2.1), (2.2) hold. There exist constant γ such that

$$\max\{\alpha,\beta\} < \gamma < 2(p+2)$$

and

$$\min\{l,k\} > \frac{1/(2\gamma)}{(\gamma/2) - 1 + 1/(2\gamma)}.$$

Then, any solution of (1.1) grows exponentially.

Proof. We define the functional

$$H(t) = E_1 - E(t). \tag{3.1}$$

From (2.1), (2.5) and Lemma 3.2, we have

$$\begin{aligned}
0 < H(0) &\le H(t) \\
&\le E_1 - E(t) \\
&= E_1 - \frac{1}{j+2}\left(\|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2}\right) - \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
&- \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 - \frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right)\|\nabla v\|^2 \\
&+ \int_\Omega F(u,v)\,dx - \frac{1}{2}(g\circ\nabla u + h\circ\nabla v) - \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha - \frac{1}{\beta}\|\nabla v\|_\beta^\beta \\
&< E_1 - \frac{1}{2}\alpha_2^2 + \frac{1}{2(p+2)}\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right) \\
&< \frac{C_1}{2(p+2)}\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right).
\end{aligned} \tag{3.2}$$

Let us define the functional

$$L(t) = H(t) + \frac{\varepsilon}{j+1}\int_\Omega \left(|u_t|^j u_t u + |v_t|^j v_t v\right)dx - \varepsilon\int_\Omega (\Delta u u_t + \Delta v v_t)\,dx, \tag{3.3}$$

where ε is a small positive constants to be determined later.
By differentiating with respect to t and using (3.3) and (1.1), we have

$$\begin{aligned}
L'(t) &= H'(t) + \varepsilon\int_\Omega \left[\left(|u_t|^j u_{tt} u + |v_t|^j v_{tt} v\right) + \frac{1}{j+1}\left(|u_t|^{j+2} + |v_t|^{j+2}\right)\right]dx \\
&+ \varepsilon\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) - \varepsilon\int_\Omega (u\Delta u_{tt} + v\Delta v_{tt})\,dx \\
&= H'(t) + \frac{\varepsilon}{j+1}\left(\|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2}\right) - \varepsilon\int_\Omega \left(u|u_t|^{m-1}u_t + v|v_t|^{r-1}v_t\right)dx \\
&+ \varepsilon\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) - \varepsilon\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) - \varepsilon\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\beta^\beta\right) \\
&+ 2\varepsilon(p+2)\int_\Omega F(u,v)dx + \varepsilon\left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \varepsilon\left(\int_0^t h(s)\,ds\right)\|\nabla v\|^2 \\
&+ \varepsilon\int_0^t g(t-s)\int_\Omega \nabla u[\nabla u(s) - \nabla u(t)]\,dxds \\
&+ \varepsilon\int_0^t h(t-s)\int_\Omega \nabla v[\nabla v(s) - \nabla v(t)]\,dxds.
\end{aligned} \tag{3.4}$$

Using Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned}
\int_0^t g(t-s)\int_\Omega \nabla u[\nabla u(s) - \nabla u(t)]\,dxds &\le \int_0^t g(t-s)\left(\int_\Omega |\nabla u(t)|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u(s) - \nabla u(t)|^2 dx\right)^{\frac{1}{2}} ds \\
&\le \int_0^t g(t-s)\|\nabla u(t)\|\,\|\nabla u(s) - \nabla u(t)\|\,ds \\
&\le \int_0^t g(t-s)\left(\lambda\|\nabla u(s) - \nabla u(t)\|^2 + \frac{1}{4\lambda}\|\nabla u(t)\|^2\right)ds \\
&\le \lambda\int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds + \frac{1}{4\lambda}\int_0^t g(t-s)\|\nabla u(t)\|^2 ds \\
&\le \lambda(g\circ\nabla u) + \frac{1}{4\lambda}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2.
\end{aligned} \tag{3.5}$$

Similarly, we obtain

$$\int_0^t h(t-s)\int_\Omega \nabla v[\nabla v(s)-\nabla v(t)]\,dxds \le \lambda(h\circ\nabla v)+\frac{1}{4\lambda}\left(\int_0^t h(s)\,ds\right)\|\nabla v(t)\|^2. \tag{3.6}$$

Inserting (3.5) and (3.6) into (3.4), we have

$$\begin{aligned}
L'(t) &\ge H'(t)+\frac{\varepsilon}{j+1}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)-\varepsilon\int_\Omega\left(u|u_t|^{m-1}u_t+v|v_t|^{r-1}v_t\right)dx\\
&+\varepsilon\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)-\varepsilon\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-\varepsilon\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\beta^\beta\right)\\
&+2\varepsilon(p+2)\int_\Omega F(u,v)dx+\varepsilon\left(\int_\Omega g(s)\,ds\right)\|\nabla u\|^2+\varepsilon\left(\int_\Omega h(s)\,ds\right)\|\nabla v\|^2\\
&+\varepsilon\lambda(g\circ\nabla u+h\circ\nabla v)+\frac{\varepsilon}{4\lambda}\left[\left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\left(\int_0^t h(s)\,ds\right)\|\nabla v\|^2\right].
\end{aligned} \tag{3.7}$$

By the definition of $E(t)$ and (3.1), we obtain

$$\begin{aligned}
\int_\Omega F(u,v)\,dx &= H(t)-E_1+\frac{1}{j+2}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)+\frac{1}{2}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\
&+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\frac{1}{2}\left(1-\int_0^t h(s)\,ds\right)\|\nabla v\|^2\\
&+\frac{1}{2}(g\circ\nabla u+h\circ\nabla v)+\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha+\frac{1}{\beta}\|\nabla v\|_\beta^\beta.
\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we get

$$\begin{aligned}
L'(t) &\ge H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)-\varepsilon\int_\Omega\left(u|u_t|^{m-1}u_t+v|v_t|^{r-1}v_t\right)dx\\
&+\varepsilon\left(1+\frac{\gamma}{2}\right)\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)+\gamma\varepsilon H(t)-\varepsilon\gamma E_1+\varepsilon(2(p+2)-\gamma)\int_\Omega F(u,v)\,dx\\
&+\varepsilon\left[\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty g(s)\,ds\right]\|\nabla u\|^2\\
&+\varepsilon\left[\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty h(s)\,ds\right]\|\nabla v\|^2\\
&+\varepsilon\left(\frac{\gamma}{2}-\lambda\right)(g\circ\nabla u+h\circ\nabla v)+\varepsilon\left(\frac{\gamma}{\alpha}-1\right)\|\nabla u\|_\alpha^\alpha+\varepsilon\left(\frac{\gamma}{\beta}-1\right)\|\nabla v\|_\beta^\beta.
\end{aligned} \tag{3.9}$$

By using the Young's inequality, we get

$$\begin{aligned}
\int_\Omega |u_t|^{m-1}u_t u\,dx &\le \frac{\delta_1^{m+1}}{m+1}\|u\|_{m+1}^{m+1}+\frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}\|u_t\|_{m+1}^{m+1}\\
&\le \frac{\delta_1}{m+1}\|u\|_{m+1}^{m+1}+\frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}H'(t)
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\int_\Omega |v_t|^{r-1}v_t v\,dx &\le \frac{\delta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1}+\frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}\|v_t\|_{r+1}^{r+1}\\
&\le \frac{\delta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1}+\frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}H'(t).
\end{aligned} \tag{3.11}$$

Since $L^{2(p+2)}(\Omega)\hookrightarrow L^{m+1}(\Omega)$ and $L^{2(p+2)}(\Omega)\hookrightarrow L^{r+1}(\Omega)$, we have

$$\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^m\|u\|_{m+1}^{m+1}\le C_2\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^{m+\frac{m+1}{2(p+2)}} \tag{3.12}$$

and

$$\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^r\|v\|_{r+1}^{r+1}\le C_3\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^{r+\frac{r+1}{2(p+2)}}. \tag{3.13}$$

We use the following algebraic inequality

$$z^{v}\leq z+1\leq\left(1+\frac{1}{a}\right)(z+a),\quad \forall z\geq 0,\ 0<v\leq 1, \tag{3.14}$$

we obtain, for $t\geq 0$,

$$\begin{aligned}\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^{m+\frac{m+1}{2(p+2)}} &\leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}+H(0)\right)\\ &\leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}+H(t)\right)\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^{r+\frac{r+1}{2(p+2)}} &\leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}+H(0)\right)\\ &\leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}+H(t)\right)\end{aligned} \tag{3.16}$$

for $d=1+\frac{1}{H(0)}$.
By (3.9)-(3.13),(3.15) and (3.16), we have

$$\begin{aligned}L'(t)\geq&\left(1+\frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}+\frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}\right)H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)\\ &-\left(\frac{\delta_1^{m+1}c_2d}{m+1}+\frac{\delta_2^{r+1}c_3d}{r+1}\right)\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)\\ &+\varepsilon\left(\gamma-\left(\frac{\delta_1^{m+1}c_2d}{m+1}+\frac{\delta_2^{r+1}c_3d}{r+1}\right)\right)H(t)\\ &+\varepsilon\left(1+\frac{\gamma}{2}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon\left(2(p+2)-\gamma(p+2)\right)\int_\Omega F(u,v)\,dx\\ &+\varepsilon\left[\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty g(s)\,ds\right]\|\nabla u\|^2\\ &+\varepsilon\left(\frac{\gamma}{2}-\lambda\right)(g\circ\nabla u+h\circ\nabla v)\\ &+\varepsilon\left[\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty h(s)\,ds\right]\|\nabla v\|^2\\ &+\varepsilon\left(\frac{\gamma}{\alpha}-1\right)\|\nabla u\|_\alpha^\alpha+\varepsilon\left(\frac{\gamma}{\beta}-1\right)\|\nabla v\|_\beta^\beta.\end{aligned}$$

By use (3.2) and since

$$\min\left\{\frac{\gamma}{\alpha}-1,\ \frac{\gamma}{\beta}-1\right\}>0$$

and

$$1+\frac{\gamma}{2}>0$$

we obtain

$$\begin{aligned}L'(t)\geq& MH'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)+\varepsilon(\gamma-K_1)H(t)\\ &+\varepsilon K_2\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\beta^\beta\right)+\varepsilon K_3\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\ &+\varepsilon\left(\frac{\gamma}{2}-\lambda\right)(g\circ\nabla u+h\circ\nabla v)+\varepsilon\left(1+\frac{\gamma}{2}\right)\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\ &+\varepsilon\left(\frac{(2(p+2)-\gamma(p+2))C_1}{2(p+2)}-K_1\right)\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right),\end{aligned}$$

where

$$\begin{aligned}M&=1+\frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}+\frac{r\delta_2^{-\frac{r+1}{r}}}{r+1},\\ K_1&=\frac{\delta_1^{m+1}c_2d}{m+1}+\frac{\delta_2^{r+1}c_3d}{r+1},\\ K_2&=\min\left\{\frac{\gamma}{\alpha}-1,\frac{\gamma}{\beta}-1\right\}\end{aligned}$$

and

$$K_3=\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\max\left(\int_0^\infty g(s)\,ds,\int_0^\infty h(s)\,ds\right).$$

Choose δ_1,δ_2 appropriate such that

$$b_1=\gamma-K_1>0,\;\; b_2=\frac{(2(p+2)-\gamma(p+2))C_1}{2(p+2)}-K_1>0 \text{ and } M>0.$$

Then, we can find positive constants b_1 and b_2 such that

$$\begin{aligned}L'(t)&\geq MH'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)\\&+\varepsilon K_2\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\beta^\beta\right)+\varepsilon\left(1+\frac{\gamma}{2}\right)\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\&+\varepsilon b_1H(t)+\varepsilon b_2\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)\geq 0.\end{aligned}$$

Because of $H'(t)\geq 0$, there exists constants $t>0$ such that

$$\begin{aligned}L'(t)&\geq \tilde{K}(H(t)+\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}+\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\beta^\beta\\&+\|\nabla u_t\|^2+\|\nabla v_t\|^2+\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)})\geq 0\end{aligned}\tag{3.17}$$

where $\tilde{K}=\min\left\{\varepsilon b_1,\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right),\varepsilon K_2,\varepsilon\left(1+\frac{\gamma}{2}\right),\varepsilon b_2\right\}$.
On the other hand, we can choose ε smaller so that

$$L(0)=H(0)+\varepsilon\int_\Omega(u_0u_1+v_0v_1)\,dx>0.\tag{3.18}$$

Furthermore, we have

$$L(t)\geq L(0),\;\; t\geq 0.\tag{3.19}$$

Next we estimate $L(t)$. Using Young's inequality, we obtain

$$\left|\int_\Omega |u_t|^{j+1}u\,dx\right|\leq\frac{\mu_1^{j+2}}{j+2}\|u\|_{j+2}^{j+2}+\frac{(j+1)\,\mu_1^{-\frac{j+2}{j+1}}}{j+2}\|u_t\|_{j+2}^{j+2},\;\forall\mu_1>0.\tag{3.20}$$

Next, using the embedding $L^{2(p+2)}(\Omega)\hookrightarrow L^{j+2}(\Omega)$, the estimate (3.20) becomes

$$\begin{aligned}\left|\int_\Omega |u_t|^{j+1}u\,dx\right|&\leq C\left(\|u\|_{2(p+2)}^{j+2}+\|u_t\|_{j+2}^{j+2}\right)\\&\leq C\left(\left(\|u\|_{2(p+2)}^{2(p+2)}\right)^{\frac{j+2}{2(p+2)}}+\|u_t\|_{j+2}^{j+2}\right).\end{aligned}$$

Since $2(p+2)>j+2$ and $H(t)>H(0)$, use the inequality (3.14), we have

$$\begin{aligned}\left|\int_\Omega |u_t|^{j+1}u\,dx\right|&\leq C\left[\left(1+\frac{1}{H(0)}\right)\left(\|u\|_{2(p+2)}^{2(p+2)}+H(0)\right)+\|u_t\|_{j+2}^{j+2}\right]\\&\leq C\left[\left(1+\frac{1}{H(0)}\right)\left(\|u\|_{2(p+2)}^{2(p+2)}+H(t)\right)+\|u_t\|_{j+2}^{j+2}\right].\end{aligned}\tag{3.21}$$

Similarly, we have

$$\left|\int_\Omega |v_t|^{j+1}v\,dx\right|\leq C\left[\left(1+\frac{1}{H(0)}\right)\left(\|v\|_{2(p+2)}^{2(p+2)}+H(t)\right)+\|v_t\|_{j+2}^{j+2}\right].\tag{3.22}$$

By Green identity and Hölder's inequality, we get

$$\begin{aligned}-\int_\Omega u_t\Delta u\,dx&=\int_\Omega\nabla u\nabla u_t\,dx\\&\leq\left(\int_\Omega(\nabla u)^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega(\nabla u_t)^2\,dx\right)^{\frac{1}{2}}\\&=\|\nabla u\|\,\|\nabla u_t\|,\end{aligned}\tag{3.23}$$

similarly

$$-\int_{\Omega} v_t \Delta v dx \leq \|\nabla v\| \|\nabla v_t\|. \tag{3.24}$$

Next, using the embedding $L^{\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ and $L^{\beta}(\Omega) \hookrightarrow L^2(\Omega)$ the estimate (3.23) and (3.24) becomes

$$\begin{cases} \|\nabla u\| \|\nabla u_t\| \leq C \|\nabla u\|_{\alpha} \|\nabla u_t\|, \\ \|\nabla v\| \|\nabla v_t\| \leq C \|\nabla v\|_{\beta} \|\nabla v_t\|. \end{cases} \tag{3.25}$$

By Young's inequality (3.25), we get

$$\begin{aligned} \|\nabla u\|_{\alpha} \|\nabla u_t\| &\leq \frac{1}{2}\left(\|\nabla u\|_{\alpha}^2 + \|\nabla u_t\|^2\right), \\ \|\nabla v\|_{\beta} \|\nabla v_t\| &\leq \frac{1}{2}\left(\|\nabla v\|_{\alpha}^2 + \|\nabla v_t\|^2\right). \end{aligned} \tag{3.26}$$

Since $\alpha \geq 2$, $\beta \geq 2$ and $H(t) > H(0)$, the inequality (3.14) yields

$$\begin{aligned} \|\nabla u\|_{\alpha}^2 &= \left(\|\nabla u\|_{\alpha}^{\alpha}\right)^{\frac{2}{\alpha}} \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|\nabla u\|_{\alpha}^{\alpha} + H(0)\right) \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right) \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \|\nabla v\|_{\beta}^2 &= \left(\|\nabla v\|_{\beta}^{\beta}\right)^{\frac{2}{\beta}} \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|\nabla v\|_{\beta}^{\beta} + H(0)\right) \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|\nabla v\|_{\beta}^{\beta} + H(t)\right). \end{aligned} \tag{3.28}$$

Combining (3.20)-(3.28), we have

$$\begin{aligned} &\left|\frac{\varepsilon}{j+1}\int_{\Omega}\left(|u_t|^j u_t u + |v_t|^j v_t v\right) dx - \varepsilon \int_{\Omega} (\Delta u u_t + \Delta v v_t)\, dx\right| \\ &\leq \mu(H(t) + \|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2} + \|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\beta}^{\beta} + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \\ &+ \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} L(t) \leq{}& C^*(H(t) + \|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2} + \|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\beta}^{\beta} + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \\ &+ \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}). \end{aligned} \tag{3.29}$$

A combination of (3.17) and (3.29) yields

$$L(t) \leq C^* L'(t) \text{ for all } t \geq 0, \tag{3.30}$$

where C^* is a some positive constants. Integrating the differential inequality (3.30) between 0 and t gives the following estimate for $L(t)$,

$$L(t) \geq L(0)\, e^{t/C^*}.$$

This completes the proof. □

4. Conclusion

In this paper, we obtained a exponential growth of solutions for a nonlinear coupled viscoelastic wave equations with nonlinear damping terms. This improves and extends many results in the literature such as (Houari [2], Pişkin [5]).

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Statistical Convergence of Nets Through Directed Sets

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Abstract

The concept of statistical convergence based on asymptotic density is introduced in this article through nets. Some possible extensions of classical results for statistical convergence of sequences are obtained in this article, with extensions to nets.

1. Introduction

The concept of statistical convergence was introduced independently by H. Fast [10] and by H. Steinhaus in [34] as an applicable concept that generalizes the classical concept of usual convergence. This convergence was studied for sequences of numbers in [11, 12, 31], for sequences of elements in uniform spaces in [4, 21], for sequences of elements in paranormed spaces in [2, 14], for sequences of elements in topological groups in [6], for sequences of elements in metric spaces in [3], for sequences of elements in topological vector spaces in [20], and for sequences of elements in topological vector lattices in [1]. There are articles [26, 27], which study statistical convergence of double sequences and generalized sequences. There are generalizations of this concept through ideals in the articles [15, 16, 17, 32]. Almost all applicable statistical convergence ideas depend on asymptotic densities of sets. These sets may be subsets of N, $N\times N$, $N\times N\times N$, ... , where N represents the set of all natural numbers. So, if the concept of asymptotic density for subsets of directed sets is introduced, then the concept of statistical convergence for nets can be introduced. This is done in the present article. For this purpose, a natural restriction is made on directed sets. The restriction is the following:
For the directed sets $(D,\leq)$ considered in this article, to each $\beta\in D$, the set $\{\alpha\in D:\alpha\leq\beta\}$ is finite and the set $\{\alpha\in D:\alpha\geq\beta\}$ is infinite.
It is assumed that all directed sets considered in this article satisfy this condition.
All directed sets considered through N, $N\times N$, $N\times N\times N,\ldots$, in earlier studies for statistical convergence satisfy this condition. Thus, a common extension is proposed in this article.
There is an article [17], which discusses statistical convergence of nets through ideals, but not through a concept of asymptotic density. The present article presents statistical convergence of nets through a concept of asymptotic density for directed sets.
The articles [7, 8, 9, 18, 19, 22, 23, 24, 25, 28, 29, 35, 36, 37, 38, 39] are related to this study of the concept. There are articles related to summability through statistical convergence (see [11, 33]) and articles for generalizations of asymptotic density (see [5]). Let us first introduce a concept of asymptotic density for our purpose.

2. Asymptotic density

A definition of asymptotic density for a special class of directed sets is presented in this section. This definition considers with the classical definition for the directed set of natural numbers.

Definition 2.1. *Let $(D,\leq)$ be a directed set that satisfies the condition mentioned above.*
To each $\alpha\in D$, let $D_\alpha=\{\beta\in D:\beta\leq\alpha\}$ and $|D_\alpha|$ denote the cardinality of D_α. The lower asymptotic density of a nonempty subset A of D is defined as the number $\liminf\limits_{\alpha\in D}\frac{|A\cap D_\alpha|}{|D_\alpha|}$ and the upper asymptotic density of A is defined as the number $\limsup\limits_{\alpha\in D}\frac{|A\cap D_\alpha|}{|D_\alpha|}$.

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If the upper and lower densities are equal, then the common number is called the asymptotic density of A and it is denoted by $\delta(A;D)$. *Thus,* $\delta(A;D) = \lim_{\alpha\in D} \frac{|A\cap D_\alpha|}{|D_\alpha|}$, *in the real interval* $[0,1]$. *If A is an empty subset, it is assumed that* $\delta(A;D) = 0$.
Here, for $x_\alpha \in \mathbb{R}$, *the real line,*

$$\liminf_{\alpha\in D} x_\alpha = \sup_{\beta\in D} \inf_{\alpha\geq\beta} x_\alpha$$

and

$$\limsup_{\alpha\in D} x_\alpha = \inf_{\beta\in D} \sup_{\alpha\geq\beta} x_\alpha.$$

Example 2.2. *Let* $D = \{(x_1,x_2,x_3) : x_i \in N,\ i = 1,2,3\}$. *Define* $\leq$ *on D by:* $(x_1,x_2,x_3) \leq (y_1,y_2,y_3)$ *if and only if* $x_1 \leq y_1$, $x_2 \leq y_2$ *and* $x_3 \leq y_3$. *Then, to each* $(y_1,y_2,y_3) \in D$, *the set* $\{(x_1,x_2,x_3) \in D : (x_1,x_2,x_3) \leq (y_1,y_2,y_3)\}$ *is finite, and it contains* $y_1 \cdot y_2 \cdot y_3$ *elements. Let* $A = \{(x,x,x) : x \in N\}$. *Then,* $\delta(A;D) = 0$.

Example 2.3. *Let* $D = N$ *be the directed set with the usual order relation. Then, to each* $\alpha \in D$, $D_\alpha = \{\beta \in D : \beta \leq \alpha\}$ *has precisely* α *elements. The asymptotic density introduced in Definition 2.1 for D coincides with the classical asymptotic density for subsets of N.*

Definition 2.4. *Let* $(D^{(1)}, \leq^{(1)})$ *and* $(D^{(2)}, \leq^{(2)})$ *be two directed sets. Let* $D = D^{(1)} \times D^{(2)}$. *Define the product order* $\leq$ *in D by:* $(x_1,x_2) \leq (y_1,y_2)$ *if and only if* $x_1 \leq^{(1)} y_1$ *and* $x_2 \leq^{(2)} y_2$. *Observe again that to each* $\alpha \in D$, *the set* $D_\alpha = \{\beta \in D : \beta \leq \alpha\}$ *is finite. This definition can be extended to any Cartesian product of a finite number of directed sets.*

Remark 2.5. *If* $A \subseteq D^{(1)}$, *and if* $\delta(A;D^{(1)})$ *exists then* $\delta(A \times D^{(2)};D) = \delta(A;D^{(1)})$; *for the notations used in the previous Definition 2.4. Moreover, if* $B \subseteq D^{(2)}$ *and* $\delta(A;D^{(1)}) = 0$ *then* $\delta(A \times B;D) = \delta(A;D^{(1)})$.

Proposition 2.6. *Let D be one among the directed sets* $N, N\times N, N\times N\times N, \cdots$, *when N is endowed with the usual order, and the other sets are endowed with the corresponding product orders. Then, to each* $\gamma \in D$,

$$\delta(\{\alpha \in D : \alpha \text{ not greater than or equal to } \gamma\};D) = 0, \text{ and hence } \delta(\{\alpha \in D : \alpha \geq \gamma\};D) = 1$$

Proof. It is easy to verify the relation $\delta(\{x \in D : \alpha \geq \gamma\};D) = 1$. □

Example 2.7. *Consider the set N with the following different order relation.* $m \leq n$ *if and only if m divides n. Then, N is a directed set with the properties mentioned in the introduction. Fix* $k \in N\backslash\{1\}$. *Let* $A = \{n \in N : n \text{ is not greater than or equal to } k\} = N\backslash\{k,\ 2k,\ 3k,\ \cdots\} = N\backslash kN$ *(say).*
If $m \in N\backslash kN$ *and if* $D_m = \{n \in N : n \leq m\}$, *then, for* $i \in D_m$, $i \in N\backslash kN$ *and hence* $A \cap D_m = D_m$. *This shows that* $\limsup_{m\to\infty} \frac{|A\cap D_m|}{|D_m|} = 1$.
If $m = k^i$ *for some* $i > 1$, *then* $D_m = \{n \in N : n \leq m\} = \{1,k,2k,\cdots,k^i\}$, *when* $A \cap D_m = \{1\}$. *This shows that* $\liminf_{m\to\infty} \frac{|A\cap D_m|}{|D_m|} = 0$. *In particular,* $\delta(A;N)$ *does not exist. However, if* $A = \{n \in N : n \text{ is not greater than or equal to } 1\} = N\backslash\{1\}$, *then* $\delta(A;N) = 1$. *Now, let* $D = N\backslash\{1\}$ *and consider the order relation defined above. To each* $\beta \in D$, *let* $D_\beta = \{\alpha \in D : \alpha \leq \beta\}$. *For a fixed* $\gamma \in D$, *let* $B = \{\alpha \in D : \alpha \geq \gamma\}$. *Then,* $\limsup_{\beta\in D} \frac{|B\cap D_\beta|}{|D_\beta|} = 1 > 0$.

Definition 2.8. *A directed set is said to satisfy the condition (*), if to each fixed* $\gamma \in D$, *for the set* $B = \{\alpha \in D : \alpha \geq \gamma\}$, *it is true that* $\limsup_{\beta\in D} \frac{|B\cap D_\beta|}{|D_\beta|} > 0$, *when* $D_\beta = \{\alpha \in D : \alpha \leq \beta\}$.

3. Statistical convergence

The classical concept of statistical convergence is generalized in this section. Some new fundamental properties are derived.

Definition 3.1. *Let* $(x_\alpha)_{\alpha\in D}$ *be a net in a topological space* (X,τ) *and let* $x \in X$. *Let us say that* $(x_\alpha)_{\alpha\in D}$ *converges statistically to x in* (X,τ), *if, to each* $U \in \tau$ *such that* $x \in U$, *the relation* $\delta(\{\alpha \in D : x_\alpha \notin U\};D) = 0$ *is true.*

Let us first verify the uniqueness of statistical limits in Hausdorff spaces.

Proposition 3.2. *Suppose* $(x_\alpha)_{\alpha\in D}$ *be a net in a Hausdorff space* (X,τ) *such that it converges statistically to x and y in X. Then,* $x = y$.

Proof. Suppose $x \neq y$. Then, there are disjoint open sets U and V such that $x \in U$ and $y \in V$. Then,

$$\{\alpha \in D : x_\alpha \notin U\} \cup \{\alpha \in D : x_\alpha \notin V\} = \{\alpha \in D : x_\alpha \notin U \cap V\} = D.$$

But, $\delta(\{\alpha \in D : x_\alpha \notin U\} \cup \{\alpha \in D : x_\alpha \notin V\};D) = 0$ and $\delta(D;D) = 1$; which is a contradiction. Therefore, $x = y$. Observe that $\delta(A \cup B;D) = 0$, whenever $\delta(A;D) = 0$ and $\delta(B;D) = 0$, for subsets A and B of D. □

Proposition 3.3. *Let* $(D^{(1)}, \leq^{(1)})$, $(D^{(2)}, \leq^{(2)})$ *and* $(D,\leq)$ *be as in Definition 2.4. Let* (X,τ_X) *and* (Y,τ_Y) *be given topological spaces. Let* τ *be the product topology on* $X \times Y$. *Let* $(x_\alpha)_{\alpha\in D^{(1)}}$ *and* $(y_\beta)_{\beta\in D^{(2)}}$ *be two nets in X and Y respectively. Then,* $((x_\alpha,y_\beta))_{(\alpha,\beta)\in D}$ *converges statistically to some* (x,y) *in* $(X\times Y,\tau)$ *if and only if* $(x_\alpha)_{\alpha\in D^{(1)}}$ *converges statistically to x in* (X,τ_X) *and* $(y_\beta)_{\beta\in D^{(2)}}$ *converges statistically to y in* (Y,τ_Y).

Proof. Suppose, $(x_\alpha)_{\alpha\in D^{(1)}}$ converges statistically to x in (X,τ_X) and $(y_\beta)_{\beta\in D^{(2)}}$ converges statistically to y in (Y,τ_Y). Fix $U\in\tau_X$ and $V\in\tau_Y$ such that $x\in U$ and $y\in V$.
Then,

$$\delta(\{\alpha\in D^{(1)}:x_\alpha\notin U\};D^{(1)})=0 \text{ and } \delta(\{\beta\in D^{(2)}:y_\beta\notin V\};D^{(2)})=0$$

By Remark 2.5,

$$\delta(\{\alpha\in D^{(1)}:x_\alpha\notin U\}\times D^{(2)}\cup D^{(1)}\times\{\beta\in D^{(2)}:y_\beta\notin V\};D)=0.$$

Thus,

$$\delta(\{(\alpha,\beta)\in D:(x_\alpha,y_\beta)\notin U\times V\};D)=0.$$

This implies that, $((x_\alpha,y_\beta))_{(\alpha,\beta)\in D}$ converges statistically to (x,y) in $(X\times Y,\tau)$. Conversely, assume that $((x_\alpha,y_\beta))_{(\alpha,\beta)\in D}$ converges statistically to (x,y) in $(X\times Y,\tau)$. Fix an open neighborhood U of x in (X,τ_X). Then,

$$\delta(\{(\alpha,\beta)\in D:(x_\alpha,y_\beta)\notin U\times Y\};D)=0.$$

So, $\delta(\{\alpha\in D^{(1)}:x_\alpha\notin U\};D^{(1)})=0$. This implies that, $(x_\alpha)_{\alpha\in D^{(1)}}$ converges statistically to x in (X,τ_X). Similarly, $(y_\beta)_{\beta\in D^{(2)}}$ converges statistically to y in (Y,τ_Y). □

Proposition 3.4. *Let (X,τ_X), (Y,τ_Y) and $(X\times Y,\tau)$ be as in the previous Proposition 3.3. Let $(x_\alpha)_{\alpha\in D}$ be a net that converges statistically to some x in (X,τ_X), for some directed set $(D,\leq)$. Let $(y_\alpha)_{\alpha\in D}$ be a net that converges statistically to some y in (Y,τ_Y). Then, $((x_\alpha,y_\alpha))_{\alpha\in D}$ converges statistically to (x,y) in $(X\times Y,\tau)$. On the other hand, if $((x_\alpha,y_\alpha))_{\alpha\in D}$ converges statistically to some (x,y) in $(X\times Y,\tau)$ then $(x_\alpha)_{\alpha\in D}$ converges statistically to x in (X,τ_X) and $(y_\alpha)_{\alpha\in D}$ converges statistically to y in (Y,τ_Y).*

Proof. Suppose $(x_\alpha)_{\alpha\in D}$ converges statistically to x and $(y_\alpha)_{\alpha\in D}$ converges statistically to y.
Let U be an open neighbourhood of x in X and V be an open neighbourhood of y in Y. Then,

$$\delta(\{\alpha\in D:x_\alpha\notin U\}\cup\{\alpha\in D:y_\alpha\notin V\};D)=0.$$

That is

$$\delta(\{\alpha\in D:(x_\alpha,y_\alpha)\notin U\times V\};D)=0.$$

So, $((x_\alpha,y_\alpha))_{\alpha\in D}$ converges statistically to (x,y).
Conversely assume that $((x_\alpha,y_\alpha))_{\alpha\in D}$ converges statistically to (x,y). Let U be an open neighbourhood of x. Then,

$$\delta(\{\alpha\in D:(x_\alpha,y_\alpha)\notin U\times Y\};D)=0.$$

That is

$$\delta(\{\alpha\in D:x_\alpha\notin U\};D)=0.$$

This implies that, $(x_\alpha)_{\alpha\in D}$ converges statistically to x. Similarly $(y_\alpha)_{\alpha\in D}$ converges statistically to y. □

Remark 3.5. *Proposition 3.3 and Proposition 3.4 can be extended to any Cartesian product of a finite number of spaces.*

Proposition 3.6. *Let (X,τ_X) and (Y,τ_Y) be topological spaces and let $f:(X,\tau_X)\to(Y,\tau_Y)$ be a function which is continuous at a point x in X. Let $(x_\alpha)_{\alpha\in D}$ be a net that converges statistically to some x in (X,τ_X). Then, $(f(x_\alpha))_{\alpha\in D}$ converges statistically to $f(x)$ in (Y,τ_Y).*

Proof. Let U be an open neighbourhood of $f(x)$ in (Y,τ_Y). Then, there is an open neighbourhood V of x in (X,τ_X) such that $f(V)\subseteq U$. Then,

$$\{\alpha\in D:f(x_\alpha)\notin U\}\subseteq\{\alpha\in D:x_\alpha\notin V\} \text{ and } \delta(\{\alpha\in D:x_\alpha\notin V\};D)=0.$$

So, $\delta(\{\alpha\in D:f(x_\alpha)\notin U\};D)=0$. This proves that $(f(x_\alpha))_{\alpha\in D}$ converges statistically to $f(x)$ in (Y,τ_Y). □

Proposition 3.7. *Let $D^{(1)},D^{(2)}$ and D be as in Proposition 3.3. Let $(x_\alpha)_{\alpha\in D^{(1)}}$ and $(y_\beta)_{\beta\in D^{(2)}}$ be two nets in a topological vector space X over the field of real numbers or the field of complex numbers. Let $(a_\alpha)_{\alpha\in D^{(1)}}$ be a net of scalars. If $(x_\alpha)_{\alpha\in D^{(1)}}$, $(y_\beta)_{\beta\in D^{(2)}}$ and $(a_\alpha)_{\alpha\in D^{(1)}}$ converge statistically to x,y and a respectively, then $(x_\alpha+y_\beta)_{(\alpha,\beta)\in D}$ and $(a_\alpha y_\beta)_{(\alpha,\beta)\in D}$ converge statistically to $x+y$ and ay respectively.*

Proof. Use Proposition 3.3 and Proposition 3.6. Observe that, it has been assumed that, the addition and the scalar multiplication in a topological vector space are jointly continuous. □

Proposition 3.8. *Let $(x_\alpha)_{\alpha\in D}$ and $(y_\alpha)_{\alpha\in D}$ be two nets in a topological vector space X; with respect to a common directed set D. Let $(a_\alpha)_{\alpha\in D}$ be a net of scalars. If $(x_\alpha)_{\alpha\in D}$, $(y_\alpha)_{\alpha\in D}$ and $(a_\alpha)_{\alpha\in D}$ converge statistically to x,y and a respectively, then $(x_\alpha+y_\alpha)_{\alpha\in D}$ and $(a_\alpha y_\alpha)_{\alpha\in D}$ converge statistically to $x+y$ and ay respectively.*

Proof. Use Proposition 3.4 and Proposition 3.6. □

Remark 3.9. *One may derive results similar to Proposition 3.7 and Proposition 3.8 for the structures, topological groups, topological rings, and topological algebras.*

4. Statistically Cauchy nets

The concept of statistically Cauchy nets is to be introduced for uniform spaces. For the concepts and notations in uniform spaces, one may refer to the book of Kelley [13] on General topology. The following definition agrees with the known definitions for statistically Cauchy sequences and statistically Cauchy double sequences (see [11, 26, 30]).

Definition 4.1. *Let $(X,\mathfrak{U})$ be a uniform space with a uniformity $\mathfrak{U}$. A net $(x_\alpha)_{\alpha\in D}$ in X is said to be statistically Cauchy if, for given $U\in\mathfrak{U}$, there is a $\gamma\in D$ such that*

$$\delta(\{\alpha\in D:(x_\alpha,x_\gamma)\notin U,\alpha\geq\gamma\};D)=0.$$

It is easy to verify that every Cauchy net is a statistically Cauchy net, and hence every converging net is a statistically Cauchy net in a uniform space. It is also possible to prove that statistical convergence implies statistical Cauchyness in a uniform space.

Proposition 4.2. *Let D be a directed set. Then, every statistically convergent net $(x_\alpha)_{\alpha\in D}$ in a uniform space is statistically Cauchy.*

Proof. Let $(x_\alpha)_{\alpha\in D}$ be a net which converges statistically to x in a uniform space $(X,\mathfrak{U})$. Fix $U\in\mathfrak{U}$. Find a symmetric $V\in\mathfrak{U}$ such that $V\circ V\subseteq U$. For this V,

$$\delta(\{\alpha\in D:(x_\alpha,x)\notin V\};D)=0$$

and hence there is a $\gamma\in D$ such that $(x_\gamma,x)\in V$. Then,

$$\{\alpha\in D:(x_\alpha,x_\gamma)\notin U\}\subseteq\{\alpha\in D:(x_\alpha,x)\notin V\}.$$

Thus,

$$\delta(\{\alpha\in D:(x_\alpha,x_\gamma)\notin U,\alpha\geq\gamma\};D)=0.$$

This proves that $(x_\alpha)_{\alpha\in D}$ is statistically Cauchy. □

Let us recall the order in product of two directed sets described in Definition 2.4.

Proposition 4.3. *Let $(x_\alpha)_{\alpha\in D}$ be a net that is statistically Cauchy in a uniform space $(X,\mathfrak{U})$. Then, for given $U\in\mathfrak{U}$, there is a $\gamma\in D$ such that*

$$\delta(\{(\alpha,\beta)\in D\times D:(x_\alpha,x_\beta)\notin U,\alpha\geq\gamma,\beta\geq\gamma\};D\times D)=0.$$

Proof. Fix $U\in\mathfrak{U}$. Find a symmetric $V\in\mathfrak{U}$ such that $V\circ V\subseteq U$. For this V, there is a $\gamma\in D$ such that $\delta(\{\alpha\in D:(x_\alpha,x_\gamma)\notin V,\alpha\geq\gamma\};D)=0$. Since

$$\begin{aligned}\{(\alpha,\beta)\in D\times D:(x_\alpha,x_\beta)\notin U,\alpha\geq\gamma,\beta\geq\gamma\}&\subseteq\{(\alpha,\beta)\in D\times D:(x_\alpha,x_\gamma)\notin V\\&\qquad \textit{or } (x_\beta,x_\gamma)\notin V,\alpha\geq\gamma,\beta\geq\gamma\}\\&\subseteq(\{\alpha\in D:(x_\alpha,x_\gamma)\notin V,\alpha\geq\gamma\}\times D)\\&\qquad\cup(D\times\{\beta\in D:(x_\beta,x_\gamma)\notin V,\beta\geq\gamma\}),\end{aligned}$$

by Remark 2.5,

$$\delta(\{(\alpha,\beta)\in D\times D:(x_\alpha,x_\beta)\notin U,\alpha\geq\gamma,\beta\geq\gamma\};D\times D)=0.$$

□

Proposition 4.4. *Let $D^{(1)},D^{(2)}$ and D be as in Proposition 3.3. Let $(X,\mathfrak{U}_X)$ and $(Y,\mathfrak{U}_Y)$ be two uniform spaces. Let $\mathfrak{U}$ be the product uniformity on $X\times Y$. Let $(x_\alpha)_{\alpha\in D^{(1)}}$ and $(y_\beta)_{\beta\in D^{(2)}}$ be two nets in X and Y respectively. Then, $((x_\alpha,y_\beta))_{(\alpha,\beta)\in D}$ is statistically Cauchy in $(X\times Y,\mathfrak{U})$ if $(x_\alpha)_{\alpha\in D^{(1)}}$ is statistically Cauchy in $(X,\mathfrak{U}_X)$ and $(y_\beta)_{\beta\in D^{(2)}}$ is statistically Cauchy in $(Y,\mathfrak{U}_Y)$. Moreover, if $D^{(1)}$ and $D^{(2)}$ satisfy the condition (*) mentioned in Definition 2.8, and $((x_\alpha,y_\beta))_{(\alpha,\beta)\in D}$ is statistically Cauchy in $(X\times Y,\mathfrak{U})$, then $(x_\alpha)_{\alpha\in D^{(1)}}$ is statistically Cauchy in $(X,\mathfrak{U}_X)$ and $(y_\beta)_{\beta\in D^{(2)}}$ is statistically Cauchy in $(Y,\mathfrak{U}_Y)$.*

Proof. The proof follows from the set relation: For $U\in\mathfrak{U}_X$, $V\in\mathfrak{U}_Y$, $\gamma_1\in D^{(1)}$ and for $\gamma_2\in D^{(2)}$, it is true that

$$\begin{aligned}\{(\alpha,\beta)\in D:((x_\alpha,x_{\gamma_1}),(y_\beta,y_{\gamma_2}))\notin U\times V,(\alpha,\beta)\geq(\gamma_1,\gamma_2)\}&=(\{\alpha\in D^{(1)}:(x_\alpha,x_{\gamma_1})\notin U,\alpha\geq\gamma_1\}\times\{\beta\in D^{(2)}:\beta\geq\gamma_2\})\\&\quad\cup(\{\alpha\in D^{(1)}:\alpha\geq\gamma_1\}\times\{\beta\in D^{(2)}:(y_\beta,y_{\gamma_2})\notin V,\beta\geq\gamma_2\}).\end{aligned}$$

□

Proposition 4.5. *Let $(X,\mathfrak{U}_X)$ and $(Y,\mathfrak{U}_Y)$ be two uniform spaces. Let $\mathfrak{U}$ be the product uniformity on $X\times Y$. Let $(x_\alpha)_{\alpha\in D}$ and $(y_\alpha)_{\alpha\in D}$ be nets in X and Y respectively. Then, $((x_\alpha,y_\alpha))_{\alpha\in D}$ is statistically Cauchy in $X\times Y$ if and only if $(x_\alpha)_{\alpha\in D}$ is statistically Cauchy in X and $(y_\alpha)_{\alpha\in D}$ is statistically Cauchy in Y.*

Proof. Suppose $(x_\alpha)_{\alpha\in D}$ and $(y_\alpha)_{\alpha\in D}$ be statistically Cauchy. Fix $U\in\mathfrak{U}_X$ and $V\in\mathfrak{U}_Y$. Then, there is a $\gamma\in D$ such that

$$\delta(\{\alpha\in D:(x_\alpha,x_\gamma)\notin U,\alpha\geq\gamma\};D)=0 \text{ and } \delta(\{\alpha\in D:(y_\alpha,y_\gamma)\notin V,\alpha\geq\gamma\};D)=0.$$

The statistically Cauchyness of $((x_\alpha,y_\alpha))_{\alpha\in D}$ follows from the relation:

$$\{\alpha\in D:(x_\alpha,x_\gamma)\notin U \quad \text{or} \quad (y_\alpha,y_\gamma)\notin V,\ \alpha\geq\gamma\} \quad \subseteq\{\alpha\in D:(x_\alpha,x_\gamma)\notin U,\alpha\geq\gamma\}\cup\{\alpha\in D:(y_\alpha,y_\gamma)\notin V,\alpha\geq\gamma\}.$$

Conversely, assume that $((x_\alpha,y_\alpha))_{\alpha\in D}$ is statistically Cauchy. Fix $U\in\mathfrak{U}$. Then, there is a $\gamma\in D$ such that

$$\delta(\{\alpha\in D:(x_\alpha,x_\gamma)\notin U,\alpha\geq\gamma\};D)=\delta(\{\alpha\in D:(x_\alpha,x_\gamma)\notin U \textit{ or } (y_\alpha,y_\gamma)\notin Y\times Y,\alpha\geq\gamma\};D)=0.$$

This shows that $(x_\alpha)_{\alpha\in D}$ is statistically Cauchy. Similarly, $(y_\alpha)_{\alpha\in D}$ is statistically Cauchy. □

Proposition 4.6. *Let $f:(X,\mathfrak{U})\to(Y,\mathfrak{V})$ be a uniformly continuous function from a uniform space $(X,\mathfrak{U})$ into a uniform space $(Y,\mathfrak{V})$. Let $(x_\alpha)_{\alpha\in D}$ be a statistically Cauchy net in $(X,\mathfrak{U})$. Then, $(f(x_\alpha))_{\alpha\in D}$ is a statistically Cauchy net in $(Y,\mathfrak{V})$.*

Proof. Fix $V\in\mathfrak{V}$. Find a $U\in\mathfrak{U}$ such that $(f(x),f(y))\in V$, whenever $(x,y)\in U$. Find a $\gamma\in D$ such that $\delta(\{\alpha\in D:(x_\alpha,x_\gamma)\notin U,\alpha\geq\gamma\};D)=0$.
Then,

$$\delta(\{\alpha\in D:(f(x_\alpha),f(x_\gamma))\notin V,\alpha\geq\gamma\};D)=0,$$

because

$$\{\alpha\in D:(f(x_\alpha),f(x_\gamma))\notin V,\alpha\geq\gamma\}\subseteq\{\alpha\in D:(x_\alpha,x_\gamma)\notin U,\alpha\geq\gamma\}.$$

□

Remark 4.7. *Let (X,τ) be a topological vector space. The usual uniformity on X implies the following: A net $(x_\alpha)_{\alpha\in D}$ is Cauchy in X if and only if for every neighbourhood U of 0 there is a $\gamma\in D$ such that*

$$\delta(\{\alpha\in D:x_\alpha-x_\gamma\notin U,\alpha\geq\gamma\};D)=0.$$

One can derive the following Proposition 4.8 and Proposition 4.9 which are similar to Proposition 3.7 and Proposition 3.8.

Proposition 4.8. *Let $D^{(1)},D^{(2)}$, D, $(x_\alpha)_{\alpha\in D^{(1)}}$, $(y_\beta)_{\beta\in D^{(2)}}$, $(a_\alpha)_{\alpha\in D^{(1)}}$ and X be as in Proposition 3.7. Let $x\in X$ and a be a scalar. If $(x_\alpha)_{\alpha\in D^{(1)}}$, $(y_\beta)_{\beta\in D^{(2)}}$ and $(a_\alpha)_{\alpha\in D^{(1)}}$ are statistically Cauchy, then $(x_\alpha+y_\beta)_{(\alpha,\beta)\in D}$, $(a_\alpha x)_{\alpha\in D^{(1)}}$ and $(ax_\alpha)_{\alpha\in D^{(1)}}$ are statistically Cauchy.*

Proof. Use Proposition 4.6 and Proposition 4.4. □

Proposition 4.9. *Let $(x_\alpha)_{\alpha\in D}$, $(y_\alpha)_{\alpha\in D}$ and X be as in Proposition 3.8. If $(x_\alpha)_{\alpha\in D}$ and $(y_\alpha)_{\alpha\in D}$ are statistically Cauchy, then $(x_\alpha+y_\alpha)_{\alpha\in D}$ is statistically Cauchy.*

Proof. Use Proposition 4.5 and Proposition 4.6. □

5. Net Spaces

Corresponding to sequence spaces, net spaces can be constructed. The following construction is similar to the construction given in [31]. The following construction uses the Propositions 3.7, 3.8, 4.8 and 4.9. Since, verifications part is a direct one, it is omitted.
Let (X,τ) be a topological vector space with the natural uniformity $\mathfrak{U}$ that induces the topology τ. Let D be a fixed directed set. Let $M=\{(x_\alpha)_{\alpha\in D}:\{x_\alpha:\alpha\in D\}\ \textit{is a bounded subset of}\ X\}$.

Let $M_{cy}=\{(x_\alpha)_{\alpha\in D}\in M:\ (x_\alpha)_{\alpha\in D}\ \textit{is statistically Cauchy}\}$.
Let $M_{ct}=\{(x_\alpha)_{\alpha\in D}\in M:\ (x_\alpha)_{\alpha\in D}\ \textit{converges statistically in}\ X\}$.
Let $M_0=\{(x_\alpha)_{\alpha\in D}\in M:\ (x_\alpha)_{\alpha\in D}\ \textit{converges statistically to zero in}\ X\}$.

To each balanced neighbourhood U of zero in X, define a function p_U on M by

$$p_U((x_\alpha)_{\alpha\in D})=\sup\{\lambda\geq0:\lambda x_\alpha\in U,\forall\alpha\in D\},$$

and define a subset N_U of M by

$$N_U=\{(x_\alpha)_{\alpha\in D}\in M:p_U((x_\alpha)_{\alpha\in D})<1\}.$$

Then, the collection of the sets of the form N_U forms a local base for M that makes M into a topological vector space under pointwise algebraic operations. Also, M_{cy} is a closed linear subspace of M. If $(X,\mathfrak{U})$ is a complete topological vector space, then M is a complete topological vector space and M_{ct} and M_0 are closed linear subspaces of M.

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Cyclic (α,β)-Admissible Mappings in Modular Spaces and Applications to Integral Equations

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Abstract

The main concern of this study is to present a generalization of Banach's fixed point theorem in some classes of modular spaces, where the modular is convex and satisfying the Δ_2-condition. In this work, the existence and uniqueness of fixed point for $(\alpha,\beta)-(\psi,\varphi)-$contractive mapping and $\alpha-\beta-\psi-$weak rational contraction in modular spaces are proved. Some examples are supplied to support the usability of our results. As an application, the existence of a solution for an integral equation of Lipschitz type in a Musielak-Orlicz space is presented.

1. Introduction and Preliminaries

It is well known fixed point theorems play important roles and have applications in mathematics analysis, particularly in differential and integral equations. One of the most popular fixed point theorem is Banach fixed point theorem [6]. By using this theorem, most authors have proved several fixed point theorems for various mappings [13, 21, 28]. Such as, Dutta and Choudhury proved $(\psi,\phi)-$contractive mappings in complete metric space [11]. Samet et al. introduced the concept of $\alpha-\varphi-$ contractive type mappings and established various fixed point theorems [32]. Later, Salimi et al. modified the concept of $\alpha-\varphi-$ contractive type mappings [31]. Alizadeh et al. [4] developed a new fixed point theorem in complete metric spaces. They introduced the concept of cyclic $(\alpha,\beta)-$admissible and $(\alpha,\beta)-(\psi,\phi)-$contractive mappings and established some fixed point results in metric spaces.
On the other hand, some authors introduced a new concept of modular vector spaces which are natural generalizations of many classical function spaces. Firstly, Nakano initiated the concept of modular spaces [26]. Later, some authors proved new fixed point theorems of Banach type in modular spaces [12, 18, 19, 22, 23, 24, 29, 33]. Then, also the concept of the fixed point theory was studied in modular metric, modular function and modular vector spaces. [1, 2, 3, 5, 8, 9, 10, 14, 15, 16, 17, 20, 30, 34].
In this work, some fixed point results as a generalization of Banach's fixed point theorem are presented using some convenient constants in the contraction assumption in modular spaces. Motivated by [4] and [25], some fixed point results for $(\alpha,\beta)-(\psi,\phi)-$contractive mappings in modular spaces are proved. Some examples are supplied in order to support the usability of our results. As an application the existence and uniqueness of solutions for an integral equation of Lipschitz type in a Musielak-Orlicz space are showed.

Definition 1.1. *[25, 27] Let X be an arbitrary vector space. A functional $\rho : X \to [0,\infty)$ is called a modular if, for any x,y in X, the following conditions hold:*

(a) $\rho(x)=0$ *if and only if* $x=0$,
(b) $\rho(-x)=\rho(x)$,
(c) $\rho(\alpha x+\beta y)\le\rho(x)+\rho(y)$, *whenever* $\alpha+\beta=1$ *and* $\alpha,\beta\ge 0$.

If (c) is replaced with $\rho(\alpha x+\beta y)\le\alpha^s\rho(x)+\beta^s\rho(y)$ where $\alpha^s+\beta^s=1$, $\alpha,\beta\ge 0$, and $s\in(0,1]$, then ρ is called s-convex modular. If $s=1$, then we say that ρ is convex modular. The following are some consequences of condition (c).

Remark 1.2. *[7]*

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(a) *For* $a,b \in \mathbb{R}$ *with* $|a| < |b|$ *we have* $\rho(ax) < \rho(bx)$ *for all* $x \in X$.

(b) *For* $a_1,...,a_n \in R^+$ *with* $\sum_{i=1}^n a_i = 1$, *we have*

$$\rho(\sum_{i=1}^n a_i x_i) = \rho(\sum_{i=1}^n x_i) \text{ for any } x_1,...,x_n \in X.$$

Remark 1.3. *[26] A modular* ρ *defines a corresponding modular space, i.e. the space is given by*

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Definition 1.4. *A sequence* $\{x_n\}$ *in modular space* X_ρ *is said to be:*

(a) ρ*-convergent to* $x \in X_\rho$ *if* $\rho(x_n - x) \to 0$ *as* $n \to \infty$.

(b) ρ*-Cauchy if* $\rho(x_n - x_m) \to 0$ *as* $n,m \to \infty$.

(c) X_ρ *is called* ρ*-complete if any* ρ*-Cauchy sequence is* ρ*-convergent.*

(d) ρ *satisfies* Δ_2*-condition if* $\rho(2x_n) \to 0$ *as* $n \to \infty$, *whenever* $\rho(x_n) \to 0$ *as* $n \to \infty$.

Definition 1.5. *[4] Let* $T : X \to X$ *be a mapping and* $\alpha, \beta : X \to \mathbb{R}^+$ *be two functions. We say that* T *is a cyclic* (α,β)*-admissible mapping if*

(i) $\alpha(x) \geq 1$ *for some* $x \in X$ *implies* $\beta(Tx) \geq 1$,
(ii) $\beta(x) \geq 1$ *for some* $x \in X$ *implies* $\alpha(Tx) \geq 1$.

Definition 1.6. *[4] Let* Ψ *be the set of continuous and increasing functions* $\psi : [0,\infty) \to [0,\infty)$ *and* Φ *be the set of lower semicontinuous functions* $\phi : [0,\infty) \to [0,\infty)$ *such that* $\phi(t) = 0$ *iff* $t = 0$. *Let* X *be a metric space and* $T : X \to X$ *be a cyclic* (α,β)*-admissible mapping. We say that* T *is a* $(\alpha,\beta)-(\psi,\phi)-$*contractive mapping if*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$$

for $x,y \in X$, *where* $\psi \in \Psi$ *and* $\phi \in \Phi$.

2. Main Results

Let Ψ and Φ be defined as in Definition 1.6. Let X_ρ be a nonempty set and $T : X_\rho \to X_\rho$ be an arbitrary mapping. We say that $x \in X_\rho$ is a fixed point of T, if $x = Tx$. We denote by $Fix(T)$ the set of all fixed points of T. In the sequel, suppose the modular ρ is convex and satisfies the Δ_2-condition.

Definition 2.1. *Let* X_ρ *be a* ρ*-complete modular space and* $T : X_\rho \to X_\rho$ *be a cyclic* $(\alpha,\beta)-$*admissible mapping. We say that* T *is a* $(\alpha,\beta)-(\psi,\phi)-$*contractive mapping if*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(\rho(Tx - Ty)) \leq \psi(\rho(x-y)) - \phi(\rho(x-y)) \tag{2.1}$$

for $x,y \in X_\rho$, *where* $\psi \in \Psi$ *and* $\phi \in \Phi$.

Theorem 2.2. *Let* X_ρ *be a* ρ*-complete modular space and* $T : X_\rho \to X_\rho$ *be a* $(\alpha,\beta)-(\psi,\phi)-$*contractive mapping. Suppose that the following conditions hold:*

(a) *there exists* $x_0 \in X_\rho$ *such that* $\alpha(x_0) \geq 1$ *and* $\beta(x_0) \geq 1$,

(b) T *is continuous, or*

(c) *if* $\{x_n\}$ *is a sequence in* X_ρ *such that* $x_n \to x$ *and* $\beta(x_n) \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\beta(x) \geq 1$,

then T *has a fixed point. Moreover, if* $\alpha(x) \geq 1$ *and* $\beta(y) \geq 1$ *for all* $x,y \in Fix(T)$, *then* T *has a unique fixed point.*

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a cyclic $(\alpha,\beta)-$admissible mapping and $\alpha(x_0) \geq 1$ then $\beta(x_1) = \beta(Tx_0) \geq 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \geq 1$. By continuing this process, we get $\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$. Again, since T is a cyclic (α,β)-admissible mapping and $\beta(x_0) \geq 1$, by the similar method, we have $\beta(x_{2n}) \geq 1$ and $\alpha(x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$. Therefore by (2.1), we have

$$\begin{aligned}\psi(\rho(x_n - x_{n+1})) &\leq \psi(\rho(x_{n-1} - x_n)) - \phi(\rho(x_{n-1} - x_n)) \\ &\leq \psi(\rho(x_{n-1} - x_n))\end{aligned} \tag{2.2}$$

and since ψ is increasing, we get

$$\rho(x_n - x_{n+1}) \leq \rho(x_{n-1} - x_n)$$

for all $n \in \mathbb{N}$. So, $\{\rho_n := \rho(x_n - x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Then, there exists $r \geq 0$ such that $\lim_{n\to\infty} \rho_n = r$. We shall show that $r = 0$. By taking the limsup on both sides of (2.2), we have

$$\lim_{n\to\infty} \rho(x_n - x_{n+1}) = 0. \tag{2.3}$$

Now, we want to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose to the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there are $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, and for $n(k) > m(k) > k$, we have

$$\rho(x_{2n(k)} - x_{2m(k)}) \geq \varepsilon \text{ and } \rho(2(x_{2n(k)-1} - x_{2m(k)})) < \varepsilon. \tag{2.4}$$

Now for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon &\leq \rho(x_{2n(k)} - x_{2m(k)}) \\ &\leq \rho(2(x_{2n(k)} - x_{2n(k)-1})) + \rho(2(x_{2n(k)-1} - x_{2m(k)})) \\ &< \rho(2(x_{2n(k)} - x_{2n(k)-1})) + \varepsilon. \end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (2.3), we get

$$\lim_{k\to\infty} \rho(x_{2n(k)} - x_{2m(k)}) = \varepsilon. \tag{2.5}$$

Since

$$\begin{aligned} \rho(x_{2n(k)+1} - x_{2m(k)+1}) &= \rho(x_{2n(k)+1} - x_{2n(k)} + x_{2n(k)} - x_{2m(k)+1}) \\ &\leq \rho(2(x_{2n(k)+1} - x_{2n(k)})) + \rho(2(x_{2n(k)} - x_{2m(k)+1})) \end{aligned}$$

and

$$\rho(2(x_{2n(k)} - x_{2m(k)+1})) = \rho(2(x_{2n(k)} - x_{2m(k)} + x_{2m(k)} - x_{2m(k)+1})) \leq \rho(4(x_{2n(k)} - x_{2m(k)}) + \rho(4(x_{2m(k)} - x_{2m(k)+1}))$$

then by taking the limit as $k \to +\infty$ in above inequality and using (2.3) and (2.5), we deduce that

$$\lim_{k\to\infty} \rho(x_{2n(k)+1} - x_{2m(k)+1}) = \varepsilon. \tag{2.6}$$

Now, by (2.1) and $\alpha(x_{2n(k)})\beta(x_{2m(k)}) \geq 1$ for all $k \in \mathbb{N}$, we get

$$\psi(\rho(x_{2n(k)+1} - x_{2m(k)+1})) \leq \psi(\rho(x_{2n(k)} - x_{2m(k)})) - \phi(\rho(x_{2n(k)} - x_{2m(k)})). \tag{2.7}$$

By taking the limsup on both sides of (2.7), applying (2.4) and (2.6), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon).$$

That is, $\varepsilon = 0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X_ρ is a complete modular space, then there is a $z \in X_\rho$ such that $x_n \to z$ as $n \to \infty$. First, we assume that T is continuous. Hence, we deduce

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So z is a fixed point of T . Now, assume that (c) holds. That is, $\alpha(x_n)\beta(z) \geq 1$. From (2.1) we have

$$\psi(\rho(x_{n+1} - Tz)) \leq \psi(\rho(x_n - z)) - \phi(\rho(x_n - z)). \tag{2.8}$$

By taking the limsup on both sides of (2.8), we get $\psi(\rho(z - Tz)) = 0$. Then $\rho(z - Tz) = 0$. i.e., $z = Tz$. To prove the uniqueness of fixed point, suppose that z and z^* are two fixed points of T . From condition (c) we have, $\alpha(z)\beta(z^*) \geq 1$, it follows from (2.1) that

$$\psi(\rho(z - Tz^*)) \leq \psi(\rho(z - z^*)) - \phi(\rho(z - z^*)).$$

So $\phi(\rho(z - z^*)) = 0$ and hence $\rho(z - z^*) = 0$ i.e., $z = z^*$. □

Corollary 2.3. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic $(\alpha, \beta)-$admissible mapping such that*

$$\alpha(x)\beta(y)\psi(\rho(Tx - Ty)) \leq \psi(\rho(x - y)) - \phi(\rho(x - y))$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Let $\alpha(x)\beta(y) \geq 1$ for $x, y \in X_\rho$. Then by (2.8), we have

$$\psi(\rho(Tx - Ty)) \leq \psi(\rho(x-y)) - \phi(\rho(x-y)).$$

This implies that the inequality (2.1) holds. Therefore, the proof follows from Theorem 2.2. $\square$

Corollary 2.4. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic (α, β)-admissible mapping such that*

$$(\alpha(x)\beta(y) + 1)^{\psi(\rho(fx - fy))} \leq 2^{\psi(\rho(x-y)) - \phi(\rho(x-y))}$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Example 2.5. *Let $X_\rho = [-2, \infty] \to \mathbb{R}$, $\rho(x) = |x|$ for all $x \in X_\rho$, and $T : X_\rho \to X_\rho$ by*

$$Tx = \begin{cases} \frac{x^2}{3} & , \quad x \in [-2, 2] \\ \sqrt{x} & , \quad otherwise. \end{cases}$$

Define $\psi, \phi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = 3t, \phi(t) = t$ and $\alpha, \beta : X_\rho \to [0, +\infty)$ by

$$\alpha(x) = \begin{cases} 1 & , \quad x \in [-2, \frac{4}{3}] \\ 0 & , \quad otherwise. \end{cases}$$

and

$$\beta(x) = \begin{cases} 1 & , \quad x \in [\frac{4}{3}, 2] \\ 0 & , \quad otherwise. \end{cases}$$

Now, we prove that the hypotheses (a) and (c) of Corollary 2.4 are satisfied by T and hence T has a fixed point. Let $\alpha(x) \geq 1$ for some $x \in X_\rho$. Then $x \in [-2, \frac{4}{3}]$ and so $Tx \in [\frac{4}{3}, 2]$. Therefore, $\beta(Tx) \geq 1$. Similarly, if $\beta(x) \geq 1$ then $\alpha(x) \geq 1$. Then T is a cyclic (α, β)-admissible mapping and that the hypotheses (a) and (c) of Corollary 2.4 hold.
Now, for all $x \in [-2, \frac{4}{3}]$ and $y \in [\frac{4}{3}, 2]$, we get

$$\begin{aligned}
(\alpha(x)\beta(y) + 1)^{\psi(\rho(fx - fy))} &= 2^{3\rho(fx - fy)} \\
&= 2^{3\left|\frac{x^2}{3} - \frac{y^2}{3}\right|} \\
&= 2^{|x-y||x+y|} \\
&\leq 2^{2|x-y|} = 2^{3|x-y| - |x-y|} \\
&= 2^{\psi(\rho(x-y)) - \phi(\rho(x-y))}
\end{aligned}$$

Otherwise, if $\alpha(x)\beta(y) = 0$, we have

$$(\alpha(x)\beta(y) + 1)^{\psi(\rho(fx - fy))} = 1 \leq 2^{\psi(\rho(x-y)) - \phi(\rho(x-y))}$$

Therefore, Corollary 2.4 implies that T has a fixed point.

Corollary 2.6. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic (α, β)-admissible mapping. Assume that there exists $\ell > 1$ such that*

$$(\psi(\rho(Tx - Ty)) + \ell)^{\alpha(x)\beta(y)} \leq \psi(\rho(x-y)) - \phi(\rho(x-y)) + \ell$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$ and $\varphi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Example 2.7. *Let $X_\rho = \mathbb{R}^+$, $\rho(x) = |x|$ for all $x \in X_\rho$, and $T : X_\rho \to X_\rho$ by*

$$Tx = \begin{cases} \frac{x^2+x}{4} & , \quad x \in [0,1] \\ 2x & , \quad otherwise. \end{cases}$$

Define $\psi, \phi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = t, \phi(t) = \frac{t}{4}$ and $\alpha, \beta : X_\rho \to [0,+\infty)$ by

$$\alpha(x) = \beta(x) = \begin{cases} 1 & , \quad x \in [0,1] \\ 0 & , \quad otherwise. \end{cases}$$

Now, we prove that the hypotheses (a) and (c) of Corollary 2.6 are satisfied by T and hence T has a fixed point. Proceeding as in the Example 2.5, we deduce that T is a cyclic (α,β)-admissible mapping and that the hypotheses (a) and (c) of Corollary 2.6 hold.
Now, for all $x \in [0,1]$ and all $y \in [0,1]$, we get

$$\begin{aligned}(\psi(\rho(Tx-Ty))+\ell)^{\alpha(x)\beta(y)} &= |Tx-Ty|+\ell \\ &\le \frac{1}{4}|x-y|\,|x+y+1|+\ell \\ &\le \frac{3}{4}|x-y|+\ell \\ &= |x-y| - \frac{1}{4}|x-y|+\ell \\ &= \psi(\rho(x-y)) - \phi(\rho(x-y))+\ell.\end{aligned}$$

Otherwise, if $\alpha(x)\beta(y) = 0$, we have

$$(\psi(\rho(Tx-Ty))+\ell)^{\alpha(x)\beta(y)} = 1 \le \psi(\rho(x-y)) - \phi(\rho(x-y))+\ell.$$

Therefore, Corollary 2.6 implies that T has a fixed point.

Definition 2.8. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic $(\alpha,\beta)-$admissible mapping. We say that T is $\alpha-\beta-\psi-$ weak rational contraction if $\alpha(x)\beta(y) \ge 1$ for some $x,y \in X_\rho$ such that*

$$\rho(Tx-Ty) \le M(x,y) - \psi(M(x,y))$$

where $\psi \in \Psi$ and

$$M(x,y) = \max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{[1+\rho(x-Tx)]\rho(y-Ty)}{\rho(x-y)+1}\}.$$

Theorem 2.9. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be $\alpha-\beta-\psi-$weak rational contraction. Assume that the following assertions hold:*

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \ge 1$ and $\beta(x_0) \ge 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \ge 1$ for all $n \in \mathbb{N}$, then $\beta(x) \ge 1$,

then T has a fixed point. Moreover, if $\alpha(x) \ge 1$ and $\beta(y) \ge 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a cyclic $(\alpha,\beta)-$admissible mapping and $\alpha(x_0) \ge 1$ then $\beta(x_1) = \beta(Tx_0) \ge 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \ge 1$. By continuing this process, we get $\alpha(x_{2n}) \ge 1$ and $\beta(x_{2n-1}) \ge 1$ for all $n \in \mathbb{N}$. Again, since T is a cyclic $(\alpha,\beta)-$admissible mapping and $\beta(x_0) \ge 1$, by the similar method, we have $\beta(x_{2n}) \ge 1$ and $\alpha(x_{2n-1}) \ge 1$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n) \ge 1$ and $\beta(x_n) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \ge 1$ for all $n \in \mathbb{N}$. Since T is $\alpha-\beta-\psi-$weak rational contraction, we get

$$\rho(x_n - x_{n+1}) \le M(x_{n-1},x_n) - \psi(M(x_{n-1},x_n)) \tag{2.9}$$

where

$$\begin{aligned}M(x_{n-1},x_n) &= \max\{\rho(x_{n-1}-x_n), \rho(x_{n-1}-Tx_{n-1}), \rho(x_n-Tx_n), \frac{[1+\rho(x_{n-1}-Tx_{n-1})]\rho(x_n-Tx_n)}{\rho(x_{n-1}-x_n)+1}\} \\ &= \max\{\rho(x_{n-1}-x_n), \rho(x_n-x_{n+1})\}.\end{aligned}$$

Now, suppose that there exists $n_0 \in \mathbb{N}$ such that $\rho(x_{n_0}-x_{n_{0+1}}) > \rho(x_{n_{0-1}}-x_{n_0})$. Therefore $M(x_{n_{0-1}},x_{n_0}) = \rho(x_{n_0}-x_{n_{0+1}})$ and so from (2.9), we get

$$\rho(x_{n_0}-x_{n_0+1}) \le \rho(x_{n_0}-x_{n_{0+1}}) - \psi(\rho(x_{n_0}-x_{n_{0+1}})).$$

This implies that $\psi(\rho(x_{n_0}-x_{n_0+1})) = 0$, i.e., $\rho(x_{n_0}-x_{n_0+1}) = 0$, which is a contradiction. Hence, $\rho(x_n-x_{n+1}) \le \rho(x_{n-1}-x_n)$ for all $n \in \mathbb{N}$. That is the sequence $\{\rho_n : \rho(x_n-x_{n+1})\}$ is decreasing and so there exists $r \ge 0$ such that $\rho_n \to r$ as $n \to \infty$. Taking the limit as $n \to \infty$ in (2.9), we have

$$r \le r - \psi(r).$$

This implies that $\psi(r)=0$. Therefore, the property of ψ implies that $r=0$. That is

$$\lim_{n\to\infty}\rho(x_{n+1}-x_n)=0. \tag{2.10}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon>0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, for $n(k)>m(k)>k$, we have

$$\rho(x_{2n(k)}-x_{2m(k)})\geq\varepsilon \text{ and } \rho(2(x_{2n(k)-1}-x_{2m(k)}))<\varepsilon. \tag{2.11}$$

For all $k\in\mathbb{N}$, we have

$$\varepsilon\leq\rho(x_{2n(k)}-x_{2m(k)})\leq\rho(2(x_{2n(k)}-x_{2n(k)-1}))+\rho(2(x_{2n(k)-1}-x_{2m(k)}))<\rho(2(x_{2n(k)}-x_{2n(k)-1}))+\varepsilon.$$

Taking the limit as $k\to\infty$ in above inequality and from (2.10), we have

$$\lim_{k\to\infty}\rho(x_{2n(k)}-x_{2m(k)})=\varepsilon. \tag{2.12}$$

Then, we get

$$\rho(x_{2n(k)+1}-x_{2m(k)+1})=\rho(x_{2n(k)+1}-x_{2n(k)}+x_{2n(k)}-x_{2m(k)+1})\leq\rho(2(x_{2n(k)+1}-x_{2n(k)}))+\rho(2(x_{2n(k)}-x_{2m(k)+1}))$$

and

$$\rho(2(x_{2n(k)}-x_{2m(k)+1}))=\rho(2(x_{2n(k)}-x_{2m(k)}+x_{2m(k)}-x_{2m(k)+1}))\leq\rho(4(x_{2n(k)}-x_{2m(k)})+\rho(4(x_{2m(k)}-x_{2m(k)+1})).$$

Taking the limit as $k\to+\infty$ in above inequality and using (2.12) and (2.11), we deduce that

$$\lim_{k\to\infty}\rho(x_{2n(k)}-x_{2m(k)+1})=\varepsilon. \tag{2.13}$$

Now, by (2.1), we get

$$\rho(x_{2n(k)+1}-x_{2m(k)+1})\leq M(x_{2n(k)}-x_{2m(k)})-\varphi(M(x_{2n(k)}-x_{2m(k)})) \tag{2.14}$$

where

$$M(x_{2n(k)}-x_{2m(k)})=\max\{\rho(x_{2n(k)}-x_{2m(k)}),\rho(x_{2n(k)}-x_{2n(k)+1}),\rho(x_{2m(k)}-x_{2m(k)+1}),\frac{[1+\rho(x_{2n(k)}-x_{2n(k)+1})]\rho(x_{2m(k)}-x_{2m(k)+1})}{\rho(x_{2n(k)}-x_{2m(k)})+1}\}.$$

Letting $k\to\infty$ in (2.14) and using (2.10), (2.12) and (2.13), we get

$$\varepsilon\leq\varepsilon-\psi(\varepsilon).$$

That is $\varepsilon=0$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since X_ρ is complete, then there exists a $z\in X_\rho$ such that $x_n\to z$. Suppose that (c) holds. That is, $\alpha(x_{2n})\beta(z)\geq1$. Since T is $\alpha-\beta-\psi-$weak rational contradiction, then we have

$$\rho(x_{2n+1}-Tz)\leq M(x_{2n},z)-\psi((x_{2n},z)) \tag{2.15}$$

where

$$M(x_{2n},z)=\max\{\rho(x_{2n}-z),\rho(x_{2n}-Tz),\rho(z-Tz),\frac{[1+\rho(x_{2n}-x_{n+1})]\rho(z-Tz)}{\rho(x_{2n}-z)+1}\}.$$

Taking the limit as $n\to\infty$ in (2.15), we have $z=Tz$. Now, let show that T has at most one fixed point. Indeed, if $x,y\in X_\rho$ be two fixed points of T, that is, $Tx=x\neq y=Ty$. From condition (c) we have, $\alpha(x)\beta(y)\geq1$, it follows that

$$\psi(\rho(x-y))\leq\psi(M(x,y))-\phi(M(x,y))$$

where

$$M(x,y)=\max\{\rho(x-y),\rho(x-Tx),\rho(y-Ty),\frac{[1+\rho(x-Tx)]\rho(y-Ty)}{\rho(x-y)+1}\}.$$

Then, we obtain

$$\psi(\rho(x-y))\leq\psi(\rho(x-y))-\phi(\rho(x-y)).$$

So $\phi(\rho(x-y))=0$ and hence, $\rho(x-y)=0$, that is, $x=y$.

□

We obtain the following corollaries from Theorem 2.9.

Corollary 2.10. *Let X_ρ be a ρ-complete modular space and $T: X_\rho \to X_\rho$ be a cyclic $(\alpha,\beta)-$admissible mapping such that*

$$\alpha(x)\beta(y)\rho(Tx-Ty) \leq M(x,y) - \psi(M(x,y)),$$

where $\psi \in \Psi$ and

$$M(x,y) = \max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{[1+\rho(x-Tx)]\rho(y-Ty)}{\rho(x-y)+1}\}.$$

Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Corollary 2.11. *Let X_ρ be a ρ-complete modular space and $T: X_\rho \to X_\rho$ be a cyclic $(\alpha,\beta)-$admissible mapping such that*

$$(\alpha(x)\beta(y)+1)^{\rho(Tx-Ty)} \leq 2^{M(x,y)-\psi(M(x,y))}$$

for all $x,y \in X_\rho$ where $\psi \in \Psi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Corollary 2.12. *Let X_ρ be a ρ-complete modular space and $T: X_\rho \to X_\rho$ be a cyclic $(\alpha,\beta)-$admissible mapping such that*

$$(\alpha x)(\beta x)+\ell)^{(\alpha x)(\beta x)} \leq M(x,y) - \psi(M(x,y)) + \ell$$

for all $x,y \in X_\rho$ where $\psi \in \Psi$ and $l > 1$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

3. Application

In this section, firstly we shall apply Corollary 2.3 to show the existence of solution of integral equation. Let φ be a Musielak-Orlicz function on a measurable space $C = ([0,1], \Lambda, \mu)$, where ρ_φ is a modular defined by

$$\rho_\varphi(u) = \int_0^1 \varphi(s, |u(s)|)ds$$

for $\in u\ \mathscr{L}^\varphi$ and $\alpha_0 > e$ and $c_0 \in [\frac{e}{\alpha_0}, 1)$. Assume that ρ_φ is convex satisfying the Δ_2-condition. Now, we investigate the existence and uniqueness of solution of integral equation:

$$u(t) = e^{-t}f + \int_0^t e^{s-t}(\int_0^1 K(\xi, u(s))d\xi)ds,$$

where $K: [0,1] \times \mathscr{L}^\varphi \to \mathscr{L}^\varphi$ is a measurable function satisfying:

(1) $\lim\limits_{\lambda \to 0^+} \int_0^1 \varphi(\xi, \lambda \left| (\int_0^1 K(s,u)ds)\xi \right| d\xi = 0$ for any $u \in \mathscr{L}^\varphi$.

(2) $\left| \int_0^1 (K(\xi, u(s)) - K(\xi, v(s)))d\xi \right| \leq k|(u-v)(s)|$ for any $u,v \in \mathscr{L}^\varphi$ with $k \in (0,1)$.

(3) We denote by $B = C([0,1], A)$ the space of all ρ-continuous function from $[0,1]$ into A which is a convex, closed, bounded subset of $\mathscr{L}^\varphi$. So, B is a closed, bounded, convex subset of $C([0,1], \mathscr{L}^\varphi)$ satisfying the Δ_2-condition.

Let $T: B \to B$ defined by

$$T(u) = \int_0^1 \frac{c_0}{e} K(s,u)ds.$$

(4) $f \in B$.

(5) There exists $u_0 \in B$ such that $\theta(u_0) \geq 0$, $\eta(u_0) \geq 0$ and

$$\theta(u) \geq 0 \text{ for some } u \in B \text{ implies } \eta(Tu) \geq 0,$$
$$\eta(u) \geq 0 \text{ for some } u \in B \text{ implies } \theta(Tu) \geq 0.$$

(6) if $\{u_n\}$ is a sequence in B such that $\theta(u_n) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to u$ as $n \to \infty$, then $\theta(u) \geq 0$.
(7) Let $\alpha, \beta : B \to [0, \infty)$ by

$$\alpha(u) = \begin{cases} 1, & \theta(u) \geq 0 \\ 0, & otherwise \end{cases} \text{ and } \beta(v) = \begin{cases} 1, & \eta(u) \geq 0 \\ 0, & otherwise. \end{cases}$$

Theorem 3.1. *Under the above assumptions (1)-(7), the integral equation has a solution in $C([0,1], \mathscr{L}^{\varphi})$.*

Proof. Firstly, we show that T is ρ-Lipschitz. By assumption (1), we have $\int_0^1 \varphi(\xi, \lambda\, |Tu(\xi)|\, d\xi \to 0$ as $\lambda \to 0^+$. Hence the definition of $\mathscr{L}^{\varphi}$, we get $Tu \in \mathscr{L}^{\varphi}$ for any $u \in \mathscr{L}^{\varphi}$.
Let $x, y \in B$, then we have

$$\begin{aligned}
\rho_f(Tu - Tv) &= \rho_f\left(\frac{c_0}{e}\left(\frac{e}{c_0}(Tu - Tv)\right)\right) \\
&\leq \frac{c_0}{e}\rho_f\left(\frac{e}{c_0}(Tu - Tv)\right) \\
&= \frac{c_0}{e}\int_0^1 \varphi\left(s, \frac{e}{c_0}\,|(Tu - Tv)(s)|\, ds\right. \\
&= \frac{c_0}{e}\int_0^1 \varphi\left(s, \frac{e}{c_0}\left|\int_0^1 (K(\xi, u(s)) - K(\xi, v(s)))d\xi\right|\right)ds.
\end{aligned}$$

Therefore by assumption (2)

$$\begin{aligned}
\rho_f(Tu - Tv) &\leq \frac{c_0}{e}\int_0^1 \varphi(s, k\,|(u - v)(s)|)ds \\
&= \frac{c_0}{e}\rho_{\varphi}(k(u - v)) \\
&= \frac{c_0}{e}k\rho_{\varphi}(u - v).
\end{aligned}$$

Then, we get T is ρ-Lipschitz (see Theorem 1.3 in [33]). Also define $\psi, \phi : C([0,1], \mathscr{L}^{\varphi}) \to C([0,1], \mathscr{L}^{\varphi})$ by

$$\psi(u) = u, \text{ and } \phi(u) = \left(1 - \frac{c_0}{e}k\right)u \text{ for } \frac{c_0}{e}k \in (0,1).$$

Consequently, for all $u, v \in B$ we have

$$\alpha(u)\beta(v)\psi(\rho_{\varphi}(Tu - Tv)) \leq \psi(\rho_{\varphi}(u - v)) - \phi(\rho_{\varphi}(u - v)).$$

It shows that all the hypotheses of Corollary 2.3 are satisfied, hence T has a solution $u \in C([0,1], \mathscr{L}^{\varphi})$. □

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A Bound for the Joint Spectral Radius of Operators in a Hilbert Space

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Abstract

We suggest a bound for the joint spectral radius of a finite set of operators in a Hilbert space. In appropriate situations that bound enables us to avoid complicated calculations and gives a new explicit stability test for the discrete time switched systems. The illustrative example is given. Our results are new even in the finite dimensional case.

1. Introduction and statement of the main result

Let $\mathscr{H}$ be a complex separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and unit operator I. By $\mathscr{B}(\mathscr{H})$ we denote the set of all bounded linear operators in $\mathscr{H}$. For an $A \in \mathscr{B}(\mathscr{H})$, $\sigma(A)$ is the spectrum, $r_s(A)$ is the spectral radius; A^* is adjoint to A, and $\|A\| = \sup_{h\in\mathscr{H}, h\neq 0} \|Ah\|/\|h\|$.
Let $\mathscr{M} = \{A_1, \ldots, A_\nu\}$ be a finite set of operators $A_j \in \mathscr{B}(\mathscr{H})$ $(j = 1, \ldots, \nu)$. Our main object is the joint spectral radius $\rho(\mathscr{M})$ of $\mathscr{M}$ defined by

$$\rho(\mathscr{M}) := \lim_{k\to\infty} \sup\{\|A_{i_k} \cdots A_{i_1}\|^{1/k} : A_i \in \mathscr{M}\},$$

cf. [1, 2]. The joint spectral radius arises naturally in a range of topics including the theory of difference equations [3], control and stability of discrete time switched systems [4, 5, 6, 7, 8, 9, 10, 11], wavelets [12], ergodic theory [13], etc.
The literature on the theory of the joint spectral radius is rather rich, cf. [14], [15], [16], [17, 18] and references therein. Mainly, the finite dimensional operators were considered and the numerical methods were developed.
In the present paper, under some restrictions, we suggest a bound for $\rho(\mathscr{M})$. In appropriate situations that bound enables us to avoid complicated calculations and gives an explicit stability test for the discrete time switched systems. The example characterizing the sharpness of our results is given. To the best of our knowledge, our results are new even in the finite dimensional case.
Let $A \in \mathscr{B}(\mathscr{H})$ with $r_s(A) < 1$. Then the discrete Lyapunov equation

$$X - A^*XA = I \tag{1.1}$$

has a positive definite self-adjoint solution $X(A)$ [19]. It can be represented by

$$X(A) = \sum_{j=0}^{\infty} (A^*)^j A^j \tag{1.2}$$

and

$$X(A) = \frac{1}{2\pi}\int_0^{2\pi} (Ie^{-i\omega} - A^*)^{-1}(Ie^{i\omega} - A)^{-1} d\omega, \tag{1.3}$$

cf. [20, Section 7.1]. We will say that $\mathscr{M}$ is Schur-Kohn stable, if $\rho(\mathscr{M}) < 1$. Now we are in a position to formulate our main result.

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Theorem 1.1. *Let there be an $A \in \mathscr{B}(\mathscr{H})$ with $r_s(A) < 1$, such that*

$$\|X(A)\|(2\|A-A_k\|\|A\| + \|A-A_k\|^2) < 1 \quad (A_k \in \mathscr{M};\ k = 1,...,\nu). \tag{1.4}$$

Then $\mathscr{M}$ is Schur-Kohn stable. Moreover,

$$\rho(\mathscr{M}) \leq \sqrt{1 - \frac{1}{\|X(A)\|}(1 - \max_{j=1,...,\nu} \|X(A)\|(2\|A-A_j\|\|A\| + \|A-A_j\|^2)}.$$

The proof of this theorem is presented in the next section. In Theorem 1.1, one can take $A = A_m$ for an $A_m \in \mathscr{M}$. Below we consider some concrete classes of operators. Note that from (1.2) and (1.3) it follows that

$$\|X(A)\| \leq \sum_{j=1}^{\infty} \|A^j\|^2 \tag{1.5}$$

and

$$\|X(A)\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|(Ie^{i\omega} - A)^{-1}\|^2 d\omega. \tag{1.6}$$

If A is normal: $AA^* = A^*A$, then $\|A\| = r_s(A)$ and (1.5) implies

$$\|X(A)\| \leq \sum_{j=0}^{\infty} r_s^{2j}(A) = \frac{1}{1 - r_s^2(A)}. \tag{1.7}$$

2. Proof of Theorem 1.1

In this section for the simplicity we put $X(A) = X$.

Lemma 2.1. *Let $A, \tilde{A} \in \mathscr{B}(\mathscr{H})$, $r_s(A) < 1$ and X be a solution of (1.1). If*

$$\|X\|(2\|A-\tilde{A}\|\|A\| + \|A-\tilde{A}\|^2) < 1,$$

then

$$(X\tilde{A}x, \tilde{A}x) \leq (1 - \frac{c_0}{\|X\|})(Xx, x) \quad (x \in \mathscr{H}),$$

where

$$c_0 := 1 - \|X\|(2\|A-\tilde{A}\|\|A\| + \|A-\tilde{A}\|^2).$$

Proof. Put $Y = \tilde{A} - A$. Then

$$X - \tilde{A}^*X\tilde{A} = X - (Y+A)^*X(Y+A) = X - A^*XA - Y^*XA - A^*XY - Y^*XY = I - Y^*XA - A^*XY - Y^*XY.$$

By (2.1)

$$\|I - Y^*XA - A^*XY - YXY\| \geq 1 - \|Y^*XA - A^*XY - Y^*XY\|, \geq 1 - \|X\|(2\|A-\tilde{A}\| + \|A-\tilde{A}\|^2) = c_0.$$

Thus,

$$X - \tilde{A}^*X\tilde{A} \geq c_0 I.$$

Hence,

$$(Xx, x) - (X\tilde{A}x, \tilde{A}x) \geq c_0(x, x) \geq c_0(\frac{X}{\|X\|}x, x) = \frac{c_0}{\|X\|}(Xx, x),$$

as claimed. □

Proof of Theorem 1.1: Define the norms

$$|x|_X = \sqrt{(Xx, x)} \quad (x \in \mathscr{H}) \text{ and } |A|_X = \sup_{x \in \mathscr{H}} \frac{|Ax|_X}{|x|_X}.$$

Due to Lemma 2.1 and (1.4) we have

$$|A_j|_X^2 \leq 1 - \frac{c_j}{\|X\|}, \tag{2.1}$$

where

$$c_j := 1 - \|X\|(2\|A-A_j\|\|A\| + \|A-A_j\|^2).$$

Put

$$a_0 := \max_j \sqrt{1 - \frac{c_j}{\|X\|}} = \sqrt{1 - \frac{1}{\|X\|}(1 - \max_j \|X\|(2\|A - A_j\|\|A\| + \|A - A_j\|^2)}$$

Then by (2.1)

$$\max_j |A_j|_X \le a_0. \tag{2.2}$$

Since X is positive definite, it is boundedly invertible. For any $T \in \mathscr{B}(\mathscr{H})$ one has

$$\frac{\|Tx\|^2}{\|x\|^2} = \frac{(X^{-1}XTx, Tx)}{(X^{-1}Xx, x)} \le \frac{(X^{-1}XTx, Tx)}{(\frac{1}{\|X\|}Xx, x)} \le \|X\|\|X^{-1}\|\frac{(XTx, Tx)}{(Xx, x)} \ (x \in \mathscr{H}).$$

So

$$\|T\|^2 \le \|X\|\|X^{-1}\||T|_X^2$$

Hence, according to (2.2),

$$\|A_{i_k} \cdots A_{i_1}\| \le (\|X\|\|X^{-1}\|)^{1/2}|A_{i_k} \cdots A_{i_1}|_X \le (\|X\|\|X^{-1}\|)^{1/2} a_0^k$$

and therefore,

$$\rho(\mathscr{M}) \le \overline{\lim}_{k\to\infty} (\|X\|\|X^{-1}\|)^{1/2k} a_0 = a_0,$$

as claimed. □

3. Concrete classes of operators

In this section we suggest estimates for $X(A)$ under various assumptions. From (1.6) it follows

$$\|X(A)\| \le \sup_{|z|=1} \|(Iz - A)^{-1}\|^2. \tag{3.1}$$

Let there be monotonically increasing non-negative continuous function $F(x)$ $(x \ge 0)$, such that $F(0) = 0$, $F(\infty) = \infty$ and

$$\|(\lambda I - A)^{-1}\| \le F(1/\text{dist}(A, \lambda)) \ \ (\lambda \notin \sigma(A)),$$

where $\text{dist}(A, \lambda) = \inf_{s \in \sigma(A)} |s - \lambda|$. If $r_s(A) < 1$ and $|z| = 1$, then obviously, $\text{dist}(A, z) \ge 1 - r_s(A)$ and therefore, $\|(Iz - A)^{-1}\| \le F(1/(1 - r_s(A)))$. Now (3.1) implies

$$\|X(A)\| \le F^2\left(\frac{1}{1 - r_s(A)}\right). \tag{3.2}$$

3.1. Operators in finite dimensional spaces

Let $\mathbb{C}^n$ $(n < \infty)$ be the complex n-dimensional Euclidean space with a scalar product $(.,.)$, the Euclidean norm $\|.\| = \sqrt{(.,.)}$ and unit matrix I, $\mathbb{C}^{n\times n}$ is the set of all $n \times n$ matrices. $\lambda_k(A), k = 1, ..., n$, are the eigenvalues of $A \in \mathbb{C}^{n\times n}$, counted with their multiplicities; $N_2(A) = (\text{trace } AA^*)^{1/2}$ is the Hilbert-Schmidt (Frobenius) norm of A. The quantity (the departure from normality of A)

$$g(A) = [N_2^2(A) - \sum_{k=1}^n |\lambda_k(A)|^2\,]^{1/2},$$

plays an essential role hereafter. The following relations are checked in [21, Section 3.1]:

$$g^2(A) \le N_2^2(A) - |\text{trace } A^2| \text{ and } g^2(A) \le \frac{N_2(A - A^*)}{2} = 2N_2^2(A_I),$$

where $A_I = (A - A^*)/2i$. If A is a normal matrix, then $g(A) = 0$.
Due to Example 3.3 from [21],

$$\|A^m\| \le \sum_{k=0}^{n-1} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}} \ \ (m = 1, 2, ...).$$

Now (1.5) implies

$$\|X(A)\| \le \xi_n(A) := \sum_{j=1}^{\infty} \left(\sum_{k=0}^{n-1} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}}\right)^2 \ \ (A \in \mathbb{C}^{n\times n}). \tag{3.3}$$

Note that if A is normal, then $g(A) = 0$ and (3.3) gives us the sharp inequality (1.7).
Theorem 1.1 and (3.3) yield the following corollary.

Corollary 3.1. *Let $\mathscr{M}$ be a finite set of $n \times n$-matrices. Let there be an $n \times n$-matrix A with $r_s(A) < 1$, such that*

$$\xi_n(A) \max_{B\in\mathscr{M}} (2\|A-B\|\|A\| + \|A-B\|^2) < 1.$$

Then $\mathscr{M}$ is Schur-Kohn stable. Moreover,

$$\rho(\mathscr{M}) \le \sqrt{1 - \frac{1}{\xi_n(A)}(1 - \xi_n(A) \max_{B\in\mathscr{M}} (2\|A-B\|\|A\| + \|A-B\|^2)}.$$

Let us point the more compact but less sharper estimate for $X(A)$. To this end put

$$\eta_n(A) := \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}(1-r_s(A))^{k+1}}.$$

By Theorem 3.2 from [21]

$$\|(A-\lambda I)^{-1}\| \le \sum_{k=0}^{n-1} \frac{g^k(A)}{(\mathrm{dist}(A,\lambda))^{k+1}\sqrt{k!}} \quad (A \in \mathbb{C}^{n\times n}, \lambda \notin \sigma(A)).$$

Making use of (3.2) we can assert that

$$\|X(A)\| \le \eta_n^2(A) \ \ (A \in \mathbb{C}^{n\times n}).$$

So in Corollary 3.1 one can replace $\xi_n(A)$ by $\eta_n^2(A)$.

3.2. Hilbert-Schmidt operators

Denote by SN_2 the ideal of Hilbert-Schmidt operators in $\mathscr{H}$ with the finite norm $N_2(A) = (\text{trace } AA^*)^{1/2}$. In the infinite dimensional case we put

$$g(A) = [N_2^2(A) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2]^{1/2},$$

where $\lambda_k(A), k = 1,2,...,$ are the eigenvalues of $A \in SN_2$, counted with their multiplicities and enumerated in the non-increasing order of their absolute values.
Since

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^2 \ge |\sum_{k=1}^{\infty} \lambda_k^2(A)| = |\text{trace } A^2|,$$

one can write

$$g^2(A) \le N_2^2(A) - |\text{trace } A^2|.$$

If A is a normal Hilbert-Schmidt operator, then $g(A) = 0$, since

$$N_2^2(A) = \sum_{k=1}^{\infty} |\lambda_k(A)|^2$$

in this case. Moreover,

$$g^2(A) \le \frac{N_2^2(A-A^*)}{2} = 2N_2^2(A_I),$$

cf. [21, Section 7.1]. Due to Corollary 7.4 from [21] for any $A \in SN_2$ we have

$$\|A^m\| \le \sum_{k=0}^{m} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}} \quad (m = 1,2,...).$$

Now (1.5) implies

$$\|X(A)\| \le \xi_\infty(A) := \sum_{j=1}^{\infty} \left(\sum_{k=0}^{m} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}} \right)^2 \quad (A \in SN_2). \tag{3.4}$$

If A is normal, then (3.4) gives us inequality (1.7).
Furthermore, by Theorem 7.1 from [21] for any $A \in SN_2$ we have

$$\|R_\lambda(A)\| \le \sum_{k=0}^{\infty} \frac{g^k(A)}{(\mathrm{dist}(A,\lambda))^{k+1}\sqrt{k!}} \quad (\lambda \notin \sigma(A)).$$

Inequality (3.2) gives us the more compact but less sharper estimate

$$\|X(A)\| \le \eta_\infty^2(A) \ \ (A \in SN_2),$$

where

$$\eta_\infty(A) := \sum_{j=0}^{\infty} \frac{g^j(A)}{\sqrt{j!}(1-r_s(A))^{j+1}}.$$

Now we can directly apply Theorem 1.1.
By the Schwarz inequality

$$(\sum_{j=0}^{\infty} \frac{(cg)^j(A)}{c^j\sqrt{j!}(1-r_s(A))^j})^2 \le \sum_{k=0}^{\infty} c^{2k} \sum_{j=0}^{\infty} \frac{g^{2j}(A)}{c^{2j} j!(1-r_s(A))^{2j}} =$$

$$\frac{1}{1-c^2} \exp\,[\frac{g^2(A)}{c^2(1-r_s(A))^2}] \;\; (c \in (0,1)).$$

Thus,

$$\|X(A)\| \le \frac{1}{(1-c^2)(1-r_s(A))^2} \exp\,[\frac{g^2(A)}{c^2(1-r_s(A))^2}] \;\; (A \in SN_2, c \in (0,1)).$$

In particular, taking $c^2 = 1/2$, we get

$$\|X(A)\| \le \hat{\eta}(A) := \frac{2}{(1-r_s(A))^2} \exp\,[\frac{2g^2(A)}{(1-r_s(A))^2}].$$

Now Theorem 1.1 implies the following corollary.

Corollary 3.2. *Let $\mathscr{M}$ be a finite set of bounded operators from $\mathscr{H}$ Let there be an $A \in SN_2$ with $r_s(A) < 1$, such that*

$$\hat{\eta}(A) \max_{B \in \mathscr{M}} (2\|A-B\|\|A\| + \|A-B\|^2) < 1.$$

Then $\mathscr{M}$ is Schur-Kohn stable. Moreover,

$$\rho(\mathscr{M}) \le \sqrt{1 - \frac{1}{\hat{\eta}(A)}(1 - \hat{\eta}(A) \max_{B \in \mathscr{M}} (2\|A-B\|\|A\| + \|A-B\|^2)}.$$

Similarly, making use of Theorems 7.2, 7.3 from [21] one can apply Theorem 1.1 to Shatten-von Neumann operators.

3.3. Non-compact non-normal operators

In this subsection we suggest a norm estimate for the solution of (1.1) under the condition

$$A_I = (A - A^*)/(2i) \in SN_2. \tag{3.5}$$

To this end introduce the quantity

$$g_I(A) := \sqrt{2}\left[N_2^2(A_I) - \sum_{k=1}^{\infty} (\Im\, \lambda_k(A))^2\right]^{1/2}.$$

Obviously, $g_I(A) \le \sqrt{2} N_2(A_I)$. Due to Example 10.2 from [21],

$$\|A^m\| \le \sum_{k=0}^{m} \frac{m! r_s^{m-k}(A) g_I^k(A)}{(m-k)!(k!)^{3/2}} \;\; (m = 1, 2, \ldots).$$

Now (1.5) implies

$$\|X(A)\| \le \xi_I(A) := \sum_{j=0}^{\infty} \left(\sum_{k=0}^{m} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}}\right)^2 \;\; (A_I \in SN_2).$$

If A is normal from this inequality we get (1.7).
Furthermore, by Theorem 9.1 from [21] under condition (4.1) we have,

$$\|R_\lambda(A)\| \le \sum_{k=0}^{\infty} \frac{g_I^k(A)}{(\mathrm{dist}(A,\lambda))^{k+1}\sqrt{k!}}$$

and

$$\|R_\lambda(A)\| \le \frac{\sqrt{e}}{\mathrm{dist}(A,\lambda)} \exp\,[\frac{g_I^2(A)}{2(\mathrm{dist}(A,\lambda))^2}] \;\; (\lambda \notin \sigma(A)).$$

Inequality (3.2) implies

$$\|X(A)\| \le \eta_I^2(A) \text{ and } \|X(A)\| \le \hat{\eta}_I^2(A) \;\; (A_I \in SN_2),$$

where

$$\eta_I := \sum_{j=0}^{\infty} \frac{g_I^j(A)}{\sqrt{j!}(1-r_s(A))^{j+1}}$$

and

$$\hat{\eta}_I := \frac{\sqrt{e}}{1-r_s(A)} \exp\left[\frac{g_I^2(A)}{2(1-r_s(A))^2}\right].$$

Now we can directly apply Theorem 1.1.
Some other classes of operators can be considered, in particular, via norm estimates for operator functions from [21].

4. Example

The following example characterizes the sharpness of Theorem 1.1.
Let $\mathscr{H} = \mathbb{C}^n$, $\mathscr{M} = \{A_1, A_2\}$ with real positive matrices matrices $A_1 = \text{diag}\,(a_k)_{k=1}^n, A_2 = \text{diag}\,(b_k)_{k=1}^n$; $(a_k, b_k \geq 0)$ and $r_s(A_1) < 1$. So $a_m = \max_k a_k < 1$. Take $A = A_1$. Since A_1 is Hermitian, according to (1.7) condition (1.4) takes the form

$$\frac{1}{1-r_s^2(A_1)}(2r_s(A_1)\|A_1 - A_2\| + \|A_1 - A_2\|^2) < 1. \tag{4.1}$$

Besides, $\|A_1 - A_2\| = \max_k |a_k - b_k|$.
Assume that $r_s(A_2) \geq 1$. Namely, $b_m \geq 1$. So $\mathscr{M}$ is Schur-Kohn unstable. Then $|a_m - b_m| = b_m - a_m \geq 1 - a_m$ and

$$\frac{1}{1-a_m^2}(2a_m|a_m - b_m| + |a_m - b_m|^2) \geq \frac{1}{1-a_m^2}(2a_m(1-a_m) + (1-a_m)^2) \geq \frac{1}{1+a_m}(2a_m + 1 - a_m) \geq 1.$$

Therefore, condition (4.1) is not fulfilled. So condition (1.4) is necessary under consideration.

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The Existence and Uniqueness of Initial-Boundary Value Problems of the Fractional Caputo-Fabrizio Differential Equations

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Abstract

In this paper, the existence and uniqueness problem of the initial and boundary value problems of the linear fractional Caputo-Fabrizio differential equation of order $\sigma \in (1,2]$ have been investigated. By using the Laplace transform of the fractional derivative, the fractional differential equations turn into the classical differential equation of integer order. Also, the existence and uniqueness of nonlinear boundary value problem of the fractional Caputo-Fabrizio differential equation has been proved. An application to mass spring damper system for this new fractional derivative has also been presented in details.

1. Introduction

The fractional differential calculus has gained much interest by the many researcher in the last decades and it has strong mathematical background and many papers are attributed to the development of it. Among them, we can cite some e.g, [1, 2, 3]. Fractional calculus has been also used for modelling physical phenomena including control systems, mechanics, viscoelasticity [4, 5, 6]. Up to now, several definition of fractional derivative has been proposed. Some frequently used definition of fractional derivative can be given as the Riemann-Lioville, Caputo, Grünwald–Letnikov [7, 8] and conformable fractional derivative [9, 10, 11, 12, 13]. Among them, the Riemann-Lioville definition requires nonlocal initial condition, so it does not reflect physical experiment while the Caputo definition allows to use the classical initial condition. In the recent years, a new definition of fractional order derivative has been defined by Caputo and Fabrizio [14] with a regular kernel. This new definition can be able to describe better heterogeneousness and systems with different scales with memory effects. The other good property of this new definition is that the real power turn into the integer by the Laplace transformation, thus the exact solution can be easily found for some cases. Some properties of this definition have been studied in [15]. Several papers are devoted to development of this new fractional derivative [16, 17]. Some applications based on this new fractional derivative can be found in the papers [18, 19, 20, 21, 22, 23]. In this paper, the previous results will be extended and the existence and uniqueness solution will be given for high order fraction derivative. As an application, a mass-spring-damper system will be analyzed basen on this new derivative. In [15], the results are presented when the fractional order $\alpha \in (0,1)$. The aim of this paper enriches these results for the case when the fractional order of $\alpha+1 \in (1,2)$. In [19], a mass spring damper motion has been studied, but the solution available only for numerical approximation using Laplace transform algorithm. More importantly, they consider the fractional order $2\alpha \in (1,2)$ when $\alpha \in (0,1)$. However, this is not true when $\alpha \in (0,1/2)$. Additionally, the Caputo-Fabrizio fractional operator does not have semigroup property. For this reason, the different cases of the fractional order also have been examined and the exact solution for each case is given for the mass spring damper equation using only the Laplace transformation.

The rest of the paper is organized as follows. In Section 2, preliminaries and previous related works have been introduced. The existence and uniqueness results for linear problems have been presented in Section 3. Some simple but important initial and boundary value problems of the fraction Caputo-Fabrizio differential equation are given in Section 4. In Section 5, the existence and uniqueness of the nonlinear

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boundary value problem of the fraction Caputo-Fabrizio differential equation have been demonstrated. Finally, an application to a mass spring damper system is given in the last section.

2. Preliminaries and Previous Results

We present some definitions and previous results of the new fractional Caputo–Fabrizio derivative that are needed in this work.

Definition 2.1. *Let $\alpha \in (0,1)$ and $f \in H^1(a,b), a<b$. The Caputo fractional derivative of the function f defined as*

$$D_C^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{\alpha} f'(t)dt.$$

Definition 2.2. *[14] Given $a<b$ and $f \in H^1(a,b)$, the fractional Caputo-Fabrizio derivative of the function of order $\alpha \in (0,1)$ is defined for $t \geq 0$*

$$^{CF}D^{\alpha} f(x) = \frac{1}{1-\alpha} \int_a^x \exp\Big(-\frac{\alpha}{1-\alpha}(x-t)\Big) f'(t)dt. \tag{2.1}$$

Definition 2.3. *[15] The Caputo-Fabrizio fractional integral of oder $\alpha \in (0,1)$ is defined as*

$$^{CF}\mathscr{I}(f)(x) = (1-\alpha)f(x) + \alpha \int_0^x f(s)ds.$$

The Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0,1)$ and $n \in \mathbb{N}$ defined as

$$^{CF}D^{\alpha+n} f(x) :=^{CF} D^{\alpha}(^{CF}D^n f(x)).$$

Theorem 2.4. *[14] Let the function $f(x)$ satisfy $f^{(k)}(a) = 0, \quad k = 1,2,\ldots,n$, then the equality*

$$^{CF}D^{\alpha}(^{CF}D^n f(x)) =^{CF} D^n(^{CF}D^{\alpha} f(x)) \tag{2.2}$$

holds.

Definition 2.5. *For $\sigma = \alpha+1$ with $\alpha \in (0,1)$, the Caupto-Fabrizio fractional derivative of order σ defined as*

$$^{CF}D^{\sigma} f(x) = \frac{1}{1-\alpha} \int_a^x \exp\Big(-\frac{\alpha}{1-\alpha}(x-t)\Big) f''(t)dt. \tag{2.3}$$

Note that the equality $^{CF}D^{\alpha}(^{CF}D^1 f(x)) = CFD^1(^{CF}D^{\alpha} f(x))$ is defined unambiguously when $f'(0) = 0$. (see [14])

Definition 2.6. *For a function $f(x)$, the Laplace transformation $F(s)$ of f is given by*

$$F(s) = \mathscr{L}\Big[f(x)\Big] = \int_0^{\infty} \exp(-sx) f(x)dx.$$

Lemma 2.7. *[14] The Laplace transform of the Caputo-Fabrizio fractional of order $\sigma = \alpha+n$ for $\alpha \in (0,1)$ and $n \in \mathbb{N}$ is given by*

$$\mathscr{L}\Big\{^{CF}D^{\sigma}(f)(x)\Big\}(s) = \frac{s^{n+1}\mathscr{L}\{f(x)\}(s) - s^n f(0) - s^{n-1}f'(0) - \cdots - f^{(n)}(0)}{s+\alpha(1-s)}.$$

3. Existence and Uniqueness of the Solution

We show the existence and uniqueness of the solution of the fractional differential equations involving the Caputo- Fabrizio fractional derivative in this section. We also derive the solution for some fractional differential equation that are important for physical applications.

Theorem 3.1. *[15] For $\alpha \in (0,1)$ and $h \in L_1(0,\infty)$, the following first order fractional differential equation*

$$^{CF}D^{\alpha}(u)(x) = h(x), \quad x \geq 0$$
$$u(0) = u_0$$

has the unique solution

$$u(x) = u_0 + (1-\alpha)(h(x) - h(0)) + \alpha \int_0^x h(s)ds.$$

Theorem 3.2. *[15] For $\alpha \in (0,1)$ and $h \in L_1(0,\infty)$, the following first order fractional differential equation*

$$^{CF}D^{\alpha}(u)(x) = \lambda u(x) + h(x), \quad \lambda \neq 0, \quad x \geq 0$$
$$u(0) = u_0$$

has the unique solution, when $\lambda(1-\alpha) = 1$

$$u(x) = -\frac{1-\alpha}{\lambda\alpha} u'(x) - \frac{\alpha}{\lambda} u(x)$$

and when $\lambda(1-\alpha) \neq 1$

$$u(x) = \frac{\lambda\alpha}{1-\lambda(1-\alpha)} \int_0^x u(s)ds + u_0 + \frac{1-\alpha}{1-\lambda(1-\alpha)}(h(x) - h(0)) + \frac{\alpha}{1-\lambda(1-\alpha)} \int_0^x h(s)ds.$$

Theorem 3.3. *[15] If $\alpha \in (0,1)$, then the function u solves the fractional differential equation*

$$^{CF}D^{\alpha}(u)(x) = 0, \quad x \geq 0$$

if and only if u is a constant function.

We study here the boundary value problem of a class of the Caputo-Fabrizio fractional differential equations of order $\sigma \in (1,2)$ on $[0,1]$

Theorem 3.4. *For $\sigma = \alpha + 1, \quad \alpha \in (0,1)$, and $g : [0,\infty) \to \mathbb{R}$ with $g \in L_1(0,\infty)$, the following boundary value problem of the fractional Caputo-Fabrizio differential equation*

$$^{CF}D^{\sigma}(u)(x) = g(x), \quad x \geq 0 \tag{3.1}$$

$$u(0) = u_0, \quad u(1) = u_1 \tag{3.2}$$

has the unique solution given by

$$\begin{aligned} u(x) &= u_0 + (u_1 - u_0)x + (1-\alpha)(1-x)\int_0^x g(t)dt + \alpha(1-x)\int_0^x tg(t)dt \\ &\quad - (1-\alpha)x\int_x^1 g(t)dt - \alpha x\int_x^1 (1-t)g(t)dt. \end{aligned}$$

Proof. Applying the Laplace operator to the equation (3.1), we get

$$\mathscr{L}\left\{^{CF}D^{\sigma}(u)(x)\right\}(s) = \mathscr{L}\left\{g(x)\right\}(s)$$

Appealing the Lemma 2.7, we are led to

$$\frac{s^2U(s) - su(0) - u'(0)}{s + \alpha(1-s)} = G(s)$$

where $U(s) = \mathscr{L}\left\{(u)(x)\right\}(s)$ and $G(s) = \mathscr{L}\left\{g(x)\right\}(s)$.
Equivalently, we can rewrite the last equation as

$$U(s) = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \frac{1-\alpha}{s}G(s) + \frac{\alpha}{s^2}G(s).$$

The inverse Laplace operator is applied to above equation to arrive at

$$u(x) = u(0) + xu'(0) + (1-\alpha)\int_0^x g(t)dt + \alpha\int_0^x (x-t)g(t)dt. \tag{3.3}$$

Taking into account the boundary conditions (3.2), we have the desired result

$$\begin{aligned} u(x) &= u_0 + (u_1 - u_0)x + (1-\alpha)(1-x)\int_0^x g(t)dt + \alpha(1-x)\int_0^x tg(t)dt \\ &\quad - (1-\alpha)x\int_x^1 g(t)dt - \alpha x\int_x^1 (1-t)g(t)dt. \end{aligned}$$

For the uniqueness, as usual, we suppose that there are two solutions of the problem, say v_1 and v_2. Then we must have

$$^{CF}D^{\sigma}(v_1)(x) - ^{CF}D^{\sigma}(v_2)(x) = ^{CF}D^{\sigma}(v_1 - v_2)(x) = ^{CF}D^{\alpha}(Dv_1 - Dv_2)(x) = 0$$

Thus, by Theorem 3.3 we get

$$Dv_1(x) = Dv_2(x).$$

This implies that $v_1(x) = v_2(x) + c$ for some constant c. But the condition $v_1(0) = v_2(0)$ leads to $c = 0$. That is $v_1(x) = v_2(x)$ for all $x \geq 0$. □

Remark 3.5. *In Theorem 3.4, if we let $h(x) := g(x) - g(0)$, then $h(0) = 0$ so that the initial value problem*

$$^{CF}D^{\sigma}(u)(x) = h(x), \quad x \geq 0$$

$$u(0) = A, \quad u'(0) = B$$

has the unique solution of much simpler form given by

$$u(x) = A + Bx + (1-\alpha)\int_0^x h(t)dt + \alpha\int_0^x (x-t)h(t)dt.$$

We further study the linear differential equation of fractional order in the sense of Caputo-Fabrizio fractional derivative, then we will work on nonlinear boundary value problems of the fractional Caputo-Fabrizio differential equations. We first give the results for the linear cases.

Theorem 3.6. *If $\sigma \in (1,2)$ and $g \in L_1(0,\infty) \cap C^1[0,\infty)$, then the following linear boundary value problem of the fractional Caputo-Fabrizio differential equation has the unique solution for all $\eta \in \mathbb{R}$.*

$$^{CF}D^{\sigma}(u)(x) = \eta u(x) + g(x), \quad \eta \neq 0, \quad x \geq 0 \tag{3.4}$$

$$u(0) = u_0, \quad u(1) = u_1 \tag{3.5}$$

Proof. The case when $\eta = 0$ is already was proved in Theorem 3.4. So, assume that $\eta \neq 0$. we see that from Theorem 3.4, the solution to (3.4) and (3.5) can be written as

$$\begin{aligned} u(x) &= u_0 + (u_1 - u_0)x + (1-\alpha)(1-x)\int_0^x \Big(\eta u(t) + g(t)\Big)dt + \alpha(1-x)\int_0^x t\Big(\eta u(t) + g(t)\Big)dt \\ &\quad - (1-\alpha)x\int_x^1 \Big(\eta u(t) + g(t)\Big)dt - \alpha x\int_x^1 (1-t)\Big(\eta u(t) + g(t)\Big)dt. \end{aligned}$$

A little algebraic manipulation reveals that

$$\begin{aligned} & u(x) + \eta x\int_0^1 (1-\alpha t)u(t)dt - \eta\int_0^x \Big(1-\alpha + x\alpha - t\alpha\Big)u(t)dt \\ &= u_0 + (u_1 - u_0)x + (1-\alpha)(1-x)\int_0^x g(t)dt + \alpha(1-x)\int_0^x tg(t)dt \qquad -(1-\alpha)x\int_x^1 g(t)dt - \alpha x\int_x^1 (1-t)g(t)dt \end{aligned} \tag{3.6}$$

Differentiating the equation (3.6) twice, we have that

$$u''(x) - (1-\alpha)\eta u'(x) = (1-\alpha)g'(x) + \alpha g(x). \tag{3.7}$$

Now we have two cases to analyze. First, we assume that $(1-\alpha)\eta = 0 \Leftrightarrow \alpha = 1$ since $\eta \neq 0$. In this case, the equation (3.7) becomes

$$u''(x) = g(x).$$

This is just a second order ordinary differential equation with solution given by

$$u(x) = -u_0 x + u_0 + u_1 x + (x-1)\int_1^0 \Big(\int_1^s g(y)dy\Big)ds + \int_1^x \Big(\int_1^s g(y)dy\Big)ds.$$

The second case when $(1-\alpha)\eta \neq 0$, we have

$$\begin{aligned} u(x) &= u_0 + \int_0^x e^{(1-\alpha)\eta t}\int_0^t e^{(1-\alpha)\eta s}\Big((1-\alpha)g'(s) + \alpha g(s)\Big)dsdt \\ &\quad + \frac{u_1 - u_0 - \int_0^1 e^{(1-\alpha)\eta t}\int_0^t e^{(1-\alpha)\eta s}\Big((1-\alpha)g'(s) + \alpha g(s)\Big)dsdt}{\int_0^1 e^{(1-\alpha)\eta t}dt}\int_0^x e^{(1-\alpha)\eta t}dt. \end{aligned}$$

□

4. Solutions of the initial and boundary value problem of the linear Caputo-Fabrizio fractional differential equations

In this section, some initial and boundary value problems of the fractional differential equation in the sense of the Caputo-Fabrizio derivative have been presented.

Example 4.1. *If $\sigma = \alpha + 1$ with $\alpha = \frac{1}{2}$ and $c_1, c_2 \in \mathbb{R}$, then the following initial value problem of fractional differential equation*

$$u''(x) + c_1^{CF}D^{\sigma}(u)(x) = c_2^{CF}D^{\alpha}u(x) + 1 - \exp(-x) \tag{4.1}$$

$$u(0) = 0, \quad u'(0) = 0 \tag{4.2}$$

has a unique solution of the form

$$u(x) = \frac{\exp(-\frac{3(2c_1+1)x}{2})\sinh\Big(\sqrt{\frac{9x(2c_1+1)^2}{4} + 2c_2}\Big)}{\sqrt{\frac{9x(2c_1+1)^2}{4} + 2c_2}}.$$

In fact, by the Laplace transformation, the equation can be written as

$$\begin{aligned} & s^2U(s) - su(0) - u'(0) + c_1\frac{s^2U(s) - su(0) - u'(0)}{(s+1)/2} - c_2\frac{s}{(s+1)/2} = \frac{s}{s+1} \\ & U(s)\Big(s^3 + (2c_1+1)s^2 - 2c_2 s\Big) = s \\ & U(s) = \frac{s}{s^3 + (2c_1+1)s^2 - 2c_2 s} \end{aligned}$$

where $U(s) = \{\mathscr{L}u(t)\}(s)$. Now, the inverse Laplace transformation gives us that

$$u(x) = \frac{\exp(-\frac{3(2c_1+1)x}{2})\sinh\Big(\sqrt{\frac{9x(2c_1+1)^2}{4} + 2c_2}\Big)}{\sqrt{\frac{9x(2c_1+1)^2}{4} + 2c_2}}$$

Example 4.2. *Consider the initial value problem*

$$
\begin{aligned}
&{}^{CF}D^{\sigma}u(x)+u(x)=0\\
&u(0)=1, \quad u'(0)=0
\end{aligned}
$$

where $\sigma=\alpha+1$ *with* $\alpha\in(0,1)$

Applying the Laplace transformation leads to have

$$U(s)(s^2+s+\alpha(1-s))=s.$$

Now, the inverse Laplace transformation gives the exact solution as follows

$$
\begin{aligned}
&u(x)=\exp(x(\alpha/2-1/2))(\cosh(x(\alpha^2/4-3\alpha/2+1/4)^{1/2})+(\sinh(x(\alpha^2/4\\
&-3\alpha/2+1/4)^{1/2})(\alpha/2-1/2))/(\alpha^2/4-3\alpha/2+1/4)^{1/2})
\end{aligned}
$$

Example 4.3. *Consider the system of fractional algebraic-differential equations*

$$
\begin{aligned}
&{}^{CF}D^{1/2}u(x)-xv(x)+u(x)-(1+x)v(t)=0\\
&v(x)=\sin x\\
&u(0)=1, \quad v(0)=0
\end{aligned}
$$

Applying the Laplace transformation, one gets

$$
\begin{aligned}
&\frac{sU(s)-1}{(s+1)/2}+V(s)+sV'(s)+U(s)-V(s)+V'(s)=0\\
&V(s)=\frac{1}{s^2+1}, \quad V'(s)=-\frac{2s}{(s^2+1)^2}\\
&U(s)=\frac{s(s+1)}{(1+s^2)^2}+\frac{1}{s+1}
\end{aligned}
$$

Now, the inverse Laplace transform gives the exact solution

$$u(x)=\frac{x+1}{2}\sin x+\frac{x}{2}\cos x+\exp(-x)$$

Example 4.4. *Consider the boundary value problem*

$$
\begin{aligned}
&{}^{CF}D^{3/2}u(x)=\lambda u(x)\\
&u(0)=0, \quad u(1)=1
\end{aligned}
$$

This is the equation given in the problem (3.4) and (3.5) with $\sigma=1+1/2$ *and* $u_0=0, u_1=1$. *Thus, the exact solution given by*

$$u(x)=\frac{1}{\int_0^1 e^{(\lambda/2)t}dt}\int_0^x e^{(\lambda/2)t}dt=\frac{e^{\lambda x/2}-1}{e^{\lambda/2}-1}$$

5. Nonlinear boundary value problems

We prove the existence and uniquness of the nonlinear boundary value problems of the Caputo-Fabrizio differential equations by the help of the Banach contraction principle.
Let $C(I)$ be the Banach space of continuous functions on $I=[0,1]$ with maximum norm

$$\|x\|=\max_{s\in[0,1]}|x(s)|, \quad x\in C(I).$$

We now state the existence and uniquness of the solution in the next theorem.

Theorem 5.1. *If* $\sigma=1+\alpha, \quad \alpha\in(0,1]$ *and* $F:[0,1]\times\mathbb{R}\to\mathbb{R}$ *is a continuous function with the property that*

$$\Big|F(x,u_1)-F(x,u_2)\Big|\le q|u_1-u_2| \quad u_1,u_2\in\mathbb{R}, \quad q>0,$$

then the boundary value problem

$$
\begin{aligned}
&{}^{CF}D^{\sigma}(u)(x)=F(x,u(x)), \quad x\ge 0 && (5.1)\\
&u(0)=u_0, \quad u(1)=u_1 && (5.2)
\end{aligned}
$$

has a unique solution in $C(I)$ *provided* $q<1$.

Proof. Let the operator $T : C(I) \to C(I)$ be given by

$$(Tu)(x) = u_0 + (u_1 - u_0)x + (1-\alpha)(1-x)\int_0^x F(t,u(t))dt + \alpha(1-x)\int_0^x tF(t,u(t))dt$$
$$-(1-\alpha)x\int_x^1 F(t,u(t))dt - \alpha x\int_x^1 (1-t)F(t,u(t))dt$$

We see that the solution for the problem (5.1) and (5.2) is the fixed point of the map T. For $u,v \in C(I)$ and $0 \le t \le 1$, we find that

$$\begin{aligned}\Big|(Tu)(x)-(Tv)(x)\Big| &= \Big|(1-\alpha)(1-x)\int_0^x \Big(F(t,u(t))-F(t,v(t)\Big)dt + \alpha(1-x)\int_0^x t\Big(F(t,u(t))-F(t,v(t)\Big)dt \\ &\quad -(1-\alpha)x\int_x^1 \Big(F(t,u(t))-F(t,v(t)\Big)dt - \alpha x\int_x^1 (1-t)\Big(F(t,u(t))-F(t,v(t)\Big)dt\Big| \\ &\le (1-\alpha)(1-x)xq\|u-v\| + \alpha(1-x)\frac{x^2}{2}q\|u-v\| + (1-\alpha)x(1-x)q\|u-v\| + \alpha x\frac{(1-x)^2}{2}q\|u-v\| \\ &= (1-x)x\frac{4-3\alpha}{2}q\|u-v\| \le \max_{x\in[0,1]}(1-x)x\frac{4-3\alpha}{2}q\|u-v\| \le \frac{4-3\alpha}{8}q\|u-v\| \le q\|u-v\|.\end{aligned}$$

Since $q<1$, the operator T is a contraction, and by the Banach contraction theorem T must have a unique fixed point that is the solution of the problem (5.1) and (5.2). □

6. An Application to a Mass-Spring-Damper System

In [19], a mass spring damper system equation has been modelled by the Caputo-Fabrizio fractional differential equation as follows

$$\frac{m}{\mu^{2(1-\alpha)}}{}^{CF}D^{2\alpha}u(x) + \frac{c}{\mu^{1-\alpha}}{}^{CF}D^{\alpha}u(x) + ku(x) = F(x), \quad \alpha \in (0,1]. \tag{6.1}$$

where μ is the dimension of second, m is the damping coefficient, c is the spring constant and $F(x)$ is the force of the system. The parameter μ is introduced because of the dimensionless quantity of the physical problem in the case of fractional derivative of the displacement. The equation (6.1) has been provided with an initial displacement, u_0, and velocity, $v_0 = 0$ for the mass m. As in [19], two cases for the forcing term will be considered. Additionally, the order of the fractional is also considered in two cases.

1 . Assume that the forcing term $F(x) = A$ for some constant A. Moreover, suppose that $\alpha \in (0,1/2)$ so that $2\alpha \in (0,1)$. Applying the Laplace transform of (6.1) leads to get

$$\frac{sU(s)-u_0}{s+2\alpha(1-s)} + c\frac{\eta^2\mu^{\alpha-1}}{k}\Big(\frac{sU(s)-u_0}{s+\alpha(1-s)}\Big) + \eta^2U(s) = A\frac{\eta^2}{k}\frac{1}{s} \tag{6.2}$$

$$U(s) = \frac{u_0(s+\alpha(1-s)+B(s+2\alpha(1-s)))}{s(s+\alpha(1-s)+s+2\alpha(1-s)\Big(B+\eta^2(s+\alpha(1-s))\Big)} \tag{6.3}$$

$$+\frac{(\eta^2/k)A(s+2\alpha(1-s))(s+\alpha(1-s))}{s^2\Big(s+\alpha(1-s)+s+2\alpha(1-s)\Big(B+\eta^2(s+\alpha(1-s))\Big)} \tag{6.4}$$

where $\frac{\eta^2}{k} = \frac{\mu^{2(1-\alpha)}}{m}$ and $B = c\frac{\eta^2}{k}\mu^{\alpha-1}$
The inverse Laplace transform yields the exact solution

$$\begin{aligned}u(x) &= (A\eta^2/k)\Big(\frac{2\alpha x}{(2\alpha\eta^2+2B+1} - \frac{-2\alpha\eta^2+2\alpha-6B+4\alpha B+1}{(2\alpha\eta^2+2B+1)^2}\exp(\frac{x(\alpha+2\alpha B-2\alpha\eta^2+4\alpha^2\eta^2-2)}{4\alpha\eta^2(\alpha-1)} \\ &(\cosh((x(4\alpha^2B^2+8\alpha^2B\eta^2+4\alpha^2B+4\alpha^2n^4-12\alpha^2\eta^2+\alpha^2-8\alpha B+8\alpha\eta^2-4\alpha+4)^{1/2})/(4\alpha\eta^2(\alpha-1)))+ \\ &(\sinh((x(4\alpha^2B^2+8\alpha^2B\eta^2+4\alpha^2B+4\alpha^2n^4-12\alpha^2\eta^2+\alpha^2-8\alpha B+8\alpha\eta^2-4\alpha+4)^{1/2})/(4\alpha\eta^2(\alpha-1))) \\ &(8\alpha^2B^2+4\alpha^2B\eta^2+8\alpha^2B+4\alpha^2n^4-14\alpha^2\eta^2+2\alpha^2-12\alpha B^2+4\alpha B\eta^2-12\alpha B+6\alpha\eta^2-3\alpha+8B^2-4B+4)) \\ &\div((-2\alpha\eta^2+2\alpha-6B+4\alpha B+1)(4\alpha^2B^2+8\alpha^2B\eta^2+4\alpha^2B+4\alpha^2n^4-12\alpha^2\eta^2+\alpha^2-8\alpha B+8\alpha\eta^2-4\alpha+4)^(1/2))) \\ &(-2\alpha\eta^2+2\alpha-6B+4\alpha B+1))/(2\alpha\eta^2+2B+1)^2)\Big) \\ &+(2B+1)/(2\alpha n^2+2B+1)-(exp((x(\alpha+2\alpha B-2\alpha\eta^2+4\alpha^2\eta^2-2))/(4\alpha\eta^2(\alpha-1)))(2B+1) \\ &(\cosh((x(4\alpha^2B^2+8\alpha^2B\eta^2+4\alpha^2B+4\alpha^2\eta^4-12\alpha^2\eta^2+\alpha^2-8\alpha B+8\alpha\eta^2-4\alpha+4)^{1/2})/(4\alpha\eta^2(\alpha-1)))+ \\ &(\sinh((x(4\alpha^2B^2+8\alpha^2B\eta^2+4\alpha^2B+4\alpha^2\eta^4-12\alpha^2\eta^2+\alpha^2-8\alpha B+8\alpha\eta^2-4\alpha+4)^{1/2})/(4\alpha\eta^2(\alpha-1))) \\ &(\alpha-2B+4\alpha B+4\alpha B^2-2\alpha\eta^2-4B^2))/((2B+1) \\ &(4\alpha^2B^2+8\alpha^2B\eta^2+4\alpha^2B+4\alpha^2\eta^4-12\alpha^2\eta^2+\alpha^2-8\alpha B+8\alpha\eta^2-4\alpha+4)^{1/2})))/(2\alpha\eta^2+2B+1)\end{aligned}$$

2 . Assume that the forcing term $F(x) = A$ for some constant A. Moreover, suppose that $\alpha \in (1/2, 1)$ so that $2\alpha \in (1, 2)$. Applying the Laplace transform of (6.1) leads to get

$$\frac{s^2U(s)-u_0}{s+\alpha(1-s)}+c\frac{\eta^2\mu^{\alpha-1}}{k}\Big(\frac{sU(s)-u_0}{s+\alpha(1-s)}\Big)+\eta^2U(s)=A\frac{\eta^2}{k}\frac{1}{s}$$
$$U(s)=\frac{u_0(s+\alpha(1-s)+B(s+2\alpha(1-s)))}{s(s+\alpha(1-s)+s+2\alpha(1-s)\Big(B+\eta^2(s+\alpha(1-s))\Big)}$$
$$+\frac{(\eta^2/k)A(s+2\alpha(1-s))(s+\alpha(1-s))}{s^2\Big(s+\alpha(1-s)+s+2\alpha(1-s)\Big(B+\eta^2(s+\alpha(1-s))\Big)}$$

where $\frac{\eta^2}{k}=\frac{\mu^{2(1-\alpha)}}{m}$ and $B=c\frac{\eta^2}{k}\mu^{\alpha-1}$

The inverse Laplace transform yields

$$u(x)=u_0\Big((\exp(-x(B/2-(\alpha\eta^2)/2+\eta^2/2))\sinh(x((\alpha^2\eta^4)/4-(\alpha B\eta^2)/2-(\alpha\eta^4)/2-\alpha\eta^2+B^2/4+(B\eta^2)/2+\eta^4/4)^{1/2})$$
$$(B+1))/((\alpha^2\eta^4)/4-(\alpha B\eta^2)/2-(\alpha\eta^4)/2-\alpha\eta^2+B^2/4+(B\eta^2)/2+\eta^4/4)^{1/2}\Big)$$
$$+(A\eta^2/k)\Big(1/\eta^2-(exp(-x(B/2-(\alpha\eta^2)/2+\eta^2/2))(\cosh(x((\alpha^2\eta^4)/4-(\alpha B\eta^2)/2-(\alpha\eta^4)/2-\alpha\eta^2+B^2/4+(B\eta^2)/2$$
$$+\eta^4/4)^{1/2})+(\sinh(x((\alpha^2\eta^4)/4-(\alpha B\eta^2)/2-(\alpha\eta^4)/2-\alpha\eta^2+B^2/4+(B\eta^2)/2+\eta^4/4)^{1/2})(B/2+(\alpha\eta^2)/2-\eta^2/2))$$
$$\div((\alpha^2\eta^4)/4-(\alpha B\eta^2)/2-(\alpha\eta^4)/2-\alpha\eta^2+B^2/4+(B\eta^2)/2+\eta^4/4)^{1/2}))/\eta^2\Big)$$

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