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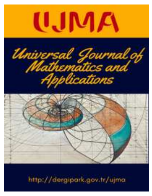

On Various g-Topology in Statistical Metric Spaces

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Abstract

The purpose of this paper is to analyze the significance of new g-topologies defined in statistical metric spaces and we prove various properties for the neighbourhoods defined by Thorp in statistical metric spaces. Also, we give a partial answer to the questions, namely "What are the necessary and sufficient conditions that the g-topology of $typeV$ to be of $typeV_D$?, the g-topology of $typeV_\alpha$ to be the g-topology of $typeV_D$? and the g-topology of $typeV_\alpha$ to be a topology?" raised by Thorp in 1962. Finally, we discuss the relations between λ_Ω-open sets in generalized metric spaces and various g-topology neighbourhoods defined in statistical metric spaces. Also, we prove weakly complete metric space is equivalent to a complete metric space if Ω satisfies the $\mathscr{V}$-property.

1. Introduction

Fréchet introduced the notion of an abstract metric space in the year 1906 [1] from which the concept of "distance" appears. The notion of distance is defined in terms of functions, points and sets. Indeed, in many situations, it is appropriate to look upon the distance concept as a statistical rather than a determinate one. More precisely, instead of associating a number to the distance d(p, q) with every pair of points p, q, one should associate a distribution function F_{pq} and for any positive number x, interpret $F_{pq}(x)$ as the probability that the distance from p to q be less than x.

Using this idea, Menger [3] defined a statistical metric space using the probability function in the year 1942. In 1943, shortly after the appearance of Menger's article, Wald [10] published an article in which he criticized Menger's generalized triangle inequality. In 1951, Menger [5] continued his study of statistical metric spaces and in [4], he studied the behaviour of probabilistic theory.

In 1960, Schweizer et. al gave some properties of neighbourhoods defined by Thorp [7]. Thorp introduced some g-topologies in a statistical metric space [9] and he studied the properties of t-function in [8]. Further, Thorp proved some results using g-topologies defined in a statistical metric space [9]. Finally, he raised some questions about the relationship between various g-topologies defined in [9].

A *statistical metric space* (SM space) [9] is an ordered pair (S,F) where S is a non-null set and F is a mapping from $S\times S$ into the set of distribution functions (that is, real-valued functions of a real variable which are everywhere defined, non decreasing, left-continuous and have infimum 0 and supremum 1).

The distribution function $F(p,q)$ associated with a pair of points p and q in S is denoted by F_{pq}. Moreover, $F_{pq}(x)$ represents the probability that the "distance" between p and q is less than x.

The functions F_{pq} are assumed to satisfy the following:
(SM-I) $F_{pq}(x)=1$ for all $x>0$ if and only if $p=q$.
(SM-II) $F_{pq}(0)=0$.
(SM-III) $F_{pq}=F_{qp}$.
(SM-IV) If $F_{pq}(x)=1$ and $F_{qr}(y)=1$, then $F_{pr}(x+y)=1$.

We often find it convenient to work with the tails of the distribution functions rather than with these distribution functions themselves. Then the *tail* [9] of F_{pq}, denoted by G_{pq}, is defined by $G_{pq}(x)=1-F_{pq}(x)$ for all $x\in\mathbb{R}$.

Let (S,F) be a statistical metric space. Then the menger inequality is,
(SM-IVm) $F_{pr}(x+y)\geq T(F_{pq}(x),F_{qr}(y))$ holds for all points $p,q,r\in S$ and for all numbers $x,y\geq 0$ where T is a 2-place function on the

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unit square satisfying:
(T-I) $0 \leq T(a,b) \leq 1$ for all $a,b,c \in [0,1]$.
(T-II) $T(c,d) \geq T(a,b)$ if $c \geq a, d \geq b$ (monotonicity) for all $a,b,c,d \in [0,1]$.
(T-III) $T(a,b) = T(b,a)$ (commutativity) for all $a,b \in [0,1]$.
(T-IV) $T(1,1) = 1$.
(T-V) $T(a,1) > 0$ for all $a > 0$.

2. Preliminaries

In this section, we recall some basic definitions in [9] and give some examples for these definitions in a statistical metric space.

Let (S,F) be a statistical metric space, $p \in S$ and u,v be positive numbers. Then $N_p(u,v) = \{q \in S \mid F_{pq}(u) > 1-v\} = \{q \in S \mid G_{pq}(u) < v\}$ [9] is called the (u,v)-*sphere* with center p.

The following Example 2.1 shows that the existence of (u,v)-sphere in a statistical metric space.

Example 2.1. Consider the SM space (S,F) where S denotes the possible outcomes of getting a tail when a coin is tossed once. Then $S = \{0,1\}$. Here $F_{pq}(u)$ is the probability that the "distance" between p and q is less than u where $u > 0$ and $p,q \in S$. Fix $p = 0$. Then

$$N_p(u,v) = \begin{cases} S & if \quad 0 < u < 1, v > 1, \\ S & if \quad u > 1, v > 0, \\ \{0\} & if \quad 0 < u < 1, 0 < v < 1, \\ S & if \quad u > 1, v > 1. \end{cases}.$$

Fix $p = 1$. Then

$$N_p(u,v) = \begin{cases} S & if \quad 0 < u < 1, v > 1, \\ S & if \quad u > 1, v > 0, \\ \{1\} & if \quad 0 < u < 1, 0 < v < 1, \\ S & if \quad u > 1, v > 1. \end{cases}.$$

For fixed positive numbers u and v, define $U(u,v)$ [9] by $U(u,v) = \{(p,q) \in S \times S \mid G_{pq}(u) < v\}$.

Example 2.2. Consider the SM space (S,F) where S denotes the possible outcomes of rolling a dice. Then $S = \{1,2,3,4,5,6\}$ and the distribution function $F_{pq}(x)$ is the probability that the "distance" between p and q is less than u where $u > 0$ and $p,q \in S$. Then $U(u,v) = \{(p,q) \in S \times S : p = q\}$ for $0 < u < 1, 0 < v < 1$. For $1 < u < 2, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 1\}$. For $2 < u < 3, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 2\}$. For $3 < u < 4, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 3\}$. For $4 < u < 5, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 4\}$. For $u > 5, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 5\} = S \times S$. Now $0 < u < 1, v > 1$. Then $U(u,v) = S \times S$. Also, $U(u,v) = S \times S$, for $u > 1, v > 1$.

For any set Z of ordered pairs of positive numbers, $\mathscr{N}(Z) = \{N_p(u,v) \mid (u,v) \in Z, p \in S\}$ and $\mathscr{U}(Z) = \{U(u,v) \mid (u,v) \in Z\}$.

A non-null collection $\{N_p\}$ of subsets $\mathscr{N}(Z)$ in a set S associated with a point $p \in S$ is a family of neighbourhoods for p if each N_p contains p. Let the family of neighbourhoods be associated with each point p of a set S. The set S and the collection of neighbourhoods is called the *g-topological space of typeV* [9].

Using the following conditions, Thorp [9] introduced new g-topologies in a statistical metric space (S,F).
N_0. *typeV*.
N_1. For each point p and each neighbourhood U_p of p, there is a neighbourhood W_p of p such that for each point q of W_p, there is a neighbourhood U_q of q contained in U_p.
N_2. For each point p and each pair of neighbourhoods U_p and W_p of p, there is a neighbourhood of p contained in the intersection of U_p and W_p.

The following are various g-topologies in a statistical metric space (S,F) defined by Thorp [9].
(a) If the conditions N_0 and N_2 are satisfied, then the collection of neighbourhoods on S is called the *g-topology of typeV$_D$*.
(b) The collection of neighbourhoods on S is called the *g-topology of typeV$_\alpha$* if the conditions N_0 and N_1 are satisfied.
(c) A g-topology is a *topology* if the conditions N_0, N_1 and N_2 are satisfied.

Let S be a set and P be a partially ordered $(<)$ set with least element 0. A *generalized écart* [9] (g-écart for short) is a mapping G from $S \times S$ into P. If a g-écart G satisfies $G(p,p) = 0$ and the set S consists of more than one point, the g-écart g-topology for S is the g-topology determined from G, and its partially ordered range set P, as follows. For each $f > 0$ in P and each $p \in S$, the f-sphere for p is $N_p(f) = \{q \in S \mid G(p,q) < f\}$. Then for each $p \in S$, the collection of f-spheres, $\mathscr{N}_p(P) = \{N_p(f) \mid f > 0 \text{ in } P\}$ is a family of neighbourhoods for p.

The g-écart associated with a statistical metric space (S,F) is the mapping G defined by $G(p,q) = G_{pq}$ [9].

Example 2.3. Let $S = \mathbb{N}$ and $P = \mathbb{N} \cup \{0\}$ be a partially ordered set with the relation $<$ where $\mathbb{N}$ denote the set of all natural numbers. Let $A = \{1,2,3\}$ be a subset of S. Define

$$G(p,q) = \begin{cases} 1 & if \quad p \notin A, q \in S, \\ 1 & if \quad p \in S, q \notin A, \\ \{0\} & if \quad p \notin A, q \notin A, \end{cases}.$$

and $p \in A, q \in A$ define $G(p,q)$ as follows: $G(1,1) = 0, G(1,2) = 2, G(1,3) = 3, G(2,1) = 4, G(2,2) = 0, G(2,3) = 6, G(3,1) = 1, G(3,2) = 2, G(3,3) = 0$. Case 1: $p \notin A, q \in S$. Then $G(p,q) = 1$. Let $f = 1$. Then $N_p(1) = \emptyset$. For $f \geq 2, N_p(f) = S$. Case 2: $p \in S, q \notin A$. Then

$G(p,q) = 1$. Let $f = 1$. Then $N_p(1) = \emptyset$ and $N_p(f) = S - A$ for $f \geq 2$. Case 3: $p \notin A, q \notin A$. Then $G(p,q) = 0$ and so $N_p(f) = S - A$ for $f > 0$. Case 4: $p \in A, q \in A$. Then $N_1(1) = N_1(2) = \{1\}; N_1(3) = \{1,2\}; N_1(f) = A$ for $f \geq 4$. Now $N_2(1) = N_2(2) = N_2(3) = N_2(4) = \{2\}; N_2(5) = N_2(6) = \{1,2\}; N_2(f) = A$ for $f \geq 7$ and $N_3(1) = \{3\}; N_3(2) = \{1,3\}; N_3(f) = A$ for $f \geq 3$.

Given a statistical metric space (S,F), for each pair of points p and r in S, the r-sphere with center p, $N_p(r)$ is defined to be the sphere $N_p(G_{pr}) = \{q \mid G_{pq} < G_{pr}\}$. The *R-g-topology* [9] for (S,F) is the structure whose family of neighbourhoods at each point p is the collection $\mathscr{N}_p(r) = \{N_p(r) \mid r \in S\}$.

Example 2.4. Consider the SM space (S,F) where $S = \mathbb{N}$ and the distribution function

$$F_{pq}(x) = \begin{cases} \frac{x}{d(p,q)} & if \quad 0 < x < d(p,q), d(p,q) \neq 0 \\ 1 & if \quad x \geq d(p,q) \end{cases}.$$

Fix $p = 1$ and $r = 2$ are in S. Let $x = \frac{1}{4}$. Then $G_{pr}(x) = 0.75$. Now $N_1(0.75) = \{1\}$.

Observation 2.5. *In a statistical metric space, $N_p(G_{pr}) = \emptyset$ if $p = r$.*

Notations 2.6. *In a SM space (S,F), we use the following notations:*
(a) Let τ denote the g-topology of $typeV$.
(b) Let τ_D denote the g-topology of $typeV_D$.
(c) Let τ_α denote the g-topology of $typeV_\alpha$.
(d) Let τ_e denote the g-écart g-topology.
(e) Let τ_R denote the R-g-topology.
(f) Each element in $\mathscr{N}(X)$ is called a τ-neighborhood.
(f) Each element in $\mathscr{N}_p(P)$ is called a τ_e-neighborhood.
(f) Each element in $\mathscr{N}_p(r)$ is called a τ_R-neighborhood.

3. Behaviour of various g-topology

In this section, we give some properties and find the relations between four types of neighborhoods in a SM space. Also, we give the answer for some of the questions raised by Throp [9].

Theorem 3.1. *Let (S,F) be a statistical metric space. Then the following hold.*
(a) *If $u_1 \leq u$ and $v_1 \leq v$, then $N_p(u_1,v_1) \subset N_p(u,v)$ where $u,v,u_1,v_1 > 0$.*
(b) *If $\kappa = \{\mathscr{N}(Z), \mathscr{U}(Z), \mathscr{N}_p(P), \mathscr{N}_p(r)\}$ and $A \in \kappa$, then there exist $B,C \in \kappa$ such that $B \subset A \subset C$.*

Proof.

(a) Let $q \in N_p(u_1,v_1)$. Then $F_{pq}(u_1) > 1 - v_1$. Since $u_1 \leq u$ and $v_1 \leq v$, $F_{pq}(u) \geq F_{pq}(u_1) > 1 - v_1 \geq 1 - v$. Thus, $F_{pq}(u) > 1 - v$.

Therefore, $q \in N_p(u,v)$. Hence $N_p(u_1,v_1) \subset N_p(u,v)$.

(b) We give the detailed proof only for $\kappa = \mathscr{N}(Z)$ and $\kappa = \mathscr{N}_p(r)$. Suppose that $\kappa = \mathscr{N}(Z)$ and $A \in \kappa$. Then $A = N_p(u,v)$ where $u,v > 0$.

Take $0 < u_1 \leq u, 0 < v_1 \leq v$ and $B = N_p(u_1,v_1)$. By (a), $B \subset A$. If $u_2 \geq u, v_2 \geq v$, then $u_2 > 0, v_2 > 0$. Define $C = N_p(u_2,v_2)$. By (a), $A \subset C$. Thus, there exist $B,C \in \mathscr{N}(Z)$ such that $B \subset A \subset C$.

Suppose that $\kappa = \mathscr{N}_p(r)$. Let $A \in \kappa$. Then $A = \{q \in S \mid G_{pq} < G_{pr}\}$ and so $A = \{q \in S \mid G_{pq}(u) < G_{pr}(u)\}$ where $u > 0$. Take $u_1 \geq u$. Define $B = \{q \in S \mid G_{pq}(u) < G_{pr}(u_1)\}$. Then $B \in \mathscr{N}_p(r)$. Let $s \in B$. Then $G_{ps}(u) < G_{pr}(u_1)$ and so $G_{ps}(u) < G_{pr}(u)$, since $G_{pr}(u_1) \leq G_{pr}(u)$. Therefore, $s \in A$. Hence $B \subset A$. Define $C = \{q \in S \mid G_{pq}(u_1) < G_{pr}(u)\}$. Then $C \in \mathscr{N}_p(r)$. Let $s \in A$. Then $G_{ps}(u) < G_{pr}(u)$ and so $G_{ps}(u_1) < G_{pr}(u)$, since $G_{ps}(u_1) \leq G_{ps}(u)$. Therefore, $s \in C$. Hence $A \subset C$.

□

From the definition of g-topology of $typeV_D$, it is observed that every g-topological space of $typeV_D$ is a g-topological space of $typeV$. The following Theorem 3.2 discusses the converse of the question that "What are the necessary and sufficient conditions that the g-topology of $typeV$ to be of $typeV_D$?" which is raised by Thorp [9].

Theorem 3.2. *Let (S,F) be a statistical metric space. Then the following hold.*
(a) *τ satisfies N_2.*
(b) *The g-topology of $typeV$ is a g-topology of $typeV_D$.*

Proof. (a) Let U_p and W_p be two neighbourhoods of p. Then $U_p = \{q \in S \mid G_{pq}(u) < v\}$ and $W_p = \{q \in S \mid G_{pq}(u_1) < v_1\}$. Define $V_p = \{q \in S \mid G_{pq}(min(u,u_1)) < min(v,v_1)\}$. Then $p \in V_p$ and so V_p is a neighbourhood of p. Since $min(u,u_1) \leq u$, $min(v,v_1) \leq v$, we have $V_p \subset U_p$, by Theorem 3.1(a). Also, $min(u,u_1) \leq u_1$ and $min(v,v_1) \leq v_1$. Therefore, $V_p \subset W_p$, by Theorem 3.1(a). Hence $V_p \subset U_p \cap W_p$. Therefore, τ satisfies N_2.

(b) By (a) and the definition of g-topology of $typeV_D$, it follows that every g-topology of $typeV$ is a g-topology of $typeV_D$.

□

The following two questions are raised by Thorp [9].

(I) "What are the necessary and sufficient conditions that the g-topology of $typeV_\alpha$ to be the g-topology of $typeV_D$?".

(II) "What conditions are both necessary and sufficient for the g-topology of $typeV_\alpha$ to be a topology?".

The following Corollary 3.3 (a) gives a necessary condition for the given space to be a g-topological space of $typeV_D$ which also gives a partial answer to the question (I) and Corollary 3.3 (b) gives the answer to the question (II).

Corollary 3.3. *Let (S,F) be a statistical metric space. Then the following hold.*

(a) *The g-topology of $typeV_\alpha$ is a g-topology of $typeV_D$.*

(b) *The g-topology of $typeV_\alpha$ is a topology and conversely.*

Proof. (a) By the definition of g-topology of $typeV_\alpha$, g-topology of $typeV_\alpha$ is a g-topology of $typeV$. Therefore, g-topology of $typeV_\alpha$ is a g-topology of $typeV_D$, by Theorem 3.2(b).

(b) By the definition of g-topology of $typeV_\alpha$, g-topology of $typeV_\alpha$ is a g-topology of $typeV$ and satisfies the condition N_1. By (a), g-topology of $typeV_\alpha$ satisfies the condition N_2, by the definition g-topology of $typeV_D$. Hence a g-topology of $typeV_\alpha$ is a topology. Converse follows from the definition of topology in a statistical metric space.

□

Theorem 3.4. *Let (S,F) be a statistical metric space. If $U \in \mathscr{U}(Z)$, then there exists $V \subset S$ such that $V \in \mathscr{N}(Z)$.*

Proof. Let $U \in \mathscr{U}(Z)$. Define $V = \{q \in S \mid (p,q) \in U\}$. Since $U \in \mathscr{U}(Z), V = \{q \in S \mid G_{pq}(u) < v\}$. Hence $V = N_p(u,v)$, by the definition of $N_p(u,v)$. Therefore, $V \in \mathscr{N}(Z)$. □

Theorem 3.5. *Let (S,F) be a statistical metric space. Then the following hold.*

(a) $\tau_e \subset \tau$.

(b) τ_e satisfies N_2.

Proof. (a) Let $p \in S$ and U be a τ_e-neighbourhood of a point p. Then $U = \{q \in S \mid G(p,q) < f\}$. Since in a statistical metric space $G(p,q) = G_{pq}, U = \{q \in S \mid G_{pq} < f\}$. Here $0 < f \in P$ where P is a partially ordered set. Then there is an element $g \in P$ such that $g < f$. Take $g(u) = v$ for all $u > 0$. Then $v > 0$. Define $V = \{q \in S \mid G_{pq}(u) < v\}$. Then V is a τ-neighbourhood such that $p \in V \subset U$. Hence $\tau_e \subset \tau$.

(b) Suppose that U_p and W_p are two neighbourhoods of p. Then $U_p = \{q \in S \mid G(p,q) < f_1\}$ and $W_p = \{q \in S \mid G(p,q) < f_2\}$. Consider $V_p = \{q \in S \mid G(p,q) < min(f_1,f_2)\}$. Then $p \in V_p$ and so V_p is a neighbourhood of p. Also, $V_p \subset U_p \cap W_p$. Therefore, τ_e satisfies N_2.

□

The following Corollary 3.6 gives a necessary condition for the given space to be a g-topological space of $typeV_D$ which also gives a partial answer to the question that "What are the necessary and sufficient conditions for τ_e to be of $typeV_D$?" raised by Thorp [9].

Corollary 3.6. *Let (S,F) be a statistical metric space. Then $\tau_e \subset \tau_D$.*

Proof. Follows from Theorem 3.5 and the definition of $typeV_D$. □

Theorem 3.7. *Let (S,F) be a statistical metric space. Then the following hold.*

(a) τ_R satisfies N_2.

(b) $\tau_R \subset \tau$.

Proof. (a) Let U_p and W_p be τ_R-neighbourhoods of p. Then $U_p = \{q \in S \mid G_{pq} < G_{pr_1}\}$ and $W_p = \{q \in S \mid G_{pq} < G_{pr_2}\}$ where $r_1, r_2 \in S$. Define $V_p = \{q \in S \mid G_{pq} < inf(G_{pr_1}, G_{pr_2})\}$. It follows that V_p is a neighbourhood of p and $p \in V_p$. Also, $V_p \subset U_p \cap W_p$. Therefore, τ_R satisfies N_2.

(b) If $t \in S$ and B is a τ_R-neighbourhood of t, then $B = \{q \mid G_{pq} < G_{pr}\}$ and so $B = \{q \mid G_{pq}(u) < G_{pr}(u)\}$ where $u > 0$. Choose an element v such that $u < v$. Take $v_1 = G_{pr}(v)$. Since $B \neq \emptyset$ we have $p \neq r$ and so $G_{pr}(v) \neq 0$ so that $v_1 > 0$. Define $B_1 = \{q \mid G_{pq}(u) < v_1\}$. Then $t \in B_1$ and B_1 is a τ-neighbourhood contained in B. Hence $\tau_R \subset \tau$.

□

The following Corollary 3.8 gives a necessary condition for the given space to be a g-topological space of $typeV_D$ which also gives a partial answer to the question that "What are the necessary and sufficient conditions for the R-g-topology to be g-topology of $typeV_D$?" raised by Thorp in [9].

Corollary 3.8. *Let (S,F) be a statistical metric space. Then $\tau_R \subset \tau_D$.*

Proof. Follows from Theorem 3.7 and the definition of $typeV_D$. □

Lemma 3.9. *A function $T : I \times I \to I$ is defined by $T(x,y) = max(x,y)$ where $I = [0,1]$. Then T satisfies the conditions (T-II) and (T-IV).*

Proof. (a) Suppose that $c \geq a, d \geq b$ where $a,b,c,d \in [0,1]$. Now $T(c,d) = max(c,d)$ and $T(a,b) = max(a,b)$. Case-1: If $T(c,d) = c$ and $T(a,b) = a$, then $T(c,d) \geq T(a,b)$. Case-2: Suppose $T(c,d) = c$ and $T(a,b) = b$. Since $b \leq d \leq c$, $T(c,d) \geq T(a,b)$. Case-3: If $T(c,d) = d$ and $T(a,b) = b$, then $T(c,d) \geq T(a,b)$. Case-4: Suppose $T(c,d) = d$ and $T(a,b) = a$. Since $a \leq c \leq d$, $T(c,d) \geq T(a,b)$. Therefore, T satisfies the condition (T-II).

(b) Now $T(1,1) = max(1,1) = 1$. Hence T satisfies the condition (T-IV).

□

The following Theorem 3.10 gives the answer to the question that "What are the necessary and sufficient conditions for the g-topology of $typeV$ to be a topology?" raised by Thorp in [9].

Theorem 3.10. *Let (S,F) be a statistical metric space with the g-topology of typeV. If SM-IVm satisfies under $T : I \times I \to I$ defined by $T(x,y) = max(x,y)$, then the g-topology on S is a topology and conversely.*

Proof. Given that (S,F) is a statistical metric space with a g-topology of $typeV$. Then by Theorem 3.2, N_0 and N_2 are satisfied. Let $p \in S$ and U_p be a neighbourhood for p. Then $U_p = \{r \in S \mid F_{pr}(u) > 1 - v\}$. Choose $u_1 = \frac{u}{2}$ and $v_1 < v$ with $0 \leq v_1 \leq 1$. Taking $W_p = \{s \in S \mid F_{ps}(u_1) > 1 - v_1\}$, we get that W_p is a neighbourhood of p. For $q \in W_p$, define $V_q = \{t \in S \mid F_{qt}(u_1) > 1 - v_1\}$ so that V_q is a neighbourhood of q. Since $q \in W_p$, $F_{pq}(u_1) > 1 - v_1$ and so $F_{qp}(u_1) > 1 - v_1$, by the condition (SM-III). Hence $p \in V_q$. If $a \in V_q$, then $F_{qa}(u_1) > 1 - v_1$. Since $p \in V_q, F_{qp}(u_1) > 1 - v_1$. By Lemma 3.9, T satisfies the condition (T-II). Thus, $T(F_{pq}(u_1), F_{qa}(u_1)) \geq T(1 - v_1, 1 - v_1)$. By (SM-IVm), $F_{pa}(u) \geq T(F_{pq}(u_1), F_{qa}(u_1))$, since $u_1 = \frac{u}{2}$ which implies that $F_{pa}(u) \geq T(1 - v_1, 1 - v_1)$ which in turn implies that $F_{pa}(u) \geq 1 - v_1$, by hypothesis. Hence $F_{pa}(u) > 1 - v$ and so $a \in U_p$. Therefore, $V_q \subset U_p$ and consequently N_1 is satisfied. Thus, g-topology of $typeV$ is a topology. Converse part follows from the definition of topology in a statistical metric space. □

The following Corollary 3.11 gives the answer to the question "What are the necessary and sufficient conditions for the g-topology of $typeV_D$ to be a topology?" raised by Thorp [9].

Corollary 3.11. *Let (S,F) be a statistical metric space with the g-topology of typeV$_D$. If SM-IVm satisfies under a function $T : I \times I \to I$ defined by $T(x,y) = max(x,y)$, then the g-topology of typeV$_D$ is a topology and conversely.*

Proof. By the definition of $typeV_D$, it follows that it is of $typeV$. By hypothesis and Theorem 3.10, g-topology of $typeV_D$ is a topology. Converse follows from the definition of topology in a statistical metric space. □

The following Corollary 3.12 gives a sufficient condition for g-topology of $typeV_D$ to be a g-topology of $typeV_\alpha$ which also gives a partial answer to the question "What conditions are both necessary and sufficient for the g-topology of $typeV_\alpha$ to be of $typeV_D$?" raised by Thorp in [9].

Corollary 3.12. *Let (S,F) be a statistical metric space with the g-topology of typeV$_D$. If SM-IVm satisfies under a function $T : I \times I \to I$ defined by $T(x,y) = max(x,y)$, then g-topology of typeV$_D$ is a g-topology of typeV$_\alpha$.*

Proof. By the definition of $typeV_D$, g-topology of $typeV_D$ is of $typeV$. As in the proof of Theorem 3.10, $typeV$ satisfies the condition N_1. Therefore, g-topology of $typeV_D$ is a g-topology of $typeV_\alpha$. □

The following Theorem 3.13 gives a necessary condition for the g-écart-g-topology to be a topology which also gives a partial answer to the questions "What are the necessary and sufficient conditions for the g-écart-g-topology to be a topology?" raised by Thorp [9].

Theorem 3.13. *Let (S,F) be a statistical metric space with g-écart-g-topology. If SM-IVm holds under a function T satisfying T-IV, T-II and $sup_{x<1} T(x,x) = 1$, then the g-écart g-topology is a topology on S.*

Proof. By Corollary 3.6, g-écart g-topology is a g-topology of $typeV_D$ and hence the conditions N_0 and N_2 are satisfied. Let $p \in S$ and U_p be a neighbourhood of p. Then $U_p = \{r \in S \mid G_{pr} < f\}$. Let f_1 be a tail with $L < f_1 < f$. If $W_p = \{s \in S \mid G_{ps} < f_1\}$, then W_p is a neighbourhood of p. Choose $q \in W_p$ and take $V_q = \{t \in S \mid G_{qt} < f_1\}$. Then V_q is a neighbourhood of q. Since $q \in W_p$, $G_{pq} < f_1$ and so $G_{qp} < f_1$ which implies that $p \in V_q$ which in turn implies that $G_{qp} < f_1$ and hence $F_{qp}(x) > 1 - f_1(x)$. Let $m \in V_q$. Then $G_{qm} < f_1$ and so $F_{qm}(x) > 1 - f_1(x)$. By T-II, $T(F_{pq}(x), F_{qm}(x)) \geq T(1 - f_1(x), 1 - f_1(x))$. Also, $F_{pm}(2x) \geq T(F_{pq}(x), F_{qm}(x))$, by SM-IVm. Hence it suffices to find a f_1 such that $T(1-f_1(x),1-f_1(x)) \geq 1-f_1(2x)$ for some x. Since $f > L$, there exists $a > 0$ such that $1-f(2a) < 1$. By hypothesis, there is a number $b < 1$ such that $T(b,b) > 1-f(2a)$. Now we define $f_1(x)$ using a and b by

$$f_1(x) = \begin{cases} 0 & \text{if} \quad x > a, \\ 1-b & \text{if} \quad 0 < x \leq a. \end{cases}$$

If $x > a$, then $T(1-f_1(x),1-f_1(x)) = T(1,1)$. Again, using T-IV, $T(1-f_1(x),1-f_1(x)) = 1$. Therefore, $T(1-f_1(x),1-f_1(x)) \geq 1-f(2x)$. If $0 < x \leq a$, then $T(1-f_1(x),1-f_1(x)) = T(b,b) > 1-f(2a) \geq 1-f(2x)$, since f is a left continuous function. Thus, $T(1-f_1(x),1-f_1(x)) > 1-f(2x)$ for $0 < x \leq a$. Hence $F_{pm}(2x) > 1-f(2x)$ for $0 < x \leq a$. Thus, $m \in U_p$ so that $V_q \subset U_p$. Therefore, N_1 is satisfied and hence g-écart g-topology is a topology. □

Theorem 3.14 below gives a necessary condition for an R-g-topology to be a topology which also gives a partial answer to the question "What are the necessary and sufficient conditions for the R-g-topology to be a topology?" raised by Thorp in [9].

Theorem 3.14. *Let (S,F) be a statistical metric space with R-g-topology. If SM-IVm satisfies under a function $T : I \times I \to I$ defined by $T(x,y) = max(x,y)$, then the R-g-topology is a topology.*

Proof. By hypothesis and Corollary 3.8, R-g-topology is a g-topology of $typeV_D$ and hence the conditions N_0 and N_2 are satisfied. Let $p \in S$ and U_p be a neighbourhood for p. Then $U_p = \{s \in S \mid G_{ps} < G_{pr}\}$. Take $0 < c \leq 1$ and define $W_p = \{t \in S \mid G_{pt} < cG_{pr}\}$. Then W_p is a neighbourhood of p. If $q \in W_p$, then $G_{pq} < cG_{pr}$ and so $G_{qp} < cG_{pr}$. Hence $p \in \{u \in S \mid G_{qu} < cG_{pr}\}$. Take $V_q = \{u \in S \mid G_{qu} < cG_{pr}\}$. Then $p \in V_q$ and V_q is a neighbourhood of q. If $n \in V_q$, then $G_{qn} < cG_{pr}$ and so $G_{qn} < G_{pr}$ so That $F_{qn} > F_{pr}$. Since $p \in V_q$, $G_{qp} < cG_{pr} < G_{pr}$ and hence $F_{qp} > F_{pr}$. By SM-IVm, $F_{pn}(x) \geq T(F_{pq}(0),F_{qn}(x)) = Max(0,F_{qn}(x)) = F_{qn}(x)$, by hypothesis and SM-II. Thus, $F_{pn}(x) \geq F_{qn}(x)$ so that $F_{pn}(x) > F_{pr}(x)$ and hence $G_{pn} < G_{pr}$. Therefore, $n \in U_p$ and so $V_q \subset U_p$. Thus, N_1 is satisfied. Therefore, the R-g-topology is a topology. □

In [6], Min introduced stack as in the following way: A collection $\mathscr{C}$ of subsets of S is called a *stack* [6] if $A \in \mathscr{C}$ whenever $B \in \mathscr{C}$ and $B \subset A$. Also, he analyzes whether a neighbourhood collections are stack or not in generalized topological spaces. Here we prove that different types of the neighbourhood collections become stack in statistical metric spaces.

Theorem 3.15. *Let (S,F) be a statistical metric space. Then $\mathscr{N}(Z)$ is a stack.*

Proof. Let $A \in \mathscr{N}(Z)$ and $A \subseteq B$. Then $A = \{q \in S \mid G_{pq}(u) < v\}$. Take $u_1 > u$ and

$$v_1 = \begin{cases} v & \text{if} \quad s \in A, \\ G_{ps}(u) & \text{if} \quad s \in B-A, \\ G_{ps}(u_1) & \text{if} \quad s \in S-B. \end{cases}$$

Then $u_1 > 0$ and $v_1 > 0$. If $U = \{q \in S \mid G_{pq}(u_1) < v_1\}$, then $U \in \mathscr{N}(Z)$. Choose $t \in B$. Then $t \in A$ or $t \in B-A$. Suppose $t \in A$. Then $G_{pt}(u) < v$. Since $u_1 > u$, $G_{pt}(u_1) < G_{pt}(u)$ which implies that $G_{pt}(u_1) < v = v_1$ and hence $t \in U$. If $t \in B-A$, then $G_{pt}(u) > v$. Since $u_1 > u$, $G_{pt}(u_1) < G_{pt}(u) = v_1$ and so $t \in U$. Hence $B \subset U$. Let $s \in U$. Then $G_{ps}(u_1) < v_1$. By the definition of v_1, $s \in A$ or $s \in B-A$. This implies that $s \in B$ which implies that $U \subset B$. Therefore, $B = U$. Since $U \in \mathscr{N}(Z), B \in \mathscr{N}(Z)$. Hence $\mathscr{N}(Z)$ is a stack. □

Theorem 3.16. *Let (S,F) be a statistical metric space. Then $\mathscr{U}(Z)$ is a stack.*

Proof. Let $A \in \mathscr{U}(Z)$ and $A \subseteq B$. Then $A = \{(p,q) \in S \times S \mid G_{pq}(u) < v\}$. Take $u_1 > u$ and

$$v_1 = \begin{cases} v & \text{if} \quad (p,q) \in A, \\ G_{pq}(u) & \text{if} \quad (p,q) \in B-A, \\ G_{pq}(u_1) & \text{if} \quad (p,q) \in S-B. \end{cases}$$

Then u_1 and $v_1 > 0$. Define $U = \{(p,q) \in S \times S \mid G_{pq}(u_1) < v_1\}$ so that $U \in \mathscr{U}(Z)$. If $(s,t) \in B$, then $(s,t) \in A$ or $(s,t) \in B-A$. If $(s,t) \in A$, then $G_{st}(u) < v$. Since $u_1 > u, G_{st}(u_1) < G_{st}(u)$ which implies that $G_{st}(u_1) < v = v_1$ and hence $(s,t) \in U$. Suppose that $(s,t) \in B-A$. Then $G_{st}(u) > v$. Since $u_1 > u, G_{st}(u_1) < G_{st}(u) = v_1$ and so $(s,t) \in U$. Hence $B \subset U$. Let $(l,m) \in U$. Then $G_{lm}(u_1) < v_1$. By definition of $v_1, (l,m) \in A$ or $(l,m) \in B-A$. This implies that $(l,m) \in B$ which implies that $U \subset B$. Therefore, $B = U$. Since $U \in \mathscr{U}(Z), B \in \mathscr{U}(Z)$. Hence $\mathscr{U}(Z)$ is a stack. □

Theorem 3.17. *Let (S,F) be a statistical metric space. Then $\mathscr{N}_p(P)$ is a stack.*

Proof. Let $A \in \mathscr{N}_p(P)$ and $A \subseteq B$. Then $A = \{q \in S \mid G(p,q) < f\}$. In a statistical metric space, $G_{pq} = G(p,q)$ so that $A = \{q \in S \mid G_{pq}(u) < f(u)\}$ where $u > 0$. Take $u_1 > u$ and

$$f_1(u_1) = \begin{cases} f(u) & \text{if} \quad s \in A, \\ G_{ps}(u) & \text{if} \quad s \in B-A, \\ G_{ps}(u_1) & \text{if} \quad s \in S-B. \end{cases}$$

Define $U = \{q \in S \mid G(p,q) < f_1\}$. Then $U \in \mathscr{N}_p(P)$. Since (S,F) is a statistical metric space, $U = \{q \in S \mid G_{pq}(u_1) < f_1(u_1)\}$. Let $t \in B$. Then $t \in A$ or $t \in B-A$. If $t \in A$, then $G_{pt}(u) < f(u)$. Since $u_1 > u$, $G_{pt}(u_1) < G_{pt}(u)$ which implies that $G_{pt}(u_1) < f(u) = f_1(u_1)$ and hence $t \in U$. If $t \in B-A$, then $G_{pt}(u) > v$. Since $u_1 > u$, $G_{pt}(u_1) < G_{pt}(u) = f_1(u_1)$ and so $t \in U$. Hence $B \subset U$. Let $s \in U$. Then $G_{ps}(u_1) < f_1(u_1)$. By definition of $v_1, s \in A$ or $s \in B-A$. This implies that $s \in B$ which implies that $U \subset B$. Therefore, $B = U$ and so $B \in \mathscr{N}_p(P)$, since $U \in \mathscr{N}_p(P)$. Hence $\mathscr{N}_p(P)$ is a stack. □

The following Theorem 3.18 shows that a neighbourhood collection $\mathscr{N}(Z)$ is closed under finite intersection in a statistical metric space.

Theorem 3.18. *Let (S,F) be a statistical metric space and $\kappa = \{\mathscr{N}(Z), \mathscr{N}_p(P)\}$. If $W_1, W_2, \ldots, W_n \in Q$ with $W_1 \cap W_2 \cap \ldots \cap W_n \neq \emptyset$, then $W_1 \cap W_2 \cap \ldots \cap W_n \in Q$ where $Q \in \kappa$.*

Proof. We will give a detailed proof only for $Q = \mathscr{N}(Z)$ where $Q \in \kappa$. Suppose that $V_1, V_2, \ldots, V_n \in Q$ with $V_1 \cap V_2 \cap \ldots \cap V_n \neq \emptyset$. Let $x \in V_1 \cap V_2 \cap \ldots \cap V_n$. Then $x \in V_i$ for $i = 1$ to n. Since V_1 and V_2 are τ-neighbourhoods containing x, there exists τ-neighbourhood W_1 containing x such that $W_1 \subset V_1 \cap V_2$, by Theorem 3.2(a). Again, W_1 and V_3 are τ-neighbourhoods containing x implies that there exists a τ-neighbourhood W_2 containing x such that $W_2 \subset W_1 \cap V_3 \subset V_1 \cap V_2 \cap V_3$. Proceeding like this, we get a τ-neighbourhood W_{n-1} containing x such that $W_{n-1} \subset W_{n-2} \cap V_n \subset V_1 \cap V_2 \cap V_3 \cap \ldots \cap V_n$. Since $W_{n-1} \in Q$ and $W_{n-1} \subset V_1 \cap V_2 \cap V_3 \cap \ldots \cap V_n$ we have $V_1 \cap V_2 \cap V_3 \cap \ldots \cap V_n \in Q$, by Theorem 3.15. $\square$

4. Relation between GMS and SM space

In this section, we find the relations between λ_Ω-open sets in generalized metric spaces and various g-topology neighbourhoods defined in statistical metric spaces. Also, we give some properties of λ_Ω-open sets, kernel and perfect kernel in generalized metric spaces.

The notion of a generalized metric space was introduced by Korczak-Kubiak et al. in [2]. Let $X \neq \emptyset$. The symbol Ω to denote the family consisting of metrics defined on subsets of X, that is, if $\rho \in \Omega$, then there exists a non-null set $A_\rho \subset X$ such that ρ is a metric on A_ρ where A_ρ is a domain of ρ and it will be denoted by $dom(\rho)$. The space (X,Ω) is called a *generalized metric space* (GMS) [2]. We will write Ω_X if we want to point out that all the metrics $\rho \in \Omega_X$ defined on X [2].

Denote λ_Ω is the family of Ω-open sets in (X,Ω), more precisely, $V \in \lambda_\Omega$ if and only if for each $x \in V$, there exist $\rho \in \Omega$ and $\varepsilon > 0$ such that $B_\rho(x,\varepsilon) \subset V$ where $B_\rho(x,\varepsilon) = \{y \in dom(\rho) : \rho(x,y) < \varepsilon\}$ [2].

Let (X,Ω) be a GMS. A *kernel* [2] of the space (X,Ω) is a finite family $\Omega_0 \subset \Omega$ with the following property: for any set $V \in \tilde{\lambda}_\Omega$, there exists $\rho \in \Omega_0$ such that $i_\rho V \neq \emptyset$. A finite family $\Omega_0 \subset \Omega$ is called a *perfect kernel* [2] of the space (X,Ω) if for any $V_1, V_2, \ldots, V_m \in \mu_\Omega$ such that $V_1 \cap V_2 \cap \ldots \cap V_m \neq \emptyset$, there exists $\rho \in \Omega_0$ such that $i_\rho(V_1 \cap V_2 \cap \ldots \cap V_m) \neq \emptyset$ [2]. Every perfect kernel is a kernel [2].

A GMS (X,Ω) is said to be a *weakly complete space* [2] if there exists a kernel $\Omega_0 \subset \Omega$ consisting of complete metrics. A GMS (X,Ω) is said to be a *complete space* [2] if there exists a perfect kernel $\Omega_0 \subset \Omega$ consisting of complete metrics. Every complete space is a weakly complete space [2].

Definition 4.1. Let (X,Ω) be a generalized metric space. Then Ω is said to satisfy $\mathscr{V}$*-property* if $\sigma_1, \sigma_2 \in \Omega$ and $x, y \in X$, then $\sigma(x,y) = max\{\sigma_1(x,y), \sigma_2(x,y)\}$ is a metric and hence $\sigma \in \Omega$.

Theorem 4.2. *Let (X,Ω) be a generalized metric space. Then λ_Ω satisfies the condition N_1.*

Proof. Let $p \in X$ and U_p be a neighbourhood of p. Then $U_p \in \tilde{\lambda}_\Omega$. Since $p \in U_p$, there is a metric $\sigma_1 \in \Omega$ and $\varepsilon_1 > 0$ such that $B_{\sigma_1}(p,\varepsilon_1) \subset U_p$. Since $B_{\sigma_1}(p,\varepsilon_1) \in \tilde{\lambda}_\Omega$, for every $q \in B_{\sigma_1}(p,\varepsilon_1)$, there exist $\sigma \in \Omega$ and $\varepsilon > 0$ such that $B_\sigma(q,\varepsilon) \subset B_{\sigma_1}(p,\varepsilon_1) \subset U_p$. Therefore, every λ_Ω satisfies the condition N_1. $\square$

Theorem 4.3. *Let (X,Ω) be a generalized metric space and Ω satisfies the $\mathscr{V}$-property. Then the following hold.*
(a) *λ_Ω satisfies N_2.*
(b) *If $W_1, W_2, \ldots, W_n \in \lambda_\Omega$ with $W_1 \cap W_2 \cap \ldots \cap W_n \neq \emptyset$, then $W_1 \cap W_2 \cap \ldots \cap W_n \in \lambda_\Omega$.*

Proof.
(a) Let $p \in X$ and $U_p, W_p \in \lambda_\Omega$. Then there exist $\sigma_1, \sigma_2 \in \Omega$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $B_{\sigma_1}(p,\varepsilon_1) \subset U_p$, $B_{\sigma_2}(p,\varepsilon_2) \subset W_p$. For $y \in X$, define $\sigma_3(x,y) = max\{\sigma_1(x,y), \sigma_2(x,y)\}$. Then $\sigma_3 \in \Omega$ and $\sigma_3(x,y) \geq \sigma_1(x,y), \sigma_3(x,y) \geq \sigma_2(x,y)$. This implies that $B_{\sigma_3}(p,\varepsilon_1) \subset B_{\sigma_1}(p,\varepsilon_1)$ and $B_{\sigma_3}(p,\varepsilon_2) \subset B_{\sigma_2}(p,\varepsilon_2)$ which implies that $B_{\sigma_3}(p,\varepsilon_1) \cap B_{\sigma_3}(p,\varepsilon_2) \subset B_{\sigma_1}(p,\varepsilon_1) \cap B_{\sigma_2}(p,\varepsilon_2)$. Choose $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$ so that $\varepsilon > 0$. Then $B_{\sigma_3}(p,\varepsilon) \subset B_{\sigma_3}(p,\varepsilon_1) \cap B_{\sigma_3}(p,\varepsilon_2)$ and so $B_{\sigma_3}(p,\varepsilon) \subset B_{\sigma_1}(p,\varepsilon_1) \cap B_{\sigma_2}(p,\varepsilon_2)$. Therefore, $B_{\sigma_3}(p,\varepsilon) \subset U_p \cap W_p$. Take $V_p = B_{\sigma_3}(p,\varepsilon)$. Then V_p is a λ_Ω-neighbourhood of p such that $V_p \subset U_p \cap W_p$. Hence λ_Ω satisfies N_2.
(b) The proof is similar to that of (a).

$\square$

Theorem 4.4. *Let (X,Ω) be a generalized metric space and Ω satisfy the $\mathscr{V}$-property. Then every kernel in (X,Ω) is a perfect kernel.*

Proof. Suppose that $\Omega_0 \subset \Omega$ is a kernel in (X,Ω). Let $W_1, W_2, W_3, \ldots, W_n \in \lambda_\Omega$ with $W_1 \cap W_2 \cap \ldots \cap W_n \neq \emptyset$. By Theorem 4.3, $W_1 \cap W_2 \cap \ldots \cap W_n \in \tilde{\lambda}_\Omega$. Since Ω_0 is a kernel, there exists a metric $\sigma_1 \in \Omega_0$ such that $i_{\sigma_1}(\cap_{i=1}^n W_i) \neq \emptyset$. Therefore, Ω_0 is a perfect kernel in (X,Ω). $\square$

Theorem 4.5. *Let (X,Ω) be a generalized metric space and Ω satisfy the $\mathscr{V}$-property. Then (X,Ω) is a weakly complete metric space if and only if (X,Ω) is a complete metric space.*

Proof. Suppose (X,Ω) is a weakly complete metric space. Then there exists a kernel $\Omega_0 \subset \Omega$ consisting of all complete metrics on X. By Theorem 4.4, Ω_0 is a perfect kernel on X. Thus, there exists a perfect kernel $\Omega_0 \subset \Omega$ consisting of all complete metrics on X. Therefore, (X,Ω) is a complete space. Since every complete metric space is a weakly complete metric space, the converse follows. $\square$

The following Theorem 4.6 gives the relations between λ_Ω-open sets and neighbourhoods defined in a statistical metric space.

Theorem 4.6. *Let (S,F) be a statistical metric space. If the distribution function $F_{pq}(x_i) = 1 - \sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S$ and $i \in \mathbb{N}$ where Ω_S is the collection of all metrics defined on S, then the following hold.*
(a) *Every τ-neighbourhood on S is a λ_{Ω_S}-open set.*
(b) *Every τ_e-neighbourhood on S is a λ_{Ω_S}-open set.*
(c) *Every τ_R-neighbourhood on S is a λ_{Ω_S}-open set.*

Proof.
(a) Let U be an arbitrary τ-neighbourhood on S. Then $U = \{q \in S \mid F_{pq}(u_1) > 1 - v_1\}$ where $u_1, v_1 > 0$. By hypothesis, $U = \{q \in S \mid 1 - \sigma_1(p,q) > 1 - v_1\} = \{q \in S \mid \sigma_1(p,q) < v_1\} = \{q \in S \mid q \in B_{\sigma_1}(p,v_1)\}$. Hence $U = B_{\sigma_1}(p,v_1)$ and so for each $x \in U$, there is a metric $\sigma \in \Omega_S$ and $\varepsilon > 0$ such that $B_\sigma(x,\varepsilon) \subset U$. Therefore, $U \in \lambda_{\Omega_S}$. Hence every τ-neighbourhood is a λ_{Ω_S}-open set.

(b) By Theorem 3.5, every τ_e-neighbourhood on S is a τ-neighbourhood on S. Therefore, by (a), every τ_e-neighbourhood on S is a λ_{Ω_S}-open set on S.
(c) Every τ_R-neighbourhood on S is a τ-neighbourhood on S, by Theorem 3.7. By (a), every τ_R-neighbourhood on S is a λ_{Ω_S}-open set on S.

□

Theorem 4.7. *Let (S,F) be a statistical metric space. If the distribution function $F_{pq}(x_i) = 1-\sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S$ and $i \in \mathbb{N}$, then the following hold.*
(a) Every λ_{Ω_S}-open set contains a τ-neighbourhood on S.
(b) Every λ_{Ω_S}-open set contains a τ_e-neighbourhood on S.

Proof. We will present the detailed proof only for (b). Let $A \in \tilde{\lambda}_{\Omega_S}$ and $x \in A$. Then there is a metric $\sigma_1 \in \Omega_S$ and $\varepsilon > 0$ such that $B_{\sigma_1}(x,\varepsilon) \subset A$. Let $y \in B_{\sigma_1}(x,\varepsilon)$. Then $\sigma_1(x,y) < \varepsilon$ implies that $1-F_{xy}(u_1) < \varepsilon$ where $u_1 > 0$, by hypothesis. Take $f(u_1) = \varepsilon$. Then $F_{xy}(u_1) > 1-f(u_1)$ and so $y \in \{z \in S \mid F_{xz}(u_1) > 1-f(u_1)\}$. Take $U = \{z \in S \mid F_{xz}(u_1) > 1-f(u_1)\}$. Then $U = \{z \in S \mid G_{xz}(u_1) < f(u_1)\}$ and $B_{\sigma_1}(x,\varepsilon) \subseteq U$. Since in a statistical metric space $G(p,q) = G_{pq}$, $U = \{z \in S \mid G(x,z) < f\}$. Therefore, U is a τ_e-neighbourhood on S. Let $t \in U$. Then $F_{xt}(u_1) > 1-f(u_1)$ and so $1-\sigma_1(x,t) > 1-f(u_1)$, by hypothesis. This implies that $\sigma_1(x,t) < f(u_1)$ which implies that $\sigma_1(x,t) < \varepsilon$, since $f(u_1) = \varepsilon$. Therefore, $t \in B_{\sigma_1}(x,\varepsilon)$. Hence $U = B_{\sigma_1}(x,\varepsilon)$. Thus, $U \subset A$. Hence A contains a τ_e-neighbourhood on S. □

The following Theorem 4.8 shows that a collection of all λ_Ω-open sets is a stack in statistical metric spaces.

Theorem 4.8. *Let (S,F) be a statistical metric space with a g-topology ν. If the distribution function $F_{pq}(x_i) = 1-\sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S, i \in \mathbb{N}$ where $\nu \in \{\tau, \tau_e\}$, then the following hold.*
(a) *The collection λ_{Ω_S} is a stack.*
(b) *If $W_1, W_2, \ldots, W_n \in \lambda_{\Omega_S}$ with $W_1 \cap W_2 \cap \ldots \cap W_n \neq \emptyset$, then $W_1 \cap W_2 \cap \ldots \cap W_n \in \lambda_{\Omega_S}$.*

Proof. We will give a detailed proof only for $\nu = \tau$.

(a) Let $A \in \lambda_{\Omega_S}$ and $A \subset B$. By hypothesis and Theorem 4.7, A contains a τ-neighbourhood W on S. This implies that $W \subset B$ which implies that $B \in \mathcal{N}(Z)$, since $\mathcal{N}(Z)$ is stack (Theorem 3.15). Therefore, $B \in \lambda_{\Omega_S}$, by hypothesis and Theorem 4.6. Hence λ_{Ω_S} is a stack.

(b) Let $V_1, V_2, \ldots, V_n \in \lambda_{\Omega_S}$ with $V_1 \cap V_2 \cap \ldots \cap V_n \neq \emptyset$. Choose $x \in V_1 \cap V_2 \cap \ldots \cap V_n$. Then there exist $\sigma_i \in \Omega_S, \varepsilon_i > 0$ such that $B_{\sigma_i}(x,\varepsilon_i) \subset V_i$ for $i = 1$ to n and so $\cap_{i=1}^n B_{\sigma_i}(x,\varepsilon_i) \subset \cap_{i=1}^n V_i$. As in the proof of Theorem 4.7, we get that $B_i = W_i$ where $B_i = B_{\sigma_i}(x,\varepsilon_i)$ and W_i is a τ-neighbourhood on S for $i = 1$ to n. Therefore, $\cap_{i=1}^n W_i \subset \cap_{i=1}^n V_i$. By Theorem 3.18, $\cap_{i=1}^n W_i$ is a τ-neighbourhood on S. Thus, $\cap_{i=1}^n W_i$ is a λ_{Ω_S}-open set, by Theorem 4.6 and hence $\cap_{i=1}^n V_i \in \lambda_{\Omega_S}$, by (a).

□

Theorem 4.9. *Let (S,F) be a statistical metric space with a g-topology τ or τ_e. If the distribution function $F_{pq}(x_i) = 1-\sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S, i \in \mathbb{N}$ and if $\Omega_0 \subset \Omega_S$ is a kernel in (S,Ω_S), then it is a perfect kernel in (S,Ω_S).*

Proof. Let (S,F) be a statistical metric space with τ. Suppose $\Omega_0 \subset \Omega_S$ is a kernel in (S,Ω_S). Let $V_1, V_2, V_3, \ldots, V_n \in \lambda_{\Omega_S}$ with $V_1 \cap V_2 \cap \ldots \cap V_n \neq \emptyset$. By hypothesis and Theorem 4.8, $V_1 \cap V_2 \cap \ldots \cap V_n \in \tilde{\lambda}_{\Omega_S}$. Since Ω_0 is a kernel, there exists a metric $\sigma_1 \in \Omega_0$ such that $i_{\sigma_1}(\cap_{i=1}^n V_i) \neq \emptyset$. Therefore, Ω_0 is a perfect kernel in (S,Ω_S).
Let (S,F) be a statistical metric space with τ_e. By the same argument as in above and Theorem 4.8, we can prove that Ω_0 is a perfect kernel in (S,Ω_S). □

Theorem 4.10. *Let (S,F) be a statistical metric space with a g-topology τ or τ_e. If the distribution function $F_{pq}(x_i) = 1-\sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S, i \in \mathbb{N}$ and if (S,Ω_S) is a weakly complete metric space, then (S,Ω_S) is a complete metric space.*

Proof. Let (S,F) be a statistical metric space with τ. Suppose (S,Ω_S) is a weakly complete space. Then there exists a kernel $\Omega_0 \subset \Omega_S$ consisting of all complete metrics on S. By hypothesis and Theorem 4.9, Ω_0 is a perfect kernel on S. Thus, there exists a perfect kernel $\Omega_0 \subset \Omega_S$ consisting of all complete metrics on S. Therefore, (S,Ω_S) is a complete metric space.
Suppose that (S,F) is a statistical metric space with τ_e. By the same argument as in above and Theorem 4.9, we can prove that (S,Ω_S) is a complete metric space. □

5. Conclusion

This article provide the basis for carrying out analysis in statistical metric spaces, in particular for the development of various g-topologies, neighbourhoods defined in a statistical metric space and also the improvement of λ_Ω-open sets in a generalized metric space. We have given more examples of the neighbourhoods defined in a statistical metric space and the special kind of relationship between various g-topologies defined by Thorp in a SM space. Also, new properties for λ_Ω-open sets in a generalized metric space have presented. We have given partial answer to the following questions raised by Thorp in statistical metric spaces:
What are the necessary and sufficient conditions that the g-topology of $typeV$ to be of $typeV_D$?
What are the necessary and sufficient conditions that the g-topology of $typeV_\alpha$ to be the g-topology of $typeV_D$?
What conditions are both necessary and sufficient for the g-topology of $typeV_\alpha$ to be a topology?

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On the Solutions of Four Second-Order Nonlinear Difference Equations

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Abstract

This paper deals with the form, the stability character, the periodicity and the global behavior of solutions of the following four rational difference equations

$$x_{n+1}=\frac{\pm 1}{x_n\left(x_{n-1}\pm 1\right)-1}$$
$$x_{n+1}=\frac{\pm 1}{x_n\left(x_{n-1}\mp 1\right)+1}.$$

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1. Introduction

Difference equation or discrete dynamical system is a diverse field which impact almost every branch of pure and applied mathematics. Lately, there has been great interest in the study of solving difference equations and systems of difference equations, see [1-20]. In these studies, the authors deal with the closed-form, stability, periodicity, boundedness and asymptotic behavior of solutions of nonlinear difference equations and systems of difference equations. There are many recent investigations and interest in the field which difference equations have been studied by several authors, as in the examples given below:
In [2], Tollu et al. considered the following difference equations

$$x_{n+1}=\frac{1}{1+x_n},\ \ y_{n+1}=\frac{1}{-1+y_n},\ \ n=0,1,..., \tag{1.1}$$

such that their solutions are associated with Fibonacci numbers.
In [6], Halim and Bayram investigated the solutions, stability character, and asymptotic behavior of the difference equation

$$x_{n+1}=\frac{\alpha}{\beta+\gamma x_{n-k}},\ n\in\mathbb{N}_0, \tag{1.2}$$

where the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonzero real numbers, such that its solutions are associated to Horadam numbers, which are generalized Fibonacci numbers.
Then, in [7] Halim considered the system of difference equations

$$x_{n+1}=\frac{1}{1+y_{n-2}},\ \ y_{n+1}=\frac{1}{1+x_{n-2}},\ \ n=0,1,..., \tag{1.3}$$

such that their solutions are associated with Fibonacci numbers, where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and the initial conditions x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, and y_0 are real numbers.
In [8], Halim and Rabago studied the systems of difference equaions

$$x_{n+1}=\frac{1}{\pm 1\pm y_{n-k}},\ \ y_{n+1}=\frac{1}{\pm 1\pm x_{n-k}},\ \ n,k\in\mathbb{N}_0, \tag{1.4}$$

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where the initial conditions x_{-k}, x_{-k+1}, ..., x_0, y_{-k}, y_{-k+1}, ..., y_0 are nonzero real numbers.
Then, in [9], the authors studied the rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma x_n x_{n-1}},\ n \in \mathbb{N}_0, \tag{1.5}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, α, β, $\gamma \in \mathbb{R}^+$ and the initial conditions nonzero real numbers and also investigated the two-dimensional case of the this equation given by

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma y_n x_{n-1}},\ y_{n+1} = \frac{\alpha y_{n-1} + \beta}{\gamma x_n y_{n-1}},\ n \in \mathbb{N}_0. \tag{1.6}$$

Also, the solutions of Eq. (1.5) and system of (1.6) are associated to generalized Padovan numbers.
As far as we examine, there is no paper dealing with the following difference equations. Hence, in this study, we study the following four difference equations

$$x_{n+1} = \frac{1}{x_n(x_{n-1}+1)-1},\quad n = 0,1,..., \tag{1.7}$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1}-1)-1},\quad n = 0,1,..., \tag{1.8}$$

$$x_{n+1} = \frac{1}{x_n(x_{n-1}-1)+1},\quad n = 0,1,..., \tag{1.9}$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1}+1)+1},\quad n = 0,1,.... \tag{1.10}$$

2. Preliminaries

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}),\quad n = 0,1,.... \tag{2.1}$$

A solution of Eq.(2.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(2.1) for all $n \geq -k$.

Definition 2.1. *A solution of Eq.(2.1) that is constant for all $n \geq -k$ is called an equilibrium solution of Eq.(2.1). If*

$x_n = \bar{x}$, *for all* $n \geq -k$

is an equilibrium solution of Eq.(2.1), then $\bar{x}$ is called an equilibrium point, or simply an equilibrium of Eq.(2.1)..

Definition 2.2 (Stability, 1)**.** *Let $\bar{x}$ an equilibrium point of Eq.(2.1).*

(a) *An equilibrium point $\bar{x}$ of Eq.(2.1) is called locally stable if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2.1) with*

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

then

$$|x_n - \bar{x}| < \varepsilon,\ \textit{for all } n \geq -k.$$

(b) *An equilibrium point $\bar{x}$ of Eq.(2.1) is called locally asymptotically stable if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2.1) with*

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + ... + |x_0 - \bar{x}| < \gamma,$$

then we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(c) *An equilibrium point $\bar{x}$ of Eq.(2.1) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1), we have*

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(d) *An equilibrium point $\bar{x}$ of Eq.(2.1) is called globally asymptotically stable if it is locally stable, and a global attractor.*

(e) *An equilibrium point $\bar{x}$ of Eq.(2.1) is called unstable if it is not locally stable.Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let*

$$q_i = \frac{\partial f}{\partial u_i}(\bar{x},\bar{x},...,\bar{x}), \;\; for\; i = 0,1,...,k$$

denote the partial derivative of $f(u_0,u_1,...,u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(2.1).

Definition 2.3. *The equation*

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + ... + q_k y_{n-k}, \; n = 0,1,... \tag{2.2}$$

is called the linearized equation of Eq.(2.1) about the equilibrium point $\bar{x}$, and the equation

$$\lambda^{k+1} - q_0\lambda^k - ... - q_{k-1}\lambda - q_k = 0 \tag{2.3}$$

is called the characteristic equation of Eq.(2.2) about $\bar{x}$.

Theorem 2.4 (The Linearized Stability Theorem, 1)**.** *Assume that the function f is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$. Then the following statements are true:*

(a) *When all the roots of characteristic equation (2.3) have absolute value less than one, then the equilibrium point $\bar{x}$ of Eq.(2.1) is locally asymptotically stable.*
(b) *If at least one root of characteristic equation (2.3) has absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(2.1) is unstable.*
(c) *The equilibrium point $\bar{x}$ of Eq.(2.1) is called hyperbolic if no root of characteristic equation (2.3) has absolute value equal to one.*
(d) *If there exists a root of characteristic equation (2.3) with absolute value equal to one, then the equilibrium $\bar{x}$ is called nonhyperbolic.*
(e) *An equilibrium point $\bar{x}$ of Eq.(2.1) is called a repeller if all roots of characteristic equation (2.3) have absolute value greater than one.*
(f) *An equilibrium point $\bar{x}$ of Eq.(2.1) is called a saddle if one of the roots of characteristic equation (2.3) is greater and another is less than one in absolute value.*

3. Main Results

In this section, we present our main results for the above mentioned difference equations. Our aim is to investigate the general solutions in explicit form of the above mentioned difference equations and the asymptotic behavior of solutions of these difference equations.

3.1. The Difference Equation (1.7)

Theorem 3.1. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1.7). Then, for $n = 0,1,2,...$, the forms of solutions $\{x_n\}_{n=-1}^{\infty}$ are given by*

$$x_{2n-1} = \frac{(1-n)x_{-1}x_0 + n}{nx_{-1}x_0 + x_0 - n} \tag{3.1}$$

$$x_{2n} = \frac{nx_{-1}x_0 + x_0 - n}{-nx_{-1}x_0 + n + 1} \tag{3.2}$$

where the initial conditions x_{-1}, $x_0 \in \mathbb{R} - F_1$, with F_1 is the forbidden set of Eq.(1.7) given by

$$F_1 = \cup_{n=-1}^{\infty}\{(x_{-1},x_0) : nx_{-1}x_0 + x_0 - n = 0 \; or \; -nx_{-1}x_0 + n + 1 = 0\}.$$

Proof. For $n = 0$ the result holds. Assume that $n > 0$ and that our assumption holds for $n-1$. That is,

$$x_{2n-3} = \frac{(2-n)x_{-1}x_0 + n - 1}{(n-1)x_{-1}x_0 + x_0 - (n-1)}$$

and

$$x_{2n-2} = \frac{(n-1)x_{-1}x_0 + x_0 - (n-1)}{(1-n)x_{-1}x_0 + n}.$$

From this and from Eq.(1.7), it follows that

$$\begin{aligned} x_{2n-1} &= \frac{1}{x_{2n-2}(x_{2n-3}+1) - 1} \\ &= \frac{1}{\frac{(n-1)x_{-1}x_0 + x_0 - (n-1)}{(1-n)x_{-1}x_0 + n}\left(\frac{(2-n)x_{-1}x_0 + n - 1}{(n-1)x_{-1}x_0 + x_0 - (n-1)} + 1\right) - 1} \\ &= \frac{(1-n)x_{-1}x_0 + n}{nx_{-1}x_0 + x_0 - n}. \end{aligned}$$

Hence, similarly, we obtain

$$\begin{aligned} x_{2n} &= \frac{1}{x_{2n-1}(x_{2n-2}+1) - 1} \\ &= \frac{1}{\frac{(1-n)x_{-1}x_0 + n}{nx_{-1}x_0 + x_0 - n}\left(\frac{(n-1)x_{-1}x_0 + x_0 - (n-1)}{(1-n)x_{-1}x_0 + n} + 1\right) - 1} \\ &= \frac{nx_{-1}x_0 + x_0 - n}{-nx_{-1}x_0 + n + 1}. \end{aligned}$$

Theorem 3.2. *The following statements are true.*

(i) *The equilibrium points of Eq.(1.7) are* $\bar{x}_1 = 1$ *and* $\bar{x}_2 = -1$.
(ii) *The positive equilibrium point of Eq.(1.7),* $\bar{x}_1 = 1$*, is nonhyperbolic point.*
(iii) *The negative equilibrium point of Eq.(1.7),* $\bar{x}_2 = -1$*, is nonhyperbolic point.*

Proof.

(i) Equilibrium points of Eq.(1.7) satisfy the equation

$$\bar{x} = \frac{1}{\bar{x}(\bar{x}+1)-1}.$$

After simplification, we have the following cubic equation

$$\bar{x}^3 + \bar{x}^2 - \bar{x} - 1 = 0. \tag{3.3}$$

The roots of the cubic equation (3.3) are -1, -1, 1. Therefore, Eq.(1.7) has two equilibra, one positive and one negative, such that

$$\bar{x}_1 = 1, \ \bar{x}_2 = -1.$$

(ii) Now, let $I = (0, \infty)$ and consider the function

$$f : I^2 \to I$$

defined by

$$f(x,y) = \frac{1}{x(y+1)-1}. \tag{3.4}$$

Then, it follows that

$$\frac{\partial f(x,y)}{\partial x} = \frac{-(y+1)}{(x(y+1)-1)^2},$$
$$\frac{\partial f(x,y)}{\partial y} = \frac{-x}{(x(y+1)-1)^2}.$$

Therefore, the linearized equation of Eq.(1.7) about the equilibrium point $\bar{x}_1 = 1$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$p = \frac{\partial f(\bar{x}_1, \bar{x}_1)}{\partial x} = -2,$$
$$q = \frac{\partial f(\bar{x}_1, \bar{x}_1)}{\partial y} = -1,$$

and the corresponding characteristic equation is

$$\lambda^2 + 2\lambda + 1 = 0.$$

Therefore, from Theorem 2.4, it is clearly seen that

$$\lambda_{1,2} = -1$$

and

$$|\lambda_1| = |\lambda_2| = 1.$$

So, $\bar{x}_1$ is nonhyperbolic point.

(iii) Similarly, from (3.4), the linearized equation of Eq.(1.7) about the equilibrium point $\bar{x}_2 = -1$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$p = \frac{\partial f(\bar{x}_2, \bar{x}_2)}{\partial x} = 0,$$
$$q = \frac{\partial f(\bar{x}_2, \bar{x}_2)}{\partial y} = 1,$$

and its characteristic equation is

$$\lambda^2 - 1 = 0.$$

Thus, it follows that

$$\lambda_{1,2} = \pm 1$$

and

$$|\lambda_1| = |\lambda_2| = 1.$$

So, $\bar{x}_2$ is nonhyperbolic point.

Theorem 3.3. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1.7). Then, the negative equilibrium point of Eq.(1.7), $\bar{x}_2$, is a global attractor.*

Proof. From Theorem 3.1, we have

$$\begin{aligned}
\lim_{n\to\infty} x_{2n-1} &= \lim_{n\to\infty} \frac{(1-n)x_{-1}x_0+n}{nx_{-1}x_0+x_0-n} \\
&= \lim_{n\to\infty} \frac{(1-n)\left(x_{-1}x_0+\frac{n}{1-n}\right)}{n\left(x_{-1}x_0+\frac{x_0}{n}-1\right)} \\
&= \lim_{n\to\infty} \frac{(1-n)\left(x_{-1}x_0-1+\frac{1}{1-n}\right)}{n\left(x_{-1}x_0+\frac{x_0}{n}-1\right)} \\
&= -1,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} \frac{nx_{-1}x_0+x_0-n}{-nx_{-1}x_0+n+1} \\
&= \lim_{n\to\infty} \frac{n\left(x_{-1}x_0+\frac{x_0}{n}-1\right)}{-n\left(x_{-1}x_0-1-\frac{1}{n}\right)} \\
&= -1.
\end{aligned}$$

Hereby, it implies

$$\lim_{n\to\infty} x_n = -1.$$

3.2. The Difference Equation (1.8)

Theorem 3.4. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1.8). Then, for $n=0,1,2,\ldots$, the forms of solutions $\{x_n\}_{n=-1}^{\infty}$ are given by*

$$x_{2n-1} = \frac{-\left((1-n)x_{-1}x_0+n\right)}{nx_{-1}x_0-x_0-n} \tag{3.5}$$

$$x_{2n} = \frac{-\left(nx_{-1}x_0-x_0-n\right)}{-nx_{-1}x_0+n+1} \tag{3.6}$$

where the initial conditions x_{-1}, $x_0 \in \mathbb{R}-F_2$, with F_2 is the forbidden set of Eq.(1.8) given by

$$F_2 = \cup_{n=-1}^{\infty}\left\{(x_{-1},x_0) : nx_{-1}x_0-x_0-n=0 \text{ or } -nx_{-1}x_0+n+1=0\right\}.$$

Proof. For $n=0$ the result holds. Assume that $n>0$ and that our assumption holds for $n-1$. That is,

$$x_{2n-3} = \frac{-\left((2-n)x_{-1}x_0+n-1\right)}{(n-1)x_{-1}x_0-x_0-(n-1)}$$

and

$$x_{2n-2} = \frac{-\left((n-1)x_{-1}x_0-x_0-(n-1)\right)}{-(n-1)x_{-1}x_0+n}.$$

From this and from Eq.(1.8), it follows that

$$\begin{aligned}
x_{2n-1} &= \frac{-1}{x_{2n-2}\left(x_{2n-3}-1\right)-1} \\
&= \frac{-1}{\frac{-((n-1)x_{-1}x_0-x_0-(n-1))}{-(n-1)x_{-1}x_0+n}\left(\frac{-((2-n)x_{-1}x_0+n-1)}{(n-1)x_{-1}x_0-x_0-(n-1)}-1\right)-1} \\
&= \frac{-\left((1-n)x_{-1}x_0+n\right)}{nx_{-1}x_0-x_0-n}.
\end{aligned}$$

Hence, similarly, we obtain

$$\begin{aligned}
x_{2n} &= \frac{-1}{x_{2n-1}\left(x_{2n-2}-1\right)-1} \\
&= \frac{-1}{\frac{-((1-n)x_{-1}x_0+n)}{nx_{-1}x_0-x_0-n}\left(\frac{-((n-1)x_{-1}x_0-x_0-(n-1))}{-(n-1)x_{-1}x_0+n}-1\right)-1} \\
&= \frac{-\left(nx_{-1}x_0-x_0-n\right)}{-nx_{-1}x_0+n+1}.
\end{aligned}$$

Theorem 3.5. *The following statements are true.*

(i) *The equilibrium points of Eq.(1.8) are $\bar{x}_1=1$ and $\bar{x}_2=-1$.*
(ii) *The positive equilibrium point of Eq.(1.8), $\bar{x}_1=1$, is nonhyperbolic point.*

(iii) *The negative equilibrium point of Eq.(1.8), $\bar{x}_2 = -1$, is nonhyperbolic point.*

Proof.

(i) Equilibrium points of Eq.(1.8) satisfy the equation

$$\bar{x} = \frac{-1}{\bar{x}(\bar{x}-1)-1}.$$

After simplification, we have the following cubic equation

$$\bar{x}^3 - \bar{x}^2 - \bar{x} + 1 = 0. \tag{3.7}$$

The roots of the cubic equation (3.7) are -1, 1, 1. Therefore, Eq.(1.8) has two equilibra, one positive and one negative, such that

$$\bar{x}_1 = 1,\ \bar{x}_2 = -1.$$

(ii) Now, let $I = (0, \infty)$ and consider the function

$$f : I^2 \to I$$

defined by

$$f(x,y) = \frac{-1}{x(y-1)-1}. \tag{3.8}$$

Then, it follows that

$$\frac{\partial f(x,y)}{\partial x} = \frac{(y-1)}{(x(y-1)-1)^2},$$
$$\frac{\partial f(x,y)}{\partial y} = \frac{x}{(x(y-1)-1)^2}.$$

Therefore, the linearized equation of Eq.(1.8) about the equilibrium point $\bar{x}_1 = 1$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$p = \frac{\partial f(\bar{x}_1, \bar{x}_1)}{\partial x} = 0,$$
$$q = \frac{\partial f(\bar{x}_1, \bar{x}_1)}{\partial y} = 1,$$

and the corresponding characteristic equation is

$$\lambda^2 - 1 = 0.$$

Therefore, from Theorem 2.4, it is clearly seen that

$$\lambda_{1,2} = \pm 1$$

and

$$|\lambda_1| = |\lambda_2| = 1.$$

So, $\bar{x}_1$ is nonhyperbolic point.

(iii) Similarly, from (3.8), the linearized equation of Eq.(1.8) about the equilibrium point $\bar{x}_2 = -1$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$p = \frac{\partial f(\bar{x}_2, \bar{x}_2)}{\partial x} = -2,$$
$$q = \frac{\partial f(\bar{x}_2, \bar{x}_2)}{\partial y} = -1,$$

and its characteristic equation is

$$\lambda^2 + 2\lambda + 1 = 0.$$

Thus, it follows that

$$\lambda_{1,2} = -1$$

and

$$|\lambda_1| = |\lambda_2| = 1.$$

So, $\bar{x}_2$ is nonhyperbolic point.

Theorem 3.6. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1.8). Then, the positive equilibrium point of Eq.(1.8), $\bar{x}_1$, is a global attractor.*

Proof. From Theorem 3.4, we obtain

$$\begin{aligned}\lim_{n\to\infty} x_{2n-1} &= \lim_{n\to\infty} \frac{-\left((1-n)x_{-1}x_0+n\right)}{nx_{-1}x_0-x_0-n}\\ &= \lim_{n\to\infty} \frac{(n-1)\left(x_{-1}x_0+\frac{n}{1-n}\right)}{n\left(x_{-1}x_0-\frac{x_0}{n}-1\right)}\\ &= \lim_{n\to\infty} \frac{(n-1)\left(x_{-1}x_0-1+\frac{1}{1-n}\right)}{n\left(x_{-1}x_0-\frac{x_0}{n}-1\right)}\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} \frac{-\left(nx_{-1}x_0-x_0-n\right)}{-nx_{-1}x_0+n+1}\\ &= \lim_{n\to\infty} \frac{-n\left(x_{-1}x_0-\frac{x_0}{n}-1\right)}{-n\left(x_{-1}x_0-1-\frac{1}{n}\right)}\\ &= 1.\end{aligned}$$

Herewith, it implies

$$\lim_{n\to\infty} x_n = 1.$$

So, the proof is complete.

3.3. The Difference Equation (1.9)

Lemma 3.7. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1.9). Then, $\{x_n\}_{n=-1}^{\infty}$ is periodic with period four.*

Proof. From Eq.(1.9),

$$\begin{aligned}x_{n+4} &= \frac{1}{x_{n+3}\left(x_{n+2}-1\right)+1}\\ &= \frac{1}{\left(\frac{1}{x_{n+2}(x_{n+1}-1)+1}\right)\left(\frac{1}{x_{n+1}(x_n-1)+1}-1\right)+1}\\ &= \frac{1}{\left(\frac{1}{\left(\frac{1}{x_{n+1}(x_n-1)+1}\right)\left(\frac{1}{x_n\left(x_{n-1}-1\right)+1}-1\right)+1}\right)\left(\frac{1}{\left(\frac{1}{x_n\left(x_{n-1}-1\right)+1}\right)(x_n-1)+1}-1\right)+1}\\ &= \frac{1}{\left(\frac{1}{\left(\frac{1}{\frac{1}{x_n\left(x_{n-1}-1\right)+1}(x_n-1)+1}\right)\left(\frac{x_n\left(1-x_{n-1}\right)}{x_nx_{n-1}-x_n+1}\right)+1}\right)\left(\frac{1-x_n}{x_nx_{n-1}}\right)+1}\\ &= \frac{1}{x_{n-1}\left(\frac{1-x_n}{x_nx_{n-1}}\right)+1}\\ &= x_n.\end{aligned}$$

Hence, the result holds.

Theorem 3.8. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1.9). Then, for $n=1,2,\ldots$,*

$$\begin{aligned}x_{4n-3} &= \frac{1}{x_{-1}x_0-x_0+1}\\ x_{4n-2} &= \frac{x_{-1}x_0-x_0+1}{x_{-1}x_0}\\ x_{4n-1} &= x_{-1}\\ x_{4n} &= x_0\end{aligned} \tag{3.9}$$

where the initial conditions $x_{-1}, x_0 \in \mathbb{R}-F_3$, with F_3 is the forbidden set of Eq.(1.9) given by

$$F_3 = \left\{(x_{-1},x_0) : x_{-1}x_0 = 0 \text{ or } x_{-1} = \frac{x_0-1}{x_0}\right\}.$$

Proof. From (1.9), for $n = 0$, the result holds. Suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$x_{4n-7} = \frac{1}{x_{-1}x_0 - x_0 + 1},$$
$$x_{4n-6} = \frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0},$$
$$x_{4n-5} = x_{-1},$$
$$x_{4n-4} = x_0.$$

Now, from Eq.(1.9), it follows that

$$x_{4n-3} = \frac{1}{x_{4n-4}\left(x_{4n-5} - 1\right) + 1} = \frac{1}{x_{-1}x_0 - x_0 + 1}.$$

From this and from Eq.(1.9), it follows that

$$x_{4n-2} = \frac{1}{x_{4n-3}\left(x_{4n-4} - 1\right) + 1} = \frac{1}{\frac{1}{x_{-1}x_0 - x_0 + 1}\left(x_0 - 1\right) + 1} = \frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0}.$$

Again from Eq.(1.9), we get

$$x_{4n-1} = \frac{1}{x_{4n-2}\left(x_{4n-3} - 1\right) + 1} = \frac{1}{\frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0}\left(\frac{1}{x_{-1}x_0 - x_0 + 1} - 1\right) + 1} = \frac{x_{-1}x_0}{x_0} = x_{-1}.$$

Similarly, from Eq.(1.9), we have

$$x_{4n-4} = \frac{1}{x_{4n-1}\left(x_{4n-2} - 1\right) + 1} = \frac{1}{x_{-1}\left(\frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0} - 1\right) + 1}$$
$$= \frac{1}{x_{-1} - 1 + \frac{1}{x_0} - x_{-1} + 1} = x_0.$$

Thus, the proof is complete.

Theorem 3.9. *Eq.(1.9) has unique positive equilibrium point $\bar{x} = 1$ and 1 is nonhyperbolic point.*

Proof. Equilibrium point of Eq.(1.9) satisfy the equation

$$\bar{x} = \frac{1}{\bar{x}\left(\bar{x} - 1\right) + 1}.$$

After simplification, we have the following cubic equation

$$\bar{x}^3 - \bar{x}^2 + \bar{x} - 1 = 0. \tag{3.10}$$

The roots of the cubic equation (3.10) are $-i$, i, 1. Therefore, the unique positive equilibrium point of Eq.(1.9) is $\bar{x} = 1$.
Now, we prove that the equilibrium point of Eq.(1.9) is nonhyperbolic.
Let $I = (0, \infty)$ and consider the function

$$f : I^2 \to I$$

defined by

$$f\left(x, y\right) = \frac{1}{x\left(y - 1\right) + 1}.$$

The linearized equation of Eq.(1.9) about the equilibrium point $\bar{x} = 1$ is

$$z_{n+1} = p z_n + q z_{n-1},$$

where

$$p = \frac{\partial f\left(\bar{x}, \bar{x}\right)}{\partial x} = 0,$$
$$q = \frac{\partial f\left(\bar{x}, \bar{x}\right)}{\partial y} = -1,$$

and the corresponding characteristic equation is

$$\lambda^2 + 1 = 0.$$

Therefore, from Theorem 2.4, it is clearly seen that

$$\lambda_{1,2} = \pm i$$

and

$$\left|\lambda_1\right| = \left|\lambda_2\right| = 1.$$

So, this completes the proof.

3.4. The Difference Equation (1.10)

Lemma 3.10. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1.10). Then, $\{x_n\}_{n=-1}^{\infty}$ is periodic with periods four.*

Proof. From Eq.(1.10),

$$\begin{aligned}
x_{n+4} &= \frac{-1}{x_{n+3}\left(x_{n+2}+1\right)+1} \\
&= \frac{-1}{\left(\frac{-1}{x_{n+2}\left(x_{n+1}+1\right)+1}\right)\left(\frac{-1}{x_{n+1}\left(x_n+1\right)+1}+1\right)+1} \\
&= \frac{-1}{\left(\frac{-1}{\left(\frac{-1}{x_{n+1}\left(x_n+1\right)+1}\right)\left(\frac{-1}{x_n\left(x_{n-1}+1\right)+1}+1\right)+1}\right)\left(\frac{-1}{\left(\frac{-1}{x_n\left(x_{n-1}+1\right)+1}\right)\left(x_n+1\right)+1}+1\right)+1} \\
&= \frac{-1}{\left(\frac{-1}{\left(\frac{-1}{\frac{-1}{x_n\left(x_{n-1}+1\right)+1}\left(x_n+1\right)+1}\right)\left(\frac{x_n\left(x_{n-1}+1\right)}{x_nx_{n-1}+x_n+1}\right)+1}\right)\left(-\frac{x_n+1}{x_nx_{n-1}}\right)+1} \\
&= \frac{-1}{x_{n-1}\left(-\frac{x_n+1}{x_nx_{n-1}}\right)+1} \\
&= x_n.
\end{aligned}$$

Hence, the result holds.

Theorem 3.11. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1.10). Then, for $n=1,2,\ldots,$*

$$\begin{aligned}
x_{4n-3} &= \frac{-1}{x_{-1}x_0+x_0+1} \\
x_{4n-2} &= \frac{-\left(x_{-1}x_0+x_0+1\right)}{x_{-1}x_0} \\
x_{4n-1} &= x_{-1} \\
x_{4n} &= x_0
\end{aligned} \tag{3.11}$$

where the initial conditions x_{-1}, $x_0 \in \mathbb{R}-F_4$, with F_4 is the forbidden set of Eq.(1.10) given by

$$F_4 = \left\{\left(x_{-1},x_0\right) : x_{-1}x_0 = 0 \text{ or } x_{-1} = \frac{-\left(x_0+1\right)}{x_0}\right\}.$$

Proof. From (1.10), for $n=0$, the result holds. Suppose that $n>0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{4n-7} &= \frac{-1}{x_{-1}x_0+x_0+1}, \\
x_{4n-6} &= \frac{-\left(x_{-1}x_0+x_0+1\right)}{x_{-1}x_0}, \\
x_{4n-5} &= x_{-1}, \\
x_{4n-4} &= x_0.
\end{aligned}$$

Now, from Eq.(1.10), it follows that

$$x_{4n-3} = \frac{-1}{x_{4n-4}\left(x_{4n-5}+1\right)+1} = \frac{-1}{x_{-1}x_0+x_0+1}.$$

From this and from Eq.(1.10), it follows that

$$x_{4n-2} = \frac{-1}{x_{4n-3}\left(x_{4n-4}+1\right)+1} = \frac{-1}{\frac{-1}{x_{-1}x_0+x_0+1}\left(x_0+1\right)+1} = \frac{-\left(x_{-1}x_0+x_0+1\right)}{x_{-1}x_0}.$$

Again from Eq.(1.10), we get

$$x_{4n-1} = \frac{-1}{x_{4n-2}\left(x_{4n-3}+1\right)+1} = \frac{-1}{\frac{-\left(x_{-1}x_0+x_0+1\right)}{x_{-1}x_0}\left(\frac{-1}{x_{-1}x_0+x_0+1}+1\right)+1} = \frac{-x_{-1}x_0}{-x_0} = x_{-1}.$$

Similarly, from Eq.(1.10), we have

$$\begin{aligned}
x_{4n} &= \frac{-1}{x_{4n-1}\left(x_{4n-2}+1\right)+1} = \frac{-1}{x_{-1}\left(\frac{-\left(x_{-1}x_0+x_0+1\right)}{x_{-1}x_0}+1\right)+1} \\
&= \frac{-1}{-x_{-1}-1-\frac{1}{x_0}+x_{-1}+1} = x_0.
\end{aligned}$$

Thus, the proof is complete.

Theorem 3.12. *Eq.(1.10) has unique negative equilibrium point $\bar{x}=-1$ and the equilibrium point -1 is nonhyperbolic point.*

Proof. Equilibrium point of Eq.(1.10) satisfy the equation

$$\bar{x}=\frac{-1}{\bar{x}(\bar{x}+1)+1}.$$

After simplification, we have the following cubic equation

$$\bar{x}^3+\bar{x}^2+\bar{x}+1=0. \tag{3.12}$$

The roots of the cubic equation (3.12) are $-i$, i, -1. Therefore, the unique negative equilibrium point of Eq.(1.10) is $\bar{x}=-1$.
Now, we demonstrate that the equilibrium point of Eq.(1.10) is nonhyperbolic.
Let $I=(0,\infty)$ and consider the function

$$f:I^2\rightarrow I$$

defined by

$$f(x,y)=\frac{-1}{x(y+1)+1}.$$

The linearized equation of Eq.(1.10) about the equilibrium point $\bar{x}=-1$ is

$$z_{n+1}=pz_n+qz_{n-1},$$

where

$$p=\frac{\partial f(\bar{x},\bar{x})}{\partial x}=0,$$
$$q=\frac{\partial f(\bar{x},\bar{x})}{\partial y}=-1,$$

and the corresponding characteristic equation is

$$\lambda^2+1=0.$$

Therefore, from Theorem 2.4, it is clearly seen that

$$\lambda_{1,2}=\pm i$$

and

$$|\lambda_{1,2}|=1.$$

Thus, the proof is complete.

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De-Moivre and Euler Formulae for Dual-Complex Numbers

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Abstract

In this study, we generalize the well-known formulae of De-Moivre and Euler of complex numbers to dual-complex numbers. Furthermore, we investigate the roots and powers of a dual-complex number by using these formulae. Consequently, we give some examples to illustrate the main results in this paper.

1. Introduction

The complex numbers have emerged from the need to solve cubic equations. First studies on complex numbers were produced by G. Cardan (1501-1576) and B. Bombelli (1526-1572). Later, Euler used the formula

$$x+iy=r(\cos\theta+i\sin\theta)$$

and he studied the root of the equation $z^n=1$. Also, he proved that a complex number can be written in the form of

$$e^{i\theta}=\cos\theta+i\sin\theta$$

[1]. Abraham de Moivre found the formula

$$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$$

and gave his own name to this formula. The developments in the number theory present us new number systems including the dual numbers which are expressed by the real and dual parts like complex numbers. This idea was first introduced by W. K. Clifford to solve some algebraic problems, [2]. Afterwards, E. Study presented different theorems with his studies on kinematics and line geometry, [3]. A dual number is a pair of real numbers which consists of the real unit $+1$ and dual unit ε satisfying $\varepsilon^2=0$ for $\varepsilon\neq 0$. Therefore, dual numbers are elements of two-dimensional real algebra $\mathbb{D}=\left\{z=x+\varepsilon y | x,y\in\mathbb{R},\varepsilon^2=0,\varepsilon\neq 0\right\}$ which is generated by 1 and ε. Similar to the complex numbers, the module of a dual number z is defined by $|z|=|x+\varepsilon y|=|x|=r$, its argument is $\theta=\frac{y}{x}$ and represented by $\arg(z)$. The set of all points which satisfy the equation $|z|=|x|=r>0$ and which are on the dual plane are $x=\pm r$ lines, [4].

This circle is called Galilean circle on a dual plane. Let S be a circle centered with O and M be a point on S. If d is OM line, and α is the angle δ_{Od}, a Galilean circle is represented by

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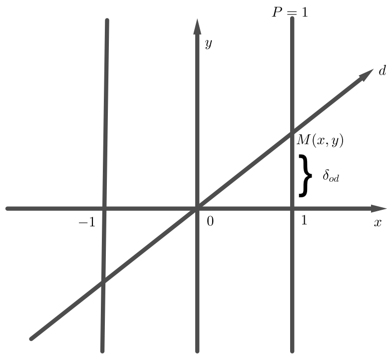

Figure 1.1: Galilean unit circle

So, one can easily see that

$$\cos g\alpha = \frac{|OP|}{|OM|} = 1 \quad , \quad \sin g\alpha = \frac{|MP|}{|OM|} = \frac{\delta_{Od}}{1} = \alpha.$$

On the other hand, exponential representation of a dual number $z = x + \varepsilon y$ is in the form of $z = xe^{\varepsilon\theta}$ where $\frac{y}{x}$ is dual angle and it is shown as $\arg(z) = \frac{y}{x} = \theta$, [5]. In addition, from the definitions of Galilean cosine and sine, we realize

$$\cos g(\theta) = 1 \text{ and } \sin g(\theta) = \tfrac{y}{x} = \theta.$$

By considering the exponential rules, we write

$$\begin{aligned}
&\cos g(x+y) = \cos g(x)\cos g(y) - \varepsilon^2 \sin g(x)\sin g(y)\\
&\sin g(x+y) = \sin g(x)\cos g(y) + \cos g(x)\sin g(y)\\
&\cos g^2(x) + \varepsilon^2 \sin g^2(x) = 1.
\end{aligned}$$

[6].
E. Cho proved that De-Moivre formula for the complex numbers is admissible for quaternions, [7]. Yaylı and Kabadayı gave De-Moivre formula for dual quaternions, [8]. This formula is also investigated for the case of hyperbolic quaternions in [9]. In this study, we first introduce dual-complex numbers and algebraic expressions on dual complex numbers. We also generalize De-Moivre and Euler formulae which are given for complex and dual numbers to dual-complex numbers. Then we have found the roots and forces of the dual-complex numbers. Finally, the obtained results are supported by examples.

2. Dual-Complex Numbers

A dual-complex number w can be written in the form of complex pair (z,t) such that $+1$ is the real unit and ε is the dual unit. Thus, we denote dual-complex numbers set by $\mathbb{DC} = \left\{ w = z + \varepsilon t \,|\, z,t \in \mathbb{C}, \varepsilon^2 = 0, \varepsilon \neq 0 \right\}$. If we consider complex numbers $z = x_1 + ix_2$ and $t = x_3 + ix_4$, we represent a dual-complex number $w = x_1 + x_2 i + x_3 \varepsilon + x_4 \varepsilon i$. Here i, ε and εi are unit vectors in three-dimensional vectors space such that i is a complex unit, ε is a dual unit, and εi is a dual-complex unit, [10]. So, the multiplication table of dual–complex numbers' base elements is given below.

x	1	i	ε	$i\varepsilon$
1	1	i	ε	$i\varepsilon$
i	i	1	$i\varepsilon$	ε
ε	ε	$i\varepsilon$	0	0
$i\varepsilon$	$i\varepsilon$	ε	0	0

Table 1: Multiplication Table of Dual-Complex Numbers

We define addition and multiplication on dual-complex numbers as follows

$$\begin{aligned}
&w_1 + w_2 = (z_1 \pm \varepsilon z_2) + (z_3 \pm \varepsilon z_4) = (z_1 \pm z_3) + \varepsilon(z_2 \pm z_4)\\
&w_1 \times w_2 = (z_1 + \varepsilon z_2) \times (z_3 + \varepsilon z_4) = z_1 z_3 + \varepsilon(z_1 z_4 + z_2 z_3)
\end{aligned}$$

where w_1 and w_2 are dual-complex numbers and $z_1, z_2, z_3, z_4 \in \mathbb{C}$. On the other hand, the division of two dual-complex numbers is $\frac{w_1}{w_2} = \frac{z_1 + \varepsilon z_2}{z_3 + \varepsilon z_4} = \frac{z_1}{z_3} + \varepsilon \frac{z_2 z_3 - z_1 z_4}{z_3{}^2}$ where $\mathscr{R}\mathrm{e}(w_2) \neq 0$. Thus, dual-complex numbers yield a commutative ring whose characteristic is 0. If we consider both algebraic and geometric properties of dual-complex numbers, we define five possible conjugations of dual-complex numbers.

These are

$$\begin{aligned}
w^{\dagger_1} &= \bar{z}+\varepsilon\bar{t} \ \text{(complex conjugation)}\\
w^{\dagger_2} &= z-\varepsilon t \ \text{(dual conjugation)}\\
w^{\dagger_3} &= \bar{z}-\varepsilon\bar{t} \ \text{(coupled conjugation)}\\
w^{\dagger_4} &= \bar{z}\left(1-\varepsilon\frac{t}{z}\right) \ \text{(dual-complex conjugation)}\\
w^{\dagger_5} &= t-\varepsilon\, z \ \text{(anti-dual conjugation)}
\end{aligned}$$

such that $w=z+\varepsilon t\in\mathbb{DC}$ is a dual-complex number, [11]. In regards to these definitions, we give the following proposition for modules of dual-complex numbers.

Proposition 2.1. *Let be a dual-complex number. Then we write*

$$|w|^2_{\dagger_1} = w\times w^{\dagger_1} = (z+t\varepsilon)(\bar{z}+\bar{t}\varepsilon) = z\bar{z}+z\bar{t}\varepsilon+\bar{z}t\varepsilon = z\bar{z}+(z\bar{t}+\bar{z}t)\,\varepsilon = |z|^2+2\varepsilon\mathscr{R}e\,(z\bar{t}) \ \in\mathbb{D}$$

$$|w|^2_{\dagger_2} = w\times w^{\dagger_2} = (z+t\varepsilon)(z-t\varepsilon) = zz-zt\varepsilon+zt\varepsilon = z^2\in\mathbb{C}$$

$$|w|^2_{\dagger_3} = w\times w^{\dagger_3} = (z+t\varepsilon)(\bar{z}-\bar{t}\varepsilon) = z\bar{z}-z\bar{t}\varepsilon+\bar{z}t\varepsilon = |z|^2-(z\bar{t}-\bar{z}t)\,\varepsilon = |z|^2-2i\varepsilon\,\mathrm{Im}\,(z\bar{t})\in\mathbb{D}$$

$$|w|^4_{\dagger_4} = w\times w^{\dagger_4} = (z+t\varepsilon)\left(\bar{z}\left(1-\frac{t}{z}\varepsilon\right)\right) = z\bar{z}-z\bar{z}\frac{t}{z}\varepsilon+\bar{z}t\varepsilon = z\bar{z}-\bar{z}t\varepsilon+\bar{z}t\varepsilon = z\bar{z} = |z|^2\in\mathbb{DC} \ (\mathscr{R}e\,(w)\neq 0)$$

$$|w|^2_{\dagger_5} = w\times w^{\dagger_5} = (z+t\varepsilon)(t-z\varepsilon) = zt+t^2\varepsilon-z^2\varepsilon = zt+\varepsilon\left(t^2-z^2\right)\in\mathbb{DC}$$

[11].

3. De-Moivre and Euler Formulae for Dual-Complex Number

Definition 3.1. *Exponential representation of a dual-complex number is $e^w = ze^{\frac{t}{z}\varepsilon}$ where $w=z+t\varepsilon\in\mathbb{DC}$ is a dual-complex number and $(z\neq 0)$, [11].*

Definition 3.2. *Let $w=z+t\varepsilon$ be a dual-complex number with the exponential representation $e^w = ze^{\frac{t}{z}\varepsilon}$. The dual-complex angle $\frac{t}{z}$ is called argument of dual-complex number and it is denoted by $\arg w = \frac{t}{z} = \varphi$, [11].*

Definition 3.3. *Let $w=z+t\varepsilon$ be a dual-complex number and φ be its principal argument. Every dual-complex number can be written in the form of $w=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$ such that $\cos g(\varphi)=1$ and $\sin g(\varphi)=\varphi$, [11].*

Theorem 3.4. *(Euler Formula) Let $w=z+t\varepsilon$ be a dual-complex number and φ be the principal argument of w. Then* $w=ze^{\varepsilon\varphi}=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$.

Proof. As it is aforementioned in Definition 3.2, the exponential representation of a dual-complex number $w=z+t\varepsilon\in\mathbb{DC}$ is $e^w = ze^{\frac{t}{z}\varepsilon}$, where dual-complex number $\frac{t}{z}$ is the principal argument φ. Thus, if we write w in the form of $w=ze^{\varepsilon\varphi}=z\left(1+\varepsilon\varphi+\frac{(\varepsilon\varphi)^2}{2!}+\frac{(\varepsilon\varphi)^3}{3!}+\ldots\right)$, from properties of the dual unit, we see that $w=ze^{\varepsilon\varphi}=z(1+\varepsilon\varphi)=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$. □

Theorem 3.5. *Let $w=z+t\varepsilon$ be a dual-complex number and $\varphi=\frac{t}{z}$. Then $\frac{1}{e^{\varepsilon\varphi}}=e^{\varepsilon(-\varphi)}$.*

Proof. If we use Euler formula for $\frac{1}{e^{\varepsilon\varphi}}$, we have

$$\frac{1}{e^{\varepsilon\varphi}} = \frac{1}{\left(1+\varepsilon\varphi+\frac{(\varepsilon\varphi)^2}{2!}+\frac{(\varepsilon\varphi)^3}{3!}+\ldots.\right)} = \frac{1}{\cos g(\varphi)+\varepsilon\sin g(\varphi)}.$$

If we multiplicate both the numerator and the denominator with $\cos g(\varphi)-\varepsilon\sin g(\varphi)$ in the last expression, we get

$$\frac{1}{e^{\varepsilon\varphi}} = \frac{1}{\cos g(\varphi)+\varepsilon\sin g(\varphi)}\frac{(\cos g(\varphi)-\varepsilon\sin g(\varphi))}{(\cos g(\varphi)-\varepsilon\sin g(\varphi))} = \frac{\cos g(\varphi)-\varepsilon\sin g(\varphi)}{\cos g^2(\varphi)}.$$

If we consider the equality $\cos g^2(\varphi)=1$, we have $\frac{1}{e^{\varepsilon\varphi}}=\cos g(\varphi)-\varepsilon\sin g(\varphi)$. Considering the last equation, we write $\frac{1}{e^{\varepsilon\varphi}}=\cos g(\varphi)-\varepsilon\sin g(\varphi)=\cos g(-\varphi)+\varepsilon\sin g(-\varphi)$. As a consequence, we get $\frac{1}{e^{\varepsilon\varphi}}=e^{\varepsilon(-\varphi)}$. □

Theorem 3.6. *(De-Moivre Formula) Let $w=z+t\varepsilon$ be a dual-complex number and $w=ze^{\varepsilon\varphi}=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$ be its polar representation. Then, the equation $w^n=(ze^{\varepsilon\varphi})^n=(z(\cos g(\varphi)+\varepsilon\sin g(\varphi))^n=z^n(\cos g(n\varphi)+\varepsilon\sin g(n\varphi))$ yields for all non-negative integers.*

Proof. Considering Galelian trigonometric identities for dual-complex number $w = z + t\varepsilon$, we will prove that $w^n = (ze^{\varepsilon\varphi})^n = (z(\cos g(\varphi) + \varepsilon \sin g(\varphi))^n = z^n (\cos g(n\varphi) + \varepsilon \sin g(n\varphi))$ is admissible by the help of induction. For $n = 2$, we have

$$\begin{aligned}\left(ze^{\varepsilon\varphi}\right)^2 &= z(\cos g(\varphi) + \varepsilon \sin g(\varphi))\, z(\cos g(\varphi) + \varepsilon \sin g(\varphi)) \\ &= z^2 \left(\cos^2 g(\varphi) + \varepsilon (\cos g(\varphi) \sin g(\varphi) + \sin g(\varphi) \cos g(\varphi))\right) \\ &= z^2 (\cos g(2\varphi) + \varepsilon \sin g(2\varphi))\end{aligned}$$

For $n = k$ non-negative integer, let $(z(\cos g(\varphi) + \varepsilon \sin g(\varphi))^k = z^k (\cos g(k\varphi) + \varepsilon \sin g(k\varphi))$ be true. For $n = k+1$, we get

$$\begin{aligned}(z(\cos g(\varphi) + \varepsilon \sin g(\varphi))^{k+1} &= z(\cos g(\varphi) + \varepsilon \sin g(\varphi))^k (z(\cos g(\varphi) + \varepsilon \sin g(\varphi)) \\ &= z^k (\cos g(k\varphi) + \varepsilon \sin g(k\varphi))\, z(\cos g(k\varphi) + \varepsilon \sin g(k\varphi)) \\ &= z^k (\cos g(k\varphi) \cos g(\varphi) + \varepsilon (\cos g(k\varphi) \sin g(\varphi) + \sin g(k\varphi) \cos g(\varphi))) \\ &= z^{k+1} (\cos g((k+1)\varphi) + \varepsilon \sin g((k+1)\varphi))\end{aligned}$$

So, the desired equality holds for $n = k+1$. This completes the proof. □

Theorem 3.7. *For the dual-complex number $w = z + t\varepsilon \in \mathbb{DC}$, the following equality yields for any integer n.*

Proof. We give the proof for non-negative integers in Theorem 3.6. Let $-n$ be a negative integer considering Theorem 3.5., we get

$$\begin{aligned}(w)^{-1} &= z^{-1} (\cos g(\varphi) - \varepsilon \sin g(\varphi)) \\ w^{-n} &= z^{-n} (\cos g(n\varphi) - \varepsilon \sin g(n\varphi)) \\ &= z^{-n} (\cos g(-n\varphi) + \varepsilon \sin g(-n\varphi))\end{aligned}$$

Thus, we see that for any integer $w^n = (ze^{\varepsilon\varphi})^n = (z(\cos g(\varphi) + \varepsilon \sin g(\varphi))^n = z^n (\cos g(n\varphi) + \varepsilon \sin g(n\varphi))$. □

Example 3.8. *Let $w = 1 + i + \varepsilon + \varepsilon i$ be a dual-complex number, we investigate $\left(w^4\right)$, 4th- degree power of w where w is written in the form of $w = z + t\varepsilon$ and $z = 1 + i$, $t = 1 + i$ are complex numbers. Seeing that argument of w is $\frac{t}{z} = \varphi$, polar representation of w is given by $w = z(\cos g(\varphi) + \varepsilon \sin g(\varphi))$. From Theorem 3.7, we have $w^4 = z^4 (\cos g(4\varphi) + \varepsilon \sin g(4\varphi))$ We gave equivalence for these Galilean trigonometric functions. So we find,*

$$w^4 = (1+i)^4 (1 + \varepsilon 4) = -4 (1 + \varepsilon 4) = -4 - 16\varepsilon$$

Example 3.9. *We find values of w^2 and w^{10} for the dual-complex number $w = 1 - i + \varepsilon + 3\varepsilon i \in \mathbb{DC}$. If we write w in the form of $w = z + t\varepsilon$, then its argument is $\frac{t}{z} = \frac{1-i}{1+3i} = \varphi$ where $z, t \in \mathbb{C}$ and $z = 1 - i, t = 1 + 3i$. Thus, the polar representation of w is $w = z(\cos g(\varphi) + \varepsilon \sin g(\varphi))$. So, we find*

$$w^2 = z^2 (\cos g(2\varphi) + \varepsilon \sin g(2\varphi)) = (1-i)^2 \left(1 + \varepsilon \frac{2(1+3i)}{(1-i)}\right) = -2i + 8\varepsilon + 4\varepsilon i$$

and

$$w^{10} = z^{10} (\cos g(10\varphi) + \varepsilon \sin g(10\varphi)) = (1-i)^{10} \left(1 + \varepsilon 10 \frac{(1+3i)}{(1-i)}\right) = -32i + 640\varepsilon + 320i$$

Theorem 3.10. *n-th degree root of w is $\sqrt[n]{w} = \sqrt[n]{z} \left(\cos g\left(\frac{\varphi}{n}\right) + \varepsilon \sin g\left(\frac{\varphi}{n}\right)\right)$ where $w = z + t\varepsilon \in \mathbb{DC}$ is a dual-complex number.*

Proof. Polar representation of $w = z + t\varepsilon \in \mathbb{DC}$ is $w = z(\cos g(\varphi) + \varepsilon \sin g(\varphi))$. From Theorem 3.7, we know that $w^n = (z.(\cos g(\varphi) + \varepsilon \sin g(\varphi))^n = z^n (\cos g(n\varphi) + \varepsilon \sin g(n\varphi))$. So, we get

$$\sqrt[n]{w} = w^{\frac{1}{n}} = z^{\frac{1}{n}} \left(\cos g\left(\frac{1}{n}\varphi\right) + \varepsilon \sin g\left(\frac{1}{n}\varphi\right)\right) = \sqrt[n]{z} \left(\cos g\left(\frac{\varphi}{n}\right) + \varepsilon \sin g\left(\frac{\varphi}{n}\right)\right)$$

This completes the proof. □

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$\mathscr{I}_2$-Convergence of Double Sequences of Functions in 2-Normed Spaces

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Abstract

In this study, we introduced the concepts of $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences of functions in 2-normed space. Also, were studied some properties about these concepts and investigated relationships between them for double sequences of functions in 2-normed spaces.

1. Introduction and Background

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [14] and Schoenberg [32]. Gökhan et al. [19] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions.
The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [25] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of $\mathbb{N}$ [14, 15]. Gezer and Karakuş [18] investigated $\mathscr{I}$-pointwise and uniform convergence and $\mathscr{I}^*$-pointwise and uniform convergence of function sequences and they examined the relation between them. Baláz et al. [4] investigated $\mathscr{I}$-convergence and $\mathscr{I}$-continuity of real functions. Das et al. [6] introduced the concept of $\mathscr{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Dündar and Altay [7, 9] studied the concepts of pointwise and uniformly $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [11] investigated some results of $\mathscr{I}_2$-convergence of double sequences of functions. Also, a lot of development have been made about double sequences of functions (see [8], [10]-[12], [18], [27], [28], [34]-[36]).
The concept of 2-normed spaces was initially introduced by Gähler [16, 17] in the 1960's. Statistical convergence and statistical Cauchy sequence of functions in 2-normed space were studied by Yegül and Dündar [39]. Also, Yegül and Dündar [40] introduced concepts of pointwise and uniform convergence, statistical convergence and statistical Cauchy double sequences of functions in 2-normed space. Sarabadan and Talebi [29] presented various kinds of statistical convergence and $\mathscr{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathscr{I}$-equistatistically convergence and study $\mathscr{I}$-equistatistically convergence of sequences of functions. Recently, Arslan and Dündar [1, 2] inroduced $\mathscr{I}$-convergence and $\mathscr{I}$-Cauchy sequences of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [3, 5, 13, 26, 30, 33]).

2. Definitions and Notations

Now, we recall the concept of density, statistical convergence, 2-normed space and some fundamental definitions and notations (See [1, 2, 4, 6, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 37, 38, 40]).
Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent.

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(ii) $\|x,y\| = \|y,x\|$.
(iii) $\|\alpha x,y\| = |\alpha|\|x,y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x,y\| = |x_1y_2 - x_2y_1|; \;\; x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.
Throughout the paper, we X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f,g be two functions from X to Y.
The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to be convergent to f if $f_n(x) \overset{\|.,.\|_Y}{\longrightarrow} f(x)$ for each $x \in X$. We write $f_n \overset{\|.,.\|_Y}{\longrightarrow} f$. This can be expressed by the formula

$$(\forall y \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x),y\| < \varepsilon.$$

A family of sets $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathscr{I}$, (ii) For each $A,B \in \mathscr{I}$ we have $A\cup B \in \mathscr{I}$, (iii) For each $A \in \mathscr{I}$ and each $B \subseteq A$ we have $B \in \mathscr{I}$.
An ideal is called nontrivial if $\mathbb{N} \notin \mathscr{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$.
A family of sets $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if
(i) $\emptyset \notin \mathscr{F}$, (ii) For each $A,B \in \mathscr{F}$ we have $A\cap B \in \mathscr{F}$, (iii) For each $A \in \mathscr{F}$ and each $B \supseteq A$ we have $B \in \mathscr{F}$.
$\mathscr{I}$ is nontrivial ideal in $\mathbb{N}$ if and only if $\mathscr{F}(\mathscr{I}) = \{M \subset \mathbb{N} : (\exists A \in \mathscr{I})(M = \mathbb{N}\backslash A)\}$ is a filter in $\mathbb{N}$.
A nontrivial ideal $\mathscr{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible ideal if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathscr{I}_2$ for each $i \in N$.
Throughout the paper we take $\mathscr{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.
It is evident that a strongly admissible ideal is admissible also.
$\mathscr{I}_2^0 = \{A \subset \mathbb{N}\times\mathbb{N} : (\exists m(A) \in \mathbb{N})(i,j \geq m(A) \Rightarrow (i,j) \notin A)\}$. Then $\mathscr{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathscr{I}_2$ is strongly admissible if and only if $\mathscr{I}_2^0 \subset \mathscr{I}_2$.
A sequence $\{f_n\}$ of functions is said to be $\mathscr{I}$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if for every $\varepsilon > 0$ and each $x \in D$,

$$\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathscr{I}.$$

In this case, we will write $f_n \overset{\mathscr{I}}{\to} f$ on D.
The sequence of functions $\{f_n\}$ is said to be $\mathscr{I}$-pointwise convergent to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$

$$A(\varepsilon,z) = \{n \in \mathbb{N} : \|f_n(x) - f(x),z\| \geq \varepsilon\} \in \mathscr{I},$$

or $\mathscr{I} - \lim_{n\to\infty} \|f_n(x) - f(x),z\|_Y = 0$, for each $x \in X$. In this case, we write $f_n \overset{\|.,.\|_Y}{\longrightarrow}_{\mathscr{I}} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall \varepsilon > 0)(\exists M \in \mathscr{I})(\forall n_0 \in \mathbb{N}\backslash M)(\forall x \in X)(\forall n \geq n_0)\|f_n(x) - f(x),z\| \leq \varepsilon.$$

The sequence of functions $\{f_n\}$ is said to be (pointwise) $\mathscr{I}^*$-convergent to f, if there exists a set $M \in \mathscr{F}(\mathscr{I})$, (i.e., $\mathbb{N}\backslash M \in \mathscr{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, such that for each $x \in X$ and each nonzero $z \in Y$

$$\lim_{k\to\infty} \|f_{n_k}(x),z\| = \|f(x),z\|$$

and we write

$$\mathscr{I}^* - \lim_{n\to\infty} \|f_n(x),z\| = \|f(x),z\| \text{ or } f_n \overset{\mathscr{I}^*}{\to} f.$$

An admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{E_1,E_2,...\}$ belonging to $\mathscr{I}_2$, there exists a countable family of sets $\{F_1,F_2,...\}$ such that $E_j\Delta F_j \in \mathscr{I}_2^0$, i.e., $E_j\Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathscr{I}_2$ (hence $F_j \in \mathscr{I}_2$ for each $j \in \mathbb{N}$).
Throughout the paper, we let $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, X and Y be two 2-normed spaces, $\{f_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$, $\{g_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ and $\{h_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ be three double sequences of functions, f, g and k be three functions from X to Y.
A double sequence $\{f_{mn}\}$ is said to be pointwise convergent to f if, for each point $x \in X$ and for each $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(x,\varepsilon)$ such that for all $m,n \geq k_0$ implies $\|f_{mn}(x) - f(x),z\| < \varepsilon$, for every $z \in Y$. In this case, we write $f_{mn} \overset{\|.,.\|_Y}{\longrightarrow} f$.
A double sequence $\{f_{mn}\}$ is said to be (pointwise) statistical convergent to f, if for every $\varepsilon > 0$, $\lim_{i,j\to\infty} \frac{1}{ij}\big|\{(m,n), m \leq i, n \leq j : \|f_{mn}(x) - f(x),z\| \geq \varepsilon\}\big| = 0$, for each (fixed) $x \in X$ and each nonzero $z \in Y$. It means that for each (fixed) $x \in X$ and each nonzero $z \in Y$, $\|f_{mn}(x) - f(x),z\| < \varepsilon$, $a.a.$ (m,n). In this case, we write

$$st - \lim_{m,n\to\infty} \|f_{mn}(x) - z\| = \|f(x),z\| \;\; or \;\; f_{mn} \overset{\|.,.\|_Y}{\longrightarrow}_{st} f.$$

The double sequences of functions $\{f_{mn}\}$ is said to be statistically Cauchy sequence, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, there exist a number $k = k(\varepsilon,z)$, $t = t(\varepsilon,z)$ such that $d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{kt}(x),z\| \geq \varepsilon\}) = 0$, for each (fixed) $x \in X$, i.e., for each nonzero $z \in Y$, $\|f_{nm}(x) - f_{kt}(x),z\| < \varepsilon$, $a.a.$ (m,n).

3. Main Results

We introduced the concepts of $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences of functions in 2-normed space. Also, were studied some properties about these concepts and investigated relationships between them for double sequences of functions in 2-normed spaces.

Definition 3.1. *$\{f_{mn}\}$ is said to be $\mathscr{I}_2$-convergent (pointwise sense) to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$*

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \in \mathscr{I}_2,$$

for each $x \in X$. This can be expressed by the formula

$$(\forall z \in Y)\ (\forall x \in X)\ (\forall \varepsilon > 0)\ (\exists H \in \mathscr{I}_2)\ (\forall (m,n) \notin H)\ \|f_{mn}(x) - f(x), z\| < \varepsilon.$$

In this case, we write

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|, \text{ or } f_{mn} \xrightarrow{\|.,.\|_Y}_{\mathscr{I}_2} f.$$

Theorem 3.2. *For each $x \in X$ and each nonzero $z \in Y$,*

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

Proof. Let $\varepsilon > 0$ be given. Since

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

for each $x \in X$ and each nonzero $z \in Y$, therefore, there exists a positive integer $k_0 = k_0(\varepsilon, x)$ such that $\|f_{mn}(x) - f(x), z\| < \varepsilon$, whenever $m, n \geq k_0$. This implies that for each nonzero $z \in Y$,

$$\begin{aligned} A(\varepsilon, z) &= \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| < \varepsilon\} \\ &\subset ((\mathbb{N} \times \{1, 2, .., k_0 - 1\}) \cup (\{1, 2, .., k_0 - 1\} \times \mathbb{N})). \end{aligned}$$

Since $\mathscr{I}_2$ be an admissible ideal, therefore

$$((\mathbb{N} \times \{1, 2, .., k_0 - 1\}) \cup (\{1, 2, .., k_0 - 1\} \times \mathbb{N})) \in \mathscr{I}_2.$$

Hence, it is clear that $A(\varepsilon, z) \in \mathscr{I}_2$ and consequently, for each nonzero $z \in Y$ we have

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

□

Theorem 3.3. *If $\mathscr{I}_2$-limit of any double sequence of functions $\{f_{mn}\}$ exists, then it is unique.*

Proof. Assume that

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x_0), z\| = \|f(x_0), z\| \text{ and } \mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x_0), z\| = \|g(x_0), z\|,$$

where $f(x_0) \neq g(x_0)$ for a $x_0 \in X$ each nonzero $z \in Y$. Since $f(x_0) \neq g(x_0)$. So we may suppose that $f(x_0) \geq g(x_0)$. Now, select $\varepsilon = \frac{f(x_0) - g(x_0)}{3}$, so that neighborhoods $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$ of points $f(x_0)$ and $g(x_0)$, respectively, are disjoints. Since for $x_0 \in X$ and each nonzero $z \in Y$

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x_0), z\| = \|f(x_0), z\| \text{ and } \mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x_0), z\| = \|g(x_0), z\|,$$

then for each nonzero $z \in Y$, we have

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x_0) - f(x_0), z\| \geq \varepsilon\} \in \mathscr{I}_2$$

and

$$B(\varepsilon, z) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x_0) - g(x_0), z\| \geq \varepsilon\} \in \mathscr{I}_2.$$

This implies that the sets

$$A^c(\varepsilon, z) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x_0) - f(x_0), z\| < \varepsilon\}$$

and

$$B^c(\varepsilon, z) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x_0) - g(x_0), z\| < \varepsilon\}$$

belongs to $\mathscr{F}(\mathscr{I}_2)$ and $A^c(\varepsilon, z) \cap B^c(\varepsilon, z)$ is nonempty set in $\mathscr{F}(\mathscr{I}_2)$ for $x_0 \in X$ and each nonzero $z \in Y$. Since $A^c(\varepsilon, z) \cap B^c(\varepsilon, z) \neq \emptyset$, we obtain a contradiction to the fact that the neighborhoods $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$ of points $f(x_0)$ and $g(x_0)$ respectively are disjoints. Hence, it is clear that for $x_0 \in X$ and each nonzero $z \in Y$,

$$\|f(x_0), z\| = \|g(x_0), z\|$$

and consequently, we have $\|f(x), z\| = \|g(x), z\|$, (i.e., $f = g$) for each $x \in X$ and each nonzero $z \in Y$. □

Theorem 3.4. *For each $x \in X$ and each nonzero $z \in Y$, If*

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathscr{I}_2 - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|g(x), z\|,$$

then
(i) $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x) + g_{mn}(x), z\| = \|f(x) + g(x), z\|$,
(ii) $\mathscr{I}_2 - \lim_{m,n\to\infty} \|cf_{mn}(x), z\| = \|cf(x), z\|$, $c \in \mathbb{R}$,
(iii) $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x)g_{mn}(x), z\| = \|f(x)g(x), z\|$.

Proof. (i) Let $\varepsilon > 0$ be given. Since

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathscr{I}_2 - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|g(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$, then

$$A\left(\frac{\varepsilon}{2}, z\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}\right\} \in \mathscr{I}_2$$

and

$$B\left(\frac{\varepsilon}{2}, z\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - g(x), z\| \geq \frac{\varepsilon}{2}\right\} \in \mathscr{I}_2$$

and by the definition of ideal we have

$$A\left(\frac{\varepsilon}{2}, z\right) \cup B\left(\frac{\varepsilon}{2}, z\right) \in \mathscr{I}_2.$$

Now, for each $x \in X$ and each nonzero $z \in Y$ we define the set

$$C(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|(f_{mn}(x) + g_{mn}(x)) - (f(x) + g(x)), z\| \geq \varepsilon\}$$

and it is sufficient to prove that $C(\varepsilon, z) \subset A\left(\frac{\varepsilon}{2}, z\right) \cup B\left(\frac{\varepsilon}{2}, z\right)$. Let $(m,n) \in C(\varepsilon, z)$, then for each $x \in X$ and each nonzero $z \in Y$, we have

$$\varepsilon \leq \|(f_{mn}(x) + g_{mn}(x)) - (f(x) + g(x)), z\| \leq \|f_{mn}(x) - f(x), z\| + \|g_{mn}(x) - g(x), z\|.$$

As both of $\{\|f_{mn}(x) - f(x), z\|, \|g_{mn}(x) - g(x), z\|\}$ can not be (together) strictly less than $\frac{\varepsilon}{2}$ and therefore either

$$\|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2} \text{ or } \|g_{mn}(x) - g(x), z\| \geq \frac{\varepsilon}{2},$$

for each $x \in X$ and each nonzero $z \in Y$. This shows that $(m,n) \in A\left(\frac{\varepsilon}{2}, z\right)$ or $(m,n) \in B\left(\frac{\varepsilon}{2}, z\right)$ and so we have

$$(m,n) \in A\left(\frac{\varepsilon}{2}, z\right) \cup B\left(\frac{\varepsilon}{2}, z\right).$$

Hence, we have

$$C(\varepsilon, z) \subset A\left(\frac{\varepsilon}{2}, z\right) \cup B\left(\frac{\varepsilon}{2}, z\right)$$

and so

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x) + g_{mn}(x), z\| = \|f(x) + g(x), z\|.$$

(ii) Let $c \in \mathbb{R}$ and $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. If $c = 0$, there is nothing to prove. We assume that $c \neq 0$. Then,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{|c|}\right\} \in \mathscr{I}_2$$

for each $x \in X$ and each nonzero $z \in Y$ and by the definition we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|cf_{mn}(x) - cf(x), z\| \geq \varepsilon\} = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{|c|}\right\}.$$

Hence, the right side of above equality belongs to $\mathscr{I}_2$ and so

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|cf_{mn}(x), z\| = \|cf(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$.
(iii) Since $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$, then for $\varepsilon = 1 > 0$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq 1\} \in \mathscr{I}_2,$$

and so

$$A=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|<1\}\in\mathscr{F}(\mathscr{I}_2).$$

Also, for any $(m,n)\in A$, $\|f_{mn}(x),z\|<1+\|f(x),z\|$ for each $x\in X$ and each nonzero $z\in Y$. Let $\varepsilon>0$ be given. Choose $\delta>0$ such that

$$0<2\delta<\frac{\varepsilon}{\|f(x),z\|+\|g(x),z\|+1}$$

for each $x\in X$ and each nonzero $z\in Y$. It follows from the assumption that

$$B=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|<\delta\}\in\mathscr{F}(\mathscr{I}_2)$$

and

$$C=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|g_{mn}(x)-g(x),z\|<\delta\}\in\mathscr{F}(\mathscr{I}_2)$$

for each $x\in X$ and each nonzero $z\in Y$. Since $\mathscr{F}(\mathscr{I}_2)$ is a filter, therefore $A\cap B\cap C\in\mathscr{F}(\mathscr{I}_2)$. Then, for each $(m,n)\in A\cap B\cap C$ we have

$$\begin{aligned}\|f_{mn}(x)g_{mn}(x)-f(x).g(x),z\|&=\|f_{mn}(x)g_{mn}(x)-f_{mn}(x)g(x)+f_{mn}(x)g(x)-f(x)g(x),z\|\\&\leq\|f_{mn}(x),z\|\|g_{mn}(x)-g(x),z\|+\|g(x),z\|\|f_{mn}(x)-f(x),z\|\\&<(\|f(x),z\|+1)\delta+(\|g(x),z\|)\delta\\&=(\|f(x),z\|+\|g(x),z\|+1)\delta\\&<\varepsilon\end{aligned}$$

and so, we have

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x).g_{mn}(x)-f(x).g(x),z\|\geq\varepsilon\}\in\mathscr{I}_2,$$

for each $x\in X$ and each nonzero $z\in Y$. This completes the proof of theorem. □

Theorem 3.5. *For each $x\in X$ and each nonzero $z\in Y$, if*
(i) $\{f_{mn}\}\leq\{g_{mn}\}\leq\{h_{mn}\}$, for every $(m,n)\in K$, where $\mathbb{N}\times\mathbb{N}\supseteq K\in\mathscr{F}(\mathscr{I}_2)$
and
(ii) $\mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|k(x),z\|$ and $\mathscr{I}_2-\lim_{m,n\to\infty}\|h_{mn}(x),z\|=\|k(x),z\|$,
then we have

$$\mathscr{I}_2-\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|k(x),z\|.$$

Proof. Let $\varepsilon>0$ be given. By condition (ii) we have

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-k(x),z\|\geq\varepsilon\}\in\mathscr{I}_2$$

and

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|h_{mn}(x)-k(x),z\|\geq\varepsilon\}\in\mathscr{I}_2,$$

for each $x\in X$ and each nonzero $z\in Y$. This implies that the sets

$$P=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-k(x),z\|<\varepsilon\}$$

and

$$R=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|h_{mn}(x)-k(x),z\|<\varepsilon\}$$

belong to $\mathscr{F}(\mathscr{I}_2)$, for each $x\in X$ and each nonzero $z\in Y$. Let

$$Q=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|g_{mn}(x)-k(x),z\|<\varepsilon\},$$

for each $x\in X$ and each nonzero $z\in Y$. It is clear that the set $P\cap R\cap K\subset Q$. Since $P\cap R\cap K\in\mathscr{F}(\mathscr{I}_2)$ and $P\cap R\cap K\subset Q$, then from the definition of filter, we have $Q\in\mathscr{F}(\mathscr{I}_2)$ and so

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|g_{mn}(x)-k(x),z\|\geq\varepsilon\}\in\mathscr{I}_2,$$

for each $x\in X$ and each nonzero $z\in Y$. Hence,

$$\mathscr{I}_2-\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|k(x),z\|.$$

□

Theorem 3.6. *For each $x\in X$ and each nonzero $z\in Y$, we let*

$$\mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|\text{ and }\mathscr{I}_2\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|g(x),z\|.$$

Then, for every $(m,n)\in K$ we have
(i) If $f_{mn}(x)\geq0$ then, $f(x)\geq0$ and
(ii) If $f_{mn}(x)\leq g_{mn}(x)$ then $f(x)\leq g(x)$, where $K\subseteq\mathbb{N}\times\mathbb{N}$ and $K\in\mathscr{F}(\mathscr{I}_2)$.

Proof. (i) Suppose that $f(x)<0$. Select $\varepsilon=-\frac{f(x)}{2}$, for each $x\in X$. Since

$$\mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|,$$

so there exists the set M such that

$$M=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|<\varepsilon\}\in\mathscr{F}(\mathscr{I}_2),$$

for each $x\in X$ and each nonzero $z\in Y$. Since $M,K\in\mathscr{F}(\mathscr{I}_2)$, then $M\cap K$ is a nonempty set in $\mathscr{F}(\mathscr{I}_2)$. So we can find out point $(m_0,n_0)\in K$ such that

$$\|f_{m_0n_0}(x)-f(x),z\|<\varepsilon.$$

Since $f(x)<0$ and $\varepsilon=-\frac{f(x)}{2}$ for each $x\in X$, then we have $f_{m_0n_0}(x)\leq 0$. This is a contradiction to the fact that $f_{mn}(x)>0$ for every $(m,n)\in K$. Hence, we have $f(x)>0$, for each $x\in X$.
(ii) Suppose that $f(x)>g(x)$. Select $\varepsilon=\frac{f(x)-g(x)}{3}$, for each $x\in X$. So that the neighborhoods $(f(x_0)-\varepsilon,f(x_0)+\varepsilon)$ and $(g(x_0)-\varepsilon,g(x_0)+\varepsilon)$ of $f(x)$ and $g(x)$, respectively, are disjoints. Since for each $x\in X$ and each nonzero $z\in Y$,

$$\mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\| \text{ and } \mathscr{I}_2-\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|g(x),z\|$$

and $\mathscr{F}(\mathscr{I}_2)$ is a filter on $\mathbb{N}\times\mathbb{N}$, therefore we have

$$A=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|<\varepsilon\}\in\mathscr{F}(\mathscr{I}_2)$$

and

$$B=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|g_{mn}(x)-g(x),z\|<\varepsilon\}\in\mathscr{F}(\mathscr{I}_2).$$

This implies that $\emptyset\neq A\cap B\cap K\in\mathscr{F}(\mathscr{I}_2)$. There exists a point $(m_0,n_0)\in K$ such that

$$\|f_{mn}(x)-f(x),z\|<\varepsilon \text{ and } \|g_{mn}(x)-g(x),z\|<\varepsilon.$$

Since $f(x)>g(x)$ and $\varepsilon=\frac{f(x)-g(x)}{3}$ for each $x\in X$, then we have

$$f_{m_0n_0}(x)>g_{m_0n_0}(x).$$

This is a contradiction to the fact $f_{mn}(x)\leq g_{mn}(x)$ for every $(m,n)\in K$. Thus, we have $f(x)\leq g(x)$, for each $x\in X$. □

Definition 3.7. *The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X,\|.,.\|)$ is said to be $\mathscr{I}_2^*$-convergent (pointwise sense) to f, if there exists a set $M\in\mathscr{F}(\mathscr{I}_2)$ (i.e., $H=\mathbb{N}\times\mathbb{N}\setminus M\in\mathscr{I}_2$) such that for each $x\in X$, each nonzero $z\in Y$ and all $(m,n)\in M$*

$$\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|$$

and we write

$$\mathscr{I}_2^*-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\| \ \ or \ \ f_{mn}\xrightarrow{\|.,.\|_Y}_{\mathscr{I}_2^*} f.$$

Theorem 3.8. *For each $x\in X$ and nonzero $z\in Y$,*

$$\mathscr{I}_2^*-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\| \textit{ implies } \mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|.$$

Proof. Since for each $x\in X$ and each nonzero $z\in Y$,

$$\mathscr{I}_2^*-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|,$$

so there exists a set $H\in\mathscr{I}_2$ such that for $M\in\mathscr{F}(\mathscr{I}_2)$ $(i.e., H=\mathbb{N}\times\mathbb{N}\setminus M\in\mathscr{I}_2)$ we have

$$\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|,\ (m,n)\in M.$$

Let $\varepsilon>0$. Then, for each $x\in X$ there exists a $k_0=k_0(\varepsilon,x)\in\mathbb{N}$ such that for each nonzero $z\in Y$, $\|f_{mn}(x)-f(x),z\|<\varepsilon$, for all $(m,n)\in M$ such that $m,n\geq k_0$. Then, clearly we have

$$\begin{aligned}A(\varepsilon,z)&=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|\geq\varepsilon\}\\&\subset H\cup[M\cap((\{1,2,3,...,(k_0-1)\}\times\mathbb{N})\cup(\mathbb{N}\times\{1,2,3,...,(k_0-1)\}))],\end{aligned}$$

for each $x\in X$, for each nonzero $z\in Y$. Since $\mathscr{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$be a strongly admissible ideal then

$$H\cup[M\cap((\{1,2,3,...,(k_0-1)\}\times\mathbb{N})\cup(\mathbb{N}\times\{1,2,3,...,(k_0-1)\}))]\in\mathscr{I}_2$$

and so, $A(\varepsilon,z)\in\mathscr{I}_2$. This implies that $\mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|$. □

Theorem 3.9. *Let $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal having the property $(AP2)$. For each $x \in X$ and nonzero $z \in Y$,*

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x),z\| = \|f(x),z\| \text{ implies } \mathscr{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x),z\| = \|f(x),z\|.$$

Proof. Let $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal having the property $(AP2)$ and $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x),z\| = \|f(x),z\|$, for each $x \in X$ and each nonzero $z \in Y$. Then, for any $\varepsilon > 0$

$$A(\varepsilon,z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x),z\| \geq \varepsilon\} \in \mathscr{I}_2$$

for each $x \in X$ and each nonzero $z \in Y$. Now, put

$$A_1(\varepsilon,z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x),z\| \geq 1\}$$

and

$$A_k(\varepsilon,z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{k} \leq \|f_{mn}(x) - f(x),z\| < \frac{1}{k-1}\}$$

for $k \geq 2$. $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in \mathscr{I}_2$ for each $i \in \mathbb{N}$. By property $(AP2)$ there exists a sequence $\{B_k\}_k \in \mathbb{N}$ of sets such that $A_j \triangle B_j$ is finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \cup_{j=1}^{\infty} B_j \in \mathscr{I}_2$.
We shall prove that, for each $x \in X$ and each nonzero $z \in Y$

$$\lim_{m,n\to\infty} \|f_{mn}(x) - f(x),z\| = \|f(x),z\|, \ (m,n) \in M,$$

for $M = \mathbb{N}\times\mathbb{N} \setminus B \in \mathscr{F}(\mathscr{I}_2)$. Let $\delta > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$. Then, we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x),z\| \geq \delta\} \subset \bigcup_{j=1}^{k} A_j.$$

Since $A_j \triangle B_j$, $j = 1,2,...,k$ are included in finite union of rows and columns, there exis

$$\left(\bigcup_{j=1}^{k} B_j\right) \cap \{(m,n) \in \mathbb{N}\times\mathbb{N} : m \geq n_0 \wedge n \geq n_0\} = \left(\bigcup_{j=1}^{k} A_j\right) \cap \{(m,n) \in \mathbb{N}\times\mathbb{N} : m \geq m_0 \wedge n \geq n_0\}.$$

If $m,n \geq n_0$ and $(m,n) \notin B$ then

$$(m,n) \notin \bigcup_{j=1}^{k} B_j \textit{ and so } (m,n) \notin \bigcup_{j=1}^{k} A_j.$$

Thus, we have $\|f_{mn}(x) - f(x),z\| < \frac{1}{k} < \delta$ for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$\lim_{m,n\to\infty} \|f_{mn}(x),z\| = \|f(x),z\|, \ (m,n) \in M$$

and so we have

$$\mathscr{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x),z\| = \|f(x),z\|$$

for each $x \in X$ and each nonzero $z \in Y$. □

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Pseudo-Valuations on UP-Algebras

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Abstract

Looking at pseudo-valuations on some classes of abstract algebras, such as BCK, BCI, BCC and KU, in this article we introduce the concept of pseudo-valuations on UP-algebras and analyze the relationship of these mappings with UP-substructures.

1. Introduction

The idea that universal algebra should be analyzed by means of pseudo-valuation was first developed by D. Busneag in 1996 [1]. This author has expanded the perception of pseudo-valuation on Hilbert's algebras [2]. Logical algebras and pseudo-valuations on them have become an object of interest for researchers in recent years. For example, Doh and Kang [3, 4] introduced in the concept of pseudo-valuation on BCK/BCI - algebras. Ghorbani in 2010 [5] determined a congruence on BCI - algebras based on pseudo-valuation and describe the obtained factorial structure generated by this congruence. Song, Roh and Jun described pseudo-valuation on BCK/BCI - algebras [12] and Song, Bordbar and Jun have described the quotient structure on such algebras generated by pseudo-valuation [13]. Jun, Lee and Song analyzed in article [8] several types of quasi-valuation maps on BCK - algebra and their interactions. Also, Mehrshad and Kouhestani were interested in pseudo-valuations on BCK - algebra [10]. Jun, Ahn and Roh. in [7] described pseudo-valuation on the BCC - algebras. Koam, Haider and Ansari described in 2019 pseudo-valuations on KU algebras [9].
The concept of UP-algebras is introduced and analyzed by Iampan in 2017 [6] as a generalization of the concept of KU - algebras. In this note, we offer one way of determining of pseudo-evaluation on PU - algebras. Apart from showing the features of this pseudo-valuation on UP-algebras, we have demonstrated how to construct a pseudo-metric space by such mapping.

2. Preliminaries

Here we give the definition of UP-algebra and some of its substructures necessary for further work.

Definition 2.1 ([6]). *An algebra $A = (A, \cdot, 0)$ of type $(2,0)$ is called a UP- algebra if it satisfies the following axioms:*
(UP-1) $(\forall x,y,z \in A)((y\cdot z)\cdot((x\cdot y)\cdot(x\cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0\cdot x = x)$,
(UP-3) $(\forall x \in A)(x\cdot 0 = 0)$, *and*
(UP-4) $(\forall x,y \in A)((x\cdot y = 0 \wedge y\cdot x = 0) \Longrightarrow x = y)$.

In A we can define a binary relation $'\leqslant'$ by

$$(\forall x,y \in A)(x \leqslant y \Longleftrightarrow x\cdot y = 0).$$

Definition 2.2 ([6]). *A non-empty subset J of a UP-algebra A is called a UP-ideal of A if it satisfies the following conditions:*
(1) $0 \in J$, *and*
(2) $(\forall x,y,z \in A)((x\cdot(y\cdot z) \in J \wedge y \in J) \Longrightarrow x\cdot z \in J)$.

Definition 2.3 ([11]). *Let A be a UP-algebra. A subset G of A is called a* proper UP-filter *of A if it satisfies the following properties:*
(3) $\neg(0 \in G)$, *and*
(4) $(\forall x,y,z \in A)((\neg(x\cdot(y\cdot z) \in G) \wedge x\cdot z \in G) \Longrightarrow y \in G)$.

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3. The concept of pseudo-valuations on UP-algebras

In this section, we introduce the concept of pseudo-valuations on UP-algebras, describe the basics properties of such pseudo-valuation and construct a pseudo-metric space based on this mapping.

Definition 3.1. *A real-valued function v on a UP-algebra A is called a pseudo-valuation on A if it satisfies the following two conditions:*
(1) $v(0) = 0$*, and*
(2) $(\forall x, y, z \in A)(v(x \cdot z) \leqslant v(x \cdot (y \cdot z)) + v(y))$.
A pseudo-valuation v on a UP-algebra A satisfying the following condition:
(3) $(\forall x \in A)(v(x) = 0 \Longrightarrow x = 0)$
is called a valuation on X.

Theorem 3.2. *Let v be a pseudo-valuation on a UP-algebra A. Then the following are valid:*
(4) $(\forall x, y \in A)(v(y) \leqslant v(x \cdot y) + v(x))$,
(5) $(\forall x, y \in A)(v(x \cdot y) \leqslant v(y))$,

Proof. If we put $x = 0$, $y = x$ and $z = y$ in formula (2), we get

$$v(y) \leqslant v(x \cdot y) + v(x).$$

Thus, formula (4) is valid.
If we put $z = y$ in formula (2), we have $v(x \cdot y) \leqslant v(x \cdot (y \cdot y)) + v(y)$ from which follows $v(x \cdot y) \leqslant v(y)$ due to the assertion (1) of Proposition 1.7 in [6], (UP-3) and (1). So, (5) is proven. □

Corollary 3.3. *Let v be a pseudo-valuation on a UP-algebra A. Then*
(6) $(\forall x, y \in A)(x \leqslant y \Longrightarrow v(y) \leqslant v(x))$.

Proof. Let x and y be arbitrary elements of a UP-algebra A such that $x \leqslant y$. Then $x \cdot y = 0$ and $v(x \cdot y) = 0$ by (1). From here follows $v(y) \leqslant v(x \cdot y) + v(x)$ according to (4). Thus $v(y) \leqslant v(x)$. Thus, any pseudo-valuation on a UP-algebra is an inversely monotone mapping. □

Corollary 3.4. *Let v be a pseudo-valuation on a UP-algebra A. Then*
(7) $(\forall x \in A)(0 \leqslant v(x))$.

Proof. Since $x \cdot 0 = 0$ according to (UP-3), i.e. as always $x \leqslant 0$ in UP-algebra A, we have $0 = v(0) \leqslant v(x)$ according to Corollary 3.3. □

Corollary 3.5. *Let v be a pseudo-valuation on a UP-algebra A. Then*
(8) $(\forall x, y \in A)(v(x \cdot y) \leqslant v(x) + v(y))$.

Proof. Let x and y be arbitrary elements of A. Thus $v(x \cdot y) \leqslant y(y)$ by (5). Thus $v(x \cdot y) \leqslant v(x) + v(y)$ by Corollary 3.4. □

Theorem 3.6. *Let v be a pseudo-valuation on a UP-algebra A. Then the set $J_v = \{x \in A : v(x) = 0\}$ is an UP-ideal of A and the set $G = \{x \in A : 0 < v(x)\}$ is a proper UP-filter of A.*

Proof. Since $v(0) = 0$, follows $0 \in J_v$.
Let x, y and z be arbitrary elements of A such that $x \cdot (y \cdot z) \in J_v$ and $y \in J_v$. Then $v(x \cdot (y \cdot z)) = 0$ and $v(y) = 0$. By (2) we have

$$v(x \cdot z) \leqslant v(x \cdot (y \cdot z)) + v(y) = 0 + 0 = 0.$$

Thus $v(x \cdot z) = 0$ according to Corollary 3.4. Hence $x \cdot z \in J_v$. So, the set J_v is a UP-ideal of UP-algebra A.
The set G is a proper UP-filter of A by Theorem 3.7 in [11]. □

Corollary 3.7. *Let v be a pseudo-valuation on a UP-algebra A. Then v is a valuation on A if and only if $J_v = \{0\}$.*

Proof. The claim follows from the definition of the concept of valuations on a UP-algebra A. □

Remark 3.8. *The previous corollary suggested that a valuation on an UP-algebra A can be defined if $\{0\}$ is a UP-ideal at A.*

Example 3.9. *For any ideal J of a UP-algebra A, define a map $v_J : A \longrightarrow \mathbb{R}$ by $(\forall x \in J)(v_J(x) = 0)$ and $(\forall x \in A \backslash J)(v_J(x) \in \mathbb{R}^+)$. Then, v_J is a pseudo-valuation of A.*

Example 3.10. *Let $A = \{0, 1, 2, 3, 4\}$ be given and an operations on it as in Example 2.2 in [6]. Then $(A, \cdot, 0)$ is a UP-algebra. It is easy to directly verified that $v : A \longrightarrow \mathbb{R}$, given with $v(0) = v(1) = v(2) = 0$, $v(3) = v(4) = 3$, is a pseudo-valuation on A.*

Theorem 3.11. *Let $f : (A, \cdot, 0_A) \longrightarrow (B, *, 0_B)$ be a homomorphism of UP-algebras. If v is a pseudo-valuation on B, then the composition $v \circ f$ is a pseudo-valuation on A.*

Proof. First, we have $(v \circ f)(0_A) = v(f(0_A)) = v(0_B) = 0$.
For any $x, y, z \in A$, we get $(v \circ f)(x \cdot z) = v(f(x \cdot z)) = v(f(x) * f(z)) \leqslant v(f(x) * (f(y) * f(z))) + v(f(y)) = (v \circ f)(x \cdot (y \cdot z)) + (v \circ f)(y)$.
Hence, $v \circ f$ is a pseudo-valuation on A. □

Lemma 3.12. *Suppose that A is a UP-algebra. Then every pseudo- valuation v on A satisfies the following inequality:*
(9) $(\forall x, y, z \in A)(v(x \cdot z) \leqslant v(x \cdot y) + v(y \cdot z))$.

Proof. From (UP-1) follows $y \cdot z \leqslant (x \cdot y) \cdot (x \cdot z)$. Thus $v(y \cdot z) \geqslant v((x \cdot y) \cdot (x \cdot z))$ by (6) and $v(y \cdot z) \geqslant v(x \cdot z) - v(x \cdot y)$ by (4). Therefore, $v(x \cdot z) \leqslant v(x \cdot y) + v(y \cdot z)$. □

Now, we define pseudo-metric on UP-algebras and show related results.

Theorem 3.13. *Let A be a UP-algebra and v be a pseudo-valuation on A. Then the mapping $d_v : A \times A \ni (x,y) \longmapsto v(x \cdot y) + v(y \cdot x) \in \mathbb{R}$ is a pseudo-metric on A.*

Proof. Clearly, $0 \leqslant d_v(x,y)$; $d_v(x,x) = 0$ and $d_v(x,y) = d_v(y,x)$ for any $x, y \in A$. For any $x, y, z \in A$ from Lemma 3.12, we get that

$$\begin{aligned} d_v(x,y) + d_v(y,z) = \\ (v(x \cdot y) + v(y \cdot x)) + (v(y \cdot z) + v(z \cdot y)) = \\ (v(x \cdot y) + v(y \cdot z)) + (v(z \cdot y) + v(y \cdot x)) \geqslant \\ v(x \cdot z) + v(z \cdot x) = d_v(x \cdot z). \end{aligned}$$

Hence (A, d_v) is a pseudo-metric space. □

4. Conclusion

The aim of this paper was to study the concept of pseudo-valuation and their induced pseudo-metrics on UP - algebras. This work can be the basis for further and deeper research of the properties of UP - algebras.

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Energy-Momentum Distribution of Six-Dimensional Geometric Model of Gravitational Field

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Abstract

Much work has been done in exploring the energy-momentum distribution of different four-dimensional spacetimes using different prescriptions. In this paper, we intend to explore the energy and momentum density of six-dimensional geometric model of the gravitational field. The model was constructed by postulating a six-dimensional spacetime manifold with a structure of spacetime of absolute parallelism. For this purpose, we consider the metric representing the geometric model and use five prescriptions, namely, Einstein, Landau-Lifshitz, Bergmann-Thomson, Papapetrou, and Möller in the framework of General Relativity. The energy and momentum turn out to be well defined and finite. The comparison of the results shows that Einstein and Bergmann-Thomson prescriptions yield same energy-momentum densities but different from the other three prescriptions. It is mentioning here that the energy vanishes in the case of Möller's prescription and the momentum densities become zero in all the cases.

1. Introduction

The theory of General Relativity (GR) is recognized as the best among all theories of gravitation, available in the literature, because many important features of the universe have been verified experimentally in the framework of this theory. Although, the issue of localization of energy and momentum is most divisive and longstanding. Huge efforts have been made to resolve this problem by the big names of Physics and Mathematics. Einstein [1] is an innovator who introduced his energy-momentum complexes for the localization of energy and momentum. By the passage of time, a number of efficient scientists, including Landau-Lifshitz [2], Møller [3], Bergmann-Thomson [4], Weinberg [5], Papapetrou [6] and Tolman [7], presented their own energy-momentum complexes. These prescriptions yield feasible results only if one uses cartesian coordinates except Møller's prescription, which is independent of coordinate choice in case of energy only.
Misner and Sharp [8] proved that the energy can be localized in the spherical coordinate system. Cooperstock and Sarracino [9] proved that if the energy can be localized in a spherically symmetric coordinate system then it surely be localized in non-spherical symmetric coordinate which are static or quasistatic. Virbhadra and his coworkers [10, 11] investigated the energy-momentum distribution of different spacetimes, such as the Kerr-Newmann, Kerr-Schild classes, Einstein-Rosen, Vaidya, and Bonnor-Vaidya spacetimes. They showed that different energy-momentum prescriptions could give the same consequences, which are coincident with the results obtained by Tod [12] using the Penrose [13] formalism in the context of quasi-local mass.
Various scientist [14, 15] described the energy of the gravitational field by presenting the Hamiltonian approach in the structure of Schwinger's condition [16]. Using this formalism, energy can be determined by an integral of a scalar density in the form of total divergence which appears as the Hamiltonian constraints of this theory. Initially, the problem of localization of energy-momentum emerged in electromagnetism and then it became a serious issue in GR due to the non-tensorial quantities. The localization of energy is discussed in some other theories such as the $f(R)$ theory and Teleparallel (TP) theory of gravity.
Mikhail et al. [17] defined the superpotential in the Møller's tetrad theory which has been used to find the energy in the teleparallel theory of gravity (TTG). Vargas [18] defined the TP version of Bergman, Einstein, and Landau-Lifshitz prescriptions and found that the total energy of the closed Friedman-Robertson-Walker universe is zero by using the last two prescriptions. This agrees with the results of GR available in

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literature [12, 13]. Later, many authors [19]-[23] used TP version of these prescriptions and showed that energy may be localized in TTG. Sharif and Jamil [24]-[32] explored the energy of different spcetimes in GR and TTG and obtained interesting results.
Multamäki et al. [33] generalized the Landau-Lifshitz energy-momentum prescription in $f(R)$ theory of gravity. They also found the energy density for the Schwarzschild de Sitter spacetime. By using generalized Landau-Lifshitz prescriptions, Sharif and Farasat [34] calculated the energy density of plane-symmetric static metric and cosmic string spacetime. Faraoni and Nadeau [35] have discussed some important $f(R)$ models along with their stability conditions. Jamil and his collaborators [36, 37] obtained the spatially homogeneous rotating solutions and locally rotating spacetimes in $f(R)$ gravity and the energy contents are obtained for non-trivial solution for particular $f(R)$ model.
Silva et al. [38] proposed a six-dimensional string-like braneworld built from a warped product between a 3-brane and the Hamilton cigar soliton space. They discussed the effects of the evolution of the transverse space on the geometric and physical quantities. The gravitational massless mode remains trapped to the brane and the width of the model depends on the evolution parameter. For the Kaluza-Klein modes, the asymptotic spectrum of mass is the same as for the thin string-like brane and the analog Schrödinger potential also changes according to the flow. Linch and Tartaglino-Mazzucchelli [39] introduced a superspace result for $N=(1,0)$ conformal supergravity in six dimensions. They formulated a locally supersymmetric and super-Weyl invariant action principle in projective superspace.
Popov [40] constructed a six-dimensional pure geometric model by postulating a six-dimensional spacetime manifold with a structure of spacetime of absolute parallelism. He established a clear relation between the Schwarzschild solution of the gravitational field of a point mass and the field of point source torsion. In the case of Teleparallel Equivalent of General Relativity (TEGR), it is equivalent to interpret the gravitation either in terms of curvature or in terms of torsion. Therefore, we try to explore the energy-momentum distribution of six-dimensional teleparallel solution by using GR prescriptions. Our aim is to explore the energy-momentum distribution of this geometric model using different complexes in the context of GR.

2. Energy-Momentum Distribution of Six Dimensional Geometric Model of Gravitational Field

We consider the line element is given in Eq.(14) of [40] representing the geometric model of a gravitational field in six dimensions, given as

$$\begin{aligned} ds^2 = & \frac{-(3y_1+a^3)^{-4/3}}{1-a/(3y_1+a^3)^{1/3}}dy_1^2 - \frac{(3y_1+a^3)^{2/3}}{1-y_2^2}dy_2^2 - (3y_1+a^3)^{2/3}(1-y_2)^2dy_3^2 \\ & + (1-\frac{a}{(3y_1+a^3)^{1/3}})dy_4^2 + \frac{1}{2y_5+\gamma}dy_5^2 + (2y_5+\gamma)dy_6^2. \end{aligned} \tag{1}$$

Here, y_1 stands for time component and rest are the space components.
The **Einstein** energy-momentum prescription is defined as [1]

$$\Theta_a^b = \frac{1}{16\pi}H^{bc}{}_{a,c}, \tag{2}$$

where $H^{bc}{}_a$ depends on metric tensor and its first order derivative, found by Freud [42] given as

$$H^{bc}{}_a = \frac{g_{ad}}{\sqrt{-g}}[-g(g^{bd}g^{ce} - g^{cd}g^{be})]_{,e}. \tag{3}$$

Here, the term Θ_0^0 stands for energy density, Θ_i^0 $(i=1,...,5)$ are the components of momentum density and the current density components are denoted by Θ_0^i. Using Eq.(1) in Eq.(3), the non-vanishing components of $H^{bc}{}_a$ turn out to be

$$\begin{aligned} & H^{01}{}_0 = -H^{10}{}_0 = 4\left(a - \left(a^3+3y_1\right)^{1/3}\right), \\ & H^{02}{}_0 = -H^{20}{}_0 = H^{12}{}_1 = -H^{21}{}_1 = \frac{2y_2}{\left(a^3+3y_1\right)^{2/3}}, \\ & H^{04}{}_0 = H^{14}{}_1 = H^{24}{}_2 = H^{35}{}_3 = 2, \\ & H^{40}{}_0 = H^{41}{}_1 = H^{42}{}_2 = H^{43}{}_3 = -2, \\ & H^{12}{}_2 = H^{13}{}_3 = -H^{21}{}_2 = -H^{31}{}_3 = -a + 2\left(a^3+3y_1\right)^{1/3}, \\ & H^{41}{}_4 = H^{51}{}_5 = -H^{14}{}_4 = -H^{15}{}_5 = 3a - 4\left(a^3+3y_1\right)^{1/3}, \\ & H^{24}{}_4 = -H^{42}{}_4 = H^{52}{}_5 = -H^{25}{}_5 = \frac{2y_2}{\left(a^3+3y_1\right)^{2/3}}. \end{aligned}$$

Making use of these values in Eq.(2), the energy and momentum densities of **Einstein's** prescription turn out to be

$$\begin{aligned} & \Theta^{00} = \frac{-3y_1 + a\left(a^2 + a\left(a^3+3y_1\right)^{1/3} + \left(a^3+3y_1\right)^{2/3}\right)}{24\pi y_1\left(a^3+3y_1\right)^{2/3}}, \\ & \Theta^{0i} = 0, \qquad (i=1,2,...,5). \end{aligned}$$

The **Landau-Lifshitz**'s prescription, in GR, is given as [2]

$$L^{ab} = \frac{1}{16\pi}l^{acbd}{}_{,cd}, \tag{4}$$

where l^{abcd} is a 4 rank tensor, given as

$$l^{abcd}=(-g)(g^{ab}g^{cd}-g^{ad}g^{cb}). \tag{5}$$

The term L^{00} yields the energy density component of the whole system and L^{i0} $(i=1,...,5)$ represents the momentum density components. In view of Eq.(1), the Eq.(5) gives the following non-vanishing components of the tensor l^{abcd}

$$\begin{aligned}
&l^{1001}=l^{0110}=-l^{1010}=-l^{0101}=\left(a^3+3y_1\right)^{4/3},\\
&l^{2002}=l^{0220}=-l^{2020}=-l^{0202}=\frac{1-y_2^2}{-a(a^3+3y_1)^{1/3}+(a^3+3y_1)^{2/3}},\\
&l^{3003}=l^{0330}=-l^{3030}=-l^{0303}=\frac{-1}{(a^3+3y_1)^{1/3}\left(-a+(a^3+3y_1)^{1/3}\right)\left(-1+y_2^2\right)},\\
&l^{4004}=l^{0440}=-l^{4040}=-l^{0404}=-\frac{\left(a^3+3y_1\right)^{1/3}(\gamma+2y_4)}{-a+(a^3+3y_1)^{1/3}},\\
&l^{5005}=l^{0550}=-l^{5050}=-l^{0505}=\frac{-1}{\left(1-\frac{a}{(a^3+3y_1)^{1/3}}\right)(\gamma+2y_4)},\\
&l^{2112}=l^{1221}=-l^{2121}=-l^{1212}=-\left(a^3+3y_1\right)^{1/3}\left(\left(a^3+3y_1\right)^{1/3}-a\right)\left(y_2^2-1\right),\\
&l^{3113}=l^{1331}=-l^{3131}=-l^{1313}=-\frac{a\left(a^3+3y_1\right)^{1/3}-\left(a^3+3y_1\right)^{2/3}}{y_2^2-1},\\
&l^{4114}=l^{1441}=-l^{4141}=-l^{1414}=\left(a^3+3y_1\right)\left(-a+\left(a^3+3y_1\right)^{1/3}\right)(\gamma+2y_4),\\
&l^{5115}=l^{1551}=-l^{5151}=-l^{1515}=\frac{\left(a^3+3y_1\right)\left(-a+\left(a^3+3y_1\right)^{1/3}\right)}{\gamma+2y_4},\\
&l^{3223}=l^{2332}=-l^{3232}=-l^{2323}=\frac{-1}{(a^3+3y_1)^{4/3}},\\
&l^{4224}=l^{2442}=-l^{4242}=-l^{2424}=-\frac{\left(-1+y_2^2\right)(\gamma+2y_4)}{(a^3+3y_1)^{2/3}},\\
&l^{5225}=l^{2552}=-l^{5252}=-l^{2525}=\frac{-1+y_2^2}{(a^3+3y_1)^{2/3}(\gamma+2y_4)},\\
&l^{4334}=l^{3443}=-l^{4343}=-l^{3434}=-\frac{\gamma+2y_4}{(a^3+3y_1)^{2/3}\left(-1+y_2^2\right)},\\
&l^{5335}=l^{3553}=-l^{5353}=-l^{3535}=\frac{-1}{(a^3+3y_1)^{2/3}\left(-1+y_2^2\right)(\gamma+2y_4)},\\
&l^{5445}=l^{4554}=-l^{5454}=-l^{4545}=-1.
\end{aligned}$$

When we use these values of l^{abcd} in Eq.(4), the energy and momentum density components of **Landau-Lifshitz's** prescription are obtained as

$$L^{00}=\frac{2a-\left(a^3+3y_1\right)^{1/3}}{8a^3\pi+24\pi y_1-8a\pi\left(a^3+3y_1\right)^{2/3}}.$$
$$L^{0i}=0,\qquad (i=1,...,5).$$

In case of **Bergmann-Thomson** prescription, the energy-momentum density components are defined as [4]

$$B^{ab}=\frac{1}{16\pi}M^{abc}{}_{,c}, \tag{6}$$

where

$$M^{abc}=g^{ad}V^{bc}{}_{d}, \tag{7}$$

and

$$V^{bc}{}_{d}=\frac{g_{de}}{\sqrt{-g}}[-g(g^{be}g^{cf}-g^{ce}g^{bf})]_{,f}. \tag{8}$$

Here, B^{00} represents the energy density of the entire system and B^{0i} $(i=1,...,5)$ stands for the momentum density components. Using Eq.(1) in Eq.(8), we get the following non-vanishing components of $V^{ab}{}_{c}$

$$\begin{aligned}
&V^{01}{}_0=-V^{10}{}_0=4\left(a-\left(a^3+3y_1\right)^{1/3}\right),\\
&V^{02}{}_0=-V^{20}{}_0=V^{12}{}_1=-V^{21}{}_1=\frac{2y_2}{(a^3+3y_1)^{2/3}},\\
&V^{04}{}_0=V^{14}{}_1=V^{24}{}_2=V^{34}{}_3=2,\\
&V^{40}{}_0=V^{41}{}_1=V^{42}{}_2=V^{43}{}_3=-2,\\
&V^{12}{}_2=V^{13}{}_3=-V^{21}{}_2=-V^{31}{}_3=-a+2\left(a^3+3y_1\right)^{1/3},\\
&V^{41}{}_4=V^{51}{}_5=-V^{14}{}_4=-V^{15}{}_5=3a-4\left(a^3+3y_1\right)^{1/3},\\
&V^{24}{}_4=-V^{42}{}_4=V^{52}{}_5=-V^{25}{}_5=-\frac{2y_2}{(a^3+3y_1)^{2/3}}.
\end{aligned}$$

Substituting these values in Eq.(7) and then in Eq.(6), the energy and momentum components of **Bergmann-Thomson's** prescription turn out to be

$$B^{00}=\frac{-3y_1+a\left(a^2+a\left(a^3+3y_1\right)^{1/3}+\left(a^3+3y_1\right)^{2/3}\right)}{24\pi y_1\left(a^3+3y_1\right)^{2/3}}.$$
$$B^{0i}=0,\qquad (i=1,...,5).$$

Now, the energy-momentum prescription of **Papapetrou** [6] is given as

$$\Omega^{ab}=\frac{1}{16\pi}N^{abcd}{}_{,cd}\,,$$

where

$$N^{abcd}=\sqrt{-g}(g^{ab}\eta^{cd}-g^{ac}\eta^{bd}+g^{cd}\eta^{ab}-g^{bd}\eta^{ac}).$$

Here, η^{ab} is the metric tensor of Minkowski spacetime. The Ω^{00} and Ω^{0i} represent the energy and momentum density components respectively. The non-vanishing components of N^{abcd} are evaluated as

$$\begin{aligned}
\mathrm{N}^{1010}&=\mathrm{N}^{0101}=\frac{1}{1-\frac{a}{(a^3+3y_1)^{1/3}}}-\left(a^3+3y_1\right)\left(-a+\left(a^3+3y_1\right)^{1/3}\right)\\
&=-\mathrm{N}^{0011}=-\mathrm{N}^{1100},\\
\mathrm{N}^{2020}&=\mathrm{N}^{0202}=-\mathrm{N}^{0022}=-\mathrm{N}^{2200}=\frac{1}{1-\frac{a}{(a^3+3y_1)^{1/3}}}+\frac{-1+y_2^2}{\left(a^3+3y_1\right)^{2/3}},\\
\mathrm{N}^{3030}&=\mathrm{N}^{0303}=-\frac{1}{1-\frac{a}{(a^3+3y_1)^{1/3}}}+\frac{1}{\left(a^3+3y_1\right)^{2/3}\left(-1+y_2^2\right)}\\
&=-\mathrm{N}^{0033}=-\mathrm{N}^{3300},\\
\mathrm{N}^{4040}&=\mathrm{N}^{0404}=-\mathrm{N}^{0044}=-\mathrm{N}^{4400}=\gamma-\frac{1}{1-\frac{a}{(a^3+3y_1)^{1/3}}}+2y_4,\\
\mathrm{N}^{5050}&=\mathrm{N}^{0505}=-\mathrm{N}^{0055}=-\mathrm{N}^{5500}=-\frac{1}{1-\frac{a}{(a^3+3y_1)^{1/3}}}+\frac{1}{\gamma+2y_4},\\
\mathrm{N}^{2121}&=\mathrm{N}^{1212}=\left(a^3+3y_1\right)\left(-a+\left(a^3+3y_1\right)^{1/3}\right)+\frac{1-y_2^2}{\left(a^3+3y_1\right)^{2/3}}\\
&=-\mathrm{N}^{1122}=-\mathrm{N}^{2211},\\
\mathrm{N}^{3131}&=\mathrm{N}^{1313}=\left(a^3+3y_1\right)\left(a-\left(a^3+3y_1\right)^{1/3}\right)-\frac{1}{\left(a^3+3y_1\right)^{2/3}\left(y_2^2-1\right)}\\
&=-\mathrm{N}^{3311}=-\mathrm{N}^{1133},\\
\mathrm{N}^{4141}&=\mathrm{N}^{1414}=\gamma+\left(a^3+3y_1\right)\left(\left(a^3+3y_1\right)^{1/3}-a\right)+2y_4\\
&=-\mathrm{N}^{1144}=-\mathrm{N}^{4411},\\
\mathrm{N}^{5151}&=\mathrm{N}^{1515}=\left(a^3+3y_1\right)\left(\left(a^3+3y_1\right)^{1/3}-a\right)+\frac{1}{\gamma+2y_4}\\
&=-\mathrm{N}^{1155}=-\mathrm{N}^{5511},\\
\mathrm{N}^{3232}&=\mathrm{N}^{2323}=-\mathrm{N}^{3322}=-\mathrm{N}^{2233}=\frac{y_2^2\left(y_2^2-2\right)}{\left(a^3+3y_1\right)^{2/3}\left(y_2^2-1\right)},\\
\mathrm{N}^{2424}&=\mathrm{N}^{4242}=-\mathrm{N}^{2244}=-\mathrm{N}^{4422}=\gamma-\frac{y_2^2-1}{\left(a^3+3y_1\right)^{2/3}}+2y_4,\\
\mathrm{N}^{2525}&=\mathrm{N}^{5252}=-\mathrm{N}^{5522}=-\mathrm{N}^{2255}==\frac{1-y_2^2}{\left(a^3+3y_1\right)^{2/3}}+\frac{1}{\gamma+2y_4},\\
\mathrm{N}^{3434}&=\mathrm{N}^{4343}=-\mathrm{N}^{3344}=\mathrm{N}^{4433}=-\gamma+\frac{y_2^2-1}{\left(a^3+3y_1\right)^{2/3}}-2y_4,\\
\mathrm{N}^{3535}&=\mathrm{N}^{5353}=-\mathrm{N}^{3355}=-\mathrm{N}^{5533}=-\frac{1}{\left(a^3+3y_1\right)^{2/3}\left(y_2^2-1\right)}-\frac{1}{\gamma+2y_4},\\
\mathrm{N}^{4545}&=\mathrm{N}^{5454}=-\mathrm{N}^{4455}=-\mathrm{N}^{5544}=-\gamma-2y_4-\frac{1}{\gamma+2y_4}.
\end{aligned}$$

After some calculations, we obtain the energy and momentum density components of **Papapetrou** prescription as

$$\Omega^{00} = \frac{1}{8\pi\left(a^3+3y_1\right)^{5/3}}\left[\left(a^3+3y_1\right) - \frac{a^2-2a\left(\left(a^3+3y_1\right)^{1/3}-a\right)}{\left(\left(a^3+3y_1\right)^{1/3}-a\right)^3}\right].$$
$$\Omega^{0i} = 0, \quad i = 1,...,5.$$

The **Møller** [3] energy-momentum pseudo-tensor M_a^b is given as

$$M_a^b = \frac{1}{8\pi}K^{bc}{}_{a,c},$$

where

$$K^{bc}{}_a = \sqrt{-g}(g_{ad,e} - g_{ae,d})g^{be}g^{cd}. \tag{9}$$

Here, M_0^0 stands for the energy density and M_i^0 $(i = 1,...5)$ for the momentum density components. The momentum four-vector is given by

$$p_a = \int\int_V\int M_a^0 dx^1 dx^2 dx^3, \tag{10}$$

where p_0 gives the energy and p_a $(a = 1,...5)$ give the momentum. Using Gauss's theorem, the total energy-momentum components may be given in the form of surface integral as

$$p_a = \frac{1}{8\pi}\int_S\int K_a^{0c}\mathbf{n_c}.\mathbf{dS},$$

where $\mathbf{n_c}$ is the outward unit normal vector over an infinitesimal surface element **dS**. When we use Eq.(1) in Eq.(10) the non-vanishing components of $K^{bc}{}_a$, come out as

$$K^{01}{}_0 = -K^{10}{}_0 = a,$$
$$K^{12}{}_2 = K^{13}{}_3 = -K^{21}{}_2 = -K^{31}{}_3 = 2a - 2\left(a^3+3y_1\right)^{1/3},$$
$$K^{23}{}_3 = -K^{32}{}_3 = \frac{2y_2}{\left(a^3+3y_1\right)^{2/3}},$$
$$K^{45}{}_5 = -K^{54}{}_5 = 2.$$

Making use of these values in Eq.(9), we get the energy-momentum density components of **Møller's** prescription, which turn out to be zero, i.e.,

$$M^{00} = 0,$$
$$M^{0i} = 0, \quad i = 1,...,5.$$

3. Discussion

The most important issue in GR is an acceptable definition of energy-momentum localization. Although, the problem of localization of energy is unresolved and controversial but numerous scientists attempted to resolve it in the context of different frames of work. For this purpose, a huge number of examples have been explored in different frames of work so that these may help us, at some stage, to make a conjecture about this problem.
It is well-known that geometry and physics are very much related to each other for any physical problem. The motivation behind this work to see what comes out by exploring the energy-momentum distribution of this six-dimensional geometric model of the gravitational field. In other words, to see how is the physics associated with this model. For this purpose, we use Einstein, Landau-Lifshitz, Bergmann-Thomson, Papapetrou and Møller prescriptions. The results obtained are given in this table.

Table 1. Energy Density (ED) Components for all Prescriptions

ED	Expressions
Θ^{00}	$\frac{-3y_1+a\left(a^2+a\left(a^3+3y_1\right)^{1/3}+\left(a^3+3y_1\right)^{2/3}\right)}{24\pi y_1\left(a^3+3y_1\right)^{2/3}}$
B^{00}	$\frac{-3y_1+a\left(a^2+a\left(a^3+3y_1\right)^{1/3}+\left(a^3+3y_1\right)^{2/3}\right)}{24\pi y_1\left(a^3+3y_1\right)^{2/3}}$
L^{00}	$\frac{2a-\left(a^3+3y_1\right)^{1/3}}{8a^3\pi+24\pi y_1-8a\pi\left(a^3+3y_1\right)^{2/3}}$
Ω^{00}	$\frac{1}{8\pi(a^3+3y_1)^{5/3}}\left[\left(a^3+3y_1\right)-\frac{a^2}{\left((a^3+3y_1)^{1/3}-a\right)^3}-\frac{2a}{\left(a-(a^3+3y_1)^{1/3}\right)^2}\right]$
M^{00}	0

This table shows that the energy density components turn out to be well defined and finite for Einstein, Bergmann-Thomson and Papapetrou prescriptions everywhere in its domain except at $y_1 = 0$ and $y_1 = -\frac{a^3}{3}$ while Papapetrou prescription becomes infinite at $y_1 = -\frac{a^3}{3}$ only. Further, it shows that the results for Einstein and Bergmann-Thomson prescriptions turned out to be same but different from the other prescriptions. It is mentioning here that the energy density becomes zero in case of Møller's prescription. Further, the momentum densities of the six-dimensional geometric model of the gravitational field are zero for all prescriptions. This adds an example of the six-dimensional model towards the solution of the problem of localization of energy and momentum in GR and helpful, at some stage, for making a conjecture about this issue. The graphical representations of the energy density are given as under

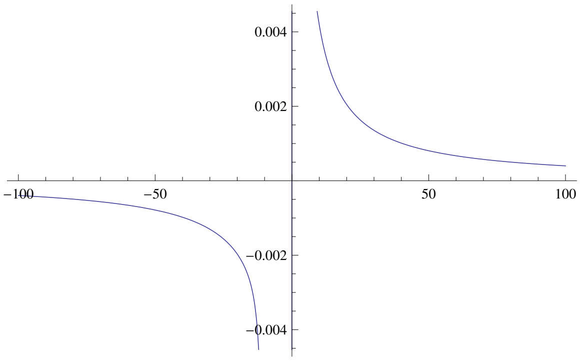

Figure 1: Einstein and Bergmann Thomson Energy density graph vs y_1

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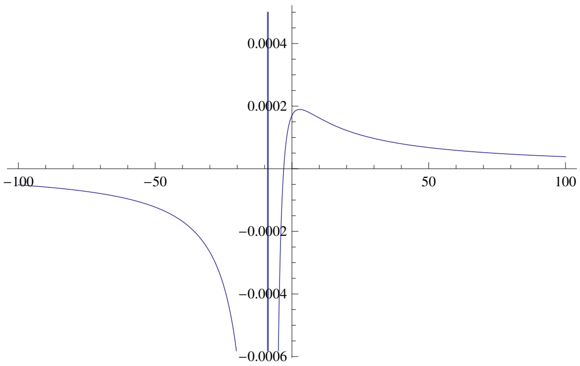

Figure 2: Landau-Lifshitz Energy density graph vs y_1

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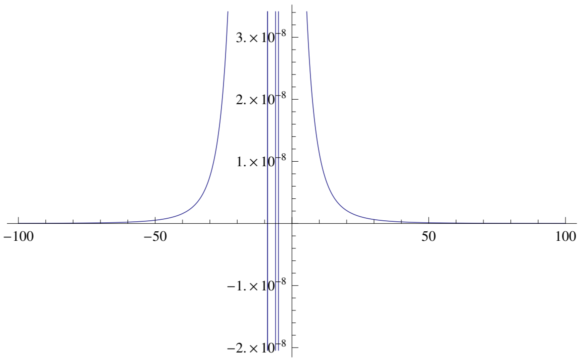

Figure 3: Papapetrou Energy density graph vs y_1

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It is mentioned here that the results of energy-momentum distribution for different spacetimes are not astonishing. On the basis of these results, we can conclude that the energy-momentum complexes, which are treated as pseudo-tensors are not covariant. This is in coherence with the equivalence principle [41] which implies that the gravitational field cannot be detected at a point. The energy-momentum complexes for a huge number of spacetimes have been discussed but no consensus has been built yet to decide which one is the best.

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On Some Properties of Integral-Type Operator in Weighted Herz Spaces with Variable Exponent Lebesgue Spaces

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Abstract

For the last quarter century a considerable number of research has been carried out on the study of Herz spaces, variable exponent Lebesgue spaces and Sobolev spaces. This studies also have played an important role in problems of elasticity, fluid dynamics, calculus of variations. Our aim in this work is to prove some properties of the integral-type operator on weighted Herz space with variable exponent Lebesgue space (VELS).

1. Introduction

Herz spaces and variable exponent spaces have played an important role in recent harmonic analysis because they have an interesting norm including both local and global properties.We refer to the book [1] for the history of Herz spaces. Based on the classical Muckenhoupt theory [2]-[4] some classes of weighted Herz spaces have been defined and the boundedness of many operators on those spaces have been proved [4]-[8] . Herz spaces can be generalized using variable exponents and many properties of them have been studied [9]-[11] . The boundedness or compactness of integral-type operators on weighted Lebesgue spaces has been obtained [12]-[19] . We prove the boundedness of the fractional maximal operator on the spaces with power weight.

2. Preliminaries, Definitions and Assertions

We use the following definitions and notations:

a) For a point $x \in R^n$ and a constant $r < 0$, we have $B(x,r) = \{y \in R^n : |x-y| < r\}$
b) The set N_0 consist of all non-negative integers. For every $k \in N_0$, we write

$$B_k = \overline{B(0,2^k)} = \{x \in R^n : |x| \leq 2^k\}$$

c) We define a family $\{C_k\}_{k=0}^{\infty}$ by

$$C_k = \begin{cases} B_0 = \{x \in R^n : |x| \leq 1\}, k = 0 \\ B_k \backslash B_{k-1} = \{x \in R^n : 2^{k-1} < |x| \leq 2^k\}, k \geq 1 \end{cases}$$

Moreover χ_k denote the charecteristic function of C_k, namely $\chi_k = \chi_{C_k}$.
Definition 2.1 (See [16]) Let ω be a weight function on R^n and let $p(.) = R^n \longrightarrow [1,\infty)$ a bounded measurable function. For all f measurable function, weighted norm Lebesgue space $L_{\omega}^{p(.)}(R^n)$ with

$$\|f\|_{L_{\omega}^{p(.)}(R^n)} = inf\{\lambda > 0 : \int_{R^n} (\frac{|f(x)|}{\lambda})\omega(x)dx \leq 1\}$$

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In the case that $\omega \equiv 1$, we simply write $L^{p(.)} = L^{p(.)}_{\omega}$ and $\|f\|_{L^{p(.)}} = \|f\|_{L^{p(.)}_{\omega}}$.

Definition 2.2(See [9], [11]) Given a, $0 < a < n$ and given an open set $\Omega \subset R^n$ the fractional maximal operator M_a is defined by

$$M_a f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{a}{n}}} \int_{B\cap\Omega} f(y)dy$$

where the supremum is taken over all balls B which contain x. When $a = 0, M_0 = M$ is the Hardy-Littlewood maximal operator.

Definition 2.3 ([9], [16])

1. for measurable function $p(.) = R^n \longrightarrow (0,\infty)$, we write

$$p^+ = ess \sup_{x\in R^n} p(x)$$

$$p^- = \{(\frac{1}{p(.)})_+\}^{-1}$$

2. The set $P(R^n)$ consist of all measurable functions $p(.)$ defined on R^n satisfying $1 < p^- \leq p^+ < \infty$.
3. The set $LH(R^n)$ consist of $p(.)$ defined on R^n satisfying the following two inequalities

$$|p(x) - p(y)| \lesssim -\frac{1}{log(|x-y|)}, |x-y| \leq \frac{1}{2}$$

$$|p(x) - p(y)| \lesssim -\frac{1}{log(e+|x|)}, x \in R^n \tag{1}$$

for some real constant p_∞. In particular a measurable function $p(.)$ is said to be log-Hölder continuous at infinity when $p(.)$ satisfies (1).

3. Main Results

Let's start with the following definitions for proof of our main theorem.

Definition 3.1. *(See [11]) Let $0 < q < \infty$, $\beta(.) \in L^\infty$, ω be a weight function on R^n and $p(.) : R^n \longrightarrow [1,\infty)$ a bounded measurable function.*
1. The set $L^{p(.)}_{loc}(\omega^{\frac{1}{p(.)}})$ consist of all measurable function f such that $f\chi_K \in L^{p(.)}(\omega^{\frac{1}{p(.)}})$ for any compact set $K \in R^n$.
2.The non-homogeneous Herz space $K^{\beta(.),q}_{p(.)}$ consist of all measurable functions $f \in L^{p(.)}_{loc}(1)$ such that

$$\|f\|_{K^{\beta(.),q}_{p(.)}} = (\sum_{k=0}^{\infty} \|2^{\beta(.)k} f\chi_K\|^q_{L^{p(.)}})\frac{1}{q} < \infty.$$

3. The critical weighted Herz space $B^{p(.)}(\omega)$ consist of all measurable functions $f \in L^{p(.)}_{loc}(\omega^{\frac{1}{p(.)}})$ such that

$$\|f\|_{B^{p(.)}(\omega)} = \sup_{k\geq 0} \|\omega(B_k)^{-\frac{1}{p(.)}} f\chi_K\|_{L^{p(.)}(\omega)} < \infty. \tag{2}$$

Proposition 3.2. *(see [9, Proposition 3.8]). Let $p(.) \in P(R^n)$, $q \in (0,\infty)$ and $\beta(.) \in L^\infty$. If $\beta(.)$ is log-Hölder continuous at infinity, then we have*

$$K^{\beta(.),q}_{p(.)} = K^{\beta_\infty,q}_{p(.)}$$

with norm equivalence.

From now on, we consider a power weight $\omega(x) = |x|^m$ with a real constant m. It is easy to see that for all $k \in N_0$ and $R > 0$,

$$\omega(B_k) \sim 2^{(mn+n-1)k}, \omega(B(0,r)) \sim r^{mn+n-1}$$

where implicit constants are independant of k and r. Proposition 3.2 can be extended to the case $B^{p(.)}(\omega)$ by the same as the proof of [9, Proposition 3.8]. Herewith, we have following a corollary.

Corollary 3.3. *(See [11]) Let $\omega(x) = |x|^m$ with a real constant m, $p(.) \in P(R^n) \cap LH(R^n)$ and $\beta(.) \in L^\infty$. If $\omega \in \aleph_{p(.)}$, we have that for all measurable function $f \in B^{p(.)}(\omega)$,*

$$\|f\|_{B^{p(.)}(\omega)} = \sup_{k\geq 0} \|2^{-\frac{k(mn+n-1)}{p(.)}} f\chi_{C_k}\|_{L^{p(.)}(\omega)} \sim \sup_{k\geq 0} \|2^{-\frac{k(mn+n-1)}{p_\infty}} f\chi_{B_k}\|_{L^{p(.)}(\omega)}$$

Proposition 3.4. *(See [17, Proposition 1.2]). Let $\omega(x)=|x|^m$ with a real constant m, $p(.)\in P(R^n)$ and $f\in L^{p(.)}_{loc}(\omega^{\frac{1}{p(.)}})$. Then $f\in B^{p(.)}(\omega)$ holds if and only if*

$$\sup_{r\geq 1}\|\frac{|f|\chi_{B(0,r)}}{\omega(B(0,r))^{\frac{1}{p(.)}}}\|_{L^{p(.)}(\omega)} \tag{3}$$

is finite. If this is the case, then the quantity (3) is equivalent to the $B^{p(.)}(\omega)$ norm (2).

Our result is the boundedness of the weighted fractional maximal operator $M_{a,\omega}$ on the space $B^{p(.)}(\omega)$.

Theorem 3.5. *Let $m\in R$ and $\omega(x)=|x|^m$. Suppose that $p(.)\in P(R^n)\cap LH(R^n)$ and $\omega\in\aleph_{p(.)}$. Then $M_{a,\omega}$ is bounded on $B^{p(.)}(\omega)$.*

Proof. Let $f\in B^{p(.)}(\omega)$. Then Corollary 3.3 we have

$$\|M_{a,\omega}f\|_{B^{p(.)}(\omega)}\sim\sup_{k\geq 0}\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega}f)\chi_{B_k}\|_{L^{p(.)}(\omega)}$$

Thus we have only to estimate $\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega}f)\chi_{B_k}\|_{L^{p(.)}(\omega)}$ for each $k\in N_0$. Let constant $k\in N_0$ then we have

$$M_{a,\omega}f(x)\leq\sup_{x\in B\subset B_{k+1}}\frac{1}{|B|^{1-\frac{a}{n}}\omega(B)}\int_B|f(y)|\omega(y)dy+\sup_{x\in B\backslash B_{k+1}\neq 0}\frac{1}{|B|^{1-\frac{a}{n}}\omega(B)}\int_B|f(y)|\omega(y)dy$$

$$=M_{a,\omega,1,k}f(x)+M_{a,\omega,2,k}f(x)$$

(a) Note that $M_{a,\omega,1,k}f(x)=M_{a,\omega,1,k}f(x)(f\chi_{B_{k+1}})\chi_{B_{k+1}}(x)$.
By [10, Theorem 2.10] we have

$$\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega,1,k}f)\chi_{B_k}\|_{L^{p(.)}(\omega)}=\|2^{-\frac{k(mn+n-1)}{p_\infty}}M_{a,\omega,1,k}(f\chi_{B_{k+1}})\chi_{B(0,2^{k+1})}\|_{L^{p(.)}(\omega)}$$

$$\lesssim\|2^{-\frac{k(mn+n-1)}{p_\infty}}f\chi_{B_{k+1}}\|_{L^{p(.)}(\omega)}$$

(b) Next we estimate the function $M_{a,\omega,2,k}f(x)$. Let $|x|<2^k$. The smallest ball centered at 0 and containing B is called $B^{'}$. Then, there exist a constant C which is depended only on the dimension n such that $|B|\geq\frac{|B^{'}|}{C}$ by a geometric consideration.

$$M_{a,\omega,2,k}f(x)\leq CM_{a,\omega}f(0).$$

We note that $\|\omega(B^{'})^{-\frac{1}{p^{'}(.)}}\chi_{B^{'}}\|_{L^{p^{'}(.)}(\omega)}=1$ because

$$\int_{R^n}(\omega(B^{'})^{-\frac{1}{p^{'}(.)}}\chi_{B^{'}})^{p^{'}(x)}\omega(x)dx=\frac{1}{\omega(B^{'})}\int_{B^{'}}\chi_{B^{'}}(x)\omega(x)dx=1$$

If we use Hölder inequality and Proposition 3.4 we have

$$\frac{1}{|B|^{1-\frac{a}{n}}\omega(B^{'})}\int_{B^{'}}|f(y)|\omega(y)dy$$

$$=\frac{1}{|B|^{1-\frac{a}{n}}}\int_{B^{'}}\frac{|f(y)\chi_{B^{'}}(y)|\omega(y)^{\frac{1}{p(y)}}}{\omega(B^{'})^{\frac{1}{p(y)}}}.\frac{\chi_{B^{'}}(y)\omega(y)^{\frac{1}{p^{'}(y)}}}{\omega(B^{'})^{\frac{1}{p^{'}(y)}}}dy\leq C\|f\|_{B^{p(.)}(\omega)}\|\omega(B^{'})^{-\frac{1}{p^{'}(.)}}\chi_{B^{'}}\|_{L^{p^{'}(.)}(\omega)}\leq C\|f\|_{B^{p(.)}(\omega)}$$

Hence, we can see that $M_{a,\omega,2,k}f(x)\lesssim\|f\|_{B^{p(.)}(\omega)}$ holds.

Combining (a) and (b) we see that

$$\|M_{a,\omega}f\|_{B^{p(.)}(\omega)}\sim\sup_{k\geq 0}\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega}f)\chi_k\|_{L^{p(.)}(\omega)}$$

$$\leq\sup_{k\geq 0}(\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega,1,k}f)\chi_k\|_{L^{p(.)}(\omega)}+\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega,2,k}f)\chi_k\|_{L^{p(.)}(\omega)})$$

$$\leq\sup_{k\geq 0}(\|2^{-\frac{k(mn+n-1)}{p_\infty}}f\chi_{B(0,2^{k+1})}\|_{L^{p(.)}(\omega)}+\|f\|_{B^{p(.)}(\omega)}\|2^{-\frac{k(mn+n-1)}{p_\infty}}\chi_k\|_{L^{p(.)}(\omega)})$$

$$\lesssim \sup_{k\geq 0}\|2^{-\frac{k(mn+n-1)}{p_\infty}} f\chi_{B(0,2^{k+1})}\|_{L^{p(\cdot)}(\omega)} + \|f\|_{B^{p(\cdot)}(\omega)} \sup_{k\geq 0}\|2^{-\frac{k(mn+n-1)}{p_\infty}}\chi_k\|_{L^{p(\cdot)}(\omega)}$$

$$\lesssim 2^{\frac{(mn+n-1)}{p_\infty}}\|f\|_{B^{p(\cdot)}(\omega)} + \|f\|_{B^{p(\cdot)}(\omega)} \lesssim \|f\|_{B^{p(\cdot)}(\omega)}.$$

Theorem 3.5 is proved. □

Conclusion. We investigated and saw that is bounded the fractional maximal operator $M_{a,\omega}$ on the spaces with power weight in view of the given conditions. This method can be applied to different operators.

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Inequalities of Hermite-Hadamard and Bullen Type for AH-Convex Functions

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Abstract

In this paper, by using an integral identity some new general inequalities of the Hermite-Hadamard and Bullen type for functions whose second derivatives in absolute value at certain power are arithmetically-harmonically convex are obtained. Some applications to special means of real numbers are also given.

1. Introduction

Definition 1.1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

valids for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$. This definition is well known in the literature.

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds.

The inequality (1.1) is known in the literature as Hermite-Hadamard integral inequality for convex functions. Moreover, it is known that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the function f. See [3, 5, 8, 9], for the generalizations, improvements and extensions of the Hermite-Hadamard integral inequality.

Theorem 1.3. *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. Then, the inequalities are obtained:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.2}$$

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The third inequality in (1.2) is known in the literature as Bullen's inequality.

Definition 1.4 ([2, 10]). *A function $f : I \subset \mathbb{R} \to (0,\infty)$ is said to be arithmetic-harmonically (AH) convex function if for all $x,y \in I$ and $t \in [0,1]$ the inequality*

$$f(tx+(1-t)y) \leq \frac{f(x)f(y)}{tf(y)+(1-t)f(x)} \tag{1.3}$$

holds. If the inequality (1.2) is reversed then the function $f(x)$ is said to be arithmetic-harmonically (AH) concave function.

Readers can find more informations on arithmetic-harmonically convex functions in [1, 2, 4, 6, 7, 10] and references therein.
In order to establish some integral inequalities of Hermite-Hadamard type for arithmetic-harmonically convex functions, the following lemma [4] will be used.

Lemma 1.5 ([4]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$, then the following identity holds:*

$$J_n(f,a,b) = \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left[\int_0^1 t(1-t) f''\left(\begin{array}{c} t\left(\frac{1+n-k}{n}a+\frac{k-1}{n}b\right) \\ +(1-t)\left(\frac{n-k}{n}a+\frac{k}{n}b\right)\end{array}\right)dt\right] \tag{1.4}$$

for all $n \in \mathbb{N}$, where

$$J_n(f,a,b) = \sum_{k=1}^{n} \frac{1}{2n}\left[f\left(a+\frac{(k-1)(b-a)}{n}\right)+f\left(a+\frac{k(b-a)}{n}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx.$$

In this study, using Hölder integral inequality and the identity (1.4) in order to provide inequality for functions whose first derivatives in absolute value at certain power are arithmetic-harmonically-convex functions.
Throughout this paper, for shortness, the following notations will be used for special means of two nonnegative numbers a,b with $b>a$:
1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \quad a,b>0,$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a,b \geq 0$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \quad a,b>0,$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a=b \end{cases} ; \ a,b>0$$

5. The p-logarithmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R}\backslash\{-1,0\} \\ a, & a=b \end{cases} ; \ a,b>0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$. In addition,

$$A_{n,k} = A_{n,k}(a,b) = \frac{1+n-k}{n}a+\frac{k-1}{n}b, n \in \mathbb{N}, k = 1,2,...,n,$$

and $B(\alpha,\beta)$ is the classical Beta function which may be defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \ \alpha,\beta>0.$$

2. Main results

Theorem 2.1. *Let $f: I \subset (0,\infty) \to (0,\infty)$ be a twice differentiable mapping on I°, $n \in \mathbb{N}$ and $a,b \in I^{\circ}$ with $a<b$ such that $f'' \in L_1[a,b]$ and $|f''|$ are an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequalities hold:*
i) If $|f''(A_{n,k+1})| - |f''(A_{n,k})| \neq 0$, then

$$|J_n(f,a,b)| \le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \frac{|f''(A_{n,k})|\,|f''(A_{n,k+1})|}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^2} \tag{2.1}$$

$$\times \left[A\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right) - |f''(A_{n,k})|\,|f''(A_{n,k+1})|\,L^{-1}\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right)\right]$$

ii) If $|f''(A_{n,k+1})| - |f''(A_{n,k})| = 0$, then

$$|J_n(f,a,b)| \le \sum_{k=1}^{n} \frac{(b-a)^2}{12n^3} |f''(A_{n,k+1})|. \tag{2.2}$$

Proof. i) Let $|f''(A_{n,k+1})| - |f''(A_{n,k})| \neq 0$. From the Lemma 1.5 and the properties of modulus, the inequality can be written:

$$\begin{aligned} |J_n(f,a,b)| &= \left| \sum_{k=1}^{n} \frac{1}{2n} \left[f\left(a + \frac{(k-1)(b-a)}{n}\right) + f\left(a + \frac{k(b-a)}{n}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ &= \left| \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \left[\int_0^1 t(1-t) f''(tA_{n,k} + (1-t)A_{n,k+1})\,dt \right] \right| \\ &\le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \left[\int_0^1 |t(1-t)|\,|f''(tA_{n,i} + (1-t)A_{n,i+1})|\,dt \right]. \end{aligned} \tag{2.3}$$

Since $|f''|$ is an arithmetic-harmonically convex function on the interval $[a,b]$, the inequality

$$|f''(tA_{n,k} + (1-t)A_{n,k+1})| \le \frac{|f''(A_{n,k})|\,|f''(A_{n,k+1})|}{t|f''(A_{n,k+1})| + (1-t)|f''(A_{n,k})|}$$

holds. By using the above inequality in (2.3), the inequality

$$|J_n(f,a,b)| = \le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \int_0^1 \frac{t(1-t)|f''(A_{n,k})|\,|f''(A_{n,k+1})|}{t|f''(A_{n,k+1})| + (1-t)|f''(A_{n,k})|} dt \tag{2.4}$$

is obtained. By changing variable as $u = t|f''(A_{n,k+1})| + (1-t)|f''(A_{n,k})|$ in the last integral, it is easily seen that

$$\begin{aligned} \int_0^1 \frac{t(1-t)}{t|f''(A_{n,k+1})| + (1-t)|f''(A_{n,k})|} dt &= \frac{1}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^3} \int_{|f''(A_{n,k})|}^{|f''(A_{n,k+1})|} \frac{\left(u - |f''(A_{n,k})|\right)\left(|f''(A_{n,k+1})| - u\right)}{u} du \\ &= \frac{\left[-\frac{u^2}{2} + \left(|f''(A_{n,k+1})| + |f''(A_{n,k})|\right)u - |f''(A_{n,k})|\,|f''(A_{n,k+1})|\ln u\right]}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^3} \Bigg|_{|f''(A_{n,k})|}^{|f''(A_{n,k+1})|} \\ &= \frac{1}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^3} \left[-\frac{|f''(A_{n,k+1})|^2 - |f''(A_{n,k})|^2}{2} \right. \\ &\quad + \left(|f''(A_{n,k+1})| + |f''(A_{n,k})|\right)\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right) \\ &\quad \left. - |f''(A_{n,k})|\,|f''(A_{n,k+1})|\left(\ln|f''(A_{n,k+1})| - \ln|f''(A_{n,k})|\right)\right] \\ &= \frac{A\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right) - |f''(A_{n,k})|\,|f''(A_{n,k+1})|\,L^{-1}\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right)}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^2} \end{aligned} \tag{2.5}$$

Substituting (2.5) in (2.4), the inequality

$$|J_n(f,a,b)| \le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \frac{|f''(A_{n,k})|\,|f''(A_{n,k+1})|}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^2}$$
$$\times \left[A\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right) - |f''(A_{n,k})|\,|f''(A_{n,k+1})|\,L^{-1}\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right)\right],$$

is obtained which is the desired result.
ii) Let $|f''(A_{n,k+1})| - |f''(A_{n,k})| = 0$. Then, substituting $|f''(A_{n,k+1})| = |f''(A_{n,k})|$ in the inequality (2.4), the following holds:

$$|J_n(f,a,b)| \le \sum_{k=1}^{n} \frac{(b-a)^2}{12n^3} |f''(A_{n,k+1})|.$$

This completes the proof of theorem. □

Corollary 2.2. *By choosing $n=1$ in Theorem 2.1, the following inequalities are obtained:*
i) If $\left|f''\left(A_{1,k+1}\right)\right|-\left|f''\left(A_{1,k}\right)\right|\neq 0$ for $k=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{2}\frac{|f''(a)||f''(b)|\left[A\left(|f''(a)|,|f''(b)|\right)-|f''(a)||f''(b)|L^{-1}\left(|f''(a)|,|f''(b)|\right)\right]}{\left(|f''(b)|-|f''(a)|\right)^2},$$

ii) If $\left|f''\left(A_{1,k+1}\right)\right|-\left|f''\left(A_{1,k}\right)\right|=0$ for $k=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{12}\left|f''(b)\right|.$$

Corollary 2.3. *By choosing $n=2$ in Theorem 2.1, the following Bullen type inequalities are obtained:*
i) If $\left|f''\left(A_{2,k+1}\right)\right|-\left|f''\left(A_{2,k}\right)\right|\neq 0$ for $k=1,2$, then

$$\begin{aligned}\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq\frac{(b-a)^2}{16}\frac{|f''(a)|\left|f''\left(\frac{a+b}{2}\right)\right|}{\left(\left|f''\left(\frac{a+b}{2}\right)\right|-|f''(a)|\right)^2}\left[A\left(|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right.\\ &\left.-|f''(a)|\left|f''\left(\frac{a+b}{2}\right)\right|-L^{-1}\left(|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right]\\ &+\frac{(b-a)^2}{16}\frac{\left|f''\left(\frac{a+b}{2}\right)\right||f''(b)|}{\left(|f''(b)|-\left|f''\left(\frac{a+b}{2}\right)\right|\right)^2}\left[A\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(b)|\right)\right.\\ &\left.-\left|f''\left(\frac{a+b}{2}\right)\right||f''(b)|-L^{-1}\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(b)|\right)\right],\end{aligned}$$

ii) If $\left|f''\left(A_{2,k+1}\right)\right|-\left|f''\left(A_{2,k}\right)\right|=0$ for $k=1,2$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{96}\left[\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right].$$

Theorem 2.4. *Let $f:I\subset(0,\infty)\to(0,\infty)$ be a twice differentiable mapping on I°, $n\in\mathbb{N}$ and $a,b\in I^\circ$ with $a<b$ such that $f''\in L_1[a,b]$ and $|f''|^q$ are an arithmetic-harmonically convex function on the interval $[a,b]$ for some fixed $q>1$, then the following inequalities hold:*
i) If $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\neq 0$, then

$$|J_n(f,a,b)|\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\frac{\left[B(p+1,p+1)\right]^{\frac{1}{p}}G^2\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)},\tag{2.6}$$

ii) If $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q=0$, then

$$|J_n(f,a,b)|\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left|f''\left(A_{n,k+1}\right)\right|.\tag{2.7}$$

where $B(\alpha,\beta)$ is the classical Beta function and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. i) Let $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\neq 0$. From the Lemma 1.5 and the properties of modulus, the following inequality can be writtten

$$|J_n(f,a,b)|\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left[\int_0^1|t(1-t)|\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|dt\right].\tag{2.8}$$

Since $|f''|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, the inequality

$$\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|^q\leq\frac{\left|f''\left(A_{n,k}\right)\right|^q\left|f''\left(A_{n,k+1}\right)\right|^q}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}\tag{2.9}$$

holds. By applying the well known Hölder integral inequality and the inequality (2.9) on (2.8), the inequality

$$\begin{aligned}|J_n(f,a,b)| &\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left(\int_0^1[t(1-t)]^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left(\int_0^1 t^p(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\frac{\left|f''\left(A_{n,k}\right)\right|^q\left|f''\left(A_{n,k+1}\right)\right|^q dt}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}\right)^{\frac{1}{q}} &(2.10)\\ &=\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\frac{\left[B(p+1,p+1)\right]^{\frac{1}{p}}G^2\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}, &(2.11)\end{aligned}$$

is obtained, where

$$\int_0^1 t^p (1-t)^p dt = B(p+1,p+1)$$

$$\int_0^1 \frac{1}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}dt = L^{-1}\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right).$$

ii) Let $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q=0$. Then, substituting $\left|f''\left(A_{n,k+1}\right)\right|^q=\left|f''\left(A_{n,k}\right)\right|^q$ in the inequality (2.10), the following inequality is found:

$$|J_n(f,a,b)| \le \sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left|f''\left(A_{n,k+1}\right)\right|.$$

This completes the proof of theorem. □

Corollary 2.5. *By choosing $n=1$ in Theorem 2.4, the following inequalities are obtained:*
i) If $\left|f''\left(A_{1,k+1}\right)\right|-\left|f''\left(A_{1,k}\right)\right|\neq 0$ for $k=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2}\frac{\left[B(p+1,p+1)\right]^{\frac{1}{p}}G^2\left(\left|f''(a)\right|,\left|f''(b)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right)},$$

ii) If $\left|f''\left(A_{1,k+1}\right)\right|-\left|f''\left(A_{1,k}\right)\right|=0$ for $k=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left|f''(b)\right|.$$

Corollary 2.6. *By choosing $n=2$ in Theorem 2.4, the following Bullen type inequalities are obtained:*
i) If $\left|f''\left(A_{2,k+1}\right)\right|-\left|f''\left(A_{2,k}\right)\right|\neq 0$ for $k=1,2$, then

$$\begin{aligned}\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{(b-a)^2}{16}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\\ &\times\left[\frac{G^2\left(\left|f''(a)\right|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''(a)\right|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)}+\frac{G^2\left(\left|f''\left(\frac{a+b}{2}\right)\right|,\left|f''(b)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''\left(\frac{a+b}{2}\right)\right|,\left|f''(b)\right|\right)}\right],\end{aligned}$$

ii) If $\left|f''\left(A_{2,k+1}\right)\right|-\left|f''\left(A_{2,k}\right)\right|=0$ for $k=1,2$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{16}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left[\left|f''\left(\frac{a+b}{2}\right)\right|+\left|f''(b)\right|\right].$$

Theorem 2.7. *Let $f: I\subset(0,\infty)\to(0,\infty)$ be a twice differentiable mapping on I°, $n\in\mathbb{N}$ and $a,b\in I^\circ$ with $a<b$ such that $f''\in L_1[a,b]$ and $|f''|^q$ are an arithmetic-harmonically convex function on the interval $[a,b]$ for some fixed $q\ge 1$, then the following inequalities hold:*
i) If $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\neq 0$, then

$$\begin{aligned}|J_n(f,a,b)| &\le \sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right)}{\left(\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\right)^{\frac{2}{q}}} \\ &\times\left[A\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)-\frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}{L\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}\right]^{\frac{1}{q}},\end{aligned} \tag{2.12}$$

ii) If $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q=0$, then

$$|J_n(f,a,b)| \le \sum_{k=1}^{n}\frac{(b-a)^2}{12n^3}\left|f''\left(A_{n,k+1}\right)\right|, \tag{2.13}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. i) Let $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\neq 0$. From the Lemma 1.5 and the properties of modulus, the inequality can be written:

$$|J_n(f,a,b)| \le \sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left[\int_0^1 |t(1-t)|\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|dt\right]. \tag{2.14}$$

Since $|f''|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, the inequality

$$\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|^q \le \frac{\left|f''\left(A_{n,k}\right)\right|^q\left|f''\left(A_{n,k+1}\right)\right|^q}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}$$

holds. By applying the last inequality and the well known power-mean integral inequality on (2.14), the inequality

$$
\begin{aligned}
|J_n(f,a,b)| &\le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left(\int_0^1 |t(1-t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t(1-t)|\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left(\int_0^1 t(1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t(1-t)\left|f''\left(A_{n,k}\right)\right|^q\left|f''\left(A_{n,k+1}\right)\right|^q}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}dt\right)^{\frac{1}{q}} \qquad (2.15)\\
&= \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left|f''\left(A_{n,k}\right)\right|\left|f''\left(A_{n,k+1}\right)\right|\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t(1-t)\,dt}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}\right)^{\frac{1}{q}} \\
&= \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right)}{\left(\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\right)^{\frac{2}{q}}} \\
&\quad\times\left[A\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)-\frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}{L\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}\right]^{\frac{1}{q}},
\end{aligned}
$$

is obtained, where

$$
\begin{aligned}
&\int_0^1 t(1-t)\,dt=\frac{1}{6},\\
&\int_0^1 \frac{t(1-t)}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}dt=\frac{1}{\left(\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\right)^2}\\
&\times\left[A\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)-\frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}{L\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}\right].
\end{aligned}
$$

ii) Let $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q=0$. Then, substituting $\left|f''\left(A_{n,k+1}\right)\right|^q=\left|f''\left(A_{n,k}\right)\right|^q$ in the inequality (2.15), the following inequality is found:

$$
\begin{aligned}
|J_n(f,a,b)| &\le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left(\int_0^1 t(1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left|f''\left(A_{n,k+1}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&=\sum_{k=1}^{n} \frac{(b-a)^2}{12n^3}\left|f''\left(A_{n,k+1}\right)\right|.
\end{aligned}
$$

This completes the proof of theorem. □

Corollary 2.8. *By choosing $n=1$ in Theorem 2.7, the following inequalities are obtained:*
i) If $\left|f''\left(A_{1,k+1}\right)\right|^q-\left|f''\left(A_{1,k}\right)\right|^q\neq 0$ for $k=1$, then

$$
\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f''(a)\right|,\left|f''(b)\right|\right)}{\left(\left|f''(b)\right|^q-\left|f''(a)\right|^q\right)^{\frac{2}{q}}}\\
&\quad\times\left[A\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right)-\frac{G^2\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right)}{L\left(\left|f''(a)\right|^q,\left|f''b\right|^q\right)}\right]^{\frac{1}{q}},
\end{aligned}
$$

ii) If $\left|f''\left(A_{1,k+1}\right)\right|^q-\left|f''\left(A_{1,k}\right)\right|^q=0$ for $k=1$, then

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{12}\left|f''(b)\right|.
$$

Corollary 2.9. *By choosing $n=2$ in Theorem 2.7, the following Bullen type inequalities are obtained:*
i) If $\left|f''\left(A_{2,k+1}\right)\right|^q-\left|f''\left(A_{2,k}\right)\right|^q\neq 0$ for $k=1,2$, then

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{16}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f''(a)\right|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)}{\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q-\left|f''(a)\right|^q\right)^{\frac{2}{q}}}\\
&\times\left[A\left(\left|f''(a)\right|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)-\frac{G^2\left(\left|f''(a)\right|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)}{L\left(\left|f''(a)\right|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)}\right]^{\frac{1}{q}}+\frac{(b-a)^2}{16}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f''\left(\frac{a+b}{2}\right)\right|,\left|f''(b)\right|\right)}{\left(\left|f''(b)\right|^q-\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{2}{q}}}\\
&\times\left[A\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right)-\frac{G^2\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right)}{L\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right)}\right]^{\frac{1}{q}},
\end{aligned}
$$

ii) If $\left|f''\left(A_{2,k+1}\right)\right|^q-\left|f''\left(A_{2,k}\right)\right|^q=0$ for $k=1,2$, then

$$
\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{16}\left[\left|f''\left(\frac{a+b}{2}\right)\right|+\left|f''(b)\right|\right].
$$

Corollary 2.10. *Taking $q=1$ in the inequality (2.12), the following inequality is obtained:*

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right)}{\left(\left|f''\left(A_{n,k+1}\right)\right|-\left|f''\left(A_{n,k}\right)\right|\right)^2} \times \left[A\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right) - \frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|,f''\left(A_{n,k+1}\right)\right)}{L\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right)}\right].$$

3. Applications for special means

If $p\in(-1,0)$ then the function $f(x)=x^p, x>0$ is an arithmetic harmonically-convex [2]. Using this function, the following propositions are obtained:

Proposition 3.1. *Let $0<a<b$ and $p\in(-1,0)$. Then, the following inequality holds:*

$$\begin{aligned}
&\frac{1}{(p+1)(p+2)}\left|\sum_{k=1}^{n}\frac{1}{n}\left[\left(A_{n,k}\right)^{p+2}+\left(A_{n,k+1}\right)^{p+2}\right]-L_{p+2}^{p+2}(a,b)\right| \\
&\leq \sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\frac{\left(A_{n,k}\right)^p\left(A_{n,k+1}\right)^p}{\left[\left(A_{n,k+1}\right)^p-\left(A_{n,k}\right)^p\right]^2}\left[A\left(\left(A_{n,k}\right)^p,\left(A_{n,k+1}\right)^p\right)-\frac{\left(A_{n,k}\right)^p\left(A_{n,k+1}\right)^p}{L\left(\left(A_{n,k}\right)^p,\left(A_{n,k+1}\right)^p\right)}\right].
\end{aligned}$$

Proof. It is known that if $p\in(-1,0)$ then the function $f(x)=\frac{x^{p+2}}{(p+1)(p+2)}, x>0$ is an arithmetic harmonically-convex function. Therefore, the assertion follows from the inequality (2.1) in the Theorem 2.1, for $f:(0,\infty)\to\mathbb{R},\ f(x)=\frac{x^{p+2}}{(p+1)(p+2)}$. □

Corollary 3.2. *Taking $n=1$ in Proposition 3.1, the following inequality is obtained:*

$$\begin{aligned}
&\frac{1}{(p+1)(p+2)}\left|A\left(\left(A_{1,1}\right)^{p+2},\left(A_{1,2}\right)^{p+2}\right)-L_{p+2}^{p+2}(a,b)\right| \\
&\leq \frac{(b-a)^2}{2}\frac{\left(A_{1,1}\right)^p\left(A_{1,2}\right)^p}{\left[\left(A_{1,2}\right)^p-\left(A_{1,1}\right)^p\right]^2}\left[A\left(\left(A_{1,1}\right)^p,\left(A_{1,2}\right)^p\right)-\frac{a^pb^p}{L(a^p,b^p)}\right],
\end{aligned}$$

that is,

$$\frac{1}{(p+1)(p+2)}\left|A\left(a^{p+1},b^{p+1}\right)-L_{p+2}^{p+2}(a,b)\right| \leq \frac{(b-a)^2}{2}\frac{a^pb^p}{\left[a^p-b^p\right]^2}\left[A\left(a^p,b^p\right)-\frac{G^{2p}(ab)}{L(a^p,b^p)}\right].$$

Proposition 3.3. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\in(-1,0)$. Then, the following inequality is obtained:*

$$\frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)}\left|\sum_{k=1}^{n}\frac{1}{n}A\left(\left(A_{n,k}\right)^{\frac{m}{q}+2},\left(A_{n,k+1}\right)^{\frac{m}{q}+2}\right)-L_{\frac{m}{q}+2}^{\frac{m}{q}+2}(a,b)\right| \leq \sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\frac{\left[B(p+1,p+1)\right]^{\frac{1}{p}}G^{\frac{2m}{q}}\left(\left(A_{n,k}\right),\left(A_{n,k+1}\right)\right)}{L^{\frac{1}{q}}\left(\left(A_{n,k}\right)^m,\left(A_{n,k+1}\right)^m\right)}.$$

Proof. The assertion follows from the inequality (2.6) in the Theorem 2.4. Let

$$f(x)=\frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)}x^{\frac{m}{q}+2},\quad x\in(0,\infty).$$

Then

$$\left|f''(x)\right|^q=x^m$$

is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.4. □

Corollary 3.4. *Taking $n=1$ in Proposition 3.3, the following inequality is obtained:*

$$\frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)}\left|A\left(a^{\frac{m}{q}+2},b^{\frac{m}{q}+2}\right)-L_{\frac{m}{q}+2}^{\frac{m}{q}+2}(a,b)\right| \leq \frac{(b-a)^2}{2}\frac{\left[B(p+1,p+1)\right]^{\frac{1}{p}}G^{\frac{2m}{q}}(a,b)}{L^{\frac{1}{q}}(a^m,b^m)}.$$

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