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Regularity of Linear Systems of Differential Equations on the Axes and Pencils of Quadratic Forms

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Abstract
It is considered linear systems of differential equations and investigated questions of regularity of these systems. To explore the regularity it is comfortable to use quadratic form whose derivative with respect to the adjoint system is positive definite. Sometimes it is possible to find such a quadratic form, the derivative of which with respect to the system is non-negative. There are examples showing that in this case we can't say anything about the exponential dichotomy of this system (that is, its regularity). The question arises whether it is possible to combine a certain set of quadratic forms to get such a form, the derivative of which with respect to the system is positive definite. This question is similar to the question that arises in the theory of control: having a set of certain data about an object, can one say something about this object as a whole. It turns out that this is possible, only a set of these quadratic forms should be special, in some sense complete. In the presented article the authors propose to write it with the help of some combination of specific symmetric matrices $S_1, S_2, \ldots$. So we have a quadratic form

$$V_p = p_1 \langle S_1(t)x, x\rangle + p_2 \langle S_2(t)x, x\rangle + \cdots + p_{k-1} \langle S_{k-1}(t)x, x\rangle + \langle S_k(t)x, x\rangle$$

It is proved that the derivative of this quadratic form is positive definite for sufficiently large parameters $p_1, \ldots, p_{k-1}$. The results are illustrated by examples.

Keywords: Adjoint system, Quadratic form, Regular, Symmetric matrices, Weakly regular
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1. Introduction

In many interesting investigations [1]-[4] it is arisen linear systems of differential equations in which we have to find the strong properties, i.e. such properties which arent changed under small perturbations. Such properties often are exponential dichotomy and trichotomy of the solutions of linear systems of differential equations. As for non-stationary systems this question is opened it is interesting to find something new in investigation of dichotomy of linear systems of differential equations. The investigation of sets of quadratic forms is a promising and relevant topic, since they provide an opportunity to answer the question of the magnitude of perturbation, which does not disturb the property of the regularity of linear systems. Consequently, we consider certain classes of systems of linear equations and try to find quadratic forms that will enable us to investigate these systems.

2. Main results

Let's consider the homogeneous system of differential equations

$$\frac{dx}{dt} = P(t)x, \tag{2.1}$$

where $x \in R^n$ and $P(t)$ is $n \times n$-dimensional matrix with scalar functions whose elements are real continuous and bounded on $R = (-\infty, +\infty)$. We will denote by $C^0(R)$ the space of functions which are continuous and bounded on R and by $C^1(R)$ – the subspace of space $C^0(R)$ of continuously differentiable functions with bounded derivative on R.

We will consider a norm of a vector $x \in R^n$ as $\|x\| = \sqrt{\langle x,x\rangle}$, where $\langle x,y\rangle = \sum_{j=1}^n x_j y_j$ – scalar product in R^n. And we will denote a norm of a matrix A as $\|A\| = \max\|Ax\|$, $\|x\| = 1$, $\|A\|_0 = \sup\|A(t)\|$, $t \in R$.

The important question about system (2.1) is its regularity on entire axis R. It is known the following definition of regularity [1]:

Definition 2.1. *The system (2.1) is called regular on R if corresponding non-homogeneous system $\frac{dx}{dt} = P(t)x + f(t)$ has unique bounded solution on R with any fixed vector function $f(t) \in C^0(R)$. If it is only known that such system has at least one solution bounded on R with any $f(t) \in C^0(R)$ then the system (2.1) is called weakly regular on R.*

It is known that the system (2.1) is regular on R if and only if there exists a quadratic form $V = \langle S(t)x, x\rangle$ where $S(t) \in C^1(R)$ – symmetric matrix whose derivative with respect to the system (2.1) is positive definite, i.e.

$$\dot{V} = \left\langle \left[\frac{dS(t)}{dt} + S(t)P(t) + P^T(t)S(t)\right]x, x\right\rangle \geq \|x\|^2 \tag{2.2}$$

and wherein the matrix $S(t)$ is non-degenerated for any $t \in R$

$$\det S(t) \neq 0 \qquad \forall t \in R. \tag{2.3}$$

In case the matrix $P(t)$ from the system (2.1) is a constant, from a weak regularity always follows the regularity. It can be if and only if real parts of all eigenvalues of matrix P are non-zeroes. Therefore, if $\det P = 0$ then the system (2.1) with constant matrix P is not regular. It turns out that there exists variable matrix $P(t)$ such that $\det P(t) \equiv 0$ $\forall t \in R$ but the system (2.1) is regular on R. The examples of such systems are:

$$\begin{cases} \frac{dx_1}{dt} = x_1(p_1\cos 2\omega t + p_2 \sin 2\omega t) + x_2(-p_2\cos 2\omega t + p_1 \sin 2\omega t - \omega), \\ \frac{dx_2}{dt} = x_1(-p_2\cos 2\omega t + p_1\sin 2\omega t + \omega) - x_2(p_1 \cos 2\omega t + p_2 \sin 2\omega t), \end{cases}$$

where parameters $p_1, p_2, \omega \in R$ are non-zero, real and $p_1^2 + p_2^2 = \omega^2$.

In this case the derivative of non-degenerated quadratic form

$$V = x_1^2 \cos 2\omega t + 2x_1x_2 \sin 2\omega t - x_2^2 \cos 2\omega t$$

with respect to this system equals $\dot{V} = 2p_1\left(x_1^2 + x_2^2\right)$.

Remark 2.2. *For some systems (2.1) there exists symmetric matrices $S(t) \in C^1(R)$ which satisfy inequality (2.2) but the condition (2.3) is not satisfied. Then the system (2.1) is not regular but adjoint system $\frac{dx}{dt} = -P^T(t)x$ is weakly regular.*

Linear operator $S(t) \in C^1(R)$ which affects on symmetric matrices we will denote $L[S]$:

$$L[S] = \frac{dS(t)}{dt} + S(t)P(t) + P^T(t)S(t) \tag{2.4}$$

Remark 2.3. *If instead of the inequality (2.2) we write $\langle L[S]x, x\rangle \geq \|Nx\|^2$, where N – some constant non-generated matrix, then we can't say anything about regularity of the system (2.1). We can see this from the example:*

$$\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = 0.$$

The derivative of quadratic form $V = x_1x_2$ with respect to this system is $\dot{V} = x_2^2$, but this system is not regular.

The question arises: If we have not a single matrix S, but some set of matrices $S_1, S_2, \ldots$, then is it possible to combine these matrices to construct the matrix S that satisfy the inequality $\langle L[S]x, x\rangle \geq \|x\|^2$? This article is devoted to investigating this question.

Theorem 2.4. *Let there exists two matrices $S_1(t), S_2(t) \in C^1(R)$ which satisfy the following inequalities*

$$\begin{cases} \langle L[S_1]Mx, Mx\rangle \geq \varepsilon_1 \|(M-N)x\|^2, \\ \langle L[S_2]Nx, Nx\rangle \geq \varepsilon_2 \|Nx\|^2 \quad , \quad \varepsilon_1, \varepsilon_2 = const > 0 \quad , \end{cases} \tag{2.5}$$

for some constant matrices M, N. Then the sum of these matrices $\bar{S} = pS_1(t) + S_2(t)$ satisfies an inequality

$$\left\langle L\left[\bar{S}\right]Mx, Mx\right\rangle \geq \sigma(p)\|Mx\|^2, \tag{2.6}$$

where $\sigma(p) = \frac{(p-\alpha\varepsilon_1)\varepsilon_2-\alpha^2}{2(p-\alpha\varepsilon_1+\varepsilon_2)}$, $(p-\alpha\varepsilon_1)\varepsilon_2 - \alpha^2 > 0$, constant α is chosen from inequality $\|L[S_2]\| \leq \alpha$.

Proof. Taking into account the linearity of the operator (2.4), we can write the left side of the inequality (2.6) in the following form:

$$\left\langle L\left[\bar{S}\right]Mx, Mx\right\rangle = p\langle L[S_1]Mx, Mx\rangle + \langle L[S_2]Mx, Mx\rangle = p\langle L[S_1]Mx, Mx\rangle + \langle L[S_2]Nx, Nx\rangle + Q, \tag{2.7}$$

where

$$\begin{aligned} Q &= \langle L[S_2]Mx, Mx\rangle - \langle L[S_2]Nx, Nx\rangle \\ &= \langle L[S_2]Mx, Mx\rangle - \langle L[S_2]Mx, Nx\rangle + \langle L[S_2]Mx, Nx\rangle - \langle L[S_2]Nx, Nx\rangle \\ &= \langle L[S_2]Mx, (M-N)x\rangle + \langle L[S_2]Nx, (M-N)x\rangle \\ &= \langle L[S_2](M-N+N)x, (M-N)x\rangle + \langle L[S_2]Nx, (M-N)x\rangle \\ &= \langle L[S_2](M-N)x, (M-N)x\rangle + 2\langle L[S_2]Nx, (M-N)x\rangle. \end{aligned}$$

From this we obtain an estimation from below

$$Q \geq -\alpha\|(M-N)x\|^2 - 2\alpha\|Nx\|\|(M-N)x\|^2. \tag{2.8}$$

Therefore using inequalities (2.5) and (2.8) from inequality (2.7), we get

$$\left\langle L\left[\bar{S}\right]Mx, Mx\right\rangle \geq (p-\alpha\varepsilon_1)\|(M-N)x\|^2 - 2\alpha\|Nx\|\|(M-N)x\| + \varepsilon_2\|Nx\|^2. \tag{2.9}$$

Let us write the quadratic form corresponding to the right side of the inequality (2.9)

$$V(x_1, x_2) = (p-\alpha\varepsilon_1)x_1^2 - 2\alpha x_1 x_2 + \varepsilon_2 x_2^2.$$

We should find its lowest value on a single circle $x_1 = \cos y$, $x_2 = \sin y$. We obtain

$$V(\cos y, \sin y) = (p-\alpha\varepsilon_1)\frac{1+\cos 2y}{2} - \alpha\sin 2y + \varepsilon_2\frac{1-\cos 2y}{2} \geq \frac{p-\alpha\varepsilon_1+\varepsilon_2}{2} - \sqrt{\left(\frac{p-\alpha\varepsilon_1-\varepsilon_2}{2}\right)^2 + \alpha^2} = \frac{(p-\alpha\varepsilon_1)\varepsilon_2-\alpha^2}{\frac{p-\alpha\varepsilon_1+\varepsilon_2}{2} + \sqrt{\left(\frac{p-\alpha\varepsilon_1-\varepsilon_2}{2}\right)^2 + \alpha^2}}.$$

Choosing sufficiently large the value of parameter $p > 0$, exactly $p > \alpha\varepsilon_1 + \frac{\alpha^2}{\varepsilon_2}$, we get

$$V(x_1, x_2) \geq \frac{(p-\alpha\varepsilon_1)\varepsilon_2 - \alpha^2}{p-\alpha\varepsilon_1+\varepsilon_2}\left(x_1^2 + x_2^2\right).$$

Therefore, from (2.9) we obtain

$$\left\langle L\left[\bar{S}\right]Mx, Mx\right\rangle \geq \frac{(p-\alpha\varepsilon_1)\varepsilon_2 - \alpha^2}{p-\alpha\varepsilon_1+\varepsilon_2}\left(\|(M-N)x\|^2 + \|Nx\|^2\right). \tag{2.10}$$

As for any matrices M and N of equal dimension the inequality

$$\|(M-N)x\|^2 + \|Nx\|^2 \geq \frac{1}{2}\|Mx\|^2$$

is fulfilled then from (2.10) we get

$$\left\langle L\left[\bar{S}\right]Mx,Mx\right\rangle \geq \frac{(p-\alpha\varepsilon_1)\,\varepsilon_2-\alpha^2}{2\,(p-\alpha\varepsilon_1+\varepsilon_2)}\,\|Mx\|^2\,.$$

□

Theorem 2.5. *Let there exists symmetric matrices $S_j(t)\in C^1(R)$, $j=\overline{1,k}$, $n\times n$-dimensional and they satisfy the following inequalities:*

$$\left\langle L[S_j]\,M_j(t)\,x,M_j(t)\,x\right\rangle \geq \left\|\left[M_j(t)-M_{j+1}(t)\right]x\right\|^2\quad,\quad j=\overline{1,(k-1)} \tag{2.11}$$

$$\langle L[S_k]\,M_k(t)\,x,M_k(t)\,x\rangle \geq \|M_k(t)\,x\|^2\,, \tag{2.12}$$

with some $n\times n$- dimensional continuous non-degenerated matrices $M_j(t)$. Then the derivative of a quadratic form

$$V_p=p_1\langle S_1(t)\,x,x\rangle+p_2\langle S_2(t)\,x,x\rangle+\cdots+p_{k-1}\langle S_{k-1}(t)\,x,x\rangle+\langle S_k(t)\,x,x\rangle$$

with respect to the system (2.1) will be positive definite for sufficiently large fixed values of parameters $p_1,p_2,\ldots,p_{k-1}$.

Proof. Let us choose and fix the constant α, that satisfy an inequalities $\left\|L[S_j]\right\|\leq\alpha$, $j=\overline{1,k}$. From the last of the inequalities (2.11)

$$\langle L[S_{k-1}]\,M_{k-1}(t)\,x,M_{k-1}(t)\,x\rangle \geq \|[M_{k-1}(t)-M_k(t)]\,x\|^2$$

and (2.12) using Theorem 2.4 with $\varepsilon_1=1$, $\varepsilon_2=1$ we get an inequality

$$\left\langle L\left[\tilde{S}\right]M_{k-1}(t)\,x,M_{k-1}(t)\,x\right\rangle \geq \sigma(p_{k-1})\,\|M_{k-1}(t)\,x\|^2\,, \tag{2.13}$$

where

$$\tilde{S}=p_{k-1}S_{k-1}(t)+S_k(t)\,,\sigma(p_{k-1})=\frac{p_{k-1}-\alpha-\alpha^2}{2\,(p_{k-1}-\alpha+1)}.$$

Then let us consider the penultimate of the inequalities (2.11)

$$\langle L[S_{k-2}]\,M_{k-2}(t)\,x,M_{k-2}(t)\,x\rangle \geq \|[M_{k-2}(t)-M_{k-1}(t)]\,x\|^2\,.$$

Together with the inequality (2.13), based on the Theorem 2.4 ($\varepsilon_1=1$, $\varepsilon_2=\sigma(p_{k-1})$) for the sum of the matrices

$$\bar{S}=p_{k-2}S_{k-2}(t)+\tilde{S}=p_{k-2}S_{k-2}(t)+p_{k-1}S_{k-1}(t)+S_k(t)\,,$$

we get an inequality

$$\left\langle L\left[\bar{S}\right]M_{k-2}(t)\,x,M_{k-2}(t)\,x\right\rangle \geq \sigma(p_{k-2},p_{k-1})\cdot\|M_{k-2}(t)\,x\|^2\,,$$

where

$$\sigma(p_{k-2},p_{k-1})=\frac{(p_{k-2}-\alpha)\,\sigma(p_{k-1})-\alpha^2}{2\,(p_{k-2}-\alpha+\sigma(p_{k-1}))}.$$

So we have the following estimation

$$\left\langle L\left[\widehat{S}\right]M_{k-3}(t)\,x,M_{k-3}(t)\,x\right\rangle \geq \sigma(p_{k-3},p_{k-2},p_{k-1})\,\|M_{k-3}(t)\,x\|^2\,,$$

$$\sigma(p_{k-3},p_{k-2},p_{k-1})=\frac{(p_{k-3}-\alpha)\,\sigma(p_{k-2},p_{k-1})-\alpha^2}{2\,(p_{k-2}-\alpha+\sigma(p_{k-2},p_{k-1}))}.$$

Continuing to receive similar estimates, in the end for the sum of the matrices

$$S_p = p_1 S_1(t) + p_2 S_2(t) + \cdots + p_{k-1} S_{k-1}(t) + S_k(t),$$

we will get

$$\left\langle L[S_p] M_1(t) x, M_1(t) x \right\rangle \geq \sigma(p_1, p_2, .., p_{k-2}, p_{k-1}) \left\| M_1(t) x \right\|^2, \tag{2.14}$$

where

$$\sigma(p_1, p_2, .., p_{k-2}, p_{k-1}) = \frac{(p_1 - \alpha)\,\sigma(p_2, .., p_{k-2}, p_{k-1}) - \alpha^2}{2\,(p_1 - \alpha + \sigma(p_2, .., p_{k-2}, p_{k-1}))}.$$

Let us denote $M_1(t)x = y$. Since the matrix $M_1(t)$ is non-degenerate then we get $\left\langle L[S_p] y, y \right\rangle \geq \sigma(p_1, p_2, .., p_{k-2}, p_{k-1}) \|y\|^2$ from the inequality (2.14) for any $y \in R^n$ with positive coefficient $\sigma(p_1, p_2, .., p_{k-2}, p_{k-1})$. That means the derivative of the quadratic form V_p with respect to the system (2.1) at certain choices of the vector of parameters $(p_1, p_2, .., p_{k-2}, p_{k-1}) = p$ will be positive definite. The Theorem 2.5 is proved. □

Let us consider an example of the application of the proved theorem.

Denote

$$a(t;\lambda) = \frac{\lambda e^{-t} - (1-\lambda) e^{t}}{\lambda e^{-t} + (1-\lambda) e^{t}}, 0 \leq \lambda \leq 1$$

and consider the system

$$\begin{cases} \frac{dx_1}{dt} = [a(t;\lambda_1) + a(t;\lambda_2) - 1] x_1 + [a(t;\lambda_1) + a(t;\lambda_2)] x_2, \\ \frac{dx_2}{dt} = [-a(t;\lambda_2) + 1] x_1 - a(t;\lambda_2) x_2, \\ \frac{dx_3}{dt} = [a(t;\lambda_2) + 1] x_1 - [a(t;\lambda_1) + a(t;\lambda_2)] x_2 - a(t;\lambda_1) x_3, \end{cases} \tag{2.15}$$

where λ_1, λ_2 – independent parameters from the closed segment $[0,1]$.

We choose the matrices S_i, M_i in the following form

$$S_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a(t;\lambda_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a(t;\lambda_1) \end{pmatrix},$$
$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculating the left sides of the inequalities (2.11) and (2.12) $(k=3)$, we get

$$L[S_1] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L[S_2] = \begin{pmatrix} 0 & \left(a^2(t;\lambda_2) - a(t;\lambda_2)\right) & 0 \\ \left(a^2(t;\lambda_2) - a(t;\lambda_2)\right) & \left(2a^2(t;\lambda_2) - \frac{da(t;\lambda_2)}{dt}\right) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L[S_3] = \begin{pmatrix} 0 & 0 & -a(t;\lambda_1)[a(t;\lambda_2)+1] \\ 0 & 0 & a(t;\lambda_1)[a(t;\lambda_1)+a(t;\lambda_2)] \\ -a(t;\lambda_1)[a(t;\lambda_2)+1] & a(t;\lambda_1)[a(t;\lambda_1)+a(t;\lambda_2)] & \left(2a^2(t;\lambda_1) - \frac{da(t;\lambda_1)}{dt}\right) \end{pmatrix}.$$

Hence it is already clear that the inequalities (2.11) and (2.12) are fulfilled. Thus, the derivative of the quadratic form $p_1(x_1x_2 + x_1x_3 + x_2x_3) - p_2 x_2^2 a(t;\lambda_2) - x_3^2 a(t;\lambda_1)$ with respect to the system (2.15) with the appropriate choice of parameters p_1, p_2 will be positively defined. This implies that the system (2.15) is regular on R.

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Delay Differential Equations in Sequence Spaces

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Abstract
The standard delay equations are newly studied in the context of classical separable Banach Sequence Spaces. As a classical solution is shown to exist, the associated semigroup and its infinitesimal generator are found, and some important properties of those operators are proven, including some spectral properties. As an application, it is shown how can these results be used to characterize the constrained null-controllability.

Keywords: Delay differential equations, Exact controllability, Sequence spaces
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1. Statement of the problem

Consider the l_p spaces for $1 < p < \infty$ consisting of all absolutely p-power summable scalar sequences, with the p-norm. Consider also the subspace of all null scalar sequences c_0 with the supremum norm.

Let X be either $l_p, 1 \leq p < \infty$ or c_0. Let also be $B(X)$ the space of all bounded linear operators on X. Consider the following equation

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{n} A_i x(t-h_i), \quad t \geq 0 \\ x(0) &= r, \\ x(\theta) &= f(\theta), \quad -h_n \leq \theta < 0, \end{aligned} \tag{1.1}$$

where $0 < h_1 < \ldots < h_n$ are the delaying points, $x(t) \in X$ for $t > 0$, $A_i \in B(X),\ i = 0, \ldots, n$ and $f : [-h_n, 0] \to X$ must also satisfy $f(0) = x(0) = r$ (a fixed vector in X), and $f(\theta) \neq 0$ for every θ such that $-h_n \leq \theta < 0$. Here, the convergence is in the norm of X, ie, $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$ if and only if $\|x_n - x\| \to 0$ as $n \to \infty$, where $\|.\|$ stands for the p-norm or the supremum norm, respectively.

The fundamental concepts of derivative and integral for scalar functions of a single variable can be extended to a function $F : [0, \infty) \to X$. We simply express F as a function of its components and do the calculus operations on those components, i.e., if $F(t) = \{f_i(t)\}_{i=1}^{\infty}$, we have $F'(t) = \{f_i'(t)\}_{i=1}^{\infty}$ and $\int_a^b F(t)\,dt = \left\{\int_a^b f_i(t)\,dt\right\}_{i=1}^{\infty}$.

In view of these definitions, it is easily checked that the basic theorems about continuity, differentiability and integrability are also valid in this case. Using standard arguments, it can also be proven that

$$\left|\left|\int_0^t x(s)\,ds\right|\right|_X \leq \int_0^t ||x(s)||_X\,ds.$$

We also have, as usual, $e^{At} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$ for $A \in B(X)$.

Note that the functions $x \in X$ we are working with should satisfy

i) $x(t) \in X$ for every $t \geq 0$.

ii) $x'(t) \in X$ for every $t \geq 0$.

iii) $g(t) = \int_a^t x(s)\,ds \in X$ for each fixed $t \geq 0$ and every $t \geq 0$.

The mapping $x(t) = \left\{ \frac{e^{\lambda t}}{i^q} \right\}_{i=1}^{\infty}$, where $\lambda \in \mathbb{C}$ and $q \geq \frac{1}{p}$ is a example of this. More generally, the same is true for $y(t) = \{g(t)\,a_i\}_{i=1}^{\infty}$, where $\{a_i\}_{i=1}^{\infty} \in X$ and g is differentiable function on $\mathbb{R}$.

In the next pages we will show that (1.1) can be rewritten as an abstract differential equation of the form

$$\begin{aligned} \dot{z}(t) &= A\,z(t) \\ z(0) &= z_0, \end{aligned} \tag{1.2}$$

where A is the infinitesimal generator of a c_0-semigroup $\{T_t\}_{t\geq 0}$ on a suitable Banach space, and we will prove some important properties of $\{T_t\}_{t\geq 0}$ and A (including some spectral properties). Finally, as an application, we will characterize the null-controllability by using some techniques from Functional Analysis and Operator Theory. The control u is constrained to lie in a separable weakly compact subset Ω of an arbitrary Banach space U.

2. Main results

In the following we will prove a standard formula for the solution of (1.1). Then, we will introduce the c_0-semigroup $\{T_t\}_{t\geq 0}$ associated to (1.1), and its infinitesimal generator A.

Theorem 2.1. *Consider the retarded differential equation (1.1). For every $r \in X$ and every $f : [-h_p, 0] \to X$ there exists a unique function x from $(0,\infty)$ to X that is absolutely continuous and satisfies the differential equation (1.1) almost everywhere. This function is called the solution of (1.1) and it satisfies*

$$x(t) = e^{A_0 t} r + \sum_{i=1}^{n} \int_0^t e^{A_0(t-s)} A_i x(s-h_i)\,ds, \text{ for } t \geq 0. \tag{2.1}$$

Proof. Notice first that, for $t \in (0, h_i)$ the term $\sum_{i=1}^{n} A_i x(t-h_i)$ equals the known function $v(t) := \sum_{i=1}^{n} A_i f(t-h_i)$. So we may reformulate the equation (1.1) on $[0, h_i]$ as

$$\begin{aligned} \dot{x}(t) &= A_0\,x(t) + v(t) \\ x(0) &= r. \end{aligned} \tag{2.2}$$

Now, we can proceed coordinatewise an apply finite dimensional theory on each coordinate. We thus find that the solution of (2.2) is given by

$$x(t) = e^{A_0 t} f + \int_0^t e^{A_0(t-s)} v(s)\,ds$$

and this equals (2.1).

Now, we will consider the case $t \geq h_1$. At a given time t, the past is known and so the delayed part $\sum_{i=1}^{n} A_i x(t-h_i)$ is also a known function. Using the same argument as before we conclude that the solution of (1.1) is unique and it satisfies (2.1). □

Lemma 2.2. *If $x(t)$ is the solution of (1.1), then the following inequalities hold*

a) $||x(t)|| \leq C_t[||r|| + ||f(\cdot)||]$,

b) $\int_0^1 ||x(t)||^q dt \leq D_t[||r||^q + ||f(\cdot)||^q]$,

where $1 \leq q < \infty$ and C_t and D_t are constants, depending only on t.

Proof. We know that for some positive constants M_0 and $\omega_0, e^{A_0 t}$ satisfies

$$||e^{A_0 t}|| \leq M_0\, e^{\omega_0 t},\ t \geq 0.$$

Define the positive constant M by $M := \max\left(||A_1||, \ldots, ||A_n|, M_0|\right)$. Then, from (2.1) it follows that

$$\begin{aligned} ||x(t)|| &\leq ||e^{A_0 t} r|| + ||\sum_{i=1}^{n} \int_0^t e^{A_0(t-s)} A_i x(s-h_i)\, ds|| \\ &\leq Me^{\omega_0 t}||r|| + \sum_{i=1}^{n} \int_0^t M\, e^{\omega_0(t-s)} M||x(s-h_i)||\, ds \\ &= Me^{\omega_0 t}||r|| + \sum_{i=1}^{n} M^2 \int_{-h_i}^{t-h_i} e^{\omega_0(t-\bar{t}-h_i)}||x(\bar{t})||\, d\bar{t} \\ &= Me^{\omega_0 t}||r|| + M^2 e^{\omega_0 t} \sum_{i=1}^{n} \int_{-h_i}^{t-h_i} e^{-\omega_0(\bar{t}+h_i)}||x(\bar{t})||\, d\bar{t}. \end{aligned} \tag{2.3}$$

We now establish the following inequalities for the last term of (2.3)

$$\begin{aligned} \sum_{i=1}^{n} \int_{-h_i}^{t-h_i} e^{-\omega_0(\bar{t}+h_i)}||x(\bar{t})||\, d\bar{t} &\leq \sum_{i=1}^{n} \int_{-h_i}^{0} e^{-\omega_0(\bar{t}+h_i)}||x(\bar{t})||\, d\bar{t} + \sum_{i=1}^{n} \int_0^t e^{-\omega_0(\bar{t}+h_i)}||x(\bar{t})||\, d\bar{t} \\ &\leq \sum_{i=1}^{n} \int_{-h_i}^{0} ||f(\bar{t})||\, d\bar{t} + \sum_{i=1}^{n} \int_0^t e^{-\omega_0 \bar{t}}||x(\bar{t})||\, d\bar{t} \text{ since } \omega_0 > 0 \\ &\leq \sum_{i=1}^{n} \int_{-h_r} ||f(\bar{t})||\, d\bar{t} + \sum_{i=1}^{n} \int_0^t e^{-\omega_0 \bar{t}}||x(\bar{t})||\, d\bar{t}. \end{aligned}$$

Now, let us fix $\theta \in [-h_n, 0)$ and let $g : [-h_n, 0] \to \mathbb{R}$ be defined by $g(\bar{t}) = \dfrac{||f(\bar{t})||}{||f(\theta)||}$. Since g is a continuous function over the compact set $[-h_n, 0]$, there exists $k > 0$ such that $||f(\bar{t})|| \leq k||f(\theta)||$ for all $\bar{t} \in [-h_n, 0]$.

From this we deduce that the former equation is lesser or equal than

$$n\, h_n\, Q||f(\cdot)|| + n \int_0^t e^{-\omega_0 \bar{t}}||x(\bar{t})||\, d\bar{t}. \tag{2.4}$$

Comparing equations (2.3) and (2.4) gives

$$||x(t)|| \leq e^{\omega_0\, t} \left[M||r|| + M^2\, n\, h_n\, Q\, ||f(\cdot)|| + M^2\, n \int_0^t e^{-\omega_0 \bar{t}}||x(\bar{t})||\, d\bar{t} \right]. \tag{2.5}$$

Setting $\alpha = M||r|| + M^2\, n\, h_n\, Q\, ||f(\cdot)||$, $\beta = n M^2$ and $y(t) = e^{-\omega_0 \bar{t}}||x(\bar{t})||$, we can reformulate (2.5) as

$$y(t) \leq \alpha + \beta \int_0^t y(\bar{t})\, d\bar{t}.$$

From Gronwall's Lemma we conclude that $y(t) \leq \alpha\, e^{\beta\, t}$. So we have

$$\begin{aligned} ||x(t)|| &\leq \alpha\, e^{(\beta+\omega_0)\, t} \\ &= e^{(\rho\, M^2+\omega_0)\, t} \left[M||r|| + M^2\, Q\, n\, h_n\, ||f(\cdot)|| \right] \\ &\leq e^{(\rho\, M^2+\omega_0)\, t} \max\left[M, M^2\, n\, h_n\, Q \right] \left[||r|| + ||f(\cdot)|| \right], \end{aligned}$$

which proves a).

It follows, from the above inequality, that

$$||x(t)||^q \leq e^{q\,(\rho\, M^2+\omega_0)\, t} \max\left[M, M^2\, Q\, n\, h_n \right]^q \left[||r|| + ||f(\cdot)|| \right]^q.$$

Let us now suppose $r \neq 0$. On the whole compact set $[-h_n, 0]$ we define

$$w(\theta) = \frac{(||r|| + ||f(\theta)||)^q}{||r||^q + ||f(\theta)||^q},$$

w is a continuous function over $[-h_n, 0]$. Then there exists $K > 0$ such that $w(\theta) \leq K$. This, in particular, is valid for $\theta \in [-h_n, 0)$ and thus, we have

$$(||r|| + ||f(\theta)||)^q \leq K\,(||r||^q + ||f(\theta)||^q)\ \theta \in [-h_n, 0).$$

Consequently

$$||x(t)||^q \leq K\, e^{q\,(\rho M^2 + \omega_0)\,t} \max\left[M, M^2 Q\, n\, h_n\right]^q [||r|| + ||f(\cdot)||]^q .$$

If $r = 0$, then $(||r|| + ||f(\cdot)||)^q = (||r||^q + ||f(\cdot)||^q)$ and we have the same estimation for $||x(t)||^q$. Integrating this inequality gives $b)$. □

Now we shall introduce the semigroup related to (1.1). Consider the space $X \oplus X$ with the usual norm $||(x_1, x_2)||_{X\oplus X} = ||x_1||_X + ||x_2||_X$.

It should be noted that l_p, $1 \leq p < \infty$ and c_0 are **prime** Banach spaces, i.e, every infinite-dimensional complemented subspace of X is isomorphic to X. From this we can deduce that $X \oplus X$ is isomorphic to X and thus, the norm $||(\cdot,\cdot)||_{X\oplus X}$ is, in fact, equivalent to $||\cdot||_X$ (see, for example,[1]).

We define the following family of operators on $X \oplus X$ for $t \geq 0$ by

$$T(t) \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} := \begin{pmatrix} x(t) \\ x(t+\cdot) \end{pmatrix}, \tag{2.6}$$

where $x(\cdot)$ is the solution of (1.1) and $x(-s) = f(-s)$ for $h_p > s > 0$.

Theorem 2.3. *The family $\{T(t)\}_{t\geq 0}$ defined by (2.6) satisfies:*

a) $T(t) \in B(X \oplus X)$ *for all* $t \geq 0$.

b) $\{T(t)\}_{t\geq 0}$ *is a* c_0*-semigroup on* $X \oplus X$

Proof. The linearity of $T(t)$ follows easily from the linearity of (1.1) and the uniqueness of its solution. We will now prove that $T(t)$ is a bounded operator.

We can suppose that x is not constantly equal to zero (otherwise the result is trivial) and let us choose t_0 such that $x(t_0) \neq 0$. For each t, let $M_t = \sup\limits_{\bar{t}\in[-h_p,0]} ||x(t+\bar{t})||_X$. Then, we have

$$||x(t+\cdot)||_X := \frac{||x(t+\cdot)||_X}{||x(t_0)||_X} ||x(t_0)||_X \leq D_t\,[||r||_X + ||f(\cdot)||_X],$$

where $D_t = \dfrac{M_t}{||x(t_0)||} C_{t_0}$ (C_{t_0} as in the previous lemma), and so

$$\left|\left| T(t) \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \right|\right|_{X\oplus X} = ||x(t)||_X + ||x(t+\cdot)||_X \leq (C_t + D_t)\,(||r||_X + ||f(\cdot)||_X).$$

The semigroup property can be proven exactly as in Theorem 2.4.4 of [2].

It only remains to prove the strong continuity. For $t < h_1$ we have

$$\left|\left| T(t) \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} - \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \right|\right| = \left|\left| e^{A_0 t} r + \sum_{i=1}^{p} \int_0^t e^{A_0(t-s)} A_i f(s-h_i)\, ds - r \right|\right|_X + ||x(t+\cdot) - f(\cdot)||_X .$$

The first term converges to zero as $t \to 0$ since

$$e^{A_0 t} r + \sum_{i=1}^{p} \int_0^t e^{A_0(t-s)} A_i f(s-h_i)\, ds$$

is continuous. Let us now prove that $||x(t+\cdot)-f(\cdot)||_X$ tends to zero as $t\to 0$. We first suppose $X=l_p$, $1\le p<\infty$. We thus have

$$\begin{aligned}
||x(t+\cdot)-f(\cdot)||_X^p &\le (||x(t+\cdot)||_X+||f(\cdot)||_X)^p\\
&\le \left(\sup_{t\in[0,h_1]}||x(t+\cdot)||+||f(\cdot)||\right)^p\\
&\le K(\cdot)\left(\sup_{t\in[0,h_1]}||x(t+\cdot)||^p+||f(\cdot)||^p\right)\\
&= \sum_{i=1}^{\infty}\left[\frac{K(\cdot)\sup_{t\in[0,h_1]}||x(t+\cdot)||^p}{2^i}+K(\cdot)|f_i(\cdot)|^p\right],
\end{aligned}$$

where $K(\cdot)$ is a constant non depending on t. Consequently, the series

$$\sum_{i=1}^{\infty}|x_i(t+\cdot)-f(\cdot)|^p$$

converges uniformly on $[0,h_1]$, according to the classical Weierstrass M Test, and so we have

$$\lim_{t\to 0}\left(\sum_{i=1}^{\infty}|x_i(t+\cdot)-f(\cdot)|^p\right)=\sum_{i=1}^{\infty}\left(\lim_{t\to 0}|x_i(t+\cdot)-f(\cdot)|^p\right)=0.$$

Let us now suppose $X=c_0$, and let $\varepsilon>0$ be given. We use the results for l_p in the particular case $p=1$. Then, there exists $\delta>0$ such that

$$|x_i(t+\cdot)-f(\cdot)|\le\sum_{i=1}^{\infty}|x_i(t+\cdot)-f(\cdot)|<\varepsilon$$

for $t\in[0,h_1)\cap(-\delta,\delta)$ and every $i\in\mathbb{N}$. Consequently we have,

$$\sup_{i\in\mathbb{N}}|x_i(t+\cdot)-f(\cdot)|<\varepsilon,\ \text{for } t\in[0,h_1)\cap(-\delta,\delta).$$

□

The following two results deal with the infinitesimal generator A. We will give a detailed description of A and prove some important properties of it. Bearing in mind this, and the former comments about $\{T(t)\}_{t\ge 0}$, it can be shown, in a classical manner, that (1.1) can be rewritten as (1.2).

Lemma 2.4. *Consider the c_0-semigroup $T(t)$ defined by (2.6) and let A denote its infinitesimal generator. For sufficiently large $\alpha\in\mathbb{R}$, the resolvent is given by*

$$(\alpha I-A)^{-1}\begin{pmatrix} r\\ f(\cdot)\end{pmatrix}=\begin{pmatrix} g(0)\\ g(\cdot)\end{pmatrix}\tag{2.7}$$

where

$$g(\theta)=e^{\alpha\theta}g(0)-\int_0^{\theta}e^{\alpha(\theta-s)}f(s)\,ds,\ \text{for } \theta\in[-h,0]\tag{2.8}$$

and

$$g(0)=[\Delta(\alpha)]^{-1}\left[r+\sum_{i=1}^{n}\int_{-h_i}^{0}e^{-\alpha(\theta+h_i)}A_i f(\theta)\,d\theta\right],\tag{2.9}$$

where

$$\Delta(\lambda)=\left[\lambda I-A_0-\sum_{i=1}^{n}e^{-\lambda h_i}A_i\right],\ \text{for } \lambda\in\mathbb{C}.$$

Furthemore, g satisfies the following relation:

$$\alpha g(0)=r+A_0\,g(0)+\sum_{i=1}^{n}A_i\,g(-h_i).$$

Proof. The proof is essentially the same as in Lemma 2.4.5 of [2]. One should only note the following: c_0 and l_p, $1 \leq p < \infty$ have Schauder bases. For X being either l_p, $1 \leq p < \infty$, or c_0, we have that every bounded linear operator $T : X \to X$ can be written as an infinite matrix M in the usual way. If $\{e_i\}_{i=1}^{\infty}$ is the standard unit vector basis of X, and $T(e_1) = \sum_{k=1}^{\infty} \alpha_{k_1} e_k$, $T(e_2) = \sum_{k=1}^{\infty} \alpha_{k_2} e_k$, etc., then

$$M = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdot & \cdot & \cdot \\ \alpha_{21} & \alpha_{22} & \cdot & \cdot & \cdot \\ \alpha_{31} & \alpha_{32} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

For more details, see [3]. It can also be proven, using standard arguments, that the equalities

$$\begin{aligned} \int_0^{\infty} e^{-\alpha t} A_0 x(t) dt &= A_0 \int_0^{\infty} e^{-\alpha t} x(t) dt, \\ \sum_{i=1}^{n} \int_0^{\infty} e^{-\alpha t} A_i x(t-h_i) dt &= \sum_{i=1}^{n} A_i g(-h_i), \\ \sum_{i=1}^{n} \int_{h_i}^{\infty} e^{-\alpha t} A_i x(t-h_i) dt &= \sum_{i=1}^{n} e^{-\alpha h_i} A_i g(0) \end{aligned}$$

from Lemma 2.4.5 of [2] remain valid for the present case. □

Theorem 2.5. *Consider the c_0-semigroup defined by (2.6). Its infinitesimal generator is given by*

$$A \begin{pmatrix} r \\ \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 r + \sum_{i=1}^{p} A_i f(-h_i) \\ \\ \dfrac{df}{d\theta}(\cdot) \end{pmatrix} \tag{2.10}$$

with domain

$$D(A) = \left\{ \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in X \oplus X : f \text{ is absolutely continuous}, \frac{df}{d\theta}(\cdot) \in X,\, f(0) = r \right\}.$$

Furthermore, the spectrum of A is discreet and is given by

$$\sigma(A) = \sigma_p(A) = \left\{ \lambda \in \mathbb{C} : \Delta(\lambda)^{-1} \text{ does not exist} \right\},$$

where $\Delta(\lambda)$ is defined in the former Lemma.

If $\lambda \in \sigma_p(A)$, then $\begin{pmatrix} r \\ e^{\lambda \cdot} r \end{pmatrix}$, where $r \neq 0$ satisfies $\Delta(\lambda) r = 0$, is an eigenvector of A, with eigenvalue λ. On the other hand, if v is an eigenvector of A with eigenvalue λ, then $v = \begin{pmatrix} r \\ e^{\lambda \cdot} r \end{pmatrix}$ with $\Delta(\lambda) r = 0$.

Proof. Denote by $\tilde{A}$ the operator

$$\tilde{A} \begin{pmatrix} r \\ \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 r + \sum_{i=1}^{p} A_i f(-h_i) \\ \\ \dfrac{df}{d\theta}(\cdot) \end{pmatrix}$$

with domain

$$D(\tilde{A}) = \left\{ \begin{pmatrix} \sigma \\ f(\cdot) \end{pmatrix} \in X \oplus X : f \text{ is absolutely continuos}, \frac{df}{d\theta}(\cdot) \in X,\, f(0) = r \right\}.$$

We have to show that the infinitesimal generator A equals $\tilde{A}$. Let α_0 be a sufficiently larger real number such that the results of the former Lemma hold. If we can show that the inverse of $(\alpha_0 I - \tilde{A})$ equals $(\alpha_0 I - A)^{-1}$, then $A = \tilde{A}$. To this end, we calculate

$$\begin{aligned}
(\alpha_0 I - \tilde{A})(\alpha_0 I - A)^{-1}\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} &= (\alpha_0 I - \tilde{A})\begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix} \text{ with } g \text{ as in the former Lemma} \\
&= \begin{pmatrix} \alpha_0 g(0) - A_0 g(0) + \sum_{i=1}^{p} A_i g(-h_i) \\ \\ \alpha_0 g(\cdot) - \dfrac{dg}{\partial\theta}(\cdot) \end{pmatrix} \\
&= \begin{pmatrix} r \\ \alpha_0 g(\cdot) - \dfrac{dg}{\partial\theta}(\cdot) \end{pmatrix} \text{ from (2.3) of the former Lemma} \\
&= \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \text{ by differentiating (1.2).}
\end{aligned}$$

So for $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in X \oplus X$ we have shown that

$$(\alpha_0 I - \tilde{A})(\alpha_0 I - A)^{-1}\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} r \\ f(\cdot) \end{pmatrix}. \tag{2.11}$$

It remains to show that

$$(\alpha_0 I - A)^{-1}(\alpha_0 I - \tilde{A})\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} r \\ f(\cdot) \end{pmatrix},$$

for $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in D(A)$.

For $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in D(A)$ we define

$$\begin{pmatrix} r_1 \\ f_1(\cdot) \end{pmatrix} := (\alpha_0 I - A)^{-1}(\alpha_0 I - \tilde{A})\begin{pmatrix} r \\ f(\cdot) \end{pmatrix}.$$

Then, from (2.11) we have that

$$(\alpha_0 I - \tilde{A})\begin{pmatrix} r_1 \\ f_1(\cdot) \end{pmatrix} = (\alpha_0 I - \tilde{A})\begin{pmatrix} r \\ f(\cdot) \end{pmatrix}.$$

So $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} r_1 \\ f_1(\cdot) \end{pmatrix}$ if and only if $(\alpha_0 I - \tilde{A})$ is injective. Suppose, on the contrary, that there exists a $\begin{pmatrix} r_0 \\ f_0(\cdot) \end{pmatrix} \in D(A)$ such that

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= (\alpha_0 I - \tilde{A})\begin{pmatrix} r_0 \\ f_0(\cdot) \end{pmatrix} = (\alpha_0 I - \tilde{A})\begin{pmatrix} f_0(0) \\ f_0(\cdot) \end{pmatrix} \\
&= \begin{pmatrix} \alpha_0 f_0(0) - A_0 f_0(0) - \sum_{i=1}^{n} A_i f_0(-h_i) \\ \\ \alpha_0 f_0(\cdot) - \dfrac{df_0}{d\theta}(\cdot) \end{pmatrix},
\end{aligned}$$

where we have used the definitions of $\tilde{A}$ and $D(\tilde{A})$ in the last two steps. Then, working coordinatewise as it has been established, we have

$$\begin{aligned}
f_0(\theta) &= f_0(0)\, e^{\alpha_0\theta} \\
\alpha_0 f_0(0) - A_0 f_0(0) - \sum_{i=1}^{n} A_i f_0(0)\, e^{-\alpha_0 h_i} &= 0.
\end{aligned}$$

Since $\left(\alpha_0 I - A_0 - \sum_{i=1}^n A_i e^{-\alpha_0 h_i}\right)$ is invertible, this implies that $f_0(0) = 0$ and thus $f_0(\cdot) = f_0(0) e^{\alpha_0 \cdot} = 0$. This is a contradiction and thus $(\alpha_0 I - \tilde{A})$ is injective. This proves the assertion that A equals $\tilde{A}$.

It remains to calculate the spectrum of A. In the previous Lemma we obtained the explicit expression (2.7) for the resolvent operator for sufficiently large $\alpha \in \mathbb{R}$ in terms of g given by (2.8) and (2.9). Denote by Q_λ the extension of (2.7) to $\mathbb{C}$

$$Q_\lambda \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} := \begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix}.$$

A simple calculation shows that if $\lambda \in \mathbb{C}$ satisfies $(\lambda I - A_0 - \sum_{i=1}^n A_i e^{-\lambda h_i})$ is invertible then Q_λ is a bounded linear operator from $X \oplus X$ to $X \oplus X$ (working coordinatewise, as ever, we have that each component is continuous). Furthemore, for these λ we have $(\lambda I - A) Q_\lambda = I$ and $(\lambda I - A)$ is injective. So, as in the first part of the proof, we conclude that $Q_\lambda = (\lambda I - A)^{-1}$, the resolvent operator of A. We have that

$$\{\lambda \in \mathbb{C} : (\lambda I - A_0 - \sum_{i=1}^n A_i e^{-\lambda h_i}) \text{ is invertible}\} \subseteq \rho(A).$$

If, on the other hand, $\left(\lambda I - A_0 - \sum_{i=1}^n A_i e^{-\lambda h_i}\right)$ is not invertible, there exists $\xi \in X, \xi \neq 0$, such that the following element of $X \oplus X$: $z_0 = \begin{pmatrix} \xi \\ e^{\lambda \cdot} \xi \end{pmatrix}$ is in $D(A)$ and

$$(\lambda_0 I - \tilde{A}) z_0 = \begin{pmatrix} \lambda \xi - A_0 \xi - \sum_{i=1}^n A_i e^{-h_i \lambda \xi} \\ \\ \lambda e^{\lambda_0} \xi - \dfrac{d}{d\theta} e^{\lambda \theta} \xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So

$$\sigma_p(A) \supset \{\lambda \in \mathbb{C} : (\lambda I - A_0 - \sum_{i=1}^n A_i e^{-\lambda h_i}) \text{ is not invertible}\}.$$

Let $v = \begin{pmatrix} r \\ f(\cdot) \end{pmatrix}$ be an eigenvector of A with eigenvalue λ. From (2.10) we obtain that for $\theta \in [-h_p, 0)$

$$\frac{df}{d\theta}(\theta) = \lambda f(\theta),$$

which gives $f(\theta) = e^{\lambda \theta f(0)}$. Since $v \in D(A)$ we have $f(0) = r$. Using the first equation of (2.10) gives

$$A_0 r + \sum_{i=1}^n A_i e^{-\lambda h_i} r = \lambda r.$$

This shows that $\Delta(\lambda) r = 0$. □

3. An application: Constrained null-controllability

We will now consider a system like the following

$$\begin{aligned} \dot{\omega}(t) &= A \omega(t) + \overline{B} u(t), t > 0 \\ \omega(0) &= \omega_0 = (f(0), f(\cdot)), \end{aligned} \tag{3.1}$$

where A is the infinitesimal generator of the semigroup $\{T(t)\}_{t \geq 0}$, X is as before, U is a Banach space, $B : U \to X$ is a bounded linear operator, $u : [0, \infty) \to U$ is a strongly measurable, essentially bounded function and $\overline{B} u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}$. Note that the homogeneous part of (3.1) is exactly (1.2). On the other hand, the mild solution of (3.1) is given by

$$\omega(t) = T(t) \omega_0 + \int_0^t T(t-s) \overline{B} u(s) \, ds.$$

Let Ω be a non-empty separable weakly compact subset of U, and let $\overline{\Omega}_r$ be defined as follows:

$$\overline{\Omega}_r = \{u \in L_U^\infty[0,r] : u \in \Omega\, a.e\}.$$

$\overline{\Omega}_r$ is called the set of *admissible controls* of (3.1), while the set

$$A_r(\omega_0) = \left\{ T(t)\,\omega_0 + \int_0^r T(r-s)\,\overline{B}\,u(s)\,ds : u \in \overline{\Omega}_r \right\}$$

is the set of *accesible points* of (3.1). The system (3.1) is controllable if $0 \in A_r(\omega_0)$.

The *controllability map* on $[0,r]$ for some $r \geq 0$ is the linear map

$$B^r : L_\infty([0,r];U) \to X \quad \text{defined by} \quad B^r u = \int_0^r T(r-s)\,\overline{B}\,u(s)\,ds.$$

Now, one says that the system is exactly controllable on $[0,r]$ if every point in X can be reached from the origin at r, i.e., if $ran(B^r) = X$.

If $ran(B^r) = X$ then $0 \in A_r(0)$. On the other hand, one can prove, using the Open Mapping Theorem, the following: if $0 \in \text{interior}(A_r(0))$, then $ran(B^r) = X$, see [4].

Next we recall a result that we will use to characterize the null controllability, see [5].

Theorem 3.1. Bárcenas-Diestel *Let X and U be Banach spaces, let $B : U \to X$ be a bounded linear operator and $A : X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on X whose dual semigroup is strongly continuous on $(0,\infty)$. Suppose Ω is a non-empty separable weakly compact convex subset of U containing 0. Then for each $T > 0$, $0 \in A_T(x_0)$ if and only if for each $x^* \in X^*$,*

$$\langle x^*, S(T)x_0\rangle + \int_0^T \max_{v\in\Omega} \langle x^*, S(T)Bv\rangle\, dt \geq 0.$$

The Bárcenas-Diestel theorem is an important and recent achievement on exact controllability. Using techniques from Banach space theory and the theory of vector measures, the authors show how to translate the question of accesibility of controls to a problem in semigroups of operators, namely, given a c_0-semigroup $\{S(t)\}_{t\geq 0}$ of operators on a Banach space X, under what conditions is the dual semigroup $\{S^*(t)\}_{t\geq 0}$ strongly continuous on $(0,\infty)$? This is the question we will try to answer in the following.

We recall that a Banach space is a *Grothendieck space* if every weakly* convergent sequence in X^* is also weakly convergent. Equivalently, X is a Grothendieck space if every linear bounded from X to any separable Banach Space is a weakly compact. Among Grothendieck spaces we will list all reflexive Banach spaces and $L^\infty(\Omega,\Sigma,\mu)$, where (Ω,Σ,μ) is a positive measure space (see for example [6]). We also recall that a bounded linear operator $T : X \to Y$, (where X and Y are Banach spaces) *factors through a Banach space* Z, if there are bounded linear operators $u : X \to Z$ and $v : Z \to Y$ such that $T = vu$.

It is proven in [7] that if X is a Banach space and $\{T(t)\}_{t\geq 0}$ a c_0-semigroup defined on X such that for every $a > 0$ there exists a Grothendieck space Y_a such that $T(a)$ factors through Y_a, then $\{T^*(t)\}_{t\geq 0}$ is strongly continuous on $(0,\infty)$. Among semigroups satisfying those assumptions (and hence having adjoints which are strongly continuous on $(0,\infty)$) we mention weakly compact semigroups, i.e, semigroups such that $T(t)$ is weakly compact for each t (see [7] for more detrails). There are many examples of weakly compact semigroups, a category that includes all compact semigroups. Moreover, in $X = l_1$, the terms "weakly compact" and "compact" are equivalent, due to the classical Schur Theorem. This will prove useful to establish our result for the non reflexive cases.

We are, in our case, working with X being either c_0 or l_p, $1 \leq p < \infty$, and we have a c_0-semigroup $\{T(t)\}_{t\geq 0}$ (and its infinitesimal generator A) defined on $X \oplus X$, which, as we have indicated before, is isomorphic to X. If $p \in (1,\infty)$, we have a reflexive Banach space, hence a Grothendieck space. Then, for every $a > 0$ there exists, in an obvious manner, a Grothendieck space Y_a (X itself) such that $T(a)$ factors through Y_a. $\{T^*(t)\}_{t\geq 0}$ is thus strongly continuous on $(0,\infty)$, and we are under the hypotheses of the Bárcenas-Diestel Theorem.

For the cases $X = l_1$ or $X = c_0$, we can additionally suppose that $\{T(t)\}_{t\geq 0}$ factors through a Grothendieck space as before (this happens, for example, if $\{T(t)\}_{t\geq 0}$ is weakly compact, as we have previously indicated). Then we can apply the Bárcenas-Diestel Theorem again.

All this can be sumarized in the following:

Theorem 3.2. *For each $r > 0, 0 \in A_r(\omega_0)$ (i.e., system (3.1) is controllable) if and only if, for each $x^* \in l_q$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$*

$$\langle x^*, T(r)\omega_0\rangle + \int_0^r \max_{v\in\Omega} \left\langle x^*, T(t)\overline{B}v(t)\right\rangle\, dt \geq 0.$$

If, additionally, we suppose that the associated semigroup satisfies that, for every $a > 0$ there exists a Grothendieck space Y_a such that $T(a)$ factors through Y_a (in particular, if it is is weakly compact) then we have similar results for $X = l_1$ and $X = c_0$.

4. Final remarks

Problems of this kind are usually set in the context of Hilbert function spaces (see, for example, [2]). But according to the Riesz-Fischer Theorem (see [8]) every separable infinite-dimensional Hilbert space H is isomorphic to l_2. For an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H and $x \in H$, the map $Tx = \{(x, e_i)\}_{i=1}^{\infty}$ is an isometry. We can thus identify any Hilbert function space with a specific sequence space, namely l_2. Hence, by studying and solving this type of problems in l_2 (as we have, in particular, done) we are, in a certain important sense, studying and solving problems set in any Hilbert function space.

On the other hand, since the function f is allowed, in the present work, to belong to l_p, $1 \leq p < \infty$ or c_0, we are able to study these classical problems (and also, in particular, their null controllability) in a considerably more general context.

In the same line of research to the ones presented here but considering other topological spaces that are not Banach spaces, we refer the reader to [9, 10, 11, 12].

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Solving FIDEs by Using Semi-Analytical Techniques

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Abstract
This paper mainly focuses on the recent advances in the semi-analytical approximated methods for solving Fredholm Integro-Differential Equations (FIDEs) of the second kind by using Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM) and Direct Homotopy Analysis Method (DHAM). Convergence analysis of the exact solution of the proposed methods is established. Moreover, we proved the uniqueness of the solution. To illustrate the methods, an example is presented.

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1. Introduction

In this paper, we consider FIDE of the form:

$$\sum_{j=0}^{k} p_j(x)\Delta^{(j)}(x) = f(x) + \lambda \int_a^b W(x,t)G(\Delta(t))dt \tag{1.1}$$

with the initial conditions

$$\Delta^{(r)}(a) = b_r, \quad r = 0,1,2,\cdots,(k-1), \tag{1.2}$$

where $\Delta^{(j)}(x)$ is the j^{th} derivative of the unknown function $\Delta(x)$ that will be determined, $W(x,t)$ is the kernel of the equation, $f(x)$ and $p_j(x)$ are analytic functions, G is nonlinear function of Δ and a, b, λ, and b_r are real finite constants.

The FIDEs arise in many scientific applications. It was also shown that these equations can be derived from boundary value problems.

The application of homotopy techniques in linear and non-linear problems has been devoted by scientists and engineers, because this method is to continuously deform a simple problem which is easy to solve into the under study problem which is difficult to solve. This method was proposed first by He in 1997 and systematical description in 2000 which is, in fact, a coupling of the traditional perturbation method and homotopy in topology [1]. This method was further developed and improved by He and applied to non-linear oscillators with discontinuities [2]. After that many researchers applied the method to various linear and non-linear problems. For example, it was applied to the quadratic Ricatti differential equation by Abbasbandy [3], to the axisymmetric flow over a stretching sheet by Ariel et al. [4], to the Helmholtz equation and fifth-order KdV equation by Rafei and Ganji [5], for the thin film flow of a fourth grade fluid down a vertical cylinder by Siddiqui et al. [6], to the non-linear Volterra-Fredholm integral equations by Hamoud and Ghadle [7], to FIDE [8], to system of Fredholm integral equations [9], Alao et al. [10] studied the ADM and the VIM on various types of integro-differential equation. Moreover, many methods for solving integro-differential equations have been studied by several authors [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by semi-analytical approximated methods as the VIM, HPM and DHAM. Moreover, we proved the existence and uniqueness results of the FIDEs.

2. Variational iteration method (VIM)

The main idea of this method is to construct a correction functional form using general Lagrange multipliers. To illustrate, we consider the following general differential equation:

$$L\Delta(t)+N\Delta(t)=f(t),$$

where L is a linear operator, N is a nonlinear operator and $f(t)$ is inhomogeneous term. According to variational iteration method [7], the terms of a sequence Δ_n are constructed such that this sequence converges to the exact solution. The terms Δ_n are calculated by a correction functional as follows:

$$\Delta_{n+1}(t)=\Delta_n(t)+\int_0^t \mu(\tau)(L\Delta_n(\tau)+N\tilde{y}(\tau)-f(\tau))d\tau. \tag{2.1}$$

The successive approximation $\Delta_n(t), n\geq 0$ of the solution $\Delta(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function Δ_0. The zeroth approximation Δ_0 may be selected using any function that just satisfies at least the initial and boundary conditions. With μ determined, several approximations $\Delta_n(t), n\geq 0$ follow immediately.

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

To obtain the approximation solution of IVP $(1.1)-(1.2)$, according to the VIM, the iteration formula (2.1) can be written as follows:

$$\Delta_{n+1}(x)=\Delta_n(x)+L^{-1}\Big[\mu(x)\Big[\sum_{j=0}^{k}p_j(x)\Delta_n^{(j)}(x)-f(x)-\lambda\int_a^b W(x,t)G(\Delta_n(t))dt\Big]\Big],$$

where L^{-1} is the multiple integration operator given as follows:

$$L^{-1}(\cdot)=\int_a^x\int_a^x\cdots\int_a^x(\cdot)dxdx\cdots dx \quad (k-times).$$

To find the optimal $\mu(x)$, we proceed as follows:

$$\begin{aligned}\delta\Delta_{n+1}(x) &= \delta\Delta_n(x)+\delta L^{-1}\Big[\mu(x)\Big[\sum_{j=0}^{k}p_j(x)\Delta_n^{(j)}(x)-f(x)-\lambda\int_a^b W(x,t)G(\Delta_n(t))dt\Big]\Big]\\ &= \delta\Delta_n(x)+\mu(x)\delta\Delta_n(x)-L^{-1}\Big[\delta\Delta_n(x)\mu'(x)\Big].\end{aligned} \tag{2.2}$$

From Eq. (2.2), the stationary conditions can be obtained as follows:

$$\mu'(x)=0, \text{ and } 1+\mu(x)|_{x=t}=0.$$

As a result, the Lagrange multipliers can be identified as $\mu(x)=-1$ and by substituting in Eq. (2.2), the following iteration formula is obtained:

$$\Delta_0(x)=L^{-1}\Big[\frac{f(x)}{p_k(x)}\Big]+\sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r, \tag{2.3}$$

$$\Delta_{n+1}(x)=\Delta_n(x)-L^{-1}\Big[\sum_{j=0}^{k}p_j(x)\Delta_n^{(j)}(x)-f(x)-\lambda\int_a^b W(x,t)G(\Delta_n(t))dt\Big], n\geq 0.$$

The term $\sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r$ is obtained from the initial conditions, $p_k(x)\neq 0$. Relation (2.3) will enable us to determine the components $\Delta_n(x)$ recursively for $n\geq 0$. Consequently, the approximation solution may be obtained by using

$$\Delta(x)=\lim_{n\to\infty}\Delta_n(x).$$

3. Homotopy perturbation method (HPM)

The homotopy perturbation method first proposed by He [1, 2]. To illustrate the basic idea of this method, we consider the following nonlinear differential equation

$$A(\Delta)-f(r)=0, \quad r\in\Omega, \tag{3.1}$$

under the boundary conditions

$$B\left(\Delta,\frac{\partial\Delta}{\partial n}\right)=0, \quad r\in\Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.

In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Eq. (3.1) therefore can be rewritten as follows [19]:

$$L(\Delta)+N(\Delta)-f(r)=0.$$

By the homotopy technique, we will construct a homotopy $v(r,p):\Omega\times[0,1]\longrightarrow\mathbb{R}$ which satisfies

$$H(v,p)=(1-p)[L(v)-L(\Delta_0)]+p[A(v)-f(r)]=0, p\in[0,1]. \tag{3.2}$$

or

$$H(v,p)=L(v)-L(\Delta_0)+pL(\Delta_0)]+p[N(v)-f(r)]=0, \tag{3.3}$$

where $p\in[0,1]$ is an embedding parameter, Δ_0 is an initial approximation of Eq.(3.1) which satisfies the boundary conditions. From Eqs.(3.2), (3.3) we have

$$\begin{aligned} H(v,0) &= L(v)-L(\Delta_0)=0,\\ H(v,1) &= A(v)-f(r)=0. \end{aligned}$$

The changing in the process of p from zero to unity is just that of $v(r,p)$ from $\Delta_0(r)$ to $\Delta(r)$. In topology this is called deformation, the $L(v)-L(\Delta_0)$, and $A(v)-f(r)$ are called homotopic. Now, assume that the solution of Eqs. (3.2) and (3.3) can be expressed as

$$v=v_0+pv_1+p^2v_2+\cdots.$$

The approximate solution of Eq.(3.1) can be obtained by setting $p=1$.

$$\Delta=\lim_{p\to 1}v=v_0+v_1+v_2+\cdots.$$

Then equating the terms with identical power of P, we obtain the following series of linear equations:

$$\begin{aligned}
P^0:\Delta_0(x) &= \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r,\\
P^1:\Delta_1(x) &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right)+\lambda L^{-1}\left(\int_a^b\frac{W(x,t)}{p_k(x)}G(\Delta_0(t))(t)dt\right)-\sum_{j=0}^{k-1}L^{-1}\left(\frac{p_j(x)}{p_k(x)}\Delta_0^{(j)}(x)\right),\\
P^2:\Delta_2(x) &= \lambda L^{-1}\left(\int_a^b\frac{W(x,t)}{p_k(x)}G(\Delta_1(t))(t)dt\right)-\sum_{j=0}^{k-1}L^{-1}\left(\frac{p_j(x)}{p_k(x)}\Delta_1^{(j)}(x)\right),\\
P^3:\Delta_3(x) &= \lambda L^{-1}\left(\int_a^b\frac{W(x,t)}{p_k(x)}G(\Delta_2(t))dt\right)-\sum_{j=0}^{k-1}L^{-1}\left(\frac{p_j(x)}{p_k(x)}\Delta_2^{(j)}(x)\right),\\
&\ \ \vdots
\end{aligned}$$

4. Direct homotopy analysis method (DHAM)

Consider FIDE (1.1) and substitute the kernel $W(x,t) = g(x)h(t)$ we obtain

$$\sum_{j=0}^{k} p_j(x)\Delta^{(j)}(x) = f(x) + \lambda g(x)\int_a^b h(t)G(\Delta(t))dt.$$

To obtain the approximate solution, we integrating (k)-times in the interval $[a,x]$ with respect to x we obtain,

$$\Delta(x) = L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r + \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}\int_a^b h(t)G(\Delta(t))dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}\Delta_n^{(j)}(x)\right),$$

Setting

$$\begin{aligned} Q &= \int_a^b h(t)G(\Delta(t))dt \\ F &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}\Delta_n^{(j)}(x)\right). \end{aligned}$$

Therefore, we can rewrite Eq. (4.1) as

$$\Delta(x) = F(x) + \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}Q\right),$$

we define the nonlinear homotopy operator as:

$$N[\Delta(x)] = \Delta(x) - F(x) - \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}Q\right),$$

The corresponding mth-order deformation equation is as follows

$$L[\Delta_m(x) - \chi_m\Delta_{m-1}(x)] = BH(x)R_m(\overrightarrow{\Delta_{m-1}(x)})$$

where

$$R_m(\overrightarrow{\Delta_{m-1}(x)}) = \Delta_{m-1}(x) - F(x)(1-\chi_m) - \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}Q\right),$$

and

$$\chi_m = \begin{cases} 1, & m > 1. \\ 0, & m \le 1. \end{cases}$$

choosing the auxiliary linear operator $L[\Delta] = \Delta$, we obtain

$$\begin{aligned} &\Delta_0(x) \quad \text{Choosing initial guess} \\ \Delta_1(x) &= BH(x)\Big[\Delta_0(x) - L^{-1}\left(\frac{f(x)}{p_k(x)}\right) - \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r - \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}\int_a^b h(t)G(\Delta_0(t))dt\right) \\ &\quad + \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}\Delta_0^{(j)}(x)\right)\Big], \\ \Delta_m(x) &= \chi_m\Delta_{m-1}(x) + BH(x)\Big[\Delta_{m-1}(x) - \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}\int_a^b h(t)G(\Delta_{m-1}(t))dt\right) \\ &\quad + \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}\Delta_{m-1}^{(j)}(x)\right)\Big], m > 1. \end{aligned}$$

with auxiliary function $H(x)$ and auxiliary parameter B.

Then, $\Delta(x) = \sum_{i=0}^{m}\Delta_i$ as the approximate solution.

5. Uniqueness results

In this section, we shall give an uniqueness results of Eq. (1.1), with the initial condition (1.2) and prove it [22, 23].

We can be written equation (1.1) in the form of:

$$\Delta(x) = L^{-1}\Big[\frac{f(x)}{p_k(x)}\Big] + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r + \lambda_1 L^{-1}\Big[\int_a^b \frac{1}{p_k(x)}W(x,t)G(\Delta_n(t))dt\Big] - L^{-1}\Big[\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\Delta^{(j)}(x)\Big].$$

we can write

$$L^{-1}\Big[\int_a^b \frac{1}{p_k(x)}W(x,t)G(\Delta_n(t))dt\Big] = \int_a^b \frac{(x-t)^k}{k!p_k(x)}W(x,t)G(\Delta_n(t))dt$$

$$\sum_{j=0}^{k-1} L^{-1}\Big[\frac{p_j(x)}{p_k(x)}\Big]\Delta^{(j)}(x) = \sum_{j=0}^{k-1}\int_a^b \frac{(x-t)^{k-1}p_j(t)}{k-1!p_k(t)}\Delta^{(j)}(t)dt.$$

We set,

$$\Psi(x) = L^{-1}\Big[\frac{f(x)}{p_k(x)}\Big] + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r.$$

Before starting and proving the main results, we introduce the following hypotheses:

(H1) There exist two constants α and $\gamma_j > 0,\ j = 0,1,\cdots,k$ such that, for any $\Delta_1, \Delta_2 \in C(J,\mathbb{R})$

$$|G(\Delta_1)) - G(\Delta_2))| \leq \alpha\,|\Delta_1 - \Delta_2|$$

and

$$\left|D^j(\Delta_1) - D^j(\Delta_2)\right| \leq \gamma_j\,|\Delta_1 - \Delta_2|,$$

we suppose that the nonlinear terms $G(\Delta(x))$ and $D^j(\Delta) = (\frac{d^j}{dx^j})\Delta(x) = \sum_{i=0}^{\infty}\gamma_{i_j}$, ($D^j$ is a derivative operator), $j = 0,1,\cdots,k$, are Lipschitz continuous.

(H2) We suppose that for all $a \leq t \leq x \leq b$, and $j = 0,1,\cdots,k$:

$$\left|\frac{\lambda(x-t)^k W(x,t)}{k!p_k(x)}\right| \leq \theta_1, \qquad \left|\frac{\lambda(x-t)^k W(x,t)}{k!}\right| \leq \theta_2,$$

and

$$\left|\frac{(x-t)^{k-1}p_j(t)}{(k-1)!p_k(t)}\right| \leq \theta_3, \qquad \left|\frac{(x-t)^{k-1}p_j(t)}{(k-1)!}\right| \leq \theta_4,$$

(H3) There exist three functions θ_3^*, θ_4^*, and $\gamma^* \in C(D,\mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R}\times\mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that:

$$\theta_3^* = \max|\theta_3|, \quad \theta_4^* = \max|\theta_4|, \quad \text{and} \quad \gamma^* = \max\left|\gamma_j\right|.$$

(H4) $\Psi(x)$ is bounded function for all x in $J = [a,b]$.

Theorem 5.1. *Assume that (H1)–(H4) hold. If*

$$0 < \psi = (\alpha\theta_1 + k\gamma^*\theta_3^*)(b-a) < 1,$$

then there exists a unique solution $\Delta(x) \in C(J)$ to IVP (1.1) − (1.2).

Proof. Let Δ_1 and Δ_2 be two different solutions of IVP (1.1) – (1.2)., then

$$\begin{aligned}\Big|\Delta_1-\Delta_2\Big| &= \Big|\int_a^b \frac{\lambda(x-t)^k W(x,t)}{p_k(x)k!}[G(\Delta_1)-G(\Delta_2))]dt \\ &\quad -\sum_{j=0}^{k-1}\int_a^b \frac{(x-t)^{k-1}p_j(t)}{p_k(t)(k-1)!}[D^j(\Delta_1)-D^j(\Delta_2))]dt\Big| \\ &\leq \int_a^b \Big|\frac{\lambda(x-t)^k W(x,t)}{p_k(x)k!}\Big|\Big|G(\Delta_1)-G(\Delta_2))\Big|dt \\ &\quad -\sum_{j=0}^{k-1}\int_a^b \Big|\frac{(x-t)^{k-1}p_j(t)}{p_k(t)(k-1)!}\Big|\Big|D^j(\Delta_1)-D^j(\Delta_2))\Big|dt \\ &\leq (\alpha\theta_1+k\gamma^*\theta_3^*)(b-a)|\Delta_1-\Delta_2|,\end{aligned}$$

we get $(1-\psi)|\Delta_1-\Delta_2|\leq 0$. Since $0<\psi<1$, so $|\Delta_1-\Delta_2|=0$. Therefore, $\Delta_1=\Delta_2$ and the proof is completed. □

6. Example

In this section, we present the semi-analytical techniques based on VIM, HPM and DHAM to solve FIDEs. To show the efficiency of the present methods for our problem in comparison with the exact solutions.

Example 6.1. *Consider the following FIDE:*

$$\Delta'(x)=e^x(1+x)-x+\int_0^1 x\Delta(t)dt,$$

with the initial condition

$$\Delta(0)=0,$$

and the the exact solution is $\Delta(x)=xe^x$.

Table 1. Numerical Results of the Example 6.1

x	Exact	VIM	HPM	DHAM
0.1	0.1105170	0.1096837	0.1103782	0.1105170
0.2	0.2442805	0.2409472	0.2437249	0.2442805
0.3	0.4049576	0.3974576	0.4037076	0.4049576
0.4	0.5967298	0.5833965	0.5945076	0.5967298
0.5	0.8243606	0.8035273	0.8208884	0.8233606
0.6	1.0932712	1.0632712	1.0882712	1.0932712
0.7	1.4096268	1.3687935	1.4028213	1.4096268
0.8	1.7804327	1.7270994	1.7715438	1.7804327
0.9	2.2136428	2.1461428	2.2023928	2.2136428

7. Discussion and conclusion

We discussed the VIM, HPM and DHAM for solving FIDEs of the second kind. To assess the accuracy of each method, the test example with known exact solution is used. In this work, the above methods have been successfully employed to obtain the approximate solution of a FIDE. The results show that these methods are very efficient, convenient and can be adapted to fit a larger class of problems. The comparison reveals that although the numerical results of these methods are similar approximately, Table 1 shows that the numerical results obtained with DHAM agree with the exact solutions.

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On Extensions of Extended Gauss Hypergeometric Function

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Abstract
The aim of this paper is to introduce a new extensions of extended Gauss hypergeometric function. Certain integral representations, transformation and summation formulas for extended Gauss hypergeometric function are presented and some special cases are also discussed.

Keywords: Extended hypergeometric function, Integral representations, Mittag-Leffler function, Summation formulas, Transformation formulas
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1. Introduction

The classical Beta function $B(x,y)$ is defined by:

$$B(x,y) = \begin{cases} \int_0^1 t^{x-1}(1-t)^{y-1}dt & , \quad (Re(x)>0\,,\ Re(y)>0) \\ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} & , \quad Re(x)>0\,,\ Re(y)>0, \end{cases} \tag{1.1}$$

where $\Gamma(x)$ is the familiar Gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad (Re(x)>0).$$

The generalized hypergeometric function ${}_pF_q$ with p numerator parameters and q denominator parameters is defined by (see [1])

$${}_pF_q\left[\begin{array}{ccc} a_1,\ldots,a_p & ; & \\ & & z \\ b_1,\ldots,b_q & ; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(b_p)_n}{(a_1)_n\ldots(b_q)_n}\frac{z^n}{n!}, \tag{1.2}$$

where $(\lambda)_n$ is the well-known Pochhammer symbol. The case $p=2$ and $q=1$ of (1.2), yields the Gauss's hypergeometric function ${}_2F_1(z)$.

The Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}[x,y]$ is defined by (see[1])

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{ll}(a_p)\ :\ (b_q)\ ;\ (c_k)\ ; & \\ & x\ ,\ y \\ (e_l)\ :\ (f_m)\ ;\ (g_n)\ ; & \end{array}\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\ \prod_{j=1}^{q}(b_j)_r\ \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(e_j)_{r+s}\ \prod_{j=1}^{m}(f_j)_r\ \prod_{j=1}^{n}(g_j)_s}\ \frac{x^r}{r!}\ \frac{y^s}{s!}.$$

In 1903, Gosta Mittag-Leffler [2] introduced the function $E_\alpha(z)$ defined as:

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha>0\ ,\ z\in\mathbb{C}.$$

In 1905, Wiman [3] defined the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ as follows:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad (\alpha,\beta\in\mathbb{C}\ ,\ Re(\alpha)>0\ ,\ Re(\beta)>0).$$

Afterward, Prabhakar [4] defined the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ as follows:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad (\alpha,\beta,\gamma\in\mathbb{C}\ ,\ Re(\alpha)>0\ ,\ Re(\beta)>0\ ,\ Re(\gamma)>0). \tag{1.3}$$

Clearly,

$$E^{1}_{\alpha,\beta}=E_{\alpha,\beta}(z),\quad E^{1}_{\alpha,1}=E_\alpha(z),\quad E^{1}_{1,1}=E_1(z)=e^z.$$

In recent years, some extensions of Beta function and Gauss hypergeometric function have been considered by several authors (see [5, 6, 7, 8, 9, 10, 11]).

The following extended Beta function and extended Gauss hypergeometric function are introduced by Chaudhry *et al*. [12] and Chaudhry *et al*. [13] respectively:

$$B(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}exp\left(\frac{-p}{t(1-t)}\right)dt,\quad (Re(p)>0,\ Re(x)>0,\ Re(y)>0)$$

and

$$F_p(a,b;c;z)=\sum_{n=0}^{\infty}\frac{B_p(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!},\quad (Re(c)>Re(b)>0,\ p\geq 0).$$

Choi *et al*. [14] introduced the extended Beta and extended Gauss hypergeometric functions as follows:

$$B(x,y;p;q)=\int_0^1 t^{x-1}(1-t)^{y-1}exp\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)dt,\quad (Re(p)>0,\ Re(q)>0) \tag{1.4}$$

and

$$F_{p,q}(a,b;c;z)=\sum_{n=0}^{\infty}\frac{B_{p,q}(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!},\quad (Re(c)>Re(b)>0,\ p,q\geq 0). \tag{1.5}$$

Rahman *et al*. [15] introduced the following extensions of (1.4) and (1.5) as follows:

$$B^{\alpha}_{p,q}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}E_\alpha\left(-\frac{p}{t}\right)E_\alpha\left(-\frac{q}{(1-t)}\right)dt,\quad (Re(\alpha)>0,\ p,q\geq 0) \tag{1.6}$$

and

$$F^{\alpha}_{p,q}(a,b;c;z)=\sum_{n=0}^{\infty}\frac{B^{\alpha}_{p,q}(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!},\quad (Re(c)>Re(b)>0,\ Re(\alpha)>0,\ p,q\geq 0).$$

Further generalizations of (1.6) are introduced by Atash *et al*. [16] and Barahmah [17] as follows:

$$B_{p,q}^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}\left(-\frac{p}{t}\right)E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right)dt,\quad (Re(\alpha)>0,\ Re(\beta)>0,\ p,q\geq 0) \tag{1.7}$$

and

$$B_{p,q}^{(\alpha,\beta,\gamma)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}^{\gamma}\left(-\frac{p}{t}\right)E_{\alpha,\beta}^{\gamma}\left(-\frac{q}{(1-t)}\right)dt,\quad (Re(\alpha)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0). \tag{1.8}$$

In the present paper, we aim to introduce new extensions for extended Gauss hypergeometric function by using (1.7) and (1.8) as follows:

$$F_{p,q}^{(\alpha,\beta)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},\quad (Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ p,q\geq 0) \tag{1.9}$$

and

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_{p,q}^{(\alpha,\beta,\gamma)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},$$
$$(Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0). \tag{1.10}$$

Clearly,

$$F_{p,q}^{(\alpha,\beta,1)}=F_{p,q}^{(\alpha,\beta)},\ F_{p,q}^{(\alpha,1,1)}=F_{p,q}^{\alpha},\ F_{p,q}^{(1,1,1)}=F_{p,q},\ F_{p,p}^{(1,1,1)}=F_p,\ F_{0,0}^{(1,1,1)}={}_2F_1.$$

Further, if we use (1.7) in (1.9) and (1.8) in (1.10), we have respectively the following integral representations:

$$F_{p,q}^{(\alpha,\beta)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-\alpha}E_{\alpha,\beta}\left(-\frac{p}{t}\right)E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right)dt,$$
$$(Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ p,q\geq 0)$$

and

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-\alpha}E_{\alpha,\beta}^{\gamma}\left(-\frac{p}{t}\right)E_{\alpha,\beta}^{\gamma}\left(-\frac{q}{(1-t)}\right)dt,$$
$$(Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0). \tag{1.11}$$

2. Transformation and summation formulas

In this section, we present some transformation and summation formulas for extended Gauss hypergeometric function (1.10) as follows:

Theorem 2.1. *For* $(Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0)$*, the following transformation formula holds true:*

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;z)=(1-z)^{-a}F_{q,p}^{(\alpha,\beta,\gamma)}(a,c-b;c;\frac{-z}{1-z}). \tag{2.1}$$

Proof. Replacing t by $(1-t)$ in (1.11) and using the following result:

$$(1-z(1-t))^{-a}=(1-z)^{-a}(1-\frac{z}{z-1}t)^{-a},$$

we obtain

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}(1-\frac{z}{z-1}t)^{-a}E_{\alpha,\beta}\left(-\frac{q}{t}\right)E_{\alpha,\beta}\left(-\frac{p}{1-t}\right)dt,$$

which, by applying (1.11) yields the desired result. □

Remark 2.2. *Replacing* z *by* $1-\frac{1}{z}$ *and* $\frac{z}{1+z}$ *in (2.1), we have respectively*

Corollary 2.3.

$$F_{p,q}^{(\alpha,\beta,\gamma)}\left(a,b;c;1-\frac{1}{z}\right)=z^{a}F_{q,p}^{(\alpha,\beta,\gamma)}(a,c-b;c;1-z). \tag{2.2}$$

Corollary 2.4.

$$F_{p,q}^{(\alpha,\beta,\gamma)}\left(a,b;c;\frac{z}{1+z}\right)=(1+z)^{a}F_{q,p}^{(\alpha,\beta,\gamma)}(a,c-b;c;-z). \tag{2.3}$$

Theorem 2.5. *For* $(Re(c-a-b)>0,\ Re(k)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0)$*, the following summation formula holds true:*

$$F_{p,q}^{(k,\beta,\gamma)}(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c-a)\Gamma(c-b)} \tag{2.4}$$
$$\times F\begin{matrix}1 & : & 1 & ; & 1\\ 0 & : & 1+k & ; & 1+k\end{matrix}\left[\begin{matrix}1+a-c & : & \gamma & ; & \gamma & ; \\ - & : & 1-b,\Delta(k;\beta) & ; & 1+a+b-c,\Delta(k;\beta) & ;\end{matrix}\ \frac{-p}{k^{k}},\frac{-q}{k^{k}}\right],$$

where $\Delta(k;\beta)$ *is k-tuple*

$$\frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k}.$$

Proof. From (1.11), we have

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;1)=\frac{1}{B(b,c-b)}\int_{0}^{1}t^{b-1}(1-t)^{c-a-b-1}E_{k,\beta}^{\gamma}\left(-\frac{p}{t}\right)E_{k,\beta}^{\gamma}\left(-\frac{q}{(1-t)}\right)dt.$$

Applying (1.3) and interchanging the order of summation and integration and then using (1.1), we obtain

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;1)=\frac{\Gamma(c)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c-b)}\times\sum_{r,s=0}^{\infty}\frac{(\gamma)_{r}(\gamma)_{s}(-p)^{r}(-q)^{s}\Gamma(b-r)\Gamma(c-a-b-s)}{(\beta)_{kr}(\beta)_{ks}r!s!}.$$

Now, using the following identities (see [1]):

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}=\frac{(-1)^{n}}{(1-\alpha)_{n}}$$

and

$$(\alpha)_{kn}=k^{kn}\prod_{j=1}^{k}\left(\frac{\alpha+j-1}{k}\right)_{n},\quad n=1,2,3,\ldots,$$

we have

$$F_{p,q}^{(k,\beta,\gamma)}(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c-a)\Gamma(c-b)}\times\sum_{r,s=0}^{\infty}\frac{(1-c+a)_{r+s}(\gamma)_{r}(\gamma)_{s}(-p)^{r}(-q)^{s}}{k^{kr}\prod_{j=1}^{k}\left(\frac{\beta+j-1}{k}\right)_{r}k^{ks}\prod_{j=1}^{k}\left(\frac{\beta+j-1}{k}\right)_{s}(1-b)_{r}(1+a+b-c)_{s}r!s!}$$
$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c-a)\Gamma(c-b)}$$
$$\times F\begin{matrix}1 & : & 1 & ; & 1\\ 0 & : & 1+k & ; & 1+k\end{matrix}\left[\begin{matrix}1+a-c & : & \gamma & ; & \gamma & ; \\ - & : & 1-b,\frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k} & ; & 1+a+b-c,\frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k} & ;\end{matrix}\ \frac{-p}{k^{k}},\frac{-q}{k^{k}}\right].$$

This completes the proof of (2.4). □

Remark 2.6. *Putting* $a=-n$ *in (2.4), we obtain*

Corollary 2.7.

$$F_{p,q}^{(k,\beta,\gamma)}(-n,b;c;1) = \frac{\Gamma(c)\Gamma(c+n-b)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c+n)\Gamma(c-b)}$$

$$\times F\, {}^{1\ :\ 1\ ;\ 1}_{0\ :\ 1+k\ ;\ 1+k} \left[\begin{array}{ccccccc} 1-n-c & : & \gamma & ; & \gamma & ; & \\ - & : & 1-b,\Delta(k;\beta) & ; & 1-n+b-c,\Delta(k;\beta) & ; & \end{array} \frac{-p}{k^k}, \frac{-q}{k^k} \right]. \tag{2.5}$$

Remark 2.8. *Putting $a=-n$ and $b=a+n$ in (2.4), we obtain*

Corollary 2.9.

$$F_{p,q}^{(k,\beta,\gamma)}(-n,a+n;c;1) = \frac{\Gamma(c)\Gamma(c-a)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c+n)\Gamma(c-a-n)}$$

$$\times F\, {}^{1\ :\ 1\ ;\ 1}_{0\ :\ 1+k\ ;\ 1+k} \left[\begin{array}{ccccccc} 1-n-c & : & \gamma & ; & \gamma & ; & \\ - & : & 1-a-n,\Delta(k;\beta) & ; & 1+a-c,\Delta(k;\beta) & ; & \end{array} \frac{-p}{k^k}, \frac{-q}{k^k} \right]. \tag{2.6}$$

Remark 2.10. *Putting $a=-n$ and $b=1-b-n$ in (2.4), we obtain*

Corollary 2.11.

$$F_{p,q}^{(k,\beta,\gamma)}(-n,1-b-n;c;1) = \frac{\Gamma(c)\Gamma(c+b-1+2n)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c+n)\Gamma(c+b-1-n)}$$

$$\times F\, {}^{1\ :\ 1\ ;\ 1}_{0\ :\ 1+k\ ;\ 1+k} \left[\begin{array}{ccccccc} 1-n-c & : & \gamma & ; & \gamma & ; & \\ - & : & b+n,\Delta(k;\beta) & ; & 2-b-c-2n,\Delta(k;\beta) & ; & \end{array} \frac{-p}{k^k}, \frac{-q}{k^k} \right]. \tag{2.7}$$

Theorem 2.12. *For $(Re(b)>0,\ Re(k)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0)$, the following summation formula holds true:*

$$F_{p,q}^{(k,\beta,\gamma)}(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1) = \frac{\Gamma(b+\frac{1}{2})\Gamma(b+n)}{\Gamma(\beta)\Gamma(\beta)\Gamma(b+\frac{n}{2})\Gamma(b+\frac{n}{2}+\frac{1}{2})}$$

$$\times F\, {}^{1\ :\ 1\ ;\ 1}_{0\ :\ 1+k\ ;\ 1+k} \left[\begin{array}{ccccccc} \frac{1}{2}-\frac{n}{2}-b & : & \gamma & ; & \gamma & ; & \\ - & : & (n/2)+(1/2),\Delta(k;\beta) & ; & 1-b-n,\Delta(k;\beta) & ; & \end{array} \frac{-p}{k^k}, \frac{-q}{k^k} \right], \tag{2.8}$$

where $\Delta(k;\beta)$ is k-tuple

$$\frac{\beta}{k}, \frac{\beta+1}{k}, \ldots, \frac{\beta+k-1}{k}.$$

The proof of the Theorem 2.12 is similar to that of the Theorem 2.5. Therefore, we omit the details.

3. Special cases

(i) Setting $\beta=\gamma=1$ in (2.1), we get the following corrected formula given by Rahman *et al*. [15]

$$F_{p,q}^{k}(a,b;c;z) = (1-z)^{-a} F_{q,p}^{k}(a,c-b;c;\frac{-z}{1-z}).$$

(ii) Setting $k=\beta=\gamma=1$ in (2.1), (2.2) and (2.3), we get a known transformation formulas of Choi *et al*. [14] for $F_{p,q}(a,b;c;z)$.
(iii) Setting $k=\beta=\gamma=1,\ p=q$ in (2.1), we get a known transformation formula of Chaudhry *et al*. [13] for $F_p(a,b;c;z)$.

(iv) Setting $k=\beta=\gamma=1,\ p=q=0$ in (2.1), we get Euler transformation [18, 1]).
(v) Setting $k=\beta=\gamma=1$ in (2.4), we get

$$F_{p,q}^{(1,1,1)}(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\psi_2[1+a-c;1-b,1+a+b-c;-p,-q], \tag{3.1}$$

where ψ_2 is the Humbert's confluent hypergeometric function [1].
By setting $p=q$ in (3.1) and using the result [1]

$$\psi_2[a;b,c;x,x]={}_3F_3\left[a,\frac{b+c}{2},\frac{b+c-1}{2};b,c,b+c-1;4x\right], \tag{3.2}$$

equation (3.1) reduces to

$$F_{p,p}^{(1,1,1)}(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_2\left[\frac{a-c+1}{2},\frac{a-c+2}{2};1-b,1+a+b-c;-4p\right]. \tag{3.3}$$

Further, setting $p=0$ in (3.3), we get the well-known Gauss summation formula (see [18])

$$F_{0,0}^{(1,1,1)}(a,b;c;1)={}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

(vi) Setting $k=\beta=\gamma=1$ in (2.5), we get

$$F_{p,q}^{(1,1,1)}(-n,b;c;1)=\frac{\Gamma(c)\Gamma(c+n-b)}{\Gamma(c+n)\Gamma(c-b)}\psi_2[1-n-c;1-b,1-n+b-c;-p,-q]. \tag{3.4}$$

Further, setting $p=q=0$ in (3.4), we get a known result (see [18])

$$F_{0,0}^{(1,1,1)}(-n,b;c;1)={}_2F_1(-n,b;c;1)=\frac{(c-b)_n}{(c)_n}.$$

(vii) Setting $k=\beta=\gamma=1$ in (2.6), we get

$$F_{p,q}^{(1,1,1)}(-n,a+n;c;1)=\frac{\Gamma(c)\Gamma(c-a)}{\Gamma(c+n)\Gamma(c-a-n)}\psi_2[1-n-c;1-a-n,1+a-c;-p,-q]. \tag{3.5}$$

Further, setting $p=q=0$ in (3.5), we get a known result (see [18])

$$F_{0,0}^{(1,1,1)}(-n,a+n;c;1)={}_2F_1(-n,a+n;c;1)=\frac{(-1)^n(1+a-c)_n}{(c)_n}.$$

(viii) Setting $k=\beta=\gamma=1$ in (2.7), we get

$$F_{p,q}^{(1,1,1)}(-n,1-b-n;c;1)=\frac{\Gamma(c)\Gamma(b+c-1+2n)}{\Gamma(c+n)\Gamma(b+c-1-n)}\psi_2[1-n-c;b+n,2-b-c-2n;-p,-q]. \tag{3.6}$$

Further, setting $p=q=0$ in (3.6), we get a known result (see [18])

$$F_{0,0}^{(1,1,1)}(-n,1-b-n;c;1)={}_2F_1(-n,1-b-n;c;1)=\frac{(-1)^n(b+c-1)_{2n}}{(c)_n(b+c-1)_n}.$$

(viiii) Setting $k=\beta=\gamma=1$ in (2.8), we get

$$F_{p,q}^{(1,1,1)}(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1)=\frac{\Gamma(b+\frac{1}{2})\Gamma(b+n)}{\Gamma(b+\frac{n}{2})\Gamma(b+\frac{n}{2}+\frac{1}{2})}\psi_2\left[\frac{1}{2}-\frac{n}{2}-b;\frac{1}{2}+\frac{n}{2},1-b-n;-p,-q\right],$$

which for $p=q$ and using the result (3.2) reduces to

$$F_{p,p}^{(1,1,1)}(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1)=\frac{\Gamma(b+\frac{1}{2})\Gamma(b+n)}{\Gamma(b+\frac{n}{2})\Gamma(b+\frac{n}{2}+\frac{1}{2})}{}_2F_2\left[\frac{3}{4}-\frac{n}{4}-\frac{b}{2},\frac{1}{4}-\frac{n}{4}-\frac{b}{2};\frac{1}{2}+\frac{n}{2},1-b-n;-4p\right]. \tag{3.7}$$

Further, setting $p=0$ in (3.7) and using Legender's duplication formula (see [18])

$$\Gamma(b)\Gamma(b+\frac{1}{2})=2^{1-2b}\sqrt{\pi}\Gamma(2b),$$

we get a known result (see [18])

$$F_{0,0}^{(1,1,1)}(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1)={}_2F_1(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1)=\frac{2^n(b)_n}{(2b)_n}.$$

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On the Cohomology of Topological Semigroups

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Abstract
In this short note, we give some new results on continuous bounded cohomology groups of topological semigroups with values in complex field. We show that the second continuous bounded cohomology group of a compact metrizable semigroup, is a Banach space. Also, we study cohomology groups of amenable topological semigroups, and we show that cohomology groups of rank greater than one of a compact left or right amenable semigroup, are trivial. Also, we give some examples and applications about topological lattices.

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1. Introduction

Homology theory is one of the most powerful tools for study of various mathematical objects. There are many kind of (co)homology theories, for instance: de Rham (co)homology for smooth manifolds, sheaf cohomology for algebraic varieties, singular (co)homology of topological spaces with values in an arbitrary ring, Čech cohomology for topological spaces, bordism homology, and Hochschild (co)homology for rings and topological algebras with values in bimodules.

The bounded cohomology theory was first defined for discrete groups by B. E. Johnson [1] and F. Trauber. Then, M. Gromov [2] extended it to topological spaces. Gromov has been proved that for every path connected manifold, the bounded cohomology group of any rank is equivalent with bounded cohomology group of fundamental group of the manifold with the same rank; for more details one can look [3]. The continuous cohomology theory for topological spaces and topological groups have been studied by many mathematicians in different approaches; see [4, 5, 6] and [7].

The bounded continuous cohomology theory for topological spaces and topological groups, generalizing both continuous cohomology and bounded cohomology theories simultaneously, has been studied by many authors such as R. Frigerio [8] and N. Monod [9]. Bounded cohomology of semigroups has also been considered by many mathematicians such as R. Brooks [10], R. I. Grigorchuk [11] and N. V. Ivanov [12].

In this paper we establish a topological bounded cohomology theory for topological semigroups, using continuous bounded cocycles.

In the section 2, we define bounded continuous cohomology group of a topological semigroup. In the section 3, we show some basic properties of the bounded continuous cohomology. In the next section, we explain the bounded continuous cohomology relation with amenability. In the last section, we give some examples of it.

2. Definition of the cohomology

For any set X, $\mathbf{B}(X)$ denotes the Banach space of all bounded complex ($\mathbb{C}$) valued maps on X with the uniform norm. If X has a topology, then $\mathbf{C}(X) \subset \mathbf{B}(X)$ denotes the Banach subspace of continuous maps. By a topological semigroup we mean a semigroup S with a topology such that the multiplication $S \times S \to S$ is jointly continuous.

Let S be a semigroup. Let $\mathscr{C}_b^0(S) = \mathbb{C}$, and for $n \geq 1$, let $\mathscr{C}_b^n(S) = \mathbf{B}(S^n)$. The elements of $\mathscr{C}_b^n(S)$ are called *bounded cochains* of the semigroup S. Let $\delta^0 : \mathscr{C}_b^0(S) \to \mathscr{C}_b^1(S)$ be the zero linear map and for $n \geq 1$, define the bounded linear map $\delta^n : \mathscr{C}_b^n(S) \to \mathscr{C}_b^{n+1}(S)$ by

$$\delta^n(f)(s_1,\ldots,s_{n+1}) = f(s_2,\ldots,s_{n+1}) + \sum_{i=1}^{n}(-1)^i f(s_1,\ldots,s_is_{i+1},\ldots,s_{n+1}) + (-1)^{n+1} f(s_1,\ldots,s_n),$$

for $f \in \mathscr{C}_b^n(S)$ and $s_1,\ldots,s_{n+1} \in S$. The linear map δ^n is called *coboundary*. It is easily checked that $\delta^{n+1}\delta^n = 0$ and thus, we have the following cochain complex of Banach spaces and bounded linear maps:

$$0 \longrightarrow \mathscr{C}_b^0(S) \xrightarrow{\delta^0} \mathscr{C}_b^1(S) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} \mathscr{C}_b^n(S) \xrightarrow{\delta^n} \cdots \tag{2.1}$$

Then, the cohomology groups of the complex (2.1) are called *bounded cohomology groups* of S and denoted by $\mathscr{H}_b^n(S) = \frac{\ker\delta^n}{\operatorname{Im}\delta^{n-1}}$. Always, the quotient vector space $\mathscr{H}_{cb}^n(S)$ is considered as a semi normed space with quotient semi norm. As any cohomology theory, we let $\mathscr{B}_b^n(S) = \operatorname{Im}\delta^{n-1}$ and $\mathscr{Z}_b^n(S) = \ker\delta^n$. The elements of $\mathscr{B}_b^n$ and $\mathscr{Z}_b^n$ are called bounded n-coboundaries and bounded n-cocycles, respectively. For more details on bounded cohomology of semigroups, see [11].

Now, suppose that S is a topological semigroup. Let $\mathscr{C}_{cb}^0(S) = \mathbb{C}$, and for $n \geq 1$, let $\mathscr{C}_{cb}^n(S) = \mathbf{C}(S^n)$. The elements of $\mathscr{C}_{cb}^n(S)$ are called *continuous bounded cochains* of the semigroup S. Then we have the following Banach subcomplex of (2.1):

$$0 \longrightarrow \mathscr{C}_{cb}^0(S) \xrightarrow{\delta^0} \mathscr{C}_{cb}^1(S) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} \mathscr{C}_{cb}^n(S) \xrightarrow{\delta^n} \cdots \tag{2.2}$$

Definition 2.1. *The cohomology groups of the complex (2.2) are called continuous bounded cohomology groups of S and denoted by $\mathscr{H}_{cb}^n(S)$.*

Analogously, we have the space of continuous bounded n-coboundaries $\mathscr{B}_{cb}^n(S)$, and the space of continuous bounded n-cocycles $\mathscr{Z}_{cb}^n$, and $\mathscr{H}_{cb}^n(S)$ is considered by the quotient semi norm.

Remark 2.2. *(I) Let S be a discrete semigroup. Consider the convolution Banach algebra $\ell^1(S)$. Then the space $\mathbb{C}$ is a Banach $\ell^1(S)$-bimodule by the symmetric action $f \cdot \lambda = \lambda \cdot f = \lambda \sum_{s \in S} f(s)$ for $f \in \ell^1(S)$ and $\lambda \in \mathbb{C}$. It is well known and easily checked that the bounded Hochschild cohomology groups of $\ell^1(S)$ with values in the bimodule $\mathbb{C}$ and the bounded cohomology groups of S are isometric isomorph. Thus, the bounded cohomology is a special case of Hochschild cohomology, see[13].*

(II) Let S be a compact Hausdorff semigroup. If we dualize cochain complex (2.2), then (by the natural isomorphism between $\mathbf{C}(X)^*$ and the Banach space of complex Borel regular measures $\mathbf{M}(X)$ for any compact Hausdorff space X) we have the chain complex

$$0 \longleftarrow \mathbb{C} \xleftarrow{(\delta^0)^*} \mathbf{M}(S) \xleftarrow{(\delta^1)^*} \mathbf{M}(S^2) \xleftarrow{(\delta^2)^*} \cdots$$

One can consider the homology of this complex as a measure homology theory (cf. [14, 15]) that is a topological version of ℓ^1-homology of discrete semigroups [11].

3. Some basic properties

Theorem 3.1. *Let S, T be topological semigroups and $\phi : S \to T$ be a continuous homomorphism. Then for every $n \geq 0$, there is a canonical continuous linear map*

$$\mathscr{H}_{cb}^n(\phi) : \mathscr{H}_{cb}^n(T) \to \mathscr{H}_{cb}^n(S).$$

Proof. For every $n \geq 1$ let $\hat{\phi}_n : \mathscr{C}^n_{cb}(T) \to \mathscr{C}^n_{cb}(S)$ be defined by

$$\hat{\phi}_n(f)(s_1, \ldots, s_n) = f(\phi(s_1), \ldots, \phi(s_n)) \quad (f \in \mathscr{C}^n_{cb}(T)).$$

Then $(\hat{\phi}_n)_n$ is a cochain map between continuous bounded cohomology complexes of T and S, i.e. the following diagram is commutative:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{C} & \xrightarrow{\delta^0} & \mathscr{C}^1_{cb}(T) & \xrightarrow{\delta^1} & \cdots & \xrightarrow{\delta^{n-1}} & \mathscr{C}^n_{cb}(T) & \xrightarrow{\delta^n} & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \hat{\phi}_1} & & & & \downarrow{\scriptstyle \hat{\phi}_n} & & \\
0 & \longrightarrow & \mathbb{C} & \xrightarrow[\delta^0]{} & \mathscr{C}^1_{cb}(S) & \xrightarrow[\delta^1]{} & \cdots & \xrightarrow[\delta^{n-1}]{} & \mathscr{C}^n_{cb}(S) & \xrightarrow[\delta^n]{} & \cdots
\end{array}$$

Thus, the standard arguments of Banach homology ([16], [17]) shows that we have a continuous linear map $\mathscr{H}^n_{cb}(\phi)$, defined by

$$\mathscr{H}^n_{cb}(\phi)(f + \mathscr{B}^n_{cb}(T)) = \hat{\phi}_n(f) + \mathscr{B}^n_{cb}(S),$$

for $f \in \mathscr{Z}^n_{cb}(T)$. □

Let $\mathscr{T}\mathscr{S}\mathscr{G}$ be the category of topological semigroups and continuous homomorphisms. Then, the above theorem shows that $\mathscr{H}^n_{cb}$ is a contravariant functor from $\mathscr{T}\mathscr{S}\mathscr{G}$ to the category of seminormed spaces and continuous linear maps. Since the category $\mathscr{T}\mathscr{S}\mathscr{G}$ has no additive properties, the computation of continuous bounded cohomology groups often are very hard. In another paper, we will consider various extensions of $\mathscr{H}^n_{cb}$ to some categories of representations of topological semigroups on topological vector spaces.

For any topological semigroup S it is trivial that $\mathscr{H}^0_{cb}(S) = \mathbb{C}$. First order cohomology groups are zero:

Theorem 3.2. *For any topological semigroup S, $\mathscr{H}^1_{cb}(S)$ is zero.*

Proof. Let $f \in \mathscr{Z}^1_{cb}(S)$ be a 1-cocycle. Then for every $s, t \in S$, we have $\delta^1(f)(s,t) = f(t) - f(st) + f(s) = 0$ and thus,

$$f(st) = f(s) + f(t).$$

In particular, for every $s \in S$ and $n \in \mathbb{N}$, we have $f(s^n) = nf(s)$. This implies that $f(s) = 0$, since f is a bounded map. Therefore $\mathscr{Z}^1_{cb}(S) = 0$ and $\mathscr{H}^1_{cb}(S)$ is zero. □

We recall a kind of limiting process: Let E be the Banach space of all bounded sequences of complex numbers with uniform norm and let $F \subset E$ be the subspace of all convergent sequences. Then, the functional $lim : F \to \mathbb{C}$ defined by $lim(a_n)_{n \in \mathbb{N}} = \lim_{n \to \infty} a_n$ is a bounded functional and thus, by the Hann-Banach theorem there is a bounded functional $\mathrm{LIM} : E \to \mathbb{C}$ that extends lim and $\|\mathrm{LIM}\| = 1$ (such functionals are called *Banach limits*).

Theorem 3.3. *Let S be a compact semigroup with a metric d that induces the topology of S and has the following property:*

- *For every $\beta > 0$, $s, t \in S$, and $i \in \mathbb{N}$ if $d(s,t) < \beta$ then $d(s^i, t^i) < \beta$.*

Then $\mathscr{H}^2_{cb}(S)$ is a Banach space.

Proof. It is enough that we prove $\delta^1(\mathscr{C}^1_{cb}(S))$ is closed in $\mathscr{C}^2_{cb}(S)$, and thus, it is sufficient to construct a bounded linear map $\gamma : \mathscr{C}^2_{cb}(S) \to \mathscr{C}^1_{cb}(S)$ such that $\gamma\delta = id_{\mathscr{C}^1_{cb}(S)}$.

Let $f \in \mathscr{C}^2_{cb}(S)$ be a 2-cochain. For every $s \in S$, consider the bounded sequence $a_n^{f,s} = n^{-1} \sum_{i=1}^{n-1} f(s^i, s)$ $(n \geq 2)$ of complex numbers and define $\gamma(f)(s) = \mathrm{LIM}(a_n^{f,s})$. Let $\alpha > 0$ be arbitrary. Since S^2 is a compact metric space and f is continuous, there is $\beta > 0$ such that if $d(t_1, t_2) < \beta$ and $d(t'_1, t'_2) < \beta$ then $|f(t_1, t'_1) - f(t_2, t'_2)| < \alpha$. This property together with $(*)$ implies that for every $s, t \in S$ and $n \in \mathbb{N}$ if $d(s,t) < \beta$ then $|a_n^{f,s} - a_n^{f,t}| < \alpha$ and thus, $|\gamma(f)(s) - \gamma(f)(t)| < \alpha$. Therefore we have proved $\gamma(f)$ is continuous and $\gamma(f) \in \mathscr{C}^1_{cb}(S)$. Also, it is easily checked that γ is a bounded linear operator.

Now, suppose that g is in $\mathscr{C}^1_{cb}(S)$. For every $s \in S$ and $i \geq 1$ we have

$$\delta^1(g)(s^i, s) = g(s^i) - g(s^{i+1}) + g(s),$$

thus, for every $n \geq 2$, $a_n^{\delta^1(g),s} = g(s) - n^{-1} g(s^n)$. Therefore we have

$$\gamma(\delta^1(g))(s) = \mathrm{LIM}(a_n^{\delta^1(g),s}) = \lim_{n \to \infty} g(s) - n^{-1} g(s^n) = g(s).$$

Thus, we have proved $\gamma\delta(g) = g$. □

It is easily checked that the arguments of the proof of Theorem 3.3, satisfy when S is a discrete semigroup:

Theorem 3.4. *Let S be a discrete semigroup. Then $\mathscr{H}_b^2(S)$ is a Banach space.*

Proposition 3.5. *Let S and T be topological semigroups and $p : S\times T \to S$ be the natural projection. Suppose that T has a unite element e. Then the linear map $\mathscr{H}^n_{cb}(p) : \mathscr{H}^n_{cb}(S) \to \mathscr{H}^n_{cb}(S\times T)$ is injective for all $n \geq 1$.*

Proof. For $n = 1$, the result follows from Theorem 3.2. Let $n \geq 2$ be fixed, and let $p^{(n)} : (S\times T)^n \to S^n$ be defined by

$$p^{(n)}((s_1,t_1),\dots,(s_n,t_n)) = (s_1,\dots,s_n)$$

for $s_1,\dots,s_n \in S, t_1,\dots,t_n \in T$. By definition of $\mathscr{H}^n_{cb}(p)$, we must prove that if $f \in \mathscr{Z}^n_{cb}(S)$ and $f\circ p^{(n)} \in \mathscr{B}^n_{cb}(S\times T)$, then f is in $\mathscr{B}^n_{cb}(S)$. Thus, consider such a n-cocycle f. There is $g \in \mathscr{C}^{n-1}_{cb}(S\times T)$ such that $\delta^{n-1}(g) = f\circ p^{(n)}$. Define $\hat{g} \in \mathscr{C}^{n-1}_{cb}(S)$ by

$$\hat{g}(s_1,\dots,s_{n-1}) = g((s_1,e),\dots,(s_{n-1},e)) \qquad (s_1,\dots,s_{n-1} \in S).$$

Then, for every $s_1,\dots,s_n \in S$, we have

$$\begin{aligned}\delta^{n-1}(\hat{g})(s_1,\dots,s_n) = \hat{g}(s_2,\dots,s_n) + \sum_{i=1}^{n-1}(-1)^i\hat{g}(s_1,\dots,s_is_{i+1},\dots,s_n) + (-1)^n\hat{g}(s_1,\dots,s_{n-1}) &= g((s_2,e),\dots,(s_n,e)) \\ &+ \sum_{i=1}^{n-1}(-1)^i g((s_1,e),\dots,(s_is_{i+1},e),\dots,(s_n,e)) + (-1)^n g((s_1,e),\dots,(s_{n-1},e)) \\ &= \delta^{n-1}(g)((s_1,e),\cdots,(s_n,e)).\end{aligned}$$

On the other hand, $\delta^{n-1}(g)((s_1,e)\dots,(s_n,e)) = f(s_1,\dots,s_n)$. Thus, we have $\delta^{n-1}(\hat{g}) = f$ and $f \in \mathscr{B}^n_{cb}(S)$. □

4. Relation with amenability

Let S be a topological semigroup. A function $f \in \mathbf{C}(S)$ is called *right uniformly continuous*, if the map $\Phi_f : S \to \mathbf{C}(S)$ defined by $\Phi_f(s) = f\cdot s$ is continuous with uniform norm of $\mathbf{C}(S)$, where $f\cdot s(x) = f(sx)$ $(x \in S)$. *Left* uniformly continuous functions are similarly defined. The space of all right (left) uniformly continuous functions is denoted by $\mathbf{RUC}(S)$ ($\mathbf{LUC}(S)$). Note that if $f \in \mathbf{RUC}(S)$ and $s \in S$, then $f\cdot s \in \mathbf{RUC}(S)$. Also, it is easily checked that $\mathbf{RUC}(S) = \mathbf{LUC}(S) = \mathbf{C}(S)$ when S is compact, and it is clear that $\mathbf{RUC}(S) = \mathbf{LUC}(S) = \mathbf{C}(S) = \mathbf{B}(S)$ when S is discrete.

A topological semigroup S is called it left amenable if there is a *left invariant mean* on $\mathbf{RUC}(S)$, i.e. a bounded linear functional m on $\mathbf{RUC}(S)$ such that $\langle m, 1_S\rangle = \|m\| = 1$ (where 1_S is the constant map on S with value 1) and for every $s \in S$ and $f \in \mathbf{RUC}(S)$, $\langle m, f\cdot s\rangle = \langle m, f\rangle$. *Right* invariant means and *right* amenable semigroups are similarly defined. A topological semigroup is called amenable if it is both left and right amenable.

It is well known and easily checked that for topological semigroups S and T, if there is a continuous homomorphisms form S onto T, and S is left (right) amenable, then T is also left (right) amenable. In particular, if S is left (right) amenable semigroup with topology τ, and τ' is another semigroup topology on S such that $\tau' \subset \tau$, then (S,τ') is left (right) amenable. Thus, any commutative topological semigroup is amenable since any commutative discrete semigroup is amenable ([18]). It is well known that any compact group is amenable ([18]), but there are compact semigroups that are not left amenable nor right amenable:

Example 4.1. *Let X and Y be two disjoint compact spaces with distinguished elements $x_0 \in X$ and $y_0 \in Y$. Define a semigroup multiplication on disjoint union space $T = X\cup Y$ by*

$$xx' = x_0, \quad yy' = y_0, \quad xy = x_0, \quad yx = y_0,$$

for every $x,x' \in X$ and $y,y' \in Y$. Then T becomes a compact semigroup. We show that T is not left amenable. Suppose m is a bounded linear functional on $\mathbf{C}(T)$ such that $\langle m, 1_T\rangle = \|m\| = 1$. For every $x \in X \subset T$ and $f \in \mathbf{C}(T)$, we have $\langle m, f\cdot x\rangle = \langle m, f(x_0)1_T\rangle = f(x_0)$, and similarly $\langle m, f\cdot y\rangle = f(y_0)$ for every $y \in Y$. Thus, m is not a left invariant mean, since there is a continuous map f on S such that $f(x_0) \neq f(y_0)$. Thus, we have proved that T is not left amenable. Let T^{op} be the opposite semigroup of T. Then T^{op} is not right amenable. Now the compact semigroup $S = T\times T^{op}$ is not left nor right amenable, since the canonical projection maps from S to T and T^{op} are continuous surjective homomorphisms.

We need the following simple topological lemma.

Lemma 4.2. *Let X be a topological space and Y be a compact space. Let $f : X \times Y \to \mathbb{C}$ be a continuous map. Then $F : X \to \mathbf{C}(Y)$, defined by $F(x)(y) = f(x,y)$ is continuous with norm topology of $\mathbf{C}(Y)$.*

Proof. Let $x_0 \in X$ and $\alpha > 0$ be arbitrary. Since f is continuous, for every $y \in Y$, there are open sets U_y, V_y in X and Y respectively, such that $(x_0, y) \in U_y \times V_y$ and $|f(x_0,y) - f(x,y')| < \alpha/2$ for every $(x,y') \in U_y \times V_y$. Since Y is compact, there are $y_1, \ldots, y_n \in Y$ such that $Y = \cup_{i=1}^n V_{y_i}$. Let W be the open set $\cap_{i=1}^n U_{y_i}$. Let $x \in W$ and $y \in Y$ be arbitrary. Then for some i $(i = 1, \cdots, n)$, y belongs to V_{y_i} and we have,

$$|f(x,y) - f(x_0,y)| \le |f(x,y) - f(x_0,y_i)| + |f(x_0,y_i) - f(x_0,y)| < \alpha/2 + \alpha/2 = \alpha.$$

Thus, we have $\|F(x) - F(x_0)\| < \alpha$ for every $x \in W$. The proof is complete. □

The proof of the following Theorem is an adaptation of the proof given in [11, Theorem 2.1] to the topological case.

Theorem 4.3. *Let S be a compact semigroup and suppose that S is left (right) amenable. Then $\mathscr{H}_{cb}^n$ is zero for every $n \ge 0$.*

Proof. Suppose that S is left amenable and let m be a left invariant mean on $\mathbf{C}(S)^*$. Similar [11], we use the notation

$$m(f) = \int_S f(s) d(s) \tag{4.1}$$

for $f \in \mathbf{C}(S)$. Thus, we have

(i) $\int_S 1_S(s) d(s) = 1$, and

(ii) $\int_S f(ts) d(s) = \int_S f(s) d(s)$ for every $f \in \mathbf{C}(S)$ and $t \in S$.

The cases $n = 0$ and $n = 1$ were considered before, thus, suppose that $n \ge 2$ and let $f \in \mathscr{Z}_{cb}^n(S)$. Then, for every $s_1, \ldots, s_{n+1} \in S$, we have,

$$\delta^n(f)(s_1, \ldots, s_{n+1}) = f(s_2, \ldots, s_{n+1}) + \sum_{i=1}^{n} (-1)^i f(s_1, \ldots, s_i s_{i+1}, \ldots, s_{n+1}) + (-1)^{n+1} f(s_1, \ldots, s_n) = 0$$

If we fix $s_1, \ldots, s_n \in S$ and integrate the above formula over the variable s_{n+1} in the sense of (4.1), then we have

$$\begin{aligned} &\int_S f(s_2, \ldots, s_{n+1}) d(s_{n+1}) + \sum_{i=1}^{n-1} (-1)^i \int_S f(s_1, \ldots, s_i s_{i+1}, \ldots, s_n, s_{n+1}) d(s_{n+1}) \\ &\qquad + (-1)^n \int_S f(s_1, \ldots, s_{n-1}, s_n s_{n+1}) d(s_{n+1}) + (-1)^{n+1} \int_S f(s_1, \cdots, s_n) d(s_{n+1}) = 0 \end{aligned} \tag{4.2}$$

By property (i),

$$\int_S f(s_1, \ldots, s_n) d(s_{n+1}) = f(s_1, \ldots, s_n), \tag{4.3}$$

and by property (ii),

$$\int_S f(s_1, \ldots, s_{n-1}, s_n s_{n+1}) d(s_{n+1}) = \int_S f(s_1, \ldots, s_{n-1}, s_{n+1}) d(s_{n+1}). \tag{4.4}$$

Let $g : S^{n-1} \to \mathbb{C}$ be defined by

$$g(s_2, \ldots, s_n) = \int_S f(s_2, \ldots, s_n, s_{n+1}) d(s_{n+1}).$$

By Lemma 4.2, the map $F : S^{n-1} \to \mathbf{C}(S)$, defined by

$$F(s_2, \ldots, s_n)(x) = f(s_2, \ldots, s_n, x) \qquad (x \in S),$$

is continuous with the norm of $\mathbf{C}(S)$. On the other hand, $\int : \mathbf{C}(S) \to \mathbb{C}$ is also continuous with the norm. Thus, the map $g = \int F$ is in $\mathscr{C}_{cb}^{n-1}(S)$. Therefore, by (4.2), (4.3) and (4.4), we have,

$$(-1)^n f(s_1, \ldots, s_n) = g(s_2, \ldots, s_n) + \sum_{i=1}^{n-1} (-1)^i g(s_1, \ldots, s_i s_{i+1}, \ldots, s_n) + (-1)^n g(s_1, \ldots, s_{n-1})$$

But the right hand side of the latter equation is $\delta^{n-1}(g)$. Thus,

$$f = \delta^{n-1}((-1)^n g).$$

Therefore we have proved $\mathscr{B}_{cb}^n(S) = \mathscr{Z}_{cb}^n(S)$ and $\mathscr{H}_{cb}^n(S) = 0$. A similar proof can be given in the case of right amenable S. □

5. Some examples

Gromov ([2]) proved that for any connected manifold X, and any $n \geq 1$, the bounded cohomology of X and the bounded cohomology of the fundamental homotopy group $\pi_1(X)$ of X coincide (for more details see [2, 10, 12], and [11]). Thus, there are many discrete groups that their bounded cohomology groups are non zero.

Let G be a discrete group and S be a topological semigroup with a unite. Suppose that for an integer $n \geq 2$, $\mathscr{H}^n_{cb}(G) \neq 0$ (for example $G = F_2$, the free group on two generators, and $n = 2$, see [11], [19]). Then by Proposition 3.5 we have $\mathscr{H}^n_{cb}(G \times S) \neq 0$.

A semigroup S is called *semilattice* if it is commutative and $ss = s$ for every $s \in S$.

Theorem 5.1. *Let S be a topological semilattice. Then $\mathscr{H}^2_{cb}(S)$ is zero.*

Proof. Let $f \in \mathscr{Z}^2_{cb}(S)$ be a 2-cocycle. We need a $g \in \mathscr{C}^1_{cb}(S)$ such that for every $s,t \in S$,

$$f(s,t) = g(s) + g(t) - g(s,t).$$

Since f is a 2-cocycle, for every $s_1, s_2, s_3 \in S$ we have

$$\delta^2(f)(s_1,s_2,s_3) = f(s_2,s_3) - f(s_1s_2,s_3) + f(s_1,s_2s_3) - f(s_1,s_2) = 0. \tag{5.1}$$

By applying (5.1) with $s_1 = s, s_2 = s, s_3 = t$, we obtain

$$f(s,s) = f(s,st) \tag{5.2}$$

and similarly

$$f(t,t) = f(t,st). \tag{5.3}$$

By applying (5.1) with $s_1 = s, s_2 = t, s_3 = st$, we obtain

$$f(s,t) = f(t,st) - f(st,st) + f(s,st) \tag{5.4}$$

Now, by (5.2), (5.3) and (5.4), we have

$$f(s,t) = f(t,t) + f(s,s) - f(st,st).$$

Thus, if we define $g(s) = f(s,s)$ $(s \in S)$, then $g \in \mathscr{C}^1_{cb}(S)$ and $\delta^1(g) = f$. □

Remark 5.2. *(a) The above result follows directly from Theorem 4.3, when S is compact.*

(b) In [20], Y. Choi proved that for any discrete semilattice S and any symmetric Banach $\ell^1(S)$ bimodule E, every Hochschild cohomology group of $\ell^1(S)$ with coefficient in E is trivial (see Remark of Section 1). Thus, for every discrete semilattice S and $n \geq 0$, $\mathscr{H}^n_b(S) = 0$.

Example 5.3. *Let (X,d) be a metric space and let for every $A \subset X$ and $\varepsilon > 0$, $N_\varepsilon(A)$ be the ε-neighborhood of A in X. Let S_X be the set of all nonempty closed bounded subsets of X. Then S_X is a metric space by the following metric that is called Hausdorff distance:*

$$d_H(C_1,C_2) = \inf\{\varepsilon > 0 : C_1 \subset N_\varepsilon(C_2) \text{ and } C_2 \subset N_\varepsilon(C_1)\},$$

for $(C_1, C_2 \in S_X)$. Also S_X is a semilattice with the multiplicaton $C_1C_2 = C_1 \cup C_2$. It is easily checked that S_X is a topological semilattice with the topology induced by d_H. Note that if (X,d) is a compact metric space then so is (S_X, d_H), [21, Lemma 5.31].

Let (X,d) and (Y,ρ) be disjoint compact metric spaces with distinguished elements $x_0 \in X$ and $y_0 \in Y$. Let S and T be compact semigroups defined in Example of Section 3, using X and Y. Define the metric d' on $T = X \cup Y$ by $d'|_{X\times X} = d$, $d'|_{Y\times Y} = \rho$ and $d'(x,y) = 1$ for $x \in X, y \in Y$. Then T together with d', satisfy the conditions of Theorem 3.3. Also, S together with the maximum metric satisfy the conditions of Theorem 3.3, but it is not left nor right amenable, and thus, we can not apply Theorem 4.3, to conclude that the cohomology groups of S are zero.

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Computable Proximity of ℓ_1-Discs on the Digital Plane

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Abstract
This paper investigates problems in the characterization of the proximity of digital discs. Based on the l_1-metric structure for the 2D digital plane and using a Jaccard-like metric, we determine numerical characters for intersecting digital discs.

Keywords: Digital discs, Jaccard like metric, l_1-metric, Proximity
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1. Introduction

In pure and applied mathematics, one of the important questions is connected to the discovery of proximal objects [1]. The objects often can be represented as sets of points and this stipulates that set-theoretic and topological methods are very useful tools in the study of proximity relations. Digital geometry is the study of geometric properties of shapes in digital pictures.

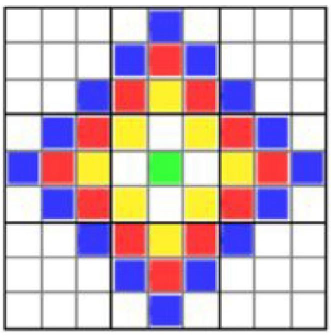


Figure 1.1. Structure of the Digital Discs

Many different computer screen images can be obtained via pixel lighting. A *pixel* is the smallest element on the digital plane and they are usually identified as points. In other words, we can describe images on the computer screen by their pixels that have digital valued coordinates, *i.e.*, a mathematical model of the computer screen is the digital plane $\mathbb{Z}^2$.

The importance of the notions of the circle and disc in Euclidean geometry is well known. In digital geometry, digital circles and digital discs have various important properties that are different from the Euclidean ones (see, *e.g.*, [2–4]). One of the reasonable realizations of metric structure on the digital plane $\mathbb{Z}^2$ can be determined via the so-called l_1 metric. This metric has the following view:

$$d(p_1,p_2)=|a_1-a_2|+|b_1-b_2|,\text{where } p_1 \text{ and } p_2 \text{ are some matched points,}$$
$$\text{with coordinates } (a_1,b_1) \text{ and } (a_2,b_2)\text{. respectively,}$$

i.e., p_1 and p_2 are pixels for our future considerations. Since we can represent pixel coordinates as digital pairs, then it is obvious that $d(p_1,p_2)\in\mathbb{Z}$ (the integers).

Based on the l_1 metric, we define a digital circle with radius r and center x (denoted by $C_d(x,r)$) as follows:

$$C_d(x,r)=\left\{z\in\mathbb{Z}^2 : d(x,z)=r\right\}.$$

Moreover, we denote by $c(C_d(x,r))$ the circumference of the circle $C_d(x,r)$ where $r\in\mathbb{N}\cup\{0\}$.

It is well-known that if r is a natural number, we have $\pi_{l_1}=\frac{c(C_d(x,r))}{\text{diam}(C_d(x,r))}=\frac{8r}{2r}=4$, where $\text{diam}(C_d(x,r))$ is the diameter of the circle $C_d(x,r)$. Using this fact, we easily obtain the following result.

Lemma 1.1. *Let $C_d(x,r)$ be a digital circle with center at point x and radius r relative to the l_1 metric. Then, for the number of pixels of $C_d(x,r)$, we have the formula*

$$card(C_d(x,r))=\frac{2c(C_d(x,r))}{\pi_{l_1}}=4r.$$

Fig. 1.1 demonstrates the structural property of the digital disc, namely,

$$D_d(x,R)=\left\{z\in\mathbb{Z}^2 \mid d(x,z)\le R\right\}, \text{ particularly:}$$
$$D_d(x,R)=\{x\}\cup\left(\bigcup_{r=1}^{R}C_d(x,r)\right), \text{ where } R\in\mathbb{N}.$$

Lemma 1.2. *If $D_d(x,R)$ is a digital disc relative to the l_1 metric d, then the number of pixels forming the disc $D_d(x,R)$ can be computed by the formula $card(D_d(x,R))=2R^2+2R+1$.*

Proof. Since $D_d(x,R)=\{x\}\cup\left(\bigcup_{r=1}^{R}C_d(x,r)\right)$, we can write

$$\text{card}(D_d(x,R))=1+\text{card}(C_d(x,1))+\text{card}(C_d(x,2))+\cdots+\text{card}(C_d(x,R)).$$

Now, applying Lemma 1.1, we get

$$\begin{aligned}\text{card}(D_d(x,R))&=1+4+8+\cdots+4R=\\&=1+4\left(\frac{1+R}{2}R\right)=\\&=2R^2+2R+1.\end{aligned}$$

□

2. How near are digital discs?

To solve a wide class of the problems of computational proximity, the Hausdorff metric is appropriate. The Hausdroff metric (denoted by $d_H(A,B)$) measures the distance between the sets A,B in the given metric space (X,d) and is defined by

$$d_H(A,B)=\max\left\{\sup_{x\in A}\inf_{y\in B}d(x,y),\sup_{y\in B}\inf_{x\in A}d(x,y)\right\}.$$

If the sets A,B are finite, we obtain the simplication of the Hausdorff metric by maxima and minima, *i.e.*,

$$d_H(A,B) = \max\left\{\max_{x\in A}\min_{y\in B} d(x,y), \max_{y\in B}\min_{x\in A} d(x,y)\right\}.$$

It is clear that even in the case of finite sets, computation of the Hausdorff distances are quite capacitive. These difficulties can be bypassed in some special cases of analytical sets. Below, we are interested in characterization of intersecting digital discs.

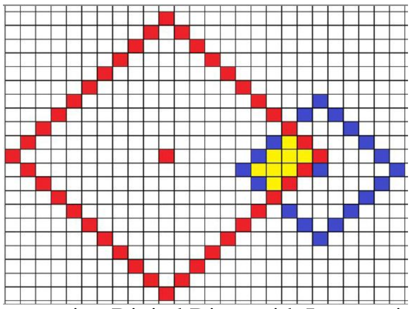


Figure 2.1. Intersecting Digital Discs with Intersecting Boundaries

Classification of images in computer science frequently need the application of Jaccard-like metrics [5], [6], [7] . We will use a simplified version to analyze proximity of intersecting digital discs. It must be especially noticed that the problem connected with the intersection of plane discs was considered from a computer science perspective in [8].

For the Jaccard-like metric m, we understand the distance function defined via the cardinality of the symmetric difference of two arbitrary nonempty finite sets A and B, *i.e.*,

$$\begin{aligned} m(A,B) &= \operatorname{card}(A \triangle B) \\ &= \operatorname{card}(A \setminus B) + \operatorname{card}(B \setminus A) \\ &= \operatorname{card}(A) + \operatorname{card}(B) - 2\operatorname{card}(A \cap B). \end{aligned}$$

It is obvious that if $\operatorname{card}(A) \neq \operatorname{card}(B)$ and both sets are finite while $A \cap B \neq \varnothing$, we get $m(A,B) \neq 0$. This raises the question of the computation of the proximity of intersecting digital discs such as the ones in Fig. 2.1.

Theorem 2.1. *Let $D_d(x,R_1)$ and $D_d(y,R_2)$ be digital discs such that $C_d(x,R_1) \cap C_d(y,R_2) \neq \varnothing$. Then*

$$m(D_d(x,R_1), D_d(y,R_2)) = 2\left(R_1^2 + R_2^2 + R_1 + R_2 - 2kn + k + n\right),$$

where k and n denote the number of pixels forming the width and height of the greatest rectangle subset of an intersection set.

Proof. Applying Lemma 1.2, we obtain the following cardinal equalities:

$$\begin{aligned} m(D_d(x,R_1), D_d(y,R_2)) &= \operatorname{card}(D_d(x,R_1)) + \operatorname{card}(D_d(y,R_2)) - 2\operatorname{card}(D_d(x,R_1) \cap D_d(y,R_2)) \\ &= 2\left(R_1^2 + R_2^2 + R_1 + R_2 + 1)\right) - 2\operatorname{card}(D_d(x,R_1) \cap D_d(y,R_2)) \\ &= 2\left(R_1^2 + R_2^2 + R_1 + R_2 + 1\right) - 2[kn + (k-1)(n-1)] \\ &= 2\left(R_1^2 + R_2^2 + R_1 + R_2 - 2kn + k + n\right) \end{aligned}$$

□

Notice that there is a situation in which two digital discs are intersecting but their boundaries are not intersecting (see, *e.g.*, Fig.2.2). Observe that in that case, we have $C_d(x,R_1-1) \cap C_d(y,R_2) \neq \varnothing$, or, equivalently, $C_d(x,R_1) \cap C_d(y,R_2-1) \neq \varnothing$.

Theorem 2.2. *Let $D_d(x,R_1)$ and $D_d(y,R_2)$ be digital discs such that $C_d(x,R_1) \cap C_d(y,R_2) = \varnothing$, but $C_d(x,R_1-1) \cap C_d(y,R_2) \neq \varnothing$. Then we have $m(D_d(x,R_1), D_d(y,R_2)) = 2\left(R_1^2 + R_2^2 + R_1 + R_2 + 1 - 2kn\right)$, where k and n denote the number of pixels forming the width and height of the greatest rectangle subset of an intersection set.*

Proof. In this case, we can easily note that $\operatorname{card}(D_d(x,R_1) \cap D_d(y,R_2)) = 2kn$. Hence, we have $\mathrm{m}(D_d(x,R_1), D_d(y,R_2)) = 2\left(R_1^2 + R_2^2 + R_1 + R_2 + 1 - 2kn\right)$. □

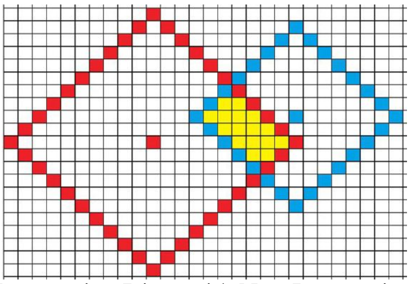
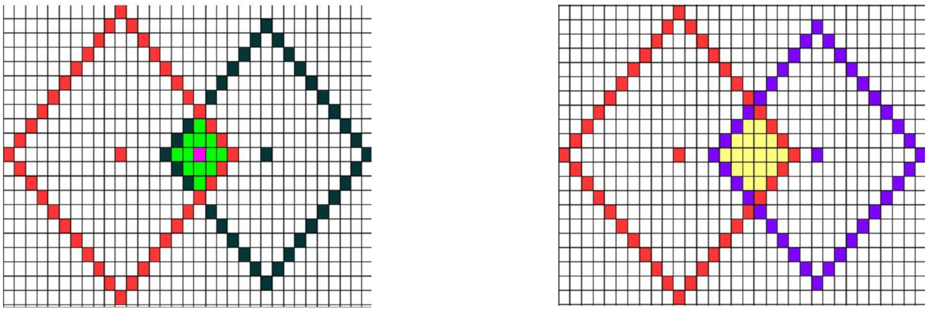


Figure 2.2. Intersecting Discs with Non-Intersecting Boundaries

Next, we need to represent the centers x and y of discs $D_d(x,R_1)$ and $D_d(y,R_2)$ by a couple of digital coordinates as follows: $x=(\alpha,\beta)$ and $y=(\gamma,\delta)$. If one of the following equalities hold $d(x,y)=|\alpha-\gamma|$ or $d(x,y)=|\beta-\delta|$, *i.e.*, the centers of the discs lie on horizontal or vertical axes (similar to the situations shown in Fig. 2.3.1 and Fig. 2.3.2), then we can measure the proximity of the discs via computation of the pixel cardinality of the intersections sets.



2.3.1: Intersecting Discs with Intersecting Boundaries

2.3.2: Intersecting Discs with Non-Intersecting Boundaries

Figure 2.3. Intersecting Discs on the Digital Plane

Theorem 2.3. *Let $D_d(x,R_1)$ and $D_d(y,R_2)$ be digital discs such that $x=(\alpha,0)$ and $y=(\gamma,0)$ with $\alpha<\gamma$ and $\gamma-\alpha\leq R_1+R_2$. If $C_d(x,R_1)\cap C_d(y,R_2)\neq\varnothing$, then*

$$m(D_d(x,R_1),D_d(y,R_2))=(R_1-R_2)^2+2(R_1+R_2+1)(\gamma-\alpha)-(\gamma-\alpha)^2.$$

Proof. Since $x=(\alpha,0)$, $y=(\gamma,0)$ and $C_d(x,R_1)\cap C_d(y,R_2)\neq\varnothing$, we claim that

$$C_d(x,R_1)\cap C_d(y,R_2)=C_d(k,r), \text{ where,}$$

$k=\left(\frac{\alpha+R_1+\gamma-R_2}{2},0\right)$ and
$r=R_1-(k-\alpha)=\frac{R_1+R_2+(\gamma-\alpha)}{2}\in\mathbb{N}\cup\{0\}$. Consequently, simplification of

$$\mathrm{m}(D_d(x,R_1),D_d(y,R_2))=2\left(R_1^2+R_2^2+R_1+R_2+1-2r^2-2r-1\right)$$

gives the needed expression

$$\mathrm{m}(D_d(x,R_1),D_d(y,R_2))=(R_1-R_2)^2+2(R_1+R_2+1)(\gamma-\alpha)-(\gamma-\alpha)^2.$$

□

Observe that Theorem 2.3 can be applied in similar cases when the intersection set of the digital discs itself is a disc.

This leads us to consider two intersecting digital discs with non-intersecting boundaries (see, *e.g.*, Fig. 2.3.2) so that both centers lie on the horizontal or vertical axes. In such cases, we obtain the following result.

Corollary 2.4. *Let $D_d(x,R_1)$ and $D_d(y,R_2)$ be intersecting digital discs that satisfy the conditions of Theorem 2.3, but $C_d(x,R_1) \cap C_d(y,R_2) = \varnothing$. Then we have*

$$m(D_d(x,R_1), D_d(y,R_2)) = 2\left(R_1^2 + R_2^2 + R_1 + R_2 - 2r_0^2 - 4r_0 + 1\right), where,$$
$$r_0 = \frac{R_1 - 1 + R_2 + (\gamma - \alpha)}{2}.$$

Recall that a boundary point x_0 of a convex set C is called a *support point* [9, p. 27].

Lemma 2.5. [Bishop-Phelps [9]] *Suppose M is a closed subspace of finite co-dimension in a topological vector space X, and that C is a convex subset of X. Suppose x_0 is a support point of $C \cup M$ in the subspace M. Then x_0 is a support point of C.*

Let A,B be nonempty sets and let bdy$A$ denote the set of boundary points of a nonempty set A. Also, let $A \overset{\wedge\wedge}{\delta} B$ denote that A and B are overlapping sets. From Lemma 2.5, we obtain

Theorem 2.6. *Let A,B be nonempty sets of digital discs. If A,B are convex sets in a subspace M of the Euclidean plane intersect, then*
1º $A \cap B$ is a convex set.
2º $A \overset{\wedge\wedge}{\delta} B$ (A and B are strongly near).
3º $x_0 \in bdy(A \cap B)$ is a support point of $A \cap B$.

Proof.

1º Immediate, since the intersection of any two convex sets is a convex set.
2º Let int$A$, int$B$ be the interior of A,B, respectively. From Theorem 2.6.1, $A \cap B \neq \varnothing$ implies int$A \cap$ int$B \neq \varnothing$. Hence, from [10, §2.3], $A \overset{\wedge\wedge}{\delta} B$.
3º Immediate from Lemma 2.5.

□

Example 2.7. *The blue and red pixels on the boundary of the intersecting discs in Fig. 2.3.1 are support points.* ■

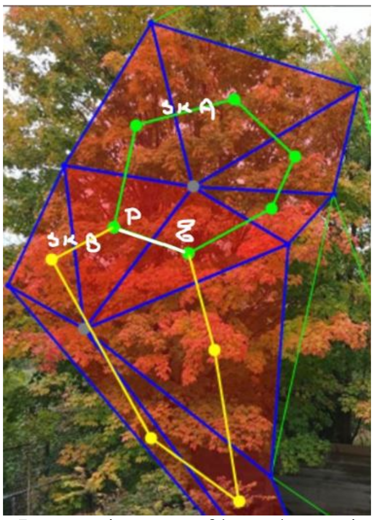


Figure 2.4. Intersecting sets of boundary points sk$A$, sk$B$

In what follows, we give an application of Theorem 2.6.2, namely, digital discs A,B are convex sets in a subspace M of the Euclidean plane intersect, provided

$A \overset{\wedge\wedge}{\delta} B$ (A and B are strongly near).

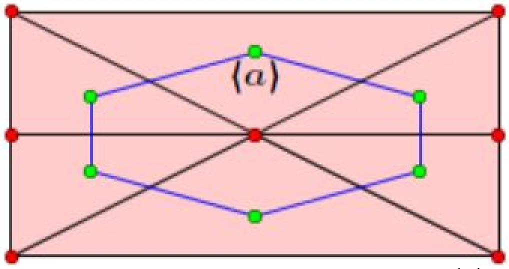


Figure 2.5. Boundary with support point $\langle a \rangle$

3. Application: Classifying triangulated digital images

Recall that an Alexandroff nerve on a triangulated 2D surface is a set of triangles with a common vertex (called the nucleus of the nerve), introduced by P. Alexandroff [Aleksandrov] [11, §31, p. 39], [12] and elaborated in [13, Vol. 3, p. 67], [14, §2.11, pp. 160-161]. Such a nerve with nucleus p is maximal, provided the number of triangles attached to p is highest [15]. It is possible for more than one Alexandroff nerve to be maximal on the same triangulated image (see, for example, Fig. 2.4). This observation leads to an application of Theorem 2.6.2 in classifying triangulated digital images.

Let $\mathrm{sk}A, \mathrm{sk}B$ be sets of boundary points on polygons whose vertexes are barycenters on an Alexandroff nerve in a triangulaged digital image *img* (see, for example, the set of boundary points that includes a support point $\langle a \rangle$ in Fig. 2.5). Also let I be a collection of triangulated digital images.

We can then derive a collection $\mathfrak{C}(I)$ of classified triangulated digital images containing intersecting support points on boundary sets on barycentric polygons on maximal Alexandroff nerves defined by

$$\mathfrak{C}(I) = \overbrace{\left\{ img \in I : \mathrm{sk}A, \mathrm{sk}B \in img \ \&\ A \overset{\wedge\!\!\wedge}{\delta} B \right\}}^{\textbf{Images containing overlapping boundary sets}}. \qquad \blacksquare$$

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Characterizations of Hayashi-Samuel Spaces via Boundary Points

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Abstract
Some new closure operators in topological spaces with ideals are a part of this paper. A comparative study of a new type of boundary point, which is defined with the help of the local function and the boundary points will be discussed through this paper. Characterizations of Hayashi-Samuel spaces are also an object of this paper.

Keywords: Ideal topological space, Hayashi-Samuel space, Local function, ψ-operator.
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1. Introduction and preliminaries

A modification of closure operator in topological space is the local function in ideal topological space. This study was introduced by Kuratowski [1] and Vaidyanathswamy [2]. An ideal [1] $\mathscr{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies the following conditions:

(1) $A \in \mathscr{I}$ and $B \subseteq A$ implies $B \in \mathscr{I}$,

(2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$.

A topological space (X,τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X,\tau,\mathscr{I})$. For a subset A of an ideal topological space $(X,\tau,\mathscr{I})$, the local function A^* is defined as: $A^* = \{x \in X : U_x \cap A \notin \mathscr{I}, U_x \in \tau(x)\}$ (where $\tau(x)$ is the collection of all open sets which contains x) and it was defined by imposing extra condition on the closure operator. As a result, the mathematicians like Samuel [3], Pavlović [4], Hayashi [5], Hashimoto [6], Janković and Hamlett [7, 8], Ekici [9, 10, 11], Hatir [12], Noiri [11, 12, 13] have reached to obtain a new topology known as $*$-topology and it is finer topology than the original topology. In an ideal topological space $(X,\tau,\mathscr{I})$, the structures-"topology" and"ideal" played important roles simultaneously. The condition $\tau \cap \mathscr{I} = \{\emptyset\}$ is a remarkable part in ideal topological space and such ideal topological space is called Hayashi-Samuel space [14]. Modak and his associates studied this ideal topological space and introduced different types of generalized open sets and operators with the help of ideals (see [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26]). The complement operator of the local function is known as ψ-operator [8, 27] and it is defined by: $\psi(A) = X \setminus (X \setminus A)^*$, for a subset A of an ideal topological space $(X,\tau,\mathscr{I})$. ψ-operator is an important part for the study of ideal topological space.

In this paper, we introduce a new type of boundary points in ideal topological spaces by using $*$ - operator. We consider a comparative study of these boundary points with the boundary points in topological spaces. We also explore the characterizations of Hayashi-Samuel space which was established in [18, 19, 24]. We also obtain more closure operators in ideal topological spaces through this paper.

2. $*$ boundary points

Boundary operator [28] is a set valued set-function and we may consider it by the following way:

Let (X,τ) be a topological space and $A \subseteq X$. The boundary operator $Bd : \wp(X) \to C(\tau)$ is defined as $Bd(A) = Cl(A) \cap Cl(X \setminus A)$, where $C(\tau)$ denotes the collection of all closed sets and $Cl(A)$ denotes the closure of A in (X,τ).

Thus boundary point of a set $A \subseteq X$ is a common point between closure of A and closure of $(X \setminus A)$.

We modify the boundary operator with the help of the local function and call it $*$-boundary operator.

Definition 2.1. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space. The operator $Bd^* : \wp(X) \to C(\tau)$, defined by: $Bd^*(A) = A^* \cap (X \setminus A)^*$, for $A \in \wp(X)$, is called $*$-boundary operator on $(X,\tau,\mathscr{I})$.*

The point $x \in Bd^*(A)$ is called $*$-boundary point of A and it is the common point of A^* and $(X \setminus A)^*$.

We start with the following example which shows that there is some common points in A^* and $(X \setminus A)^*$.

Example 2.2. *Let $X = \mathbb{R}$, $\mathbb{R}_u$ be the usual topology on $\mathbb{R}$ and $\mathscr{I} = \{\emptyset\}$. Then $\mathbb{Q}^* = Cl(\mathbb{Q}) = \mathbb{R}$ and $(\mathbb{R} \setminus \mathbb{Q})^* = Cl(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$. This shows that there are common points between $\mathbb{Q}^*$ and $(\mathbb{R} \setminus \mathbb{Q})^*$.*

We know that boundary points of a set depends on the topology. For this, if we consider the indiscrete topology on $\mathbb{R}$, then $Bd(\mathbb{Q}) = \mathbb{R}$, where $\mathbb{Q}$ denotes the set of all rational numbers. But if we consider the discrete topology on $\mathbb{R}$, then $Bd(\mathbb{Q}) = \emptyset$.

$*$-boundary point of a set depends on not only the topology but the ideal also.

Followings examples show the role of ideal in $*$-boundary points:

Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$.

(i) If we take $\mathscr{I} = \{\emptyset\}$, then $Bd^*(A) = Bd(A)$.

(ii) If the ideal $\mathscr{I} = \wp(X)$, $Bd^*(A) = \emptyset$.

Note that in discrete topological space, boundary points of any set is always empty. But in any ideal topological space, if the ideal is the collection of all subsets of the set then $*$-boundary points of any set is always empty.

(iii) When the ideal $\mathscr{I} = \mathscr{I}_f$, the ideal of finite subsets of X, then $Bd^*(A)$ is the ω-accumulation points of A and $X \setminus A$.

(iv) If one choose the ideal $\mathscr{I} = \mathscr{I}_c$, the ideal of countable subsets of X, then A^* is precisely the set of condensation points of A and boundary points accordingly.

(v) Let $\mathscr{I}_n$ be the collection of all nowhere dense subsets of (X,τ), then $\mathscr{I}_n$ is an ideal on X. If we take $\mathscr{I} = \mathscr{I}_n$, then $A^* = Cl(Int(Cl(A)))$ and $Bd^*(A) = Cl(Int(Cl(A))) \setminus Int(Cl(Int(A)))$.

(vi) Let (X,τ) be a topological space and $\mathscr{I}_m$ be the collection of all meager sets (or sets of first category). Then it forms an ideal on X and A^* is set the points of second category of A.

Note that for a subset $A \subseteq X$ in a topological space (X,τ) with an ideal $\mathscr{I}$, $x \in Bd^*(A)$ implies $U_x \notin \mathscr{I}$ for all $U_x \in \tau(x)$ but converse statement is not true in general.

Example 2.3. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{a,b\}\}$ and $\mathscr{I} = \{\emptyset, \{a\}\}$. Then $(\{b\})^* = \{b,c\}$ and all open sets containing a do not belongs to $\mathscr{I}$ but $a \notin Bd^*(\{b\})$.*

One of the characterizations of $*$-boundary point is:

Theorem 2.4. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then $x \in Bd^*(A)$ if and only if $x \in A^* \setminus \psi(A)$.*

Similar characterization of boundary point is:

Theorem 2.5. *[28] Let (X,τ) be a topological space and $A \subseteq X$. Then $x \in Bd(A)$ if and only if $x \in Cl(A) \setminus Int(A)$, where $Int(A)$ denotes the interior of A.*

Theorem 2.6. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then $Bd^*(A) = \emptyset$ if and only if $A^* \subseteq \psi(A)$.*

Similar characterization of boundary point is:

Theorem 2.7. *[28] Let (X,τ) be a topological space and $A \subseteq X$. Then $Bd(A) = \emptyset$ if and only if A is both open and closed.*

Note that $()^*$ is not a closure operator and ψ is not an interior operator, but both $A^* \setminus \psi(A)$ and $Cl(A) \setminus \psi(A)$ are closed set. In this regards, $A \cap \psi(A)$ is an interior operator [8] and $A \cup A^*$ is a closure operator [7] and both the operators induce the same topology which is above $*$-topology [7].

Corollary 2.8. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space and $A \subseteq X$. Then $Bd^*(A) = \emptyset$ if and only if $A^* = \psi(A)$.*

Proof. Proof is obvious from Theorem 2.6 and the following lemma. □

Lemma 2.9. *[16] Let $(X,\tau,\mathscr{I})$ be A be a Hayashi-Samuel space and $A \subseteq X$. Then $\psi(A) \subseteq A^*$.*

Theorem 2.10. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then $Bd^*(A) = (X\setminus A)^*$ if and only if $X\setminus A^* \subseteq \psi(A)$.*

Proof. Suppose $Bd^*(A) = (X\setminus A)^*$. Then $A^* \cap (X\setminus A)^* = (X\setminus A)^*$ implies $(X\setminus A)^* \subseteq A^*$. Therefore $X\setminus A^* \subseteq \psi(A)$.
Proof of the converse part is obvious. □

Theorem 2.11. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and A be a $\mathscr{I}$-dense subset of X. Then $Bd^*(A) = (X\setminus A)^*$.*

Proof. Obvious from definition of $\mathscr{I}$-dense set (A subset A of X is said to be $\mathscr{I}$-dense [14] if $A^* = X$). □

Now we look how the $*$-boundary operator gives new closure operator:

Theorem 2.12. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A,B \subseteq X$. Then following statements hold:*

1. $Bd^*(\emptyset) = \emptyset$.
2. $Bd^*(X) = \emptyset$.
3. $Bd^*(I) = \emptyset$, *if* $I \in \mathscr{I}$.
4. $Bd^*(A)$ *is a closed set in* (X,τ).
5. $Bd^*(A\cup B) \subseteq Bd^*(A)\cup Bd^*(B)$.
6. $Bd^*(A)\cup Bd^*(B) = [A\cap Bd^*(B)]\cup[Bd^*(A\cup B)]\cup[Bd^*(A)\cap B]$.
7. $Bd^*(A) = A^*\setminus \psi(A)$.
8. $Cl^*(A) = Bd^*(A)\cup\psi(A)\cup A$ *(Cl^* denotes the closure operator of $*$-topology).*
9. $Bd^*(A) = \emptyset$ *implies* $Int^*(A) \supseteq A\cap A^*$ *(Int^* denotes the interior operator of $*$-topology).*
10. $Bd^*(Bd^*(A)) \subseteq Bd^*(A)$.
11. $Bd^*(A) = (X\setminus A)^*\setminus\psi(X\setminus A)$.
12. $Bd^*(X\setminus A) = Bd^*(A)$.
13. $Bd^*(A) \subseteq Bd_{\tau^*(\mathscr{I})}(A) \subseteq Bd(A)$ *($Bd_{\tau^*(\mathscr{I})}(A)$ denotes the set of all boundary points of A with respect to $*$-topology).*
14. $X\setminus Bd^*(A) = \psi(X\setminus A)\cup\psi(A)$.
15. $X = \psi(X\setminus A)\cup\psi(A)\cup Bd^*(A) = \psi(X\setminus A)\cup\psi(A)\cup Bd^*(X\setminus A)$.

Proof. The proofs of 1., 2., 3. and 4. are obvious.

5. $Bd^*(A\cup B) = (A\cup B)^*\cap(X\setminus A\cup B)^* = (A\cup B)^*\cap[(X\setminus A)\cap(X\setminus B)]^* \subseteq (A^*\cup B^*)\cap[(X\setminus A)^*\cap(X\setminus B)^*] = [[(X\setminus A)^*\cap(X\setminus B)^*]\cap A^*]\cup[[(X\setminus A)^*\cap(X\setminus B)^*]\cap B^*] \subseteq [A^*\cap(X\setminus A)^*]\cup[B^*\cap(X\setminus B)^*] = Bd^*(A)\cup Bd^*(B)$.

6. Note that $[A\cap Bd^*(B)]\cup[Bd^*(A)\cap B]\cup[Bd^*(A\cup B)] \subseteq Bd^*(B)\cup Bd^*(B)\cup Bd^*(A\cup B) = Bd^*(A)\cup Bd^*(B)$ (from 4.).

Again $Bd^*(A)\cup Bd^*(B) \subseteq Bd^*(A)\cup Bd^*(B)\cup Bd^*(A\cup B)\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)] = [(A^*\cap(X\setminus A)^*\cup(B^*\cap(X\setminus B)^*]\cup Bd^*(A\cup B)\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)] \subseteq [(A^*\cup B^*)\cup Bd^*(A\cup B)]\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)] = [(A\cup B)^*\cap(X\setminus(A\cup B)^*]\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)] = Bd^*(A\cup B)\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)]$.

7. $Bd^*(A) = A^*\cap(X\setminus A)^* = A^*\cap(X\setminus\psi(A)) = A^*\setminus\psi(A)$.

8. $Bd^*(A)\cup\psi(A)\cup A = (A^*\setminus\psi(A))\cup\psi(A)\cup A = A^*\cup A = Cl^*(A)$.

9. Given that $Bd^*(A) = \emptyset$. Then $A^* \subseteq \psi(A)$ and hence $A\cap A^* \subseteq Int^*(A)$.

10. $Bd^*(Bd^*(A)) = Bd^*[A^*\cap(X\setminus A)^*] = [A^*\cap(X\setminus A)^*]^*\cap(X\setminus[A^*\cap(X\setminus A)^*])^* \subseteq A^{**}\cap(X\setminus A)^{**} \subseteq A^*\cap(X\setminus A)^* = Bd^*(A)$.

11. $Bd^*(A) = (X\setminus A)^*\cap A^* = (X\setminus A)^*\cap[X\setminus\psi(X\setminus A)] = (X\setminus A)^*\setminus\psi(X\setminus A)$.

12. The proof of 12. is obvious from definition.

13. The proof is obvious from the fact $A^* \subseteq Cl^*(A) \subseteq Cl(A)$, for any subset A of X.

14. $X\setminus Bd^*(A) = (X\setminus A^*)\cup[X\setminus(X\setminus A)^*] = \psi(X\setminus A)\cup\psi(A)$.

15. The proof of 15. is obvious from definition. □

Definition 2.13. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A,B\subseteq X$. The operator $k_1:\wp(X)\to\wp(X)$ on X is defined by*

$$k_1(A)=A\cup T_1(A),$$

where $T_1:\wp(X)\to\wp(X)$ is an operator which satisfies the following conditions:
(i) $T_1(\emptyset)=\emptyset$,
(ii) $T_1(A\cup B)\subseteq T_1(A)\cup T_1(B)$,
(iii) $Cl^*(A)=T_1(A)\cup\psi(A)\cup A$,
(iv) $T_1(T_1(A))\subseteq T_1(A)$.

Then, k_1 is a closure operator on X,and $T_1(A)=Bd^*(A)$ for every subset A of X, in which the topology is induced by k.
The operator $k_1(A)=A\cup T_1(A)$, satisfies the following conditions:
(i) $k_1(\emptyset)=\emptyset\cup T_1(\emptyset)=\emptyset$;
(ii) $A\subseteq A\cup T_1(A)=k_1(A)$;
(iii) $k_1(k_1(A))=k_1(A\cup T_1(A))=A\cup T_1(A)\cup T_1(A\cup T_1(A))\subseteq A\cup T_1(A)\cup T_1(A)\cup T_1(T_1(A))\subseteq A\cup T_1(A)\cup T_1(A)=k_1(A)$;
(iv) $k_1(A\cup B)=A\cup B\cup T_1(A\cup B)\subseteq A\cup B\cup T_1(A)\cup T_1(B)=k_1(A)\cup k_1(B)$ and $k_1(A)\cup k_1(B)=A\cup T_1(A)\cup B\cup T_1(B)=A\cup B\cup(A\cap T_1(B))\cup T_1(A\cup B)\cup(B\cap T_1(A))\subseteq A\cup B\cup T_1(A\cup B)=k_1(A\cup B)$.
Recall the following lemma:

Lemma 2.14. *[20] An ideal topological space $(X,\tau,\mathscr{I})$ is Hayashi-Samuel if and only if, for each $O\in\tau$, $O^*=Cl(O)$.*

Theorem 2.15. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. Then for each open set U, $Bd^*(U)\subseteq U^*\setminus U$.*

Proof. $Bd^*(U)=U^*\cap(X\setminus U)^*\subseteq Cl(U)\cap Cl(X\setminus U)=U^*\cap(X\setminus Int(U))=U^*\setminus U$, since the space is Hayashi-Samuel. □

We recall the following theorem:

Theorem 2.16. *[19] Let $(X,\tau,\mathscr{I})$ be an ideal topological space. Then, the following properties are equivalent:*

1. *$\tau\cap\mathscr{I}=\{\emptyset\}$;*
2. *$I\in\mathscr{I}$, then $Int(I)=\emptyset$;*
3. *for every $G\in\tau$, $G\subseteq G^*$;*
4. *$X=X^*$;*
5. *if $O\in\tau$, then $O^*=Cl(O)$.*

Theorem 2.17. *An ideal topological space $(X,\tau,\mathscr{I})$ is Hayashi-Samuel if and only if, for each closed set $A\subseteq X$, $Bd^*(A)=A^*\setminus Int(A)$.*

Proof. $Bd^*(A)=A^*\cap(X\setminus A)^*=A^*\cap Cl(X\setminus A)=A^*\setminus Int(A)$, since the space is Hayashi-Samuel.
From the given condition, we have $Bd^*(X)=X^*\setminus Int(X)$. Then $\emptyset=X^*\setminus Int(X)$ (from Theorem 2.12) implies $X^*=X$. Thus, $\tau\cap\mathscr{I}=\{\emptyset\}$. □

Theorem 2.18. *An ideal topological space $(X,\tau,\mathscr{I})$ is Hayashi-Samuel if and only if, for each open set $U\subseteq X$, $Bd^*(U)=Bd(U)$.*

Proof. Suppose $(X,\tau,\mathscr{I})$ is Hayashi-Samuel. Then for $U\in\tau$, $Bd^*(U)=U^*\cap(X\setminus U)^*=Cl(U)\cap Cl(X\setminus U)=Bd(U)$.
Conversely suppose that $Bd^*(U)=Bd(U)$. Then $U^*\cap(X\setminus U)^*=Cl(U)\cap Cl(X\setminus U)$ implies $U^*\cap(X\setminus\psi(U))=Cl(U)\setminus U$. Thus $U^*\setminus\psi(U)=Cl(U)\setminus U$ implies $Cl(U)\setminus U$. Thus $U^*\setminus\psi(U)=Cl(U)\setminus U$ implies $Cl(U)\setminus U\subseteq U^*\setminus U$ (since for open set U, $U\subseteq\psi(U)$ [8]). This implies that $Cl(U)\subseteq U^*$ and hence $U\subseteq Cl(U)\subseteq U^*$. Thus $U\subseteq U^*$. Therefore, $(X,\tau,\mathscr{I})$ is Hayashi-Samuel. □

Corollary 2.19. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space. Then, the following properties are equivalent:*

1. *$\tau\cap\mathscr{I}=\{\emptyset\}$;*
2. *$I\in\mathscr{I}$, then $Int(I)=\emptyset$;*
3. *for every $G\in\tau$, $G\subseteq G^*$;*

4. $X = X^*$;

5. if $O \in \tau$, then $O^* = Cl(O)$;

6. $Bd^*(A) = A^* \setminus Int(A)$;

7. for each $U \in \tau$, $Bd^*(U) = Bd(U)$.

Theorem 2.20. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then for A, $B \subseteq X$, $Bd^*(A) \cup Bd^*(B) = Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$.*

Proof. We have:
(a) $Bd^*(A \cap B) = Bd^*(X \setminus (A \cap B)) = Bd^*[(X \setminus A) \cup (X \setminus B)] \subseteq Bd^*(X \setminus A) \cup Bd^*(X \setminus B) = Bd^*(A) \cup Bd^*(B)$.
(b) $Bd^*(A \setminus B) = Bd^*[A \cap (X \setminus B)] \subseteq Bd^*(A) \cup Bd^*(X \setminus B) = Bd^*(A) \cup Bd^*(B)$.
(c) $Bd^*(B \setminus A) \subseteq Bd^*(A) \cup Bd^*(B)$.
Thus from (a), (b) and (c) $Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A) \subseteq Bd^*(A) \cup Bd^*(B)$.
Further, we have $Bd^*(A) \cup Bd^*(B) = Bd^*[(A \setminus B) \cup (A \cap B)] \cup Bd^*[(B \setminus A) \cup (A \cap B)] \subseteq Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$.
Therefore, $Bd^*(A) \cup Bd^*(B) = Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$. □

Theorem 2.21. *Let A and B be subsets of a topological space (X, τ) with an ideal $\mathscr{I}$. Then the following properties hold:*
(1) $Bd^*(A) \cup Bd^*(B) = Bd^*(A \cap B) \cup Bd^*(A \setminus B) \cup Bd^*(A \cup B)$.
(2) $Bd^*(A) \cup Bd^*(B) = Bd^*(A \cup B) \cup Bd^*(B \setminus A) \cup Bd^*(A \cap B)$.
(3) $Bd^*(A) \cup Bd^*(B) = Bd^*(A \setminus B) \cup Bd^*(B \setminus A) \cup Bd^*(A \cap B)$.
(4) $Bd^*(A) \cup Bd^*(A \Delta B) = Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$ (Δ denotes the symmetric difference).
(5) $Bd^*(B) \cup Bd^*(A \Delta B) = Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$.

Proof. (1) If we put $X \setminus B$ in the relation of the Theorem 2.20 instead of B, then we get,

$$Bd^*(A) \cup Bd^*(X \setminus B) = Bd^*(A \setminus (X \setminus B)) \cup Bd^*(A \cap (X \setminus B)) \cup Bd^*((X \setminus B) \setminus A).$$

This implies that

$$Bd^*(A) \cup Bd^*(B) = Bd^*(A \cap B) \cup Bd^*(A \setminus B) \cup Bd^*(A \cup B).$$

(2) If we put $X \setminus A$ in the relation of the Theorem 2.20 instead of A, then we get,

$$Bd^*(X \setminus A) \cup Bd^*(B) = Bd^*((X \setminus A) \setminus B) \cup Bd^*((X \setminus A) \cap B) \cup Bd^*(B \setminus (X \setminus A)).$$

This implies that

$$Bd^*(A) \cup Bd^*(B) = Bd^*(A \cup B) \cup Bd^*(B \setminus A) \cup Bd^*(A \cap B).$$

(3) If we put $X \setminus A$ instead of A and $X \setminus B$ instead of B in the relation of the Theorem 2.20 we get,
$Bd^*(X \setminus A) \cup Bd^*(X \setminus B) = Bd^*[(X \setminus A) \setminus (X \setminus B)] \cup Bd^*[(X \setminus A) \cap (X \setminus B)] \cup Bd^*[(X \setminus B) \setminus (X \setminus A)]$.
This implies that

$$Bd^*(A) \cup Bd^*(B) = Bd^*(B \setminus A) \cup Bd^*(A \cup B) \cup Bd^*(A \setminus B).$$

(4) From Theorem 2.20,
$Bd^*(A) \cup Bd^*(A \Delta B) = Bd^*[A \setminus (A \Delta B)] \cup Bd^*[A \cap (A \Delta B)] \cup Bd^*[(A \Delta B) \setminus A] = Bd^*(A \cap B) \cup Bd^*(A \setminus B) \cup Bd^*(B \setminus A) = Bd^*(B) \cup Bd^*(A \Delta B)$.
(5) The proof of (5) is obvious from (4). □

We have from Theorem 2.21, the union of any two distinct elements of $\{Bd^*(A),\ Bd^*(B),\ Bd^*(A \Delta B)\}$ is equal to the union of any three distinct elements of $\{Bd^*(A \cup B),\ Bd^*(A \cap B),\ Bd^*(A \setminus B),\ Bd^*(B \setminus A)\}$

Definition 2.22. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. The operator $()^{*-} : \wp(X) \to \wp(X)$ is defined as:*

$$A^{*-} = A^* \setminus A, \text{ for } A \subseteq X.$$

Theorem 2.23. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A, $B \subseteq X$, then following conditions hold:*

1. $\emptyset^{*-} = \emptyset$;
2. $A \cap A^{*-} = \emptyset$;
3. $(A \cup B)^{*-} = (A^{*-} \setminus B) \cup (B^{*-} \setminus A)$;
4. $(A^{*-})^{*-} \subseteq A$.

Proof. The proof of 1. and 2. are obvious from definition.
3. $(A \cup B)^{*-} = (A \cup B)^* \setminus (A \cup B) = (A^* \cup B^*) \setminus (A \cup B) = [(A^* \setminus A) \setminus B] \cup [(B^* \setminus B) \setminus A] = (A^{*-} \setminus B) \cup (B^{*-} \setminus A)$.
4. $(A^{*-})^{*-} = (A^{*-})^* \setminus A^{*-} = (A^* \setminus A)^* \setminus (A^* \setminus A) \subseteq (A^*)^* \setminus (A^* \setminus A) \subseteq A^* \setminus (A^* \setminus A) \subseteq A$. □

Definition 2.24. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A, B \subseteq X$. The operator $k_2 : \wp(X) \to \wp(X)$ on X is defined by*

$$k_2(A) = A \cup T_2(A),$$

where $T_2 : \wp(X) \to \wp(X)$ is an operator which satisfies the following conditions:
(i) $T_2(\emptyset) = \emptyset$,
(ii) $A \cap T_2(A) = \emptyset$,
(iii) $T_2(A \cup B) = (T_2(A) \setminus B) \cup (T_2(B) \setminus A)$,
(iv) $T_2(T_2(A)) \subseteq A$.

The operator k_2 satisfies the following conditions:
(i) $k_2(\emptyset) = \emptyset \cup T_2(\emptyset) = \emptyset$;
(ii) $A \subseteq A \cup T_2(A) = k_2(A)$;
(iii) $k_2(A \cup B) = (A \cup B) \cup T_2(A \cup B) = (A \cup B) \cup (T_2(A) \setminus B) \cup (T_2(B) \setminus A) = A \cup T_2(A) \cup B \cup T_2(B) = k_2(A) \cup k_2(B)$;
(iv) $k_2(k_2(A)) = k_2(A) \cup T_2(k_2(A)) = k_2(A) \cup T_2(A \cup T_2(A)) = k_2(A) \cup (T_2(A) \setminus T_2(A)) \cup (T_2(T_2(A)) \setminus A) = k_2(A)$.
Thus, the operator k_2 is a closure operator on $(X, \tau, \mathscr{I})$.

Theorem 2.25. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A,\ B \subseteq X$. Then, the following conditions hold:*

1. $A^{*-} \cup B^{*-} = (A \cap B^{*-}) \cup (A \cup B)^{*-} \cup (A^{*-} \cap B)$;
2. $(A^*)^{*-} = \emptyset$;
3. *A is $*$-open [5] if and only if $A^{*-} = Bd^*(A)$.*

Proof. 1. Note that $A^* \subseteq (A \cup B)^*$ if and only if $(A^* \setminus A) \setminus B \subseteq (A \cup B)^* \setminus (A \cup B)$ if and only if $A^{*-} \setminus B \subseteq (A \cup B)^{*-}$. Therefore, $(A^{*-} \setminus B) \cup (A^{*-} \cap B) \subseteq (A \cup B)^{*-} \cup (A^{*-} \cap B)$ and $A^{*-} \subseteq (A \cup B)^{*-} \cup (A^{*-} \cap B)$. Analogously, $B^{*-} \subseteq (A \cup B)^{*-} \cup (B^{*-} \cap A)$. So $A^{*-} \cup B^{*-} \subseteq (A \cup B)^{*-} \cup (B^{*-} \cap A) \cup (A^{*-} \cap B)$.

For the reverse inclusion we will only show that $(A \cup B)^{*-} \subseteq A^{*-} \cup B^{*-}$. Note that $(A \cup B)^* \setminus (A \cup B) \subseteq (A^* \setminus A) \cup (B^* \setminus B)$. Thus $(A \cup B)^{*-} \subseteq A^{*-} \cup B^{*-}$. This implies that $(A \cup B)^{*-} \cup (A \cap B^{*-}) \cup (B \cap A^{*-}) \subseteq A^{*-} \cup B^{*-}$.
2. Note that $(A^*)^{*-} = (A^*)^* \setminus A^* \subseteq A^* \setminus A^* = \emptyset$. □

Theorem 2.26. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then a subset A of X is $*$-closed [5] if and only if $A^{*-} = \emptyset$.*

Proof. Suppose A is $*$-closed. Then $A \cup A^* \subseteq A$ and hence $A^* \subseteq A$. Now $A^{*-} = A^* \setminus A = \emptyset$.
Conversely suppose that $A^{*-} = \emptyset$. Then $A^* \setminus A = \emptyset$ implies $A^* \subseteq A$. Thus $A \cup A^* = A$. So A is $*$-closed. □

Theorem 2.27. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$. If $A^{*-} = X$, then A is $\mathscr{I}$-dense.*

Proof. Given that $A^{*-} = X$, then $A^* \setminus A = X$. Thus $X \subseteq A^*$. □

Converse of the above theorem need not hold in general:

Example 2.28. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Then $(\{a, c\})^* = X$, but $(\{a, c\})^* \setminus \{a, c\} \neq X$.*

Definition 2.29. *We define the operator $()^{*\psi}$ on an ideal topological space $(X, \tau, \mathscr{I})$ in the following way: for a subset A of X, $A^{*\psi} = A \setminus \psi(A)$.*

Theorem 2.30. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A,\ B \subseteq X$. Then following conditions hold:*

1. $X^{*\psi} = \emptyset$;

2. $A^{*\psi} \subseteq A$;

3. $(A\cap B)^{*\psi} = (A^{*\psi}\cap B)\cup(A\cap B^{*\psi})$;

4. $(A^{*\psi})^{*\psi} = A^{*\psi}$, *if the space is Hayashi-Samuel.*

Proof. The proofs of 1. and 2. hold trivially.

3. $(A\cap B)^{*\psi} = (A\cap B)\setminus\psi(A\cap B) = (A\cap B)\cap[X\setminus\psi(A)\cap\psi(B)] = [A\cap(X\setminus\psi(A))\cap B]\cup[A\cap B\cap(X\setminus\psi(B))] = (A^{*\psi}\cap B)\cup(A\cap B^{*\psi})$.

4. $(A^{*\psi})^{*\psi} = A^{*\psi}\setminus\psi[A^{*\psi}] = (A\setminus\psi(A))\setminus\psi[A\setminus\psi(A)] = (A\setminus\psi(A))\setminus\psi(A\setminus[X\setminus(X\setminus A)^*]) = (A\setminus\psi(A))\setminus\psi[A\setminus X\cup(X\setminus A)^*] = (A\setminus\psi(A))\setminus\emptyset = A^{*\psi}$. □

Definition 2.31. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A,B\subseteq X$. The operator $Int_\psi:\wp(X)\to\wp(X)$ on X is defined by*

$$Int_\psi(A) = A\setminus T_3(A),$$

where $T_3:\wp(X)\to\wp(X)$ is an operator which satisfies the following conditions:
(i) $T_3(X)=\emptyset$,
(ii) $T_3(A)\subseteq A$,
(iii) $T_3(A\cap B) = (T_3(A)\cap B)\cup(A\cap T_3(B))$,
(iv) $T_3(T_3(A)) = T_3(A)$, if the space is Hayashi-Samuel.

The operator Int_ψ satisfies the following conditions:
(i) $Int_\psi(X) = X\setminus T_3(X) = X$;
(ii) $Int_\psi(A) = A\setminus T_3(A)\subseteq A$;
(iii) $Int_\psi(A\cap B) = (A\cap B)\setminus T_3(A\cap B) = (A\cap B)\setminus[(T_3(A)\cap B)\cup(T_3(B)\cap A)] = [A\setminus T_3(A)]\cap(B\setminus T_3(B)) = Int_\psi(A)\cap Int_\psi(B)$ (from (ii));
(iv) $Int_\psi(Int_\psi(A)) = Int_\psi[A\setminus T_3(A)] = [A\setminus T_3(A)]\setminus T_3[A\setminus T_3(A)] = [A\setminus T_3(A)]\setminus T_3(A\cap T_3(A)^c)\supseteq(A\setminus T_3(A))\setminus T_3(A)\cap T_3(X\setminus T_3(A))$ (from (iii)) $\supseteq(A\setminus T_3(A))\setminus(T_3(A)\cap(X\setminus T_3(A))$ (from 3. of Theorem 2.30) $= Int_\psi(A)$.
This shows that Int_ψ is an interior operator on X.

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On the Bicomplex k-Fibonacci Quaternions

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Abstract
In this paper, bicomplex k-Fibonacci quaternions are defined. Also, some algebraic properties of bicomplex k-Fibonacci quaternions are investigated. For example, the summation formula, generating functions, Binet's formula, the Honsberger identity, the d'Ocagne's identity, Cassini's identity, Catalan's identity for these quaternions are given. In the last part, a different way to find $n-th$ term of the bicomplex k-Fibonacci quaternion sequence was given using the determinant of a tridiagonal matrix.

Keywords: Bicomplex Fibonacci quaternion, Bicomplex k-Fibonacci quaternion, Bicomplex number, k-Fibonacci number, Tridiagonal matrix
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1. Introduction

In 2007, the k-Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined by Falcon and Plaza [1, 2] as follows

$$\left\{\begin{array}{rl} F_{k,0}= & 0,\ F_{k,1}=1 \\ F_{k,n+1}= & k\,F_{k,n}+F_{k,n-1},\ \ n\geq 1 \\ or & \\ \{F_{k,n}\}_{n\in\mathbb{N}}= & \{0,1,k,k^2+1,k^3+2k,k^4+3k^2+1,\ldots\}. \end{array}\right\}$$

Here, k is a positive real number.

In 2015, Ramirez [3] defined the the k-Fibonacci and the k-Lucas quaternions as follows:

$$D_{k,n}=\{F_{k,n}+\mathbf{i}\,F_{k,n+1}+\mathbf{j}\,F_{k,n+2}+\mathbf{k}\,F_{k,n+3}\,|\,F_{k,n},\ n-th\ k\text{-Fibonacci number}\},$$

and

$$P_{k,n}=\{L_{k,n}+\mathbf{i}\,L_{k,n+1}+\mathbf{j}\,L_{k,n+2}+\mathbf{k}\,L_{k,n+3}\,|\,L_{k,n},\ n-th\ k\text{-Lucas number}\}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy the multiplication rules

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-1\,,\ \ \mathbf{i}\,\mathbf{j}=-\mathbf{j}\mathbf{i}=\mathbf{k}\,,\quad \mathbf{j}\,\mathbf{k}=-\mathbf{k}\mathbf{j}=\mathbf{i}\,,\quad \mathbf{k}\,\mathbf{i}=-\mathbf{i}\mathbf{k}=\mathbf{j}\,.$$

In 1892, bicomplex numbers were introduced by Corrado Segre, for the first time [4]. In 1991, G. Baley Price, the bicomplex numbers gave in the book based on multicomplex spaces and functions [5]. The set of bicomplex numbers can be expressed by a basis $\{1\,,i\,,j\,,i\,j\}$ as,

$$\mathbb{BC}=\{q=q_1+iq_2+jq_3+ijq_4\mid q_1,q_2,q_3,q_4\in\mathbb{R}\}$$

where i, j and $i\,j$ satisfy the conditions

$$i^2 = -1,\ j^2 = -1,\ i\,j = j\,i. \tag{1.1}$$

In 2019, bicomplex k-pell quaternions were introduced by Catarino Paula, [6] as follows

$$\mathbb{BC}^P_{k,n} = \{P_{k,n} + i\,P_{k,n+1} + j\,P_{k,n+2} + i\,j\,P_{k,n+3} \,|\, P_{k,n},\ n-th\ k\text{-Pell number}\},$$

where i, j and $i\,j$ satisfy the conditions

$$i^2 = -1,\ j^2 = -1,\ i\,j = j\,i.$$

The aim of this study is to define bicomplex k-Fibonacci quaternions with k-Fibonacci number and bicomplex number and to give their algebraic properties.

2. The bicomplex k-Fibonacci numbers

Definition 2.1. The bicomplex k-Fibonacci and k-Lucas numbers can be define by with the basis $\{1, i, j, i\,j\}$, where $i,\ j$ *and* $i\,j$ satisfy the conditions

$$i^2 = -1,\ j^2 = -1,\ i\,j = j\,i,\ (i\,j)^2 = 1.$$

as

$$\begin{aligned}\mathbb{BC}F_{k,n} =& (F_{k,n} + i\,F_{k,n+1}) + j\,(F_{k,n+2} + i\,F_{k,n+3})\\ =& F_{k,n} + i\,F_{k,n+1} + j\,F_{k,n+2} + i\,j\,F_{k,n+3}\end{aligned}$$

and

$$\begin{aligned}\mathbb{BC}L_{k,n} =& (L_{k,n} + i\,L_{k,n+1}) + j\,(L_{k,n+2} + i\,L_{k,n+3})\\ =& L_{k,n} + i\,L_{k,n+1} + j\,L_{k,n+2} + i\,j\,L_{k,n+3}.\end{aligned} \tag{2.1}$$

For two bicomplex k-Fibonacci numbers, addition and subtraction are defined by the following:

$$\mathbb{BC}F_{k,n} \pm \mathbb{BC}F_{k,m} = (F_{k,n} \pm F_{k,m}) + i\,(F_{k,n+1} \pm F_{k,m+1}) + j\,(F_{k,n+2} \pm F_{k,m+2}) + i\,j\,(F_{k,n+3} \pm F_{k,m+3})$$

and multiplication of by

$$\begin{aligned}\mathbb{BC}F_{k,n} \times \mathbb{BC}F_{k,m} =& (F_{k,n}F_{k,m} - F_{k,n+1}F_{k,m+1} - F_{k,n+2}F_{k,m+2} - F_{k,n+3}F_{k,m+3})\\ &+i\,(F_{k,n}F_{k,m+1} + F_{k,n+1}F_{k,m} - F_{k,n+2}F_{k,m+3} - F_{k,n+3}F_{k,m+2})\\ &+j\,(F_{k,n}F_{k,m+2} + F_{k,n+2}F_{k,m} - F_{k,n+1}F_{k,m+3} - F_{k,n+3}F_{k,m+1})\\ &+i\,j\,(F_{k,n}F_{k,m+3} + F_{k,n+3}F_{k,m} + F_{k,n+1}F_{k,m+2} + F_{k,n+2}F_{k,m+1})\\ =& \mathbb{BC}F_{k,m} \times \mathbb{BC}F_{k,n}.\end{aligned}$$

3. The bicomplex k-Fibonacci quaternions and some basic properties

In 2018, the bicomplex Fibonacci quaternions defined by Aydın Torunbalcı [7] as follows

$$Q_{F\,n} = F_n + i\,F_{n+1} + j\,F_{n+2} + i\,j\,F_{n+3}$$

where quaternionic units satisfy the rules Eq. 1.1. In this section, firstly the bicomplex k-Fibonacci quaternions will be defined.

Definition 3.1. The bicomplex k-Fibonacci quaternions are defined by using the bicomplex numbers and k-Fibonacci numbers as follows

$$\mathbb{BC}^{F_{k,n}} = F_{k,n} + i\,F_{k,n+1} + j\,F_{k,n+2} + i\,j\,F_{k,n+3} \tag{3.1}$$

where quaternionic units satisfy the rules Eq. 1.1.

Let $\mathbb{BC}^{F_{k,n}}$ and $\mathbb{BC}^{F_{k,m}}$ be two bicomplex k-Fibonacci quaternions. For two bicomplex k-Fibonacci quaternions, addition and subtraction are defined in the obvious way,

$$\begin{aligned}\mathbb{BC}^{F_{k,n}} \pm \mathbb{BC}^{F_{k,m}} = &\ (F_{k,n} + i\,F_{k,n+1} + j\,F_{k,n+2} + i\,j\,F_{k,n+3}) \pm (F_{k,m} + i\,F_{k,m+1} + j\,F_{k,m+2} + i\,j\,F_{k,m+3}) \\ = &\ (F_{k,n} \pm F_{k,m}) + i\,(F_{k,n+1} \pm F_{k,m+1}) + j\,(F_{k,n+2} \pm F_{k,m+2}) + i\,j\,(F_{k,n+3} \pm F_{k,m+3}).\end{aligned}$$

and multiplication by

$$\begin{aligned}\mathbb{BC}^{F_{k,n}} \times \mathbb{BC}^{F_{k,m}} = &\ [F_{k,n}\,F_{k,m} - F_{k,n+1}\,F_{k,m+1} - F_{k,n+2}\,F_{k,m+2} + F_{k,n+3}\,F_{k,m+3}] \\ &+ i\,[F_{k,n}\,F_{k,m+1} + F_{k,n+1}\,F_{k,m} - F_{k,n+2}\,F_{k,m+3} - F_{k,n+3}\,F_{k,m+2}] \\ &+ j\,[F_{k,n}\,F_{k,m+2} - F_{k,n+1}\,F_{k,m+3} + F_{k,n+2}\,F_{k,m} - F_{k,n+3}\,F_{k,m+1}] \\ &+ i\,j\,[F_{k,n}\,F_{k,m+3} + F_{k,n+1}\,F_{k,m+2} + F_{k,n+2}\,F_{k,m+1} + F_{k,n+3}\,F_{k,m}] \\ = &\ \mathbb{BC}^{F_{k,m}} \times \mathbb{BC}^{F_{k,n}}.\end{aligned}$$

The different conjugations for bicomplex k-Fibonacci quaternions are presented as follows:

$$(\mathbb{BC}^{F_{k,n}})^{*1} = F_{k,n} - i\,F_{k,n+1} + j\,F_{k,n+2} - i\,j\,F_{k,n+3},$$

$$(\mathbb{BC}^{F_{k,n}})^{*2} = F_{k,n} + i\,F_{k,n+1} - j\,F_{k,n+2} - i\,j\,F_{k,n+3},$$

$$(\mathbb{BC}^{F_{k,n}})^{*3} = F_{k,n} - i\,F_{k,n+1} - j\,F_{k,n+2} + i\,j\,F_{k,n+3}.$$

Therefore, the norm of the bicomplex k-Fibonacci quaternion $\mathbb{BC}^{F_{k,n}}$ is defined in three different ways as follows

$$\begin{aligned}N_(\mathbb{BC}^{F_{k,n}})^{*1} = &\ \|\mathbb{BC}^{F_{k,n}} \times (\mathbb{BC}^{F_{k,n}})^{*1}\|^2 \\ = &\ |(F_{k,n}^2 + F_{k,n+1}^2) - (F_{k,n+2}^2 + F_{k,n+3}^2) + 2\,j\,(F_{k,n}\,F_{k,n+2} + F_{k,n+1}\,F_{k,n+3})| \\ = &\ |F_{k,2n+1} - F_{k,2n+5} + 2\,j\,F_{k,2n+3}| = \mathbb{BC}^{F_{k,n}}\,(\mathbb{BC}^{F_{k,n}})^{*1},\end{aligned}$$

$$\begin{aligned}N_(\mathbb{BC}^{F_{k,n}})^{*2} = &\ \|\mathbb{BC}^{F_{k,n}} \times (\mathbb{BC}^{F_{k,n}})^{*2}\|^2 \\ = &\ |(F_{k,n}^2 - F_{k,n+1}^2) + (F_{k,n+2}^2 - F_{k,n+3}^2) + 2\,i\,F_{k,n}\,F_{k,n+1} + k\,F_{k,2n+3}| = \mathbb{BC}^{F_{k,n}}\,(\mathbb{BC}^{F_{k,n}})^{*2},\end{aligned}$$

$$\begin{aligned}N_(\mathbb{BC}^{F_{k,n}})^{*3} = &\ \|\mathbb{BC}^{F_{k,n}} \times (\mathbb{BC}^{F_{k,n}})^{*3}\|^2 \\ = &\ |(F_{k,n}^2 + F_{k,n+1}^2) + (F_{k,n+2}^2 + F_{k,n+3}^2) + 2\,i\,j\,(F_{k,n}\,F_{k,n+3} - F_{k,n+1}\,F_{k,n+2})| \\ = &\ |F_{k,2n+1} + F_{k,2n+5} + 2\,i\,j\,(-1)^{n+1}\,k| = \mathbb{BC}^{F_{k,n}}\,(\mathbb{BC}^{F_{k,n}})^{*3}.\end{aligned}$$

In the following theorem, some properties related to the bicomplex k-Fibonacci quaternions are given.

Theorem 3.2. *Let $\mathbb{BC}^{F_{k,n}}$ be the bicomplex k-Fibonacci quaternion. In this case, we can give the following relations:*

$$\mathbb{BC}^{F_{k,n+2}} = \mathbb{BC}^{F_{k,n}} + k\,\mathbb{BC}^{F_{k,n+1}}, \tag{3.2}$$

$$\begin{aligned}(\mathbb{BC}^{F_{k,n}})^2 + (\mathbb{BC}^{F_{k,n+1}})^2 = &\ \mathbb{BC}^{F_{k,2n+1}} + (k\,F_{k,2n+6} - F_{k,2n+3}) + i\,(F_{k,2n+2} - 2\,F_{k,2n+6}) \\ &+ j\,(F_{k,2n+3} - 2\,F_{k,2n+5}) + i\,j\,(3\,F_{k,2n+4}),\end{aligned} \tag{3.3}$$

$$\begin{aligned}(\mathbb{BC}^{F_{k,n+1}})^2 - (\mathbb{BC}^{F_{k,n-1}})^2 = &\ k\,[\mathbb{BC}^{F_{k,2n}} - F_{k,2n+2} + k\,F_{k,2n+5} + i\,(F_{k,2n+1} - 2\,F_{k,2n+5}) \\ &+ j\,(-F_{k,2n+2} - 2\,k\,F_{k,2n+3}) + i\,j\,(3\,F_{k,2n+3})],\end{aligned} \tag{3.4}$$

$$\mathbb{BC}^{F_{k,n+1}} + \mathbb{BC}^{F_{k,n-1}} = \mathbb{BC}^{L_{k,n}}, \tag{3.5}$$

$$\mathbb{BC}^{F_{k,n+2}} - \mathbb{BC}^{F_{k,n-2}} = k\,\mathbb{BC}^{L_{k,n}}. \tag{3.6}$$

Proof. (3.2): By the Eq.(3.1) we get,

$$\begin{aligned}\mathbb{BC}^{F_{k,n}}+k\,\mathbb{BC}^{F_{k,n+1}} &= (F_{k,n}+kF_{k,n+1})+i\,(F_{k,n+1}+kF_{k,n+2})+j\,(F_{k,n+2}+kF_{k,n+3})+i\,j\,(F_{k,n+3}+kF_{k,n+4})\\ &= F_{k,n+2}+i\,F_{k,n+3}+j\,F_{k,n+4}+i\,j\,F_{k,n+5}\\ &= \mathbb{BC}^{F_{k,n+2}}.\end{aligned}$$

(3.3): By the Eq.(3.1) we get,

$$\begin{aligned}(\mathbb{BC}^{F_{k,n}})^2+(\mathbb{BC}^{F_{k,n+1}})^2 &= (F_{k,2n+1}-F_{k,2n+3}-F_{k,2n+5}+F_{k,2n+7})+2\,i\,(F_{k,2n+2}-F_{k,2n+6})\\ &\quad +2\,j\,(F_{k,2n+3}-F_{k,2n+5})+2\,i\,j\,(2\,F_{k,2n+4})\\ &= (F_{k,2n+1}+i\,F_{k,2n+2}+j\,F_{k,2n+3}+i\,j\,F_{k,2n+4})-F_{k,2n+3}-F_{k,2n+5}+F_{k,2n+7}\\ &\quad +i\,(F_{k,2n+2}-2\,F_{k,2n+6})+j\,(F_{k,2n+3}-2\,F_{k,2n+5})+i\,j\,(3\,F_{k,2n+4})\\ &= \mathbb{BC}^{F_{k,2n+1}}+(k\,F_{k,2n+6}-F_{k,2n+3})\\ &\quad +i\,(F_{k,2n+2}-2\,F_{k,2n+6})+j\,(F_{k,2n+3}-2\,F_{k,2n+5})+i\,j\,(3\,F_{k,2n+4}).\end{aligned}$$

(3.4): By the Eq.(3.1) we get,

$$\begin{aligned}(\mathbb{BC}^{F_{k,n+1}})^2-(\mathbb{BC}^{F_{k,n-1}})^2 &= [(F^2_{k,n+1}-F^2_{k,n-1})-(F^2_{k,n+2}-F^2_{k,n})-(F^2_{k,n+3}-F^2_{k,n+1})+(F^2_{k,n+4}-F^2_{k,n+2})]\\ &\quad +2\,i\,[(F_{k,n+1}F_{k,n+2}-F_{k,n-1}F_{k,n})-(F_{k,n+3}F_{k,n+4}-F_{k,n+1}F_{k,n+2})]\\ &\quad +2\,j\,[(F_{k,n+1}F_{k,n+3}-F_{k,n-1}F_{k,n+1})-(F_{k,n+2}F_{k,n+4}-F_{k,n}F_{k,n+2})]\\ &\quad +2\,i\,j\,[(F_{k,n+1}F_{k,n+4}-F_{k,n-1}F_{k,n+2})+(F_{k,n+2}F_{k,n+3}-F_{k,n}F_{k,n+1})]\\ &= k\,(F_{k,2n}-k\,F_{k,2n+2}-k\,F_{k,2n+4}+k\,F_{k,2n+6})\\ &\quad +2\,i\,(k\,F_{k,2n+1}-k\,F_{k,2n+5})+2\,j\,(-k^2\,F_{k,2n+3})+2\,i\,j\,(2\,k\,F_{k,2n+3})\\ &= k\,[\mathbb{BC}^{F_{k,2n}}-F_{k,2n+2}+k\,F_{k,2n+5}+i\,(F_{k,2n+1}-2\,F_{k,2n+5})\\ &\quad +j\,(-F_{k,2n+2}-2\,k\,F_{k,2n+3})+i\,j\,(3\,F_{k,2n+3})].\end{aligned}$$

(3.5) and (3.6):Proof of equalities can easily be done using Eq.(2.1). ☐

Theorem 3.3. *Let $\mathbb{BC}^{F_{k,n}}$ be the bicomplex k-Fibonacci quaternion. Then, we have the following identities*

$$\begin{aligned}&\sum_{s=1}^{n}\mathbb{BC}^{F_{k,s}}=\frac{1}{k}\,(\mathbb{BC}^{F_{k,n+1}}+\mathbb{BC}^{F_{k,n}}-\mathbb{BC}^{F_{k,1}}-\mathbb{BC}^{F_{k,0}}),\\ &\sum_{s=1}^{n}\mathbb{BC}^{F_{k,2s-1}}=\frac{1}{k}(\mathbb{BC}^{F_{k,2n}}-\mathbb{BC}^{F_{k,0}}),\\ &\sum_{s=1}^{n}\mathbb{BC}^{F_{k,2s}}=\frac{1}{k}(\mathbb{BC}^{F_{k,2n+1}}-\mathbb{BC}^{F_{k,1}}).\end{aligned}\tag{3.7}$$

Proof. Proof can be easily done using sums of series following
$\sum_{i=1}^{n}F_{k,i}=\frac{1}{k}(F_{k,n+1}+F_{k,n}-1)$, $\sum_{i=1}^{n}F_{k,2i+1}=\frac{1}{k}F_{k,2n+2}$ and $\sum_{i=1}^{n}F_{k,2i}=\frac{1}{k}(F_{2n+1}-1)$ [1] . ☐

4. Generating functions and Binet's formula

In this section, the generating functions and the Binet's formula of the bicomplex k-Fibonacci quaternions will be defined.

Theorem 4.1. *Let $\mathbb{BC}^{F_{k,n}}$ be the bicomplex k-Fibonacci quaternion. For the generating function for these quaternions is as follows:*

$$g_{\mathbb{BC}^{F_{k,n}}}(t)=\sum_{s=1}^{n}\mathbb{BC}^{F_{k,n}}\,t^n=\frac{\mathbb{BC}^{F_{k,0}}+(\mathbb{BC}^{F_{k,1}}-k\,\mathbb{BC}^{F_{k,0}})\,t}{1-k\,t-t^2}$$

Proof. Using the definition of generating function, we obtain

$$g_{\mathbb{BC}^{F_{k,n}}}(t)=\mathbb{BC}^{F_{k,0}}+\mathbb{BC}^{F_{k,1}}\,t+\ldots+\mathbb{BC}^{F_{k,n}}\,t^n+\ldots.$$

Multiplying both sides of Eq.(3.30) and using Eq.(3.7), we have

$$(1-k\,t-t^2)\,g_{\mathbb{BC}^{F_{k,n}}}(t)=\mathbb{BC}^{F_{k,0}}+(\mathbb{BC}^{F_{k,1}}-k\,\mathbb{BC}^{F_{k,0}})\,t\,.$$

Thus, the proof is completed. ☐

Theorem 4.2. *Let $\mathbb{BC}^{F_{k,n}}$ be the bicomplex k-Fibonacci quaternion. For $n \geq 1$, Binet's formula for these quaternions is as follows:*

$$\mathbb{BC}^{F_{k,n}} = \frac{1}{\alpha-\beta}\left(\hat{\alpha}\,\alpha^n - \hat{\beta}\,\beta^n\right)$$

where

$$\hat{\alpha} = 1 + i\,\alpha + j\,\alpha^2 + i\,j\,\alpha^3, \quad \alpha = \tfrac{k+\sqrt{k^2+4}}{2},$$

$$\hat{\beta} = 1 + i\,\beta + j\,\beta^2 + i\,j\,\beta^3, \quad \beta = \tfrac{k-\sqrt{k^2+4}}{2},$$

$$\alpha+\beta = k\,, \quad \alpha-\beta = \sqrt{k^2+4}, \quad \alpha\beta = -1.$$

Proof. Using the Binet formula for k-Fibonacci number [2], we obtain

$$\begin{aligned}
\mathbb{BC}^{F_{k,n}} &= F_{k,n} + i\,F_{k,n+1} + j\,F_{k,n+2} + i\,j\,F_{k,n+3} \\
&= \tfrac{\alpha^n-\beta^n}{\sqrt{k^2+4}} + i\,(\tfrac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{k^2+4}}) + j\,(\tfrac{\alpha^{n+2}-\beta^{n+2}}{\sqrt{k^2+4}}) + i\,j\,(\tfrac{\alpha^{n+3}-\beta^{n+3}}{\sqrt{k^2+4}}) \\
&= \tfrac{\alpha^n\,(1+i\,\alpha+j\,\alpha^2+i\,j\,\alpha^3)-\beta^n\,(1+i\,\beta+j\,\beta^2+i\,j\,\beta^3)}{\sqrt{k^2+4}} \\
&= \tfrac{1}{\sqrt{k^2+4}}(\,\hat{\alpha}\,\alpha^n - \hat{\beta}\,\beta^n).
\end{aligned}$$

where $\hat{\alpha} = 1 + i\,\alpha + j\,\alpha^2 + i\,j\,\alpha^3$, $\hat{\beta} = 1 + i\,\beta + j\,\beta^2 + i\,j\,\beta^3$. □

5. Some identities for bicomplex k-Fibonacci quaternions

Theorem 5.1. ***Honsberger Identity***
For $n, m \geq 0$ the Honsberger identity for the bicomplex k-Fibonacci quaternions is given by

$$\begin{aligned}
\mathbb{BC}^{F_{k,n}}\,\mathbb{BC}^{F_{k,m}} + \mathbb{BC}^{F_{k,n+1}}\,\mathbb{BC}^{F_{k,m+1}} = \; & \mathbb{BC}^{F_{k,n+m+1}} - F_{k,n+m+3} + k\,F_{k,n+m+6} + i\,(F_{k,n+m+2} - 2\,F_{k,n+m+6}) \\
& + j\,(F_{k,n+m+3} - 2\,F_{k,n+m+5}) + i\,j\,(3\,F_{k,n+m+4}).
\end{aligned} \tag{5.1}$$

Proof. (5.1): By the Eq.(3.1) we get,

$$\begin{aligned}
\mathbb{BC}^{F_{k,n}}\,\mathbb{BC}^{F_{k,m}} + \mathbb{BC}^{F_{k,n+1}}\,\mathbb{BC}^{F_{k,m+1}} &= (F_{k,n+m+1} + i\,F_{k,n+m+2} + j\,F_{k,n+m+3} + i\,j\,F_{k,n+m+4}) - F_{k,n+m+3} + k\,F_{k,n+m+6} \\
&\quad + i\,(F_{k,n+m+2} - 2\,F_{k,n+m+6}) + j\,(F_{k,n+m+3} - 2\,F_{k,n+m+5}) + i\,j\,(3\,F_{k,n+m+4}) \\
&= \mathbb{BC}^{F_{k,n+m+1}} - F_{k,n+m+3} + k\,F_{k,n+m+6} + i\,(F_{k,n+m+2} - 2\,F_{k,n+m+6}) \\
&\quad + j\,(F_{k,n+m+3} - 2\,F_{k,n+m+3}) + i\,j\,(3\,F_{k,n+m+4}).
\end{aligned}$$

where the identity $F_{k,n}F_{k,m} + F_{k,n+1}F_{k,m+1} = F_{k,n+m+1}$ was used [1]. □

Theorem 5.2. ***D'Ocagne's Identity***
For $n, m \geq 0$ the D'Ocagne's identity for the bicomplex k-Fibonacci quaternions is given by

$$\mathbb{BC}^{F_{k,n}}\,\mathbb{BC}^{F_{k,m+1}} - \mathbb{BC}^{F_{k,n+1}}\,\mathbb{BC}^{F_{k,m}} = (-1)^m\,F_{k,n-m}\,[\,2\,(k^2+2)\,j + (k^3+2k)\,i\,j\,]. \tag{5.2}$$

Proof. (5.2): By the Eq.(3.1) we get,

$$\begin{aligned}
\mathbb{BC}^{F_{k,n}}\,\mathbb{BC}^{F_{k,m+1}} - \mathbb{BC}^{F_{k,n+1}}\,\mathbb{BC}^{F_{k,m}} = \; & [\,(F_{k,n}F_{k,m+1} - F_{k,n+1}F_{k,m}) - (F_{k,n+1}F_{k,m+2} - F_{k,n+2}F_{k,m+1}) \\
& \quad -(F_{k,n+2}F_{k,m+3} - F_{k,n+3}F_{k,m+2}) + (F_{k,n+3}F_{k,m+4} - F_{k,n+4}F_{k,m+3})\,] \\
& + i\,[\,(F_{k,n}F_{k,m+2} - F_{k,n+1}F_{k,m+1}) + (F_{k,n+1}F_{k,m+1} - F_{k,n+2}F_{k,m}) \\
& \quad -(F_{k,n+3}F_{k,m+3} - F_{k,n+4}F_{k,m+2})\,] \\
& + j\,[\,(F_{k,n}F_{k,m+3} - F_{k,n+1}F_{k,m+2}) + (F_{k,n+2}F_{k,m+1} - F_{k,n+3}F_{k,m}) \\
& \quad -(F_{k,n+1}F_{k,m+4} - F_{k,n+2}F_{k,m+3}) - (F_{k,n+3}F_{k,m+2} - F_{k,n+4}F_{k,m+1})\,] \\
& + i\,j\,[\,(F_{k,n}F_{k,m+4} - F_{k,n+1}F_{k,m+3}) + (F_{k,n+1}F_{k,m+3} - F_{k,n+2}F_{k,m+2}) \\
& \quad +(F_{k,n+2}F_{k,m+2} - F_{k,n+3}F_{k,m+1}) + (F_{k,n+3}F_{k,m+1}F_{k,n+4}F_{k,m})\,] \\
= \; & (-1)^m\,F_{k,n-m}\,[\,2\,(k^2+2)\,j + (k^3+2k)\,i\,j\,].
\end{aligned}$$

where the identity $F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n}$ is used [1]. □

Theorem 5.3. ***Catalan's Identity***
Let $\mathbb{BC}^{F_{k,n+r}}$ be the bicomplex k-Fibonacci quaternion. For $n \geq 1$, Catalan's identity for $\mathbb{BC}^{F_{k,n+r}}$ is as follows:

$$\mathbb{BC}^{F_{k,n+r-1}}\,\mathbb{BC}^{F_{k,n+r+1}} - (\mathbb{BC}^{F_{k,n+r}})^2 = (-1)^{n+r}\,[2(k^2+2)\,j + (k^3+2k)\,i\,j]. \tag{5.3}$$

Proof. (5.3): By using (3.1) we get

$$\begin{aligned}
\mathbb{BC}^{F_{k,n+r-1}}\,\mathbb{BC}^{F_{k,n+r+1}} - (\mathbb{BC}^{F_{k,n+r}})^2 &= (F_{k,n+r-1}F_{k,n+r+1} - F_{k,n+r}^2) - (F_{k,n+r}F_{k,n+r+2} - F_{k,n+r+1}^2) \\
&\quad (F_{k,n+r+1}F_{k,n+r+3} - F_{k,n+r+2}^2) + (F_{k,n+r+2}F_{k,n+r+4} - F_{k,n+r+3}^2) \\
&\quad + i\,[\,(F_{k,n+r-1}F_{k,n+r+2}) - (F_{k,n+r}F_{k,n+r+1}) - (F_{k,n+r+1}F_{k,n+r+4} - F_{k,n+r+2}F_{k,n+r+3}) \\
&\quad + j\,[\,(F_{k,n+r-1}F_{k,n+r+3} - F_{k,n+r}F_{k,n+r+2}) - (F_{k,n+r}F_{k,n+r+4} - F_{k,n+r+1}F_{k,n+r+3}) \\
&\quad\quad + (F_{k,n+r+1}F_{k,n+r+1} - F_{k,n+r+2}F_{k,n+r}) - (F_{k,n+r+2}F_{k,n+r+2} - F_{k,n+r+3}F_{k,n+r+1})\,] \\
&\quad + i\,j\,[\,(F_{k,n+r-1}F_{k,n+r+4} - F_{k,n+r}F_{k,n+r+3}) \\
&\quad\quad + (F_{k,n+r}F_{k,n+r+3} - F_{k,n+r+1}F_{k,n+r+2}) + (F_{k,n+r+2}F_{k,n+r+1} - F_{k,n+r+3}F_{k,n+r})\,] \\
&= (-1)^{n+r}\,[\,2\,(k^2+2)\,j + (k^3+2k)\,i\,j\,]
\end{aligned}$$

where the identity of the k-Fibonacci numbers $F_{k,n+r-1}F_{k,n+r+1} - F_{k,n+r}^2 = (-1)^{n+r}$ is used [2]. Furthermore;

$$\begin{cases}
F_{k,n+r-1}\,F_{k,n+r+2} + F_{k,n+r}\,F_{k,n+r+1} = (-1)^{n+r}\,k, \\
F_{k,n+r-1}\,F_{k,n+r+3} - F_{k,n+r}\,F_{k,n+r+2} = (-1)^{n+r}\,(k^2+1), \\
F_{k,n+r+1}\,F_{k,n+r+3} - F_{k,n+r}\,F_{k,n+r+4} = (-1)^{n+r}\,(k^2+1), \\
F_{k,n+r-1}\,F_{k,n+r+4} - F_{k,n+r}\,F_{k,n+r+3} = (-1)^{n+r}\,(k^3+2k), \\
F_{k,n+r}\,F_{k,n+r+3} - F_{k,n+r+1}\,F_{k,n+r+2} = (-1)^{n+r+1}\,k, \\
F_{k,n+r+2}\,F_{k,n+r+1} - F_{k,n+r+3}\,F_{k,n+r} = (-1)^{n+r}\,k\,.
\end{cases}$$

are used. □

Theorem 5.4. ***Cassini's Identity***
Let $\mathbb{BC}^{F_{k,n}}$ be the bicomplex k-Fibonacci quaternion. For $n \geq 1$, Cassini's identity for $\mathbb{BC}^{F_{k,n}}$ is as follows:

$$\mathbb{BC}^{F_{k,n-1}}\,\mathbb{BC}^{F_{k,n+1}} - (\mathbb{BC}^{F_{k,n}})^2 = (-1)^{n}\,[2(k^2+2)\,j + (k^3+2k)\,i\,j]. \tag{5.4}$$

Proof. (5.4): By using (3.1) we get

$$\begin{aligned}
\mathbb{BC}^{F_{k,n-1}}\,\mathbb{BC}^{F_{k,n+1}} - (\mathbb{BC}^{F_{k,n}})^2 = &\quad [\,(F_{k,n-1}F_{k,n+1} - F_{k,n}^2) - (F_{k,n}F_{k,n+2} - F_{k,n+1}^2) \\
&\quad\quad - (F_{k,n+1}F_{k,n+3} - F_{k,n+2}^2) + (F_{k,n+2}F_{k,n+4} - F_{k,n+3}^2)\,] \\
&\quad + i\,[\,(F_{k,n-1}F_{k,n+2} - F_{k,n}F_{k,n+1}) - (F_{k,n+1}F_{k,n+4} - F_{k,n+2}F_{k,n+3})\,] \\
&\quad + j\,[\,(F_{k,n-1}F_{k,n+3} - F_{k,n}F_{k,n+2}) - (F_{k,n}F_{k,n+4} - F_{k,n+1}F_{k,n+3}) \\
&\quad\quad + (F_{k,n+1}F_{k,n+1} - F_{k,n+2}F_{k,n}) - (F_{k,n+2}F_{k,n+2} - F_{k,n+3}F_{k,n+1})\,] \\
&\quad + i\,j\,(F_{k,n-1}F_{k,n+4} - F_{k,n}F_{k,n+3}) \\
= &\quad (-1)^n\,[2(k^2+2)\,j + (k^3+2k)\,i\,j].
\end{aligned}$$

where the identities of the k-Fibonacci numbers $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$ [2]. Furthermore;

$$\begin{cases}
F_{k,n-1}F_{k,n+2} - F_{k,n}F_{k,n+1} = k\,(-1)^n \\
F_{k,n-1}F_{k,n+3} - F_{k,n}F_{k,n+2} = (k^2+1)\,(-1)^n\,, \\
F_{k,n+1}F_{k,n+3} - F_{k,n}F_{k,n+4} = (k^2+1)\,(-1)^n\,, \\
F_{k,n-1}F_{k,n+4} - F_{k,n}F_{k,n+3} = (k^3+2k)\,(-1)^n
\end{cases}$$

are used. □

6. An application of bicomplex k-Fibonacci quaternions in tridiagonal matrices

In this section, we give another method to obtain the n-th term of bicomplex k-Fibonacci quaternion sequence as the calculation of a tridiagonal matrix [8].

Theorem 6.1. *Let x_n be any second-order linear sequence, defined recursively as*

$$x_{n+1} = A x_n + B x_{n-1},\, n \geq 1$$

with the values $x_0 = C$, $x_1 = D$. Then,for all $n \geq 0$,

$$x_n = det \begin{pmatrix} C & D & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & B & 0 & \dots & 0 & 0 \\ 0 & -1 & A & B & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A & B \\ 0 & 0 & 0 & 0 & \dots & -1 & A \end{pmatrix}$$

Theorem 6.2. *Now, let's consider the second-order linear sequence $\mathbb{BC}^{F_{k,n+1}} = \mathbb{BC}^{F_{k,n}} + k\,\mathbb{BC}^{F_{k,n-1}}$. Using the previous theorem and taking $A = 1$, $B = k$, $C = \mathbb{BC}^{F_{k,0}}$, $D = \mathbb{BC}^{F_{k,1}}$ the following determinant was obtained.*

$$\mathbb{BC}^{F_{k,n}} = det \begin{pmatrix} \mathbb{BC}^{F_{k,0}} & \mathbb{BC}^{F_{k,1}} & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & k & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & k & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & k \end{pmatrix}$$

Proof. For $n \geq 0$, using the above theorem, we have

$$\mathbb{BC}^{F_{k,0}} = \left| \;\mathbb{BC}^{F_{k,0}}\; \right| = \mathbb{BC}^{F_{k,0}},$$

$$\mathbb{BC}^{F_{k,1}} = \begin{vmatrix} \mathbb{BC}^{F_{k,0}} & \mathbb{BC}^{F_{k,1}} \\ -1 & 0 \end{vmatrix} = \mathbb{BC}^{F_{k,1}},$$

$$\begin{aligned} \mathbb{BC}^{F_{k,2}} &= \begin{vmatrix} \mathbb{BC}^{F_{k,0}} & \mathbb{BC}^{F_{k,1}} & 0 \\ -1 & 0 & 1 \\ 0 & -1 & k \end{vmatrix} \\ &= \mathbb{BC}^{F_{k,0}} \begin{vmatrix} 0 & 1 \\ -1 & k \end{vmatrix} - \mathbb{BC}^{F_{k,1}} \begin{vmatrix} -1 & 1 \\ 0 & k \end{vmatrix} \\ &= \mathbb{BC}^{F_{k,0}} + k\,\mathbb{BC}^{F_{k,1}} = \mathbb{BC}^{F_{k,2}}, \end{aligned}$$

$$\begin{aligned} \mathbb{BC}^{F_{k,3}} &= \begin{vmatrix} \mathbb{BC}^{F_{k,0}} & \mathbb{BC}^{F_{k,1}} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & k & 1 \\ 0 & 0 & -1 & k \end{vmatrix} \\ &= \mathbb{BC}^{F_{k,0}} \begin{vmatrix} 0 & 1 & 0 \\ -1 & k & 1 \\ 0 & -1 & k \end{vmatrix} - \mathbb{BC}^{F_{k,1}} \begin{vmatrix} -1 & 1 & 0 \\ 0 & k & 1 \\ 0 & -1 & k \end{vmatrix} \\ &= \mathbb{BC}^{F_{k,0}}(-1) \begin{vmatrix} -1 & 1 \\ 0 & k \end{vmatrix} - \mathbb{BC}^{F_{k,1}}(-1) \begin{vmatrix} k & 1 \\ -1 & k \end{vmatrix} \\ &= k\,\mathbb{BC}^{F_{k,0}} + \mathbb{BC}^{F_{k,1}}(1+k^2) \\ &= \mathbb{BC}^{F_{k,1}} + (k^2\,\mathbb{BC}^{F_{k,1}} + k\,\mathbb{BC}^{F_{k,0}}) = k\,\mathbb{BC}^{F_{k,2}} + \mathbb{BC}^{F_{k,1}} = \mathbb{BC}^{F_{k,3}} \end{aligned}$$

In this way, $\mathbb{BC}^{F_{k,n}}$ n-th term is obtained by calculating the determinant nxn. □

7. Conclusion

In this paper, a number of new results on bicomplex k-Fibonacci quaternions were derived. Furthermore, a different way to find the n-th term of Bicomplex k-Fibonacci quaternion sequence was expressed using the determinant of a tridiagonal matrix.

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