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A NEW FAMILY OF LIFETIME DISTRIBUTIONS IN TERMS OF CUMULATIVE HAZARD RATE FUNCTION

OMID KHARAZMI AND SHAHLA JAHANGARD

Abstract. In the present paper, a new family of lifetime distributions is introduced according to cumulative hazard rate function, the well-known concept in survival analysis and reliability engineering. Some important properties of proposed model including survival function, quantile function, hazard function, order statistic and some results of stochastic ordering are obtained in general setting. An especial case of this new family is introduced by considering Weibull distribution as the parent distribution; in addition estimating unknown parameters of specialized model will be examined from the perspective of Bayesian and classic statistics. Moreover, three examples of real data sets: complete, right-censored and progressively type-I interval-censored data are studied; point and interval estimations of all parameters are obtained. Finally, the superiority of proposed model in terms of parent Weibull distribution over other fundamental statistical distributions is shown via complete real observations.

1. Introduction

The statistical distribution theory has been widely explored by researchers in recent years. Given the fact that the data from our surrounding environment follow various statistical models, is necessary to extract and develop appropriate high-quality models.

Recently, Nadarajah and Haghighi (2011) have introduced a new model of lifetime distributions, which the researchers refer it as NH distribution. It is an extended form of exponential distribution and attracted the attention of some researchers. We refer the reader to (Lemonte (2013), Dey et al. (2017) and Kumar et al. (2017)). This model has a number of desirable features and is comprehensively studied by the authors. For example, whenever the data contains zero values, NH model can be a strong competitor for other well-known lifetime distributions such

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Key words and phrases. Cumulative hazard rate function, Bayesian estimation, maximum likelihood estimation, progressively type-I interval-censored data.

as gamma, Weibull and generalized exponential distribution. The cumulative distribution function (*cdf*) and probability distribution function (*pdf*) related to NH model is given respectively as;

$$F(x) = 1 - e^{1-(1+\lambda x)^{\alpha}},\ x > 0$$

and

$$f(x) = \alpha\lambda(1+\lambda x)^{\alpha-1}e^{1-(1+\lambda x)^{\alpha}},\ x > 0$$

where the parameters $\alpha > 0$ controls the shapes of the distribution and the parameter $\lambda > 0$ is the scale parameter. It is easy to see that the NH model has increasing, decreasing and constant hazard shapes.

In the present paper, we introduce a New family of Lifetime distributions based on the Cumulative Hazard rate quantity of a parent distribution G, so-called $NLCH - G$ distribution. One of our main motivation to introduce this new category of distributions is that, when the parent distribution G be exponential, the proposed model reduced to NH distribution.

The cumulative hazard rate function is a prominent concept in topics of survival analysis and reliability engineering and plays an important role in this area of science. Suppose that X be a random variable with density function f and cumulative distribution function F, then hazard rate and cumulative hazard rate functions are defined;

$$h(x) = \frac{f(x)}{R(x)}$$

and

$$H(x) = -\log R(x) = e^{-\int_0^x h(t)dt},$$

respectively, where $R(x) = 1 - F(x)$ denotes the survival function of X (Barlow and Proschan (1975)).

In the next, we first obtain the fundamental and statistical properties of $NLCH-G$ in general setting and then we propose an especial case of $NLCH - G$ model by considering Weibull distribution instead of the parent distribution G. It is referred as $NLCH - Weibull$ (or $NLCH - W$) distribution. We provide a comprehensive discussion about statistical and reliability properties of new $NLCH - W$ model. Furthermore, we consider Maximum likelihood, Bayesian and bootstrap estimation procedures in order to estimate the unknown parameters of the new model for complete, right-censored and progressively type-I interval-censored data sets. In the Bayesian discussion, we consider different types of symmetric and asymmetric loss functions such as squared error, absolute value, Linear Exponential ($LINEX$) and generalized entropy to estimate three unknown parameters of $NLCH - W$ model. Since the parameter space for all three parameters is positive, we use gamma priors distributions. Bayesian %95 credible and highest posterior density (HPD) intervals (see Chen et al. (1999)) are provided for each parameter of proposed model. In addition, the asymptotic confidence intervals and parametric and non-parametric bootstrap confidence intervals are calculated in order to compare with corresponding Bayesian intervals.

The rest of the paper organized as follows. In the section 2, a new category of lifetime distributions is introduced based on the fundamental quantity $H(x)$ and then the main statistical and reliability properties are obtained in general setting. In section 3, by considering the Weibull distribution as the base distribution, a new model is presented according to the general model discussed in section 1 and its prominent characteristics are studied. This new model refer as $NLCH-W$ distribution. In section 4, we examine the inferential procedures for estimation unknown parameters of the $NLCH-W$ model. In this Section, we provide discussions about three important estimation methods maximum likelihood, Bayesian and bootstrap. Here we use four well-known loss functions like squared error, absolute value, LINEX and generalized entropy. Application and numerical analysis of three real data sets (complete, right-censored and progressively type-I interval-censored) are presented in section 5. Finally, in section 6 the paper is concluded.

2. New model and properties

In this section, first we introduce a new category of lifetime distributions and then we obtain main statistical and reliability properties of the proposed family in general setting.

Definition 2.1. *A random variable X is said to have $NLCH-G$ distribution if its probability distribution function (pdf) is given by*

$$f(x;\alpha,\gamma)=\alpha h(x)\big(\gamma+H(x)\big)^{\alpha-1}e^{\gamma-(\gamma+H(x))^{\alpha}},\ x>0,\ \alpha>0,\ \gamma>0, \quad (1)$$

and its cumulative distribution function (cdf) is given by

$$F(x;\alpha,\gamma)=1-e^{\gamma-(\gamma+H(x))^{\alpha}},\ x>0,\ \alpha>0,\ \gamma>0 \quad (2)$$

where, $H(x)$ is cumulative hazard function of baseline distribution $G(x)$ and $h(x)=\frac{\partial H(x)}{\partial x}$.

The corresponding survival function of (1) is given as

$$R(x;\alpha,\gamma)=e^{\gamma-(\gamma+H(x))^{\alpha}},\ x>0,\ \alpha>0,\ \gamma>0. \quad (3)$$

Remark 2.2. *Let $\alpha=1$, then we get $F(x;\alpha=1,\gamma)=G(x)$.*

In the following theorem we investigate the connection between NH and $NLCH-G$ models.

Theorem 2.3. *Suppose that the random variable X be a continuous random variable with cumulative hazard rate function $H(x)$, and the random variable Y has NH distribution with parameter α and λ. Then the transformed variable $Z=H^{-1}(\lambda Y)$ has a density with pdf (1) as parameter $\gamma=1$. $H^{-1}(.)$ is inverse function of $H(.)$.*

Proof. Using the method of distribution function we have;

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(H^{-1}(\lambda Y) \leq z) \\ &= P(Y \leq \frac{1}{\lambda} H(z)) \\ &= 1 - e^{1-(1+H(z))^{\alpha}}, \end{aligned}$$

so the proof is completed. □

Following this section, we get some fundamental properties of proposed model such as hazard rate function, survival function, quantile function and order statistic distribution. It is seen that all of these measures have closed expression in terms of quantity $H(x)$.

2.1. **Hazard Rate Function.** The hazard rate is a key concept in analysis of reliability and measuring the aging process. Knowing shape and behavior of the hazard rate in reliability theory, risk analysis, and so on, is very important. The hazard rate function of the $NLCH-G$ distribution is given as

$$\begin{aligned} h_F(x, \alpha, \gamma) &= \frac{f(x; \alpha, \gamma)}{R(x; \alpha, \gamma)} \\ &= \alpha h(x)(\gamma + H(x))^{\alpha-1}. \end{aligned} \tag{4}$$

Remark 2.4. *In fact the hazard rate function of new model is a weighted version of baseline hazard with weight* $w(x) = \alpha(\gamma + H(x))^{\alpha-1}$.

Lemma 2.5. *By considering (4), we have*

- *if* $r(x)$ *is increasing and* $\alpha \geq 1$ *then* $r_F(x, \alpha, \gamma)$ *is increasing.*
- *if* $r(x)$ *is decreasing and* $\alpha \leq 1$ *then* $r_F(x, \alpha, \gamma)$ *is decreasing.*

Proof. The proof is straightforward. □

In the following lemma we provide a result about stochastic order in hazard function to compare proposed model and baseline distribution. First we recall the following definition. The random variable X is said to be less than variable Y in hazard rate order, $X \leq_{hr} Y$, if $h_X(x) \geq h_Y(x)$, for all x in the union of supports of X and Y, where $h_X(x)(h_Y(x))$ is the hazard rate of $X(Y)$. For more details see Shaked and Shanthikumar (2007).

Lemma 2.6. *Let* X_F *and* X_G *be two random variables corresponding with proposed model (1) and distribution* G *respectively, then under the condition* $\gamma \geq 1$

- *if* $\alpha > 1$ *then* $X_F \leq_{hr} X_G$.
- *if* $\alpha < 1$ *then* $X_G \leq_{hr} X_F$.

Proof. The proof is straightforward. □

2.2. **Random variate generation.** One important quantity for each probabilistic model is to have the data generator function based on an explicit formula, because the simulation studies researchers are more satisfied with the data generator functions of a given form. For generating random variables from the $NLCH-G$ distribution, we use the inverse transformation method. The quantile of order p of the $NLCH - G$ distribution is

$$x_p = F^{-1}(p; \alpha, \gamma) = H^{-1}((\gamma - \log(1-p))^{\frac{1}{\alpha}} - \gamma). \tag{5}$$

where $H^{-1}(x)$ is inverse function of quantity $H(x)$. Let U be a random variable generated from a uniform distribution on$(0, 1)$, then

$$X = H^{-1}((\gamma - \log(1-U))^{\frac{1}{\alpha}} - \gamma) \tag{6}$$

is a random variable generated from the $NLCH-G$ distribution by the probability integral transform.

2.3. **Order statistics.** Order statistics have applications in various directions such as statistical inference, reliability engineering, quality control and etc. Let X_1, X_2, $\ldots$, X_n be a random sample from $NLCH - G$ distribution. Let $X_{i:n}$ denote the ith order statistic. Then the *pdf* of $X_{i:n}$ is given by

$$\begin{aligned} g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} g(x)[G(x)]^{i-1}[\bar{G}(x)]^{n-i} \\ &= \frac{n! e^{n-i+1}}{(i-1)!(n-i)!} \alpha h(x)\big(\gamma + H(x)\big)^{\alpha-1} e^{(n-i+1)(\gamma+H(x))^{\alpha}} \\ &\quad \times \big(1 - e^{\gamma-(\gamma+H(x))^{\alpha}}\big)^{i-1} \end{aligned}$$

3. NLCH-Weibull (NLCH-W) model

Without loss of generality let parameter $\gamma = 1$ and consider the Weibull distribution as a parent distribution with *cdf* function $F(x; \beta, \lambda) = 1 - e^{-\lambda x^{\beta}}, x > 0, \beta > 0, \lambda > 0$. By replacing this model in relation (3), the *pdf* of the $NLCH - W$ is given as

$$f(x; \alpha, \beta, \lambda) = \alpha\lambda\beta x^{\beta-1}(1 + \lambda x^{\beta})^{\alpha-1} e^{1-(1+\lambda x^{\beta})^{\alpha}} \tag{7}$$

and its *cdf* is given by

$$F(x; \alpha, \lambda) = 1 - e^{1-(1+\lambda x^{\beta})^{\alpha}}. \tag{8}$$

Remark 3.1. *If $\alpha = 1$, we attain the pdf of Weibull distribution and If $\beta = 1$, we get NH distribution respectively.*

3.1. **Density shape.** It is easy to investigate that the shape of $NLCH-W$ is unimodal and

- if $\beta > 1$ then $\lim_{x\to 0} f(x) = 0$,
- if $\beta < 1$ then $\lim_{x\to 0} f(x) = \infty$,

and

$$\lim_{x\to\infty} f(x) = 0. \tag{9}$$

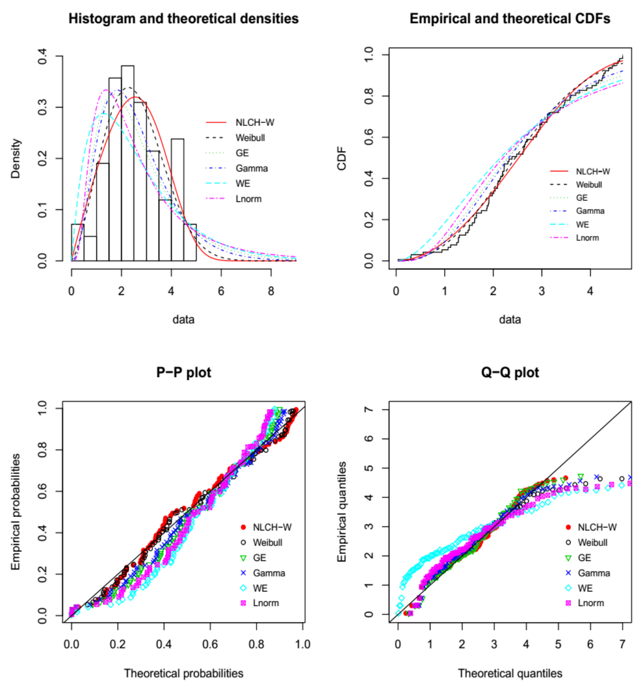

FIGURE 1. The graphs of *pdf* (a) and hazard rate function (b, c and d) of the $NLCH-W$ distribution for some selected values of parameters.

In the next section, we consider the hazard shape of $NLCH-W$ distribution.

3.2. **Hazard rate function of NLCH-W distribution.** The hazard rate function of $NLCH-W$ distribution is

$$\begin{aligned} h(x) &= \frac{f(x)}{1-F(x)} \\ &= \alpha\lambda\beta x^{\beta-1}(1+\lambda x^{\beta})^{\alpha-1}. \end{aligned}$$

Determining the behavior of the hazard rate is very important in various applications, especially in reliability theory. It can easily be shown that the proposed model (7) has a variety of hazard shapes. The hazard rate function allows for constant, monotonically increasing, monotonically decreasing, unimodal and bathtub shaped hazard rates. In summary, different types of hazard rates are as follows.

- if $\beta > 1$ and $\alpha\beta > 1$ then $h(x)$ is monotonically increases with $h(0) = 0$.
- if $\beta < 1$ and $\alpha\beta < 1$ then $h(x)$ is monotonically decreases with $h(0) = \infty$.
- if $\beta > 1$ and $\alpha\beta < 1$ then $h(x)$ is bathtub shape.
- if $\beta < 1$ and $\alpha\beta > 1$ then $h(t)$ is upside down bathtub shape.
- if $\beta = 1$ and $\alpha = 1$ then $h(t)$ is constant.

Some shapes of *pdf* and hazard function for the selected values of parameters is given in Figure 1.

4. Estimation procedures

Nowadays, three methods of maximum likelihood estimation, Bayesian and bootstrap procedures are of particular importance in the theory of statistical inference undoubtedly. In this section, we describe each of these methods separately for estimating the parameters α, β and λ of the $NLCH-W$ distribution. For all methods we consider the case when all three parameters are unknown.

4.1. **Maximum likelihood estimation.** The maximum likelihood procedure is one of the most common methods for obtaining an estimator for an unknown parameter in classic statistical inference. The likelihood function is a function that written based on the mechanism of the observations occurrence. Here, the structure of the likelihood function is expressed for three modes of observations including complete data, right-censored and progressive interval-censored data sets.

4.1.1. *Maximum likelihood estimation for complete data set.* Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $NLCH-W$ distribution. The likelihood function is given for equation (7) by

$$L(\underline{x}, \alpha, \beta, \lambda) = \prod_{i=1}^{n} \alpha\lambda\beta x_i^{\beta-1}(1+\lambda x_i^{\beta})^{\alpha-1} e^{1-(1+\lambda x_i^{\beta})^{\alpha}}. \tag{10}$$

So, the log-likelihood function is written as

$$\begin{aligned}\ell(\underline{x}, \alpha, \beta, \lambda) &= \log L(\underline{x}, \alpha, \beta, \lambda) = n[\log\alpha + \log\lambda + \log\beta] + (\beta-1)\sum_{i=1}^{n}\log x_i \\ &+ (\alpha-1)\sum_{i=1}^{n}\log(1+\lambda x_i^{\beta}) + n - \sum_{i=1}^{n}(1+\lambda x_i^{\beta})^{\alpha}.\end{aligned}$$

The normal equations are derived by differentiation of the log-likelihood function with respect to parameters α, β and λ.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1 + \lambda x_i^{\beta}) - \sum_{i=1}^{n} (1 + \lambda x_i^{\beta}) \log(1 + \lambda x_i^{\beta}),$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} (1 + \lambda x_i^{\beta}) \log x_i + (\alpha - 1) \sum_{i=1}^{n} \frac{\lambda x_i^{\beta} \log x_i}{1 + \lambda x_i^{\beta}} - \sum_{i=1}^{n} \lambda x_i^{\beta} (1 + \lambda x_i^{\beta})^{\alpha} \log x_i,$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \frac{x_i^{\beta}}{1 + \lambda x_i^{\beta}} - \alpha \sum_{i=1}^{n} x_i^{\beta} (1 + \lambda x_i^{\beta})^{\alpha - 1}.$$

Setting these differentiations equal to zero and solving for α, β and λ, then we can obtain the maximum likelihood estimator MLE of parameters α, β and λ.

4.1.2. *Maximum likelihood estimation for right-censored data set.* Let $(X_1, \delta_1), (X_2, \delta_2), \ldots, (X_n, \delta_n)$ be a right-censored random sample of size n from $NLCH-W$ distribution. Where δ_i is a censoring indicator variable, that is, $\delta_i = 1$ for an observed survival time and $\delta_i = 0$ for a right-censored survival time. In the case $NLCH-W$ distribution the likelihood function and the corresponding log-likelihood are given as

$$L(\underline{x}, \underline{\delta}, \alpha, \beta, \lambda) = \prod_{i=1}^{n} \left(\alpha \lambda \beta x_i^{\beta - 1} (1 + \lambda x_i^{\beta})^{\alpha - 1} e^{1 - (1 + \lambda x_i^{\beta})^{\alpha}} \right)^{\delta_i} \left(e^{1 - (1 + \lambda x_i^{\beta})^{\alpha}} \right)^{1 - \delta_i} \tag{11}$$

and

$$\begin{aligned} \ell(\underline{x}, \underline{\delta}, \alpha, \beta, \lambda) &= \log L(\underline{x}, \underline{\delta}, \alpha, \beta, \lambda) \\ &= [\log \alpha + \log \lambda + \log \beta] \sum_{i=1}^{n} \delta_i + (\beta - 1) \sum_{i=1}^{n} \delta_i \log x_i \\ &\quad + (\alpha - 1) \sum_{i=1}^{n} \delta_i \log(1 + \lambda x_i^{\beta}) + n - \sum_{i=1}^{n} (1 + \lambda x_i^{\beta})^{\alpha}, \end{aligned}$$

respectively. Analogous above results the normal equations can be derived in the case right-censored sample data.

4.1.3. *Maximum likelihood estimation for progressively type-I interval-censored data set.* Let n items to be applied on a life testing simultaneously at time $t = 0$ and suppose that m pre-specified times $t_1 < t_2 < ... < t_m$, where t_m is scheduled time to terminate the experiment, be determined . At the ith inspection time, t_i , the number, X_i , of failures within $(t_i - 1, t_i]$ is recorded and R_i surviving items are randomly removed from the life testing for $i = 1, 2, ..., m$. Therefore, a progressively type-I interval-censored sample can be denoted as $S = (X_i, R_i, t_i)$ and sample size is $n = \sum_{i=1}^{m}(X_i + R_i)$. The likelihood function of density (1) based on progressively type-I interval-censored sample

$$S = (X_i, R_i, t_i), i = 1, 2, ..., n$$

is given as

$$L(S, \alpha, \beta, \lambda) = \prod_{i=1}^{m} \left[e^{1-(1+\lambda t_{i-1}^{\beta})^{\alpha}} - e^{1-(1+\lambda t_i^{\beta})^{\alpha}} \right]^{X_i} \left[1 - e^{1-(1+\lambda t_i^{\beta})^{\alpha}} \right]^{R_i}. \quad (12)$$

The log-likelihood function is given as;

$$\begin{aligned} \ell(S, \alpha, \beta, \lambda) &= \log L(S, \alpha, \beta, \lambda) = \sum_{i=1}^{n} X_i \log \left[e^{1-(1+\lambda t_{i-1}^{\beta})^{\alpha}} - e^{1-(1+\lambda t_i^{\beta})^{\alpha}} \right] \\ &+ \sum_{i=1}^{n} R_i \log \left[1 - e^{1-(1+\lambda t_i^{\beta})^{\alpha}} \right]. \end{aligned}$$

Also, here we can derive normal equations for corresponding log-likelihood function similar complete and right-censored samples data. In practical due to the non-linearity of corresponding normal equations in three cases that discussed above, we use numerical algorithms to extract MLEs estimators.

4.2. **Bootstrap estimation.** The uncertainty in parameters of the fitted distribution can be estimated by parametric (re-sampling from the fitted distribution) or non-parametric (re-sampling with replacement from the original data set) bootstraps re-sampling Efron and Tibshirani (1994). These two parametric and non-parametric bootstrap procedures for complete data set are described as follows.

Parametric bootstrap procedure:

(1) Estimate θ (vector of unknown parameters), say $\hat{\theta}$, by using the MLE procedure based on a random sample.
(2) Generate a bootstrap sample $\{X_1^*, \ldots, X_m^*\}$, using θ and obtain the bootstrap estimate of θ, say $\widehat{\theta}^*$, from the bootstrap sample based on the MLE procedure.
(3) Repeat step 2 $NBOOT$ times.
(4) Order $\widehat{\theta}^*{}_1, \ldots, \widehat{\theta}^*{}_{NBOOT}$ as $\widehat{\theta}^*{}_{(1)}, \ldots, \widehat{\theta}^*{}_{(NBOOT)}$. Then obtain γ-quantiles and $100(1-\alpha)\%$ confidence intervals of parameters.

In case of the $NLCH - W$ distribution, the parametric bootstrap estimators (PBs) of α, β and λ, say $\hat{\alpha}_{PB}, \hat{\beta}_{PB}$ and $\hat{\lambda}_{PB}$, respectively.

Non-parametric bootstrap procedure:

(1) Generate a bootstrap sample $\{X_1^*, \ldots, X_m^*\}$, with replacement from original data set. Obtain the bootstrap estimate of θ with MLE procedure, say $\widehat{\theta^*}$ using the bootstrap sample.
(2) Repeat step 2 $NBOOT$ times.
(3) Order $\widehat{\theta^*}_1, \ldots, \widehat{\theta^*}_{NBOOT}$ as $\widehat{\theta^*}_{(1)}, \ldots, \widehat{\theta^*}_{(NBOOT)}$. Then obtain γ-quantiles and $100(1-\alpha)\%$ confidence intervals of parameters.

In case of the $NLCH-W$ distribution, the non-parametric bootstrap estimators (NPBs) of α, β and λ, say $\hat{\alpha}_{NPB}, \hat{\beta}_{NPB}$ and $\hat{\lambda}_{NPB}$, respectively.
Analogous algorithms can be expressed for bootstrap estimation of right-censored sample data.

4.3. **Bayesian estimation.** Bayesian inference procedure for censored data have been taken into consideration by many statistical researchers, especially researchers in the field of survival analysis and reliability engineering. In this section, a complete sample data and two widely used types of censored observations, right-censored and progressively type-I interval-censored observations are analyzed through Bayesian point of view. We assume that the parameters α, β and λ of $NLCH-W$ distribution have independent prior distributions as

$$\alpha \sim Gamma(a,b), \beta \sim Gamma(c,d), \lambda \sim Gamma(e,f),$$

where a,b,c,d,e and f are positive. Hence, the joint prior density function is formulated as follow:

$$\pi(\alpha,\beta,\lambda) = \frac{b^a d^c f^e}{\Gamma(a)\Gamma(c)\Gamma(e)} \alpha^{a-1}\beta^{c-1}\lambda^{e-1}e^{-(b\alpha+d\beta+f\lambda)}. \tag{13}$$

In the Bayesian estimation, according to that we do not know the actual value of the parameter, we may be adversely affected by loss when we choose an estimator. This loss can be measured by a function of the parameter and corresponding estimator. Four well-known loss functions and associated Bayesian estimators are presented as:

- Squared error loss function and Bayesian estimator
  $L(\gamma(\theta), d(\underline{x})) = (\gamma(\theta) - d(\underline{x}))^2$
  $d_B(\underline{x}) = E(\gamma(\theta)|d(\underline{x}))$
- Absolute value loss function and Bayesian estimator
  $L(\gamma(\theta), d(\underline{x})) = |\gamma(\theta) - d(\underline{x})|$
  $d_B(\underline{x}) = Median(\gamma(\theta)|d(\underline{x}))$
- LINEX loss function and Bayesian estimator
  $L(\gamma(\theta), d(\underline{x})) = \left[e^{c(\gamma(\theta)-d(\underline{x}))} - (\gamma(\theta) - d(\underline{x})) - 1\right]$
  $d_B(\underline{x}) = -\frac{\log E\left(e^{-c\gamma(\theta)}|d(\underline{x})\right)}{c}$

- Generalized entropy loss function and Bayesian estimator

$$L(\gamma(\theta), d(\underline{x})) = \left[(\tfrac{\gamma(\theta)}{d(\underline{x})})^c - \log(\tfrac{\gamma(\theta)}{d(\underline{x})}) - 1 \right]$$

$$d_B(\underline{x}) = \left(E(\gamma^{-c}(\theta)|\underline{x}) \right)^{-\frac{1}{c}}.$$

For more details see Calabria and Pulcini (1996).

In the next, we provide the posterior probability distributions in three modes: complete, right-censored and progressively type-I interval-censored data sets. Let we define the function φ as

$$\varphi(\alpha, \beta, \lambda) = \alpha^{a-1}\beta^{c-1}\lambda^{e-1}e^{-(b\alpha+d\beta+f\lambda)},\ \alpha > 0,\ \beta > 0,\ \lambda > 0.$$

The joint posterior distribution in terms of a given likelihood function $L(data)$ and joint prior distribution $\pi(\alpha, \beta, \lambda)$ defined as

$$\pi^*(\alpha, \beta, \lambda|data) \propto \pi(\alpha, \beta, \lambda)L(data). \tag{14}$$

Hence, we get joint posterior density of parameters α, β and λ for complete sample data by combining the likelihood function (10) and joint prior density (13). Therefore, the joint posterior density function is given by

$$\pi^*(\alpha, \beta, \lambda|\underline{x}) = K\varphi(\alpha, \beta, \lambda) \prod_{i=1}^{n} \alpha\beta\lambda x_i^{\beta-1}(1 + \lambda x_i^{\beta})^{\alpha-1}e^{-(1+\lambda x_i^{\beta})^{\alpha}} \tag{15}$$

where K is given as

$$K^{-1} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \varphi(\alpha, \beta, \lambda) \prod_{i=1}^{n} \alpha\beta\lambda x_i^{\beta-1}(1 + \lambda x_i^{\beta})^{\alpha-1}e^{-(1+\lambda x_i^{\beta})^{\alpha}} d\alpha d\beta d\lambda.$$

Furthermore, by using likelihood functions (11), (12) and joint prior distribution (13) the joint posterior probability distribution functions for right-censored $(\underline{x}, \underline{\delta})$ and progressively type-I interval-censored sample data $(S = (X_i, R_i, t_i), i = 1, 2, ..., n)$ presented respectively with

$$\pi^*(a, \beta, \lambda|\underline{x}, \underline{\delta}) = M\varphi(\alpha, \beta, \lambda) \prod_{i=1}^{n} \left(\alpha\lambda\beta x_i^{\beta-1}(1 + \lambda x_i^{\beta})^{\alpha-1} \right)^{\delta_i} e^{-(1+\lambda x_i^{\beta})^{\alpha}}$$

and

$$\pi^*(a, \beta, \lambda|S) = Z\varphi(\alpha, \beta, \lambda) \prod_{i=1}^{n} \left[e^{1-(1+\lambda t_{i-1}^{\beta})^{\alpha}} - e^{1-(1+\lambda t_i^{\beta})^{\alpha}} \right]^{X_i} \left[1 - e^{1-(1+\lambda t_i^{\beta})^{\alpha}} \right]^{R_i},$$

where M is given as

$$M^{-1} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \varphi(\alpha, \beta, \lambda) \prod_{i=1}^{n} \left(\alpha\lambda\beta x_i^{\beta-1}(1 + \lambda x_i^{\beta})^{\alpha-1} \right)^{\delta_i} e^{-(1+\lambda x_i^{\beta})^{\alpha}} d\alpha d\beta d\lambda.$$

and Z is given as

$$\begin{aligned} Z^{-1} &= \int_0^\infty \int_0^\infty \int_0^\infty \varphi(\alpha,\beta,\lambda) \\ &\times \prod_{i=1}^{n} \left[e^{1-(1+\lambda t_{i-1}^{\beta})^{\alpha}} - e^{1-(1+\lambda t_{i}^{\beta})^{\alpha}} \right]^{X_i} \left[1 - e^{1-(1+\lambda t_{i}^{\beta})^{\alpha}} \right]^{R_i} d\alpha d\beta d\lambda. \end{aligned}$$

Here we interested in obtaining Bayesian estimators for three sample data sets (complete, right-censored and type-I progressive interval-censored data sets) under the four loss functions described above. As it is observed, there are no closed forms for the Bayes estimators. It is possible to simulated posterior sample data sets by using Gibbs sampling method and Metropolis-Hasting algorithm. Thus, by applying MCMC algorithm the corresponding Bayes estimators, Bayesian credible and HPD intervals are calculated.

5. Application of NLCH-W distribution on the real datasets

This section aims to show applications of the $NLCH-W$ model under the methods discussed in the section 4 via real data examples. In order to achieve this target, we consider three real data sets to illustrate the application of proposed distribution in real world and the superiority of this model to some other useful classic models. Furthermore, in this section, we provide Bayesian and bootstrap analysis of parameter estimation of $NLCH-W$ model for three real data sets. The following data sets contain three modes of real world observations: complete, right-censored and progressively type-I interval-censored.

Complete data set: Failure times of 84 Aircraft Windshield

We consider the data of service times for a particular model windshield. These data were recently studied by Ramos et al. (2013). The data consist of 84 observations.
0.040 1.866 2.385 3.443 0.301 1.876 2.481 3.467 0.309 1.899 2.610 3.478 0.557 1.911 2.625 3.578 0.943 1.912 2.632 3.595 1.070 1.914 2.646 3.699 1.124 1.981 2.661 3.779 1.248 2.010 2.688 3.924 1.281 2.038 2.82 3 4.035 1.281 2.085 2.890 4.121 1.303 2.089 2.902 4.167 1.432 2.097 2.934 4.240 1.480 2.135 2.962 4.255 1.505 2.154 2.964 4.278 1.506 2.190 3.000 4.305 1.568 2.194 3.103 4.376 1.615 2.223 3.114 4.449 1.619 2.224 3.117 4.485 1.652 2.229 3.166 4.570 1.652 2.300 3.344 4.602 1.757 2.324 3.376 4.663

Right-censored data set: Lifetimes of 30 devices

Meeker and Escobar (2014) represented observed lifetimes of 30 devices that includes eight censored observations. 2 10 13 23 23 28 30 65 80 88 106 143 147 173 181 212 245 247 261 266 275 293 300+ 300+ 300+ 300+ 300+ 300+ 300+ 300+
The + sign indicates right-ensored observations.

Progressively type-I interval-censored data set: 112 patients with plasma cell myeloma

Table 1 contains a typical progressively type-I interval-censored data that devoted to 112 patients with plasma cell myeloma treated at the National Cancer Institute (see Carbone et al. (1967)).

TABLE 1. Progressively type-I interval-censored data set

Interval in months	Number at risk	Number of withdrawals
[0,5.5)	112	1
[5.5,10.5)	93	1
[10.5,15.5)	76	3
[15.5,20.5)	55	0
[20.5,25.5)	45	0
[25.5,30.5)	34	1
[30.5, 40.5)	25	2
[40.5,50.5)	10	3
[50.5,60,6)	3	2
[60.5,∞)	0	0

5.1. **MLE, bootstrap and Bayesian estimation of NLCH-W model and comparing with other models in case complete data set.** First, we fit the proposed distribution to the complete real data set by MLE method and compare the results with the gamma, Weibull, log-normal ($Lnorm$), generalized exponential (GE) and weighted exponential (WE) distributions with respective densities

$$f_{gamma}(x) = \frac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x}$$

$$f_{Weibull}(x) = \frac{\beta}{\lambda^{\beta}}x^{\beta-1}e^{-(\frac{x}{\lambda})^{\beta}}$$

$$f_{Lnorm}(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{\frac{-(\log x-\mu)^2}{2\sigma^2}}$$

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}$$

$$f_{WE}(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1-e^{-\alpha\lambda x}).$$

Table 2 includes the MLEs of parameters, log-likelihood and Akaike information criterion (AIC) for $NLCH-W$ distribution and the mentioned above distributions in the case complete real data set. The results of Table 2 shows that, the $NLCH-W$

Table 2: The $MLEs$ of parameters for complete data set.

Model	Estimation	Log-likelihood	AIC
NLCH-W	$(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$=(3.874,1.938,0.024)	-128.052	262.105
gamma	$(\hat{\alpha}, \hat{\lambda})$=(3.492,1.365)	-136.937	277.874
Weibull	$(\hat{\beta}, \hat{\lambda})$=(2.374,2.863)	-130.053	264.107
Lnorm	$(\hat{\mu}, \hat{\sigma})$=(0.789,0.687)	-153.920	311.840
WE	$(\hat{\alpha}, \hat{\lambda})$=(0.002,0.781)	-143.025	290.049
GE	$(\hat{\alpha}, \hat{\lambda})$=(3.562 ,0.758)	-139.841	283.681

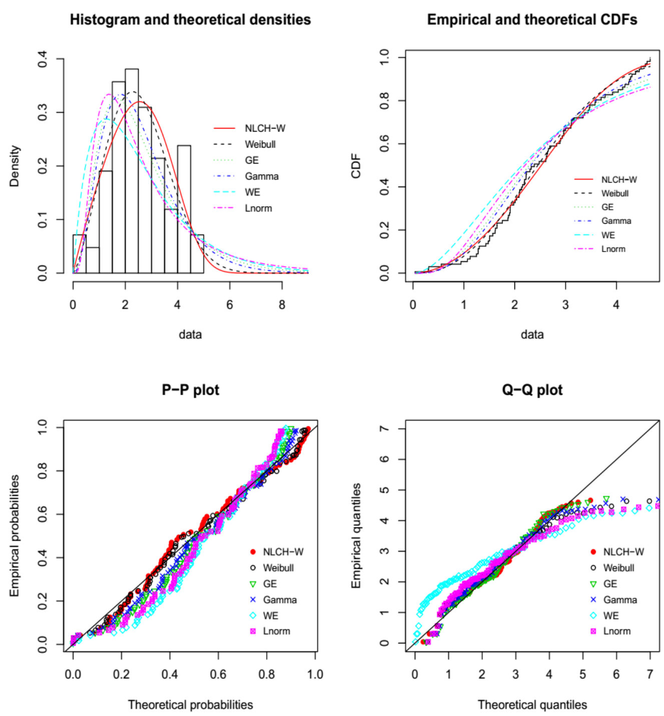

FIGURE 2. Histogram and fitted density plots, the plots of empirical and fitted cdfs, $P-P$ plots and $Q-Q$ plots for the complete data set.

distribution provides the best fit for the complete data set as it has lower AIC statistic than the other competitor models. The histogram of data set, fitted *pdf* of the $NLCH-W$ distribution and fitted *pdf*s of other competitor distributions for the real data set are plotted in Figure 2. Also, the plots of empirical and fitted *cdf*s functions, $P-P$ plots and $Q-Q$ plots for the $NLCH-W$ and other fitted distributions are displayed in Figure 2. These plots also support the results in Table 2.

In the rest of this subsection, we provide Bayesian and Bootstrap estimation results. It is clear from the equation (15) that there is no closed form for the Bayesian estimators under the four loss functions described in subsection 4.3, so we suggest using an MCMC procedure based on 1000 replicates to compute Bayesian estimators. The corresponding Bayesian point and interval estimation provided in Table 3. The posterior samples extracted by using Gibbs sampling technique. Moreover, we provide the posterior summary plots. These plots confirm that the sampling process is of prime quality and convergence is occurred.

Also, here we obtain point and %95 confidence interval estimation of parameters of the $NLCH-W$ distribution by parametric and non-parametric bootstrap methods for complete real data set. We provide results of bootstrap estimation based on 10000 bootstrap replicates in Table 3. It is interesting to look at the joint distribution of the bootstrapped values in a scatter plot in order to understand the potential structural correlation between parameters (see Figures 3 and 4).

Table 3: Bayesian and bootstrap estimation of parameters of $NLCH-W$ for complete data set.

Estimation procedure	Bootstrap estimation of parameters		
Parametric Bootstrap	$\hat{a}_{PB}$	$\hat{\beta}_{PB}$	$\hat{\lambda}_{PB}$
Point estimation	4.517	1.981	0.019
Confidence interval	(0.74, 37.944)	(1.584, 2.973)	(0.003, 0.105)
Non-Parametric Bootstrap	$\hat{a}_{NPB}$	$\hat{\beta}_{NPB}$	$\hat{\lambda}_{NPB}$
point estimation	21.481	1.859	0.005
confidence interval	(0.589, 49.932)	(1.507, 3.260)	(0.0022, 0.088)
Bayesian procedure	Bayesian estimation of parameters		
Loss function	$\hat{a}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
Squared error	3.934	1.952	0.022
Absolute value	3.905	1.955	0.022
LINEX ($c=-0.5$)	4.059	1.959	0.021
Generalized entropy ($c=-0.5$)	3.904	1.949	0.022
Bayesian Interval	$\hat{a}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
Credible interval	(3.444, 4.386)	(1.845, 2.052)	(0.019, 0.024)
HPD	(2.550, 5.266)	(1.624, 2.247)	(0.015, 0.029)

By analyzing the results of the present table, we can see that the estimated values of parameters are similar for both Bayesian and bootstrap procedures in terms of point and interval (quantile bootstrap, %95 credible and HPD intervals) estimation. In addition, by comparing this results with $MLEs$ estimation of parameters

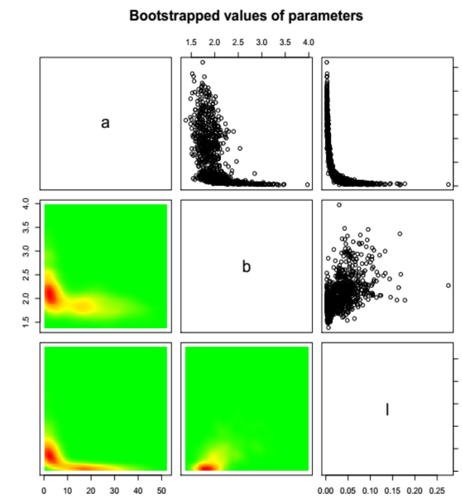

FIGURE 3. Parametric bootstrapped values of parameters of the $NLCH-W$ distribution for the complete data.

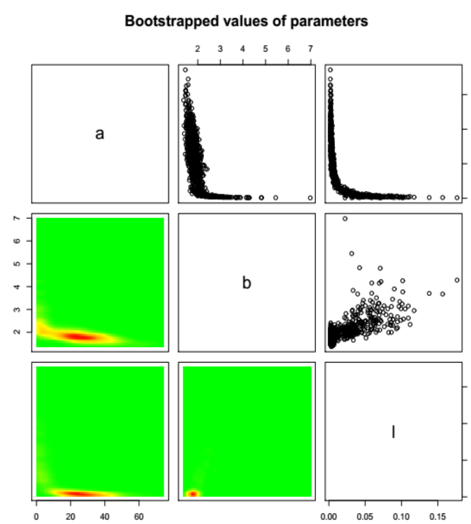

FIGURE 4. Non-parametric bootstrapped values of parameters of the $NLCH-W$ distribution for the complete data.

of $NLCH-W$ in Table 2, it can be seen that, in general the estimation results are

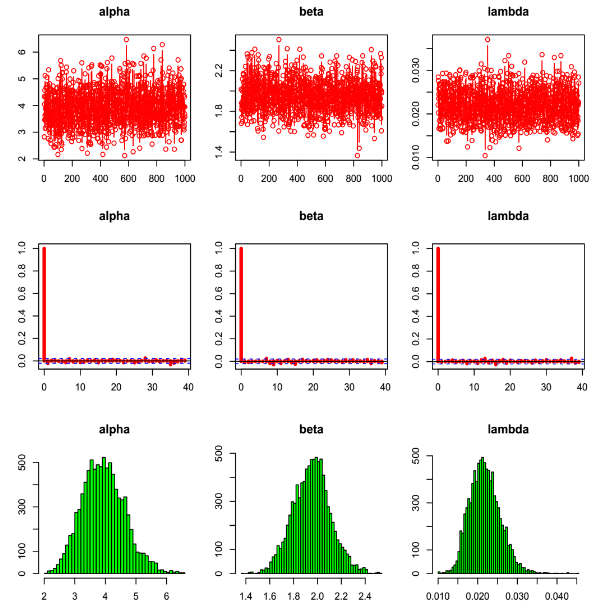

FIGURE 5. Plots of Bayesian analysis and performance of Gibbs sampling for complete data set. Top panel: trace plots; Middle panel: autocorrelation plots; Bottom panel: histograms of each parameter of $NLCH - W$ distribution.

similar under three estimation procedures that described in section 5. Figures 3 and 4 relate to the parametric and non-parametric bootstrap estimation of parameters α, β and λ. Also, Figure 5 relates to the Bayesian analysis process, including history (Trace plot), autocorrelation function(acf) and histogram of three parameters samples drown from posterior distribution (15). These plots show that convergence was reached, no autocorrelation problems were encountered and the density of the posterior is extracted.

5.2. **MLE, bootstrap and Bayesian estimation in case right-censored data set.** Here, we provide the MLE, non-parametric bootstrap and Bayesian estimation of α, β and λ, the parameters of $NLCH - W$ distribution for right-censored data set that given at the beginning of section 5. In order to compare different estimation results, we also provide interval estimation (%95 asymptotic confidence, quantile bootstrap, %95 credible and HPD intervals) of parameters under the three estimation procedures that considered in section 4. Table 4 shows the corresponding results for right-censored data set. In addition, the plots of empirical and theoretical $cdfs$ and diagrams of the Bayesian analysis process are provided in Figure 6 and

Figure 7, respectively. Associated Bayesian procedure plots show that convergence was reached and no autocorrelation problems there exist. Also, Figure 6 represent the estimated lower and upper bound of cumulative probability.

Table 4: MLE, Bayesian and bootstrap estimation of parameters of $NLCH-W$ distribution for right-censored data set.

Estimation procedure	Maximum likelihood estimation		
MLE's	$\hat{a}$	$\hat{\beta}$	$\hat{\lambda}$
Point estimation	3.344	0.835	0.002
Confidence interval	(0, 9.603)	(0.477, 1.192)	(0.001, 0.0034)
LL	-142.259		
AIC	290.517		
Estimation procedure	Bootstrap estimation of parameters		
Non-Parametric Bootstrap	$\hat{a}_{NPB}$	$\hat{\beta}_{NPB}$	$\hat{\lambda}_{NPB}$
point estimation	1.070	0.835	0.007
confidence interval	(0.116, 9.822)	(0.579, 1.831	(0.002, 0.054)
Bayesian procedure	Bayesian estimation of parameters		
Loss function	$\hat{a}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
Squared error	3.296	0.876	0.0022
Absolute value	3.129	0.873	0.002
LINEX ($c=-0.5$)	3.790	0.876	0.0022
Generalized entropy ($c=-0.5$)	3.175	0.872	0.0021
Bayesian Interval	$\hat{a}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
Credible interval	(2.335, 4.027)	(0.800, 0.949)	(0.001 0.0028)
HPD	(1.145, 5.937)	(0.677, 1.089)	(0.0005, 0.004)

5.3. **MLE and Bayesian estimation in the case progressively type-I interval-censored data set.** Analogous two previous subsections, here we provide a summary of numerical analysis of progressively type-I interval-censored data set based on the Bayesian and maximum likelihood methods described in section 5. Table 5 is devoted to estimation of parameters. This table provides the Bayesian estimators and %95 credible and HPD intervals for each parameter of proposed $NLCH-W$ model. In addition, the maximum likelihood estimators are calculated in order to compare with corresponding Bayesian estimators under the different loss functions. Plots of history, acf plots and histogram of posterior samples of each parameter of proposed distribution provided in Figures 8. These figures show that the simulation processes of Gibbs algorithm has good quality and convergence is occurred.

6. Conclusion

In this article, a new model of lifetime distributions is introduced and main properties of it are obtained. One of the interesting and important properties of proposed family is that, it results the Nadarajah and Haghighi (2011) famous

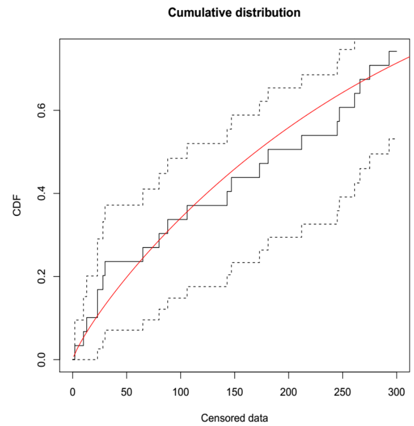

FIGURE 6. Plots of *cdf* of $NLCH - W$ distribution for right-censored data set.

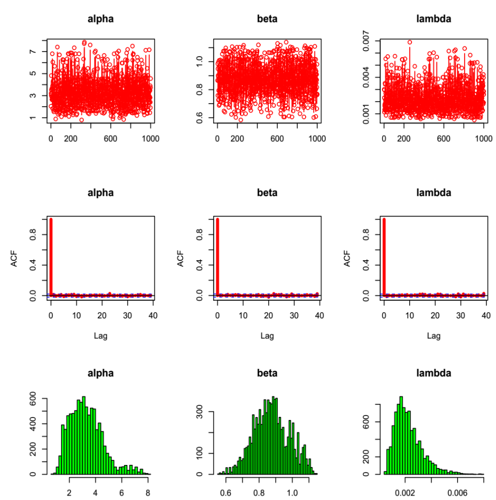

FIGURE 7. Plots of Bayesian analysis and performance of Gibbs sampling for right-censored data set. Top panel: trace plots; Middle panel: autocorrelation plots; Bottom panel: histograms of each parameter of $NLCH - W$ distribution.

distribution, as an especial case, when the parent distribution is exponential. An especial example of this family is introduced by considering Weibull distribution as the base distribution. We also show that the proposed distribution has variability

Table 5: Bayesian estimation of parameters of $NLCH-W$ for progressively type-I interval-censored data set.

Estimation method			
Maximum likelihood estimation	$\hat{a}$	$\hat{\beta}$	$\hat{\lambda}$
MLE's	0.996	1.228	0.019
LL	230.340		
AIC	466.681		
Bayesian estimation			
Loss function	$\hat{a}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
Squared error	1.005	1.333	0.019
Absolute value	0.939	1.322	0.018
LINEX ($c = 0.5$)	0.976	1.328	0.019
Generalized entropy ($c = 0.5$)	0.924	1.321	0.018
Bayesian Interval	$\hat{a}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
Credible interval	(0.758, 1.181)	(1.226,1.427)	(0.015,0.022)
HPD	(0.429,1.728)	(1.048,1.623)	(0.009,0.029)

of hazard rate shapes such as increasing, decreasing, bathtub shape and upside-down bathtub shapes. Classic and Bayesian inferences for three cases of real data such as complete, right-censored and progressively type-I interval-censored data sets are investigated. Bayesian estimators under the four well-known loss functions are presented. Numerical results of maximum likelihood, Bayesian and bootstrap procedures for each set of real data are presented in separate tables. From a practical point of view, the distribution introduced in this study was shown to be better than some common statistical distributions for some real data sets applied as an example.

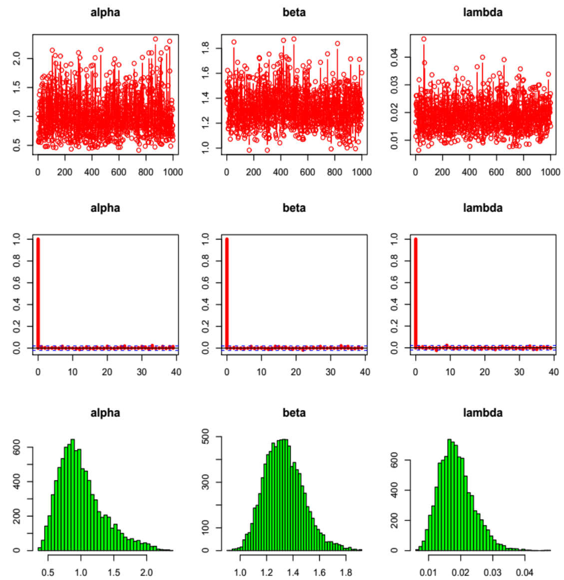

FIGURE 8. Plots of Bayesian analysis and performance of Gibbs sampling for progressively type-I interval-censored data set. Top panel: trace plots; Middle panel: autocorrelation plots; Bottom panel: histograms of each parameter of $NLCH-W$ distribution.

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APPROXIMATE TEST FOR TESTING A NULL VARIANCE RATIO IN THE UNBALANCED ONE-WAY RANDOM MODEL

SEVGI DEMIRCIOĞLU AND BILGEHAN GÜVEN

Abstract. The approximate test for testing the significance of the random effect is presented in the unbalanced one-way random model in which both random effects and errors are from nonnormal universes. The test is based on the asymptotic distribution of the F-ratio. Under the condition that the number of groups tends to infinity while the average of powers of the group sizes is bounded, the asymptotic distribution of the F statistic is obtained. Robustness of the proposed test is given.

1. Introduction

We derive the approximate test for testing the significance of the random effect in the unbalanced one-way random effects model where both random effects and errors are from nonnormal universes. To derive the approximate test, we first obtain the asymptotic distribution of the F-ratio.

In literature there are two different methods to obtain the asymptotic distribution of the F-ratio. Akritas and Arnold (2000) and Akritas and Papadatos (2004) obtained asymptotic normality of the F-ratio from the difference $MS_\tau - MSE$ and from the fact that MSE converges in probability to constant. Here, MS_τ and MSE are the mean square for the random effects and errors respectively. Westfall (1988) first derived the joint asymptotic distribution of MS_τ and MSE and then used the delta method to obtain asymptotic normality of the F-ratio.

To get the asymptotic distribution of the F-ratio, we use the method of Westfall and establish the following asymptotic condition. The number of groups is large while the average of powers of the group sizes is bounded. This asymptotic condition may be viewed as modification of the asymptotic condition established by Wesfall (1987, 1988). He assumed that the number of groups is large while the group sizes are from a finite set of positive integers.

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Also it is implicitly shown that the presented approximate test is robust for the size of the test in the balanced model does not depend on the fourth moment of the error term for the balanced case. The size of the test in the non normal case is same as it in the normal case.

This paper differs from the previous studies in three ways. A new asymptotic condition is established by modifying Westfall's asymptotic condition. Robustness of the asymptotic distribution of the F-ratio is analytically shown. Different distributions having positive, null and negative kurtosis are used in simulations.

This paper is organized as follows: Sec. 2 demonstrates the asymptotic condition and its consequences. Sec. 3 gives the asymptotic distribution of the F-ratio under the established asymptotic condition. Sec. 4 proposes the approximate test for testing significance of the random effects. Sec. 5 shows that the approximate test is robust. In Sec. 6 some numerical and simulated results are given to examine the accuracy of the approximate test.

Throughout the paper we shall use the following notations. If d_N is a sequence of N and r is a real number then $d_N = o(N^r)$ if $N^{-r}d_N \to 0$ as $N \to \infty$ and $d_N = O(N^r)$ if $N^{-r}d_N$ has a nonzero finite limit as $N \to \infty$.

2. The Model and Asymptotic

The unbalanced one-way random effects model is:

$$Y_{ij} = \mu + \tau_i + e_{ij} \qquad i = 1, 2, \dots t \qquad j = 1, 2, \dots, n_i \tag{1}$$

where μ is an overall mean, τ_i and e_{ij} are random variables with zero means and variances σ_τ^2 and σ^2 respectively. The model is appropriate for analyzing data involving t random treatments. The number of observation is N where $N = \sum_{i=1}^{t} n_i$.

We shall address the problem of testing $H_0: \quad \rho = 0$ vs. $H_1: \quad \rho > 0$ where the ratio of variances ρ is defined as $\rho = \sigma_\tau^2/\sigma^2$. The statistic for testing H_0 is based on

$$F_N = MS_\tau / MSE \tag{2}$$

where $MS_\tau = (t-1)^{-1} SS_\tau$ and $MSE = (N-t)^{-1} SSE$. SS_τ and SSE are the sum of squares for treatment and for error respectively and they are defined as

$$SS_\tau = \sum_{i=1}^{t} n_i (\overline{Y}_{i.} - \overline{Y}_{..})^2 \qquad \text{and} \qquad SSE = \sum_{i=1}^{t} \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_{i.})^2 \tag{3}$$

with $\overline{Y}_{i.} = n_i^{-1} \sum_{j=1}^{n_i} Y_{ij}$ and $\overline{Y}_{..} = N^{-1} \sum_{i=1}^{t} \sum_{j=1}^{n_i} Y_{ij}$. Under the normality of the random effects and the error terms, the test rejects H_0 when $F_N > F_{t-1,N-t,\alpha}$ where $F_{\nu_1,\nu_2,\alpha}$ denotes the $1-\alpha$ quantile of the F distribution with degrees of freedom ν_1 and ν_2.

When the random effects and error terms are from nonnormal universes, the approximate distribution of F_N is used for testing problem presented above. With the moment conditions that $E|\tau_i|^{4+\delta} < \infty$ and $E|e_{ij}|^{4+\delta} < \infty$ for some positive δ we establish the following asymptotic condition.

Asymptotic Condition. Consider a sequence of the model (1). The number of groups t tends to infinity in such a way that the average of n_1^p, n_2^p, …, n_t^p is bounded where $p \geq 1$. So there exists a real number $M > 0$ such that

$$t^{-1} \sum_{i=1}^{t} n_i^p < M$$

for all t. It is ensured by finite group sizes.

We are free to put in order the levels of the random effect among the $(t+1)$ levels. The group sizes can be ordered in the ascending order,i.e., $n_i \leq n_{i+1}$. Then

$$\frac{\sum_{i=1}^{t+1} n_i^p}{t+1} - \frac{\sum_{i=1}^{t} n_i^p}{t} = \frac{t n_{t+1}^p - \sum_{i=1}^{t} n_i^p}{t(t+1)}$$

where $t n_{t+1}^p > \sum_{i=1}^{t} n_i^p$. The sequences $t^{-1} \sum_{i=1}^{t} n_i^p$ of t are bounded and monotone and than they have a finite limit as $t \to \infty$. The positive monotone sequence $t^{-1} \sum_{i=1}^{t} (1/n_i^p)$ are bounded from both left by 0 and right by $t^{-1} \sum_{i=1}^{t} n_i^p$ So it has a finite limit as $t \to \infty$.

We have shown that $(1/t) \sum_{i=1}^{t} n_i$ has a finite limit as $t \to \infty$ where $(1/t) \sum_{i=1}^{t} n_i = N/t$. Then $t/N = O(1)$ implying that t and N are of the same order. So t can be replaced by N.

Thus we are ready to define the following limits appearing in calculation of the asymptotic covariance matrix. They are:

$$a = \lim_{N \to \infty} (t/N), \qquad \gamma_p = \lim_{N \to \infty} (1/N) \sum_{i=1}^{t} n_i^p \quad \text{for} \quad p = -1, 2. \tag{4}$$

where $a \in (0, 1)$ since $0 < t < N$.

3. Asymptotic Distribution of F_N

In this section we derive the asymptotic distribution of F_N in Eq. (2) where a variance ratio ρ is considered to be positive. The derivation of the asymptotic distribution of F_N is based on obtaining the joint asymptotic distribution of $\sqrt{N}(MS_\tau, MSE)$ and then applying the delta method.

Lemma 3.1. *Suppose the asymptotic condition established in Sec. 3. holds. Then the covariance matrix of* $\sqrt{N}(MS_\tau, MSE)'$ *is:*

$$\boldsymbol{ACOV} = 2\sigma^4 \begin{bmatrix} (\gamma_2\rho^2 + 2\rho + a)/a^2 & 0 \\ 0 & 1/(1-a) \end{bmatrix} + k_\tau \sigma^4 \begin{bmatrix} \gamma_2\rho^2/a^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ k_e \sigma^4 \begin{bmatrix} \gamma_{-1}/a^2 & (a - \gamma_{-1})/a(1-a) \\ (a - \gamma_{-1})/a(1-a) & (1 - 2a + \gamma_{-1})/(1-a)^2 \end{bmatrix} \tag{5}$$

as $N \to \infty$ where k_τ and k_e are the kurtosis of the underlying distributions of τ_i and e_{ij} defined as $k_\tau = E|\tau_i|^4/\sigma^4 - 3$ and $k_e = E|e_{ij}|^4/\sigma^4 - 3$, the limits a and γ_p for $p = -1, 2$ are in Eq. (4).

Proof. We first derive the asymptotic covariance matrix of $N^{-1/2}(SS_\tau, SSE)^{'}$. Let $\boldsymbol{Y}_i = (Y_{i1}, Y_{i2}, \dots, Y_{in_i})^{'}$, $\boldsymbol{Y} = (\boldsymbol{Y}_1^{'}, \boldsymbol{Y}_2^{'}, \dots, \boldsymbol{Y}_t^{'})^{'}$. We follow Searle's notation (see Searle 1987, p 212-213). SS_τ and SSE in Eq. (3) can be expressed in a matrix notation as $SS_\tau = \boldsymbol{Y}^{'}\boldsymbol{Q}_1\boldsymbol{Y}$ and $SSE = \boldsymbol{Y}^{'}\boldsymbol{Q}_2\boldsymbol{Y}$ where symmetric idempotent matrices $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are:

$$\boldsymbol{Q}_1 = \{_d(1/n_i)\boldsymbol{J}_{n_i}\}_{i=1}^t - (1/N)\boldsymbol{J}_N \quad \text{and} \quad \boldsymbol{Q}_2 = \boldsymbol{I}_N - \{_d(1/n_i)\boldsymbol{J}_{n_i}\}_{i=1}^t. \tag{6}$$

Here $\boldsymbol{I}_m$ and $\boldsymbol{J}_m$ are matrices of identity and ones of the order $m \times m$ respectively.

The model (1) is in a matrix notation as $\boldsymbol{Y} = \boldsymbol{1}_N\mu + \boldsymbol{U}\boldsymbol{\tau} + \boldsymbol{e}$ where $\boldsymbol{1}_m$ denotes a vector of ones of the order $m \times 1$, $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots \tau_t)^{'}$ and $\boldsymbol{e}$ is defined similarly to $\boldsymbol{Y}$. The matrix $\boldsymbol{U}$ of the order $N \times t$ is defined as

$$\boldsymbol{U} = \{_d\boldsymbol{1}_{n_i}\}_{i=1}^{i=t}. \tag{7}$$

It follows that SS_τ and SSE are rewritten as

$$SS_\tau = (\boldsymbol{\tau}^{'}, \boldsymbol{e}^{'}) \begin{bmatrix} \boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U} & \boldsymbol{U}^{'}\boldsymbol{Q}_1 \\ \boldsymbol{Q}_1\boldsymbol{U} & \boldsymbol{Q}_1 \end{bmatrix} \begin{pmatrix} \boldsymbol{\tau} \\ \boldsymbol{e} \end{pmatrix}, \quad SSE = (\boldsymbol{\tau}^{'}, \boldsymbol{e}^{'}) \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{\tau} \\ \boldsymbol{e} \end{pmatrix}. \tag{8}$$

From Eqs. (6) and (7), the matrix $\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U}$ of the order $t \times t$ is of the form

$$\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U} = \begin{cases} n_i - (1/N)n_i^2, & \text{if} \quad i = j \\ -(1/N)n_in_j & \text{if} \quad i \neq j \end{cases} \tag{9}$$

and the matrix $\boldsymbol{U}^{'}\boldsymbol{Q}_1$ of the order $t \times N$ is equal to $\{\boldsymbol{B}_{ij}\}_{i=1,j=1}^{i=t,j=t}$ where the matrix $\boldsymbol{B}_{ij}$ of the order $1 \times n_j$ is of the form

$$\boldsymbol{B}_{ij} = \begin{cases} (1 - \frac{1}{N}n_i)\boldsymbol{1}_{n_i}^{'} & \text{if} \quad i = j \\ -\frac{1}{N}n_i\boldsymbol{1}_{n_j}^{'} & \text{if} \quad i \neq j \end{cases} \tag{10}$$

Using Lemma 1 of Westfall (1987) that simplifies calculation of covariance between quadratic forms in a vector of mean zero random variables, we get

$$\begin{aligned} Var(SS_\tau) = \sigma^4[&2\rho^2 tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U})^2 + 4\rho tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U}) + 2tr(\boldsymbol{Q}_1)^2 \\ &+ \rho^2 k_\tau tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U}diag(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U})) + k_e tr(\boldsymbol{Q}_1 diag(\boldsymbol{Q}_1))], \end{aligned} \tag{11}$$

$$Var(SSE) = \sigma^4[2tr(\boldsymbol{Q}_2)^2 + k_e tr(\boldsymbol{Q}_2 diag(\boldsymbol{Q}_2))], \tag{12}$$

and

$$Cov(SS_\tau, SSE) = \sigma^4 k_e tr(\boldsymbol{Q}_1 diag(\boldsymbol{Q}_2)). \tag{13}$$

Using Eqs. (6) and (9), we get the following traces

$$tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U})^2 = \sum_{i=1}^t n_i^2 + b_N, \quad tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U}) = N + c_N, \tag{14}$$

$$tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U}diag(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U})) = \sum_{i=1}^{t} n_i^2 + d_N,\ tr(\boldsymbol{Q}_1)^2 = t-1,\ tr(\boldsymbol{Q}_2)^2 = N-t, \quad (15)$$

$$tr(\boldsymbol{Q}_1 diag(\boldsymbol{Q}_1)) = \sum_{i=1}^{t}(1/n_i) + e_N,\ tr(\boldsymbol{Q}_2 diag(\boldsymbol{Q}_2)) = N - 2t + \sum_{i=1}^{t}(1/n_i), \quad (16)$$

$$tr(\boldsymbol{Q}_1 diag(\boldsymbol{Q}_2)) = t - \sum_{i=1}^{t}(1/n_i) + f_N \quad (17)$$

where

$$b_N = -(2/N)\sum_{i=1}^{t} n_i^3 + (1/N^2)\sum_{i=1}^{t} n_i^4 + (1/N^2)\sum_{i=1}^{t}\sum_{j=1}^{t} n_i^2 n_j^2,$$

$$c_N = -(1/N)\sum_{i=1}^{t} n_i^2,\ d_N = -(2/N)\sum_{i=1}^{t} n_i^3,$$

$$e_N = (-2t+1)/N, \quad f_N = -(1/N)\sum_{i=1}^{t} n_i + (t/N).$$

Then the sequences b_N, c_N, d_N, e_N and f_N are all $o(N)$.

From the asymptotic condition given in Sec.2. and Eqs. (14)-(17), we get

$$\lim_{N\to\infty}(1/N)Var(SS_\tau) = \sigma^4[2\rho^2\gamma_2 + 4\rho + 2a + k_\tau\rho^2\gamma_2 + k_e\gamma_{-1}], \quad (18)$$

$$\lim_{N\to\infty}(1/N)Var(SSE) = \sigma^4[2(1-a) + k_e(1 - 2a + \gamma_{-1})] \quad (19)$$

and

$$\lim_{N\to\infty}(1/N)Cov(SS_\tau, SSE) = \sigma^4(a - \gamma_{-1}) \quad (20)$$

where $Var(SS_\tau)$, $Var(SSE)$ and $Cov(SS_\tau, SSE)$ are given in Eqs. (11), (12) and (13) respectively and the limits a and γ_p for $lp-1, 2$ are defined by Eq. (4). From these, the covariance matrix of $N^{-1/2}(SS_\tau, SSE)^{'}$ is:

$$\boldsymbol{\Delta} = \sigma^4\begin{bmatrix} 2\rho^2\gamma_2 + 4\rho + 2a & 0 \\ 0 & 2(1-a) \end{bmatrix} + k_\tau\sigma^4\begin{bmatrix} \rho^2\gamma_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ k_e\sigma^4\begin{bmatrix} \gamma_{-1} & a - \gamma_{-1} \\ a - \gamma_{-1} & 1 - 2a + \gamma_{-1} \end{bmatrix} \quad (21)$$

as $N \to \infty$. We have the equality $\sqrt{N}(MS_\tau, MSE)^{'} = \boldsymbol{\Lambda}_N N^{-1/2}(SS_\tau, SSE)^{'}$ where $\boldsymbol{\Lambda}_N = diag(N/(t-1), N/(N-t))$. $\boldsymbol{\Lambda}_N$ converges to $\boldsymbol{\Gamma}$ as $N \to \infty$ where $\boldsymbol{\Gamma} = diag(1/a, 1/(1-a))$. Thus, the asymptotic covariance matrix of $\sqrt{N}(MS_\tau, MSE)^{'}$ denoted by $\boldsymbol{ACOV}$ is equal to $\boldsymbol{\Gamma\Delta\ \Gamma}$ and its explicit form is given in Eq. (5). $\square$

Theorem 3.2. *The sequences in random vector*

$$\sqrt{N}(MS_\tau - [1+\rho a^{-1}]\sigma^2, MSE - \sigma^2)'$$

converges in distribution to the bivariate normal distribution with zero-mean vector and the covariance matrix $\boldsymbol{ACOV}$ *given in Eq. (5).*

Proof. Define Q_N as $Q_N = SS_\tau + SSE$. Then Q_N is written as $\boldsymbol{Y}'\boldsymbol{P}\boldsymbol{Y}$ where from Eqs. (6) and (8), the matrix $\boldsymbol{P}$ can be written as

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{U}'\boldsymbol{Q}_1\boldsymbol{U} & \boldsymbol{U}'\boldsymbol{Q}_1 \\ \boldsymbol{Q}_1\boldsymbol{U} & \boldsymbol{I}_N - (1/N)\boldsymbol{J}_N \end{bmatrix}$$

Let $\boldsymbol{P} = \{\boldsymbol{P}_{ij}\}_{i=1,j=1}^{i=t,j=t}$. Then with the aid of Eqs. (9) and (10), the $(n_i+1)\times(n_j+1)$ symmetric submatrix $\boldsymbol{P}_{ij}$ of $\boldsymbol{P}$ is written as

$$\boldsymbol{P}_{ii} = \begin{bmatrix} n_i - (1/N)n_i^2 & (1-(1/N))\mathbf{1}_{n_i}' \\ (1-(1/N))\mathbf{1}_{n_i} & \boldsymbol{I}_{n_i} - (1/N)\boldsymbol{J}_{n_i} \end{bmatrix} \qquad \text{if} \quad i = j$$

and

$$\boldsymbol{P}_{ij} = -(1/N)\begin{bmatrix} n_i n_j & \mathbf{1}_{n_i}' \\ \mathbf{1}_{n_i} & \boldsymbol{J}_{n_i\times n_j} \end{bmatrix} \qquad \text{if} \quad i \neq j.$$

Define $\boldsymbol{\epsilon}_i$ as $\boldsymbol{\epsilon}_i' = (\tau_i, e_{i1}, e_{i2}, \ldots, e_{in_i})$. Using the projection method for quadratic forms (see Akritas and Papadatos (2004), van der Vaart (1998) ch. 11), Q_N is decomposed as $Q_N = U_N - V_N$ where

$$U_N = \sum_{i=1}^{t} \boldsymbol{\epsilon}_i'\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i \quad \text{and} \quad V_N = \sum_{i=1}^{t}\sum_{j\neq i, j=1}^{t} \boldsymbol{\epsilon}_i'\boldsymbol{P}_{ij}\boldsymbol{\epsilon}_j$$

It should be noted that U_N is the sum of independent but not identical random variables and U_N and V_N are uncorrelated.

Observe that

$$E|\boldsymbol{\epsilon}_i'\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i - E[\boldsymbol{\epsilon}_i'\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i]| = tr(\boldsymbol{P}_{ii}E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i' - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i']|)$$
$$\leq tr(\boldsymbol{P}_{ii}\boldsymbol{P}_{ii})^{1/2}(E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i' - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i']|E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i' - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i']|)^{1/2}$$

where the inequality is acquired by using Cauchy-Schwartz inequality. The moment conditions $E|\tau_i|^{4+\delta} < \infty$ and $E|e_{ij}|^{4+\delta} < \infty$ for some positive δ ensure that there exists a finite and positive M such that $(E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i' - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i']|E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i' - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i']|)^{1/2} \leq M^{1/2}$. On the other hand, $tr(\boldsymbol{P}_{ii}\boldsymbol{P}_{ii}) = (1-(n_i/N))^2(1+n_i)^2 +_i^n -1 \leq 6n_i^4$. It follows from these that

$$\sum_{i=1}^{t}[E|\boldsymbol{\epsilon}_i'\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i - E[\boldsymbol{\epsilon}_i'\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i]|^{2+\delta} \leq M^{1+\delta/2}6^{1+\delta/2}\sum_{i=1}^{t} n_i^{4+2\delta},$$

where $\sum_{i=1}^{t} n_i^{4+2\delta} = O(N)$. Therefore

$$\sum_{i=1}^{t}[E|\boldsymbol{\epsilon}_i'\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i - E[\boldsymbol{\epsilon}_i'\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i]|^{2+\delta} = o(N^b) \qquad (22)$$

for $b > 1$ when either the small or large n_i assumption holds. Let $c_N^2 = Var(U_N)$ where $c_N = \sum_{i=1}^t Var(\boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i)$. By using Lemma 1 of Westfall (1987), c_N^2 is calculated and it is equal to

$$c_N^2 = \sigma^4[2\rho^2\sum_{i=1}^t(n_i - (1/N)n_i^2)^2 + 4\rho\sum_{i=1}^t n_i(1-(n_i/N))^2 + 2\sum_{i=1}^t n_i(1-(1/N))^2$$

$$+k_\tau\rho^2\sum_{i=1}^t(n_i-(1/N)n_i^2)^2 + k_e\sum_{i=1}^t n_i(1-(n_i/N))^2].$$

Then, using the asymptotic condition in Sec. 2, the following limit is obtained

$$\lim_{N\to\infty}(1/N)c_N^2 = (1/N)Var(U_N) = \sigma^4[2\rho^2\gamma_2 + 4\rho + 2 + k_\tau\rho^2\gamma_2 + k_e]. \tag{23}$$

and consequently $c_N^2 = O(N)$. The facts that Eq. (22) and $c_N^{2+\delta} = O(N^{1+\delta/2})$ together imply that the Liapounov, condition as applied to $\boldsymbol{\epsilon}_1^{'}\boldsymbol{P}_{11}\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2^{'}\boldsymbol{P}_{22}\boldsymbol{\epsilon}_2$, $\ldots$, $\boldsymbol{\epsilon}_t^{'}\boldsymbol{P}_{tt}\boldsymbol{\epsilon}_t$, holds. Thus $Var(U_N)^{-1/2}(U_N - E[U_N])$ converges in distribution to $N(0,1)$.

The expression $\lim_{N\to\infty}(1/N)Var(Q_N)$ can be obtained by Eqs. (18)-(20) since $Q_N = SS_\tau + SSE$ and it is equal to Eq. (23). From the facts that $Q_N = U_N + V_N$ and $Cov(U_N, V_N) = 0$, we get $\lim_{N\to\infty}(1/N)Var(V_N) = \lim_{N\to\infty}(1/N)[Var(Q_N) - Var(U_N)] = 0$. Consequently U_N converges in probability to 0. Thus $Var(Q_N)^{-1/2}$ $(Q_N - E[Q_N])$ converges in distribution to $N(0,1)$ if $Var(U_N)^{-1/2}$ $(U_N - E[U_N])$ converges in distribution to $N(0,1)$. Let $\boldsymbol{SS} = (SS_\tau, SSE)^{'}$ and $\boldsymbol{MS} = (MS_\tau, MSE)^{'}$. Then if $Var(Q_N)^{-1/2}(Q_N - E[Q_N])$ converges in distribution to $N(0,1)$ where $\boldsymbol{\Delta}$ is in Eq. (21). $\sqrt{N}(\boldsymbol{MS} - E[\boldsymbol{MS}])^{'}$ converges in distribution to $N_2(\boldsymbol{0}, \boldsymbol{ACOV})$ if $N^{-1/2}(\boldsymbol{SS} - E[\boldsymbol{SS}])^{'}$ converges in distribution to $N_2(\boldsymbol{0}, \boldsymbol{\Delta})$ where $\boldsymbol{ACOV}$ is in Eq. (5). It should be noted that $E[\boldsymbol{MS}] = (\sigma^2[1 + \rho(N - 1/N\sum_{i=1}^t n_i^2)/(t-1)], \sigma^2)^{'}$ and $E[\boldsymbol{MS}]$ converges to $E[\boldsymbol{\Gamma}] = (\sigma^2[1+\rho a^{-1}], \sigma^2)^{'}$ as $N \to \infty$. This completes the proof of Theorem 3.2 $\square$

Theorem 3.3. *Suppose the asymptotic condition established in Sec. 2 holds. Then*

$$\sqrt{N}(F_N - [1+\rho a^{-1}])$$

converges in distribution to normal distribution with 0*-mean and variance* σ_F^2 *as* $N \to \infty$ *where* F_N *is as in Eq. (2),* σ_F^2 *is:*

$$\sigma_F^2 = \frac{2(\rho^2\gamma_2 + 2\rho + a)}{a^2} + \frac{2(1+\rho a^{-1})}{(1-a)} + k_\tau\frac{\rho^2\gamma_2}{a^2}$$

$$+ k_e(\frac{\gamma_{-1}}{a^2} - \frac{2(a-\gamma_{-1})(1+\rho a^{-1})}{a(1-a)} + \frac{(1-2a+\gamma_{-1})(1+\rho a^{-1})^2}{(1-a)^2}), \tag{24}$$

and the limits a *and* γ_p *for* $p = -1, 2$ *are in Eq. (4).*

Proof. Let $\bigtriangledown F_N$ denote the vector of the partial derivatives of F_N with respect to MS_τ and MSE at their expectations. Then $\bigtriangledown F_N = (1/\sigma^2, -[1+\rho a^{-1}]/\sigma^2)^{'}$. From the delta method, $\sqrt{N}(F_N - [1+\rho a^{-1}])$ converges in distribution to normal distribution with zero mean and the variance $\sigma_F^2 = \bigtriangledown^{'} F_N \boldsymbol{ACOV} \bigtriangledown F_N$ where $\boldsymbol{ACOV}$ is in (5). The explicit form of σ_F^2 is given in Eq. (24). □

4. The Proposed Test

The α sized approximate test rejects $H_0 : \rho = 0$ when $F_N > u_\alpha$ where F_N is in Eq. (2) and u_α is the upper $1-\alpha$ quantile of the asymptotic null distribution of F_N. Then, we have

$$P(F_N > u_\alpha | \rho = 0) = \alpha$$

The asymptotic null distribution of $\sqrt{N}(F_N - 1)$ determined from Theorem 3.3 is the normal distribution with zero mean and variance σ_0^2 where it is written as

$$\sigma_0^2 = \frac{2}{a(1-a)} + k_e \frac{\gamma_{-1} - a^2}{a^2(1-a)^2} \tag{25}$$

after some algebraic operation on Eq. (24). One finds u_α and it is given by

$$u_\alpha = \frac{\sigma_0}{\sqrt{N}} z_\alpha + 1 \tag{26}$$

where z_α is the upper $1-\alpha$ quantile of the standard normal distribution.

Finally the approximate power of the proposed test for a finite sample size is:

$$P(F_N > u_\alpha | \rho > 0) = 1 - \Phi(\frac{\sqrt{N}(u_\alpha - [1+\rho a^{-1}])}{\sigma_F}) \tag{27}$$

where Φ denotes the cumulative standard normal distribution, σ_F^2 and u_α are in Eqs. (24) and (26) respectively.

5. Robustness of the Test

The robustness of the asymptotic distribution of the F_N statistic is valid only for the balanced models and it is defined as follows. The asymptotic null distribution of F_N does not depend on the fourth moment of error.

Corollary 5.1. *The asymptotic null distribution of F_N does not depend on the fourth moment of error in the balanced models.*

Proof. To show this, it is enough to show that the asymptotic null variance σ_0^2 in Eq. (25) is free of the kurtosis k_e of error. When $n_i = n$ for all i, where n is fixed, we have $N = tn$ and then

$$\gamma_{-1} - a^2 = \lim_{N\to\infty} \{(1/N)\sum_{i=1}^{t} 1/n_i - t^2/N^2\} = 1/n^2 - 1/n^2 = 0.$$

where γ_{-1} and a are in Eq. (4). The coefficient of k_e appearing in the asymptotic null variance σ_0^2 is equal to 0. So σ_0^2 does not include k_e. □

As indicated by (Akritas and Arnold 2000, p.221), (Scheffe 1959, p.344), and Güven (2014) the asymptotic null distribution of F_N is asymptotically robust with respect to departure from normality of error. So, for the balanced case, the size of the test is asymptotically robust to nonnormal error.

6. Numerical and Simulation Study

The power values of the approximate test are compared with the simulated power values for some selected distributions to τ_i and e_{ij} in order to check accuracy of the power of the approximate test.

A power value of the approximate test is obtained from Eq. (27) for a given positive variance ratio ρ after calculation of the upper percentile point u_α in Eq. (26) for a given α and of variance σ_F^2 in (24). The limit values a, γ_{-1} and γ_2 appearing in σ_F^2 are replaced with their sample encounter values.

The simulated power value is the ratio of the number of generated F_N value in (2) exceeding u_α to the number of simulation runs. Generation of the F_N value is as follow

1)Set μ equal to any constant.

2)Generate τ_i for $i = 1, 2, \ldots, t$ from one of three different distributions: $\sqrt{\rho}N(0,1)$, $\sqrt{\rho}(exp(1)-1)$ and $\sqrt{\rho}U(-\sqrt{3},\sqrt{3})$ for a given ρ where $\rho = 0.0, 0.5, 0.7, 1.0, 1.5, 1.7$.

3) Generate e_{ij} for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, n_i$ from one of three different distributions: $N(0,1)$, $exp(1) - 1$ and $U(-\sqrt{3},\sqrt{3})$. The generation of e_{ij}'s is separated from the generation of τ_i's.

It should be noted that the distributions $N(0,1)$, $exp(1) - 1$ and $U(-\sqrt{3},\sqrt{3})$ have zero mean and unit variance. Also note that the distributions $N(0,1)$, $exp(1) - 1$ and $U(-\sqrt{3},\sqrt{3})$ have the null (0), positive (6) and negative ($-6/5$) kurtosis respectively.

4) Generated Y_{ij}, values $i = 1, 2, \ldots t$ and $j = 1, 2, \ldots n_i$ are obtained where $Y_{ij} = \mu + \tau_i + e_{ij}$ and then the ratio F_N is obtained.

Two different design are considered. One is a small n_i design for which $t = 20$, $n_1 = \ldots = n_5 = 2$, $n_6 = \ldots = n_{10} = 3$, $n_{11} = \ldots = n_{15} = 4$ and $n_{16} = \ldots = n_{20} = 5$. The other one is a large n_i design for which $t = 4$, $n_1 = n_2 = 20$ and $n_3 = n_4 = 25$.

Simulation is based on 1000 runs. In each run, F_N is calculated from generating data. The number of F_N exceeding u_α is divided by 1000 to get a power value of the approximate test. The simulated level of significance of the test is obtained in getting simulated power value of the approximate test when $\rho = 0$. Simply we skip the step 2 in generation of F_N It is equivalently to simulate the level of significance of the test for testing hypothesis of no fixed treatment effects in the one-way ANOVA model.

In Table 1. through 6, sizes and power values of the approximate test are very closer to simulated sizes and power values of the test for small values of ρ. However,the differences between them values slightly increase as the value of ρ increases. It is also observed that both approximated and simulated power values of the test are higher for a large n_i design than for a small n_i design. So according to the simulation results, the test is more appropriate for a small variance ratio and large group sizes.

Table 1 and 4 are for the null kurtosis case while the rest of the tables are for either the positive or negative kurtosis case. It is not detected any significant rise or decline of power values of the approximate test in departing from the null kurtosis case. In comparison Tables 2 and 5 with Table 3 and 6, the power values of the test are higher for the negative kurtosis case than for the positive kurtosis case.

7. Conclusion

In the present paper we establish the approximate test for the hypothesis of zero variance ratio in the unbalanced one way random effects model from non normal universes. As shown in Sec. 4. calculation of both the upper percentile point and a power value of the test can easily be accomplished. The test is robust for the balanced one way random effects model. In the balanced case the null distribution of the test statistics F_N ratio does not depend on the fourth moment of the error term.

The differences between the calculated and generated sizes and power values are closer to a small design and lower variance ratios than a large design and higher variance ratios. It follows that the approximate test is more accurate for a small design and lower variance ratios. It is not detected any significant rise or descend of the power from null to non null kurtosis. Thus, departing from null kurtosis does not have an impact to the power of the approximate test.

Table 1. Approximation to power values of the α sized test for a small n_i design and the null kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
0	0	0.01	0.63	0.76	0.85	0.92	0.93
		(0.03)	(0.61)	(0.77)	(0.89)	(0.97)	(0.99)
		0.05	0.73	0.83	0.89	0.94	0.95
		(0.08)	(0.74)	(0.86)	(0.94)	(0.99)	(0.99)
		0.10	0.78	0.86	0.91	0.95	0.95
		(0.13)	(0.80)	(0.90)	(0.96)	(0.99)	(0.99)

Table 2. Approximation to power values of the α sized test for a small n_i design and the positive kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
6	0	0.01	0.60	0.70	0.77	0.82	0.84
		(0.03)	(0.54)	(0.69)	(0.82)	(0.91)	(0.93)
		0.05	0.69	0.76	0.81	0.85	0.85
		(0.08)	(0.68)	(0.79)	(0.88)	(0.94)	(0.96)
		0.10	0.73	0.78	0.83	0.86	0.86
		(0.13)	(0.74)	(0.84)	(0.91)	(0.97)	(0.97)
0	6	0.01	0.52	0.65	0.76	0.84	0.86
		(0.06)	(0.50)	(0.68)	(0.84)	(0.93)	(0.95)
		0.05	0.64	0.74	0.82	0.87	0.89
		(0.17)	(0.66)	(0.80)	(0.91)	(0.97)	(0.98)
		0.10	0.70	0.78	0.83	0.86	0.86
		(0.25)	(0.74)	(0.86)	(0.93)	(0.98)	(0.99)
6	6	0.01	0.51	0.62	0.71	0.77	0.79
		(0.05)	(0.45)	(0.58)	(0.71)	(0.85)	(0.89)
		0.05	0.62	0.70	0.76	0.80	0.82
		(0.08)	(0.58)	(0.69)	(0.81)	(0.91)	(0.93)
		0.10	0.67	0.73	0.78	0.82	0.83
		(0.12)	(0.65)	(0.76)	(0.86)	(0.93)	(0.95)

Table 3. Approximation to power values of the α sized test for a small n_i design and the negative kurtosis case where the numbers in parentheses are simulated values

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
-6/5	0	0.01	0.63	0.78	0.88	0.95	0.96
		(0.03)	(0.62)	(0.80)	(0.92)	(0.99)	(0.99)
		0.05	0.75	0.85	0.92	0.96	0.97
		(0.08)	(0.76)	(0.89)	(0.96)	(0.99)	(0.99)
		0.10	0.80	0.88	0.93	0.97	0.97
		(0.13)	(0.83)	(0.98)	(0.91)	(0.99)	(0.99)
0	-6/5	0.01	0.66	0.78	0.88	0.93	0.95
		(0.03)	(0.63)	(0.80)	(0.92)	(0.98)	(0.98)
		0.05	0.76	0.85	0.91	0.95	0.96
		(0.07)	(0.77)	(0.89)	(0.96)	(0.99)	(0.99)
		0.10	0.80	0.88	0.93	0.96	0.96
		(0.13)	(0.83)	(0.92)	(0.97)	(0.98)	(0.99)
-6/5	-6/5	0.01	0.67	0.81	0.91	0.96	0.97
		(0.03)	(0.64)	(0.85)	(0.94)	(0.99)	(0.99)
		0.05	0.77	0.87	0.94	0.97	0.98
		(0.07)	(0.80)	(0.91)	(0.97)	(0.99)	(0.99)
		0.10	0.82	0.90	0.95	0.98	0.98
		(0.13)	(0.86)	(0.94)	(0.98)	(0.99)	(0.99)

Table 4. Approximations to power values of the α sized test for a large n_i design and the null kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
0	0	0.01	0.86	0.88	0.89	0.90	0.90
		(0.05)	(0.89)	(0.93)	(0.96)	(0.98)	(0.98)
		0.05	0.87	0.88	0.89	0.90	0.90
		(0.10)	(0.93)	(0.96)	(0.97)	(0.99)	(0.99)
		0.10	0.88	0.86	0.91	0.95	0.95
		(0.14)	(0.93)	(0.96)	(0.97)	(0.99)	(0.99)

Table 5. Approximation to power values of the α sized test for a large n_i design and the positive kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
6	0	0.01	0.72	0.73	0.74	0.75	0.75
		(0.03)	(0.75)	(0.81)	(0.85)	(0.90)	(0.91)
		0.05	0.73	0.74	0.74	0.75	0.75
		(0.10)	(0.80)	(0.84)	(0.88)	(0.92)	(0.93)
		0.10	0.73	0.74	0.75	0.75	0.75
		(0.14)	(0.82)	(0.86)	(0.89)	(0.92)	(0.93)
0	6	0.01	0.85	0.86	0.88	0.88	0.89
		(0.06)	(0.87)	(0.91)	(0.94)	(0.97)	(0.98)
		0.05	0.86	0.87	0.88	0.89	0.89
		(0.10)	(0.90)	(0.93)	(0.96)	(0.98)	(0.99)
		0.10	0.86	0.88	0.88	0.89	0.89
		(0.14)	(0.91)	(0.94)	(0.97)	(0.99)	(0.99)
6	6	0.01	0.72	0.73	0.74	0.74	0.74
		(0.06)	(0.76)	(0.82)	(0.87)	(0.90)	(0.92)
		0.05	0.73	0.73	0.74	0.74	0.75
		(0.10)	(0.81)	(0.85)	(0.89)	(0.93)	(0.94)
		0.10	0.73	0.74	0.74	0.75	0.75
		(0.14)	(0.83)	(0.87)	(0.91)	(0.94)	(0.95)

Table 6. Approximations to power values of the α sized test for a large n_i design and the negative kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
-6/5	0	0.01	0.93	0.95	0.96	0.97	0.97
		(0.05)	(0.92)	(0.95)	(0.97)	(0.99)	(0.99)
		0.05	0.94	0.96	0.96	0.97	0.97
		(0.10)	(0.94)	(0.97)	(0.98)	(0.99)	(0.99)
		0.10	0.95	0.96	0.97	0.97	0.97
		(0.14)	(0.96)	(0.97)	(0.91)	(0.99)	(0.99)
0	-6/5	0.01	0.86	0.88	0.89	0.90	0.90
		(0.05)	(0.89)	(0.93)	(0.96)	(0.97)	(0.98)
		0.05	0.87	0.89	0.90	0.90	0.91
		(0.10)	(0.92)	(0.95)	(0.96)	(0.98)	(0.99)
		0.10	0.88	0.89	0.90	0.91	0.91
		(0.15)	(0.94)	(0.96)	(0.97)	(0.99)	(0.99)
-6/5	-6/5	0.01	0.94	0.95	0.97	0.97	0.97
		(0.05)	(0.92)	(0.95)	(0.98)	(0.99)	(0.99)
		0.05	0.95	0.96	0.97	0.97	0.98
		(0.10)	(0.95)	(0.96)	(0.98)	(0.99)	(0.99)
		0.10	0.95	0.96	0.97	0.98	0.98
		(0.15)	(0.95)	(0.97)	(0.99)	(0.99)	(0.99)

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STUDY OF A GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL VIA CONVEX FUNCTIONS

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Abstract. In this paper estimations in general form of sum of left and right sided Riemann-Liouville (RL) fractional integrals for convex functions are studied. Also some similar fractional inequalities for functions whose derivatives in absolute value are convex, have been obtained. Associated fractional integral inequalities provide the bounds of different known fractional inequalities. These results may be useful in in the study of uniqueness solutions of fractional differential equations and fractional boundary value problems.

1. Introduction

Fractional calculus is applied in almost all disciplines of engineering and modern sciences. Since nineteenth century it has been acknowledged significantly and several new directions and subjects are invented. For example fractional geometry, fractional differential equations and fractional dynamics are due to fractional calculus.

Fractional integral operators play a leading and keen role in the development of fractional calculus. A first formulation of a fractional integral operator is due to a continuous study of well renowned mathematicians and physicist. This fractional integral is well known as Riemann-Liouville (RL) fractional integral operator. After its existence there have been introduced many other fractional integral and fractional derivative operators.

Now a days scientists in their diverse fields are working in the environment of fractional calculus and new directions of respective fields are developing rapidly. Theory of convex functions is the subject of mathematics that connects the mathematical analysis with other branches of science and engineering. Convex functions play an important role in the advancement of optimization theory, majorization theory, probability and statistics. A real valued function $f : [a, b] \to \mathbb{R}$ is said to

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be convex if $f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$ holds for all $x, y \in [a,b]$ and $t \in [0,1]$. If $-f$ is convex, then f is said to be concave on $[a,b]$.

Convex functions are very close to the theory of inequalities. Many known and useful inequalities are consequences of convex functions. Some very natural inequalities for example Jensen inequality, Hadamard inequality interpret convex functions beautifully. Fractional integral inequalities occur by default in the study of convex and related functions due to applications of definitions of fractional integral as well as fractional derivative operators. Fractional integral inequalities are very useful in the study of fractional partial as well as ordinary differential equations. These are used to establish the uniqueness and bounds of their solutions. For detailed study suggested references are [15, 13, 9, 14, 11, 12, 17].

In this paper we study a general form of Riemann-Liouville (RL) fractional integrals via convex functions. Therefore it is need to give definitions of used fractional integrals. We start with the definition of Riemann-Liouville (RL) fractional integral.

Definition 1. *Let $f \in L[a,b]$. Then the left-sided and right-sided Riemann Liouville fractional integrals of order $\alpha > 0$ with $a \ge 0$ are defined as:*

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

where $\Gamma(.)$ is the Gamma function.

A slight generalization of (RL) fractional integral is Riemann-Liouville (k-RL) k-fractional integral (see, [5]).

Definition 2. *Let $f \in L[a,b]$. Then the k-fractional integrals of order $\alpha, k > 0$ with $a \ge 0$ are defined as:*

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a \tag{1}$$

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b, \tag{2}$$

where $\Gamma_k(.)$ is the k-Gamma function (see, [2]).

A more general definition of (RL) fractional integral is the Riemann-Liouville fractional integral with respect an increasing function (see, [9]).

Definition 3. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b).*

The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\alpha > 0$ are defined as:

$$I^{\alpha}_{g,a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}g'(t)f(t)dt,\ x>a$$

$$I^{\alpha}_{g,b_-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1}g'(t)f(t)dt,\ \ x<b.$$

A k-analogue of above definition is defined in the next definition [10].

Definition 4. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\alpha, k > 0$ are defined as:*

$$I^{\alpha,k}_{g,a^+}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (g(x)-g(t))^{\frac{\alpha}{k}-1}g'(t)f(t)dt,\ x>a \tag{3}$$

$$I^{\alpha,k}_{g,b_-}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (g(t)-g(x))^{\frac{\alpha}{k}-1}g'(t)f(t)dt,\ \ x<b. \tag{4}$$

This is a compact form of a several fractional integral operators which are independently defined by the researchers in recent decade. The following lemma comprises on the formation of some particular fractional integrals which one can obtain from (3) and (4).

Remark 5. *In the above Definition 4.*
(i) If we take $k = 1$, then we get the Definition 3 of Riemann-Liouville fractional integrals with respect to an increasing function.
(ii) If we take $g(x) = x$, then we get the Definition 2 of Riemann-Liouville k-fractional integrals.
(iii) If we take $g(x) = x$ and $k = 1$, then we get the Definition 1 of Riemann-Liouville fractional integrals.
(iv) If we take $g(x) = \frac{x^\rho}{\rho}$, $\rho > 0$ and $k = 1$, then we get the definition of Katugampola fractional integrals given in [1].
(v) If we take $g(x) = \frac{x^{\tau+s}}{\tau+s}$ and $k = 1$, then we get the definition of generalized conformable fractional integrals defined by T. U. Khan et al. in [8].
(vi) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (3) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (4), then we get the definition of conformable (k,s)- fractional integrals defined by Habib et al. in [6].
(vii) If we take $g(x) = \frac{x^{1+s}}{1+s}$, then we get the definition of generalized conformable fractional integrals defined by Sarikaya et al. in [16].
(viii) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (3) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (4) with $k = 1$, then we get the definition of conformable fractional integrals defined by F. Jarad et al. in [7].

The paper is organized as follows:

In Section 2, bounds of sum of the left and right sided (RL) k-fractional integrals in general form defined in Definition 4 have been established. Some related similar results are also obtained. These results are achieved by means of monotonicity and convexity properties. The presented results are useful in the study of fractional differential equations and fractional boundary value problems. Also they provide the estimations of Riemann-Liouville fractional integrals which are published in [4] and some results of [3]. In Section 3 applications are discussed.

2. Main Results

The first result provides the estimates of sum of the left and right sided general (RL) fractional integral defined in Definition 4. Convexity and monotonicity properties of real valued functions are used.

Theorem 6. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be real valued functions with $a < b$. Also let f be positive convex, and g be differentiable and strictly increasing function with $g' \in L[a, b]$. Then for $\alpha, \beta \geq k$, the following estimate is valid*

$$\begin{aligned}
&k\Gamma_k(\alpha) I^{\alpha,k}_{g,a^+} f(x) + k\Gamma_k(\beta) I^{\beta,k}_{g,b^-} f(x) \qquad (5)\\
&\leq \frac{(g(x)-g(a))^{\frac{\alpha}{k}-1}}{x-a}\left[(x-a)(f(x)g(x)-f(a)g(a)) - (f(x)-f(a))\int_a^x g(t)dt\right]\\
&+ \frac{(g(b)-g(x))^{\frac{\beta}{k}-1}}{b-x}\left[(b-x)(f(b)g(b)-f(x)g(x)) - (f(b)-f(x))\int_x^b g(t)dt\right].
\end{aligned}$$

Proof. Since the function g is differentiable and strictly increasing, therefore for $x \in [a, b]$ and $t \in [a, x]$, $(g(x)-g(t))^{\frac{\alpha}{k}-1} \leq (g(x)-g(a))^{\frac{\alpha}{k}-1}, \alpha \geq k$. Also $g'(x) > 0$ hence $g'(t)(g(x)-g(t))^{\frac{\alpha}{k}-1} \leq g'(t)(g(x)-g(a))^{\frac{\alpha}{k}-1}$. From convexity of f, we have $f(t) \leq \frac{x-t}{x-a} f(a) + \frac{t-a}{x-a} f(x)$. From the last two inequalities one can has the following integral inequality

$$\begin{aligned}
&\int_a^x (g(x)-g(t))^{\frac{\alpha}{k}-1} f(t) g'(t) dt\\
&\leq \frac{(g(x)-g(a))^{\frac{\alpha}{k}-1}}{x-a}\left[f(a)\int_a^x (x-t)g'(t)dt + f(x)\int_a^x (t-a)g'(t)dt\right].
\end{aligned}$$

By using the Definition 4, we get

$$\begin{aligned}
&k\Gamma_k(\alpha) I^{\alpha,k}_{g,a^+} f(x) \qquad (6)\\
&\leq \frac{((g(x)-g(a))^{\frac{\alpha}{k}-1}}{x-a}\left[(x-a)(f(x)g(x)-f(a)g(a)) - (f(x)-f(a))\int_a^x g(t)dt\right].
\end{aligned}$$

Now on the other hand for $x \in [a, b]$, $t \in [x, b]$, $g'(t)(g(x)-g(t))^{\frac{\beta}{k}-1} \leq g'(t)(g(b)-g(x))^{\frac{\beta}{k}-1}, \beta \geq k$. From convexity of f we have $f(t) \leq \frac{t-x}{b-x} f(b) + \frac{b-t}{b-x} f(x)$.

Multiplying the last two inequalities and integrating over $[x,b]$ one can has the integral inequality, which in addition with (6) constitutes the required estimate.

$$k\Gamma_k(\beta)I^{\beta,k}_{g,b^-}f(x)$$
$$\leq \frac{(g(b)-g(x))^{\frac{\beta}{k}-1}}{b-x}\left[(b-x)(f(b)g(b)-f(x)g(x))-(f(b)-f(x))\int_x^b g(t)dt\right].$$

□

A special case of above theorem is stated in the following corollary which gives [4, Theorem 1] for $k=1$, g as identity function.

Corollary 7. *If the assumptions of Theorem 6 hold, then the following fractional integral inequality holds*

$$k\Gamma_k(\alpha)\left(I^{\alpha,k}_{g,a^+}f(x)+I^{\alpha,k}_{g,b^-}f(x)\right) \tag{7}$$
$$\leq \frac{((g(x)-g(a))^{\frac{\alpha}{k}-1}}{x-a}\left[(x-a)(f(x)g(x)-f(a)g(a))-(f(x)-f(a))\int_a^x g(t)dt\right]$$
$$+\frac{((g(b)-g(x))^{\frac{\alpha}{k}-1}}{b-x}\left[(b-x)(f(b)g(b)-f(x)g(x))-(f(b)-f(x))\int_x^b g(t)dt\right].$$

Next theorem is the modulus fractional inequality that derives some known results.

Theorem 8. *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be real valued functions with $a<b$. Also let f be differentiable and $|f'|$ is convex, and g be also differentiable and strictly increasing with $g'\in L[a,b]$. Then for $\alpha,\beta\geq 0$ and $k>0$, the following modulus fractional inequality holds*

$$\Big|\Gamma_k(\alpha+k)I^{\alpha,k}_{g,a^+}f(x)+\Gamma_k(\beta+k)I^{\beta,k}_{g,b^-}f(x) \tag{8}$$
$$-\left((g(x)-g(a))^{\frac{\alpha}{k}}f(a)+(g(b)-g(x))^{\frac{\beta}{k}}f(b)\right)\Big|$$
$$\leq\frac{(g(x)-g(a))^{\frac{\alpha}{k}}(x-a)|f'^{\frac{\beta}{k}}(b-x)|f'(b)|}{2}$$
$$+|f'(x)|\frac{\left((g(x)-g(a))^{\frac{\alpha}{k}}(x-a)+(g(b)-g(x))^{\frac{\beta}{k}}(b-x)\right)}{2}.$$

Proof. From convexity of $|f'|$, we have $|f'(t)|\leq\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|$ which gives $f'(t)\leq\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|$. Since the function g is differentiable and strictly increasing therefore we have $g(x)-g(t))^{\frac{\alpha}{k}}\leq(g(x)-g(a))^{\frac{\alpha}{k}}$, where as $x\in[a,b]$ and $t\in[a,x]$, $\alpha\geq 0$, $k>0$.
The product of last two inequalities give

$$(g(x)-g(t))^{\frac{\alpha}{k}}f'(t)\leq\frac{(g(x)-g(a))^{\frac{\alpha}{k}}}{x-a}\left((x-t)|f'(a)|+(t-a)|f'(x)|\right).$$

Integrating with respect to t over $[a,x]$, we have

$$\begin{aligned}
&\int_a^x (g(x)-g(t))^{\frac{\alpha}{k}} f'(t)dt \qquad (9)\\
&\leq \frac{(g(x)-g(a))^{\frac{\alpha}{k}}}{x-a}\left[|f'(a)|\int_a^x (x-t)dt + |f'(x)|\int_a^x (t-a)dt\right]\\
&= (g(x)-g(a))^{\frac{\alpha}{k}}(x-a)\left[\frac{|f'(a)|+|f'(x)|}{2}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^x \Big(g(x)-g(t)\Big)^{\frac{\alpha}{k}} f'^{\frac{\alpha}{k}}\Big|_a^x + \frac{\alpha}{k}\int_a^x \Big(g(x)-g(t)\Big)^{\frac{\alpha}{k}-1} f(t)g'(t)dt\\
= \quad & -f(a)\Big(g(x)-g(a)\Big)^{\frac{\alpha}{k}} + \Gamma_k(\alpha+k)I_{g,a^+}^{\alpha,k}f(x).
\end{aligned}$$

Therefore (9) takes the form

$$\Gamma_k(\alpha+k)I_{g,a^+}^{\alpha,k}f(x)-f(a)(g(x)-g(a))^{\frac{\alpha}{k}} \leq (g(x)-g(a))^{\frac{\alpha}{k}}(x-a)\left[\frac{|f'(a)|+|f'(x)|}{2}\right]. \qquad (10)$$

Also from convexity of $|f'|$ one can has $f(t) \geq -\left(\frac{x-t}{x-a}|f'(a)| + \frac{t-a}{x-a}|f'(x)|\right)$ and following the same procedure as we did to get (10) next inequality holds

$$f(a)(g(x)-g(a))^{\frac{\alpha}{k}}-\Gamma_k(\alpha+k)I_{g,a^+}^{\alpha,k}f(x) \leq (g(x)-g(a))^{\frac{\alpha}{k}}(x-a)\left[\frac{|f'(a)|+|f'(x)|}{2}\right]. \qquad (11)$$

Inequalities (10) and (11) provide the modulus inequality

$$\begin{aligned}
&\left|\Gamma_k(\alpha+k)I_{g,a^+}^{\alpha,k}f(x)-f(a)(g(x)-g(a))^{\frac{\alpha}{k}}\right| \qquad (12)\\
&\leq (g(x)-g(a))^{\frac{\alpha}{k}}(x-a)\left[\frac{|f'(a)|+|f'(x)|}{2}\right].
\end{aligned}$$

On the other hand from convexity of $|f'|$ we have $|f'(t)| \leq \frac{t-x}{b-x}|f'(b)| + \frac{b-t}{b-x}|f'(x)|$, for $x \in [a,b]$ and $t \in [x,b]$ and $\beta \geq 0$, $k > 0$. Also the inequality $(g(t)-g(x))^{\frac{\beta}{k}} \leq (g(b)-g(x))^{\frac{\beta}{k}}$ holds true for function g. Following the same way as we have done to obtain (12) the following inequality holds

$$\left|\Gamma_k(\beta+k)I_{g,b^-}^{\beta,k}f(x)-f(b)(g(b)-g(x))^{\frac{\beta}{k}}\right| \leq (g(b)-g(x))^{\frac{\beta}{k}}(b-x)\left[\frac{|f'(b)|+|f'(x)|}{2}\right]. \qquad (13)$$

From inequalities (12) and (13) via triangular inequality we get (8) which is required. $\square$

A special case of above theorem is stated in the following corollary which gives [4, Theorem 2] for $k=1$, g as identity function.

Corollary 9. *If the assumptions of Theorem 8 hold, then the following fractional integral inequality holds*

$$\begin{aligned}
&\left|\Gamma_k(\alpha+k)(I^{\alpha,k}_{g,a^+}f(x)+I^{\alpha,k}_{g,b^-}f(x))-\big((g(x)-g(a))^{\frac{\alpha}{k}}f(a)+(g(b)-g(x))^{\frac{\alpha}{k}}f(b)\big)\right|\\
&\leq \frac{(g(x)-g(a))^{\frac{\alpha}{k}}(x-a)|f'^{\frac{\alpha}{k}}(b-x)|f'(b)|}{2}\\
&+|f'(x)|\frac{\big((g(x)-g(a))^{\frac{\alpha}{k}}(x-a)+(g(b)-g(x))^{\frac{\alpha}{k}}(b-x)\big)}{2}.
\end{aligned}$$

The following lemma is useful to prove the next result.

Lemma 10. [4] *Let $f:[a,b]\longrightarrow\mathbb{R}$, be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right)\leq f(x)\quad x\in[a,b]. \tag{14}$$

Theorem 11. *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be real valued functions with $a<b$. Also let f be positive convex and symmetric about $\frac{a+b}{2}$, and g be differentiable and strictly increasing with $g'\in L[a,b]$. Then for $\alpha,\beta\geq 0$ and $k>0$, we have the following fractional inequality*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\Big[\Gamma_k(\alpha+k)I^{\alpha,k}_{g,a^+}g(b)-\Gamma_k(\beta+k)I^{\beta,k}_{g,b^-}g(a) \qquad (15)\\
&-(g(b)-g(a))^{\frac{\alpha}{k}}g(a)+(g(b)-g(a))^{\frac{\beta}{k}}g(b)\Big]\\
&\leq \Gamma_k(\alpha+1)I^{\alpha+1,k}_{g,a^+}f(b)+\Gamma_k(\beta+1)I^{\beta+1,k}_{g,b^-}f(a)\\
&\leq \frac{((g(b)-g(a))^{\frac{\beta}{k}}+(g(b)-g(a))^{\frac{\alpha}{k}})}{b-a}\\
&\times\left[(b-a)(f(b)g(b)-f(a)g(a))-(f(b)-f(a))\int_a^b g(x)dx\right].
\end{aligned}$$

Proof. Since the function g is differentiable and strictly increasing therefore $(g(x)-g(a))^{\frac{\beta}{k}}\leq(g(b)-g(a))^{\frac{\beta}{k}}$, where as $x\in[a,b]$, $\beta\geq 0$, $k>0$. Also $g'(x)>0$ hence the inequality $g'(x)(g(x)-g(a))^{\frac{\beta}{k}}\leq g'(x)(g(b)-g(a))^{\frac{\beta}{k}}$ holds true. From convexity of f , $f(x)\leq\frac{x-a}{b-a}f(b)+\frac{b-x}{b-a}f(a)$. The product of last two inequalities is integrated over $[a,b]$ to get

$$\begin{aligned}
&\int_a^b(g(x)-g(a))^{\frac{\beta}{k}}f(x)g'(x)dx\\
&\leq\frac{(g(b)-g(a))^{\frac{\beta}{k}}}{b-a}\left[f(b)\int_a^b(x-a)g'(x)dx+f(a)\int_a^b(b-x)g'(x)dx\right].
\end{aligned}$$

By using Definition (4) we get

$$\Gamma_k(\beta+1)I^{\beta+1,k}_{g,b^-}f(a) \tag{16}$$
$$\leq \frac{(g(b)-g(a))^{\frac{\beta}{k}}}{b-a}\left[(b-a)(f(b)g(b)-f(a)g(a))-(f(b)-f(a))\int_a^b g(x)dx\right].$$

Now for $x\in[a,b]$, $t\in[x,b]$ and $\alpha\geq 0$, $k>0$, the inequality $g'(x)(g(b)-g(x))^{\frac{\beta}{k}}\leq g'(x)(g(b)-g(a))^{\frac{\beta}{k}}$ holds true. Following the same way as we have done to get (16) the following inequality can be obtained

$$\Gamma_k(\alpha+1)I^{\alpha+1,k}_{g,a^+}f(b) \tag{17}$$
$$\leq \frac{(g(b)-g(a))^{\frac{\alpha}{k}}}{b-a}\left[(b-a)(f(b)g(b)-f(a)g(a))-(f(b)-f(a))\int_a^b g(x)dx\right].$$

From (16) and (17), we get

$$\Gamma_k(\alpha+1)I^{\alpha+1,k}_{g,a^+}f(b)+\Gamma_k(\beta+1)I^{\beta+1,k}_{g,b^-}f(a) \tag{18}$$
$$\leq \frac{((g(b)-g(a))^{\frac{\beta}{k}}+(g(b)-g(a))^{\frac{\alpha}{k}})}{b-a}$$
$$\times\left[(b-a)(f(b)g(b)-f(a)g(a))-(f(b)-f(a))\int_a^b g(x)dx\right].$$

Using Lemma 10 and multiplying (14) with $(g(x)-g(a))^{\frac{\beta}{k}}g'(x)$, then integrating over $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\int_a^b (g(x)-g(a))^{\frac{\beta}{k}}g'(x)dx\leq\int_a^b (g(x)-g(a))^{\frac{\beta}{k}}g'(x)f(x)dx. \tag{19}$$

By using Definition 3, we get

$$f\left(\frac{a+b}{2}\right)\left[(g(b)-g(a))^{\frac{\beta}{k}}g(b)-\Gamma_k(\beta+k)I^{\beta,k}_{g,b^-}g(a)\right]\leq\Gamma_k(\beta+1)I^{\beta+1,k}_{g,b^-}f(a). \tag{20}$$

Similarly, using Lemma 10 and multiplying (14) with $(g(b)-g(x))^{\frac{\beta}{k}}g'(x)$, then integrating over $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\left[\Gamma_k(\alpha+k)I^{\alpha,k}_{g,a^+}g(b)-(g(b)-g(a))^{\frac{\beta}{k}}g(a)\right]\leq\Gamma_k(\alpha+1)I^{\alpha+1,k}_{g,a^+}f(b). \tag{21}$$

From (20) and (21) the following inequality holds which with (18) constitute (15).

$$f\left(\frac{a+b}{2}\right)\Big[\Gamma_k(\alpha+k)I^{\alpha,k}_{g,a^+}g(b)-\Gamma_k(\beta+k)I^{\beta,k}_{g,b^-}g(a)$$
$$-(g(b)-g(a))^{\frac{\alpha}{k}}g(a)+(g(b)-g(a))^{\frac{\beta}{k}}g(b)\Big]$$

$$\leq \Gamma_k(\alpha+1) I_{g,a^+}^{\alpha+1,k} f(b) + \Gamma_k(\beta+1) I_{g,b^-}^{\beta+1,k} f(a).$$

□

A special case of above theorem is stated in the following corollary which gives [4, Theorem 3] for $k = 1$, g as identity function.

Corollary 12. *If the assumptions of Theorem 11 hold, then the following fractional integral inequality holds*

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right)\left[\Gamma_k(\alpha+k)\left(I_{g,a^+}^{\alpha,k} g(b) - I_{g,b^-}^{\alpha,k} g(a)\right)\right. \\
& \left. -(g(b)-g(a))^{\frac{\alpha}{k}} g(a) + (g(b)-g(a))^{\frac{\alpha}{k}} g(b)\right] \\
& \leq \Gamma_k(\alpha+1)\left(I_{g,a^+}^{\alpha+1,k} f(b) + I_{g,b^-}^{\alpha+1,k} f(a)\right) \\
& \leq \frac{((g(b)-g(a))^{\frac{\alpha}{k}} + (g(b)-g(a))^{\frac{\alpha}{k}})}{b-a} \\
& \times\left[(b-a)(f(b)g(b) - f(a)g(a)) - (f(b)-f(a))\int_a^b g(x)dx\right].
\end{aligned}$$

3. Applications

In this section we give applications of the results proved in the previous section. First we apply Theorem 6 and get the following result.

Theorem 13. *Under the assumptions of Theorem 6, we have*

$$\begin{aligned}
& k\Gamma_k(\alpha) I_{g,a^+}^{\alpha,k} f(b) + k\Gamma_k(\beta) I_{g,b^-}^{\beta,k} f(a) \\
& \leq \left(\frac{(g(b)-g(a))^{\frac{\alpha}{k}-1} + (g(b)-g(a))^{\frac{\beta}{k}-1}}{b-a}\right) \\
& \times\left((b-a)(f(b)g(b) - f(a)g(a)) - (f(b)-f(a))\int_a^b g(t)dt\right).
\end{aligned} \tag{22}$$

Proof. If we take $x = a$ and $x = b$ in (5), then adding resulting inequalities, we get (22). □

Corollary 14. *If we take $\alpha = \beta$ in (22), then we have the following fractional integral inequality*

$$\begin{aligned}
& k\Gamma_k(\alpha)(I_{g,a^+}^{\alpha,k} f(b) + I_{g,b^-}^{\alpha,k} f(a)) \\
& \leq \frac{2(g(b)-g(a))^{\frac{\alpha}{k}-1}}{b-a}\left((b-a)(f(b)g(b) - f(a)g(a)) - (f(b)-f(a))\int_a^b g(t)dt\right).
\end{aligned} \tag{23}$$

Corollary 15. *If we take $\alpha = k = 1$ and $g(x) = x$ in (23), then we get the following inequality*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{24}$$

Next we apply Theorem 8 to obtain required results.

Theorem 16. *Under the assumptions of Theorem 8, we have*

$$\begin{aligned}
&\Bigg|\Gamma_k(\alpha+k)I_{g,a^+}^{\alpha,k}f\left(\frac{a+b}{2}\right)+\Gamma_k(\beta+k)I_{g,b^-}^{\beta,k}f\left(\frac{a+b}{2}\right) \qquad (25)\\
&-\left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\alpha}{k}}f(a)+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\beta}{k}}f(b)\right)\Bigg|\\
&\leq\frac{\left(\frac{b-a}{2}\right)\left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\alpha}{k}}|f'(a)|+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\beta}{k}}|f'(b)|\right)}{2}\\
&+\left|f'\left(\frac{a+b}{2}\right)\right|\frac{\left(\frac{b-a}{2}\right)\left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\alpha}{k}}+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\beta}{k}}\right)}{2}.
\end{aligned}$$

Proof. If we take $x=\frac{a+b}{2}$ in (8), then resulting inequality (25) can be obtained. □

Corollary 17. *If we take $\alpha = \beta$ in (25), then we have the following fractional integral inequality*

$$\begin{aligned}
&\Bigg|\Gamma_k(\alpha+k)\left(I_{g,a^+}^{\alpha,k}f\left(\frac{a+b}{2}\right)+I_{b^-}^{\alpha,g}f\left(\frac{a+b}{2}\right)\right) \qquad (26)\\
&-\left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\alpha}{k}}f(a)+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\alpha}{k}}f(b)\right)\Bigg|\\
&\leq\frac{\left(\frac{b-a}{2}\right)\left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\alpha}{k}}|f'(a)|+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\alpha}{k}}|f'(b)|\right)}{2}\\
&+\left|f'\left(\frac{a+b}{2}\right)\right|\frac{\left(\frac{b-a}{2}\right)\left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\alpha}{k}}+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\alpha}{k}}\right)}{2}.
\end{aligned}$$

Corollary 18. *If we take $\alpha = k = 1$ and $g(x) = x$ in (26), then we get the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(a)+f(b)}{2}\right|\leq\frac{b-a}{8}\left[|f'(a)|+|f'(b)|+2\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{27}$$

It is interesting to note that for $f'\left(\frac{a+b}{2}\right)=0$ (27) produce [3, Theorem 2.2]. If $f'(x)\leq 0$, then (27) provides the refinement of [3, Theorem 2.2].

Concluding Remarks

This paper gives estimates of (RL) fractional integral in general form by means of convex functions. These estimates provide the estimations of (RL) and (kRL) fractional integrals and also for all fractional integrals comprises in Remark 1. Some related fractional inequalities are also obtained for differentiable functions having convex derivatives in absolute value. Applications of Theorem 6 and Theorem 8 are given by connecting some known results. By applying Theorem 11 similar results can be established which are left for the reader.

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FURTHER INEQUALITIES FOR THE GENERALIZED k-g-FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

SILVESTRU SEVER DRAGOMIR

Abstract. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b]$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b),$$

where the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some new inequalities for the k-g-fractional integrals of functions of bounded variation.Examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

1. Introduction

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

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As a simple example, if $k(t) = t^{\alpha-1}$ then for $\alpha \in (0,1)$ the function k is defined on $(0,\infty)$ and $K(t) := \frac{1}{\alpha}t^{\alpha}$ for $t \in [0,\infty)$. If $\alpha \geq 1$, then k is defined on $[0,\infty)$ and $K(t) := \frac{1}{\alpha}t^{\alpha}$ for $t \in [0,\infty)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f : (a,b) \to \mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x) = \int_a^x k(g(x) - g(t))\, g'(t) f(t)\, dt, \ x \in (a,b] \tag{1}$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x) = \int_x^b k(g(t) - g(x))\, g'(t) f(t)\, dt, \ x \in [a,b). \tag{2}$$

If we take $k(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned} S_{k,g,a+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x [g(x) - g(t)]^{\alpha-1} g'(t) f(t)\, dt \\ &=: I^{\alpha}_{a+,g}f(x), \ a < x \leq b \end{aligned} \tag{3}$$

and

$$\begin{aligned} S_{k,g,b-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b [g(t) - g(x)]^{\alpha-1} g'(t) f(t)\, dt \\ &=: I^{\alpha}_{b-,g}f(x), \ a \leq x < b, \end{aligned} \tag{4}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [23, p. 100].

For $g(t) = t$ in (4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [23, p. 111]

$$H^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t}, \ 0 \leq a < x \leq b \tag{5}$$

and

$$H^{\alpha}_{b-}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t}, \ 0 \leq a < x < b. \tag{6}$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R^{\alpha}_{a+}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{f(t)\, dt}{(x-t)^{1-\alpha}\, t^{\alpha+1}}, \ 0 \leq a < x \leq b \tag{7}$$

and

$$R^{\alpha}_{b-}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{f(t)\, dt}{(t-x)^{1-\alpha}\, t^{\alpha+1}}, \ 0 \leq a < x < b. \tag{8}$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x [\exp(\beta x) - \exp(\beta t)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{9}$$

for $a < x \le b$ and

$$E_{b-,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{10}$$

for $a \le x < b$.

If we take $g(t) = t$ in (1) and (2), then we can consider the following *k-fractional integrals*

$$S_{k,a+} f(x) = \int_a^x k(x-t) f(t)\, dt, \; x \in (a,b] \tag{11}$$

and

$$S_{k,b-} f(x) = \int_x^b k(t-x) f(t)\, dt, \; x \in [a,b). \tag{12}$$

In [26], Raina studied a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k, \;\; |x| < R, \text{ with } R > 0 \tag{13}$$

for ρ, $\lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f(x) := \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}(w(x-t)^{\rho}) f(t)\, dt, \; x > a \tag{14}$$

where ρ, $\lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}_{\rho,\lambda,b-;w}^{\sigma} f(x) := \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}(w(t-x)^{\rho}) f(t)\, dt, \; x < b \tag{15}$$

where ρ, $\lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}(wt^{\rho})$ we re-obtain the definitions of (14) and (15) from (11) and (12).

In [24], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}_{a+}^{\alpha} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\, dt, \; x > a \tag{16}$$

and

$$\mathcal{T}_{b-}^{\alpha} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\, dt, \; x < b \tag{17}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha} \exp\left(-\frac{1-\alpha}{\alpha} t\right)$, $t \in \mathbb{R}$ we re-obtain the definitions of (16) and (17) from (11) and (12).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}_{g,a+}^{\alpha} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x) - g(t))\right\} g'(t) f(t) \, dt, \ x > a \tag{18}$$

and

$$\mathcal{T}_{g,b-}^{\alpha} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t) - g(x))\right\} g'(t) f(t) \, dt, \ x < b \tag{19}$$

where $\alpha \in (0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}_{g,a+}^{\alpha} f(x) := \int_a^x (g(x) - g(t))^{\alpha-1} \ln(g(x) - g(t)) \, g'(t) f(t) \, dt, \tag{20}$$

for $0 < a < x \leq b$ and

$$\mathcal{L}_{g,b-}^{\alpha} f(x) := \int_x^b (g(t) - g(x))^{\alpha-1} \ln(g(t) - g(x)) \, g'(t) f(t) \, dt, \tag{21}$$

for $0 < a \leq x < b$, where $\alpha > 0$. These are obtained from (11) and (12) for the kernel $k(t) = t^{\alpha-1} \ln t$, $t > 0$.

For $\alpha = 1$ we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t)) \, g'(t) f(t) \, dt, \ 0 < a < x \leq b \tag{22}$$

and

$$\mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x)) \, g'(t) f(t) \, dt, \ 0 < a \leq x < b. \tag{23}$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t) \, dt, \ 0 < a < x \leq b, \tag{24}$$

$$\mathcal{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t) \, dt, \ 0 < a \leq x < b, \tag{25}$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t) \, dt, \ 0 < a < x \leq b \tag{26}$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t) \, dt, \ 0 < a \leq x < b. \tag{27}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[17], [21]-[34] and the references therein.

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} & S_{k,g,a+,b-}f(x) \quad (28) \\ & := \frac{1}{2}\left[S_{k,g,a+}f(x)+S_{k,g,b-}f(x)\right] \\ & = \frac{1}{2}\left[\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt+\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt\right] \end{aligned}$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

We also define the function $\mathbf{K}:[0,\infty)\to[0,\infty)$ by

$$\mathbf{K}(t):=\begin{cases} \int_0^t |k(s)|\,ds \text{ if } 0<t, \\ \\ 0 \text{ if } t=0. \end{cases}$$

In the recent paper [19] we obtained the following result for functions of bounded variation:

Theorem 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the Ostrowski type inequality*

$$\begin{aligned} & \left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\ & \le \frac{1}{2}\left[\int_x^b |k(g(t)-g(x))|\bigvee_x^t(f)\,g'(t)\,dt+\int_a^x |k(g(x)-g(t))|\bigvee_t^x(f)\,g'(t)\,dt\right] \\ & \qquad \le \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\ +\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right] \\ & \le \frac{1}{2}\begin{cases} \max\left\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b(f); \\ \\ \left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right] \end{cases} \quad (29) \end{aligned}$$

and the trapezoid type inequality

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right|$$

$$\leq \frac{1}{2}\left[\int_a^x |k(g(x)-g(t))| \bigvee_a^t (f)\, g'(t)\, dt + \int_x^b |k(g(t)-g(x))| \bigvee_t^b (f)\, g'(t)\, dt\right]$$

$$\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x)) \bigvee_x^b (f) + \mathbf{K}(g(x)-g(a)) \bigvee_a^x (f)\right]$$

$$\leq \frac{1}{2}\begin{cases} \max\left\{\mathbf{K}(g(b)-g(x)), \mathbf{K}(g(x)-g(a))\right\} \bigvee_a^b (f); \\ \\ \left[\mathbf{K}^p(g(b)-g(x)) + \mathbf{K}^p(g(x)-g(a))\right]^{1/p} \\ \times\left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\mathbf{K}(g(b)-g(x)) + \mathbf{K}(g(x)-g(a))\right] \\ \times\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases} \tag{30}$$

for any $x \in (a,b)$, *where* $\bigvee_c^d (f)$ *denoted the total variation on the interval* $[c,d]$.

Observe that

$$S_{k,g,x+}f(b) = \int_x^b k(g(b)-g(t))\, g'(t)\, f(t)\, dt,\ x \in [a,b) \tag{31}$$

and

$$S_{k,g,x-}f(a) = \int_a^x k(g(t)-g(a))\, g'(t)\, f(t)\, dt,\ x \in (a,b]. \tag{32}$$

We can define also the mixed operator

$$\begin{aligned} &\breve{S}_{k,g,a+,b-}f(x) \\ &:= \frac{1}{2}\left[S_{k,g,x+}f(b) + S_{k,g,x-}f(a)\right] \\ &= \frac{1}{2}\left[\int_x^b k(g(b)-g(t))\, g'(t)\, f(t)\, dt + \int_a^x k(g(t)-g(a))\, g'(t)\, f(t)\, dt\right] \end{aligned} \tag{33}$$

for any $x \in (a,b)$.

In this paper we establish some inequalities for the k-g-fractional integrals of functions with bounded variation $f : [a,b] \to \mathbb{C}$ that provide error bounds in approximating the composite operators $S_{k,g,a+,b-}f$ and $\breve{S}_{k,g,a+,b-}f$ in terms of the *double trapezoid rule*

$$\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x)) + \frac{f(a)+f(x)}{2}K(g(x)-g(a))\right],\ x \in (a,b).$$

Examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

2. Further Inequalities for Functions of BV

The following two parameters representation for the operators $S_{k,g,a+,b-}$ and $\breve{S}_{k,g,a+,b-}$ hold [20]:

Lemma 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then*

$$\begin{aligned} S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\gamma K(g(b)-g(x))+\lambda K(g(x)-g(a))\right] \\ &+\frac{1}{2}\int_a^x k(g(x)-g(t))\,g'(t)\left[f(t)-\lambda\right]dt \\ &+\frac{1}{2}\int_x^b k(g(t)-g(x))\,g'(t)\left[f(t)-\gamma\right]dt \end{aligned} \tag{34}$$

and

$$\begin{aligned} \breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\lambda K(g(b)-g(x))+\gamma K(g(x)-g(a))\right] \\ &+\frac{1}{2}\int_a^x k(g(t)-g(a))\,g'(t)\left[f(t)-\gamma\right]dt \\ &+\frac{1}{2}\int_x^b k(g(b)-g(t))\,g'(t)\left[f(t)-\lambda\right]dt \end{aligned} \tag{35}$$

for $x\in(a,b)$ and for any λ, $\gamma\in\mathbb{C}$.

Proof. We have, by taking the derivative over t and using the chain rule, that

$$\left[K(g(x)-g(t))\right]'=K'(g(x)-g(t))\,(g(x)-g(t))'=-k(g(x)-g(t))\,g'(t)$$

for $t\in(a,x)$ and

$$\left[K(g(t)-g(x))\right]'=K'(g(t)-g(x))\,(g(t)-g(x))'=k(g(t)-g(x))\,g'(t)$$

for $t\in(x,b)$.

Therefore, for any λ, $\gamma\in\mathbb{C}$ we have

$$\begin{aligned} &\int_a^x k(g(x)-g(t))\,g'(t)\left[f(t)-\lambda\right]dt \\ &=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt-\lambda\int_a^x k(g(x)-g(t))\,g'(t)\,dt \\ &=S_{k,g,a+}f(x)+\lambda\int_a^x \left[K(g(x)-g(t))\right]'dt \\ &=S_{k,g,a+}f(x)+\lambda\left[K(g(x)-g(t))\right]\Big|_a^x=S_{k,g,a+}f(x)-\lambda K(g(x)-g(a)) \end{aligned} \tag{36}$$

and

$$\int_x^b k(g(t)-g(x))\,g'(t)\,[f(t)-\gamma]\,dt \tag{37}$$
$$= \int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt - \gamma\int_x^b k(g(t)-g(x))\,g'(t)\,dt$$
$$= S_{k,g,b-}f(x) - \gamma\int_x^b [K(g(t)-g(x))]'\,dt$$
$$= S_{k,g,b-}f(x) - \gamma\,[K(g(t)-g(x))]\big|_x^b = S_{k,g,b-}f(x) - \gamma K(g(b)-g(x))$$

for $x \in (a,b)$.

If we add the equalities (36) and (37) and divide by 2 then we get the desired result (34).

Moreover, by taking the derivative over t and using the chain rule, we have that

$$[K(g(b)-g(t))]' = K'(g(b)-g(t))\,(g(b)-g(t))' = -k(g(b)-g(t))\,g'(t)$$

for $t \in (x,b)$ and

$$[K(g(t)-g(a))]' = K'(g(t)-g(a))\,(g(t)-g(a))' = k(g(t)-g(a))\,g'(t)$$

for $t \in (a,x)$.

For any λ, $\gamma \in \mathbb{C}$ we have

$$\int_x^b k(g(b)-g(t))\,g'(t)\,[f(t)-\lambda]\,dt \tag{38}$$
$$= \int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt - \lambda\int_x^b k(g(b)-g(t))\,g'(t)\,dt$$
$$= S_{k,g,x+}f(b) + \lambda\int_x^b [K(g(b)-g(t))]'\,dt$$
$$= S_{k,g,x+}f(b) - \lambda K(g(b)-g(x))$$

and

$$\int_a^x k(g(t)-g(a))\,g'(t)\,[f(t)-\gamma]\,dt \tag{39}$$
$$= \int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt - \gamma\int_a^x k(g(t)-g(a))\,g'(t)\,dt$$
$$= \int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt - \gamma\int_a^x [K(g(t)-g(a))]'\,dt$$
$$= \int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt - \gamma K(g(x)-g(a))$$

for $x \in (a,b)$.

If we add the equalities (38) and (39) and divide by 2 then we get the desired result (35). $\square$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a, b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Using the g-mean of two numbers we can introduce

$$\begin{aligned} P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f(M_g(a,b)) \\ &= \frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)f(t)\,dt \\ &+ \frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)f(t)\,dt. \end{aligned} \tag{40}$$

Using the representation (34) we have

$$\begin{aligned} P_{k,g,a+,b-}f &= K\left(\frac{g(b)-g(a)}{2}\right)\frac{\gamma+\lambda}{2} \\ &+ \frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)\left[f(t)-\lambda\right]dt \\ &+ \frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)\left[f(t)-\gamma\right]dt \end{aligned} \tag{41}$$

for any $\lambda, \gamma \in \mathbb{C}$.

Also, if

$$\begin{aligned} \breve{P}_{k,g,a+,b-}f &:= \breve{S}_{k,g,a+,b-}f(M_g(a,b)) \\ &= \frac{1}{2}\int_{M_g(a,b)}^b k(g(b)-g(t))\,g'(t)f(t)\,dt \\ &+ \frac{1}{2}\int_a^{M_g(a,b)} k(g(t)-g(a))\,g'(t)f(t)\,dt. \end{aligned} \tag{42}$$

then by (35) we get

$$\breve{P}_{k,g,a+,b-}f = K\left(\frac{g(b)-g(a)}{2}\right)\frac{\gamma+\lambda}{2} + \frac{1}{2}\int_a^{M_g(a,b)} k(g(t)-g(a))\,g'(t)[f(t)-\gamma]\,dt + \frac{1}{2}\int_{M_g(a,b)}^{b} k(g(b)-g(t))\,g'(t)[f(t)-\lambda]\,dt \tag{43}$$

for any λ, $\gamma \in \mathbb{C}$.

Theorem 3. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the double trapezoid inequalities*

$$\begin{aligned}
&\Big|S_{k,g,a+,b-}f(x) \\
&\quad -\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(a)+f(x)}{2}K(g(x)-g(a))\right]\Big| \\
&\qquad\le \frac{1}{4}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right] \\
&\le \frac{1}{4}\begin{cases}
\max\left\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b(f); \\[2ex]
\left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\
\textit{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]
\end{cases}
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
&\Big|\breve{S}_{k,g,a+,b-}f(x) \\
&\quad -\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(a)+f(x)}{2}K(g(x)-g(a))\right]\Big| \\
&\qquad\le \frac{1}{4}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right]
\end{aligned}$$

$$\leq \frac{1}{4}\begin{cases} \max\left\{\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right),\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right\}\bigvee_a^b\left(f\right); \\ \\ \left[\mathbf{K}^p\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}^p\left(g\left(x\right)-g\left(a\right)\right)\right]^{1/p}\left(\left(\bigvee_a^x\left(f\right)\right)^q+\left(\bigvee_x^b\left(f\right)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right]\left[\frac{1}{2}\bigvee_a^b\left(f\right)+\frac{1}{2}\left|\bigvee_a^x\left(f\right)-\bigvee_x^b\left(f\right)\right|\right] \end{cases} \tag{45}$$

for $x\in(a,b)$.

Proof. Using the identity (34) for $\lambda=\frac{f(a)+f(x)}{2}$ and $\gamma=\frac{f(x)+f(b)}{2}$ we have

$$\begin{aligned} &S_{k,g,a+,b-}f\left(x\right) \\ &=\frac{1}{2}\left[\frac{f\left(x\right)+f\left(b\right)}{2}K\left(g\left(b\right)-g\left(x\right)\right)+\frac{f\left(a\right)+f\left(x\right)}{2}K\left(g\left(x\right)-g\left(a\right)\right)\right] \\ &+\frac{1}{2}\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-\frac{f\left(a\right)+f\left(x\right)}{2}\right]dt \\ &+\frac{1}{2}\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)\left[f\left(t\right)-\frac{f\left(x\right)+f\left(b\right)}{2}\right]dt \end{aligned} \tag{46}$$

for $x\in(a,b)$.

Since f is of bounded variation, then

$$\begin{aligned} \left|f\left(t\right)-\frac{f\left(a\right)+f\left(x\right)}{2}\right| &= \left|\frac{f\left(t\right)-f\left(a\right)+f\left(t\right)-f\left(x\right)}{2}\right| \\ &\leq\frac{1}{2}\left[\left|f\left(t\right)-f\left(a\right)\right|+\left|f\left(x\right)-f\left(t\right)\right|\right]\leq\frac{1}{2}\bigvee_a^x\left(f\right) \end{aligned}$$

and

$$\begin{aligned} \left|f\left(t\right)-\frac{f\left(x\right)+f\left(b\right)}{2}\right| &= \left|\frac{f\left(t\right)-f\left(x\right)+f\left(t\right)-f\left(b\right)}{2}\right| \\ &\leq\frac{1}{2}\left[\left|f\left(t\right)-f\left(x\right)\right|+\left|f\left(b\right)-f\left(t\right)\right|\right]\leq\frac{1}{2}\bigvee_x^b\left(f\right) \end{aligned}$$

for $x\in(a,b)$.

Using the equality (46) we have

$$\begin{aligned} &\left|S_{k,g,a+,b-}f\left(x\right)\right. \\ &\qquad\left.-\frac{1}{2}\left[\frac{f\left(x\right)+f\left(b\right)}{2}K\left(g\left(b\right)-g\left(x\right)\right)+\frac{f\left(a\right)+f\left(x\right)}{2}K\left(g\left(x\right)-g\left(a\right)\right)\right]\right| \\ &\qquad\qquad\leq\frac{1}{2}\left|\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-\frac{f\left(a\right)+f\left(x\right)}{2}\right]dt\right| \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left|\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)\left[f\left(t\right)-\frac{f\left(x\right)+f\left(b\right)}{2}\right]dt\right| \\
&\leq \frac{1}{2}\int_a^x \left|k\left(g\left(x\right)-g\left(t\right)\right)\right|\left|f\left(t\right)-\frac{f\left(a\right)+f\left(x\right)}{2}\right|g'\left(t\right)dt \\
&+\frac{1}{2}\int_x^b \left|k\left(g\left(t\right)-g\left(x\right)\right)\right|\left|f\left(t\right)-\frac{f\left(x\right)+f\left(b\right)}{2}\right|g'\left(t\right)dt
\end{aligned}
$$

$$
\leq \frac{1}{4}\left[\bigvee_a^x\left(f\right)\int_a^x\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|g'\left(t\right)dt+\bigvee_x^b\left(f\right)\int_x^b\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|g'\left(t\right)dt\right] =: B\left(x\right) \quad (47)
$$

for $x\in\left(a,b\right)$.

We have, by taking the derivative over t and using the chain rule, that

$$
\left[\mathbf{K}\left(g\left(x\right)-g\left(t\right)\right)\right]'=\mathbf{K}'\left(g\left(x\right)-g\left(t\right)\right)\left(g\left(x\right)-g\left(t\right)\right)'=-\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|g'\left(t\right)
$$

for $t\in\left(a,x\right)$ and

$$
\left[\mathbf{K}\left(g\left(t\right)-g\left(x\right)\right)\right]'=\mathbf{K}'\left(g\left(t\right)-g\left(x\right)\right)\left(g\left(t\right)-g\left(x\right)\right)'=\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|g'\left(t\right)
$$

for $t\in\left(x,b\right)$.

Then

$$
\int_a^x\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|g'\left(t\right)dt=-\int_a^x\left[\mathbf{K}\left(g\left(x\right)-g\left(t\right)\right)\right]'dt=\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)
$$

and

$$
\int_x^b\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|g'\left(t\right)dt=\int_x^b\left[\mathbf{K}\left(g\left(t\right)-g\left(x\right)\right)\right]'dt=\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right).
$$

Therefore

$$
\begin{aligned}
B\left(x\right)&=\frac{1}{4}\left[\bigvee_a^x\left(f\right)\int_a^x\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|g'\left(t\right)dt+\bigvee_x^b\left(f\right)\int_x^b\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|g'\left(t\right)dt\right] \\
&=\frac{1}{4}\left[\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\bigvee_a^x\left(f\right)+\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)\bigvee_x^b\left(f\right)\right].
\end{aligned}
$$

The last part of (44) is obvious by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n\geq 0$

$$
mc+nd\leq\begin{cases}\max\left\{m,n\right\}\left(c+d\right); \\ \\ \left(m^p+n^p\right)^{1/p}\left(c^q+d^q\right)^{1/q} \text{ with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases}
$$

Using the identity (35) for $\lambda=\frac{f(x)+f(b)}{2}$ and $\gamma=\frac{f(x)+f(a)}{2}$ we also have

$$
\left|\check{S}_{k,g,a+,b-}f\left(x\right)\right.
$$

$$-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(x)+f(a)}{2}K(g(x)-g(a))\right]\Bigg|$$

$$\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\left|f(t)-\frac{f(x)+f(a)}{2}\right| g'(t)\,dt$$

$$+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\left|f(t)-\frac{f(x)+f(b)}{2}\right| g'(t)\,dt$$

$$\leq \frac{1}{4}\bigvee_a^x(f)\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt+\frac{1}{4}\bigvee_x^b(f)\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt$$

$$=: C(x).$$

We also have, by taking the derivative over t and using the chain rule, that

$$[\mathbf{K}(g(b)-g(t))]'=\mathbf{K}'(g(b)-g(t))(g(b)-g(t))'=-|k(g(b)-g(t))|\,g'(t)$$

for $t\in(x,b)$ and

$$[\mathbf{K}(g(t)-g(a))]'=\mathbf{K}'(g(t)-g(a))(g(t)-g(a))'=|k(g(t)-g(a))|\,g'(t)$$

for $t\in(a,x)$.

Therefore

$$\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt=\mathbf{K}(g(x)-g(a))$$

and

$$\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt=\mathbf{K}(g(b)-g(x))$$

giving that

$$C(x)=\frac{1}{4}\bigvee_a^x(f)\,\mathbf{K}(g(x)-g(a))+\frac{1}{4}\bigvee_x^b(f)\,\mathbf{K}(g(b)-g(x))$$

for $x\in(a,b)$, and the inequality (45) is thus proved. □

Corollary 4. *With the assumptions of Theorem 3 we have*

$$\left|P_{k,g,a+,b-}f-\frac{1}{2}K\left(\frac{g(b)-g(a)}{2}\right)\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right| \tag{48}$$

$$\leq \frac{1}{4}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_a^b(f)$$

and

$$\left|\breve{P}_{k,g,a+,b-}f-\frac{1}{2}K\left(\frac{g(b)-g(a)}{2}\right)\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right| \tag{49}$$

$$\leq \frac{1}{4}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_a^b(f).$$

If we take $x=\frac{a+b}{2}$ in (44) and (45), then we get

$$\begin{aligned}
&\left|S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)-\frac{f\left(\frac{a+b}{2}\right)+f(b)}{4}K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\right.\\
&\qquad\left.-\frac{f(a)+f\left(\frac{a+b}{2}\right)}{4}K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right|\\
&\le\frac{1}{4}\left[\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\bigvee_a^{\frac{a+b}{2}}(f)+\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\bigvee_{\frac{a+b}{2}}^{b}(f)\right]\\
&\le\frac{1}{4}\begin{cases}
\max\left\{\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right),\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right\}\bigvee_a^b(f);\\[2ex]
\left[\mathbf{K}^p\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}^p\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]^{1/p}\\
\left(\left(\bigvee_a^{\frac{a+b}{2}}(f)\right)^q+\left(\bigvee_{\frac{a+b}{2}}^b(f)\right)^q\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left[\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]\\
\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^b(f)\right|\right]
\end{cases}
\end{aligned}\tag{50}$$

and

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)-\frac{f\left(\frac{a+b}{2}\right)+f(b)}{4}K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\right.\\
&\qquad\left.-\frac{f(a)+f\left(\frac{a+b}{2}\right)}{4}K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right|\\
&\le\frac{1}{4}\left[\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\bigvee_a^{\frac{a+b}{2}}(f)+\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\bigvee_x^{b}(f)\right]
\end{aligned}$$

$$\leq \frac{1}{4}\begin{cases} \max\left\{\mathbf{K}\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right),\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right\}\bigvee_{a}^{b}\left(f\right); \\ \\ \left[\mathbf{K}^{p}\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}^{p}\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right]^{1/p} \\ \left(\left(\bigvee_{a}^{\frac{a+b}{2}}\left(f\right)\right)^{q}+\left(\bigvee_{\frac{a+b}{2}}^{b}\left(f\right)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\mathbf{K}\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right] \\ \left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{\frac{a+b}{2}}\left(f\right)-\bigvee_{\frac{a+b}{2}}^{b}\left(f\right)\right|\right] \end{cases} \tag{51}$$

for $x\in\left(a,b\right)$.

We use the classical Lebesgue p-norms defined as

$$\left\|h\right\|_{[c,d],\infty}:=\operatorname*{essup}_{s\in[c,d]}\left|h\left(s\right)\right|$$

and

$$\left\|h\right\|_{[c,d],p}:=\left(\int_{c}^{d}\left|h\left(s\right)\right|^{p}ds\right)^{1/p},\ p\geq1.$$

Using Hölder's integral inequality we have for $t>0$ that

$$K\left(t\right)=\int_{0}^{t}\left|k\left(s\right)\right|ds\leq\begin{cases} t\left\|k\right\|_{[0,t],\infty}\ \text{ if } k\in L_{\infty}\left[0,t\right] \\ \\ t^{1/p}\left\|k\right\|_{[0,t],q}\ \text{ if } k\in L_{q}\left[0,t\right],\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

Therefore by the first inequality in (44) and (45) we get for $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned} &\Bigg|S_{k,g,a+,b-}f\left(x\right) \\ &\quad-\frac{1}{2}\left[\frac{f\left(x\right)+f\left(b\right)}{2}K\left(g\left(b\right)-g\left(x\right)\right)+\frac{f\left(a\right)+f\left(x\right)}{2}K\left(g\left(x\right)-g\left(a\right)\right)\right]\Bigg| \\ &\qquad\leq\frac{1}{4}\bigvee_{a}^{x}\left(f\right)\begin{cases}\left(g\left(x\right)-g\left(a\right)\right)\left\|k\right\|_{[0,g(x)-g(a)],\infty} \\ \\ \left(g\left(x\right)-g\left(a\right)\right)^{1/p}\left\|k\right\|_{[0,g(x)-g(a)],q}\end{cases} \\ &\qquad\quad+\frac{1}{4}\bigvee_{x}^{b}\left(f\right)\begin{cases}\left(g\left(b\right)-g\left(x\right)\right)\left\|k\right\|_{[0,g(b)-g(x)],\infty} \\ \\ \left(g\left(b\right)-g\left(x\right)\right)^{1/p}\left\|k\right\|_{[0,g(b)-g(x)],q}\end{cases} \end{aligned} \tag{52}$$

and

$$\begin{aligned} &\Bigg|\breve{S}_{k,g,a+,b-}f\left(x\right) \\ &\quad-\frac{1}{2}\left[\frac{f\left(x\right)+f\left(b\right)}{2}K\left(g\left(b\right)-g\left(x\right)\right)+\frac{f\left(a\right)+f\left(x\right)}{2}K\left(g\left(x\right)-g\left(a\right)\right)\right]\Bigg| \end{aligned}$$

$$\leq \frac{1}{4} \bigvee_{a}^{x} (f) \begin{cases} (g(x) - g(a)) \|k\|_{[0, g(x)-g(a)], \infty} \\ \\ (g(x) - g(a))^{1/p} \|k\|_{[0, g(x)-g(a)], q} \end{cases}$$

$$+ \frac{1}{4} \bigvee_{x}^{b} (f) \begin{cases} (g(b) - g(x)) \|k\|_{[0, g(b)-g(x)], \infty} \\ \\ (g(b) - g(x))^{1/p} \|k\|_{[0, g(b)-g(x)], q} \end{cases} \tag{53}$$

for $x \in (a, b)$.

From (48) and (49) we also have for p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\left| P_{k,g,a+,b-} f - \frac{1}{2} K \left(\frac{g(b) - g(a)}{2} \right) \left[f(M_g(a, b)) + \frac{f(a) + f(b)}{2} \right] \right|$$

$$\leq \frac{1}{4} \bigvee_{a}^{b} (f) \begin{cases} \left(\frac{g(b)-g(a)}{2} \right) \|k\|_{\left[0, \frac{g(b)-g(a)}{2}\right], \infty} \\ \\ \left(\frac{g(b)-g(a)}{2} \right)^{1/p} \|k\|_{\left[0, \frac{g(b)-g(a)}{2}\right], q} \end{cases} \tag{54}$$

and

$$\left| \breve{P}_{k,g,a+,b-} f - \frac{1}{2} K \left(\frac{g(b) - g(a)}{2} \right) \left[f(M_g(a, b)) + \frac{f(a) + f(b)}{2} \right] \right|$$

$$\leq \frac{1}{4} \bigvee_{a}^{b} (f) \begin{cases} \left(\frac{g(b)-g(a)}{2} \right) \|k\|_{\left[0, \frac{g(b)-g(a)}{2}\right], \infty} \\ \\ \left(\frac{g(b)-g(a)}{2} \right)^{1/p} \|k\|_{\left[0, \frac{g(b)-g(a)}{2}\right], q} . \end{cases} \tag{55}$$

3. Applications for Generalized Riemann-Liouville Fractional Integrals

If we take $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha - 1}$, where Γ is the *Gamma function*, then

$$S_{k,g,a+} f(x) = I_{a+,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{a}^{x} [g(x) - g(t)]^{\alpha - 1} g'(t) f(t) \, dt$$

for $a < x \leq b$ and

$$S_{k,g,b-} f(x) = I_{b-,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{b} [g(t) - g(x)]^{\alpha - 1} g'(t) f(t) \, dt$$

for $a \leq x < b$, which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ as defined in [23, p. 100].

We consider the mixed operators

$$I_{g,a+,b-}^{\alpha} f(x) := \frac{1}{2} \left[I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) \right] \tag{56}$$

and

$$\breve{I}_{g,a+,b-}^{\alpha}f\left(x\right):=\frac{1}{2}\left[I_{x+,g}^{\alpha}f\left(b\right)+I_{x-,g}^{\alpha}f(a)\right] \tag{57}$$

for $x\in\left(a,b\right)$.

We observe that for $\alpha>0$ we have

$$K\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}s^{\alpha-1}ds=\frac{t^{\alpha}}{\alpha\Gamma\left(\alpha\right)}=\frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)},\ t\geq0.$$

If we use the inequalities (44) and (45) we get

$$\begin{aligned}
&\Big|I_{g,a+,b-}^{\alpha}f\left(x\right)\\
&\quad-\frac{1}{2\Gamma\left(\alpha+1\right)}\left[\frac{f\left(x\right)+f\left(b\right)}{2}\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}+\frac{f\left(a\right)+f\left(x\right)}{2}\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\right]\Big|\\
&\qquad\leq\frac{1}{4\Gamma\left(\alpha+1\right)}\left[\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\bigvee_{a}^{x}\left(f\right)+\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}\bigvee_{x}^{b}\left(f\right)\right]\\
&\qquad\qquad\leq\frac{1}{4\Gamma\left(\alpha+1\right)}\\
&\times\begin{cases}
\left[\frac{g(b)-g(a)}{2}+\left|g\left(x\right)-\frac{g(b)+g(a)}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}\left(f\right); \\[2ex]
\left[\left(g\left(b\right)-g\left(x\right)\right)^{p\alpha}+\left(g\left(x\right)-g\left(a\right)\right)^{p\alpha}\right]^{1/p}\left(\left(\bigvee_{a}^{x}\left(f\right)\right)^{q}+\left(\bigvee_{x}^{b}\left(f\right)\right)^{q}\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left[\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}+\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\right]\left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right]
\end{cases}
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
&\Big|\breve{I}_{g,a+,b-}^{\alpha}f\left(x\right)\\
&\quad-\frac{1}{2\Gamma\left(\alpha+1\right)}\left[\frac{f\left(x\right)+f\left(b\right)}{2}\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}+\frac{f\left(a\right)+f\left(x\right)}{2}\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\right]\Big|\\
&\qquad\leq\frac{1}{4\Gamma\left(\alpha+1\right)}\left[\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\bigvee_{a}^{x}\left(f\right)+\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}\bigvee_{x}^{b}\left(f\right)\right]\\
&\qquad\qquad\leq\frac{1}{4\Gamma\left(\alpha+1\right)}
\end{aligned}$$

$$\times\begin{cases}\left[\frac{g(b)-g(a)}{2}+\left|g\left(x\right)-\frac{g(b)+g(a)}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}\left(f\right); \\ \\ \left[\left(g\left(b\right)-g\left(x\right)\right)^{p\alpha}+\left(g\left(x\right)-g\left(a\right)\right)^{p\alpha}\right]^{1/p}\left(\left(\bigvee_{a}^{x}\left(f\right)\right)^{q}+\left(\bigvee_{x}^{b}\left(f\right)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}+\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\right]\left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right]\end{cases} \tag{59}$$

for $x\in(a,b)$.

From (48) and (49) we get

$$\left|I_{g,a+,b-}^{\alpha}f\left(M_{g}\left(a,b\right)\right)-\frac{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}{2^{\alpha+1}\Gamma\left(\alpha+1\right)}\left[f\left(M_{g}\left(a,b\right)\right)+\frac{f\left(a\right)+f\left(b\right)}{2}\right]\right| \leq\frac{1}{2^{\alpha+2}\Gamma\left(\alpha+1\right)}\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}\bigvee_{a}^{b}\left(f\right) \tag{60}$$

and

$$\left|\breve{I}_{g,a+,b-}^{\alpha}f\left(M_{g}\left(a,b\right)\right)-\frac{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}{2^{\alpha+1}\Gamma\left(\alpha+1\right)}\left[f\left(M_{g}\left(a,b\right)\right)+\frac{f\left(a\right)+f\left(b\right)}{2}\right]\right| \leq\frac{1}{2^{\alpha+2}\Gamma\left(\alpha+1\right)}\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}\bigvee_{a}^{b}\left(f\right). \tag{61}$$

4. Example for an Exponential Kernel

For α, $\beta\in\mathbb{R}$ we consider the kernel $k\left(t\right):=\exp\left[\left(\alpha+\beta i\right)t\right]$, $t\in\mathbb{R}$. We have

$$K\left(t\right)=\frac{\exp\left[\left(\alpha+\beta i\right)t\right]-1}{\left(\alpha+\beta i\right)},\ \text{if } t\in\mathbb{R}$$

for α, $\beta\neq0$.

Also, we have

$$\left|k\left(s\right)\right|:=\left|\exp\left[\left(\alpha+\beta i\right)s\right]\right|=\exp\left(\alpha s\right)\ \text{ for } s\in\mathbb{R}$$

and

$$\mathbf{K}\left(t\right)=\int_{0}^{t}\exp\left(\alpha s\right)ds=\frac{\exp\left(\alpha t\right)-1}{\alpha}\ \text{if } 0<t,$$

for $\alpha\neq0$.

Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We consider the operator

$$\mathcal{H}_{g,a+,b-}^{\alpha+\beta i}f\left(x\right):=\frac{1}{2}\int_{a}^{x}\exp\left[\left(\alpha+\beta i\right)\left(g\left(x\right)-g\left(t\right)\right)\right]g'\left(t\right)f\left(t\right)dt \tag{62}$$

$$+\frac{1}{2}\int_{x}^{b}\exp\left[\left(\alpha+\beta i\right)\left(g\left(t\right)-g\left(x\right)\right)\right]g'\left(t\right)f\left(t\right)dt$$

for $x\in(a,b)$.

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}
&\kappa_{h,a+,b-}^{\alpha+\beta i}f\left(x\right) \\
&:=\mathcal{H}_{\ln h,a+,b-}^{\alpha+\beta i}f\left(x\right) \\
&=\frac{1}{2}\left[\int_{a}^{x}\left(\frac{h\left(x\right)}{h\left(t\right)}\right)^{\alpha+\beta i}\frac{h'\left(t\right)}{h\left(t\right)}f\left(t\right)dt+\int_{x}^{b}\left(\frac{h\left(t\right)}{h\left(x\right)}\right)^{\alpha+\beta i}\frac{h'\left(t\right)}{h\left(t\right)}f\left(t\right)dt\right],
\end{aligned}\tag{63}$$

for $x\in(a,b)$.

Using the inequality (44) we have for $x\in(a,b)$

$$\begin{aligned}
&\left|\mathcal{H}_{g,a+,b-}^{\alpha+\beta i}f\left(x\right)-\frac{1}{2}\frac{f\left(x\right)+f\left(b\right)}{2}\frac{\exp\left[\left(\alpha+\beta i\right)\left(g\left(b\right)-g\left(x\right)\right)\right]-1}{\left(\alpha+\beta i\right)}\right. \\
&\qquad\left.-\frac{f\left(a\right)+f\left(x\right)}{2}\frac{\exp\left[\left(\alpha+\beta i\right)\left(g\left(x\right)-g\left(a\right)\right)\right]-1}{\left(\alpha+\beta i\right)}\right| \\
&\leq\frac{1}{4}\left[\frac{\exp\left(\alpha\left(g\left(x\right)-g\left(a\right)\right)\right)-1}{\alpha}\bigvee_{a}^{x}\left(f\right)+\frac{\exp\left(\alpha\left(g\left(b\right)-g\left(x\right)\right)\right)-1}{\alpha}\bigvee_{x}^{b}\left(f\right)\right] \\
&\leq\frac{1}{4}\begin{cases}
\max\left\{\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha},\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right\}\bigvee_{a}^{b}\left(f\right); \\
\\
\left[\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^{p}+\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^{p}\right]^{1/p} \\
\times\left(\left(\bigvee_{a}^{x}\left(f\right)\right)^{q}+\left(\bigvee_{x}^{b}\left(f\right)\right)^{q}\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left[\frac{\exp(\alpha(g(x)-g(a)))+\exp(\alpha(g(b)-g(x)))-2}{\alpha}\right] \\
\times\left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right]
\end{cases}
\end{aligned}\tag{64}$$

and if we take $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we get

$$\left|\kappa_{h,a+,b-}^{\alpha+\beta i}f\left(x\right)-\frac{1}{2}\left[\frac{f\left(x\right)+f\left(b\right)}{2}\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha+\beta i}-1}{\left(\alpha+\beta i\right)}\right.\right.$$

$$
\left.\left. -\frac{f(a)+f(x)}{2}\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\right]\right|
$$

$$
\leq \frac{1}{4}\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\bigvee_a^x(f)+\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\bigvee_x^b(f)\right]
$$

$$
\leq \frac{1}{4}\begin{cases} \max\left\{\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha},\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right\}\bigvee_a^b(f); \\ \\ \left[\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\right)^p+\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right)^p\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}+\left(\frac{h(b)}{h(x)}\right)^{\alpha}-2}{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]. \end{cases} \tag{65}
$$

If we take if we take $x_h := h^{-1}\left(\sqrt{h(a)h(b)}\right) = h^{-1}(G(h(a),h(b))) \in (a,b)$, where G is the geometric mean, then from (65) we get

$$
\left|\bar{\kappa}_{h,a+,b-}^{\alpha+\beta i}f-\frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha+\beta i}{2}}-1}{2(\alpha+\beta i)}\left[f\left(h^{-1}(G(h(a),h(b)))\right)+\frac{f(a)+f(b)}{2}\right]\right|
$$

$$
\leq \frac{1}{4}\frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha}{2}}-1}{\alpha}\bigvee_a^b(f), \tag{66}
$$

where $\bar{\kappa}_{h,a+,b-}^{\alpha+\beta i}f = \kappa_{h,a+,b-}^{\alpha+\beta i}f(x_h)$.

Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Also define

$$
\begin{aligned} &\breve{\mathcal{H}}_{g,a+,b-}^{\alpha}f(x) \\ &:= \frac{1}{2}\int_x^b \exp[\alpha(g(b)-g(t))]g'(t)f(t)\,dt \\ &+\frac{1}{2}\int_a^x \exp[\alpha(g(t)-g(a))]g'(t)f(t)\,dt \end{aligned} \tag{67}
$$

for any $x\in(a,b)$.

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned} & \breve{\kappa}_{h,a+,b-}^{\alpha} f(x) \\ & := \breve{\mathcal{H}}_{\ln h,a+,b-}^{\alpha} f(x) \\ & = \frac{1}{2}\left[\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\, dt + \int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\, dt\right], \end{aligned} \tag{68}$$

for any $x \in (a,b)$.

Using the inequality (45) we have for $x \in (a,b)$ that

$$\begin{aligned} & \left| \breve{\mathcal{H}}_{g,a+,b-}^{\alpha+\beta i} f(x) - \frac{1}{2}\left[\frac{f(x)+f(b)}{2} \frac{\exp\left[(\alpha+\beta i)(g(b)-g(x))\right]-1}{(\alpha+\beta i)} \right.\right. \\ & \qquad \left.\left. - \frac{f(a)+f(x)}{2} \frac{\exp\left[(\alpha+\beta i)(g(x)-g(a))\right]-1}{(\alpha+\beta i)}\right]\right| \\ & \le \frac{1}{4}\left[\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha} \bigvee_a^x (f) + \frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha} \bigvee_x^b (f)\right] \\ & \le \frac{1}{4} \begin{cases} \max\left\{\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}, \frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right\} \bigvee_a^b (f); \\ \\ \left[\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p + \left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p\right]^{1/p} \\ \times \left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{\exp(\alpha(g(x)-g(a)))+\exp(\alpha(g(b)-g(x)))-2}{\alpha}\right] \\ \times \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases} \end{aligned} \tag{69}$$

and if we take $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we get

$$\begin{aligned} & \left| \breve{\kappa}_{h,a+,b-}^{\alpha+\beta i} f(x) - \frac{1}{2}\left[\frac{f(x)+f(b)}{2} \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)} \right.\right. \\ & \qquad \left.\left. - \frac{f(a)+f(x)}{2} \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\right]\right| \\ & \le \frac{1}{4}\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha} \bigvee_a^x (f) + \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha} \bigvee_x^b (f)\right] \end{aligned}$$

$$\leq \frac{1}{4}\begin{cases} \max\left\{\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}, \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right\}\bigvee_{a}^{b}(f); \\ \left[\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\right)^{p}+\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right)^{p}\right]^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}+\left(\frac{h(b)}{h(x)}\right)^{\alpha}-2}{\alpha}\right]\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]. \end{cases} \tag{70}$$

If we take if we take $x_h = h^{-1}(G(h(a), h(b))) \in (a,b)$, where G is the geometric mean, then from (65) we get

$$\left|\bar{\ell}_{h,a+,b-}^{\alpha+\beta i} f - \frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha+\beta i}{2}}-1}{2(\alpha+\beta i)}\left[f\left(h^{-1}(G(h(a), h(b)))\right)+\frac{f(a)+f(b)}{2}\right]\right|$$
$$\leq \frac{1}{4}\frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha}{2}}-1}{\alpha}\bigvee_{a}^{b}(f), \tag{71}$$

where $\bar{\ell}_{h,a+,b-}^{\alpha+\beta i} f = \breve{\kappa}_{h,a+,b-}^{\alpha+\beta i} f(x_h)$.

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GENERALIZED FRACTIONAL MAXIMAL OPERATOR ON GENERALIZED LOCAL MORREY SPACES

A. KUCUKASLAN, V.S. GULIYEV, AND A. SERBETCI

ABSTRACT. In this paper, we study the boundedness of generalized fractional maximal operator M_ρ on generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$ and generalized Morrey spaces $M_{p,\varphi}$, including weak estimates. Firstly, we prove the Spanne type boundedness of M_ρ from the space $LM_{p,\varphi_1}^{\{x_0\}}$ to another $LM_{q,\varphi_2}^{\{x_0\}}$, $1 < p < q < \infty$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to the weak space $WLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$ and $1 < q < \infty$. Secondly, we prove the Adams type boundedness of M_ρ from the space $M_{p,\varphi^{\frac{1}{p}}}$ to another $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to the weak space $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$ and $1 < q < \infty$. In all cases the conditions for the boundedness of M_ρ are given in terms of supremal-type integral inequalities on $(\varphi_1, \varphi_2, \rho)$ and (φ, ρ), which do not assume any assumption on monotonicity of $\varphi_1(x, r)$, $\varphi_2(x, r)$ and $\varphi(x, r)$ in r.

1. INTRODUCTION

The classical Morrey spaces $M_{p,\lambda}$ were first introduced by Morrey in [21] to study the local behavior of solutions to second order elliptic partial differential equations. The generalized Morrey spaces $M_{p,\varphi}$ are obtained by replacing r^λ in the definition of the Morrey space. During the last decades various classical operators, such as maximal, singular and potential operators were widely investigated in both in classical, generalized Morrey spaces and generalized local Morrey spaces. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operators on these spaces, we refer the readers to [1, 9, 15, 16, 20, 22].

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For a measurable function $\rho : (0,\infty) \to (0,\infty)$ the generalized fractional maximal operator M_ρ and the generalized fractional integral operator I_ρ are defined by

$$M_\rho f(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)|dy,$$

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)dy$$

for any suitable function f on $\mathbb{R}^n$. If $\rho(t) \equiv t^\alpha$, then $M_\alpha \equiv M_{t^\alpha}$ is the fractional maximal operator and $I_\alpha \equiv I_{t^\alpha}$ is the Riesz potential.

Spanne [24] and Adams [1] studied boundedness of the Riesz potential in Morrey spaces. Their results can be summarized as follows.

Theorem A. *(Spanne, but published by Peetre [24]) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\mu}$ and for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\mu}$.*

Theorem B. *(Adams [1]) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Nakai [22] proved the boundedness of the operators I_ρ and M_ρ from the generalized Morrey spaces M_{p,φ_1} to the spaces M_{q,φ_2} for suitable functions φ_1 and φ_2. The boundedness of M_ρ and I_ρ from the generalized Morrey spaces M_{p,φ_1} to the spaces M_{q,φ_2} is studied by Nakai [23], Eridani [10], Gunawan [18], Eridani, Gunawan and Nakai [12], Sawano, Sugano, Tanaka [25], Eridani, Gunawan, Nakai, Sawano [11], Guliyev, Ismayilova, Kucukaslan, Serbetci [17], Kucukaslan, Hasanov, Aykol [19].

In particular, the following statement containing both Theorem A and Theorem B was proved in [3, 4].

Theorem C. *([3, 4]) Let $1 \le p < q < \infty$, $0 < \lambda, \mu < n$ and*

$$0 < \alpha = \frac{n-\lambda}{p} - \frac{n-\mu}{q} < \frac{n}{p}.$$

Then, for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\mu}$, and, for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\mu}$.

In [3, 4] it was also proved that, under the assumptions of Theorem C, the operator I_α, for $p > 1$, is bounded from the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ to $LM_{q,\mu}^{\{x_0\}}$, and, for $p = 1$ from $LM_{1,\lambda}^{\{x_0\}}$ to the weak local Morrey space $WLM_{q,\mu}^{\{x_0\}}$.

Since, for some $c > 0$, $\big(M_\alpha f\big)(x) \le c\big(I_\alpha(|f|)\big)(x)$, $x \in \mathbb{R}^n$, it follows that in Theorems A, B, C the operator I_α can be replaced by the operator M_α (including also the case $p = q$). For the operator M_α Theorem C was, in fact, earlier proved in [5, 6].

Guliyev [14] proved the Spanne and Adams type boundedness of I_α from the spaces $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$ without any assumption on monotonicity of φ_1, φ_2. Paper [7] should be mentioned where for $\alpha = n\big(\frac{1}{p} - \frac{1}{q}\big)$ necessary and sufficient conditions of φ_1 and φ_2 are obtained. In [17], by using the method given in [13] the Spanne and Adams type boundedness of the operator I_ρ from the generalized local Morrey space $LM^{\{x_0\}}_{p,\varphi_1}$ to another one $LM^{\{x_0\}}_{q,\varphi_2}$ were proved.

The main goal of this paper is to show that the boundedness of the generalized fractional maximal operator M_ρ in generalized local Morrey spaces $LM^{\{x_0\}}_{p,\varphi}$ and generalized Morrey spaces $M_{p,\varphi}$ can be obtained under weaker assumptions on ρ, namely in terms of the so-called supremal operators. More precisely, we find sufficient conditions, in supremal terms, on the functions $(\varphi_1, \varphi_2, \rho)$ which ensure the boundedness of the operator M_ρ from one generalized local Morrey space $LM^{\{x_0\}}_{p,\varphi_1}$ to another $LM^{\{x_0\}}_{q,\varphi_2}$ for $1 < p < q < \infty$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2}$ for $p = 1$ and $1 < q < \infty$. We also find conditions on the pair (φ, ρ) which ensure the Adams type boundedness of M_ρ from the spaces $M_{p,\varphi^{\frac{1}{p}}}$ to another $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to the weak space $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$ and $1 < q < \infty$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. Therefore $|B(x, r)| = w_n r^n$, where w_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Definition 2.1. *Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L^{\mathrm{loc}}_p(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x, r)^{-1} \, |B(x, r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^{\mathrm{loc}}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x, r)^{-1} \, |B(x, r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$, the weak Morrey space $WM_{p,\lambda}$ respectively, under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},\ WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 2.2. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM_{p,\varphi} \equiv LM_{p,\varphi}(\mathbb{R}^n)$ the generalized local (central) Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{LM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1}\, |B(0,r)|^{-\frac{1}{p}}\, \|f\|_{L_p(B(0,r))}.$$

Also by $WLM_{p,\varphi} \equiv WLM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized local (central) Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1}\, |B(0,r)|^{-\frac{1}{p}}\, \|f\|_{WL_p(B(0,r))} < \infty.$$

Definition 2.3. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi}}.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized local Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi}} < \infty.$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ and weak local Morrey space $WLM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LM_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}},\ WLM_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 2.4. *Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all non-negative functions on $(0,\infty)$. We define a cone $\mathbb{A}$ by the set of the functions $\varphi \in \mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and such that $\lim_{t\to 0+} \varphi(t) = 0$, briefly*

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Definition 2.5. [8] *Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by*

$$(\overline{S}_u g)(r) := \|u(t)g(t)\|_{L_\infty(r,\infty)},\ r \in (0,\infty).$$

Let v be a non-negative measurable function on $(0,\infty)$. We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t>0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \sup_{t>0} v(t)g(t)$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. The following lemma is proved analogously to Lemma 5.2 in [8].

Lemma 2.1. [8] *Let v_1 and v_2 be weights and $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t>0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty.$$

The following lemma was proved in [17].

Lemma 2.2. [17] *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t>0$. Then the identity operator I is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty.$$

3. Spanne type result for the operator M_ρ in the spaces $LM^{\{x_0\}}_{p,\varphi}$

We assume that

$$\sup_{1\le t<\infty} \frac{\rho(t)}{t^n} < \infty, \tag{3.1}$$

so that the fractional maximal functions $M_\rho f$ are well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x) = \frac{\chi_{\mathbb{R}^n\setminus B(0,1)}(x)}{|x|^{2n}}.$$

In addition, we shall also assume that ρ satisfies the growth condition: there exist constants $C_1>0$ and $0<2k_1<k_2<\infty$ such that

$$\sup_{r<s\le 2r} \frac{\rho(s)}{s^n} \le C_1 \sup_{k_1 r<t<k_2 r} \frac{\rho(t)}{t^n},\ r>0. \tag{3.2}$$

This condition is weaker than the usual doubling condition for the function $\frac{\rho(t)}{t^n}$: there exists a constant $C_2>0$ such that

$$\frac{1}{C_2}\frac{\rho(t)}{t^n} \le \frac{\rho(r)}{r^n} \le C_2 \frac{\rho(t)}{t^n},$$

whenever r and t satisfy $r,\, t>0$ and $\frac{1}{2} \le \frac{r}{t} \le 2$.

Remark 3.1. *Typical examples of $\rho(t)$ that we envisage are, for $0 < \alpha < n$*

$$\rho(t) \equiv \begin{cases} t^\alpha \log(e/t), & 0 < t \le 1 \\ \frac{t^\alpha}{\log(et)}, & 1 \le t < \infty \end{cases}$$

and, for $c > 0$

$$\rho(t) \equiv \begin{cases} t^\alpha, & 0 < t \le 1 \\ e^c e^{-ct^2}, & 1 \le t < \infty. \end{cases}$$

The second one is used to control the Bessel potential (see also [26]*).*

The boundedness of the operator I_ρ in the spaces $L_p(\mathbb{R}^n)$ can be found in [11]. Let $\frac{\rho(t)}{t^n}$ be almost decreasing, that is, there exists a constant C such that $\frac{\rho(t)}{t^n} \le C\ \frac{\rho(s)}{s^n}$ for $s < t$. In this case we get

$$\begin{aligned} M_\rho f(x) &= \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)| dy \\ &\lesssim \sup_{t>0} \int_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} \, |f(y)| dy \\ &= \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} \, |f(y)| dy = I_\rho(|f|)(x). \end{aligned}$$

For proving our main results, we need the following estimate.

Lemma 3.3. *If $B_0 := B(x_0, r_0) \subset B(x, r)$ and ρ satisfies the doubling condition. Then $\rho(r_0) \lesssim M_\rho \chi_{B_0}(x)$ for every $x \in B_0$.*

Proof. Let ρ satisfy the doubling condition, then

$$\mathcal{M}_\rho f(x) \lesssim M_\rho f(x), \tag{3.3}$$

where $\mathcal{M}_\rho(f)(x) = \sup\limits_{B \ni x} \frac{\rho(r_B)}{|B|} \int_B |f(y)| dy$ and r_B is the center of the ball B.

Now let $x \in B_0$. By using (3.3), we get

$$\begin{aligned} M_\rho \chi_{B_0}(x) \gtrsim \mathcal{M}_\rho \chi_{B_0}(x) &= \sup_{B \ni x} \frac{\rho(r_B)}{|B|} \, |B \cap B_0| \\ &\gtrsim \frac{\rho(r_0)}{|B_0|} \, |B_0 \cap B_0| = \rho(r_0). \end{aligned}$$

□

The following lemma is valid.

Lemma 3.4. *Let $1 \le p < q < \infty$.*

(1) *The condition*

$$\rho(r) \le C r^{\frac{n}{p} - \frac{n}{q}} \tag{3.4}$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$. Moreover, if $p > 1$, then the condition (3.4) *is sufficient for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.*

(2) *If ρ satisfies the doubling condition, then the condition* (3.4) *is necessary for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ and from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$.*

(3) *If ρ satisfies the doubling condition and the supremal regularity condition*

$$\sup_{r<t<\infty} \rho(t)\, t^{-\frac{n}{p}} \leq C\rho(r) r^{-\frac{n}{p}}$$

holds for all $r > 0$, where $C > 0$ does not depend on r, then the condition (3.4) *is necessary and sufficient for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$. Moreover, if $p > 1$, then the condition* (3.4) *is necessary and sufficient for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.*

Proof. (1) Suppose ρ satisfies the condition (3.4). Then

$$M_\rho f(x) \lesssim M_{\frac{n}{p}-\frac{n}{q}} f(x). \tag{3.5}$$

Since the operator $M_{\frac{n}{p}-\frac{n}{q}}$ is bounded from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ and for $p > 1$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, then from (3.5) we get the statement (1).

(2) Now we shall prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.3, we have $\rho(r_0) \lesssim M_\rho \chi_{B_0}(x)$. Therefore, we have

$$\begin{aligned}\rho(r_0) &\lesssim r_0^{-\frac{n}{q}} \|M_\rho \chi_{B_0}\|_{WL_q(B_0)} \lesssim r_0^{-\frac{n}{q}} \|M_\rho \chi_{B_0}\|_{WL_q(\mathbb{R}^n)} \\ &\lesssim r_0^{-\frac{n}{q}} \|\chi_{B_0}\|_{L_p(\mathbb{R}^n)} \lesssim r_0^{\frac{n}{p}-\frac{n}{q}}\end{aligned}$$

and for $p > 1$

$$\begin{aligned}\rho(r_0) &\lesssim r_0^{-\frac{n}{q}} \|M_\rho \chi_{B_0}\|_{L_q(B_0)} \lesssim r_0^{-\frac{n}{q}} \|M_\rho \chi_{B_0}\|_{L_q(\mathbb{R}^n)} \\ &\lesssim r_0^{-\frac{n}{q}} \|\chi_{B_0}\|_{L_p(\mathbb{R}^n)} \lesssim r_0^{\frac{n}{p}-\frac{n}{q}}\end{aligned}$$

holds for every $r_0 > 0$, hence the proof of statement (2) is completed.

(3) From the first and second statements the third statement of the lemma follows. □

The following lemma is valid.

Lemma 3.5. *Let $1 \leq p < q < \infty$ and let $\rho(t)$ satisfy the conditions* (3.1), (3.2) *and* (3.4). *Then the inequality*

$$\|M_\rho f\|_{WL_q(B(x_0,r))} \lesssim \|f\|_{L_p(B(x_0,2r))} + r^{\frac{n}{q}} \sup_{t>r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}}$$

holds for any ball $B(x_0, r)$ and for all $f_p^{\mathrm{loc}}(\mathbb{R}^n)$.

If $p > 1$, then the inequality

$$\|M_\rho f\|_{L_q(B(x_0,r))} \lesssim \|f\|_{L_p(B(x_0,2r))} + r^{\frac{n}{q}} \sup_{t>r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}} \tag{3.6}$$

holds for any ball $B(x_0, r)$ and for all $f_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $1 \le p < q < \infty$ and let $\rho(t)$ satisfy the conditions (3.1), (3.2) and (3.4). For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Hence

$$\|M_\rho f\|_{WL_q(B)} \le \|M_\rho f_1\|_{WL_q(B)} + \|M_\rho f_2\|_{WL_q(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $M_\rho f_1 \in WL_q(\mathbb{R}^n)$ and by Lemma 3.4 M_ρ is bounded from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$. Thus it follows that

$$\|M_\rho f_1\|_{WL_q(B)} \le \|M_\rho f_1\|_{WL_q(\mathbb{R}^n)} \le C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

Let x be an arbitrary point from B. If $B(x,t) \cap {}^{\complement}(2B) \ne \emptyset$, then $t > r$. Indeed, if $y \in B(x,t) \cap {}^{\complement}(2B)$, then $t > |x-y| \ge |x_0 - y| - |x_0 - x| > 2r - r = r$.

On the other hand, $B(x,t) \cap {}^{\complement}(2B) \subset B(x_0, 2t)$. Indeed, $y \in B(x,t) \cap {}^{\complement}(2B)$, then we get $|x_0 - y| \le |x-y| + |x_0 - x| < t + r < 2t$.

Hence

$$\begin{aligned} M_\rho f_2(x) &= \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)\cap {}^{\complement}(2B)} |f(y)|dy \\ &\lesssim \sup_{t>r} \frac{\rho(2t)}{(2t)^n} \int_{B(x_0,2t)} |f(y)|dy \\ &= \sup_{t>2r} \frac{\rho(t)}{t^n} \int_{B(x_0,t)} |f(y)|dy. \end{aligned}$$

Therefore, for all $x \in B$ we have

$$M_\rho f_2(x) \lesssim \sup_{t>2r} \frac{\rho(t)}{t^n} \int_{B(x_0,t)} |f(y)|dy. \tag{3.7}$$

Thus

$$\begin{aligned} \|M_\rho f\|_{WL_q(B)} \le \|M_\rho f\|_{L_q(B)} &\lesssim \|f\|_{L_p(2B)} + |B|^{\frac{1}{q}} \sup_{t>2r} \|f\|_{L_1(B(x_0,t))} \frac{\rho(t)}{t^n} \\ &\lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{q}} \sup_{t>2r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}}. \end{aligned} \tag{3.8}$$

Let $p > 1$. From the (p,q) boundedness of M_ρ and (3.4) it follows that:

$$\|M_\rho f_1\|_{L_q(B)} \le \|M_\rho f_1\|_{L_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_p(\mathbb{R}^n)} = \|f\|_{L_p(2B)}. \tag{3.9}$$

Then by (3.8) and (3.9) we get the inequality (3.6). $\square$

The following theorem is one of the main results of the paper in which we get the Spanne type boundedness of the generalized fractional maximal operator M_ρ in the generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$.

Theorem 3.1. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < q < \infty$, and let the function ρ satisfy the conditions* (3.1), (3.2) *and* (3.4). *Let also (φ_1, φ_2) satisfy the conditions*

$$\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}} \le C \, \varphi_2\big(x_0, \frac{t}{2}\big) \, t^{\frac{n}{q}}, \tag{3.10}$$

$$\sup_{t>r} \Big(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}} \Big) \frac{\rho(t)}{t^{\frac{n}{p}}} \le C \, \varphi_2(x_0, r),$$

where C does not depend on x_0 and r. Then the operator M_ρ is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $WLM^{\{x_0\}}_{q,\varphi_2}$ and for $p > 1$ from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$. Moreover,

$$\|M_\rho f\|_{WLM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}},$$

and for $p > 1$

$$\|M_\rho f\|_{LM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.$$

Proof. Let the function ρ satisfy the conditions (3.1), (3.2), (3.4), and also (φ_1, φ_2) satisfy the conditions (3.10) and (3.11). By Lemmas 2.1, 2.2 and 3.5 we have

$$\begin{aligned}
\|M_\rho f\|_{WLM^{\{x_0\}}_{q,\varphi_2}} &\lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} r^{-\frac{n}{q}} \|f\|_{L_p(B(x_0,2r))} \\
&\quad + \sup_{r>0} \varphi_2(x_0, r)^{-1} \sup_{t>r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}} \\
&\approx \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}
\end{aligned}$$

and for $p > 1$

$$\begin{aligned}
\|M_\rho f\|_{LM^{\{x_0\}}_{q,\varphi_2}} &\lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} r^{-\frac{n}{q}} \|f\|_{L_p(B(x_0,2r))} \\
&\quad + \sup_{r>0} \varphi_2(x_0, r)^{-1} \sup_{t>r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}} \\
&\approx \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.
\end{aligned}$$

□

In the following corollary we get the boundedness of the generalized fractional maximal operator M_ρ on generalized Morrey spaces $M_{p,\varphi}$.

Corollary 3.1. *Let $1 \le p < q < \infty$, the function ρ satisfy the conditions* (3.1), (3.2) *and* (3.4). *Let also (φ_1, φ_2) satisfy the following conditions*

$$\operatorname*{ess\,inf}_{r<t<\infty} \varphi_1(x, t) t^{\frac{n}{p}} \le C \, \varphi_2\big(x, \frac{r}{2}\big) \, r^{\frac{n}{q}},$$

$$\sup_{t>r} \Big(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s) s^{\frac{n}{p}} \Big) \frac{\rho(t)}{t^{\frac{n}{p}}} \le C \, \varphi_2(x, r),$$

where C does not depend on x and r. Then the operator M_ρ is bounded from M_{p,φ_1} to WM_{q,φ_2} and for $p > 1$ from M_{p,φ_1} to M_{q,φ_2}.

In the case $\rho(t) = t^\alpha$ from Theorem 3.1 we get new Spanne type result for fractional maximal operator M_α on generalized local Morrey spaces.

Corollary 3.2. *Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 \le p < q < \infty$ and $1/p - 1/q = \alpha/n$. Let also (φ_1, φ_2) satisfy the condition*

$$\sup_{t>r} \Big(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}} \Big) t^{-\frac{n}{q}} \le C\, \varphi_2(x_0, r), \tag{3.11}$$

where C does not depend on r. Then the operator M_α is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{q,\varphi_2}$ for $p = 1$.

Also in the case $\rho(t) = t^\alpha$ and $\varphi(x,t) = t^{\frac{\lambda-n}{p}}$, $0 < \lambda < n$ from Theorem 3.1 we get local Morrey space variant of Theorem A.

Corollary 3.3. *Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\alpha = \frac{n}{p} - \frac{n}{q}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$, the operator M_α is bounded from $LM^{\{x_0\}}_{p,\lambda}$ to $LM^{\{x_0\}}_{q,\mu}$ and for $p = 1$, M_α is bounded from $LM^{\{x_0\}}_{1,\lambda}$ to $WLM^{\{x_0\}}_{q,\mu}$.*

Remark 3.2. *For this case $\alpha = \frac{n}{p} - \frac{n}{q}$ necessary and sufficient conditions for the boundedness of I_α from M_{p,φ_1} to M_{q,φ_2} are obtained in* [4].

4. Adams type result for the operator M_ρ in the spaces $M_{p,\varphi}$

The following theorem was proved in [2].

Theorem D. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{r<t<\infty} t^{-\frac{n}{p}} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, s^{\frac{n}{p}} \le C\, \varphi_2(x,r),$$

where C does not depend on x and r. Then the operator M is bounded from M_{p,φ_1} to WM_{p,φ_2} and for $p > 1$, the operator M is bounded from M_{p,φ_1} to M_{p,φ_2}.

The following theorem is another main result of the paper, in which we get the Adams type boundedness of the generalized fractional maximal operator M_ρ in the generalized Morrey spaces $M_{p,\varphi}$.

Theorem 4.2. *Let $1 \le p < q < \infty$, $\frac{\rho(t)}{t^n}$ be almost decreasing, and let $\rho(t)$ satisfy the condition* (3.2) *and the inequality*

$$\int_0^{k_2 r} \frac{\rho(s)}{s} ds \le C\rho(r),$$

where k_2 is given by the condition (3.2) *and C does not depend on $r > 0$. Let also $\varphi(x,t)$ satisfy the conditions*

$$\sup_{r<t<\infty} t^{-n} \operatorname*{ess\,inf}_{t<s<\infty} \varphi(x,s)\, s^n \le C\, \varphi(x,r), \tag{4.1}$$

and

$$\rho(r)\varphi(x,r)+\sup_{t>r}\rho(t)\varphi(x,t)\le C\varphi(x,r)^{\frac{p}{q}}, \tag{4.2}$$

where C does not depend on $x\in\mathbb{R}^n$ and $r>0$.

Then the operator M_ρ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ and for $p>1$ from $M_{p,\varphi}$ to $M_{q,\varphi^{\frac{1}{q}}}$.

Proof. Let $x_0\in\mathbb{R}^n$, $1\le p<q<\infty$ and $f\in M_{p,\varphi^{\frac{1}{p}}}$. Write $f=f_1+f_2$, where $B=B(x,r)$, $f_1=f\chi_{2B}$ and $f_2=f\chi_{\complement_{(2B)}}$. Then we have

$$M_\rho f(x)\le M_\rho f_1(x)+M_\rho f_2(x).$$

For $M_\rho f_1(y)$, $y\in B(x,r)$, following Hedberg's trick (see for instance [27], p. 354), we obtain

$$\begin{aligned}
M_\rho f_1(y)&=\sup_{t>0}\frac{\rho(t)}{t^n}\int_{B(y,t)\cap B(x,2r)}|f(z)|dz\\
&\lesssim\sup_{t>0}\int_{B(y,t)\cap B(x,2r)}\frac{\rho(|y-z|)}{|y-z|^n}|f(z)|dz\\
&\approx\sup_{t>0}\sum_{k=-\infty}^{0}\int_{B(y,t)\cap\left(B(x,2^{k+1}r)\setminus B(x,2^kr)\right)}\frac{\rho(|y-z|)}{|y-z|^n}|f(z)|dz\\
&\lesssim\sup_{t>0}\sum_{k=-\infty}^{0}\int_{2^kk_1r}^{2^kk_2r}\frac{\rho(s)}{s^{n+1}}ds\int_{B(y,t)\cap B(x,2^{k+1}r)}|f(z)|dz\\
&\approx Mf(x)\sup_{t>0}\sum_{k=-\infty}^{0}\int_{2^kk_1r}^{2^kk_2r}\frac{\rho(s)}{s}ds\\
&=Mf(x)\int_0^{k_2r}\frac{\rho(s)}{s}ds\lesssim Mf(x)\rho(r).
\end{aligned}\tag{4.3}$$

For $M_\rho f_2(y)$, $y\in B(x,r)$ from (3.7) we have

$$\begin{aligned}
M_\rho f_2(y)&\lesssim\sup_{t>2r}\frac{\rho(t)}{t^n}\int_{B(x,t)}|f(z)|dz\\
&\lesssim\sup_{t>2r}\|f\|_{L_p(B(x,t))}\frac{\rho(t)}{t^{\frac{n}{p}}}.
\end{aligned}\tag{4.4}$$

Then from condition (4.2) and inequalities (4.3), (4.4) for all $y\in B(x,r)$ we get

$$\begin{aligned}
M_\rho f(y)&\lesssim\rho(r)\,Mf(x)+\sup_{t>r}\|f\|_{L_p(B(x,t))}\frac{\rho(t)}{t^{\frac{n}{p}}}\\
&\le\rho(r)\,Mf(x)+\|f\|_{M_{p,\varphi^{\frac{1}{p}}}}\sup_{t>r}\varphi(x,t)\rho(t).
\end{aligned}\tag{4.5}$$

Thus, by (4.2) and (4.5) we obtain

$$M_\rho f(y) \lesssim \min\left\{\varphi(x,t)^{\frac{p}{q}-1} Mf(x), \varphi(x,t)^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}\right\}$$

$$\lesssim \sup_{s>0} \min\left\{s^{\frac{p}{q}-1} Mf(x), s^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}\right\} = (Mf(x))^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}},$$

where we have used that the supremum is achieved when the minimum parts are balanced. Hence for all $y \in B(x,r)$, we have

$$M_\rho f(y) \lesssim (Mf(x))^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}}.$$

Consequently the statement of the theorem follows in view of the boundedness of the maximal operator M in $M_{p,\varphi^{\frac{1}{p}}}$ provided by Theorem D in virtue of condition (4.1).

$$\begin{aligned}
\|M_\rho f\|_{WM_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|M_\rho f\|_{WL_q(B(x,t))} \\
&\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|_{WL_p(B(x,t))}^{\frac{p}{q}} \\
&= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{n}{p}} \|Mf\|_{WL_p(B(x,t))}\right)^{\frac{p}{q}} \\
&= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \|Mf\|_{WM_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}} \\
&\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}},
\end{aligned}$$

and

$$\begin{aligned}
\|M_\rho f\|_{M_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|M_\rho f\|_{L_q(B(x,t))} \\
&\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|_{L_p(B(x,t))}^{\frac{p}{q}} \\
&= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{n}{p}} \|Mf\|_{L_p(B(x,t))}\right)^{\frac{p}{q}} \\
&= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \|Mf\|_{M_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}} \\
&\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}},
\end{aligned}$$

if $1 < p < q < \infty$. □

In the case $\rho(t) = t^\alpha$ from Theorem 4.2 we get the Adams type result on generalized Morrey spaces (see [16, Theorem 5.7, p. 182]).

In the case $\rho(t) = t^\alpha$, $\varphi(x,t) = t^{\lambda-n}$, $0 < \lambda < n$ from Theorem 4.2 we get the following Adams's result for the fractional maximal operator.

Corollary 4.4. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then for $p > 1$, the operator M_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1$, M_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Remark 4.3. *Note that, the condition* (3.1) *is weaker than the following condition which was given in* [17] *for I_ρ:*

$$\int_1^\infty \frac{\rho(t)}{t^n}\frac{dt}{t} < \infty. \tag{4.6}$$

For example, the function

$$\rho(t) = \frac{t^n}{\log(e+t)},\ t > 0$$

satisfies (3.1)*, but not* (4.6)*. This example shows that the function ρ satisfies Theorems 3.1 and 4.2, but does not satisfy the assumptions of Theorems 16 and 22 in* [17]*. In other words, the condition (3.1) which satisfies our main theorems, is better (more general and comprehensive) than the condition (4.8) which satisfies the main theorems were given in [17].*

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SPECTRAL PROPERTIES OF THE SECOND ORDER DIFFERENCE EQUATION WITH SELFADJOINT OPERATOR COEFFICIENTS

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Abstract. In this paper, we consider the second order difference equation defined on the whole axis with selfadjoint operator coefficients. The main objective of this study is to obtain the continuous and discrete spectrum of the discrete operator which is generated by this difference equation. To achieve this, we first obtain the Jost solutions of this equation explicitly and then examine the analytical and asymptotic properties of these solutions. With the help of these properties, we find the continuous and discrete spectrum of this operator. Finally we obtain a sufficient condition which ensures that this operator has a finite number of eigenvalues.

1. Introduction

Difference equations play a very important role on modelling of problems related to physics, chemistry, biology, finance, economics, probability, engineering etc. Difference equations also arise when approximating continuous models and differential equations using numerical methods. Selfadjoint differential operators such as Sturm-Liouville, Dirac and Klein-Gordon operators are used in functional analysis and quantum mechanics and the spectral analysis of these operators have been studied (see [14, 17, 18]). There are also many studies on the spectral analysis of both selfadjoint and non-selfadjoint discrete operators defined by difference equations (see [1, 2, 3] and references therein). Besides, spectral analysis of the selfadjoint differential and difference equations with matrix coefficients are studied in [6, 8, 10]. In particular, in [4] the authors investigated the spectral properties of the discrete operator generated by selfadjoint matrix-valued difference equation of second order defined on the half-axis. Namely, they considered the discrete operator

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L_0 generated by the difference equation with matrix coefficients

$$A_{n-1}Y_{n-1}+B_nY_n+A_nY_{n+1}=\lambda Y_n, n\in\mathbb{N}, \tag{1}$$

and the boundary condition $Y_0=0$, where $A_n\,(n\in\mathbb{N}\cup\{0\})$ and $B_n\,(n\in\mathbb{N})$ are $m\times m$ selfadjoint matrices $(m<\infty)$, $detA_n\neq 0\,(n\in\mathbb{N}\cup\{0\})$ and λ is a spectral parameter. The domain of this operator is denoted by $l_2\left(\mathbb{N},\mathbb{C}^m\right)$ which is the Hilbert space of all vector sequences $Y=\left(Y_n\right)_{n\in\mathbb{N}}$ such that $Y_n\in\mathbb{C}^m$ and $\sum_{n=1}^{\infty}\|Y_n\|^2<\infty$. The inner product in $l_2\left(\mathbb{N},\mathbb{C}^m\right)$ is defined as

$$(Y,Z):=\sum_{n=1}^{\infty}\left(Y_n,Z_n\right).$$

Note that Equation (1) can be written in Sturm-Liouville form

$$\Delta\left(A_{n-1}\Delta Y_{n-1}\right)+Q_nY_n=\lambda Y_n, n\in\mathbb{N},$$

where $Q_n=A_{n-1}+A_n+B_n$ and Δ is the forward difference operator. The authors obtained the continuous and discrete spectrum of L_0 [4]. Further, in [7] the authors considered the same difference equation with non-selfadjoint matrix coefficients and examined the continuos spectrum, eigenvalues and spectral singularities of the resulting non-selfadjoint discrete operator. They proved the finiteness of the eigenvalues and spectral singularities of the operator under the condition

$$\sum_{n=1}^{\infty}n\left(\|I-A_n\|+\|B_n\|\right)<\infty.$$

Furthermore, in [5] the authors extended the results in [4] to the whole axis by considering the Equation (1) for $n\in\mathbb{Z}$. They obtained the Jost solutions of this equation and also the discrete and continuous spectrum of the discrete operator generated by this equation. They proved that the operator has a finite number of eigenvalues and spectral singularities if the coefficients satisfy

$$\sum_{n=-\infty}^{\infty}|n|\left(\|I-A_n\|+\|B_n\|\right)<\infty.$$

Let H be a separable Hilbert space $(\dim H\leq\infty)$ and $L_2\left(\mathbb{R}_+,H\right)$ denote the space of vector-valued functions $f(x)$ $(0\leq x<\infty)$ which are strongly-integrable in each finite subinterval of $[0,\infty)$ such that $\int_0^{\infty}|f(x)|^2\,dx<\infty$. Consider the differential expression in $L_2\left(\mathbb{R}_+,H\right)$

$$l_0(Y)=-Y^{''}+Q(x)Y,\ 0<x<\infty, \tag{2}$$

where $Q(x)$ is a selfadjoint, completely continuous operator in H for each $x\in(0,\infty)$. In [9, 12, 13, 15], the authors have studied the discrete spectrum of the Sturm-Liouville operator l_0 generated by (2) and the boundary condition $Y\left(0\right)=0$.

In this paper, we consider the discrete analogue of the operator l_0 and call it the discrete Sturm-Liouville operator which will be denoted by L hereafter. We investigate the spectral properties of the discrete Sturm-Liouville operator L on the whole axis with selfadjoint operator coefficients. In particular, we find Jost solutions of L and obtain the continuous and point spectrum of L. We also show that L has a finite number of eigenvalues under a condition on the coefficients.

2. Some properties and Jost solutions of the operator L

In this section we specify the properties of the discrete Sturm-Liouville operator on the whole axis. Let H be a separable Hilbert space and $H_1 = l_2(\mathbb{N}, H)$ denote the space of vector sequences $y = (y_n)_{n\in\mathbb{N}}$ $(y_n \in H,\ n \in \mathbb{N})$ such that $\|y\|_1 := \sum\limits_{n=-\infty}^{\infty} \|y_n\|_H^2 < \infty$. H_1 is a Hilbert space with inner product

$$(y, z)_1 = \sum_{n=-\infty}^{\infty} (y_n, z_n)_H \,.$$

Consider the difference expression in H_1

$$l(y)_n = A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1},\ n \in \mathbb{Z}, \tag{3}$$

where $A_n, B_n\,(n \in \mathbb{Z})$ are selfadjoint operators in H and $A_n - I, B_n\,(n \in \mathbb{Z})$ are completely continuous operators in H. We also assume A_n is invertible for each $n \in \mathbb{Z}$. We consider the operator L generated by (3). We can also define the operator L by using the infinite Jacobi matrix

$$(J)_{ij} = \begin{cases} B_i, & i = j, \\ A_{i-1}, & i = j+1, \\ A_i, & i = j-1, \\ 0, & \text{otherwise} \end{cases}$$

It is obvious that the operator L is selfadjoint in H_1. We will examine the difference equation

$$A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N}. \tag{4}$$

We shall also consider the equation

$$A_{n-1}Y_{n-1} + B_n Y_n + A_n Y_{n+1} = \lambda Y_n,\ n \in \mathbb{N}, \tag{5}$$

where Y_n is an operator sequence i.e, Y_n is an operator in H for each $n \in \mathbb{N}$.

Lemma 1. *Every sequence of solutions of (4) can be represented as an operator sequence which satisfies (5). Conversely, one can construct a sequence of vector sequences which satisfies (4) for a given operator solution of (5).*

Proof. Since H is a separable Hilbert space, there exists an orthonormal basis $(u_m)_{m\in\mathbb{N}}$. Suppose vector sequences $y_m = \left(y_m^i\right)_{i\in\mathbb{Z}}$ satisfy Equation (4) for each $m \in \mathbb{N}$. We can construct an operator sequence $Y = (Y_n)_{n\in\mathbb{Z}}$ such that $Y_n u_m = (y_m^n)_{n\in\mathbb{Z}}$ for every $m \in \mathbb{N}$. It is obvious that $Y_n u_m = y_m$ and Y satisfies the Equation (5).

Conversely, suppose an operator sequence $Y = (Y_n)_{n\in\mathbb{Z}}$ satisfies (5). Let $z_m := (z_m^n)_{n\in\mathbb{Z}} = Y_n u_m$ for every $m \in \mathbb{N}$. Then it is clear that $z_m = (z_m^n)_{n\in\mathbb{Z}}$ satisfy Equation (4) for every $m \in \mathbb{N}$. $\square$

Note that from Lemma 1, we have one-to-one correspondence between the operator solutions of Equation (5) and sequences of solutions of Equation (4). Hence we can consider and examine both equations.
Let us assume

$$\sum_{n=-\infty}^{\infty} \left(\|I - A_n\| + \|B_n\|\right) < \infty. \tag{6}$$

Let $E(z) := (E_n(z))_{n\in\mathbb{Z}}$ and $F(z) := (F_n(z))_{n\in\mathbb{Z}}$ denote the operator solutions of the equation

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = \left(z + \frac{1}{z}\right)Y_n,\ n \in \mathbb{Z}, \tag{7}$$

satisfying the conditions

$$\lim_{n\to\infty} E_n(z)z^{-n} = I,\ z \in D_0 := \{z \in \mathbb{C} :\ |z| = 1\},$$

and

$$\lim_{n\to\infty} F_n(z)z^{n} = I,\ z \in D_0,$$

respectively. $E(z)$ and $F(z)$ are called the Jost solutions of Equation (7). Note that these solutions are bounded.

Theorem 2. *Under the condition (6), the solutions $E(z)$ and $F(z)$ exist and have the representations*

$$\begin{aligned} E_n(z) &= z^nI + \\ &\sum_{k=n+1}^{\infty} \frac{z^{k-n} - z^{n-k}}{z - z^{-1}} \left[(I - A_{k-1})\, E_{k-1}(z) - B_kE_k(z) + (I - A_k)\, E_{k+1}(z)\right], \end{aligned}$$

$$\begin{aligned} F_n(z) &= z^{-n}I + \\ &\sum_{k=-n+1}^{\infty} \frac{z^{k+n} - z^{-n-k}}{z - z^{-1}} \left[(I - A_{k-1})\, F_{k-1}(z) - B_kF_k(z) + (I - A_k)\, F_{k+1}(z)\right]. \end{aligned}$$

Now, suppose that

$$\sum_{n=-\infty}^{\infty} |n| \left(\|I - A_n\| + \|B_n\|\right) < \infty, \tag{8}$$

holds.

Theorem 3. *Under the condition (8), the Jost solutions* $(E_n(z)), (F_n(z))$ $(n \in \mathbb{Z})$ *have the represantations* ********************

$$E_n(z) = T_n z^n \left[I + \sum_{m=1}^{\infty} K_{n,m} z^m \right], \quad n \in \mathbb{Z},$$
$$F_n(z) = R_n z^{-n} \left[I + \sum_{-\infty}^{m=-1} L_{n,m} z^{-m} \right], \quad n \in \mathbb{Z},$$

where $T_n, R_n, K_{n,m}$ *and* $L_{n,m}$ *are obtained in terms of* A_n *and* B_n. *Further*

$$\|K_{n,m}\| \le c \sum_{p=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(\|I - A_p\| + \|B_p\|\right), m \in \mathbb{Z}_+,$$
$$\|L_{n,m}\| \le d \sum_{-\infty}^{p=n+\left[\left|\frac{m}{2}\right|\right]} \left(\|I - A_p\| + \|B_p\|\right), m \in \mathbb{Z}_-,$$

hold where $c, d > 0$ *are constants. Thus,* $(E_n(z))$ *and* $(F_n(z))$ *have analytic continuations from* D_0 *to* $D_1 := \{z \in \mathbb{C} : |z| < 1\} \setminus \{0\}$.

Theorem 4. *Under the condition (8), the Jost solutions satisfy the following asymptotic relations for* $z \in D := \{z \in \mathbb{C} : |z| \le 1\} \setminus \{0\}$

$$E_n(z) = z^n \left[I + o(1)\right], \; n \to \infty,$$
$$F_n(z) = z^{-n} \left[I + o(1)\right], \; n \to -\infty.$$

Remark 5. *The proofs of above theorems are omitted since they are similar to the matrix coefficient case which have been obtained in* [4, 5].

3. Continuous and discrete spectrum of L

Let us introduce the equation

$$Y_{n-1}A_{n-1} + Y_n B_n + Y_{n+1} A_n = \left(z + z^{-1}\right) Y_n, \; n \in \mathbb{N}. \tag{9}$$

It can be shown similarly that equation (9) has a solution $H(z) := (H_n(z))_{n \in \mathbb{Z}}$ such that

$$\lim_{n \to \infty} H_n(z) z^n = I, \; z \in D_0$$

holds. Indeed, the solution $H(z)$ is the adjoint of the operator solution $F(z)$ i.e., $H_n(z) = (F_n(z))^*, \; n \in \mathbb{Z}$.

Definition 6. *Let* U_n *and* V_n *be operator solutions of the Equations (5) and (9), respectively. The Wronskian of* U_n *and* V_n *is defined by*

$$(W[U, V])_n := V_{n-1} A_{n-1} U_n - V_n A_{n-1} U_{n-1}.$$

Lemma 7. *Let* U_n *be an operator solution of (7) and* V_n *be an operator solution of (9). Then, the Wronskian of these solutions is constant i.e., independent of* n.

Proof. We have the equalities

$$A_{n-1}U_{n-1} + B_nU_n + A_nU_{n+1} = \left(z + z^{-1}\right)U_n,$$

$$V_{n-1}A_{n-1} + V_nB_n + V_{n+1}A_n = \left(z + z^{-1}\right)V_n.$$

If we multiply the first equality with V_n from the left and the second equality with $-U_n$ from the right we get

$$V_nA_{n-1}U_{n-1} - V_{n-1}A_{n-1}U_n + V_nA_nU_{n+1} - V_{n+1}A_nU_n = 0, \tag{10}$$

by adding two equalities. Let $H_n := (W[U,V])_n = V_{n-1}A_{n-1}U_n - V_nA_{n-1}U_{n-1}$. From (10), we have

$$\Delta H_n = H_{n+1} - H_n = V_nA_nU_{n+1} - V_{n+1}A_nU_n - V_{n-1}A_{n-1}U_n + V_nA_{n-1}U_{n-1} = 0,$$

which implies $W[U,V]$ is constant. □

From Lemma 7 it easily follows that

$$W[E(z), H(z)] = G_0(z)A_0E_1(z) - G_1(z)A_0E_0(z).$$

Let us define $T(z) := W[E(z), H(z)]$ for $z \in D$. $T(z)$ is called the Jost function of L. Now we obtain the continuous spectrum of L.

Theorem 8. *Under the condition (8), the continuous spectrum of L is $\sigma_c(L) = [-2,2]$.*

Proof. Let L_0 and L_1 denote the operators generated in $H_1 = l_2(\mathbb{Z}, H)$ by the difference expressions

$$L_0(y)_n = y_{n-1} + y_{n+1},\ n \in \mathbb{Z},$$

and

$$L_1(y)_n = (A_{n-1} - I)\,y_{n-1} + B_ny_n + (A_n - I)\,y_{n+1},\ n \in \mathbb{Z},$$

respectively. We can also define the operators L_0 and L_1 by using the infinite Jacobi matrices

$$(J_0)_{ij} = \begin{cases} I, & i = j+1 \ \mathit{or}\ i = j-1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$(J_1)_{ij} = \begin{cases} B_i, & i = j, \\ A_i - I, & i = j-1, \\ A_{i-1} - I, & i = j+1, \\ 0, & \text{otherwise} \end{cases}$$

respectively. We have $L = L_0 + L_1, L_0 = L_0^*$ and $\sigma(L_0) = \sigma_c(L_0) = [-2,2]$ (see [19]). It is well known that L_1 is a compact operator iff L_1 is bounded and the set $R = \{L_1y : \|y\|_1 \leq 1\}$ is compact in H_1. It is obvious that L_1 is bounded. Moreover, if we use the compactness criteria in l_p spaces (see [16] (p. 167)) we obtain the compactness of R. Indeed, let $\|y\|_1 \leq 1$. Then (8) implies that for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\sum_{i=n+1}^{\infty} \left(\|(A_i - I)\| + \|B_i\|\right) < \frac{\epsilon}{C}.$$

Now we have

$$\begin{aligned}
\sum_{i=n+1}^{\infty} \|(L_1 y)_i\|_H^2 &= \sum_{i=n+1}^{\infty} \|(A_{i-1} - I)\, y_{i-1} + B_i y_i + (A_i - I)\, y_{i+1}\|_H^2 \\
&\leq \sum_{i=n+1}^{\infty} \left(\|(A_{i-1} - I)\|^2 \|y_{i-1}\|_H^2 + \|B_i\|^2 \|y_i\|_H^2 + \|(A_i - I)\|^2 \|y_{i+1}\|^2\right) \\
&\leq \|y\|_1^2 \sum_{i=n+1}^{\infty} \left(\|(A_{i-1} - I)\|^2 + \|B_i\|^2 + \|(A_i - I)\|^2\right) \\
&\leq \sum_{i=n+1}^{\infty} \left(2\,\|(A_i - I)\|^2 + \|B_i\|^2\right) \\
&\leq \sum_{i=n+1}^{\infty} \left(C_1\,\|(A_i - I)\| + C_2\,\|B_i\|\right) \\
&\leq \sum_{i=n+1}^{\infty} C\left(\|(A_i - I)\| + \|B_i\|\right) \\
&< \varepsilon,
\end{aligned}$$

where

$$C_1 = \frac{1}{2}\sup_{i\in\mathbb{N}} \|(A_i - I)\|, \quad C_2 = \sup_{i\in\mathbb{N}} \|B_i\|$$

and $C = C_1 + C_2$. Thus, we proved L_1 is a compact operator in H_1. By Weyl Theorem of Compact Perturbation [11], we have

$$\sigma_c(L) = \sigma_c(L_0) = [-2, 2]\,.$$

□

Since the operator L is selfadjoint, all eigenvalues of L are real. Note that from the definition of discrete spectrum and Theorem 8 we have

$$\sigma_d(L) \subset (-\infty, -2] \cup [2, \infty)\,. \tag{11}$$

Further, from the definition of eigenvalues we find

$$\sigma_d(L) = \left\{\lambda : \ \lambda = z + \frac{1}{z},\ z \in (-1, 0) \cup (0, 1)\,,\ T(z) \text{ is not invertible}\right\}.$$

Theorem 9. *Under the condition (8), L has a finite number of eigenvalues.*

Proof. From (11), it follows that the limit points of the set $\sigma_d(L)$ could only be $\pm 2, \pm\infty$. If $\lambda = \pm\infty$ is a limit point of $\sigma_d(L)$ then it implies that L is unbounded operator which gives a contradiction. On the other hand, if $\lambda = 2$ is a limit point of $\sigma_d(L)$ then there exists an eigenvalue in the neighbourhood $[2-\varepsilon, 2)$ for sufficiently small $\varepsilon > 0$. From Theorem 8 we have $\sigma_c(L) = [-2, 2]$ and it is well known that for a selfadjoint operator $\sigma_d(L) \nsubseteq \sigma_c(L)$. Hence there can't be any eigenvalue in $[2-\varepsilon, 2)$ which means $\lambda = 2$ is not a limit point of $\sigma_d(L)$. Similarly, $\lambda = -2$ can not be a limit point of $\sigma_d(L)$. As a result, the set of eigenvalues has no limit point and therefore should have a finite number of elements by Bolzano-Weierstrass Theorem. □

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COMPACTNESS AND STABILITY IN DIFRAMES

ESRA KORKMAZ AND RIZA ERTÜRK

Abstract. The concept of diframe was introduced as a generalization of ditopological texture spaces. The purpose of this paper is to present the results of a study on the concepts of compactness and stability in the setting of diframes. Further, the bitopological concepts of locally compactness and locally stability are extended to diframes.

1. Introduction

The theory of bitopological spaces is based on the notion of open sets, and the closed sets can be obtained easily by using the set complementation. As distinct from bitopologies, a ditopological texture space is defined on a suitable subfamily of subsets, which is not necessarily complemented. It can be considered as a structure in which the open and closed sets play an equal role. Diframes were defined in [1] as a generalization of ditopological texture spaces. Briefly, it is a 3-tuple $L = (L_e, L_{fr}, L_{cf})$, where L_e is both a frame and a coframe, $L_{fr} \subseteq L_e$ is a subset closed under arbitrary joins and finite meets and $L_{cf} \subseteq L_e$ is a subset closed under arbitrary meets and finite joins. As is well known, point-free topology has a wide range of applications, including logic, topos theory and theoretical computer science. The motivation behind the notion of diframe is to provide a point-free perspective on the theory of ditopological texture spaces. We obtained a larger family of lattices by weakening the property of complete distributivity. This paper is self-contained but may also be considered as a continuation of the article [2], in which we developed the diframe versions of the separation axioms and relations between these axioms. In this study, we are interested in the notions of compactness, stability, local compactness, local stability and their duals in diframes.

The present paper is divided into 5 sections. In Section 2, we present some necessary preliminaries including the concept of compactness in ditopological texture spaces and the separation axioms in diframes. Section 3 is devoted to the study

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of compactness and stability in diframes. The questions of whether these properties are hereditary, and whether they are preserved by any reasonable kind of homomorphisms are discussed. As will be seen in the sequel, stability is a property relating the frame L_{fr} and the coframe L_{cf}. Hence we replace compactness by stability to obtain diframe versions of topological results relating separation axioms and compactness. In this section, we also give a generalization of Alexander subbase theorem. In section 4, we define the concepts of locally compactness and locally stability in terms of suitable binary relations. For bitopological versions of these concepts, we refer the reader to the comprehensive paper of Kopperman [3]. As expected, the approach of Kopperman is based on the notion of neighbourhood and hence it is dependent on points. Some of our results are parallel to those in [3] but sometimes we need to impose some extra conditions. Finally, in Section 5, we conclude the paper and discuss our future work.

2. Preliminaries

In this section, we briefly recall some definitions and results of ditopological texture spaces, (co)frames and diframes which will be used throughout the paper. We refer the reader to [4, 10] for details concerning lattice and frame theory and [6, 7, 8] for details concerning ditopological texture spaces.

Ditopological Texture Spaces: Let S be a set and S be a subset of the powerset $\mathcal{P}(S)$ with the following properties:

(1) $(S, \subseteq)$ is a complete, completely distributive lattice containing S as a top element and $\emptyset$ as a bottom element.
(2) S is point separating.
(3) Arbitrary meet coincides with intersection and finite joins coincide with union in this lattice.

The pair (S, S) is known as a *texture space.*

A *dichotomous topology*, (briefly, *ditopology*) on a texture (S, S) is a pair (τ, κ) of generally unrelated subsets of S satisfying

(T_1) $S, \emptyset \in \tau$,
(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$,
(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$,
(CT_1) $S, \emptyset \in \kappa$,
(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$,
(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

Loosely speaking, a ditopology is a structure in which the open and closed sets play an equal role.

Galois Adjunctions: A pair of monotone functions $f : L \to M$, $g : M \to L$ between partially ordered sets is called Galois adjoint if the following condition is satisfied for all $x \in L$ and $y \in M$: $f(x) \leq y \Leftrightarrow x \leq g(y)$. This fact is referred to by saying that f is a left adjoint to g, or g is a right adjoint to f. Our notation for the adjoints is that of [10], that is, we will denote this adjunction by $f = g^*$ or $g = f_*$.

One can show that a suprema (resp., infima) preserving map between complete lattices has a right (resp., left) adjoint.

Let (f, g) be a Galois adjunction.

(1) If L and M are complete lattices, then f preserves finite joins and g preserves finite meets.
(2) f is one-one if and only if g is onto.
(3) If f is onto, then $fg = id$, and if f is one-one, then $gf = id$.

Frames and coFrames A frame (resp., a coframe) is a complete lattice with the property that binary meet (resp., join) distributes over arbitrary join (resp., meet) and a frame (resp., a coframe) homomorphism is a function between frames (resp., a coframes) preserving arbitrary joins (resp., meets) and finite meets (resp., joins).

Denote by **Frm** the category of frames, and by **Loc** its opposite category. The regular subobjects of objects of **Loc**, sublocales, have various kinds of characterizations. Here we just recall two of them that we shall exploit in the sequel.

Let L be a frame and let $S \subseteq L$ be a subset closed under arbitrary meets. Then S is called a *sublocale* of L provided that $(x \to s) \in S$ for all $s \in S$ and $x \in L$. Similarly, if M is a coframe and $S \subseteq M$ is a subset closed under arbitrary joins then S is called a *subcolocale* of M if $(s \leftarrow x) \in S$ for all $s \in S$ and $x \in M$. Here, "$\to$" and "$\leftarrow$" denote the Heyting and co- Heyting algebra operation, respectively.

A sublocale can also be represented by a nucleus which is a monotone, idempotent, inflationary map preserving finite meets. Note that these two characterizations of a sublocale are equivalent. According to [4], a sublocale S is said to be *flat* if it is closed under finite joins, or equivalently, if v_S preserves finite joins.

Dually, a conucleus $t : M \to M$ on a coframe M is a monotone, idempotent map preserving finite joins and satisfying $t(x) \leq x$ for all $x \in M$. One can easily show that for a given subcolocale $S \subseteq M$, $t_S(a) = \bigvee\{s \in S : s \leq a\}$ is a conucleus, and conversely, for every conucleus $t : M \to M$, $t(M)$ is a subcolocale.

Diframes: A *diframe* is a 3-tuple $L = (L_e, L_{fr}, L_{cf})$ with the following conditions:

(1) L_e is a complete lattice satisfying
$$x \wedge (\bigvee Y) = \bigvee\{x \wedge y : y \in Y\} \text{ and } x \vee (\bigwedge Y) = \bigwedge\{x \vee y : y \in Y\}$$
for any $x \in L_e$ and any subset $Y \subseteq L_e$.
(2) $L_{fr} \subseteq L_e$ is closed under arbitrary joins and finite meets.
(3) $L_{cf} \subseteq L_e$ is closed under arbitrary meets and finite joins.

Notice that L_e is both a frame and a coframe, L_{fr} is a frame, and L_{cf} is a coframe.

Example 2.1. Consider the family $\Omega_{\text{reg}}(\mathbb{R})$ of regular open sets of $(\mathbb{R}, \tau_s)$, where τ_s is the usual topology on $\mathbb{R}$. If $L_{fr} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ and $L_e = L_{cf} = \Omega_{\text{reg}}(\mathbb{R})$ then $L = (L_e, L_{fr}, L_{cf})$ is a diframe.

Let $\varphi : L_e \to M_e$ be a map preserving arbitrary join and finite meets and satisfying $\varphi[L_{fr}] \subseteq M_{fr}$, and let $\psi : L_e \to M_e$ be a map preserving arbitrary meets and finite joins and satisfying $\psi[L_{cf}] \subseteq M_{cf}$. Then the pair (φ, ψ) is called a *diframe homomorphism*.

Diframes and diframe homomorphisms form a category **diFrm**. The dual category of **diFrm** is denoted by **diLoc**, and the objects of **diLoc** are referred to as *dilocales*.

By a base of a diframe, we mean a subset $\beta \subseteq L_{fr}$ such that for every $a \in L_{fr}$ there exists a $\beta_a \subseteq \beta$ with $a = \bigvee \beta_a$. Dually, a *cobase* is a subset $\beta \subseteq L_{cf}$ such that every $k \in L_{cf}$ can be expressed as a meet of some elements of β.

A subset $\delta \subseteq L_{fr}$ (resp., $\delta \subseteq L_{cf}$) is called a *subbase* (resp., subcobase) of L if the set of finite meets (resp., joins) of δ is a base (resp., cobase) of L.

A diframe homomorphism $(\varphi, \psi) : (L_e, L_{fr}, L_{cf}) \to (M_e, M_{fr}, M_{cf})$ is called

(1) *onto* (resp., *one-one*) if both φ and ψ are onto (resp., one-one),
(2) *open* (resp., *co- open*) if $\psi^*(a) \in L_{fr}$ (resp., $\varphi_*(a) \in L_{fr}$) for all $a \in M_{fr}$,
(3) *closed* (resp., *co- closed*) if $\psi^*(k) \in L_{cf}$ (resp., $\varphi_*(k) \in L_{cf}$) for all $k \in M_{cf}$.

Let us recall the non-full subcategory **hdiFrm** of **diFrm** introduced in [1]. The objects of **hdiFrm** are diframes, and the morphisms are mappings $\varphi : (L_e, L_{fr}, L_{cf}) \to (M_e, M_{fr}, M_{cf})$ preserving arbitrary meets and joins, and satisfying the properties $\varphi[L_{fr}] \subseteq M_{fr}$, $\varphi[L_{cf}] \subseteq M_{cf}$.

If φ is one-one and onto then the concept of openness (resp., closedness) coincides with the concept of co- opennness (resp., co- closedness). A **hdiFrm** isomorphism is an open, closed, one-one and onto **hdiFrm** morphism.

Recall that by a subdilocale of a diframe L, we mean a triple $S = (S_e, S_{fr}, S_{cf})$ where $S_e \subseteq L_e$ is both a sublocale and a subcolocale of L_e, $S_{fr} = v_{S_e}(L_{fr}) \subseteq S_e$ and $S_{cf} = t_{S_e}(L_{cf}) \subseteq S_e$.

Note that $S_e \subseteq L_e$ is obviously a flat sublocale, and hence the nucleus v_{S_e} preserves finite joins. Similarly, by defining a co- flat subcolocale as subcolocale closed under finite meets, we obtain that the co- nucleus t_{S_e} preserves finite meets.

In a diframe $L = (L_e, L_{fr}, L_{cf})$, we have the *closure* and *interior* of $a \in L_e$ given by the formulas $[a] = \bigwedge\{c \in L_{cf} : a \leq c\}$ and $]a[= \bigvee\{b \in L_{fr} : b \leq a\}$, respectively.

Now, we briefly present the separation axioms in diframes. A comprehensive discussion on their basic properties, characterizations and the implications between them can be found in our previous work [2].

A diframe $L = (L_e, L_{fr}, L_{cf})$ is called

(1) T_0 if given $a \in L_e$, there exist $c_i^j \in L_{fr} \cup L_{cf}$, $i \in I$, $j \in J$ such that $a = \bigvee_{j \in J} \bigwedge_{i \in I} c_i^j$,
(2) *co-*T_0 if given $a \in L_e$, there exist $c_i^j \in L_{fr} \cup L_{cf}$, $i \in I$, $j \in J$ such that $a = \bigwedge_{j \in J} \bigvee_{i \in I} c_i^j$,
(3) R_0 if every element of L_{fr} can be written as a supremum of some elements of L_{cf},

(4) *co-*R_0 if every element of L_{cf} can be written as a infimum of some elements of L_{fr},
(5) R_1 if for all $a \in L_{fr}$, $a = \bigvee_{j\in J} \bigwedge_{i\in I} c_i^j = \bigvee_{j\in J} \bigwedge_{i\in I} [c_i^j]$ where $c_i^j \in L_{fr}$,
(6) *co-* R_1 if for all $a \in L_{cf}$, $a = \bigwedge_{j\in J} \bigvee_{i\in I} k_i^j = \bigwedge_{j\in J} \bigvee_{i\in I}]k_i^j[$ where $k_i^j \in L_{cf}$.

Recall the following relations defined on L_e. Let $D = \{k/2^n : k, n \in \mathbb{N}, k = 0, \dots 2^n\}$ denote the set of dyadic rationals.

(1) $a \prec_{fr}$ b, if $a, b \in L_{fr}$ and if there exists $c \in L_{cf}$ such that $a \leq c \leq b$.
(2) $f \prec_{cf} k$, if $f, k \in L_{cf}$ and if there exists $a \in L_{fr}$ such that $f \leq a \leq k$.
(3) $a \prec\!\!\prec_{fr} b$ if $a, b \in L_{fr}$ and if there exists $a_q \in L_{fr}$ with $q \in D$ and satisfying

$$a_0 = a,\ a_1 = b, \text{ and } a_q \prec_{fr} a_r \text{ if } q < r.$$

(4) $k \prec\!\!\prec_{cf} f$ if $k, f \in L_{cf}$ and if there exist $k_q \in L_{cf}$ with $q \in D$ and satisfying

$$k_0 = k,\ k_1 = f, \text{ and } k_q \prec_{cf} k_r \text{ if } q < r.$$

A diframe $L = (L_e, L_{fr}, L_{cf})$ is called

(1) *regular* if $a = \bigvee\{x \in L_{fr} : x \prec_{fr} a\}$ for all $a \in L_{fr}$,
(2) *co- regular* if $c = \bigwedge\{x \in L_{cf} : c \prec_{cf} x\}$ for all $c \in L_{cf}$,
(3) *completely regular* if $a = \bigvee\{x \in L_{fr} : x \prec\!\!\prec_{fr} a\}$ for all $a \in L_{fr}$,
(4) *completely co- regular* if $c = \bigwedge\{x \in L_{cf} : c \prec\!\!\prec_{cf} x\}$ for all $c \in L_{cf}$,
(5) *normal* if for any $c \in L_{cf}$ and $a \in L_{fr}$ such that $c \leq a$ there exists $b \in L_{fr}$ with $c \leq b \leq [b] \leq a$.

Finally, we recall the definition of a Urysohn relation given in [9]: A Urysohn relation on a partially ordered set $(L, \leq)$ is a binary relation $\lhd$ satisfying

$(U1)$ If $a \lhd b$ then $a \leq b$,
$(U2)$ $a \leq b \lhd c \leq d$ implies $a \lhd d$,
$(U3)$ $a \lhd b$ implies the existence of $c \in L$ with $a \lhd c \lhd b$.

As was shown in [2], a diframe is completely regular if and only if there exists a Urysohn relation $\lhd$ on L_e with the following conditions:

(1) $a \lhd b$ implies $[a] \leq]b[$,
(2) $a = \bigvee\{x \in L_{fr} : x \lhd a\}$ for every $a \in L_{fr}$.

3. Compactness and Stability in Diframes

The notion of compactness for bitopological spaces has several versions in the literature. By adopting the definiton of Kopperman [3], Brown and Diker [6] generalized the notion of compactness to ditopological texture spaces. It was also studied by Brown and Gohar [7]. Here, we extend this concept to a broader setting.

Definition 3.1. Let $L = (L_e, L_{fr}, L_{cf})$ be a diframe and $a \in L_e$. Then a subset $G \subseteq L_{fr}$ is called a *cover of* a if $a \leq \bigvee G$. A subset K of L_{cf} is said to be a *co-cover of* a if $\bigwedge K \leq a$.

Definition 3.2. Let $L = (L_e, L_{fr}, L_{cf})$ be a diframe and $a \in L_e$.

(1) a is called *compact* (resp., Lindelöf) if for every cover G of a, there is a finite (resp., countable) $H \subseteq G$ such that $a \leq \bigvee H$.
(2) a is called *co- compact* (resp., co- Lindelöf) if for every co- cover K of a, there is a finite (resp., countable) $F \subseteq K$ such that $\bigwedge F \leq a$.
(3) L is *compact* if the top element $1 \in L_e$ is compact, and it is Lindelöf if $1 \in L_e$ is Lindelöf.
(4) L is *co- compact* if the bottom element $0 \in L_e$ is co- compact, and it is called co- Lindelöf if $0 \in L_e$ is co- Lindelöf.

Note that for each property $\mathtt{P}$, $L = (L_e, L_{fr}, L_{cf})$ is said to be bi- $\mathtt{P}$ if it is $\mathtt{P}$ and *co*-$\mathtt{P}$.

Remark 3.3. *Obviously, (co-)compact implies (co-)Lindelöf but the reverse implication is not necessarily true. If X is a countable set, $L_e = L_{fr} = \mathcal{P}(X)$ and $L_{cf} = \{X, \emptyset\}$ then the diframe $L = (L_e, L_{fr}, L_{cf})$ is Lindelöf but not compact.*

Proposition 3.4. *Every subdilocale of a compact (resp., co- compact) diframe is compact (resp., co- compact).*

Proof. It is clear since $1_{S_e} = 1_{L_e}$ and $S_e \subseteq L_e$ is closed under arbitrary suprema. □

Lemma 3.5. *Let $L = (L_e, L_{fr}, L_{cf})$, $M = (M_e, M_{fr}, M_{cf})$ be diframes, and let $(\varphi, \psi) : L \to M$ be a one-one, onto diframe homomorphism. Then the following statements hold:*

(1) *If (φ, ψ) is an open (resp., co- open) homomorphism, then for all $b \in M_{fr}$ there exists $a \in L_{fr}$ such that $\psi(a) = b$ (resp., $\varphi(a) = b$).*
(2) *If (φ, ψ) is a closed (resp., co- closed) homomorphism, then for all $k \in M_{cf}$ there exists $f \in L_{cf}$ such that $\psi(f) = k$ (resp., $\varphi(f) = k$).*

Proof. (1) Suppose that $(\varphi, \psi) : L_e \to M_e$ is a one-one, onto, open diframe homomorphism and $b \in M_{fr}$. Since ψ is onto, there is an $a \in L_e$ with $\psi(a) = b$. On the other hand, ψ being one-one yields $\psi^*\psi = 1_{L_e}$, and hence $\psi^*\psi(a) = a = \psi^*(b)$. Since (φ, ψ) is open, $a = \psi^*(b) \in L_{fr}$.

The remaining assertions can be proved similarly. □

Proposition 3.6. *Suppose $L = (L_e, L_{fr}, L_{cf})$ and $M = (M_e, M_{fr}, M_{cf})$ are diframes and $(\varphi, \psi) : L \to M$ is a one-one, onto diframe homomorphism.*

(1) *If $(\varphi, \psi) : L \to M$ is co- open then L is compact iff M is compact.*
(2) *If $(\varphi, \psi) : L \to M$ is closed then L is co- compact iff M is co- compact.*

Proof. (1) Let L be a compact diframe and $B \subseteq M_{fr}$ be a cover of 1_{M_e}. By Lemma 3.5, for each $b_i \in B$, there is an $a_i \in L_{fr}$ with $\varphi(a_i) = b_i$. Since $\varphi(1_{L_e}) = 1_{M_e} = \bigvee_{i \in I} \varphi(a_i) = \varphi(\bigvee_{i \in I} a_i)$ and φ is one-one, we have $1_{L_e} = \bigvee_{i \in I} a_i$. Now, compactness of L gives $k_1, \ldots, k_n \in I$ such that $1_{L_e} = \bigvee_{k=1}^{n} a_{i_k}$. Applying the map

φ to both sides of the equation gives

$$1_{M_e} = \varphi(1_{L_e}) = \varphi(\bigvee_{k=1}^{n} a_{i_k}) = \bigvee_{k=1}^{n} \varphi(a_{i_k}) = \bigvee_{k=1}^{n} b_{i_k}$$

and hence M is compact.

Conversely, suppose M is compact and $A = \{a_i : i \in I\} \subseteq L_{fr}$ is a cover of 1_{L_e}, that is, $1_{L_e} = \bigvee_{i\in I} a_i$. Since φ preserves arbitrary joins, we have $\varphi(1_{L_e}) = 1_{M_e} = \varphi(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} \varphi(a_i)$. Then, by compactness of M, there is a finite subset $\{a_{i_k} : k = 1, \ldots, n\}$ of A such that

$$\varphi(1_{L_e}) = 1_{M_e} = \bigvee_{k=1}^{n} \varphi(a_{i_k}) = \varphi(\bigvee_{k=1}^{n} a_{i_k}).$$

Thus, φ being one-one implies $1_{L_e} = \bigvee_{k=1}^{n} a_{i_k}$, and hence L is compact. □

We now give a generalization of Alexander subbase theorem, the proof of which runs as the same as the one given in [7].

Theorem 3.7. *Let $L = (L_e, L_{fr}, L_{cf})$ be a diframe and δ be a subbase (resp., subcobase) of L. Then L is compact (resp., co- compact) if and only if for every cover (resp., co- cover) $A \subseteq \delta$ there exists a finite cover (resp., co- cover) $B \subseteq A$.*

Proof. We just give the sketch of the proof. As we mentioned before, the idea repeats that of [7, Theorem 2.14].

The implication "⇒" is clear by definition of compactness. For the reverse implication, assume that $A \subseteq L_{fr}$ is a subset such that no finite subset of A covers 1. We claim that A is not a cover of 1. Now let G be the collection of all subsets $B \subseteq L_{fr}$ such that $A \subseteq B$, and B has no finite subset covering 1. Then $(G, \subseteq)$ is a poset and it has a maximal element H by Zorn's Lemma. Moreover, H satisfies the properties given below:

(1) Given any $a \in L_{fr}$ with $a \notin H$, there exists $\{a_i : 1 \leq i \leq n\} \subseteq H$ such that $a \vee (\bigvee_{i=1}^{n} a_i) = 1$.
(2) For every subset $\{a_i : a_i \notin H, 1 \leq i \leq n\} \subseteq L_{fr}$ we have $\bigwedge_{i=1}^{n} a_i \notin H$.
(3) For every subset $C = \{a_i : 1 \leq i \leq n\}$ of L_{fr} and every $b \in H$ with $\bigwedge_{i=1}^{n} a_i \leq b$, there exists an $a_j \in C$ such that $a_j \in H$.

We also know that no finite subset of $\delta \cap H$ covers 1 since $\delta \cap H \subseteq H$. By using the properties $(1) - (3)$, we see that $\bigvee H = \bigvee(\delta \cap H)$. Now, if H is a cover of 1 then $\bigvee(\delta \cap H) = 1$, which contradicts with the assumption. Thus H, and hence A, is not a cover of 1. □

As can be easily seen from the definitions, (co-) compactness is not a property relating L_{fr} and L_{cf}. Thus we need the following concepts that relate the frame L_{fr} and the coframe L_{cf}.

Definition 3.8. Let $L = (L_e, L_{fr}, L_{cf})$ be a diframe. Then L is called

(1) *stable* if every element of L_{cf} other than 1 is compact,
(2) *co- stable* if every $0 \neq a \in L_{fr}$ is co- compact,

Example 3.9. Consider the the diframe $L = (L_e, L_{fr}, L_{cf})$ of Example 2.1.

(1) L is not compact since the cover $\{(-\infty, a+n) : n \in \mathbb{N}\}$ of $\mathbb{R}$ does not have a finite subset covering $\mathbb{R}$. Further, L is not co- compact. Indeed, the co-cover $\{(a - \frac{1}{n}, a + \frac{1}{n}) : n \in \mathbb{N}\}$ of $0_{L_e} = \emptyset$ proves our claim.
(2) L is not co- stable. Indeed, for any $(-\infty, b) \in L_{fr}$, we have
$$\bigwedge_{n\in\mathbb{N}} (a, b + \frac{1}{n}) = int\big(\bigcap_{n\in\mathbb{N}} (a, b + \frac{1}{n})\big) = int(a, b] = (a, b) \subseteq (-\infty, b)$$
but there is no finite $F \subseteq \{(a, b + \frac{1}{n}) : n \in \mathbb{N}\}$ such that $\bigwedge F \subseteq (-\infty, b)$. (Here, "int" denotes the interior operator.) Moreover, one can easily show that L is not stable.

The following example shows that compactness does not imply stability, and vice versa.

Example 3.10. (1) Let $\Omega(\mathbb{R})$ be the open set lattice of countable complement topology on $\mathbb{R}$. If $L_e = \mathcal{P}(\mathbb{R})$, $L_{fr} = \Omega(\mathbb{R})$ and $L_{cf} = \{\emptyset, \mathbb{R}\}$ then (L_e, L_{fr}, L_{cf}) is a stable, non-compact diframe.
(2) Let $\mathbb{I}$ be the unit interval equipped with the usual topology. If $L_{fr} = \Omega(\mathbb{I})$ and $L_{cf} = \{\emptyset, [0, \frac{1}{2}), \mathbb{I}\}$, then the diframe $(\mathcal{P}(\mathbb{I}), L_{fr}, L_{cf})$ is obviously compact. But it is not stable since the element $[0, \frac{1}{2}) \in L_{cf}$ is not compact. Indeed, the cover $\big\{[0, \frac{1}{2} - \frac{1}{n}) : n \geq 2, n \in \mathbb{N}\big\}$ proves our claim.

The bitopological version of the next proposition was proved in [3]. In our case, we shall impose a stronger condition on diframe L' because of the lack of complete distributivity in diframes. We replace the property of being R_0 by that of being regular. Here, it is worth reminding the reader that our R_0 and R_1 are given, respectively, as pseudo-Hausdorff (pH) and weak symmetry (ws) in [3].

Proposition 3.11. *If $L = (L_e, L_{fr}, L_{cf})$ is a co- R_0, stable diframe and $L' = (L_e, L_{fr}, {L_{cf}}')$ is regular, then $L_{cf} \subseteq {L_{cf}}'$. Dually, if $L = (L_e, L_{fr}, L_{cf})$ is R_0, co-stable and $L' = (L_e, {L_{fr}}', L_{cf})$ is co- regular then $L_{fr} \subseteq {L_{fr}}'$.*

Proof. Let $k \in L_{cf}$. The case $k = 1$ being obvious, we assume $k \neq 1$. Since L is co- R_0, there exist $a_i \in L_{fr}$ such that $k = \bigwedge_{i\in I} a_i$. By regularity of L', $a_i = \bigvee_{j\in J}\{x_{ij} \in L_{fr} : x_{ij} \prec_{fr} a_i\}$. Moreover, $x_{ij} \prec_{fr} a_i$ implies the existence of $k_{ij} \in {L_{cf}}'$ such that $x_{ij} \leq k_{ij} \leq a_i$. Now we have $k \leq \bigvee_{j\in J} x_{ij}$ for all $i \in I$, and hence by stability of L, there is a finite $J_0 \subseteq J$ with $k \leq \bigvee_{j\in J_0} x_{ij} \leq \bigvee_{j\in J_0} k_{ij}$. Thus,
$$k \leq \bigwedge_{i\in I} \bigvee_{j\in J_0} k_{ij} \leq \bigwedge_{i\in I} a_i \leq k$$
and hence $k = \bigwedge_{i\in I} \bigvee_{j\in J_0} k_{ij} \in {L_{cf}}'$.

The dual statement can be proved in a dual manner. □

The property of being R_0 (resp., co- R_0) is generally not inherited by subdilocales but it is hereditary if the diframe is co- stable (resp., stable):

Proposition 3.12. *Every subdilocale of a (co-)stable diframe is (co-)stable.*

Proof. Obvious, since the joins in S coincide with the joins in L. □

The following two propositions establish the connection between (co-)stability and separation axioms.

Proposition 3.13. *Every stable regular diframe is normal. Dually, every co- stable co- regular diframe is normal.*

Proof. Let $c \leq a$ for some $c \in L_{cf}$, $a \in L_{fr}$. We assume $a \neq 1$ since the case $a = 1$ is trivial. By regularity, $c \leq a = \bigvee_{i \in I}\{x_i \in L_{fr} : x_i \prec_{fr} a\}$. Since $1 \neq c \in L_{cf}$ is a compact element by stability of L, we have $c \leq \bigvee_{i=1}^{n}\{x_i \in L_{fr} : x_i \prec_{fr} a\}$. If $x_i \prec_{fr} a$ there exists a $k_i \in L_{cf}$ such that $x_i \leq k_i \leq a$. Thus, by setting $b = \bigvee_{i=1}^{n}]k_i[$ we obtain

$$c \leq \bigvee_{i=1}^{n} x_i \leq b \text{ and } [b] = [\bigvee_{i=1}^{n}]k_i[] \leq [\bigvee_{i=1}^{n} k_i] \leq \bigvee_{i=1}^{n} k_i \leq a.$$

□

Proposition 3.14. (1) *A R_1 co- stable diframe is regular.*
(2) *A co- R_1 stable diframe is co- regular.*

Proof. (1) Suppose that L is a R_1, co- stable diframe, and take any $a \in L_{fr}$. The case $a = 0$ is trivial, so let $a \neq 0$. By R_1, $a \in L_{fr}$ can be expressed as $a = \bigvee_{i \in I} \bigwedge_{j \in J} c_i^j = \bigvee_{i \in I} \bigwedge_{j \in J}[c_i^j]$ for $c_i^j \in L_{fr}$. Since, for all i, $\bigwedge_{j \in J}[c_i^j] \leq a$ and L is co- stable, there is a finite subset $J_0 \subseteq J$ of indices such that $\bigwedge_{j \in J_0}[c_i^j] \leq a$. Set $x_i = \bigwedge_{j \in J_0} c_i^j$ for all $i \in I$. Then $\bigwedge_{j \in J_0} c_i^j \leq \bigwedge_{j \in J_0}[c_i^j] \leq a$ and $\bigwedge_{j \in J_0}[c_i^j] \in L_{cf}$, and hence $x_i \prec_{fr} a$ for all $i \in I$. Therefore,

$$\bigvee_{i \in I} \bigwedge_{j \in J_0} c_i^j \leq a \leq \bigvee_{i \in I} \bigwedge_{j \in J} c_i^j \leq \bigvee_{i \in I} \bigwedge_{j \in J_0} c_i^j$$

that is, $a = \bigvee_{i \in I}\{x_i \in L_{fr} : x_i \prec_{fr} a\}$, which shows that L is regular.

The proof of (2) can be done similarly. □

Corollary 3.15. *Every R_1 (resp., co- R_1) bi- stable (i.e., stable and co- stable) diframe is normal.*

Proof. L is regular by Proposition 3.14 and hence the statement follows from Proposition 3.13. □

We end this section by discussing the preservation of (co-)stability under certain morphisms.

Proposition 3.16. *Let $(\varphi, \psi) : (L_e, L_{fr}, L_{cf}) \to (M_e, M_{fr}, M_{cf})$ be an onto, one-one diframe homomorphism.*

(1) *If (φ, ψ) is a co- open, co- closed homomorphism and L is stable then M is stable.*
(2) *If (φ, ψ) is an open, closed homomorphism and L is co- stable then M is co- stable.*

Proof. Suppose that L is stable, $1_{M_e} \neq k \in M_{cf}$ and $\{b_i : i \in I\} \subseteq M_{fr}$ is a cover of k. By Lemma 3.5, there exists $1_{L_e} \neq f \in L_{cf}$ with $\varphi(f) = k$ and $a_i \in L_{fr}$ with $\varphi(a_i) = b_i$ for all $i \in I$. Then we have $\varphi(f) \leq \bigvee_{i \in I} \varphi(a_i) = \varphi(\bigvee_{i \in I} a_i)$, and hence $\varphi_* \varphi(f) \leq \varphi_* \varphi(\bigvee_{i \in I} a_i)$ since φ_* is an order preserving map. Now, φ being onto implies $\varphi_* \varphi(f) = id$, and hence, by stability of L, we have $f \leq \bigvee_{k=1}^{n} a_{i_k}$. Thus we obtain

$$\varphi(f) \leq \varphi(\bigvee_{k=1}^{n} a_{i_k}) = \bigvee_{k=1}^{n} \varphi(a_{i_k}) = \bigvee_{k=1}^{n} b_{i_k}$$

which shows that M is stable. □

Proposition 3.17. *Let M be a stable (resp., co- stable) diframe and $\varphi : L \to M$ be a one-one hdiFrm morphism. Then L is a stable (resp., co- stable) diframe.*

Proof. Suppose that M is stable. Take any element $1_{L_e} \neq f \in L_{cf}$ and any cover $\{a_i \in L_{fr} : i \in I\}$ of f. Then $\varphi(f) \leq \varphi(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} \varphi(a_i)$. Since $1_{M_e} \neq \varphi(f) \in M_{cf}$ and $\varphi(a_i) \in M_{fr}$ for all $i \in I$, stability of M gives $\varphi(f) \leq \bigvee_{k=1}^{n} \varphi(a_{i_k}) = \varphi(\bigvee_{k=1}^{n} a_{i_k})$. Thus, applying φ_* on both sides we obtain $f \leq \bigvee_{k=1}^{n} a_{i_k}$. □

4. Locally Compact Diframes

In this section, we introduce two main concepts, that of locally compactness and locally stability in diframes. As pointed out in the introduction, their bitopological versions use the notion of neighbourhood which is a point-based structure. Hence, we first define the following binary relations on L_e.

Definition 4.1. Let $L = (L_e, L_{fr}, L_{cf})$ be a diframe and $x, y \in L_e$. Then,

(1) $x \ll_c y$ iff there exists a compact $k \in L_e$ with $x \leq k \leq y$.
(2) $x \ll_{cc} y$ iff there exists a co- compact $a \in L_e$ with $x \leq a \leq y$.
(3) $x \ll_s y$ iff there exists a compact $k \in L_{cf}$ with $x \leq k \leq y$.
(4) $x \ll_{cs} y$ iff there exists a co- compact $a \in L_{fr}$ with $x \leq a \leq y$.

Remark 4.2. *It is an immediate consequence of the definitions that $x \ll_s y$ implies $x \ll_c y$ for $x, y \in L_e$ and $x \ll_s y$ implies $x \prec_{fr} y$ for $x, y \in L_{fr}$. On the other hand, it is obvious that $x \in L_e$ is compact (resp., co- compact) iff $x \ll_c x$ (resp., $x \ll_{cc} x$), and in particular, L is compact (resp., co- compact) iff $1 \ll_c 1$ (resp., $0 \ll_{cc} 0$).*

Note that the following concepts have no counterparts in the theory of ditopological texture spaces.

Definition 4.3. A diframe L is called

(1) *locally compact* if $a = \bigvee\{x \in L_{fr} : x \ll_c a\}$ for all $a \in L_{fr}$,
(2) *locally co- compact* if $k = \bigwedge\{x \in L_{cf} : k \ll_{cc} x\}$ for all $k \in L_{cf}$,
(3) *locally stable* if $a = \bigvee\{x \in L_{fr} : x \ll_s a\}$ for all $a \in L_{fr}$,
(4) *locally co- stable* if $k = \bigwedge\{x \in L_{cf} : k \ll_{cs} x\}$ for all $k \in L_{cf}$.

Example 4.4. The diframe in Example 2.1 is neither locally compact nor locally co- compact.

Proposition 4.5. *Each subdilocale of a locally (co-) compact diframe is locally (co-) compact.*

Proof. Let S be a subdilocale of a locally compact diframe L and let $a \in S_{fr}$. Then, $a = \bigvee\{x \in L_{fr} : x \ll_c a\}$. If $x \ll_c a$, then there is a compact $k \in L_e$ satisfying $x \leq k \leq a$, and then $v_{S_e}(x) \leq v_{S_e}(k) \leq v_{S_e}(a) = a$ by monotonicity of v_{S_e}. Thus, it suffices to show that $v_{S_e}(k)$ is compact, which yields

$$a = \bigvee\{v_{S_e}(x) \in S_{fr} : v_{S_e}(x) \ll_c a\}$$

and completes the proof.

Let $\{b_i \in S_{fr} : i \in I\}$ be a cover of $v_{S_e}(k)$, that is, $k \leq v_{S_e}(k) \leq \bigvee_{i\in I} b_i$. By compactness of k, we obtain a finite $I_0 \subseteq I$ with $k \leq \bigvee_{i\in I_0} b_i$. Hence, applying the nucleus v_{S_e} and using the fact that S_e is a flat sublocale yield

$$v_{S_e}(k) \leq v_{S_e}(\bigvee_{i\in I_0} b_i) = \bigvee_{i\in I_0} v_{S_e}(b_i) = \bigvee_{i\in I_0} b_i$$

Thus, $v_{S_e}(k)$ is compact. □

Proposition 4.6. *A (co-)regular, (co-)stable, (co-)compact diframe is locally (co-) stable.*

Proof. Let $a \in L_{fr}$. The case $a = 1$ is clear by compactness of L. So, assume that $a \neq 1$. Then, by regularity, $a \in L_{fr}$ can be writen as $a = \bigvee\{x \in L_{fr} : x \prec_{fr} a\}$. If $x \prec_{fr} a$ then there exists a $k \in L_{cf}$ with $x \leq k \leq a$. Moreover, $k \neq 1$ since $k \leq a$ and $a \neq 1$. Hence, k is compact since L is a stable diframe. Thus we obtain $x \ll_s a$ and

$$a \leq \bigvee\{x \in L_{fr} : x \prec_{fr} a\} \leq \bigvee\{x \in L_{fr} : x \ll_s a\} \leq a$$

which shows that L is locally stable.

The dual proof is analogous. □

Proposition 4.7. *Let L be a diframe.*

(1) *L is locally stable iff it is regular and locally compact.*
(2) *L is locally co- stable iff it is co- regular and locally co- compact.*

Proof. (1) The sufficiency is immediate by Remark 4.2. Thus, we only prove the necessity.

Suppose that L is a regular and locally compact diframe and take an arbitrary $a \in L_{fr}$. Then $a \in L_{fr}$ can be expressed as $a = \bigvee\{x \in L_{fr} : x \ll_c a\}$. Further, if $x \ll_c a$ then there exists a compact $k \in L_e$ such that $x \le k \le a$.

Claim 1: $[k] \le a$.

By regularity of L, we have $k \le a = \bigvee_{i\in I}\{x_i \in L_{fr} : x_i \prec_{fr} a\}$. If $x_i \prec_{fr} a$, then there is an $f_i \in L_{cf}$ such that $x_i \le f_i \le a$. Hence, there exists a finite $I_0 \subseteq I$ with $k \le \bigvee_{i\in I_0} x_i \le \bigvee_{i\in I_0} f_i \le a$ by compactness of k. Thus, we obtain

$$[k] \le [\bigvee_{i\in I_0} f_i] = \bigvee_{i\in I_0} [f_i] = \bigvee_{i\in I_0} f_i \le a$$

Claim 2: $[k] \in L_{cf}$ is a compact element.

Let $\{a_i \in L_{fr} : i \in I\}$ be a cover of $[k]$. By regularity of L, each a_i can be expressed as $a_i = \bigvee_{j\in I}\{x_{ij} \in L_{fr} : x_{ij} \prec_{fr} a_i\}$. If $x_{ij} \prec_{fr} a_i$ then there is an $f_{ij} \in L_{cf}$ with $x_{ij} \le f_{ij} \le a_i$. Moreover, the expression $k \le \bigvee_{i\in I} a_i = \bigvee_{i\in I}\bigvee_{j\in I} x_{ij}$, together with the fact that k is compact, implies the existence of finite subsets $I_0 \subseteq I$ and $J_0 \subseteq J$ such that $k \le \bigvee_{i\in I_0}\bigvee_{j\in J_0} x_{ij}$. Therefore,

$$[k] \le \bigvee_{i\in I_0}\bigvee_{j\in J_0} [x_{ij}] \le \bigvee_{i\in I_0}\bigvee_{j\in J_0} f_{ij} \le \bigvee_{i\in I_0} a_i$$

and hence $[k]$ is compact.

Now we can conclude that, in a regular diframe, $x \le k \le a$ and k being compact imply $x \le [k] \le a$ and $[k]$ is compact. Thus,

$$a = \bigvee\{x \in L_{fr} : x \ll_c a\} = \bigvee\{x \in L_{fr} : x \ll_s a\}$$

and hence L is locally stable. □

Proposition 4.8. *Every locally (co-)stable diframe is completely (co-)regular.*

Proof. Let L be a locally stable diframe. We claim that the relation

$$a \lhd b \text{ if there exists a compact } k \in L_e \text{ such that } [a] \le k \le]b[$$

is a Urysohn relation satisfying the following properties:

(1) $a \lhd b$ implies $[a] \le]b[$,

(2) for every $a \in L_{fr}$, $a = \bigvee\{x \in L_{fr} : x \lhd a\}$.

$(U1)$ and $(U2)$ are obvious by definition of the given relation.

For $(U3)$, let $a \lhd b$. Then there exists a compact $k \in L_e$ with $[a] \le k \le]b[$ and further, by locally stability of L, $]b[= \bigvee_{i\in I}\{c_i \in L_{fr} : c_i \ll_s]b[\}$. But then there exists a finite subset $I_0 \subseteq I$ of indices such that $k \le \bigvee_{i\in I_0} c_i$ since k is compact. Moreover, $c_i \ll_s]b[$ implies that there exists a compact $k_i \in L_{cf}$ with $c_i \le k_i \le]b[$.

Now set $d = \bigvee_{i\in I_0}[c_i]$. We have,

$$[a] \leq k \leq \bigvee_{i\in I_0} c_i \leq \,]\bigvee_{i\in I_0} c_i[\, \leq \,]\bigvee_{i\in I_0}[c_i][\, =]d[$$

and hence $a \lhd d$. On the other hand,

$$[d] = \bigvee_{i\in I_0}[c_i] \leq \bigvee_{i\in I_0} k_i \leq]b[$$

and $\bigvee_{i\in I_0} k_i$ is compact since k_i is compact for all $i \in I_0$. Hence we have $d \lhd b$.

Now it remains to show the properties (1) and (2). The first one is clear by definition. For (2), let $a \in L_{fr}$. Then by locally stability of L, it can be written as $a = \bigvee\{x \in L_{fr} : x \ll_s a\}$. If $x \ll_s a$ then there is a compact $k \in L_{cf}$ with $x \leq k \leq a$ and hence $[x] \leq k \leq a =]a[$. Thus,

$$a = \bigvee\{x \in L_{fr} : x \ll_s a\} \leq \bigvee\{x \in L_{fr} : x \lhd a\} \leq a$$

that is, $a = \bigvee\{x \in L_{fr} : x \lhd a\}$. □

Proposition 4.9. *Let $L = (L_e, L_{fr}, L_{cf})$, $M = (M_e, M_{fr}, M_{cf})$ be diframes and $(\varphi, \psi) : L \to M$ be a one-one, onto diframe homomorphism. Then the following statements hold:*

1. *If (φ, ψ) is co- open then L is locally compact iff M is locally compact.*
2. *If (φ, ψ) is closed then L is locally co- compact iff M is locally co- compact.*

Proof. (1) Suppose that L is locally compact and take any $b \in M_{fr}$. First, by Lemma 3.5, there is an $a \in L_{fr}$ such that $\varphi(a) = b$ and it can be expressed as $a = \bigvee\{x \in L_{fr} : x \ll_c a\}$ by locally compactness of L. If $x \ll_c a$ then there exist a compact $k \in L_e$ with $x \leq k \leq a$. Then, $\varphi(x) \leq \varphi(k) \leq \varphi(a) = b$ since φ preserves order.

Now we claim that $\varphi(k)$ is compact. Let $\{b_i \in M_{fr} : i \in I\}$ be an arbitrary cover of $\varphi(k)$. Then, for every $i \in I$ there exists $a_i \in L_{fr}$ such that $\varphi(a_i) = b_i$, and hence

$$\varphi(k) \leq \bigvee_{i\in I} b_i = \bigvee_{i\in I} \varphi(a_i) = \varphi(\bigvee_{i\in I} a_i).$$

Applying the map φ_* and then using the compactness of k, we get $k \leq \bigvee_{k=1}^n a_{i_k}$. Thus,

$$\varphi(k) \leq \varphi(\bigvee_{k=1}^n a_{i_k}) = \bigvee_{k=1}^n \varphi(a_{i_k}) = \bigvee_{k=1}^n b_{i_k}$$

and hence $\varphi(k)$ is compact.

Now, using the claim it is easy to see that

$$b = \varphi(a) = \bigvee\{\varphi(x) \in M_{fr} : \varphi(x) \ll_c b\}$$

which means that M is locally compact.

Conversely, assume that M is locally compact. Given any $a \in L_{fr}$, $\varphi(a) \in M_{fr}$ and hence we have $\varphi(a) = \bigvee\{y \in M_{fr} : y \ll_c \varphi(a)\}$. Moreover, for each $y \in M_{fr}$, there is an $x \in L_{fr}$ such that $\varphi(x) = y$. If $\varphi(x) \ll_c \varphi(a)$ then there exists a compact $k \in M_e$ with $\varphi(x) \leq k \leq \varphi(a)$. Now we obtain $x \leq \varphi_*(k) \leq a$. We can easily show that $\varphi_*(k)$ is compact. Thus $a = \bigvee\{x \in L_{fr} : x \ll_c a\}$.

The second one can be proved in a similar manner. □

Proposition 4.10. *Let $(\varphi, \psi) : (L_e, L_{fr}, L_{cf}) \to (M_e, M_{fr}, M_{cf})$ be an onto, one-one diframe homomorphism.*

(1) *If (φ, ψ) is a co- open, co- closed diframe homomorphism and M is locally stable then L is locally stable.*
(2) *If (φ, ψ) is an open, closed diframe homomorphism and M is locally co-stable then L is locally co- stable.*

Proof. (2) Assume that M is locally co- stable and take an arbitrary $f \in L_{cf}$. Then we have $\psi(f) \in M_{cf}$ and it can be written as $\psi(f) = \bigwedge\{y \in M_{cf} : \psi(f) \ll_{cs} y\}$. For all y, there is an $x \in L_{cf}$ with $\psi(x) = y$, and if $\psi(f) \ll_{cs} \psi(x)$ then we have a co- compact $b \in M_{fr}$ such that $\psi(f) \leq b \leq \psi(x)$. Moreover, for $b \in M_{fr}$, there is an $a \in L_{fr}$ with $\psi(a) = b$. As in the previous proof, one can see that $a \in L_{fr}$ is co- compact. Thus, we obtain $f = \bigwedge\{x \in L_{cf} : f \ll_{cs} x\}$, which completes the proof. □

Proposition 4.11. *The image of a locally stable (resp., locally co- stable) diframe under a one-one, onto, open (resp., closed) hdiFrm morphism is locally stable (resp., locally co- stable).*

Proof. This can be proved easily in a similar way used in the proof of the previous propositions. □

5. Conclusion

In this paper we have introduced the concept of compactness in diframes. Then we have defined stable, locally compact and locally stable diframes and investigated the relations between separation axioms and these properties. As a future work, other topological and bitopological structures such as paracompactness, connectedness and uniformities etc. can be constructed on diframes.

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SOME NEW RESULTS ON CONVERGENCE, STABILITY AND DATA DEPENDENCE IN $n-$NORMED SPACES

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Abstract. We introduce a new contractive condition and a new iterative method in $n-$ normed space setting. We employ both of these to study convergence, stability, and data dependence. The results presented here extend and improve some recent results announced in the existing literature.

1. Introduction

The theory of n-normed spaces has been introduced by Misiak [1] as a generalization of the theory of 2-normed spaces due to Gähler [2]. Since then, much effort has been devoted to the development of the theory of $n-$normed spaces. See, e.g. [3-5] and references therein. We recall some basic facts as follows.

Definition 1. *([1]) Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|\cdot,\ldots,\cdot\| : X^n \to \mathbb{R}$ which satisfies the following conditions:*

(nN_1) $\|x_1,\ldots,x_n\| = 0$ *iff* $x_1,\ldots,x_n$ *are linearly dependent,*
(nN_2) $\|x_1,\ldots,x_n\|$ *is invariant under any permutation,*
(nN_3) $\|\alpha x_1,\ldots,x_n\| = |\alpha| \, \|x_1,\ldots,x_n\|$ *for every* $\alpha \in \mathbb{R}$,
(nN_4) $\|x_1 + x_1', x_2 \ldots, x_n\| \leq \|x_1, x_2,\ldots,x_n\| + \|x_1', x_2,\ldots,x_n\|$.
is called an $n-$norm on X. The pair $(X, \|\cdot,\ldots,\cdot\|)$ is called an $n-$normed spaces.

Example 2. *([3]) (i) Let $X = \mathbb{R}^n$ with the following Euclidean $n-$norm:*

$$\|x_1,\ldots,x_n\|_E = abs\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i1},\ldots,x_{in}) \in \mathbb{R}^n$ for each $i = \overline{1,n}$. Then, the pair $(\mathbb{R}^n, \|x_1,\ldots,x_n\|_E)$ is an $n-$normed space.

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(ii) Let $(X, \|\cdot, \ldots, \cdot\|)$ *be an* $n-$*normed space of dimension* $d \geq n \geq 2$ *and* $\{u_1, u_2, \ldots, u_n\}$ *be a linearly independent set in* X. *A function* $\|\cdot, \ldots, \cdot\|_\infty$ *on* X^{n-1} *defined by*

$$\|x_1, \ldots, x_{n-1}\|_\infty = \max\left\{\|x_1, \ldots, x_n, u_i\| : i = \overline{1, n}\right\}$$

is an $(n-1)$ *norm on* X *w.r.t.* $\{u_1, u_2, \ldots, u_n\}$.

Definition 3. *(*[3]*) Let* X *be a* $n-$*normed linear space and* $\{x_n\}_{n=0}^\infty$ *a sequence in* X. *We say that* $\{x_n\}_{n=0}^\infty$ *converge to some* $x \in X$ *if*

$$\lim_{n\to\infty} \|x_n - x, u_2, \cdots, u_n\| = 0,$$

for all $u_2, \cdots, u_n \in X$.

The following iterative methods were studied in [6], [7], [8], and [9] respectively,

$$\begin{cases} s_0 \in X, \\ s_{n+1} = a_n s_n + b_n T_1 s_n + c_n T_2 s_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{1}$$

$$\begin{cases} p_0 \in X, \\ p_{n+1} = (1-\alpha_n) q_n + \alpha_n T q_n, \\ q_n = (1-\beta_n) p_n + \beta_n T p_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{2}$$

$$\begin{cases} p_0 \in X, \\ p_{n+1} = (1-\alpha_n) q_n + \alpha_n T_1 q_n, \\ q_n = (1-\beta_n) p_n + \beta_n T_2 p_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3}$$

$$\begin{cases} x_0 \in X, \\ x_{n+1} = a_n x_n + b_n T_1 y_n + c_n T_2 x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_3 x_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4}$$

where T, T_1, T_2 and T_3 are self maps of an ambient space X and $\{a_n\}, \{b_n\}, \{c_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}_{n=0}^\infty$ are real sequences in $[0,1]$ satisfying certain control conditions.

Inspired by the above iterative methods, we introduce the following iterative method.

$$\begin{cases} x_0 \in X, \\ x_{n+1} = a_n y_n + b_n T_1 y_n + c_n T_2 y_n, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_3 x_n, \ \forall n \in \mathbb{N}, \end{cases} . \tag{5}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset [0,1]$ are real sequences satisfying $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, and $\sum\limits_{n=0}^{\infty} b_n = \infty$.

Remark 4. *If* $T_3 = I$ *(Identity operator), then iterative method* (5) *reduces to iterative method* (1). *If* $T_1 = I$ *(Identity operator),* $T_2 = T_3 = T$, *then iterative method* (5) *reduces to iterative method* (1.2). *If* $T_1 = I$, *then the iterative method* (5) *reduces to iterative method* (1.3). *Note that* (1.4) *and* (5) *are of independent interest and so we would like to deal with both of these separately. However, it is worth noticing that* (1.4) *does not reduce to* (1.3) *but* (5) *does. Thus, in this sense,* (5) *is more general than* (1.4).

Recently, Dutta [3] introduced a generalized Z-type contractive condition as follows: Let K be nonempty, closed, convex subset of real linear $n-$normed space X and $T : K \to K$ a self map. There exists a constant $L \geq 0$ such that for all $x, y, u_2, \cdots, u_n \in K$, we have

$$\begin{aligned} &\|Tx - Ty, u_2, \cdots, u_n\| \\ &\leq e^{L\|x-Tx,u_2,\cdots,u_n\|} \times (2\delta \|x - Tx, u_2, \cdots, u_n\| + \delta \|x - y, u_2, \cdots, u_n\|), \end{aligned} \tag{6}$$

where $\delta \in [0, 1)$ and e^x denotes the exponential function of $x \in K$.

In [3], some convergence results have been constructed for fixed point of the mappings satisfying condition (6) via iterative schemes (1.1), (1.2) and (1.3).

In this paper, we introduce the following contractive condition: Let $(X, \|\cdot, \ldots, \cdot\|)$ be an $n-$normed space, $T : X \to X$ a selfmap of X, with a fixed point q such that for all $x, y, u_2, \cdots, u_n \in X$ and for some $\delta \in [0, 1)$, we have

$$\|q - Ty, u_2, \cdots, u_n\| \leq \delta \|q - y, u_2, \cdots, u_n\|. \tag{7}$$

This is similar to the condition introduced by [10] and can be obtained from (6) when $x = q$ is a fixed point. We may call this kind of operators quasi-contractive operators.

Following important observation will be used in the sequel.

$$\begin{aligned} \|Tx - Ty, u_2, \cdots, u_n\| &\leq \|Tx - q, u_2, \cdots, u_n\| + \|q - Ty, u_2, \cdots, u_n\| \\ &\leq \delta (\|x - q, u_2, \cdots, u_n\| + \|q - y, u_2, \cdots, u_n\|) \\ &\leq \delta \|x - y, u_2, \cdots, u_n\| + 2\delta \|q - y, u_2, \cdots, u_n\|. \end{aligned} \tag{8}$$

In our opinion, it is better to work with the contractive condition defined by (7) than with (6) because, as remarked above, if we suppose that T has a fixed point, then (6) implies (7) and by using it we can avoid doing unnecessary calculations.

In this paper, we first prove some convergence results for the mappings satisfying condition (7) via iterative methods (1.4) and (5). Next, we show that the iterative methods (1.4) and (5) are stable with respect to (T_1, T_2, T_3). Finally, we prove some data dependence results for the iterative methods (1.4) and (5).

We close this section with the following couple of results useful in proving our main results.

Lemma 5. [11] *Let $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:*

$$\sigma_{n+1} \leq (1 - \lambda_n) \sigma_n + \rho_n,$$

where $\lambda_n \in (0, 1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\frac{\rho_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} \sigma_n = 0$.

Lemma 6. [12] *Let $\{\sigma_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has satisfied the inequality*

$$\sigma_{n+1} \leq (1 - \mu_n) \sigma_n + \mu_n \gamma_n,$$

where $\mu_n \in (0,1)$, *for all* $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \mu_n = \infty$ *and* $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. *Then the following inequality holds.*

$$0 \leq \limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \gamma_n.$$

2. Convergence Results

For the sake of simplicity, from now on we assume that X is a $n-$normed linear space, T_1, T_2 and T_3 are self-maps of X satisfying the contractive condition (7) with the set of fixed points $F_{T_1}, F_{T_2}, F_{T_3}$ respectively, and $\bigcap\limits_{i=1}^{3} F_{Ti} \neq \emptyset$.

Theorem 7. *Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence generated by iterative method* (5) *with real sequences* $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset (0,1)$ *satisfying* $a_n + b_n + c_n = 1$ *for all* $n \in \mathbb{N}$, *and* $\sum\limits_{n=0}^{\infty} b_n = \infty$ (*or* $\sum\limits_{n=0}^{\infty} c_n = \infty$). *Suppose that* $q \in \bigcap\limits_{i=1}^{3} F_{Ti} \neq \emptyset$. *Then the iterative sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* q.

Proof. First we prove that $q \in \bigcap\limits_{i=1}^{3} F_{Ti}$ is the unique common fixed point of T_1, T_2 and T_3. Suppose that there exists another common fixed point $q^* \in \bigcap\limits_{i=1}^{3} F_{Ti}$. Then from (7), we have

$$\|q - q^*\| = \|T_i q - q^*\| \leq \delta \|q - q^*\| \text{ for each } i = 1, 2, 3,$$

which implies that $q = q^*$ as $\delta \in [0,1)$.

Next, we prove that $x_n \to q$.

Using (5) and (7), we get

$$\begin{aligned}
\|x_{n+1} - q, u_2, \cdots, u_n\| &= \|a_n y_n + b_n T_1 y_n + c_n T_2 y_n - q, u_2, \cdots, u_n\| \\
&= \|a_n (y_n - q) + b_n (T_1 y_n - q) + c_n (T_2 y_n - q), u_2, \cdots, u_n\| \\
&\leq a_n \|y_n - q, u_2, \cdots, u_n\| + b_n \|T_1 y_n - q, u_2, \cdots, u_n\| \\
&\quad + c_n \|T_2 y_n - q, u_2, \cdots, u_n\| \\
&\leq [a_n + (b_n + c_n)\delta] \|y_n - q, u_2, \cdots, u_n\| \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - q, u_2, \cdots, u_n\| &= \|\alpha_n x_n + (1-\alpha_n) T_3 x_n - q, u_2, \cdots, u_n\| \\
&\leq \alpha_n \|x_n - q, u_2, \cdots, u_n\| + (1-\alpha_n) \|T_3 x_n - q, u_2, \cdots, u_n\| \\
&\leq [\alpha_n + (1-\alpha_n)\delta] \|x_n - q, u_2, \cdots, u_n\| \\
&\leq [\alpha_n + 1 - \alpha_n] \|x_n - q, u_2, \cdots, u_n\| \\
&= \|x_n - q, u_2, \cdots, u_n\|. \qquad (10)
\end{aligned}$$

Substituting (2.2) into (2.1)

$$\|x_{n+1}-q,u_2,\cdots,u_n\| \le [1-(b_n+c_n)(1-\delta)]\,\|x_n-q,u_2,\cdots,u_n\|. \tag{11}$$

Since $\delta \in [0,1)$ and $a_n+b_n+c_n=1$ for all $n\in\mathbb{N}$,

$$0 \le 1-(b_n+c_n)(1-\delta) < 1. \tag{12}$$

Also, the assumption $\sum\limits_{n=0}^{\infty} b_n = \infty$ (or $\sum\limits_{n=0}^{\infty} c_n = \infty$) implies $\sum\limits_{n=0}^{\infty}(b_n+c_n)=\infty$. Hence, an application of Lemma 1 to (2.3) lead us to $\lim_{n\to\infty} x_n = q$. □

Remark 8. *If $T_1 = I$ (Identity operator), $T_2 = T_3 = T$, then the iterative method* (5) *reduces to the iterative method (1.2). If $T_1 = I$, then the iterative method* (5) *reduce to the iterative method (1.3). Having regard to these facts, we conclude that Theorem 1 is a generalization and extension of both (*[3]*, Theorem 3 and Theorem 4).*

Theorem 9. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by iterative method (1.4) with real sequences $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\}\subset(0,1)$ satisfying $a_n+b_n+c_n=1$ for all $n\in\mathbb{N}$, and $\sum\limits_{n=0}^{\infty} b_n=\infty$ (or $\sum\limits_{n=0}^{\infty} c_n=\infty$). Suppose that $q\in\bigcap\limits_{i=1}^{3}F_{Ti}\neq\emptyset$. Then the iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to q.*

Proof. The proof is quite similar to that of Theorem 1 above, and is thus omitted. □

Remark 10. *If $T_3 = I$ (Identity operator), then the iterative method (1.4) reduce to the iterative method (1.1). Thus, we conclude that Theorem 2 is a generalization and extension (*[3]*, Theorem 2).*

3. Stability Results

One of the most studied problems in fixed point theory is the stability of fixed points iterative methods. The initiator of this kind of study seems to be Urabe [13]. Later on, Ostrowski [14] has also put his efforts in this field. However, a formal definition for the stability of general iterative methods was apparently given by Harder and Hicks [15]. Continuing this trend, in the last three decades, a large literature has emerged and developed dealing with the stability of various well-known iterative methods for different classes of operators (see [10, 13-21] and references therein). Below we reformulate the definition of stability given by Harder and Hicks [15] in the context of $n-$normed spaces.

Definition 11. *Let X be a $n-$normed space, T a self map of X, and $\{x_n\}_{n=0}^{\infty}\subset X$ a sequence defined by*

$$x_{n+1} = f(T,x_n),\ n=0,1,\ldots, \tag{13}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point q of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set

$$\varepsilon_n = \|y_{n+1} - f(T, y_n), u_2, \cdots, u_n\|, \ n = 0, 1, \ldots.$$

Then, iteration procedure (3.1) is said to be $T-$stable or stable with respect to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} y_n = q$.

Theorem 12. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by the iterative method (5) with real sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset (0,1)$ satisfying $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, and $\sum_{n=0}^{\infty} b_n = \infty$ (or $\sum_{n=0}^{\infty} c_n = \infty$). Suppose that $q \in \bigcap_{i=1}^{3} F_{Ti} \neq \emptyset$. Let $\{r_n\}_{n=0}^{\infty} \subset X$ be any sequence and define a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ in $\mathbb{R}^+$ by*

$$\begin{cases} \varepsilon_n = \|r_{n+1} - a_n v_n - b_n T_1 v_n - c_n T_2 v_n, u_2, \cdots, u_n\|, \\ v_n = \alpha_n r_n + (1-\alpha_n) T_3 r_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{14}$$

Then, $\{x_n\}_{n=0}^{\infty}$ is stable with respect to (T_1, T_2, T_3).

Proof. Assume that $\lim_{n\to\infty} \varepsilon_n = 0$. In order to prove that $\{x_n\}_{n=0}^{\infty}$ is stable with respect to (T_1, T_2, T_3), it suffices to prove that $\lim_{n\to\infty} r_n = q$.

It follows from (5) and (7) that

$$\begin{aligned} \|r_{n+1} - q, u_2, \cdots, u_n\| &\leq \|r_{n+1} - a_n v_n - b_n T_1 v_n - c_n T_2 v_n, u_2, \cdots, u_n\| \\ &\quad + \|a_n v_n + b_n T_1 v_n + c_n T_2 v_n - q, u_2, \cdots, u_n\| \\ &\leq \varepsilon_n + a_n \|v_n - q, u_2, \cdots, u_n\| \\ &\quad + b_n \|T_1 v_n - q, u_2, \cdots, u_n\| + c_n \|T_2 v_n - q, u_2, \cdots, u_n\| \\ &\leq \varepsilon_n + [a_n + (b_n + c_n)\delta] \|v_n - q, u_2, \cdots, u_n\|, \end{aligned} \tag{15}$$

$$\begin{aligned} \|v_n - q, u_2, \cdots, u_n\| &\leq \alpha_n \|r_n - q, u_2, \cdots, u_n\| + (1-\alpha_n) \|T_3 r_n - q, u_2, \cdots, u_n\| \\ &\leq [\alpha_n + (1-\alpha_n)\delta] \|r_n - q, u_2, \cdots, u_n\|. \end{aligned} \tag{16}$$

Substituting (3.4) in (3.3), we get

$$\|r_{n+1} - q, u_2, \cdots, u_n\| \leq \varepsilon_n + [a_n + (b_n + c_n)\delta][\alpha_n + (1-\alpha_n)\delta] \|r_n - q, u_2, \cdots, u_n\|. \tag{17}$$

Since $\delta \in [0,1)$ and $\alpha_n \in [0,1]$ for all $n \in \mathbb{N}$,

$$\alpha_n + (1-\alpha_n)\delta < 1. \tag{18}$$

Using (3.6) in (3.5), we obtain

$$\begin{aligned} \|r_{n+1} - q, u_2, \cdots, u_n\| &\leq \varepsilon_n + [a_n + (b_n + c_n)\delta] \|r_n - q, u_2, \cdots, u_n\| \\ &= \varepsilon_n + [1 - (b_n + c_n)(1-\delta)] \|r_n - q, u_2, \cdots, u_n\|. \end{aligned}$$

Now using similar arguments as in the proof of Theorem 1, we obtain $\lim_{n\to\infty} r_n = q$ and hence the result. $\square$

Theorem 13. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by the iterative method (1.4) with real sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset (0,1)$ satisfying $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, and $\sum_{n=0}^{\infty} b_n = \infty$ (or $\sum_{n=0}^{\infty} c_n = \infty$). Suppose that $q \in \bigcap_{i=1}^{3} F_{Ti} \neq \emptyset$. Let $\{r_n\}_{n=0}^{\infty} \subset X$ be any sequence and define a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ in $\mathbb{R}^+$ by*

$$\begin{cases} \varepsilon_n = \|r_{n+1} - a_n r_n - b_n T_1 v_n - c_n T_2 r_n, u_2, \cdots, u_n\| \\ v_n = \alpha_n r_n + (1-\alpha_n) T_3 r_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{19}$$

Then, $\{x_n\}_{n=0}^{\infty}$ is stable with respect to (T_1, T_2, T_3).

Proof. The proof is quite similar to that of Theorem 3 above, and is thus omitted. □

Remark 14. *If $T_2 = I$ (Identity operator) and $T_1 = T_3 = T$, the iterative method (1.4) reduces to Ishikawa iterative method* [22]. *If $T_1 = T_3 = I$, then the iterative method (1.4) reduce to Mann iterative method* [23]. *Having regard to these facts, we conclude that Theorem 4 is a generalization and extension of both (*[10]*, Theorem 2.2) and (*[16]*, Theorem 2).*

4. Data Dependence Results

In some cases, it is difficult or may even be impossible to find a fixed point of a certain mapping. In such cases, instead of computing the fixed point of this mapping, we approximate this mapping with another one whose fixed point can be easily obtained. Thus we have an estimation for the approximate location of the fixed point of this mapping without actually computing it. For this reason, the topic of data dependency of fixed points has a great importance both from numerical and theoretical perspectives. Consequently, the study of data dependence of fixed points in a normed space setting has attracted several researchers; (see [12, 17, 25] and references therein).

Definition 15. *Let X be a $n-$normed space, $T, \widetilde{T} : X \to X$ two operators. We say that $\widetilde{T}$ is an approximate operator of T if for all $x, u_2, \cdots, u_n \in X$ and for a fixed $\varepsilon > 0$, we have*

$$\left\|Tx - \widetilde{T}x, u_2, \cdots, u_n\right\| \leq \varepsilon.$$

Theorem 16. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the iterative method* (5) *associated to T_1, T_2 and T_3 with a common fixed point $q \in \bigcap_{i=1}^{3} F_{Ti} \neq \emptyset$, and $\{\widetilde{x}_n\}_{n=0}^{\infty}$ be the iterative sequence generated by*

$$\begin{cases} \widetilde{x}_0 \in X, \\ \widetilde{x}_{n+1} = a_n \widetilde{y}_n + b_n \widetilde{T}_1 \widetilde{y}_n + c_n \widetilde{T}_2 \widetilde{y}_n \\ \widetilde{y}_n = \alpha_n \widetilde{x}_n + (1-\alpha_n) \widetilde{T}_3 \widetilde{x}_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{20}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset [0,1]$ *are real sequences satisfying* $a_n + b_n + c_n = 1$ *and* $\frac{1}{2-\delta} \leq b_n$ *(or* $\frac{1}{2-\delta} \leq c_n$ *) for all* $n \in \mathbb{N}$. *Suppose that for fixed* $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ *and for all* $x, u_2, \cdots, u_n \in X$, *we have* $\left\| T_1 x - \widetilde{T}_1 x, u_2, \cdots, u_n \right\| \leq \varepsilon_1$, $\left\| T_2 x - \widetilde{T}_2 x, u_2, \cdots, u_n \right\| \leq \varepsilon_2$, $\left\| T_3 x - \widetilde{T}_3 x, u_2, \cdots, u_n \right\| \leq \varepsilon_3$.

If $q^* \in \bigcap_{i=1}^{3} F_{\widetilde{T}i} \neq \emptyset$ *such that* $\widetilde{x}_n \to q^*$ *as* $n \to \infty$, *then we have*

$$\|q - q^*, u_2, \cdots, u_n\| \leq \frac{3\varepsilon}{1-\delta},$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

Proof. It follows from (5), (4.1), (7), and (8) that

$$\begin{aligned} \|x_{n+1} - \widetilde{x}_n, u_2, \cdots, u_n\| &\leq a_n \|y_n - \widetilde{y}_n, u_2, \cdots, u_n\| \\ &\quad + b_n \left\| T_1 y_n - \widetilde{T}_1 \widetilde{y}_n, u_2, \cdots, u_n \right\| \\ &\quad + c_n \left\| T_2 y_n - \widetilde{T}_2 \widetilde{y}_n, u_2, \cdots, u_n \right\|, \end{aligned} \tag{21}$$

$$\begin{aligned} \left\| T_1 y_n - \widetilde{T}_1 \widetilde{y}_n, u_2, \cdots, u_n \right\| &\leq \|T_1 y_n - T_1 \widetilde{y}_n, u_2, \cdots, u_n\| \\ &\quad + \left\| T_1 \widetilde{y}_n - \widetilde{T}_1 \widetilde{y}_n, u_2, \cdots, u_n \right\| \\ &\leq 2\delta \|y_n - q, u_2, \cdots, u_n\| \\ &\quad + \delta \|y_n - \widetilde{y}_n, u_2, \cdots, u_n\| + \varepsilon_1, \end{aligned} \tag{22}$$

$$\begin{aligned} \left\| T_2 y_n - \widetilde{T}_2 \widetilde{y}_n, u_2, \cdots, u_n \right\| &\leq 2\delta \|y_n - q, u_2, \cdots, u_n\| \\ &\quad + \delta \|y_n - \widetilde{y}_n, z_2, \ldots, z_n\| + \varepsilon_2, \end{aligned} \tag{23}$$

$$\|y_n - q, u_2, \cdots, u_n\| \leq [\alpha_n + (1-\alpha_n)\delta] \|x_n - q, u_2, \cdots, u_n\|, \tag{24}$$

$$\begin{aligned} \|y_n - \widetilde{y}_n, u_2, \cdots, u_n\| &\leq \alpha_n \|x_n - \widetilde{x}_n, u_2, \cdots, u_n\| \\ &\quad + (1-\alpha_n) \left\| T_3 x_n - \widetilde{T}_3 \widetilde{x}_n, u_2, \cdots, u_n \right\|, \end{aligned}$$

$$\begin{aligned} \left\| T_3 x_n - \widetilde{T}_3 \widetilde{x}_n, u_2, \cdots, u_n \right\| &\leq 2\delta \|x_n - q, u_2, \cdots, u_n\| \\ &\quad + \delta \|x_n - \widetilde{x}_n, u_2, \cdots, u_n\| + \varepsilon_3. \end{aligned} \tag{25}$$

Combining (4.2)-(4.6)

$$\begin{aligned} \|x_{n+1} - \widetilde{x}_n, u_2, \cdots, u_n\| &\leq [a_n + (b_n + c_n)\delta] [\alpha_n + \delta(1-\alpha_n)] \|x_n - \widetilde{x}_n, u_2, \cdots, u_n\| \\ &\quad + \{(b_n + c_n) [\alpha_n + (1-\alpha_n)\delta] \\ &\quad + (1-\alpha_n) [a_n + (b_n + c_n)\delta]\} 2\delta \|x_n - q, u_2, \cdots, u_n\| \\ &\quad + b_n \varepsilon_1 + c_n \varepsilon_2 + (1-\alpha_n) [a_n + (b_n + c_n)\delta] \varepsilon_3, \end{aligned} \tag{26}$$

Since $\delta \in [0,1)$ and $\alpha_n, a_n, b_n, c_n \in [0,1]$ for all $n \in \mathbb{N}$,we have

$$\begin{cases} 1-\alpha_n \leq 1, \\ b_n \leq b_n + c_n, \\ c_n \leq b_n + c_n. \end{cases} \tag{27}$$

Using (2.4), (3.6), (4.8) and assumption $\frac{1}{2-\delta} \leq b_n$ (which implies $\frac{1}{2-\delta} \leq b_n + c_n$) for all $n \in \mathbb{N}$, (4.7) becomes

$$\begin{aligned} \|x_{n+1} - \widetilde{x}_n, u_2, \cdots, u_n\| \leq {} & [1-(b_n+c_n)(1-\delta)] \|x_n - \widetilde{x}_n, u_2, \cdots, u_n\| \\ & + (b_n+c_n)(1-\delta) \\ & \times \frac{\{(2-\alpha_n) 2\delta \|x_n - q, u_2, \cdots, u_n\| + \varepsilon_1 + \varepsilon_2 + \varepsilon_3\}}{1-\delta}. \end{aligned} \tag{28}$$

Define

$$\begin{aligned} \sigma_n &= \|x_n - \widetilde{x}_n, u_2, \cdots, u_n\|, \\ \mu_n &= (b_n+c_n)(1-\delta) \in (0,1), \\ \gamma_n &= \frac{(2-\alpha_n) 2\delta \|x_n - q, u_2, \cdots, u_n\| + \varepsilon_1 + \varepsilon_2 + \varepsilon_3}{1-\delta}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

As the assumption $\frac{1}{2-\delta} \leq b_n$ (or $\frac{1}{2-\delta} \leq c_n$) implies $\sum_{n=0}^{\infty} b_n = \infty$ (or $\sum_{n=0}^{\infty} c_n = \infty$), we have $\sum_{n=0}^{\infty} (b_n + c_n) = \infty$ as in the proof of Theorem 1. Thus all conditions in Lemma 2 are satisfied by (4.9). Also, from Theorem 1, we know that $x_n \to q$ as $n \to \infty$. Hence, we have

$$\|q - q^*, u_2, \cdots, u_n\| \leq \frac{3\varepsilon}{1-\delta},$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. □

Theorem 17. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the iterative method (1.4) associated to T_1, T_2 and T_3 with a common fixed point $q \in \bigcap_{i=1}^{3} F_{Ti} \neq \emptyset$, and $\{\widetilde{x}_n\}_{n=0}^{\infty}$ be the iterative sequence generated by*

$$\begin{cases} \widetilde{x}_0 \in X, \\ \widetilde{x}_{n+1} = a_n \widetilde{x}_n + b_n \widetilde{T}_1 \widetilde{y}_n + c_n \widetilde{T}_2 \widetilde{x}_n, \\ \widetilde{y}_n = \alpha_n \widetilde{x}_n + (1-\alpha_n) \widetilde{T}_3 \widetilde{x}_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{29}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset [0,1]$ are real sequences satisfying $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, and $\sum_{n=0}^{\infty} b_n = \infty$ (or $\sum_{n=0}^{\infty} c_n = \infty$). Suppose that for fixed $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and for all $x, u_2, \cdots, u_n \in X$, we have

$$\left\| T_1 x - \widetilde{T}_1 x, u_2, \cdots, u_n \right\| \leq \varepsilon_1, \left\| T_2 x - \widetilde{T}_2 x, u_2, \cdots, u_n \right\| \leq \varepsilon_2,$$

$$\left\|T_3x - \widetilde{T}_3x, u_2, \cdots, u_n\right\| \quad \leq \quad \varepsilon_3.$$

If $q^* \in \bigcap_{i=1}^{3} F_{\widetilde{T}i} \neq \emptyset$ *such that* $\widetilde{x}_n \to q^*$ *as* $n \to \infty$*, then we have*

$$\|q - q^*, u_2, \cdots, u_n\| \leq \frac{(2+\delta)\,\varepsilon}{1-\delta},$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

Proof. The proof is quite similar to that of Theorem 5 above, and is thus omitted. □

Remark 18. *If* $a_n = 0$ *and* $T_1 = T_2 = T_3 = T$*, then iterative method (1.3) reduce to* $S-$*iterative method* [24]*. Also, keeping in mind Remark 3, Theorem 6 generalizes both (*[12]*, Theorem 3.2) and (*[25]*, Theorem 4).*

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CLASSICAL AND STRONGLY CLASSICAL 2-ABSORBING SECOND SUBMODULES

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Abstract. In this paper, we will introduce the concept of classical (resp. strongly classical) 2-absorbing second submodules of modules over a commutative ring as a generalization of 2-absorbing (resp. strongly 2-absorbing) second submodules and investigate some basic properties of these classes of modules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. Further, $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [11]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [18]. In this case $Ann_R(S)$ is a prime ideal of R.

The notion of 2-absorbing ideals as a generalization of prime ideals was introduced and studied in [7]. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The authors in [10] and [15], extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule N of M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [12].

In [5], the present authors introduced the dual notion of 2-absorbing submodules (that is, *2-absorbing (resp. strongly 2-absorbing) second submodules*) of M and

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investigated some properties of these classes of modules. A non-zero submodule N of M is said to be a *2-absorbing second submodule of* M if whenever $a, b \in R$, L is a completely irreducible submodule of M, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in Ann_R(N)$. A non-zero submodule N of M is said to be a *strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$.

In [14], the authors introduced the notion of classical 2-absorbing submodules as a generalization of 2-absorbing submodules and studied some properties of this class of modules. A proper submodule N of M is called *classical 2-absorbing submodule* if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$ [14].

The purpose of this paper is to introduce the concepts of classical and strongly classical 2-absorbing second submodules of an R-module M as dual notion of classical 2-absorbing submodules and provide some information concerning these new classes of modules. We characterize classical (resp. strongly classical) 2-absorbing second submodules in Theorem 2.3 (resp. Theorem 3.4). Also, we consider the relationship between classical 2-absorbing and strongly classical 2-absorbing second submodules in Examples 3.9, 3.10, and Propositions 3.11. Theorem 2.14 (resp. Theorem 3.15) of this paper shows that if M is an Artinian R-module, then every non-zero submodule of M has only a finite number of maximal classical (resp. strongly classical) 2-absorbing second submodules. Further, among other results, we investigate strongly classical 2-absorbing second submodules of a finite direct product of modules in Theorem 3.19.

2. Classical 2-absorbing second submodules

We frequently use the following basic fact without further comment.

Remark 2.1. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Definition 2.2. Let N be a non-zero submodule of an R-module M. We say that N is a *classical 2-absorbing second submodule of* M if whenever $a, b, c \in R$, L is a completely irreducible submodule of M, and $abcN \subseteq L$, then $abN \subseteq L$ or $bcN \subseteq L$ or $acN \subseteq L$. We say M is a *classical 2-absorbing second module* if M is a classical 2-absorbing second submodule of itself.

Theorem 2.3. *Let M be an R-module and N be a non-zero submodule of M. Then the following statements are equivalent:*

(a) *N is a classical 2-absorbing second submodule of M;*
(b) *For every $a, b \in R$ and completely irreducible submodule L of M with $abN \not\subseteq L$, $(L :_R abN) = (L :_R aN) \cup (L :_R bN)$;*
(c) *For every $a, b \in R$ and completely irreducible submodule L of M with $abN \not\subseteq L$, $(L :_R abN) = (L :_R aN)$ or $(L :_R abN) = (L :_R bN)$;*

(d) *For every $a, b \in R$, every ideal I of R, and completely irreducible submodule L of M with $abIN \subseteq L$, either $abN \subseteq L$ or $aIN \subseteq L$ or $bIN \subseteq L$;*
(e) *For every $a \in R$, every ideal I of R, and completely irreducible submodule L of M with $aIN \not\subseteq L$, $(L :_R aIN) = (L :_R IN)$ or $(L :_R aIN) = (L :_R aN)$;*
(f) *For every $a \in R$, ideals I, J of R, and completely irreducible submodule L of M with $aIJN \subseteq L$, either $aIN \subseteq L$ or $aJN \subseteq L$ or $IJN \subseteq L$;*
(g) *For ideals I, J of R, and completely irreducible submodule L of M with $IJN \not\subseteq L$, $(L :_R IJN) = (L :_R IN)$ or $(L :_R IJN) = (L :_R JN)$;*
(h) *For ideals I_1, I_2, I_3 of R, and completely irreducible submodule L of M with $I_1I_2I_3N \subseteq L$, either $I_1I_2N \subseteq L$ or $I_1I_3N \subseteq L$ or $I_2I_3N \subseteq L$;*
(i) *For each completely irreducible submodule L of M with $N \not\subseteq L$, $(L :_R N)$ is a 2-absorbing ideal of R.*

Proof. $(a) \Rightarrow (b)$ Let $t \in (L :_R abN)$. Then $tabN \subseteq L$. Since $abN \not\subseteq L$, $atN \subseteq L$ or $btN \subseteq L$ as needed.

$(b) \Rightarrow (c)$ This follows from the fact that if an ideal is the union of two ideals, then it is equal to one of them.

$(c) \Rightarrow (d)$ Let for some $a, b \in R$, an ideal I of R, and completely irreducible submodule L of M, $abIN \subseteq L$. Then $I \subseteq (L :_R abN)$. If $abN \subseteq L$, then we are done. Assume that $abN \not\subseteq L$. Then by part (c), $I \subseteq (L :_R bN)$ or $I \subseteq (L :_R aN)$ as desired.

$(d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h)$ The proofs are similar to that of the previous implications.

$(h) \Rightarrow (a)$ Trivial.

$(h) \Leftrightarrow (i)$ This is straightforward. □

We recall that an R-module M is said to be a *cocyclic module* if $Soc_R(M)$ is a large and simple submodule of M [19]. (Here $Soc_R(M)$ denotes the sum of all minimal submodules of M.) A submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R-module [12].

Corollary 2.4. Let N be a classical 2-absorbing second submodule of a cocyclic R-module M. Then $Ann_R(N)$ is a 2-absorbing ideal of R.

Proof. This follows from Theorem 2.3 $(a) \Rightarrow (i)$, because (0) is a completely irreducible submodule of M. □

Example 2.5. For any prime integer p, let $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module and $G_i = \langle 1/p^i + \mathbb{Z} \rangle$ for $i \in \mathbb{N}$. Then G_i is not a classical 2-absorbing second submodule of M for each integers $i \geq 3$.

Lemma 2.6. Every 2-absorbing second submodule of M is a classical 2-absorbing second submodule of M.

Proof. Let N be a 2-absorbing second submodule of M, $a, b, c \in R$, L a completely irreducible submodule of M, and $abcN \subseteq L$. Then $abN \subseteq (L :_M c)$. Thus $aN \subseteq$

$(L :_M c)$ or $bN \subseteq (L :_M c)$ or $abN = 0$ because by [6, Lemma 2.1], $(L :_M c)$ is a completely irreducible submodule of M. Hence $acN \subseteq L$ or $bcN \subseteq L$ or $abN \subseteq L$ as needed. $\square$

Example 2.7. Consider $M = \mathbb{Z}_{pq} \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module, where p, q are prime integers. Then M is a classical 2-absorbing second module which is not a strongly 2-absorbing second module.

Proposition 2.8. Let N be a classical 2-absorbing second submodule of an R-module M. Then we have the following.

(a) If $a \in R$, then $a^n N = a^{n+1} N$, for all $n \geq 2$.

(b) If L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $\sqrt{(L :_R N)}$ is a 2-absorbing ideal of R.

Proof. (a) It is enough to show that $a^2 N = a^3 N$. It is clear that $a^3 N \subseteq a^2 N$. Let L be a completely irreducible submodule of M such that $a^3 N \subseteq L$. Since N is a classical 2-absorbing second submodule, $a^2 N \subseteq L$. This implies that $a^2 N \subseteq a^3 N$.

(b) Assume that $a, b, c \in R$ and $abc \in \sqrt{(L :_R N)}$. Then there is a positive integer t such that $a^t b^t c^t N \subseteq L$. By hypotheses, N is a classical 2-absorbing second submodule of M, thus $a^t b^t N \subseteq L$ or $b^t c^t N \subseteq L$ or $a^t c^t N \subseteq L$. Therefore, $ab \in \sqrt{(L :_R N)}$ or $bc \in \sqrt{(L :_R N)}$ or $ac \in \sqrt{(L :_R N)}$. $\square$

Theorem 2.9. *Let N be a submodule of an R-module M. Then we have the following.*

(a) *If N is a classical 2-absorbing second submodule of M, then IN is a classical 2-absorbing second submodule of M for all ideals I of R with $I \not\subseteq Ann_R(N)$.*

(b) *If N is a classical 2-absorbing submodule of M, then $(N :_R I)$ is a classical 2-absorbing submodule of M for all ideals I of R with $I \not\subseteq (N :_R M)$.*

(c) *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. If $\acute{N}$ is a classical 2-absorbing second submodule of $f(M)$, then $f^{-1}(\acute{N})$ is a classical 2-absorbing second submodule of M.*

Proof. (a) Let I be an ideal of R with $I \not\subseteq Ann_R(N)$, $a, b, c \in R$, L be a completely irreducible submodule of M, and $abcIN \subseteq L$. Then $acN \subseteq L$ or $cbIN \subseteq L$ or $abIN \subseteq L$ by Theorem 2.3 $(a) \Rightarrow (d)$. If $cbIN \subseteq L$ or $abIN \subseteq L$, then we are done. If $acN \subseteq L$, then $acIN \subseteq acN$ implies that $acIN \subseteq L$, as needed. Since $I \not\subseteq Ann_R(N)$, we have IN is a non-zero submodule of M.

(b) Use the technique of part (a) and apply [14, Theorem 2].

(c) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = ff^{-1}(\acute{N}) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a, b, c \in R$, L be a completely irreducible submodule of M, and $abcf^{-1}(\acute{N}) \subseteq L$. Then

$$abc\acute{N} = abc(f(M) \cap \acute{N}) = abcff^{-1}(\acute{N}) \subseteq f(L).$$

By [5, Lemma 3.14], $f(L)$ is a completely irreducible submodule of $f(M)$. Thus as $\acute{N}$ is a classical 2-absorbing second submodule, $ab\acute{N} \subseteq f(L)$ or $bc\acute{N} \subseteq f(L)$ or

$ac\acute{N} \subseteq f(L)$. Therefore, $abf^{-1}(\acute{N}) \subseteq f^{-1}f(L) = L$ or $bcf^{-1}(\acute{N}) \subseteq f^{-1}f(L) = L$ or $acf^{-1}(\acute{N}) \subseteq f^{-1}f(L) = L$, as desired. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [8].

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [2].

Corollary 2.10. Let M be an R-module. Then we have the following.

(a) If M is a multiplication classical 2-absorbing second R-module, then every non-zero submodule of M is a classical 2-absorbing second submodule of M.

(b) If M is a comultiplication module and the zero submodule of M is a classical 2-absorbing submodule, then every proper submodule of M is a classical 2-absorbing submodule of M.

Proof. This follows from parts (a) and (b) of Lemma 2.9. □

Proposition 2.11. Let M be an R-module and $\{K_i\}_{i\in I}$ be a chain of classical 2-absorbing second submodules of M. Then $\sum_{i\in I} K_i$ is a classical 2-absorbing second submodule of M.

Proof. Let $a, b, c \in R$, L be a completely irreducible submodule of M, and $abc \sum\limits_{i\in I} K_i \subseteq L$. Assume that $ab \sum_{i\in I} K_i \not\subseteq L$ and $ac \sum_{i\in I} K_i \not\subseteq L$. Then there are $m, n \in I$ where $abK_n \not\subseteq L$ and $acK_m \not\subseteq L$. Hence, for every $K_n \subseteq K_s$ and every $K_m \subseteq K_d$ we have that $abK_s \not\subseteq L$ and $acK_d \not\subseteq L$. Therefore, for each submodule K_h such that $K_n \subseteq K_h$ and $K_m \subseteq K_h$, we have $bcK_h \subseteq L$. Hence $bc \sum_{i\in I} K_i \subseteq L$, as needed. □

Definition 2.12. We say that a classical 2-absorbing second submodule N of an R-module M is a *maximal classical 2-absorbing second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a classical 2-absorbing second submodule T of M such that $N \subset T \subset K$.

Lemma 2.13. Let M be an R-module. Then every classical 2-absorbing second submodule of M is contained in a maximal classical 2-absorbing second submodule of M.

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.11. □

Theorem 2.14. *Let M be an Artinian R-module. Then every non-zero submodule of M has only a finite number of maximal classical 2-absorbing second submodules.*

Proof. Suppose that there exists a non-zero submodule N of M such that it has an infinite number of maximal classical 2-absorbing second submodules. Let S be a submodule of M chosen minimal such that S has an infinite number of maximal classical 2-absorbing second submodules because M is an Artinian R-module. Then S is not a classical 2-absorbing second submodule. Thus there exist $a, b, c \in R$ and a completely irreducible submodule L of M such that $abcS \subseteq L$ but $abS \not\subseteq L$, $acS \not\subseteq L$, and $bcS \not\subseteq L$. Let V be a maximal classical 2-absorbing second submodule of M contained in S. Then $abV \subseteq L$ or $acV \subseteq L$ or $bcV \subseteq L$. Thus $V \subseteq (L :_M ab)$ or $V \subseteq (L :_M ac)$ or $V \subseteq (L :_M bc)$. Therefore, $V \subseteq (L :_S ab)$ or $V \subseteq (L :_S ac)$ or $V \subseteq (L :_S bc)$. By the choice of S, the modules $(L :_S ab)$, $(L :_S ac)$, and $(L :_S bc)$ have only finitely many maximal classical 2-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module S, which is a contradiction. □

3. Strongly classical 2-absorbing second submodules

Definition 3.1. Let N be a non-zero submodule of an R-module M. We say that N is a *strongly classical 2-absorbing second submodule of* M if whenever $a, b, c \in R$, L_1, L_2, L_3 are completely irreducible submodules of M, and $abcN \subseteq L_1 \cap L_2 \cap L_3$, then $abN \subseteq L_1 \cap L_2 \cap L_3$ or $bcN \subseteq L_1 \cap L_2 \cap L_3$ or $acN \subseteq L_1 \cap L_2 \cap L_3$. We say M is a *strongly classical 2-absorbing second module* if M is a strongly classical 2-absorbing second submodule of itself.

Clearly every strongly classical 2-absorbing second submodule is a classical 2-absorbing second submodule.

Question 3.2. Let M be an R-module. Is every classical 2-absorbing second submodule of M a strongly classical 2-absorbing second submodule of M?

Example 3.3. The $\mathbb{Z}$-module $\mathbb{Z}$ has no strongly classical 2-absorbing second submodule.

Theorem 3.4. *Let M be an R-module and N be a non-zero submodule of M. Then the following statements are equivalent:*

(a) *N is strongly classical 2-absorbing second;*
(b) *If $a, b, c \in R$, K is a submodule of M, and $abcN \subseteq K$, then $abN \subseteq K$ or $bcN \subseteq K$ or $acN \subseteq K$;*
(c) *For every $a, b, c \in R$, $abcN = abN$ or $abcN = acN$ or $abcN = bcN$;*
(d) *For every $a, b \in R$ and submodule K of M with $abN \not\subseteq K$, $(K :_R abN) = (K :_R aN) \cup (K :_R bN)$;*
(e) *For every $a, b \in R$ and submodule K of M with $abN \not\subseteq K$, $(K :_R abN) = (K :_R aN)$ or $(K :_R abN) = (K :_R bN)$;*
(f) *For every $a, b \in R$, every ideal I of R, and submodule K of M with $abIN \subseteq K$, either $abN \subseteq K$ or $aIN \subseteq K$ or $bIN \subseteq K$;*
(g) *For every $a \in R$, every ideal I of R, and submodule K of M with $aIN \not\subseteq K$, $(K :_R aIN) = (K :_R IN)$ or $(K :_R aIN) = (K :_R aN)$;*
(h) *For every $a \in R$, ideals I, J of R, and submodule K of M with $aIJN \subseteq K$, either $aIN \subseteq K$ or $aJN \subseteq K$ or $IJN \subseteq K$;*

(i) *For ideals I, J of R, and submodule K of M with $IJN \not\subseteq K$, $(K :_R IJN) = (K :_R IN)$ or $(K :_R IJN) = (K :_R JN)$;*

(j) *For ideals I_1, I_2, I_3 of R, and submodule K of M with $I_1I_2I_3N \subseteq K$, either $I_1I_2N \subseteq K$ or $I_1I_3N \subseteq K$ or $I_2I_3N \subseteq K$;*

(k) *For each submodule K of M with $N \not\subseteq K$, $(K :_R N)$ is a 2-absorbing ideal of R.*

Proof. $(a) \Rightarrow (b)$ Let $a, b, c \in R$, K is a submodule of M, and $abcN \subseteq K$. Assume on the contrary that $abN \not\subseteq K$, $bcN \not\subseteq K$, and $acN \not\subseteq K$. Then there exist completely irreducible submodules L_1, L_2, L_3 of M such that K is a submodule of them but $abN \not\subseteq L_1$, $bcN \not\subseteq L_2$, and $acN \not\subseteq L_3$. Now we have $abcN \subseteq L_1 \cap L_2 \cap L_3$. Thus by part (a), $abN \subseteq L_1 \cap L_2 \cap L_3$ or $bcN \subseteq L_1 \cap L_2 \cap L_3$ or $acN \subseteq L_1 \cap L_2 \cap L_3$. Therefore, $abN \subseteq L_1$ or $bcN \subseteq L_2$ or $acN \subseteq L_3$ which are contradictions.

$(b) \Rightarrow (c)$ Let $a, b, c \in R$. Then $abcN \subseteq abcN$ implies that $abN \subseteq abcN$ or $bcN \subseteq abcN$ or $acN \subseteq abcN$ by part (b). Thus $abN = abcN$ or $bcN = abcN$ or $acN = abcN$ because the reverse inclusions are clear.

$(c) \Rightarrow (d)$ Let $t \in (K :_R abN)$. Then $tabN \subseteq K$. Since $abN \not\subseteq K$, $atN \subseteq K$ or $btN \subseteq K$ as needed.

$(d) \Rightarrow (e)$ This follows from the fact that if an ideal is the union of two ideals, then it is equal to one of them.

$(e) \Rightarrow (f)$ Let for some $a, b \in R$, an ideal I of R, and submodule K of M, $abIN \subseteq K$. Then $I \subseteq (K :_R abN)$. If $abN \subseteq K$, then we are done. Assume that $abN \not\subseteq K$. Then by part (d), $I \subseteq (K :_R bN)$ or $I \subseteq (K :_R aN)$ as desired.

$(g) \Rightarrow (h) \Rightarrow (i) \Rightarrow (h) \Rightarrow (j)$ Have proofs similar to that of the previous implications.

$(j) \Rightarrow (a)$ Trivial.

$(j) \Leftrightarrow (k)$ This is straightforward. $\square$

Let N be a submodule of an R-module M. Then Theorem 3.4 $(a) \Leftrightarrow (c)$ shows that N is a strongly classical 2-absorbing second submodule of M if and only if N is a strongly classical 2-absorbing second module.

Corollary 3.5. Let N be a strongly classical 2-absorbing second submodule of an R-module M and I be an ideal of R. Then $I^nN = I^{n+1}N$, for all $n \geq 2$.

Proof. It is enough to show that $I^2N = I^3N$. By Theorem 3.4, $I^2N = I^3N$. $\square$

Example 3.6. Clearly every strongly 2-absorbing second submodule is a strongly classical 2-absorbing second submodule. But the converse is not true in general. For example, consider $M = \mathbb{Z}_{\ntriangleleft} \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module. Then M is a strongly classical 2-absorbing second module. But M is not a strongly 2-absorbing second module.

A non-zero submodule N of an R-module M is said to be a *weakly second submodule* of M if $rsN \subseteq K$, where $r, s \in R$ and K is a submodule of M, implies either $rN \subseteq K$ or $sN \subseteq K$ [1].

Proposition 3.7. Let M be an R-module. Then we have the following.

(a) If M is a comultiplication R-module and N is a strongly classical 2-absorbing second submodule of M, then N is a strongly 2-absorbing second submodule of M.

(b) If N_1, N_2 are weakly second submodules of M, then $N_1 + N_2$ is a strongly classical 2-absorbing second submodule of M.

(c) If N is a strongly classical 2-absorbing second submodule of M, then IN is a strongly classical 2-absorbing second submodule of M for all ideals I of R with $I \not\subseteq Ann_R(N)$.

(d) If M is a multiplication strongly classical 2-absorbing second R-module, then every non-zero submodule of M is a classical 2-absorbing second submodule of M.

(e) If M is a strongly classical 2-absorbing second R-module, then every non-zero homomorphic image of M is a classical 2-absorbing second R-module.

Proof. (a) By Theorem 3.4 $(a) \Rightarrow (k)$, $Ann_R(N)$ is a 2-absorbing ideal of R. Now the result follows from [5, Theorem 3.10].

(b) Let N_1, N_2 be weakly second submodules of M and $a, b, c \in R$. Since N_1 is a weakly second submodule, we may assume that $abcN_1 = aN_1$. Likewise, assume that $abcN_2 = bN_2$. Hence $abc(N_1 + N_2) = ab(N_1 + N_2)$ which implies $N_1 + N_2$ is a classical 2-absorbing second submodule by Theorem 3.4 $(c) \Rightarrow (a)$.

(c) Use the technique of the proof of Theorem 2.9 (a).

(d) This follows from part (c).

(e) This is straightforward. □

For a submodule N of an R-module M the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [9] and [3]).

Theorem 3.8. *Let M be a finitely generated comultiplication R-module. If N is a strongly classical 2-absorbing second submodule of M, then $sec(N)$ is a strongly 2-absorbing second submodule of M.*

Proof. Let N be a strongly classical 2-absorbing second submodule of M. By Proposition 3.7 (a), $Ann_R(N)$ is a 2-absorbing ideal of R. Thus by [7, Theorem 2.1], $\sqrt{Ann_R(N)}$ is a 2-absorbing ideal of R. By [4, Theorem 2.12], $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$. Therefore, $Ann_R(sec(N))$ is a 2-absorbing ideal of R. Now the result follows from [5, Theorem 3.10]. □

The following examples show that the two concepts of classical 2-absorbing submodules and strongly classical 2-absorbing second submodules are different in general.

Example 3.9. The submodule $2\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is a classical 2-absorbing submodule which is not a strongly classical 2-absorbing second module.

Example 3.10. The submodule $\langle 1/p + \mathbb{Z}\rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a a strongly classical 2-absorbing second module which is not a classical 2-absorbing submodule of $\mathbb{Z}_{p^\infty}$.

A commutative ring R is said to be a *u-ring* provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a *um-ring* is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them [16].

In the following proposition, we investigate the relationships between strongly classical 2-absorbing second submodules and classical 2-absorbing submodules.

Proposition 3.11. Let M be a non-zero R-module. Then we have the following.

(a) If M is a finitely generated strongly classical 2-absorbing second R-module, then the zero submodule of M is a classical 2-absorbing submodule.
(b) If M is a multiplication strongly classical 2-absorbing second R-module, then the zero submodule of M is a classical 2-absorbing submodule.
(c) Let R be a *um*-ring. If M is a Artinian R-module and the zero submodule of M is a classical 2-absorbing submodule, then M is a strongly classical 2-absorbing second R-module.
(d) Let R be a *um*-ring. If M is a comultiplication R-module and the zero submodule of M is a classical 2-absorbing submodule, then M is a strongly classical 2-absorbing second R-module.

Proof. (a) Let $a, b, c \in R$, $m \in M$, and $abcm = 0$. By Theorem 3.4, we can assume that $abcM = acM$. Since M is finitely generated, by using [13, Theorem 76], $Ann_R(abM) + Rc = R$. It follows that $(0 :_M abc) = (0 :_M ab)$. This implies that $abm = 0$, as needed.

(b) Let $a, b, c \in R$, $m \in M$, and $abcm = 0$. Then by Theorem 3.4, we can assume that $abcM = acM$. Thus

$$0 = abc((0 :_M abc) :_R M)M = (((0 :_M abc) :_R M)M)ab.$$

Since M is a multiplication module, $((0 :_M abc) :_R M)M = (0 :_M abc)$. Therefore, $(0 :_M abc)ab = 0$. It follows that $(0 :_M abc) \subseteq (0 :_M ab)$. Thus $(0 :_M abc) = (0 :_M ab)$ because the reverse inclusion is clear. Hence $abm = 0$, as required.

(c) Let $a, b, c \in R$. Then by [14, Theorem 4], we can assume that $(0 :_M abc) = (0 :_M ab)$. Hence $(0 :_{M/(0:_M ab)} c) = 0$. Since M is Artinian, it follows that $cM + (0 :_M ab) = M$. Therefore, $abcM = abM$. Thus by Theorem 3.4 $(c) \Rightarrow (a)$, M is a classical 2-absorbing second R-module.

(d) Let $a, b, c \in R$. Then by [14, Theorem 4], we can assume that $(0 :_M abc) = (0 :_M ab)$. Since M is a comultiplication R-module, this implies that

$$M = ((0 :_M abc) :_M Ann_R(abcM) = ((0 :_M ab) :_M Ann_R(abcM)) = (abcM :_M ab).$$

It follows that $abM \subseteq abcM$. Thus $abM = abcM$ because the reverse implication is clear and this completed the proof. $\square$

Proposition 3.12. Let M be an R-module and $\{K_i\}_{i\in I}$ be a chain of strongly classical 2-absorbing second submodules of M. Then $\sum_{i\in I} K_i$ is a strongly classical 2-absorbing second submodule of M.

Proof. Use the technique of Proposition 2.11. □

Definition 3.13. We say that a strongly classical 2-absorbing second submodule N of an R-module M is a *maximal strongly classical 2-absorbing second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a strongly classical 2-absorbing second submodule T of M such that $N \subset T \subset K$.

Lemma 3.14. Let M be an R-module. Then every strongly classical 2-absorbing second submodule of M is contained in a maximal strongly classical 2-absorbing second submodule of M.

Proof. This is proved easily by using Zorn's Lemma and Proposition 3.12. □

Theorem 3.15. *Let M be an Artinian R-module. Then every non-zero submodule of M has only a finite number of maximal strongly classical 2-absorbing second submodules.*

Proof. Use the technique of Theorem 2.14 any apply Lemma 3.14. □

Theorem 3.16. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.*

(a) *If N is a strongly classical 2-absorbing second submodule of M, then $f(N)$ is a strongly classical 2-absorbing second submodule of $\acute{M}$.*
(b) *If $\acute{N}$ is a strongly classical 2-absorbing second submodule of $f(M)$, then $f^{-1}(\acute{N})$ is a strongly classical 2-absorbing second submodule of M.*

Proof. (a) Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $a, b, c \in R$. Then by Theorem 3.4 $(a) \Rightarrow (c)$, we can assume that $abcN = abN$. Thus

$$abcf(N) = f(abcN) = f(abN) = abf(N).$$

Hence $f(N)$ is a strongly classical 2-absorbing second submodule of $\acute{M}$ by Theorem 3.4 $(c) \Rightarrow (a)$.

(b) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = ff^{-1}(\acute{N}) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a, b, c \in R$, K be a submodule of M, and $abcf^{-1}(\acute{N}) \subseteq K$. Then

$$abc\acute{N} = abc(f(M) \cap \acute{N}) = abcff^{-1}(\acute{N}) \subseteq f(K).$$

Thus as $\acute{N}$ is a strongly classical 2-absorbing second submodule, $ab\acute{N} \subseteq f(K)$ or $bc\acute{N} \subseteq f(K)$ or $ac\acute{N} \subseteq f(K)$. Therefore, $abf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$ or $bcf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$ or $acf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$, as desired. □

Let R_i be a commutative ring with identity and M_i be an R_i-module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 3.17. *Let $R = R_1 \times R_2$ be a decomposable ring and let $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a non-zero submodule of M. Then the following conditions are equivalent:*

(a) *N is a strongly classical 2-absorbing second submodule of M;*

(b) *Either $N_1 = 0$ and N_2 is a strongly classical 2-absorbing second submodule of M_2 or $N_2 = 0$ and N_1 is a strongly classical 2-absorbing second submodule of M_1 or N_1, N_2 are weakly second submodules of M_1, M_2, respectively.*

Proof. $(a) \Rightarrow (b)$. Suppose that N is a strongly classical 2-absorbing second submodule of M such that $N_2 = 0$. From our hypothesis, N is non-zero, so $N_1 \neq 0$. Set $\acute{M} = M_1 \times 0$. One can see that $\acute{N} = N_1 \times 0$ is a strongly classical 2-absorbing second submodule of $\acute{M}$. Also observe that $\acute{M} \cong M_1$ and $\acute{N} \cong N_1$. Thus N_1 is a strongly classical 2-absorbing second submodule of M_1. Suppose that $N_1 \neq 0$ and $N_2 \neq 0$. We show that N_1 is a weakly second submodule of M_1. Since $N_2 \neq 0$, there exists a completely irreducible submodule L_2 of M_2 such that $N_2 \not\subseteq L_2$. Let $abN_1 \subseteq K$ for some $a, b \in R_1$ and submodule K of M_1. Thus $(a, 1)(b, 1)(1, 0)(N_1 \times N_2) = abN_1 \times 0 \subseteq K \times L_2$. So either $(a, 1)(b, 1)(N_1 \times N_2) = abN_1 \times N_2 \subseteq K \times L_2$ or $(a, 1)(1, 0)(N_1 \times N_2) = aN_1 \times 0 \subseteq K \times L_2$ or $(b, 1)(1, 0)(N_1 \times N_2) = bN_1 \times 0 \subseteq K \times L_2$. If $abN_1 \times N_2 \subseteq K \times L_2$, then $N_2 \subseteq L_2$, a contradiction. Hence either $aN_1 \subseteq K$ or $bN_1 \subseteq K$ which shows that N_1 is a weakly second submodule of M_1. Similarly, we can show that N_2 is a weakly second submodule of M_2.

$(b) \Rightarrow (a)$. Suppose that $N = N_1 \times 0$, where N_1 is a strongly classical 2-absorbing (resp. weakly) second submodule of M_1. Then it is clear that N is a strongly classical 2-absorbing (resp. weakly) second submodule of M. Now, assume that $N = N_1 \times N_2$, where N_1 and N_2 are weakly second submodules of M_1 and M_2, respectively. Hence $(N_1 \times 0) + (0 \times N_2) = N_1 \times N_2 = N$ is a strongly classical 2-absorbing second submodule of M, by Proposition 3.7 (b). □

Lemma 3.18. Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a decomposable ring and $M = M_1 \times M_2 \cdots \times M_n$ be an R-module where for every $1 \leq i \leq n$, M_i is an R_i-module, respectively. A non-zero submodule N of M is a weakly second submodule of M if and only if $N = \times_{i=1}^{n} N_i$ such that for some $k \in \{1, 2, ..., n\}$, N_k is a weakly second submodule of M_k, and $N_i = 0$ for every $i \in \{1, 2, ..., n\} \setminus \{k\}$.

Proof. $(\Rightarrow)$ Let N be a weakly second submodule of M. We know $N = \times_{i=1}^{n} N_i$ where for every $1 \leq i \leq n$, N_i is a submodule of M_i, respectively. Assume that N_r is a non-zero submodule of M_r and N_s is a non-zero submodule of M_s for some $1 \leq r < s \leq n$. Since N is a weakly second submodule of M,

$$(0, \cdots, 0, 1_{R_r}, 0, \cdots, 0)(0, \cdots, 0, 1_{R_s}, 0, \cdots, 0)N = (0, \cdots, 0, 1_{R_r}, 0, \cdots, 0)N$$

or

$$(0,\cdots,0,1_{R_r},0,\cdots,0)(0,\cdots,0,1_{R_s},0,\cdots,0)N=(0,\cdots,0,1_{R_s},0,\cdots,0)N.$$

Thus $N_r=0$ or $N_s=0$. This contradiction shows that exactly one of the N_i's is non-zero, say N_k. Now, we show that N_k is a weakly second submodule of M_k. Let $a,b\in R_k$. Since N is a weakly second submodule of M,

$$(0,\cdots,0,a,0,\cdots,0)(0,\cdots,0,b,0,\cdots,0)N=(0,\cdots,0,a,0,\cdots,0)N$$

or

$$(0,\cdots,0,a,0,\cdots,0)(0,\cdots,0,b,0,\cdots,0)N=(0,\cdots,0,b,0,\cdots,0)N$$

Thus $abN_k=aN_k$ or $abN_k=bN_k$ as needed.

($\Leftarrow$) This is clear. $\square$

Theorem 3.19. *Let $R=R_1\times R_2\times\cdots\times R_n$ ($2\le n<\infty$) be a decomposable ring and $M=M_1\times M_2\cdots\times M_n$ be an R-module, where for every $1\le i\le n$, M_i is an R_i-module, respectively. Then for a non-zero submodule N of M the following conditions are equivalent:*

(a) *N is a strongly classical 2-absorbing second submodule of M;*

(b) *Either $N=\times_{i=1}^{n}N_i$ such that for some $k\in\{1,2,...,n\}$, N_k is a strongly classical 2-absorbing second submodule of M_k, and $N_i=0$ for every $i\in\{1,2,...,n\}\setminus\{k\}$ or $N=\times_{i=1}^{n}N_i$ such that for some $k,m\in\{1,2,...,n\}$, N_k is a weakly second submodule of M_k, N_m is a weakly second submodule of M_m, and $N_i=0$ for every $i\in\{1,2,...,n\}\setminus\{k,m\}$.*

Proof. We use induction on n. For $n=2$ the result holds by Theorem 3.17. Now suppose that the result is valid when $K=M_1\times\cdots\times M_t$ for each $t<n$. We show that the result holds when $M=K\times M_n$. By Theorem 3.17, N is a strongly classical 2-absorbing second submodule of M if and only if either $N=L\times 0$ for some strongly classical 2-absorbing second submodule L of K or $N=0\times L_n$ for some strongly classical 2-absorbing second submodule L_n of M_n or $N=L\times L_n$ for some weakly second submodule L of K and some weakly second submodule L_n of M_n. Note that by Lemma 3.18, a non-zero submodule L of K is a weakly second submodule of K if and only if $L=\times_{i=1}^{n-1}N_i$ such that for some $k\in\{1,2,...,n-1\}$, N_k is a weakly second submodule of M_k and $N_i=0$ for every $i\in\{1,2,...,n-1\}\setminus\{k\}$. Hence the claim is proved. $\square$

Example 3.20. Let R be a Noetherian ring and let $E=\oplus_{m\in Max(R)}E(R/m)$. Then for each 2-absorbing ideal P of R, $(0:_E P)$ is a strongly classical 2-absorbing second submodule of E.

Proof. By using [17, p. 147], $Hom_R(R/P,E)\neq 0$. Now since $(0:_E P)\cong Hom_R(R/P,E)$, $(0:_E P)$ is a strongly 2-absorbing second submodule of E by [5, Theorem 3.27]. Now the result follows from Example 3.6. $\square$

Theorem 3.21. *Let R be a um-ring and M be an R-module. If E is an injective R-module and N is a classical 2-absorbing submodule of M such that $Hom_R(M/N, E) \neq 0$, then $Hom_R(M/N, E)$ is a strongly classical 2-absorbing second R-module.*

Proof. Let $a, b, c \in R$. Since N is a classical 2-absorbing submodule of M, we can assume that $(N :_M abc) = (N :_M ab)$ by [14, Theorem 4]. Since E is an injective R-module, by replacing M with M/N in [1, Theorem 3.13 (a)], we have $Hom_R(M/(N :_M r), E) = rHom_R(M/N, E)$ for each $r \in R$. Therefore,

$$abcHom_R(M/N, E) = Hom_R(M/(N :_M abc), E) =$$
$$Hom_R(M/(N :_M ab), E) = abHom_R(M/N, E),$$

as needed □

Theorem 3.22. *Let M be a strongly classical 2-absorbing second R-module and F be a right exact linear covariant functor over the category of R-modules. Then $F(M)$ is a strongly classical 2-absorbing second R-module if $F(M) \neq 0$.*

Proof. This follows from [1, Lemma 3.14] and Theorem 3.4 $(a) \Rightarrow (c)$. □

Corollary 3.23. Let M be an R-module, S be a multiplicative subset of R and N be a strongly classical 2-absorbing second submodule of M. Then $S^{-1}N$ is a strongly classical 2-absorbing second submodule of $S^{-1}M$ if $S^{-1}N \neq 0$.

Proof. This follows from Theorem 3.22. □

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TRANSMUTED GUMBEL UNIVARIATE EXPONENTIAL DISTRIBUTION

MONIREH HAMELDARBANDI AND MEHMET YILMAZ

Abstract. A functional composition of the distribution function of one probability distribution with the inverse distribution function of another is called the transmutation map. The present paper is purported to show how the transmuted distribution can be obtained by using the convex combination of failure probability of two-component systems. The transmuted Gumbel univariate exponential distribution is presented by changing convex combination parameter. This new distribution is defined and studied. Some mathematical properties of this distribution including the generating function and ordinary moments are derived. The survival, hazard rate and mean residual life functions are discussed. Finally, three applications to real data are presented.

1. Introduction

In the present paper, we will start by examining two-component (series and parallel) systems. The failure probabilities of these systems will be found and a new distribution is obtained by applying convex combinations to these probabilities as these can be ordered within themselves. In the process of proposing this distribution, the lifetimes of the components of the system which are the random variables are considered to be both dependent on each other and non-identical. If the random variables that represent the lifetimes of two components are identical and independent, then this proposed distribution will emerge in the transmuted model, which is one of the important families in the pertinent-literature in recent years. The transmuted family has been introduced by [27] for the first time and the theory of transmuted distribution is clearly defined by [28]. This method has led to the development of new and more flexible distributions by many authors, proposing many different distributions and pioneering the modeling of many real data sets with these distributions. Aryal and Tsokos [4] and [5] studied the two forms of

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the transmuted distributions. These scholars provided the mathematical characterization of transmuted extreme value and transmuted Weibull distributions and their applications to analyze real data sets. Aryal [6] proposed the transmuted log-logistic distribution and discussed various properties of this distribution. Merovci [19] introduced the transmuted Lindley distribution and applied it to bladder cancer data; Merovci [20] proposed the transmuted exponentiated exponential distribution; Merovci and Elbatal [21] studied the transmuted Lindley-geometric distribution. Ashour and Eltehiwy [7] discussed the applications of Transmuted Lomax Distribution and Ashour and Eltehiwy [8] proposed the transmuted exponentiated Lomax distribution. More recently, the transmuted exponentiated modified Weibull distribution has been suggested by [13] having its applications in real data. Hussian [16] obtained the transmuted exponentiated gamma distribution and discussed their various properties and applications. Elbatal et al. [11] discussed as various estimation methods for the transmuted exponentiated Fréchet distribution. Abd El Hady [1] obtained an extended Weibull distribution as the exponentiated transmuted Weibull distribution and discussed its various properties and applications. Merovci and Puka [22] introduced the transmuted Pareto distribution. Elbatal and Aryal [12] studied the transmuted additive Weibull distribution; Merovci [23] proposed the transmuted Rayleigh distribution and discussed their various properties. In the second part of this article, the new family will be introduced and the survival and hazard rate functions of the model under study will be found. The third part of this article contains some main definitions as Gumbel Bivariate Exponential Distribution and Gumbel Univariate Exponential Distribution. Later, the baseline distributions of the proposed distribution will be taken as exponential distribution and the proposed distribution is called the transmuted Gumbel univariate exponential (TGUE) distribution. In the subsequent subsections, the analytical shapes of the probability density, survival, cumulative hazard rate, hazard rate and mean residual life functions of the TGUE distribution are presented. Statistical properties including moment generating function and moments, maximum likelihood estimates and the information matrix, random number generation, Rényi entropy and order statistics of the TGUE distribution are discussed in other subsections of Section 3. Finally, in order to demonstrate the usefulness of the proposed distribution, three real data applications are presented in the application section.

2. The New Family

In recent literature, the transmuted family of lifetime distributions have attracted the attention of the researchers for modeling the lifetime data. Firstly, two-component (series and parallel) systems will be introduced. Let T_1 and T_2 be random variables that represent the lifetime of the components. Throughout this paper, the marginal distribution functions of T_1 and T_2 are represented by $F_{T_1}(.)$ and $F_{T_2}(.)$, and the joint distribution and the joint survival functions of T_1 and

T_2 are indicated by $F_{T_1,T_2}(.,.)$ and $S_{T_1,T_2}(.,.) = 1 - F_{T_1}(.) - F_{T_2}(.) + F_{T_1,T_2}(.,.)$, respectively. The series system success requires that the two parts operate successfully at the same time. System failure occurs if either one or more components fail. Then, the random variable T_{min} that stands for the series system lifetime is defined as $T_{min} = min\{T_1, T_2\}$. Hence, the probability of the failure of the series system is given by

$$P(T_{min} \le t) = 1 - P(T_1 > t, T_2 > t) = 1 - S_{T_1,T_2}(t,t) = F_{T_1}(t) + F_{T_2}(t) - F_{T_1,T_2}(t,t)$$

Parallel system is such a system that functions when at least one of its components works and the failure of all the components is necessary for the system's failure to occur. Accordingly, $T_{max} = max\{T_1, T_2\}$ stands for the parallel system lifetime. Then, the probability of the failure of the parallel system is given by

$$P(T_{max} \le t) = P(max\{T_1, T_2\} \le t) = P(T_1 \le t, T_2 \le t) = F_{T_1,T_2}(t,t).$$

According to axiomatic properties of probability, component lifetimes T_1 and T_2 can be ordered stochastically as $T_{min} \le_{st} T_i \le_{st} T_{max},\ i = 1, 2$. Namely, we have $P(T_{max} \le t) \le P(T_i \le t) \le P(T_{min} \le t)$. Then, the lower and the upper bounds for $F_{T_i}(t)$ can be written as follows:

$$F_{T_1,T_2}(t,t) \le F_{T_i}(t) \le F_{T_1}(t) + F_{T_2}(t) - F_{T_1,T_2}(t,t). \tag{1}$$

In that case, $F_{T_i}(t)$ can be represented as a convex combination of failure probabilities series and parallel systems. Then, we have

$$\begin{aligned} & \lambda(F_{T_1}(t) + F_{T_2}(t) - F_{T_1,T_2}(t,t)) + (1-\lambda) F_{T_1,T_2}(t,t) \\ = \;& \lambda(F_{T_1}(t) + F_{T_2}(t)) + (1-2\lambda) F_{T_1,T_2}(t,t), \end{aligned}$$

where the combination parameter $\lambda \in [0,1]$. This latter well-defined statement can derive numerous univariate distribution functions with respect to combination parameter λ.

In the latter equation, if the distributions of random variables T_1 and T_2 are assumed to be identical, namely, $F_{T_1}(t) = F_{T_2}(t)$, then the new distribution with the parameter set Θ is given by

$$G(t;\Theta) = 2\lambda F_{T_i}(t) + (1-2\lambda) F_{T_1,T_2}(t,t).$$

If transformation $\lambda = \frac{\delta+1}{2}$ is done, range will change from $[0,1]$ to $[-1,1]$. So, for $|\delta| \le 1$, the distribution function can be written as

$$\begin{aligned} G(t;\Theta) &= (1+\delta) F_{T_i}(t) - \delta F_{T_1,T_2}(t,t) \qquad (2) \\ &= (1+\delta) F_{T_i}(t) - \delta(2F_{T_i}(t) + S_{T_1,T_2}(t,t) - 1) \\ &= (1-\delta)(1 - S_{T_i}(t)) + \delta(1 - S_{T_1,T_2}(t,t)). \end{aligned}$$

So, if the distributions of random variables T_1 and T_2 are taken independent, namely, $F_{T_1,T_2}(t,t) = (F_{T_i}(t))^2$ in the first equation of (2), we can obtain the transmuted distribution constructed by the quadratic rank transmutation method of [27] which has become very popular in the recent years.

In particular, for $\delta = 0$ it gives the baseline distribution $F_{T_i}(t)$, for $\delta = -1$, it gives the distribution of the maximum of dependent two random variables with joint distribution function $F_{T_1,T_2}(t,t)$ and for $\delta = 1$, $2F_{T_i}(t) - F_{T_1,T_2}(t,t)$ is the distribution of the minimum of two random variables T_1 and T_2 with identically distributed.

Theorem 1. *The probability density function (p.d.f.) of T is represented in terms of the conditional hazard rates of the component lifetimes T_1 and T_2 as*

$$g(t;\Theta) = (1-\delta) f_{T_i}(t) + \delta S_{T_1,T_2}(t,t)\left(\psi_1(t) + \psi_2(t)\right), \tag{3}$$

where $\psi_1(t)$ and $\psi_2(t)$ denote the failure rates of the corresponding components, given that both components are alive at time t.

Proof. The p.d.f. of this distribution can be obtained with derivation of distribution function defined in (2) as follows

$$g(t;\Theta) = \frac{d}{dt}G(t;\Theta) = (1-\delta)\left(\frac{-d}{dt}S_{T_i}(t)\right) + \delta\left(\frac{-d}{dt}S_{T_1,T_2}(t,t)\right)$$

and the result in (3) will be obtained from the following method.

$$\begin{aligned}
\frac{-d}{dt}S_{T_1,T_2}(t,t) &= \frac{-d}{dt}\int_t^{\infty}\int_t^{\infty} f_{T_1,T_2}(u,v)\,dvdu \\
&= -\int_t^{\infty} f_{T_1,T_2}(u,t)\,du - \int_t^{\infty} f_{T_1,T_2}(t,v)\,dv \\
&= -f_{T_2}(t)\,Pr(T_1 \le tT_2 = t) - f_{T_1}(t)\,Pr(T_2 \le tT_1 = t) \\
&= \frac{-d}{dt_2}S_{T_1,T_2}(t,t_2)\Big|_{t_2=t} + \frac{-d}{dt_1}S_{T_1,T_2}(t_1,t)\Big|_{t_1=t} \\
&= \psi_2(t)\,S_{T_1,T_2}(t,t) + \psi_1(t)\,S_{T_1,T_2}(t,t)
\end{aligned}$$

where $\psi_1(t)$ and $\psi_2(t)$ denote the failure rates of the corresponding components, given that both components are alive at time t and defined as follows:

$$\begin{aligned}
\psi_1(t) &= \lim_{\Delta t\to 0^+}\frac{Pr(t < T_1 \le t+\Delta t\,|T_1 > t,\ T_2 > t)}{\Delta t} = \frac{\frac{-d}{dt_1}S_{T_1,T_2}(t_1,t)\Big|_{t_1=t}}{S(t,t)}, \quad t \ge 0 \\
\psi_2(t) &= \lim_{\Delta t\to 0^+}\frac{Pr(t < T_2 \le t+\Delta t\,|T_1 > t,\ T_2 > t)}{\Delta t} = \frac{\frac{-d}{dt_2}S_{T_1,T_2}(t,t_2)\Big|_{t_2=t}}{S(t,t)}. \quad t \ge 0
\end{aligned}$$

(See [26] and see [17]). □

2.1. **Survival and Hazard Rate Functions of Proposed Distribution.** The survival function denoted by $S(t;\Theta)$ of this distribution is defined as follows,

$$\begin{aligned}
S(t;\Theta) &= 1 - G(t;\Theta) = 1 - (1+\delta)F_{T_i}(t) + \delta F_{T_1,T_2}(t,t) \\
&= 1 - (1-\delta)(1 - S_{T_i}(t)) - \delta(1 - S_{T_1,T_2}(t,t))
\end{aligned}$$

$$= \quad (1-\delta) S_{T_i}(t) + \delta S_{T_1,T_2}(t,t).$$

The hazard rate function (hrf) corresponding to (2) and (3) is given by

$$\begin{aligned}
h(t;\Theta) &= \frac{g(t;\Theta)}{S(t;\Theta)} = \frac{(1-\delta) h_{T_1}(t) S_{T_i}(t) + \delta S_{T_1,T_2}(t,t)(\psi_1(t)+\psi_2(t))}{(1-\delta) S_{T_i}(t) + \delta S_{T_1,T_2}(t,t)} \\
&= \psi_1(t) + \psi_2(t) + \frac{(1-\delta) S_{T_i}(t)(h_{T_1}(t) - (\psi_1(t)+\psi_2(t)))}{(1-\delta) S_{T_i}(t) + \delta S_{T_1,T_2}(t,t)} \\
&= h_{T_1}(t) + \frac{\delta S_{T_1,T_2}(t,t)((\psi_1(t)+\psi_2(t)) - h_{T_1}(t))}{(1-\delta) S_{T_i}(t) + \delta S_{T_1,T_2}(t,t)} \\
&= w_1(t) h_{T_1}(t) + w_2(t)(\psi_1(t)+\psi_2(t)),
\end{aligned}$$

where $w_1(t) = \frac{(1-\delta) S_{T_i}(t)}{(1-\delta) S_{T_i}(t) + \delta\ S_{T_1,T_2}(t,t)}$ and $w_1(t) + w_2(t) = 1$. Thus, the hrf can be written as a weighted sum of the hrf of the random variable T_1 and sum of the conditional failure rates of the corresponding components $(\psi_1(t) + \psi_2(t))$.

In the next section, we will introduce a bivariate version of the exponential distribution named the Gumbel bivariate exponential distribution. On the basis of this, the Gumbel univariate exponential distribution is defined and examined. Then, the transmuted Gumbel univariate exponential distribution is taken as a special case for the proposed distribution and some mathematical properties are studied.

3. Special Case: Transmuted Gumbel Univariate Exponential (TGUE) Distribution

We will first introduce distributions related to setting-up a special case. Then the baseline distribution is defined and we study on some reliability properties such as survival, cumulative hazard rate, hazard rate and mean residual life functions. Moment generating function and moments of proposed distribution are analyzed. ML estimation of model parameters are performed and asymptotic distribution of the parameters are obtained in terms of observed Fisher Information and then asymptotic confidence intervals are also obtained. General expressions for the Rényi entropy is presented. Furthermore, general results for the order statistics of the TGUE random variables are derived.

3.1. **Gumbel Bivariate and Univariate Exponential Distribution.**

3.1.1. *Gumbel Bivariate Exponential Distribution.* Exponential distribution plays a central role in life testing, reliability and analyses of survival or lifetime data. The Gumbel bivariate exponential (GBE) distribution introduced by [15] is the most popular model for analyzing lifetime data and its survival function is

$$S_{T_1,T_2}(t_1,t_2) = e^{-(\alpha_1 t_1 + \alpha_2 t_2 + \beta t_1 t_2)},\ t_1, t_2 > 0, \tag{4}$$

where α_1 and α_2 are the scale parameters representing the characteristic life and also positive, β is dependency parameter and $0 \leq \beta \leq \alpha_1\alpha_2$. The marginal survival functions of T_1 and T_2 respectively are $e^{-\alpha_1 t_1}$ and $e^{-\alpha_2 t_2}$. Hence T_1 and T_2 have exponential marginals. The p.d.f. of the three-parameter GBE distribution corresponding to (4) is given by

$$\begin{aligned} f_{T_1,T_2}(t_1,t_2) &= \frac{\partial^2}{\partial t_1 \partial t_2} S_{T_1,T_2}(t_1,t_2) = \frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial t_2} e^{-(\alpha_1 t_1+\alpha_2 t_2+\beta t_1 t_2)}\right) \\ &= (\alpha_1+\beta t_2)(\alpha_2+\beta t_1)\, e^{-(\alpha_1 t_1+\alpha_2 t_2+\beta t_1 t_2)}, \quad t_1,t_2>0. \end{aligned}$$

3.1.2. *Gumbel Univariate Exponential Distribution.* By letting $\alpha_1 = \alpha_2$ and considering the diagonal section of $S_{T_1,T_2}(t_1,t_2)$ i.e., $t_1 = t_2 = t$ in the survival function of GBE distribution defined in (4). Then the random vector (T_1, T_2) has the Gumbel univariate exponential (GUE) distribution, and the survival function of the GUE distribution can be written as follows

$$S_{T_1,T_2}(t,t) = e^{-\left(2\alpha t+\beta t^2\right)}, \quad t>0,\ \alpha>0,\ 0 \leq \beta \leq \alpha^2. \tag{5}$$

By using the known relation between $S_{T_1,T_2}(t,t)$ and $F_{T_1,T_2}(t,t)$, the distribution function of the GUE random variable is given by

$$F_{T_1,T_2}(t,t) = 1 - 2S_{T_i}(t) + S_{T_1,T_2}(t,t) = 1 - 2e^{-\alpha t} + e^{-\left(2\alpha t+\beta t^2\right)},$$

and its p.d.f. of the GUE random variable reduces to

$$\begin{aligned} f_{T_1,T_2}(t,t) &= 2\alpha e^{-\alpha t} - (2\alpha+2\beta t)\, e^{-\left(2\alpha t+\beta t^2\right)} \\ &= 2\alpha\left(e^{-\alpha t} - e^{-\left(2\alpha t+\beta t^2\right)}\right) - 2\beta t e^{-\left(2\alpha t+\beta t^2\right)}. \end{aligned}$$

The moment generating function of the GUE random variable is given as follows

$$M_T(k) = \int_0^\infty e^{kT} f_{T_1,T_2}(t,t)\,dt = \frac{\alpha+k}{\alpha-k} - \frac{k}{2}\sqrt{\frac{\pi}{\beta}}\, e^{\frac{(2\alpha-k)^2}{4\beta}} erfc\left(\frac{2\alpha-k}{\sqrt{\beta}}\right),$$

where $erfc$ is a complementary error function and $k < \alpha$.

Especially, the first four moments of the GUE random variable T are given as

$$\begin{aligned} E(T) &= 2\left(-te^{-\alpha t}\Big|_0^\infty + \int_0^\infty e^{-\alpha t}dt + te^{-\left(2\alpha t+\beta t^2\right)}\Big|_0^\infty - \int_0^\infty e^{-\left(2\alpha t+\beta t^2\right)}dt\right) \\ &= 2\left(\frac{-1}{\alpha}e^{-\alpha t}\Big|_0^\infty - e^{\frac{\alpha^2}{\beta}} \int_0^\infty e^{-\left(\sqrt{\beta}\left(t+\frac{\alpha}{\beta}\right)\right)^2} dt\right) = \frac{2}{\alpha} - \xi(\alpha,\beta), \\ E\left(T^2\right) &= \frac{4}{\alpha^2} - \frac{1}{\beta} + \frac{1}{\beta}\xi(\alpha,\beta), \\ E\left(T^3\right) &= \frac{12}{\alpha^3} - \frac{3\alpha}{\beta^2} + \frac{3}{4\beta}\left(1+\frac{2\alpha^2}{\beta}\right)\xi(\alpha,\beta), \end{aligned}$$

$$E\left(T^4\right) = \frac{48}{\alpha^4} - \frac{1}{2\beta^2}\left(4+12\alpha-\frac{4\alpha^2}{\beta}\right) + \left(3\frac{\alpha}{\beta^2}e^{\frac{\alpha^2}{\beta}} + 2\left(\frac{\alpha}{\beta}\right)^3\xi\left(\alpha,\beta\right)\right)$$

where $\xi\left(\alpha,\beta\right) = \sqrt{\frac{\pi}{\beta}}e^{\frac{\alpha^2}{\beta}}erfc\left(\frac{\alpha}{\sqrt{\beta}}\right)$.

3.1.3. *Transmuted Gumbel Univariate Exponential Distribution.* The transmuted Gumbel univariate exponential (TGUE) distribution is an extended model to analyze more complex data. T_1 and T_2 have a exponential distribution with the same shape parameter α and random vector (T_1, T_2) has a Gumbel univariate exponential distribution with α and β parameters, then we can write

$$\begin{cases} S_{T_i}\left(t\right) = e^{-\alpha t}, & F_{T_i}\left(t\right) = 1-e^{-\alpha t} \\ S_{T_1,T_2}\left(t,t\right) = e^{-\left(2\alpha t+\beta t^2\right)}, & F_{T_1,T_2}\left(t,t\right) = 1-2e^{-\alpha t}+e^{-\left(2\alpha t+\beta t^2\right)}. \end{cases}$$

By using equation (2) and (5), the distribution function of the TGUE random variable with the parameter space $\Theta = \left\{(\alpha,\beta,\delta):\ \alpha>0,\ \beta<\alpha^2,\ -1\le\delta\le1\right\}$, can be obtained as

$$\begin{aligned} G\left(t;\Theta\right) &= \left(1-\delta\right)\left(1-S_{T_i}\left(t\right)\right)+\delta\left(1-S_{T_1,T_2}\left(t,t\right)\right), \\ &= \left(1-\delta\right)\left(1-e^{-\alpha t}\right)+\delta\left(1-e^{-\left(2\alpha t+\beta t^2\right)}\right) \\ &= 1-\left(1-\delta\right)e^{-\alpha t}-\delta e^{-\left(2\alpha t+\beta t^2\right)}. \end{aligned} \tag{6}$$

Henceforth, the p.d.f. corresponding to (3) and (6) becomes

$$g\left(t;\Theta\right) = \frac{d}{dt}G\left(t;\Theta\right) = \left(1-\delta\right)f_{T_i}\left(t\right)+\delta S_{T_1,T_2}\left(t,t\right)\left(\psi_1\left(t\right)+\psi_2\left(t\right)\right),$$

where $\psi_1\left(t\right) = \frac{\frac{-d}{dt_1}e^{-\left(\alpha t_1+\alpha t+\beta t_1 t\right)}\Big|_{t_1=t}}{e^{-\left(2\alpha t+\beta t^2\right)}} = \alpha+\beta t$, $\psi_2\left(t\right) = \alpha+\beta t$ and $\beta\le\alpha^2$. Consequently, the p.d.f. of the TGUE random variable can be written as follows

$$\begin{aligned} g\left(t;\Theta\right) &= \left(1-\delta\right)\alpha e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^2\right)}\left(\alpha+\beta t+\alpha+\beta t\right), \\ &= \left(1-\delta\right)\alpha e^{-\alpha t}+\delta\left(2\alpha+2\beta t\right)e^{-\left(2\alpha t+\beta t^2\right)}. \end{aligned} \tag{7}$$

The shapes of the p.d.f. of the TGUE random variable can be analyzed as follows

$$g'\left(t;\Theta\right) = -\left(1-\delta\right)\alpha^2 e^{-\alpha t}-\delta(2\alpha+2\beta t)^2 e^{-\left(2\alpha t+\beta t^2\right)},$$

by examining this derivation, it is clear that when $0\le\delta\le1$, $g'\left(t;\Theta\right)<0$ is obtained and we can say that the p.d.f. is decreasing. Also, in order for p.d.f. to be unimodal, it must be $-1\le\delta\le0$.

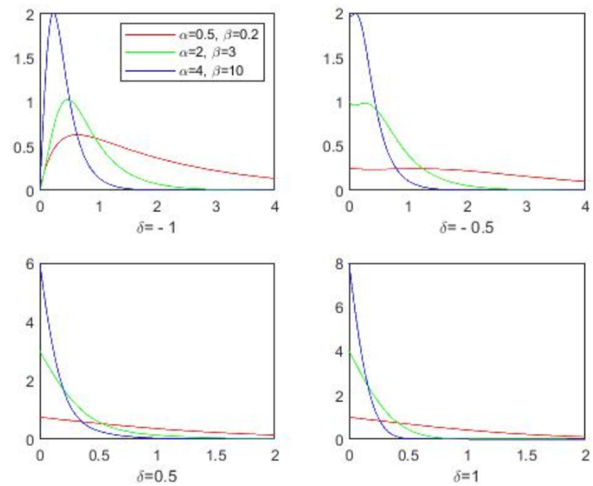

FIGURE 1. Plots of the TGUE Probability Density Function

3.1.4. *Survival, Cumulative Hazard Rate and Hazard Rate Functions of the TGUE Distribution.* The survival function of the TGUE random variable is given by

$$S\left(t;\Theta\right)=1-G\left(t;\Theta\right)=\left(1-\delta\right)e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^{2}\right)}. \tag{8}$$

Many generalized probability models have been proposed in reliability literature through the fundamental relationship between the cumulative hazard function $H\left(t;\Theta\right)$ and the survival function $S\left(t;\Theta\right)$ is given by

$$H\left(t;\Theta\right)=-\log S\left(t;\Theta\right)=-\log\left(\left(1-\delta\right)e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^{2}\right)}\right). \tag{9}$$

Thus, we find the cumulative hazard function of the TGUE random variable and this function describes how the risk of a particular outcome changes with time. We know

$$H\left(0;\Theta\right)=0,\lim_{t\rightarrow\infty}H\left(t;\Theta\right)=\infty,H\left(t;\Theta\right)$$

is increasing for all $t\geq 0$.

The other characteristic of a random variable is the hrf. By using (7) and (8), this function is given as follows

$$\begin{aligned} h\left(t;\Theta\right) &= \frac{g\left(t;\Theta\right)}{S\left(t;\Theta\right)}=\frac{\left(1-\delta\right)\alpha e^{-\alpha t}+\delta\left(2\alpha+2\beta t\right)e^{-\left(2\alpha t+\beta t^{2}\right)}}{\left(1-\delta\right)e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^{2}\right)}} \qquad (10)\\ &= \frac{\alpha\left(1-\delta\right)e^{\alpha t+\beta t^{2}}+2\delta\left(\alpha+\beta t\right)}{\left(1-\delta\right)e^{\alpha t+\beta t^{2}}+\delta} \\ &= \left(2\alpha+2\beta t\right)-\frac{\left(1-\delta\right)\left(\alpha+2\beta t\right)e^{-\alpha t}}{\left(1-\delta\right)e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^{2}\right)}}. \end{aligned}$$

The hrf of the TGUE random variable has the following properties:

$$h\left(0;\Theta\right)=\left(1+\delta\right)\alpha,$$

$$\delta\neq 1: \lim_{t\to\infty}h\left(t;\Theta\right)=\lim_{t\to\infty}\frac{\left(1-\delta\right)\alpha e^{-\alpha t}+\delta\left(2\alpha+2\beta t\right)e^{-\left(2\alpha t+\beta t^{2}\right)}}{\left(1-\delta\right)e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^{2}\right)}}=\alpha,$$

$$\delta=1: \lim_{t\to\infty}h\left(t;\Theta\right)=\lim_{t\to\infty}\left(2\alpha+2\beta t\right)=\infty.$$

The hazard rate function will be examined in the extreme values of the parameters:

(1) If $\delta=0$, the hrf is the same as the exponential distribution;

$$h\left(t;\Theta\right)=\alpha$$

(2) If $\delta=1$, the hrf is the same as the linear hazard rate function;

$$h\left(t;\Theta\right)=2\left(\alpha+\beta t\right)$$

(3) If $\beta=0$, the hrf is the same as the transmuted exponential distribution;

$$h\left(t;\Theta\right)=\frac{\left(1-\delta\right)\alpha e^{-\alpha t}+2\delta\alpha}{\left(1-\delta\right)e^{-\alpha t}+\delta}.$$

Let's investigate the monotonicity of hrf,

$$h'\left(t;\Theta\right)=\frac{-\delta\left(1-\delta\right)\left(\alpha+2\beta t\right)^{2}e^{-\left(3\alpha t+\beta t^{2}\right)}}{\left(\left(1-\delta\right)e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^{2}\right)}\right)^{2}}.$$

It is clear from above derivation, when $-1\leq\delta\leq 0$, the hazard rate function is increasing, that is, $h'\left(t;\Theta\right)\geq 0$. When $0\leq\delta\leq 1$, the hazard rate function is decreasing ($h'\left(t;\Theta\right)\leq 0$). Some possible shapes of hrf for selected parameter value are shown in the following figures.

Figure 3.2 shows the hrf defined in (10) with different choices of parameters. This distribution has an increasing hrf for $-1\leq\delta\leq 0$. If $0\leq\delta\leq 1$, the hrf is decreasing.

3.1.5. *Mean Residual Life Function of the TGUE Random Variable.* In this section, we will find the mean residual life (mrl) function of the TGUE random variable which is another important characteristic of a random variable.

$$m\left(t;\Theta\right)=E\left(T-t\,|T>t\,\right)=\int_{0}^{\infty}\left(k-t\right)dP\left(T\leq k\,|T>t\,\right)$$

$$=\frac{\int_{t}^{\infty}S\left(k;\Theta\right)dk}{S\left(t;\Theta\right)}=\frac{\left(1-\delta\right)\frac{1}{\alpha}e^{-\alpha t}-\frac{\delta}{2}\sqrt{\frac{\pi}{\beta}}e^{\frac{\alpha^{2}}{\beta}}erfc\left(\frac{\alpha+\beta t}{\sqrt{\beta}}\right)}{\left(1-\delta\right)e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^{2}\right)}}. \tag{11}$$

The mrl function of the TGUE random variable has the following properties:

(1) If $\delta=0$, the mrl function is the same as the exponential distribution;

$$m\left(t;\Theta\right)=\frac{1}{\alpha}$$

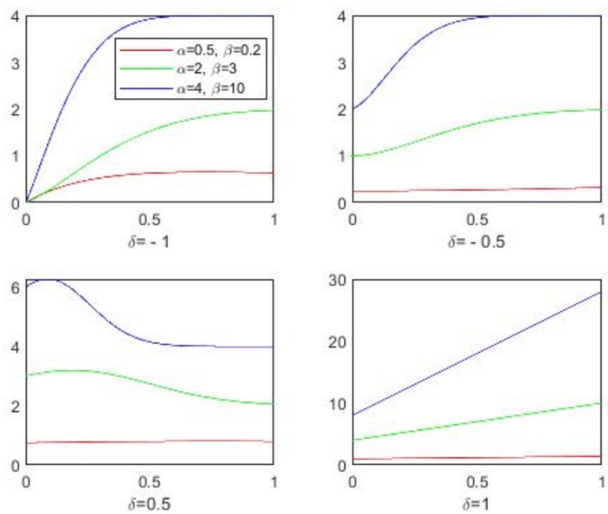

FIGURE 2. Plots of the TGUE Hazard Rate Function

(2) If $\delta = 1$, the mrl function is;

$$m(t;\Theta) = -\frac{1}{2}\sqrt{\frac{\pi}{\beta}} erfc\left(\frac{\alpha+\beta t}{\sqrt{\beta}}\right) e^{\frac{\alpha^2}{\beta}+2\alpha t+\beta t^2},$$

and some possible shapes of (11) for selected parameter values is showed in the following figures.

3.1.6. *3.5. Moment Generating Function and moments of the TGUE Random Variable.* In this section, we derive the moment generating function and first four moments for the TGUE distribution. Let T have the TGUE distribution, then the moment generating function of T is given by

$$M_T(k) = E\left(e^{kT}\right) = (1-\delta)\frac{\alpha}{\alpha-k} + \delta\left(1+\frac{k}{2}\sqrt{\frac{\pi}{\beta}} e^{\frac{(2\alpha-k)^2}{4\beta}} erfc\left(\frac{2\alpha-k}{\sqrt{\beta}}\right)\right),\ k<\alpha$$

The expressions for the expected value and variance are

$$E(T) = (1-\delta)\frac{1}{\alpha} - \delta\left(e^{\frac{\alpha^2}{\beta}}\int_0^\infty e^{-\left(\sqrt{\beta}\ \left(t+\ \frac{\alpha}{\beta}\ \right)\right)^2} dt\right) = (1-\delta)\frac{1}{\alpha} - \delta\xi(\alpha,\beta),$$

$$E\left(T^2\right) = (1-\delta)\frac{2}{\alpha^2} + \delta\left(\frac{1}{\beta} - \frac{1}{\beta}\xi(\alpha,\beta)\right),$$

$$Var(T) = (1-\delta)\frac{2}{\alpha^2} + \delta\left(\frac{1}{\beta} - \frac{1}{\beta}\xi(\alpha,\beta)\right) - \left((1-\delta)\frac{1}{\alpha} - \delta\xi(\alpha,\beta)\right)^2.$$

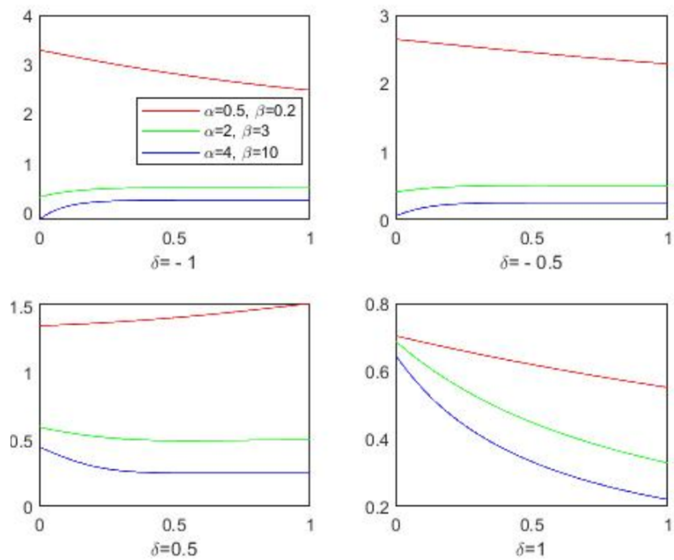

FIGURE 3. Plots of the TGUE Mean Residual Life Function

Finally, the 3^{th} and 4^{th} moments of the TGUE random variable are obtained as

$$E\left(T^3\right) = (1-\delta)\frac{6}{\alpha^3} + \delta\left(\frac{3\alpha}{\beta^2} - \frac{3}{4\beta}\left(1+\frac{2\alpha^2}{\beta}\right)\xi\left(\alpha,\beta\right)\right),$$

$$E\left(T^4\right) = (1-\delta)\frac{24}{\alpha^4} + \delta\left(\frac{1}{2\beta^2}\left(4+12\alpha-\frac{4\alpha^2}{\beta}\right) + \frac{\alpha}{\beta^2}\left(3e^{\frac{\alpha^2}{\beta}} + \frac{2\alpha^2}{\beta}\xi\left(\alpha,\beta\right)\right)\right) .$$

3.1.7. *Estimation by Maximum Likelihood and the Information Matrix of the TGUE Distribution.* Let $(t_1, t_2, \cdots, t_n)$ be sample values from this distribution with parameters α, β *and* δ. The likelihood function for $\Theta = \{\alpha, \beta, \delta\}$ is given by

$$L\left(\Theta; t_1, t_2, \cdots, t_n\right) = \prod_{i=1}^{n}\left((1-\delta)\,\alpha e^{-\alpha t_i} + \delta\left(2\alpha + 2\beta t_i\right)e^{-\left(2\alpha t_i + \beta t_i^2\right)}\right).$$

Throughout this subsection, the log-likelihood function is denoted by $l = \log L\left(\Theta; t_1, t_2, \cdots, t_n\right)$ for brevity. We differentiate l with respect to α, β and δ as follows

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{n}\frac{-\left(1-\delta\right)\alpha^2 e^{-\alpha t_i} + 2\delta\left(1 - 2\alpha t_i - 2\beta t_i^2\right)e^{-\left(2\alpha t_i + \beta t_i^2\right)}}{g\left(t_i; \Theta\right)}, \tag{12}$$

$$\frac{\partial l}{\partial \beta}=\sum_{i=1}^{n}\frac{2\delta t_i\left(1-\alpha t_i-\beta {t_i}^2\right)e^{-\left(2\alpha t_i+\beta t_i^2\right)}}{g\left(t_i;\Theta\right)}, \tag{13}$$

$$\frac{\partial l}{\partial \delta}=\sum_{i=1}^{n}\frac{-\alpha e^{-\alpha t_i}+\left(2\alpha+2\beta t_i\right)e^{-\left(2\alpha t_i+\beta t_i^2\right)}}{g\left(t_i;\Theta\right)}. \tag{14}$$

The maximum likelihood estimators as $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\delta}$ are obtained by equating these three equations (12), (13) and (14) to zero and solving the equations simultaneously. For these three parameters, we will get the second order derivatives of logarithms of the likelihood function for obtaining the elements of the Fisher-Information Matrix.

$$I_{\alpha\alpha}=\frac{\partial^2 l}{\partial \alpha^2}=-\sum_{\imath=1}^{n}\frac{(1-\delta)^2e^{-2\alpha t_i}-2\delta\left(1-\delta\right)\left(\alpha\beta t_i^3+\left(\alpha^2+2\beta\right)t_i^2-2\right)e^{-\left(3\alpha t_i+\beta t_i^2\right)}+4\delta^2e^{-2\left(2\alpha t_i+\beta t_i^2\right)}}{\left(g\left(t_i;\Theta\right)\right)^2}$$

$$I_{\beta\beta}=\frac{\partial^2 l}{\partial \beta^2}=-\sum_{\imath=1}^{n}\frac{2\delta\left(1-\delta\right)t_i^3\alpha\left(2-\alpha t_i-\beta t_i^2\right)e^{-\left(3\alpha t_i+\beta t_i^2\right)}+4\delta^2t_i^2e^{-2\left(2\alpha t_i+\beta t_i^2\right)}}{\left(g\left(t_i;\Theta\right)\right)^2},$$

$$I_{\delta\delta}=\frac{\partial^2 l}{\partial \delta^2}=-\sum_{\imath=1}^{n}\left(\frac{\alpha e^{-\alpha t_i}+2\left(\alpha+\beta t_i\right)e^{-\left(2\alpha t_i+\beta t_i^2\right)}}{g\left(t_i;\Theta\right)}\right)^2,$$

$$I_{\alpha\beta}=I_{\beta\alpha}=\frac{\partial^2 l}{\partial \beta\partial\alpha}$$

$$=-\sum_{\imath=1}^{n}\left(\frac{2\delta\left(1-\delta\right)t_i\left(1+\alpha t_i-\left(\alpha^2+\beta\right)t_i^2-\alpha\beta t_i^3\right)e^{-\left(3\alpha t_i+\beta t_i^2\right)}+4\delta^2t_ie^{-2\left(2\alpha t_i+\beta t_i^2\right)}}{\left(g\left(t_i;\Theta\right)\right)^2}\right),$$

$$I_{\alpha\delta}=I_{\delta\alpha}=\frac{\partial^2 l}{\partial \delta\partial\alpha}=-\sum_{\imath=1}^{n}\frac{2\left(\left(\alpha+\beta\right)+\alpha t_i\right)t_ie^{-\left(3\alpha t_i+\beta t_i^2\right)}}{\left(g\left(t_i;\Theta\right)\right)^2},$$

$$I_{\beta\delta}=I_{\delta\beta}=\frac{\partial^2 l}{\partial \delta\partial\beta}=-\sum_{\imath=1}^{n}\frac{-2\alpha t_i\left(1-\alpha t_i-\beta t_i^2\right)e^{-\left(3\alpha t_i+\beta t_i^2\right)}}{\left(g\left(t_i;\Theta\right)\right)^2}.$$

Thus, Fisher information matrix, $I_n\left(\Theta\right)$ of sample size n for Θ is as follows:

$$I_n\left(\Theta\right)=-E\begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\delta}\\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\delta}\\ I_{\delta\alpha} & I_{\delta\beta} & I_{\delta\delta}\end{pmatrix}$$

Inverse of the Fisher-information matrix of single observation, i.e., $I_1^{-1}\left(\Theta\right)$ indicates asymptotic variance-covariance matrix of maximum likelihood estimates of Θ. Hence, the distribution of maximum likelihood estimator for Θ is asymptotically normal with mean Θ and variance-covariance matrix $I_1^{-1}\left(\Theta\right)$. Namely,

$$\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\delta}\end{bmatrix}\sim AN\left(\begin{bmatrix}\alpha\\ \beta\\ \delta\end{bmatrix},\frac{I_1^{-1}\left(\Theta\right)}{n}\right) \tag{15}$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and covariance of these ML estimators for these parameters.

We can approximate $100\,(1-\gamma)\,\%$ confidence intervals for α, β and δ by using (15) are obtained respectively as

$$\left[\hat{\alpha} - z_{1-\frac{\gamma}{2}}\sqrt{\frac{I^{-1}_{1_{\alpha\alpha}}}{n}}, \hat{\alpha} + z_{1-\frac{\gamma}{2}}\sqrt{\frac{I^{-1}_{1_{\alpha\alpha}}}{n}}\right],$$

$$\left[\hat{\beta} - z_{1-\frac{\gamma}{2}}\sqrt{\frac{I^{-1}_{1_{\beta\beta}}}{n}}, \hat{\beta} + z_{1-\frac{\gamma}{2}}\sqrt{\frac{I^{-1}_{1_{\beta\beta}}}{n}}\right],$$

$$\left[\hat{\delta} - z_{1-\frac{\gamma}{2}}\sqrt{\frac{I^{-1}_{1_{\delta\delta}}}{n}}, \hat{\delta} + z_{1-\frac{\gamma}{2}}\sqrt{\frac{I^{-1}_{1_{\delta\delta}}}{n}}\right],$$

where $z_{1-\frac{\gamma}{2}}$ is the upper 100γ the quantile of the standard normal distribution, and $I^{-1}_{1..}$ denotes respective diagonal elements of I^{-1}_{1}.

3.1.8. *Random Number Generation from the TGUE Distribution.* Remember the distribution function defined in section 2,

$$G\,(t) = \lambda\,(F_{T_1}\,(t) + F_{T_2}\,(t) - F_{T_1,T_2}\,(t,t)) + (1-\lambda)\,F_{T_1,T_2}\,(t,t)$$

where $0 \leq \lambda \leq 1$. Again, emphasize that $G\,(t)$ represents a two-component mixture distribution, where the distribution functions of the T_{min} and T_{max} are the components of this mixture, respectively. To generate a random number from $G(t)$, we apply the reference Gentle [14] pp.125. Accordingly, a random number V is generated from uniform distribution on (0,1) to decide which of the components are chosen. As a result, when $V \leq \lambda$, the random number will be generated from $F_{T_{min}}\,(t)$ by equating as $F_{T_{min}}\,(t) = V$. Otherwise, namely $V > \lambda$, the random number will be generated from the distribution of T_{max} by equating $F_{T_{max}}\,(t) = V$.

First of all, we will consider how to produce component lifetimes. By citing the method given in Gentle [14] pp.109, these component lifetimes will be generated with the help of the conditional distribution function. Namely, $F_{T_1,T_2}\,(t_1,t_2)$ can be expressed as the product of the cdf of T_1 and the conditional cdf of T_2 with given $T_1 = t_1$, i.e. $F_{T_1,T_2}\,(t_1,t_2) = F_{T_1}\,(t_1)\,F_{T_2|T_1}\,(t_2)$.

In the first step, a random number U_1 is generated from the uniform distribution on the interval $(0,1)$. Then we generate the lifetime of the first component $t_1 = F^{-1}_{T_1}\,(U_1)$. In the second step, again we generate a uniformly distributed random variable U_2 (independent of U_1) on $(0,1)$. Therefore, the lifetime of the second component can be generated by equating $t_2 = F^{-1}_{T_2|T_1=t_1}\,(U_2)$. Hence, the random number from the TGUE is generated as for $V \leq \lambda$, $t =$

$min\hat{a}\{t_1, t_2\}$ and for $V > \lambda$, $t = max\hat{a}\{t_1, t_2\}$. Then, according to the above-mentioned steps, $t_1 = \frac{-1}{\alpha}\ln(1-U_1)$ and $t_2 = -\frac{\alpha}{\beta} - \frac{W_{-1}(\eta)}{\alpha+\beta t_1}$ are generated where $-(1-U_2)\left(\frac{\alpha^2}{\beta} + \alpha t_1\right)e^{-\left(\frac{\alpha^2}{\beta}+\alpha t_1\right)} = \eta$. Here $W_{-1}(.)$ denotes the lower part of Lambert W-function whose domain is $\left[-e^{-1}, 0\right)$ and range $(-\infty, -1]$. A more detailed inference about generating second component lifetime is given in the appendix.

3.1.9. *Rényi Entropy of the TGUE Distribution.* The entropy of a random variable is a measure of variation of the uncertainty, see [25]. Then the Rényi entropy function of the random variable T with p.d.f. (7) is defined by

$$I_R(\rho) = \frac{1}{1-\rho}\log\int_0^\infty (g(t;\Theta))^\rho dt, \tag{16}$$

where $\rho > 0$, $\rho \neq 1$. We have the following series representation of $(g(t;\Theta))^\rho$ by applying the generalized Binomial theorem to obtain Rényi entropy for proposed distribution. Accordingly,

$$(g(t;\Theta))^\rho = \left((1-\delta)\alpha e^{-\alpha t} + \delta(2\alpha+2\beta t)e^{-\left(2\alpha t+\beta t^2\right)}\right)^\rho.$$

$(g(t;\Theta))^\rho$ can be written as an infinite series representation as follows.

$$\begin{aligned}(g(t;\Theta))^\rho &= \sum_{j=0}^\infty \binom{\rho}{j}\left((1-\delta)\alpha e^{-\alpha t}\right)^{\rho-j}\left(\delta(2\alpha+2\beta t)e^{-\left(2\alpha t+\beta t^2\right)}\right)^j \\ &= \sum_{j=0}^\infty \binom{\rho}{j}(1-\delta)^{\rho-j}\alpha^{\rho-j}e^{-(\rho-j)\alpha t}\delta^j(2\alpha+2\beta t)^j\ e^{-j\left(2\alpha t+\beta t^2\right)} \\ &= \sum_{j=0}^\infty \binom{\rho}{j}(1-\delta)^{\rho-j}\delta^j\alpha^{\rho-j}(2\alpha+2\beta t)^j e^{-(\rho+j)\alpha t-j\beta t^2}\end{aligned}$$

In the latter equation, the statement $e^{-(\rho+j)\alpha t-j\beta t^2}$ is rearranged as; $e^{-j\beta\left(t+\frac{(\rho+j)\alpha}{2j\beta}\right)^2+\frac{(\rho+j)^2\alpha^2}{4j\beta}}$ and if the Binomial theorem is applied in $(2\alpha+2\beta t)^j$, we can write

$$\begin{aligned}(g(t;\Theta))^\rho &= \sum_{j=0}^\infty\sum_{l=0}^j \binom{\rho}{j}\binom{j}{l}(1-\delta)^{\rho-j}\delta^j 2^j\alpha^{\rho-j}\alpha^{j-l}\beta^l t^l e^{-j\beta\left(t+\frac{(\rho+j)\alpha}{2j\beta}\right)^2+\frac{(\rho+j)^2\alpha^2}{4j\beta}} \\ &= \sum_{j=0}^\infty\sum_{l=0}^j \binom{\rho}{j}\binom{j}{l}(1-\delta)^{\rho-j}\delta^j 2^j\alpha^{\rho-l}\beta^l e^{\frac{(\rho+j)^2\alpha^2}{4j\beta}} t^l e^{-j\beta\left(t+\frac{(\rho+j)\alpha}{2j\beta}\right)^2}\end{aligned}$$

Then, the Rényi entropy can be written as follows

$$I_R(\rho) = \frac{1}{1-\rho}\log\left[\sum_{j=0}^\infty\sum_{l=0}^j \binom{\rho}{j}\binom{j}{l}(1-\delta)^{\rho-j}\delta^j 2^j\alpha^{\rho-l}\beta^l e^{\frac{(\rho+j)^2\alpha^2}{4j\beta}} \int_0^\infty t^l e^{-j\beta\left(t+\frac{(\rho+j)\alpha}{2j\beta}\right)^2} dt\right]$$

if the transformation $z = j\beta\left(t + \frac{(\rho+j)\alpha}{2j\beta}\right)^2$ is done in above integral,

$$\int_0^\infty t^l e^{-j\beta\left(t+\frac{(\rho+j)\alpha}{2j\beta}\right)^2} dt = \int_{\frac{(\rho+j)^2\alpha^2}{4j\beta}}^\infty \frac{1}{2\sqrt{j\beta z}}\left(\sqrt{\frac{z}{j\beta}} - \frac{(\rho+j)\,\alpha}{2j\beta}\right)^l e^{-z}dz$$

and the Binomial expansion is applied for $\left(\sqrt{\frac{z}{j\beta}} - \frac{(\rho+j)\alpha}{2j\beta}\right)^l$ again, then the equality

$$\left(\sqrt{\frac{z}{j\beta}} - \frac{(\rho+j)\,\alpha}{2j\beta}\right)^l = \sum_{k=0}^{l}\left(\begin{array}{c} l \\ k \end{array}\right)\left(\sqrt{\frac{z}{j\beta}}\right)^k\left(\frac{-(\rho+j)\,\alpha}{2j\beta}\right)^{l-k}$$

is obtained, then

$$\int_{\frac{(\rho+j)^2\alpha^2}{4j\beta}}^\infty \frac{1}{2\sqrt{j\beta z}}\left(\sqrt{\frac{z}{j\beta}} - \frac{(\rho+j)\,\alpha}{2j\beta}\right)^l e^{-z}dz$$

$$= \frac{1}{2}\sum_{k=0}^{l}\left(\begin{array}{c} l \\ k \end{array}\right)\left(\frac{-(\rho+j)\,\alpha}{2j\beta}\right)^{l-k} j\beta^{-\ \frac{k+1}{2}} \int_{\frac{(\rho+j)^2\alpha^2}{4j\beta}}^\infty z^{\frac{k-1}{2}} e^{-z}dz$$

Thus, the last integral can be expressed in terms of incomplete Gamma function as follows,

$$\int_{\frac{(\rho+j)^2\alpha^2}{4j\beta}}^\infty z^{\frac{k-1}{2}} e^{-z}dz = \Gamma\left(\frac{k+1}{2}, \frac{(\rho+j)^2\alpha^2}{4j\beta}\right)$$

Now, we obtain an explicit equality for $I_R(\rho)$ as follows,

$$I_R(\rho) = \frac{1}{1-\rho}\log\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{k=0}^{l}\left(\begin{array}{c} \rho \\ j \end{array}\right)\left(\begin{array}{c} j \\ l \end{array}\right)\left(\begin{array}{c} l \\ k \end{array}\right)(-1)^{l-k}(1-\delta)^{\rho-j}\delta^j 2^{j+k-l-1}$$

$$\alpha^{\rho-k}\beta^{\frac{k-1}{2}}(\rho+j)^{l-k} j^{\frac{k-1}{2}-l} e^{\frac{(\rho+j)^2\alpha^2}{4j\beta}}\ \Gamma\left(\frac{k+1}{2}, \frac{(\rho+j)^2\alpha^2}{4j\beta}\right)$$

3.1.10. *Order Statistics of the TGUE Distribution.* The order statistics are among the most basic tools in non-parametric statistics and inference. Also, the order statistics arise in the analysis of reliability of a system and it can represent the lifetimes of components of a reliability system. Let $T_{(1)}, T_{(2)}, \ldots, T_{(n)}$ denote the order statistics of a random sample $T_1, T_2, \ldots, T_n$ from a continuous population with p.d.f. $g(t;\Theta)$ and distribution function $G(t;\Theta)$, then the p.d.f. of j^{th} order statistics $T_{(j)}$ for $j = 1, 2, \ldots,\ n$ is given by

$$f_{T_{(j)}}(t;\Theta) = \frac{n!}{(j-1)!\,(n-j)!} g(t;\Theta)\,[G(t;\Theta)]^{j-1}[1-G(t;\Theta)]^{n-j}$$

$$\frac{n!}{(j-1)!\,(n-j)!}\left((1-\delta)\,\alpha e^{-\alpha t} + \delta\,(2\alpha+2\beta t)\,e^{-\left(2\alpha t+\beta t^2\right)}\right)$$

$$\times\Big(1-(1-\delta)\,e^{-\alpha t}-\delta e^{-\left(2\alpha t+\beta t^{2}\right)}\Big)^{j-1}\Big((1-\delta)\,e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^{2}\right)}\Big)^{n-j},$$

therefore, the p.d.f. of the first order statistics $T_{(1)}$ is given by

$$\begin{aligned} f_{T_{(1)}}(t;\Theta) &= n\Big((1-\delta)\,\alpha e^{-\alpha t}+\delta\,(2\alpha+2\beta t)\,e^{-\left(2\alpha t+\beta t^{2}\right)}\Big) \\ &\quad\times\Big[(1-\delta)\,e^{-\alpha t}+\delta e^{-\left(2\alpha t+\beta t^{2}\right)}\Big]^{n-1}, \end{aligned}$$

and the p.d.f. of the last order statistics $T_{(n)}$ is given

$$\begin{aligned} f_{T_{(n)}}(t;\Theta) &= n\Big((1-\delta)\,\alpha e^{-\alpha t}+\delta\,(2\alpha+2\beta t)\,e^{-\left(2\alpha t+\beta t^{2}\right)}\Big) \\ &\quad\times\Big(1-(1-\delta)\,e^{-\alpha t}-\delta e^{-\left(2\alpha t+\beta t^{2}\right)}\Big)^{j-1}. \end{aligned}$$

Note that $\delta = 0$ yields the order statistics of the exponential distribution with parameter α and when $\delta = 1$ yields the order statistics of the TGUE distribution with parameter (α, β).

4. Numerical Examples

In this section, we provide three data analyses in order to assess the goodness-of-fit of the TGUE distribution. The following tables show goodness-of-fit measures for the different distributions.

Data Set 1. (Wheaton River Flood Data) The data consist of the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0, 1.9, 2.8. Firstly, these data were analyzed by [10]. Later on, Beta-Pareto (BP) distribution was applied to these data by [2]. Merovcia and Pukab [22] made a comparison between Pareto (P) and Transmuted Pareto (TP) distribution. They showed that better model is the transmuted Pareto distribution. Bourguignon et al. [9] proposed Kumaraswamy Pareto (Kw-P) distribution. Tahir [30] have proposed Weibull-Pareto (WP) distribution and made a comparison with Beta Exponentiated Pareto (BEP) distribution. Nasiru and Luguterah [24] have proposed a different type of Weibull-Pareto (NWP) distribution. Exponential Modified Discrete Lindley (EMDL) distribution was applied to these data by [31]. We fit data to TGUE distribution and get parameter estimates as $\hat{\alpha} = 0.0672, \hat{\beta} = 0.2972, \hat{\delta} = 0.1976$ï˘. According to the model selection criteria (AIC) tabulated in Table 5.1, TGUE takes the first place amongst 9 proposed models.

Table 5.1. K-S test values, -2LL, AIC and BIC for TGUE, P, TP, EP, BP, Kw-P, WP, BEP, BGP and EMDL distributions

Model	K-S	-2LL	AIC	BIC
TGUE	0.089	496.3	502.3	509.1
EMDL	0.116	503.6	507.6	512.1
P	0.456	606.1	610.1	610.4
TP	0.389	572.4	578.4	580.9
EP	0.199	574.6	578.6	583.2
BP	0.175	567.4	573.4	580.3
Kw-P	0.170	542.4	548.4	555.3
WP	–	498.8	502.8	507.3
BEP	–	496.1	504.1	513.2

Data set 2. (Bladder Cancer Application) The second data set on the remission times (in months) of a random sample of 128 bladder cancer patients Lee and Wang [18] is given by 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39,10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. In this section, we test the performance of the TGUE distribution and show it to be an improved model as compared to some of its sub-models such as transmuted inverse Rayleigh distribution (TIRD), transmuted inverted exponential distribution (TIED), inverse Weibull distribution (IWD) and transmuted inverse Weibull distribution (IWD). It is clear from Table 5.2 that the TGUE model provides better fits than other models to this data sets. For the TGUE distribution parameter estimates are $\hat{\alpha} = 0.0485, \hat{\beta} = 0.0057, \hat{\delta} = 0.7745$ï˘ and this distribution has the lower AIC, BIC and K-S values.

Table 5.2. K-S test values, -2LL, AIC and BIC for TGUE, TIW, TIE, IW and TIR distributions

Model	K-S	-2LL	AIC	BIC
TGUE	0.065	824.2	830.1	838.6
TIW	0.119	877.0	879.4	879.7
TIE	0.155	885.6	889.6	889.8
IW	0.131	888.0	892.0	892.2
TIR	0.676	1420.4	1424.4	1424.6

Data set 3. (Bank B Data) The data set represents the waiting times (in minutes) before customer service of 60 bank customers in Bank B. This data set is given as: 0.1, 0.2, 0.3, 0.7, 0.9, 1.1, 1.2, 1.8, 1.9, 2.0, 2.2, 2.3, 2.3, 2.3, 2.5, 2.6, 2.7, 2.7, 2.9, 3.1, 3.1, 3.2, 3.4, 3.4, 3.5, 3.9, 4.0, 4.2, 4.5, 4.7, 5.3, 5.6, 5.6, 6.2, 6.3, 6.6, 6.8, 7.3, 7.5, 7.7, 7.7, 8.0, 8.0, 8.5, 8.5, 8.7, 9.5, 10.7, 10.9, 11.0, 12.1, 12.3, 12.8, 12.9, 13.2, 13.7, 14.5, 16.0, 16.5, 28.0. This data was analyzed by [3] and was also used by [29]. They fit this data to Lindley (L) and generalized Lindley (GL) distributions. We fit data to TGUE distribution and get parameter estimates as $\hat{\alpha} = 0.185, \hat{\beta} = 0.472, \hat{\delta} = -0.222$. According to the model selection criteria tabulated in Table 5.3, it is said that TUGE takes first place in amongst 3 proposed models.

Table 5.3. K-S test values, -2LL AIC and BIC for TGUE, L and Exp distributions

Model	**K-S**	**-2LL**	**AIC**	**BIC**
TGUE	0.067	336.777	342.777	349.060
L	0.080	338.203	340.203	341.759
GL	0.068	338.026	342.026	341.582

In the above three tables, it is clear that the values of the Akaike information criterion (AIC) and Bayesian information criterion (BIC) are smaller for the TGUE distribution compared to those values of the other models; the new distribution is a very competitive model to these data.

5. Conclusion

In this article, we propose a new model of transmuted distribution so-called the transmuted Gumbel univariate exponential distribution. The subject distribution is generated by using the convex combination of failure probabilities of two-component series and systems and taking the Gumbel univariate exponential distribution as the base distribution. Some mathematical and statistical properties including explicit expressions for the probability density, survival, cumulative hazard rate, hazard rate and mean residual life functions, also, moment generating function and moments are addressed. The estimation of parameters is approached by the maximum likelihood method. According to K-S values in Numerical Examples Section, the applications of the transmuted Gumbel univariate exponential distribution to real data show that the new distribution can be used to provide better fits than the other distributions. We hope that this new distribution may attract wider applications in the lifetime literature. Taking bivariate distributions will guide to derivation of many new univariate distributions.

6. Appendix

Conditional cdf of T_2 with given $T_1 = t_1$ is given by

$$\begin{aligned} F_{T_2|T_1}(t_2) &= \frac{\frac{\partial}{\partial t_1} F_{T_1,T_2}(t_1,t_2)}{f_{T_1}(t_1)} = \frac{\frac{\partial}{\partial t_1}\left(1 - e^{-\alpha t_1} - e^{-\alpha t_2} + e^{-\alpha(t_1+t_2)-\beta t_1 t_2}\right)}{\alpha e^{-\alpha t_1}} \\ &= \frac{\alpha e^{-\alpha t_1} - (\alpha + \beta t_2)\, e^{-\alpha(t_1+t_2)-\beta t_1 t_2}}{\alpha e^{-\alpha t_1}} \\ &= 1 - \left(1 + \frac{\beta}{\alpha} t_2\right) e^{-(\alpha+\beta t_1) t_2}. \end{aligned}$$

Hence, by equating $F_{T_2|T_1}(t_2) = U_2$ where U_2 is uniformly distributed random variable on the interval $(0,1)$ we have a non linear equation to get solution for t_2 as follows,

$$1 - \left(1 + \frac{\beta}{\alpha} t_2\right) e^{-(\alpha+\beta t_1) t_2} = U_2. \tag{17}$$

To solve the above equation for t_2, we use Lambert W- function which is defined as the solution of the equation $W(z)\, e^{W(z)} = z$, where z is the complex number. If z is any real number, then this equation has a solution on $\left[-e^{-1}, +\infty\right)$.
In equation (17), if the expression $1 + \frac{\beta}{\alpha} t_2$ is taken as z, we can write

$$z e^{-\left(\frac{\alpha^2}{\beta} + \alpha t_1\right) z} e^{\left(\frac{\alpha^2}{\beta} + \alpha t_1\right)} = 1 - U_2.$$

Multiplying both sides of equation above by $-\left(\frac{\alpha^2}{\beta} + \alpha t_1\right)$, above expression can be simplified as follows,

$$-\left(\frac{\alpha^2}{\beta} + \alpha t_1\right) z e^{-\left(\frac{\alpha^2}{\beta} + \alpha t_1\right) z} = -(1 - U_2)\left(\frac{\alpha^2}{\beta} + \alpha t_1\right) e^{-\left(\frac{\alpha^2}{\beta} + \alpha t_1\right)}.$$

Substituting $-\left(\frac{\alpha^2}{\beta} + \alpha t_1\right) z = W(z)$, we have the Lambert equation

$$W(z)\, e^{-W(z)} = \eta,$$

where $\eta = -(1 - U_2)\left(\frac{\alpha^2}{\beta} + \alpha t_1\right) e^{-\left(\frac{\alpha^2}{\beta} + \alpha t_1\right)}$. Hence, the solution for $W(z)$ is

$$-\left(\frac{\alpha^2 + \alpha\beta t_1}{\beta}\right) z = W_{-1}(\eta).$$

So, t_2 is found as follows

$$t_2 = -\frac{\alpha}{\beta} - \frac{1}{\alpha + \beta t_1} W_{-1}(\eta).$$

To show the uniqueness of the solution for t_2 we take into account the well known inequality $e^{-(z+1)} \geq -z$ and replacing z with $-(\alpha + \beta t_1)\frac{\alpha}{\beta}$, then $\eta \geq -\frac{1}{e}$.

This result guarantees that η belongs to domain of negative branch of Lambert W-function.

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ANALYSIS OF THE RAYLEIGH WAVE FIELD DUE TO A TANGENTIAL LOAD APPLIED ON THE SURFACE OF A COATED ELASTIC HALF-SPACE

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Abstract. The paper deals with 3D dynamic response of a coated elastic half space subject to in-plane surface loading. The problem is formulated by a pair of elliptic equations over the interior and a two dimensional singularly perturbed hyperbolic equation expressed in terms of shear wave potentials along the interface. As an example, a point force acting one of the in-plane axis is considered and the integral solution of the normal displacement along the interface is derived through the use of the relation between the wave potentials.

1. Introduction

Propagation of surface waves has been the focus of intensive research since its introduction by the monumental work of Rayleigh [1]. Rayleigh waves, therefore, have been extensively studied by scientists and engineers due to their applicability to acoustic, seismology, electromagnetism, among others. One of the most important contribution to the Rayleigh wave was made by Friedlander who presented the Rayleigh wave field for an elastic half plane in terms of arbitrary plane harmonic functions [2]. In a later publication, Chadwick extended Friedlander's analysis and expressed the Rayleigh wave field in terms of a *single* harmonic function via a relation between the wave potentials at the surface of the elastic half-plane [3]. This relationship was, then, extended to three dimensions in [4].

The significance of the surface waves on an elastic half-plane or half-space motivates an alternative analysis under more general assumptions, which may help to extract the Rayleigh wave contribution. Therefore, recent studies have generally focused on employing approximate models to derive the Rayleigh wave contribution which is often hidden in the problem formulation, see, e.g., [5]–[8]. In [8], an explicit model for the Rayleigh and Bleustein-Gulyaev surface waves was presented. The derivations were based on perturbing in slow time the self-similar solution for

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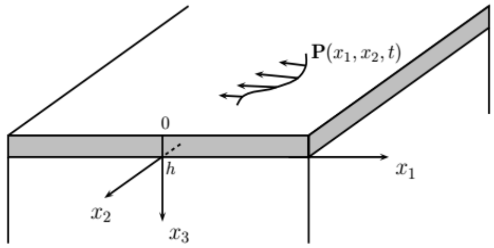

FIGURE 1. Tangential loading on the surface of coated half-space

homogeneous surface waves given in [2] and [3]. Thus, the developed models for the surface waves consisted of hyperbolic equation on the surface with two elliptic equations in the interior domain. The formulation in [8] was later generalized to the three dimensional linear, isotropic, coated elastic half-space taking into account the effect of a thin coating [9]. The hyperbolic-elliptic model for surface wave on an orthorhombic half space was presented in [10]. A surface wave of arbitrary profile in anisotropic half-space was constructed by means of the Stroh formalism in [11]. We also mention [12] which considered surface waves in a coated half-space with a clamped surface because of the applicability of the hyperbolic-elliptic model for surface wave in high frequency domain. Analysis of the Rayleigh field in a three dimensional elastic half space subject to in-plane surface loading was given in [13]. In addition we cite [14] summarizing the asymptotic model for Rayleigh and Rayleigh type waves and [15] which includes a recent composite model joining both low-frequency and high-frequency models. Along with the latest advancement of technology, moving load problems also find various modern industrial applications in modern engineering, see [16]-[19]. The developed hyperbolic elliptic models have also been utilized for investigation of the near-resonant regimes of moving loads on elastic and coated elastic half-spaces, see e.g [20]-[24].

The organization of the paper is described as follows. Section 2 contains the statement of the problem, presenting the governing equations together with the boundary conditions. In Section 3 an asymptotic model for the Rayleigh wave field in the case of an elastic half-space coated with a thin layer is developed. An illustrative example for the derived model is given in Section 4. In the last section numerical computations based on the derived approximate formulae are presented.

2. Statement of the problem

Consider a 3D homogeneous isotropic elastic half-space coated by a thin layer of constant thickness h, loaded with a tangential force of amplitude P, see Figure 1. The equations of motion in 3D elasticity are written as (see, e.g. [25])

$$(\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u} = \rho\mathbf{u}_{tt}, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, ρ is the volume density and Δ is the 3D Laplace operator.

The constitutive relations for a linear isotropic elastic solid are given by

$$\sigma_{ij} = \lambda\delta_{ij}\mathrm{div}\mathbf{u} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i = 1, 2, 3, \tag{2}$$

where σ_{ij} are the components of the Cauchy stress tensor, λ and μ are the Lamé constants and δ_{ij} is the Kronecker delta.

The boundary conditions at the surface $x_3 = 0$ of the coating are specified as

$$(\sigma_{13}, \sigma_{23}, \sigma_{33}) = -\mathbf{P}. \tag{3}$$

The tangential load may be decomposed through the Helmholtz theorem [25] as

$$\mathbf{P}(x_1, x_2, t) = (\boldsymbol{\nabla} P_0 + \boldsymbol{\nabla} \times \tilde{\mathbf{P}}) = \left(P^{(1)}, P^{(2)}, 0\right), \tag{4}$$

where

$$\tilde{\mathbf{P}} = (0, 0, P_1) \text{ and } \left(P^{(1)}, P^{(2)}\right) = \left(\frac{\partial P_0}{\partial x_1} + \frac{\partial P_1}{\partial x_2}, \frac{\partial P_0}{\partial x_2} - \frac{\partial P_1}{\partial x_1}\right). \tag{5}$$

All of the equations above describe the substrate $x_3 \geqslant h$. In the case of coating, $0 \leq x_3 \leq h$, subscript "0" is used for the material parameters, e.g. ρ_0, λ_0, μ_0 etc.

Our first aim is to state the boundary conditions at the interface $x_3 = h$. To this end, taking into consideration the effective boundary conditions presented in [9], the boundary conditions at the surface $x_3 = 0$ may be carried on the surface of the substrate. As a result, the boundary conditions at $x_3 = h$ can be written as

$$\begin{aligned} \sigma_{i3} = \mu\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right) = \rho_0 h \Bigg\{ \frac{\partial^2 u_i}{\partial t^2} - c_{20}^2 \Bigg(& \frac{\partial^2 u_3}{\partial x_j^2} + 4(1 - \kappa_0^{-2})\frac{\partial^2 u_i}{\partial x_i^2} \\ & + (3 - 4\kappa_0^{-2})\frac{\partial^2 u_j}{\partial x_i \partial x_j}\Bigg)\Bigg\} - P^{(i)}, \end{aligned} \tag{6}$$

and

$$\sigma_{33} = \lambda\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j}\right) + (\lambda + 2\mu)\frac{\partial u_3}{\partial x_3} = \rho_0 h \frac{\partial^2 u_3}{\partial t^2}, \quad i \neq j = 1, 2 \tag{7}$$

where c_{10}, c_{20} are the longitudinal and transverse wave speeds, ρ_0 is the density of the coating and $\kappa_0 = c_{10}/c_{20}$.

3. Asymptotic Model

In this section an asymptotic model is established for the substrate governed by equation (1) and subject to the effective boundary conditions (6)-(7) at the surface $x_3 = h$. First, on applying the Radon transform to the equations of motion (1), see [26], the problem is reduced to a two-dimensional one and, then, the explicit model, derived for an elastic half-plane in [8], is applied to the reduced two-dimensional boundary value problem. Thus, an elliptic-hyperbolic model may be developed for the considered problem.

Let us apply the Radon transform to eqs. (1), (6) and (7), resulting, respectively, in

$$
\begin{aligned}
&\left[(\lambda+\mu)\cos^2\alpha+\mu\right]\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi^2}+\mu\frac{\partial^2 u_1^{(\alpha)}}{\partial x_3^2}\\
&\qquad+(\lambda+\mu)\cos\alpha\left(\sin\alpha\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi^2}+\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi\partial x_3}\right)=\rho\frac{\partial^2 u_1^{(\alpha)}}{\partial t^2},\\
&\left[(\lambda+\mu)\sin^2\alpha+\mu\right]\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi^2}+\mu\frac{\partial^2 u_2^{(\alpha)}}{\partial x_3^2},\\
&\qquad+(\lambda+\mu)\sin\alpha\left(\cos\alpha\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi^2}+\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi\partial x_3}\right)=\rho\frac{\partial^2 u_2^{(\alpha)}}{\partial t^2},\\
&(\lambda+\mu)\left(\cos\alpha\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi\partial x_3}+\sin\alpha\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi\partial x_3}\right)+\mu\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi^2},\\
&\qquad+(\lambda+2\mu)\frac{\partial^2 u_3^{(\alpha)}}{\partial x_3^2}=\rho\frac{\partial^2 u_3^{(\alpha)}}{\partial t^2},
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
\sigma_{13}^{(\alpha)}&=\mu\left(\frac{\partial u_1^{(\alpha)}}{\partial x_3}+\cos\alpha\frac{\partial u_3^{(\alpha)}}{\partial\chi}\right)=\rho_0 h\left[\frac{\partial^2 u_1^{(\alpha)}}{\partial t^2}-c_{20}^2\left(\sin^2\alpha\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi^2}\right.\right.\\
&\quad\left.\left.+4\left(1-\kappa_0^{-2}\right)\cos^2\alpha\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi^2}+\left(3-4\kappa_0^{-2}\right)\sin\alpha\cos\alpha\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi^2}\right)\right]\\
&\quad-\left(\cos\alpha\frac{\partial P_0^{(\alpha)}}{\partial\chi}+\sin\alpha\frac{\partial P_1^{(\alpha)}}{\partial\chi}\right),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\sigma_{23}^{(\alpha)}&=\mu\left(\frac{\partial u_2^{(\alpha)}}{\partial x_3}+\sin\alpha\frac{\partial u_3^{(\alpha)}}{\partial\chi}\right)=\rho_0 h\left[\frac{\partial^2 u_2^{(\alpha)}}{\partial t^2}-c_{20}^2\left(\cos^2\alpha\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi^2}\right.\right.\\
&\quad\left.\left.+4\left(1-\kappa_0^{-2}\right)\sin^2\alpha\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi^2}+\left(3-4\kappa_0^{-2}\right)\sin\alpha\cos\alpha\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi^2}\right)\right]\\
&\quad-\left(\sin\alpha\frac{\partial P_0^{(\alpha)}}{\partial\chi}-\cos\alpha\frac{\partial P_1^{(\alpha)}}{\partial\chi}\right),
\end{aligned}
\tag{10}
$$

$$
\sigma_{33}^{(\alpha)}=\lambda\left(\cos\alpha\frac{\partial u_1^{(\alpha)}}{\partial\chi}+\sin\alpha\frac{\partial u_2^{(\alpha)}}{\partial\chi}\right)+(\lambda+2\mu)\frac{\partial u_3^{(\alpha)}}{\partial x_3}=\rho_0 h\frac{\partial^2 u_3^{(\alpha)}}{\partial t^2}.
$$

Here, the Radon transform is defined as

$$u_k^{(\alpha)}(\chi,\alpha,x_3,t) = \int\limits_{-\infty}^{\infty} u_k(\chi\cos\alpha - \zeta\sin\alpha, \chi\sin\alpha + \zeta\cos\alpha, x_3, t)\, d\zeta,$$

where

$$\chi = x_1\cos\alpha + x_2\sin\alpha, \quad \zeta = -x_1\sin\alpha + x_2\cos\alpha$$

with the angle α varying on the interval $0 \le \alpha \le 2\pi$, see Figure 2. The original

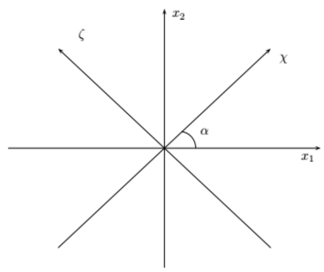

FIGURE 2. Rotation of Cartesian frame

displacements may be written in terms of the transformed displacements as

$$u_1 = u_\chi^\alpha\cos(\alpha) - u_\zeta^\alpha\sin(\alpha), \quad u_2 = u_\chi^\alpha\sin(\alpha) + u_\zeta^\alpha\cos(\alpha). \tag{11}$$

Assuming that the surface wave field is not distributed by anti-plane motion, it can be emphasized that $u_\zeta^\alpha = 0$, see [9]. Substituting eq. (11) into eq. (8) and taking into account the assumption above , eq. (8) takes the following plane problem form:

$$\begin{aligned}
&(\lambda+2\mu)\frac{\partial^2 u_\chi^{(\alpha)}}{\partial\chi^2} + \mu\frac{\partial^2 u_\chi^{(\alpha)}}{\partial x_3^2} + (\lambda+\mu)\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi\partial x_3} = \rho\frac{\partial^2 u_\chi^{(\alpha)}}{\partial t^2},\\
&(\lambda+\mu)\frac{\partial^2 u_\chi^{(\alpha)}}{\partial\chi\partial x_3} + \mu\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi^2} + (\lambda+2\mu)\frac{\partial^2 u_3^{(\alpha)}}{\partial x_3^2} = \rho\frac{\partial^2 u_\chi^{(\alpha)}}{\partial t^2},
\end{aligned} \tag{12}$$

with the boundary conditions

$$\begin{aligned}
\sigma_{\chi 3}^{(\alpha)} &= \mu\left(\frac{\partial u_\chi^{(\alpha)}}{\partial x_3} + \frac{\partial u_3^{(\alpha)}}{\partial\chi}\right) = \mu_0 h\left[c_{20}^{-2}\frac{\partial^2 u_\chi^{(\alpha)}}{\partial t^2} - 4\left(1-\kappa_0^{-2}\right)\frac{\partial^2 u_\chi^{(\alpha)}}{\partial\chi^2}\right] - \frac{\partial P_0^{(\alpha)}}{\partial\chi},\\
\sigma_{33}^{(\alpha)} &= \lambda\frac{\partial u_\chi^{(\alpha)}}{\partial\chi} + (\lambda+2\mu)\frac{\partial u_3^{(\alpha)}}{\partial x_3} = \rho_0 h\frac{\partial^2 u_3^{(\alpha)}}{\partial t^2}.
\end{aligned} \tag{13}$$

It is well known that the displacement vector $\mathbf{u}$ may expressed through the sum of gradient of scaler potential ϕ and the curl of a vector potential ψ, that is

$$\mathbf{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\times\boldsymbol{\psi}, \tag{14}$$

where $\boldsymbol{\psi} = (-\psi_2, \psi_1, 0)$, see [25] and [9]. Thus, the transformed displacement components can be expressed in terms of the transformed wave potentials as

$$u_{\chi}^{(\alpha)} = \frac{\partial \phi^{(\alpha)}}{\partial \chi} - \frac{\partial \psi^{(\alpha)}}{\partial x_3} \qquad \text{and} \qquad u_3^{(\alpha)} = \frac{\partial \phi^{(\alpha)}}{\partial x_3} + \frac{\partial \psi^{(\alpha)}}{\partial \chi}. \tag{15}$$

On inserting the transformed potentials into displacement and boundary equations (12) and (13) we obtain

$$\begin{aligned} \frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} + \frac{\partial^2 \phi^{(\alpha)}}{\partial x_3^2} - \frac{1}{c_1^2}\frac{\partial^2 \phi^{(\alpha)}}{\partial t^2} = 0, \\ \frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} + \frac{\partial^2 \psi^{(\alpha)}}{\partial x_3^2} - \frac{1}{c_2^2}\frac{\partial^2 \psi^{(\alpha)}}{\partial t^2} = 0, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mu\left[2\frac{\partial^2 \phi^{(\alpha)}}{\partial \chi \partial x_3} + \frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} - \frac{\partial^2 \psi^{(\alpha)}}{\partial x_3^2}\right] &= \mu_0 h \left[c_{20}^{-2}\left(\frac{\partial^3 \phi^{(\alpha)}}{\partial \chi \partial t^2} - \frac{\partial^3 \psi^{(\alpha)}}{\partial x_3 \partial t^2}\right)\right. \\ &\quad \left. -4\left(1-\kappa_0^{-2}\right)\left(\frac{\partial^3 \phi^{(\alpha)}}{\partial \chi^3} - \frac{\partial^3 \psi^{(\alpha)}}{\partial x_3 \partial \chi^2}\right)\right] - \frac{\partial P_0^{(\alpha)}}{\partial \chi}, \\ \mu\left[\left(\kappa^2-2\right)\frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} + \kappa^2\frac{\partial^2 \phi^{(\alpha)}}{\partial x_3^2} + 2\frac{\partial^2 \psi^{(\alpha)}}{\partial \chi \partial x_3}\right] &= \mu_0 h c_{20}^{-2}\left(\frac{\partial^3 \phi^{(\alpha)}}{\partial x_3 \partial t^2} + \frac{\partial^3 \psi^{(\alpha)}}{\partial \chi \partial t^2}\right). \end{aligned} \tag{17}$$

where c_1 and c_2 are the longitudinal and shear wave speeds and $\kappa = c_1/c_2$. It can easily be seen from the boundary equations that the surface wave is only induced by gradient part of the applied load P_0.

The considered three dimensional problem of elasticity, thus, is reduced to a two dimensional plane problem with the help of the Radon transform. We can now employ the explicit model dealing with the wave propagation along the surface of the two dimensional elastic half-plane with the Rayleigh wave speed, see [8]. Following the same asymptotic methodology performed in [8] and [9], the wave equations (16) are cast into a pair of elliptic equations in the interior of the half plane, given by

$$\begin{aligned} \frac{\partial^2 \phi^{(\alpha)}}{\partial x_3^2} + k_1^2\frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} = 0, \\ \frac{\partial^2 \psi^{(\alpha)}}{\partial x_3^2} + k_2^2\frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} = 0. \end{aligned} \tag{18}$$

Similarly, the boundary conditions (17) give a partial differential equation along the surface $x_3 = h$

$$\frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} - \frac{1}{c_R^2}\frac{\partial^2 \psi^{(\alpha)}}{\partial t^2} + \frac{bh}{k_2}\frac{\partial^3 \psi^{(\alpha)}}{\partial \chi^2 \partial x_3} = \frac{\left(1+k_2^2\right)}{2\mu B}\frac{\partial P_0^{(\alpha)}}{\partial \chi}, \tag{19}$$

with the relation between the wave potentials on the surface expressed through

$$\frac{\partial \psi^{(\alpha)}}{\partial \chi} = -\frac{2}{1+k_2^2}\frac{\partial \phi^{(\alpha)}}{\partial x_3}, \qquad \text{or} \qquad \frac{\partial \phi^{(\alpha)}}{\partial \chi} = \frac{2}{1+k_2^2}\frac{\partial \psi^{(\alpha)}}{\partial x_3}, \tag{20}$$

where $k_i^2 = 1 - c_R^2/c_i^2$; $i = 1, 2$, $m = \mu_0/\mu$ and

$$\begin{aligned} B &= \left(1-k_1^2\right)\frac{k_2}{k_1} + \left(1-k_2^2\right)\frac{k_1}{k_2} - \left(1-k_2^4\right), \\ b &= \frac{m}{2B}\left(1-k_2^2\right)\left[\left(1-k_{20}^2\right)(k_1+k_2) - 4k_2\left(1-\kappa_0^{-2}\right)\right]. \end{aligned} \tag{21}$$

By taking the inverse Radon transform of eqns. (18)-(20) we arrive at the asymptotic formulation given by two elliptic equations in the interior

$$\frac{\partial^2 \phi}{\partial x_3^2} + k_1^2 \Delta_2 \phi = 0, \qquad \frac{\partial^2 \psi_i}{\partial x_3^2} + k_1^2 \Delta_2 \psi_i = 0, \qquad i = 1, 2, \tag{22}$$

with the surface equation along the plane $x_3 = h$

$$\Delta_2 \psi_i - \frac{1}{c_R^2}\frac{\partial^2 \psi_i}{\partial t^2} - bh\sqrt{-\Delta_2}\,\Delta_2 \psi_i = \frac{\left(1+k_2^2\right)}{2\mu B}\frac{\partial P_0}{\partial x_i}, \quad i = 1, 2, \tag{23}$$

and the relations between the potentials

$$\frac{\partial \phi}{\partial x_i} = \frac{2}{1+k_2^2}\frac{\partial \psi_i}{\partial x_3}, \ (i = 1, 2), \qquad \frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} = -\frac{2}{1+k_2^2}\frac{\partial \phi}{\partial x_3}, \text{ at } x_3 = h, \tag{24}$$

where $\Delta_2 = \partial_1^2 + \partial_2^2$ and $\sqrt{-\Delta_2}$ is a pseudo differential operator, see [9].

4. Illustrative Example

In this section, the dynamic response of the coated elastic half space, which is loaded by a tangential force, is evaluated within the framework of the asymptotic formulation derived in the previous section. Consider a tangential point load acting along the x_1 axis given by

$$\mathbf{P} = (M\delta(x_1)\delta(x_2)\delta(t), 0, 0). \tag{25}$$

From the decomposition introduced in (5), $\mathbf{P}$ may be written as

$$(M\delta(x_1)\delta(x_2)\delta(t), 0, 0) = \left(\frac{\partial P_0}{\partial x_1} + \frac{\partial P_1}{\partial x_2}, \frac{\partial P_0}{\partial x_2} - \frac{\partial P_1}{\partial x_1}, 0\right),$$

resulting in

$$\Delta_2 P_0 = M\delta'(x_1)\delta(x_2)\delta(t), \tag{26}$$

where $\Delta_2 = \partial_1^2 + \partial_2^2$ is two dimensional Laplacian. On using the well-known fundamental solution for two dimensional Laplace operator, [27],

$$E(x_1, x_2) = \frac{1}{4\pi} \ln\left(x_1^2 + x_2^2\right), \tag{27}$$

P_0 may be expressed as the convolution of eq. (26) with the fundamental solution (27), namely

$$P_0(x_1, x_2, t) = E(x_1, x_2) * M\delta'(x_1)\delta(x_2)\delta(t) = \frac{M}{2\pi}\frac{x_1}{x_1^2 + x_2^2}\delta(t). \tag{28}$$

Thus, the hyperbolic equations on the surface $x_3 = h$ are written as

$$\Delta_2\psi_1 - \frac{1}{c_R^2}\frac{\partial^2\psi_1}{\partial t^2} - bh\sqrt{-\Delta_2}\Delta_2\psi_1 = M_0\frac{x_2^2 - x_1^2}{\left(x_1^2 + x_2^2\right)^2}\delta(t), \tag{29}$$

$$\Delta_2\psi_2 - \frac{1}{c_R^2}\frac{\partial^2\psi_2}{\partial t^2} - bh\sqrt{-\Delta_2}\Delta_2\psi_2 = -2M_0\frac{x_1x_2}{\left(x_1^2 + x_2^2\right)^2}\delta(t), \tag{30}$$

where

$$M_0 = \frac{M(1 + k_2^2)}{4\pi\mu B}. \tag{31}$$

Introducing scaled variables

$$\eta_1 = \frac{x_1}{b\,h}, \quad \eta_2 = \frac{x_2}{b\,h}, \quad \tau = \frac{c_R}{b\,h}\,t, \tag{32}$$

the surface equations (29) and (30) take the forms

$$\frac{\partial^2\psi_1}{\partial\eta_1^2} + \frac{\partial^2\psi_1}{\partial\eta_2^2} - \frac{\partial^2\psi_1}{\partial\tau^2} - \sqrt{-\left(\frac{\partial^2}{\partial\eta_1^2} + \frac{\partial^2}{\partial\eta_2}\right)}\left(\frac{\partial^2\psi_1}{\partial\eta_1^2} + \frac{\partial^2\psi_1}{\partial\eta_2^2}\right) = \frac{M_0c_R}{bh}\frac{\eta_2^2 - \eta_1^2}{\left(\eta_1^2 + \eta_2^2\right)^2}\delta(\tau), \tag{33}$$

$$\frac{\partial^2\psi_2}{\partial\eta_1^2} + \frac{\partial^2\psi_2}{\partial\eta_2^2} - \frac{\partial^2\psi_2}{\partial\tau^2} - \sqrt{-\left(\frac{\partial^2}{\partial\eta_1^2} + \frac{\partial^2}{\partial\eta_2}\right)}\left(\frac{\partial^2\psi_2}{\partial\eta_1^2} + \frac{\partial^2\psi_2}{\partial\eta_2^2}\right) = \frac{-2M_0c_R}{bh}\frac{\eta_1\eta_2}{\left(\eta_1^2 + \eta_2^2\right)^2}\delta(\tau). \tag{34}$$

Let us first consider the surface equation of shear potential ψ_1. Applying a double Fourier and a Laplace transform to eq. (33) result in

$$\psi_1^{FFL} = -\frac{M_0c_R}{bh}\frac{\xi_1^2}{\left(\xi_1^2 + \xi_2^2\right)\left(s^2 + \left(\xi_1^2 + \xi_2^2\right)\left(1 - \sqrt{\xi_1^2 + \xi_2^2}\right)\right)} \tag{35}$$

where (ξ_1, ξ_2) and s are the Fourier and Laplace transform parameters, respectively. Taking the inverse double Fourier and Laplace transforms gives

$$\begin{aligned}\psi_1(\eta_1, \eta_2, \tau) = -\frac{M_0 c_R}{4\pi^2 bh} \Bigg\{ &\int\limits_{\rho>1} \frac{\xi_1^2 \; \mathrm{e}^{-\rho\sqrt{\rho-1}\,\tau}}{2\rho^3\sqrt{\rho-1}} \mathrm{e}^{i\boldsymbol{\rho}\cdot\mathbf{r}} d\boldsymbol{\rho} \\ &+ \int\limits_{\rho<1} \frac{\xi_1^2 \; \sin\left(\rho\sqrt{1-\rho}\,\tau\right)}{\rho^3\sqrt{1-\rho}} \mathrm{e}^{i\boldsymbol{\rho},\cdot\mathbf{r}} d\boldsymbol{\rho} \Bigg\},\end{aligned} \tag{36}$$

where $\mathbf{r} = (\eta_1, \eta_2) = (r\cos\theta, r\sin\theta)$ and $\boldsymbol{\rho} = (\xi_1, \xi_2) = (\rho\cos\omega, \rho\sin\omega)$ with $|\mathbf{r}| = r$ and $|\boldsymbol{\rho}| = \rho$. The above integral may then be written as

$$\begin{aligned}\psi_1(r, \theta, \tau) = -\frac{M_0 c_R}{4\pi^2 bh} \Bigg\{ &\int\limits_1^\infty \frac{\mathrm{e}^{-\rho\sqrt{\rho-1}\,\tau}}{2\sqrt{\rho-1}} \int\limits_0^{2\pi} \cos^2\omega \; \mathrm{e}^{ir\rho\cos(\omega-\theta)} d\omega d\rho \\ &+ \int\limits_0^1 \frac{\sin\left(\rho\sqrt{1-\rho}\,\tau\right)}{\sqrt{1-\rho}} \int\limits_0^{2\pi} \cos^2\omega \; \mathrm{e}^{ir\rho\cos(\omega-\theta)} d\omega d\rho \Bigg\}.\end{aligned} \tag{37}$$

Using the trigonometric relation for $\cos^2\omega$ the first integral in the above equation can be written as

$$\int\limits_0^{2\pi} \cos^2\omega \; \mathrm{e}^{ir\rho\cos(\omega-\theta)} d\omega d\rho = \frac{1}{4} \int\limits_0^{2\pi} \left(\mathrm{e}^{2i\omega} + \mathrm{e}^{-2i\omega} + 2\right) \; \mathrm{e}^{ir\rho\cos(\omega-\theta)} d\omega d\rho. \tag{38}$$

Changing the variable $\omega - \theta = \gamma$, the first integral on the right hand side of (38) takes the form

$$\int\limits_0^{2\pi} \mathrm{e}^{2i\omega} \; \mathrm{e}^{ir\rho\cos(\omega-\theta)} d\omega = -\mathrm{e}^{2i\theta} \int\limits_{\theta_0}^{2\pi+\theta_0} \mathrm{e}^{i(2\gamma - r\rho\sin\gamma)} d\gamma = -2\pi \mathrm{e}^{2i\theta} J_2(r\rho), \tag{39}$$

where $\theta_0 = -\theta - \pi/2$ and $J_2(r\rho)$ is Bessel function of the first kind defined as

$$J_n(x) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \mathrm{e}^{i(x\sin\gamma - n\gamma)} d\gamma.$$

Similarly, the second and third integrals on the right hand side of (38) are written, respectively, as

$$\int\limits_0^{2\pi} \mathrm{e}^{-i2\omega} \; \mathrm{e}^{ir\rho\cos(\omega-\theta)} d\omega = -2\pi \mathrm{e}^{-2i\theta} J_{-2}(r\rho), \tag{40}$$

and

$$2\int_0^{2\pi} \mathrm{e}^{ir\rho\cos(\omega-\theta)}d\omega = 4\pi J_0(r\rho). \tag{41}$$

The shear potential ψ_1, thus, can be expressed in terms of Bessel functions of the first kind as

$$\begin{aligned}\psi_1(r,\theta,\tau) = \frac{M_0 c_R}{4\pi\, bh}\Bigg\{ &\int_0^1 \frac{\sin\left(\rho\sqrt{1-\rho}\,\tau\right)}{\sqrt{1-\rho}}\left(\cos 2\theta J_2(r\rho) - J_0(r\rho)\right)d\rho \\ &+ \int_1^\infty \frac{\mathrm{e}^{-\rho\sqrt{\rho-1}\,\tau}}{2\sqrt{\rho-1}}\left(\cos 2\theta J_2(r\rho) - J_0(r\rho)\right)d\rho\Bigg\}.\end{aligned} \tag{42}$$

Repeating an almost identical procedure, the shear potential ψ_2 may be put in the form

$$\begin{aligned}\psi_2(r,\theta,\tau) = \frac{M_0 c_R}{4\pi\, bh}\sin 2\theta\Bigg\{ &\int_0^1 \frac{\sin\left(\rho\sqrt{1-\rho}\,\tau\right)}{\sqrt{1-\rho}} J_2(r\rho)d\rho \\ &+ \int_1^\infty \frac{\mathrm{e}^{-\rho\sqrt{\rho-1}\,\tau}}{2\sqrt{\rho-1}} J_2(r\rho)d\rho\Bigg\}.\end{aligned} \tag{43}$$

It is known from eq. (14) that the normal displacement at the surface $x_3 = h$ may be given in terms the wave potentials by

$$u_3|_{x_3=h} = \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi_2}{\partial x_1} + \frac{\partial\psi_1}{\partial x_2},$$

which can be expressed in terms of the new variables as

$$u_3|_{x_3=h} = \frac{1-k_2^2}{2bh}\left(\cos\theta\frac{\partial\psi_1}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi_1}{\partial\theta} + \sin\theta\frac{\partial\psi_2}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\psi_2}{\partial\theta}\right). \tag{44}$$

On using the relation between the potentials (24), it is possible to write the scaled normal displacement from the related derivative of the integral expressions of ψ_1 and ψ_2 as

$$U_3(r,\theta,\tau) = \cos\theta\left\{\int_0^1 \frac{\sin\left(\rho\sqrt{1-\rho}\,\tau\right)}{\sqrt{1-\rho}}\rho J_1(r\rho)d\rho + \int_1^\infty \frac{\mathrm{e}^{-\rho\sqrt{\rho-1}\,\tau}}{2\sqrt{\rho-1}}\rho J_1(r\rho)d\rho\right\}, \tag{45}$$

where

$$U_3 = \frac{4\pi b^2 h^2}{M c_R(1-k_2^2)}u_3. \tag{46}$$

As it is formidable to calculate the integrals in equation (45) analytically we employ numerical integration schemes to illustrate the surface displacement.

5. Numerical Results

In this section numerical illustrations of the scaled longitudinal displacement U_3 defined in (45) are presented. Fig. 3 shows the variation of the vertical displacement U_3 on the variable r depending on θ at $\tau = 1$. As might be expected, the amplitude of the normal displacement U_3 decreases away from the surface load. Another important point is that the dispersive effect of the coating causes smoothing of the discontinuities, arising in the uncoated half-space problem, see [13]. It should also be emphasized that the normal displacement becomes zero at $\theta = \pi/2, 3\pi/2$ because of the definition of U_3, see (45). Since we concerned with the tangential load applied on the surface, a load applied perpendicular to the surface results in nonzero displacement. The variation of the displacement U_3 on the angle θ for several values of the polar distance r at $\tau = 0.01$ is depicted in Fig. 4. Similar to the previous case, the magnitude of displacement decreases for the larger values of r. It is also observed that the displacement becomes zero at $\theta = \pi/2$ and $\theta = 3\pi/2$ because the applied load, then, becomes perpendicular.

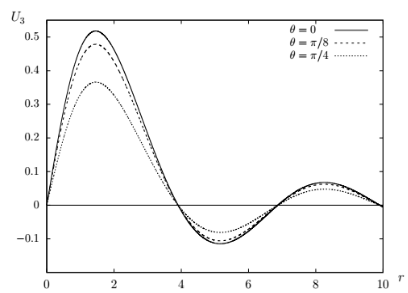

FIGURE 3. The scaled vertical displacement U_3 versus r

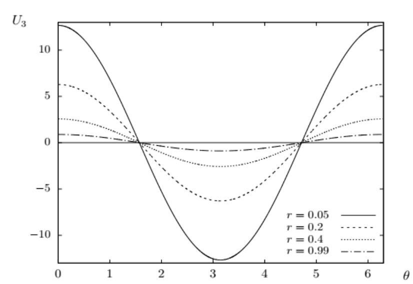

FIGURE 4. The scaled vertical displacement U_3 versus θ

6. Conclusions

In this paper, a 3D problem for a coated elastic half-space loaded by a tangential force along the surface is investigated. A long wave model for the coated half space derived in [9] and an asymptotic model for the in-plane surface wave of elastic half-space derived in [13] have been extended to the case of an in-plane loading for a three-dimensional elastic half-space coated by a thin layer. The problem is, then, formulated by two elliptic equations in the shear potentials ψ_1 and ψ_2 over the interior and a two-dimensional hyperbolic equation singularly perturbed by a pseudo-differential operator given along the interface $x_3 = h$, see (24). The longitudinal and shear potentials are also related at $x_3 = h$. It can be seen from the established model that the rotational part of the tangential load does not have any effect on the boundary equation clearly seen in (23). An integral solution of the normal displacement is expressed for the illustrative examples of point load acting on one of the in-plane axis. This solution shows that there is no displacement for a load applied perpendicular to the the surface of the half-space. It can be also observed that the presence of a coating results in smoothing the singularities arising in the corresponding problem of an uncoated half space, see [13].

The proposed approach may be generalized to more complicated structure including the effects of pre-stress, anisotropy, layered structures and viscosity, see [28]–[30]. The obtained asymptotic formulation may also be applied to in-plane moving load problems, see e.g. [20] and [21].

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ON RICCI PSEUDO-SYMMETRIC SUPER QUASI-EINSTEIN HERMITIAN MANIFOLDS

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Abstract. The present paper deals the study of a Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold and a holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold.

1. Introduction

An even dimensional differentiable manifold M^n is said to be a Hermitian manifold if a complex structure J of type (1, 1) and a pseudo-Riemannian metric g of the manifold satisfy [13, 20]

$$J^2 = -I, \tag{1.1}$$

and

$$g(JX, JY) = g(X, Y), \tag{1.2}$$

where $X, Y \in \chi(M)$ and $\chi(M)$ is Lie algebra of vector fields on the manifold.

The notion of an Einstein manifold was introduced by Albert Einstein in differential geometry and mathematical physics. An Einstein manifold is a Riemannian or pseudo-Riemannian manifold $(M^n, g)(n \geq 2)$ in which Ricci tensor be a scalar multiple of the Riemannian metric i.e.

$$S(X, Y) = \alpha g(X, Y), \tag{1.3}$$

where S denote the Ricci tensor of the manifold $(M^n, g)(n \geq 2)$ and α is a non-zero scalar. According to [3], equation (1.3) is called the Einstein metric condition. An Einstein manifold plays a important role in the study of Riemannian geometry and general theory of relativity.

From the equation (1.3), we get

$$r = n\alpha. \tag{1.4}$$

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In 2000, M. C. Chaki and R.K. Maity [8] introduced a new type of a non-flat Riemannian manifold whose non-zero Ricci tensor satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1.5}$$

and they called it a quasi-Einstein manfold, where α, β are scalars such that $\beta \neq 0$ and A is a non-zero 1-form associated with unit vector field ρ defined by $g(X,\rho) = A(X)$, for every vector field X. ρ is also called generator of the manifold. An n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$.
Contraction of the equation (1.5), gives

$$r = \alpha\, n + \beta. \tag{1.6}$$

From the equations (1.2) and (1.5), we can easily write

$$\begin{aligned} S(X,\rho) &= (\alpha+\beta)A(X), \quad S(\rho,\,\rho) = (\alpha+\beta),\\ g(J\rho,\,\rho) &= 0 \quad \textit{and} \quad S(J\rho,\,\rho) = 0. \end{aligned} \tag{1.7}$$

A quasi-Einstein manifold came in existence during the study of exact solutions of Einstein fields equations as well as considerations of a quasi-umbilical hypersurfaces of semi-Euclidean space. The Walker-space time is an example of a quasi-Einstein manifold. Also a quasi-Einstein manifolds can be taken as a model of the perfect fluid space time in general theory of relativity [17].

In 2001, M. C. Chaki [9] introduced the notion of generalised quasi-Einstein manifolds. A Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a generalised quasi-Einstein manifold if a non-zero Ricci tensor of type (0, 2) satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma C(X)C(Y), \tag{1.8}$$

where α, β and γ are scalars such that $\beta \neq 0$, $\gamma \neq 0$ and A, C are non-vanishing 1-forms associated with two orthogonal unit vectors ρ and μ by

$$\begin{aligned} &g(X,\rho) = A(X),\, g(X,\mu) = C(X),\\ &g(\rho,\,\rho) = g(\mu,\,\mu) = 1. \end{aligned} \tag{1.9}$$

An n-dimensional generalised quasi-Einstein manifold is denoted by $G(QE)_n$.
After contraction of equation (1.8), we get

$$r = \alpha\, n + \beta + \gamma. \tag{1.10}$$

From the equations (1.2), (1.8) and (1.9), we can easily write

$$\begin{aligned} &S(X,\,\rho) = (\alpha+\beta)A(X),\, S(X,\,\mu) = (\alpha+\gamma)C(X), S(\mu,\,\mu) = \alpha+\gamma,\\ &S(\rho,\,\rho) = \alpha+\beta,\;\, g(J\rho,\,\rho) = g(J\mu,\mu) = 0, \;\, \textit{and} \quad S(J\mu,\,\mu) = S(J\rho,\,\rho) = 0. \end{aligned} \tag{1.11}$$

Also in 2004, M. C. Chaki [10] introduced the notion of super quasi-Einstein manifolds. A Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a super quasi-Einstein manifold if a non-zero Ricci tensor of type (0, 2) satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)C(Y) + C(X)A(Y)] + \delta D(X,Y), \tag{1.12}$$

where α, β, γ and δ are non-zero scalars, A, C are non-vanishing 1-forms defined as (1.9) and ρ, μ are orthogonal unit vector fields, D is symmetric tensor of type (0, 2) with zero trace which satisfies the condition

$$D(X,\rho) = 0, \forall\ X. \tag{1.13}$$

An n-dimensional super quasi-Einstein manifold is denoted by $S(QE)_n$.
From the equations (1.2), (1.9), (1.12) and (1.13), we can easily write

$$\begin{aligned} &S(X,\,\rho) = (\alpha+\beta)A(X) + \gamma C(X),\ S(X,\,\mu) = \alpha C(X) + \gamma A(X),\\ &S(\mu,\,\mu) = \alpha + \delta D(\mu,\mu),\ S(\rho,\,\rho) = \alpha + \beta + \delta D(\rho,\rho),\ \ g(J\rho,\,\rho) = g(J\mu,\mu) = 0,\\ &S(J\mu,\,\mu) = \gamma A(J\mu) + \delta D(J\mu,\mu),\ \ S(J\rho,\,\rho) = \gamma C(J\rho) + \delta D(J\rho,\rho). \end{aligned} \tag{1.14}$$

In 2009, A. A. Shaikh [1] introduced the notion of pseudo quasi-Einstein manifold. A semi-Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a pseudo quasi-Einstein manifold if a non-zero Ricci tensor of type (0, 2) satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \delta D(X,Y), \tag{1.15}$$

where α, β, and δ are non-zero scalars and A is a non-zero 1-form defined by $g(X,\rho) = A(X)$. ρ denotes the unit vector called the generator of the manifold and D is symmetric tensor of type (0, 2) with zero trace defined as (1.13). An n-dimensional pseudo quasi-Einstein manifold is denoted by $P(QE)_n$.
From the equations (1.2), (1.9) (1.13) and (1.15), we can easily write

$$\begin{aligned} &S(X,\,\rho) = (\alpha+\beta)A(X),\ S(\rho,\,\rho) = \alpha+\beta,\\ &g(J\rho,\,\rho) = 0 \ \textit{and}\ S(J\rho,\,\rho) = \delta D(J\rho,\,\rho). \end{aligned} \tag{1.16}$$

2. Semi-symmetric and Ricci pseudo-symmetric manifold

Let (M^n, g) be a Riemannian manifold and ∇ be the Levi-Civita connection on (M^n, g) then, a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). The locally symmetric manifold has been studied by different geometers through different aproaches and different notion have been developed e.g., a semi-symmetric manifold by *Szabò* [18], recurrent manifold by Walker [2], conformally recurrent manifold by Adati and Miyazawa [16].

According to Z. I. *Szabò*[18], if the manifold M satisfies the condition

$$(R(X,Y).R)(U,V)W = 0, \quad X,Y,U,V,W \in \chi(M) \tag{2.1}$$

for all vector fields X and Y, then the manifold is called a semi-symmetric manifold. For a (0, k)- tensor field T on M, $k \geq 1$ and a symmetric (0, 2)-tensor field A on

M, the $(0, k+2)$-tensor fields R.T and Q(A, T) are defined by

$$\begin{aligned}(R.T)(X_1, ...X_k; X, Y) = -T(R(X, Y)X_1, X_2, ...X_k)\\ - .... - T(X_1, ...X_{k-1}, R(X, Y)X_k),\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}Q(A, T)(X_1, ...X_k; X, Y) = -T((X \wedge_A Y)X_1, X_2, ...X_k)\\ - .... - T(X_1, ...X_{k-1}, (X \wedge_A Y)X_k),\end{aligned} \tag{2.3}$$

where $X \wedge_A Y$ is the endomorphism given by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y. \tag{2.4}$$

Definition 2.1. ([19]) An n-dimensional Riemannian manifold (M^n, g) is said to be Ricci pseudo-symmetric if and only if the tensors R.S and Q(g, S) are linearly dependent, i.e.

$$(R(X, Y).S)(Z, W) = L_S Q(g, S)(Z, W; X, Y), \tag{2.5}$$

holds on U_S, where $U_S = [x \in M : S \neq \frac{r}{n} g \;\; at \;\; x]$ and L_S is a certain function on U_S.

The above developments allow to several authors for the generaliseation of the notion of quasi Einstein manifolds. In this process generalized quasi-Einstein manifolds are studied by Prakasha and Venkatesha [7] and N(k)-quasi Einstein manifolds are studied by [6, 11]. In 2012, S. K. Hui and R. S. Lemence [15] discussed generalised quasi-Einstein manifold addmitting a W_2- curvature tensor and they proved that if a W_2- curvature tensor satisfies $W_2.S = 0$, then either the associated scalars β and γ are equal or the curvature tensor R satisfies a definite condition. D. G. Prakasha and H. Venkatesha [7] studied some results on generalised quasi-Einstein manifolds and they proved that in generalised quasi-Einstein manifold if a conharmonic curvature tensor satisfies $L.S = 0$, then either M is a nearly quasi-Einstein manifold $N(QE)_n$ or the curvature tensor R satisfies a definite condition. Recently B. B. Chaturvedi and B. K. Gupta [4] studied Bochner Ricci semi-symmetric Hermitian manifold and B. K. Gupta, B. B. Chaturvedi and M. A. Lone [5] studied Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold. We have gone through the above developments in quasi-Einstein manifold $(QE)_n$, generalised quasi- Einstein manifold $G(QE)_n$, a super quasi-Einstein manifold and decide to study Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold and holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold.

3. Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold

The notion of Bochner curvature tensor was introduced by S. Bochner [14]. The Bochner curvature tensor B is defined by

$$\begin{aligned} B(Y,Z,U,V) =&R(Y,Z,U,V) - \frac{1}{2(n+2)}\Big\{S(Y,V)g(Z,U) - S(Y,U)g(Z,V) \\ &+ g(Y,V)S(Z,U) - g(Y,U)S(Z,V) + S(JY,V)g(JZ,U) \\ &- S(JY,U)g(JZ,V) + S(JZ,U)g(JY,V) - g(JY,U)S(JZ,V) \\ &- 2S(JY,Z)g(JU,V) - 2g(JY,Z)S(JU,V)\Big\} \\ &+ \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V) \\ &+ g(JZ,U)g(JY,V) - g(JY,U)g(JZ,V) - 2g(JY,Z)g(JU,V)\Big\}, \end{aligned} \tag{3.1}$$

where r is a scalar curvature of the manifold.
In a Hermitian manifold a Bochner curvature tensor satisfies the condition

$$B(X,Y,U,V) = -B(X,Y,V,U). \tag{3.2}$$

We introduce the following:

Definition 3.1. A Hermitian manifold is said to be a super quasi-Einstein Hermitian manifold if it satisfies the equation (1.12). Throughout this paper, we denote the super quasi-Einstein Hermitian manifold by $S(QEH)_n$.

Definition 3.2. An even dimensional Hermitian manifold (M^n, g) is said to be a Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if and only if the tensors B.S and Q(g, S) are linearly dependent i.e.

$$(B(X,Y).S)(Z,U) = L_S Q(g,S)(Z,U;X,Y), \tag{3.3}$$

holds on U_S, where $U_S = [x \in M : S \neq \frac{r}{n}g \;\; at \;\; x]$ and L_S is a certain function on U_S. If we take a Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold, then from equation (3.3) and (1.12), we have

$$\begin{aligned} &S(B(X,Y)Z,U) + S(Z,B(X,Y)U) \\ &= L_S[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)]. \end{aligned} \tag{3.4}$$

Using equation (1.12) in equation (3.4), we get

$$\begin{aligned}
&\alpha[g(B(X,Y)Z,U)+g(Z,B(X,Y)U)]\\
&+\beta[A(B(X,Y)Z)A(U)+A(B(X,Y)U)A(Z)]\\
&+\gamma[A(B(X,Y)Z)C(U)+A(U)C(B(X,Y)Z)\\
&+A(Z)C(B(X,Y)U)+A(B(X,Y)U)C(Z)]\\
&+\delta[D(B(X,Y)Z,U)+D(Z,B(X,Y)U)]\\
&=L_S\Big(\beta[g(Y,Z)A(X)A(U)-g(X,Z)A(Y)A(U)\\
&+g(Y,U)A(X)A(Z)-g(X,U)A(Y)A(Z)]\\
&+\gamma[g(Y,Z)[(A(X)C(U)+A(U)C(X)]-g(X,Z)[A(Y)C(U)+A(U)C(Y)]\\
&+g(Y,U)[A(X)C(Z)+A(Z)C(X)]-g(X,U)[A(Y)C(Z)+A(Z)C(Y)]]\\
&+\delta[g(Y,Z)D(X,U)-g(X,Z)D(Y,U)+g(Y,U)D(X,Z)-g(X,U)D(Y,Z)]\Big).
\end{aligned} \tag{3.5}$$

Using equation (3.2) in equation (3.5) we infer

$$\begin{aligned}
&\beta[A(B(X,Y)Z)A(U)+A(B(X,Y)U)A(Z)]\\
&+\gamma[A(B(X,Y)Z)C(U)+A(U)C(B(X,Y)Z)\\
&+A(Z)C(B(X,Y)U)+A(B(X,Y)U)C(Z)]\\
&+\delta[D(B(X,Y)Z,U)+D(Z,B(X,Y)U)]\\
&=L_S\Big\{\beta[g(Y,Z)A(X)A(U)-g(X,Z)A(Y)A(U)\\
&+g(Y,U)A(X)A(Z)-g(X,U)A(Y)A(Z)]\\
&+\gamma[g(Y,Z)[(A(X)C(U)+A(U)C(X)]-g(X,Z)[A(Y)C(U)+A(U)C(Y)]\\
&+g(Y,U)[A(X)C(Z)+A(Z)C(X)]-g(X,U)[A(Y)C(Z)+A(Z)C(Y)]]\\
&+\delta[g(Y,Z)D(X,U)-g(X,Z)D(Y,U)\\
&+g(Y,U)D(X,Z)-g(X,U)D(Y,Z)]\Big\}.
\end{aligned} \tag{3.6}$$

Now putting $U=Z=\rho$, we have

$$2\gamma\Big\{B(X,Y,\rho,\mu)-L_S[C(X)A(Y)-C(Y)A(X)]\Big\}=0. \tag{3.7}$$

This implies either $\gamma=0$ or

$$B(X,Y,\rho,\mu)=L_S[C(X)A(Y)-C(Y)A(X)]. \tag{3.8}$$

If $\gamma=0$, then from equation (1.12), we obtain

$$S(X,Y)=\alpha g(X,Y)+\beta A(X)A(Y)+\delta D(X,Y). \tag{3.9}$$

This is the condition of a pseudo quasi-Einstein manifold.
Thus we conclude:

Theorem 3.3. *A Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold is either a Bochner Ricci pseudo-symmetric pseudo quasi-Einstein Hermitian manifold or*

$$B(X,Y,\rho,\mu) = L_S[C(X)A(Y) - C(Y)A(X)].$$

From equation (3.7), we can also conclude:

Corollary 3.1. *In a Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if $\gamma \neq 0$ then $B(X,Y,\rho,\mu) = 0$ if and only if the vector fields ρ and μ corresponding to 1-forms A and C respectively are codirectional.*

4. Bochner flat Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold with $(B(X,Y).S)(Z,U) = L_S Q(g,S)(Z,U;X,Y)$

If we take a Bochner flat curvature tensor then from equation (3.1), we have

$$\begin{aligned}
R(Y,Z,U,V) &= \frac{1}{2(n+2)}\Big\{S(Y,V)g(Z,U) - S(Y,U)g(Z,V) \\
&+ g(Y,V)S(Z,U) - g(Y,U)S(Z,V) + S(JY,V)g(JZ,U) \\
&- S(JY,U)g(JZ,V) + S(JZ,U)g(JY,V) - g(JY,U)S(JZ,V) \\
&- 2S(JY,Z)g(JU,V) - 2g(JY,Z)S(JU,V)\Big\} \\
&- \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V) \\
&+ g(JZ,U)g(JY,V) - g(JY,U)g(JZ,V) - 2g(JY,Z)g(JU,V)\Big\}.
\end{aligned} \tag{4.1}$$

From equations (2.5) and (4.1), we infer

$$\begin{aligned}
&\frac{1}{(2n+4)}\Big\{S(QY,V)g(Z,U) - g(Y,U)S(QZ,V) + S(QJY,V)g(JZ,U) \\
&- g(JY,U)S(JQZ,V) - 2g(JY,Z)S(JQU,V) \\
&+ S(QY,U)g(Z,V) - g(Y,V)S(QZ,U) + S(QJY,U)g(JZ,V) \\
&- g(JY,V)S(JQZ,U) - 2g(JY,Z)S(JQV,U)\Big\} \\
&- \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)S(Y,V) - g(Y,U)S(Z,V) + g(JZ,U)S(JY,V) \\
&- g(JY,U)S(JZ,V) + g(Z,V)S(Y,U) - g(Y,V)S(Z,U) \\
&+ g(JZ,V)S(JY,U) - g(JY,V)S(JZ,U)\Big\} \\
&= L_S[g(Z,U)S(Y,V) - g(Y,U)S(Z,V) + g(Z,V)S(Y,U) - g(Y,V)S(Z,U)].
\end{aligned} \tag{4.2}$$

If we take λ be an eigen value of Q and JQ corresponding to eigen vectors X and JX respectively then $QX = \lambda X$ and $QJX = \lambda JX$ i.e. $S(X,U) = \lambda g(X,U)$ (where the manifold is not Einstein) and hence

$$S(QX,U) = \lambda S(X,U) \quad and \quad S(QJX,U) = \lambda S(JX,U). \tag{4.3}$$

Using equation (4.3) in equation (4.2), we infer

$$\begin{aligned}&\left(\frac{\lambda}{(2n+4)} - \frac{r}{(2n+2)(2n+4)}\right)\Big\{S(Y,V)g(Z,U) - g(Y,U)S(Z,V)\\ &+ S(Y,U)g(Z,V) - g(Y,V)S(Z,U) + S(JY,V)g(JZ,U)\\ &- S(JZ,V)g(JY,U) + S(JY,U)g(JZ,V) - g(JY,V)S(JZ,U)\Big\}\\ &= L_S[g(Z,U)S(Y,V) - g(Y,U)S(Z,V) + g(Z,V)S(Y,U) - g(Y,V)S(Z,U)].\end{aligned} \tag{4.4}$$

Now putting $V = U = \rho$, we get

$$\begin{aligned}&\left(\frac{\lambda}{(2n+4)} - \frac{r}{(2n+2)(2n+4)}\right)\Big([S(Y,\rho)g(Z,\rho) - g(Y,\rho)S(Z,\rho)\\ &+ S(JY,\rho)g(JZ,\rho) - S(JZ,\rho)g(JY,\rho)]\Big)\\ &= L_S[g(Z,\rho)S(Y,\rho) - g(Y,\rho)S(Z,\rho)].\end{aligned} \tag{4.5}$$

Now using equations (1.9) and (1.14) in equation (4.5), we have

$$\begin{aligned}&\gamma\left(\left(\frac{\lambda}{(2n+4)} - \frac{r}{(2n+2)(2n+4)}\right) - L_S\right)[C(Y)A(Z) - C(Z)A(Y)]\\ &= \gamma\left(\frac{\lambda}{(2n+4)} - \frac{r}{(2n+2)(2n+4)}\right)[A(JY)C(JZ) - A(JZ)C(JY)].\end{aligned} \tag{4.6}$$

If we take $\lambda = \frac{r}{(2n+2)}$ and $\gamma \neq 0$, then

$$A(Z)C(Y) = A(Y)C(Z). \tag{4.7}$$

If we take $\lambda = \frac{r}{(2n+2)}$ and $\gamma \neq 0$, then from equations (1.2) and (1.9), equation (4.7) imply $g(Z,\rho)\,g(Y,\mu) = g(Y,\rho)\,g(Z,\mu)$, therefore we can say that the vector fields ρ and μ corresponding to 1-forms A and C respectively are codirectional.
Thus we conclude:

Theorem 4.1. *In a Bochner flat Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if $\frac{r}{(2n+2)}$ is an eigen value of the Ricci operator Q and JQ and $\gamma \neq 0$ then the vector fields ρ and μ corresponding to 1-forms A and C respectively are codirectional.*

5. Holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold

The holomorphically projective curvature tensor is defined by [20]

$$\begin{aligned} P(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ + S(JX,Z)g(JY,W) - S(JY,Z)g(JX,W)]. \end{aligned} \tag{5.1}$$

This tensor has the following properties

$$P(X,Y,Z,W) = -P(Y,X,Z,W), \quad P(JX,JY,Z,W) = P(X,Y,Z,W). \tag{5.2}$$

Now we introduce the following:

Definition 5.1. An even dimensional Hermitian manifold (M^n, g) is said to be a holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if the holomorphically projective curvature tensor of the manifold satisfies $P.S = 0$, i.e.

$$(P(X,Y).S)(Z,W) = L_S Q(g,S)(Z,W;X,Y). \tag{5.3}$$

for all $X, Y, Z, W \in \chi(M^n)$.

If we take a holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold, then from the equations (1.12) and (5.1), we have

$$\begin{aligned} &\alpha[P(X,Y,Z,W) + P(X,Y,W,Z)] \\ &+ \beta[A(P(X,Y)Z)A(W) + A(Z)A(P(X,Y)W)] \\ &+ \gamma[A(P(X,Y)Z)C(W) + A(W)C(P(X,Y)Z) \\ &+ A(Z)C(P(X,Y)W) + C(Z)A(P(X,Y)W)] \\ &+ \delta[D(P(X,Y)Z,W) + D(Z,P(X,Y)W)] \\ &= L_S[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)]. \end{aligned} \tag{5.4}$$

Now putting $Z = W = \rho$ in equation (5.4) and using equation (1.14), we have

$$\begin{aligned} &(\alpha+\beta)P(X,Y,\rho,\rho) + \gamma P(X,Y,\rho,\mu) \\ &= \gamma L_S[A(Y)C(X) - A(X)C(Y)]. \end{aligned} \tag{5.5}$$

Using $Z = W = \rho$ in equation (5.1), we have

$$\begin{aligned} P(X,Y,\rho,\rho) = -\frac{\gamma}{n-2}[C(Y)A(X) - A(Y)C(X) \\ + C(JX)A(JY) - C(JY)A(JX)]. \end{aligned} \tag{5.6}$$

Similarly putting $Z = \rho$ and $W = \mu$ in equation (5.1), we get

$$\begin{aligned} P(X,Y,\rho,\mu) = R(X,Y,\rho,\mu) - \frac{(\alpha+\beta)}{n-2}[A(Y)C(X) - C(Y)A(X) \\ + C(JX)A(JY) - C(JY)A(JX)]. \end{aligned} \tag{5.7}$$

Using equations (5.6) and (5.7) in (5.5), we get

$$\gamma[R(X,Y,\rho,\mu)-L_S[A(Y)C(X)-A(X)C(Y)]]=0, \tag{5.8}$$

this implies that either $\gamma=0$ or $R(X,Y,\rho,\mu)=L_S[A(Y)C(X)-A(X)C(Y)]$. If $\gamma=0$ then from equation (1.12), we get the condition of a pseudo quasi-Einstein manifolds.

Thus we can conclude:

Theorem 5.2. *A holomorphically projectively Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold is either a holomorphically projective Ricci semi-symmetric pseudo quasi-Einstein Hermitian manifold or*

$$R(X,Y,\rho,\mu)=L_S[A(Y)C(X)-A(X)C(Y)].$$

Corollary 5.1. *In a holomorphically projectively Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if $\gamma\neq 0$ then $R(X,Y,\rho,\mu)=0$ if and only if the vector fields ρ and μ corresponding to one form A and C respectively are codirectional.*

Putting $Z=\rho$ and $W=\mu$ in equation (5.4), we get

$$\begin{aligned}&\alpha[P(X,Y,\rho,\mu)+P(X,Y,\mu,\rho)]+\beta P(X,Y,\mu,\rho)\\&+\gamma[P(X,Y,\rho,\rho)+P(X,Y,\mu,\mu)]=\beta L_S[A(X)C(Y)-A(Y)C(X)].\end{aligned} \tag{5.9}$$

Putting $Z=U=\mu$ in (5.1), we get

$$\begin{aligned}&P(X,Y,\mu,\mu)\\&=-\frac{\gamma}{n-2}[A(Y)C(X)-C(Y)A(X)+C(JY)A(JX)-C(JX)A(JY)].\end{aligned} \tag{5.10}$$

Adding equations (5.6) and (5.10), we get

$$P(X,Y,\mu,\mu)+P(X,Y,\rho,\rho)=0, \tag{5.11}$$

from equations (5.9) and (5.11), we have

$$\alpha[P(X,Y,\rho,\mu)+P(X,Y,\mu,\rho)]+\beta P(X,Y,\mu,\rho)=\beta L_S[A(X)C(Y)-A(Y)C(X)]. \tag{5.12}$$

From equations (5.1), (5.7) and (5.12), we have

$$\beta[R(X,Y,\mu,\rho)-L_S[A(X)C(Y)-A(Y)C(X)]]=0. \tag{5.13}$$

This implies either $\beta=0$ or $R(X,Y,\mu,\rho)=L_S[A(X)C(Y)-A(Y)C(X)]$.

Theorem 5.3. *In a holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if $\beta\neq 0$ then $R(X,Y,\mu,\rho)=0$ if and only if the vector fields ρ and μ corresponding to one form A and C respectively are codirectional.*

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GENERALIZED PROJECTIVE CURVATURE TENSOR OF NEARLY COSYMPLECTIC MANIFOLD

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Abstract. In this paper, we concentrated our attention on geometry of generalized projective tensor of nearly cosymplectic manifold. In particular, we studied the flatness property of generalized projective tensor. This property helped us to find the necessary and sufficient condition that nearly cosymplectic manifold is a generalized Einstein manifold.

1. Introduction

One of the important curvature tensors is the projective tensor. According to this importance, many authors focused on its geometrical properties. Kirichenko [11] proved that nontrivial projective-recurrent K-space of maximal rank is 6-dimensional manifold of constant curvature tensor. Abood [3] studied the projective tensor of nearly Kähler manifold. Abood and Mohammed [4] proved that almost Kähler manifold is a Kähler manifold if it is a projective parakähler manifold. Shashikala and Venkatesha [20] studied the generalized pseudo-projective Φ-recurrent $N(k)$-contact metric manifold. Later on, Abood and Abd Ali [1] found the necessary condition that Viasman-Grey manifold has flat generalized projective tensor. Abood and Abd Ali [2] studied the projective-recurrent Viasman-Gray manifold. Finally, Atceken, Yildirim and Dirik [6], [7], [21], [22] studied certain curvature tensors including the pseudo-projective on some contact metric manifolds.

In this paper, we obtain some results on generalized projective tensor when it's act on nearly cosymplectic manifold. In particular, we found the necessary and sufficient conditions that nearly cosymplectic manifold is generalized Einstein manifold.

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2. Preliminaries

Let M be a smooth manifold of dimension $2n+1$ greater than 3, $X(M)$ be the module of smooth vector fields on M, $X^c(M)$ be the complexification of the module $X(M)$ and $T_p^c(M)$ be the complexification of tangent space $T_p(M)$ at the point $p \in M$.

An almost contact manifold (AC-manifold) is the set (M, η, ξ, Φ, g), where η is differential 1-form called a *contact form*, ξ is a vector field called a *characteristic*, Φ is endomorphism of $X(M)$ called a *structure endomorphisim* and $g = \langle ., . \rangle$ is the Riemannian metric on M. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1, \Phi(\xi) = 0, \eta \circ \Phi = 0, \Phi^2 = -id + \eta \otimes \xi,$$

and $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y);\ X, Y \in X(M)$ [8].

In the module $X^c(M)$, define two endomorphisms σ and $\bar{\sigma}$ as follows: $\sigma = \frac{1}{2}(id - \sqrt{-1}\Phi)$ and $\bar{\sigma} = -\frac{1}{2}(id + \sqrt{-1}\Phi)$, then we can define two projections as follows:

$$\Pi = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 - \sqrt{-1}\Phi) \textit{ and } \bar{\Pi} = \bar{\sigma} \circ \ell = \frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi),$$

where $\sigma \circ \Phi = \Phi \circ \sigma = i\sigma$ and $\bar{\sigma} \circ \Phi = \Phi \circ \bar{\sigma} = -i\bar{\sigma}$. Therefore, If we denote $\mathrm{Im}\Pi = D_\Phi^{\sqrt{-1}}$ and $\mathrm{Im}\bar{\Pi} = D_\Phi^{-\sqrt{-1}}$, then

$$X^c(M) = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}} \oplus D_\Phi^0,$$

where $D_\Phi^{\sqrt{-1}}$, $D_\Phi^{-\sqrt{-1}}$ and D_Φ^0 are proper submodules of the endomorphism Φ with proper values $\sqrt{-1}, -\sqrt{-1}$ and 0 respectively [13].

At each point $p \in M$, we can construct a frame in $T_p^c(M)$ by the form $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$, where $\varepsilon_a = \sqrt{2}\sigma_p(e_p)$, $\varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_p)$ and $\varepsilon_0 = \xi_p$. The frame $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$ is called A-frame [16].

The principle fiber of all A-frames with structure group $\{1\} \times U(n)$ is called an G-adjoined structure space.

The matrices of the AC-structure Φ_p and Riemannian metric g_p in A-frame are given by the following forms:

$$(\Phi_j^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & o \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix},\ (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix} \qquad (2.1)$$

where I_n is the identity matrix of order n [14].

An almost contact manifold is called a nearly cosymplectic manifold (NC-manifold) if the equality $\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0;\ X, Y \in X(M)$ holds [9].

The following theorem explains the structure equations of NC-manifold in the G-adjoined structure space.

Theorem 2.1. [15] *In the G-adjoined structure space, the structure equations of NC-manifold are given by the following forms:*

(1) $d\omega^a = \omega^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c + \frac{3}{2}C^{ab}\omega_b \wedge \omega$;
(2) $d\omega_a = -\omega^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c + \frac{3}{2}C_{ab}\omega^b \wedge \omega$;
(3) $d\omega = C^{bc}\omega_b \wedge \omega_c + C_{bc}\omega^b \wedge \omega^c$;
(4) $d\omega^a_b = \omega^a_c \wedge \omega^c_b + [A^{ad}_{bc} - 2B^{adh}B_{hbc} + \frac{3}{2}C^{ad}C_{bc}]\omega^c \wedge \omega_d$,

where $B^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},\hat{c}}$, $C^{ab} = \sqrt{-1}\Phi^a_{0,\hat{b}}$, $C_{ab} = -\sqrt{-1}\Phi^{\hat{a}}_{b,0}$ *and* $B_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,c}$. *The tensors B, C and A are called the first, second and third structure tensors respectively.*

Definition 2.1. [17] *A Riemann-Christoffel tensor R of a smooth manifold M is a tensor of type* $(4,0)$ *which is defined by*

$$R(X,Y,Z,W) = g(R(Z,W)Y,X),$$

where $R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$, *and has the following properties:*

(1) $R(X,Y,Z,W) = -R(Y,X,Z,W)$;
(2) $R(X,Y,Z,W) = -R(X,Y,W,Z)$;
(3) $R(X,Y,Z,W) = R(Z,W,X,Y)$;
(4) $R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0$.

The components of Riemann-Christoffel tensor of NC-manifold are given in theorem below.

Lemma 2.1. [15] *In the G-adjoined structure space, the components of Riemann-Christoffel tensor of NC-manifold have the following forms:*

(1) $R_{\hat{a}bcd} = 0$;
(2) $R_{abcd} = -2B_{ab[cd]}$;
(3) $R_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd}$;
(4) $R_{\hat{a}0b0} = C^{ac}C_{bc}$;
(5) $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}$.

The other components of Riemann-Christoffel tensor R can be obtained by the property of symmetry for R or equal to zero.

Definition 2.2. [10] *A generalized Riemannian curvature tensor G_R on NC-manifold M is a tensor of type* $(4,0)$ *which is defined as the following form:*
$G_R(X,Y,Z,W) = \frac{1}{16}\{3[R(X,Y,Z,W) + R(\Phi X,\Phi Y,Z,W) + R(X,Y,\Phi Z,\Phi W) + R(\Phi X,\Phi Y,\Phi Z,\Phi W)] - R(X,Z,\Phi W,\Phi Y) - R(\Phi X,\Phi Z,W,Y) - R(X,W,\Phi Y,\Phi Z) - R(\Phi X,\Phi W,Y,Z) + R(\Phi X,Z,\Phi W,Y) + R(X,\Phi Z,W,\Phi Y) + R(\Phi X,W,Y,\Phi Z) + R(X,\Phi W,\Phi Y,Z)\}$,
where $R(X,Y,Z)$ *is the Riemann-Christoffel tensor,* $X,Y,Z,W \in T_p(M)$ *and has the following properties:*

(1) $G_R(X,Y,Z,W) = -G_R(Y,X,Z,W) = -G_R(X,Y,W,Z)$;
(2) $G_R(X,Y,Z,W) = G_R(Z,W,X,Y)$;
(3) $G_R(X,Y,Z,W) + G_R(X,Z,W,Y) + G_R(X,W,Y,Z) = 0$;

(4) $G_R(X, \Phi X, \Phi X, X) = R(X, \Phi X, \Phi X, X)$.

Definition 2.3. [18] *A tensor G_r of type $(2,0)$ which is defined as $(G_r)_{ij} = (G_R)^k_{ijk}$ is called a generalized Ricci tensor.*

Remark 2.1. [18] *A generalized Ricci tensor is symmetric , this follows form the properties of symmetry of generalized Riemannian curvature tensor. This mean $(G_r)_{ij} = (G_r)_{ji}$.*

Definition 2.4. *A generalized projective tensor G_P is a tensor of type $(4,0)$ which is defined as the form:*

$$(G_P)_{ijkl} = (G_R)_{ijkl} - \frac{1}{2n}[(G_r)_{ik}g_{jl} - (G_r)_{jk}g_{il}].$$

Definition 2.5. [11] *Let M be an AC-manifold, an Φ-holomorphic sectional curvature (ΦHS-curvature) of a manifold M in the direction $X \in X(M), X \neq 0$ is a function $H(X)$ which is defined as:*

$$H(X) = \langle R(X, \Phi X)X, \Phi X\rangle \, \|X\|^{-4}$$

Definition 2.6. [11] *An AC-manifold is called a manifold of point constant ΦHS-curvature if*

$$\langle R(X, \Phi X)X, \Phi X\rangle = c\|X\|^4,$$

where $c \in C^\infty(M)$, for all $X \in X(M)$

Theorem 2.2. [11] *An AC-manifold is a manifold of point constant ΦHS-curvature C_0 if and only if, on the G-adjoined structure, the following equation holds:*

$$R^{(ad)}_{(bc)} = \frac{C_0}{2}\tilde{\delta}^{ad}_{bc}, \tag{2.2}$$

where $C_0 \in C^\infty(M)$ and $\tilde{\delta}^{ad}_{bc} = \delta^a_b\delta^d_c + \delta^a_c\delta^d_b$.

Definition 2.7. [19] *A Riemannian manifold is called an Einstein manifold, if the Ricci tensor satisfies the equation $r_{ij} = eg_{ij}$, where e is an cosmological constant.*

Similar to the above definition, we can introduce the following definition.

Definition 2.8. *A Riemannian manifold is called a generalized Einstein manifold, if the generalized Ricci tensor satisfies the equation $(G_r)_{ij} = (G_e)g_{ij}$, where G_e is a generalized cosmological constant.*

3. The main results

In this section, we calculated the components of the generalized Riemannian curvature tensor. Moreover, the necessary and sufficient condition that a nearly cosymplectic manifold is generalized Einstein manifold has been found.

Lemma 3.1. *In the G-adjoined structure space, the components of the generalized Riemannian curvature tensor of NC-manifold are given by the following forms:*

(1) $(G_R)_{\hat{a}b\hat{c}d} = -A^{ac}_{bd}$;
(2) $(G_R)_{\hat{a}bc\hat{d}} = -\frac{1}{2}[A^{ad}_{bc} - 3B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}]$.

And the others are conjugate to the above components or equal to zero.

Proof: By using the Lemma 2.1 and Definition 2.2, we compute the components of generalized projective tensor as the following:
1) Put $i = \hat{a}, j = b, k = \hat{c}$ and $l = d$, we have

$$(G_R)_{\hat{a}b\hat{c}d} = \frac{1}{16}\{3[R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d}] + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} -$$

$$R_{\hat{a}db\hat{c}} + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} - R_{\hat{a}db\hat{c}}\}$$

Making use of the properties of G_R, we get

$$(G_R)_{\hat{a}b\hat{c}d} = -A^{ac}_{bd}.$$

2) Put $i = \hat{a}, j = b, k = \hat{c}$ and $l = d$, we obtain

$$(G_R)_{\hat{a}b\hat{c}d} = \frac{1}{16}\{3[R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d}] + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} -$$

$$R_{\hat{a}db\hat{c}} + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} - R_{\hat{a}db\hat{c}}\}$$

According the the Definition 2.2, consequently we deduce

$$(G_R)_{\hat{a}bc\hat{d}} = -\frac{1}{2}[A^{ad}_{bc} - 3B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}].$$

By the same manner, we can get the other components.

Lemma 3.2. *In the G-adjoined structure space, the components of the generalized Ricci tensor of NC-manifold are given as the following form:*

$$(G_r)_{\hat{a}b} = \frac{1}{2}A^{ac}_{cb} + 3B^{ach}B_{hcb} + \frac{5}{3}C^{ac}C_{cb}.$$

And the others are conjugate to the above component or equal to zero.

Proof: By using the Lemma 2.1 and Definition 2.3, directly we obtain the above components.

Lemma 3.3. *In the G-adjoined space, the components of the generalized projective tensor of NC-manifold take the following forms:*

(1) $(G_P)_{\hat{a}\hat{b}cd} = -\frac{1}{2n}((\frac{1}{2}A^{af}_{fc} + 3B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc})\delta^b_d) - (\frac{1}{2}A^{bf}_{fc} - 3B^{bfh}B_{hfc} - \frac{5}{3}C^{bf}C_{fc})\delta^a_d$;
(2) $(G_P)_{\hat{a}b\hat{c}d} = -A^{ac}_{bd} + \frac{1}{2n}(\frac{1}{2}A^{cf}_{fb} + 3B^{cfh}B_{hfb} + \frac{5}{3}C^{cf}C_{fb})\delta^a_d$;
(3) $(G_P)_{\hat{a}bc\hat{d}} = -\frac{1}{2n}(\frac{1}{2}A^{ad}_{bc} + 3B^{adh}B_{hbc} + \frac{5}{3}C^{ad}C_{bc}) - \frac{1}{2}(A^{af}_{fc} + 3B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc})\delta^d_b$.

The remaining components are obtained by taking the conjugated operation to the above components or are identical equal to zero.

Proof:
1. Put $i=\hat{a},\ j=\hat{b},\ c$ and $l=d$.
According to the Definition 2.4, we obtain

$$(G_P)_{\hat{a}\hat{b}cd}=(G_\Re)_{\hat{a}\hat{b}cd}-\tfrac{1}{2n}[(G_r)_{\hat{a}c}g_{\hat{b}d}-(G_r)_{\hat{b}c}g_{\hat{a}d}]$$

By using the Lemmas 3.1, 3.2 and the matrices (2.1), we have

$$(G_P)_{\hat{a}\hat{b}cd}=$$
$$-\tfrac{1}{2n}((\tfrac{1}{2}A^{af}_{fc}+3B^{afh}B_{hfc}+\tfrac{5}{3}C^{af}C_{fc})\delta^b_d)-(\tfrac{1}{2}A^{bf}_{fc}-3B^{bfh}B_{hfc}-\tfrac{5}{3}C^{bf}C_{fc})\delta^a_d.$$

2. Put $i=\hat{a},\ j=b,\ k=\hat{c}$ and $l=d$.
Harmonize to the Definition 2.4, we get

$$(G_P)_{\hat{a}b\hat{c}d}=(G_\Re)_{\hat{a}b\hat{c}d}-\tfrac{1}{2n}[(G_r)_{\hat{a}\hat{c}}g_{bd}-(G_r)_{b\hat{c}}g_{\hat{a}d}]$$

Taking into account the Lemmas 3.1, 3.2 and the matrices (2.1), we obtain

$$(G_P)_{\hat{a}b\hat{c}d}=-A^{ac}_{bd}+\tfrac{1}{2n}(\tfrac{1}{2}A^{cf}_{fb}+3B^{cfh}B_{hfb}+\tfrac{5}{3}C^{cf}C_{fb})\delta^a_d.$$

By the same technique, we can compute the other components.

Theorem 3.1. *If M is vanishing generalized projectively NC-manifold, then the necessary and sufficient condition for M to be vanishing generalized Ricci tensor is the holomorphic tensor vanishes.*

Proof: Let M be vanishing generalized projectively NC-manifold.
According to the Lemma 3.3, we have

$$(G_\Re)_{\hat{a}b\hat{c}d}+\frac{1}{2n}[(G_r)_{b\hat{c}}g_{\hat{a}d}]=0 \tag{3.1}$$

If M is vanishing generalized Ricci tensor, then directly we get

$$A^{ac}_{bd}=0.$$

Conversely, if M is vanishing holomorphic tensor, then we have

$$\frac{1}{2n}[(G_r)_{b\hat{c}}\delta^a_d]=0 \tag{3.2}$$

Contracting (3.2) by the indices (b,a) , it follows that

$$(G_r)_{d\hat{c}}=0.$$

Theorem 3.2. *An NC-manifold has vanishing holomorphic tensor if and only if, M is a manifold of vanishing generalized projective tensor.*

Proof: Let M be NC-manifold with vanishing generalized projective tensor. Making use of the Lemma 3.3, we obtain

$$-A^{ac}_{bd}+\frac{1}{2n}(\frac{1}{2}A^{cf}_{fb}+3B^{cfh}B_{hfb}+\frac{5}{3}C^{cf}C_{fb})\delta^a_d=0 \tag{3.3}$$

Symmetrizing and then antisymmetrizing the equation (3.3) by the indices (c, f), we get

$$A^{ac}_{bd} = 0.$$

Conversely, let M be NC-manifold with vanishing holomorphic tensor, then the equation (3.3) takes the following formula:

$$(G_P)_{\hat{a}b\hat{c}d} = \frac{1}{2n}(3B^{cfh}B_{hfb} + \frac{5}{3}C^{cf}C_{fb})\delta^a_d \tag{3.4}$$

Symmetrizing and then antisymmetrizing the equation (3.4) by the indices (c, f), we deduce

$$(G_p)_{\hat{a}b\hat{c}d} = 0.$$

Lemma 3.4. *An NC-manifold has Φ-invariant generalized Ricci tensor if and only if,*

$$\Phi \circ G_r = G_r \circ \Phi.$$

Theorem 3.3. *Let M be NC-manifold. Then M has Φ-invariant generalized Ricci tensor if and only if, $(G_r)^{\hat{a}}_b = 0$ hold in the G-adjoined structure space.*

Proof: Suppose that M is Φ- invariant generalized Ricci tensor.
According to the Lemma 3.4, we have

$$\Phi \circ G_r = G_r \circ \Phi$$

By the G-adjoined structure space, the above equation becomes

$$(\Phi \circ G_r)^i_j = (G_r \circ \Phi)^i_j$$

This means

$$\Phi^i_k (G_r)^k_j = (G_r)^i_k \Phi^k_j \tag{3.5}$$

Put $i = \hat{a}$ and $j = b$, then the equation (3.5) becomes

$$\Phi^{\hat{a}}_c (G_r)^c_b + \Phi^{\hat{a}}_{\hat{c}} (G_r)^{\hat{c}}_b + \Phi^{\hat{a}}_0 (G_r)^0_b = (G_r)^{\hat{a}}_c \Phi^c_b + (G_r)^{\hat{a}}_{\hat{c}} \Phi^{\hat{c}}_b + (G_r)^{\hat{a}}_0 \Phi^0_b$$

By using (2.1), we have

$$(G_r)^{\hat{a}}_b = 0.$$

Theorem 3.4. *Suppose that M is NC-manifold with vanishing generalize projective tensor and Φ-invariant generalized Ricci tensor. Then the necessary and sufficient condition for M to be generalized Einstein manifold is $A^{bc}_{bd} = \frac{Ge}{2n}\delta^c_d$, where Ge is a generalized Cosmological constant.*

Proof: Let M be NC-manifold with vanishing generalized projective tensor.
According to the Lemma 3.3, we have

$$-A^{ac}_{bd} + \frac{1}{2n}[(G_r)_{b\hat{c}} g_{\hat{a}d}] = 0 \tag{3.6}$$

Making use of the Definition 2.8, the equation (3.6) becomes

$$A^{ac}_{bd} = \frac{Ge}{2n}\delta^c_b\delta^a_d \tag{3.7}$$

Contracting the equation (3.7) by the indices (b, a), it follows that

$$A^{bc}_{bd} = \frac{Ge}{2n}\delta^c_d \tag{3.8}$$

Conversely,
Contracting the equation (3.6) by the indices (b, a), we have

$$-A^{bc}_{bd} + \frac{1}{2n}[(G_r)_{d\hat{c}}] = 0 \tag{3.9}$$

Combining the equations (3.8) and (3.9), we conclude

$$(G_r)^c_d = Ge\delta^c_d$$

Therefore, by the Definition 2.8 and Theorem 3.3, M is generalized Einstein manifold.

Theorem 3.5. *Suppose that M is NC-manifold with vanishing generalized Riemannian curvature tensor and Φ-invariant generalized Ricci tensor. If M is a generalized Einstein manifold then $Ge = \frac{10}{3}C^{bc}C_{bd}$.*

Proof: Let M be NC-manifold with vanishing generalized Riemannian curvature tensor. Then from Lemma 3.1 we have

$$-\frac{1}{2}[A^{ad}_{bc} - 3B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}] = 0 \tag{3.10}$$

By symmetrization and antisymmetrizatin the equation (3.10) by the induces (d, h) we get

$$A^{ad}_{bc} - \frac{5}{3}C^{ad}C_{bc} = 0 \tag{3.11}$$

Contracting the equation (3.11)by the induces $(b, a), (d, c)$ and (c, d), we deduce

$$A^{bc}_{bd} - \frac{5}{3}C^{bc}C_{bd} = 0 \tag{3.12}$$

Since M is generalized Einstein manifold, then from the Theorem 3.4, the equation (3.12) becomes

$$\frac{Ge}{2n}\delta^c_d - \frac{5}{3}C^{bc}C_{bd} = 0 \tag{3.13}$$

Contracting the equation (3.13)by the induces (d, a), implies

$$Ge = \frac{10}{3}C^{bc}C_{bd}.$$

Theorem 3.6. [5] *Suppose that M is NC-manifold. Then the necessary and sufficient condition that M is a manifold of point constant ΦHS-curvature C_0 is*

$$A^{ad}_{bc} = B^{adh}B_{hbc} + \frac{5}{3}C^{ad}C_{bc} + \frac{C_0}{2}\tilde{\delta}^{ad}_{bc}.$$

Theorem 3.7. *Suppose that M is NC-manifold of point constant ΦHS-curvature C_0 and vanishing generalized projective tensor with Φ-invariant generalized Ricci tensor, then $C^{af}C_{fc} = -\frac{C_0(n+1)}{10}\delta^a_c$.*

Proof: Let M be NC-manifold of ΦHS-curvature tensor and vanishing generalized projective tensor.
According to the Lemma 3.3, we have

$$-\frac{1}{2n}((\frac{1}{2}A^{af}_{fc} + 3B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc})\delta^b_d) - (\frac{1}{2}A^{bf}_{fc} - 3B^{bfh}B_{hfc} - \frac{5}{3}C^{bf}C_{fc})\delta^a_d \quad (3.14)$$

By using Theorem 3.6, the equation (3.14) becomes

$$-\frac{1}{2n}(\frac{1}{2}(B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc} + \frac{c_0}{2}\tilde{\delta}^{af}_{fc}) + 3B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc})\delta^b_d - \frac{1}{2}(-2B^{bfh}B_{hfc} + \frac{c_0}{2}\tilde{\delta}^{bf}_{fc})\delta^a_d = 0 \quad (3.15)$$

Symmetrizing and then antisymmetrizing the equation (3.15) by the indices (b, f) and (f, h), we conclude that

$$C^{af}C_{fc} = -\frac{C_0(n+1)}{10}\delta^a_c.$$

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A TYCHONOFF THEOREM FOR GRADED DITOPOLOGICAL TEXTURE SPACES

RAMAZAN EKMEKÇI

ABSTRACT. In this paper, initial and product graded ditopologies are formulated and accordingly it is shown that **dfGDitop** is a topological structure over **dfTex** $\times$ **dfTex**. By means of spectrum idea, (di)compactness in graded ditological texture spaces is defined as a generalization of (di)compactness in ditopological case and its relation with the ditopological case is investigated. Moreover, the relations between graded difilters and dicompactness of graded ditological texture spaces are studied.

1. INTRODUCTION

The idea "graded ditopology" has been introduced in [7] by Brown and Šostak. This new structure is more comprehensive than ditopologies basically given in [2, 3] and fuzzy topologies given independently by Šostak in [11] and Kubiak in [10]. Unlike ditopological case, in graded ditopologies, openness and closedness are given by means of independent grading functions.

In this work, we formulate the initial and product graded ditopologies and then we show that **dfGDitop** (given in Theorem 1.15) is a topological structure over **dfTex** $\times$ **dfTex**. Note that **dfTex** is the category of textures and difunctions between them [4]. Also **dfTex** $\times$ **dfTex** is the product category whose objects are all pairs of textures $((S,\mathscr{S}),(V,\mathscr{V}))$ and morphisms are all pairs of difunctions $((f,F),\ (h,H))$ from $((S,\mathscr{S}),\ (V,\mathscr{V}))$ to $((S',\mathscr{S}'),\ (V',\mathscr{V}'))$ with $(f,F):(S,\mathscr{S})\to(S',\mathscr{S}')$, $(h,H):(V,\mathscr{V})\to(V',\mathscr{V}')$. By using spectral theory as in [12, 13], we define (di)compactness in graded ditopological texture spaces as a generalization of (di)compactness in ditopological case and then a Tychonoff Theorem for that spaces is proved. The relationship between dicompactness spectrum and diconvergence (diclustering) spectrum is also studied.

Textures: [2] For a set S, a subset $\mathscr{S}\subseteq\mathscr{P}(S)$ is called a texturing on S if it is a point separating (i.e. for all $s,t\in S,\ s\neq t$ there exists a set $A\in\mathscr{S}$ such that $s\in A,\ t\notin A$ or $s\notin A,\ t\in A$), completely distributive, complete lattice with respect to inclusion which

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contains $\emptyset$, S and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ with unions $\bigcup$. In this case $(S,\mathscr{S})$ is called a texture space or simply a texture.

For any texture $(S,\mathscr{S})$, many properties are conveniently defined in terms of the $p-sets$

$$P_s = \bigcap\{A \in \mathscr{S} \mid s \in A\}$$

and the $q-sets$

$$Q_s = \bigvee\{A \in \mathscr{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S, s \notin P_u\}.$$

A texture $(S,\mathscr{S})$ is called plain if $P_s \nsubseteq Q_s$ for all $s \in S$ or equivalently $A = \bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$ for all $A_i \in \mathscr{S}, i \in I$.

In general, a texturing of S need not be closed under set complementation, but there may exist a mapping $\sigma : \mathscr{S} \to \mathscr{S}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ for all $A,B \in \mathscr{S}$. In this case σ is called a complementation on $(S,\mathscr{S})$ and $(S,\mathscr{S},\sigma)$ is said to be a complemented texture.

For any set $A \in \mathscr{S}$, the core of A (denoted by $A^\flat$) is defined by

$$A^\flat = \bigcap\Big\{\bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathscr{S},\ A = \bigvee\{A_i \mid i \in I\}\Big\}.$$

Product of textures: [3, 4, 5] Let $(S_j,\mathscr{S}_j)$, $j \in J$ be textures, $S = \prod_{j\in J} S_j$ and $A_k \in \mathscr{S}_k$ for some $k \in J$. If we write

$$E(k,A_k) = \prod_{j\in J} Y_j \text{ where } Y_j = \begin{cases} A_j & \text{if } j = k \\ S_j & \text{otherwise} \end{cases}$$

then the product texturing $\mathscr{S} = \bigotimes_{j\in J} \mathscr{S}_j$ of S consists of arbitrary intersections of elements of the set

$$\varepsilon = \{\bigcup_{j\in J_1} E(j,A_j) \mid J_1 \subseteq J,\ A_j \in \mathscr{S}_j \text{ for } j \in J_1\}.$$

Consider two textures $(S,\mathscr{S})$ and $(V,\mathscr{V})$. The p-sets and q-sets of the product texture $(S\times V, \mathscr{P}(S)\otimes\mathscr{V})$ will be denoted by $\overline{P}_{(s,v)}$, $\overline{Q}_{(s,v)}$ respectively.

Definition 1.1. [4] *Let $(S,\mathscr{S})$ and $(V,\mathscr{V})$ be textures. Then*

(1) *$r \in \mathscr{P}(S)\otimes\mathscr{V}$ is called a relation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ if it satisfies*
   R1 *$r \nsubseteq \overline{Q}(s,v)$, $P_{s'} \nsubseteq Q_s \Rightarrow r \nsubseteq \overline{Q}(s',v)$.*
   R2 *$r \nsubseteq \overline{Q}(s,v) \Rightarrow \exists s' \in S$ such that $P_s \nsubseteq Q_{s'}$ and $r \nsubseteq \overline{Q}(s',v)$.*

(2) *$R \in \mathscr{P}(S)\otimes\mathscr{V}$ is called a co-relation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ if it satisfies*
   CR1 *$\overline{P}(s,v) \nsubseteq R$, $P_s \nsubseteq Q_{s'} \Rightarrow \overline{P}(s',v) \nsubseteq R$.*
   CR2 *$\overline{P}(s,v) \nsubseteq R \Rightarrow \exists s' \in S$ such that $P_{s'} \nsubseteq Q_s$ and $\overline{P}(s',v) \nsubseteq R$.*

(3) *A pair (r,R), where r is a relation and R a co-relation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ is called a direlation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$.*

For a texture $(S,\mathscr{S})$ the identity direlation $(i_{(S,\mathscr{S})}, I_{(S,\mathscr{S})})$ is defined by

$$i_{(S,\mathscr{S})} = \bigvee\{\overline{P}(s,s) \mid s \in S\} \text{ and } I_{(S,\mathscr{S})} = \bigcap\{\overline{Q}(s,s) \mid s \in S^\flat\}.$$

For $A \subseteq S$, $r^{\to}A = \bigcap\{Q_v \mid \forall s, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow A \subseteq Q_s\}$ is called the A-section of r and $R^{\to}A = \bigvee\{P_v \mid \forall s, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow P_s \subseteq A\}$ is called the A-section of R.

For $B \subseteq V$, $r^{\leftarrow}B = \bigvee\{P_s \mid \forall v, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow P_v \subseteq B\}$ is called the B-presection of r and $R^{\leftarrow}B = \bigcap\{Q_s \mid \forall v, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow B \subseteq Q_v\}$ is called the B-presection of R.

Lemma 1.2. [4] *Let r, r_1, r_2 be relations, R, R_1, R_2 co-relations on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ with $r_1 \subseteq r_2$, $R_1 \subseteq R_2$, $A, A_1, A_2 \in \mathscr{S}$, $A_1 \subseteq A_2$, $B, B_1, B_2 \in \mathscr{V}$, $B_1 \subseteq B_2$.*

(1) *$r \not\subseteq \overline{Q}_{(s,v)} \Leftrightarrow \overline{P}_{(v,s)} \not\subseteq r^{\leftarrow}$ and $\overline{P}_{(s,v)} \not\subseteq R \Leftrightarrow R^{\leftarrow} \not\subseteq \overline{Q}_{(v,s)}$ for all $s \in S$, $v \in V$.*
(2) *$(r^{\leftarrow})^{\leftarrow} = r$ and $(R^{\leftarrow})^{\leftarrow} = R$*
(3) *For a second direlation (m,M) from $(S,\mathscr{S})$ to $(V,\mathscr{V})$, $(r,R) \sqsubseteq (m,M) \Leftrightarrow (r,R)^{\leftarrow} \sqsubseteq (m,M)^{\leftarrow}$*
(4) *$r^{\to}\emptyset = \emptyset$, $A \subseteq r^{\leftarrow}(r^{\to}A)$, $r^{\to}(r^{\leftarrow}B) \subseteq B$*
(5) *$R^{\to}S = V$, $R^{\leftarrow}(R^{\to}A) \subseteq A$, $B \subseteq R^{\to}(R^{\leftarrow}B)$*
(6) *$r_1^{\to}A_1 \subseteq r_2^{\to}A_2$, $R_1^{\to}A_1 \subseteq R_2^{\to}A_2$, $r_2^{\leftarrow}B_1 \subseteq r_1^{\leftarrow}B_2$, $R_2^{\leftarrow}B_1 \subseteq R_1^{\leftarrow}B_2$.*

Proposition 1.3. [4] *If (r,R) is a direlation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ then $r^{\to}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} r^{\to}A_i$, $R^{\to}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} R^{\to}A_i$, $r^{\leftarrow}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} r^{\leftarrow}B_j$ and $R^{\leftarrow}(\bigvee_{j\in J} B_j) = \bigvee_{j\in J} R^{\leftarrow}B_j$ for any $A_i \in \mathscr{S}$, $B_j \in \mathscr{V}$, $i \in I$, $j \in J$.*

Definition 1.4. [4] *Let (f,F) be a direlation from $(S,\mathscr{S})$ to $(V,\mathscr{V})$. Then (f,F) is called a difunction from $(S,\mathscr{S})$ to $(V,\mathscr{V})$ if it satisfies the following two conditions:*

(DF1) *For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \Rightarrow \exists v \in V$ with $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s',v)} \not\subseteq F$.*
(DF2) *For $v, v' \in V$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s,v')} \not\subseteq F \Rightarrow P_{v'} \not\subseteq Q_v$.*

It is clear that (i_S, I_S) is a difunction on $(S,\mathscr{S})$ and we call it the identity difunction on $(S,\mathscr{S})$. (f,F) is called surjective if

$$\forall v, v' \in V\ P_v \not\subseteq Q_{v'} \Rightarrow \exists s \in S \text{ with } f \not\subseteq \overline{Q}_{(s,v')} \text{ and } \overline{P}_{(s,v)} \not\subseteq F.$$

Proposition 1.5. [4] *Let (f,F) be a difunction on $(S,\mathscr{S})$ to $(V,\mathscr{V})$.*

(1) *$f^{\leftarrow}B = F^{\leftarrow}B$ for each $B \in \mathscr{V}$.*
(2) *$f^{\leftarrow}\emptyset = F^{\leftarrow}\emptyset = \emptyset$ and $f^{\leftarrow}V = F^{\leftarrow}V = S$.*
(3) *$A \subseteq F^{\leftarrow}(f^{\to}A)$ and $f^{\to}(F^{\leftarrow}B) \subseteq B$ for all $A \in \mathscr{S}$, $B \in \mathscr{V}$.*
(4) *If (f,F) is surjective then $F^{\to}(f^{\leftarrow}B) = B = f^{\to}(F^{\leftarrow}B)$ for all $B \in \mathscr{V}$.*

Definition 1.6. [2] *A dichotomous topology, or ditopology for short, on a texture $(S,\mathscr{S})$ is a pair (τ,κ) of subsets of $\mathscr{S}$, where the set of open sets τ satisfies*

(T_1) $S, \emptyset \in \tau$
(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$
(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$

and the set of closed sets κ satisfies

(CT_1) $S, \emptyset \in \kappa$
(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$
(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no a priori relation between the open and closed sets.

Definition 1.7. [5] *Let* $(S_k,\mathscr{S}_k,\tau_k,\kappa_k)$, $k=1,2$ *be ditopological texture spaces and* $(f,F):(S_1,\mathscr{S}_1)\to(S_2,\mathscr{S}_2)$ *a difunction.* (f,F) *is called continuous if*

$$F^{\leftarrow}A\in\tau_1,\ \ \forall A\in\tau_2$$

and cocontinuous if

$$f^{\leftarrow}A\in\kappa_1,\ \ \forall A\in\kappa_2.$$

The difunction (f,F) *is called bicontinuous if it is both continuous and cocontinuous.*

Theorem 1.8. [5] *Ditopological texture spaces and bicontinuous difunctions form a category denoted by* **dfDiTop**.

For $s=(s_j)\in S$, $P_s=\bigcap_{j\in J}E(j,P_{s_j})=\prod_{j\in J}P_{s_j}$. The jth-projection difunction $(\pi_j,\Pi_j):(S,\mathscr{S})\to(S_j,\mathscr{S}_j)$ is defined by

$$\pi_j=\bigvee\{\overline{P}_{(s,s_j)}\mid s=(s_j)\in S\},\ \Pi_j=\bigcap\{\overline{Q}_{(s,s_j)}\mid s=(s_j)\in S^{\flat}\}$$

and it is surjective by [6].

For ditopological texture spaces $(S_j,\mathscr{S}_j,\tau_j,\kappa_j)$, $j\in J$, the product ditopology on the product texture $(S,\mathscr{S})$ has subbase $\{E(j,G)\mid G\in\tau_j,\ j\in J\}$, cosubbase $\{E(j,K)\mid K\in\kappa_j,\ j\in J\}$.

Proposition 1.9. [5] *Let* (π_j,Π_j) *be the jth-projection difunction of the product texture* $(S,\mathscr{S})$ *of the textures* $(S_j,\mathscr{S}_j)$, $j\in J$. *Then*

(1) *If* $A_i\in\mathscr{S}_i$, $i\in J$ *and* $A_i\neq\emptyset$, $i\neq j$ *then* $\pi_j^{\rightarrow}\bigcap_{j\in J}E(i,A_i)=A_j$.
(2) *If* $A_i\in\mathscr{S}_i$, $i\in J$ *and* $A_i\neq S_i$, $i\neq j$ *then* $\Pi_j^{\rightarrow}\bigcup_{j\in J}E(i,A_i)=A_j$.

Proposition 1.10. [15] *Let* $(S,\mathscr{S})$ *be the product texturing of the textures* $(S_j,\mathscr{S}_j)$, $j\in J$. $(S,\mathscr{S})$ *is plain if and only if* $(S_j,\mathscr{S}_j)$ *is plain for all* $j\in J$.

Definition 1.11. [6] *Let* $(S,\mathscr{S},\tau,\kappa)$ *be a ditopological texture space and* $A\in\mathscr{S}$. *Then*

(1) *A is called compact if whenever* $\{G_i\mid i\in I\}$ *is an open cover of A (i.e.* $\forall i\in I\ G_i\in\tau$ *and* $A\subseteq\bigvee_{i\in I}G_i$*) then there is a finite subset J of I with* $A\subseteq\bigvee_{i\in J}G_i$. *In particular* $(S,\mathscr{S},\tau,\kappa)$ *is called compact if S is compact.*
(2) *A is called cocompact if whenever* $\{K_i\mid i\in I\}$ *is a closed cocover of A (i.e.* $\forall i\in I\ K_i\in\kappa$ *and* $\bigcap_{i\in I}K_i\subseteq A$*) then there is a finite subset J of I with* $\bigcap_{i\in J}K_i\subseteq A$. *In particular* $(S,\mathscr{S},\tau,\kappa)$ *is called cocompact if* $\emptyset$ *is compact.*
(3) $(S,\mathscr{S},\tau,\kappa)$ *is called stable if every* $K\in\kappa$ *with* $K\neq S$ *is compact.*
(4) $(S,\mathscr{S},\tau,\kappa)$ *is called costable if every* $G\in\kappa$ *with* $G\neq\emptyset$ *is cocompact.*
(5) $(S,\mathscr{S},\tau,\kappa)$ *is called dicompact if it is compact, cocompact, stable and costable.*

Theorem 1.12. [6] *A product of non-empty ditopological texture space is dicompact if and only if each component space is dicompact.*

Graded Ditopological Texture Spaces: [7] Let $(S,\mathscr{S})$, $(V,\mathscr{V})$ be textures and consider $\mathscr{T},\mathscr{K}:\mathscr{S}\to\mathscr{V}$ satisfying

(GT_1) $\mathscr{T}(S)=\mathscr{T}(\emptyset)=V$
(GT_2) $\mathscr{T}(A_1)\cap\mathscr{T}(A_2)\subseteq\mathscr{T}(A_1\cap A_2)\ \forall A_1,A_2\in\mathscr{S}$
(GT_3) $\bigcap_{j\in J}\mathscr{T}(A_j)\subseteq\mathscr{T}(\bigvee_{j\in J}A_j)\ \forall A_j\in\mathscr{S},\ j\in J$

and

(GCT_1) $\mathscr{K}(S)=\mathscr{K}(\emptyset)=V$
(GCT_2) $\mathscr{K}(A_1)\cap\mathscr{K}(A_2)\subseteq\mathscr{K}(A_1\cup A_2)\ \forall A_1,A_2\in\mathscr{S}$
(GCT_3) $\bigcap_{j\in J}\mathscr{K}(A_j)\subseteq\mathscr{K}(\bigcap_{j\in J}A_j)\ \forall A_j\in\mathscr{S},\ j\in J$

Then $\mathscr{T}$ is called a $(V,\mathscr{V})$-graded topology, $\mathscr{K}$ a $(V,\mathscr{V})$-graded cotopology and $(\mathscr{T},\mathscr{K})$ a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$ and for any graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ and for each $v\in V$ it is defined that

$$\mathscr{T}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathscr{T}(A)\},\ \mathscr{K}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathscr{K}(A)\}.$$

Then $(\mathscr{T}^v,\mathscr{K}^v)$ is a ditopology on $(S,\mathscr{S})$ for each $v\in V$. That is, if $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is any graded ditopological texture space, then there exists a ditopology $(\mathscr{T}^v,\mathscr{K}^v)$ on the texture space $(S,\mathscr{S})$ for each $v\in V$.

If $(S,\mathscr{S},\sigma)$ is a complemented texture and $(\mathscr{T},\mathscr{K})$ a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$, then $(\mathscr{K}\circ\sigma,\mathscr{T}\circ\sigma)$ is also a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$. $(\mathscr{T},\mathscr{K})$ is called complemented if $(\mathscr{T},\mathscr{K})=(\mathscr{K}\circ\sigma,\mathscr{T}\circ\sigma)$.

Let $(\mathscr{T}_k,\mathscr{K}_k)$, $k=1,2$ be $(V,\mathscr{V})$-graded ditopologies on $(S,\mathscr{S})$. $(\mathscr{T}_1,\mathscr{K}_1)$ said to be coarser than $(\mathscr{T}_2,\mathscr{K}_2)$ and $(\mathscr{T}_2,\mathscr{K}_2)$ said to be finer than $(\mathscr{T}_1,\mathscr{K}_1)$ if $\mathscr{T}_1(A)\subseteq\mathscr{T}_2(A)$, $\mathscr{K}_1(A)\subseteq\mathscr{K}_2(A)$ for all $A\in\mathscr{S}$ [8].

Example 1.13. [7] *Let $(S,\mathscr{S},\tau,\kappa)$ be a ditopological texture space and $(V,\mathscr{V})$ the discrete texture on a singleton. Take $(V,\mathscr{V})=(1,\mathscr{P}(1))$ (The notation 1 denotes the set $\{0\}$) and define $\tau^g:\mathscr{S}\to\mathscr{P}(1)$ by $\tau^g(A)=1\Leftrightarrow A\in\tau$. Then τ^g is a $(V,\mathscr{V})$-graded topology on $(S,\mathscr{S})$. Likewise, κ^g defined by $\kappa^g(A)=1\Leftrightarrow A\in\kappa$ is a $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$ and (τ^g,κ^g) is called the graded ditopology on $(S,\mathscr{S})$ corresponding to ditopology (τ,κ).*

Definition 1.14. [7] *Let $(S_k,\mathscr{S}_k,\mathscr{T}_k,\mathscr{K}_k,V_k,\mathscr{V}_k)$, $k=1,2$ be graded ditopological texture spaces, $(f,F):(S_1,\mathscr{S}_1)\to(S_2,\mathscr{S}_2)$, $(h,H):(V_1,\mathscr{V}_1)\to(V_2,\mathscr{V}_2)$ difunctions. For the pair $((f,F),(h,H))$, (f,F) is called continuous with respect to (h,H) if*

$$H^{\leftarrow}\mathscr{T}_2(A)\subseteq\mathscr{T}_1(F^{\leftarrow}A)\ \textit{for all}\ A\in\mathscr{S}_2$$

and cocontinuous with respect to (h,H) if

$$h^{\leftarrow}\mathscr{K}_2(A)\subseteq\mathscr{K}_1(f^{\leftarrow}A)\ \textit{for all}\ A\in\mathscr{S}_2.$$

The difunction (f,F) is called bicontinuous with respect to (h,H) if it is both continuous and cocontinuous with respect to (h,H).

Theorem 1.15. [7] *The class of graded ditopological texture spaces and relatively bicontinuous difunction pairs between them form a category denoted by* **dfGDitop**.

2. PRODUCT GRADED DITOPOLOGY

Throughout the paper we denote the finite subset of a index set J by J_0 and the finite subfamily of a family $\mathscr{U}$ by $\mathscr{U}_0$.

Theorem 2.1. *Let $(S,\mathscr{S})$, $(V,\mathscr{V})$ be textures, $(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)_{j\in J}$ be graded ditopological texture spaces and $(f_j,F_j):(S,\mathscr{S})\to(S_j,\mathscr{S}_j)$, $(h_j,H_j):(V,\mathscr{V})\to(V_j,\mathscr{V}_j)$, $(j\in J)$ be difunctions. Then the mappings $\mathscr{T},\mathscr{K}:\mathscr{S}\to\mathscr{V}$ defined by*

$$\mathscr{T}(A)=\bigvee\{\bigcap_{j\in J_0}H_j^{\leftarrow}\mathscr{T}_j(G_j)\mid A=\bigcap_{j\in J_0}F_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}$$

$$\mathscr{K}(A)=\bigvee\{\bigcap_{j\in J_0}h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid A=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}$$

for all $A\in\mathscr{S}$ form a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$. $(\mathscr{T},\mathscr{K})$ is the coarsest $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$ that makes (f_j,F_j) bicontinuous with respect to (h_j,H_j) for each $j\in J$.

Proof. Firstly, we show that $\mathscr{K}$ is a $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$:
(i) Since $S=f_j^{\leftarrow}S_j$ and $h_j^{\leftarrow}V_j=V$ for all $j\in J$ by Proposition 1.5 (2), if we take a $j_0\in J$ then we have $\mathscr{K}(S)=\bigvee\{\bigcap_{j\in J_0}h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid S=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}\supseteq h^{\leftarrow}\mathscr{K}_{j_0}(S_{j_0})=h^{\leftarrow}V_{j_0}=V$ and so $\mathscr{K}(S)=V$.

On the other hand, since $\emptyset=f_j^{\leftarrow}\emptyset$ and $h_j^{\leftarrow}\emptyset=\emptyset$ for all $j\in J$ by Proposition 1.5 (2), if we take a $j_0\in J$ then we have $\mathscr{K}(\emptyset)=\bigvee\{\bigcap_{j\in J_0}h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid\emptyset=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}\supseteq h^{\leftarrow}\mathscr{K}_{j_0}(\emptyset)=h^{\leftarrow}V_{j_0}=V$ (by (GCT_1)) and so $\mathscr{K}(\emptyset)=V$.
(ii) Let $A,B\in\mathscr{S}$ be given. If there is no $G_j\in\mathscr{S}_j$ such that $A=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j$ or $B=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j$ for a finite $J_0\subseteq J$ then $\mathscr{K}(A)\cap\mathscr{K}(B)=\emptyset\subseteq\mathscr{K}(A\cup B)$. So, let $A=\bigcup_{j\in J_1}f_j^{\leftarrow}G_j$ and $B=\bigcup_{j\in J_2}f_j^{\leftarrow}L_j$ for some finite subsets $J_1,J_2\subseteq J$ and for some $G_j,L_j\in\mathscr{S}_j$. If we redefine

$$G_j=\begin{cases}G_j, & j\in J_1\\ \emptyset, & j\in J_2\end{cases}$$

$$L_j=\begin{cases}L_j, & j\in J_2\\ \emptyset, & j\in J_1\end{cases}$$

then we have $\bigcup_{j\in J_1}f_j^{\leftarrow}G_j=\bigcup_{j\in J_1\cup J_2}f_j^{\leftarrow}G_j$ and $\bigcup_{j\in J_2}f_j^{\leftarrow}L_j=\bigcup_{j\in J_1\cup J_2}f_j^{\leftarrow}L_j$ by the fact that $f_j^{\leftarrow}\emptyset=\emptyset$ for all $j\in J$. Similarly, since $\mathscr{K}_j(\emptyset)=V_j$ and $h_j^{\leftarrow}V_j=V$ for all $j\in J$, we have $\bigcap_{j\in J_1}h_j^{\leftarrow}\mathscr{K}_j(G_j)=\bigcap_{j\in J_1\cup J_2}h_j^{\leftarrow}\mathscr{K}_j(G_j)$ and $\bigcap_{j\in J_2}h_j^{\leftarrow}\mathscr{K}_j(L_j)=\bigcap_{j\in J_1\cup J_2}h_j^{\leftarrow}\mathscr{K}_j(L_j)$. Thus we get

$$\begin{aligned}&\mathscr{K}(A)\cap\mathscr{K}(B)\\ &=\bigvee\{\bigcap_{j\in J_1}h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid A=\bigcup_{j\in J_1}f_j^{\leftarrow}G_j\}\cap\bigvee\{\bigcap_{j\in J_2}h_j^{\leftarrow}\mathscr{K}_j(L_j)\mid B=\bigcup_{j\in J_2}f_j^{\leftarrow}L_j\}\\ &=\bigvee\{\bigcap_{j\in J_1}h_j^{\leftarrow}\mathscr{K}_j(G_j)\cap\bigcap_{j\in J_2}h_j^{\leftarrow}\mathscr{K}_j(L_j)\mid A=\bigcup_{j\in J_1}f_j^{\leftarrow}G_j,\ B=\bigcup_{j\in J_2}f_j^{\leftarrow}L_j\}\end{aligned}$$

$$= \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}\mathscr{K}_j(G_j)\cap \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}\mathscr{K}_j(L_j) \mid A = \bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}G_j,\ B = \bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}L_j\}$$
$$= \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}\mathscr{K}_j(G_j)\cap h_j^{\leftarrow}\mathscr{K}_j(L_j) \mid A = \bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}G_j,\ B = \bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}L_j\}$$
$$= \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}(\mathscr{K}_j(G_j)\cap \mathscr{K}_j(L_j)) \mid A = \bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}G_j,\ B = \bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}L_j\}$$
$$\subseteq \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}(\mathscr{K}_j(G_j\cup L_j)) \mid A\cup B = \bigcup_{j\in J_1\cup J_2} (f_j^{\leftarrow}G_j\cup f_j^{\leftarrow}L_j) = \bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}(G_j\cup L_j)\}$$
$$= \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}\mathscr{K}_j(M_j) \mid A\cup B = \bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}M_j\} = \mathscr{K}(A\cup B)$$

(iii) Let $A_i \in \mathscr{S}$ for all $i\in I$ where I is a nonempty index set. If for some $i\in I$, A_i can not be written as $A_i = \bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i$ where J_i is a finite subset of J then $\bigcap_{i\in I}\mathscr{K}(A_i) = \emptyset \subseteq \mathscr{K}(\bigcap_{i\in I}A_i)$. So, for each $i\in I$ let $A_i = \bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i$ for some $G_j^i \in \mathscr{S}_j$, $j\in J_i$. If we redefine

$$G_j^i = \begin{cases} G_j^i, & j\in J_i \\ \emptyset, & j\notin J_i \end{cases}$$

then considering $f_j^{\leftarrow}\emptyset = \emptyset$, $\mathscr{K}_j(\emptyset) = V_j$ (by (GCT_1)), $h_j^{\leftarrow}V_j = V$ for all $j\in J$ and "$j\notin \bigcap_{i\in I}J_i \Rightarrow \bigcap_{i\in I}G_j^i = \emptyset$" we have

$$\bigcap_{i\in I}A_i = \bigcap_{i\in I}\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i = \bigcap_{i\in I}\bigcup_{j\in J} f_j^{\leftarrow}G_j^i = \bigcup_{j\in J}\bigcap_{i\in I} f_j^{\leftarrow}G_j^i$$
$$= \bigcup_{j\in J} f_j^{\leftarrow}(\bigcap_{i\in I}G_j^i) = \bigcup_{j\in\bigcap_{i\in I}J_i} f_j^{\leftarrow}(\bigcap_{i\in I}G_j^i)$$

and

$$\bigcap_{i\in I}(\bigcap_{j\in J_i} h_j^{\leftarrow}\mathscr{K}_j(G_j^i)) = \bigcap_{i\in I}(\bigcap_{j\in J} h_j^{\leftarrow}\mathscr{K}_j(G_j^i)) = \bigcap_{j\in J} h_j^{\leftarrow}\bigcap_{i\in I}\mathscr{K}_j(G_j^i)$$
$$\subseteq \bigcap_{j\in J} h_j^{\leftarrow}\mathscr{K}_j(\bigcap_{i\in I}G_j^i) = \bigcap_{j\in\bigcap_{i\in I}J_i} h_j^{\leftarrow}\mathscr{K}_j(\bigcap_{i\in I}G_j^i)$$

Therefore we get

$$\bigcap_{i\in I}\mathscr{K}(A_i) = \bigcap_{(i\in I)}(\bigvee_{(A_i=\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i)} (\bigcap_{(j\in J_i)} h_j^{\leftarrow}\mathscr{K}_j(G_j^i)))$$
$$= \bigvee_{(A_i=\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i,\ i\in I)} \bigcap_{(i\in I)}(\bigcap_{(j\in J_i)} h_j^{\leftarrow}\mathscr{K}_j(G_j^i))$$
$$\subseteq \bigvee_{(\bigcap_{i\in I}A_i=\bigcap_{i\in I}\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i)} (\bigcap_{(j\in\bigcap_{i\in I}J_i)} h_j^{\leftarrow}\mathscr{K}_j(\bigcap_{i\in I}G_j^i))$$

$$= \bigvee_{(\bigcap_{i\in I} A_i = \bigcup_{j\in(\bigcap_{i\in I} J_i)} f_j^{\leftarrow}(\bigcap_{i\in I} G_j^i))} \Big(\bigcap_{(j\in\bigcap_{i\in I} J_i)} h_j^{\leftarrow} \mathscr{K}_j(\bigcap_{i\in I} G_j^i)\Big)$$

$$= \bigvee_{(\bigcap_{i\in I} A_i = \bigcup_{j\in(\bigcap_{i\in I} J_i)} f_j^{\leftarrow} B_j)} \Big(\bigcap_{(j\in\bigcap_{i\in I} J_i)} h_j^{\leftarrow} \mathscr{K}_j(B_j)\Big) = \mathscr{K}(\bigcap_{i\in I} A_i)$$

So $\mathscr{K}$ is a $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$. By the definition of $\mathscr{K}$, (f_j,F_j) is cocontinuous with respect to (h_j,H_j) for each $j\in J$.

Let $\mathscr{K}'$ be a $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$ that makes (f_j,F_j) cocontinuous with respect to (h_j,H_j) for each $j\in J$. Then $G_j\in\mathscr{S}_j$ implies $h_j^{\leftarrow}\mathscr{K}_j(G_j)\subseteq\mathscr{K}'(f_j^{\leftarrow}G_j)$ for each $j\in J$. So $A=\bigcup_{j\in J_0} f_j^{\leftarrow}G_j \Rightarrow \bigcap_{j\in J_0} h_j^{\leftarrow}\mathscr{K}_j(G_j)\subseteq\bigcap_{j\in J_0}\mathscr{K}' f_j^{\leftarrow}G_j\subseteq\mathscr{K}'(\bigcup_{j\in J_0} f_j^{\leftarrow}G_j)=\mathscr{K}'(A)$ for all $A\in\mathscr{S}$. Hence,

$$\mathscr{K}(A)=\bigvee\{\bigcap_{j\in J_0} h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid A=\bigcup_{j\in J_0} f_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}\subseteq\mathscr{K}'(A)$$

for all $A\in\mathscr{S}$. Therefore $\mathscr{K}$ is the coarsest $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$ that makes (f_j,F_j) cocontinuous with respect to (h_j,H_j) for each $j\in J$.

Similarly it can be shown that $\mathscr{T}$ is the coarsest $(V,\mathscr{V})$-graded topology on $(S,\mathscr{S})$ that makes (f_j,F_j) cocontinuous with respect to (h_j,H_j) for each $j\in J$. □

Now, referring to [1], we investigate the outcomes of Theorem 2.1 in categorical aspects. If we consider the forgetful functor $\mathfrak{U}:$ **dfGDitop** $\to$ **dfTex** $\times$ **dfTex** then (**dfGDitop**$,\mathfrak{U}$) is a concrete category over **dfTex** $\times$ **dfTex**.

Theorem 2.2. *The source*

$$(((f_j,F_j),(h_j,H_j)):(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})\to(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J})$$

in **dfGDitop** *is initial if and only if* $(\mathscr{T},\mathscr{K})$ *is the graded ditopology defined as in Theorem 2.1.*

Proof. Let the source

$$(((f_j,F_j),(h_j,H_j)):(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})\to(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J})$$

be initial. For each $j\in J$; (f_j,F_j) is bicontinuous with respect to (h_j,H_j) because $((f_j,F_j),(h_j,H_j))$ is a morphism in **dfGDitop**. So, $H_j^{\leftarrow}\mathscr{T}_j(G_j)\subseteq\mathscr{T}(F_j^{\leftarrow}G_j)$ and $h_j^{\leftarrow}\mathscr{K}_j(G_j)\subseteq\mathscr{K}(f_j^{\leftarrow}G_j)$ for all $G_j\in\mathscr{S}_j$, $j\in J$. If we denote the graded ditopology defined in Theorem 2.1 by $(\mathscr{T}^\star,\mathscr{K}^\star)$ then we have

$$A=\bigcap_{j\in J_0} F_j^{\leftarrow}G_j\Rightarrow\bigcap_{j\in J_0} H_j^{\leftarrow}\mathscr{T}_j(G_j)\subseteq\bigcap_{j\in J_0}\mathscr{T}(F_j^{\leftarrow}G_j)\subseteq\mathscr{T}(\bigcap_{j\in J_0} F_j^{\leftarrow}G_j)=\mathscr{T}(A)$$

and so, $\mathscr{T}^\star(A)\subseteq\mathscr{T}(A)$ for all $A\in\mathscr{S}$, i.e. $\mathscr{T}^\star\subseteq\mathscr{T}$.

Since $((i_S,I_S),(i_V,I_V))$ in **dfTex** $\times$ **dfTex** makes the right hand diagram commutative, it lifts to a morphism in **dfGDitop** such that the left hand diagram is commutative.

$$\begin{tikzcd}
(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V}) \arrow[dr, "((f_j,F_j),(h_j,H_j))"] & \\
(S,\mathscr{S},\mathscr{T}^\star,\mathscr{K}^\star,V,\mathscr{V}) \arrow[u, "((i_S,I_S),(i_V,I_V))"] \arrow[r, "((f_j,F_j),(h_j,H_j))"'] & (S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)
\end{tikzcd}
\qquad
\begin{tikzcd}
((S,\mathscr{S}),(V,\mathscr{V})) \arrow[dr, "((f_j,F_j),(h_j,H_j))"] & \\
((S,\mathscr{S}),(V,\mathscr{V})) \arrow[u, "((i_S,I_S),(i_V,I_V))"] \arrow[r, "((f_j,F_j),(h_j,H_j))"'] & ((S_j,\mathscr{S}_j),(V_j,\mathscr{V}_j))
\end{tikzcd}$$

Since $((i_S,I_S),(i_V,I_V))$ is a morphism in **dfGDitop**, (i_S,I_S) is bicontinuous with respect to (i_V,I_V). Hence $I_V^{\leftarrow}\mathscr{T}(A) \subseteq \mathscr{T}^\star(I_S^{\leftarrow}A) \Rightarrow \mathscr{T}(A) \subseteq \mathscr{T}^\star(A)$ for all $A \in \mathscr{S}$, i.e. $\mathscr{T} \subseteq \mathscr{T}^\star$. Therefore we get $\mathscr{T} = \mathscr{T}^\star$. Similarly it can be shown that $\mathscr{K} = \mathscr{K}^\star$

Now we will show that

$$(((f_j,F_j),(h_j,H_j)) : (S,\mathscr{S},\mathscr{T}^\star,\mathscr{K}^\star,V,\mathscr{V}) \to (S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J})$$

is initial. Let $((k,K),(l,L))$ be a morphism in **dfTex** $\times$ **dfTex** that makes the right hand diagram commutative.

$$\begin{tikzcd}
(S,\mathscr{S},\mathscr{T}^\star,\mathscr{K}^\star,V,\mathscr{V}) \arrow[dr, "((f_j,F_j),(h_j,H_j))"] & \\
(S',\mathscr{S}',\mathscr{T}',\mathscr{K}',V',\mathscr{V}') \arrow[u, "((k,K),(l,L))"] \arrow[r, "((f'_j,F'_j),(h'_j,H'_j))"'] & (S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)
\end{tikzcd}
\qquad
\begin{tikzcd}
((S,\mathscr{S}),(V,\mathscr{V})) \arrow[dr, "((f_j,F_j),(h_j,H_j))"] & \\
((S',\mathscr{S}'),(V',\mathscr{V}')) \arrow[u, "((k,K),(l,L))"] \arrow[r, "((f'_j,F'_j),(h'_j,H'_j))"'] & ((S_j,\mathscr{S}_j),(V_j,\mathscr{V}_j))
\end{tikzcd}$$

Then, by using Proposition 1.3, Proposition 1.5 (1) and (GT_2) we have

$$\begin{aligned}
L^{\leftarrow}(\mathscr{T}^\star(A)) &= L^{\leftarrow}(\bigvee\{\bigcap_{j\in J_0} H_j^{\leftarrow}\mathscr{T}_j(G_j) \mid A = \bigcap_{j\in J_0} F_j^{\leftarrow}G_j,\ J_0 \subseteq J\}) \\
&= \bigvee\{\bigcap_{j\in J_0} L^{\leftarrow}(H_j^{\leftarrow}\mathscr{T}_j(G_j)) \mid A = \bigcap_{j\in J_0} F_j^{\leftarrow}G_j,\ J_0 \subseteq J\} \\
&= \bigvee\{\bigcap_{j\in J_0} (H_j \circ L)^{\leftarrow}\mathscr{T}_j(G_j) \mid A = \bigcap_{j\in J_0} F_j^{\leftarrow}G_j,\ J_0 \subseteq J\} \\
&= \bigvee\{\bigcap_{j\in J_0} H_j'^{\leftarrow}\mathscr{T}_j(G_j) \mid A = \bigcap_{j\in J_0} F_j^{\leftarrow}G_j,\ J_0 \subseteq J\} \\
&\subseteq \bigcap_{j\in J_0} \mathscr{T}'(F_j'^{\leftarrow}G_j) = \bigcap_{j\in J_0} \mathscr{T}'((F_j \circ K)^{\leftarrow}G_j) \\
&= \bigcap_{j\in J_0} \mathscr{T}'^{\leftarrow} \circ F_j^{\leftarrow}G_j) \subseteq \mathscr{T}'^{\leftarrow}(\bigcap_{j\in J_0} F_j^{\leftarrow}G_j)) = \mathscr{T}'^{\leftarrow}A)
\end{aligned}$$

for all $A \in \mathscr{S}$. Hence (k,K) is continuous with respect to (l,L). Similarly it can be shown that (k,K) is cocontinuous with respect to (l,L) and so (k,K) is bicontinuous with respect to (l,L). Therefore $((k,K),(l,L))$ is a morphism in **dfGDitop**, i.e. the left hand diagram is commutative. $\square$

Definition 2.3. *The graded ditopology constructed in Theorem 2.1 is called the initial $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$ induced by*

$$((f_j,F_j),(h_j,H_j))_{j\in J} \ \textit{and} \ (S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J}.$$

Corollary 2.4. **dfGDitop** *is a topological structure over* **dfTex** $\times$ **dfTex** *with respect to the functor* $\mathfrak{U}$.

Proof. Let $(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)\in Ob$**dfGDitop** and $((f_j,F_j),(h_j,H_j)):((S,\mathscr{S}),(V,\mathscr{V}))\longrightarrow((S_j,\mathscr{S}_j),(V_j,\mathscr{V}_j))$ is a morphism in **dfTex** $\times$ **dfTex** for all $j\in J$. If $(\mathscr{T},\mathscr{K})$ is the graded ditopology defined in Theorem 2.1 then, considering Theorem 2.2, $(((f_j,F_j),(h_j,H_j)):(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})\to(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J})$ is the unique initial source, which satisfies

$$\begin{aligned}\mathfrak{U}((((f_j,F_j),(h_j,H_j)):(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})\to(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J}))\\ =((f_j,F_j),(h_j,H_j)):((S,\mathscr{S}),(V,\mathscr{V}))\longrightarrow((S_j,\mathscr{S}_j),(V_j,\mathscr{V}_j))_{j\in J}).\end{aligned}$$

□

Definition 2.5. *Let $(S,\mathscr{S})$ and $(V,\mathscr{V})$ be the product textures of the textures $(S_j,\mathscr{S}_j)_{j\in J}$ and $(V_j,\mathscr{V}_j)_{j\in J}$ respectively then the initial $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$ induced by the projection difunctions $(\pi_j^S,\prod_j^S):(S,\mathscr{S})\to(S_j,\mathscr{S}_j)$ and $(\pi_j^V,\prod_j^V):(V,\mathscr{V})\to(V_j,\mathscr{V}_j)$ is called the product graded ditopology of $(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)_{j\in J}$.*

Example 2.6. *Let (τ,κ) be the product ditopology of $(S_j,\mathscr{S}_j,\tau_j,\kappa_j)_{j\in J}$. For each $j\in J$, if we take $(V_j,\mathscr{V}_j)=(1,\mathscr{P}(1))$ then $(S_j,\mathscr{S}_j,\tau_j^g,\kappa_j^g,V_j,\mathscr{V}_j)$ is a graded ditopological texture space where $\tau_j^g(A)=1\Leftrightarrow A\in\tau_j$ and $\kappa_j^g(A)=1\Leftrightarrow A\in\kappa_j, A\in\mathscr{S}_j$. So, the product graded ditopology $(\mathscr{T},\mathscr{K})$ of $(S_j,\mathscr{S}_j,\tau_j^g,\kappa_j^g,V_j,\mathscr{V}_j)_{j\in J}$ equals the graded ditopology (τ^g,κ^g) corresponding to ditopology (τ,κ). Indeed, for all $A\in\mathscr{S}$, by the definition of $\mathscr{T}$ and (GT_3) we have*

$$\begin{aligned}\tau^g(A)=1\Leftrightarrow A\in\tau\\ \Leftrightarrow A=\bigvee_{i\in I}B_i,\ B_i=\bigcap_{j\in J_i}(\pi_j^{\leftarrow}G_j),\ J_i\subseteq J\ finite,\ G_j\in\tau_j \textit{ for a index set } I\\ \Leftrightarrow A=\bigvee_{i\in I}B_i,\ B_i=\bigcap_{j\in J_i}(\pi_j^{\leftarrow}G_j),\ J_i\subseteq J\ finite,\ \tau_j^g(G_j)=1 \textit{ for a index set } I\\ \Leftrightarrow A=\bigvee_{i\in I}B_i,\ \mathscr{T}(B_i)=1 \textit{ for a index set } I\ \Leftrightarrow\mathscr{T}(A)=1\end{aligned}$$

3. Compactness in graded ditopological texture spaces

A. P. Šostak has developed the spectral approach for the study of various topological properties of fuzzy topological spaces in [12]. Accordingly, we use this effective approach to study compactness notion (in accordance with fuzzy idea) in graded ditopological texture spaces as a generalization of compactness in ditopological texture spaces.

Definition 3.1. *Let $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ be a graded ditopological texture space and $A\in\mathscr{S}$. The families defined by*

$$\mathscr{C}(A)=\{P_v\in\mathscr{V}\mid[\mathscr{U}\subseteq\mathscr{T}^v,\ A\subseteq\bigvee\mathscr{U}]\Rightarrow\exists\mathscr{U}_0\subseteq\mathscr{U}\ :A\subseteq\bigvee\mathscr{U}_0\}$$

$$\mathscr{C}^*(A) = \{P_v \in \mathscr{V} \mid [\mathscr{U} \subseteq \mathscr{K}^v,\ \bigwedge \mathscr{U} \subseteq A] \Rightarrow \exists \mathscr{U}_0 \subseteq \mathscr{U}\ : \bigwedge \mathscr{U}_0 \subseteq A\}$$

where $\mathscr{U}_0$ denotes a finite subfamily of $\mathscr{U}$, are called compactness and co-compactness spectrums of $A \in \mathscr{S}$ respectively. In particular, the compactness spectrum and the co-compactness spectrum of $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ are $\mathscr{C}(S)$ and $\mathscr{C}^*(\emptyset)$ respectively.

Proposition 3.2. *If $(S,\mathscr{S},\mathscr{T},\mathscr{K},\sigma,V,\mathscr{V})$ is a complemented graded ditopological texture space then $\mathscr{C}(A) = \mathscr{C}^*(\sigma(A))$ for all $A \in \mathscr{S}$. In particular, $\mathscr{C}(S) = \mathscr{C}^*(\emptyset)$ i.e. the compactness and co-compactness spectrums of a complemented graded ditopological texture space are equal.*

Proof. Since $A \subseteq \bigvee \mathscr{U} \Leftrightarrow \sigma(A) \supseteq \sigma(\bigvee \mathscr{U}) = \bigwedge \sigma(\mathscr{U}) \Leftrightarrow \bigwedge \sigma(\mathscr{U}) \subseteq \sigma(A)$ and $U \in \mathscr{T}^v \Leftrightarrow \sigma(U) \in \mathscr{K}^v$ for all $U \in \mathscr{S}$ we get

$$\mathscr{C}(A) = \{P_v \in \mathscr{V} \mid [\mathscr{U} \subseteq \mathscr{T}^v,\ A \subseteq \bigvee \mathscr{U}] \Rightarrow \exists \mathscr{U}_0 \subseteq \mathscr{U}\ : A \subseteq \bigvee \mathscr{U}_0\}$$

$$= \{P_v \in \mathscr{V} \mid [\sigma(\mathscr{U}) \subseteq \mathscr{K}^v,\ \bigwedge \sigma(\mathscr{U}) \subseteq \sigma(A)] \Rightarrow \exists \sigma(\mathscr{U}_0) \subseteq \sigma(\mathscr{U})\ : \bigwedge \sigma(\mathscr{U}_0) \subseteq \sigma(A)\}$$

$$= \{P_v \in \mathscr{V} \mid [\mathscr{U}' \subseteq \mathscr{K}^v,\ \bigwedge \mathscr{U}' \subseteq \sigma(A)] \Rightarrow \exists \mathscr{U}_0' \subseteq \mathscr{U}\ : \bigwedge \mathscr{U}_0' \subseteq \sigma(A)\} = \mathscr{C}^*(\sigma(A))$$

where $\mathscr{U}' = \sigma(\mathscr{U})$ and $\mathscr{U}_0' = \sigma(\mathscr{U}_0)$. In particular, since $S = \sigma(\emptyset)$ we have $\mathscr{C}(S) = \mathscr{C}^*(\emptyset)$. □

Theorem 3.3. *Let $(S_k,\mathscr{S}_k,\mathscr{T}_k,\mathscr{K}_k,V_k,\mathscr{V}_k)_{k=1,2}$ be graded ditopological texture spaces and let $(f,F):(S_1,\mathscr{S}_1) \to (S_2,\mathscr{S}_2)$, $(h,H):(V_1,\mathscr{V}_1) \to (V_2,\mathscr{V}_2)$ be difunctions. For all $A \in \mathscr{S}_1$*

(1) *If (f,F) is continuous with respect to (h,H) then,*

$$P_{v_1} \in \mathscr{C}_1(A) \Rightarrow P_{v_2} \in \mathscr{C}_2(f^{\to}A)$$

(2) *If (f,F) is cocontinuous with respect to (h,H) then,*

$$P_{v_1} \in \mathscr{C}_1^*(A) \Rightarrow P_{v_2} \in \mathscr{C}_2^*(F^{\to}A)$$

where $P_{v_1} \in \mathscr{V}_1$, $P_{v_2} \in \mathscr{V}_2$ with $P_{v_1} \subseteq h^{\leftarrow}P_{v_2}$.

Proof. Let $P_{v_1} \in \mathscr{C}_1(A)$ and $P_{v_1} \subseteq h^{\leftarrow}P_{v_2}$. If $\mathscr{U} \subseteq \mathscr{T}_2^{v_2}$ and $f^{\to}A \subseteq \bigvee \mathscr{U}$ then $A \subseteq F^{\leftarrow}(f^{\to}A) \subseteq F^{\leftarrow}(\bigvee \mathscr{U}) = \bigvee F^{\leftarrow}\mathscr{U} = \bigvee_{U \in \mathscr{U}} F^{\leftarrow}U$. Moreover, $P_{v_1} \subseteq h^{\leftarrow}P_{v_2} \subseteq h^{\leftarrow}(\mathscr{T}_2(\mathscr{U})) \subseteq \mathscr{T}_1(F^{\leftarrow}\mathscr{U})$ since (f,F) is continuous with respect to (h,H). Now, because of $P_{v_1} \in \mathscr{C}_1(A)$ there exists a finite subfamily $F^{\leftarrow}(\mathscr{U}_0) \subseteq F^{\leftarrow}(\mathscr{U})$ such that $A \subseteq \bigvee F^{\leftarrow}(\mathscr{U}_0)$. This follows $f^{\to}A \subseteq f^{\to} \bigvee F^{\leftarrow}(\mathscr{U}_0) = \bigvee f^{\to}(F^{\leftarrow}(\mathscr{U}_0)) \subseteq \bigvee \mathscr{U}_0$. Hence $P_{v_2} \in \mathscr{C}_2(f^{\to}A)$.

The proof of (2) is similar. □

Corollary 3.4. *Let the difunction (f,F) in Theorem 3.3 be surjective.*

(1) *If (f,F) is continuous with respect to (h,H) then,*

$$P_{v_1} \in \mathscr{C}_1(S_1) \Rightarrow P_{v_2} \in \mathscr{C}_2(S_2)$$

(2) *If* (f,F) *is cocontinuous with respect to* (h,H) *then,*

$$P_{v_1} \in \mathscr{C}_1^*(\emptyset) \Rightarrow P_{v_2} \in \mathscr{C}_2^*(\emptyset)$$

where $P_{v_1} \in \mathscr{V}_1$, $P_{v_2} \in \mathscr{V}_2$ *with* $P_{v_1} \subseteq h^{\leftarrow} P_{v_2}$.

Proof. Immediate from $f^{\rightarrow} S_1 = S_2$ and $F^{\rightarrow}\emptyset = \emptyset$. □

Corollary 3.5. *Let* $(S_j, \mathscr{S}_j, \mathscr{T}_j, \mathscr{K}_j, V_j, \mathscr{V}_j)_{j \in J}$ *be non-empty graded ditopological texture spaces and* $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ *their product graded ditopological texture space. Then for all* $j \in J$*;*

(1) $P_v \in \mathscr{C}(S) \Rightarrow P_{v_j} \in \mathscr{C}_j(S_j)$
(2) $P_v \in \mathscr{C}^*(\emptyset) \Rightarrow P_{v_j} \in \mathscr{C}_j^*(\emptyset)$

where $P_v = \prod_{j \in J} P_{v_j} \in \mathscr{V}$ *and* $P_{v_j} \in \mathscr{V}_j$.

Proof. We have $P_v \subseteq \pi_j^{V\leftarrow}(\pi_j^{V\rightarrow} P_v) = \pi_j^{V\leftarrow}(P_{v_j})$ for all $j \in J$ and $v \in V$ by Proposition 1.5 (3). So, the proof follows from Corollary 3.4. □

Theorem 3.6. ***(Tychonoff Theorem)*** *Let* $(S_j, \mathscr{S}_j, \mathscr{T}_j, \mathscr{K}_j, V_j, \mathscr{V}_j)_{j \in J}$ *be non-empty graded ditopological texture spaces and* $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ *their product graded ditopological texture space. If* $(V_j, \mathscr{V}_j)_{j \in J}$ *are plain textures then;*

(1) $P_v \in \mathscr{C}(S) \Leftrightarrow \forall j \in J\ P_{v_j} \in \mathscr{C}_j(S_j)$
(2) $P_v \in \mathscr{C}^*(\emptyset) \Leftrightarrow \forall j \in J\ P_{v_j} \in \mathscr{C}_j^*(\emptyset)$

where $P_v = \prod_{j \in J} P_{v_j} \in \mathscr{V}$ *and* $P_{v_j} \in \mathscr{V}_j$.

Proof. The necessity comes from Corollary 3.5. For sufficiency let $P_v \in \mathscr{V}$ and $P_{v_j} = \pi_j^{v\rightarrow} P_v \in \mathscr{C}_j(S_j)$ for all $j \in J$. If $\mathscr{U} \subseteq \mathscr{T}^v$ and $S = \bigvee \mathscr{U}$ then we get for all $j \in J$

$$S_j = \pi_j^{s\rightarrow}(S) = \pi_j^{s\rightarrow}(\bigvee \mathscr{U}) = \bigvee_{U \in \mathscr{U}} \pi_j^{s\rightarrow} U$$

On the other hand, since $\mathscr{U} \subseteq \mathscr{T}^v$, $U \in \mathscr{U}$ implies

$$P_v \subseteq \mathscr{T}(U) = \bigvee \{ \bigcap_{j \in J_0} \Pi_j^{v\leftarrow} \mathscr{T}_j(G_j) \mid U = \bigcap_{j \in J_0} \Pi_j^{s\leftarrow} G_j,\ J_0 \subseteq J,\ J_0 \text{ is finite} \}$$

Since $(V_j, \mathscr{V}_j)_{j \in J}$ are plain, $(V, \mathscr{V})$ is also plain by Proposition 1.10. Hence, $P_v \subseteq \bigcap_{j \in J_0} \Pi_j^{v\leftarrow} \mathscr{T}_j(G_j^U)$ for some $U = \bigcap_{j \in J_0} \Pi_j^{s\leftarrow} G_j^U$ with $G_j^U \in \mathscr{S}_j$, $j \in J_0$. From Proposition 1.5 (3) we have

$$P_v \subseteq \bigcap_{j \in J_0} \Pi_j^{v\leftarrow} \mathscr{T}_j(G_j^U) \Rightarrow \forall j \in J_0\ \ P_v \subseteq \Pi_j^{v\leftarrow} \mathscr{T}_j(G_j^U)$$

$$\Rightarrow \forall j \in J_0\ \ P_{v_j} = \pi_j^{v\rightarrow} P_v \subseteq \pi_j^{v\rightarrow}(\Pi_j^{v\leftarrow} \mathscr{T}_j(G_j^U)) \subseteq \mathscr{T}_j(G_j^U) \Rightarrow P_{v_j} \subseteq \mathscr{T}_j(G_j^U).$$

Since $U = \bigcap_{j \in J_0} \Pi_j^{s\leftarrow} G_j^U = \bigcap_{j \in J_0} E(j, G_j^U)$ we get $\pi_j^{s\rightarrow} U = \pi_j^{s\rightarrow}(\bigcap_{j \in J_0} E(j, G_j^U)) = G_j^U$ by Proposition 1.9 (1). So, considering (1) we have $S_j = \bigvee_{U \in \mathscr{U}} \pi_j^{s\rightarrow} U = \bigvee_{U \in \mathscr{U}} G_j^U$. Since

$G_j^U \in \mathscr{T}_j^{\nu_j}$ and $P_{v_j} \in \mathscr{C}_j(S_j)$ we get

$$\exists \mathscr{U}_0 \subseteq \mathscr{U} \ : \ S_j \subseteq \bigvee_{U \in \mathscr{U}_0} G_j^U \ (j \in J_0).$$

Thus, $S = \Pi_j^{s\leftarrow} S_j \subseteq \Pi_j^{s\leftarrow}(\bigvee_{U \in \mathscr{U}_0} G_j^U) \subseteq \bigvee_{U \in \mathscr{U}_0} \Pi_j^{s\leftarrow} G_j^U$ for all $j \in J_0$ and so,

$$\bigotimes_{j \in J} S_j = S \subseteq \bigcap_{j \in J_0} (\bigvee_{U \in \mathscr{U}_0} \Pi_j^{s\leftarrow} G_j^U) = \bigvee_{U \in \mathscr{U}_0} (\bigcap_{j \in J_0} \Pi_j^{s\leftarrow} G_j^U) = \bigvee_{U \in \mathscr{U}_0} (\bigcap_{j \in J_0} E(j, G_j^U)).$$

By the definition of $E(j, G_j^U)$;

$$j \notin J_0 \Rightarrow S_j = \pi_j^{s\rightarrow} S \subseteq \pi_j^{s\rightarrow}(\bigvee_{U \in \mathscr{U}_0} \bigcap_{j \in J_0} E(j, G_j^U)) = \pi_j^{s\rightarrow}(\bigvee_{U \in \mathscr{U}_0} U)$$

$$j \in J_0 \Rightarrow S_j = \pi_j^{s\rightarrow} S \subseteq \pi_j^{s\rightarrow}(\bigvee_{U \in \mathscr{U}_0} \bigcap_{j \in J_0} E(j, G_j^U)) = \pi_j^{s\rightarrow}(\bigvee_{U \in \mathscr{U}_0} U)$$

and hence, $S_j = \pi_j^{s\rightarrow} S \subseteq \pi_j^{s\rightarrow}(\bigvee_{U \in \mathscr{U}_0} U)$ for all $j \in J$. That means $S \subseteq \bigvee_{U \in \mathscr{U}_0} U$ and as a result $P_v \in \mathscr{C}(S)$. □

Definition 3.7. *For a graded ditopological texture space $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$, the families defined by*

$$\Omega = \{P_v \in \mathscr{V} \mid [A \in \mathscr{S}, \ A \neq S] \Rightarrow [P_v \subseteq \mathscr{K}(A) \Rightarrow P_v \in \mathscr{C}(A)]\}$$

$$\Omega^* = \{P_v \in \mathscr{V} \mid [A \in \mathscr{S}, \ A \neq \emptyset] \Rightarrow [P_v \subseteq \mathscr{T}(A) \Rightarrow P_v \in \mathscr{C}^*(A)]\}$$

are called stableness and costableness spectrums of $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ respectively.

Proposition 3.8. *For a complemented graded ditopological texture space*

Proposition 3.9. *For a complemented graded ditopological texture space $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, \sigma, V, \mathscr{V})$, $\Omega = \Omega^*$.*

Proof. Let $P_v \in \Omega^*$ and $A \in \mathscr{S}$, $A \neq S$. Then, $P_v \subseteq \mathscr{K}(A) \Rightarrow P_v \subseteq (\mathscr{T} \circ \sigma)(A) \Rightarrow P_v \subseteq \mathscr{T}(\sigma(A))$. Since $\sigma(A) \neq \emptyset$ and $P_v \in \Omega^*$ we have $P_v \in \mathscr{C}^*(\sigma(A)) = \mathscr{C}(A)$ by Proposition 3.2. So, $P_v \in \Omega$. The other direction of the proof can be shown similarly. □

Proposition 3.10. *Let $(S_k, \mathscr{S}_k, \mathscr{T}_k, \mathscr{K}_k, V_k, \mathscr{V}_k)_{k=1,2}$ be graded ditopological texture spaces with stableness (costableness) spectrums Ω_1, Ω_2 (Ω_1^*, Ω_2^*) respectively. If $(f,F) : (S_1, \mathscr{S}_1) \rightarrow (S_2, \mathscr{S}_2)$, $(h,H) : (V_1, \mathscr{V}_1) \rightarrow (V_2, \mathscr{V}_2)$ are surjective difunctions and (f,F) is bicontinuous with respect to (h,H) then $P_{v_1} \in \Omega_1 \Rightarrow P_{v_2} \in \Omega_2$ and $P_{v_1} \in \Omega_1^* \Rightarrow P_{v_2} \in \Omega_2^*$ where $v_1 \in V_1$, $v_2 \in V_2$ with $P_{v_1} \subseteq h^{\leftarrow} P_{v_2}$.*

Proof. Let (f,F) be bicontinuous with respect to (h,H) and $P_{v_1} \in \Omega_1$ with $P_{v_1} \subseteq h^{\leftarrow} P_{v_2}$. For a set $A \in \mathscr{S}_2$ with $A \neq S_2$ we have; $P_{v_2} \subseteq \mathscr{K}_2(A) \Rightarrow P_{v_1} \subseteq h^{\leftarrow} P_{v_2} \subseteq h^{\leftarrow} \mathscr{K}_2(A) \subseteq \mathscr{K}_1(f^{\leftarrow} A)$ by the bicontinuity of (f,F) with respect to (h,H). On the other hand, $f^{\leftarrow} A \neq S_1$ since (f,F) is surjective and $A \neq S_2$. So, $P_{v_1} \subseteq \mathscr{K}_1(f^{\leftarrow} A)$ and $P_{v_1} \in \Omega_1$ imply $P_{v_1} \in \mathscr{C}_1(f^{\leftarrow} A)$.

Now, by using Theorem 3.3 we have $P_{v_2} \in \mathscr{C}_2(f^{\rightarrow}(f^{\leftarrow}A))$. Since (f,F) is surjective we get $f^{\rightarrow}(f^{\leftarrow}A) = f^{\rightarrow}(F^{\leftarrow}A) = A$ by Proposition 1.5 (4). Therefore we have $P_{v_2} \in \mathscr{C}_2(A)$ and that means $P_{v_2} \in \Omega_2$. □

Corollary 3.11. *Let $(S_j, \mathscr{S}_j, \mathscr{T}_j, \mathscr{K}_j, V_j, \mathscr{V}_j)_{j\in J}$ be non-empty graded ditopological texture spaces with stableness (costableness) spectrums Ω_j, (Ω_j^*) respectively and $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ their product graded ditopological texture space with stableness (costableness) spectrum Ω, (Ω^*) respectively. Then for all $j \in J$;*

(1) $P_v \in \Omega \Rightarrow P_{v_j} \in \Omega_j$
(2) $P_v \in \Omega^* \Rightarrow P_{v_j} \in \Omega_j^*$

where $P_v = \prod_{j\in J} P_{v_j} \in \mathscr{V}$ and $P_{v_j} \in \mathscr{V}_j$.

Proof. We have $P_v \subseteq \pi_j^{v\leftarrow}(\pi_j^{v\rightarrow} P_v) = \pi_j^{v\leftarrow}(P_{v_j})$ for all $j \in J$ and $v \in V$ by Proposition 1.5 (3). So, the proof follows from Proposition 3.10. □

The other direction of Corollary 3.11 (i.e. $\forall j \in J\ P_{v_j} \in \Omega_j \Rightarrow P_v \in \Omega$ and $\forall j \in J\ P_{v_j} \in \Omega_j^* \Rightarrow P_v \in \Omega^*$) is an open problem for now as in the ditopological case in [6]. So we use the method which based on the relationship between ditopological and graded ditopological case to prove Theorem 3.16.

Definition 3.12. *For a graded ditopological texture space $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$, the family defined by*

$$\mathscr{DC} = \mathscr{C}(S) \cap \mathscr{C}^*(\emptyset) \cap \Omega \cap \Omega^*$$

is called dicompactness spectrum of $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$.

Example 3.13. *Let $(S, \mathscr{S}, \tau, \kappa)$ be a ditopological texture space and $(V, \mathscr{V}) = (1, \mathscr{P}(1))$ the discrete texture on a singleton. If $(S, \mathscr{S}, \tau, \kappa)$ is compact (cocompact, dicompact) then for the graded ditopological texture space $(S, \mathscr{S}, \tau^g, \kappa^g, V, \mathscr{V})$, $P_v \in \mathscr{C}(S)$ ($P_v \in \mathscr{C}^*(S)$, $P_v \in \mathscr{DC}$) respectively for all $v \in V$, i.e. $v = 0$.*

Proposition 3.14. *Let $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ be a graded ditopological texture space. Then the following hold:*

(1) $P_v \in \mathscr{C}(S) \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is compact*
(2) $P_v \in \mathscr{C}^*(\emptyset) \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is cocompact*
(3) $P_v \in \Omega \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is stable*
(4) $P_v \in \Omega^* \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is costable*
(5) $P_v \in \mathscr{DC} \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is dicompact*

Example 3.15. *Let $(S, \mathscr{S} = \mathscr{P}(S))$ and $(V, \mathscr{V} = \mathscr{P}(V))$ be discrete textures where $S \neq \emptyset$ and $V = \{a, b, c\}$. Then the mappings $\mathscr{T}, \mathscr{K} : \mathscr{S} \to \mathscr{V}$ defined by*

$$\mathscr{T}(A) = \begin{cases} V, & A = \emptyset \text{ or } A = S \\ \{a\}, & \text{otherwise} \end{cases}$$

$$\mathscr{K}(A) = \begin{cases} V, & A = \emptyset \text{ or } A = S \\ \{b\}, & \text{otherwise} \end{cases}$$

for all $A \in \mathscr{S}$ form a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$. We have $\mathscr{T}^a = \mathscr{S} = \mathscr{P}(S)$, $\mathscr{T}^b = \mathscr{T}^c = \{S,\emptyset\}$, $\mathscr{K}^b = \mathscr{S} = \mathscr{P}(S)$, $\mathscr{K}^a = \mathscr{K}^c = \{S,\emptyset\}$. If S is finite then we have $\mathscr{C}(S) = \mathscr{C}^*(\emptyset) = \Omega = \Omega^* = \mathscr{D}\mathscr{C} = \{P_a,P_b,P_c\} = \{\{a\},\{b\},\{c\}\}$.

If S is infinite then for an infinite subset $A \subseteq S$, $\mathscr{U} = \{P_x \mid x \in A\}$ implies $A \subseteq \bigvee \mathscr{U} = \bigvee_{x\in A}\{x\}$ however there is no finite subfamily $\mathscr{U}_0$ of $\mathscr{U}$ such that $A \subseteq \bigvee \mathscr{U}_0$. So we get $\mathscr{C}(S) = \Omega = \{P_b,P_c\}$. Similarly, for a subset $A \subseteq S$, if $S\setminus A$ is infinite then $\mathscr{U} = \{(S\setminus A)\setminus P_x \mid x \in (S\setminus A)\}$ implies $\bigwedge \mathscr{U} = \bigwedge_{x\in(S\setminus A)}((S\setminus A)\setminus P_x) = \emptyset \subseteq A$ however there is no finite subfamily $\mathscr{U}_0$ of $\mathscr{U}$ such that $\bigwedge \mathscr{U}_0 \subseteq A$. Hence we get $\mathscr{C}^*(\emptyset) = \Omega^* = \{P_a,P_c\}$. Therefore, if S is infinite then $\mathscr{D}\mathscr{C} = \mathscr{C}(S) \cap \mathscr{C}^*(\emptyset) \cap \Omega \cap \Omega^* = \{P_c\} = \{\{c\}\}$ is obtained.

Theorem 3.16. *Let $(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)_{j\in J}$ be non-empty graded ditopological texture spaces and $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ their product graded ditopological texture space. If $(V_j,\mathscr{V}_j)_{j\in J}$ are plain textures then;*

$$P_v \in \mathscr{D}\mathscr{C} \Leftrightarrow \forall j \in J\ P_{v_j} \in \mathscr{D}\mathscr{C}_j$$

where $P_v = \prod_{j\in J} P_{v_j} \in \mathscr{V}$ and $P_{v_j} \in \mathscr{V}_j$.

Proof. ($\Rightarrow$): It is obvious from Theorem 3.6 and Corollary 3.11.
($\Leftarrow$): Let $P_{v_j} \in \mathscr{D}\mathscr{C}_j$ for all $j \in J$ where $P_v = \prod_{j\in J} P_{v_j} \in \mathscr{V}$ and $P_{v_j} \in \mathscr{V}_j$. Then ditopological texture spaces $(S_j,\mathscr{S}_j,\mathscr{T}_j^{v_j},\mathscr{K}_j^{v_j})_{j\in J}$ are dicompact by Proposition 3.14. So, their product ditopological texture space $(S,\mathscr{S},\mathscr{T}_v,\mathscr{K}_v)$ is dicompact by Theorem 1.12.

Now, we show that $\mathscr{T}_v = \mathscr{T}^v$. Take $A \in \mathscr{T}_v$ then $A = \bigvee_{B\in\beta'} B$ where $\beta' \subseteq \beta$ and β is the base for the ditopology $(\mathscr{T}_v,\mathscr{K}_v)$. On the other hand, if $B \in \beta'$ there exists a finite subset $J_0 \subseteq J$ with $B = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j$ such that "$G_j \in \mathscr{T}_j^{v_j}$ for all $j \in J_0$". This follows,

$$\forall j \in J_0\ P_{v_j} \subseteq \mathscr{T}_j(G_j) \Rightarrow \forall j \in J_0\ \Pi_j^{V\leftarrow} P_{v_j} \subseteq \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)$$

$$\Rightarrow P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} P_{v_j} \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)$$

because $P_v \subseteq \Pi_j^{V\leftarrow}(\pi_j^{V\rightarrow} P_v) = \Pi_j^{V\leftarrow}(P_{v_j})$ for all $j \in J$ and $v \in V$ by Proposition 1.5 (3). Thus we have $P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)$ and $B = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j$ where $J_0 \subseteq J$ is a finite subset. So we get $P_v \subseteq \mathscr{T}(B)$ by the definition of $\mathscr{T}$. Using GT_3 we obtain that $\mathscr{T}(A) = \mathscr{T}(\bigvee_{B\in\beta'} B) \supseteq \bigcap_{B\in\beta'} \mathscr{T}(B) \supseteq P_v$ and so $A \in \mathscr{T}^v$.

If we take $A \in \mathscr{T}^v$ then $P_v \subseteq \mathscr{T}(A)$. Since $(V_j,\mathscr{V}_j)_{j\in J}$ are plain, $(V,\mathscr{V})$ is also plain by Proposition 1.10. Hence, considering the definition of $\mathscr{T}$ we have:

$$\exists J_0 \subseteq J \text{ finite with } A = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j\ :\ P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j).$$

Besides, considering Proposition 1.5 (3) we get

$$P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j) \Rightarrow \forall j \in J_0\ P_v \subseteq \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)$$

$$\Rightarrow \forall j \in J_0\ P_{\nu_j} = \pi_j^{V\rightarrow} P_\nu \subseteq \pi_j^{V\rightarrow}(\Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)) \subseteq \mathscr{T}_j(G_j).$$

Thus we obtain that

$$\exists J_0 \subseteq J \text{ finite with } A = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j \ :\ \text{“}\forall j \in J_0\ P_{\nu_j} \subseteq \mathscr{T}_j(G_j)\text{”}$$

$$\Rightarrow \exists J_0 \subseteq J \text{ finite with } A = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j \ :\ \text{“}\forall j \in J_0\ G_j \in \mathscr{T}_j^{\nu_j}\text{”} \Rightarrow A \in \mathscr{T}_\nu$$

Similarly it can be shown that $\mathscr{K}_\nu = \mathscr{K}^\nu$. That means $(S,\mathscr{S},\mathscr{T}^\nu,\mathscr{K}^\nu)$ is dicompact and so, $P_\nu \in \mathscr{DC}$. □

Note that in Theorem 3.16, the textures $(S_j,\mathscr{S}_j)_{j\in J}$ don't have to be plain unlike the textures $(V_j,\mathscr{V}_j)_{j\in J}$. It is an open problem whether Theorem 3.16 is still true in case $(V_j,\mathscr{V}_j)_{j\in J}$ are not plain.

4. Graded Difilters and dicompactness spectrum

Difilters on Textures: [14] Let $(S,\mathscr{S})$ be a texture. $\mathscr{F} \subseteq \mathscr{S}$ is called a filter on $(S,\mathscr{S})$ if (i) $\mathscr{F} \neq \emptyset$, (ii) $\emptyset \notin \mathscr{F}$, (iii) $F \in \mathscr{F},\ F \subseteq F' \in \mathscr{S} \Rightarrow F' \in \mathscr{F}$ and (iv) $F_1,F_2 \in \mathscr{F} \Rightarrow F_1 \cap F_2 \in \mathscr{F}$. $\mathscr{G} \subseteq \mathscr{S}$ is called a cofilter on $(S,\mathscr{S})$ if (i) $\mathscr{G} \neq \emptyset$, (ii) $S \notin \mathscr{G}$, (iii) $G \in \mathscr{G},\ G \supseteq G' \in \mathscr{S} \Rightarrow G' \in \mathscr{G}$, and (iv) $G_1,G_2 \in \mathscr{G} \Rightarrow G_1 \cup G_2 \in \mathscr{G}$. If $\mathscr{F}$ is a filter and $\mathscr{G}$ is a cofilter on $(S,\mathscr{S})$ then $\mathscr{F}\times\mathscr{G}$ is called a difilter on $(S,\mathscr{S})$. A difilter $\mathscr{F}\times\mathscr{G}$ on $(S,\mathscr{S})$ is called regular if $\mathscr{F}\cap\mathscr{G} = \emptyset$.

Proposition 4.1. [14] *If $\mathscr{F}\times\mathscr{G}$ is a difilter on $(S,\mathscr{S},\tau,\kappa)$ then*

(a) *$\mathscr{F}$ converges to $s \in S^\flat$ ($\mathscr{F} \to s$) $\Leftrightarrow$ “$G \in \tau,\ G \nsubseteq Q_s \Rightarrow G \in \mathscr{F}$”*
(b) *$\mathscr{G}$ converges to s ($\mathscr{G} \to s$) $\Leftrightarrow$ “$K \in \kappa,\ P_s \nsubseteq K \Rightarrow K \in \mathscr{G}$”*
(c) *$\mathscr{F}\times\mathscr{G}$ is diconvergent if $\mathscr{F} \to s$ and $\mathscr{G} \to s'$ for some $s,s' \in S$ with $P_{s'} \nsubseteq Q_s$.*

A difilter $\mathscr{F}\times\mathscr{G}$ on $(S,\mathscr{S},\tau,\kappa)$ is said to be diclustering if $A \in \mathscr{F} \Rightarrow P_{s'} \subseteq [A]$ and $B \in \mathscr{G} \Rightarrow]B[\subseteq Q_s$ for some $s,s' \in S$ with $P_{s'} \nsubseteq Q_s$.

Theorem 4.2. [14] *A regular difilter $\mathscr{F}\times\mathscr{G}$ on $(S,\mathscr{S})$ is maximal if and only if $\mathscr{F}\cup\mathscr{G} = S$.*

Proposition 4.3. [14] *A maximal regular difilter is diconvergent if and only if it is diclustering.*

Theorem 4.4. [14] *A ditopological texture space $(S,\mathscr{S},\tau,\kappa)$ is dicompact if and only if every regular difilter on $(S,\mathscr{S},\tau,\kappa)$ is diclustering if and only if every maximal regular difilter on $(S,\mathscr{S},\tau,\kappa)$ is diconvergent.*

Graded difilters : [9] Let $(S,\mathscr{S})$ and $(V,\mathscr{V})$ be textures. A mapping $\mathfrak{F} : \mathscr{S} \to \mathscr{V}$ is called a $(V,\mathscr{V})$-graded filter on $(S,\mathscr{S})$ if (i) $\mathfrak{F}(\emptyset) = \emptyset$, (ii) $A_1 \subseteq A_2 \Rightarrow \mathfrak{F}(A_1) \subseteq \mathfrak{F}(A_2)$ and (iii) $\mathfrak{F}(A_1)\wedge\mathfrak{F}(A_2) \subseteq \mathfrak{F}(A_1\cap A_2)$. A mapping $\mathfrak{G} : \mathscr{S} \to \mathscr{V}$ is called a $(V,\mathscr{V})$-graded cofilter on $(S,\mathscr{S})$ if (i) $\mathfrak{G}(S) = \emptyset$, (ii) $A_1 \subseteq A_2 \Rightarrow \mathfrak{G}(A_2) \subseteq \mathfrak{G}(A_1)$ and (iii) $\mathfrak{G}(A_1)\wedge\mathfrak{G}(A_2) \subseteq \mathfrak{G}(A_1\cup A_2)$. If $\mathfrak{F}$ is a $(V,\mathscr{V})$-graded filter and $\mathfrak{G}$ $(V,\mathscr{V})$-graded cofilter on $(S,\mathscr{S})$ then the pair $(\mathfrak{F},\mathfrak{G})$ is called a $(V,\mathscr{V})$-graded difilter on $(S,\mathscr{S})$.

$(\mathfrak{F},\mathfrak{G})$ is called regular if $\mathfrak{F}\wedge\mathfrak{G}=\emptyset$ i.e. $\mathfrak{F}(A)\wedge\mathfrak{G}(A)=\emptyset$ for all $A\in\mathscr{S}$.

The diconvergent graded difilters were defined in [9]. To avoid a long preliminaries we will give the following equivalent proposition instead of the definition.

Proposition 4.5. [9] *If $(\mathfrak{F},\mathfrak{G})$ is a $(V,\mathscr{V})$-graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ then*

(a) *$\mathfrak{F}$ converges to s ($\mathfrak{F}\to s$) $\Leftrightarrow$ "$A\not\subseteq Q_s \Rightarrow \mathscr{T}(A)\subseteq\mathfrak{F}(A)$"*
(b) *$\mathfrak{G}$ converges to s ($\mathfrak{G}\to s$) $\Leftrightarrow$ "$P_s\not\subseteq A \Rightarrow \mathscr{K}(A)\subseteq\mathfrak{G}(A)$"*
(c) *For $s,s'\in S$, $(\mathfrak{F},\mathfrak{G})$ is diconvergent if $P_{s'}\not\subseteq Q_s$, $\mathfrak{F}\to s$ and $\mathfrak{G}\to s'$.*

Let $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ be a graded ditopological texture space, $A\in\mathscr{S}$ and $v\in V$. The set $\bigcap\{B\in\mathscr{S}|A\subseteq B,\ P_v\subseteq\mathscr{K}(B)\}\in\mathscr{S}$ is called v-closure of A and denoted by $[A]^v$. The set $\bigvee\{B\in\mathscr{S}|B\subseteq A,\ P_v\subseteq\mathscr{T}(B)\}\in\mathscr{S}$ is called v-interior of A and denoted by $]A[^v$. Note that for each $v\in V$, $[A]^v$ ($]A[^v$) is the closure (the interior) of A in the ditopological texture space $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$.

A regular graded difilter $(\mathfrak{F},\mathfrak{G})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is called diclustering if for all $A\in\mathscr{S}$, $v\in\mathfrak{F}(A)\Rightarrow P_s\subseteq[A]^v$ and $v\in\mathfrak{G}(A)\Rightarrow]A[^v\subseteq Q_{s'}$ for some $s,s'\in S$ with $P_s\not\subseteq Q_{s'}$.

Proposition 4.6. [9] *Let $(\mathfrak{F},\mathfrak{G})$ be a regular $(V,\mathscr{V})$-graded difilter on $(S,\mathscr{S})$. For the statements*

(1) *$(\mathfrak{F},\mathfrak{G})$ is a maximal regular $(V,\mathscr{V})$-graded difilter*
(2) *$\mathfrak{F}\vee\mathfrak{G}=V$ (i.e. $\forall A\in\mathscr{S},\mathfrak{F}(A)\vee\mathfrak{G}(A)=\mathfrak{F}(A)\cup\mathfrak{G}(A)=V$)*

$(1)\Leftarrow(2)$ *and in case of $(V,\mathscr{V})$ is discrete,* $(1)\Rightarrow(2)$ *are hold.*

if $(\mathfrak{F},\mathfrak{G})$ be a (regular) $(V,\mathscr{V})$-graded difilter on a texture $(S,\mathscr{S})$ then the families

$$\mathfrak{F}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathfrak{F}(A)\},\ \ \mathfrak{G}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathfrak{G}(A)\}$$

form a (regular) difilter $\mathfrak{F}^v\times\mathfrak{G}^v$ on $(S,\mathscr{S})$ for each $v\in V$ [9].

Proposition 4.7. [9] *Let $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ be a graded ditopological texture space. Then, for the statements*

(a) *Every regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is diclustering.*
(b) *Every maximal regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is diconvergent.*

the implication $(b)\Rightarrow(a)$ and in case of $(V,\mathscr{V})$ is discrete, $(a)\Rightarrow(b)$ are hold.

Definition 4.8. *Let $(\mathfrak{F},\mathfrak{G})$ be a regular graded difilter on a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Then the family defined by*

$$\begin{aligned}Dcl(\mathfrak{F},\mathfrak{G})=\{P_v\mid \exists s,s'\in S \text{ with } P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}\\ [v\in\mathfrak{G}(A)\Rightarrow]A[^v\subseteq Q_{s'} \text{ and } v\in\mathfrak{F}(A)\Rightarrow P_s\subseteq[A]^v]\}\end{aligned}$$

is called diclustering spectrum of $(\mathfrak{F},\mathfrak{G})$.

Example 4.9. *Let $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ be a graded ditopological texture space and $v\in V$. If $\mathscr{F}\times\mathscr{G}$ is a a regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ then the mappings defined by*

$$\mathfrak{F}_{\mathscr{F}}(A)=\begin{cases} V, & A\in\mathscr{F}\\ \emptyset, & A\notin\mathscr{F}\end{cases}$$

$$\mathfrak{G}_{\mathscr{G}}(A)=\begin{cases} V, & A\in\mathscr{G}\\ \emptyset, & A\notin\mathscr{G}\end{cases}$$

for all $A\in\mathscr{S}$ form a regular graded difilter $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Moreover, $\mathfrak{F}^v_{\mathscr{F}}=\mathscr{F}$ and $\mathfrak{G}^v_{\mathscr{G}}=\mathscr{G}$.

Proposition 4.10. *For a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$, the following equation holds:*

$$\mathscr{DC}=\bigcap\{Dcl(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})\text{ is a regular graded difilter}\}$$

Proof. Let $P_v\in\bigcap\{Dcl(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a regular graded difilter$\}$ and take a regular difilter $\mathscr{F}\times\mathscr{G}$ on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$. Then $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ is a regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Since $P_v\in\bigcap\{Dcl(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a regular graded difilter$\}$ we have $P_v\in Dcl(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$. So we get

$$\exists s,s'\in S\text{ with }P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}\ [v\in\mathfrak{G}_{\mathscr{G}}(A)\Rightarrow]A[^v\subseteq Q_{s'}\text{ and }v\in\mathfrak{F}_{\mathscr{F}}(A)\Rightarrow P_s\subseteq[A]^v].$$

This follows that "$A\in\mathfrak{G}^v_{\mathscr{G}}\Rightarrow]A[^v\subseteq Q_{s'}$" and "$A\in\mathfrak{F}^v_{\mathscr{F}}\Rightarrow P_s\subseteq[A]^v$". Thus we have "$A\in\mathscr{G}\Rightarrow]A[\subseteq Q_{s'}$", "$A\in\mathscr{F}\Rightarrow P_s\subseteq[A]$" and this implies that $\mathscr{F}\times\mathscr{G}$ is diclustering. Since every regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is diclustering, $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is dicompact by Theorem 4.4 and that means $P_v\in\mathscr{DC}$.

On the other hand, let $P_v\in\mathscr{DC}$ and take a regular graded difilter $(\mathfrak{F},\mathfrak{G})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Then $(\mathfrak{F}^v\times\mathfrak{G}^v)$ is a regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$. Since $P_v\in\mathscr{DC}$, $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is dicompact and so, $(\mathfrak{F}^v\times\mathfrak{G}^v)$ is diclustering by Theorem 4.4. That means "$A\in\mathfrak{G}^v\Rightarrow]A[\subseteq Q_{s'}$", "$A\in\mathfrak{F}^v\Rightarrow P_s\subseteq[A]$" for some $s,s'\in S$ with $P_s\not\subseteq Q_{s'}$. Thus we get,

$$\exists s,s'\in S\text{ with }P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}\ [v\in\mathfrak{G}(A)\Rightarrow]A[^v\subseteq Q_{s'}\text{ and }v\in\mathfrak{F}(A)\Rightarrow P_s\subseteq[A]^v].$$

Hence we get $P_v\in\bigcap\{Dcl(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a regular graded difilter$\}$. □

Lemma 4.11. *Let $(V,\mathscr{V})$ be a discrete texture. If $(\mathfrak{F},\mathfrak{G})$ is a maximal regular graded difilter on a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ then the regular difilter $\mathfrak{F}^v\times\mathfrak{G}^v$ on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is maximal for all $v\in V$.*

Proof. Let $(\mathfrak{F},\mathfrak{G})$ be a maximal regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ and $v\in V$. Since $(V,\mathscr{V})$ is a discrete texture we have $\mathfrak{F}\vee\mathfrak{G}=V$ by Proposition 4.6. We also know that $\mathfrak{F}^v\times\mathfrak{G}^v$ is a regular difilter. Consider $\mathfrak{F}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathfrak{F}(A)\}$ and $\mathfrak{G}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathfrak{G}(A)\}$. Since $\mathfrak{F}\vee\mathfrak{G}=V$ and $(V,\mathscr{V})$ is discrete we get

$$A\in\mathscr{S}\Rightarrow P_v\subseteq\mathfrak{F}(A)\vee\mathfrak{G}(A)=\mathfrak{F}(A)\cup\mathfrak{G}(A)\Rightarrow P_v\subseteq\mathfrak{F}(A)\text{ or }P_v\subseteq\mathfrak{G}(A)$$
$$\Rightarrow A\in\mathfrak{F}^v\text{ or }A\in\mathfrak{G}^v\Rightarrow A\in\mathfrak{F}^v\cup\mathfrak{G}^v$$

for all $A\in\mathscr{S}$. That means $\mathfrak{F}^v\cup\mathfrak{G}^v=\mathscr{S}$ and so $\mathfrak{F}^v\times\mathfrak{G}^v$ is maximal by Theorem 4.2. □

Lemma 4.12. *Let $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ be a graded ditopological texture space and $v \in V$. If $\mathscr{F}\times\mathscr{G}$ is a maximal regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ then the regular graded difilter $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is maximal.*

Proof. Let $\mathscr{F}\times\mathscr{G}$ be a maximal regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$. Then we have $\mathscr{F}\cup\mathscr{G}=\mathscr{S}$. So, for all $A\in\mathscr{S}$ we get $A\in\mathscr{F}$ or $A\in\mathscr{G}$. This follows $\mathfrak{F}_{\mathscr{F}}(A)=V$ or $\mathfrak{G}_{\mathscr{G}}(A)=V$. That means $\mathfrak{F}_{\mathscr{F}}\vee\mathfrak{G}_{\mathscr{G}}=V$. Hence $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ is maximal by Proposition 4.6. □

Definition 4.13. *Let $(\mathfrak{F},\mathfrak{G})$ be a graded difilter on a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. The family defined by*

$$Dcn(\mathfrak{F},\mathfrak{G})=\{P_v \mid \exists s,s'\in S \text{ with } P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}$$
$$[(A\in\mathscr{T}^v,\ A\not\subseteq Q_s)\Rightarrow A\in\mathfrak{F}^v \text{ and } (A\in\mathscr{K}^v,\ P_{s'}\not\subseteq A)\Rightarrow A\in\mathfrak{G}^v]\}$$

is called diconvergence spectrum of $(\mathfrak{F},\mathfrak{G})$.

Proposition 4.14. *Let $(V,\mathscr{V})$ be a discrete texture. For a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$, the following equation holds:*

$$\mathscr{DC}=\bigcap\{Dcn(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G}) \text{ is a maximal regular graded difilter}\}$$

Proof. Let $P_v\in\bigcap\{Dcn(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a maximal regular graded difilter$\}$ and $\mathscr{F}\times\mathscr{G}$ be a maximal regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$. Then $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ is a maximal regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ by Lemma 4.12.

Since $P_v\in\bigcap\{Dcn(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a maximal regular graded difilter$\}$ we have $P_v\in Dcn(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$. This follows

$$\exists s,s'\in S \text{ with } P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}$$
$$[(A\in\mathscr{T}^v,\ A\not\subseteq Q_s)\Rightarrow A\in\mathfrak{F}^v_{\mathscr{F}} \text{ and } (A\in\mathscr{K}^v,\ P_{s'}\not\subseteq A)\Rightarrow A\in\mathfrak{G}^v_{\mathscr{G}}].$$

Therefore we get "$(A\in\mathscr{T}^v,\ A\not\subseteq Q_s)\Rightarrow A\in\mathscr{F}$" and "$(A\in\mathscr{K}^v,\ P_{s'}\not\subseteq A)\Rightarrow A\in\mathscr{G}$". Considering Proposition 4.1, $\mathscr{F}\times\mathscr{G}$ is diconvergent and so, $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is dicompact by Theorem 4.4. Hence we get $P_v\in\mathscr{DC}$.

On the other hand, let $P_v\in\mathscr{DC}$ and take a maximal regular graded difilter $(\mathfrak{F},\mathfrak{G})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Then $\mathfrak{F}^v\times\mathfrak{G}^v$ is a maximal regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ by Lemma 4.11. Besides, $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is dicompact since $P_v\in\mathscr{DC}$. So, $\mathfrak{F}^v\times\mathfrak{G}^v$ is diconvergent by Theorem 4.4. Thus we have

$$\exists s,s'\in S \text{ with } P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}$$
$$[(A\in\mathscr{T}^v,\ A\not\subseteq Q_s)\Rightarrow A\in\mathfrak{F}^v \text{ and } (A\in\mathscr{K}^v,\ P_{s'}\not\subseteq A)\Rightarrow A\in\mathfrak{G}^v].$$

Hence we get $P_v\in Dcn(\mathfrak{F},\mathfrak{G})$ i.e.,

$$P_v\in\bigcap\{Dcn(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G}) \text{ is a maximal regular graded difilter}\}.$$

□

Corollary 4.15. *Let $(V,\mathscr{V})$ be a discrete texture. If $(\mathfrak{F},\mathfrak{G})$ is a maximal regular graded difilter on a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Then*

$$Dcn(\mathfrak{F},\mathfrak{G})=Dcl(\mathfrak{F},\mathfrak{G})$$

Proof. Considering Lemma 4.11 and Proposition 4.3 we have

$$\begin{gathered}
P_v \in Dcn(\mathfrak{F},\mathfrak{G}) \Leftrightarrow \text{“}\exists s,s' \in S \text{ with } P_s \not\subseteq Q_{s'} \ : \ \forall A \in \mathscr{S} \\
[(A \in \mathscr{T}^v,\ A \not\subseteq Q_s) \Rightarrow A \in \mathfrak{F}^v \text{ and } (A \in \mathscr{K}^v,\ P_{s'} \not\subseteq A) \Rightarrow A \in \mathfrak{G}^v]\text{”} \\
\Leftrightarrow \text{“}\mathfrak{F}^v \to s,\ \mathfrak{G}^v \to s' \text{ and } P_s \not\subseteq Q_{s'}\text{”} \\
\Leftrightarrow \text{“}\exists s,s' \in S \text{ with } P_s \not\subseteq Q_{s'} \ : \ \forall A \in \mathscr{S}\ [A \in \mathfrak{G}^v \Rightarrow]A[^v \subseteq Q_{s'} \text{ and } A \in \mathfrak{F}^v \Rightarrow P_s \subseteq [A]^v]\text{”} \\
\Leftrightarrow P_v \in Dcl(\mathfrak{F},\mathfrak{G}).
\end{gathered}$$

□

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A SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR LINEAR HYPERBOLIC PDE'S WITH HIGH FREQUENCIES

ŞUAYIP TOPRAKSEVEN

Abstract. The main purpose of this paper is to describe a space-time discontinuous Galerkin (DG) method based on an extended space-time approximation space for the linear first order hyperbolic equation that contains a high frequency component. We extend the space-time DG spaces of tensor-product of polynomials by adding trigonometric functions in space and time that capture the oscillatory behavior of the solution. We construct the method by combining the basic framework of the space-time DG method with the extended finite element method. The basic principle of the method is integrating the features of the partial differential equation with the standard space-time spaces in the approximation. We present error analysis of the proposed space-time DG method for the linear first order hyperbolic problems. We show that the new space-time DG approximation has an improvement in the convergence compared to the space-time DG schemes with tensor-product polynomials. Numerical examples verify the theoretical findings and demonstrate the effects of the proposed method.

1. Introduction

In computational acoustics, the medium frequency regime and multiscale wave propagation governed by the wave equation have been gained a constant interest in last decades. When multiscale wave propagation presents a high frequency component, developing an efficient numerical methods for these classes of problem is a challenging task. Some example of high frequency problems include the high-intensity focused ultrasound (HIFU) treatment of cancer [1], coupled atomistic continuum modeling in nanomaterials [2] and tunneling in quantum mechanics [3]. The reason for inefficiency of the existing methods is that the standard numerical methods such as the finite element (FEM) or discontinuous Galerkin (DG) methods based on semi-discrete approach require a very fine mesh in the discretization

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in both space and time, and this leads to huge computational cost and makes the numerical methods inefficient. Moreover, these methods based on semi-discrete approach may not suitable for multiscale approximations in the temporal domain. These issues on the standard numerical techniques have lead to high order methods that solve wave propagation phenomena in the time domain. One promising approach that has gained considerable popularity is space-time approximation spaces in which the time domain is also discretized. In these methods, two approaches have been proposed during the last decades. The first approach is called the time continuous space-time Galerkin methods (TCG) that do not require continuity in time. This approach generalizes the semi-discrete discretization to time domain with continuous time functions. The detailed explanations of such methods are given in [4]. The drawback of these methods is high computational cost because of discretization of whole domain. The second approach is based on space-time discontinuous Galerkin methods that use standard polynomials spaces to discretize the problem in space and time, while temporal domains are divided into time slab and discontinuities and jumps are allowed in time. In each slab, TCG method is applied and the next slab uses the information from the previous slab. This second approach is more robust and efficient than the first one. The wave equation can be discretized by a space-time setting in two ways. One way is to discretize the wave equation directly in a one field formulation with only one unknown as [5] and [6] . The second way is to convert the second order equation to a system of first order equations as done in [7] and [8]. Using this second formulation, a *priori* and a *posteriori* error estimates have been proved in [9] using linear interpolation. This approach clearly increases the unknowns in the resulting systems. Error estimates to prove convergence of the methods have been derived by French [6] and Hughes and Hulbert [5]. In the latter work, Galerkin least-squares stabilization terms are added for convergence analysis. In [6], the weighted inner product is included for the stability. A space-time DG method in which discontinuities and jumps are allowed both in space and time have been developed in [10] and recently proposed in [11] and [12] with discontinuous Petrov-Galerkin method in temporal domain for linear hyperbolic systems. Furthermore, many applications require boundary movement such as Stefan problems and water waves. In such problems, the mesh points also move in order to capture boundary movement. These movements in mesh points make the numerical scheme in efficient or need more complicated numerical discretization. In this case, it is natural to consider the space-time discontinuous Galerkin approach. Analysis and survey of space-time DG method for hyperbolic and parabolic conservation laws on time dependent domains are explained in details in [13]. Recently, space-time methods have become popular for the time dependent problems discussed in [14] and [15]. An application of this method to the compressible Vaiver-Stokes equations is discussed in [17]. Space-time DG method for the advection-diffusion equation has been given in details in [18] and [19]. This method also has been successively applied to nonconservative hyperbolic PDEs as models for dispersed multiphase flows in [20]. Furthermore, space-time DG methods have been proposed for the nonlinear water waves in [21] as well.

The medium or high frequency in wave propagation has been dealt with high order numerical methods including the ultra-weak variational method [22] that is a special case of the Trefftz-DG formulation for the wave equation [23] and the discontinuous enrichment method [24]. In these methods, the approximation space is enriched by the solution of the equation under consideration. In [24], the DG space is extended by solutions of the homogeneous differential equation that capture the high frequency in the solutions. In the same direction, an enriched space-time FEM for the first-order hyperbolic systems with discontinuities in both space and time has been studied by Chessa and Belytschko [25]. This enriched space-time approach is based on the extended FEM studied in [26]. These methods are based on the partition of unity approach developed in [27]. Motivated by these approaches, in this paper we propose a high-order accurate space-time DG method that is well-suited for first order linear hyperbolic problem with high frequency components. We construct the extended space-time DG space by enriching space-time DG space with the trigonometric functions in space and time. These trigonometric-function spaces intuitively capture the high frequency solutions and should be used to the highly oscillatory problems. This extended space time DG method is an extension to an extended DG method presented in [28]. We will show global convergence in error estimates. Our error analysis based on the DG method proposed by [29].

The outline of this paper is as follows: Section 2 describes the mathematical analysis, formulations, and an introduction to a space-time DG method for scalar hyperbolic linear equation with high-frequency components. The basic properties of the proposed space-time DG method, the geometry of the space-time domain and elements and the space-time formulation of the problem have been explained and discussed and general solution form for linear hyperbolic equation with high frequency components is also given in Section 2. In Section 3, we introduce preliminaries and notations and recall some basic facts on DG methods for linear hyperbolic equations. We present our extended DG method for linear hyperbolic equation with high frequency components and our special interpolation operators have been given in Section 4. Stability and error analysis are given in Section 5. In Section 6 numerical example is given to show that our theoretical results agree with numerical results. Finally we explain some conclusion and future direction in Section 7.

2. Space-Time Formulation with Trigonometric Functions

The basic principle of an extended DG method is to enrich the DG space by special functions that are, generally, the solution of homogeneous differential equation. The linear hyperbolic equation has the solution of the form $h(x \pm t)$ in one dimension. In Section 2.2, we show that if the initial condition has a high frequency component, then the homogeneous differential equation will also have a high frequency functions. This observation suggests that the enrichment shape functions consist of the polynomials and the trigonometric functions in the space $E := span\{\sin(x \pm t), \cos(x \pm t)\}$. A similar idea has been proposed for the wave equation in [30] and Trefftz DG method in [23].

2.1. **Problem Statement.** In this paper, we consider a scalar hyperbolic equation in an open domain Ω with boundary $\partial\Omega$

$$\text{Find } U = U(x,t) \text{ so that}$$
$$\begin{cases} \mathcal{L}U(x,t) + c(x,t)U(x,t) = g(x,t) \quad \text{on } Q_T, \quad 0 < c_0 \le c \le c_1, \\ U(x,0) = f(x). \end{cases} \tag{1}$$

Here, $\mathcal{L}U := \frac{\partial U}{\partial t} + \gamma \frac{\partial U}{\partial x}$, $Q_T = \Omega \times (0,T]$ and c_0 and c_1 are constants with $\gamma \in (0,1]$ and U denotes a scalar quantity, and t represents time with T the final time. This problem has been chosen purely for its simplicity. This analysis can be easily extended to more general hyperbolic and scalar conservation law problems.

We propose a space-time DG method based on extended DG approximation space for the equation (1). In this method, we directly consider the domain $Q_T \subset \mathbb{R}^2$ in which spatial and temporal variables are not distinguished and a point $\hat{x} \in Q_T$ has coordinates (x_0, x_1) with $x_0 = t$ representing a time variable and $x = x_1$ space variable. Thus, we define the space-time domain as the open domain $Q_T \subset R^2$. For space-time discretization, we need space-time slabs and elements. To do this, we partition the time interval $I = (0,T]$ into an ordered time levels $0 = t_0 < t_1 < \cdots < t_N = T$. Let $I_n = (t_n, t_{n+1})$ so that $I = \cup_n I_n$ with the time length $\Delta t = t_{n+1} - t_n$. Let $\Omega(t_n)$ denote the space-time domain at the time level $t = t_n$. Then, we define space-time slabs as $Q_T^n = Q_T \cap I_n$. We divide further $\Omega(t_n)$ into non-overlapping spatial elements K^n and similarly we divide the spatial domain $\Omega(t_{n+1})$ into elements K^{n+1}. We then connect the elements K^n and K^{n+1} to obtain space-time element $\mathcal{K}^n$ by using linear interpolation in time. We also describe the tessellation of the space-time slab $T_h^n = \cup_n \mathcal{K}^n$ and all space-time elements $T_h = \cup_n T_h^n$ in Q_T. By $\partial\mathcal{K}$ we denote the boundary of the space-time element $\mathcal{K}$. These space-time elements can be mapped to reference element (square or rectangle) by a suitable map, e.g., see [13] for construction of such a map. Figure 1 show a sketch of the space-time slab in Q_T.

In this paper, we require c and g are slowly varying smooth functions with bounded derivatives of many orders while f has the high frequency components. For instance, if $f(z) = cos(\omega z)$, then the solution has the form of:

$$U(x,t) = S(x,t) + R(x,t)\cos\big(\omega(x-t)\big), \tag{2}$$

where the frequency ω is a large number in absolute value. We further assume the functions $S(x,t)$ and $R(x,t)$ are slowly-varying functions of x and t in the sense that they have many derivatives all of which have norms that are moderately sized in space.

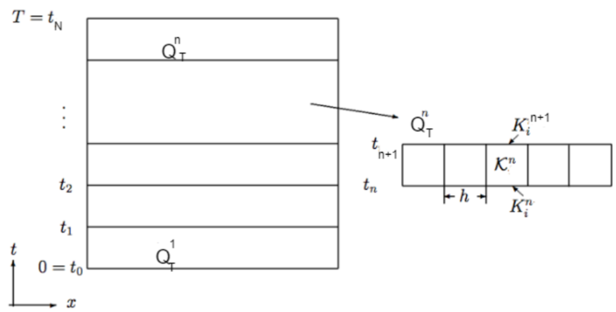

FIGURE 1. Space-time slab in space-time domain Q_T. On the right, the rectangular mesh is an example of structured discretizations in space and time.

We can assume that the forcing term $g(x,t)$ has also frequency components and we can show a similar solution form to (2). However, in this case extending the DG space with trigonometric functions is not easy task since we should extend the approximation space in all characteristic lines. For example, if we let $g(x,t) = \sin(\beta x) + \cos(\eta t)$ with $\beta, \eta >> 1$, then the solution form looks like $U(x,t) = S_1(x,t)\sin(\beta(x-t)) + S_2(x,t)\cos(\eta t) + R(x,t)\cos(\omega(x-t))$ so that S_1, S_2 and R do not have high frequency component. Therefore, enriching the space-time DG space is not easy job in this simple example. As an application of this phenomena, we consider high-intensity focused ultrasound (HIFU) treatment of cancer that uses sound wave. Tumors in body tissues are destroyed when HIFU is focused onto them. The initial condition in partial differential equation will generally help to determine high frequency shape to destroy tumor. In Figure 2, high frequency components in the initial condition determine acoustic pressure (high frequency shape) that heats and destroys the tumor.

We define an interpolation based on the assumption (2) in the error analysis of the proposed method. Hence, we prove this assumption in the next subsection.

2.2. **General Form of the Solutions.** In this section, we give the explicit solution form of the following problem:

$$\begin{cases} \text{Find } U = U(x,t) \text{ on } \mathbb{R} \times [0,T] \text{ so} \\ \quad \partial U/\partial t + \partial U/\partial x + cU = g \qquad 0 < c_0 \le c(x,t) \le c_1, \\ \text{with } U(x,0) = f(x). \end{cases} \tag{3}$$

The variable change to characteristic lines helps transform the PDE to an infinite set of ODE's. Let $x = t + x_0$ where $x_0 \in \mathbf{R}$ and define

$$\tilde{U}(t) = U(t + x_0, t).$$

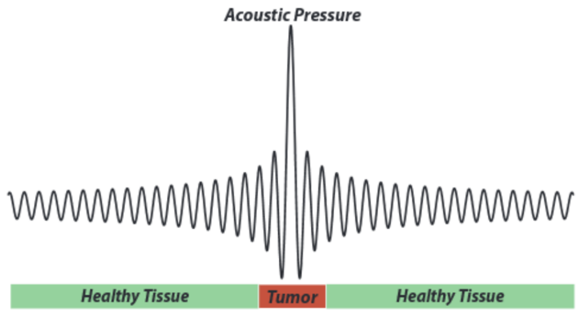

FIGURE 2. High frequency sound waves are concentrated on body tissues and tumor heats up and dies.

Then, we find that

$$\tilde{U}' = U_t + U_x = -cU + g,$$

or, with $\tilde{C}(t) = c(t + x_0, t)$ and $\tilde{G}(t) = g(t + x_0, t)$ we have

$$\tilde{U}' + \tilde{C}\tilde{U} = \tilde{G} \quad \text{and} \quad \tilde{U}(0) = f(x_0).$$

We multiply this equation by an integrating factor $\tilde{\mu}$ and find

$$\tilde{\mu}(t) = \exp\left(\int_0^t \tilde{C}(s)\ ds\right) \quad \Rightarrow \quad \left(\tilde{\mu}(t)\tilde{U}(t)\right)' = \tilde{\mu}(t)\tilde{G}(t).$$

This first order linear BVP (in $\tilde{U}$) can now be solved and we find that

$$\tilde{U}(t) = \frac{1}{\tilde{\mu}(t)}\left(f(x_0) + \int_0^t \tilde{\mu}(s)\tilde{G}(s)\ ds\right).$$

So, now we unwind the variable change to produce a solution for U. Note that $x_0 = x - t$ and, thus, letting

$$\mathcal{I}(x,t) = 1/\tilde{\mu}(t) = \exp\left(-\int_0^t c(s + (x-t), s)\ ds\right),$$

we have

$$U(x,t) = f(x-t)\mathcal{I}(x,t) + \mathcal{I}(x,t)\int_0^t \mathcal{I}(x,s)g(s + (x-t), s)\ ds.$$

If we now assume that g and c are slowly varying smooth functions with bounded derivatives of many orders while f has the high frequency components; that is, say,

$$f(z) = \cos(\omega z) \quad \text{for } \omega >> 1,$$

then, we have

$$U(x,t) = S(x,t) + R(x,t)\cos(\omega(x-t)).$$

This proves the assumption (2).

3. Preliminaries and Notations

The Sobolev space, $W^{m,p}(K)$, for a domain K, consists of functions with m derivatives in the $L^p(K)$ norm. We will use the following notation for Sobolev space semi-norms and norms for $1 \le p < \infty$

$$|v|_{m,p,K} = \left(\sum_{|\alpha|=m} \|D^\alpha v\|^p_{L^p(K)} \right)^{1/p} \quad \text{and} \quad \|v\|_{m,p,K} = \left(\sum_{|\alpha|\le m} \|D^\alpha v\|^p_{L^p(K)} \right)^{1/p}, \tag{4}$$

and when $p = \infty$

$$\|v\|_{m,\infty,K} = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^\infty(K)}, \tag{5}$$

where $D^\alpha v = \dfrac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ with $|\alpha| = \sum_{k=1}^n \alpha_k$ for $\alpha_k \ge 0, \quad k = 1, \dots, n$, and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and the standard Lebesgue space $L^p(K)$ norms

$$\|v\|_{L^p(K)} = \Big(\int_K |v|^p \, dx \Big)^{1/p} \quad \text{for} \quad 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(K)} = ess\, sup_{x\in K} |v(x)|.$$

For simplicity, we occasionally denote $\|.\|_{0,2,K}$ by $\|.\|$.

We will primarily be working with the Hilbert space $H^m(K) = W^{m,2}(K)$.

Let $P_r(K)$ be the space of polynomials with degree $\le q$ in K. We also assume there is an interpolation operator [[31], Theorem 4.4.4]

$$\pi_h : W^{q+1,p}(K) \to P_r(K)$$

for which

$$\|(I - \pi_h)v\|_{\ell,p,K} \le Ch^{r-\ell} \|v\|_{r,p,K}, \qquad (0 \le \ell \le r \le q+1). \tag{6}$$

and $\pi_h \xi = \xi$ for $\xi \in P_r(K)$. Moreover, if $v \in C(\bar\Omega)$ then $\pi_h v$ is continuous on $\bar\Omega$ as well.

The inverse inequality [[31], Theorem 4.5.11] for functions $\chi \in P_r(K)$ states that there exists $C > 0$, which is independent of h, so that

$$\|\chi\|_{\ell,p,K} \le Ch^{m-\ell+1/p-1/r} \|\chi\|_{m,r,K}, \quad m \le \ell, \quad 1 \le p \le \infty, \quad \text{and} \quad 1 \le r \le \infty. \tag{7}$$

The *arithmetic-geometric mean* inequality states that for scalars a and b,

$$|ab| \le \delta a^2 + C_\delta b^2, \tag{8}$$

where $C_\delta = 1/(4\delta)$ and $\delta > 0$.

To be able to easily present our results and compare with previous works, we follow the paper by C. Johnson and J. Pitkaranta [29]. Given a piecewise smooth function v write $v^n(.) = v^-(., nh)$ and the approximate solution u is computed successively on the strips $S_n = \{x \in \Omega : (n-l)h < t < nh\}, n = l, ..., N$ so that $\|u^n - U^n\|$ is the error on each time level $t = nh$.

Let $n_K = (n_x^K, n_t^K)$ represent the outward pointing unit normal vector on ∂K with space coordinate n_x^K and the time coordinate n_t^K. Let $\beta := (1, \gamma)$ and $\partial Q_T := \Gamma$. The inflow boundary is defined

$$\Gamma\_ := \{\hat{x} \in \Gamma : n_K \cdot \beta < 0\} = \{(x,t) : x = 0 \quad \text{or} \quad t = 0\}.$$

In an element K, its inflow boundary $\partial K\_$ and its outflow $\partial K_+ = \partial K \backslash \partial K\_$ is similarly defined by

$$\partial K\_ = \{x \in \partial K : n_K \cdot \beta < 0\},$$
$$\partial K_+ = \{x \in \partial K : n_K \cdot \beta > 0\}.$$

Space-time DG space is then defined as

$$V_h := \{v \in L_2(Q_T) : v|_K \in P_r(K), \quad \forall K \in T_h\}. \tag{9}$$

where $P_r(K)$ denotes the space of polynomials of maximum degree at most r in (x,t). Functions in V_h are allowed to be discontinuous at discrete time level. For $K \in T_h$, and a piecewise smooth function v, we define the jump operator by

$$[u](x) = \lim_{s \to 0^+} \big(u(x+s) - u(x-s)\big)$$

when $x \in \partial K\_ \subset \mathcal{E}$, interior faces, and $[u](x) := u(x)$ when $x \in \Gamma\_ \cap \partial K\_$. The jump of v across $\partial K\_ \backslash \Gamma$ defined similarly by

$$[v]_K := v_K^+ - v_K^-,$$

where v_K^+ the trace of v on ∂K taken from within the element K and v_K^- is the exterior trace of u. Note that the sign of the jump depends on the direction of the flow. The average of a function u is defined by

$$\{u\} = \frac{1}{2}(u|_{K_1} + u|_{K_2}) \qquad \text{on} \quad \partial K_1 \cap \partial K_2.$$

We define the equivalent space-time DG method for (1) by summing over $K \in T_h$: Find $u \in V_h$ so that

$$a(u,v) = \ell(v), \qquad \forall v \in V_h, \tag{10}$$

where

$$a(u,v) = \sum_K \big((\mathcal{L}u + cu, v)_K - \langle [u], v^+ \rangle_{\partial K\_}\big), \tag{11}$$

and

$$\ell(v) = (g, v), \qquad \langle u, v \rangle_{\partial K} = \int_{\partial K} |n_K \cdot \beta| uv \, ds. \tag{12}$$

where $\beta = (1, \gamma)$ and n_K is the outward unit normal to ∂K.

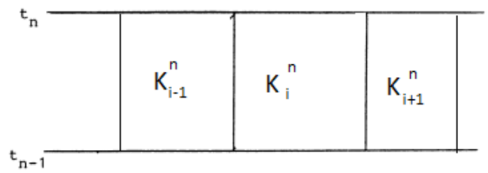

FIGURE 3. The order of the space-time elements on which u is computed.

Note that, for differentiable functions u and v, we have the following integration by parts formula

$$(\mathcal{L}u, v)_K = -(u, \mathcal{L}v)_K + \langle u, v\rangle_{\partial K_+} - \langle u, v\rangle_{\partial K_-}.$$

Equivalently, using this formula we may write this as

$$a(u, v) = \sum_K (-(u, \mathcal{L}v + cv)_K - \langle u^-, [v]\rangle_{\Gamma_h} + \langle u, v\rangle_{\Gamma_+}, \tag{13}$$

where $\Gamma_h := \left(\bigcup_K \partial K\right)\backslash\Gamma$.

Now, we have the Galerkin orthogonality relation by replacing u by the exact solution U in (11)

$$a(u - U, v) = 0, \quad \forall v \in V_h. \tag{14}$$

Let us recall the DG method for (1). Given a finite element partitioning $\mathcal{J}_h := \{K\}$ of Q_T, we look for a solution u defined on Q_T such that for all $K \in \mathcal{J}_h$ and $u|_K \in P_r(K)$ so that

$$\int_K (\mathcal{L}u + cu)v dx + \int_{\partial K_-} M_K[u]_K v^+ \, ds = (g, v)_K \quad \forall v \in P_r(K), \tag{15}$$

where $M_K := |n_K.\beta|$.

As shown in [29], u is uniquely determined by (15) and it is possible to compute u successively on each K starting at the inflow boundary Γ_- where u is given. Then it is possible to find the numerical solution u successively on one time level after another computing space-time element by element by starting for each strip on the left. The order of elements on which u will be computed is shown in Figure 3. Thus, given g and u^- on inflow boundary, we can solve u locally in each K as shown below. For detailed proof, we refer the reader to [29].

Below, we denote by C , a positive constant which may take different values on different occurrences.

Lemma 1. [29] *Assume that $g \in L_2(Q_T)$ and $f \in L_2(\Gamma_-)$ are given in (1). Then u is determined by (15) and the following local stability holds for each K*

$$\|u\|_K + h^{\frac{1}{2}}\|u^+\|_{\partial K_-} + h^{\frac{1}{2}}\|u^-\|_{\partial K_+} \leq C\{h\|g\|_K + h^{\frac{1}{2}}\|M_K u^-\|_{\partial K_-}\}.$$

Let us introduce a norm $|||.|||_h$ for the error analysis :

$$|||u|||_h^2 = \sum_K h||\mathcal{L}u||_K^2 + |u|_h^2.$$

$$|u|_h^2 = ||u||^2 + \frac{1}{2}\big\langle [u], [u] \big\rangle_{\partial K_-} + \frac{1}{2}\big\langle u^-, u^- \big\rangle_{\Gamma_+},$$

$$|||u|||_h^2 = \sum_K h||\mathcal{L}u||_K^2 + |u|_h^2.$$

Then, we have the following a generalization of the *Poincare-Friedrichs* inequality [31]:

$$\forall w \in H^1(K), \quad \|w\|_{0,2,Q_T} \leq C|||w|||_h. \tag{16}$$

4. Space-time Discontinuous Galerkin Discretization

In this section, we discuss the space-time DG method based on the extended space-time approximation spaces and that combines the framework of space-time DG with XFEM for the linear hyperbolic equation. We enrich the space-time DG space by adding a range of Fourier-series components to handle the high-frequency terms in the exact solution. As shown in Section 2.1, if the initial condition has only one high frequency component, then the solution form given by (2). In the same direction, if we assume that the initial condition has L high frequency components, that is, $f(z) = \sum_{\ell=1}^{L} \cos(\omega_\ell z)$ with $\omega_\ell >> 1$, then we wil have the solution form of

$$U(x,t) = S(x,t) + \sum_{\ell=1}^{L} R(x,t)\cos(\omega_\ell(x-t)). \tag{17}$$

Thus, our "enriched space" $X_h^r(Q_T)$ consists of the functions of the form

$$X_h^r(Q_T) = \{\psi = s + \sum_{\ell=1}^{L} (a_\ell \mathrm{cos}(n_\ell(x-t)) + b_\ell \mathrm{sin}(n_\ell(x-t)))\}, \tag{18}$$

where $n_1, \ldots, n_L$ are integers and s as well as the a_i's and the b_i's are all elements of the space-time DG space V_h (9). More precisely, these s, a_ℓ and b_ℓ functions are tensor-product of piecewise discontinuous polynomials of degree at most r in x and t variables. Note that these functions are allowed to be discontinuous at the nodal points both in space and time and continuous in each element. This enriched space provides good approximations to the solutions of (1) if the range of high frequencies are known *a priori*.

For a high-frequency component of (17), we have, by using a simple trigonometric identities,

$$\begin{aligned}R(x,t)\cos(\omega_\ell y) &= R(x,t)\cos(n_\ell y + (\omega_\ell - n_\ell)y)\\ &= R(x,t)\,\cos((\omega_\ell - n_\ell)y)\,\cos(n_\ell y) - R(x,t)\,\sin((\omega_\ell - n_\ell)y)\,\sin(n_\ell y),\end{aligned}$$

where $y := x - t$ and n_ℓ is an integer and can be chosen between n_0 and n_L with $0 \le \omega_\ell - n_\ell \le 1$. The key idea is that the functions

$$\alpha_\ell(x,t) = R(x,t)\,\cos((\omega_\ell - n_\ell)(x-t)),$$

and

$$\beta_\ell(x,t) = -R(x,t)\,\sin((\omega_\ell - n_\ell)(x-t))$$

oscillate slowly since their frequencies are small and can be well approximated by functions in V_h.

We now directly approximate the form (2) using interpolation. Let

$$U_A(x,t) = \pi_h S(x,t) + \sum_{\ell=n_0}^{n_L} \left[(\pi_h\alpha_\ell)(x,t)\cos(n_\ell(x-t)) + (\pi_h\beta_\ell)(x,t)\sin(n_\ell(x-t))\right]. \tag{19}$$

Note that $U_A \in X_h^r(Q_T)$ and we have

$$\begin{aligned}U(x,t) - U_A(x,t) = (I-\pi_h)S(x,t) + \sum_{\ell=n_1}^{n_\ell} \Big[&(I-\pi_h)\alpha_\ell(x-t)\cos(n_\ell(x-t))\\ &+ (I-\pi_h)\beta_\ell(x-t)\sin(n_\ell(x-t))\Big].\end{aligned}$$

So, using (6), there is constant C, independent of h and ω_ℓ, we have

$$\|U - U_A\|_{m,p,K} \le Ch^{r+1-m}\omega_L^m, \qquad 0 \le m \le p \le r+1. \tag{20}$$

Also, since U is continuous, it follows that U_A is continuous as well.

We remark that if $U \in P_r(Q_T)$, that is U is a polynomials, then our special interpolation agrees with the interpolation operator π_h so that in this case we have

$$U_A = \pi_h U. \tag{21}$$

Furthermore, we have, by the trace inequality,

$$\|U - U_A\|_{0,2,\partial K} \le Ch,^{r+\frac{1}{2}} \qquad 0 \le m \le 2 \le r+1. \tag{22}$$

To construct our space-time approximation solution on the extended space, we perform the space-time discretization of the linear hyperbolic equation (1). Thus, we define the space-time DG scheme : Find $u \in X_h^r$ so that

$$a(u,v) = \ell(v) \quad \forall v \in X_h^r. \tag{23}$$

First, we show that the discrete problem (23) is stable from which the existence and uniqueness of the problem follows. Then we prove the bilinear form $a(.,.)$ is coercive and continuous. For short reference, we take $M_K = M$.

Lemma 2. *The solution u to the problem (23) satisfies the following stability estimates*

$$|||u|||_h^2 \leq C\big(\|g\|^2 + \|Mf\|_{\Gamma_-}^2\big). \tag{24}$$

Proof. Taking $v = u + \delta\mathcal{L}u$ when $\delta = Ch$ for some constant C in (23) we have, for each $K \in T_h$

$$a(u, u + \delta\mathcal{L}u) = (\mathcal{L}u + cu, u + \delta\mathcal{L}u) - \int_{\partial K_-} [u]u^+ M\, ds = (g, u + \delta\mathcal{L}u)_K. \tag{25}$$

Now, using Green's formula we have

$$2(\mathcal{L}u, u) = \int_{\partial K_+} (u^-)^2 M\, ds - \int_{\partial K_-} (u^+)^2 |M|\, ds,$$

and since $c \geq c_0 > 0$, we get

$$2a_K(u, u + \delta\mathcal{L}u) \geq 2\delta(\mathcal{L}u, \mathcal{L}u)_K + 2c_0(u, u)_K + (1 + \delta c_0)\big(\int_{\partial K_+} (u^-)^2 M\, ds - \int_{\partial K_-} (u^+)^2 |M|\, ds\big) + 2\int_{\partial K_-} [u]u^+ |M|\, ds.$$

Since every side of interior element boundary ∂K_+ agrees with a side of $\partial K'_-$ for an neighbour element K', we have

$$\sum_K \int_{\partial K_+} (u^-)^2 M\, ds = \sum_K \int_{\partial K_-} (u^-)^2 |M|\, ds + \int_{\Gamma_+} (u^-)^2 M\, ds + \int_{\Gamma_-} (u^-)^2 M\, ds, \tag{26}$$

and consequently if we take $\delta = Ch$ for some constant C with $h \leq 1$ and using the fact that $1 + Ch \geq 1$

$$2a(u, u + Ch\mathcal{L}u) \geq 2Ch \sum_K (\mathcal{L}u, \mathcal{L}u)_K + 2c_0 \sum_K (u, u)_K + \sum_K \int_{\partial K_-} \Big(\big((u^+)^2 - 2u^+u^- + (u^-)^2\big)|M|\, ds\Big) + \int_{\Gamma_+} (u^-)^2 M\, ds - \int_{\Gamma_-} (u^-)^2 |M|\, ds.$$

Thus we obtain

$$a(u, u + Ch\mathcal{L}u) \geq Ch \sum_K \|\mathcal{L}u\|^2 + c_0\|u\|^2 + \frac{1}{2}\sum_K \int_{\partial K_-} [u]^2 |M|\, ds + \frac{1}{2}\int_{\Gamma_+} (u^-)^2 M\, ds - \frac{1}{2}\int_{\Gamma_-} (u^-)^2 |M|\, ds. \tag{27}$$

Now we estimate the right-hand side of (25). Applying the Cauchy-Schwarz and the arithmetic-geometric inequalities we get

$$(g, u + Ch\mathcal{L}u) \leq \|g\|^2 + \frac{1}{4}\|u\|^2 + \frac{Ch}{4}\sum_K \|\mathcal{L}u\|_K^2. \tag{28}$$

Using the fact that $u_{\_} = f$ on $\Gamma_{\_}$ and combining (27) and (28), the desired result follows. $\square$

In particular this estimate shows the uniqueness and existence of a solution to (23)

Now we prove the improved stability estimate.

Lemma 3. *The following improved stability holds for $\delta = Ch$ with suitable constant C*

$$a(u, u + \delta\mathcal{L}u) \geq C\Big(|||u|||_h^2 - \int_{\Gamma_{\_}} (u^-)^2|M|\, ds\Big). \tag{29}$$

Proof. From (27) and the definition of the norm (3) , the result easily follows. $\square$

5. Error Analysis of the Space-time DG Method

We can now state and prove the basic global error estimate for our space-time DG method (23).

Theorem 4. *If u satisfies (23) and U satisfies (1), then we have the following error estimate*

$$|||e|||_h \leq C\omega h^{r+\frac{1}{2}}, \tag{30}$$

where C does not depend on ω and h.

Proof. Let $U_A \in X_h^r$ be the special interpolation of U defined by (19). Let us write

$$\eta := U - U_A, \qquad \theta := u - U_A, \quad e = \theta - \eta.$$

Using Lemma 3 with $u = e$ and $\delta = Ch$ and the orthogonality property (14) with $v = \theta$ and the fact that $e_{\_} = 0$ on $\Gamma_{\_}$, we have

$$C|||e|||_h^2 \leq a(e, e + Ch\mathcal{L}u) = a(e, e) + Cha(e, \mathcal{L}\eta) := T_1 + T_2. \tag{31}$$

In order to bound T_1, we first prove that

$$a(e, e) = |e|_h^2. \tag{32}$$

By Green's formula for each K

$$2(\mathcal{L}e, e)_K = \int_{\partial K_+} (e^-)^2 M\, ds - \int_{\partial K_{\_}} (e^+)^2|M|\, ds,$$

and thus

$$\begin{aligned} 2a(e, e) = \sum_K \Big\{ &\int_{\partial K_+} (e^-)^2 M\, ds - \int_{\partial K_{\_}} (e^+)^2|M|\, ds \\ &+ 2\int_{\partial K_{\_}} (e^+ - e^-)e^+|M|\, ds\Big\} + 2\|e\|^2. \end{aligned}$$

Now using the identity (26) with $u = e$ along with the fact that $e^- = 0$ on Γ_- we obtain that

$$2a(e,e) = \sum_K \Big\{ \int_{\partial K_-} \big((e^+)^2 - 2e^+e^- + (e^-)^2\big)|M|\,ds \Big\} + \int_{\Gamma_+} (e^-)^2 M\,ds + 2\|e\|^2,$$

which proves the result (32).

We next bound the term T_2. To end this, we bound the bilinear form $a(e, \mathcal{L}\eta)$. So using the Cauchy-Schwarz and the arithmetic-geometric inequalities we have

$$\begin{aligned} a(e,\mathcal{L}\eta) &= \sum_K \Big\{ \big(\mathcal{L}e + ce, \mathcal{L}\eta\big)_K - \int_{\partial K_-} [e]\eta^+ M\,ds \Big\} \\ &\le \frac{1}{2}\sum_K \|\mathcal{L}e\|_K^2 + \frac{1}{2}\|\mathcal{L}\eta\|^2 + \frac{h^{-1}}{2}\|e\|^2 + \frac{h}{2}\|\mathcal{L}\eta\|^2 \\ &\quad + \sum_K \Big(\frac{1}{4h}\int_{\partial K_-} [e]^2|M|\,ds + h\int_{\partial K_-} \eta^2|M|\,ds \Big), \end{aligned}$$

thus we find that

$$\begin{aligned} Cha(e,\mathcal{L}\eta) &\le \frac{h}{2}\sum_K \|\mathcal{L}e\|_K^2 + \frac{h}{2}\|\mathcal{L}\eta\|^2 + \frac{1}{2}\|e\|^2 + \frac{h^2}{4}\|\mathcal{L}\eta\|^2 \\ &\quad + \sum_K \Big(\frac{1}{4}\int_{\partial K_-} [e]^2|M|\,ds + h^2\int_{\partial K_-} \eta^2|M|\,ds \Big), \end{aligned}$$

and

$$Cha(e,\mathcal{L}\eta) \le \frac{1}{2}|||e|||_h^2 + \frac{h}{2}\|\mathcal{L}\eta\|^2 + \frac{h^2}{4}\|\mathcal{L}\eta\|^2 + \sum_K h^2\int_{\partial K_-} \eta^2|M|\,ds.$$

Using the bounds (20) and (22) and the fact that the number of elements is $O(h^{-2})$ we can bound the right hand-side by

$$\begin{aligned} Cha(e,\mathcal{L}\eta) &\le \frac{1}{2}|||e|||_h^2 + \frac{h}{2}C\omega^2h^{2r} + \frac{h^2}{4}C\omega^2h^{2r} + \sum_K h^2Ch^{2r+1} \\ &\le \frac{1}{2}|||e|||_h^2 + C\omega^2h^{2r+1} + C\omega^2h^{2r+2} + Ch^{-2}h^2h^{2r+1} \\ &\le \frac{1}{2}|||e|||_h^2 + C\omega^2h^{2r+1}. \end{aligned} \tag{33}$$

Finally inserting (32) and (33) into (31), it follows that

$$|||e|||_h^2 \le C\omega^2h^{2r+1},$$

or

$$|||e|||_h \le C\omega h^{r+\frac{1}{2}}.$$

This finishes the proof of the error estimate (30). □

Remark 5. *Typically, standard DG (without enrichment) with approximation u_{DG} would have (see, for example, ([7])*

$$|||e|||_h^2 \leq Ch^{r+\frac{1}{2}} \|U\|_{r+1,2,Q}.$$

Since $\|U\|_{r+1,2,Q} \sim \omega^{r+1}$ this error is dramatically larger than our error estimates (30).

6. Numerical Results

In this section, we will demonstrate some numerical experiments to verify our theoretical findings. Let us consider the scalar hyperbolic equation. We take $Q_T = \Omega \times (0,T] = [0,2\pi] \times (0,T]$ and the initial condition

$$u(x,0) = \sin(\omega x), \quad \omega = 100.$$

We choose the boundary conditions so that the exact solution is given by

$$u(x,t) = \sin(\omega(x-t)).$$

In this example, the initial condition has only one high frequency component so we take $L = 1$ and $n_1 \approx \omega$. For simplicity, we consider here only structured discretizations in space and time and choose $h = 2^{-\ell}$, $\quad \ell = 3,4,5,6,7$ and we use linear tensor-product polynomials, that is, $r = 1$. Thus we have 2 shape functions in the (unenriched) space-time DG space, and we have 6 shape functions (2 unenriched and 4 enriched) in the extended space-time DG space for the reference element $I^2 = (0,1) \times (0,1)$. Matrix integrals are all done on a reference element by using 10 Gauss-Lobatto points numerical integration. The most simple shape functions of maximum degree r in the reference element can be given by

$$\phi(\eta_0, \eta_1) = \eta_0^{r_0} \eta_1^{r_1}, \quad r = r_0 + r_1.$$

These shape functions give better conditioned mass and stiffness matrices, and make the computations relatively easier. Define the transformation

$$\begin{aligned} &G_K^n : (0,1)^2 \to \mathcal{K}^n \\ &G_K^n(\eta_0, \eta_1) = (x,t), \end{aligned} \tag{34}$$

where

$$(x,t) = (\frac{1}{2}(t_n + t_{n+1}) - \frac{1}{2}(t_n - t_{n+1})\eta_0, \frac{1}{2}(1-\eta_0)\xi_0 + \frac{1}{2}(1+\eta_0)\xi_1)$$

with ξ_0 and ξ_1 are linear finite element shape functions that are the images of η_1 to the elements K^n and K^{n+1}, respectively, by a suitable mapping. An example of such a mapping, F_K^n can be given as

$$\begin{aligned} &F_K^n : I^2 \to K^n \\ &F_K^n(\eta_1) = \sum_{k=1}^{8} x_k(K^n)\chi_k(\eta_1), \end{aligned}$$

where $x_k(K^n)$ is the vertices of the element K^n and $\chi(\eta_1)$ is the standard linear finite element shape functions defined on I^2. Thus, the space-time tessellation consists of the union of all the partitioning of the space-time slabs. For more detailed discussions of such mappings and other basis functions, see [13] and [14]. The numerical results are shown in Table 1 and Table 2. The observed convergence rates (OCR) of the proposed method in L_2 and the energy norms are given at $T=1$ and $T=2$. The observed convergence rate R_1 in L_2 norm is computed by the formula $R_1 = \log(||e_{2h}||/||e_h||)/\log(2)$ and the observed convergence rate R_2 in the energy norm is computed by the formula $R_2 = \log(|||e_{2h}|||_h/|||e_h|||_h)/\log(2)$ where $e_h = u - U$ is the error on the mesh. It is known that optimal convergence is observed only by using suitable chosen meshes. The loss of order $h^{1/2}$ in the order of convergence of L_2 norm is still under discussion, e.g., see [32]. In practice, the optimal convergence h^{r+1} is achieved when polynomials of degree at most r used even if there is no uniform requirement on the chosen meshes. See [33] for the computational results for conforming triangulations for an example of this issue . Thus, typically there is a gap of order $h^{1/2}$ between computed convergence rate and the optimal convergence rate in DG methods.

h	$\|e_h\|_{L_2,T=1}$	R_1	$\|e_h\|_{L_2,T=2}$	R_1
1/8	0.2542	-	0.3653	-
1/16	0.7461e-1	1.739	1.070e-1	1.771
1/32	0.2193e-1	1.7664	0.288e-1	1.893
1/64	0.5993e-2	1.8715	0.768e-2	1.9068
1/128	0.1480e-2	2.0176	0.195e-2	1.9776

TABLE 1. The errors and the order of convergence of the space-time DG for the first order polynomial approximation ($r=1$) at $T=1$ and $T=2$ in L_2 norm.

h	$\|\|\|e_h\|\|\|_{h,T=1}$	R_2	$\|\|\|e_h\|\|\|_{h,T=2}$	R_2
1/8	1.0532	-	1.2471	-
1/16	3.792e-1	1.473	4.584e-1	1.484
1/32	1.346e-1	1.494	1.563e-1	1.506
1/64	4.721e-2	1.511	5.491e-2	1.506
1/128	1.657e-2	1.510	1.943e-2	1.498

TABLE 2. The errors and the order of convergence of the space-time DG for the first order polynomial approximation ($r=1$) at $T=1$ and $T=2$ in the energy norm.

The results in Table 2 clearly indicate that the numerical results are in good agreement with the theoretical findings and show that the proposed method convergences with the expected $(r+1/2)$-th order of convergence when the polynomial space of order r is used without any mesh refinement.

7. Conclusion

In this paper, we presented a space-time discontinuous Galerkin method for the scalar hyperbolic problems that contain high frequency components. We extend the space-time approximation space with trigonometric functions to capture the oscillatory behavior of the solutions. We applied discontinuous Galerkin methodology in both space and time and derived a stable space-time DG scheme. Thus, the method can be seen as a space-time framework of extended DG method. The key feature of the method is that it uses the solutions of PDE under consideration. Furthermore, the choice of DG space enriched by the solutions of the governing differential equation enables an efficient evaluation of integral terms. The proposed method here performs well when compared to standard space-time DG method. With conventional space-time DG method, one needs to refine the mesh size to get an acceptable accuracy for high frequency component. This leads to the computational costs in each space-time slab for solving the resulting system. We showed optimal a priori error estimates in a mesh dependent space-time DG norm. Additionally, we gave a numerical experiments to verify the theoretical findings. An extension of the analysis for an extended space-time solutions for the linear hyperbolic problems or conservation laws in two and three dimensional computational domains will be considered in the future.

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INTRODUCTION TO TEMPORAL INTUITIONISTIC FUZZY APPROXIMATE REASONING

FATIH KUTLU, FERIDE TUĞRUL, AND MEHMET ÇITIL

Abstract. In this study; temporal intuitionistic fuzzy negation, temporal intuitionistic fuzzy triangular norm and temporal intuitionistic fuzzy triangular conorm have been researched. The aim of this study is to define negator, t-norm and t-conorms, which is the generalization of negation, conjunctions and disconjunctions in the temporal intuitionistic fuzzy sets and to examine the De Morgan relations between these concepts. The thing to note here is that conjunctions generalized with $t-$norm and $t-$conorm is changed depending on time. We will carry concept of implication and coimplication to temporal intuitionistic fuzzy sets. With the new implication definitions, a causal structure will be established which will match the variable structure of the systems depending on the position and time variables. It is evident that successful results will be achieved in this type of system, which is being dealt with by this new structure.

1. Introduction

The notion of fuzzy logic was firstly defined by Zadeh in 1965 [10]. Then; intuitionistic fuzzy sets (shortly IFS) were defined by K.Atanassov in 1986 [1]. Intuitionistic fuzzy sets form a generalization of the notion of fuzzy sets. The concept of temporal intuitionistic fuzzy sets is defined by Atanassov in 1991 [2]. In this concept; the membership and non-membership degrees are described based on the time-moment and time-element. The temporal intuitionistic fuzzy set theory create a new perspective in various application areas such as: Weather, economy, image, video processing, etc.

In this study, firstly definition of temporal intuitionistic fuzzy sets has been given. Then, temporal intuitionistic fuzzy negation, temporal intuitionistic fuzzy triangular norm and temporal intuitionistic fuzzy triangular conorm have been researched. The aim of this study is to define negator, t-norm and t-conorms,

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which is the generalization of negation, conjunctions and disconjunctions in the temporal intuitionistic fuzzy sets and to examine the De Morgan relations between these concepts. The thing to note here is that conjunctions generalized with $t-$norm and $t-$conorm is changed depending on time. The changing conjunctive idea that depends on time has a meaning only when the connected objects change depending on time. Therefore these conjunctions can be used on temporal intuitionistic fuzzy sets.

In this study; we will carry concept of implication and coimplication to temporal intuitionistic fuzzy sets. The definition of the intuitionistic implication is based on the notation from fuzzy set theory introduced by Fodor, Roubens [26]. These concepts, which are used to establish the IF-THEN structure with a clearer reasoning in the fuzzy set and in the intuitionistic fuzzy set theory, are known to be the basic elements in the systems studied by fuzzy and intuitionistic fuzzy set theories. With the new implication definitions given below, a causal structure will be established which will match the variable structure of the systems depending on the position and time variables. It is evident that successful results will be achieved in this type of system, which is being dealt with by this new structure. When these two concepts are established, the necessity of satisfying the "modus ponens" conditions in the classical logic will be taken into consideration. At this point, implications and coimplication definitions will be moved to the temporal intuitionistic fuzzy set space in the studies light, which has been done previously and successfully in practice. Many researchers have been researched in this field ([8],[12],[13],[22],[23],[24],[25],[27])

2. Preliminaries

Definition 1. [1] *An intuitionistic fuzzy set on a non-empty set X given by a set of ordered triples $A = \{(x, \mu_A(x), \eta_A(x)) : x \in X\}$ where $\mu_A(x) : X \to I = [0, 1]$, $\eta_A(x) : X \to I$, are functions such that $0 \le \mu(x) + \eta(x) \le 1$ for all $x \in X$. For $x \in X$, $\mu_A(x)$ and $\eta_A(x)$ represent the degree of membership and degree of non-membership of x to A respectively. For each $x \in X$; intuitionistic fuzzy index of x in A is defined as follows $\pi_A(x) = 1 - \mu_A(x) - \eta_A(x)$. π_A is the called degree of hesitation or indeterminacy. Let denote the set of all intuitionistic fuzzy sets defined on X by IFS^X*

Definition 2. [1] *Let $A, B \in IFS^X$. Then,*
(i) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ and $\eta_A(x) \ge \eta_B(x)$ for $\forall x \in X$,
(ii) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$,
(iii) $\bar{A} = \{(x, \eta_A(x), \mu_A(x)) : x \in X\}$,
(iv) $\bigcap A_i = \left\{\left(x, \wedge\mu_{A_i}(x), \vee\eta_{A_i}(x)\right) : x \in X\right\}$,
(v) $\bigcup A_i = \left\{\left(x, \vee\mu_{A_i}(x), \wedge\eta_{A_i}(x)\right) : x \in X\right\}$.

Definition 3. [2] *Let X be an universe and T be a non-empty time set. We call the elements of T as "time moments". Based on the definition of IFS, a temporal* *intuitionistic fuzzy set (TIFS) is defined as the following:*

$$A(T) = \{(x, \mu_A(x,t), \eta_A(x,t)) : X \times T\}$$

where:
a. $A \subseteq X$ *is a fixed set,*
b. $\mu_A(x,t) + \eta_A(x,t) \leq 1$ *for every* $(x,t) \in X \times T$,
c. $\mu_A(x,t)$ *and* $\eta_A(x,t)$ *are the degrees of membership and non-membership, respectively, of the element* $x \in X$ *at the time moment* $t \in T$.

For brevity, we write A instead of $A(T)$. The hesitation degree of an TIFS is defined as $\pi_A(x,t) = 1 - \mu_A(x,t) - \eta_A(x,t)$. Obviously, every ordinary IFS could be regarded as TIFS for which T is a singleton set. All operations and operators on IFS could be defined for TIFSs.

By $TIFS^{(X,T)}$, we denote to the set of all temporal intuitionistic fuzzy sets defined on X and time set T. Obviously, each intuitionistic fuzzy sets could be expressed as temporal intuitionistic fuzzy set via a singular time set. In additionally, all operations and operators defined for intuitionistic fuzzy sets could be defined for temporal intuitionistic fuzzy sets.

Definition 4. [2] *Let*

$$A(T') = \{(x, \mu_A(x,t), \eta_A(x,t)) : X \times T'\}$$

and

$$B(T'') = \{(x, \mu_B(x,t), \eta_B(x,t)) : X \times T''\}$$

where T' *and* T'' *have finite number of distinct time-elements or they are time intervals. Then;*
$A(T') \cap B(T'') =$

$$\left\{(x,\ min\left(\bar{\mu}_A(x,t), \bar{\mu}_B(x,t)\right),\ max\left(\bar{\eta}_A(x,t), \bar{\eta}_B(x,t)\right)) : (x,t) \in X \times \left(T' \cup T''\right)\right\}$$

and
$A(T') \cup B(T'') =$

$$\left\{(x,\ max\left(\bar{\mu}_A(x,t), \bar{\mu}_B(x,t)\right),\ min\left(\bar{\eta}_A(x,t), \bar{\eta}_B(x,t)\right)) : (x,t) \in X \times \left(T' \cup T''\right)\right\}$$

Also from definition of subset in intuitionistic fuzzy sets, subsets of temporal intuitionistic fuzzy sets can be defined as the following:

$$A(T') \subseteq B(T'') \Leftrightarrow \bar{\mu}_A(x,t) \geq \bar{\mu}_B(x,t) \text{ and } \bar{\eta}_A(x,t) \leq \bar{\eta}_B(x,t)$$

for every $(x,t) \in X \times (T' \cup T'')$ *where*

$$\bar{\mu}_A(x,t) = \begin{cases} \mu_A(x,t), & if \quad t \in T' \\ 0, & if \quad t \in T'' - T' \end{cases}$$

$$\bar{\mu}_B(x,t) = \begin{cases} \mu_B(x,t), & if \quad t \in T'' \\ 0, & if \quad t \in T' - T'' \end{cases}$$

$$\bar{\eta}_A(x,t) = \begin{cases} \eta_A(x,t), & if \quad t \in T' \\ 1, & if \quad t \in T'' - T' \end{cases}$$

$$\bar{\eta}_B(x,t) = \begin{cases} \eta_B(x,t), & \text{if} \quad t \in T'' \\ 1, & \text{if} \quad t \in T' - T'' \end{cases}$$

It is obviously seen that if $T' = T''$*;* $\bar{\mu}_A(x,t) = \mu_A(x,t)$*,* $\bar{\mu}_B(x,t) = \mu_B(x,t)$*,* $\bar{\eta}_A(x,t) = \eta_A(x,t)$*,* $\bar{\eta}_B(x,t) = \eta_B(x,t)$*. [2]*

Let J *be an index set and* T_i *is a time set for each* $i \in J$*. Let define that* $T = \bigcup_{i\in J} T_i$*. Now we extend union and intersection of temporal intuitionistic fuzzy sets to the family* $F = \left\{A_i(T_i) = \left(x, \mu_{A_i}(x,t), \eta_{A_i}(x,t)\right) : x \in X \times T_i,\ i \in J\right\}$ *as:*

$$\begin{aligned} \bigcup_{i\in J} A(T_i) &= \left\{\left(x,\ \max_{i\in J}\left(\bar{\mu}_{A_i}(x,t)\right), \min_{i\in J}\left(\bar{\eta}_{A_i}(x,t)\right) : (x,t) \in X \times T\right)\right\}, \\ \bigcap_{i\in J} A(T_i) &= \left\{\left(x,\ \min_{i\in J}\left(\bar{\mu}_{A_i}(x,t)\right), \max_{i\in J}\left(\bar{\eta}_{A_i}(x,t)\right) : (x,t) \in X \times T\right)\right\} \end{aligned}$$

where

$$\bar{\mu}_{A_i}(x,t) = \begin{cases} \mu_{A_i}(x,t), & \text{if} \quad t \in T_i \\ 0, & \text{if} \quad t \in T - T_i \end{cases}$$

and

$$\bar{\eta}_{A_i}(x,t) = \begin{cases} \eta_{A_i}(x,t), & \text{if} \quad t \in T_i \\ 1, & \text{if} \quad t \in T - T_i \end{cases}.$$

Definition 5. *The set of all intuitionistic fuzzy pair is defined as*

$$IFP^* = \{(x,y) \in [0,1] \times [0,1];\ x + y \leq 1\}$$

The order relation $\leq$ *on this set is defined by* $(x_1,y_1) \leq (x_2,y_2) \Leftrightarrow x_1 \leq x_2,\ y_1 \geq y_2$ *for* $\forall (x_1,y_1), (x_2,y_2) \in IFP^*$*. Also* $\widetilde{1} = (1,0)$ *and* $\widetilde{0} = (0,1)$*.*

Let $x : T \to [0,1]$, $y : T \to [0,1]$ *are functions such that* $x(t) + y(t) \leq 1$ *for each time moment* $t \in T$*. Then temporal intuitionistic pair set on time set* T *defined as follows:*

$$TIFP_T^* = \{(x(t), y(t)) :\ t \in T\}$$

$0_T, 1_T \in TIFP_T^*$ *which are defined such as* $0_T = (x_{0_T}(t), y_{0_T}(t)) = (0,1)$ *and* $1_T = (x_{1_T}(t), y_{1_T}(t)) = (1,0)$ *for each time moment* $t \in T$ *and are called overall zero and overall one. On the other hand* $0_t, 1_t \in TIFP_T^*$*, which are defined such as* $0_t = (x_{0_t}(t), y_{0_t}(t)) = (0,1)$ *and* $1_t = (x_{1_t}(t), y_{1_t}(t)) = (1,0)$ *for a fixed time moment* $t \in T$*, are called temporal zero and temporal one at time moment* t*.*

3. Temporal Intuitionistic Fuzzy Negation, t-norm and t-conorm

In this section firstly; we will carry negation, $t-$norm and $t-$conorm definitions to temporal intuitionistic fuzzy sets. Then, the basic relations between these definitions will be researched.

Definition 6. *Let T be a time set, the decreasing mapping $N_t : TIFP_T^* \times T \rightarrow TIFP_T^*$ which is satisfied following the condition $N_t(0_t, t) = 1_t$ and $N_t(1_t, t) = 0_t$ at fixed time moment $t \in T$ is called temporal intuitionistic fuzzy negation at fixed time moment t .*

Definition 7. *If N_t is satisfied*
a. $N_t(N_t(a(t), t), t) = a(t)$ for all time moment $t \in T$ and all $a(t) \in TIFP_T^*$, it is called temporal intuitionistic fuzzy strong negation at time moment t,
b. $x(t) = 0_t \Leftrightarrow N_t(x(t), t) = 1_t$ for fixed time moment $t \in T$, it is called temporal intuitionistic fuzzy non-filling negation at time moment t,
c. $x(t) = 1_t \Leftrightarrow N_t(x, t) = 0_t$ for $t \in T$ and all $a \in IF^*$, it is called temporal intuitionistic fuzzy non-vanishing negation at time moment t.

Remark 1. *According to the this definition, it would be seen that the negation operator may change with the time parameter. It would be more correct to define temporal intuitionistic fuzzy negation on a temporal intuitionistic fuzzy pair, even if it is true with a classical approach which defined with intuitionistic fuzzy pair. Despite the fact that the cases to be handled by the negation operator can change according to the time makes it necessary for the negation operator to change depending on the time.*

Definition 8. *The mapping $N_t : TIFP_T^* \times T \rightarrow [0,1]$ defined by $N_t((x_1(t), x_2(t)), t) = (x_2, x_1)$ for all $(x_1, x_2) \in IF^*$ is called standard temporal intuitionistic fuzzy negator.*

The following proposition is also valid for temporal intuitionistic fuzzy negations as well as fuzzy and intuitionistic fuzzy negations.

Proposition 1. *The equation $N_t\ (N_t(0_t, t), t) = 0_t$ is satisfied for any temporal intuitionistic fuzzy strong negator N_t.*

Proof. From the temporal intuitionistic fuzzy negation definition;

$$N_t(0_t, t) = 1_t,\ N_t(1_t, t) = 0_t,\ N_t(N_t(0_t, t), t) = 0_t.$$

□

Definition 9. *Let T be a time set. If the mapping $T_t : (TIFP_T^* \times TIFP_T^{**}) \times T \rightarrow TIFP_T^*$ is satisfied following condition for a fixed time moment $t \in T$, it is called temporal intuitionistic fuzzy triangular norm ($t-$norm) at time moment t:*

- T1. *$T_t((x(t), y(t)), t) = T_t((y(t), x(t)), t)$ for every $x, y \in TIFP_T^*$ at fixed the time moment $t \in T$ (symmetry),*
- T2. *$T_t((x_1(t), y_1(t)), t) \leq T_t((x_2(t), y_2(t)), t)$ for every $x_1(t),\ y_1(t),\ x_2(t),\ y_2(t) \in TIFP_T^*$ such that $x_1(t) \leq x_2(t)$ and $y_1(t) \leq y_2(t)$ at fixed the time moment $t \in T$(monotonicity),*
- T3. *$T_t((T_t((x(t), y(t)), t)), z(t), t) = T_t((x(t), T_t((z(t), y(t)), t)), t)$ for every $x(t), y(t), z(t) \in TIFP_T^*$ at fixed the time moment $t \in T$(associativity),*

T4. $T_t\left(\left(x\left(t\right), 1_t\right), t\right) = x\left(t\right)$ *for every* $x\left(t\right) \in TIFP_T^*$ *(boundary condition).*

Definition 10. *Let T be a time set. If the mapping $S_t : (TIFP_T^* \times TIFP_T^*) \times T \to TIFP_T^*$ is satisfied following condition at time moment $t \in T$ and , it is called temporal triangular conorm (or $s-$norm) at time moment t :*

S1. $S_t\left(\left(x\left(t\right), y\left(t\right)\right), t\right) = S_t\left(\left(y\left(t\right), x\left(t\right)\right), t\right)$ *for every* $x, y \in TIFP_T^*$ *at fixed the time moment* $t \in T$ *(symmetry),*

S2. $S_t\left(\left(x_1\left(t\right), y_1\left(t\right)\right), t\right) \leq S_t\left(\left(x_2\left(t\right), y_2\left(t\right)\right), t\right)$ *for every* $x_1\left(t\right),\ y_1\left(t\right),\ x_2\left(t\right),\ y_2\left(t\right) \in TIFP_T^*$ *such that* $x_1\left(t\right) \leq x_2\left(t\right)$ *and* $y_1\left(t\right) \leq y_2\left(t\right)$ *at fixed the time moment* $t \in T$ *(monotonicity),*

S3. $S_t\left(\left(S_t\left(\left(x\left(t\right), y\left(t\right)\right), t\right)\right), z\left(t\right), t\right) = S_t\left(\left(x\left(t\right), S_t\left(\left(z\left(t\right), y\left(t\right)\right), t\right)\right), t\right)$ *for every* $x\left(t\right), y\left(t\right), z\left(t\right) \in TIFP_T^*$ *at fixed the time moment* $t \in T$ *(associativity),*

S4. $S_t\left(\left(x\left(t\right), 0_t\right), t\right) = x\left(t\right)$ *for every* $x\left(t\right) \in TIFP_T^*$ *at fixed the time moment* $t \in T$ *(boundary condition).*

The thing to note here is that conjunctions generalized with $t-$norm and $t-$conorm is changed depending on time. The changing conjunctive idea that depends on time has a meaning only when the connected objects change depending on time. Therefore these conjunctions could be used on temporal intuitionistic fuzzy sets.

Proposition 2. *Let*

$$A = \{(x, \mu_A\left(x,t\right), \eta_A\left(x,t\right)) : (x,t) \in X \times T'\}$$

and

$$B = \{(x, \mu_B\left(x,t\right), \eta_B\left(x,t\right)) : (x,t) \in X \times T''\}$$

be two TIFSs where T' and T'' are time set. Then the following mappings are $t-$norm and $t-$conorm for $(x,t) \in X \times T' \cup T''$:

(1) $T_{\min}^t\left[\left(A,B\right), t\right] = \left(\min\left(\bar{\mu}_A\left(x,t\right), \bar{\mu}_B\left(x,t\right)\right), \max\left(\bar{\eta}_A\left(x,t\right), \bar{\eta}_B\left(x,t\right)\right)\right)$,

(2) $T_0^t\left[\left(A,B\right), t\right] = \begin{cases} \left(\bar{\mu}_A\left(x,t\right), \bar{\eta}_A\left(x,t\right)\right) & , \quad \left(\mu_B\left(x,t\right), \eta_B\left(x,t\right)\right) = \widetilde{1} \\ \left(\bar{\mu}_B\left(x,t\right), \bar{\eta}_B\left(x,t\right)\right) & , \quad \left(\mu_A\left(x,t\right), \eta_A\left(x,t\right)\right) = \widetilde{1} \\ \widetilde{0} & , \quad otherwise \end{cases}$,

(3) $T_1^t\left[\left(A,B\right), t\right] = \left(\max\left\{0, \left(\bar{\mu}_A\left(x,t\right) + \bar{\mu}_B\left(x,t\right)\right)\right\}, \min\left\{1, \bar{\eta}_A\left(x,t\right) + \bar{\eta}_B\left(x,t\right)\right\}\right)$,

(4) $T_2^t\left[\left(A,B\right), t\right] = \left(\bar{\mu}_A\left(x,t\right)\bar{\mu}_B\left(x,t\right), \bar{\eta}_A\left(x,t\right) + \bar{\eta}_B\left(x,t\right) - \bar{\eta}_A\left(x,t\right)\bar{\eta}_B\left(x,t\right)\right)$,

(5) $T_3^t\left[\left(A,B\right), t\right] =$

$$\left(\log_t\left(1 + \frac{\left(t^{\left(\bar{\mu}_A(x,t)\right)} - 1\right)\left(t^{\left(\bar{\mu}_B(x,t)\right)} - 1\right)}{t-1}\right), 1 - \log_t\left(1 + \frac{\left(t^{\left(1-\bar{\eta}_A(x,t)\right)} - 1\right)\left(t^{\left(1-\bar{\eta}_B(x,t)\right)} - 1\right)}{t-1}\right)\right),$$

(6) $S_{\max}^t\left[\left(A,B\right), t\right] = \left(\max\left(\bar{\mu}_A\left(x,t\right), \bar{\mu}_B\left(x,t\right)\right), \min\left(\bar{\eta}_A\left(x,t\right), \bar{\eta}_B\left(x,t\right)\right)\right)$,

(7) $S_0^t\left[\left(A,B\right), t\right] = \begin{cases} \left(\bar{\mu}_A\left(x,t\right), \bar{\eta}_A\left(x,t\right)\right) & , \quad B = 1_t \\ \left(\bar{\mu}_B\left(x,t\right), \bar{\eta}_B\left(x,t\right)\right) & , \quad A = 0_t \\ \widetilde{1} & , \quad otherwise \end{cases}$,

(8) $S_1^t\left[\left(A,B\right), t\right] = \left(\min\left\{1, \bar{\mu}_A\left(x,t\right) + \bar{\mu}_B\left(x,t\right)\right\}, \max\left\{0, \left(\bar{\eta}_A\left(x,t\right) + \bar{\eta}_B\left(x,t\right)\right)\right\}\right)$,

(9) $S_2^t\left[\left(A,B\right), t\right] = \left(\bar{\mu}_A\left(x,t\right) + \bar{\mu}_B\left(x,t\right) - \bar{\mu}_A\left(x,t\right)\bar{\mu}_B\left(x,t\right), \bar{\eta}_A\left(x,t\right)\bar{\eta}_B\left(x,t\right)\right)$,

(10) $S_3^t\left[(A,B),t\right]=$

$$\left(1-\log_t\left(1+\frac{\left(t^{\left(1-\bar{\mu}_A(x,t)\right)}-1\right)\left(t^{\left(1-\bar{\mu}_B(x,t)\right)}-1\right)}{t-1}\right),\log_t\left(1+\frac{\left(t^{\left(\bar{\eta}_A(x,t)\right)}-1\right)\left(t^{\left(\bar{\eta}_B(x,t)\right)}-1\right)}{t-1}\right)\right)$$

Proposition 3. *Following inequalities are satisfied for each T^t temporal intuitionistic fuzzy $t-$norm and S^t temporal intuitionistic fuzzy $t-$conorm*

(1) $T_0^t \leq T^t \leq T_{\min}^t$,
(2) $S_{\max}^t \leq S^t \leq S_0^t$.

Proof. 1. Let's prove on a single T time set without disturbing the generality. Firstly; let's show that $T_0^t \leq T^t$.

In case of $\left(\mu_B(x,t),\eta_B(x,t)\right)=\widetilde{1}$ or $\left(\mu_A(x,t),\eta_A(x,t)\right)=\widetilde{1}$ (let's accept $\left(\mu_B(x,t),\eta_B(x,t)\right)=\widetilde{1}$ without loss of the generality) the following equation is easily obtained.

$$T_0^t\left[(A,B),t\right]=\left(\mu_A(x,t),\eta_A(x,t)\right)=T^t\left[(A,B),t\right]$$

In other cases, because of $T_0^t\left[(A,B),t\right]=\widetilde{0}$, $T_0^t\left[(A,B),t\right]\leq T^t\left[(A,B),t\right]$ inequality is clearly obtained. Let's show that $T^t\left[(A,B),t\right]\leq T_{\min}^t\left[(A,B),t\right]$. Because of $T^t\left[(A,B),t\right]\leq T^t\left[\left(A,\widetilde{1}\right),t\right]=\left(\mu_A(x,t),\eta_A(x,t)\right)$ and

$$\begin{aligned}
T^t\left[(A,B),t\right] &= T^t\left[(B,A),t\right]\leq T^t\left[\left(B,\widetilde{1}\right),t\right]=\left(\mu_B(x,t),\eta_B(x,t)\right)\\
T^t\left[(A,B),t\right] &\leq \left(\min\left(\bar{\mu}_A(x,t),\,\bar{\mu}_B(x,t)\right),\,\max\left(\bar{\eta}_A(x,t),\,\bar{\eta}_B(x,t)\right)\right)\\
&= T_{\min}^t\left[(A,B),t\right]
\end{aligned}$$

inequality is easily obtained. The other expression could be similarly proven. □

Definition 11. *As stated in* [9], *Let $T^*: TIFP_T^*\times TIFP_T^*\to[0,1]$ and $S^*: TIFP_T^*\times TIFP_T^*\to[0,1]$ be respectively intuitionistic fuzzy $t-$ norm and $t-$ conorm on $TIFP_T^*$ and at fixed time moment $t\in T$ such that*

$$T^*\left(x(t),y(t)\right)\leq N\left(S^*\left(\left(N\left(x(t)\right),\ N\left(y(t)\right)\right)\right)\right)$$

where N is intuitionistic fuzzy standard negation. Then the mapping T_t defined as follows

$$T_t\left((A,B),t\right)=\left(T^*\left(\bar{\mu}_A(x,t),\,\bar{\mu}_B(x,t)\right),S^*\left(\bar{\eta}_A(x,t),\,\bar{\eta}_B(x,t)\right)\right)$$

is a temporal intuitionistic fuzzy $t-$ norm and it is called t-representable temporal intuitionistic fuzzy $t-$ norm.

Similarly; the mapping S_t defined as follows

$$S_t\left((A,B),t\right)=\left(S^*\left(\bar{\mu}_A(x,t),\,\bar{\mu}_B(x,t)\right),T^*\left(\bar{\eta}_A(x,t),\,\bar{\eta}_B(x,t)\right)\right)$$

is a temporal intuitionistic fuzzy $t-$ conorm and it is called t-representable temporal intuitionistic fuzzy $t-$ conorm.

Looking at the definitions of $t-$norm and $t-$conorm given above, it is seen that they are $t-$representable temporal intuitionistic fuzzy $t-$norm and $t-$conorm. The temporal intuitionistic fuzzy De Morgan triplet defined with approach described in [10] as follows:

Definition 12. *A triplet (S_t, T_t, N_t) is called temporal intuitionistic fuzzy De Morgan triplet if T_t is temporal intuitionistic fuzzy $t-$ norm, S_t is temporal intuitionistic fuzzy $t-$ conorm, N_t is temporal intuitionistic fuzzy negator and if they fulfill De Morgan's law*

$$S_t\left(\left(A,B\right),t\right)=N_t\left(T_t\left(\left(N_t\left(A,t\right),N_t\left(B,t\right)\right),t\right),t\right)$$

or equivalently

$$T_t\left(\left(A,B\right),t\right)=N_t\left(S_t\left(\left(N_t\left(A,t\right),N_t\left(B,t\right)\right),t\right),t\right).$$

Proposition 4. *$T^t_{\min}$ and $S^t_{\max}$ together with N_t generate a De Morgan Triplet.*

Proof.

$$\begin{aligned}
S^t_{\max}\left(\left(A,B\right),t\right) &= N_t\left(T^t_{\min}\left(N_t\left(A,t\right),N_t\left(B,t\right)\right),t\right)\\
S^t_{\max}\left[\left(A,B\right),t\right] &= \left(\max\left(\bar{\mu}_A\left(x,t\right),\bar{\mu}_B\left(x,t\right)\right),\min\left(\bar{\eta}_A\left(x,t\right),\bar{\eta}_B\left(x,t\right)\right)\right)\\
T^t_{\min}\left[\left(A,B\right),t\right] &= \left(\min\left(\bar{\mu}_A\left(x,t\right),\bar{\mu}_B\left(x,t\right)\right),\max\left(\bar{\eta}_A\left(x,t\right),\bar{\eta}_B\left(x,t\right)\right)\right)\\
T^t_{\min}\left[\left(N_t\left(A\right),N_t\left(B\right),t\right)\right] &= \left(\min\left(\bar{\eta}_A\left(x,t\right),\bar{\eta}_B\left(x,t\right)\right),\max\left(\bar{\mu}_A\left(x,t\right),\bar{\mu}_B\left(x,t\right)\right)\right)
\end{aligned}$$

$$N_t\left(T^t_{\min}\left(N_t\left(A,t\right),N_t\left(B,t\right),t\right),t\right)=\left(\max\left(\bar{\mu}_A\left(x,t\right),\bar{\mu}_B\left(x,t\right),\min\left(\bar{\eta}_A\left(x,t\right),\bar{\eta}_B\left(x,t\right)\right)\right)\right)$$

$$S^t_{\max}\left(\left(A,B\right),t\right)=N_t\left(T^t_{\min}\left(N_t\left(A,t\right),N_t\left(B,t\right),t\right),t\right)$$

□

Proposition 5. *T^t_i and S^t_i $(i=1,2,3)$ together with N_t generate a De Morgan Triplet.*

Proof. for $i=1$;
T^t_1 and S^t_1 together with N_t generate a De Morgan Triplet.

$$\begin{aligned}
S^t_1\left(\left(A,B\right),t\right) &= N_t\left(T^t_1\left(N_t\left(A\right),N_t\left(B\right)\right)\right)\\
S^t_1\left[\left(A,B\right),t\right] &= \left(\min\left\{1,\bar{\mu}_A\left(x,t\right)+\bar{\mu}_B\left(x,t\right)\right\},\max\left\{0,\left(\bar{\eta}_A\left(x,t\right)+\bar{\eta}_B\left(x,t\right)\right)\right\}\right)\\
T^t_1\left[\left(A,B\right),t\right] &= \left(\max\left\{0,\left(\bar{\mu}_A\left(x,t\right)+\bar{\mu}_B\left(x,t\right)\right)\right\},\min\left\{1,\bar{\eta}_A\left(x,t\right)+\bar{\eta}_B\left(x,t\right)\right\}\right)
\end{aligned}$$

$$T^t_1\left(N_t\left(A\right),N_t\left(B\right)\right)=\left(\max\left\{0,\left(\bar{\eta}_A\left(x,t\right)+\bar{\eta}_B\left(x,t\right)\right)\right\},\min\left\{1,\bar{\mu}_A\left(x,t\right)+\bar{\mu}_B\left(x,t\right)\right\}\right)$$

$$N_t\left(T^t_1\left(N_t\left(A\right),N_t\left(B\right)\right)\right)=\left(\min\left\{1,\bar{\mu}_A\left(x,t\right)+\bar{\mu}_B\left(x,t\right)\right\},\max\left\{0,\left(\bar{\eta}_A\left(x,t\right)+\bar{\eta}_B\left(x,t\right)\right)\right\}\right)$$

Consequently (for $i=1$);
$S^t_1\left(\left(A,B\right),t\right)=N_t\left(T^t_1\left(N_t\left(A\right),N_t\left(B\right)\right)\right)$
for $i=2$;
T^t_2 and S^t_2 together with N_t generate a De Morgan Triplet.
$S^t_2\left(\left(A,B\right),t\right)=N_t\left(T^t_2\left(N_t\left(A\right),N_t\left(B\right)\right)\right)$
$S^t_2\left[\left(A,B\right),t\right]=\left(\bar{\mu}_A\left(x,t\right)+\bar{\mu}_B\left(x,t\right)-\bar{\mu}_A\left(x,t\right)\bar{\mu}_B\left(x,t\right),\bar{\eta}_A\left(x,t\right)\bar{\eta}_B\left(x,t\right)\right)$

$T_2^t\left[(A,B),t\right] = \left(\bar{\mu}_A(x,t)\,\bar{\mu}_B(x,t), \bar{\eta}_A(x,t) + \bar{\eta}_B(x,t) - \bar{\eta}_A(x,t)\ \bar{\eta}_B(x,t)\right)$
$T_2^t\left(N_t(A), N_t(B)\right) = \left(\bar{\eta}_A(x,t)\,\bar{\eta}_B(x,t), \bar{\mu}_A(x,t) + \bar{\mu}_B(x,t) - \bar{\mu}_A(x,t)\ \bar{\mu}_B(x,t)\right)$

$$N_t\left(T_2^t\left(N_t(A), N_t(B)\right)\right) = \left(\bar{\mu}_A(x,t) + \bar{\mu}_B(x,t) - \bar{\mu}_A(x,t)\ \bar{\mu}_B(x,t), \bar{\eta}_A(x,t)\,\bar{\eta}_B(x,t)\right)$$

Consequently (for $i = 2$);
$S_2^t\left((A,B),t\right) = N_t\left(T_2^t\left(N_t(A), N_t(B)\right)\right)$
for $i = 3$; T_3^t and S_3^t together with N_t generate a De Morgan Triplet.
$S_3^t\left[(A,B),t\right] = S_3^t\left((A,B),t\right) = N_t\left(T_3^t\left(N_t(A), N_t(B)\right)\right)$

$$\left(1 - \log_t\left(1 + \frac{\left(t^{(1-\bar{\mu}_A(x,t))} - 1\right)\left(t^{(1-\bar{\mu}_B(x,t))} - 1\right)}{t-1}\right), \log_t\left(1 + \frac{\left(t^{(\bar{\eta}_A(x,t))} - 1\right)\left(t^{(\bar{\eta}_B(x,t))} - 1\right)}{t-1}\right)\right)$$

$T_3^t\left[(A,B),t\right] =$

$$\left(\log_t\left(1 + \frac{\left(t^{(\bar{\mu}_A(x,t))} - 1\right)\left(t^{(\bar{\mu}_B(x,t))} - 1\right)}{t-1}\right), 1 - \log_t\left(1 + \frac{\left(t^{(1-\bar{\eta}_A(x,t))} - 1\right)\left(t^{(1-\bar{\eta}_B(x,t))} - 1\right)}{t-1}\right)\right)$$

$T_3^t\left(N_t(A), N_t(B)\right) =$

$$\left(\log_t\left(1 + \frac{\left(t^{(\bar{\eta}_A(x,t))} - 1\right)\left(t^{(\bar{\eta}_B(x,t))} - 1\right)}{t-1}\right), 1 - \log_t\left(1 + \frac{\left(t^{(1-\bar{\mu}_A(x,t))} - 1\right)\left(t^{(1-\bar{\mu}_B(x,t))} - 1\right)}{t-1}\right)\right)$$

$N_t\left(T_3^t\left(N_t(A), N_t(B)\right)\right) =$

$$\left(1 - \log_t\left(1 + \frac{\left(t^{(1-\bar{\mu}_A(x,t))} - 1\right)\left(t^{(1-\bar{\mu}_B(x,t))} - 1\right)}{t-1}\right), \log_t\left(1 + \frac{\left(t^{(\bar{\eta}_A(x,t))} - 1\right)\left(t^{(\bar{\eta}_B(x,t))} - 1\right)}{t-1}\right)\right)$$

Consequently (for $i = 3$);
$S_3^t\left((A,B),t\right) = N_t\left(T_3^t\left(N_t(A), N_t(B)\right)\right)$ □

4. Temporal Intuitionistic Fuzzy Implicator

In this section firstly; we will carry concepts of implication and coimplication to temporal intuitionistic fuzzy sets. These concepts, which are used to establish the IF-THEN structure with a clearer reasoning in the fuzzy set and in the intuitionistic fuzzy set theory, are known to be the basic elements in the systems studied by fuzzy and intuitionistic fuzzy set theories. With the new implication definitions given below, a causal structure will be established which will match the variable structure of the systems depending on the position and time variables. It is evident that successful results will be achieved in this type of system, which is being dealt with by this new structure. When these two concepts are established, the necessity of satisfying the "modus ponens" conditions in the classical logic will be taken into consideration. At this point, definitions of implication and coimplication will be moved to the temporal intuitionistic fuzzy set space in the studies light, which has been done previously and successfully in practice.

Definition 13. *If a function* $I_t : (TIFP_T^* \times TIFP_T^*) \times T \to IFP^*$ *is satisfied* *following condition, I is called temporal intuitionistic fuzzy implication at time moment t*

I-1: *(Boundary Conditions):*
a: $I_t\left(\left(0_t, a\left(t\right)\right), t\right) = \widetilde{1}$ *for all* $a\left(t\right) \in TIFP_t^*$ *at fixed time moment* t,
b: $I_t\left(\left(a\left(t\right), 1_t\right), t\right) = \widetilde{1}$ *for all* $a\left(t\right) \in TIFP_t^*$ *at fixed time moment* t,
c: $I_t\left(\left(1_t, 0_t\right), t\right) = \widetilde{0}$,
I-2: I_t *is decreasing in first variable i.e.*
If $x \leq y$ *then* $I_t\left(\left(y, z\right), t\right) \leq I_t\left(\left(x, z\right), t\right)$ *for each* $x = \left(x_1\left(t\right), x_2\left(t\right)\right)$, $y = \left(y_1\left(t\right), y_2\left(t\right)\right)$, $z = \left(z_1\left(t\right), z_2\left(t\right)\right) \in IF^* \times T$ *and time moment* $t \in T$,
I-3: I_t *is increasing in second variable i.e.*
If $y \leq z$ *then* $I_t\left(\left(x, y\right), t\right) \leq I_t\left(\left(x, z\right), t\right)$ *for each* $x = \left(x_1\left(t\right), x_2\left(t\right)\right)$, $y = \left(y_1\left(t\right), y_2\left(t\right)\right)$, $z = \left(z_1\left(t\right), z_2\left(t\right)\right) \in IF^* \times T$ *and time moment* $t \in T$.

As this definition shows, the intuitionistic fuzzy pairs to be subjected to the implication process need to change depending on the time. For this reason, the following implication examples will be given based on membership and non-membership values in temporal intuitionistic fuzzy sets. These implications have been obtained by modifying existing implications in the literature according to temporal intuitionistic fuzzy sets.

Proposition 6. *Let*

$$A\left(T'\right) = \left\{\left(x, \mu_A\left(x,t\right), \eta_A\left(x,t\right)\right) : X \times T'\right\}$$

and

$$B\left(T''\right) = \left\{\left(x, \mu_B\left(x,t\right), \eta_B\left(x,t\right)\right) : X \times T''\right\}$$

where T' and T'' have finite number of distinct time-elements or they are time intervals. Then the followings are temporal intuitionistic fuzzy implication at time moment $t \in T = T' \cup T''$.

1. *Kleene- Dienes:*

$$I_t^1\left(\left(\left(\mu_A\left(x,t\right), \eta_A\left(x,t\right)\right), \left(\mu_B\left(x,t\right), \eta_B\left(x,t\right)\right)\right), t\right) = \left(\max\left\{\bar{\eta}_A\left(x,t\right), \bar{\mu}_B\left(x,t\right)\right\}, \min\left\{\bar{\mu}_A\left(x,t\right), \bar{\eta}_B\left(x,t\right)\right\}\right)$$

(This implication is defined by Parvathi and Geeta in [14]*)*

2. *Reichenbach:*

$$I_t^2\left(\left(\left(\mu_A\left(x,t\right), \eta_A\left(x,t\right)\right), \left(\mu_B\left(x,t\right), \eta_B\left(x,t\right)\right)\right), t\right) = \left(\bar{\eta}_A\left(x,t\right) + \bar{\mu}_B\left(x,t\right) - \bar{\eta}_A\left(x,t\right)\bar{\mu}_B\left(x,t\right), \bar{\mu}_A\left(x,t\right)\bar{\eta}_B\left(x,t\right)\right)$$

3. *Gödel:*

$$I_t^3\left(\left(\left(\mu_A\left(x,t\right), \eta_A\left(x,t\right)\right), \left(\mu_B\left(x,t\right), \eta_B\left(x,t\right)\right)\right), t\right) = \begin{cases} (1,0) & , \quad 1 - \bar{\eta}_A\left(x,t\right) \leq \bar{\mu}_B\left(x,t\right) \\ \left(\bar{\mu}_B\left(x,t\right), \bar{\eta}_B\left(x,t\right)\right) & , \quad 1 - \bar{\mu}_A\left(x,t\right) \leq \bar{\eta}_B\left(x,t\right) \\ \left(\bar{\mu}_B\left(x,t\right), 0\right) & , \quad \textit{otherwise} \end{cases}$$

4. *Lukasiewicz:*
$I_t^4\left(\left(\left(\mu_A(x,t)\,,\,\eta_A(x,t)\right),\left(\mu_B(x,t)\,,\,\eta_B(x,t)\right)\right),t\right)=$

$$\left(\min\left\{1,\bar{\eta}_A(x,t)+\bar{\mu}_B(x,t)\right\},\max\left\{0,\left(\bar{\mu}_A(x,t)+\bar{\eta}_A(x,t)-1\right)\right\}\right)$$

5. *Yager:*
$I_t^5\left(\left(\left(\mu_A(x,t)\,,\,\eta_A(x,t)\right),\left(\mu_B(x,t)\,,\,\eta_B(x,t)\right)\right),t\right)=$

$$\left(\left(\bar{\mu}_B(x,t)\right)^{1-\bar{\eta}_A(x,t)},1-\left(1-\bar{\eta}_B(x,t)\right)^{\bar{\mu}_A(x,t)}\right)$$

6. *Mamdani:* $I_t^6\left(\left(\left(\mu_A(x,t)\,,\,\eta_A(x,t)\right),\left(\mu_B(x,t)\,,\,\eta_B(x,t)\right)\right),t\right)=$

$$\left(\min\left\{1-\bar{\eta}_A(x,t)\,,\bar{\mu}_B(x,t)\right\},\max\left\{1-\bar{\mu}_A(x,t)\,,\bar{\eta}_B(x,t)\right\}\right)$$

If I_t is a implication and N_t is a temporal fuzzy strong negation at time moment t then the function

$$\widetilde{I}_t\left(\left(\left(x_1,x_2\right),\left(y_1,y_2\right)\right),t\right)=I_t\left(N_t\left(\left(y_1,y_2\right),t\right),N_t\left(\left(x_1,x_2\right),t\right),t\right)$$

is an implication at time moment t.

Proof. Let I_t be a temporal intuitionistic fuzzy implication at time moment $t\in T$. Then we should show that the mapping $\widetilde{I}_t$ satisfy the conditions I1,I2,I3.

I1:

a. $\widetilde{I}_t\left(\left(0_t,a(t)\right),t\right)=I_t\left(N_t\left(a(t)\,,t\right),N_t\left(0_t,t\right)\right)=I_t\left(\left(N_t\left(a(t)\,,t\right),1_t\right),t\right)$. Since I_t satisfy the condition I-1(a) I-1(b), it is obtained that $I_t\left(\left(N_t\left(a(t)\,,t\right),1_t\right),t\right)=\widetilde{1}$. So it is obtained that $\widetilde{I}_t\left(\left(0_t,a(t)\right),t\right)=\widetilde{1}$ for all $a(t)=\left(a_1(t)\,,a_2(t)\right)\in IF^*$ at fixed time moment t.

b. $\widetilde{I}_t\left(\left(a(t)\,,1_t\right),t\right)=I_t\left(\left(N_t\left(1_t,t\right),N_t\left(a(t)\,,t\right)\right),t\right)=I_t\left(\left(0_t,N_t\left(a(t)\,,t\right)\right),t\right)$. Since I_t satisfy the condition I-1(a), it is obtained that $I_t\left(\left(\left(0_t,N_t\left(a(t)\,,t\right)\right),t\right)\right)=\widetilde{1}$. So it is obtained that $\widetilde{I}_t\left(\left(a(t)\,,0_t\right),t\right)=\widetilde{1}$ for all $a(t)=\left(a_1(t)\,,a_2(t)\right)\in IF^*$ at fixed time moment t.

c. Since I_t satisfy the condition I-1(c), the following equation is obtained as: $\widetilde{I}_t\left(\left(0_t,1_t\right),t\right)=I_t\left(\left(N_t\left(0_t,t\right),N_t\left(1_t,t\right)\right),t\right)=I_t\left(\left(1_t,0_t\right),t\right)=\widetilde{0}$

I2: Let $x(t)=\left(x_1(t)\,,x_2(t)\right)$ and $y(t)=\left(y_1(t)\,,y_2(t)\right)$ are two temporal intuitionistic fuzzy pair such that $x(t)\leq y(t)$ at the time moment t. Since I_t satisfy the condition I3 and $N_t\left(y(t)\right)\leq N_t\left(x(t)\right)$, it is clearly obtained that

$$\widetilde{I}_t\left(\left(y(t)\,,z(t)\right),t\right)=I_t\left(N_t\left(\left(z_1,z_2\right),t\right),N_t\left(\left(y_1,y_2\right),t\right)\right)$$
$$\leq I_t\left(N_t\left(\left(z_1,z_2\right),t\right),N_t\left(\left(x_1,x_2\right),t\right)\right)=\widetilde{I}_t\left(\left(x(t)\,,z(t)\right),t\right).$$

I3: Let $y(t)=\left(y_1(t)\,,y_2(t)\right)$ and $z(t)=\left(z_1(t)\,,z_2(t)\right)$are two temporal intuitionistic fuzzy pair such that $y(t)\leq z(t)$ at the time moment t. Since I_t satisfy the condition I2 and $N_t\left(z(t)\right)\leq N_t\left(y(t)\right)$, it is clearly obtained that

$$\widetilde{I}_t\left(\left(x(t)\,,y(t)\right),t\right)=I_t\left(N_t\left(\left(y_1,y_2\right),t\right),N_t\left(\left(x_1,x_2\right),t\right)\right)$$
$$\leq I_t\left(N_t\left(\left(z_1,z_2\right),t\right),N_t\left(\left(x_1,x_2\right),t\right)\right)=\widetilde{I}_t\left(\left(x(t)\,,z(t)\right),t\right).$$

□

As stated in [15] , the coimplication, which is the dual of the implication concept, is transferred to temporal intuitionistic fuzzy sets as follows.

Definition 14. *If a function* $I_t^c : (TIFP_t^* \times TIFP_t^*) \times T \to IFP^*$*is satisfied following condition,* I_t^c *is called temporal intuitionistic fuzzy coimplication at time moment t*

CI-1: *(Boundary Conditions):*

a: $I_t^c((a(t), 0_t), t) = \widetilde{0}$ *for all* $a(t) = (a_1(t), a_2(t)) \in IF^*$ *at time moment* t,

b: $I_t^c((1_t, a(t)), t) = \widetilde{0}$ *for all* $a(t) = (a_1(t), a_2(t)) \in IF^*$ *at time moment* t,

c: $I_t^c((0_t, 1_t), t) = \widetilde{1}$,

CI-2: I_t^c *is decreasing in first variable i.e.*
If $x \le y$ *then* $I_t^c((y, z), t) \le I_t^c((x, z), t)$ *for each* $x = (x_1(t), x_2(t))$, $y = (y_1(t), y_2(t))$, $z = (z_1(t), z_2(t)) \in IF^* \times T$ *and time moment* $t \in T$,

CI-3: I_t^c *is increasing in second variable i.e.*
If $y \le z$ *then* $I_t^c((x, y), t) \le I_t^c((x, z), t)$ *for each* $x = (x_1(t), x_2(t))$, $y = (y_1(t), y_2(t))$, $z = (z_1(t), z_2(t)) \in IF^* \times T$ *and time moment* $t \in T$.

The relationship between temporal intuitionistic fuzzy implication and temporal intuitionistic fuzzy coimplication is shown below.

Proposition 7. *A function* $I_t^c : (TIFP_t^* \times TIFP_t^*) \times T \to IFP^*$ *is a temporal coimplication at time moment t if and only if the function*

$$I_t((x(t), y(t)), t) = N_t((I_t^c(N_t(x(t), t), N_t(y(t)), t)), t)$$

is a temporal intuitionistic fuzzy implication at time moment t for any temporal intuitionistic fuzzy strong negation N_t *and each* $x(t) = (x_1(t), x_2(t))$, $y(t) = (y_1(t), y_2(t)) \in TIFP_t^*$.

Proof. $\Rightarrow$: Let I_t^c be a coimplication at time moment $t \in T$. Then we should show that the conditions I1,I2,I3 are satisfied.

I1:

a. $I_t((0_t, a(t)), t) = N_t(I_t^c(N_t(0_t, t), N_t(a(t)), t))$. From CI-1(b), it is obtained that
$N_t(I_t^c(1_t, N_t(a(t)), t)) = N_t\left(\widetilde{0}, t\right) = \widetilde{1}$. So it is obtained that $I_t((0_t, a(t)), t) = \widetilde{1}$
for all $a(t) = (a_1(t), a_2(t)) \in IF^*$ at time moment t.

b. $I_t((a(t), 1_t), t) = N_t(I_t^c(N_t(a(t), t), N_t(1_t, t), t)) = N_t(I_t^c(N_t(a(t), t), 0_t, t))$. From CI-1(a), it is obtained that $N_t(I_t^c((N_t(a(t)), 0_t), t)) = N_t(0_t, t) = \widetilde{1}$ for all $a(t) = (a_1(t), a_2(t)) \in IF^*$ at fixed time moment t.

c. From CI-1(c), $I_t((1_t, 0_t), t) = N_t(I_t^c(N_t(1_t, t), N_t(0_t, t), t)) = N_t(I_t^c(0_t, 1_t, t))$ $= N_t\left(\widetilde{1}, t\right) = \widetilde{0}$

I2: Let $x(t)$ and $y(t) \in TIFP_t^*$ such that $x(t) \leq y(t)$ at the time moment t, From CI-2 and $N_t(y(t), t) \leq N_t(x(t), t)$, the inequality

$$I_t^c((N_t(x(t), t), N_t(z(t), t)), t) \leq I_t^c((N_t(y(t), t), N_t(z(t), t)), t)$$

is satisfied for any $z(t) \in TIFP_t^*$ at fixed time moment t. Since N_t is temporal intuitionistic fuzzy strong negation at the time moment t, the inequality

$$N_t(I_t^c((N_t(y(t), t), N_t(z(t), t)), t)) \leq N_t(((N_t(x(t), t), N_t(z(t), t)), t))$$

is obtained. So it is clearly understood that the inequality

$$I_t((y(t), z(t)), t) \leq I_t((x(t), z(t)), t)$$

is satisfied at the time moment t with the above assumptions.

I3: Let be $y(t)$ and $z(t) \in TIFP_t^*$ such that $y(t) \leq z(t)$ at fixed time moment t. From CI-3 and $N_t(z(t), t) \leq N_t(y(t), t)$, it is obtained that

$$I_t^c((N_t(x(t), t), N_t(z(t)), t), t) \leq I_t^c((N_t(x(t), t), N_t(y(t), t)), t).$$

Since N_t is temporal intuitionistic strong negation at the time moment t, the inequality

$$N_t(I_t^c((N_t(x(t), t), N_t(y(t), t)), t), t) \leq N_t(I_t^c((N_t(x(t), t), N_t(z(t), t)), t))$$

is obtained. So it is clearly understood that the inequality

$$I_t(x(t), y(t)) \leq I_t(x(t), z(t))$$

is satisfied at the time moment t with the above assumptions. □

Theorem 1. *Let S_t be a temporal intuitionistic fuzzy t-conorm and N_t be a temporal intuitionistic fuzzy strong negation at time moment t. Then, the mapping defined as $I_{S_t}((x(t), y(t)), t) = S_t((N_t(x(t), t), y(t)), t)$ for each $x(t), y(t) \in IF^*$ is a temporal intuitionistic fuzzy implication.*

Proof. **I1-**

a. $I_{S_t}((0_t, x(t)), t) = S_t((N_t(0_t, t), y(t)), t) = S_t((1_t, y(t)), t) = \widetilde{1}$,

b. $I_{S_t}((x(t), 1_t), t) = S_t((N_t(a_t, t), 1_t), t) = \widetilde{1}$,

c. $I_{S_t}((1_t, 0_t), t) = S_t((N_t(1_t, t), 1_t), t) = S_t((0_t, 0_t), t) = \widetilde{0}$.

I2- Let be $x(t)$ and $y(t) \in TIFP_t^*$ such that $x(t) \leq y(t)$ at the time moment t. Then $N_t(y(t), t) \leq N_t(x(t), t)$. From *S2*, $S_t((N_t(y(t), t), z(t)), t) \leq S_t((N_t(x(t), t), z(t)), t)$ for each $z(t) \in TIFP_t^*$. Thus

$$I_{S_t}((y(t), z(t)), t) \leq I_{S_t}((x(t), z(t)), t).$$

I3- Let be $y(t)$ and $z(t) \in TIFP_t^*$ such that $y(t) \leq z(t)$ at the time moment t. From *S2*, $I_{S_t}((x(t), y(t)), t) = S_t((N_t(x(t), t), y(t)), t) \leq S_t((N_t(x(t), t), z(t)), t)$ $= I_{S_t}((x(t), z(t)), t)$ for each $z(t) \in TIFP_t^*$. Thus it is obtained that

$$I_{S_t}((x(t), y(t)), t) \leq I_{S_t}((x(t), z(t)), t).$$

□

Definition 15. *Let S_t be a temporal intuitionistic fuzzy t-conorm and N_t be a temporal intuitionistic fuzzy strong negation at fixed time moment t. Then I_{S_t} : $(TIFP_T^* \times TIFP_T^*) \times T \rightarrow IFP^*$ is called temporal intuitionistic fuzzy $S-$implication.*

Example 1. *I_t^1 is a $S-$implication produced with $S_{\max}^t$ and temporal intuitionistic fuzzy standard negation*

Theorem 2. *Let T_t be a temporal intuitionistic fuzzy t-norm and N_t be a temporal intuitionistic fuzzy strong negation at time moment t. Let define the family of TIFPs such as $Z_{(x(t),y(t))} = \{z(t) = (z_x(t), z_y(t)) \in TIFP_T^* : T_t((x(t), z(t)), t) \leq y(t)\}$ for each $x(t), y(t) \in TIFP_T^*$. Then, the mapping defined as $I_{T_t}((x(t), y(t)), t) = (\sup(z_x), \inf(z_y))$ is a temporal intuitionistic fuzzy implication.*

Proof. **I1-**

a. Since $T_t((0_t, z(t)), t) = 0_t \leq y(t)$ for each $z(t) = (z_x(t), z_y(t)) \in TIFP_T^*$, So 1_t can be chosen as $z(t)$. Then, it is obtained that $I_{T_t}((0_t, y(t)), t) = \widetilde{1}$,

b. From T4, $T_t((x(t), 1_t), t) = x(t) \leq 1_t$. So 1_t can be chosen as $z(t)$. Then, it is obtained that $I_{T_t}((0_t, x(t)), t) = \widetilde{1}$.

c. Since *the equation* $T_t((1_t, z(t)), t) = z(t) \leq 0_t$ has a only one solution as $z(t) = 0_t$,it is clearly understood that $I_{T_t}((1_t, 0_t), t) = \widetilde{1}$.

I2- Let be $x(t)$ and $y(t) \in TIFP_t^*$ such that $x(t) \leq y(t)$ at the time moment t. We must show that $I_{T_t}((y(t), z(t)), t) \leq I_{T_t}((x(t), z(t)), t)$. From T2, The inequality $T_t((x(t), z^*(t)), t) \leq z(t)$ is satisfied for each $z^*(t) \in TIFP_T^*$ which satisfy the inequality $T_t((y(t), z^*(t)), t) \leq z(t)$.Then $Z_{(y(t),z(t))} \subseteq Z_{(x(t),z(t))}$. Then it is clearly understood from the definition of I_{T_t}

$$I_{T_t}((y(t), z(t)), t) \leq I_{T_t}((x(t), z(t)), t).$$

I3- Let be $y(t)$ and $z(t) \in TIFP_t^*$ such that $y(t) \leq z(t)$ at the time moment t. We must show that $I_{T_t}((x(t), y(t)), t) \leq I_{T_t}((x(t), z(t)), t)$. The inequality $T_t((x(t), z^*(t)), t) \leq z(t)$ is satisfied for each $z^*(t) \in TIFP_T^*$ which satisfy the inequality $T_t((x(t), z^*(t)), t) \leq y(t)$.Then $Z_{(x(t),z(t))} \subseteq Z_{(y(t),z(t))}$. Then it is clearly understood from the definition of I_{T_t}

$$I_{T_t}((x(t), y(t)), t) \leq I_{T_t}((x(t), z(t)), t).$$

□

Definition 16. *Let T_t be a temporal intuitionistic fuzzy t-conorm and N_t be a temporal intuitionistic fuzzy strong negation at fixed time moment t. Then I_{T_t} : $(TIFP_T^* \times TIFP_T^*) \times T \rightarrow IFP^*$ is called temporal intuitionistic fuzzy $R-$implication.*

Proposition 8. *Let I_{T_t} be a temporal intuitionistic fuzzy $R-$implication produced any T_t temporal intuitionistic fuzzy t-conorm and N_t temporal intuitionistic fuzzy strong negation at fixed time moment t. Then $I_{T_t}((x(t), x(t)), t) = \widetilde{1}$ for each $x(t) \in TIFP_t^*$.*

Proof. From T4, $T_t\left(\left(x\left(t\right),1_t\right),t\right)=x\left(t\right)$. Then it is understood that $1_t\in Z_{(x(t),x(t))}$. So $I_{T_t}\left(x\left(t\right),x\left(t\right)\right)=\widetilde{1}$ □

Remark 2. *The concepts, which we have given in our work until this section, have always been defined for a single time moment. If these concepts are defined in all of their clusters when these concepts are defined, these concepts are called the overall intuitionistic fuzzy (negation, t-norm, t-conorm, implication and coimplication). It is often essential to produce a final conclusion from a concept that is overall intuitionistic fuzzy. This could be done using the aggregation function. The following theorem offers a way for this final conclusion.*

Theorem 3. *Let $T=\{t_1,t_2,...t_n\}$ be a finite time set which has $n\geq 2$ elements, N_{t_i} be a overall intuitionistic fuzzy negation and $f:\left(TIFP_T^*\right)^n\rightarrow IFP^*$ $(n\geq 2)$ be a function satisfied following conditions:*

1. $f\left(0_T,0_T,...,0_T\right)=\widetilde{0}$ *and* $f\left(1_T,1_T,...,1_T\right)=\widetilde{1}$
2. $f\left(a\left(t_1\right),a\left(t_2\right),...,a\left(t_n\right)\right)\leq f\left(b\left(t_1\right),b\left(t_2\right),...,b\left(t_n\right)\right)$ *for any pair* $(a\left(t_1\right),$ $a\left(t_2\right),\ \ldots,a\left(t_n\right))$ *and* $\left(b\left(t_1\right),b\left(t_2\right),...,b\left(t_n\right)\right)$ *of* $n-$*tuples in* $\left(TIFP_T^*\right)^n$ *such that* $a\left(t_i\right)\leq b\left(t_i\right)$ $(i\in\{1,2,...,n\})$
3. f *is a continuous function.*

Then the mapping $N:TIFP_T^*\rightarrow IFP^*$ defined as

$$N\left(x\left(t_i\right)\right)=f\left(N_{t_1}\left(x\left(t_i\right),t_1\right),N_{t_2}\left(x\left(t_i\right),t_2\right),...,N_{t_n}\left(x\left(t_i\right),t_n\right)\right)$$

$(i\in\{1,2,...,n\})$ *is a intuitionistic fuzzy negation on $TIFP_T^*$*

Proof. For every $x\left(t_i\right),y\left(t_i\right)\in TIFP_T^*$ and $i\in\{1,2,...,n\}$ such that $x\left(t_i\right)\leq y\left(t_i\right)$, the inequality $N_{t_j}\left(y\left(t_i\right),t_j\right)\leq N_{t_j}\left(x\left(t_i\right),t_j\right)$ is obtained for each $i,j\in\{1,2,...,n\}$ from the definition of overall intuitionistic fuzzy negation. Then, following inequality is clearly obtained from the definition of f for each $i\in\{1,2,...,n\}$:

$$\begin{aligned}N\left(y\left(t_i\right)\right)&=f\left(N_{t_1}\left(y\left(t_i\right),t_1\right),N_{t_2}\left(y\left(t_i\right),t_2\right),...,N_{t_n}\left(y\left(t_i\right),t_n\right)\right)\\&\leq f\left(N_{t_1}\left(x\left(t_i\right),t_1\right),N_{t_2}\left(x\left(t_i\right),t_2\right),...,N_{t_n}\left(x\left(t_i\right),t_n\right)\right)=N\left(x\left(t_i\right)\right)\end{aligned}$$

Hence it is clearly understood that N is decreasing. On the other hand,

$$\begin{aligned}N\left(0_T\left(t_i\right)\right)&=f\left(N_{t_1}\left(0_T\left(t_i\right),t_1\right),N_{t_2}\left(0_T\left(t_i\right),t_2\right),...,N_{t_n}\left(0_T\left(t_i\right),t_n\right)\right)\\&=f\left(\widetilde{1},\widetilde{1},...,\widetilde{1}\right)=\widetilde{1},\\N\left(1_T\left(t_i\right)\right)&=f\left(N_{t_1}\left(1_T\left(t_i\right),t_1\right),N_{t_2}\left(1_T\left(t_i\right),t_2\right),...,N_{t_n}\left(1_T\left(t_i\right),t_n\right)\right)\\&=f\left(\widetilde{0},\widetilde{0},...,\widetilde{0}\right)=\widetilde{0}.\end{aligned}$$

□

Theorem 4. *Let $T=\{t_1,t_2,...t_n\}$ be a finite time set which has $n\geq 2$ elements, T_{t_i} be a overall intuitionistic fuzzy $t-$norm and $f:\left(TIFP_T^*\right)^n\rightarrow IFP^*$ $(n\geq 2)$ be a function satisfied following conditions:*

1. $f\left(a\left(t_i\right),a\left(t_i\right),...,a\left(t_i\right)\right)=a\left(t_i\right)$ *for* $a\left(t_i\right)\in TIFP_T^*$,

(2) $f(a(t_1), a(t_2), ..., a(t_n)) \leq f(b(t_1), b(t_2), ..., b(t_n))$ *for any pair* $(a(t_1), a(t_2), ..., a(t_n))$ *and* $(b(t_1), b(t_2), ..., b(t_n))$ *of* $n-$*tuples in* $(TIFP_T^*)^n$ *such that* $a(t_i) \leq b(t_i)$ $(i \in \{1, 2, ..., n\})$.

(3) f *is a continuous function.*

Then the mapping $T : TIFP_T^* \to IFP^*$ *defined as*

$$T(x(t_i), y(t_i)) = f(T_{t_1}((x(t_i), y(t_i)), t_1), T_{t_2}((x(t_i), y(t_i)), t_2), ..., T_{t_n}((x(t_i), y(t_i)), t_n))$$

$(i \in \{1, 2, ..., n\})$ *is a intuitionistic fuzzy* $t-$*norm on* $TIFP_T^*$.

Proof. T1. Since the equation $T_{t_j}((x(t_i), y(t_i)), t_j) = T_{t_j}((x(t_i), y(t_i)), t_j)$ holds for every $x, y \in TIFP_T^*$ and $i, j \in \{1, 2, ..., n\}$, the following equation

$$\begin{aligned} & T((x(t_i), y(t_i))) \\ = \ & f(T_{t_1}((x(t_i), y(t_i)), t_1), T_{t_2}((x(t_i), y(t_i)), t_2), ..., T_{t_n}((x(t_i), y(t_i)), t_n)) \\ = \ & f(T_{t_1}((y(t_i), x(t_i)), t_1), T_{t_2}((y(t_i), x(t_i)), t_2), ..., T_{t_n}((y(t_i), x(t_i)), t_n)) \\ = \ & T((y(t_i), x(t_i))) \end{aligned}$$

is obtained for each $i \in \{1, 2, ..., n\}$.

T2. Since T_{t_j} is a overall intuitionistic fuzzy $t-$norm, the inequality

$$T_{t_j}((x_1(t_i), y_1(t_i)), t_j) \leq T_{t_j}((x_2(t_i), y_2(t_i)), t_j)$$

is satisfied for every $i, j \in \{1, 2, ..., n\}$ and every $x_1(t_i), y_1(t_i), x_2(t_i), y_2(t_i) \in TIFP_T^*$ such that $x_1(t_i) \leq x_2(t_i)$ and $y_1(t_i) \leq y_2(t_i)$.

From the definition of f , the following inequality is obtained:

$$\begin{aligned} & f(T_{t_1}((x_1(t_i), y_1(t_i)), t_1), T_{t_2}((x_1(t_i), y_1(t_i)), t_2), ..., T_{t_n}((x_1(t_i), y_1(t_i)), t_n)) \\ \leq \ & f(T_{t_1}((x_2(t_i), y_2(t_i)), t_1), T_{t_2}((x_2(t_i), y_2(t_i)), t_2), ..., T_{t_n}((x_2(t_i), y_2(t_i)), t_n)) \end{aligned}$$

Then it is obtained that $T((x_1(t_i), y_1(t_i))) \leq T((x_2(t_i), y_2(t_i)))$.

T3. Since T_{t_j} is a overall intuitionistic fuzzy $t-$norm, the equality

$$T_{t_j}\left(\left(T_{t_j}((x(t_i), y(t_i)), t_j)\right), z(t_i), t_j\right) = T_{t_j}\left(\left(x(t_i), T_{t_j}((z(t_i), y(t_i)), t_j)\right), t_j\right)$$

is satisfied for every $i, j \in \{1, 2, ..., n\}$ and every $x(t_i), y(t_i) \in TIFP_T^*$. Then we must show that

$$T((T((x(t_i), y(t_i)), t_j)), z(t_i), t_j) = T((x(t_i), T((z(t_i), y(t_i)), t_j)), t_j)$$

Let $T((x(t_i), y(t_i))) = a(t_i)$, $T((z(t_i), y(t_i))) = b(t_i)$. Hence the following equation is obtained

$$T_{t_j}((a(t_i), z(t_i)), t_j) = T_{t_j}((x(t_i), b(t_i)), t_j)$$

Then

$$\begin{aligned} & T(a(t_i), z(t_i)) \\ = \ & f(T_{t_1}((a(t_i), z(t_i)), t_1), T_{t_2}((a(t_i), z(t_i)), t_2), ..., T_{t_n}((a(t_i), x(t_i)), t_n)) \\ = \ & f(T_{t_1}((x(t_i), b(t_i)), t_1), T_{t_2}((x(t_i), b(t_i)), t_2), ..., T_{t_n}((x(t_i), b(t_i)), t_n)) \end{aligned}$$

$$= \quad T\left(x\left(t_i\right), b\left(t_i\right)\right)$$

T4. Since T_{t_j} is a overall intuitionistic fuzzy $t-$norm, the equality

$$T_{t_j}\left(\left(x\left(t_i\right), 1_t\right), t_j\right) = x\left(t_i\right)$$

for every $x\left(t_i\right) \in TIFP_T^*$ and for every $i, j \in \{1, 2, ..., n\}$. Then it is easily obtained that

$$\begin{aligned} & f\left(T_{t_1}\left(\left(x\left(t_i\right), 1_t\right), t_1\right), T_{t_2}\left(\left(x\left(t_i\right), 1_t\right), t_2\right), ..., T_{t_n}\left(\left(x\left(t_i\right), 1_t\right), t_n\right)\right) \\ = \quad & f\left(x\left(t_i\right), x\left(t_i\right), ..., x\left(t_i\right)\right) = x\left(t_i\right) \end{aligned}$$

□

Theorem 5. *Let $T = \{t_1, t_2, ...t_n\}$ be a finite time set which has $n \geq 2$ elements, S_{t_i} be a overall intuitionistic fuzzy $s-norm$ and $f : \left(TIFP_T^*\right)^n \rightarrow IFP^*$ $(n \geq 2)$ be a function satisfied following conditions:*

(1) *$f\left(a\left(t_i\right), a\left(t_i\right), ..., a\left(t_i\right)\right) = a\left(t_i\right)$ for $a\left(t_i\right) \in TIFP_T^*$,*
(2) *$f\left(a\left(t_1\right), a\left(t_2\right), ..., a\left(t_n\right)\right) \leq f\left(b\left(t_1\right), b\left(t_2\right), ..., b\left(t_n\right)\right)$ for any pair $(a\left(t_1\right), a\left(t_2\right), ..., a\left(t_n\right))$ and $\left(b\left(t_1\right), b\left(t_2\right), ..., b\left(t_n\right)\right)$ of $n-$tuples in $\left(TIFP_T^*\right)^n$ such that $a\left(t_i\right) \leq b\left(t_i\right)$ $(i \in \{1, 2, ..., n\})$,*
(3) *f is a continuous function.*

Then the mapping $S : TIFP_T^* \rightarrow IFP^*$ defined as

$$\begin{aligned} & S\left(x\left(t_i\right), y\left(t_i\right)\right) = \\ & \quad f\left(S_{t_1}\left(\left(x\left(t_i\right), y\left(t_i\right)\right), t_1\right), S_{t_2}\left(\left(x\left(t_i\right), y\left(t_i\right)\right), t_2\right), ..., S_{t_n}\left(\left(x\left(t_i\right), y\left(t_i\right)\right), t_n\right)\right) \end{aligned}$$

$(i \in \{1, 2, ..., n\})$ is a intuitionistic fuzzy $s-norm$ on $TIFP_T^*$.

Proof. It could be proven as previous theorem. □

Theorem 6. *Let $T = \{t_1, t_2, ...t_n\}$ be a finite time set which has $n \geq 2$ elements, I_{t_i} be a overall intuitionistic fuzzy implication and $f : \left(TIFP_T^*\right)^n \rightarrow IFP^*$ $(n \geq 2)$ be a function satisfied following conditions:*

(1) *$f\left(a\left(t_i\right), a\left(t_i\right), ..., a\left(t_i\right)\right) = a\left(t_i\right)$ for $a\left(t_i\right) \in TIFP_T^*$,*
(2) *$f\left(a\left(t_1\right), a\left(t_2\right), ..., a\left(t_n\right)\right) \leq f\left(b\left(t_1\right), b\left(t_2\right), ..., b\left(t_n\right)\right)$ for any pair $(a\left(t_1\right), a\left(t_2\right), ..., a\left(t_n\right))$ and $\left(b\left(t_1\right), b\left(t_2\right), ..., b\left(t_n\right)\right)$ of $n-$tuples in $\left(TIFP_T^*\right)^n$ such that $a\left(t_i\right) \leq b\left(t_i\right)$ $(i \in \{1, 2, ..., n\})$,*
(3) *f is a continuous function.*

Then the mapping $I : TIFP_T^* \rightarrow IFP^*$ defined as

$$\begin{aligned} & I\left(x\left(t_i\right), y\left(t_i\right)\right) = \\ & \quad f\left(I_{t_1}\left(\left(x\left(t_i\right), y\left(t_i\right)\right), t_1\right), I_{t_2}\left(\left(x\left(t_i\right), y\left(t_i\right)\right), t_2\right), ..., I_{t_n}\left(\left(x\left(t_i\right), y\left(t_i\right)\right), t_n\right)\right) \end{aligned}$$

$(i \in \{1, 2, ..., n\})$ is a intuitionistic fuzzy implication on $TIFP_T^*$.

Proof. **I-1**: (Boundary Conditions):

a. Since $I_t\left(\left(0_T, a\left(t_i\right)\right), t\right)=\widetilde{1}$ for all $a(t) \in TIFP_t^*$ and every time moment $t \in T$,The equation is satisfied

$$\begin{aligned} I\left(0_T, a\left(t_i\right)\right) &= f\left(I_{t_1}\left(\left(0_T, a\left(t_i\right)\right), t_1\right), I_{t_2}\left(\left(0_T, a\left(t_i\right)\right), t_2\right), \ldots, I_{t_n}\left(\left(0_T, a\left(t_i\right)\right), t_n\right)\right) \\ &= f\left(1_T, 1_T, \ldots, 1_T\right)=\widetilde{1} \end{aligned}$$

b. Since $I\left(\left(a\left(t_i\right), 1_T\right), t\right)=\widetilde{1}$ for all $a\left(t_i\right) \in TIFP_t^*$ and every time moment $t \in T$,The equation is satisfied

$$\begin{aligned} I\left(a\left(t_i\right), 1_T\right) &= f\left(I_{t_1}\left(\left(a\left(t_i\right), 1_T\right), t_1\right), I_{t_2}\left(\left(a\left(t_i\right), 1_T\right), t_2\right), \ldots, I_{t_n}\left(\left(a\left(t_i\right), 1_T\right), t_n\right)\right) \\ &= f\left(1_T, 1_T, \ldots, 1_T\right)=\widetilde{1} \end{aligned}$$

c. Since $I_t\left(\left(1_T, 0_T\right), t\right)=\widetilde{0}$ for every time moment $t \in T$,The equation is satisfied

$$\begin{aligned} I\left(1_T, 0_T\right) &= f\left(I_{t_1}\left(\left(1_T, 0_T\right), t_1\right), I_{t_2}\left(\left(1_T, 0_T\right), t_2\right), \ldots, I_{t_n}\left(\left(1_T, 0_T\right), t_n\right)\right) \\ &= f\left(0_T, 0_T, \ldots, 0_T\right)=\widetilde{0} \end{aligned}$$

I-2: Since I_t is decreasing in first variable, the inequality $I_t\left(y\left(t_i\right), z\left(t_i\right), t\right) \leq I_t\left(x\left(t_i\right), z\left(t_i\right), t\right)$ is satisfied at every time moment t and each $x=\left(x_1(t), x_2(t)\right)$, $y=\left(y_1(t), y_2(t)\right)$, $z=\left(z_1(t), z_2(t)\right) \in TIFP_t^*$ such that $x \leq y$. As the definition of f, the following inequality is obtained such that:

$$\begin{aligned} & I\left(\left(y\left(t_i\right), z\left(t_i\right)\right), t\right) \\ = & f\left(I_{t_1}\left(\left(y\left(t_i\right), z\left(t_i\right)\right), t_1\right), I_{t_2}\left(\left(y\left(t_i\right), z\left(t_i\right)\right), t_2\right), \ldots, I_{t_n}\left(\left(y\left(t_i\right), z\left(t_i\right)\right), t_n\right)\right) \\ \leq & f\left(I_{t_1}\left(\left(x\left(t_i\right), z\left(t_i\right)\right), t_1\right), I_{t_2}\left(\left(x\left(t_i\right), z\left(t_i\right)\right), t_2\right), \ldots, I_{t_n}\left(\left(x\left(t_i\right), z\left(t_i\right)\right), t_n\right)\right) \\ = & I\left(\left(x\left(t_i\right), z\left(t_i\right)\right), t\right) \end{aligned}$$

I-3: Since I_t is increasing in second variable, the inequality $I_t\left(y\left(t_i\right), x\left(t_i\right), t\right) \leq I_t\left(z\left(t_i\right), x\left(t_i\right), t\right)$ is satisfied at every time moment t and each $x=\left(x_1(t), x_2(t)\right)$, $y=\left(y_1(t), y_2(t)\right)$, $z=\left(z_1(t), z_2(t)\right) \in TIFP_t^*$ such that $y \leq z$. As the definition of f, the following inequality is obtained such that:

$$\begin{aligned} & I\left(\left(y\left(t_i\right), x\left(t_i\right)\right), t\right) \\ = & f\left(I_{t_1}\left(\left(y\left(t_i\right), x\left(t_i\right)\right), t_1\right), I_{t_2}\left(\left(y\left(t_i\right), x\left(t_i\right)\right), t_2\right), \ldots, I_{t_n}\left(\left(y\left(t_i\right), x\left(t_i\right)\right), t_n\right)\right) \\ \leq & f\left(I_{t_1}\left(\left(z\left(t_i\right), x\left(t_i\right)\right), t_1\right), I_{t_2}\left(\left(z\left(t_i\right), x\left(t_i\right)\right), t_2\right), \ldots, I_{t_n}\left(\left(z\left(t_i\right), x\left(t_i\right)\right), t_n\right)\right) \\ = & I\left(\left(z\left(t_i\right), x\left(t_i\right)\right), t\right) \end{aligned}$$

$\square$

Theorem 7. *Let $T=\{t_1, t_2, \ldots t_n\}$ be a finite time set which has $n \geq 2$ elements, $I_{t_i}^c$ be an overall intuitionistic fuzzy coimplication and $f:\left(TIFP_T^*\right)^n \rightarrow IFP^*$ $(n \geq 2)$ be a function satisfied following conditions:*

(1) *$f\left(a\left(t_i\right), a\left(t_i\right), \ldots, a\left(t_i\right)\right)=a\left(t_i\right)$ for $a\left(t_i\right) \in TIFP_T^*$,*
(2) *$f\left(a\left(t_1\right), a\left(t_2\right), \ldots, a\left(t_n\right)\right) \leq f\left(b\left(t_1\right), b\left(t_2\right), \ldots, b\left(t_n\right)\right)$ for any pair $\left(a\left(t_1\right), a\left(t_2\right), \ldots, a\left(t_n\right)\right)$ and $\left(b\left(t_1\right), b\left(t_2\right), \ldots, b\left(t_n\right)\right)$ of $n-$tuples in $\left(TIFP_T^*\right)^n$ such that $a\left(t_i\right) \leq b\left(t_i\right)$ $(i \in\{1,2, \ldots, n\})$,*

(3) *f is a continuous function.*

Then the mapping $I^C : TIFP_T^* \to IFP^*$ defined as

$$\begin{aligned} & I^C\left(x\left(t_i\right), y\left(t_i\right)\right) \\ = \quad & f\left(I_{t_1}^c\left(\left(x\left(t_i\right), y\left(t_i\right)\right), t_1\right), I_{t_2}^c\left(\left(x\left(t_i\right), y\left(t_i\right)\right), t_2\right), ..., I_{t_n}^c\left(\left(x\left(t_i\right), y\left(t_i\right)\right), t_n\right)\right) \end{aligned}$$

$(i \in \{1, 2, ..., n\})$ *is an intuitionistic fuzzy coimplication on $TIFP_T^*$.*

5. Conclusion

It is understood from the definitions and theorems given in the whole article, from the judgments obtained from a temporal system, that a conclusion judgment could be obtained by aggregation functions. This provides a way for crisp outlets to be obtained from temporal intuitionistic fuzzy systems. In this study; temporal intuitionistic fuzzy negation, temporal intuitionistic fuzzy triangular norm and temporal intuitionistic fuzzy triangular conorm have been researched. The aim of this study is to define negator, t-norm and t-conorms, which is the generalization of negation, conjunctions and disconjunctions in the temporal intuitionistic fuzzy sets and to examine the De Morgan relations between these concepts. The thing to note here is that conjunctions generalized with $t-$norm and $t-$conorm is changed depending on time. we will carry concept of implication and coimplication to temporal intuitionistic fuzzy sets. With the new implication definitions, a causal structure will be established which will match the variable structure of the systems depending on the position and time variables. It is evident that successful results will be achieved in this type of system, which is being dealt with by this new structure.

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GENERALIZED BURNSIDE ALGEBRA OF TYPE B_n

HASAN ARSLAN AND HIMMET CAN

ABSTRACT. In this paper, we firstly give an alternative method to determine the size of $C(S_n)$ which is the set of elements of type S_n in a finite Coxeter system (W_n, S_n) of type B_n. We also show that all cuspidal classes of W_n are actually the conjugacy classes $\mathcal{K}_\lambda$ for every $\lambda \in \mathcal{DP}^+(n)$. We then define the generalized Burnside algebra $HB(W_n)$ for W_n and construct a surjective algebra morphism between $HB(W_n)$ and Mantaci-Reutenauer algebra $\mathcal{MR}(W_n)$. We obtain a set of orthogonal primitive idempotents e_λ, $\lambda \in \mathcal{DP}(n)$ of $HB(W_n)$, that is, all the characteristic class functions of W_n. Finally, we give an effective formula to compute the number of elements of all the conjugacy classes $\mathcal{K}_\lambda$, $\lambda \in \mathcal{DP}(n)$ of W_n.

1. INTRODUCTION

Solomon's descent algebra of a finite Coxeter system (W, S) was introduced by Solomon in 1976 in [11]. In 1992, Bergeron, Bergeron, Howlett and Taylor elegantly reconstructed the Solomon's descent algebra for a finite Coxeter system by using the group structure of Coxeter group and also they introduced a family of orthogonal primitive idempotents of the Solomon's descent algebra by lifting orthogonal primitive idempotents of parabolic Burnside algebra in [1].

Let W_n be the Coxeter group of type B_n. As a convention, throughout this paper, we denote by $HB(W_n)$, $\mathcal{MR}(W_n)$, $\mathcal{SC}(n)$ and $\mathcal{DP}(n)$ the generalized Burnside algebra of type B_n, the Mantaci-Reutenauer algebra, the set of all signed compositions of n and the set of all double partitions of n, respectively.

Mantaci-Reutenauer algebra $\mathcal{MR}(W_n)$, which is a subalgebra of the group algebra $\mathbb{Q}W_n$ and contains the Solomon's descent algebras of type A_n and B_n, was firstly constructed in [10]. In [2], Bonnafé and Hohlweg reconstructed $\mathcal{MR}(W_n)$ by the methods which depend more on the structure of W_n as a Coxeter group. Bonnafé studied the representation theory of Mantaci-Reutenauer algebra in [3].

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In Section 3, we prove that for every positive signed composition A of n, the parabolic closure of the reflection subgroup W_A is W_n. As a result of this, we obtain that the number of all elements of type S_n is equal to $\sum_{\lambda\in\mathcal{DP}^+(n)}|\mathcal{K}_\lambda|$ and realize that all cuspidal classes of W_n are the conjugacy classes $\mathcal{K}_\lambda$ for $\lambda\in\mathcal{DP}^+(n)$.

In Section 4, we introduce the Burnside algebra $HB(W_n)$ generated by isomorphism classes of reflection subgroups of W_n corresponding to signed compositions of n. We call $HB(W_n)$ *generalized Burnside algebra* of type B_n. Generalized Burnside algebra $HB(W_n)$ is isomorphic to the algebra $\mathbb{Q}\mathrm{Irr}W_n$ generated by the irreducible characters of W_n. Then we construct a set of orthogonal primitive idempotents of $HB(W_n)$. These orthogonal primitive idempotents are actually all the characteristic class functions of the Coxeter group W_n. We determine the coefficient of the sign character ε_n of W_n in the expression of the each orthogonal primitive idempotent of $HB(W_n)$ in terms of irreducible characters of W_n. We get a formula to compute the number of elements of all the conjugacy classes $\mathcal{K}_\lambda$, $\lambda\in\mathcal{DP}(n)$ of W_n.

2. Preliminaries

2.1. **Hyperoctahedral group.** Let (W_n,S_n) denote a Coxeter group of type B_n and write its generating set as $S_n=\{t,s_1,\cdots,s_{n-1}\}$. Any element w of W_n acts by the permutation on the set $X_n=\{-n,\cdots,-1,1,\cdots,n\}$ such that for every $i\in I_n$, $w(-i)=-w(i)$. The Dynkin diagram of W_n is as follows:

$$B_n\ :\ \overset{t}{\circ}\Leftarrow\overset{s_1}{\circ}-\overset{s_2}{\circ}-\cdots-\overset{s_{n-1}}{\circ}.$$

If $J\subset S_n$, the subgroup W_J generated by J is called a *standard parabolic subgroup* of W_n. A *parabolic subgroup* of W_n is a subgroup of W_n conjugate to W_J for some $J\subset S_n$. Let $t_1:=t$ and $t_i:=s_{i-1}t_{i-1}s_{i-1}$ for each i, $2\le i\le n$. Put $T_n:=\{t_1,\cdots,t_n\}$. It is well-known that there are the following relations between the elements of S_n and T_n:

(1) $t_i^2=1, s_j^2=1$ for all $i,j,\ 1\le i\le n,\ 1\le j\le n-1$;
(2) $ts_1ts_1=s_1ts_1t$;
(3) $s_is_{i+1}s_i=s_{i+1}s_is_{i+1}$ for all $i,\ 1\le i\le n-2$;
(4) $ts_i=s_it,\ 1<i\le n-1$;
(5) $s_is_j=s_js_i$ for $|i-j|>1$;
(6) $t_it_j=t_jt_i$ for $1\le i,j\le n$.

We denote by $l:W_n\to\mathbb{N}$ the length function attached to S_n. Let $\mathcal{T}_n$ denote the reflection subgroup of W_n generated by T_n. It is also clear that $\mathcal{T}_n$ is a normal subgroup of W_n. Now let $S_{-n}=\{s_1,\cdots,s_{n-1}\}$ and let W_{-n} denote the reflection subgroup of W_n generated by S_{-n}, where W_{-n} is isomorphic to the symmetric group Ξ_n of degree n. Thus $W_n=W_{-n}\ltimes\mathcal{T}_n$.

Let $\{e_1,\cdots,e_n\}$ be the canonical basis of the Euclidian space $\mathbb{R}^n$ over $\mathbb{R}$. Let

$$\Psi_n^+=\{e_i:1\le i\le n\}\cup\{e_j+\lambda e_i:\lambda\in\{-1,1\}\text{ and }1\le i<j\le n\}.$$

Then Ψ_n is a root system of type B_n. For further information about the Coxeter groups of type B_n, see [8], [9].

A *signed composition* of n is an expression of n as a finite sequence $A = (a_1, \cdots, a_k)$ whose each part consists of non-zero integers such that $\sum_{i=1}^{k} |a_i| = n$. Put $|A| = \sum_{i=1}^{k} |a_i|$. We write $\mathcal{SC}(n)$ to denote the set of all signed compositions of n.

Let $A = (a_1, \cdots, a_k) \in \mathcal{SC}(n)$. A is said to be *positive*(resp. *negative*) if $a_i > 0$ (resp. $a_i < 0$) for every $i \geq 1$. If $a_i < 0$ for every $i \geq 2$, then A is called *parabolic*. Let define $A^+ = (|a_1|, \cdots, |a_r|)$. Then A^+ is a positive signed composition of n. The set of positive signed compositions of n is denoted by $\mathcal{SC}^+(n)$.

A *double partition* $\mu = (\mu^+; \mu^-)$ of n consists of a pair of partitions μ^+ and μ^- such that $|\mu| = |\mu^+| + |\mu^-| = n$. If the number of positive parts of n (resp. negative parts of n) is equal to zero, then we write $\emptyset$ instead of μ^+ (resp. μ^-). We denote the set of all double partitions of n by $\mathcal{DP}(n)$. We define $\mathcal{DP}^+(n) = \{\mu = (\mu^+; \mu^-) \in \mathcal{DP}(n) \ : \ \mu^- = \emptyset\}$. For $\mu = (\mu^+; \mu^-) \in \mathcal{DP}(n)$, $\hat{\mu} := \mu^+ * -\mu^-$ is the signed composition obtained by appending the sequence of components of μ^+ to that of $-\mu^-$ [2].

Now let $A \in \mathcal{SC}(n)$. If μ^+ (resp. μ^-) is rearrangement of the positive parts (resp. absolute value of negative parts) of A in decreasing order, then $\boldsymbol{\lambda}(A) := (\mu^+; \mu^-)$ is a double partition of n and also $\boldsymbol{\lambda}(\hat{\mu}) = \mu$ for every $\mu \in \mathcal{DP}(n)$ [2]. In [2], Bonnafé and Hohlweg constructed some reflection subgroups of W_n corresponding to signed compositions of n as an analogue to Ξ_n as follows: For each $A = (a_1, \cdots, a_k) \in \mathcal{SC}(n)$, the reflection subgroup W_A of W_n is generated by S_A, which is

$$\begin{aligned} S_A = &\{s_p \in W_{-n} \ : \ |a_1| + \cdots + |a_{i-1}| + 1 \leq p \leq |a_1| + \cdots + |a_i| - 1\} \\ &\cup \{t_{|a_1|+\cdots+|a_{j-1}|+1} \in T_n\} \mid a_j > 0\} \subset S_n^{'} \end{aligned}$$

where $S_n' = \{s_1 \cdots s_{n-1}, t_1, t_2, \cdots, t_n\}$. By the definition of S_A, there exists an isomorphism $W_A \cong W_{a_1} \times \cdots \times W_{a_k}$ [2]. By taking into account the definition of the generating set S_A and the isomorphism $W_A \cong W_{a_1} \times \cdots \times W_{a_r}$, for i, $1 \leq i \leq r$ if $a_i > 0$ then we have rank $W_{a_i} = a_i$ and if $a_i < 0$ then we have rank $W_{a_i} = |a_i| - 1$. Therefore, we get

$$\text{rank} W_A = |S_A| = n - ng(A),$$

where $ng(A)$ denotes the number of negative parts of A. Because of $\sum_{i=1}^{r} |a_i| = n$, we obtain $\text{rank} W_A = |S_A| \leq n$.

For $A, B \in \mathcal{SC}(n)$, we write $A \subset B$ if $W_A \subset W_B$, where $\subset$ is a partial ordering relation on $\mathcal{SC}(n)$ [2]. For $A \in \mathcal{SC}(n)$ let cox_A be a Coxeter element of W_A in terms of generating set S_A. For $B, B^{'} \subset A$, we write $B \equiv_A B'$ if W_B is conjugate to $W_{B'}$ under W_A and also cox_B and $\text{cox}_{B'}$ are conjugate to each other in W_A if and only if $B \equiv_A B^{'}$ [3]. We write $B \equiv_n B^{'}$ if W_B is conjugate to $W_{B'}$ under W_n. This equivalence is a special case for these kind of reflection subgroups of W_n, because this statement is not true for every reflection subgroup of W_n. Although some two reflection subgroups R and R' of W_n contain W_n-conjugate Coxeter elements cox_R and $\mathrm{cox}_{R'}$ respectively, these subgroups are not able to W_n-conjugate to each other [6]. For every element w of W_n, there exists a unique $\lambda \in \mathcal{DP}(n)$ such that w is W_n-conjugate to $\mathrm{cox}_{\hat{\lambda}}$ [3]. Let $\mathcal{K}_\lambda$ be the conjugacy class of W_n corresponding to $\lambda \in \mathcal{DP}(n)$. Since the number of conjugacy classes of W_n is equal to $|\mathcal{DP}(n)|$, thus we may split up W_n into $|\mathcal{DP}(n)|$ conjugacy classes. In [3], Bonnafé showed that for $A, B \in \mathcal{SC}(n)$, W_A is conjugate to W_B in W_n if and only if $\boldsymbol{\lambda}(A) = \boldsymbol{\lambda}(B)$.

For a subset X of W_n, we denote by $\mathrm{Fix}(X) = \{v \in \mathbb{R}^n : \forall x \in X,\ x(v) = v\}$ the subspace of $\mathbb{R}^n$ fixed by X and let write $W_{\mathrm{Fix}(X)} = \{w \in W_n : \forall v \in \mathrm{Fix}(X),\ w(v) = v\}$ for the stabilizer of $\mathrm{Fix}(X)$ in W_n. By [6], the set $W_{\mathrm{Fix}(X)}$ is called the *parabolic closure* of X and it is denoted by $A(X)$. For any $w \in W_n$, if we take $X = \{w\}$ then we write $\mathrm{Fix}(w)$ and $A(w)$ instead of $\mathrm{Fix}(\{w\})$ and $A(\{w\})$, respectively. By [1], w is said to be an element of *type* J if there exists a $J \subset S_n$ such that $A(w)$ is conjugate to W_J under W_n.

2.2. **Mantaci-Reutenauer algebra.** For any $A \in \mathcal{SC}(n)$, we set

$$D_A = \{x \in W_n \ : \ \forall\ s \in S_A,\ l(xs) > l(x)\}.$$

By [2] and [7], D_A is the set of distinguished coset representatives of W_A in W_n. Let

$$d_A = \sum_{w \in D_A} w \in \mathbb{Q}W_n,$$

and let

$$\mathcal{MR}(W_n) = \bigoplus_{A \in \mathcal{SC}(n)} \mathbb{Q}d_A.$$

For every $A \in \mathcal{SC}(n)$, from [2] $\Phi_n \ : \ \mathcal{MR}(W_n) \to \mathbb{Q}\mathrm{Irr}W_n$ is a surjective algebra morphism such that $\Phi_n(d_A) = \mathrm{Ind}_{W_A}^{W_n} 1_A$, where 1_A stands for the trivial character of W_A. It is well-known from [2] that the radical of $\mathcal{MR}(W_n)$ is $\mathrm{Ker}\Phi_n = \bigoplus_{A,B \in \mathcal{SC}(n),\ A \equiv_n B} \mathbb{Q}(d_A - d_B)$

By [2], for $A, B \in \mathcal{SC}(n)$, the set of distinguished double coset representatives is defined as $D_{AB} = D_A^{-1} \cap D_B$ and for any $x \in D_{AB}$,

$$W_A \cap {}^xW_B = W_{A \cap {}^xB}.$$

For $A, B \in \mathcal{SC}(n)$, let define [3] the sets $D_{AB}^{\subseteq} = \{x \in D_{AB} :\ {}^{x^{-1}}W_A \subset W_B\}$ and $D_{AB}^{\equiv} = \{x \in D_{AB} : W_A =^x W_B\}$.

The following proposition proved by Bonnafé in [3] gives the ring multiplication structure in $\mathcal{MR}(W_n)$.

Proposition 1 ([3])**.** *Let A and B be any two signed composition of n. Then,*

i. *There is a map $f_{AB} : D_{AB} \to \mathcal{SC}(n)$ satisfying the following conditions:*
   - *For every $x \in D_{AB}$, $f_{AB}(x) \subset B$ and $f_{AB}(x) \equiv_B {}^{x^{-1}}A \cap B$.*
   - *$d_A d_B - \sum_{x \in D_{AB}} d_{f_{AB}(x)} \in \mathcal{MR}_{\subsetneq_{\boldsymbol{\lambda}} A}(W_n) \cap \mathcal{MR}_{\prec B}(W_n) \cap Ker\Phi_n$.*

ii. *If A parabolic or B is semi-positive, then $f_{AB}(x) = {}^{x^{-1}}A \cap B$ for $x \in D_{AB}$ and $d_A d_B = \sum_{x \in D_{AB}} d_{x^{-1}A \cap B}$.*
iii. $\tau_{\boldsymbol{\lambda}(A)}(d_B) = |D^{\subseteq}_{AB}|$.
iv. $D^{\equiv}_{AB} = \{x \in W_n \ : \ S_A = {}^{x} S_B\}$.
v. $\mathcal{W}(B) = \{w \in W_n : {}^{w} S_B = S_B\}$.
vi. *$\mathcal{W}(B)$ is a subgroup of $N_{W_n}(W_B)$.*
vii. $N_{W_n}(W_B) = \mathcal{W}(B) \ltimes W_B$.

In the Proposition 1, the symbols $\subset_{\boldsymbol{\lambda}}$ and $\prec$ denote a pre-order and an ordering defined on $\mathcal{SC}(n)$, respectively. If $A \equiv_n B$, then it is clear $D^{\equiv}_{AB} = D^{\subseteq}_{AB}$ and $\mathcal{W}(A) = D^{\subseteq}_{AA}$. Thus $\mathcal{MR}(W_n)$ is called *Mantaci-Reutenauer algebra* of W_n.

For $\lambda \in \mathcal{DP}(n)$, the map $\tau_\lambda \ : \ \mathcal{MR}(W_n) \to \mathbb{Q}, \ x \mapsto \Phi_n(x)(\text{cox}_{\hat{\lambda}})$ is independent of the choice of $\text{cox}_{\hat{\lambda}} \in \mathcal{K}_\lambda$ and it is also an algebra morphism [2].

3. Some Properties of Coxeter group of type B_n

Let $A \in \mathcal{SC}(n)$ and let $l_A : W_A \to \mathbb{N}$ be the length function of W_A in terms of its generating set S_A. When A is not a parabolic signed composition of n, the value $l_A(w)$ is not equal to $l(w)$ for some $w \in W_A$. The following lemma gives a relation between these two length functions. The proof of the following lemma is clear from the fact that $l(t_i) = 2i - 1$ for i, $1 \le i \le n$.

Lemma 2. *Let $A \in \mathcal{SC}(n)$. Then for every $w \in W_A$*

$$l(w) \equiv l_A(w) \ (mod\ 2).$$

Let ε_n and ε_A be the sign character of W_n and W_A, respectively. As a result of the previous lemma, we get

$$\varepsilon_n(w) = (-1)^{l(w)} = (-1)^{l_A(w)} = \varepsilon_A(w).$$

Since the restriction of ε_n to W_A, that is $\text{res}^{W_n}_{W_A}\varepsilon_n$, is an irreducible character of W_A for every $A \in \mathcal{SC}(n)$ and Lemma 2, then we have $\text{res}^{W_n}_{W_A}\varepsilon_n = \varepsilon_A$.

Example 3. For a concrete example, let $A = (-2, 3, -1, -3, 1) \in \mathcal{SC}(10)$. Then $S_A = \{s_1\} \cup \{t_3, s_3, s_4\} \cup \{s_7, s_8\} \cup \{t_{10}\} \subset S'_{10}$ and $S'_A = W_A \cap S'_{10} = \{s_1\} \cup \{t_3, s_3, s_4, t_4, t_5\} \cup \{s_7, s_8\} \cup \{t_{10}\}$. Thus $W_A \cong W_{-2} \times W_3 \times W_{-1} \times W_{-3} \times W_1$. For $w = s_7 t_3 s_3 s_1 t_{10} \in W_A$, $l_A(w) = 5$ and also

$$w = s_7 t_3 s_3 s_1 t_{10} = s_7 s_2 s_1 t_1 s_1 s_2 s_3 s_1 s_9 s_8 s_7 s_6 s_5 s_4 s_3 s_2 s_1 t_1 s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 \in W_{10},$$

so $l(w) = 27$. It follows that $l(w) \equiv l_A(w) \equiv 1(mod\ 2)$.

Proposition 4. *If $B \in \mathcal{SC}^+(n)$, then the parabolic closure of W_B is $A(W_B) = W_n$.*

Proof. Since $B \in \mathcal{SC}^+(n)$, we have $\mathcal{T}_n \le W_B$ and so $w_n \in W_B$. By considering w_n as a linear map $-id_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$, we obtain $\text{Fix}(w_n) = \{\vec{0}\}$. Thus, the parabolic closure of w_n is $A(w_n) = W_{\text{Fix}(w_n)} = W_n$. Because of the relation $w_n \in W_B \subset A(\text{cox}_B) = A(W_B)$, we get $w_n \in A(\text{cox}_B)$. By [11], the inclusion $A(w_n) \subset$

$A(\mathrm{cox}_B) = A(W_B)$ holds. If we take into account the fact that $A(w_n) = W_n$, then we have $A(W_B) = W_n$. This completes the proof. $\square$

As a consequence of Proposition 4, if $B \in \mathcal{SC}^+(n)$, then the parabolic closure of W_B is W_n and each element of $\mathcal{K}_{\boldsymbol{\lambda}(B)}$ is of type S_n.

Lemma 5. *Let A be a signed composition of n. Then w_n belongs to W_A if and only if $A \in \mathcal{SC}^+(n)$.*

Proof. When A is a positive signed composition of n, we can easily see from the proof of Proposition 4 that w_n is an element of W_A. Conversely, let w_n be in W_A. We suppose that $A = (a_1, \cdots, a_i, \cdots, a_r)$ is not a positive signed composition of n. Then there exists $a_i < 0$ for some i, $1 \le i \le r$. Thus from the definition of W_A, we obtain $t_{|a_1|+\cdots+|a_{i-1}|+1}, \cdots, t_{|a_1|+\cdots+|a_i|} \notin S_A^{'} = W_A \cap S_n^{'}$. Hence for any $x \in W_A$ and $e_{|a_1|+\cdots+|a_{i-1}|+1} + \cdots + e_{|a_1|+\cdots+|a_i|} \in \mathbb{R}^n$, we have $x(e_{|a_1|+\cdots+|a_{i-1}|+1} + \cdots + e_{|a_1|+\cdots+|a_i|}) = e_{|a_1|+\cdots+|a_{i-1}|+1} + \cdots + e_{|a_1|+\cdots+|a_i|}$ and so $e_{|a_1|+\cdots+|a_{i-1}|+1} + \cdots + e_{|a_1|+\cdots+|a_i|} \in \mathrm{Fix}(W_A)$. This is a contradiction, because the subspace $\mathrm{Fix}(W_A)$ consists of only $\vec{0}$. Therefore, we get $A \in \mathcal{SC}^+(n)$. $\square$

Theorem 6. *If the set $\mathcal{C}(S_n)$ denotes the set of all elements of W_n of type S_n, then we have*

$$\mathcal{C}(S_n) = \coprod_{\lambda \in \mathcal{DP}^+(n)} \mathcal{K}_\lambda. \tag{1}$$

Proof. For each $\lambda \in \mathcal{DP}^+(n)$, we have $\hat{\lambda} \in \mathcal{SC}^+(n)$. From Proposition 4, for every element of $\mathcal{K}_\lambda$ is of type S_n and so the reverse inclusion holds. Now let $w \in \mathcal{C}(S_n)$. Then w is W_n-conjugate to cox_A for some $A \in \mathcal{SC}(n)$. Thus we get $A(w) = A(\mathrm{cox}_A) = A(W_A) = W_n$. From here, for every $x \in W_n$ and every $v \in \mathrm{Fix}(W_A)$ we obtain $x(v) = v$. In particular, if we take $w_n = -id_{\mathcal{R}^n} \in W_n$, then it is seen that $\mathrm{Fix}(W_A)$ includes just $\{\vec{0}\}$. Thus w_n is an element of W_A. Otherwise, if $A \notin \mathcal{SC}^+(n)$, then from the proof of Lemma 5 we get $\mathrm{Fix}(W_A) \neq \{\vec{0}\}$, which is a contradiction. Hence $A \in \mathcal{SC}^+(n)$. By taking the definition of $\boldsymbol{\lambda}$ into account, we get a $\lambda \in \mathcal{DP}^+(n)$ such that $\boldsymbol{\lambda}(A) = \lambda$. Thus w belongs to $\mathcal{K}_\lambda$ and so it is seen that the inclusion $\mathcal{C}(S_n) \subset \coprod_{\lambda \in \mathcal{DP}^+(n)} \mathcal{K}_\lambda$ satisfies. It is required. $\square$

Since the exponents of W_n are in turn $1, 3, \cdots, 2n-1$, then from [1] the number of elements of type S_n is equal to the product of exponents of W_n and so $|\mathcal{C}(S_n)| = 1 \cdot 3 \cdots 2n - 1$. By the equation (1), we obtain the formula

$$|\mathcal{C}(S_n)| = \sum_{\mu \in \mathcal{DP}^+(n)} |\mathcal{K}_\mu|.$$

Thus Theorem 6 gives us an alternative method to compute the number of elements of type S_n. We will give a formula in Corollary 19 to find the number of elements of every conjugacy class $\mathcal{K}_\lambda$, $\lambda \in \mathcal{DP}(n)$ of W_n. Moreover, we will give an example for Theorem 6 in Section 5.

A conjugacy class of a finite Coxeter group W which does not contain an element of a proper standard parabolic subgroup of W is called a *cuspidal class* of W [8].

Corollary 7. *Let A be a positive signed composition of n. Then the conjugacy class $\mathcal{K}_{\boldsymbol{\lambda}(A)}$ is a cuspidal class of W_n.*

If we consider the proof of Proposition 4 and Corollary 7, then all cuspidal classes of W_n are the conjugacy classes $\mathcal{K}_{\boldsymbol{\lambda}(A)}$ for every $A \in \mathcal{SC}^+(n)$. From Theorem 6, the set $\mathcal{C}(S_n)$ is disjoint union of cuspidal classes of W_n. Therefore, each element of W_n of type S_n belongs to a unique cuspidal class of W_n.

4. Generalized Burnside Algebra of W_n

Let A, B be any two signed compositions of n. Then, we have that

$$A \equiv_n B \;\Leftrightarrow\; W_A \sim_{W_n} W_B \;\Leftrightarrow\; [W/W_A] = [W/W_B]$$

where $[W/W_A]$ represents the isomorphism class of W_n-set W/W_A. The orbits of W_n on $W/W_A \times W/W_B$ are of the form $(W_A x, W_B)$ where $x \in D_{AB}$. The stabilizer of $(W_A x, W_B)$ in W_n is ${}^{x^{-1}}W_A \cap W_B = W_{{}^{x^{-1}}A \cap B}$. Therefore

$$[W/W_A].[W/W_B] = [W/W_A \times W/W_B] = \sum_{x \in D_{AB}} [W/W_{{}^{x^{-1}}A \cap B}].$$

Thus, we are now in a position to give the following definition.

Definition 8. *The generalized Burnside algebra* of W_n is $\mathbb{Q}$-spanned by the set $\{[W/W_A] : A \in \mathcal{SC}(n)\}$ and it is denoted by $\mathrm{HB}(W_n)$.

From part (i) of Proposition 1 and the structure of $\mathrm{Ker}(\Phi_n)$, the ring multiplication rule in $\mathcal{MR}(W_n)$ may be expressed by

$$d_A d_B = \sum_{x \in D_{AB}} d_{f_{AB}(x)} + \sum\nolimits_{N \equiv_n N'} a_{NN'}(d_N - d_{N'}),$$

where $a_{NN'} \in \mathbb{Z}$; $N, N' \subsetneq_{\boldsymbol{\lambda}} A$; $N, N' \prec B$; $f_{AB}(x) \subset B$ and $f_{AB}(x) \equiv_B {}^{x^{-1}}A \cap B$.

Now we define

$$\psi \;:\; \mathcal{MR}(W_n) \to HB(W_n),\; d_A \mapsto [W/W_A].$$

Thus ψ is well-defined and surjective linear map. By considering the structure of $\mathrm{Ker}\Phi_n$ and $f_{AB}(x) \equiv_B {}^{x^{-1}}A \cap B \Rightarrow W_{f_{AB}(x)} \sim_{W_B} W_{{}^{x^{-1}}A \cap B}$, we get

$$\begin{aligned}
\psi(d_A d_B) &= \psi\Big(\sum_{x \in D_{AB}} d_{f_{AB}(x)} + \sum\nolimits_{N \equiv_n N'} a_{NN'}(d_N - d_{N'})\Big) \\
&= \sum_{x \in D_{AB}} [W/W_{f_{AB}(x)}] \\
&= \psi(d_A)\psi(d_B).
\end{aligned}$$

Then the map ψ is an algebra morphism. Since $\dim_{\mathbb{Q}}\mathrm{HB}(W_n) = \dim_{\mathbb{Q}}\mathbb{Q}\mathrm{Irr}W_n = |\mathcal{DP}(n)|$, then there is an algebra isomorphism between $\mathrm{HB}(W_n)$ and $\mathbb{Q}\mathrm{Irr}W_n$ such that

$$\mathrm{HB}(W_n) \to \mathbb{Q}\mathrm{Irr}W_n,\ [W/W_A] \mapsto \mathrm{Ind}_{W_A}^{W_n} 1_A.$$

Now let $\lambda, \mu \in \mathcal{DP}(n)$ and let $\varphi_\lambda = \mathrm{Ind}_{W_{\hat{\lambda}}}^{W_n} 1_{\hat{\lambda}}$. From part (iii) of Proposition 1, $\varphi_\lambda(\mathrm{cox}_{\hat{\lambda}}) = \tau_\lambda(d_{\hat{\lambda}}) = |D^{\subset}_{\hat{\lambda}\hat{\lambda}}| \neq 0$ and $\tau_\lambda(d_{\hat{\mu}}) = 0$ if $\lambda \nsubseteq \mu$. Thus the matrix $(\tau_\lambda(d_{\hat{\lambda}}))_{\lambda,\mu\in\mathcal{DP}(n)}$ is lower diagonal. Then $(\varphi_\lambda(\mathrm{cox}_{\hat{\mu}}))_{\lambda,\mu}$ is upper diagonal and also has positive diagonal entries. Therefore $(\varphi_\lambda(\mathrm{cox}_{\hat{\mu}}))_{\lambda,\mu}$ is invertible and we write $(u_{\lambda\mu})_{\lambda,\mu\in\mathcal{DP}(n)}$ for the inverse of $(\varphi_\lambda(\mathrm{cox}_{\hat{\mu}}))_{\lambda,\mu}$. We define

$$e_\lambda = \sum_{\mu\in\mathcal{DP}(n)} u_{\lambda\mu}\varphi_\mu.$$

By definition of e_λ and $(\varphi_\lambda(\mathrm{cox}_{\hat{\mu}}))^{-1} = (u_{\lambda\mu})$, we obtain that

$$e_\lambda(\mathrm{cox}_{\hat{\mu}}) = \sum_{\gamma\in\mathcal{DP}(n)} u_{\lambda\gamma}\varphi_\gamma(\mathrm{cox}_{\hat{\mu}}) = \delta_{\lambda,\mu}.$$

Hence the set $\{e_\lambda : \lambda \in \mathcal{DP}(n)\}$ is a collection of orthogonal primitive idempotents of $\mathrm{HB}(W_n)$. Consequently, we have $HB(W_n) = \oplus_{\lambda\in\mathcal{DP}(n)}\mathbb{Q}e_\lambda$.

For each $A \in \mathcal{SC}(n)$,

$$s_A : HB(W_n) \to \mathbb{Q},\ s_A([X]) = |{}^{W_A}X|$$

is an algebra map, where ${}^{W_A}X = \{x \in X : W_A x = x\}$. Since $HB(W_n)$ is semisimple and commutative algebra, then all algebra maps $HB(W_n) \to \mathbb{Q}$ are of the form s_A for every $A \in \mathcal{SC}(n)$. The proof of the following lemma is immediately seen from [5].

Lemma 9. *For $A, B \in \mathcal{SC}(n)$, we have that*

$$s_A = s_B \ \Leftrightarrow\ \boldsymbol{\lambda}(A) = \boldsymbol{\lambda}(B).$$

Thus the dual basis of $HB(W_n)$ is $\{s_{\hat{\lambda}} : \lambda \in \mathcal{DP}(n)\}$. For any $\lambda,\ \mu \in \mathcal{DP}(n)$, we have the following equality

$$s_{\hat{\lambda}}(e_\mu) = \delta_{\lambda,\mu}, \tag{2}$$

and so any element x in $HB(W_n)$ can be expressed as $x = \sum_{\lambda\in\mathcal{DP}(n)} s_{\hat{\lambda}}(x)e_\lambda$.

Let A be a signed composition of n. Induction and restriction of characters give rise to two maps between $HB(W_A)$ and $HB(W_n)$. For any $A, B \in \mathcal{SC}(n)$ such that $B \subset A$, we have $\mathrm{Ind}_{W_A}^{W_n}([W_A/W_B]) = [W_n/W_B]$.

Definition 10. Let $A, B \in \mathcal{SC}(n)$ be such that $B \subset A$. The *restriction* is a linear map

$$\mathrm{res}_{W_B}^{W_A} : HB(W_A) \to HB(W_B),\ \mathrm{res}_{W_B}^{W_A}([W_A/W_C]) = \sum_{d\in W_A\cap D_{CB}} [W_B/W_{B\cap {}^{d^{-1}}C}].$$

Before going into a further discussion of the restriction and induced character theories of generalized Burnside algebra, we shall first mention the number of elements of the conjugacy class of W_A in W_n.

Proposition 11. *Let $A \in \mathcal{SC}(n)$ and $\boldsymbol{\lambda}(A) = \lambda$. The number of all reflection subgroups of W_n which are conjugate to W_A is*

$$|[W_A]| = |D_A| \cdot u_{\lambda,\lambda}.$$

Proof. Put $[W_A] = \{{}^xW_A : x \in W_n\}$. Now we note that $xW_Ax^{-1} = yW_Ay^{-1}$ if and only if $x \in yN_{W_n}(W_A)$. Thus, the number of distinct conjugates of W_A in W_n is $[W_n : N_{W_n}(W_A)]$. Since also $N_{W_n}(W_A) = \mathcal{W}(A) \ltimes W_A$, we have

$$|[W_A]| = \frac{|W_n|}{|\mathcal{W}(A)| \cdot |W_A|} = \frac{|D_A|}{|\mathcal{W}(A)|}.$$

Furthermore, from the fact that $\tau_{\boldsymbol{\lambda}(A)}(d_A) = |D^{\subset}_{AA}| = |\mathcal{W}(A)|$ and $\varphi_\lambda(\mathrm{cox}_{\hat{\lambda}}) = \tau_{\boldsymbol{\lambda}(A)}(d_A) = \frac{1}{u_{\lambda,\lambda}}$, as desired. $\square$

Example 12. We consider the set $D_{(2,1)} = \{1, s_2, s_1s_2\}$ consisting of the distinguished coset representatives of reflection subgroup $W_{(2,1)}$ in W_3. The number of all reflection subgroups conjugate to $W_{(2,1)}$ in W_3 is

$$|[W_{(2,1)}]| = |D_{(2,1)}| \cdot u_{(2,1;\emptyset),(2,1;\emptyset)} = 3 \cdot 1 = 3.$$

These are explicitly $W_{(2,1)}$, $W_{(1,2)}$ and ${}^{s_2}W_{(2,1)} = \langle s_2s_1s_2, t_1, t_2\rangle$. We note that the reflection subgroup ${}^{s_2}W_{(2,1)}$ does not coincide with any subgroup of W_3 corresponding to any signed composition of 3.

Remark 13. *For $A, B \in \mathcal{SC}(n)$ such that $B \subset A$ and for any $x \in HB(W_n)$, by using the definition of s_A one can see that there exists the relation $s^A_B(res^{W_n}_{W_A}(x)) = s_B(x)$.*

We can now give the following proposition.

Proposition 14. *Let be $A, B \in \mathcal{SC}(n)$ and let $A_1, A_2, \cdots, A_r$ be representatives of the W_A-equivalent classes of subsets of A, which are W_n-equivalent to B. Then,*

$$res^{W_n}_{W_A} e_B = \sum_{i=1}^{r} e^A_{A_i}.$$

If B is not W_n-equivalent to any subset of A then $res^{W_n}_{W_A} e_B = 0$.

Proof. Since $\mathrm{res}^{W_n}_{W_A} e_B$ is an element of $HB(W_A)$, then we have

$$\mathrm{res}^{W_n}_{W_A} e_B = \sum_{A_i \subset A} s^A_{A_i}(\mathrm{res}^{W_n}_{W_A}(e_B)) e^A_{A_i}.$$

Then by using Remark 13 and the relation (2), we get

$$\mathrm{res}^{W_n}_{W_A} e_B = \sum_{A_i \subset A} s_{A_i}(e_B) e^A_{A_i}$$

$$= \sum_{\substack{A_i \subset A \\ A_i \equiv_A B}} e^A_{A_i}$$
$$= \sum_{i=1}^{r} e^A_{A_i}.$$

□

Proposition 15. *Let $A, B \in \mathcal{SC}(n)$ and let $B \subset A$. Then we have*

$$Ind^{W_n}_{W_A} e^A_B = \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B)|} \cdot e_B.$$

Proof. Firstly, we assume that $A = B$ and cox_A is a Coxeter element of W_A. Since the image of cox_A under permutation character of W_n on the cosets of W_A is $|\mathcal{W}(A)|$ then it follows from the fact that

$$x^{-1}\mathrm{cox}_A x \in W_A \Leftrightarrow x \in N_{W_n}(W_A).$$

Thus we obtain

$$\begin{aligned}\mathrm{Ind}^{W_n}_{W_A} e^A_A(\mathrm{cox}_A) &= |D_A \cap N_{W_n}(W_A)| \\ &= |\mathcal{W}(A)|.\end{aligned}$$

As $\mathrm{Ind}^{W_n}_{W_A} e^A_A$ takes value zero except for the elements conjugate to cox_A and so we get

$$\mathrm{Ind}^{W_n}_{W_A} e^A_A = |\mathcal{W}(A)| e_A.$$

By transitivity of induced characters, we generally get

$$\begin{aligned}\mathrm{Ind}^{W_n}_{W_A} e^A_B &= \mathrm{Ind}^{W_n}_{W_A} \left(\frac{1}{|W_A \cap \mathcal{W}(B)|} |W_A \cap \mathcal{W}(B)| e^A_B\right) \\ &= \mathrm{Ind}^{W_n}_{W_A} \left(\frac{1}{|W_A \cap \mathcal{W}(B)|} \mathrm{ind}^{W_A}_{W_B} e^B_B\right) \\ &= \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B)|} e_B.\end{aligned}$$

□

Furthermore, there is also the equality $\mathrm{Ind}^{W_n}_{W_A} e^A_B = |N_{W_n}(W_B) : N_{W_A}(W_B)| e_B$.

Theorem 16. *Let $A, B \in \mathcal{SC}(n)$ be such that $\boldsymbol{\lambda}(B) \subset \boldsymbol{\lambda}(A)$. If $B_1, B_2, \cdots, B_r$ are the representatives of the W_A-equivalence classes of subsets of A which are W_n-equivalent to B, then for $cox_B \in W_n$,*

$$Ind^{W_n}_{W_A} 1_A(cox_B) = \sum_{i=1}^{r} \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B_i)|}.$$

Proof. Let $A, B \in \mathcal{SC}(n)$. If $A \equiv_n B$ then it is easy to prove that $|\mathcal{W}(A)| = |\mathcal{W}(B)|$. We write $1_A = \sum_E e_E^A$, where $E \in \mathcal{SC}(n)$ runs over W_A-conjugacy classes of subsets of A. From Proposition 15, we have

$$\mathrm{Ind}_{W_A}^{W_n} 1_A = \sum_E \mathrm{Ind}_{W_A}^{W_n} e_E^A \Rightarrow \mathrm{Ind}_{W_A}^{W_n} 1_A = \sum_E \frac{|\mathcal{W}(E)|}{|W_A \cap \mathcal{W}(E)|} \cdot e_E.$$

Since each B_i is W_n-equivalent to B, then $e_E(\mathrm{cox}_B) = 1$ if and only if $E \equiv_{W_A} B_i$. Thus we obtain that

$$\mathrm{Ind}_{W_A}^{W_n} 1_A(\mathrm{cox}_B) = \sum_{i=1}^{r} \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B_i)|}.$$

Hence the theorem is proved. □

Theorem 17 and Proposition 18 give us a useful computation to determine the coefficient of the sign character ε_n in the expression of the orthogonal primitive idempotent $e_\lambda,\ \lambda \in \mathcal{DP}(n)$ in terms of irreducible characters of W_n.

Theorem 17. $u_{(n;\emptyset),(\emptyset;1,\cdots,1)} = \frac{(-1)^n}{2n}$.

Proof. Let $\chi_{reg} : W_n \to \mathbb{Z}$ be the regular character of W_n. For $A = (-1, \cdots, -1)$ it is satisfied $\mathrm{Ind}_{W_A}^{W_n} 1_A = \chi_{reg}$. The character ε_n is contained in χ_{reg} with the property that its coefficient is just 1, thus we have

$$\langle \mathrm{Ind}_{W_A}^{W_n} 1_A, \varepsilon_n \rangle = 1.$$

Now let $A \neq (-1, \cdots, -1)$. By using Frobenius Reciprocity and the formula $\mathrm{res}_{W_A}^{W_n} \varepsilon_n = \varepsilon_A$, it is obtained that $\langle \mathrm{Ind}_{W_A}^{W_n} 1_A, \varepsilon_n \rangle = 0$. If w is conjugate to cox_{W_n} under W_n, then we have $e_{(n;\emptyset)}(w) = 1$ and $\varepsilon_n(w) = \varepsilon_n(\mathrm{cox}_{W_n}) = (-1)^{l(w)} = (-1)^n$. Let $\mathrm{ccl}_{W_n}(\mathrm{cox}_{W_n})$ denote the conjugacy class of cox_{W_n} in W_n. By considering the formula $|\mathrm{ccl}_{W_n}(\mathrm{cox}_{W_n})| = \frac{|W_n|.n}{2N}$ in [4], we obtain

$$\langle e_{(n;\emptyset)}, \varepsilon_n \rangle = \frac{(-1)^n}{2n}.$$

On the other hand, $\langle e_{(n;\emptyset)}, \varepsilon_n \rangle = \sum_{\mu \in \mathcal{DP}(n)} u_{(n;\emptyset)\mu} \langle \varphi_\mu, \varepsilon_n \rangle = u_{(n;\emptyset),(\emptyset;1,\cdots,1)}$ and so the proof is completed. □

Proposition 18. *For $\lambda \in \mathcal{DP}(n)$ and $\lambda \neq (n;\emptyset)$, then we have*

$$u_{\lambda,(\emptyset;1,\cdots,1)} = (-1)^{|S_{\hat{\lambda}}|} \cdot \frac{|\mathcal{K}_\lambda|}{|W_n|}.$$

Proof. Since the sign character is constant on the conjugacy classes, then we have

$$\begin{aligned}\langle e_\lambda, \varepsilon_n \rangle &= \frac{1}{|W_n|} \sum_{w \in \mathcal{K}_\lambda} (-1)^{l(w)} \ (\mathrm{rank} W_{\hat{\lambda}} = |S_{\hat{\lambda}}|) \\ &= (-1)^{|S_{\hat{\lambda}}|} \cdot \frac{|\mathcal{K}_\lambda|}{|W_n|}.\end{aligned}$$

Note that $\langle \varphi_\mu, \varepsilon_n \rangle$ has value 1 for $\mu = (\emptyset; 1, \cdots, 1)$ and zero for the others. Henceforth, we obtain $\langle e_\lambda, \varepsilon_n \rangle = \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \langle \varphi_\mu, \varepsilon_n \rangle = u_{\lambda,(\emptyset;1,\cdots,1)}$. Eventually, we have $u_{\lambda,(\emptyset;1,\cdots,1)} = (-1)^{|S_{\hat{\lambda}}|} \cdot \frac{|\mathcal{K}_\lambda|}{|W_n|}$. □

Notice that calculation of the inner product $\langle e_\lambda, 1_{W_n} \rangle$ leads to the following corollary.

Corollary 19. *Let $\lambda \in \mathcal{DP}(n)$. Then*

$$|W_n| \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda,\mu} = |\mathcal{K}_\lambda|.$$

By means of Corollary 19 and the matrix $(u_{\lambda\mu})_{\lambda,\mu \in \mathcal{DP}(n)}$, one can readily determine the sizes of all the conjugacy classes of W_n.

Theorem 20. *Let $A \in \mathcal{SC}(n)$ and $\lambda \in \mathcal{DP}(n)$. Then*

$$\sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} a_{\hat{\mu}A(-1,\cdots,-1)} = (-1)^{|S_{\hat{\lambda}}|} \frac{|\mathcal{K}_\lambda \cap W_A|}{|W_A|},$$

where $a_{\hat{\mu}A(-1,\cdots,-1)} = |\{x \in D_{\hat{\mu}A} : {}^{x^{-1}}\hat{\mu} \cap A = (-1, \cdots, -1)\}|$.

Proof. The term $d_{(-1,\cdots,-1)}$ in the multiplication $d_{\hat{\mu}} d_A$ lies in the summand $\sum_{x \in D_{\hat{\mu}A}} d_{f_{\hat{\mu}A}(x)}$ from the structure of $\mathrm{Ker}\Phi_n$ and part (i) of Proposition 1. If we write the coefficient of $d_{(-1,\cdots,-1)}$ in this summand as $a_{\hat{\mu}A(-1,\cdots,-1)}$, and so we get

$$a_{\hat{\mu}A(-1,\cdots,-1)} = |\{x \in D_{\hat{\mu}A} : f_{\hat{\mu}A}(x) = (-1, \cdots, -1)\}|.$$

By using part (i) of Proposition 1 along with the fact $f_{\hat{\mu}A}(x) \equiv_A {}^{x^{-1}}\hat{\mu} \cap A$, it is seen that there is the equivalence ${}^{x^{-1}}\hat{\mu} \cap A \equiv_A (-1, \cdots, -1)$. Since no element in $\mathcal{SC}(n)$ is congruent to $(-1, \cdots, -1)$ except for $(-1, \cdots, -1)$, it then follows that ${}^{x^{-1}}\hat{\mu} \cap A = (-1, \cdots, -1)$. Hence we have deduced the equality $a_{\hat{\mu}A(-1,\cdots,-1)} = |\{x \in D_{\hat{\mu}A} : {}^{x^{-1}}\hat{\mu} \cap A = (-1, \cdots, -1)\}|$ holds. Therefore, by Frobenius Reciprocity and Mackey Theorem, we have

$$\begin{aligned}
\langle e_\lambda, \mathrm{ind}_{W_A}^{W_n} \varepsilon_A \rangle &= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \sum_{x \in D_{\hat{\mu}A}} \langle \mathrm{ind}_{W_{x^{-1}\hat{\mu} \cap A}}^{W_A} 1_{x^{-1}\hat{\mu} \cap A}, \varepsilon_A \rangle \\
&= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \sum_{\substack{x \in D_{\hat{\mu}A} \\ {}^{x^{-1}}\hat{\mu} \cap A = (-1,\cdots,-1)}} 1_{x^{-1}\hat{\mu} \cap A} \\
&= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} a_{\hat{\mu}A(-1,\cdots,-1)}.
\end{aligned}$$

Also, $\varepsilon_n(w)$ is the same value for every $w \in \mathcal{K}_\lambda$ and so $\varepsilon_n(w) = \varepsilon_n(\mathrm{cox}_{\hat{\lambda}}) = (-1)^{|S_{\hat{\lambda}}|}$. Therefore, by Lemma 2, we have

$$\langle e_\lambda, \mathrm{ind}_{W_A}^{W_n} \varepsilon_A \rangle = \frac{1}{|W_A|} \sum_{w \in \mathcal{K}_\lambda \cap W_A} (-1)^{l_A(w^{-1})}$$

$$= \frac{1}{|W_A|} \sum_{w \in \mathcal{K}_\lambda \cap W_A} (-1)^{l(w)} = \frac{1}{|W_A|} (-1)^{|S_{\hat{\lambda}}|} |\mathcal{K}_\lambda \cap W_A|$$

Putting these two results together, we see that theorem is proved. □

5. Example

We consider the Coxeter group W_3. For all $\lambda, \mu \in \mathcal{DP}(3)$, by means of the character table of $\mathcal{MR}(W_3)$ in [3], we can write the values $\varphi_\lambda(\mathrm{cox}_{\hat{\mu}})$ as in the following table:

	$c_{(3;\emptyset)}$	$c_{(\emptyset;3)}$	$c_{(2,1;\emptyset)}$	$c_{(2;1)}$	$c_{(1;2)}$	$c_{(\emptyset;2,1)}$	$c_{(1,1,1;\emptyset)}$	$c_{(1,1;1)}$	$c_{(1;1,1)}$	$c_{(\emptyset;1,1,1)}$
$\varphi_{(3;\emptyset)}$	1	1	1	1	1	1	1	1	1	1
$\varphi_{(\emptyset;3)}$	0	2	0	0	0	4	0	0	0	8
$\varphi_{(2,1;\emptyset)}$	0	0	1	1	1	1	3	3	3	3
$\varphi_{(2;1)}$	0	0	0	2	0	2	0	2	4	6
$\varphi_{(1;2)}$	0	0	0	0	2	2	0	0	4	12
$\varphi_{(\emptyset;2,1)}$	0	0	0	0	0	4	0	0	0	24
$\varphi_{(1,1,1;\emptyset)}$	0	0	0	0	0	0	6	6	6	6
$\varphi_{(1,1;1)}$	0	0	0	0	0	0	0	4	8	12
$\varphi_{(1;1,1)}$	0	0	0	0	0	0	0	0	8	24
$\varphi_{(\emptyset;1,1,1)}$	0	0	0	0	0	0	0	0	0	48

The matrices $(u_{\lambda,\mu})_{\lambda,\mu \in \mathcal{DP}(n)}$ is

$$\begin{pmatrix}
1 & -1/2 & -1 & 0 & 0 & 1/2 & 1/3 & 0 & 0 & -1/6 \\
0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 1/6 \\
0 & 0 & 1 & -1/2 & -1/2 & 1/4 & -1/2 & 1/4 & 1/4 & -1/8 \\
0 & 0 & 0 & 1/2 & 0 & -1/4 & 0 & -1/4 & 0 & 1/8 \\
0 & 0 & 0 & 0 & 1/2 & -1/4 & 0 & 0 & -1/4 & 1/8 \\
0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & -1/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & -1/4 & 1/8 & -1/48 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & -1/4 & 1/16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & -1/16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/48
\end{pmatrix}.$$

For $\lambda = (3;\emptyset)$, $(2,1;\emptyset)$, $(1,1,1;\emptyset) \in \mathcal{DP}(3)$, the size of $\mathcal{K}_\lambda$ is calculated by means of Corollary 19 and matrix $(u_{\lambda,\mu})_{\lambda,\mu \in \mathcal{DP}(n)}$ the above. Since $|\mathcal{K}_{(3;\emptyset)}| = 8$, $|\mathcal{K}_{(2,1;\emptyset)}| = 6$ and $|\mathcal{K}_{(1,1,1;\emptyset)}| = 1$, then we have found that the number of elements of type S_3 is $|\mathcal{C}(S_3)| = 15$.

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EXISTENCE OF FIXED POINTS IN QUASI METRIC SPACES

SUMIT CHANDOK AND SAURABH MANRO

Abstract. In this paper, we obtain some new fixed point theorems for two pairs of weakly compatible mappings in the framework of non-symmeterical quasi metric spaces. Several interesting corollaries are also deduced. The results obtained extend various well known results of the literature in the setting of quasi metric space. We also construct an example to demonstrate the usability of the proved results.

1. Introduction and Preliminaries

In 2002, Aamri and Moutaawakil [1] introduced the notion of property E.A. in metric spaces and proved various results in the area of fixed point theory. Later on, using the idea of property E.A., Liu et al. [19] defined common (E.A.) property and proved various common fixed point theorems under strict contractive conditions.

In 2006, Mustafa and Sims [21] introduced a new notion of generalized metric space, called G-metric space, by showing that most of the results concerning Dhage's D-metric spaces [10] are invalid. After then, many authors studied fixed and common fixed points in $G-$metric spaces, see [3, 4, 5, 7, 8, 11, 20, 21, 22, 23, 24].

Here, we give preliminaries and basic definitions which are helpful in the sequel.

Definition 1.1. (see [2, 18]) A quasi-metric on a non-empty set X is a function $q : X \times X \to [0, \infty)$ satisfying the following properties:

(q1) $q(x, y) = 0$ if and only if $x = y$;
(q2) $q(x, y) \leq q(x, z) + q(z, y)$, for all $x, y, z \in X$.

In such a case, the pair (X, q) is called a quasi-metric space.

For symmetry, convergence, Cauchy sequence, completeness, continuity in quasi-metric space see [2].

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Example 1.2. (see [2]) Let X be a subset of $\mathbb{R}$ containing $[0,1]$ and define, for all $x,y \in X$,

$$q(x,y) = \begin{cases} x-y, & \text{if } x \geq y, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X,q) is a quasi-metric space.

Definition 1.3. (see [2, 21]) Let X be a nonempty set and $G : X \times X \times X \to [0,+\infty)$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ if $x = y = z$,
(G2) $0 < G(x,x,y)$, for all $x,y \in X$ with $x \neq y$,
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

For more details about Symmetric G-metric, $G-$ Cauchy sequence, continuity of G function, $G-$ completeness, one may refers to paper [21].

Let (X,G) be a G-metric space. Then

(a) (X,G) is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.
(b) The pair (S,T) of self mappings of a G-metric space (X,G) is said to be weakly compatible if they commute at their coincidence points.
(c) The pair (S,T) of self mappings of a G-metric space (X,G) is said to satisfy the property E.A if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Tx_n = t$, for some $t \in X$.
(d) Two pairs (A,S) and (B,T) of self mappings of a $G-$ metric space are said to satisfy the common (E.A.) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Ty_n = z$ for some $z \in X$.

Recently, Chandok et al. [6] used the concept of a C-class functions which cover a large class of contractive conditions.

Definition 1.4. A continuous function $F : [0,\infty)^2 \to \mathbb{R}$ is called *C-class* function if for any $s,t \in [0,\infty)$, the following conditions hold:

(1) $F(s,t) \leq s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0,0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ will denote the class of all C- functions.

Example 1.5. (see [6]) The following examples show that the class $\mathcal{C}$ is nonempty:

(1) $F(s,t) = s - t$.

(2) $F(s,t) = ms$, for some $m \in (0,1)$.
(3) $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,\infty)$.
(4) $F(s,t) = \log(t + a^s)/(1+t)$, for some $a > 1$.
(5) $F(s,t) = \ln(1 + a^s)/2$, for $e > a > 1$.
Indeed $F(s,t) = s$ implies that $s = 0$.

Throughout this paper, we suppose that Ψ denote the class of all real valued continuous non-decreasing functions $\psi : [0,\infty) \to [0,\infty)$ satisfying $\psi(t) = 0$ if and only if $t = 0$ and Φ denote the class of all real valued continuous non-decreasing functions $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(t) > 0$ if $t > 0$.

In this paper, by using C- class functions, we prove some new common fixed point theorems for two pairs of weakly compatible mappings in the framework of quasi metric spaces. Several interesting corollaries are also deduced. The results obtained extend various well known results of the literature in the setting of quasi metric space. We also construct an example to demonstrate the usability of the proved results.

2. Main results

Throughout this section, we assume that $\phi \in \Phi, \psi \in \Psi$ and F is a C-class function.

Also, we assume that (X,G) is a G-metric space and define $d_G : X \times X \to [0,\infty)$ by $d_G(x,y) = G(x,y,y)$. Using Lemma 3.3.1 of [2], every G-metric G induces a quasi-metric d_G in the sense of Kunzi [18] in such a way that $\tau(G) = \tau(d_G)$.

Now, we prove the main result of this section.

Theorem 2.1. *Let (X,d_G) be a quasi metric space and let A,B,S,T be four self-mappings on set X such that:*

(i) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*
(ii) *for all $x,y \in X$,*

$$\psi(d_G(Ax,By)) \le F(\psi(M(x,y)), \phi(M(x,y))),$$

where $M(x,y,y) = \max\{d_G(Sx,Ty), d_G(Sx,By), d_G(Ty,By)\}$;
(iii) *one of $A(X), B(X), S(X)$ or $T(X)$ is a closed subset of X.*

Further assume that one of the pairs (A,S) or (B,T) satisfies the property E.A. Then the pairs (A,S) and (B,T) have a coincidence point. Moreover, if (A,S) and (B,T) are weakly compatible then A,B,S and T have a unique common fixed point in X.

Proof. If the pair (B,T) satisfies the property E.A., then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} Bx_n = \lim_{n\to+\infty} Tx_n = t$, for some $t \in X$. Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence $\lim_{n\to+\infty} Sy_n = t$. We shall show that $\lim_{n\to+\infty} Ay_n = t$. From (ii), take $x =$

$y_n, y = x_n$,
we have

$$\psi(d_G(Ay_n, Bx_n)) \leq F(\psi(M(y_n, x_n)), \phi(M(y_n, x_n))) \leq \psi(M(y_n, x_n))$$

where

$$M(y_n, x_n) = \max\{d_G(Sy_n, Tx_n), d_G(Sy_n, Bx_n), d_G(Tx_n, Bx_n)\}.$$

Taking the limit as $n \to +\infty$ (upper limit) and using the definition of F, ϕ and ψ, we have

$$\psi(\lim_{n\to+\infty} d_G(Ay_n, t)) \leq \psi(0) = 0.$$

So, $\psi(\lim_{n\to+\infty}(d_G(Ay_n, t))) = 0$. Thus, $\lim_{n\to+\infty} d_G(Ay_n, t) = 0$ and so $\lim_{n\to+\infty} Ay_n = t$. Thus we have

$$\lim_{n\to+\infty} Ay_n = \lim_{n\to+\infty} Bx_n = \lim_{n\to+\infty} Sy_n = \lim_{n\to+\infty} Tx_n = t.$$

Suppose that $S(X)$ is a closed subset of X. Then $t = Su$ for some $u \in X$. Now, we shall show that $Au = Su = t$. From (ii), take $x = u$, $y = x_n$, we have

$$\psi(d_G(Au, Bx_n)) \leq F(\psi(M(u, x_n)), \phi(M(u, x_n)))$$

where

$$M(u, x_n) = \max\{d_G(Su, Tx_n), d_G(Su, Bx_n), d_G(Tx_n, Bx_n)\}.$$

Taking the limit as $n \to +\infty$, and using the definition of F, ϕ and ψ, we have

$$\psi(d_G(Au, Su)) \leq F(\psi(d_G(t, t)), \phi(d_G(t, t))) \leq \psi(d_G(t, t)) = \psi(0) = 0.$$

This gives, $Au = Su$. Thus u is a coincidence point of the pair (A, S). The weak compatibility of A and S implies that $ASu = SAu$ and hence $AAu = ASu = SAu = SSu$. As $A(X) \subseteq T(X)$, there exists $v \in X$ such that $Au = Tv$. We claim that $Tv = Bv$. By (ii), take $x = u$, $y = v$, we have

$$\psi(d_G(Au, Bv)) \leq F(\psi(M(u, v)), \phi(M(u, v))),$$

where

$$M(u, v) = \max\{d_G(Su, Tv), d_G(Su, Bv), d_G(Tv, Bv)\};$$

or

$$M(u, v) = \max\{0, d_G(Au, Bv), d_G(Au, Bv)\} = d_G(Au, Bv).$$

Using the definition of F, ϕ and ψ, we have

$$\psi(d_G(Au, Bv)) \leq F(\psi(d_G(Au, Bv)), \phi(d_G(Au, Bv))) \leq \psi(d_G(Au, Bv)).$$

This gives, $\psi(d_G(Au, Bv)) = 0$ or $\phi(d_G(Au, Bv)) = 0$. This implies that $Au = Bv$ and hence $Tv = Bv$. It follows that also the pair (B, T) has a coincidence point. Thus we have $Au = Su = Tv = Bv$.

Now, if B and T are weakly compatible, then we obtain $BTv = TBv = TTv = BBv$ and this shows that Au is a common fixed point of A, B, S and T. Again from (ii), take $x = Au$, $y = v$, we have

$$\psi(d_G(AAu, Au)) = \psi(d_G(AAu, Bv)) \leq F(\psi(M(Au, v)), \phi(M(Au, v))),$$

where

$$M(Au, v) = \max\{d_G(SAu, Tv), d_G(SAu, Bv), d_G(Tv, Bv)\};$$

$$M(Au, v) = \max\{d_G(AAu, Bv), d_G(AAu, Bv), 0\} = d_G(AAu, Bv).$$

Again using the definition of F, ϕ and ψ, we have

$$\psi(d_G(AAu, Au)) \leq F(\psi(d_G(AAu, Au)), \phi(d_G(AAu, Au))) \leq \psi(d_G(AAu, Au)).$$

This implies $Au = AAu = Bv$. Therefore, $Au = AAu = SAu$ is a common fixed point of A and S. Similarly, one can prove that Bv is a common fixed point of B and T. Since $Au = Bv$, we deduce that Au is a common fixed point of A, B, S and T.

Now, we have to show that the common fixed point is unique. Suppose to the contrary that w and z ($w \neq z$), are two common fixed points of A, B, S and T. Then, from (ii) and using the definition of F, ϕ, ψ, we have

$$\psi(d_G(Az, Bw)) \leq F(\psi(M(z, w)), \phi(M(z, w))),$$

where

$$\begin{aligned} M(z, w) &= \max\{d_G(Sz, Tw), d_G(Sz, Bw), d_G(Tw, Bw)\} \\ &= \max\{d_G(z, w), d_G(z, w), d_G(w, w)\} \\ &= d_G(z, w). \end{aligned}$$

Therefore, we have

$$\psi(d_G(z, w)) \leq F(\psi(d_G(z, w)), \phi(d_G(z, w))) \leq \psi(d_G(z, w)).$$

This gives, $w = z$. Therefore, A, B, S and T have a unique common fixed point.

Clearly proceeding on the foregoing lines, one can easily obtain the same conclusion in case (instead of $S(X)$) one of $A(X), B(X)$ or $T(X)$ is a closed subset of X, and in case (instead of (B, T)) (A, S) satisfies the property E.A. □

Corollary 2.2. *Let (X, d_G) be a quasi metric space and $A, B, S, T : X \to X$ be four mappings such that:*

(i) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*
(ii) *for all $x, y \in X$*

$$d_G(Ax, By) \leq F(\psi(M(x, y)), \phi(M(x, y))),$$

where $M(x, y) = \max\{d_G(Sx, Ty), d_G(Sx, By), d_G(Ty, By)\}$;
(iii) *one of $A(X), B(X), S(X)$ or $T(X)$ is a closed subset of X.*

Suppose that one of the pairs (A, S) or (B, T) satisfies the property E.A. Then the pairs (A, S) and (B, T) have a coincidence point. Further, if (A, S) and (B, T) are weakly compatible then A, B, S and T have a unique common fixed point in X.

If we assume $S = T$ in the above Theorem 2.1, we deduce the following result involving three self-mappings.

Corollary 2.3. *Let (X, d_G) be a quasi metric space and $A, B, S : X \to X$ be three mappings such that:*

(i) *$A(X) \subseteq S(X)$ and $B(X) \subseteq S(X)$;*

(ii) *for all $x, y \in X$,*

$$\psi(d_G(Ax, By)) \leq F(\psi(M(x, y)), \phi(M(x, y))),$$

where $M(x, y) = \max\{d_G(Sx, Sy), d_G(Sx, By), d_G(Sy, By)\}$;

(iii) *one of $A(X), B(X)$ or $S(X)$ is a closed subset of X.*

Suppose that one of the pairs (A, S) or (B, S) satisfies the property E.A. Then the pairs (A, S) and (B, S) have a coincidence point. Further, if (A, S) and (B, S) are weakly compatible then A, B and S have a unique common fixed point in X.

Example 2.4. Let $F(s, t) = \frac{99s}{100}, X = [0, 2]$ and $G : X \times X \times X \to [0, +\infty)$ be defined by $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$, for all $x, y, z \in X$. Define also $A, B, S : X \to X$ by $Ax = 1$, $Bx = 2 - x$ and $Sx = x$ for all $x \in X$ and $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = 20t$ for all $t \geq 0$. Clearly, the hypotheses (i) and (iii) of Corollary 2.3 hold trivially. Moreover, the pair (A, S) satisfies the property E.A. by taking sequence $x_n = \frac{n+1}{n}$. Here we show only that the hypothesis (ii) of Corollary 2.3 holds. In fact for all $x, y \in X$ we have, $d_G(Ax, By) = G(Ax, By, By) = G(1, 2 - y, 2 - y) = |1 - y|$, $d_G(Sx, Sy) = G(Sx, Sy, Sy) = G(x, y, y) = |x - y|$, $d_G(Sx, By) = G(Sx, By, By) = G(x, 2 - y, 2 - y) = |2 - x - y|$, $d_G(Sy, By) = G(Sy, By, By) = G(y, 2 - y, 2 - y) = 2|1 - y|$ and consequently

$$\psi(d_G(Ax, By)) = \psi(G(Ax, By, By)) \leq F(\psi(M(x, y)), \phi(M(x, y))).$$

Further, it implies that

$$|y - 1| \leq \frac{99}{100} M(x, y),$$

where

$$M(x, y) = \max\{d_G(Sx, Sy), d_G(Sx, By), d_G(Sy, By)\} = \max\{|x - y|, |x + y - 2|, 2|y - 1|\},$$

which is true. Then, by the Corollary 2.3, the pairs (A, S) and (B, S) have a coincidence point, that is, $u = 1$. Moreover, since (A, S) and (B, S) are weakly compatible, $u = 1$ is the unique common fixed point of A, B and S in X.

Corollary 2.5. *Let (X, d_G) be a quasi metric space and $A, T : X \to X$ be two mappings such that:*

(i) *for all $x, y \in X$,*

$$d_G(Ax, Ty) \leq F(\psi(M(x, y)), \phi(M(x, y))),$$

where $M(x, y) = \max\{d_G(Tx, Ty), d_G(Tx, Ay), d_G(Ty, Ay)\}$;

(iii) *$T(X)$ is a closed subset of X.*

Suppose that the pair (A, T) satisfies the property E.A. Then the pair (A, S) has a coincidence point. Further, if pair (A, S) is weakly compatible then A and T have a unique common fixed point in X.

Remark 2.6. Theorem 2.1 still remains true if $M(x,y)$ in condition (ii) is replaced by:

$$M_1(x,y) = \max\{d_G(Sx,Ty), d_G(Ax,Sx), d_G(By,Ty)\}.$$

By taking, $F(s,t) = s - t$, we obtain the following corollary.

Corollary 2.7. *Let (X, d_G) be a quasi metric space and A, B, S, T be four self-mappings on set X such that:*

(i) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*

(ii) *for all $x, y \in X$,*

$$\psi(d_G(Ax,By)) \le \psi(M(x,y)) - \phi(M(x,y)),$$

where $M(x,y) = \max\{d_G(Sx,Ty), d_G(Sx,By), d_G(Ty,By)\}$;

(iii) *one of $A(X), B(X), S(X)$ and $T(X)$ is a closed subset of X.*

Suppose that one of the pairs (A,S) and (B,T) satisfies the property E.A. Then the pairs (A,S) and (B,T) have a coincidence point. Moreover, if (A,S) and (B,T) are weakly compatible then A, B, S and T have a unique common fixed point in X.

Theorem 2.8. *Let A, B, S, T be four self-mappings on a quasi metric space (X, d_G) satisfying the condition (ii) of Theorem 2.1 and*

(a) *the pair (A,S) and (B,T) share the common (E.A.) property,*

(b) *$S(X)$ and $T(X)$ are closed subsets of X.*

Then the pairs (A,S) and (B,T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.

Proof. In view of (a), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Ty_n = z$ for some $z \in X$.
Since $S(X)$ is a closed subset of X, therefore, there exists a point $u \in X$ such that $z = Su$. We claim that $Au = z$. By (ii), take $x = u, y = y_n$,

$$\psi(d_G(Au, By_n)) \le F(\psi(M(u,y_n)), \phi(M(u,y_n))) \le \psi(M(u,y_n)),$$

where

$$M(u,y_n) = \max\{d_G(Su,Ty_n), d_G(Su,By_n), d_G(Ty_n,By_n)\}.$$

Taking the limit as $n \to +\infty$ (upper limit) and using the definition of F, ϕ and ψ, we have

$$\psi(d_G(Au,z)) \le F(0,0) \le \psi(0) = 0.$$

This gives, $Au = z = Su$ which shows that u is a coincidence point of the pair (A,S).
Since $T(X)$ is also a closed subset of X, therefore $\lim_{n\to+\infty} Ty_n = z$ in $T(X)$ and hence there exists $v \in X$ such that $Tv = z = Au = Su$. Now, we show that $Bv = z$. By using inequality (ii) of Theorem 2.1, take $x = u, y = v$, we have

$$\psi(d_G(Au,Bv)) \le F(\psi(M(u,v)), \phi(M(u,v))),$$

where

$$M(u,v) = \max\{d_G(Su,Tv), d_G(Su,Bv), d_G(Tv,Bv)\}.$$

Using the definition of F, ϕ and ψ, we have

$$\psi(d_G(z,Bv)) \leq F(\psi(d_G(z,Bv)), \phi(d_G(z,Bv)) \leq \psi(d_G(z,Bv)).$$

This gives, $\psi(d_G(z,Bv)) = 0$ or $\phi(d_G(z,Bv) = 0$. Hence $Bv = z = Tv$ which shows that v is a coincidence point of the pair (B,T).
Since the pairs (A,S) and (B,T) are weakly compatible and $Au = Su, Bv = Tv$, therefore, $Az = ASu = SAu = Sz, Bz = BTv = TBv = Tz$.

Next, we show that $Az = z$. Again, by using inequality (ii) of Theorem 2.1, take $x = z$ and $y = v$ we have

$$\psi(d_G(Az,Bv)) \leq F(\psi(M(z,v)), \phi(M(z,v))),$$

where

$$M(z,v) = \max\{d_G(Sz,Tv), d_G(Sz,Bv), d_G(Tv,Bv)\}.$$

Again by using the definition of F, ϕ and ψ, we have

$$\psi(d_G(Az,z)) \leq F(\psi(d_G(Az,z)), \phi(d_G(Az,z))) \leq \psi(d_G(Az,z)).$$

This gives, $Az = z = Sz$.
Similarly, one can prove that $Bz = Tz = z$. Hence, $Az = Bz = Sz = Tz$, and z is common fixed point of A, B, S and T.
For uniqueness, let z and w be two common fixed points of A, B, S and T. Then by using inequality (ii) of Theorem 2.1, we have

$$\psi(d_G(Az,Bw)) \leq F(\psi(M(z,w)), \phi(M(z,w))),$$

where

$$M(z,w) = \max\{d_G(Sz,Tw), d_G(Sz,Bw), d_G(Ty,Bw)\}.$$

Again by using the definition of F, ϕ and ψ, we easily get $z = w$. □

Remark 2.9. For different variants of $F(s,t)$ as in Example 1.5, we have various variants of our proved results for two or three or four self mappings with E.A. property or common (E.A.) property.

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CONFIDENCE REGIONS FOR BIVARIATE PROBABILITY DENSITY FUNCTIONS USING POLYGONAL AREAS

ORHAN KESEMEN, EDA ÖZKUL, AND ÜLKÜ ÜNSAL

Abstract. In this study, a polygonal approach is suggested to generalize the notion of the confidence region of the univariate probability density function for the bivariate probability density function. The equal density approach is used to demonstrate that confidence regions can be polygonal shapes. The bisection method is the preferred method in finding the equal density value that reveals the desired confidence coefficient. Confidence regions estimate not only bivariate unimodal probability functions but also bivariate multimodal probability functions. An approach is enhanced to estimate these confidence regions for probability density functions which are defined as rectangular, polygonal and infinite expanse areas. In order to show the applicable of the proposed method, four different examples are analyzed. The results show that the confidence region is found no matter how complex the distribution function. In addition, the proposed method gives more efficient results for multimodal probability density functions.

1. Introduction

In statistics, a confidence interval is an estimation of a parameter which represents the population within an acceptable range. Confidence interval was firstly identified, and its validity was proven by Neyman [1]. Tate and Klett [2] determined the optimal confidence interval and they estimated the optimal confidence interval for a normal distribution. Then, Dunn [3, 4] presented several procedures for determining the rectangular confidence regions. Chew [5] compiled the formulas for confidence, prediction, and tolerance regions for the multivariate normal distribution for the various cases of known and unknown mean vector and covariance matrix. Sidak [6] proved the validity of the rectangular confidence regions for the means of multivariate normal distributions given by Dunn [3, 4]. Hu and Yang [7] proposed a distribution-free approach, based on a few basic geometrical principles, to determine the confidence region for two or more variables. Also, they analyzed

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some biological data sets to demonstrate the use of the proposed method for genomics. Mammen and Polonik [8] constructed a confidence region for a density level set using kernel density estimators. Martin [9] described an approach, based on random sets, to construct exact confidence regions that attain the nominal coverage probability. Rambaud et al. [10] used the confidence regions to determine the characteristics for diagnosis of pneumonia in children younger than 5 years. Harrar and Xu [11] developed methods to construct confidence regions for level differences in the multi-dimensional cases and they applied it to the profile analysis.

This paper proposes the confidence regions for bivariate probability density functions using polygonal areas. The aim of this study is to estimate a more accurate confidence region by utilizing the equal density approach.

2. Confidence Interval

In a probability density function, infinite confidence interval might be defined that gives same confidence coefficient. Three different approaches are improved for choosing the most convenience in these intervals. These approaches are

a) The equally tailed confidence interval,
b) The shortest confidence interval,
c) The equal density confidence interval.

These methods show the same limits in symmetric distribution, such as normal distribution. However, the shortest confidence interval and the equal density confidence interval estimate the same limits while the equally tailed confidence interval estimates different limits in asymmetric distributions [8, 12, 13, 14]

Suppose $\mathbf{X} = \{X_1, X_2, ..., X_n\}$ are random samples that are taken from an i.i.d. This distribution's probability density function is $f(x;\theta,...)$ and the parameter θ is defined $\theta \in \mathbb{R}$. The values $\mathbf{x} = \{x_1, x_2, ..., x_n\}$ are the observed values of $\mathbf{X}$. Interval estimation of θ for these observed values are executed by means of the two bound values L and U. These two limit values must satisfy the condition $L \leqslant U$ for all x values. As a result, interval $[L, U]$ should involve the parameter θ with a certain probability $(1-\alpha)$. This probability value is called a confidence coefficient. This confidence coefficient constitutes the confidence interval when it is used in a format such as (1).

$$Pr(L \leqslant \theta \leqslant U) = 1 - \alpha \tag{1}$$

Here, the value α is the significance coefficient which is determined by the researcher and it ranges from $(0, 1)$.

2.1. **The Equally Tailed Confidence Interval.**

The equally tailed confidence interval is used commonly in the literature. Statistic $\theta^* \sim \Phi(x)$, which is obtained from a sample, is an estimation of the parameter θ of the discussed population. If the following condition is satisfied, this interval is called the equally tailed confidence interval.

$$Pr\left[L \leqslant \theta \leqslant U \mid Pr(\theta \geqslant U)\right] = 1 - \alpha \tag{2}$$

According to this condition, the probabilities of parameter θ, being less than the lower bound and being greater than the upper bound, are equal. Therefore, in this interval, the right and the left side of the confidence region are equal to each other.

The parameter θ of the population with the $(1 - \alpha)$ confidence coefficient can be calculated with the inverse distribution function below.

$$Pr\left[\Phi^{-1}\left(\alpha/2; \theta^*, ...\right) \leqslant \theta \leqslant \Phi^{-1}\left(1 - \alpha/2; \theta^*, ...\right)\right] = 1 - \alpha \tag{3}$$

2.2. **The Shortest Confidence Interval.** The shortest confidence interval is defined as (4).

$$Pr\left[L \leqslant \theta \leqslant U \mid \min(U - L)\right] = 1 - \alpha \tag{4}$$

There are infinite confidence limits with the same confidence level. In this method, confidence limits which have the minimum confidence width $(U - L)$ is preferred. Although the shortest confidence interval gives a more precise estimation, it is not preferred due to difficulty of its calculation.

2.3. **The Equal Density Confidence Interval.** The equal density method presents a different approach to the shortest confidence method. The basis of this approach is the conditional equality which is given as follows.

$$Pr\left[L \leqslant \theta \leqslant U \mid \phi(L; \theta, ...) = \phi(U; \theta, ...)\right] = 1 - \alpha \tag{5}$$

According to (5), the probability density values are equal in the shortest confidence limits with the $(1 - \alpha)$ confidence coefficient.

This approach is novel because it determines a cutting level in the y axis rather than research shortest confidence interval in the x axis, in order to calculate the shortest confidence interval. According to the cutting level, the confidence limits are determined as the roots of the following equation.

$$\phi(x; \theta, ...) - \zeta = 0 \tag{6}$$

This method is superior because it determines the confidence interval within a multimodal probability density function [14].

3. Confidence Region in Polygonal Area

The univariate confidence region is defined as a region that is restricted by the two bounds, (L, U), of the probability density function. These bounds can be determined by employing the afore-mentioned methods. However, the confidence region for bivariate probability density function cannot be estimated by using the equally-tailed confidence interval. In this case, the confidence region with the smallest area can be searched. Although there are many regions with the same volume, the confidence region with the smallest area can be found by utilizing two objective optimization techniques [13]. Since there is no relationship between two objective functions in multi objective optimization problems, several solutions with the same $(1-\alpha)$ confidence coefficient are selected to solve the problem, and the region with the smallest area is considered the confidence region. Although the selected region is likely the desired region, it is possible for deviations to exist. The aim of this study is to estimate a more accurate confidence region by utilizing the equal density approach.

3.1. **Determining the Search Area for the Confidence Region.** It is often impossible to determine a function, which defines the bounds of the region, that estimates a confidence region for bivariate arbitrary probability density functions. To solve this problem, this study considers the confidence region to be polygonal region. A polygonal region is a closed region which is formed by lines with combining clockwise or counterclockwise set points. The determination of this region on an infinite plane is often not possible due to the required computation time and memory usage. To solve this problem, an approximate search area is determined according to the probability density function within various situations, and thus the confidence region can be estimated with greater accuracy in this region. The determination of the search area for some probability density functions is given below.

3.1.1. *The Search Area for the Rectangular Definition Region.* In Figure 1(a), the probability density function of the discussed statistics is defined in a polygonal (rectangular) region $\left(\Phi(x,y) : [a,b]\times[c,d] \to \mathbb{R}^2\right)$. Its search area $\left(\Omega = \{P_i = (x_i, y_i)\,,\ i = 1, 2, \ldots, 4\}\right)$ is described in Figure 1(b).

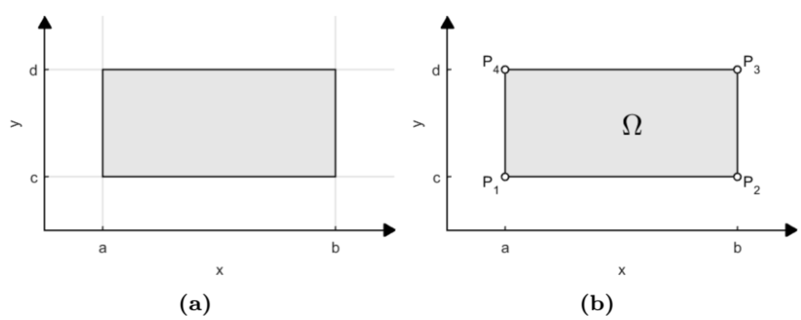
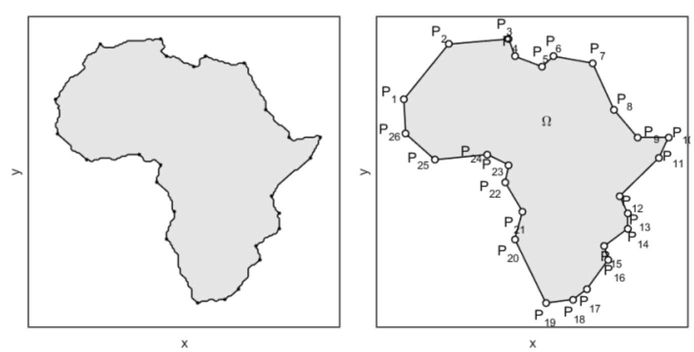


FIGURE 1. Determination of the search area of the rectangular definition region, (a) Rectangular definition region, (b) Search area.

3.1.2. *The Search Area for the Arbitrary Definition Region.* In Figure 2(a), the probability density function of the discussed statistics is defined in an arbitrarily restricted area. This area can be taken from digital images, maps, satellite images etc. There are many applications of bivariate density functions that are in a region restricted by polygonal area. These applications include the analysis of pollution or crime rate in a city, of the earthquake risk distribution and frequency in a country, of the density of a certain tree species in a forest [15], of the intensity of the dispersion of harmful insects in a field, of the traffic density in a particular area and of the location and transmission of an epidemic illness in a country. In addition, there could be multiple piecewise density functions of the examined area in the polygonal region. In this regard, the applications of bivariate density functions extend to many physically and politically divided cities (eg. Belfast, Beirut, Jerusalem, Mostar and Nicosia). That said, the data in the politically divided cities changes significantly over time. In this case, it may not be possible to evaluate an entire city in the same way that a region is evaluated. Each example may need statistical analysis based on a probability density function that is defined within a region with arbitrarily determined limits. A polygonal structure should be used to define an arbitrary region. Manual or automatic identification can determine the nodes of the polygonal area. In automatic identification, the examined region must be converted to a digital image. The nodes of the objects in the digital images are automatically determined by using the dominant point detection algorithm [16]. In this case, the search area, $\Omega = \{P_i = (x_i, y_i)\,,\ i = 1, 2, \ldots, N_\Omega\}$, is defined as a polygonal region consisting of N nodes.



(a) **(b)**

FIGURE 2. Determination of the search area of the polygonal definition region (a) Arbitrary definition region (African content) (b) Converting the arbitrary region to the polygonal search area via dominant points.

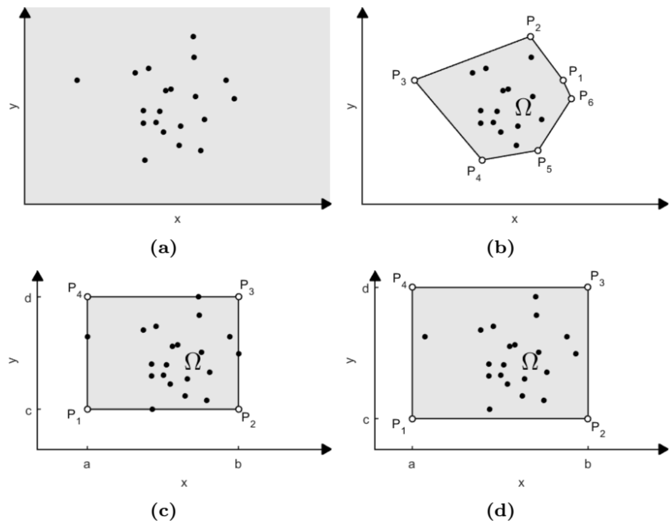


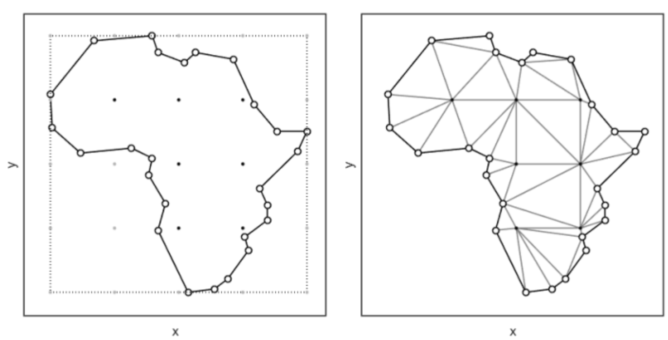
FIGURE 3. Determination of the search area via the points which are randomly generated from the distribution function, (a) Randomly generated points in an infinite definition region (b) Definition of the polygonal search area consisting of the random points (c) Determination of the rectangular search area on the boundaries of the random points (d) Determination of the expanded search area.

3.1.3. *The Search Area for the Infinite Definition Region.* The probability density functions are often defined in an infinite or semi-infinite space. In this study, the search area is determined as a finite region which has a higher confidence coefficient instead of an infinite region due to the difficulty of searching within an infinite region. In this regard, the proposed automatic solution requires generating a sufficiently random number(point) from the probability density function in this region (Figure 3(a)). This study proposes two approaches to determine the search area via these random points [17].

The first approach defines a polygonal search area $\Omega = \{P_i = (x_i, y_i)\,,\ i = 1, 2, \ldots, N_\Omega\}$ (Figure 3(b)) in which each point in this convex polygon surrounds random points called nodes The second approach determines a rectangular search area (Figure 3(c)) consisting of bounds of randomly generated numbers $(\Omega : [\min(X), \max(X)] \times [\min(Y), \max(Y)])$. However, in order to find the confidence region within the desired level for each of these two approaches, generated numbers must be numerous or selected search area must be expanded (Figure 3(c)).

3.2. **Triangulation of the Search Area.** In this study, a polygonal approach is proposed to find the confidence region within the search area. The equal density approach can be only used to find the coordinates of the nodes of the polygonal confidence region because there is no tail definition for equally area or equally volume in the two-variable expression. In addition, the term of the shortest region converts to the two objective optimization problem. In this optimization problem, the reliability of computation is weak because there is no relationship between objective functions. As the equal density approach creates polygonal closed curves similar to contour curves, it becomes simpler to apply as compared to the other approaches. In order to more quickly reach the solution in an infinite space, the search area is divided into small triangles and then the nodes of the polygonal confidence region are sought in the edges of these triangles.

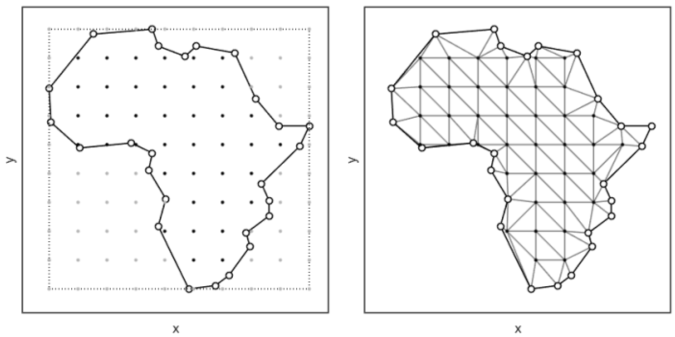
If the search area has a polygonal structure, grid points that are created around the polygon equidistantly in rectangular area are selected via the minimum and the maximum coordinates of the polygon nodes. These points are then recorded in a list (Figure 4(a)). Grid points that fall out of the polygon are removed from the list. The remaining grid points and nodes within the polygon are recorded on the same list, and obtained point set are divided into small triangles by using the Delaunay triangulation algorithm [18] (Figure 4(b)). If the polygonal confidence region needs to be more sensitively bound, then the region should be gridded more closely and divided into more small triangles (Figure 5).



(a) **(b)**

FIGURE 4. Dividing the polygonal area into small triangles, (a) Choosing the grid points with the size 5×5 grid; (b) Triangulation of the polygonal area.

When the number of grids increases, the sensitivity of confidence region also increases. Computation time and memory usage also increase. Therefore, the choice of the number of grids belongs to the researcher. However, it can be easily applied if the search area has a rectangular shape.



(a) **(b)**

FIGURE 5. Dividing the polygonal area into small triangles, (a) Choosing the grid points with the size 10×10 grid; (b) Triangulation of the polygonal area.

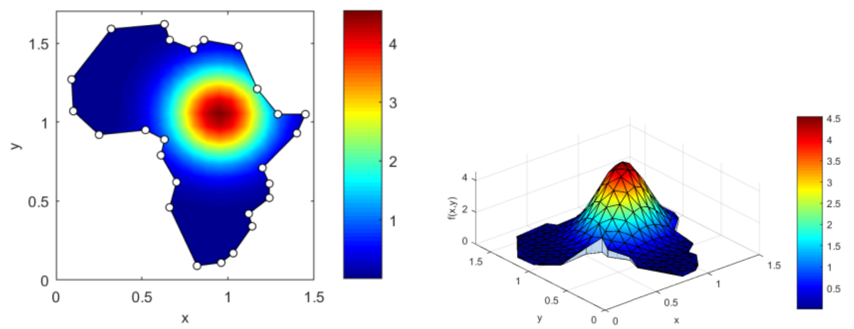
There are many studies in the literature about the optimal grid size for calculating numerical integration [19, 20, 21, 22]. Since this study is based on statistical distributions, it is necessary to investigate whether the error is significant or not. Selected statistical distributions can be used in many polygonal areas with different parameters. Also, distributions in the real life may be mixed or truncated distributions. In that case, it must be multiplied by a constant c, for the summation of the distribution value to be 1. Therefore, the constant c depends on the number of the selected grids. The best approach is to select the number the grid size according to the researcher's purpose. Nevertheless, the following equation can be given as an appropriate approach to determine the number of grids in each case.

$$I = \arg \min_{i=1,2,\ldots} \left| \frac{\Phi_i - \Phi_{i-1}}{\Phi_i} - \epsilon \right| \tag{7}$$

This equation gives the number of grids of two regions. Φ_i is the total volume of the two grids and ϵ is the tolerance value.

3.3. **Computing the Probability in the Polygonal Area.** The volume between the surface of the bivariate probability density function, defined in polygonal area and x-y plane, gives the sum of the probability value in the polygonal area. Hence, the probability value is calculated for each triangle within the triangulated polygonal region in order to estimate this volume. Analytical and numerical approaches for the calculation of the probability in the polygonal region are improved by Kesemen et. al [15]

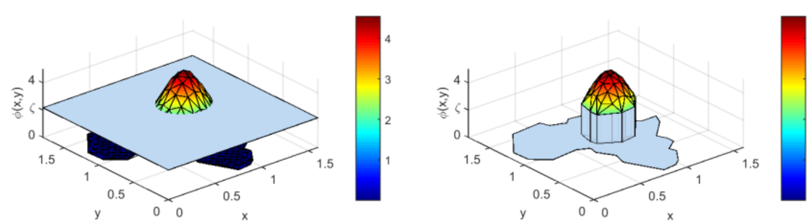
3.4. **Determining the Polygonal Area for the Cutting Level.** In this study, the equal density approach is used to determine the confidence region from the probability density function which is defined in the polygonal area. To explain the equal density approach, the model, shown in Figure 6, is chosen as an example. Figure 6(a) shows the values of two variable probability density functions in a polygonal area according to color changes. These color changes are shown in the surface form in Figure 6(b).



(a) **(b)**

FIGURE 6. Normal distribution which is limited by the African continent, (a) Contour format; (b) Surface format.

Figure 7(a) shows the intersection of a plane at ζ level. It is selected parallel to the x-y plane of the model given in Figure 6(b), with the probability density function placed in a polygonal area. The probability density function, which is intersected by ζ plane, returns to an area of zero under the plane. The area over the plane remains the same (Figure 7(b)).



(a) **(b)**

FIGURE 7. Cutting region, (a)Applying the cutting level to the probability density function; (b) Partial probability density function in the cutting region.

Each line segment between two nodes in the triangulated search area is labeled. Each labeled line segment and its neighbor nodes are recorded in a list. If the line segment is one of the edges of the polygon, it has only one neighbor node (Figure 8(a)). If not, it has two neighbor nodes (Figure 8(b)).

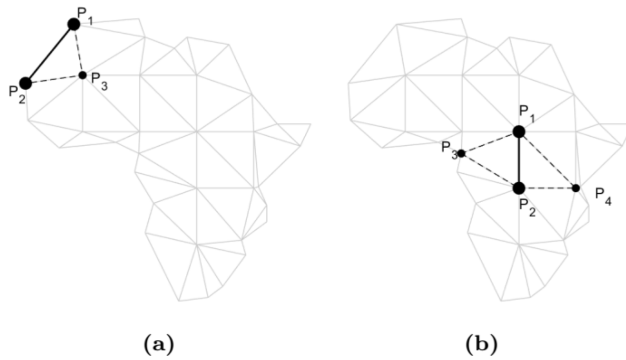


FIGURE 8. Labeling line segments, (a) Edge line segment and its neighbor nodes; (b) Inner line segment and its neighbor nodes.

The bold line segment in Figure 8(a) is between P_1 and P_2. Since it is an edge of the polygon, it has only one neighbor node (P_3). The bold line segment in Figure 8(b) is between P_1 and P_2. Since it is not an edge of the polygon, it has two neighbor nodes (P_3 and P_4).

Two-dimensional linear interpolation is used for the determination of the cutting points of all lines cut by the ζ, which is chosen by employing the equal density approach. According to interpolation, if the cutting point of each line is on the line segment, it is recorded as the inner polygon node (Figure 9(a), Figure 10(a), Figure 11(a)). These nodes $Q_\zeta : \{q_i = (x_i,\ y_i)\ ,\ i = 1, 2, \ldots, N_Q\}$ are listed by the line segment labels. The inner polygon nodes must be labeled in order to form a polygon. For this, in this study, the neighbor-tracking approach is used. According to this approach, labeling is performed by tracking the cutting nodes on the line segments (the dashed lines in Figure 11) in the neighbor nodes starting from an initial point (Figure 9(b), Figure 10(b), Figure 11(b)). If all cutting nodes are on the inner lines of the polygon, an inner polygon is formed by starting a random node and tracking the desired direction (Figure 9).

However, some cutting nodes fall on the line segment (Figure 10). Any edge-cutting node is chosen as the initial point. Then, tracking is performed from the initial point to the inner cutting nodes. When an edge-cutting node is reached, the tracking process continues until it reaches the initial cutting point. Thus, the cutting polygon is determined. After tracking the second edge cutting node, the edge node with the highest density value is most preferred. In Figure 10(b), this node is shown in a square by label 20.

If the cutting points cut the edges piece by piece (Figure 11(a)), tracking is performed from any edge cutting point to the inner nodes (Figure 11(a)). If an

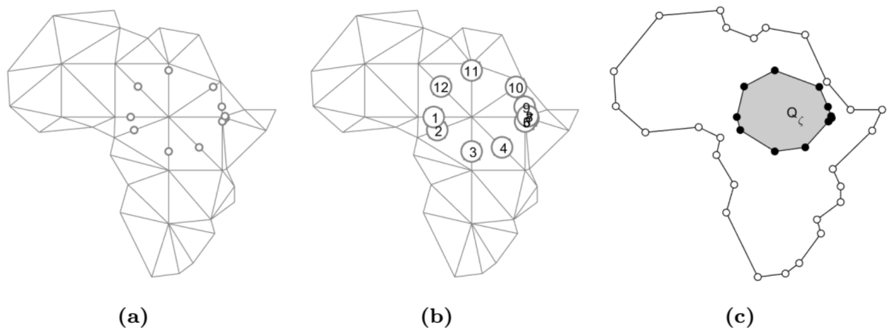


FIGURE 9. Inner polygon nodes according to selected cutting level, (a) Inner polygon nodes; (b) Sequentially labeled inner polygon nodes; (c) Inner polygon area.

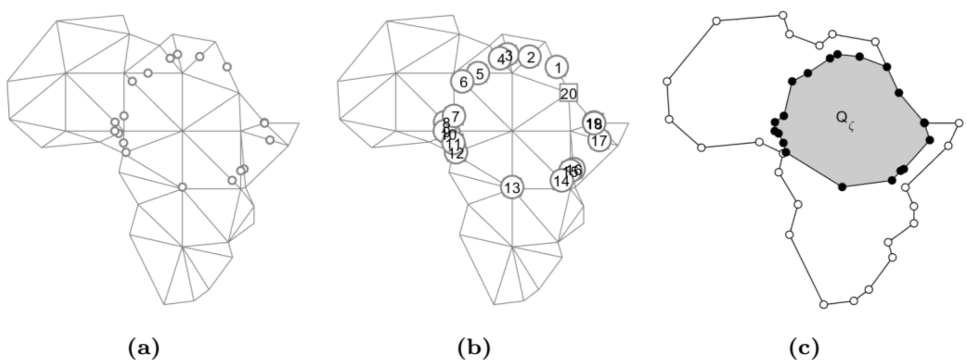


FIGURE 10. Inner polygon nodes according to selected cutting level, (a) Inner polygon nodes; (b) Sequentially labeled inner polygon nodes; (c) Inner polygon area.

edge node is reached in the tracking process, the tracking is continued on the edge node (Figure 11(a)) until another edge node is reached. If the reached node is not on the tracking list, tracking is performed on the other nodes (Figure 11(a)). This process is continued until the edge cutting node is on the tracking list (Figure 11).

Although the tracking process is finished, there are still unlabeled cutting nodes remaining. This shows that there is more than one confidence region. In this case, independent regions are determined by performing an independent tracking process to the unlabeled nodes.

3.5. **Finding the Confidence Region within the Search Area.** There is a relationship between the probability value (P) of each level at the cutting region

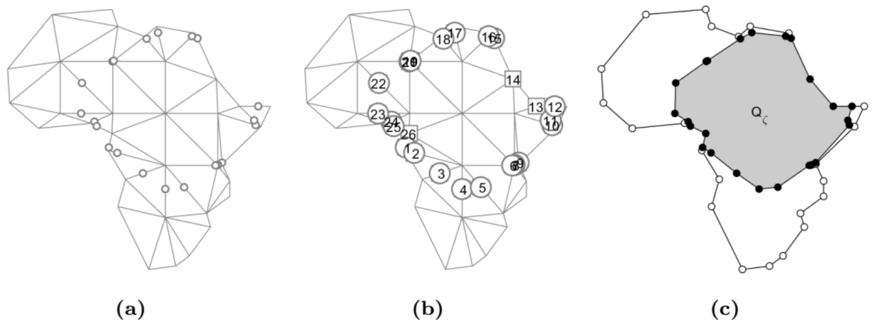


FIGURE 11. Inner polygon nodes according to selected cutting level, (a) Inner polygon nodes; (b) Sequentially labeled inner polygon nodes; (c) Inner polygon area.

and the cutting level (ζ). Based on this relationship, the probability value can be written as a function of ζ below.

$$P(\zeta) = \Pr\left[(X,Y) \in Q_\zeta \mid \phi\left(x_i, y_i\right) = \zeta,\ i = 1, 2, \ldots, N_Q\right] \tag{8}$$

In this approach, the bisection method is preferred in order to find the density level (ζ) which gives the desired probability value $(1-\alpha)$. According to the bisection method, two initial values, given below, are determined as the minimum and maximum value of the probability density value of all nodes.

$$\begin{aligned} \zeta_a &= \min\{\phi\left(x_i, y_i\right)\}, (i = 1, 2, ..., N_T) \\ \zeta_b &= \max\{\phi\left(x_i, y_i\right)\}, (i = 1, 2, ..., N_T) \end{aligned} \tag{9}$$

The probabilities of these levels are expected to be as follows.

$$\begin{aligned} P\left(\zeta_a\right) &= 1 \\ P\left(\zeta_b\right) &= 0 \end{aligned} \tag{10}$$

This only occurs if the distribution function of the polygonal region is the uniform distribution. In this case, polygon points are collapsed inward or expanded outward until the desired probability value is obtained. Eventually, polygonal bounds of the region, which provide the desired confidence coefficient within a polygonal search area, are determined.

Algorithm 1. *Algorithm of the determination of the polygonal confidence*

Step 1.	*Determine polygon nodes of the search area,*
Step 2.	*Divide the polygonal search area into small triangles by gridding the area,*
Step 3.	*Label the line segment between two neighbor nodes, forming the triangles,*
Step 4.	*Determine the lower and upper initial cutting levels* (ζ_a, ζ_b) *as in Equation (9),*
Step 5.	*Determine the search level* ζ_c*,* $\zeta_c = (\zeta_a + \zeta_b)/2$
Step 6.	*Determine the locations of the line segments which are cut by search level,*
Step 7.	*Determine the cutting region* Q_ζ *via cutting points,*
Step 8.	*Apply the triangulation algorithm to the cutting region after the gridding process,*
Step 9.	*Calculate the probability value* $P(\zeta_c)$ *of the triangulated cutting region,*
Step 10.	*If* $\lvert P(\zeta_c) - (1-\alpha)\rvert < \epsilon$*, stop the process,*
Step 11.	*Update the lower and the upper levels* *If* $P(\zeta_c) < (1-\alpha)$ $\zeta_b = \zeta_c$ *else,* $\zeta_a = \zeta_c$
Step 12.	*Go to Step 5.*

4. Experimental Results

In this study, three different examples are used to measure the performance of the proposed method. The first example aims to determine the confidence region for the standard normal distribution that is defined in an infinite space. In this example, the performance of the proposed method is shown by comparing the polygonal confidence regions. The second example searches the confidence region for the quadratic gamma distribution that is defined in a semi-infinite space. In the third example, the performance of the proposed method for multimodal probability density function is also analyzed. The fourth example examines the degree of influence by the northern Anatolian fault line of Sivas province.

For this study, a computer with an Intel® Core i7 2.40GHz processor was used, and MATLAB® was used for the application. For all models, the contour that passes the through the middle of the initially selected minimum and maximum contours are determined in each iteration and then, the cumulative distribution value of its region is calculated. Until the obtained value converges the desired confidence level, the iterations are continued. The last found contour is assumed the desired contour.

Example 1. *This example determines the confidence region with the confidence coefficients* $\alpha = 0.01, 0.025, 0.05, 0.1$ *for the bivariate standard normal distribution function defined in an infinite space. Its probability density function is given in Equation (11).*

TABLE 1. Confidence coefficients for the significance level $\alpha = 0.01$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.99 - P(\zeta_c) \mid$
1	0.000000	0.152609	0.076304	0.515705	0.474295
2	0.000000	0.076304	0.038152	0.762730	0.227270
3	0.000000	0.038152	0.019076	0.882182	0.107818
4	0.000000	0.019076	0.009538	0.940484	0.049516
5	0.000000	0.009538	0.004769	0.970907	0.019093
6	0.000000	0.004769	0.002385	0.985380	0.004620
7	0.000000	0.002385	0.001192	0.992541	0.002541
8	0.001192	0.002385	0.001788	0.988984	0.001016
9	0.001192	0.001788	0.001490	0.990809	0.000809

TABLE 2. Confidence coefficients for the significance level $\alpha = 0.025$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.975 - P(\zeta_c) \mid$
1	0.000000	0.152609	0.076304	0.515705	0.459295
2	0.000000	0.076304	0.038152	0.762730	0.212270
3	0.000000	0.038152	0.019076	0.882182	0.092818
4	0.000000	0.019076	0.009538	0.940484	0.034516
5	0.000000	0.009538	0.004769	0.970907	0.004093
6	0.000000	0.004769	0.002385	0.985380	0.010380
7	0.002385	0.004769	0.003577	0.977459	0.002459
8	0.003577	0.004769	0.004173	0.974140	0.000860

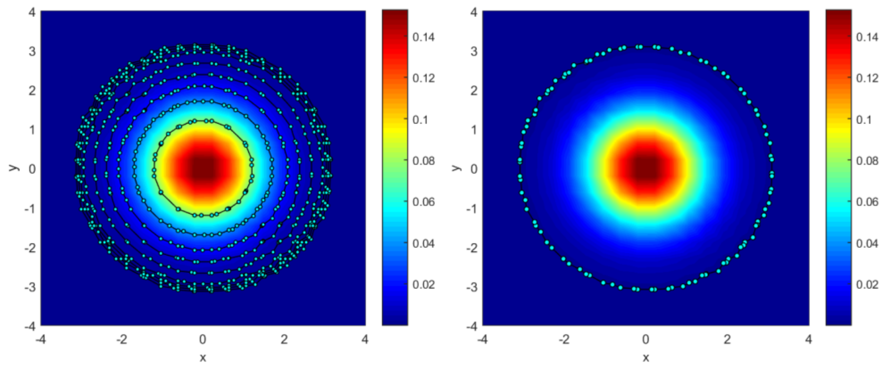
$$\phi(x, y) = \frac{1}{2\pi} e^{\left(-\frac{x^2+y^2}{2}\right)} \tag{11}$$

The search area is determined as $\Omega : [-4, 4] \times [-4, 4]$*. Some probability values may be outside of the region due to the conversion of the probability density function, which is defined in an infinite space to the search area in a finite space. Therefore, the probability value in the search area is calculated as* 0.998*. This probability value can be sufficient for the search area. The tolerance value* (ϵ) *is determined as* 1×10^{-3} *and the region is also gridded according to the number of* 20×20 *grids. Table 1 shows the simulation results for the significance level* 0.01*.*

According to Table 1, the confidence coefficient 0.99 *was reached in the* 9^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 12(a). Figure 12(b) demonstrates the optimum confidence region for* $\alpha = 0.01$*.*

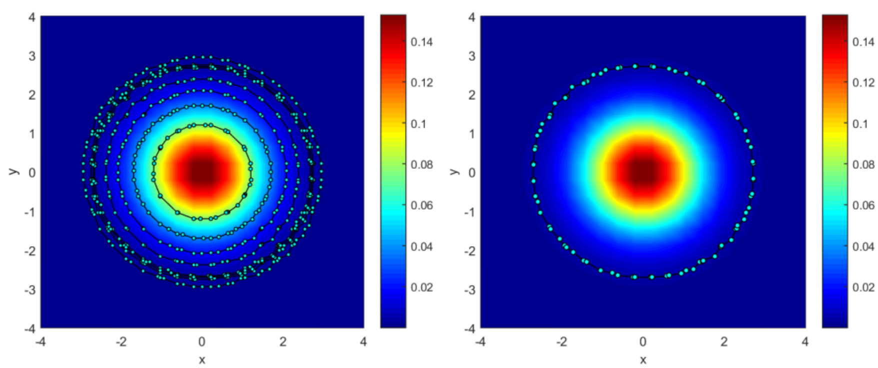
The simulation results for the significance level 0.025 *are given in Table 2.*

According to Table 2, the confidence coefficient 0.975 *was reached in the* 8^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 13(a). Figure 13(b) demonstrates the optimum confidence region for* $\alpha = 0.025$*.*



(a) **(b)**

FIGURE 12. Finding the confidence region for $\alpha = 0.01$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.



(a) **(b)**

FIGURE 13. Finding the confidence region for $\alpha = 0.025$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

Table 3 shows the simulation results for the significance level 0.05.

According to Table 3, the confidence coefficient 0.95 *was reached in the* 9^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 14(a). Figure 14(b) demonstrates the optimum confidence region for* $\alpha = 0.05$.

Table 4 shows the simulation results for the significance level 0.10.

TABLE 3. Confidence coefficients for the significance level $\alpha = 0.05$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.95 - P(\zeta_c) \mid$
1	0.000000	0.152609	0.076304	0.515705	0.434295
2	0.000000	0.076304	0.038152	0.762730	0.187270
3	0.000000	0.038152	0.019076	0.882182	0.067818
4	0.000000	0.019076	0.009538	0.940484	0.009516
5	0.000000	0.009538	0.004769	0.970907	0.020907
6	0.004769	0.009538	0.007154	0.956182	0.006182
7	0.007154	0.009538	0.008346	0.948216	0.001784
8	0.007154	0.008346	0.007750	0.952225	0.002225
9	0.007750	0.008346	0.008048	0.950226	0.000226

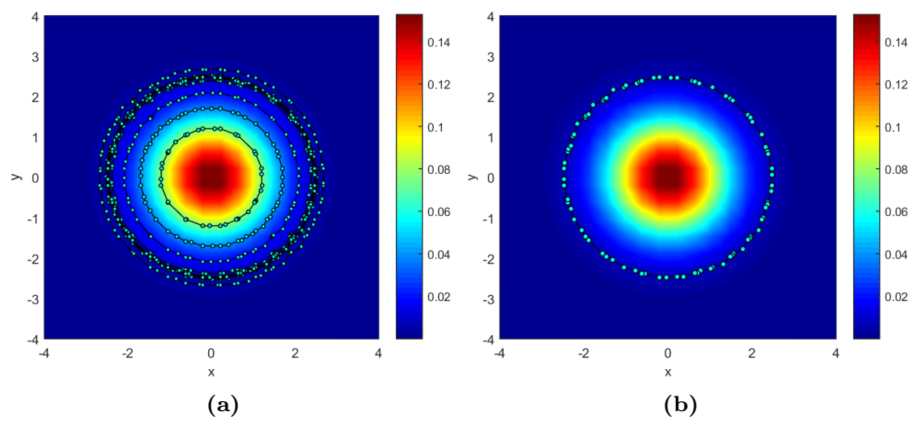


FIGURE 14. Finding the confidence region for $\alpha = 0.05$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

According to Table 4, the confidence coefficient 0.90 *was reached in the* 9^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 15(a). Figure 15(b) demonstrates the optimum confidence region for* $\alpha = 0.10$.

In order to measure the performance of the proposed method, the confidence regions for the standard normal distribution must be determined according to the given confidence coefficients. As shown in Equation (11), the desired confidence region is symmetrical in all directions. According to each of the three approaches mentioned in Section 2, the confidence region has a circular structure. Thus, the first parameter of the circle is the center point, $(0,0)$, *as seen in Equation (11). To find the other parameter, the radius, the probability density function should be converted to the polar coordinate system from the Cartesian coordinate system. This*

TABLE 4. Confidence coefficients for the significance level $\alpha = 0.10$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.90 - P(\zeta_c) \mid$
1	0.000000	0.152609	0.076304	0.515705	0.384295
2	0.000000	0.076304	0.038152	0.762730	0.137270
3	0.000000	0.038152	0.019076	0.882182	0.017818
4	0.000000	0.019076	0.009538	0.940484	0.040484
5	0.009538	0.019076	0.014307	0.912242	0.012242
6	0.014307	0.019076	0.016692	0.897226	0.002774
7	0.014307	0.016692	0.015499	0.904739	0.004739
8	0.015499	0.016692	0.016095	0.901006	0.001006
9	0.016095	0.016692	0.016394	0.899079	0.000921

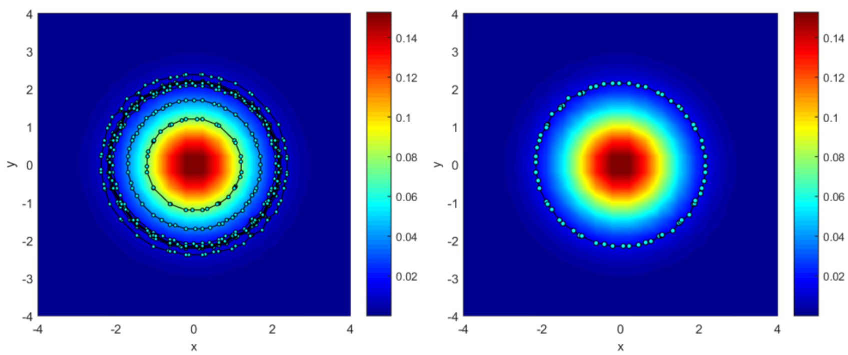


FIGURE 15. Finding the confidence region for $\alpha = 0.10$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

function is converted as (12).

$$\Phi(r,\theta) = \frac{1}{2\pi}\int_0^{\theta}\int_0^{r} e^{-\frac{r^2}{2}}\ r\ dr d\theta \tag{12}$$

Equation (12) is updated to the following equation by integrating θ into the interval $[0, 2\pi]$. θ is integrated due to the equality of the changes in each direction of the probability density function.

$$\Phi(r, 2\pi) = \Phi_R(r) = \int_0^{r} e^{-\frac{r^2}{2}} r\ dr \tag{13}$$

The distribution function based on r is found below.

$$\Phi_R(r) = 1 - e^{-\frac{r^2}{2}} \tag{14}$$

TABLE 5. Radius sizes and relative absolute error

α	r_α	E_α	$100\,(1-E_\alpha)$
0.01	3.0349	0.0197	98.0306
0.025	2.7162	0.0168	98.3199
0.05	2.4477	0.0144	98.5593
0.1	2.1460	0.0162	98.3779

The desired confidence region Ω *is shown as (15).*

$$\begin{aligned}\Pr\left((X,Y)\in\Omega\right) &= 1-\alpha \\ &= \Phi_R\left(r\right)-\Phi_R(0)\end{aligned} \tag{15}$$

In this equation, $\Phi_R\left(0\right)=0$. *Thus, Equation (16) is obtained by updating the equation above.*

$$\begin{aligned}\Phi_R\left(r\right) &= 1-\alpha \\ &= 1-e^{-\frac{r^2}{2}}\end{aligned} \tag{16}$$

The following equation is obtained when the equation above is updated.

$$r_\alpha=\sqrt{-2\log\left(\alpha\right)} \tag{17}$$

Through this equation, radius sizes with different α *values are given Table 5.*

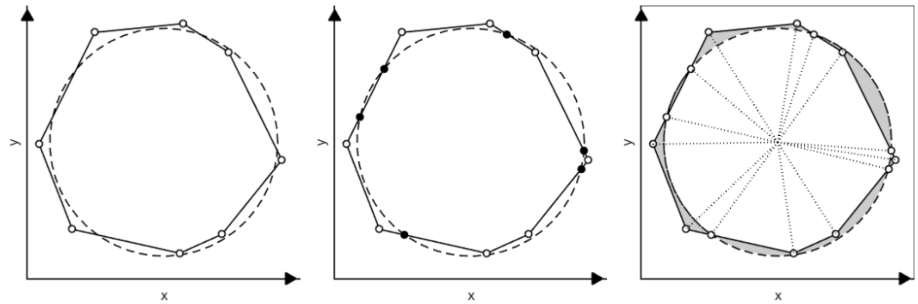
In order to measure the performance of the proposed method, the radius that results from Equation (17) should be compared with the confidence region determined by employing the proposed method. In order to do this, the circle and the polygon are superimposed in Figure 16(a). Also, to calculate more clearly the differences between two shapes, the points on the circle are added to the polygon (Figure 16(b)). The sectors are formed by the drawing of line segments from each polygon node to the center of the circle. Then, the relative absolute error is calculated by determining the ratio of the sum of the subtraction of the sectors and triangles with the entire area of the circle (Figure 16(c)). It is demonstrated as follows.

$$E_\alpha=\frac{\sum_{i=1}^{n}\left|Sector_i-Triangle_i\right|}{\pi r_\alpha^2} \tag{18}$$

The relative absolute error and percent performance rate for all significance values are shown in Table 5.

Example 2. *In this example, the aim is to determine the confidence regions for bivariate Gamma distribution within a semi-infinite space. The probability density function of the Gamma distribution is given as (19)*

$$\phi\left(x,y\right)=\frac{1}{4}x^2y^2\ e^{-(x+y)} \tag{19}$$



(a) **(b)** **(c)**

FIGURE 16. The calculation of the relative absolute error value, (a) The theoretical confidence region boundary (dashed curve) with the polygonal confidence region boundary (solid lines); (b) Determining the junction points of the boundaries (black points) and recording them to polygonal array; (c) The determination of sectors that draw lines from all points of the polygon to the center of the circle and the determination of the differences with the sector and the triangle (shaded area).

TABLE 6. Confidence coefficients for the significance level $\alpha = 0.01$

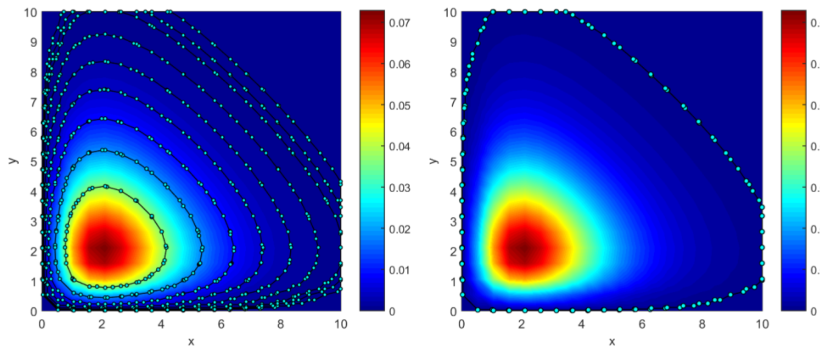
Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.99 - P(\zeta_c) \mid$
1	0.000000	0.072871	0.036436	0.466596	0.523404
2	0.000000	0.036436	0.018218	0.720246	0.269754
3	0.000000	0.018218	0.009109	0.854392	0.135608
4	0.000000	0.009109	0.004554	0.922488	0.067512
5	0.000000	0.004554	0.002277	0.958070	0.031930
6	0.000000	0.002277	0.001139	0.976851	0.013149
7	0.000000	0.001139	0.000569	0.987064	0.002936
8	0.000000	0.000569	0.000285	0.991079	0.001079
9	0.000285	0.000569	0.000427	0.989236	0.000764

Table 6 shows the simulation results for the significance level 0.01.

According to Table 6, the confidence coefficient 0.99 *was reached in the* 9^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 17(a). Figure 17(b) demonstrates the optimum confidence region for* $\alpha = 0.01$.

The simulation results for the significance level 0.025 are given in Table 7.

According to Table 7, the confidence coefficient 0.975 *was reached in the* 9^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 18(a). Figure 18(b) demonstrates the optimum confidence region for* $\alpha = 0.025$.



(a) (b)

FIGURE 17. Finding the confidence region for $\alpha = 0.01$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

TABLE 7. Confidence coefficients for the significance level $\alpha = 0.025$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\lvert 0.975 - P(\zeta_c) \rvert$
1	0.000000	0.072871	0.036436	0.466596	0.508404
2	0.000000	0.036436	0.018218	0.720246	0.254754
3	0.000000	0.018218	0.009109	0.854392	0.120608
4	0.000000	0.009109	0.004554	0.922488	0.052512
5	0.000000	0.004554	0.002277	0.958070	0.016930
6	0.000000	0.002277	0.001139	0.976851	0.001851
7	0.001139	0.002277	0.001708	0.967380	0.007620
8	0.001139	0.001708	0.001423	0.972095	0.002905
9	0.001139	0.001423	0.001281	0.974489	0.000511

Table 8 shows the simulation results for the significance level 0.05.

According to Table 8, the confidence coefficient 0.95 *was reached in the* 10^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 19(a). Figure 19(b) demonstrates the optimum confidence region for* $\alpha = 0.05$.

Table 9 shows the simulation results for the significance level 0.10.

According to Table 9, the confidence coefficient 0.90 was reached in the 10^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 20(a). Figure 20(b) demonstrates the optimum confidence region for* $\alpha = 0.10$.

Example 3. *This example determines the confidence regions for mixture normal distribution based on the African continent as a polygonal area. Based on the pixel*

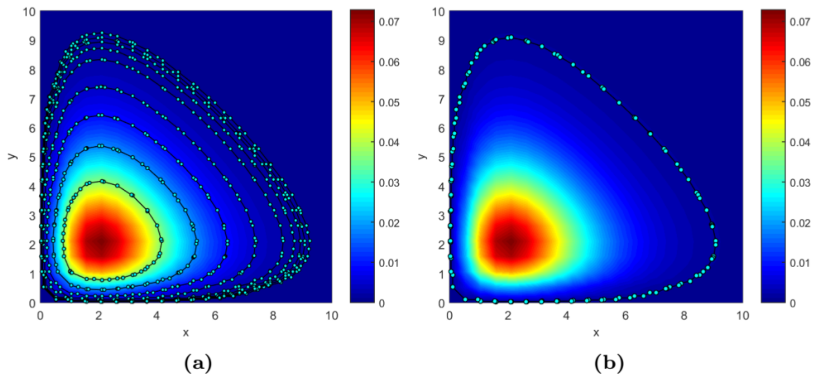


FIGURE 18. Finding the confidence region for $\alpha = 0.025$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

TABLE 8. Confidence coefficients for the significance level $\alpha = 0.05$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.95 - P(\zeta_c) \mid$
1	0.000000	0.072871	0.036436	0.466596	0.483404
2	0.000000	0.036436	0.018218	0.720246	0.229754
3	0.000000	0.018218	0.009109	0.854392	0.095608
4	0.000000	0.009109	0.004554	0.922488	0.027512
5	0.000000	0.004554	0.002277	0.958070	0.008070
6	0.002277	0.004554	0.003416	0.940103	0.009897
7	0.002277	0.003416	0.002847	0.948978	0.001022
8	0.002277	0.002847	0.002562	0.953394	0.003394
9	0.002562	0.002847	0.002704	0.951218	0.001218
10	0.002704	0.002847	0.002775	0.950116	0.000116

length of the African continent, which is taken as a digital image, the probability density function is given below.

$$\phi(x, y) = \frac{1}{3300} e^{-\frac{(x-120)^2+(y-60)^2}{1000}} + \frac{1}{4000} e^{-\frac{(x-50)^2+(y-125)^2}{700}} \tag{20}$$

Table 10 shows the simulation results for the significance level 0.01.

According to Table 10, the confidence coefficient 0.99 *was reached in the* 4^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 21(a). Figure 21(b) demonstrates the optimum confidence region for* $\alpha = 0.01$.

The simulation results for the significance level 0.025 *are given in Table 11.*

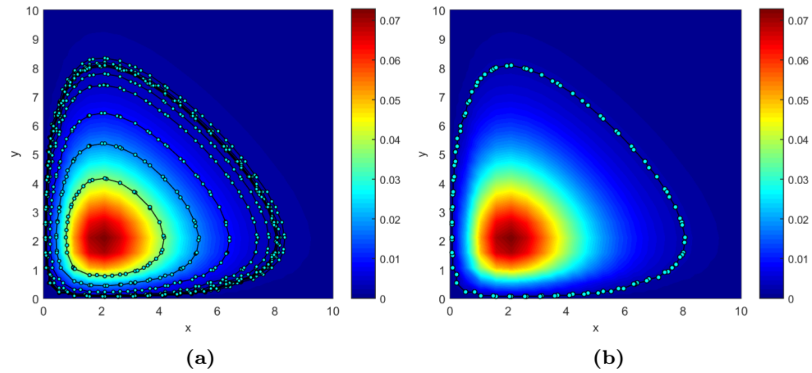


FIGURE 19. Finding the confidence region for $\alpha = 0.05$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

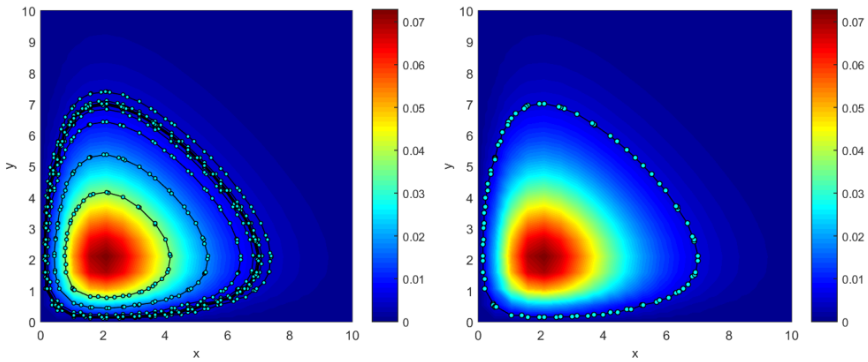
TABLE 9. Confidence coefficients for the significance level $\alpha = 0.10$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.90 - P(\zeta_c) \mid$
1	0.000000	0.072871	0.036436	0.466596	0.433404
2	0.000000	0.036436	0.018218	0.720246	0.179754
3	0.000000	0.018218	0.009109	0.854392	0.045608
4	0.000000	0.009109	0.004554	0.922488	0.022488
5	0.004554	0.009109	0.006832	0.888147	0.011853
6	0.004554	0.006832	0.005693	0.905481	0.005481
7	0.005693	0.006832	0.006262	0.896776	0.003224
8	0.005693	0.006262	0.005978	0.901152	0.001152
9	0.005978	0.006262	0.006120	0.898973	0.001027
10	0.005978	0.006120	0.006049	0.900064	0.000064

TABLE 10. Confidence coefficients for the significance level $\alpha = 0.01$

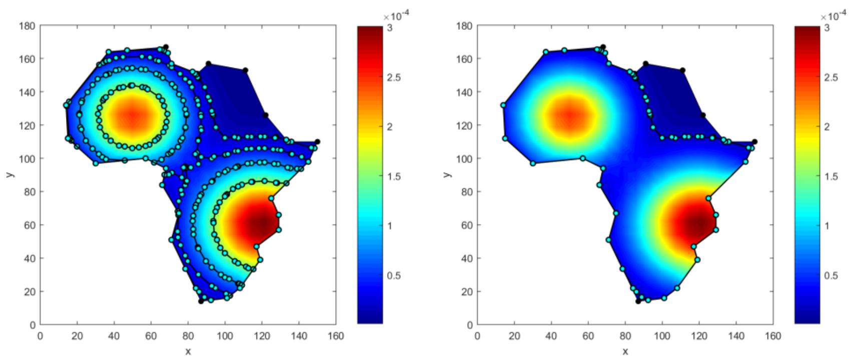
Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.99 - P(\zeta_c) \mid$
1	0.000000	0.000301	0.000151	0.513829	0.476171
2	0.000000	0.000151	0.000076	0.811344	0.178656
3	0.000000	0.000076	0.000038	0.947257	0.042743
4	0.000000	0.000038	0.000019	0.990765	0.000765

According to Table 11, the confidence coefficient 0.975 *was reached in the* 5^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 22(a). Figure 22(b) demonstrates the optimum confidence region for* $\alpha = 0.025$.



(a) **(b)**

FIGURE 20. Finding the confidence region for $\alpha = 0.10$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.



(a) **(b)**

FIGURE 21. Finding the confidence region for $\alpha = 0.01$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

Table 12 shows the simulation results for the significance level 0.05.

According to Table 12, the confidence coefficient 0.95 *was reached in the* 9^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 23(a). Figure 23(b) demonstrates the optimum confidence region for* $\alpha = 0.05$.

Table 13 shows the simulation results for the significance level 0.10.

TABLE 11. Confidence coefficients for the significance level $\alpha = 0.025$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.975 - P(\zeta_c) \mid$
1	0.000000	0.000301	0.000151	0.513829	0.461171
2	0.000000	0.000151	0.000076	0.811344	0.163656
3	0.000000	0.000076	0.000038	0.947257	0.027743
4	0.000000	0.000038	0.000019	0.990765	0.015765
5	0.000019	0.000038	0.000029	0.975359	0.000359

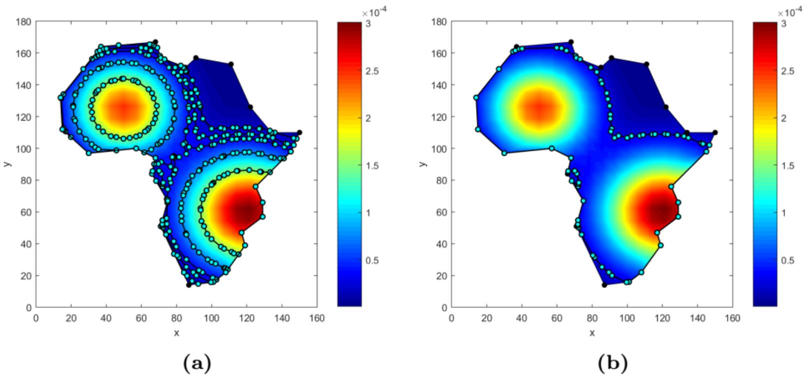
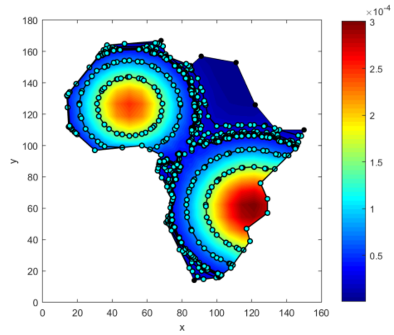


FIGURE 22. Finding the confidence region for $\alpha = 0.025$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

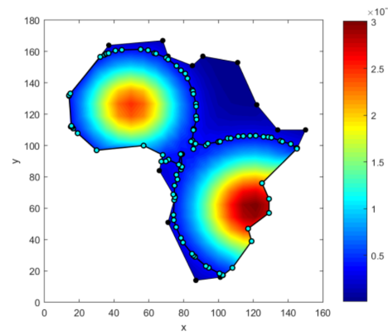
TABLE 12. Confidence coefficients for the significance level $\alpha = 0.05$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.95 - P(\zeta_c) \mid$
1	0.000000	0.000301	0.000151	0.513829	0.436171
2	0.000000	0.000151	0.000076	0.811344	0.138656
3	0.000000	0.000076	0.000038	0.947257	0.002743
4	0.000000	0.000038	0.000019	0.990765	0.040765
5	0.000019	0.000038	0.000029	0.975359	0.025359
6	0.000029	0.000038	0.000033	0.963739	0.013739
7	0.000033	0.000038	0.000036	0.956269	0.006269
8	0.000036	0.000038	0.000037	0.951965	0.001965
9	0.000037	0.000038	0.000037	0.949513	0.000487

According to Table 13, the confidence coefficient 0.90 *was reached in the* 9^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 24(a). Figure 24(b) demonstrates the optimum confidence region for* $\alpha = 0.10$*.*



(a)



(b)

FIGURE 23. Finding the confidence region for $\alpha = 0.05$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

TABLE 13. Confidence coefficients for the significance level $\alpha = 0.10$

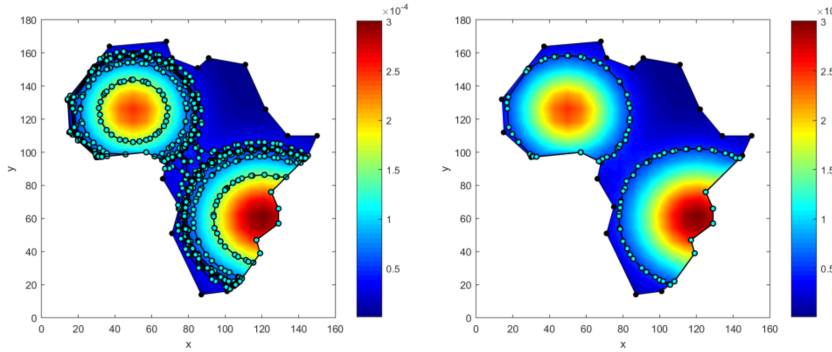
Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\lvert 0.90 - P(\zeta_c) \rvert$
1	0.000000	0.000301	0.000151	0.513829	0.386171
2	0.000000	0.000151	0.000076	0.811344	0.088656
3	0.000000	0.000076	0.000038	0.947257	0.047257
4	0.000038	0.000076	0.000057	0.881265	0.018735
5	0.000038	0.000057	0.000047	0.914899	0.014899
6	0.000047	0.000057	0.000052	0.898041	0.001959
7	0.000047	0.000052	0.000050	0.906344	0.006344
8	0.000050	0.000052	0.000051	0.902225	0.002225
9	0.000051	0.000052	0.000051	0.900138	0.000138

Example 4. *In this example, earthquake risk distribution of Sivas province is modeled linearly according to North Anatolian fault line* [23] *and its confidence region is estimated. Based on the pixel length of the Sivas province, which is taken as a digital image, the probability density function is given below.*

$$\phi(x,y) = \frac{155 - \left|(x-150)\cos\left(\frac{17\pi}{12}\right) + (y-160)\sin\left(\frac{17\pi}{12}\right)\right|}{1.6 \times 10^6} \tag{21}$$

In (21), the decreasing earthquake intensity risk in the direction parallel to the fault line is modeled as a linear decreasing probability density function with respect to the fault line.

Table 14 shows the simulation results for the significance level 0.01.



(a) **(b)**

FIGURE 24. Finding the confidence region for $\alpha = 0.10$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

TABLE 14. Confidence coefficients for the significance level $\alpha = 0.01$

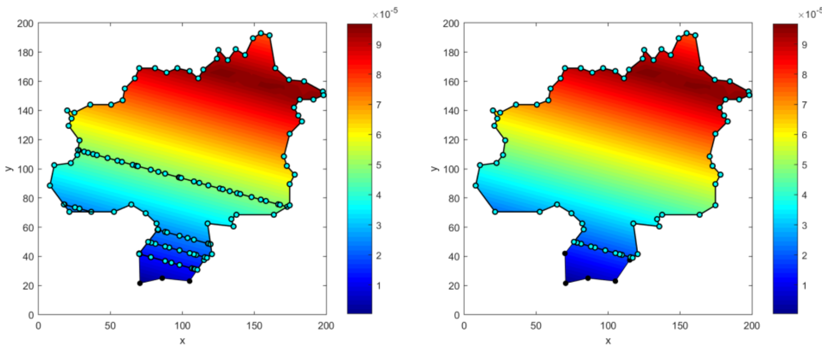
Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.99 - P(\zeta_c) \mid$
1	0.000000	0.000097	0.000049	0.805538	0.184462
2	0.000000	0.000049	0.000024	0.980396	0.009604
3	0.000000	0.000024	0.000012	0.995995	0.005995
4	0.000012	0.000024	0.000018	0.989925	0.000075

TABLE 15. Confidence coefficients for the significance level $\alpha = 0.025$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.975 - P(\zeta_c) \mid$
1	0.000000	0.000097	0.000049	0.805538	0.169462
2	0.000000	0.000049	0.000024	0.980396	0.005396
3	0.000024	0.000049	0.000037	0.919918	0.055082
4	0.000024	0.000037	0.000031	0.958831	0.016169
5	0.000024	0.000031	0.000028	0.971512	0.003488
6	0.000024	0.000028	0.000026	0.976528	0.001528
7	0.000026	0.000028	0.000027	0.974098	0.000902

According to Table 14, the confidence coefficient 0.99 *was reached in the* 4^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 25(a). Figure 25(b) demonstrates the optimum confidence region for* $\alpha = 0.01$.

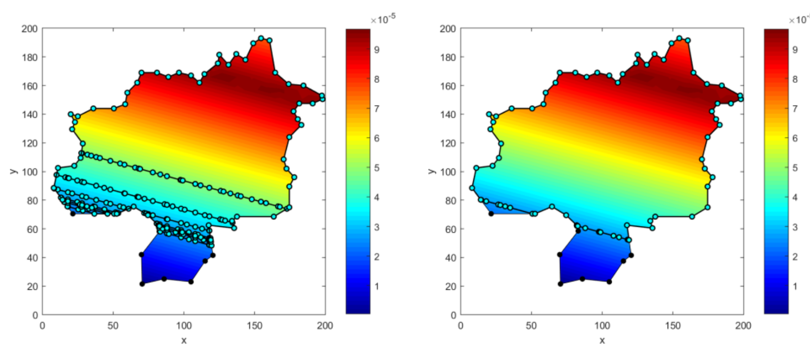
The simulation results for the significance level 0.025 *are given in Table 15.*



(a) **(b)**

FIGURE 25. Finding the confidence region for $\alpha = 0.01$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

According to Table 15, the confidence coefficient 0.975 *was reached in the* 7^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 26(a). Figure 26(b) demonstrates the optimum confidence region for* $\alpha = 0.025$*.*



(a) **(b)**

FIGURE 26. Finding the confidence region for $\alpha = 0.025$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

Table 16 shows the simulation results for the significance level 0.05*.*

According to Table 16, the confidence coefficient 0.95 *was reached in the* 6^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 27(a). Figure 27(b) demonstrates the optimum confidence region for* $\alpha = 0.05$*.*

TABLE 16. Confidence coefficients for the significance level $\alpha = 0.05$

Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.95 - P(\zeta_c) \mid$
1	0.000000	0.000097	0.000049	0.805538	0.144462
2	0.000000	0.000049	0.000024	0.980396	0.030396
3	0.000024	0.000049	0.000037	0.919918	0.030082
4	0.000024	0.000037	0.000031	0.958831	0.008831
5	0.000031	0.000037	0.000034	0.941625	0.008375
6	0.000031	0.000034	0.000032	0.950717	0.000717

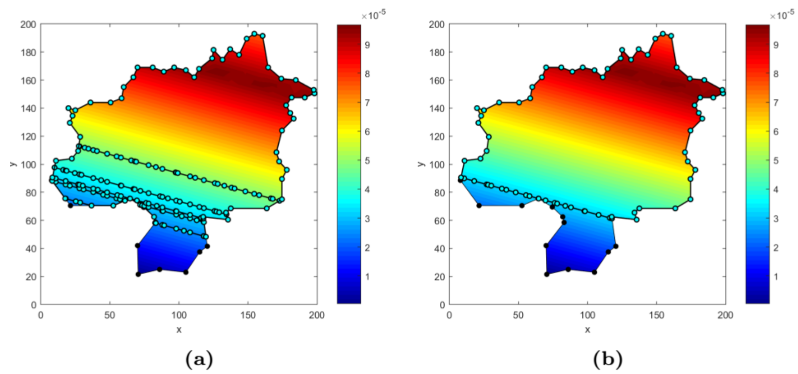


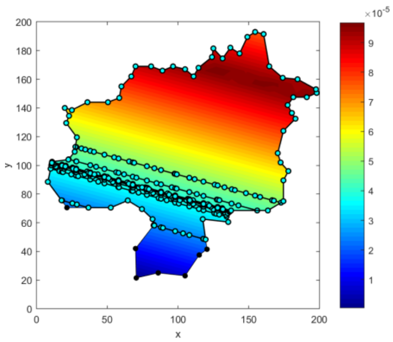
FIGURE 27. Finding the confidence region for $\alpha = 0.05$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

TABLE 17. Confidence coefficients for the significance level $\alpha = 0.10$

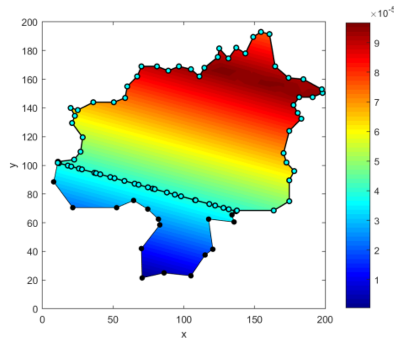
Iter. No.	ζ_a	ζ_b	ζ_c	$P(\zeta_c)$	$\mid 0.90 - P(\zeta_c) \mid$
1	0.000000	0.000097	0.000049	0.805538	0.094462
2	0.000000	0.000049	0.000024	0.980396	0.080396
3	0.000024	0.000049	0.000037	0.919918	0.019918
4	0.000037	0.000049	0.000043	0.869624	0.030376
5	0.000037	0.000043	0.000040	0.896214	0.003786
6	0.000037	0.000040	0.000038	0.908369	0.008369
7	0.000038	0.000040	0.000039	0.902369	0.002369
8	0.000039	0.000040	0.000039	0.899311	0.000689

Table 17 shows the simulation results for the significance level 0.10.

According to Table 17, the confidence coefficient 0.90 *was reached in the* 8^{th} *iteration. The confidence regions obtained from each iteration are given in Figure 28(a). Figure 28(b) demonstrates the optimum confidence region for* $\alpha = 0.10$.



(a)



(b)

FIGURE 28. Finding the confidence region for $\alpha = 0.10$, (a) The confidence regions obtained from all iterations; (b) The optimum confidence region.

5. CONCLUSIONS

This study proposes a practical method to in the attempt to modify the univariate probability density functions into the bivariate probability density functions. In the proposed method, the polygonal area is firstly separated into grids and then triangulation is applied. If the number of grids increases the performance of method can increase as well. According to the examples, the performance of the proposed method is considerably high. In addition, the proposed method provides a solution for infinite, semi-infinite and polygonal restricted areas. According to this method, the confidence region is found no matter how complex the distribution function. Confidence regions cannot be as precisely calculated in many applications. Therefore, the proposed method gives more efficient results for multimodal probability density functions.

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LIGHTLIKE HYPERSURFACES AND LIGHTLIKE FOCAL SETS WITH RESPECT TO BISHOP FRAME IN 4-DIMENSIONAL MINKOWSKI SPACE $\mathbb{E}_1^4$

ZEHRA OZDEMIR

Abstract. In this article, light-like hypersurfaces which are derived by null Cartan curves are examined and discussed. The singularities of lightlike hypersurfaces and light-like focal sets are investigated by using the Bishop frame on the Null Cartan curves. We obtain that the types of these singularities and the order of contact between the null Cartan curves are closely related to the Bishop curvatures of the null Cartan curves. Moreover, two examples of light-like hypersurfaces and light-like focal sets are given to illustrate our theoretical results.

1. Introduction

In 4-dimensional Minkowski space, due to the causal character there are three categories of vectors, namely, space-like, time-like and light-like (null) ones. Therefore, hypersurfaces of a Lorentzian manifold (M, g) can be of three types(see [16]): Space-like, time-like and light-like (null) hypersurfaces. Especially, the geometry of light-like hypersurfaces becomes more difficult and is completely different from that of the space-like and time-like hypersurfaces. In the case of light-like (degenerate, null) hypersurfaces, the situation is totally different. The normal bundle $TM^\perp$ is a rank-one distribution on $M : TM^\perp \subset TM$. It is also coincides with the radical distribution $RadTM = TM \cap TM^\perp$. Therefore, the induced metric g is degenerate on M and it has a constant rank n. Therefore, these hypersurfaces are usually used in modeling objects that are difficult to understand. In particular, light-like hypersurfaces are of interest to physicists because Kerr black holes, and various horizons can be modeled with these hypersurfaces [2, 4, 10, 11, 13, 17, 18, 20, 24]. Moreover, these hypersurfaces are used in the electromagnetism theory [19, 21]. Nersessian and Ramos have shown that there is a geometric particle-model based on the geometry of null curves in Minkowski 4-space [14]. Moreover, they studied

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the geometric particle model related to the null curves in Minkowski 3-space [15]. Duggal et.al. gave various fundamental works for the differential geometry theory of light-like submanifolds [5, 6, 7, 8].

On the other hand, the use of differential geometry in the singularity theory was first demonstrated by Thom in 1965. This study provides a connection between physics and geometry. Then, in 1998, Akivis et al. investigate the singular points of light-like hypersurfaces of the de Sitter space S^{n+1} [1]. The singularities of light-like surface and hypersurfaces have been studied in [22, 23].

In this study, the singularities of the hypersurfaces are defined by using the Bishop frame of the null Cartan curve in Minkowski 4−space. We will classify singular points of light-like hypersurfaces and light-like focal sets. We have also shown that the types of these singularities are closely related to the curvatures of the Bishop Cartan curvatures. Finally, we visualized light-like hypersurfaces and light-like focal set to demonstrate our theoretical results.

2. Preliminaries

The 4-dimensional Minkowski space-time is a real vector space $\mathbb{E}_1^4$ furnished with a symmetric non-degenerate (0,2) tensor field g with constant index. The metric tensor g on $\mathbb{E}_1^4$ with the signature (-,+,+,+) has the form

$$g(x, y) = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

for any two $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}_1^4$.

A non-zero vector x of $\mathbb{E}_1^4$ is called space-like if $g(x, x) > 0$, time-like if $g(x, x) < 0$ and null if $g(x, x) = 0$. Any two vectors $x, y \in \mathbb{E}_1^4$ are called orthogonal if $g(x, y) = 0$. Any two null vectors are called orthogonal on the condition that they are linearly dependent.

Let $\alpha : I \to \mathbb{E}_1^4;\ w \to \alpha(w)$ be a smooth curve in $\mathbb{E}_1^4$. Then the tangent vector of the curve denoted by

$$t = \frac{d\alpha}{dw}.$$

The curve α is said to be a null (isotropic or light-like) curve iff locally at each point it satisfies

$$g(\frac{d\alpha}{dw}, \frac{d\alpha}{dw}) = 0.$$

The null curve parameterized by the pseudo-arc parameter s denoted by

$$s(w) = \int_0^w g(\alpha''(w), \alpha''(w))dw.$$

is called as a null Cartan Curve. The Cartan frame $\{t, n, b_1, b_2\}$ along the non-geodesic null Cartan curve α satisfies the following Cartan frame equations

$$\begin{aligned} t' &= k_1 n \\ n' &= -k_2 t + k_1 b_1 \end{aligned}$$

$$\begin{aligned} b_1' &= -k_2 n + k_3 b_2 \\ b_2' &= k_3 t \end{aligned}$$

here $k_1(s)$, $k_2(s)$ and $k_3(s)$ are Cartan curvature functions and the first Cartan curvature $k_1(s) = 1$ in pseudo-arc parameter s. The null Cartan curve is called a null Cartan cubic on the condition that the second Cartan curvature satisfies $k_2(s) = 0$. Moreover, we have the following relations

$$\begin{aligned} g(t,t) &= g(b_1,b_1) = 0, g(n,n) = g(b_2,b_2) = 1, \\ g(t,n) &= g(t,b_2) = g(n,b_1) = g(b_1,b_2) = 0, g(t,b_1) = -1 \end{aligned}$$

[9].

Definition 1. *The Bishop frame $\{t_1, n_1, n_2, n_3\}$ of a null Cartan curve in $\mathbb{E}_1^4$ is positively oriented pseudo-orthonormal frame. It contains a tangential vector field t_1, two relatively parallel space-like normal vector fields n_1 and n_3, and a relatively parallel light-like transversal vector field n_2. These vectors have satisfy the following conditions*

$$\begin{aligned} g(t_1,t_1) &= g(n_2,n_2) = 0, \ g(n_1,n_1) = g(n_3,n_3) = 1, \\ g(t_1,n_1) &= g(n_1,n_2) = g(n_1,n_3) = g(n_2,n_3) = 0, g(t_1,n_2) = -1 \end{aligned}$$

[12].

Theorem 2. *Let α be a null Cartan curve in $\mathbb{E}_1^4$ with the Cartan curvatures $k_1(s) = 1$, $k_2(s)$ and $k_3(s) = 0$. Then the Bishop frame $\{t_1, n_1, n_2, n_3\}$ and the Cartan frame $\{t, n, b_1, b_2\}$ of α have the following relation*

$$\begin{bmatrix} t_1 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\sigma_2 & 1 & 0 & 0 \\ \frac{\sigma_2^2}{2} & -\sigma_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{2.1}$$

and the Bishop frame derivative formulas are given as

$$\begin{bmatrix} t_1' \\ n_1' \\ n_2' \\ n_3' \end{bmatrix} = \begin{bmatrix} \sigma_2 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & -\sigma_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

where the first Bishop curvature $\sigma_1(s) = 1$, the third Bishop curvature $\sigma_3(s) = 0$ and the second Bishop curvature satisfies the following differential equation

$$\sigma_2'(s) = -\frac{1}{2}\sigma_2^2(s) - k_2(s)$$

[12].

Particularly, the relation for cross products of the Bishop frame vector fields are

$$\begin{aligned} t_1 \times n_1 \times n_2 &= -n_3;\ t_1 \times n_1 \times n_3 = -t_1; \\ t_1 \times n_3 \times n_2 &= n_1;\ n_1 \times n_2 \times n_3 = -n_2 \end{aligned}$$

[12].

Theorem 3. *Let α be a null Cartan curve in $\mathbb{E}_1^4$ with the Cartan curvatures $k_1(s) = 1$, $k_2(s)$ and $k_3(s) \neq 0$. Then the Bishop frame vectors $\{t_1, n_1, n_2, n_3\}$ and the Cartan frame vectors $\{t, n, b_1, b_2\}$ of α are given by the following relation*

$$\begin{bmatrix} t_1 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\sigma_1\sigma_2 - \sigma_3\sqrt{\sigma_1'^2 + \sigma_3'^2} & \sigma_1 & 0 & \sigma_3 \\ \frac{\sigma_2^2 + \sigma_1'^2 + \sigma_3'^2}{2} & -\sigma_2 & 1 & -\sqrt{\sigma_1'^2 + \sigma_3'^2} \\ \sigma_2\sigma_3 - \sigma_1\sqrt{\sigma_1'^2 + \sigma_3'^2} & -\sigma_3 & 0 & \sigma_1 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{2.2}$$

where the Bishop frame derivative formulas are presented as

$$\begin{bmatrix} t_1' \\ n_1' \\ n_2' \\ n_3' \end{bmatrix} = \begin{bmatrix} \sigma_2 & \sigma_1 & 0 & -\sigma_3 \\ 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & -\sigma_2 & 0 \\ 0 & 0 & -\sigma_3 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

where the first Bishop curvature $\sigma_1(s) = \sin\phi(s)$, the second Bishop curvature satisfies the differential equation

$$\sigma_2(s) = \frac{k_3(s) - \phi''(s)}{\phi'(s)}, \phi'(s) \neq 0$$

the third Bishop curvature $\sigma_3(s) = \cos\phi(s)$ and the function $\phi(s)$ satisfies the differential equation

$$2\phi'(\phi''' - k_3') + 2\phi''(k_3 - \phi'') + \phi'^4 - (k_3 - \phi'')^2 - 2k_2\phi'^2 = 0$$

where $\phi(s) \notin \{\frac{\pi}{2} + k\pi, k\pi\}$, $k \in \mathbb{Z}$ [12].

Definition 4. *The map $\mathbb{DG}_C^{\pm}(u, \theta)$ is called as de Sitter Gauss image of $C = \alpha(I)$ with respect to Bishop frame in $\mathbb{E}_1^4$ and defined as*

$$\mathbb{DG}_C = U \times \mathbb{R} \to \mathbb{S}_1^3;\ \mathbb{DG}_C(s, \eta, \theta) = \eta t_1(s) + \cos\theta n_1(s) + \sin\theta n_3(s).$$

Definition 5. *The light-like hypersurfaces along C with respect to Bishop frame in $\mathbb{E}_1^4$ defined by*

$$\mathbb{LH}_C : U \times \mathbb{R} \to \mathbb{E}_1^4;\ \mathbb{LH}_\alpha(s, \eta, \theta) = \alpha(s) + \mathbb{DG}_C(s, \eta, \theta).$$

In the following section we derive the light-like hypersurfaces along C and investigate the singularities of the light-like hypersurfaces.

3. Lightlike Hypersurfaces and Singularities

Let $\alpha : I \to \mathbb{E}_1^4$ be a null Cartan curve with the Bishop frame apparatus $\{t_1, n_1, n_2, n_3\}$. Then the light-like distance squared function is defined as

$$d(p, \xi) = g(\xi - \alpha(s), \xi - \alpha(s)) - 1$$

here $p = \alpha(s)$ for any fixed $\xi_0 \in \mathbb{E}_1^4$, we have

$$\zeta(p) = \zeta\xi_0(p) = d(p, \xi_0).$$

If we take derivative of the last equation we get

$$\zeta'(p) = -2g(t_1(s), \xi_0 - \alpha(s)).$$

Then, we calculate the discriminant set of the light-like squared function d as follows

$$\mathbb{D}_d = \{\xi = \alpha(s) + \eta t_1(s) + \cos\theta n_1(s) + \sin\theta n_3(s) : \theta \in [0, 2\pi),\ s \in I,\ \eta \in \mathbb{R}\}.$$

It is called as image of the light-like hypersurface along C. The second derivative of the function $\zeta(p)$ calculated

$$\begin{aligned} \zeta''(p) &= -2g(T_1'(s), \xi_0 - \alpha(s))) \\ &= -2g((\sigma_2(s)t_1(s) + \sigma_1(s)n_1(s) - \sigma_3(s)n_3(s)), \cos\theta n_1(s) + \sin\theta n_3(s))) \\ &= -2\sigma_1(s)\cos\theta + 2\sigma_3(s)\sin\theta \end{aligned}$$

from the Bishop frame equation which give following two case

Case 1. $\sigma_3(s) = 0$,

$$\zeta(p) = \zeta'(p) = \zeta''(p) = 0$$

iff

$$\cos\theta = 0.$$

Case 2. $\sigma_3(s) \neq 0$,

$$\zeta(p) = \zeta'(p) = \zeta''(p) = 0$$

iff

$$\tan\theta = \frac{\sigma_1(s)}{\sigma_3(s)}.$$

Using the Bishop curvature equation we obtain

$$\theta = \phi(s) + k\pi,\ k \in \mathbb{Z}.$$

Therefore, singular points of the light-like hypersurfaces are points satisfy

(i) If $\cos\theta = 0$ then the singular point of the light-like hypersurface is $\xi_0 = \alpha(s_0) + \eta_0 t_1(s) \mp n_3(s)$ for $\eta_0 \in \mathbb{R}$. Then we obtain the light-like focal sets as

$$\mathbb{LFS}_C^{\pm} = \left\{\xi = \alpha(s) +\ \mathbb{DG}_C(s, \eta, \pi \pm \frac{\pi}{2}) :\ s \in I,\ \eta \in \mathbb{R}\right\}$$

(ii) If $\theta = \phi + k\pi$, $k \in \mathbb{Z}$ then the singular points of the light-like hypersurfaces are points $\xi_0 = \alpha(s_0) + \eta_0 t_1(s) + \cos\theta n_1(s) + \sin\theta n_3(s)$.

Then we obtain the light-like focal sets as

$$\{\mathbb{LFS}_C = \xi = \alpha(s) +\ \mathbb{DG}_C(s, \eta, \phi(s) + k\pi,) : k \in \mathbb{Z},\ s \in I,\ \eta \in \mathbb{R}\}$$

Using the above characterizations we obtain the following proposition.

Proposition 6. *Let α be a curve null Cartan curve with the Bishop frame $\{t_1, n_1, n_2, n_3\}$. Then we have following three condition*
1. $h(p) = \zeta'(p) = 0$ iff there exist $\theta_0 \in [0, 2\pi)$ and $\eta_0 \in \mathbb{R}$ such that

$$\xi_0 - p_0 = \eta t_1(s_0) + \cos\theta n_1(s_0) + \sin\theta n_3(s_0).$$

2. i. If $\sigma_3(s) = 0$, then we have,
$\zeta(p) = \zeta'(p) = \zeta''(p) = 0$ iff there exist $\theta_0 = \pi \pm \frac{\pi}{2}$ and $\eta_0 \in \mathbb{R}$ such that

$$\xi_0 - p_0 = \eta_0 t_1(s_0) \pm n_3(s_0).$$

ii. If $\sigma_3(s) \neq 0$, then we have,
$\zeta(p) = \zeta'(p) = \zeta''(p) = 0$ iff there exist $\theta_0 = \phi(s) + k\pi$, $k \in \mathbb{Z}$ and $\eta_0 \in \mathbb{R}$ such that

$$\xi_0 - p_0 = \eta t_1(s) + \cos(\phi(s) + k\pi) n_1(s) + \sin(\phi(s) + k\pi) n_3(s).$$

3.i. If $\sigma_3(s) = 0$, then we have,
$\zeta(p) = \zeta'(p) = \zeta''(p) = \zeta'''(p) = 0$ iff there exist $\theta_0 = \pi \pm \frac{\pi}{2}$ and $\eta_0 = 0$ such that

$$\xi_0 - p_0 = \pm n_3(s_0).$$

ii. If $\sigma_3(s) \neq 0$, then we have,
$\zeta(p) = \zeta'(p) = \zeta''(p) = \zeta'''(p) = 0$ iff there exist $\theta_0 = \phi_0 = \frac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$ and $\eta_0 = 0$ such that

$$\xi_0 - p_0 = \eta t_1(s) + \cos(\frac{\pi}{4} + 2k\pi) n_1(s) + \sin(\frac{\pi}{4} + 2k\pi) n_3(s).$$

4. If $\zeta(p) = \zeta'(p) = \zeta''(p) = \zeta'''(p)$ then we have $\zeta^{(4)}(p) \neq 0$.

Definition 7. *Let $\alpha : I \to \mathbb{E}_1^4$ be a null Bishop Cartan curve in $\mathbb{E}_1^4$. Then, the pseudo-sphere that have five-point contact with α is said to be the osculating pseudo-sphere of α* [3].

Corollary 8. *Let $\alpha : I \to \mathbb{E}_1^4$ be a null Bishop Cartan curve in $\mathbb{E}_1^4$. Then the curve α has not fife-point contact with osculating pseudo-sphere.*

Proposition 9. *If $\zeta_{\xi_0}(s_0)$ has A_k-singularity at s_0 ($k = 1, 2, 3, 4$) then it is a versal unfolding of $\zeta_{\xi_0}(s_0)$.*

Proof. Let we give

$$\alpha(s) = (x_1(s), x_2(s), x_3(s), x_4(s)),\ \xi(s) = (\xi_1, \xi_2, \xi_3, \xi_4)$$

in $\mathbb{E}_1^4$. Then we know that if $h(s)$ has A_1 singularity at s_0 then we have

$$\delta_1 = (-2(\xi_1 - x_1(s_0)), -2(\xi_2 - x_2(s_0)), -2(\xi_3 - x_3(s_0)), -2(\xi_4 - x_4(s_0)))$$

Since $\xi - \alpha(s) \in \mathbb{S}^3_{1,\xi}$we calculate that $rank\delta_1 = 1$
Let we assume that $h_{\xi_0}(s_0)$ has A_k-singularity at s_0 $(k = 2, 3, 4)$ then we have the following matrix form

$$\delta_2 = \begin{bmatrix} (\xi_1 - x_1(s_0) & \xi_2 - x_2(s_0) & \xi_3 - x_3(s_0) & \xi_4 - x_4(s_0) \\ x_1'(s_0) & x_2'(s_0) & x_3'(s_0) & x_4'(s_0) \\ x_1''(s_0) & x_2''(s_0) & x_3''(s_0) & x_4''(s_0) \\ x_1'''(s_0) & x_2'''(s_0) & x_3'''(s_0) & x_4'''(s_0) \end{bmatrix}$$

this give following two condition
i. If $\sigma(3) \neq 0$ then we have

$$\xi - \alpha(s) = \eta t_1(s) + \cos(\phi(s) + k\pi)n_1(s) + \sin(\phi(s) + 2\pi)n_3(s).$$

The determinant of the matrix δ_2 calculated as

$$\begin{aligned}
\det \delta_2 &= g((\xi - \alpha(s)) \times \alpha'(s) \times \alpha''(s), \alpha'''(s)) \\
&= g(\eta t_1(s) + \cos(\phi(s) + k\pi)n_1(s) \\
&\quad + \sin(\phi(s) + k\pi)n_3(s)) \times t_1(s) \times (\sigma_2(s)t_1(s) + \sigma_1(s)n_1(s) - \sigma_3(s)n_3(s)) \\
&\quad , (\sigma_2'(s) + \sigma_2^2(s))t_1(s) + (\sigma_1'(s) + \sigma_1(s)\sigma_2(s))n_1(s) \\
&\quad +(\sigma_1^2(s) + \sigma_3^2(s))n_2(s) - (\sigma_3'(s) + \sigma_3(s)\sigma_2(s))n_3(s)) \\
&= g(-\sigma_3 \cos(\phi(s) + k\pi)n_1(s) \times T_1(s) \times n_3(s) \\
&\quad +\sigma_1 \sin(\phi(s) + k\pi)n_3(s) \times t_1(s) \times n_1(s), \\
&\quad , (\sigma_2'(s) + \sigma_2^2(s))t_1(s) + (\sigma_1'(s) + \sigma_1(s)\sigma_2(s))n_1(s) \\
&\quad +(\sigma_1^2(s) + \sigma_3^2(s))n_2(s) - (\sigma_3'(s) + \sigma_3(s)\sigma_2(s))n_3(s)) \\
&= g(-\sigma_3 \cos(\phi(s) + k\pi)t_1(s) - \sigma_1 \sin(\phi(s) + k\pi)t_1(s) \\
&\quad , (\sigma_2'(s) + \sigma_2^2(s))t_1(s) + (\sigma_1'(s) + \sigma_1(s)\sigma_2(s))n_1(s) \\
&\quad +(\sigma_1^2(s) + \sigma_3^2(s))n_2(s) - (\sigma_3'(s) + \sigma_3(s)\sigma_2(s))n_3(s))
\end{aligned}$$

from the Bishop curvature equation we calculated as

$$\det \delta_2 = \cos(k\pi) = \pm 1.$$

ii. If $\sigma(3) = 0$ then we have

$$\xi - \alpha(s) = \eta t_1(s) \pm n_3(s).$$

The determinant of the matrix δ_2 calculated as

$$\begin{aligned}
\det \delta_2 &= g((\xi - \alpha(s)) \times \alpha'(s) \times \alpha''(s), \alpha'''(s)) \\
&= g((\eta t_1(s) \pm n_3(s)) \times t_1 \times (\sigma_2 t_1 + \sigma_1 N_1) \\
&\quad , (\sigma_2' + \sigma_2^2)t_1 + (\sigma_1' + \sigma_1\sigma_2)n_1 + \sigma_1^2 n_2)
\end{aligned}$$

from the Bishop curvature equation we calculated as

$$\det \delta_2 = \pm 1.$$

This gives $\det \delta_2 \neq 0$. These complete the proof. $\square$

Now we may give the following main theorem.

Let $\alpha : I \to \mathbb{E}_1^4$ be a null Cartan curve with the Cartan Bishop frame $\{t_1, n_1, n_2, n_3\}$ in $\mathbb{E}_1^4$. Then the lightlike hypersurfaces $\mathbb{LHC}(s, \mu, \theta) = \alpha(s) + \mu t(s) + \cos\theta\ n_1(s) + \sin\theta\ n_3(s)$, satisfy the following assertions:
1.The pseudosphere $\mathbb{S}^3_{1,\xi_0}$ and the null Bishop Cartan curve α have at least a two-point contact.
2. *i.* The pseudosphere $\mathbb{S}^3_{1,\xi_0}$ and the null Bishop Cartan curve α have a three-point contact iff there exist $\theta_0 = \pi \pm \frac{\pi}{2}$ and $\eta_0 \in \mathbb{R}$ such that

$$\xi_0 - p_0 = \eta_0 t_1(s_0) \pm n_3(s_0).$$

where the third Bishop curvature $\sigma_3(s) = 0$.
ii. the pseudosphere $\mathbb{S}^3_{1,\xi_0}$ and the null Bishop Cartan curve α have a three-point contact iff there exist $\theta_0 = \phi(s) + k\pi$, $k \in \mathbb{Z}$ and $\eta_0 \in \mathbb{R}$ such that

$$\xi_0 - p_0 = \eta t_1(s) + \cos(\phi(s) + k\pi) n_1(s) + \sin(\phi(s) + k\pi) n_3(s)$$

where the third Bishop curvature $\sigma_3(s) \neq 0$.
Remark 1. Under this condition the light-like focal set $\mathbb{LFS}^{\pm}_C$ is non-singular. In addition, the light-like hypersurfaces is locally diffeomorphic to $C(2,3) \times \mathbb{R}^2$ at ξ_0.
3. *i.* The pseudosphere $\mathbb{S}^3_{1,\xi_0}$ and the null Bishop Cartan curve α have a four-point contact on the condition that there exist $\theta_0 = \pi \pm \frac{\pi}{2}$ and $\eta_0 = 0$ such that

$$\xi_0 - p_0 = \pm n_3(s_0).$$

where the third Bishop curvature $\sigma_3(s) = 0$.
ii. The pseudosphere $\mathbb{S}^3_{1,\xi_0}$ and the null Bishop Cartan curve α have a four-point contact on the condition that there exist $\theta_0 = \phi_0 = \frac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$ and $\eta_0 = 0$ such that

$$\xi_0 - p_0 = \eta t_1(s) + \cos(\frac{\pi}{4} + 2k\pi) n_1(s) + \sin(\frac{\pi}{4} + 2k\pi) n_3(s)$$

where the third Bishop curvature $\sigma_3(s) \neq 0$.
Remark 2. Under this condition the critical value set of the $\mathbb{LFS}_C$ is a regular curve and the focal set $\mathbb{LFS}_C$ is locally diffeomorphic to $C(2,3,4) \times \mathbb{R}^2$. Also, the light-like hypersurfaces is locally diffeomorphic to $SW \times \mathbb{R}$ at ξ_0.
4. The null Bishop Cartan curve α and pseudosphere $\mathbb{S}^3_{1,\xi_0}$ not have five-point contact.

In this section we provide two examples in $\mathbb{E}_1^4$. The first example is given for the case of $\sigma_3(s) \neq 0$ and the second example is for the case of $\sigma_3(s) = 0$, to verify the given theory.

4. Applications

Example 10. *Let α be a following parameterized curve in $\mathbb{E}_1^4$*

$$\alpha(s) = (\sinh \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}s}{\sqrt{2}}, \cosh \frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \cos \frac{\sqrt{3}s}{\sqrt{2}})$$

[12]. *Then the parallel transport frame of the curve α calculated as follows:*

$$t_1 = (\frac{1}{\sqrt{2}} \cosh \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos \frac{\sqrt{3}s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sinh \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sin \frac{\sqrt{3}s}{\sqrt{2}})$$

$$n_1 = \begin{pmatrix} (-\frac{1}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}})\cosh\frac{s}{\sqrt{2}} + \frac{1}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}}\sinh\frac{s}{\sqrt{2}} \\ +\frac{3}{2\sqrt{3}}\cos\frac{\sqrt{3}s}{\sqrt{2}}\sinh\frac{s}{\sqrt{2}}, \\ (-\frac{1}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}})\cos\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}}\sin\frac{\sqrt{3}s}{\sqrt{2}} \\ +\frac{1}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}}\sin\frac{\sqrt{3}s}{\sqrt{2}}, \\ (-\frac{1}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}})\sinh\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\frac{\sqrt{3}s}{\sqrt{2}}\cosh\frac{s}{\sqrt{2}} \\ +\frac{3}{2\sqrt{3}}\cos\frac{\sqrt{3}s}{\sqrt{2}}\cosh\frac{s}{\sqrt{2}}, \\ (-\frac{1}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}})\sin\frac{\sqrt{3}s}{\sqrt{2}} + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}}\cos\frac{\sqrt{3}s}{\sqrt{2}} \\ -\frac{1}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}}\cos\frac{\sqrt{3}s}{\sqrt{2}} \end{pmatrix}$$

$$n_2 = (\sqrt{2}\cosh\frac{s}{\sqrt{2}} - \sqrt{2}\sinh\frac{s}{\sqrt{2}}, 0, \sqrt{2}\sinh\frac{s}{\sqrt{2}} - \sqrt{2}\cosh\frac{s}{\sqrt{2}})$$

$$n_3 = \begin{pmatrix} (\frac{1}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}})\cosh\frac{s}{\sqrt{2}} - \frac{1}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}}\sinh\frac{s}{\sqrt{2}} \\ +\frac{3}{2\sqrt{3}}\sin\frac{\sqrt{3}s}{\sqrt{2}}\sinh\frac{s}{\sqrt{2}}, \\ (\frac{1}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}})\cos\frac{\sqrt{3}s}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}}\sin\frac{\sqrt{3}s}{\sqrt{2}} \\ +\frac{1}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}}\sin\frac{\sqrt{3}s}{\sqrt{2}}, \\ (\frac{1}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}})\sinh\frac{s}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cos\frac{\sqrt{3}s}{\sqrt{2}}\cosh\frac{s}{\sqrt{2}} \\ +\frac{3}{2\sqrt{3}}\sin\frac{\sqrt{3}s}{\sqrt{2}}\cosh\frac{s}{\sqrt{2}}, \\ (\frac{1}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}})\sin\frac{\sqrt{3}s}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}s}{\sqrt{2}}\cos\frac{\sqrt{3}s}{\sqrt{2}} \\ -\frac{1}{2}\sin\frac{\sqrt{3}s}{\sqrt{2}}\cos\frac{\sqrt{3}s}{\sqrt{2}} \end{pmatrix}$$

We can give the light-like hypersurface as the form $\mathbb{LHC}(s,\mu,\theta) = \alpha(s) + \mu T_1(s) + \cos\theta\, n_1(s) + \sin\theta\, n_3(s)$. When $\mu = 0$, it is a ruled hypersurface derived by using the null parallel transport frame. We draw the projections of surface $\mathbb{LHC}(s,0,\theta)$ on 3-dimensional space. Also, we plotted the projections of the critical value focal set $\mathbb{LFC}(s,0,\frac{\pi}{4})$ on 3 dimensional space illustrated in Figure 2. Thus, we may provide the clue for the image of the light-like hypersurface $\mathbb{LHC}(s,0,\theta)$ with view of the null parallel transport frame via these projections.

Remark 2. The yellow-orange parts correspond to the Focal set $\mathbb{LFS}_C(s,\eta,\frac{\pi}{4})$, $s \in I$, $\eta \in \mathbb{R}$, the red parts correspond to the critical value set $\mathbb{LFC}(s,0,\frac{\pi}{4})$.

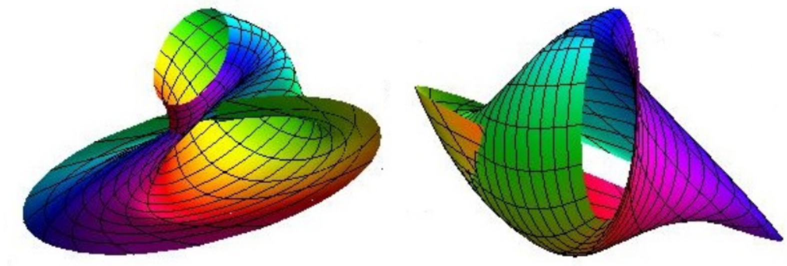

FIGURE 1. Projections of surface $\mathbb{LHC}(s,0,\theta)$ on 3-dimensional space.

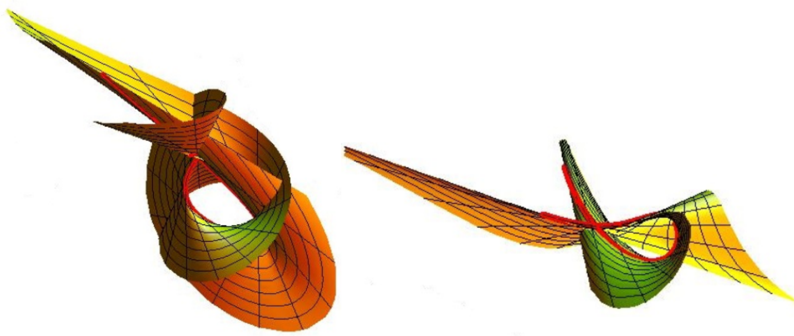

FIGURE 2. Projections of critical value set $\mathbb{LFC}(s,0,\frac{\pi}{4})$ on 3-dimensional space.

Example 11. *Let α be a following parameterized curve in $\mathbb{E}_1^4$*

$$\alpha(s)=\left(\begin{array}{c}\frac{1}{6}s^3+\frac{1}{2}s,\\ \frac{1}{6}s^3-\frac{1}{2}s,\\ \frac{-\frac{1}{5}s^2+\frac{4}{5}s^2\tan(\frac{1}{2}\ln s)+\frac{1}{5}\tan^2(\frac{1}{2}\ln s)}{1+\tan^2(\frac{1}{2}\ln s)},\\ \frac{\frac{2}{5}s^2+\frac{2}{5}s^2\tan(\frac{1}{2}\ln s)-\frac{2}{5}\tan^2(\frac{1}{2}\ln s)}{1+\tan^2(\frac{1}{2}\ln s)}\end{array}\right)$$

[23]. *Then the parallel transport frame of the curve α calculated as follows:*

$$t=\frac{\sqrt{2}}{2}(\frac{1}{2}s^2+\frac{1}{2},\frac{1}{2}s^2-\frac{1}{2},s\sin(\ln s),s\cos(\ln s))$$

$$n_1 = \begin{pmatrix} (\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})(\frac{1}{2}s^2+\frac{1}{2})+\frac{\sqrt{2}}{2}s, \\ (\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})(\frac{1}{2}s^2-\frac{1}{2})+\frac{\sqrt{2}}{2}s, \\ (\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})s\sin(\ln s)+\frac{\sqrt{2}}{2}(\cos(\ln s)+\sin(\ln s)), \\ (\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})s\cos(\ln s)+\frac{\sqrt{2}}{2}(\cos(\ln s)-\sin(\ln s) \end{pmatrix}$$

$$n_2 = \begin{pmatrix} (\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})^2(\frac{1}{4}s^2+\frac{1}{4})-(\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})\frac{\sqrt{2}}{2}s-\frac{5\sqrt{2}}{8}-\frac{\sqrt{2}}{8s^2}, \\ (\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})^2(\frac{1}{4}s^2-\frac{1}{4})-(\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})\frac{\sqrt{2}}{2}s-\frac{5\sqrt{2}}{8}+\frac{\sqrt{2}}{8s^2}, \\ (\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})^2\frac{s\sin(\ln s)}{2}-(\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})\frac{\sqrt{2}}{2}(\cos(\ln s)+\sin(\ln(s))) \\ -\frac{\sqrt{2}}{4s}(2\cos(\ln s)-\sin(\ln s)), \\ (\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})^2\frac{s\cos(\ln s)}{2}-(\frac{\sqrt{6}s^{\sqrt{3}-1}}{c+s^{\sqrt{3}}})\frac{\sqrt{2}}{2}(\cos(\ln s)-\sin(\ln(s)) \\ +\frac{\sqrt{2}}{4s}(2\sin(\ln s)+\cos(\ln s)) \end{pmatrix}$$

$$n_3 = \frac{\sqrt{2}}{4}(s-\frac{1}{s}, s+\frac{1}{s}, -2\cos(\ln s)), 2\sin(\ln s)$$

We can give the light-like hypersurface as the form $\mathbb{LHC}(s,\mu,\theta) = \alpha(s)+\mu T_1(s)+\cos\theta\ N_1(s)+\sin\theta\ N_3(s)$, *when* $\mu = 0$,*is a ruled hypersurface generated by using the null parallel transport frame. We draw the projections of surface* $\mathbb{LHC}(s,0,\theta)$ *on 3-dimensional space. Also, we illustrated the projections of the critical value Focal set* $\mathbb{LFC}^{\div}(s,0,\pi\pm\frac{\pi}{2})$ *on 3 dimensional space illustrated in Figure 3 and Figure 4. Thus, we can obtain the information for the image of the hypersurface* $\mathbb{LHC}(s,0,\theta)$ *with view of the null parallel transport frame via these projections.*

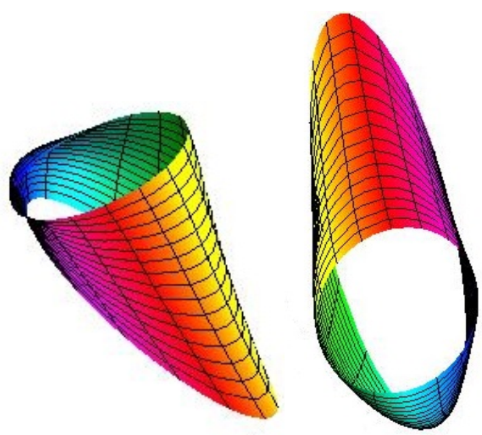

FIGURE 3. Projections of the hypersurface $\mathbb{LHC}(s,0,\theta)$ on 3-dimensional space.

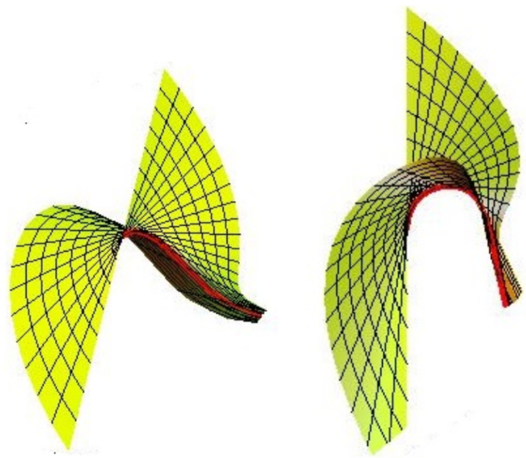

FIGURE 4. Projections of focal Set $\mathbb{LFS}_C^{+}(s,\eta,\pi+\frac{\pi}{2})$ on 3-dimensional space.

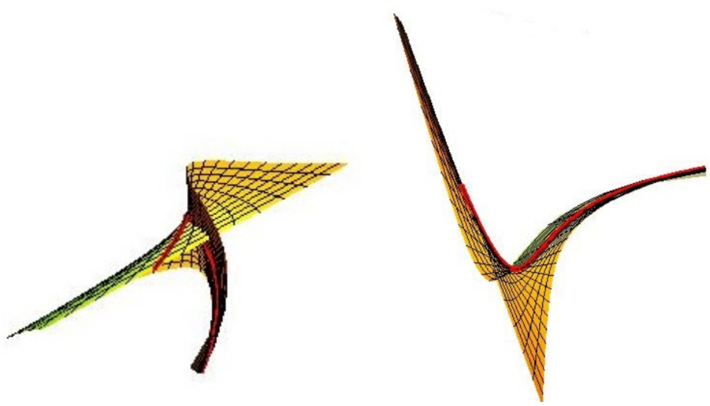

FIGURE 5. Projection of $\mathbb{LFS}_C^{-}(s,\eta,\pi-\frac{\pi}{2})$ on 3-dimensional space.

Remark 4. The yellow-orange parts correspond to the focal set $\mathbb{LFS}_C^{\pm}(s,\eta,\pi\pm\frac{\pi}{2})$, $s\in I$, $\eta\in\mathbb{R}$, the red parts correspond to the critical value focal set $\mathbb{LFS}_C^{\pm}(s,\eta,\pi\pm\frac{\pi}{2})$.

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PRO-$\mathcal{C}$ COMPLETIONS OF CROSSED SQUARES OF COMMUTATIVE ALGEBRAS

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Abstract. In this paper we give the explicit construction of a pro-$\mathcal{C}$ completion functor which is defined in the category of crossed squares of commutative algebras. Afterwards, we study some functorial properties of this pro-$\mathcal{C}$ completion process.

1. Introduction

A profinite group [9] occurs in a wide range of problems related to number theory, commutative algebra, algebraic geometry and algebraic topology. Although the category of profinite groups forms a natural extension of the category of finite groups, it carries a richer structure. Because it has some categorical objects and constructions which do not exist in finite case; e.g. projective limits and free products. The existence of such constructions in extended category leads to the definition of profinite analogues of the usual constructions of combinatorial group theory, such as free groups and presentations of group by generators and relations.

The theory of crossed modules [10] plays an important role in combinatorial and cohomological group theory. Profinite crossed modules are introduced in [7]. They examined the pro-$\mathcal{C}$ completion of crossed modules for a full class of finite groups $\mathcal{C}$. The crossed square version of this completion process is given in [3]. The definition of crossed modules over a commutative algebra is given in [8]; also see [5] for the general case. Crossed squares in the category of commutative algebras studied in [1, 4].

2. Preliminaries

In this paper, k will be a fixed commutative ring with $1 \neq 0$ for abstract cases and k will be a fixed commutative profinite ring with $1 \neq 0$ for topological cases.

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All k-algebras will be commutative and associative. $\mathcal{C}$ will denote a class of finite k-algebras which is closed under the formation of subalgebras, homomorphic images, finite products and which contains at least one non-trivial algebra. Pro-$\mathcal{C}$ algebras are profinite algebras whose finite quotients are in $\mathcal{C}$. The class $\mathcal{C}$ will be assumed to be full in the sense that $\mathcal{C}$ must also be closed under extension of algebras.

Throughout this paper we denote an action of P on M by $p\,m$, where P and M are k-algebras. We recall the definition of a crossed module, and its pro-$\mathcal{C}$ analogue.

A crossed module [8] is a k-algebra homomorphism $\partial\colon M \to P$ together with an action of P on M such that following two Peiffer relations hold (for all $m, m' \in M$ and $p \in P$):

$$\begin{array}{ll} CM1) & \partial(pm) = p\partial(m), \\ CM2) & \partial(m)m' = mm'. \end{array}$$

We denote such a crossed module by (M, P, ∂).

If (M, P, ∂) and (M', P', ∂') are two crossed modules, a crossed module morphism $(\phi, \psi)\colon (M, P, \partial) \to (M', P', \partial')$ is a tuple which consists of k-algebra homomorphisms, $\phi : M \to M'$, $\psi : P \to P'$ such that $\psi\partial = \partial'\phi$ and $\phi(pm) = \psi(p)\phi(m)$. Thus we get the category of crossed modules, denoted by **XMod** .

There are special classes of morphisms, those in which $P = P'$ and ψ is the identity morphism. For fixed P, a morphism $(\phi, id_P) : (M, P, \partial) \to (M', P, \partial')$ will be called a morphism of crossed modules over P. Then we have a subcategory $\mathbf{XMod}/P$ of **XMod**.

Let (M, P, ∂) be a crossed module. (M_1, P_1, ∂_1) is a subcrossed module of (M, P, ∂) if:

i) M_1 is a subalgebra of M and P_1 is a subring of P,
ii) $\partial_1 = \partial_{|M_1}$, the restriction of ∂ to M_1,
iii) The action of P_1 on M_1 is induced by the action of P on M.

A subcrossed module (M_1, P_1, ∂_1) of (M, P, ∂) is a crossed ideal if:

i) P_1 is an ideal of P and M_1 is an ideal of M,
ii) $pm_1 \in M_1$, for all $p \in P, m_1 \in M_1$,
iii) $p_1 m \in M_1$, for all $p_1 \in P_1, m \in M$.

A pro-$\mathcal{C}$ crossed module (M, P, ∂) is a crossed module in which M and P are pro-$\mathcal{C}$ k-algebras, ∂ is a continuous k-algebra homomorphism and the action of P on M is a continuous P-action [6].

A morphism of pro-$\mathcal{C}$ crossed modules

$$(\phi, \psi) : (M, P, \partial) \to (M', P', \partial')$$

is a morphism of the underlying crossed modules in which both ϕ and ψ are continuous morphisms of pro-$\mathcal{C}$ k-algebras. Thus we get the categories **Pro-$\mathcal{C}$.XMod** and similarly $\mathbf{Pro\text{-}}\mathcal{C}.\mathbf{XMod}/P$ for a fixed codomain P; we therefore obtain the forgetful functor:

$$\mathcal{U}_{\mathbf{XMod}} : \mathbf{Pro\text{-}}\mathcal{C}.\mathbf{XMod} \to \mathbf{XMod}.$$

Recall from [8] that a cat^1-algebra is a triple (E, s, t), where E is an k-algebra and s, t are endomorphisms of E satisfying the following conditions:

i) $st = t$ and $ts = s$
ii) $[Kers, Kert] = 0$.

It is well known that there is an equivalence of between the categories $\mathbf{XMod}$ and $\mathbf{Cat^1}(\mathbf{Alg})$.

A pro-$\mathcal{C}$ cat^1-algebra is a cat^1-algebra (E, s, t) in which E is a pro-$\mathcal{C}$ algebra and s and t are continuous endomorphisms of E. A morphism of pro-$\mathcal{C}$ cat^1-algebra is a morphism

$$\phi : (E, s, t) \to (E', s', t')$$

of the underlying cat^1-algebras such that ϕ: $E \to E'$ is a continuous morphism of pro-$\mathcal{C}$ algebras. Thus we get the category of pro-$\mathcal{C}$ cat^1-algebras, denoted by $\mathbf{Pro\text{-}}\mathcal{C}.\mathbf{Cat^1}(\mathbf{Alg})$. There is a forgetful functor:

$$\mathcal{U}_{\mathbf{CAlg}} : \mathbf{Pro\text{-}}\mathcal{C}.\mathbf{Cat^1}(\mathbf{Alg}) \to \mathbf{Cat^1}(\mathbf{Alg})$$

It is proven in [6] that, there exists an equivalence of categories $\mathbf{Pro\text{-}}\mathcal{C}.\mathbf{XMod}$ and $\mathbf{Pro\text{-}}\mathcal{C}.\mathbf{Cat^1}(\mathbf{Alg})$ compatible with the forgetful functors, in the sense of the equivalence between $\mathbf{XMod}$ and $\mathbf{Cat^1}(\mathbf{Alg})$.

3. Crossed Squares and their Pro-$\mathcal{C}$ Analogue

3.1. **Crossed squares.** The following definition is due to [1].

A crossed square of commutative algebras is a commutative diagram:

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

together with actions of P on L, M and N (there are thus actions of N on L and M via μ', and of M on L and N via μ) and a function $h : M \times N \to L$ such that:

1) The maps $\lambda, \lambda', \mu, \mu'$ and the composite $\mu\lambda = \mu'\lambda'$ are crossed modules,
2) The maps λ, λ' preserve the action of P,
3) $kh(m, n) = h(km, n) = h(m, kn)$,
4) $h(m + m', n) = h(m, n) + h(m', n)$,
5) $h(m, n + n') = h(m, n) + h(m, n')$,
6) $ph(m, n) = h(pm, n) = h(m, pn)$,
7) $\lambda h(m, n) = m\mu'(n)$,
8) $\lambda' h(m, n) = \mu(m)n$,

9) $h(\lambda(l),n) = l\mu'(n)$,
10) $h(m,\lambda'(l)) = \mu(m)l$

for all $n, n' \in N,\ m, m' \in M,\ p \in P,\ l \in L,\ k \in k$. We denote such a crossed square by (L, M, N, P).

Let μ and μ' are normal subalgebra inclusions and $L = M \cap N$, with h is given by the multiplication in P, i.e., $h(m,n) = mn$. Then, we have the crossed square:

$$\begin{tikzcd}
M \cap N \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\mu'"'] & P
\end{tikzcd}$$

A morphism of crossed square

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) : (L_1, M_1, N_1, P_1) \to (L_2, M_2, N_2, P_2),$$

consists of homomorphisms:

$$\begin{array}{ll} \Phi_1 : L_1 \to L_2, & \Phi_2 : M_1 \to M_2, \\ \Phi_3 : N_1 \to N_2, & \Phi_4 : P_1 \to P_2, \end{array}$$

such that the diagram commutes:

$$\begin{tikzcd}
 & \boldsymbol{L_1} \arrow[rr, "\lambda_1"] \arrow[d, "\lambda_1'"] \arrow[ddl, "\Phi_1"'] & & \boldsymbol{M_1} \arrow[d, "\mu_1"] \arrow[ddl, "\Phi_2"] \\
 & \boldsymbol{N_1} \arrow[rr, "\mu_1'"] \arrow[ddl, "\Phi_3"] & & \boldsymbol{P_1} \arrow[ddl, "\Phi_4"] \\
\boldsymbol{L_2} \arrow[rr, "\lambda_2"] \arrow[d, "\lambda_2'"'] & & \boldsymbol{M_2} \arrow[d, "\mu_2"'] & \\
\boldsymbol{N_2} \arrow[rr, "\mu_2'"] & & \boldsymbol{P_2} &
\end{tikzcd}$$

and

$$\Phi_1 h(m_1, n_1) = h(\Phi_2(m_1), \Phi_3(n_1)),$$

for all $m_1 \in M_1, n_1 \in N_1$, and the homomorphisms Φ_1, Φ_2, Φ_3 are Φ_4-equivariant.

Thus we get the category of crossed squares, denoted by $\mathbf{Crs^2}$.

There are special classes of morphisms, those in which $P_1 = P_2$ and Φ_4 is the identity morphism. For a fixed P, such a morphism

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, id) : (L_1, M_1, N_1, P) \to (L_2, M_2, N_2, P),$$

will be called a morphism of crossed squares over P, yields a subcategory $\mathbf{Crs^2}/P$.

A crossed square

$$\begin{tikzcd} L_1 \arrow[r, "\lambda_1"] \arrow[d, "\lambda_1'"'] & M_1 \arrow[d, "\mu_1"] \\ N_1 \arrow[r, "\mu_1'"'] & P_1 \end{tikzcd}$$

is a subcrossed square of

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

if,

i) L_1 is a subalgebra of L, M_1 is a subring of M, N_1 is a subring of N, P_1 is a subring of P,

ii) λ_1 is restriction of λ to L_1, μ_1 is restriction of μ to M_1, λ_1' is restriction of λ' to L_1, μ_1' is restriction of μ' to N_1,

iii) Actions of P_1 on L_1, M_1 and N_1 are induced by the actions of P on L, M and N, respectively.

iv) $h_1 : M_1 \times N_1 \to L_1$ is the restriction of $h : M \times N \to L$ to $M_1 \times N_1$.

A subcrossed square

$$\begin{tikzcd} L_1 \arrow[r, "\lambda_1"] \arrow[d, "\lambda_1'"'] & M_1 \arrow[d, "\mu_1"] \\ N_1 \arrow[r, "\mu_1'"'] & P_1 \end{tikzcd}$$

is an ideal of

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

if,

i) P_1 is an ideal of P, M_1 is an ideal of M, N_1 is an ideal of N,

ii) For all $p \in P$, $l_1 \in L_1$, $m_1 \in M_1$ and $n_1 \in N_1$,

$$\begin{aligned} &pl_1 \in L_1, \\ &pm_1 \in M_1, \\ &pn_1 \in N_1. \end{aligned}$$

iii) For all $p_1 \in P_1$, $l \in L$, $m \in M$ and $n \in N$,

$$\begin{aligned} &p_1 l \in L_1, \\ &p_1 m \in M_1, \\ &p_1 n \in N_1. \end{aligned}$$

Let

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

be a crossed square and

$$\begin{tikzcd} L_1 \arrow[r, "\lambda_1"] \arrow[d, "\lambda_1'"'] & M_1 \arrow[d, "\mu_1"'] \\ N_1 \arrow[r, "\mu_1'"'] & P_1 \end{tikzcd}$$

be an ideal of (L, M, N, P). Let $\overline{\lambda}$ is induced by λ, $\overline{\mu}$ is induced by μ, $\overline{\lambda'}$ is induced by λ', $\overline{\mu'}$ is induced by μ'. Then there are actions of P/P_1 on L/L_1, M/M_1 and N/N_1 given by

$$\begin{aligned} (p + P_1)(l + L_1) &= (pl) + L_1, \\ (p + P_1)(m + M_1) &= pm + M_1, \\ (p + P_1)(n + N_1) &= pn + N_1 \end{aligned}$$

and N/N_1 on L/L_1 and M/M_1 via $\overline{\mu'}$, i.e.,

$$\begin{aligned} (n + N_1)(l + L_1) &= \mu'(n)l + L_1, \\ (n + N_1)(m + M_1) &= \mu'(n)m + M_1 \end{aligned}$$

and then M/M_1 acts on L/L_1 and N/N_1 via $\overline{\mu}$, i.e.,

$$\begin{aligned} (m + M_1)(l + L_1) &= \mu(m)l + L_1, \\ (m + M_1)(n + N_1) &= \mu(m)n + N_1. \end{aligned}$$

for all $p \in P$, $l \in L$, $m \in M$ and $n \in N$. The conditions for (L_1, M_1, N_1, P_1) to be ideal in (L, M, N, P) ensure that the actions are well defined. Let $h_1 : M_1 \times N_1 \to L_1$ is defined by $h(m + M_1, n + N_1) = h(m, n) + L_1$. It is clear that:

$$\begin{tikzcd} L/L_1 \arrow[r, "\overline{\lambda}"] \arrow[d, "\overline{\lambda'}"'] & M/M_1 \arrow[d, "\overline{\mu}"] \\ N/N_1 \arrow[r, "\overline{\mu'}"'] & P/P_1 \end{tikzcd}$$

is a crossed square. It is called the quotient crossed square of (L, M, N, P) by (L_1, M_1, N_1, P_1) and denoted by $(L, M, N, P)/(L_1, M_1, N_1, P_1)$.

3.2. **Pro-$\mathcal{C}$ crossed squares.** A pro-$\mathcal{C}$ crossed square of algebras

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

is a crossed square in which L, M, N and P are pro-$\mathcal{C}$ algebras, $\lambda, \lambda', \mu, \mu'$ are continuous homomorphisms, all the actions are continuous and the h-map is continuous.

A morphism

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) : (L_1, M_1, N_1, P_1) \to (L_2, M_2, N_2, P_2)$$

of pro-$\mathcal{C}$ crossed squares is a morphism of the underlying crossed squares in which all the maps Φ_1, Φ_2, Φ_3 and Φ_4 are continuous morphisms of pro-$\mathcal{C}$ algebras. Thus we get the categories $\mathbf{Pro}\text{-}\mathcal{C}.\mathbf{Crs^2}$ and similarly $\mathbf{Pro}\text{-}\mathcal{C}.\mathbf{Crs^2}/P$ for a fixed codomain P; and we also get the forgetful functor:

$$\mathcal{U}_{\mathbf{Crs^2}} : \mathbf{Pro}\text{-}\mathcal{C}.\mathbf{Crs^2} \to \mathbf{Crs^2}.$$

Recall the corresponding situation for k-algebras; the forgetful functor

$$\mathcal{U}_{\mathbf{Alg}} : \mathbf{Pro}\text{-}\mathcal{C}.\ \mathbf{Alg} \to \mathbf{Alg}$$

has a left adjoint, known as the pro-$\mathcal{C}$ completion functor, which we will denote by a " $\widehat{\ }$ " .

This is defined as follows:

Let P be a k-algebra and let $\Omega(P)$ be the directed set of finite index ideals W of P with $P/W \in \mathcal{C}$, then

$$\widehat{P} = \varprojlim_{W \in \Omega(P)} P/W.$$

We will sometimes write $W_{\text{fin}} P$ as indicating that $W \in \Omega(P)$. This notation is useful in as it is more suggestive of the actual concept involved, but can also become somewhat cumbersome so we will use both notations.

We wish to see if the crossed square forgetful functor

$$\mathcal{U}_{\mathbf{Crs^2}} : \mathbf{Pro}\text{-}\mathcal{C}.\mathbf{Crs^2} \to \mathbf{Crs^2}$$

also has a left adjoint. The obvious approach using some idea of finite index ideals is technically messy so we use an equivalence formulation involving Loday's notion of cat^2-algebras.

4. Cat^2-algebras and their Pro-$\mathcal{C}$ Analogue

4.1. **Cat^2-algebras.** A cat^2-algebra [1] is a 5-tuble (E, s_1, t_1, s_2, t_2) where (E, s_i, t_i), $i = 1, 2$ are cat^1-algebras and

$$s_i s_j = s_j s_i,\ t_i t_j = t_j t_i,\ s_i t_j = t_j s_i$$

for $i, j = 1, 2$, $i \neq j$.

If (E, s_1, t_1, s_2, t_2) and $(E', s'_1, t'_1, s'_2, t'_2)$ are cat^2-algebras a cat^2-algebra morphism:

$$\phi : (E, s_1, t_1, s_2, t_2) \to (E', s'_1, t'_1, s'_2, t'_2)$$

is an algebra homomorphism $\phi : E \to E'$ such that:

$$\begin{aligned} s'_1\phi &= \phi s_1, \\ t'_1\phi &= \phi t_1, \\ s'_2\phi &= \phi s_2, \\ t'_2\phi &= \phi t_2. \end{aligned}$$

Thus we get the category of cat^2-algebras, denoted by $\mathbf{Cat^2}(\mathbf{Alg})$.

Proposition 1. *There is an equivalence of categories between the category of cat^2 -algebras and that of crossed squares.*

Proof. The cat^1-algebra (E, s_1, t_1) will give us a crossed module

$$\partial : C \to B$$

with $C = \mathrm{Ker}s$, $B = \mathrm{Im}s$ and $\partial = t \mid C$, but as the two cat^1 -algebra structures are independent, (E, s_2, t_2) restricts to give cat^1-algebra structures on C and B and makes ∂ a morphism of cat^1-algebras. Thus we get a morphism of crossed modules

$$\begin{tikzcd} \mathrm{Ker}s_1 \cap \mathrm{Ker}s_2 \arrow[r] \arrow[d] & \mathrm{Im}s_1 \cap \mathrm{Ker}s_2 \arrow[d] \\ \mathrm{Ker}s_1 \cap \mathrm{Im}s_2 \arrow[r] & \mathrm{Im}s_1 \cap \mathrm{Im}s_2 \end{tikzcd}$$

where each morphism is a crossed module for the natural action, i.e. multiplication in E. It remains to produce an h-map, but it is given by the multiplication within E since if $x \in \mathrm{Ker}s_2 \cap \mathrm{Im}s_1$ and $y \in \mathrm{Im}s_2 \cap \mathrm{Ker}s_1$ then $xy \in \mathrm{Ker}s_1 \cap \mathrm{Ker}s_2$. It is easy to check the crossed squares axioms.

Conversely, if

$$\begin{tikzcd} L \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & R \end{tikzcd}$$

is a crossed square, then we can think of it as a morphism of crossed modules

$$\begin{tikzcd} L \to N \arrow[d] \\ M \to R. \end{tikzcd}$$

Using the equivalence between crossed modules and cat^1-algebras this gives a morphism

$$\partial : (L \rtimes N, s, t) \longrightarrow (M \rtimes R, s', t')$$

of cat^1-algebras. There is an action of $M \rtimes R$ on $L \rtimes N$ given by

$$(m, r)(l, n) = (rl + \partial(m)l + h(m, n), rn + mn)$$

for all $(m, r) \in M \rtimes R$ and $(l, n) \in L \rtimes N$. Using this action, we thus form its associated cat^1-algebra with algebra $(L \rtimes N) \rtimes (M \rtimes R)$ and induced endomorphisms s_1, t_1, s_2, t_2. □

4.2. Pro-$\mathcal{C}$ cat^2-algebras.

Definition 2. *A pro-$\mathcal{C}$ cat^2-algebra is a cat^2 -algebra (E, s_1, t_1, s_2, t_2) in which E is a pro-$\mathcal{C}$ algebra and s_1, s_2, t_1 and t_2 are continuous endomorphisms of E.*

A morphism of pro-$\mathcal{C}$ cat^2-algebra

$$\phi : (E, s_1, t_1, s_2, t_2) \to (E', s'_1, t'_1, s'_2, t'_2)$$

is a morphism of the underlying cat^2-algebras such that ϕ is a continuous morphism of pro-$\mathcal{C}$ algebras. Thus we get the category of pro-$\mathcal{C}$ cat^2-algebras, denoted by $\mathbf{Pro}$ $\mathbf{\text{-}}\mathcal{C}.\mathbf{Cat^2}(\mathbf{Alg})$.

There is a forgetful functor:

$$\mathcal{U}_{\mathbf{C^2Alg}} : \mathbf{Pro}\text{–}\mathcal{C}.\mathbf{Cat^2}(\mathbf{Alg}) \to \mathbf{Cat^2}(\mathbf{Alg}).$$

Theorem 3. *There exists an equivalence of categories* $\mathbf{Pro}$-$\mathcal{C}.\mathbf{Crs^2}$ *and* $\mathbf{Pro}$-$\mathcal{C}.\mathbf{Cat^2}(\mathbf{Alg})$ *compatible with the forgetful functors, in the sense of the equivalence between* $\mathbf{Crs^2}$ *and* $\mathbf{Cat^2}(\mathbf{Alg})$*, i.e. the following diagram commutes:*

$$\begin{tikzcd} \mathbf{Pro}\text{-}\mathcal{C}.\mathbf{Crs^2} \arrow[r, "\cong"] \arrow[d, "\mathcal{U}_{\mathbf{Crs^2}}"'] & \mathbf{Pro}\text{-}\mathcal{C}.\mathbf{Cat^2}(\mathbf{Alg}) \arrow[d, "\mathcal{U}_{\mathbf{C^2Alg}}"] \\ \mathbf{Crs^2} \arrow[r, "\cong"'] & \mathbf{Cat^2}(\mathbf{Alg}) \end{tikzcd}$$

Proof. In fact, if

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

is a pro-$\mathcal{C}$ crossed square, then $E = (L \rtimes M) \rtimes (N \rtimes P)$ is a pro-$\mathcal{C}$ algebra and the endomorphisms s_1, s_2, t_1 and t_2, given before, are continuous, so resulting (E, s_1, t_1, s_2, t_2) is a pro-$\mathcal{C}$ cat^2-algebra. Similarly if (E, s_1, t_1, s_2, t_2) is a pro-$\mathcal{C}$ cat^2 -algebra then

$$\begin{tikzcd} Kers_1 \cap Kers_2 \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & Ims_1 \cap Kers_2 \arrow[d, "\mu"] \\ Kers_1 \cap Ims_2 \arrow[r, "\mu'"] & Ims_1 \cap Ims_2 \end{tikzcd}$$

is a pro-$\mathcal{C}$ crossed square. $\square$

This lemma will enable us to prove the existence of a left adjoint for

$$\mathcal{U}_{\mathbf{Crs^2}} : \textbf{Pro-}\mathcal{C}.\mathbf{Crs^2} \longrightarrow \mathbf{Crs^2}$$

by constructing one for

$$\mathcal{U}_{\mathcal{C}^2\mathbf{Alg}} : \textbf{Pro -}\mathcal{C}.\mathbf{Cat^2}(\mathbf{Alg}) \longrightarrow \mathbf{Cat^2}(\mathbf{Alg}).$$

This latter construction will need projective limit within **Pro-**$\mathcal{C}$.$\mathbf{Cat^2}(\mathbf{Alg})$ and so we will briefly look at their construction as it sheds more light on the pro-$\mathcal{C}$ completion functor that will result from their use.

5. The Completion Process

Given a projective system $F : I \to \mathbf{Cat^2}(\mathbf{Alg})$, one notes that F is a projective system of algebras together with four endomorphisms of projective systems, $s_1, s_2, t_1, t_2 : F \to F$ satisfying

$$s_i s_j = s_j s_i,\ t_i t_j = t_j t_i,\ s_i t_j = t_j s_i$$

for $i, j = 1, 2,\ i \neq j$ and $[Kers_1, Kert_1] = 0, [Kers_2, Kert_2] = 0$. We form $\varprojlim F$ by taking the limit of this underlying system of pro-$\mathcal{C}$ algebras together with the induced endomorphism $\varprojlim s$ and $\varprojlim t$. Writing the result as $(\overline{F}, \overline{s_1}, \overline{t_1}, \overline{s_2}, \overline{t_2})$, we only need to check the conditions $[Ker\overline{s}_1, Ker\overline{t_1}] = 0$ and $[Ker\overline{s}_2, Ker\overline{t_2}] = 0$. However $\overline{F}$ can be realized as a subalgebra of the product $\prod_{i \in I} F(i)$ and

$$\overline{s_1}((x_i)) = (s_1(i)x_i, \overline{t_1}(x_i)) = (t_1(i)x_i),$$

similarly for $\overline{s_2}$, $\overline{t_1}$ and $\overline{t_2}$. So the commutator subalgebras $[Ker\overline{s_1}, Ker\overline{t_1}]$ and $[Ker\overline{s_2}, Ker\overline{t_2}]$ are trivial for each i in I.

Proposition 4. *A pro-$\mathcal{C}$ completion functor from the category* $\mathbf{Cat^2}(\mathbf{Alg})$ *to the category* $\mathbf{Pro\text{-}}\mathcal{C}.\mathbf{Cat^2}(\mathbf{Alg})$ *exists, (i.e. the forgetful functor* $\mathcal{U}_{\mathcal{C}^2\mathbf{Alg}}$ *has a left adjoint).*

Proof. An exact sequence

$$0 \longrightarrow (E', s'_1, t'_1, s'_2, t'_2) \xrightarrow{u} (E, s_1, t_1, s_2, t_2) \xrightarrow{v} (E'', s''_1, t''_1, s''_2, t''_2) \longrightarrow 0$$

of cat^2-algebras is an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of the underlying algebras and continuous maps compatible with the source and target maps. In this situation, we say that the cat^2-algebra $(E'', s''_1, t''_1, s''_2, t''_2)$ is the quotient of (E, s_1, t_1, s_2, t_2) by the ideal $(E', s'_1, t'_1, s'_2, t'_2)$. The latter is of finite index in (E, s_1, t_1, s_2, t_2) if E' is finite.

Given any cat^2-algebra (E, s_1, t_1, s_2, t_2) the set of its ideals $(I, s'_1, t'_1, s'_2, t'_2)$ of finite index with $E/I \in \mathcal{C}$ is directed by the inclusion so we can form an inverse system of finite quotient of (E, s_1, t_1, s_2, t_2) and take its limit within the category of pro-$\mathcal{C}$ cat^2-algebras. (As usual one considers each finite cat ^{2}-algebra as a pro-$\mathcal{C}$ one having the discrete topology.)

Thus we define a pro-$\mathcal{C}$ completion functor:

$$\widetilde{\ }: \mathbf{Cat^2}(\mathbf{Alg}) \to \mathbf{Pro\text{-}}\mathcal{C}.\mathbf{Cat^2}(\mathbf{Alg}) \tag{1}$$

by

$$\widetilde{(E,s_1,t_1,s_2,t_2\ )} = \varprojlim\{\text{finite quotients of } (E, s_1, t_1, s_2, t_2) \text{ by } (I, s'_1, t'_1, s'_2, t'_2)\}.$$

Categorically this functor is left adjoint to the forgetful functor from $\mathbf{Pro\text{-}}\mathcal{C}.\mathbf{Cat^2}(\mathbf{Alg})$ to $\mathbf{Cat^2}(\mathbf{Alg})$. □

Proposition 5. *A pro-$\mathcal{C}$ completion functor from* $\mathbf{Crs^2}$ *to* $\mathbf{Pro}\text{-}\mathcal{C}.\mathbf{Crs^2}$ *exists (i.e. the forgetful functor* $\mathcal{U}_{\mathbf{Crs^2}}$ *has a left adjoint).*

Proof. In the diagram

$$\begin{tikzcd}
\mathbf{Pro\text{-}}\mathcal{C}.\mathbf{Crs^2} \arrow[r, "\cong"] \arrow[d, "\mathcal{U}_{\mathbf{Crs^2}}"'] & \mathbf{Pro\text{-}}\mathcal{C}.\mathbf{Cat^2}(\mathbf{Alg}) \arrow[d, "\mathcal{U}_{\mathbf{C^2Alg}}"] \\
\mathbf{Crs^2} \arrow[r, "\cong"'] & \mathbf{Cat^2}(\mathbf{Alg})
\end{tikzcd}$$

we already found that (1) is the left adjoint functor to right vertical functor. This induces a left adjoint functor to left vertical functor via the equivalence of categories. □

Remark 6. *One can attempt to use the functors defining the two equivalence to give an "explicit" description of this pro-$\mathcal{C}$ completion functor, but in what follows we shall merely use its existence and the universal property that it satisfies to compare it with the individual algebras involved.*

Notation 7. *We denote the pro-$\mathcal{C}$ completion of the crossed square ($\widetilde{(L,M,N,P)}$ or less accurately, $(\widetilde{L}, \widetilde{M}, \widetilde{N}, \widetilde{P})$),as follows:*

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd} \quad by \quad \begin{tikzcd} \widetilde{L} \arrow[r, "\widetilde{\lambda}"] \arrow[d, "\widetilde{\lambda'}"'] & \widetilde{M} \arrow[d, "\widetilde{\mu}"] \\ \widetilde{N} \arrow[r, "\widetilde{\mu'}"'] & \widetilde{P} \end{tikzcd}$$

It is natural to compare this pro-$\mathcal{C}$ completion $(\widetilde{L}, \widetilde{M}, \widetilde{N}, \widetilde{P})$ with the pro-$\mathcal{C}$ completions $\widehat{L}, \widehat{M}, \widehat{N}, \widehat{P}$ and $\widehat{\lambda}, \widehat{\mu}, \widehat{\lambda'}, \widehat{\mu'}$ of the individual pieces of data involved. One may even wonder why

$$\begin{tikzcd} \widehat{L} \arrow[r, "\widehat{\lambda}"] \arrow[d, "\widehat{\lambda'}"'] & \widehat{M} \arrow[d, "\widehat{\mu}"] \\ \widehat{N} \arrow[r, "\widehat{\mu'}"'] & \widehat{P} \end{tikzcd}$$

is not itself always the same as,

$$\begin{tikzcd} \widetilde{L} \arrow[r, "\widetilde{\lambda}"] \arrow[d, "\widetilde{\lambda'}"'] & \widetilde{M} \arrow[d, "\widetilde{\mu}"] \\ \widetilde{N} \arrow[r, "\widetilde{\mu'}"'] & \widetilde{P} \end{tikzcd}$$

To start the study of this problem we first look at $\widetilde{P}$.

Proposition 8. *For any crossed square (L, M, N, P), we have $\widetilde{P} \cong \widehat{P}$.*

Proof. This follows from an adjoint functor argument: There is a forgetful functor

$$R : \mathbf{Crs^2} \to \mathbf{Alg}$$

given by $R(L, M, N, P) = P$ also an analogous one

$$R_{p\mathcal{C}} : \mathbf{Pro\text{-}\mathcal{C}.Crs^2} \to \mathbf{Pro\text{-}\mathcal{C}.Alg}.$$

These have left adjoints L and $L_{p\mathcal{C}}$ defined by $L(P) = (P, P, P, P)$

$$\begin{tikzcd} P \arrow[r, "id"] \arrow[d, "id"'] & P \arrow[d, "id"] \\ P \arrow[r, "id"'] & P \end{tikzcd}$$

with the h-map given by $h(p,p') = pp'$ for all $p,p' \in P$ and similarly for $L_{p\mathcal{C}}$.

Then, we get the diagram:

$$\begin{tikzcd} \textbf{Pro-}\mathcal{C}.\mathbf{Crs^2} \arrow[r, shift left, "R_p"] \arrow[d, shift left, "\mathcal{U}_{\mathbf{Crs^2}}"] & \textbf{Pro-}\mathcal{C}.\mathbf{Cat^2(Alg)} \arrow[l, shift left, "L_p"] \arrow[d, shift left, "\mathcal{U}_{\mathbf{C^2Alg}}"] \\ \mathbf{Crs^2} \arrow[r, shift left, "R"] \arrow[u, shift left, "(\widetilde{\ })"] & \mathbf{Cat^2(Alg)} \arrow[l, shift left, "L"] \arrow[u, shift left, "(\widetilde{\ })"] \end{tikzcd}$$

The right adjoint diagram commutes, so the left adjoint diagram commutes up to isomorphism, i.e.

$$(\widetilde{P,P,P,P}) \simeq (\widehat{P}, \widehat{P}, \widehat{P}, \widehat{P}),$$

but better we have a sequence of isomorphisms: for a pro-$\mathcal{C}$ algebra H,

$$\begin{aligned} &\textbf{Pro-}\mathcal{C}.\mathbf{Alg}(R_{p\mathcal{C}}(\widetilde{L,M,N,P}), H) \\ &\quad\cong \textbf{Pro-}\mathcal{C}.\mathbf{Crs^2}((\widetilde{L,M,N,P}), L_{p\mathcal{C}}(H)) \\ &\quad\cong \mathbf{Crs^2}((L,M,N,P), U_{\mathbf{Crs^2}} L_{p\mathcal{C}}(H)) \\ &\quad\cong \mathbf{Crs^2}((L,M,N,P), LU_{\mathcal{C}^2\mathbf{Alg}}(H)) \text{ by observation} \\ &\quad\cong \mathbf{Alg}(R(L,M,N,P), U_{\mathbf{Alg}}(H)) \\ &\quad\cong \mathbf{Alg}(P, U_{\mathbf{Alg}}(H)) \\ &\quad\cong \textbf{Pro-}\mathcal{C}.\mathbf{Alg}(\widehat{P}, H) \end{aligned}$$

as required; hence $\widehat{P} \cong \widetilde{P}$, independent from L, M, N are. □

In order to study the conditions that yields the isomorphism between $\widetilde{L}, \widetilde{M}, \widetilde{N}$ and $\widehat{L}, \widehat{M}, \widehat{N}$, respectively; we introduce a condition called "cofinality".

Let (L, P, ∂) be a crossed module and write $\Omega_P(L)$ for the directed subset of $\Omega(L)$ the set of finite index ideals of L, consisting of those $L_1 \in \Omega(L)$, $L/L_1 \in \mathcal{C}$, which are P-invariant. We will say that (L, M, N, P) satisfies the cofinal condition if $\Omega_P(L)$ is cofinal in $\Omega(L)$. It was shown in [6] that if (L, P, ∂) satisfies the cofinality condition, then $\widetilde{L} \cong \widehat{L}$.

Let

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"] & P \end{tikzcd}$$

be a crossed square and write $\Omega_P(L)$ for the directed subset of $\Omega(L)$ the set of finite index ideals of L, given above.

We say that (L, M, N, P) satisfies the cofinal condition if $\Omega_P(L)$ is cofinal in $\Omega(L)$, $\Omega_P(M)$ is cofinal in $\Omega(M)$ and $\Omega_P(N)$ is cofinal in $\Omega(N)$. Note that $\Omega_P(L) \subseteq \Omega_M(L)$ and $\Omega_P(L) \subseteq \Omega_N(L)$ so if $\Omega_P(L)$ is cofinal in $\Omega(L)$, then $\Omega_M(L)$ and $\Omega_N(L)$ are cofinal in $\Omega(L)$.

Proposition 9. *If $P \in \mathcal{C}$, then any crossed square*

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

satisfies the cofinality condition.

Proof. Given any $L_1 \in \Omega(L)$, let

$$L_1' = \bigcap_{p \in P} pL_1$$

Then, L_1' is P-invariant and as P in $\mathcal{C}$, L_1' is of finite index $L/L_1' \in \mathcal{C}$. As L_1' is contained in L_1, so $\Omega_P(L)$ is cofinal in $\Omega(L)$. Similarly, we can show that $\Omega_P(M)$ and $\Omega_P(N)$ are cofinal in $\Omega(L)$. This completes the proof. □

Remark 10. *Let C be k-algebra, let $\mathcal{M}(C)$ be the commutative k-algebra of multipliers of C. Recall that a multiplier of C is a linear mapping $\lambda : C \to C$ such that for all $c, c' \in C$; $\lambda(cc') = \lambda(c)c'$, see* [2] *for more details.*

Theorem 11. *If (L, M, N, P)*

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

satisfies the cofinality condition, then $\widetilde{L} \cong \widehat{L}, \widetilde{M} \cong \widehat{M}, \widetilde{N} \cong \widehat{N}$.

Proof. Since $\Omega_P(M)$ is cofinal in $\Omega(M)$ and $\Omega_P(N)$ is cofinal in $\Omega(N)$, $\widehat{M} \cong \widetilde{M}$ and $\widehat{N} \cong \widetilde{N}$. On the other hand since $\Omega_P(L)$ is cofinal in $\Omega(L)$, $\widehat{L} \cong \widetilde{L}$.

To check the axioms we need an explicit description of $\widehat{\lambda} : \widehat{L} \to \widehat{M}$, $\widehat{\mu} : \widehat{M} \to \widehat{P}$, $\widehat{\lambda'} : \widehat{L} \to \widehat{N}$, $\widehat{\mu'} : \widehat{N} \to \widehat{P}$, $\widehat{\mu\lambda} = \widehat{\mu'\lambda'} : \widehat{M} \to \widehat{P}$ and the h-map $\widehat{h} : \widehat{M} \times \widehat{N} \to \widehat{L}$. Given $U_{\text{fin}} P$, there is a composed homomorphism $M \to P \to P/U$.

Take K_U to be its kernel then since μ is P-equivariant and P/U is finite, it follows that K_U is in $\Omega_P(M)$ and that $U \subseteq \mathcal{M}_P(M/K_U)$. Similarly, there is a composed homomorphism $L \to M \to M/K_U$. Take T_U to be its kernel then since λ is M -equivariant and P/U is finite, it follows that T_U is in $\Omega_M(L)$ and that $K_U \subseteq \mathcal{M}_P(L/T_U)$. On the other hand there is also composed homomorphism $L \to P \to P/U$.

Take H_U to be its kernel then since $\mu\lambda$ is P-equivariant and P/U is finite, it follows that H_U is in $\Omega_P(L)$ and that $U \subseteq \mathcal{M}_P(L/H_U)$. Similarly there are composed homomorphisms $N \to P \to P/U$, $L \to N \to N/K'_U$ and $L \to P \to P/U$ and kernels K'_U, T'_U, H'_U of these morphisms respectively. It is easy to show that $H_U = T_U = T'_U = H'_U$. These observations readily imply that $\widehat{\lambda}$, $\widehat{\mu}$, $\widehat{\lambda'}$, $\widehat{\mu'}$, $\widehat{\mu\lambda} = \widehat{\mu'\lambda'}$ and the h-map $\widehat{h}$, defined by

$$\widehat{\lambda}_U(lT_U) = \lambda l_U K_U, \qquad \widehat{\mu}_U(mK_U) = \mu m_U U,$$

$$\widehat{\lambda'}_U(lT_U) = \lambda' l_U K'_U, \qquad \widehat{\mu'}_U(nK'_U) = \mu n_U U,$$

$$\widehat{\mu\lambda}_U(lH_U) = (\mu\lambda) l_U U, \qquad \widehat{\mu'\lambda'}_U(lH_U) = (\mu'\lambda') l_U U,$$

$$\widehat{h}_U(mK_U, nK'_U) = h(m,n)_U H_U.$$

It is clear that $\widehat{\mu\lambda}_U = \widehat{\mu}_U \widehat{\lambda}_U = \widehat{\mu'}_U \widehat{\lambda'}_U = \widehat{\mu'\lambda'}_U$. Rest of the proof follows from the crossed square axioms of (L, M, N, P) and the descriptions of $\widehat{\lambda}$, $\widehat{\mu}$, $\widehat{\lambda'}$, $\widehat{\mu'}$, $\widehat{\mu\lambda} = \widehat{\mu'\lambda'}$, $\widehat{h}$ and the $\widehat{P}$-action. □

Corollary 12. *If P is in $\mathcal{C}$, and*

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

is a crossed square then

$$\begin{tikzcd} \widehat{L} \arrow[r, "\widehat{\lambda}"] \arrow[d, "\widehat{\lambda'}"'] & \widehat{M} \arrow[d, "\widehat{\mu}"] \\ \widehat{N} \arrow[r, "\widehat{\mu'}"'] & \widehat{P} \end{tikzcd}$$

is a crossed square, which is the pro-$\mathcal{C}$ completion of (L, M, N, P).

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LIGHTLIKE HYPERSURFACES WITH PLANAR NORMAL SECTIONS IN $\mathbb{R}^4_1$

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Abstract. In the present paper our aim is to investigate lightlike hypersurfaces of $\mathbb{R}^4_1$ having degenerate or non-degenerate planar normal sections. Firstly, we prove that lightlike hypersurfaces in $\mathbb{R}^4_1$ always have degenerate planar normal sections. Then we examine the conditions for lightlike hypersurfaces in $\mathbb{R}^4_1$ to have non-degenerate planar normal sections and obtain some characterizations for such lightlike hypersurfaces.

1. Introduction

In Euclidean spaces, B.Y. Chen [2] initiated the study of surfaces with planar normal sections. After this, an important literature has been created on such surfaces and submanifolds (for example, see [2], [6], [7], [9],[8]). The semi-Riemannian adaptation of such surfaces was done by Y. H. Kim [7]. Recently, the authors ([12], [11]) introduced lightlike surfaces with planar normal sections in Minkowski 3-space and halflightlike submanifolds of $\mathbb{R}^4_2$ having degenerate and non-degenerate planar normal sections (see also [13]).

By a similar manner in [12] and [11] we define the normal section of a lightlike hypersurface $\acute{N}$ in $\mathbb{R}^4_1$ and non-degenerate planar normal sections as follows:

For a point p in a lightlike hypersurface $\acute{N}$ of $\mathbb{R}^4_1$ and a lightlike vector ξ such that the radical space $Rad(T\acute{N}) = Span\{\xi\}$, the vector ξ and transversal space $tr(T\acute{N})$ to $\acute{N}$ at p determine a 2-dimensional subspace $E(p, \xi)$ in $\mathbb{R}^4_1$ through p. The intersection $\acute{N} \cap E(p, \xi)$ gives rise to a lightlike curve α in a neighborhood of p, which we call normal section of $\acute{N}$ at the point p in the direction of ξ. If each normal section α at p in the direction of ξ satisfies $\alpha' \wedge \alpha'' \wedge \alpha''' = 0$, for each $p \in \acute{N}$, then we say that $\acute{N}$ has degenerate pointwise planar normal sections.

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On the other hand, let w be a non-degenerate vector tangent to $\acute{N}$ at p such that $w \in S(T\acute{N}) = Sp\{u, v\}$, where $S(T\acute{N})$ is the screen distribution of $\acute{N}$. Then the vector w and transversal space $tr(T\acute{N})$ to $\acute{N}$ at p determine a 2- dimensional subspace $E(p, w)$ in $\mathbb{R}^4_1$ through p. From the intersection of $\acute{N}$ and $E(p, w)$, we have a non-degenerate curve α in a neighborhood of p which is called the normal section of $\acute{N}$ at p in the direction of w. In this case, if $\alpha' \wedge \alpha'' \wedge \alpha''' = 0$ is satisfied, for each point p in $\acute{N}$, where α is a normal section of $\acute{N}$ at p in the direction of w, then $\acute{N}$ is said to have non-degenerate pointwise planar normal sections.

In this paper, we study lightlike hypersurfaces in $\mathbb{R}^4_1$ having degenerate and nondegenerate planar normal sections. We prove that every lightlike hypersurfaces of $\mathbb{R}^4_1$ has degenerate planar normal sections. Also we obtain some results for a lightlike hypersurface with non-degenerate planar normal sections. We prove that a lightlike hypersurface $\acute{N}$ in $\mathbb{R}^4_1$ has non-degenerate planar normal sections if and only if it is either screen conformal and totally umbilical or totally geodesic. We also obtain a characterization for non-umbilical screen conformal lightlike hypersurface with non-degenerate planar normal sections.

2. Preliminaries

Let $(\breve{N}, \breve{g})$ be an $(n+2)$-dimensional semi-Riemannian manifold with the indefinite metric $\breve{g}$ of index $q \in \{1, ..., n+1\}$ and $\acute{N}$ be a hypersurface of $\breve{N}$. We denote the tangent space at $x \in \acute{N}$ by $T_x\acute{N}$. Then

$$T_x\acute{N}^{\perp} = \{V_x \in T_x\breve{N} \mid \breve{g}_x(V_x, W_x) = 0, \forall W_x \in T_x\acute{N}\}$$

and

$$Rad\, T_x\acute{N} = T_x\acute{N} \cap T_x\acute{N}^{\perp}.$$

Then, $\acute{N}$ is called a lightlike hypersurface of $\breve{N}$ if $Rad\, T_x\acute{N} \neq \{0\}$, for any $x \in \acute{N}$. Thus $T\acute{N}^{\perp} = \bigcap_{x \in \acute{N}} T_x\acute{N}^{\perp}$ becomes a 1- dimensional distribution $Rad\, T\acute{N}$ on $\acute{N}$. Then there exists a vector field $\xi \neq 0$ on $\acute{N}$ such that

$$g(\xi, X) = 0, \quad \forall X \in \Gamma(T\acute{N}),$$

where g is the induced degenerate metric tensor on $\acute{N}$. We denote the algebra of differential functions on $\acute{N}$ by $F(\acute{N})$ and the $F(\acute{N})$-module of differentiable sections of a vector bundle E over $\acute{N}$ by $\Gamma(E)$.

A complementary vector bundle $S(T\acute{N})$ of $T\acute{N}^{\perp} = Rad\, T\acute{N}$ in $T\acute{N}$ defined by

$$T\acute{N} = Rad\, T\acute{N} \oplus_{orth} S(T\acute{N}), \tag{1}$$

is called a screen distribution on $\acute{N}$. It follows from the equation above that $S(T\acute{N})$ is a non-degenerate distribution. Moreover, since we assume that $\acute{N}$ is para-compact, there always exists a screen $S(T\acute{N})$. Thus, along $\acute{N}$ we have

$$T\breve{N}_{|\acute{N}} = S(T\acute{N}) \oplus_{orth} S(T\acute{N})^{\perp}, \quad S(T\acute{N}) \cap S(T\acute{N})^{\perp} \neq \{0\}, \tag{2}$$

that is, $S(T\acute{N})^{\perp}$ is the orthogonal complement to $S(T\acute{N})$ in $T\breve{N}\mid_{\acute{N}}$. Note that $S(T\acute{N})^{\perp}$ is also a non-degenerate vector bundle of rank 2. However, it includes $T\acute{N}^{\perp} = Rad\,T\acute{N}$ as its sub-bundle.

Let $(\acute{N}, g, S(T\acute{N}))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\breve{N}, \breve{g})$. Then there exists a unique vector bundle $tr(T\acute{N})$ of rank 1 over $\acute{N}$, such that for any non-zero section ξ of $T\acute{N}^{\perp}$ on a coordinate neighborhood $U \subset \acute{N}$, there exists a unique section N of $tr(T\acute{N})$ on U satisfying: $T\acute{N}^{\perp}$ in $S(T\acute{N})^{\perp}$ and take $V \in \Gamma(F\mid_U), V \neq 0$. Then $\breve{g}(\xi, V) \neq 0$ on U, otherwise $S(T\acute{N})^{\perp}$ would be degenerate at a point of U [5]. Define a vector field

$$N = \frac{1}{\breve{g}(V,\xi)}\left\{V - \frac{\breve{g}(V,V)}{2\breve{g}(V,\xi)}\xi\right\},$$

on U where $V \in \Gamma(F\mid_U)$ such that $\breve{g}(\xi, V) \neq 0$. Then we have

$$\breve{g}(N,\xi) = 1,\ \breve{g}(N,N) = 0,\ \breve{g}(N,W) = 0,\ \forall W \in \Gamma(S(T\acute{N})|_U). \tag{3}$$

Moreover, from (1) and (2) we have the following decomposition:

$$T\breve{N}\mid_{\acute{N}} = S(T\acute{N}) \oplus_{orth} (T\acute{N}^{\perp} \oplus tr(T\acute{N})) = T\acute{N} \oplus tr(T\acute{N}). \tag{4}$$

Locally, suppose $\{\xi, N\}$ is a pair of sections on $U \subset \acute{N}$ satisfying (3). Define a symmetric $F(U)$-bi-linear form B and a 1-form τ on U. Hence on U, for $X, Y \in \Gamma(T\acute{N}\mid_U)$, we write

$$\begin{aligned} \breve{\nabla}_X Y &= \breve{\nabla}_X Y + B(X,Y)N, &(5)\\ \breve{\nabla}_X N &= -A_N X + \tau(X)N, &(6)\end{aligned}$$

which are called local Gauss and Weingarten formula, respectively. Since $\breve{\nabla}$ is a metric connection on $\breve{N}$, it is easy to see that

$$B(X,\xi) = 0, \forall X \in \Gamma(T\acute{N}\mid_U). \tag{7}$$

Consequently, the second fundamental form of $\acute{N}$ is degenerate [5]. Define a local 1-from η by

$$\eta(X) = \breve{g}(X,N), \forall \in \Gamma(T\acute{N}\mid_U). \tag{8}$$

Let P denote the projection morphism of $\Gamma(T\acute{N})$ on $\Gamma(S(T\acute{N}))$ with respect to the decomposition (1). We obtain

$$\begin{aligned} \breve{\nabla}_X PY &= \breve{\nabla}^*_X PY + C(X,PY)\xi, &(9)\\ \breve{\nabla}_X \xi &= -A^*_\xi X + \varepsilon(X)\xi = -A^*_\xi X - \tau(X)\xi, &(10)\end{aligned}$$

where $\breve{\nabla}^*_X PY$ and $A^*_\xi X$ belong to $\Gamma(S(T\acute{N})), \breve{\nabla}$ and $\breve{\nabla}^*$ are linear connections on $\Gamma(S(T\acute{N}))$ and $T\acute{N}^{\perp}$, respectively, h^* is a $\Gamma(T\acute{N}^{\perp})$-valued $F(\acute{N})$-bi-linear form on $\Gamma(T\acute{N})\times\ \Gamma(S(T\acute{N}))$ and A^*_ξ is $\Gamma(S(T\acute{N}))$-valued $F(\acute{N})$-linear operator on $\Gamma(T\acute{N})$.

We call them the screen fundamental form and screen shape operator of $S(T\acute{N})$, respectively. Define

$$\begin{aligned} C(X,PY) &= \breve{g}(h^*(X,PY),N), && (11)\\ \varepsilon(X) &= \breve{g}(\breve{\nabla}^{*t}_X\xi,N), \forall X,Y\in\Gamma(T\acute{N}). && (12)\end{aligned}$$

One can easily show that $\varepsilon(X) = -\tau(X)$. Here, $C(X,PY)$ is called the local screen fundamental form of $S(T\acute{N})$. Precisely, the two local second fundamental forms of $\acute{N}$ and $S(T\acute{N})$ are related to their shape operators by

$$\begin{aligned} B(X,Y) &= \breve{g}\left(Y,A^*_\xi X\right), && (13)\\ A^*_\xi\xi &= 0, && (14)\\ \breve{g}\left(A^*_\xi PY,N\right) &= 0, && (15)\\ C(X,PY) &= \breve{g}(PY,A_N X), && (16)\\ \breve{g}(N,A_N X) &= 0. && (17)\end{aligned}$$

A lightlike hypersurface $(\acute{N},g,S(T\acute{N}))$ of a semi-Riemannian manifold is called totally umbilical[5] if there is a smooth function ϱ, such that

$$B(X,Y) = \varrho g(X,Y), \forall X,Y\in\Gamma(T\acute{N}), \tag{18}$$

where ϱ is non-vanishing smooth function on a neighborhood U in $\acute{N}$.

A lightlike hypersurface $(\acute{N},g,S(T\acute{N}))$ of a semi-Riemannian manifold is called screen locally conformal if the shape operators A_N and A^*_ξ of $\acute{N}$ and $S(T\acute{N})$, respectively, are related by

$$A_N = \varphi A^*_\xi, \tag{19}$$

where φ is non-vanishing smooth function on a neighborhood U in $\acute{N}$. Therefore, it follows that for any $X,Y\in\Gamma(S(T\acute{N}))$ and $\xi\in Rad\,T\acute{N}$ we have

$$C(X,\xi) = 0. \tag{20}$$

For details about screen conformal lightlike hypersurfaces, we refer [1] and [5] .

3. Planar Normal Sections of Lightlike Hypersurfaces in $\mathbb{R}^4_1$

Let $\acute{N}$ be a lightlike hypersurface of $\mathbb{R}^4_1$. Now we shall investigate lightlike hypersurfaces with degenerate planar normal sections. If α is a null curve, for a point p in $\acute{N}$, we have

$$\begin{aligned} \alpha'(s) &= \xi, && (21)\\ \alpha''(s) &= \breve{\nabla}_\xi\xi = -\tau(\xi)\xi, && (22)\\ \alpha'''(s) &= -\left[\xi(\tau(\xi)) + \tau^2(\xi)\right]\xi. && (23)\end{aligned}$$

Then, α''' is a linear combination of α' and α''. Thus from (21), (22) and (23), we conclude $\alpha'''\wedge\alpha''\wedge\alpha' = 0$.

Hence we give

Corollary 1. *Every lightlike hypersurface of $\mathbb{R}_1^4$ has degenerate planar normal sections.*

Let $\acute{N}$ be a lightlike hypersurface of $\mathbb{R}_1^4$. For a point p in $\acute{N}$ and a spacelike vector $w \in S(T\acute{N}) = Sp\{u,v\}$, where u, v are unit spacelike vectors tangent to $\acute{N}$ at p, the vector w and transversal space $tr(T\acute{N})$ to $\acute{N}$ at p determine a 2-dimensional subspace $E(p,w)$ in $\mathbb{R}_1^4$ through p. The intersection of $\acute{N}$ and $E(p,w)$ gives a spacelike curve α in a neighborhood of p, which is called the normal section of $\acute{N}$ at p in the direction of w.

Now, we shall research the conditions for a lightlike hypersurface of $\mathbb{R}_1^4$ to have non-degenerate planar normal sections.

Let $(\acute{N}, g, S(T\acute{N}))$ be a totally umbilical and screen conformal lightlike hypersurface of $\left(\mathbb{R}_1^4, \breve{g}\right)$. In this case $S(T\acute{N})$ is integrable [1]. We denote integral hypersurface of $S(T\acute{N})$ by $\acute{N}'$. Then, using (6), (11) and (19) we find

$$\begin{aligned} C(w,w)\xi + B(w,w)N &= \breve{g}(w,w)\{\rho\xi + \beta N\} \\ &= \lambda\{\rho\xi + \beta N\},\ \lambda = a^2 + b^2, \end{aligned} \tag{24}$$

where $\lambda, \rho, \beta \in \mathbb{R}$. In this case, we obtain

$$\alpha'(s) = w, \tag{25}$$

$$\alpha''(s) = \breve{\nabla}^*_w w + C(w,w)\xi + B(w,w)N, \tag{26}$$

$$\alpha''(s) = \breve{\nabla}^*_w w + \rho\xi + \beta N, \tag{27}$$

and

$$\begin{aligned} \alpha'''(s) = {} & \breve{\nabla}^*_w \breve{\nabla}^*_w w + C(w, \breve{\nabla}^*_w w)\xi + w(C(w,w))\xi \\ & - C(w,w) A^*_\xi w + w(B(w,w))N \\ & - B(w,w) A_N w + B(w, \breve{\nabla}^*_w w)N, \end{aligned} \tag{28}$$

which implies

$$\begin{aligned} \alpha'''(s) = {} & \breve{\nabla}^*_w \breve{\nabla}^*_w w + C(w, \breve{\nabla}^*_w w)\xi \\ & + B(w, \breve{\nabla}^*_w w)N - \rho A^*_\xi w - \beta A_N w. \end{aligned} \tag{29}$$

Here $\breve{\nabla}^*$ and $\breve{\nabla}$ are linear connections on $S(T\acute{N})$ and $\Gamma(T\acute{N})$, respectively and $\alpha'(s) = w = au + bv$, $a, b \in \mathbb{R}$. Since $\acute{N}$ is a totally umbilical screen conformal lightlike hypersurface, we find

$$C(w, \breve{\nabla}^*_w w)\xi + B(w, \breve{\nabla}^*_w w)N = g(w, \breve{\nabla}^*_w w)\{\rho_1\xi + \beta_1 N\}, \tag{30}$$

where $\rho_1, \beta_1 \in \mathbb{R}$. On the other hand we write

$$\breve{\nabla}^*_w w = a^2 \breve{\nabla}^*_u u + ab\breve{\nabla}^*_u v + ab\breve{\nabla}^*_v u + b^2 \breve{\nabla}^*_v v \tag{31}$$

and

$$g(w, \breve{\nabla}^*_w w) = a^3 g(u, \breve{\nabla}^*_u u) + a^2 b g(u, \breve{\nabla}^*_u v) + a^2 b g(u, \breve{\nabla}^*_v u) + ab^2 g(u, \breve{\nabla}^*_v v)$$

$$+a^2bg(v,\breve{\nabla}^*_u v)+ab^2g(v,\breve{\nabla}^*_u v)+ab^2g(v,\breve{\nabla}^*_v u)+b^3g(v,\breve{\nabla}^*_v v).$$

Since $\breve{g}(u,u)=\breve{g}(v,v)=1$ and $\breve{g}(u,v)=0$, then by a direct computation, we obtain

$$\breve{\nabla}^*_u u=\lambda_1 v,\ \breve{\nabla}^*_v u=\lambda_2 v, \tag{32}$$

$$\lambda_1=-\lambda_3, \tag{33}$$

$$\lambda_2=-\lambda_4, \tag{34}$$

$$\breve{\nabla}^*_u v=\lambda_3 u,\ \breve{\nabla}^*_v v=\lambda_4 u, \tag{35}$$

where $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{R}$. Hence, from (32)-(35) we get

$$g(w,\breve{\nabla}^*_w w)=0$$

and

$$C(w,\breve{\nabla}^*_w w)\xi+B(w,\breve{\nabla}^*_w w)N=0.$$

Therefore, we obtain

$$\begin{aligned} C(w,\breve{\nabla}^*_w w) &= 0,\\ B(w,\breve{\nabla}^*_w w) &= 0.\end{aligned}$$

Since $\acute{N}$ is screen conformal, we find

$$\begin{aligned} \alpha'(s) &= w,\\ \alpha''(s) &= \lambda(\rho\xi+\beta N),\\ \alpha'''(s) &= -\lambda\rho A^*_\xi w-\lambda\beta A_N w,\end{aligned}$$

where $\rho,\beta\neq 0$. Then, we have

$$\alpha'''(s)=tA^*_\xi w,\ t=-2\lambda\rho.$$

Hence, we obtain

$$B(w,w)=g\left(A^*_\xi w,w\right)=\beta g(w,w)=g(\beta w,w),$$

which implies $A^*_\xi w=\beta w$, that is, α' and α''' are linearly dependent and so $\acute{N}$ has non-degenerate planar normal sections.

Assume that $\acute{N}$ is a totally geodesic lightlike hypersurface of $\mathbb{R}^4_1$. Then, we have $B=0$, $A^*_\xi=0$. Hence, from (25)-(28), we write

$$\alpha'(s) = w, \tag{36}$$

$$\alpha''(s) = \breve{\nabla}^*_w w, \tag{37}$$

$$\alpha'''(s) = \breve{\nabla}^*_w\breve{\nabla}^*_w w. \tag{38}$$

Since $\alpha',\alpha'',\alpha'''\in\Gamma(S(T\acute{N}))$ and $\dim(S(T\acute{N}))=2$, we have $\alpha'''(s)\wedge\alpha''(s)\wedge\alpha'(s)=0$.

Conversely, we assume that $\acute{N}$ has non-degenerate planar normal sections. Then, from (25), (26) and (28) we obtain

$$w \wedge \begin{pmatrix} \breve{\nabla}^*_w w + C(w,w)\xi \\ +B(w,w)N \end{pmatrix} \wedge \begin{pmatrix} \breve{\nabla}^*_w \breve{\nabla}^*_w w + C(w, \breve{\nabla}^*_w w)\xi + w(C(w,w))\xi \\ -C(w,w)A^*_\xi w + w(B(w,w))N \\ -B(w,w)A_N w + B(w, \breve{\nabla}^*_w w)N \end{pmatrix} = 0.$$

Since $w = au + bv$, $a, b \in \mathbb{R}$, for the sake of simplicity, we choose $u = (0,1,0,0)$ and $v = (0,0,1,0)$, which give

$$\breve{\nabla}^*_w w = \left(0, ab\lambda_3 + b^2\lambda_4, a^2\lambda_1 + ab\lambda_2, 0\right). \tag{39}$$

If we take $a = b = 1$, from (32)-(34), we obtain

$$\breve{\nabla}^*_w w = (0, -(\lambda_1 + \lambda_2), \lambda_1 + \lambda_2, 0),$$

which yields that w and $\breve{\nabla}^*_w w$ are linearly dependent. Thus we find

$$w \wedge \breve{\nabla}^*_w w = 0 \tag{40}$$

for any $a, b \in \mathbb{R}$. Moreover, if we take $a, b \in \{-1, 1\}$, we have

$$\breve{\nabla}^*_w w = (0, b(a\lambda_1 + b\lambda_2), a(a\lambda_1 + b\lambda_2), 0),$$

namely, in any case w and $\breve{\nabla}^*_w w$ are linearly dependent.

From (31), we find

$$\begin{aligned} \breve{\nabla}^*_w \breve{\nabla}^*_w w \quad = \quad & a^3\lambda_1\lambda_3 u + a^2 b\lambda_1\lambda_3 v + a^2 b\lambda_2\lambda_3 u + ab^2\lambda_4\lambda_1 v \\ & + a^2 b\lambda_1\lambda_4 u + a^2 b\lambda_2\lambda_3 v + ab^2\lambda_4\lambda_2 u + b^3\lambda_4\lambda_2 v. \end{aligned}$$

Here, for simplicity, if we take $a = b = 1$ then we obtain

$$\breve{\nabla}^*_w \breve{\nabla}^*_w w = \left(0, \lambda_1^2 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2^2, \lambda_1^2 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2^2, 0\right),$$

which yields

$$w \wedge \breve{\nabla}^*_w \breve{\nabla}^*_w w = 0. \tag{41}$$

Then we have

$$w \wedge (C(w,w)\xi + B(w,w)N) \wedge \Big(\breve{\nabla}_w (C(w,w)\xi + B(w,w)N)\Big) = 0. \tag{42}$$

Thus $C(w,w)\xi + B(w,w)N = 0$ or $\breve{\nabla}_w (C(w,w)\xi + B(w,w)N) = 0$. If $C(w,w)\xi + B(w,w)N = 0$, then $C = B = 0$, at $p \in \acute{N}$, which implies that $\acute{N}$ is totally geodesic and totally umbilical. If $\breve{\nabla}_w (C(w,w)\xi + B(w,w)N) = 0$, then we have

$$w(C(w,w))\xi + w(B(w,w))N - C(w,w)A^*_\xi w - B(w,w)A_N w = 0. \tag{43}$$

Hence $C(w,w)A^*_\xi w + B(w,w)A_N w = 0$, we find

$$A^*_\xi w = -\frac{B(w,w)}{C(w,w)} A_N w, \tag{44}$$

at $p \in \acute{N}$, which shows that $\acute{N}$ is a screen conformal lightlike hypersurface.

Consequently, we have the following.

Theorem 2. *Let $\acute{N}$ be a lightlike hypersurface of $\mathbb{R}^4_1$. Then $\acute{N}$ has non-degenerate planar normal sections if and only if either $\acute{N}$ is totally umbilical and screen conformal or $\acute{N}$ is totally geodesic.*

Proof. Assume that $\acute{N}$ is a totally umbilical and screen conformal lightlike hypersurface of $\mathbb{R}^4_1$. Then we have $A^*_\xi w = \beta w$, $\beta \in \mathbb{R}$. By using (25), (27) and (29), we obtain

$$\alpha''' (s) \wedge \alpha'' (s) \wedge \alpha' (s) = 0.$$

If we consider that $\acute{N}$ is totally geodesic, then, we have $C = B = 0$ and from (36)–(38), we see that w, $\breve{\nabla}^*_w w$ and $\breve{\nabla}^*_w \breve{\nabla}^*_w w$ belong to $S(T\acute{N})$. Since $\dim(S(T\acute{N})) = 2$, we conclude that $\alpha', \alpha'', \alpha'''$ are linearly dependent.

Conversely, we assume that $\acute{N}$ has non-degenerate planar normal sections. Then, from (42)–(44) we complete the proof. $\square$

Theorem 3. *Let $(\acute{N}, g, S(T\acute{N}))$ be a screen conformal non-umbilical lightlike hypersurface of $\mathbb{R}^4_1$. Then, for $T(w,w) = C(w,w)\xi + B(w,w)N$, the following statements are equivalent:*

(1) $(\breve{\nabla}_w T)(w,w) = 0$, *for every spacelike vector $w \in S(T\acute{N})$,*
(2) $\breve{\nabla} T = 0$,
(3) *$\acute{N}$ has non-degenerate planar normal sections and each normal section at p has one of its vertices at p.*

Note that, by the vertex of curve $\alpha(s)$ we mean a point p on α such that its curvature κ satisfies $\frac{d\kappa^2(p)}{ds} = 0$, where $\kappa^2 = \langle \alpha''(s), \alpha''(s) \rangle$.

Proof. From (25), (26), we have

$$(\breve{\nabla}_w T)(w,w) = \breve{\nabla}_w T(w,w),$$

which shows $(\breve{\nabla}_w T)(w,w) = 0$ if and only if $\breve{\nabla} T = 0$.

Assume that $\breve{\nabla} T = 0$. Then $\acute{N}$ is totally geodesic and Theorem 2 implies that $\acute{N}$ has (pointwise) planar normal sections. Let the $\alpha(s)$ be a normal section of $\acute{N}$ at p in a given direction $w \in S(T\acute{N})$. Then (25) shows that the curvature $\kappa(s)$ of $\alpha(s)$ satisfies

$$\begin{aligned} \kappa^2(s) &= \langle \alpha''(s), \alpha''(s) \rangle \\ &= 2C(w,w)B(w,w) \\ &= \langle T(w,w), T(w,w) \rangle, \end{aligned} \tag{45}$$

where $w = \alpha'(s)$. Therefore we find

$$\frac{d\kappa^2(p)}{ds} = \left\langle \breve{\nabla}_w T(w,w), T(w,w) \right\rangle = \left\langle (\breve{\nabla}_w T)(w,w), T(w,w) \right\rangle. \tag{46}$$

Since $\breve{\nabla}_w T(w,w) = 0$, this implies

$$\frac{d\kappa^2(0)}{ds} = 0,$$

at $p = \alpha(0)$. Thus p is a vertex of the normal section $\alpha(s)$.

If $\acute{N}$ has planar normal sections, then by using Theorem 2 we have

$$T(w,w) \wedge (\breve{\nabla}_w T)(w,w) = 0. \tag{47}$$

If p is a vertex of $\alpha(s)$, then we have

$$\frac{d\kappa^2(0)}{ds} = 0.$$

Thus, since $\acute{N}$ has planar normal sections, using (46) we find

$$\begin{array}{rcl} \alpha'(s) \wedge \alpha''(s) \wedge \alpha'''(s) & = & w \wedge (\breve{\nabla}^*_w w + T(w,w)) \\ & & \wedge(\breve{\nabla}^*_w \breve{\nabla}^*_w w + tT(w,w) + (\breve{\nabla}_w T)(w,w)) = 0, \end{array}$$

which yields

$$T(w,w) \wedge (\breve{\nabla}_w T)(w,w) = 0$$

and

$$\Big\langle (\breve{\nabla}_w T)(w,w), T(w,w) \Big\rangle = 0. \tag{48}$$

Combining (47) and (48) we obtain either $(\breve{\nabla}_w T)(w,w) = 0$ or $T(w,w) = 0$. Let us define $U = \Big\{ w \in S(T\acute{N}) \mid T(w,w) = 0 \Big\}$. If $int(U) \neq \varnothing$, we obtain $(\breve{\nabla}_w T)(w,w) = 0$ on $int(U)$. Thus, by continuity we have $\breve{\nabla} T = 0$. $\square$

Considering those obtained results above with [12], we give the following example.

Example 4. *Let $\mathbb{R}^4_1$ be the space $\mathbb{R}^4$ endowed with the semi-Euclidean metric*

$$\breve{g}(x,y) = -u_0 v_0 + \sum_{a=1}^{3} u_a v_a, \qquad u = \sum_{a=0}^{3} u_a \frac{\partial}{\partial u_a}.$$

Consider the null cone of $\mathbb{R}^4_1$ given by

$$\wedge^3_0 = \left\{ (u_0, u_1, u_2, u_3) \mid -u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0, u_0,\ u_1,\ u_2,\ u_3 \in \mathbb{R} \right\}.$$

The radical bundle of null cone is

$$\xi = u_0 \frac{\partial}{\partial u_0} + u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_3}$$

and screen distribution is spanned by

$$w = -u_2 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2} - u_3 \frac{\partial}{\partial u_3}.$$

Then the lightlike transversal vector bundle is given by

$$Itr(T\wedge_0^3) = Span\left\{N = \frac{1}{2(u_0)^2}\left(-u_0\frac{\partial}{\partial u_0} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3}\right)\right\}.$$

Let $\wedge_0^3$ be a lightlike hypersurfaces of $\mathbb{R}_1^4$. For a point p in $\wedge_0^3$ and a lightlike vector ξ which spans the radical distribution of a lightlike hypersurface, the vector ξ and transversal space $tr(T\wedge_0^3)$ to $\wedge_0^3$ at p determine a 2- dimensional subspace $E(p,\xi)$ in $\mathbb{R}_1^4$ through p. The intersection of $\wedge_0^3$ and $E(p,\xi)$ gives a lightlike curve α in a neighborhood of p, which is called the normal section of $\wedge_0^3$ at the point p in the direction of ξ. Therefore, we have

$$\begin{aligned}
\breve{\nabla}_\xi \xi &= u_0\frac{\partial}{\partial u_0} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} \\
\breve{\nabla}_\xi \breve{\nabla}_\xi \xi &= u_0\frac{\partial}{\partial u_0} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3}.
\end{aligned}$$

Then, we obtain

$$\alpha'''(s) \wedge \alpha''(s) \wedge \alpha'(s) = 0$$

which shows that null cone has degenerate planar normal sections.

On the other hand, by direct computations, we find

$$\breve{\nabla}_{\breve{\xi}} w = \breve{\nabla}_\xi w = w$$

and

$$A_N w = \frac{1}{2(u_0)^2} A_{\xi}^* w.$$

Namely, $\wedge_0^3$ is a screen conformal lightlike hypersurface of $\mathbb{R}_1^4$ [5].

Now, for a point p in $\wedge_0^3$ and a non-degenerate vector w tangent to $\wedge_0^3$ at p ($w \in S(T\wedge_0^3)$), the vector w and transversal space $tr(T\wedge_0^3)$ to $\acute{N}$ at p determine a 2- dimensional subspace $E(p,w)$ in $\mathbb{R}_1^4$ through p. The intersection of $\wedge_0^3$ and $E(p,w)$ gives a non-degenerate curve α in a neighborhood of p. Therefore, we have

$$\begin{aligned}
\alpha' &= w = -u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_3}, \\
\alpha'' &= \breve{\nabla}_w w + B(w,w)N \\
&= \frac{1}{2}u_0\frac{\partial}{\partial u_0} - \frac{3}{2}u_1\frac{\partial}{\partial u_1} - \frac{3}{2}u_2\frac{\partial}{\partial u_2} - \frac{3}{2}u_3\frac{\partial}{\partial u_3}, \\
\alpha''' &= \breve{\nabla}_w \breve{\nabla}_w w + w(B(w,w))N + B(w,w)\breve{\nabla}_w N \\
&= \breve{\nabla}_w \breve{\nabla}_w w + B(w,\breve{\nabla}_w w)N + w(B(w,w))N - B(w,w)A_N w.
\end{aligned}$$

Using $A_N w$ in α''' we find

$$\alpha''' = -\frac{1}{2}\left(-u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_3}\right).$$

Therefore α''' and α' are linearly dependent at $p \in \wedge_0^3$ and we have

$$\alpha' \wedge \alpha'' \wedge \alpha''' = 0.$$

Namely, $\wedge_0^3$ has non-degenerate planar normal sections.

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ON SOME PROPERTIES OF GENERALIZED STRUVE FUNCTION

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ABSTRACT. The main purpose of this investigation is to present some monotonic and log-concavity properties of the generalized Struve function. By using Hadamard product representation of the generalized Struve function, we investigate the sign of this function on some sets. Also, we determine an interval such that the generalized Struve function is decreasing in this interval. Moreover, we show that generalized Struve function is strictly logaritmically concave on some intervals. In addition, we prove that a function related to generalized Struve function is increasing function on $\mathbb{R}$.

1. Introduction and Preliminaries

In the last three decades many geometric and monotonic properties of some special functions like Bessel, Struve, Lommel, Mittag-Leffler, Wright functions and their generalizations were investigated by many authors. In general, by using the properties of zeros of the special functions many mathematicians studied about univalence, starlikeness, convexity and close-to-convexity of the mentioned functions. In addition, some authors focused on the monotonicity and log-convexity properties of the special functions by using their integral representations and some earlier results on analytic functions. For more information about these investigations the readers are referred to the papers [1, 2, 3, 4, 5, 8, 9, 10, 11] and references therein. Some inequalities which were obtained via above special functions and monotonic properties of these functions are intensively used in engineering sciences, mathematical physics, probability and statistics, and economics. Especially, it is known that the logarithmic concavity and logarithmic convexity properties have an important role in economics. Information on the logarithmic concavity and logarithmic convexity in the economic can be found in [7] and its references, comprehensively. In this study, motivated by the some earlier studies, our main goal is to give some monotonic and log-concavity properties of the generalized Struve functions.

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It is well-known that many special functions can be defined by using familiar gamma function. That is why, we want to remember the definition of gamma function. The Euler's gamma or classical gamma function Γ is defined by the following improper integral, for $x > 0$:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

On the other hand, the definition of logarithmic concavity of a function can be given as follow:

Definition 1 ([7]). *A function f is said to be log-concave on interval (a, b) if the function $\log f$ is a concave function on (a, b).*

The log-concavity of the function f on the interval (a, b) can be shown by using one of the following two conditions:
i. $\frac{f'}{f}$ monotone decreasing on (a, b).
ii. $\log f'' < 0$.
Also the following lemma due to Biernacki and Krzyż (see [6]) will be used in order to prove some monotonic properties of the mentioned functions.

Lemma 2. *Consider the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$, where $a_n \in \mathbb{R}$ and $b_n > 0$ for all $n \in \{0, 1, \dots\}$, and suppose that both converge on $(-r, r), r > 0$. If the sequence $\{\frac{a_n}{b_n}\}_{n\geq 0}$ is increasing(decreasing), then the function $x \mapsto \left(\frac{f(x)}{g(x)}\right)$ is also increasing(decreasing) on $(0, r)$.*

It is important to note that the above result remains true for the even or odd functions.

2. Main Results

In this section, we are going to discuss some properties like monotonicity and log-concavity of the generalized Struve function by using its product representation which is known as Hadamard product or Weierstrassian decomposition. The generalized Struve function has the following series representation (see [12]):

$$S^q_{p,b,c,\delta}(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma(qn + \frac{p}{\delta} + \frac{b+2}{2})} \left(\frac{x}{2}\right)^{2n+p+1} \tag{1}$$

for $q \in \mathbb{N}, p, b, c \in \mathbb{C}$ and $\delta > 0$. The author studied some geometric properties such as starlikeness and convexity of generalized Struve function in [12]. Also, the author showed that the zeros of the generalized Struve function are all real. In the same paper, by using Hadamard's theorem an infinite product representation of the generalized Struve function was given as follow (see [12, Lemma 2.1]):

$$S^q_{p,b,c,\delta}(x) = \frac{\left(\frac{x}{2}\right)^{p+1}}{\Gamma\left(\frac{p}{\delta} + \frac{b+2}{2}\right)} \prod_{n\geq 1} \left(1 - \frac{x^2}{{}_q x^2_{p,b,c,\delta,n}}\right), \tag{2}$$

where ${}_qx_{p,b,c,\delta,n}$ denotes the n-th positive zero of the generalized Struve function $S^q_{p,b,c,\delta}(x)$.

Theorem 3. *Let b, c, δ, q are positive real numbers , $p > -1$ and ${}_qx_{p,b,c,\delta,n}$ denote the nth positive zero of the generalized Struve function $S^q_{p,b,c,\delta}(x)$. In addition, consider the following sets:*

$$\Delta_1 = \bigcup_{n\geq 1} \left({}_qx_{p,b,c,\delta,2n-1}, {}_q x_{p,b,c,\delta,2n}\right), \Delta_2 = \bigcup_{n\geq 1} \left({}_qx_{p,b,c,\delta,2n}, {}_q x_{p,b,c,\delta,2n+1}\right)$$

and

$$\Delta_3 = [0, {}_q x_{p,b,c,\delta,1}) \cup \Delta_2.$$

Then, the generalized Struve function

$$\Psi^q_{p,b,c,\delta}(x) = \left(\frac{2}{x}\right)^{p+1} \Gamma\left(\frac{p}{\delta} + \frac{b+2}{2}\right) S^q_{p,b,c,\delta}(x) = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{p}{\delta} + \frac{b+2}{2}\right)(-c)^n}{n!\Gamma\left(nq + \frac{p}{\delta} + \frac{b+2}{2}\right)} \left(\frac{x}{2}\right)^{2n} \tag{3}$$

satisfies the next properties:
a. *the function $x \mapsto \Psi^q_{p,b,c,\delta}(x)$ is negative on Δ_1 and positive on Δ_3,*
b. *the function $x \mapsto \Psi^q_{p,b,c,\delta}(x)$ is decreasing on $[0, {}_q x_{p,b,c,\delta,1})$,*
c. *the function $x \mapsto \Psi^q_{p,b,c,\delta}(x)$ is strictly log-concave on Δ_3.*

Proof. **a.** By considering the infinite product representation of the generalized Struve function $S^q_{p,b,c,\delta}(x)$ which is given by (2), we can easily see that the function $\Psi^q_{p,b,c,\delta}(x)$ can be written as the following product representation:

$$\Psi^q_{p,b,c,\delta}(x) = \prod_{n\geq 1} \left(1 - \frac{x^2}{{}_qx^2_{p,b,c,\delta,n}}\right). \tag{4}$$

In order to determine the sign of the function $x \mapsto \Psi^q_{p,b,c,\delta}(x)$ on the mentioned sets, we rewrite the function $x \mapsto \Psi^q_{p,b,c,\delta}(x)$ as

$$\Psi^q_{p,b,c,\delta}(x) = \chi_n \tau_n,$$

where

$$\chi_n = \prod_{n\geq 1} \frac{{}_qx_{p,b,c,\delta,n} + x}{{}_qx^2_{p,b,c,\delta,n}} \text{ and } \tau_n = \prod_{n\geq 1} \left({}_qx_{p,b,c,\delta,n} - x\right).$$

It can be easily seen that $\chi_n > 0$ for all $x \in [0, \infty)$. On the other hand, since

$$0 <_q x_{p,b,c,\delta,1} <_q x_{p,b,c,\delta,2} < \cdots <_q x_{p,b,c,\delta,n} < \cdots$$

it can be said that, if $x \in ({}_qx_{p,b,c,\delta,2n-1}, {}_q x_{p,b,c,\delta,2n})$, then the first $(2n-1)$ terms of τ_n are strictly negative and remained terms are strictly positive. Also, if $x \in ({}_qx_{p,b,c,\delta,2n}, {}_q x_{p,b,c,\delta,2n+1})$, then the first $2n$ terms of τ_n are strictly negative and the rest is strictly positive. In addition, for $x \in [0, {}_q x_{p,b,c,\delta,1})$ all the terms of τ_n are strictly positive. As a consequence, the function $x \mapsto \Psi^q_{p,b,c,\delta}(x)$ is negative on Δ_1 and it is positive on Δ_3.

b. We know from the previous part of this theorem that the function $x \mapsto \Psi^q_{p,b,c,\delta}(x)$ is positive on the interval $[0,_q x_{p,b,c,\delta,1})$. Now, taking logarithmic derivative of (4) implies that

$$\begin{aligned}\frac{d}{dx}\left[\log \Psi^q_{p,b,c,\delta}(x)\right] &= \frac{\left(\Psi^q_{p,b,c,\delta}(x)\right)'}{\Psi^q_{p,b,c,\delta}(x)} \\ &= \frac{d}{dx}\left[\log \prod_{n\geq 1}\left(1-\frac{x^2}{{}_qx^2_{p,b,c,\delta,n}}\right)\right] \\ &= \sum_{n=1}^{\infty}\frac{2x}{x^2-_q x^2_{p,b,c,\delta,n}}.\end{aligned}$$

As a result, we get

$$\left(\Psi^q_{p,b,c,\delta}(x)\right)' = \Psi^q_{p,b,c,\delta}(x)\sum_{n=1}^{\infty}\frac{2x}{x^2-_q x^2_{p,b,c,\delta,n}} < 0$$

for all $x \in [0,_q x_{p,b,c,\delta,1})$. So, the function $x \mapsto \Psi^q_{p,b,c,\delta}(x)$ is decreasing on $[0,_q x_{p,b,c,\delta,1})$.

c. To show the log-concavity of the function $x \mapsto \Psi^q_{p,b,c,\delta}(x)$, it is enough that

$$\frac{d^2}{dx^2}\left[\log \Psi^q_{p,b,c,\delta}(x)\right] < 0$$

for all $x \in \Delta_3$. Now, by using the Hadamard product representation of the function $\Psi^q_{p,b,c,\delta}(x)$ which is given by (4) we deduce

$$\begin{aligned}\frac{d^2}{dx^2}\left[\log \Psi^q_{p,b,c,\delta}(x)\right] &= \frac{d^2}{dx^2}\left[\log \prod_{n\geq 1}\left(1-\frac{x^2}{{}_qx^2_{p,b,c,\delta,n}}\right)\right] \\ &= \frac{d}{dx}\left[\frac{d}{dx}\sum_{n=1}^{\infty}\log\left(1-\frac{x^2}{{}_qx^2_{p,b,c,\delta,n}}\right)\right] \\ &= \frac{d}{dx}\sum_{n=1}^{\infty}\frac{-2x}{{}_qx^2_{p,b,c,\delta,n}-x^2} \\ &= -2\sum_{n=1}^{\infty}\frac{{}_qx^2_{p,b,c,\delta,n}+x^2}{\left({}_qx^2_{p,b,c,\delta,n}-x^2\right)^2} \\ &< 0\end{aligned}$$

for $x \in \Delta_3$. So, the conclusion follows. □

Theorem 4. *Let b, c, δ, q are positive real numbers , $p > -1$ and ${}_qx_{p,b,c,\delta,n}$ denote the nth positive zero of the generalized Struve function $S^q_{p,b,c,\delta}(x)$. Then, the function $x \mapsto S^q_{p,b,c,\delta}(x)$ is strictly log-concave on $(0,_q x_{p,b,c,\delta,1}) \cup \Delta_2$.*

Proof. By using the fact that the product of two strictly log-concave function is also strictly log-concave, it is possible to prove the log-concavity of the generalized Struve function $S^q_{p,b,c,\delta}(x)$ on Δ_3. Because of this, we consider the function $S^q_{p,b,c,\delta}(x)$ in the following form:

$$S^q_{p,b,c,\delta}(x) = \frac{1}{\Gamma\left(\frac{p}{\delta} + \frac{b+2}{2}\right)} \left(\frac{x}{2}\right)^{p+1} \Psi^q_{p,b,c,\delta}(x).$$

We known from part **c.** of Theorem 3 that the generalized Struve function $\Psi^q_{p,b,c,\delta}(x)$ is strictly log-concave on Δ_3. In addition, since

$$\frac{d^2}{dx^2}\left(\log\left(\frac{x}{2}\right)^{p+1}\right) = \frac{d^2}{dx^2}\left((p+1)\log\left(\frac{x}{2}\right)\right) = -\frac{p+1}{x^2} < 0$$

for $p > -1$, the function $x \mapsto \left(\frac{x}{2}\right)^{p+1}$ is strictly log-concave on $(-\infty, 0) \cup (0, \infty)$. As a result, the function $S^q_{p,b,c,\delta}(x)$ is strictly log-concave on $(0,_q x_{p,b,c,\delta,1}) \cup \Delta_2$ as a product of two strictly log-concave functions. □

Now, let define the function $x \mapsto h^q_{p,b,\delta}(x)$ by putting $c = -1$ in (3). It is easily seen that the function $h^q_{p,b,\delta}(x)$ has the following infinite sum representation:

$$h^q_{p,b,\delta}(x) = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{p}{\delta} + \frac{b+2}{2}\right)}{n!4^n\Gamma\left(nq + \frac{p}{\delta} + \frac{b+2}{2}\right)} x^{2n}. \tag{5}$$

By using the Lemma 2 we have the following:

Theorem 5. *The function*

$$x \mapsto \frac{x\left(h^q_{p,b,\delta}(x)\right)'}{h^q_{p,b,\delta}(x)}$$

is increasing on $(0, \infty)$ for $p, b, \delta, q \in \mathbb{R}^+$.

Proof. By using the infinite sum representation of the function $h^q_{p,b,\delta}(x)$ which is given by (5), it can be written that

$$\frac{x\left(h^q_{p,b,\delta}(x)\right)'}{h^q_{p,b,\delta}(x)} = \frac{\sum_{n=0}^{\infty} A_n x^{2n}}{\sum_{n=0}^{\infty} B_n x^{2n}},$$

where

$$A_n = \frac{2n\Gamma(\frac{p}{\delta} + \frac{b+2}{2})}{n!4^n\Gamma(nq + \frac{p}{\delta} + \frac{b+2}{2})} \text{ and } B_n = \frac{\Gamma(\frac{p}{\delta} + \frac{b+2}{2})}{n!4^n\Gamma(nq + \frac{p}{\delta} + \frac{b+2}{2})}.$$

Cauchy-Hadamard theorem for power series implies that the both series $\sum_{n=0}^{\infty} A_n x^{2n}$ and $\sum_{n=0}^{\infty} B_n x^{2n}$ are convergent for all $x \in \mathbb{R}$, since

$$\lim_{n\to\infty}\left|\frac{A_n}{A_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{B_n}{B_{n+1}}\right| = \infty.$$

Moreover, we can say that $A_n \in \mathbb{R}$ and $B_n > 0$ for all $n = 0, 1, 2, \ldots$. On the other hand, if we consider the sequence

$$\mathcal{H}_n = \frac{A_n}{B_n} = 2n,$$

then we deduce

$$\frac{\mathcal{H}_{n+1}}{\mathcal{H}_n} = \frac{n+1}{n} > 1.$$

So the sequence $\{\mathcal{H}_n\}_{n\geq 0}$ is increasing. As a result, by applying the Lemma 2 to the function $x \mapsto \frac{x\left(h^q_{p,b,\delta}(x)\right)'}{h^q_{p,b,\delta}(x)}$ the proof is completed. □

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INTEGRABILITY OF THE DISTRIBUTIONS OF GCR-LIGHTLIKE SUBMANIFOLDS OF (ε)-SASAKIAN MANIFOLDS

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Abstract. We study GCR-lightlike submanifolds of (ε)-Sasakian manifolds and derived some important structural characteristics equations for further uses. We also obtain some necessary and sufficient conditions for the integrability of various distributions of GCR-lightlike submanifolds of (ε)-Sasakian manifolds.

1. Introduction

As a generalization of complex and totally real submanifolds, Cauchy-Riemann (CR)-submanifolds of Kaehler manifolds were introduced by Bejancu [1] in 1978 and further studied by many authors on using positive definite metric. In [3], Duggal introduced the geometry of CR-submanifolds with Lorentz metric and showed mutual interplay between the Cauchy-Riemann structure and physical spacetime geometry. In [4], Duggal showed the interaction of Lorentz CR-submanifolds with relativity and also studied a new class of CR-submanifolds. Later on, Duggal and Bejancu [5] introduced the concept of CR-lightlike submanifolds of indefinite Kaehler manifolds but which excluded the complex and totally real subcases, therefore Duggal and Sahin [6] introduced Screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds which included complex and screen real subcases but there was no inclusion relation between SCR and CR classes. Thus as an umbrella of complex, real hypersurfaces, screen real and CR-lightlike submanifolds, Duggal and Sahin [7] introduced Generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Kaehler manifolds and further studied by [10–15]. Since there are significant applications of contact geometry in thermodynamics, optics, mechanics and many more. Therefore, Duggal and Sahin [8] introduced the geometry of (GCR)-lightlike submanifolds of indefinite Sasakian manifolds and further studied by [16–18]. Recent developments in the geometry of GCR-lightlike submanifolds motivated us to extend this work. Kumar et al. [9] contributed in the study of

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(ε)-Sasakian manifolds and our aim of this paper is to study GCR-lightlike submanifolds of (ε)-Sasakian manifolds.

2. Preliminaries

2.1. **(ε)-Sasakian Manifolds.** Assume that $\bar{M}$ is a $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ, η, V), where ϕ is a $(1,1)$-type tensor field, η is a 1-form and V is a vector field on $\bar{M}$, called the characteristic vector field, satisfying

$$\phi^2 X = -X + \eta(X) V, \quad \eta(V) = 1, \tag{2.1}$$

$$\eta(\phi X) = 0, \quad \phi(V) = 0, \quad rank\phi = 2n, \tag{2.2}$$

then $\bar{M}$, with the triple (ϕ, η, V) is called an almost contact manifold. If there exists a semi-Riemannian metric $\bar{g}$ such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \varepsilon\eta(X)\eta(Y), \quad \forall X, Y \in T\bar{M}, \tag{2.3}$$

$$\eta(X) = \varepsilon\bar{g}(X, V), \quad \bar{g}(V, V) = \varepsilon, \quad \forall X \in T\bar{M}, \tag{2.4}$$

for any vector fields X, Y on $\bar{M}$, where $\varepsilon = \mp 1$, then $(\phi, \eta, V, \bar{g})$ is called an (ε)-almost contact metric structure on $\bar{M}$. If $d\eta(X, Y) = \bar{g}(\phi X, Y)$, then (ε)-almost contact metric structure is called an (ε)-contact metric structure and $\bar{M}$ endowed with this structure is called an (ε)- contact metric manifold. Furthermore, if the (ε)-contact metric structure is normal, that is, if satisfying

$$[\phi X, \phi Y] + \phi^2 [X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -2d\eta(X, Y) V, \tag{2.5}$$

then (ε)-contact metric structure is called an (ε)-Sasakian structure and $\bar{M}$ endowed with this structure is called as an (ε)-Sasakian manifold [2].

Remark 1. *From the relations $g(V, V) = \varepsilon$ and $\varepsilon = \mp 1$, it is clear that the vector field V can never be null. If $\varepsilon = -1$ and the index of $\bar{g}$ is odd; then $\bar{M}$ is called a time-like Sasakian manifold. If $\varepsilon = 1$ and the index of $\bar{g}$ is even; then $\bar{M}$ is called a space-like Sasakian manifold. In particular, if $\varepsilon = -1$ and the index of $\bar{g}$ is either zero or one; then $\bar{M}$ is said to be a usual Sasakian manifold or a Lorentz-Sasakian manifold, respectively.*

Theorem 1 ([2, Theorem 3]). *The necessary and sufficient conditions for an (ε)-almost contact metric structure $(\phi, \eta, V, \bar{g})$ to be an (ε)-Sasakian structure is*

$$\left(\bar{\nabla}_X \phi\right) Y = \bar{g}(X, Y) V - \varepsilon\eta(Y) X, \quad \forall X, Y \in T\bar{M}, \tag{2.6}$$

for any vector fields X, Y on $\bar{M}$, where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to $\bar{g}$. Moreover, we also have

$$\bar{\nabla}_X V = -\varepsilon\phi X, \tag{2.7}$$

for any $X \in T\bar{M}$.

2.2. **Lightlike Submanifolds.** Suppose that $\left(\bar{M}^{m+n}, \bar{g}\right)$ is a semi-Riemann manifold and M^m is its immersed submanifold. Then, M^m is called a lightlike submanifold; if the metric g on M induced from $\bar{g}$ has a radical distribution $Rad\,(TM)$ of rank r, for $1 \leq r \leq m$, for details see [5]. Then, its semi-Riemannian complementary distribution in TM, denoted by $S(TM)$, is known as the screen distribution and it follows that $TM = Rad\,(TM) \perp S\,(TM)$. The orthogonal complementary of $Rad\,(TM)$ in $TM^{\perp}$, denoted by $S\left(TM^{\perp}\right)$, is also a semi-Riemannian bundle and known as a screen transversal bundle of M. Since $S(TM)$ is a non-degenerate vector subbundle of $T\bar{M}|_M$; then, we have $T\bar{M}|_M = S(TM) \perp S(TM)^{\perp}$ where $S(TM)^{\perp}$ is the complementary orthogonal vector bundle of $S(TM)$ in $T\bar{M}|_M$. Then, clearly we have $S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}$. If (M, g) is an r-lightlike submanifold of $(\bar{M}, \bar{g})$; then, for the local basis $\{\xi_i\}_{i=1}^{r}$ of $Rad\,(TM)$ on a coordinate neighbourhood $\mathcal{U}$ of M, there exist smooth sections $\{N_i\}_{i=1}^{r}$ of $S(TM^{\perp})^{\perp}|_{\mathcal{U}}$ such that $\bar{g}\,(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, for any $i, j \in \{1, \ldots, r\}$. Then, there exists a vector subbundle of $S(TM^{\perp})^{\perp}$ spanned by $\{N_i\}_{i=1}^{r}$, known as the lightlike transversal vector bundle of M and denoted by $ltr(TM)$. Consider a vector bundle $tr(TM) = ltr(TM) \perp S(TM^{\perp})$, which is a complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$ and known as the transversal vector bundle of M. Thus, we have the following decomposition

$$T\bar{M}|_M = TM \oplus tr(TM) = S(TM) \perp \{Rad(TM) \oplus ltr(TM)\} \perp S(TM^{\perp}).$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$; then using above decomposition, the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h\,(X, Y)\,, \;\; \forall X, Y \in \Gamma\,(TM)\,, \tag{2.8}$$

$$\bar{\nabla}_X U = -A_U X + \nabla_x^t U, \; \forall X \in \Gamma\,(TM)\,, \; U \in \Gamma\,(tr\,(TM))\,, \tag{2.9}$$

where $\{\nabla_X Y, \; A_U X\}$ and $\left\{h\,(X, Y)\,, \; \nabla_X^t U\right\}$ are the elements of $\Gamma\,(TM)$ and $\Gamma\,(tr\,(TM))$, respectively. Here ∇ and ∇^t are the linear connections on TM and $tr\,(TM)$, respectively and the linear operator A_U on M is called the shape operator and the symmetric bilinear form h on TM is called the second fundamental form.

Consider projection morphisms $\mathcal{L}$ and $\mathcal{S}$ of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then particularly Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h^{\ell}\,(X, Y) + h^{s}\,(X, Y)\,, \tag{2.10}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\ell}\,(N) + D^{s}\,(X, N)\,, \tag{2.11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla_X^{s}\,(W) + D^{\ell}\,(X, W)\,, \tag{2.12}$$

for any $X, Y \in \Gamma\,(TM)$, $N \in \Gamma\,(\ell tr\,(TM))$ and $W \in \Gamma\left(S\left(TM^{\perp}\right)\right)$, where $h^l(X, Y) = \mathcal{L}(h(X, Y))$ and $h^s(X, Y) = \mathcal{S}(h(X, Y))$ are the lightlike second fundamental form and the screen second fundamental form of M, respectively. It should be noted that $D^l : \Gamma(TM) \times \Gamma(S(TM^{\perp})) \to \Gamma(ltr(TM))$ and $D^s : \Gamma(TM) \times \Gamma(ltr(TM)) \to \Gamma(S(TM^{\perp}))$ are $\mathcal{F}(M)$-bilinear mappings. ∇^{ℓ} and ∇^{s} are the lightlike and the screen transversal connection on M, respectively. In the consequence of (2.8), (2.10), (2.11) and (2.12), we have

$$g\,(A_W X, \, Y) = \bar{g}\,(h^{s}\,(X, \, Y)\,, W) + \bar{g}\left(Y, D^{\ell}\,(X, W)\right), \tag{2.13}$$

$$g\left(A_N X, \bar{P}Y\right) = \bar{g}\left(N, \nabla_X \bar{P}Y\right), \tag{2.14}$$

where $\bar{P}$ is the projection of TM on $S(TM)$. Furthermore, we also have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*\left(X, \bar{P}Y\right), \tag{2.15}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.16}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where ∇^* and ∇^{*t} are the linear connections on $S(TM)$ and $Rad(TM)$, respectively. h^* and A^* are $\Gamma(Rad(TM))$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as second fundamental forms of distributions $S(TM)$ and $Rad(TM)$, respectively. By the virtue of (2.16) and (2.17), we have

$$\bar{g}\left(h^*\left(X, \bar{P}Y\right), N\right) = g\left(A_N X, \bar{P}Y\right) \tag{2.17}$$

3. GCR-lightlike submanifolds of (ε)-Sasakian manifolds

Definition 1. *Suppose $\left(\bar{M}, \bar{g}\right)$ is an (ε)-Sasakian manifold and $(M, g, S(TM))$ is its real lightlike submanifold, where V is tangent to M. Then, M is called a GCR-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:*

- (A) *There exist two subbundles D_1 and D_2 of $Rad(TM)$; such that $Rad(TM) = D_1 \oplus D_2$, where $\phi(D_1) = D_1$ and $\phi(D_2) \subset S(TM)$.*
- (B) *There exist two subbundles D_0 and $\bar{D}$ of $S(TM)$; such that $S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V$ and $\phi(\bar{D}) = L \perp S$,*

where D_0 is invariant non-degenerate distribution on M, $\{V\}$ is one dimensional distribution spanned by V, L and S are vector subbundles of $ltr(TM)$ and $S(TM^\perp)$, respectively. Then, the tangent bundle TM of M is decomposed as $TM = \{D \oplus \bar{D} \oplus \{V\}\}$, where $D = Rad(TM) \oplus D_0 \oplus \phi(D_2)$.

Suppose $(M, g, S(TM))$ is a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, any $X \in TM$ can be written as

$$X = P_1X + P_2X + P_0X + \phi P_2X + QX + \eta(X)V, \tag{3.1}$$

where P_1X, P_2X, P_0X, ϕP_2X and QX belong to the distributions D_1, D_2, D_0, ϕD_2 and $\bar{D}$, respectively. Assume that L^1 represents the orthogonal complement of the vector subbundle L in $\ell tr(TM)$; then using the definition of GCR-lightlike submanifold, for any $N \in \Gamma(\ell tr(TM))$, we have

$$\phi N = TN + CN, \tag{3.2}$$

where $TN \in \Gamma(\phi L)$ is the tangential part of ϕN and $CN \in \Gamma\left(L^\perp\right)$ is the transversal part of ϕN. Similarly, suppose that $S^\perp$ represents the orthogonal complement of the vector subbundle S in $S\left(TM^\perp\right)$; then for any $W \in \Gamma\left(S\left(TM^\perp\right)\right)$, we have

$$\phi W = TW + CW, \tag{3.3}$$

where $TW \in \Gamma(\phi S)$ is the tangential part of ϕW and $CW \in \Gamma\left(S^\perp\right)$ is the screen transversal part of ϕW. Using (3.1), we obtain

$$\phi X = \phi(P_1X) + \phi(P_2X) + \phi(P_0X) - P_2X + \phi QX, \tag{3.4}$$

where $\phi QX \in \Gamma(L \perp S)$ and we can write

$$\phi QX = LX + SY, \tag{3.5}$$

where $LX \in \Gamma(L)$ and $SY \in \Gamma(S)$. So, we have

$$\begin{aligned} U(X,Y) &= \nabla_X(\phi P_1 Y) + \nabla_X(\phi P_2 Y) - \nabla_X(P_2 Y) \\ &\quad + \nabla_X(\phi P_0 Y) - A_{LY}X - A_{SY}X, \end{aligned} \tag{3.6}$$

for any $X, Y \in TM$.

Lemma 1. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, for any $X, Y \in TM$ the following equalities hold*

$$P_1 U(X,Y) - \phi P_1 \nabla_X Y = -\varepsilon\eta(Y) P_1 X, \tag{3.7}$$

$$P_2 U(X,Y) + P_2 \nabla_X Y = -\varepsilon\eta(Y) P_2 X, \tag{3.8}$$

$$P_0 U(X,Y) - \phi P_0 \nabla_X Y = -\varepsilon\eta(Y) P_0 X, \tag{3.9}$$

$$\phi P_2 U(X,Y) - \phi P_2 \nabla_X Y = -\varepsilon\eta(Y) \phi P_2 X, \tag{3.10}$$

$$QU(X,Y) - QTh^\ell(X,Y) - QTh^s(X,Y) = -\varepsilon\eta(Y) QX, \tag{3.11}$$

$$\{\eta(U(X,Y)) - \bar{g}(X,Y)\} V = -\varepsilon\eta(Y)\eta(X) V, \tag{3.12}$$

$$\begin{aligned} &\nabla^\ell_X(LY) + D^\ell(X, SY) - L\nabla_X Y + h^\ell(X, \phi P_1 Y) \\ &\quad + h^\ell(X, \phi P_2 Y) + h^\ell(X, \phi P_0 Y) - h^\ell(X, P_2 Y) - Ch^\ell(X,Y) = 0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\nabla^s_X(SY) - S\nabla_X Y + D^s(X, LY) + h^S(X, \phi P_1 Y) \\ &\quad + h^S(X, \phi P_2 Y) - h^s(X, P_2 Y) + h^s(X, \phi P_0 Y) - Ch^s(X,Y) = 0. \end{aligned} \tag{3.14}$$

Proof. Let $Y \in \Gamma(TM)$; then using (3.4) and (3.5), it follows that $\phi(P_1 Y) - P_2 Y$, $\phi(P_2 Y) + \phi P_0 Y$, LY and SY belong to $Rad(TM)$, $S(TM)$, $\ell tr(TM)$ and $S(TM^\perp)$, respectively. Also for any $X, Y \in \Gamma(TM)$, it is known that

$$(\bar{\nabla}_X \phi) Y = \bar{\nabla}_X(\phi Y) - \phi(\bar{\nabla}_X Y). \tag{3.15}$$

Using (2.10), (2.11), (2.12) and (3.4) in (3.15) and afterwards applying (3.6), we obtain

$$\begin{aligned} (\bar{\nabla}_X \phi) Y &= (P_1 U(X,Y) - \phi P_1 \nabla_X Y) + (P_2 U(X,Y) + P_2 \nabla_X Y) \\ &\quad + (P_0 U(X,Y) - \phi P_0 \nabla_X Y) + \phi(P_2 U(X,Y) - \phi P_2 \nabla_X Y) \\ &\quad + \left(QU(X,Y) - Th^\ell(X,Y) - Th^s(X,Y)\right) + \eta(U(X,Y)) V \\ &\quad + \Big(\nabla^\ell_X(LY) + D^\ell(X, SY) - L\nabla_X Y + h^\ell(X, \phi P_1 Y) + h^\ell(X, \phi P_2 Y) \\ &\quad + h^\ell(X, \phi P_0 Y) \ - h^\ell(X, P_2 Y) - Ch^\ell(X,Y)\big) \\ &\quad + (\nabla^s_X(SY) - S\nabla_X Y + h^s(X, \phi P_1 Y), \end{aligned} \tag{3.16}$$

for any $X, Y \in \Gamma(TM)$. Also from (2.6) and (3.1), it follows that

$$\begin{aligned} (\bar{\nabla}_X \phi) Y &= -\varepsilon\eta(Y)(P_1 X) - \varepsilon\eta(Y)(P_2 X) - \varepsilon\eta(Y)(P_0 X) - \varepsilon\eta(Y)(\phi P_2 X) \\ &\quad - \varepsilon\eta(Y) QX + (\bar{g}(X,Y) - \varepsilon\eta(X)\eta(Y)) V. \end{aligned} \tag{3.17}$$

By using (3.16) and (3.17), it is easy to obtain

$$\begin{aligned}\left(\bar{\tilde{N}}_X f\right) Y &= \left(P_1 U(X,Y) - f P_1 \tilde{N}_X Y\right) + P_2\left(U(X,Y) + \tilde{N}_X Y\right)\\ &+ \left(P_0 U(X,Y) - f P_0 \tilde{N}_X Y\right) + f P_2\left(U(X,Y) - \tilde{N}_X Y\right)\\ &+ QU(X,Y) - Th^l(X,Y) - Th^s(X,Y) + \eta(U(X,Y))V\\ &+ \left(\tilde{N}^l_X(LY) + D^l(X,SY) - L\tilde{N}_X Y + h^l(X, fP_1Y) + h^l(X, fP_2Y)\right.\\ &+ h^l(X, fP_0Y) - h^l(X, P_2Y) - Ch^l(X,Y))\\ &+ \left(\tilde{N}^s_X(SY) - S\tilde{N}_X Y + h^s(X, fP_1Y)\right) + h^s(X, fP_2Y)\\ &- h^s(X, P_2Y) + h^s(X, fP_0Y) + D^s(X, LY) - Ch^s(X,Y)).\end{aligned}$$

Then, (3.7) to (3.14) follow on comparing the components of the vector bundles D_1, D_2, D_0, ϕD_2, $\bar{D}$, $\{V\}$, $\ell tr(TM)$ and $S\left(TM^{\perp}\right)$, respectively. □

Lemma 2. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, for any $X \in \Gamma(TM)$ and $N \in \ell tr(TM)$, the following relations hold*

$$P_1\nabla_X(TN) - P_1A_{CN}X + \phi P_1(A_NX) = 0, \tag{3.18}$$
$$P_2\nabla_X(TN) - P_2A_{CN}X - P_2(A_NX) = 0, \tag{3.19}$$
$$P_0\nabla_X(TN) - P_0A_{CN}X + \phi P_0(A_NX) = 0, \tag{3.20}$$
$$\phi P_2(\nabla_X(TN)) - \phi P_2(A_{CN}X) + \phi(P_2A_NX) = 0, \tag{3.21}$$
$$Q\nabla_X(TN) - QA_{CN}X - T\nabla^\ell_X N - TD^s(X,N) = 0, \tag{3.22}$$
$$\eta(\nabla_X TN - A_{CN}X) = \bar{g}(P_1X, N) + \bar{g}(P_2X, N), \tag{3.23}$$
$$h^\ell(X, TN) + \nabla^\ell_X(CN) - C\nabla^\ell_X N + LA_NX = 0, \tag{3.24}$$
$$h^s(X, TN) + D^s(X, CN) - CD^s(X,N) + SA_NX = 0. \tag{3.25}$$

Proof. Let $X \in \Gamma(TM)$ and $N \in \Gamma(\ell tr(TM))$, then we have

$$\begin{aligned}\left(\bar{\nabla}_X\phi\right)N &= \bar{\nabla}_X(TN) + \bar{\nabla}_X(CN)\\ &+ \phi(A_NX) - \phi\left(\nabla^\ell_X(N)\right) - \phi(D^s(X,N)).\end{aligned}\tag{3.26}$$

Further on using the equations (2.10) and (2.11) in (3.26), we get

$$\begin{aligned}\left(\bar{\nabla}_X\phi\right)N &= \nabla_X(TN) + h^\ell(X,TN) + h^S(X,TN) - A_{CN}X\\ &+ \nabla^\ell_X(CN) + D^s(X,CN) + \phi(A_NX)\\ &- \phi\left(\nabla^\ell_X(N)\right) - \phi D^s(X,N)\end{aligned}\tag{3.27}$$

Using (3.1) to (3.3), we also have

$$\begin{aligned}\nabla_X(TN) &= P_1\nabla_X(TN) + P_2\nabla_X(TN) + P_0\nabla_X(TN)\\ &+ \phi P_2\nabla_X(TN) + Q\nabla_X(TN) + \eta(\nabla_X(TN))V,\end{aligned}\tag{3.28}$$

$$\begin{aligned}A_{CN}X &= P_1A_{CN}X + P_2A_{CN}X + P_0A_{CN}X\\ &+ \phi P_2A_{CN}X + QA_{CN}X + \eta(A_{CN}X)V,\end{aligned}\tag{3.29}$$

$$\begin{aligned}\phi\left(A_N X\right) &= \phi\left(P_1 A_N X\right)+\phi\left(P_2 A_N X\right) \\ &\quad +\phi\left(P_0 A_N X\right)-P_2\left(A_N X\right)+\phi Q\left(A_N X\right),\end{aligned} \tag{3.30}$$

$$\phi\left(\nabla_X^{\ell} N\right)=T\left(\nabla_X^{\ell} N\right)+C\left(\nabla_X^{\ell} N\right), \tag{3.31}$$

$$\phi\left(D^s(X, N)\right)=T D^s(X, N)+C D^s(X, N). \tag{3.32}$$

On using the equations from (3.28) to (3.32) in equation (3.27), we get

$$\begin{aligned}\left(\bar{\nabla}_X \phi\right) N &= \left\{P_1 \nabla_X T N-P_1 A_{C N} X+\phi P_1\left(A_N X\right)\right\} \\ &\quad +\left\{P_2\left(\nabla_X T N\right)-P_2\left(A_{C N} X\right)-P_2\left(A_N X\right)\right\} \\ &\quad +\left\{P_0 \nabla_X T N-P_0 A_{C N} X+\phi P_0 A_N X\right\} \\ &\quad +\left\{\phi P_2\left(\nabla_X T N\right)-\phi P_2\left(A_{C N} X\right)+\phi P_2\left(A_N X\right)\right\} \\ &\quad +\left\{Q \nabla_X T N-Q A_{C N} X-T \nabla_X^{\ell} N-T D^s(X, N)\right\} \\ &\quad +\left\{\eta\left(\nabla_X T N\right)-\eta\left(A_{C N} X\right)\right\} V \\ &\quad +\left\{h^{\ell}(X, T N)+\nabla_X^{\ell}(C N)-C \nabla_X^{\ell} N+L A_N X\right\} \\ &\quad +\left\{h^s(X, T N)+D^s(X, C N)-C D^s(X, N)+S A_N X\right\},\end{aligned} \tag{3.33}$$

which implies $\phi Q\left(A_N X\right)=L\left(A_N X\right)+S\left(A_N X\right)$, where $L\left(A_N X\right) \in \Gamma(L)$ and $S\left(A_N X\right) \in \Gamma(S)$. Also using (2.6), we have

$$\left(\bar{\nabla}_X \phi\right) N=\bar{g}\left(P_1 X, N\right) V+\bar{g}\left(P_2 X, N\right) V. \tag{3.34}$$

On using (3.33) in (3.34), we obtain

$$\begin{aligned}\bar{g}\left(P_1 X, N\right) V+\bar{g}\left(P_2 X, N\right) V &= \left\{P_1 \nabla_X T N-P_1 A_{C N} X+\phi P_1\left(A_N X\right)\right\} \\ &\quad +\left\{P_2\left(\nabla_X T N\right)-P_2\left(A_{C N} X\right)-P_2\left(A_N X\right)\right\} \\ &\quad +\left\{P_0 \nabla_X T N-P_0 A_{C N} X+\phi P_0 A_N X\right\} \\ &\quad +\left\{\phi P_2\left(\nabla_X T N\right)-\phi P_2\left(A_{C N} X\right)+\phi P_2\left(A_N X\right)\right\} \\ &\quad +\left\{Q \nabla_X T N-Q A_{C N} X-T \nabla_X^{\ell} N-T D^s(X, N)\right\} \\ &\quad +\left\{\eta\left(\nabla_X T N\right)-\eta\left(A_{C N} X\right)\right\} V \\ &\quad +\left\{h^{\ell}(X, T N)+\nabla_X^{\ell}(C N)-C \nabla_X^{\ell} N+L A_N X\right\} \\ &\quad +\left\{h^s(X, T N)+D^s(X, C N)-C D^s(X, N)+S A_N X\right\}.\end{aligned}$$

Then, the relations from (3.18) to (3.25) are obtained on comparing the components of the vector bundles D_1, D_2, D_0, ϕD_2, $\bar{D}$, $\{V\}$, $\ell tr(TM)$ and $S\left(TM^{\perp}\right)$, respectively. □

Lemma 3. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, for any $X \in \Gamma(TM)$ and $W \in \Gamma(S\left(TM^{\perp}\right))$, the following relations hold*

$$P_1\left\{\nabla_X T W-A_{C W} X+\phi\left(A_N X\right)\right\}=0, \tag{3.35}$$

$$P_2\left\{\nabla_X T W-A_{C W} X-A_W X\right\}=0, \tag{3.36}$$

$$P_0\left\{\nabla_X T W-A_{C W} X+\phi\left(A_W X\right)\right\}=0, \tag{3.37}$$

$$\phi P_2\left\{\nabla_X TW - A_{CW}X + A_W X\right\} = 0, \tag{3.38}$$

$$Q\nabla_X TW - QA_{CW}X - T\nabla^s_X W - TD^\ell(X, W) = 0, \tag{3.39}$$

$$\eta\left(\nabla_X TW - A_{CW}X\right) = 0, \tag{3.40}$$

$$h^\ell(X, TW) - CD^\ell(X, W) + D^\ell(X, CW) + LA_W X = 0, \tag{3.41}$$

$$h^s(X, TW) + \nabla^s_X(CW) - C\left(\nabla^s_X W\right) + SA_W X = 0. \tag{3.42}$$

Proof. Let $X \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$; then using (2.12) and (3.3), it follows that

$$\left(\bar\nabla_X\phi\right)W = \bar\nabla_X(TW) + \bar\nabla_X(CW) + \phi\left(A_W X\right) - \phi\left(\nabla^s_X W\right) - \phi\left(D^\ell(X, W)\right). \tag{3.43}$$

Then, further using (2.10), (2.12), (3.1), (3.3) and (3.4) in equation (3.43), we obtain

$$\begin{aligned}\left(\bar\nabla_X\phi\right)W \;=\;& P_1\left\{\nabla_X(TW) - \left(A_{CW}X\right) + \phi\left(A_W X\right)\right\}\\ &+P_2\left\{\nabla_X(TW) - A_{CW}X - A_W X\right\}\\ &+P_0\left\{\nabla_X(TW) - A_{CW}X + \phi\left(A_W X\right)\right\}\\ &+\phi P_2\left\{\nabla_X TW - A_{CW}X + A_W X\right\}\\ &+\left\{Q\nabla_X TW - QA_{CW}X - T\nabla^s_X W - TD^\ell(X, W)\right\}\\ &+\left\{\eta\left(\nabla_X TW\right) - \eta\left(A_{CW}X\right)\right\}V\\ &+\left\{h^\ell(X, TW) + D^\ell(X, CW) - CD^\ell(X, W) + LA_N X\right\}\\ &+\left\{h^s(X, TW) - C\nabla^s_X W + \nabla^s_X(CW) + S\left(A_N X\right)\right\}.\end{aligned} \tag{3.44}$$

In consequence of (2.6), we know that $\left(\bar\nabla_X\phi\right)W = 0$; then the relations from (3.35) to (3.42) follow immediately on comparing the components of the vector bundles D_1, D_2, D_0, ϕD_2, $\bar D$, $\{V\}$, $\ell tr(TM)$ and $S\left(TM^\perp\right)$, respectively. $\square$

Lemma 4. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar M$. Then, for any $X \in D$ and $Y \in \bar D$, we have the following relations*

$$\nabla_X V = -\varepsilon\phi X, \;\; h^\ell(X, V) = 0, \;\; h^s(X, V) = 0, \tag{3.45}$$

$$\nabla_Y V = 0, \;\; h^\ell(Y, V) = -\varepsilon LY, \;\; h^s(Y, V) = -\varepsilon SY, \tag{3.46}$$

$$\nabla_V V = 0, \;\; h^\ell(V, V) = 0, \; h^s(V, V) = 0. \tag{3.47}$$

Proof. The proof follows immediately by using (2.10), (3.1) and (3.4) in (2.6). $\square$

4. Integrability of the distributions

Theorem 2. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar M$. Then, necessary and sufficient conditions for the radical distribution $Rad(TM)$ to be integrable are the following*

(1) $h^\ell(X, \phi Y) = h^\ell(Y, \phi X)$, $h^s(X, \phi Y) = h^s(Y, \phi X)$, $\forall X, Y \in Rad(TM)$.
(2) $\bar g\left(h^*(X, \phi Y), \phi\tilde Z\right) = \bar g\left(h^*(Y, \phi X), \phi\tilde Z\right)$, $\forall X, Y \in D_2$, $\phi\tilde Z \in \ell tr(TM)$.
(3) $\bar g\left(\nabla^{*t}_X\phi Y, \phi\tilde Z\right) = \bar g\left(\nabla^{*t}_Y\phi X, \phi\tilde Z\right)$, $\forall X, Y \in D_1$, $\phi\tilde Z \in \ell tr(TM)$.

(4) $\bar{g}\left(h^*(X,\phi Y),\phi\tilde{Z}\right)=\bar{g}\left(\nabla^{*t}_Y(\phi X),\phi\tilde{Z}\right), \forall X\in D_1, Y\in D_2, \phi\tilde{Z}\in \ell tr(TM)$.
$\bar{g}\left(h^*(Y,\phi X),\phi\tilde{Z}\right)=\bar{g}\left(\nabla^{*t}_X(\phi Y),\phi\tilde{Z}\right), \forall X\in D_2, Y\in D_1, \phi\tilde{Z}\in \ell tr(TM)$.
(5) $\bar{g}\left(A^*_{\phi Y}X,\phi Z_0\right)=\bar{g}\left(A^*_{\phi X}Y,\phi Z_0\right), \forall X,Y\in D_1, Z_0\in D_0$.
(6) $\bar{g}\left(\nabla^*_X\phi Y,\phi Z_0\right)=\bar{g}\left(\nabla^*_Y(\phi X),\phi Z_0\right), \forall X,Y\in D_2,\ Z_0\in D_0$.
(7) $\bar{g}\left(\nabla^*_X(\phi Y),\phi Z_0\right)=-\bar{g}\left(A^*_{\phi X}Y,\phi Z_0\right), \forall X\in D_1, Y\in D_2,\ Z_0\in D_0$.

Proof. (1) Assume that the radical distribution $RadTM$ is integrable; then, this implies that $[X,Y]\in RadTM$ for any $X,Y\in RadTM$. Using (3.13), we have $h^\ell(X,\phi Y)=L\nabla_XY+Ch^\ell(X,Y)$ and $h^\ell(Y,\phi X)=L\nabla_YX+Ch^\ell(Y,X)$ this further implies that $h^\ell(X,\phi Y)-h^\ell(Y,\phi X)=L(\nabla_XY-\nabla_YX)=L[X,Y]$. If $[X,Y]\in RadTM$, then $L[X,Y]=0$, therefore we get

$$h^\ell(X,\phi Y)=h^\ell(Y,\phi X). \tag{4.1}$$

Similarly as a consequence of (3.14), we also have

$$h^s(X,\phi Y)=h^s(Y,\phi X), \tag{4.2}$$

and $\bar{g}([X,Y],V)=2\varepsilon g(Y,\phi X)=0$. Now using (2.6) for any $X,Y\in RadTM$ and $\tilde{Z}\in\bar{D}$, we get

$\bar{g}\left(\bar{\nabla}_X(\phi Y)+h^\ell(X,\phi Y)+h^s(X,\phi Y)-\nabla_Y(\phi X)-h^\ell(Y,\phi X)-h^s(Y,\phi X)\phi\tilde{Z}\right)=0,$

and then by using (4.1) and (4.2), we obtain

$$0=\bar{g}\left(\nabla_X(\phi Y),\phi\tilde{Z}\right)-\bar{g}\left(\nabla_Y(\phi X),\phi\tilde{Z}\right). \tag{4.3}$$

For $\nabla_X(\phi Y)$, $\nabla_Y(\phi X)\in\Gamma(TM)$ and $\phi\tilde{Z}\in S\left(TM^\perp\right)$, (4.3) is satisfied.

(2) Let $\phi X,\phi Y\in S(TM)$ and $\phi\tilde{Z}\in\ell tr(TM)$, then applying (2.16) in (4.3), we get

$$\begin{aligned}0 &= \bar{g}\left(\nabla^*_X(\phi Y),\phi\tilde{Z}\right)-\bar{g}\left(\nabla^*_Y(\phi X),\phi\tilde{Z}\right)\\ &\quad+\bar{g}\left(h^*(X,\phi Y),\phi\tilde{Z}\right)-\bar{g}\left(h^*(Y,\phi X),\phi\tilde{Z}\right)\end{aligned} \tag{4.4}$$

where $\nabla^*_X(\phi Y),\nabla^*_Y(\phi X)\in S(TM)$ and $h^*(X,\phi Y),\ h^*(Y,\phi X)\in Rad(TM)$. Hence, (4.4) becomes

$$\bar{g}\left(h^*(X,\phi Y),\phi\tilde{Z}\right)=\bar{g}\left(h^*(Y,\phi X),\phi\tilde{Z}\right). \tag{4.5}$$

(3) Similarly, for $\phi X,\phi Y\in S(TM)$ and $\phi\tilde{Z}\in\ell tr(TM)$, using (2.17) in (4.3), we get

$$0=\bar{g}\left(A^*_{\phi X}Y,\phi\tilde{Z}\right)-\bar{g}\left(A^*_{\phi Y}X,\phi\tilde{Z}\right)-\bar{g}\left(A^{*t}_Y\phi X,\phi\tilde{Z}\right)+\bar{g}\left(A^{*t}_X\phi Y,\phi\tilde{Z}\right), \tag{4.6}$$

where $A^*_{\phi X}Y,A^*_{\phi Y}X\in S(TM)$ and $\nabla^{*t}_Y\phi X,\nabla^{*t}_X\phi Y\in Rad(TM)$. Hence, (4.6) becomes

$$\bar{g}\left(\nabla^{*t}_X\phi Y,\phi\tilde{Z}\right)=\bar{g}\left(\nabla^{*t}_Y\phi X,\phi\tilde{Z}\right). \tag{4.7}$$

(4) Let $\phi X \in Rad\,(TM)$, $\phi Y \in S\,(TM)$ and $\phi\tilde{Z} \in \ell tr\,(TM)$; then using (2.16) and (2.17) in (4.3), we get

$$\bar{g}\left(h^*\left(X, \phi Y\right), \phi\tilde{Z}\right) = \bar{g}\left(\nabla_Y^{*t}\left(\phi X\right), \phi\tilde{Z}\right). \tag{4.8}$$

Similarly, by using $\phi X \in S\,(TM)$, $\phi Y \in Rad\,(TM)$ and $\phi\tilde{Z} \in \ell tr\,(TM)$, with the help of (2.16) and (2.17) in (4.3), it follows that

$$\bar{g}\left(h^*\left(Y, \phi X\right), \phi\tilde{Z}\right) = \bar{g}\left(\nabla_X^{*t}\left(\phi Y\right), \phi\tilde{Z}\right). \tag{4.9}$$

(5) For $X, Y \in D_1$ and $Z_0 \in D_0$, using (2.6) and (2.17), we have the following relation immediately

$$\bar{g}\left(A^*_{\phi X} Y, \phi Z_0\right) = \bar{g}\left(A^*_{\phi Y} X, \phi Z_0\right). \tag{4.10}$$

(6) For $X, Y \in D_2$ and $Z_0 \in D_0$, by the use of (2.6) and (2.16), we have

$$\bar{g}\left(\nabla_X^*\left(\phi Y\right), \phi Z_0\right) = \bar{g}\left(\nabla_Y^*\left(\phi X\right), \phi Z_0\right). \tag{4.11}$$

(7) Finally, let $X \in D_1$, $Y \in D_2$ and $Z_0 \in D_0$; then, using (2.6), (2.16) and (2.17), we obtain

$$\bar{g}\left(\nabla_X^*\left(\phi Y\right), \phi Z_0\right) = -\bar{g}\left(A^*_{\phi X} Y, \phi Z_0\right). \tag{4.12}$$

If we take $X, Y \in D_2$ and $Z \in \phi D_2$, then on applying(2.17) and (2.16), we get

$$\bar{g}\left(h^*\left(X, \phi Y\right), \phi Z\right) - \bar{g}\left(h^*\left(Y, \phi X\right), \phi Z\right) = 0, \tag{4.13}$$

where $\bar{g}\left(h^*\left(X, \phi Y\right), \phi Z\right) = 0 = \bar{g}\left(h^*\left(Y, \phi X\right), \phi Z\right)$, for all $\phi Z \in D_2$, this implies (4.13) holds. For $X, Y \in D_1$ and $Z \in \phi D_2$ with the help of (2.6) and (2.17) it is possible to get $g\left(\nabla_X^{*t}\phi Y, \phi Z\right) = g\left(\nabla_Y^{*t}\phi X, \phi Z\right) = 0$. For $X \in D_1$, $Y \in D_2$ and $Z \in \phi D_2$ by using (2.6) and (2.16) and (2.17), we get $\bar{g}\left(h^*\left(X, \phi Y\right), \phi Z\right) = 0 = \bar{g}\left(h^*\left(Y, \phi X\right), \phi Z\right)$, for all $\phi Z \in D_2$. $\square$

Theorem 3. *Let $(M, g, S\,(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, the distribution D_0 is never integrable.*

Proof. Assume that the distribution D_0 is integrable, then $g\left(\left[X, Y\right], V\right) = 0$ for any $X, Y \in D_0$. By using (2.6), we obtain $\bar{g}\left(\left[X, Y\right], V\right) = 2\varepsilon\bar{g}\left(Y, \phi X\right)$ for $X, Y \in D_0$, then by using the above relation, we have $\bar{g}\left(Y, \phi X\right) = 0$. Since D_0 is non-degenerate then $\bar{g}\left(Y, \phi X\right) \neq 0$. This leads to a contradiction and hence the assertion follows. $\square$

Lemma 5. *Let $(M, g, S\,(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, $[X, V] \in D \oplus \{V\}$, for any $X \in D$.*

Proof. Let $X \in D$ and $Y \in \bar{D}$; then using (3.47), we get $\bar{g}\left(\left[X, V\right], Y\right) = \varepsilon\bar{g}\left(X, \phi Y\right) - \bar{g}\left(\nabla_V X, Y\right)$. Particularly, on taking $\phi Y \in S\left(TM^{\perp}\right)$, we have $\bar{g}\left(\left[X, V\right], Y\right) = -\bar{g}\left(\nabla_V X, Y\right)$ and further on putting $X = \phi X$, then using (2.6), (2.12), (2.13) and (3.47), we obtain $\bar{g}\left(\left[\phi X, V\right], Y\right) = \bar{g}\left(h^s\left(V, X\right), \phi Y\right) = 0$. In particular, if we take $\phi Y \in \ell tr\,(TM)$ then by using (2.6), (2.15) and (2.11), we get $\bar{g}\left(\left[\phi X, V\right], Y\right) = \bar{g}\left(\phi Y, \nabla_V X\right) = 0$. We know that if $X \in D$ then this implies that $\phi X \in D$ therefore $\bar{g}\left(\left[X, V\right], Y\right) = 0$ implies that $[X, V] \in D \oplus \{V\}$ for $X \in D$. $\square$

Theorem 4. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, necessary and sufficient conditions for $D \oplus \{V\}$ to be integrable are $h^{\ell}(X, \phi Y) = h^{\ell}(\phi X, Y)$ and $h^{s}(X, \phi Y) = h^{s}(\phi X, Y)$, for any $X, Y \in D \oplus \{V\}$.*

Proof. Assume that $D \oplus \{V\}$ is integrable then $[X, Y] \in D \oplus \{V\}$, for any $X, Y \in D \oplus \{V\}$. We know that for any $X, Y \in D \oplus \{V\}$, we can write $X = PX + \eta(X) V$ and $Y = PY + \eta(Y) V$, where $PX, PY \in D$. Using these relations, we obtain $[X, Y] - \eta(Y)[PX, V] - \eta(X)[V, PY] = [PX, PY]$. Since $[X, Y]$, $[PX, V]$, $[V, PY] \in D \oplus \{V\}$ then $[PX, PY] \in D \oplus \{V\}$. Thus $Q[PX, PY] = 0$ implies $L[PX, PY] = 0 = S[PX, PY]$. On using these equalities in (3.13), our assertion follows. □

Theorem 5. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, necessary and sufficient condition for the distribution $\bar{D}$ to be integrable is that $A_{\phi X} Y = A_{\phi Y} X$ for $X, Y \in \bar{D}$.*

Proof. Let $\phi X \in S(TM^{\perp})$, $Y \in \bar{D}$ and $Z \in S(TM)$, then using (2.6), (2.10) and (2.13), it follows that

$$g(A_{\phi X} Y, Z) = -\bar{g}\left(\bar{\nabla}_Z(\phi Y), X\right). \tag{4.14}$$

For $\phi Y \in S(TM^{\perp})$, by the use of (2.13) in (4.14), we have

$$g(A_{\phi X} Y, Z) = g(A_{\phi Y} X, Z). \tag{4.15}$$

Also for $\phi Y \in \ell tr(TM)$, by using (2.17) in (4.14), we have

$$g(A_{\phi X} Y, Z) = g(A_{\phi Y} X, Z). \tag{4.16}$$

Hence by the use of (4.15) and (4.16), for any $Z \in S(TM)$, we obtain $A_{\phi X} Y = A_{\phi Y} X$. If $\phi X \in \ell tr(TM)$, $Y \in \bar{D}$ and $Z \in S(TM)$ then from (2.6), (2.16) and (2.17), it follows that $g(A_{\phi X} Y, Z) = \bar{g}(\nabla_Z X, \phi Y)$. Furthermore, particularly on taking $\phi Y \in \ell tr(TM)$ and using (2.16), we obtain $g(A_{\phi X} Y, Z) = \bar{g}(h^*(X, Z), \phi Y) = g(A_{\phi Y} X, Z)$. If particularly we take $\phi X \in S(TM^{\perp})$, $Y \in \bar{D}$ and $Z \in Rad(TM)$, then using (2.6), (2.10), (2.12) and (2.13), we get

$$\begin{aligned} g(A_{\phi X} Y, Z) &= \bar{g}(h^s(Y, Z), \phi X) + \bar{g}\left(Z, D^{\ell}(Y, \phi X)\right) \\ &= \bar{g}(h^s(X, Z), \phi Y) + \bar{g}\left(Z, D^{\ell}(Y, \phi X)\right). \end{aligned} \tag{4.17}$$

Now, for any $\phi Y \in S(TM^{\perp})$, using (3.13), we get $D^{\ell}(X, \phi Y) = D^{\ell}(Y, \phi X)$. Then, using (4.17) for any $Z \in S(TM)$, we obtain $g(A_{\phi X} Y, Z) = \bar{g}(A_{\phi Y} X, Z)$.

If $\phi X \in \ell tr(TM)$, $Y \in \bar{D}$ and $Z \in S(TM)$; then using (2.6), (2.16) and (2.17), it yields

$$\begin{aligned} g(A_{\phi X} Y, Z) &= \bar{g}(\phi X, \nabla^*_Y Z) + \bar{g}\left(\nabla^{\ell}_Y(\phi X), Z\right) \\ &= -\bar{g}(X, \nabla_Y(\phi Z)) + \bar{g}\left(\nabla^{\ell}_Y(\phi X), Z\right). \end{aligned} \tag{4.18}$$

On applying (2.16) and (2.17) in (4.18) we get $g(A_{\phi X} Y, Z) = \bar{g}\left(\nabla^{\ell}_Y(\phi X), Z\right)$ and further on taking $\phi Y \in \ell tr(TM)$ and using (3.13), we obtain $g(A_{\phi X} Y, Z) =$

$\bar{g}\left(\nabla^{\ell}_{\phi X}(Y), Z\right) = g\left(A_{\phi Y}X, Z\right)$. This implies that on considering $Z \in Rad\,(TM)$, we have $A_{\phi X}Y = A_{\phi Y}X$.

Conversely, let $X, Y \in \bar{D}$; then using $\phi P\nabla_X Y = \phi\left(P_1\nabla_X Y\right) + \phi\left(P_2\nabla_X Y\right) + \phi\left(P_0\nabla_X Y\right) - P_2\nabla_X Y$ and applying (2.6) and (2.10),by the use of the equations from (3.2) to (3.5), it is possible to have

$$\begin{aligned} \bar{\nabla}_X \phi Y &= \bar{g}(X,Y)V + \phi P\nabla_X Y + L\nabla_X Y + S\nabla_X Y \\ &\quad + Th^{\ell}(X,Y) + Ch^{\ell}(X,Y) + Th^{s}(X,Y) + Ch^{s}(X,Y). \quad (4.19) \end{aligned}$$

For $\phi X, \phi Y \in \ell tr\,(TM)$, by using (2.11) in (4.19), we also have

$$\begin{aligned} -A_{\phi Y}X + \nabla^{\ell}_X(\phi Y) + D^{s}(X,\phi Y) &= \bar{g}(X,Y)V + \phi P\nabla_X Y + L\nabla_X Y \\ &\quad + S\nabla_X Y + Th^{\ell}(X,Y) + Ch^{\ell}(X,Y) \\ &\quad + Th^{s}(X,Y) + Ch^{s}(X,Y). \quad (4.20) \end{aligned}$$

On separating the tangential and transversal components of (4.20), we obtain

$$A_{\phi Y}X = -\bar{g}(X,Y)V - \phi P\nabla_X Y - Th^{\ell}(X,Y) - Th^{s}(X,Y), \quad (4.21)$$

$$\nabla^{\ell}_X(\phi Y) + D^{s}(X,\phi Y) = L\nabla_X Y + S\nabla_X Y + Ch^{\ell}(X,Y) + Ch^{s}(X,Y). \quad (4.22)$$

From (4.21), we get $A_{\phi Y}X - A_{\phi X}Y = -\phi P[X,Y]$. Since $A_{\phi Y}X = A_{\phi X}Y$; then we get $P[X,Y] = 0$ and this implies that $[X,Y] \in \bar{D} \oplus \{V\}$. Therefore, $\bar{g}([X,Y],V) = -g(Y,\nabla_X V) + g(X,\nabla_Y V) = 0$, for any $X, Y \in \bar{D}$, $\phi X, \phi Y \in \ell tr\,(TM)$ and $[X,Y] \in \bar{D}$. Similarly, for any $\phi X, \phi Y \in S\left(TM^{\perp}\right)$, by using (2.12) in (4.19), we obtain

$$\begin{aligned} -A_{\phi Y}X + \nabla^{s}_X(\phi Y) + D^{\ell}(X,\phi Y) &= \bar{g}(X,Y)V + \phi P\nabla_X Y + L\nabla_X Y \\ &\quad + S\nabla_X Y + Th^{\ell}(X,Y) + Th^{s}(X,Y) \\ &\quad + Ch^{\ell}(X,Y) + Ch^{s}(X,Y). \quad (4.23) \end{aligned}$$

On separating the tangential and transversal components of (4.23), we get

$$A_{\phi Y}X = -\bar{g}(X,Y)V - \phi P\nabla_X Y - Th^{\ell}(X,Y) - Th^{s}(X,Y), \quad (4.24)$$

$$\nabla^{s}_X(\phi Y) + D^{\ell}(X,\phi Y) = L\nabla_X Y + S\nabla_X Y + Ch^{\ell}(X,Y) + Ch^{s}(X,Y). \quad (4.25)$$

From (4.24), we get $A_{\phi Y}X - A_{\phi X}Y = -\phi P[X,Y]$. Since $A_{\phi Y}X = A_{\phi X}Y$; then $P[X,Y] = 0$. We know that $\bar{g}([X,Y],V) = 0$; therefore, for any $X, Y \in \bar{D}$, $\phi X, \phi Y \in S\left(TM^{\perp}\right)$, it follows that $[X,Y] \in \bar{D}$. Hence, the proof is complete. □

Theorem 6. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, the distribution D defines a totally geodesic foliation in M if $Th(X,Y) = 0$ for any $X, Y \in \Gamma(D)$.*

Proof. From the Definition 1, for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D)$, $W \in \Gamma(S)$, we have $g(\nabla_X Y, \phi Z) = g(\nabla_X Y, \phi W) = 0$. Particularly, from (2.6) and (2.10), for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D_1) \subset Rad\,(TM)$, it follows that

$$g(\nabla_X Y, \phi Z) = -g\left(h^{\ell}(X,\phi Y), Z\right) = 0. \quad (4.26)$$

Similarly, using (2.10) and (2.3), for any $X, Y \in \Gamma(D)$, $W \in \Gamma(S)$, we have

$$g(\nabla_X Y, \phi W) = -g\left(h^{s}(X,\phi Y), W\right) = 0. \quad (4.27)$$

Thus, from (4.26) and (4.27), it is clear that if the distribution D defines a totally geodesic foliation in M then $h^s(X, \phi Y)$ and $h^\ell(X, \phi Y)$ have no components in S and L, respectively. Thus, using these results with (3.2) and (3.3), the proof is complete. □

Theorem 7. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, the distribution $\bar{D}$ defines a totally geodesic foliation in M if and only if $A_{\phi Y}X \in \Gamma(\bar{D})$ for any $X, Y \in \Gamma(\bar{D})$.*

Proof. For the elements $X, Y \in \Gamma(\bar{D})$, by using (3.4), we obtain $\phi(\bar{\nabla}_X Y) = \phi(P\bar{\nabla}_X Y) + \phi(Q\bar{\nabla}_X Y)$. If we set $\phi(P\bar{\nabla}_X Y) = T(\bar{\nabla}_X Y)$ and $\phi(Q\bar{\nabla}_X Y) = C\bar{\nabla}_X Y$; then, by the use of (2.6), (2.8) and (2.9), we further have $-A_{\phi Y}X = T\nabla_X Y - Th(X, Y)$ for any $X, Y \in \Gamma(\bar{D})$. Assume that the distribution $\bar{D}$ is a totally geodesic foliation in M; then, it follows that $A_{\phi Y}X = -Th(X, Y)$. Therefore, $A_{\phi Y}X \in \Gamma(\bar{D})$ for any $X, Y \in \Gamma(\bar{D})$. Conversely, let $A_{\phi Y}X \in \Gamma(\bar{D})$, for any $X, Y \in \Gamma(\bar{D})$ then this implies that $T\nabla_X Y = 0$ and hence $\nabla_X Y \in \Gamma(\bar{D})$. □

Definition 2. *A GCR-lightlike submanifold M is called D-geodesic if $h(X, Y) = 0$, for any $X, Y \in \Gamma(D)$. Using the decomposition of the transversal vector bundle, GCR-lightlike submanifold M is said to be a D-geodesic if $h^\ell(X, Y) = 0$ and $h^s(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. Also, M is said to be a mixed geodesic if $h^\ell(X, Y) = 0$ and $h^s(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(\bar{D})$.*

Theorem 8. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, the following assertions are equivalent*

(1) *M is mixed totally geodesic.*
(2) *$\nabla^s_D(\phi\bar{D}) \subset \phi\bar{D}$ and $A_{\phi\bar{D}}D \subset D$.*

Proof. Choose $Y \in \bar{D}$ such that $\phi Y \in S(TM^\perp)$; then, there exists a $W \in S(TM^\perp)$ such that $\phi W = TW = Y$. Let $X \in D$ and $W \in S(TM^\perp)$; then, we have $h^s(X, Y) = C(\nabla^s_X W) - S(A_W X)$. Using the hypothesis that M is a mixed totally geodesic; then, for $X \in D$ and $Y \in \bar{D}$, $h^s(X, Y) = 0$ holds and we further obtain $C(\nabla^s_X W) = S(A_W X)$ where $S(A_W X) \in \Gamma(S) \subset S(TM^\perp)$ and $C(\nabla^s_X W) \in \Gamma(S^\perp) \subset S(TM^\perp)$. For any $\nabla^s_X W \in S(TM^\perp)$, on using (3.3), we have

$$\phi(C\nabla^s_X W) = -\nabla^s_X W - \phi(T\nabla^s_X W), \tag{4.28}$$

where $\nabla^s_X W \in S(TM^\perp)$ and $\phi(T\nabla^s_X W) \in \phi\bar{D} \subset S(TM^\perp)$. Using (4.28), it follows that $C\nabla^s_X W \in \{S(TM^\perp) - \phi\bar{D}\}$. Since $S(A_W X) \in \phi\bar{D}$; then, using (4.27), we get $C(\nabla^s_X W) = 0$ and $S(A_W X) = 0$. Thus, from (4.27) and (4.28), we have $\nabla^s_X W = \phi(T\nabla^s_X W), \nabla^s_X W \in \phi\bar{D}$ and $A_W X \in D$, for any $X \in D$ and $W \in \phi\bar{D}$. Consequently, we obtain that $\nabla^s_D\phi\bar{D} \subset \phi\bar{D}$ and $A_{\phi\bar{D}}D \subset D$.

Next, choose $Y \in \bar{D}$ such that there exists a $N \in \ell tr(TM)$ such that $\phi N = TN = Y$, $CN = 0$. Using (3.26), for any $X \in D$ and $N \in \ell tr(TM)$, it follows that $h^\ell(X, Y) = C\nabla^\ell_X N - LA_N X$. Assume that M is the mixed totally geodesic; then, $h^\ell(X, Y) = 0$ where $X \in D$, $Y \in \bar{D}$, therefore we further obtain

$$C\nabla^\ell_X N = LA_N X, \tag{4.29}$$

where $C\left(\nabla^{\ell}_X N\right) \in \Gamma\left(L^{\perp}\right) \subset \ell tr\,(TM)$ and $LA_N X \in \Gamma\,(L) \subset \ell tr\,(TM)$. For $\nabla^{\ell}_X N \in \ell tr\,(TM)$, from (3.2), we also have

$$\phi\left(C\nabla^{\ell}_X N\right) = -\nabla^{\ell}_X N - \phi\left(T\nabla^{\ell}_X N\right), \tag{4.30}$$

where $\nabla^{\ell}_X N \in \ell tr\,(TM)$ and $\phi\left(T\nabla^{\ell}_X N\right) \in \phi\bar{D} \subset \ell tr\,(TM)$. From (4.30), it is obvious that $\phi\left(C\nabla^{\ell}_X N\right) \notin \bar{D}$, this implies that $C\nabla^{\ell}_X N \notin \phi\bar{D}$, that is, $C\nabla^{\ell}_X N \in \left\{\ell tr\,(TM) - \phi\bar{D}\right\}$. Since $L\,(A_N X) \in \phi D$ therefore from (4.29), we get $C\nabla^{\ell}_X N = 0$ and $LA_N X = 0$. As a conclusion, we obtain $\nabla^{\ell}_X N \in \phi\bar{D}$ and $A_N X \in D$, for any $X \in D$ and $N \in \phi\bar{D} \subset \ell tr\,(TM)$. Consequently, we have $\nabla^{\ell}_D \phi\bar{D} \subset \phi\bar{D}$ and $A_{\phi\bar{D}} D \subset D$. Hence the proof is complete. □

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MORE IDENTITIES FOR FIBONACCI AND LUCAS QUATERNIONS

NURETTIN IRMAK

Abstract. In this paper, we define the associate matrix as

$$F = \begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix}.$$

By the means of the matrix F, we give several identities about Fibonacci and Lucas quaternions by matrix methods. Since there are two different determinant definitions of a quaternion square matrix (whose entries are quaternions), we obtain different Cassini identities for Fibonacci and Lucas quaternions apart from Cassini identities that given in the papers [5] and [7].

1. Introduction

The quaternions were described by Irish mathematicians Sir William and Rowan Hamilton as a extension of a complex number. The set of quaternion is defined by

$$H = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_n \in \mathbb{Z},\ n = 0, 1, 2, 3\}$$

where $i^2 = j^2 = k^2 = -1 = ijk$. This imply that $ij = k = -ji,\ jk = i = -kj$ and $ki = j = -ik$. The set of all quaternions form are associate but not commutative algebra. We can write

$$q = a_0 + u$$

where $u = ia_1 + ja_2 + ka_3$. The conjugate of the quaternion q is denoted by q^* and defined by $q^* = q - u$. Namely, the conjugate of the quaternion q is $q^* = a_0 - ia_1 - ja_2 - ka_3$.

For $n \geq 2$, the Fibonacci and Lucas sequences are defined as

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1$$

and

$$L_n = L_{n-1} + L_{n-2},\ L_0 = 2,\ L_1 = 1,$$

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respectively. There are lots of amazing identities belongs to Fibonacci and Lucas numbers. For the details, we refer the book of T. Koshy ([1]).

A. F. Horadam [2] defined nth Fibonacci and Lucas quaternions as follows,

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

and

$$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}.$$

The conjugates of these quaternions are given by

$$\widetilde{Q}_n = F_n - iF_{n+1} - jF_{n+2} - kF_{n+3}$$

and

$$\widetilde{K}_n = L_n - iL_{n+1} - jL_{n+2} - kL_{n+3}.$$

There are several researchers who focus on this Fibonacci and Lucas quaternions. Swamy [4] gave interesting identities for Fibonacci quaternions. Iyer [3] established some relations about Fibonacci and Lucas quaternions. Binet formula and generating functions of Fibonacci quaternions was given by Halıcı [5]. Akyiğit et al. [6] gave the definition of split Fibonacci quaternions together with their properties. Afterwards, they gave Fibonacci generalized quaternions and they used the well-known identities related to the Fibonacci and Lucas numbers to obtain the relations regarding these quaternions in [7]. Another type generalization was given by Tan et. al. [8], [9]. They defined bi-periodic Fibonacci and Lucas quaternions.

We use the determinant of quaternion matrix whose entries are quaternions. Since the set of all quaternions are not commutative, we give the definitions of the determinant of a quaternions matrix. Let A be a quaternion square matrix. Denote A by,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{ij} \in H$ for $i = 1, 2$ and $j = 1, 2$. The determinant of A, $\det A$, is defined by

$$\begin{aligned} \det A &= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned} \tag{1}$$

The above definition is called rule "multiplication from above to down below". Since the set of all quaternion is not commutative, another product direction can be defined. Namely, the definition

$$\begin{aligned} \det A &= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= a_{22}a_{11} - a_{21}a_{12} \end{aligned} \tag{2}$$

is called the rule "multiplication from down below to above" (For details, see the book [11], section 9.11.)

In this paper, we present some novel identities between Fibonacci and Lucas quaternions by using matrix method. Thanks to the this method, identities belongs to Fibonacci and Lucas quaternions can be obtained easily together with the properties of matrices. Before going further, we define the following two matrices,

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; F = \begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix}. \tag{3}$$

It is known that

$$U^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \tag{4}$$

2. Main Results

First theorem is about the Cassini identity belongs to Fibonacci and Lucas quaternions. We obtain the two versions of Cassini identity since there are two different determinant definitions of quaternion matrix. The first type Cassini identity for Fibonacci quaternions was given by Halıcı ([5], Theorem 3.4) and Akyiğit et. al. [7] gave the first type Cassini identity for Fibonacci and Lucas generalized quaternions. For both identities, they used the determinant definition of (2). We get the second type Cassini identity for the definition of (1).

Theorem 1. *(First type Cassini identity) For* $n \geq 1$, *the identities*

$$Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n (2Q_1 - 3k)$$

and

$$K_{n-1}K_{n+1} - K_n^2 = 5(-1)^{n-1} (2Q_1 - 3k).$$

Proof. By the matrices in (3) and (4), we obtain that

$$\begin{aligned} U^n F &= \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix} && (5) \\ &= \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}. && (6) \end{aligned}$$

Using the second definition of determinant (2) for the equation (5), we get

$$\begin{aligned} Q_{n-1}Q_{n+1} - Q_n^2 &= \left(F_{n+1}F_{n-1} - F_n^2\right) \\ &\quad \times \left((1+j+k)(1+i+2j+3k) - (i+j+2k)^2\right) \\ &= (-1)^n (2Q_1 - 3k) \end{aligned}$$

as claimed. Since $F_{n-1} + F_{n+1} = L_n$, then we write

$$\begin{aligned} \left(U^{n-1} + U^{n+1}\right) F &= \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} \begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix} \\ &= \begin{pmatrix} K_{n+1} & K_n \\ K_n & K_{n-1} \end{pmatrix}. \end{aligned}$$

Applying the equation (2) gives that

$$\begin{aligned} K_{n-1}K_{n+1}-K_n^2 &= \left(L_{n+1}L_{n-1}-L_n^2\right) \\ &\quad \times \Big(\left(1+j+k\right)\left(1+i+2j+3k\right)-\left(i+j+2k\right)^2\Big) \\ &= 5\left(-1\right)^{n-1}\left(2Q_1-3k\right). \end{aligned}$$

□

The first type Cassini identity for the generalized bi-periodic Fibonacci quaternions was given by Tan et. al [10].

Theorem 2. *(Second type Cassini identity) For* $n \geq 1$, *the identities*

$$Q_{n+1}Q_{n-1}-Q_n^2=\left(-1\right)^n\left(2+2j+5k\right)$$

and

$$K_{n+1}K_{n-1}-K_n^2=5\left(-1\right)^{n-1}\left(2+2j+5k\right).$$

Proof. Since the equation (5) holds, then the definition (1) yields that

$$\begin{aligned} Q_{n+1}Q_{n-1}-Q_n^2 &= \left(F_{n-1}F_{n+1}-F_n^2\right) \\ &\quad \times \Big(\left(1+i+2j+3k\right)\left(1+j+k\right)-\left(i+j+2k\right)^2\Big) \\ &= \left(-1\right)^n\left(2+2j+5k\right). \end{aligned}$$

Similarly, one can see that

$$K_{n+1}K_{n-1}-K_n^2=5\left(-1\right)^{n-1}\left(2+2j+5k\right)$$

holds. □

Theorem 3. *For* $n, m \geq 1$ *integers, then*

$$Q_{m+n}=F_{m+1}Q_n+F_mQ_{n-1}$$

follows.

Proof. Since $U^{m+n}F=U^m\left(U^nF\right)$ holds, then we obtain that

$$\begin{aligned} U^{m+n}F &= \begin{pmatrix} Q_{m+n+1} & Q_{m+n} \\ Q_{m+n} & Q_{m+n-1} \end{pmatrix} \\ &= U^mU^nF \\ &= \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}. \end{aligned} \tag{7}$$

Equating the first row second column entries of two matrix in (7) is yields that

$$Q_{m+n}=F_{m+1}Q_n+F_mQ_{n-1}.$$

□

Since $F_{m+2}+F_m=F_{m+1}$ and $Q_{m+n+1}+Q_{m+n-1}=K_{m+n}$ holds, we get the following identity as a corollary.

Corollary 1. *For positive integers m and n, we get*

$$K_{m+n} = L_{m+1}Q_n + L_m Q_{n-1} \tag{8}$$

Theorem 4. *For $n, m \geq 1$ integers, we have*

$$Q_{m+1}Q_{n+1} + Q_m Q_n = Q_{m+n+1} + iQ_{m+n+2} + jQ_{m+n+3} + kQ_{m+n+4}.$$

Proof. The fact $F\left(U^{m+n}F\right) = \left(FU^m\right)\left(U^n F\right)$ yields that

$$\begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix}\begin{pmatrix} Q_{m+n+1} & Q_{m+n} \\ Q_{m+n} & Q_{m+n-1} \end{pmatrix} \tag{9}$$

$$= \begin{pmatrix} Q_{m+1} & Q_m \\ Q_m & Q_{m-1} \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix} \tag{10}$$

If we equalize the pivot elements in the equation (9), we obtain claimed result. □

The equation (8) give the following system,

$$\begin{pmatrix} K_{m+n+1} & K_{m+n} \\ K_{m+n} & K_{m+n-1} \end{pmatrix} = \begin{pmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}. \tag{11}$$

If we multiply the equation (11) with the matrix F from left side, we get that

$$\begin{aligned} & \begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix}\begin{pmatrix} K_{m+n+1} & K_{m+n} \\ K_{m+n} & K_{m+n-1} \end{pmatrix} \\ = & \left(\begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix}\begin{pmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{pmatrix}\right)\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix} \\ = & \begin{pmatrix} K_{m+1} & K_m \\ K_m & K_{m-1} \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}. \end{aligned}$$

Equating the first row and second column element, we obtain the following theorem.

Theorem 5. *For $m, n \geq 1$, we get*

$$K_{m+n+1} + iK_{m+n+2} + jK_{m+n+3} + kK_{m+n+4} = K_{m+1}Q_{n+1} + K_n Q_n.$$

Let define the conjugate matrix of F as

$$\widetilde{F} = \begin{pmatrix} 1-i-2j-3k & i-j-2k \\ i-j-2k & 1-j-k \end{pmatrix}.$$

We present the first and second type Cassini identities for the conjugate Fibonacci and Lucas quaternions.

Theorem 6. *The identities*

$$\begin{aligned} \widetilde{Q}_{n-1}\widetilde{Q}_{n+1} - \widetilde{Q}_n^2 &= (-1)^n(2-2j-5k) \\ \widetilde{Q}_{n+1}\widetilde{Q}_{n-1} - \widetilde{Q}_n^2 &= (-1)^n\left(2\widetilde{Q}_1 + 3k\right) \\ \widetilde{K}_{n-1}\widetilde{K}_{n+1} - \widetilde{K}_n^2 &= 5(-1)^{n-1}(2-2j-5k) \end{aligned}$$

and

$$\widetilde{K}_{n+1}\widetilde{K}_{n-1}-\widetilde{K}_n^2=5\,(-1)^{n-1}\left(2\widetilde{Q}_1+3k\right)$$

hold for $n\geq 1$.

Proof. The identity

$$\begin{aligned}\widetilde{F}U^n &= \begin{pmatrix}1-i-2j-3k & i-j-2k\\ i-j-2k & 1-j-k\end{pmatrix}\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}\\ &= \begin{pmatrix}\widetilde{Q}_{n+1} & \widetilde{Q}_n\\ \widetilde{Q}_n & \widetilde{Q}_{n-1}\end{pmatrix}.\end{aligned}$$

We get the first and second identities after taking the determinant under applying the rules (2) and (1), respectively. By the way of the proofs of Theorem 1 and Theorem 2, one can see the other identities easily. □

By using the conjugate matrix of F together with the matrix F, we get the following identities. We equalize the first row and second column element to obtain these identities,

Theorem 7. *For the integers* m *and* n,

1) The identity $\widetilde{F}\,(U^{m+n}F)=\left(\widetilde{F}U^m\right)(U^nF)$ yields that

$$Q_{m+n+1}-iQ_{m+n+2}-jQ_{m+n+3}-kQ_{m+n+4}=\widetilde{Q}_{m+1}Q_{n+1}+\widetilde{Q}_mQ_n.$$

2) The identity $\left(\widetilde{F}U^{m+n}\right)F=\left(\widetilde{F}U^m\right)(U^nF)$ gives

$$\widetilde{Q}_{m+n+1}+i\widetilde{Q}_{m+n+2}+j\widetilde{Q}_{m+n+3}+k\widetilde{Q}_{m+n+4}=\widetilde{Q}_{m+1}Q_{n+1}+\widetilde{Q}_mQ_n.$$

3) By the identities

$$\left(FU^{m+n}\right)\widetilde{F}=(FU^m)\left(U^n\widetilde{F}\right)\text{ and }F\left(Q^{m+n}\widetilde{F}\right)=(FQ^m)\left(Q^n\widetilde{F}\right),$$

$$\begin{aligned}\widetilde{Q}_{m+n+1}+i\widetilde{Q}_{m+n+2}+j\widetilde{Q}_{m+n+3}+k\widetilde{Q}_{m+n+4} &= Q_{m+1}\widetilde{Q}_{n+1}+Q_m\widetilde{Q}_n\\ Q_{m+n+1}-iQ_{m+n+2}-jQ_{m+n+3}-kQ_{m+n+4} &= Q_{m+1}\widetilde{Q}_{n+1}+Q_m\widetilde{Q}_n\end{aligned}$$

hold, respectively.

3. Open Question

There are several divisibility identities for Fibonacci and Lucas number. For m,n positive integers, the well-known identities are

$$n|m\Longleftrightarrow F_n|F_m$$

and

$$n=km\Leftrightarrow L_m|L_n\quad\text{where } k \text{ is odd integer.}$$

Which conditions are sufficient and necessary for the elements $\frac{Q_m}{Q_n}$ and $\frac{K_m}{K_n}$ to be Fibonacci and Lucas quaternions?

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ON NEW BÉZIER BASES WITH SCHURER POLYNOMIALS AND CORRESPONDING RESULTS IN APPROXIMATION THEORY

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Abstract. A new type Bézier bases with λ shape parameters have been defined [30, Ye et al., 2010]. We slightly modify these bases to establish new Bézier bases with Schurer polynomials and λ shape parameters. We construct a new type Schurer operators via defined new Bézier-Schurer bases. Also, we study statistical convergence properties of these operators and obtain an estimate for the rate of weighted A-statistical convergence. Moreover, we prove two Voronovskaja-type theorems including a Voronovskaja-type approximation theorem using weighted A-statistical convergence.

1. Extended Bézier bases

In computer aided geometric design and computer graphics parametric representations of surfaces and curves have extensively been used for modeling miscellaneous surfaces. It is important which basis functions are used if we want to preserve the shape of the curve or surface when we demonstrate a parametric surface or curve. This is why Bernstein-Bézier curve and surface representation have an important role in computer graphics. Bernstein basis functions are used to construct classical Bézier curves since they have a simple structure to use. They have also received attention for their utility in the meshing of curved geometries and the numerical solution of partial differential equations. We refer to [15, 22, 29] for recent computer graphics studies including Bézier curves or bases.

A new type Bézier bases with shape parameters λ were defined by Ye et al. in 2010 [30]. We slightly modify these bases to establish new Bézier bases with Schurer polynomials, which were defined in [25], and shape parameters λ.

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Let $d \geq 0$ be a given integer and shape parameters $\lambda \in [-1,1]$. We define the following Bézier-Schurer bases

$$\begin{aligned}
\tilde{s}_{n,0}(\lambda;x) &= s_{n,0}(x) - \frac{\lambda}{n+d+1} s_{n+1,1}(x),\\
\tilde{s}_{n,i}(\lambda;x) &= s_{n,i}(x) + \frac{\lambda}{(n+d)^2-1}\left[(n+d-2i+1)s_{n+1,i}(x)\right.\\
&\quad \left.-(n+d-2i-1)s_{n+1,i+1}(x)\right] \ (i=1,2\ldots,n+d-1),\\
\tilde{s}_{n,n+d}(\lambda;x) &= s_{n,n+d}(x) - \frac{\lambda}{n+d+1} s_{n+1,n+d}(x),
\end{aligned} \tag{1}$$

where fundamental Schurer polynomials $s_{n,i}(x)$ of degree $n+d$ defined as

$$s_{n,i}(x) = \binom{n+d}{i} x^i \ (1-x)^{n+d-i} \quad (i=0,1,\ldots,n+d).$$

Lemma 1. *New Bézier-Schurer bases have partition of unity property.*

Proof. It is enough to show the equality $\sum_{i=0}^{n+d} \tilde{s}_{n,i}(\lambda,x) = 1$ holds.

$$\begin{aligned}
\sum_{i=0}^{n+d} \tilde{s}_{n,i}(\lambda,x) &= s_{n,0}(x) - \frac{\lambda}{n+d+1} s_{n+1,1}(x) + s_{n,n+d}(x) - \frac{\lambda}{n+d+1} s_{n+1,n+d}(x)\\
&+ \sum_{i=1}^{n+d-1}\left[s_{n,i}(x) + \lambda\left(\frac{n+d-2i+1}{(n+d)^2-1} s_{n+1,i}(x)\right.\right.\\
&\left.\left.-\frac{n+d-2i-1}{(n+d)^2-1} s_{n+1,i+1}(x)\right)\right]\\
&= s_{n,0}(x) + s_{n,1}(x) + \cdots + s_{n,n+d}(x)\\
&+ \lambda\left(\frac{n+d-1}{(n+d)^2-1} s_{n+1,1}(x) - \frac{n+d-3}{(n+d)^2-1} s_{n+1,2}(x)\right)\\
&+ \lambda\left(\frac{n+d-3}{(n+d)^2-1} s_{n+1,2}(x) - \frac{n+d-5}{(n+d)^2-1} s_{n+1,3}(x)\right) + \cdots\\
&+ \lambda\left(-\frac{n+d-5}{(n+d)^2-1} s_{n+1,n+d-2}(x) + \frac{n+d-3}{(n+d)^2-1} s_{n+1,n+d-1}(x)\right)\\
&+ \lambda\left(-\frac{n+d-3}{(n+d)^2-1} s_{n+1,n+d-1}(x) + \frac{n+d-1}{(n+d)^2-1} s_{n+1,n+d}(x)\right)\\
&- \frac{\lambda}{n+d+1} s_{n+1,1}(x) - \frac{\lambda}{n+d+1} s_{n+1,n+d}(x).
\end{aligned}$$

Since Schurer operators satisfy the equality $\sum_{i=0}^{n+d} s_{n,i}(x) = 1$ we get the desired result. □

Rest of the paper is organized as follows: In Section 2, λ-Schurer operators are constructed and corresponding approximation results are obtained. In Section 3, some statistical approximation properties of defined operators are studied and an estimate for the rate of weighted A-statistical convergence is established. In Section 4, two Voronovskaja-type theorems including a Voronovskaja-type approximation theorem using weighted A-statistical convergence are proved. Final section of the paper is devoted to give some concluding remarks including some future studies.

2. λ-Schurer operators and corresponding results in approximation theory

A new type λ-Bernstein operators have been introduced by Cai et al. in [6] based on Bézier bases defined by Ye et al. in [30]. We refer to [5, 6, 20, 23, 26] for recent studies about λ-Bernstein type operators and [13, 14, 28] for some Schuer type operators.

Considering a given non-negative integer d, we introduce λ-Schurer operators $S^{\lambda}_{n,d}(f;x) : C[0,1+d] \longrightarrow C[0,1]$

$$S^{\lambda}_{n,d}(f;x) = \sum_{i=0}^{n+d} \tilde{s}_{n,i}(\lambda;x)\ f\left(\frac{i}{n}\right) \tag{2}$$

for any $n \in \mathbb{N}$, where new Bézier-Schurer bases $\tilde{s}_{n,i}(\lambda;x)$ are defined in (1).

Lemma 2. *We have following results for λ-Schurer operators:*

$$\begin{aligned}
S^{\lambda}_{n,d}(t;x) &= \frac{n+d}{n}x + \frac{1-2x+x^{n+d+1}-(1-x)^{n+d+1}}{n(n+d-1)}\lambda;\\
S^{\lambda}_{n,d}(t^2;x) &= \frac{(n+d)^2}{n^2}x^2 + \frac{n+d}{n^2}x(1-x)\\
&\quad + \frac{2(n+d)x-1-4(n+d)x^2+(2(n+d)+1)x^{n+d+1}+(1-x)^{n+d+1}}{n^2(n+d-1)}\lambda;
\end{aligned}$$

Proof. Using definition of operators (2) and Bézier-Schurer bases $\tilde{s}_{n,i}(\lambda;x)$ (1), we write

$$\begin{aligned}
&S^{\lambda}_{n,d}(t;x) = \sum_{i=0}^{n+d} \frac{i}{n}\tilde{s}_{n,i}(\lambda;x)\\
&= \frac{n+d}{n}s_{n,n+d}(x) - \frac{n+d}{n}\frac{\lambda}{n+d+1}s_{n+1,n+d}(x)\\
&\ + \sum_{i=0}^{n+d-1}\frac{i}{n}\left[s_{n,i}(x) + \lambda\left(\frac{n+d-2i+1}{(n+d)^2-1}s_{n+1,i}(x) - \frac{n+d-2i-1}{(n+d)^2-1}s_{n+1,i+1}(x)\right)\right]\\
&= \sum_{i=0}^{n+d}\frac{i}{n}s_{n,i}(x) + \lambda\left(\varphi_1(n,d,x) - \varphi_2(n,d,x)\right),
\end{aligned}$$

where

$$\varphi_1(n,d,x) = \sum_{i=0}^{n+d} \frac{i}{n} \frac{n+d-2i+1}{(n+d)^2-1} s_{n+1,i}(x);$$
$$\varphi_2(n,d,x) = \sum_{i=1}^{n+d-1} \frac{i}{n} \frac{n+d-2i-1}{(n+d)^2-1} s_{n+1,i+1}(x).$$

Now we compute the expressions $\varphi_1(n,d,x)$ and $\varphi_2(n,d,x)$.

$$\begin{aligned}
\varphi_1(n,d,x) &= \frac{1}{n+d-1} \sum_{i=0}^{n+d} \frac{i}{n} s_{n+1,i}(x) - \frac{2}{(n+d)^2-1} \sum_{i=0}^{n+d} \frac{i^2}{n} s_{n+1,i}(x) \\
&= \frac{x(n+d+1)}{n(n+d-1)} \sum_{i=0}^{n+d-1} s_{n,i}(x) - \frac{2x}{n(n+d-1)} \sum_{i=0}^{n+d-1} s_{n,i}(x) \\
&- \frac{2x^2(n+d)}{n(n+d-1)} \sum_{i=0}^{n+d-2} s_{n-1,i}(x) \\
&= -\frac{(1-x^{n+d})(x(n+d)+x-2x)}{n(n+d-1)} - \frac{2x^2(n+d)(1-x^{n+d-1})}{n(n+d-1)} \\
&= \frac{x-x^{n+d+1}}{n} - \frac{2(n+d)(x^2-x^{n+d+1})}{n(n+d-1)}.
\end{aligned}$$

$$\begin{aligned}
\varphi_2(n,d,x) &= \frac{n+d-1}{n((n+d)^2-1)} \sum_{i=1}^{n+d-1} i \; s_{n+1,i+1}(x) \\
&- \frac{2}{n((n+d)^2-1)} \sum_{i=1}^{n+d-1} i^2 \; s_{n+1,i+1}(x) \\
&= \frac{2x}{n(n+d-1)} \sum_{i=1}^{n+d-1} s_{n,i}(x) - \frac{2x^2(n+d)}{n(n+d-1)} \sum_{i=0}^{n+d-2} s_{n-1,i}(x) \\
&+ \frac{x}{n} \sum_{i=1}^{n+d-1} s_{n,i}(x) \\
&- \frac{2}{n((n+d)^2-1)} \sum_{i=1}^{n+d-1} s_{n+1,i+1}(x) - \frac{1}{n(n+d+1)} \sum_{i=1}^{n+d-1} s_{n+1,i+1}(x) \\
&= \frac{2x-2x(1-x)^{n+d}-x^{n+d+1}}{n(n+d-1)} - \frac{2(n+d)(x^2-x^{n+d+1})}{n(n+d-1)} \\
&- \frac{2-(1-x)^{n+d+1}-2x(n+d+1)(1-x)^{n+d}-2x^{n+d+1}}{n((n+d)^2-1)}
\end{aligned}$$

$$+\frac{x-x^{n+d+1}}{n}-\frac{1-(1-x)^{n+d+1}-x(n+d+1)(1-x)^{n+d}-x^{n+d+1}}{n(n+d-1)}.$$

We obtain the result for $S^{\lambda}_{n,d}(t;x)$ combining the results obtained for $\varphi_1(n,d,x)$ and $\varphi_2(n,d,x)$ since Schurer operators are linear, and Schurer operators and fundamental Schurer bases satisfy the following equality:

$$\sum_{i=1}^{n+d}\frac{i}{n}s_{n,i}(x)=\left(1+\frac{d}{n}\right)x.$$

We again use the definition of operators (2), Bézier-Schurer bases $\tilde{s}_{n,i}(\lambda;x)$ (1) and the following relations to prove the second part of the lemma:

$$\begin{aligned}
S^{\lambda}_{n,d}(t^2;x)&=\sum_{i=0}^{n+d}\frac{i^2}{n^2}\tilde{s}_{n,i}(\lambda;x)=\frac{(n+d)^2}{n^2}s_{n,n+d}(x)-\frac{(n+d)^2}{n^2}\frac{\lambda}{n+d+1}s_{n+1,n+d}(x)\\
&+\sum_{i=0}^{n+d-1}\frac{i^2}{n^2}\left[s_{n,i}(x)+\lambda\left(\frac{n+d-2i+1}{(n+d)^2-1}s_{n+1,i}(x)-\frac{n+d-2i-1}{(n+d)^2-1}s_{n+1,i+1}(x)\right)\right]\\
&=\sum_{i=0}^{n+d}\frac{i^2}{n^2}s_{n,i}(x)+\lambda\left(\varphi_3(n,d,x)-\varphi_4(n,d,x)\right),
\end{aligned}$$

where

$$\varphi_3(n,d,x)=\sum_{i=0}^{n+d}\frac{i^2}{n^2}\frac{n+d-2i+1}{(n+d)^2-1}s_{n+1,i}(x);$$

$$\varphi_4(n,d,x)=\sum_{i=1}^{n+d-1}\frac{i^2}{n^2}\frac{n+d-2i-1}{(n+d)^2-1}s_{n+1,i+1}(x).$$

Now we compute the expressions $\varphi_3(n,d,x)$ and $\varphi_4(n,d,x)$.

$$\begin{aligned}
\varphi_3(n,d,x)&=\frac{1}{n+d-1}\sum_{i=0}^{n+d}\frac{i^2}{n^2}s_{n+1,i}(x)-\frac{2}{(n+d)^2-1}\sum_{i=0}^{n+d}\frac{i^3}{n^2}s_{n+1,i}(x)\\
&=\frac{(n+d)(n+d+1)x^2}{n^2(n+d-1)}\sum_{i=0}^{n+d-2}s_{n-1,i}(x)+\frac{x}{n^2}\sum_{i=0}^{n+d-1}s_{n,i}(x)\\
&-\frac{2(n+d)x^3}{n^2}\sum_{i=0}^{n+d-3}s_{n-2,i}(x)-\frac{6(n+d)x^2}{n^2(n+d-1)}\sum_{i=0}^{n+d-2}s_{n-1,i}(x)\\
&=\frac{(n+d)(n+d+1)(x^2-x^{n+d+1})}{n^2(n+d-1)}+\frac{x-x^{n+d+1}}{n^2}\\
&-\frac{2(n+d)(x^3-x^{n+d+1})}{n^2}-\frac{6(n+d)(x^2-x^{n+d+1})}{n^2(n+d-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(n+d)(x^{n+d+1}-x^3)}{n^2} + \frac{x - x^{n+d+1}}{n^2} \\
&+ \frac{(n+d)^2 - 5(n+d)(x^2 - x^{n+d+1})}{n^2(n+d-1)}.
\end{aligned}
$$

$$
\begin{aligned}
\varphi_4(n,d,x) &= \frac{1}{n+d+1}\sum_{i=1}^{n+d-1}\frac{i^2}{n^2}\, s_{n+1,i+1}(x) - \frac{2}{(n+d)^2-1}\sum_{i=1}^{n+d-1}\frac{i^3}{n^2}\, s_{n+1,i+1}(x) \\
&= \frac{x^2(n+d)}{n^2}\sum_{i=0}^{n+d-2} s_{n-1,i}(x) - \frac{x}{n^2}\sum_{i=1}^{n+d-1} s_{n,i}(x) \\
&+ \frac{1}{n^2(n+d+1)}\sum_{i=1}^{n+d-1} s_{n+1,i+1}(x) + \frac{2(n+d)x^3}{n^2}\sum_{i=0}^{n+d-3} s_{n-2,i}(x) \\
&+ \frac{2x}{n^2(n+d-1)}\sum_{i=1}^{n+d-1} s_{n,i}(x) - \frac{2}{n^2((n+d)^2-1)}\sum_{i=1}^{n+d-1} s_{n+1,i+1}(x) \\
&= \frac{x^2(n+d)(1-x^{n+d-1})}{n^2} - \frac{x(1-x^{n+d})}{n^2} \\
&+ \frac{1-(1-x)^{n+d+1} - x(n+d+1)(1-x)^{n+d} - x^{n+d+1}}{n^2(n+d-1)} \\
&- \frac{2(n+d)x^3(1-x^{n+d-2})}{n^2} - \frac{2x(1-x^{n+d})}{n^2(n+d-1)} \\
&+ \frac{2 - 2(1-x)^{n+d+1} - 2x(n+d+1)(1-x)^{n+d} - 2x^{n+d+1}}{n^2((n+d)^2-1)}.
\end{aligned}
$$

We get $S^{\lambda}_{n,d}(t^2;x)$ combining $\varphi_3(n,d,x)$ and $\varphi_4(n,d,x)$ since Schurer operators and fundamental Schurer bases satisfy the following equality:

$$
\sum_{i=1}^{n+d}\frac{i^2}{n^2}s_{n,i}(x) = \frac{n+d}{n^2}\left\{(n+d)x^2 + x(1-x)\right\}.
$$

□

Corollary 3. *We have the following relations for $S^{\lambda}_{n,d}(t-x;x)$ and $S^{\lambda}_{n,d}((t-x)^2;x)$:*

$$
\begin{aligned}
S^{\lambda}_{n,d}(t-x;x) &= \frac{d}{n}x + \frac{1-2x+x^{n+d+1}-(1-x)^{n+d+1}}{n(n+d-1)}\lambda; \\
S^{\lambda}_{n,d}((t-x)^2;x) &= \frac{d^2}{n^2}x^2 + \frac{n+d}{n^2}x(1-x) - \frac{2x^{n+d+2}-2x(1-x)^{n+d+1}}{n(n+d-1)}\lambda \\
&\quad + \frac{2dx - 1 - 4dx^2 + (2(n+d)+1)x^{n+d+1} + (1-x)^{n+d+1}}{n^2(n+d-1)}\lambda;
\end{aligned}
$$

Corollary 4. *We have the following relations for $S_{n,d}^{\lambda}(t-x;x)$ and $S_{n,d}^{\lambda}((t-x)^2;x)$:*

$$\lim_{n\to\infty} n\ S_{n,d}^{\lambda}(t-x;x) = dx;$$
$$\lim_{n\to\infty} n\ S_{n,d}^{\lambda}((t-x)^2;x) = x(1-x).$$

Remark 5. *We have the following results for λ-Schurer operators and Bézier-Schurer bases:*

- *If we take $d=0$, Bézier-Schurer bases (1) reduce to the classical Bézier bases defined in* [30].
- *If we take $\lambda=0$, λ-Schurer operators (2) reduce to the classical Schurer operators defined in* [25].
- *If we take $d,\ \lambda=0$, λ-Schurer operators (2) with Bézier-Schurer bases (1) reduce to the classical Bernstein operators defined in* [3].

The following theorem gives the uniform convergence property of λ-Schurer operators (2) by the well-known Bohman-Korovkin-Popoviciu theorem:

Theorem 6. *Let $f \in C[0,1+d]$, then we have*

$$\lim_{n\to\infty} S_{n,d}^{\lambda}(f;x) = f(x)$$

uniformly on $[0,1]$, where $C[0,1+d]$ denotes the space of all real-valued continuous functions on $[0,1+d]$ endowed with the norm $\|f\|_{C[0,1]} = \sup_{x\in[0,1+d]}|f(x)|$.

We achieve a global approximation formula in terms of Ditzian-Totik uniform modulus of smoothness of first and second order for λ-Schurer operators (2), and give a local direct estimate of the rate of convergence by Lipschitz-type function involving two parameters.

Definition 7. *Global approximation formula in terms of Ditzian-Totik uniform modulus of smoothness of first and second order defined by*

$$\omega_{\xi}(f,\zeta) := \sup_{0<|h|\leq\zeta}\ \sup_{x,x+h\xi(x)\in[0,1]} \{|f(x+h\xi(x)) - f(x)|\};$$

$$\omega_2^{\tau}(f,\zeta) := \sup_{0<|h|\leq\zeta}\ \sup_{x,x\pm h\tau(x)\in[0,1]} \{|f(x+h\tau(x)) - 2f(x) + f(x-h\tau(x))|\},$$

respectively, where τ is an admissible step-weight function on $[a,b]$, i.e. $\tau(x) = [(x-a)(b-x)]^{1/2}$ if $x\in[a,b]$, [7]. *We write AC for absolutely continuous functions, then K-functional is*

$$K_{2,\tau(x)}(f,\zeta) = \inf_{g\in W^2(\tau)} \left\{||f-g||_{C[0,1]} + \zeta||\tau^2 g''||_{C[0,1]} : g\in C^2[0,1+d]\right\},$$

where $\zeta>0$, $W^2(\tau) = \{g\in C[0,1+d] : g'^2 g'' \in C[0,1+d]\}$ and $C^2[0,1+d] = \{g\in C[0,1+d] : g', g'' \in C[0,1+d]\}$.

Remark 8. *It is known by* [9] *that there exists an absolute constant $C>0$, such that*

$$C^{-1}\omega_2^{\tau}(f,\sqrt{\zeta}) \le K_{2,\tau(x)}(f,\zeta) \le C\omega_2^{\tau}(f,\sqrt{\zeta}). \tag{3}$$

First we obtain global approximation formula in terms of Ditzian-Totik uniform modulus of smoothness of first and second order.

Theorem 9. *Let $f\in C[0,1+d]$, $x\in[0,1]$ and $\lambda\in[-1,1]$. Then for $C>0$, λ-Schurer operators (2) verify*

$$|S_{n,d}^{\lambda}(f;x)-f(x)| \le C\omega_2^{\tau}\Big(f,\frac{\sqrt{\alpha_{n,\lambda}(x)+\beta_{n,\lambda}^2(x)}}{2\tau(x)}\Big)+\omega_{\xi}\Big(f,\frac{\beta_{n,\lambda}(x)}{\xi(x)}\Big),$$

where $\beta_{n,\lambda}(x)=S_{n,d}^{\lambda}(t-x;x)$ and $\alpha_{n,\lambda}(x)=S_{n,d}^{\lambda}((t-x)^2;x)$ are given in Corollary 3, and $\tau(x)$ $(\tau\neq 0)$ is an admissible step-weight function of Ditzian-Totik modulus of smoothness such that τ^2 is concave.

Proof. Let $f\in C[0,1+d]$, $x\in[0,1]$ and $\lambda\in[-1,1]$. Defining the operators

$$\breve{S}_{n,d}^{\lambda}(f;x)=S_{n,d}^{\lambda}(f;x)+f(x)-f\Big(x+\frac{d}{n}x+\frac{1-2x+x^{n+d+1}-(1-x)^{n+d+1}}{n(n+d-1)}\lambda\Big) \tag{4}$$

we see that $\breve{S}_{n,d}^{\lambda}(1;x)=1$ and $\breve{S}_{n,d}^{\lambda}(t;x)=x$, that is $\breve{S}_{n,d}^{\lambda}(t-x;x)=0$.

Let $u=\rho x+(1-\rho)t$, $\rho\in[0,1]$. Since τ^2 is concave on $[0,1]$, it follows that $\tau^2(u)\ge\rho\tau^2(x)+(1-\rho)\tau^2(t)$ and

$$\frac{|t-u|}{\tau^2(u)} \le \frac{\rho|x-t|}{\rho\tau^2(x)+(1-\rho)\tau^2(t)} \le \frac{|t-x|}{\tau^2(x)}. \tag{5}$$

Hence the following inequalities hold:

$$\begin{aligned}|\breve{S}_{n,d}^{\lambda}(f;x)-f(x)| &\le |\breve{S}_{n,d}^{\lambda}(f-g;x)|+|\breve{S}_{n,d}^{\lambda}(g;x)-g(x)|+|f(x)-g(x)| \\ &\le 4\|f-g\|_{C[0,1+d]}+|\breve{S}_{n,d}^{\lambda}(g;x)-g(x)|.\end{aligned} \tag{6}$$

Applying Taylor's formula we obtain

$$\begin{aligned}&|\breve{S}_{n,d}^{\lambda}(g;x)-g(x)| \\ &\le S_{n,d}^{\lambda}\Big(\Big|\int_x^t |t-u|\ |g''(u)|du\Big|;x\Big)+\Big|\int_x^{x+\beta_{n,\lambda}(x)} \big|x+\beta_{n,\lambda}(x)-u\big|\ |g''(u)|\ du\Big| \\ &\le \|\tau^2 g''\|_{C[0,1+d]}S_{n,d}^{\lambda}\Big(\Big|\int_x^t \frac{|t-u|}{\tau^2(u)}du\Big|;x\Big) \\ &\le +\|\tau^2 g''\|_{C[0,1+d]}\ \Big|\int_x^{x+\beta_{n,\lambda}(x)} \frac{|x+\beta_{n,\lambda}(x)-u|}{\tau^2(u)}\ du\Big| \\ &\le \tau^{-2}(x)\|\tau^2 g''\|_{C[0,1+d]}S_{n,d}^{\lambda}((t-x)^2;x)+\tau^{-2}(x)\|\tau^2 g''\|_{C[0,1+d]}\beta_{n,\lambda}^2(x).\end{aligned} \tag{7}$$

By definition of K-functional with relation (3) and inequalities (6)-(7), we have

$$|\breve{S}_{n,d}^{\lambda}(f;x)-f(x)| \leq 4\|f-g\|_{C[0,1+d]}+\tau^{-2}(x)\|\tau^2 g''\|_{C[0,1+d]}\big(\alpha_{n,\lambda}(x)+\beta_{n,\lambda}^2(x)\big)$$
$$\leq C\omega_2^{\tau}\bigg(f,\frac{\sqrt{\alpha_{n,\lambda}(x)+\beta_{n,\lambda}^2(x)}}{2\tau(x)}\bigg).$$

Also, by Ditzian-Totik uniform modulus of smoothness of first order we have

$$|f(x+\beta_{n,\lambda}(x))-f(x)| = \bigg|f\bigg(x+\xi(x)\frac{\beta_{n,\lambda}(x)}{\xi(x)}\bigg)-f(x)\bigg| \leq \omega_{\xi}\bigg(f,\frac{\beta_{n,\lambda}(x)}{\xi(x)}\bigg).$$

Therefore, following inequality, which completes the proof, holds:

$$|S_{n,d}^{\lambda}(f;x)-f(x)| \leq |\breve{S}_{n,d}^{\lambda}(f;x)-f(x)|+\big|f(x+\beta_{n,\lambda}(x))-f(x)\big|$$
$$\leq C\omega_2^{\tau}\bigg(f,\frac{\sqrt{\alpha_{n,\lambda}(x)+\beta_{n,\lambda}^2(x)}}{2\tau(x)}\bigg)+\omega_{\xi}\bigg(f,\frac{\beta_{n,\lambda}(x)}{\xi(x)}\bigg).$$

□

Theorem 10. *The following inequality holds:*

$$|S_{n,d}^{\lambda}(f;x)-f(x)| \leq |\beta_{n,\lambda}(x)|\;|f'(x)|+2\sqrt{\alpha_{n,\lambda}(x)}w\big(f',\sqrt{\alpha_{n,\lambda}(x)}\;\big)$$

for $f\in C^1[0,1+d]$ and $x\in[0,1]$, where $\alpha_{n,\lambda}(x)$ and $\beta_{n,\lambda}(x)$ are given in Theorem 9.

Proof. For any $t\in[0,1]$ and $x\in[0,1]$ we have

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t (f'(u)-f'(x))du.$$

Applying operators $S_{n,d}^{\lambda}(f;x)$ to both sides of (**??**), we have

$$S_{n,d}^{\lambda}(f(t)-f(x);x)=f'(x)S_{n,d}^{\lambda}(t-x;x)+S_{n,d}^{\lambda}\bigg(\int_x^t (f'(u)-f'(x))du;x\bigg).$$

The following inequality holds for any $\zeta>0$, $u\in[0,1]$ and $f\in C[0,1+d]$:

$$|f(u)-f(x)| \leq w(f,\zeta)\bigg(\frac{|u-x|}{\zeta}+1\bigg),$$

With above inequality we get

$$\bigg|\int_x^t (f'(u)-f'(x))du\bigg| \leq w(f',\zeta)\bigg(\frac{(t-x)^2}{\zeta}+|t-x|\bigg).$$

Hence we have

$$\begin{aligned}|S_{n,d}^{\lambda}(f;x)-f(x)| \quad &\leq \quad |f'(x)|\;|S_{n,d}^{\lambda}(t-x;x)| \\ &\qquad +w(f',\zeta)\bigg\{\frac{1}{\zeta}S_{n,d}^{\lambda}((t-x)^2;x)+S_{n,d}^{\lambda}(t-x;x)\bigg\}.\end{aligned}$$

Applying Cauchy-Schwarz inequality on the right hand side of above inequality, we have

$$\begin{aligned}|S^{\lambda}_{n,d}(f;x)-f(x)| &\leq f'(x)|\beta_{n,\lambda}(x)| \\ &\quad +w(f',\zeta)\left\{\frac{1}{\zeta}\sqrt{S^{\lambda}_{n,d}((t-x)^2;x)}+1\right\}\sqrt{S^{\lambda}_{n,d}(|t-x|;x)}.\end{aligned}$$

□

3. Some statistical approximation theorems

In this section, we use weighted mean matrix method to establish statistical approximation properties of λ-Schurer operators. We also give an estimate for the rate of weighted A-statistical convergence of λ-Schurer operators.

Statistical convergence was first introduced in [8] and [27]. A new characterization in terms of weighted regular matrix and a Korovkin type approximation theorem through statistically weighted A-summable sequences of real or complex numbers have been given by Mohiuddine et al. [16, 17]. For further results in weighted statistical approximation theory we refer to [11, 12] and for statistical approximation papers to [2] .

All the following notions, notations and definitions which can be found in [2, 8, 11, 12, 17, 27] are needed for the results of this part.

Definition 11. *Natural density of K is denoted by $\zeta(K)=\lim_{n\to\infty}\frac{1}{n}|K_n|$ provided that limit exists, where $K_n=\{k\leq n : k\in K\}$, $K\subseteq \mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and vertical bars denote cardinality of the enclosed set. A sequence $x=(x_n)$ of numbers is called statistically convergent to a number L, denoted by st-$\lim_{n\to\infty} x = L$, if, for each $\epsilon>0$, $\zeta\{n : n\in\mathbb{N}$ and $|x_n-L|\geqq\epsilon\}=0$.*

Definition 12. *A-transform of x denoted by $Ax := \{(Ax)_n\}$ is defined as $(Ax)_n=\sum_{k=0}^{\infty}a_{nk}x_k$ for a given non-negative infinite summability matrix $A=(a_{nk}), n,k\in\mathbb{N}$. It is provided defined series converges for every $n\in\mathbb{N}_0$. If $\lim_{n\to\infty}(Ax)_n=L$ whenever $\lim_{n\to\infty}x_n=L$, we say that A is a regular method. Then sequence $x=(x_n)$ is said to be A-statistically convergent to L, denoted by st_A-$\lim_{n\to\infty}x_n=L$, provided that for each $\epsilon>0$, $\lim_{n\to\infty}\sum_{k:|x_k-L|\geqq\epsilon}a_{nk}=0$.*

Remark 13. *We have the following results for A-statistical convergence concept:*

- *If we take $A=(C_1)$, the Cesaro matrix of order 1, A-statistical convergence becomes ordinary statistical convergence which was introduced in* [10].
- *If we take $A=I$, the identity matrix, A-statistical convergence becomes classical convergence.*
- *Every convergent sequence is statistically convergent to the same limit but not conversely.*

Definition 14. [16] *Assume that $q=(q_n)$ is a sequence of non-negative numbers so that $q_0>0$ and $Q_n=\sum_{k=0}^{n}q_k\to\infty$ as $n\to\infty$. Then $x=(x_n)$ is called* *weighted A-statistically convergent to L, if, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{m:|x_m - L| \geqq \epsilon} a_{km} = 0.$$

In this case, we write $S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} x_n = L$.

Remark 15. [16] *The weighted A-statistical convergence generalizes A-statistical convergence, which we recover by putting $q_n = 1$ for all $n \in \mathbb{N}$.*

We now give main results related to statistical approximation of operators in (2).

Theorem 16. *Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix for $n, k \in \mathbb{N}$ and $q = (q_n)$ be a sequence of non-negative numbers such that $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ as $n \to \infty$. For any $f \in C[0, 1 + d]$, we have*

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(f) - f\|_{C[0,1]} = 0.$$

Proof. Consider sequence of functions $e_j(x) = x^j$, where $j \in \{0, 1, 2\}$ and $x \in [0, 1]$. It is sufficient to satisfy

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(e_j; x) - e_j\|_{C[0,1]} = 0, \quad j = 0, 1, 2.$$

From Lemma 2, it is clear that

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(e_0; x) - e_0\|_{C[0,1]} = 0. \tag{8}$$

We have

$$\begin{aligned} \|S_{n,d}^{\lambda}(e_1; x) - e_1\|_{C[0,1]} &= \sup_{x\in[0,1]} \left| \frac{d}{n}x + \frac{1 - 2x + x^{n+d+1} - (1-x)^{n+d+1}}{n(n+d-1)}\lambda \right| \\ &\leq \frac{d}{n} + \frac{4}{n(n+d-1)} \end{aligned}$$

by Lemma 2. We choose a number $\epsilon > 0$ for a given $\epsilon' > 0$ such that $\epsilon < \epsilon'$. If we define following sets:

$$\begin{aligned} \Delta &:= \left\{ n \in \mathbb{N} : \|S_{n,d}^{\lambda}(e_1; x) - e_1\|_{C[0,1]} \geqq \epsilon' \right\}, \\ \Delta_1 &:= \left\{ n \in \mathbb{N} : \frac{d}{n} + \frac{4}{n(n+d-1)} \geqq \epsilon - \epsilon' \right\}, \end{aligned}$$

we see that the inclusion $\Delta \subset \Delta_1$ holds and

$$\frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{m\in\Delta} a_{km} \leq \frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{m\in\Delta_1} a_{km} \qquad \text{for all } n \in \mathbb{N}. \tag{9}$$

So we have

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(e_1; x) - e_1\|_{C[0,1]} = 0 \tag{10}$$

as $n \to \infty$ in (9). By Lemma 2 we have

$$\begin{aligned}\|S_{n,d}^{\lambda}(e_2;x) - e_2\|_{C[0,1]} = \sup_{x\in[0,1]} \Bigg| & \frac{2nd+d^2}{n^2}x^2 + \frac{n+d}{n^2}x(1-x) \\ & + \frac{2(n+d)x - 1 - 4(n+d)x^2 +}{n^2(n+d-1)}\lambda \\ & + \frac{(2(n+d)+1)x^{n+d+1} + (1-x)^{n+d+1}}{n^2(n+d-1)}\lambda\Bigg| \\ & \leq \frac{2nd+d^2}{n^2} + \frac{2n+2d}{n^2} + \frac{8(n+d)+2}{n^2(n+d-1)}.\end{aligned}$$

We also obtain

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(e_2;x) - e_2\|_{C[0,1]} = 0 \tag{11}$$

since

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty}\left[\frac{2nd+d^2}{n^2} + \frac{2n+2d}{n^2} + \frac{8(n+d)+2}{n^2(n+d-1)}\right] = 0.$$

Combining (8), (10) and (11), we get desired result. □

We now estimate rate of weighted A-statistical convergence of operators $S_{n,d}^{\lambda}(f;x)$.

Definition 17. *Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix and let $q = (q_n)$ be a sequence of non-negative numbers such that $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ as $n \to \infty$. Also let (u_n) be a positive non-decreasing sequence. We say that a sequence $x = (x_n)$ is weighted A-statistically convergent to L with the rate $o(u_n)$ if*

$$\lim_{n\to\infty} \frac{1}{u_n Q_n} \sum_{k=0}^{n} q_k \sum_{m:|x_m - L| \geqq \epsilon} a_{km} = 0.$$

This relation is denoted by $[stat_A, q_n] - o\ (u_n) = x_n - L$.

Theorem 18. *Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix. Assume that following condition yields:*

$$w(f,h_n) = [stat_A, q_n] - o\ (u_n) \text{ on } [0,1], \text{ where } h_n = \sqrt{\|S_{n,d}^{\lambda}((s-x)^2;x)\|_{C[0,1+d]}}.$$

Then for every bounded $f \in C[0,1+d]$ we have

$$\|S_{n,d}^{\lambda}(f) - f\|_{C[0,1]} = [stat_A, q_n] - o\ (u_n).$$

Proof. Let $f \in C[0,1+d]$, then we have

$$|S_{n,d}^{\lambda}(f;x) - f(x)| \leq |S_{n,d}^{\lambda}\big(|f(t) - f(x)|;x\big) + A\ |S_{n,d}^{\lambda}(1;x) - 1|$$

$$\leq \omega(f,\zeta)S_{n,d}^{\lambda}\bigg(\frac{|t-x|}{\zeta}+1;x\bigg)$$
$$=\omega(f,\zeta)S_{n,d}^{\lambda}(1;x)+\omega(f,\zeta)\frac{1}{\zeta^2}S_{n,d}^{\lambda}\left((t-x)^2;x\right)$$

for any $x,s\in[0,1]$, where $A=\sup_{x\in[0,1]}|f(x)|$. Let $\zeta:=h_n$ for all $n\in\mathbb{N}$. Taking supremum over $x\in[0,\infty)$ on both sides, we obtain

$$\|S_{n,d}^{\lambda}(f)-f\|_{C[0,1]}\leq\omega(f,h_n)+\omega(f,h_n)\frac{1}{h_n^2}\|S_{n,d}^{\lambda}((t-x)^2;x)\|_{C[0,1+d]}=2\omega(f,h_n).$$

We define the following sets for a given $\epsilon>0$:

$$\mathcal{S}=\left\{n:\|S_{n,d}^{\lambda}(f)-f\|_{C[0,1]}\geq\epsilon\right\}\text{ and }\mathcal{E}=\left\{n:\omega(f,h_n)\geq\frac{\epsilon}{2}\right\}.$$

It is easy to see the following inequality holds:

$$\frac{1}{u_nQ_n}\sum_{k=0}^{n}\sum_{m\in\mathcal{S}}q_ka_{km}\leq\frac{1}{u_nQ_n}\sum_{k=0}^{n}\sum_{m\in\mathcal{E}}q_ka_{km}.$$

Hence we are led to the fact that

$$\|S_{n,d}^{\lambda}(f)-f\|_{C[0,1]}=[stat_A,q_n]-o\left(u_n\right)$$

by the hypothesis, as asserted by Theorem 18. □

4. Voronovskaja-type approximation theorems

Two Voronovskaja-type theorems are established in this part: A quantitative Voronovskaja-type theorem and a Voronovskaja-type approximation theorem by $\bar{S}_{n,d}^{\lambda}(f;x)$ family of linear operators using the notion of weighted A-statistical convergence.

Theorem 19. *Let (x_n) be a sequence of real numbers such that $S_{\mathbf{A}}^{\widetilde{N}}-\lim_{n\to\infty}x_n=0$, where $A=(a_{nk})$ is a weighted non-negative regular summability matrix. Also let $\bar{S}_{n,d}^{\lambda}(f;x)$ be a sequence of positive linear operators acting from $C_B[0,1+d]$ into $C[0,1+d]$ defined by*

$$\bar{S}_{n,d}^{\lambda}(f;x)=(1+x_n)S_{n,d}^{\lambda}(f;x).$$

Then for every $f\in C_B[0,1+d]$ we have

$$S_{\mathbf{A}}^{\widetilde{N}}-\lim_{n\to\infty}n\big\{\bar{S}_{n,d}^{\lambda}(f;x)-f(x)\big\}=xd\ f'(x)+\frac{x(1-x)}{2}\ f''(x),$$

where $f',f''\in C_B[0,1+d]$.

Proof. Let $x\in[0,1]$ and $f''\in C_B[0,1+d]$. Applying $\bar{S}_{n,d}^{\lambda}(f;x)$ to both sides of Taylor's expansion theorem, we have

$$\bar{S}_{n,d}^{\lambda}(f;x)-f(x)\quad=\quad f^{'}(x)\bar{S}_{n,d}^{\lambda}(t-x;x)+\frac{f^{''}(x)}{2}\bar{S}_{n,d}^{\lambda}((t-x)^2;x)$$

$$+\bar{S}^{\lambda}_{n,d}((t-x)^2 r_x(t);x),$$

which yields to

$$n\{\bar{S}^{\lambda}_{n,d}(f;x)-f(x)\} = nf^{'}(x)(1+x_n)S^{\lambda}_{n,d}(t-x;x)$$
$$+\frac{n}{2}f^{''}(x)(1+x_n)S^{\lambda}_{n,d}((t-x)^2;x)+n(1+x_n)S^{\lambda}_{n,d}((t-x)^2 r_x(t);x).$$

We also have from Corollary 3

$$S^{\lambda}_{n,d}(t-x;x) \le \frac{d}{n}x + \frac{1+2x+x^{n+d+1}+(1-x)^{n+d+1}}{n(n+d-1)} := E(n,d,x),$$

and again from Corollary 3

$$S^{\lambda}_{n,d}((t-x)^2;x) \le \frac{d^2}{n^2}x^2 + \frac{n+d}{n^2}x(1-x) + \frac{2x^{n+d+2}+2x(1-x)^{n+d+1}}{n(n+d-1)}$$
$$+\frac{2dx+1+4dx^2+(2(n+d)+1)x^{n+d+1}+(1-x)^{n+d+1}}{2n(n+d-1)}\lambda := F(n,d,x).$$

Hence we have

$$\Bigg|n\{\bar{S}^{\lambda}_{n,d}(f;x)-f(x)\} - f^{'}(x)dx - f^{'}(x)\frac{1-2x+x^{n+d+1}-(1-x)^{n+d+1}}{n+d-1}\lambda$$
$$- f^{''}(x)\left(\frac{d^2}{2n}x^2+\frac{n+d}{2n}x(1-x)-\frac{x^{n+d+2}-x(1-x)^{n+d+1}}{n+d-1}\lambda\right.$$
$$\left.+\frac{2dx-1-4dx^2+(2(n+d)+1)x^{n+d+1}+(1-x)^{n+d+1}}{2n(n+d-1)}\lambda\right)\Bigg|$$
$$= nf^{'}(x)x_n S^{\lambda}_{n,d}(t-x;x) + \frac{n}{2}f^{''}(x)x_n S^{\lambda}_{n,d}((t-x)^2;x)$$
$$+n(1+x_n)S^{\lambda}_{n,d}((t-x)^2 r_x(t);x)$$
$$\le x_n\{f^{'}(x)\bar{E}(n,d,x)+\frac{f^{''}(x)}{2}\bar{F}(n,d,x)\}+n(1+x_n)S^{\lambda}_{n,d}((t-x)^2 r_x(t);x)$$
$$\le x_n\{\sup_{x\in[0,1]}|f'(x)|\bar{E}(n,d,x)+\frac{1}{2}\sup_{x\in[0,1]}|f''(x)|\bar{F}(n,d,x)\}$$
$$+n(1+x_n)S^{\lambda}_{n,d}((t-x)^2 r_x(t);x),$$

where $\bar{E}(n,d,x) = n\ E(n,d,x)$ and $\bar{F}(n,d,x) = n\ F(n,d,x)$. Since we have

$$S^{\widetilde{N}}_{\mathbf{A}} - \lim_{n\to\infty} n\ (S^{\lambda}_{n,d}((t-x)^2 r_x(t);x)) = 0$$

and $S^{\widetilde{N}}_{\mathbf{A}} - \lim_{n\to\infty} x_n = 0$, we get desired result. □

A quantitative Voronovskaja-type theorem for $S^{\lambda}_{n,d}(f;x)$ is established using Ditzian-Totik modulus of smoothness defined as

$$\omega_\tau(f,\zeta) := \sup_{0<|h|\le\zeta}\left\{\left|f\left(x+\frac{h\tau(x)}{2}\right)-f\left(x-\frac{h\tau(x)}{2}\right)\right|, x\pm\frac{h\tau(x)}{2}\in[0,1]\right\},$$

where $\tau(x) = (x(1-x))^{1/2}$ and $f \in C[0,1+d]$, and corresponding Peetre's K-functional is defined by

$$K_\tau(f,\zeta) = \inf_{g\in W_\tau[0,1+d]} \big\{||f-g|| + \zeta||\tau g'^1[0,1+d], \zeta > 0\big\},$$

where $W_\tau[0,1+d] = \{g: \ g \in AC_{loc}[0,1+d], \ \|\tau g'\| < \infty\}$ and $AC_{loc}[0,1+d]$ is the class of absolutely continuous functions defined on $[a,b] \subset [0,1+d]$. There exists a constant $C > 0$ such that

$$K_\tau(f,\zeta) \le C\ \omega_\tau(f,\zeta).$$

Theorem 20. *Let $f, f', f'' \in C[0,1+d]$, then we have*

$$\left| S^\lambda_{n,d}(f;x) - f(x) - \beta_{n,\lambda}(x)f'(x) - \frac{\alpha_{n,\lambda}(x)+1}{2} f''(x)\right| \le \frac{C}{n}\tau^2(x)\omega_\tau\left(f'', \frac{1}{\sqrt{n}}\right)$$

for every $x \in [0,1]$ and sufficiently large n, where C is a positive constant, $\alpha_{n,\lambda}(x)$ and $\beta_{n,\lambda}(x)$ are defined in Theorem 9.

Proof. Consider following equality

$$f(t) - f(x) - (t-x)f'(x) = \int_x^t (t-u)f''(u)du$$

for $f \in C[0,1+d]$. It means we have

$$f(t) - f(x) - (t-x)f'(x) - \frac{f''(x)}{2}\big((t-x)^2+1\big) \le \int_x^t (t-u)[f''(u) - f''(x)]du. \tag{12}$$

Applying $S^\lambda_{n,d}(f;x)$ to both sides of (12), we obtain

$$\begin{aligned}&\left| S^\lambda_{n,d}(f;x) - f(x) - S^\lambda_{n,d}((t-x);x)f'(x) - \frac{f''(x)}{2}\big(S^\lambda_{n,d}((t-x)^2;x) + S^\lambda_{n,d}(1;x)\big)\right| \\ &\le S^\lambda_{n,d}\left(\Big|\int_x^t |t-u|\ |f''(u) - f''(x)|\ du\Big|; x\right). \end{aligned}\tag{13}$$

The quantity in right hand side of (13) can be estimated as

$$\left|\int_x^t |t-u|\ |f''(u)-f''(x)|\ du\right| \le 2\|f''^2 + 2\|\tau g'^{-1}(x)|t-x|^3, \tag{14}$$

where $g \in W_\tau[0,1+d]$. There exists $C > 0$ such that

$$S^\lambda_{n,d}((t-x)^2;x) \le \frac{C}{2n}\tau^2(x) \quad \text{and} \quad S^\lambda_{n,d}((t-x)^4;x) \le \frac{C}{2n^2}\tau^4(x) \tag{15}$$

for sufficiently large n. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left| S^\lambda_{n,d}(f;x) - f(x) - S^\lambda_{n,d}((t-x);x)f'(x) - \frac{f''(x)}{2}\big(S^\lambda_{n,d}((t-x)^2;x) + S^\lambda_{n,d}(1;x)\big)\right| \\ &\le 2\|f''-g\|S^\lambda_{n,d}((t-x)^2;x) + 2\|\tau g'^{-1}(x)S^\lambda_{n,d}(|t-x|^3;x)\end{aligned}$$

$$\leq \frac{C}{n}x(1-x)\|f''-g\| + 2\|\tau g'^{-1}(x)\{S^{\lambda}_{n,d}((t-x)^2;x)\}^{1/2}\{S^{\lambda}_{n,d}((t-x)^4;x)\}^{1/2}$$
$$\leq \frac{C}{n}\tau^2(x)\Big\{\|f''^{-1/2}\|\tau g'\|\Big\}$$

by (13)–(15). Taking infimum on the right-hand side over all $g \in W_\tau[0,1+d]$, we deduce

$$\left|S^{\lambda}_{n,d}(f;x) - f(x) - \beta_{n,\lambda}(x)f'(x) - \frac{\alpha_{n,\lambda}(x)+1}{2}f''(x)\right| \leq \frac{C}{n}\tau^2(x)\omega_\tau\left(f'', \frac{1}{\sqrt{n}}\right).$$

□

Finally we obtain the following theorem applying Taylor's expansion theorem and as an immediate consequence of Lemma (2), Corollary (3) and Corollary (4):

Theorem 21. *Let $f \in C_B[0,1+d]$, then for each $x \in [0,1]$*

$$\lim_{n\to\infty} n\big\{S^{\lambda}_{n,d}(f;x) - f(x)\big\} = xd\ f'(x) + \frac{x(1-x)}{2}\ f''(x)$$

uniformly on $[0,1]$, where $f', f'' \in C_B[0,1+d]$

As an immediate consequence of Theorem 20 we have the following result.

Corollary 22. *Let $f \in C[0,1+d]$, then*

$$\lim_{n\to\infty} n\left[S^{\lambda}_{n,d}(f;x) - f(x) - \beta_{n,\lambda}(x)f'(x) - \frac{\alpha_{n,\lambda}(x)+1}{2}f''(x)\right] = 0,$$

where $f', f'' \in C_B[0,1+d]$, and $\alpha_{n,\lambda}(x)$ and $\beta_{n,\lambda}(x)$ are defined in Theorem 9.

5. Concluding Remarks

A Korovkin type approximation theorem via K_a-convergence on weighted spaces is studied by Yıldız et al. in [31] and a new concept, statistical e-convergence, is introduced by Sever and Talo in [18, 24, 32]. As a future work we may study the approximation properties of operators defined in this article and other Bernstein type operators using those convergence types. The results of the paper will also be extended to λ-Schurer-Kantorovich and λ-Schurer-Stancu operators using λ-Bézier-Schurer bases defined in (1).

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SIMPLE CRITERIA FOR UNIVALENCE AND COEFFICIENT BOUNDS FOR A CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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AND EBRAHIM ANALOUEI ADEGANI

Abstract. In the first part of this work we present several new geometric properties of analytic functions by applying the differential subordination. In addition, several results in the geometric functions theory pointed out. In the second part we find upper bounds for coefficients of functions in class $\mathcal{B}_{\Sigma}^{q;\mu}(\beta,\lambda,h)$ which is defined by fractional q-calculus operators.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}:=\{z\in\mathbb{C}:|z|<1\}$, and denote by $\mathcal{S}$ the class of all functions of $\mathcal{A}$ which are univalent in $\mathbb{U}$.

For two functions f and F which are analytic in $\mathbb{U}$, we say that the function f *is subordinate to* F in $\mathbb{U}$, and write $f(z)\prec F(z)$, if there exists a *Schwarz function* ω, which is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$, such that $f(z)=F(\omega(z))$ for all $z\in\mathbb{U}$.

By *Schwarz's lemma* we have $|\omega(z)|\le|z|$, $z\in\mathbb{U}$, which concludes that $\omega(\mathbb{U})\subset\mathbb{U}$. Since $\omega(0)=0$ and $\omega(\mathbb{U})\subset\mathbb{U}$ it follows that if $f(z)\prec F(z)$, then $f(0)=F(0)$ and $f(\mathbb{U})\subset F(\mathbb{U})$. In particular, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z)\prec F(z)\Leftrightarrow f(0)=F(0)\quad\text{and}\quad f(\mathbb{U})\subset F(\mathbb{U}).$$

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First, Miller and Mocanu [18] in 1978 introduced the method of differential subordinations and then in recent years several authors obtained several applications in the geometric functions theory by using differential subordination, see for example [5, 7, 8, 9, 12, 13, 15, 20].

We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are *starlike of order* α in $\mathbb{U}$, as follows:

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathbb{U},\ 0 \leq \alpha < 1 \right\},$$

and, in particular, $\mathcal{S}^* := \mathcal{S}^*(0)$ is the class of *starlike functions* in the unit disk $\mathbb{U}$.

Also, we denote by $\mathcal{C}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are *close-to-convex of order* α if there exists a function $g \in \mathcal{S}^*$ such that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > \alpha,\ z \in \mathbb{U},\ 0 \leq \alpha < 1.$$

In particular, $\mathcal{C} := \mathcal{C}(0)$ is the class of *close-to-convex functions* in the unit disk $\mathbb{U}$. It is well-known that $\mathcal{S}(\alpha) \subset \mathcal{S}$ and $\mathcal{C}(\alpha) \subset \mathcal{S}$, for all $0 \leq \alpha < 1$.

It is well known that every function $f \in \mathcal{S}$ contains a disk of radius $\dfrac{1}{4}$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by $f^{-1}\left(f(z)\right) = z$ $(z \in \mathbb{U})$, and $f\left(f^{-1}(w)\right) = w$ $\left(|w| < r_0(f);\ r_0(f) \geq \dfrac{1}{4}\right)$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots =: w + \sum_{n=2}^{\infty} b_n w^n. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$, and let Σ denote the class of bi-univalent functions in $\mathbb{U}$. In recent years many authors made an effort to introduce various subclasses of the bi-univalent function class Σ, see for example [10, 22, 23, 27].

Purohit and Raina [25] (see also [22]) defined a *fractional q-differential operator* Ω_q^{μ} (by using the definitions of the fractional q-calculus operators) for a function f of the form (1.1) by

$$\Omega_q^{\mu} f(z) = z + \sum_{n=2}^{\infty} \Psi_q^{n}(\mu) a_n z^n = \frac{\Gamma_q(2-\mu)}{\Gamma_q(2)} z^{\mu-1} D_{q,z}^{\mu} f(z),\ z \in \mathbb{U}, \tag{1.3}$$

where

$$\Theta_n := \Psi_n^{q}(\mu) = \frac{\Gamma_q(2-\mu)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\mu)},\ -\infty < \mu < 2,\ 0 < q < 1,$$

where $D_{q,z}^{\mu} f$ in (1.3) represents, respectively, a fractional q-integral of f of order μ when $-\infty < \mu < 0$, and a fractional q-derivative of f of order μ when $0 \leq \mu < 2$.

We note that $\Omega_q^0 f(z) = f(z)$ and $\lim\limits_{q\to 1^-} \Omega_q^\mu f(z) = \Omega^\mu f(z)$ (see Owa and Srivastava [24], Aouf and Dziok [6] and Srivastava and Aouf [26]).

Definition 1.1. [22] Let $h : \mathbb{U} \to \mathbb{C}$ be a convex (univalent) function such that

$$h(0) = 1 \quad \text{and} \quad \operatorname{Re} h(z) > 0,\ z \in \mathbb{U}.$$

A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma^{q,\mu}(\beta, \lambda, h)$ if the following conditions are satisfied:

$$e^{i\beta}\left(\frac{z^{1-\lambda}(\Omega_q^\mu f(z))'}{[\Omega_q^\mu f(z)]^{1-\lambda}}\right) \prec h(z)\cos\beta + i\sin\beta$$

and

$$e^{i\beta}\left(\frac{w^{1-\lambda}(\Omega_q^\mu g(w))'}{[\Omega_q^\mu g(w)]^{1-\lambda}}\right) \prec h(w)\cos\beta + i\sin\beta,$$

where $\beta \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$, $\lambda \geq 0$, $z, w \in \mathbb{U}$, and where $g = f^{-1}$ is given by (1.2).

The following lemmas will be used in prove the main result.

Lemma 1.1. [19] *Let* $p(z) = 1 + \sum\limits_{n\geq m}^{\infty} c_n z^n$, $c_m \neq 0$, *be an analytic function in* $|z| < 1$ *with* $p(0) = 1$. *If there exists a point* z_0, *with* $|z_0| < 1$, *such that*

$$\operatorname{Re} p(z) > 0 \quad \textit{for} \quad |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0,$$

then we have

$$z_0 p'(z_0) = \begin{cases} ikp(z_0), & \textit{when} \quad p(z_0) \neq 0 \\ -l/2, & \textit{when} \quad p(z_0) = 0 \end{cases}$$

for some $k \geq m$, $l \geq m$.

Lemma 1.2. [11, p. 190] *Let* u *be analytic function in the unit disk* $\mathbb{U}$, *with* $u(0) = 0$, *and* $|u(z)| < 1$ *for all* $z \in \mathbb{D}$, *with the power series expansion*

$$u(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}.$$

Then, $|c_n| \leq 1$ *for all* $n = 1, 2, 3, \ldots$. *Furthermore,* $|c_n| = 1$ *for some* n *(*$n = 1, 2, 3, \ldots$*) if and only if* $u(z) = e^{i\theta} z^n$, $\theta \in \mathbb{R}$.

Lemma 1.3. [14] *Let the function* w *be a* Schwarz function *with the power series expansion given by* $w(z) = \sum\limits_{n=1}^{\infty} w_n z^n$, $z \in \mathbb{U}$. *Then, for every complex number* s, *the next inequality holds:*

$$\left|w_2 - s w_1^2\right| \leq 1 + (|s| - 1)\left|w_1^2\right|.$$

In Section 2, the paper aims in presenting several new geometric properties of analytic functions by applying the differential subordinations, and in addition, several special results are pointed out. In Section 3 we use the *Faber polynomial* expansion techniques to derive bounds for the coefficients $|a_n|$ for the functions of the class $\mathcal{B}_{\Sigma}^{q,\mu}(\beta,\lambda,h)$, that our results generalize and improve some of the previously ones. In the literature, several authors used the Faber polynomial expansions under certain conditions to determine the general coefficient bounds of $|a_n|$ for the analytic bi-univalent functions (see, for example, [16, 17, 30]).

2. Sufficient Conditions for Univalence and Starlikeness

In the following section we study differential subordinations and several sufficient conditions for the univalence, starlikeness and close-to-convexity of functions $f \in \mathcal{A}$.

Theorem 2.1. *Let p be an analytic function in $\mathbb{U}$, with $p(0) = 1$ and $p'(0) \neq 0$, that satisfies*

$$\operatorname{Re}\left[\frac{p(z)+\dfrac{zp'(z)}{p(z)}}{a+p^2(z)+zp'(z)}\right]^2 > 0,\ z \in \mathbb{U}, \quad \textit{for some} \quad a \in \mathbb{R}\setminus\{-1\}. \tag{2.1}$$

Then,

$$\operatorname{Re} p(z) > 0,\ z \in \mathbb{U}.$$

Proof. If $a = 0$, using the fact that $p(0) = 1$ it is easy to prove that the assumption (2.1) implies $\operatorname{Re} p(z) > 0$, $z \in \mathbb{U}$, and therefore we will assume that $a \neq 0$. Also, since the inequality (2.1) holds for $z_* = 0$, it is necessary to assume that $a \neq -1$.

Supposing that there exists a point $z_0 \in \mathbb{U}$ such that

$$\operatorname{Re} p(z) > 0, \quad \text{for} \quad |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0,$$

it follows that

$$p(z_0) = i\lambda,\ \lambda \in \mathbb{R}.$$

Hence, according to Lemma 1.1 for $m = 1$, we have

$$z_0p'(z_0) = \begin{cases} ikp(z_0), & \text{when} \quad p(z_0) \neq 0 \\ -l/2, & \text{when} \quad p(z_0) = 0 \end{cases} = \begin{cases} -k\lambda, & \text{when} \quad \lambda \neq 0 \\ -l/2, & \text{when} \quad \lambda = 0, \end{cases}$$

for some $k \geq 1$, $l \geq 1$.

(i) For the case $p(z_0) \neq 0$ suppose that

$$a + p^2(z_0) + zp'(z_0) = a - \lambda^2 - k\lambda = 0. \tag{2.2}$$

(α) If

$$p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}=i(\lambda+k)\neq 0,$$

then $z_0\in\mathbb{U}$ will be a double pole for the function

$$\left[\frac{p(z)+\dfrac{zp'(z)}{p(z)}}{a+p^2(z)+zp'(z)}\right]^2$$

and therefore, in any neighborhood $U(z_0;\rho):=\{z\in\mathbb{C}:|z|<\rho\}\subset\mathbb{U}$ of the pole z_0 there exists at least a $z_\rho\in U(z_0;\rho)$ such that

$$\operatorname{Re}\left[\frac{p(z_\rho)+\dfrac{z_\rho p'(z_\rho)}{p(z_\rho)}}{a+p^2(z_\rho)+z_\rho p'(z_\rho)}\right]^2<0,$$

which contradicts the assumption (2.1).

(β) If

$$p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}=i(\lambda+k)=0,$$

from this relation and (2.2) it follows that $a=0$, which contradicts our assumption.

Therefore, from (α) and (β) we deduce that the assumption (2.1) implies that the function

$$\left[\frac{p(z)+\dfrac{zp'(z)}{p(z)}}{a+p^2(z)+zp'(z)}\right]^2,$$

is analytic in $\mathbb{U}$, and

$$\operatorname{Re}\left[\frac{p(z_0)+\dfrac{z_0p'(z_0)}{p(z_0)}}{a+p^2(z_0)+zp'(z_0)}\right]^2=\operatorname{Re}\left(\frac{i\lambda+ik}{a-\lambda^2-k\lambda}\right)^2=-\left(\frac{\lambda+k}{a-\lambda^2-k\lambda}\right)^2\leq 0,$$

which is a contradiction with the assumption (2.1).

(ii) For the case $p(z_0)=0$ it follows that $z_0p'(z_0)$ is a negative real number, and the function $\dfrac{zp'(z)}{p(z)}$ has a simple pole at z_0. Since $p(0)=1$, then $z_0\in\mathbb{U}\setminus\{0\}$ will be at least a double pole for the function

$$\left[\frac{p(z)+\dfrac{zp'(z)}{p(z)}}{a+p^2(z)+zp'(z)}\right]^2$$

and therefore, in any neighborhood $U(z_0;\rho) := \{z \in \mathbb{C} : |z| < \rho\} \subset \mathbb{U}$ of the pole z_0 there exists at least a $z_\rho \in U(z_0;\rho)$ such that have

$$\operatorname{Re}\left[\frac{p(z_\rho)+\dfrac{z_\rho p'(z_\rho)}{p(z_\rho)}}{a+p^2(z_\rho)+z_\rho p'(z_\rho)}\right]^2 < 0,$$

which contradicts the assumption (2.1).

Concluding, from the above cases it follows that $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{U}$, and the proof of the theorem is complete. □

For $f \in \mathcal{A}$ and $p := f'$ the above theorem leads to the following result which gives sufficient condition for the close-to-convexity (univalence) of the function f:

Corollary 2.1. *If $f \in \mathcal{A}$, with $f''(0) \neq 0$, and satisfies*

$$\operatorname{Re}\left[\frac{f'(z)+\dfrac{zf''(z)}{f'(z)}}{a+[f'^2+zf''(z)}\right]^2 > 0,\ z \in \mathbb{U}, \quad \textit{for some} \quad a \in \mathbb{R}\setminus\{-1\},$$

then

$$\operatorname{Re} f'(z) > 0,\ z \in \mathbb{U}.$$

For $f \in \mathcal{A}$ and $p(z) := \dfrac{zf'(z)}{f(z)}$, then $p'(0) \neq 0$ is equivalent to $f''(0) \neq 0$, and Theorem 2.1 leads to the following result which gives a sufficient starlikeness (univalence) condition:

Corollary 2.2. *If $f \in \mathcal{A}$, with $f''(0) \neq 0$, and satisfies*

$$\operatorname{Re}\left[\frac{1+\dfrac{zf''(z)}{f'(z)}}{a+\dfrac{zf'(z)}{f(z)}+\dfrac{z^2f''(z)}{f(z)}}\right]^2 > 0,\ z \in \mathbb{U}, \quad \textit{for some} \quad a \in \mathbb{R}\setminus\{-1\},$$

then

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{U}.$$

Theorem 2.2. *Let p be an analytic function in $\mathbb{U}$, with $p(0)=1$, $p'(0)\neq 0$, that satisfies*

$$\operatorname{Re}\left[\frac{p(z)+\dfrac{zp'(z)}{p(z)}}{a+\dfrac{zp'(z)}{p^2(z)}}\right]^2>0,\ z\in\mathbb{U},\quad \textit{for some}\quad a\in\mathbb{R}\setminus\{0\}. \tag{2.3}$$

Then,

$$\operatorname{Re}p(z)>0,\ z\in\mathbb{U}.$$

Proof. Since the inequality (2.3) holds for $z_*=0$ it is necessary to assume that $a\neq 0$. For $a=1$, using the fact that $p(0)=1$ it is easy to prove that the assumption (2.3) implies our conclusion, and thus we will assume that $a\neq 1$.

Supposing that there exists a point $z_0\in\mathbb{U}$ such that

$$\operatorname{Re}p(z)>0,\quad \text{for}\quad |z|<|z_0|$$

and

$$\operatorname{Re}p(z_0)=0,$$

it follows that

$$p(z_0)=i\lambda,\ \lambda\in\mathbb{R}.$$

Now, using Lemma 1.1 for $m=1$, we have

$$z_0p'(z_0)=\begin{cases} ikp(z_0), & \text{when}\quad p(z_0)\neq 0\\ -l/2, & \text{when}\quad p(z_0)=0\end{cases}=\begin{cases} -k\lambda, & \text{when}\quad \lambda\neq 0\\ -l/2, & \text{when}\quad \lambda=0,\end{cases}$$

for some $k\geq 1$, $l\geq 1$.

(i) For the case $p(z_0)\neq 0$, that is $\lambda\neq 0$, suppose that

$$a+\frac{z_0p'(z_0)}{p^2(z_0)}=a+\frac{k}{\lambda}=0. \tag{2.4}$$

(α) If

$$p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}=i(\lambda+k)\neq 0,$$

then $z_0\in\mathbb{U}$ will be a double pole for the function

$$\left[\frac{p(z)+\dfrac{zp'(z)}{p(z)}}{a+\dfrac{zp'(z)}{p^2(z)}}\right]^2$$

and therefore, in any neighborhood $U(z_0;\rho) := \{z \in \mathbb{C} : |z| < \rho\} \subset \mathbb{U}$ of the pole z_0 there exists at least a $z_\rho \in U(z_0;\rho)$ such that

$$\operatorname{Re}\left[\frac{p(z_\rho) + \dfrac{z_\rho p'(z_\rho)}{p(z_\rho)}}{a + \dfrac{z_\rho p'(z_\rho)}{p^2(z_\rho)}}\right]^2 < 0,$$

which contradicts the assumption (2.3).

(β) If

$$p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} = i(\lambda + k) = 0,$$

from this relation and (2.4) it follows that $a = 1$, which contradicts our assumption.

Therefore, from (α) and (β) we deduce that the assumption (2.3) implies that the function

$$\left[\frac{p(z) + \dfrac{zp'(z)}{p(z)}}{a + \dfrac{zp'(z)}{p^2(z)}}\right]^2,$$

is analytic in $\mathbb{U}$, and

$$\operatorname{Re}\left[\frac{p(z_0) + \dfrac{z_0 p'(z_0)}{p(z_0)}}{a + \dfrac{z_0 p'(z_0)}{p^2(z_0)}}\right]^2 = \operatorname{Re}\left(\frac{i\lambda + ik}{a + \dfrac{k}{\lambda}}\right)^2 = -\left(\frac{\lambda + k}{a + \dfrac{k}{\lambda}}\right)^2 \leq 0,$$

which is a contradiction with the assumption (2.3).

(ii) For the case $p(z_0) = 0$, since

$$\left[\frac{p(z) + \dfrac{zp'(z)}{p(z)}}{a + \dfrac{zp'(z)}{p^2(z)}}\right]^2 = \left[\frac{p^3(z) + zp(z)p'(z)}{ap^2(z) + zp'(z)}\right]^2,$$

it follows that

$$\operatorname{Re}\left[\frac{p^3(z_0) + z_0 p(z_0)p'(z_0)}{ap^2(z_0) + z_0 p'(z_0)}\right]^2 = 0,$$

which contradicts the assumption (2.3).

Thus, from the above cases it follows that $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{U}$. □

For $f \in \mathcal{A}$ and $p := f'$, and for $p(z) := \dfrac{zf'(z)}{f(z)}$, Theorem 2.2 reduces to the following two results which represent sufficient condition for the close-to-convexity and starlikeness, respectively:

Corollary 2.3. *If $f \in \mathcal{A}$, with $f''(0) \neq 0$, and satisfies*

$$\operatorname{Re}\left[\frac{f'(z)+\dfrac{zf''(z)}{f'(z)}}{a+\dfrac{zf''(z)}{[f'^2}}\right]^2>0,\ z\in\mathbb{U},\quad \textit{for some}\quad a\in\mathbb{R}\setminus\{0\},$$

then,

$$\operatorname{Re} f'(z)>0,\ z\in\mathbb{U}.$$

Corollary 2.4. *If $f \in \mathcal{A}$, with $f''(0) \neq 0$, and satisfies*

$$\operatorname{Re}\left[\frac{1+\dfrac{zf''(z)}{f'(z)}}{a+\dfrac{f(z)}{zf'(z)}\left(1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}\right)}\right]^2>0,\ z\in\mathbb{U},\quad \textit{for some}\quad a\in\mathbb{R}\setminus\{0\},$$

then,

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0,\ z\in\mathbb{U}.$$

Theorem 2.3. *Let p be an analytic function in $\mathbb{U}$, with $p(0)=1$, $p'(0)\neq 0$, that satisfies*

$$\operatorname{Re}\left[\frac{p(z)\,[a+zp'(z)]}{a+p^2(z)+zp'(z)}\right]^2>0,\ z\in\mathbb{U},\quad \textit{for some}\quad a\in\left(-\infty,\frac{1}{2}\right)\setminus\{-1,0\}. \tag{2.5}$$

Then

$$\operatorname{Re} p(z)>0,\ z\in\mathbb{U}.$$

Proof. First, since the assumption (2.5) holds for $z_*=0$, it is necessary to assume that $a\neq 0$ and $a\neq -1$. If we suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$\operatorname{Re} p(z)>0,\quad \text{for}\quad |z|<|z_0|$$

and

$$\operatorname{Re} p(z_0)=0,$$

it follows that

$$p(z_0)=i\lambda,\ \lambda\in\mathbb{R}.$$

Hence, according to Lemma 1.1 for $m=1$, we have

$$z_0p'(z_0)=\begin{cases} ikp(z_0), & \text{when}\quad p(z_0)\neq 0\\ -l/2, & \text{when}\quad p(z_0)=0\end{cases}=\begin{cases} -k\lambda, & \text{when}\quad \lambda\neq 0\\ -l/2, & \text{when}\quad \lambda=0,\end{cases}$$

for some $k\geq 1$, $l\geq 1$.

(i) For the case $p(z_0)\neq 0$, that is $\lambda\neq 0$, suppose that

$$a+p^2(z_0)+zp'(z_0)=a-\lambda^2-k\lambda=0. \tag{2.6}$$

(α) If

$$p(z_0)\,[a+z_0p'(z_0)]=i\lambda(a-k\lambda)\neq 0,$$

then $z_0 \in \mathbb{U}$ will be a double pole for the function

$$\left[\frac{p(z)\,[a+zp'(z)]}{a+p^2(z)+zp'(z)}\right]^2$$

and therefore, in any neighborhood $U(z_0;\rho) := \{z \in \mathbb{C} : |z| < \rho\} \subset \mathbb{U}$ of the pole z_0 there exists at least a $z_\rho \in U(z_0;\rho)$ such that have

$$\operatorname{Re}\left[\frac{p(z_\rho)\,[a+z_\rho p'(z_\rho)]}{a+p^2(z_\rho)+z_\rho p'(z_\rho)}\right]^2 < 0,$$

which contradicts the assumption (2.5).

(β) If

$$p(z_0)\,[a+z_0p'(z_0)] = i\lambda(a-k\lambda) = 0,$$

hence $a = k\lambda$, and from (2.6) it follows that $-\lambda^2 = 0$, that contradicts the fact $\lambda \neq 0$.

Therefore, from (α) and (β) we deduce that the assumption (2.1) implies that the function

$$\left[\frac{p(z)\,[a+zp'(z)]}{a+p^2(z)+zp'(z)}\right]^2$$

is analytic in $\mathbb{U}$, and

$$\operatorname{Re}\left[\frac{p(z_0)\,[a+z_0p'(z_0)]}{a+p^2(z_0)+z_0p'(z_0)}\right]^2 = \operatorname{Re}\left[\frac{i\lambda(a-k\lambda)}{a-\lambda^2-k\lambda}\right]^2 = -\left[\frac{\lambda(a-k\lambda)}{a-\lambda^2-k\lambda}\right]^2 \leq 0,$$

which is a contradiction with the assumption (2.5).

(ii) For the case $p(z_0) = 0$, using the fact that $a < \dfrac{1}{2}$ we have

$$a + p^2(z_0) + z_0p'(z_0) = a - \frac{l}{2} \neq 0,$$

hence

$$\operatorname{Re}\left[\frac{p(z_0)\,[a+z_0p'(z_0)]}{a+p^2(z_0)+z_0p'(z_0)}\right]^2 = 0,$$

which contradicts the assumption (2.5).

From the two which discussed above it follows that $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{U}$. $\square$

Taking $f \in \mathcal{A}$ and $p := f'$, and $p(z) := \dfrac{zf'(z)}{f(z)}$ in Theorem 2.3 we obtain the next two special cases that represent sufficient condition for the close-to-convexity and starlikeness, respectively:

Corollary 2.5. *If $f \in \mathcal{A}$, with $f''(0) \neq 0$, and satisfies*

$$\operatorname{Re}\left[\frac{f'(z)[a+zf''(z)]}{a+[f'^2+zf''(z)}\right]^2 > 0,\ z \in \mathbb{U}, \quad \text{for some} \quad a \in \left(-\infty, \frac{1}{2}\right) \setminus \{-1, 0\},$$

then

$$\operatorname{Re} f'(z) > 0,\ z \in \mathbb{U}.$$

Corollary 2.6. *If $f \in \mathcal{A}$, with $f''(0) \neq 0$, and satisfies*

$$\operatorname{Re}\left[\frac{a + \dfrac{zf'(z)}{f(z)}\left(1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}\right)}{1 + a\dfrac{f(z)}{zf'(z)} + \dfrac{zf''(z)}{f'(z)}}\right]^2 > 0,\ z \in \mathbb{U},$$

for some $a \in \left(-\infty, \dfrac{1}{2}\right) \setminus \{-1, 0\}$, *then*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{U}.$$

Theorem 2.4. *Let p be an analytic function in $\mathbb{U}$, with $p(0) = 1$, $p'(0) \neq 0$, that satisfies*

$$\operatorname{Re}\left[\frac{p(z)\left[a + zp'(z)\right]}{a + \dfrac{zp'(z)}{p^2(z)}}\right]^2 > 0,\ z \in \mathbb{U}, \quad \text{for some} \quad a \in \mathbb{R}. \tag{2.7}$$

Then

$$\operatorname{Re} p(z) > 0,\ z \in \mathbb{U}.$$

Proof. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\operatorname{Re} p(z) > 0, \quad \text{for} \quad |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0.$$

By using Lemma 1.1 for $m = 1$, it follows that

$$p(z_0) = i\lambda,\ \lambda \in \mathbb{R},$$

and

$$z_0 p'(z_0) = \begin{cases} ikp(z_0), & \text{when} \quad p(z_0) \neq 0 \\ -l/2, & \text{when} \quad p(z_0) = 0 \end{cases} = \begin{cases} -k\lambda, & \text{when} \quad \lambda \neq 0 \\ -l/2, & \text{when} \quad \lambda = 0, \end{cases}$$

for some $k \geq 1$, $l \geq 1$.

(i) For the case $p(z_0) \neq 0$, that is $\lambda \neq 0$, suppose that

$$a + \frac{z_0 p'(z_0)}{p^2(z_0)} = a + \frac{k}{\lambda} = 0. \tag{2.8}$$

(α) If

$$p(z_0)\left[a + z_0 p'(z_0)\right] = i\lambda(a - k\lambda) \neq 0,$$

then $z_0 \in \mathbb{U}$ will be a double pole for the function

$$\left[\frac{p(z)\left[a+zp'(z)\right]}{a+\dfrac{zp'(z)}{p^2(z)}}\right]^2$$

and therefore, in any neighborhood $U(z_0;\rho) := \{z \in \mathbb{C} : |z| < \rho\} \subset \mathbb{U}$ of the pole z_0 there exists at least a $z_\rho \in U(z_0;\rho)$ such that have

$$\left[\frac{p(z_\rho)\left[a+z_\rho p'(z_\rho)\right]}{a+\dfrac{z_\rho p'(z_\rho)}{p^2(z_\rho)}}\right]^2 < 0,$$

which contradicts the assumption (2.7).

(β) If

$$p(z_0)\left[a+z_0p'(z_0)\right] = i\lambda(a-k\lambda) = 0,$$

then $a = k\lambda$ and from (2.8) it follows that $k = 0$ or $\lambda^2 = -1$, which contradicts the facts $k \geq 1$ and $\lambda \in \mathbb{R}$.

Therefore, from (α) and (β) we deduce that the assumption (2.7) implies that the function

$$\left[\frac{p(z)\left[a+zp'(z)\right]}{a+\dfrac{zp'(z)}{p^2(z)}}\right]^2,$$

is analytic in $\mathbb{U}$, and

$$\operatorname{Re}\left[\frac{p(z_0)\left[a+z_0p'(z_0)\right]}{a+\dfrac{z_0p'(z_0)}{p^2(z_0)}}\right]^2 = \operatorname{Re}\left[\frac{i\lambda(a-k\lambda)}{a+\dfrac{k}{\lambda}}\right]^2 = -\left[\frac{\lambda(a-k\lambda)}{a+\dfrac{k}{\lambda}}\right]^2 \leq 0,$$

which is a contradiction with the assumption (2.7).

(ii) For the case $p(z_0) = 0$, since

$$\left[\frac{p(z)\left[a+zp'(z)\right]}{a+\dfrac{zp'(z)}{p^2(z)}}\right]^2 = \left[\frac{p^3(z)\left[a+zp'(z)\right]}{ap^2(z)+zp'(z)}\right]^2,$$

it follows that

$$\operatorname{Re}\left[\frac{p^3(z_0)\left[a+z_0p'(z_0)\right]}{ap^2(z_0)+z_0p'(z_0)}\right]^2 = 0,$$

which contradicts the assumption (2.7).

Concluding, from the two cases we discussed above it follows that $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{U}$. $\square$

Replacing $p := f'$, and $p(z) := \dfrac{zf'(z)}{f(z)}$ where $f \in \mathcal{A}$ in Theorem 2.4 we obtain the next two special cases that represent sufficient condition for the close-to-convexity and starlikeness, respectively:

Corollary 2.7. *If $f \in \mathcal{A}$, with $f''(0) \neq 0$, and satisfies*

$$\operatorname{Re}\left[\frac{f'(z)[a + zf''(z)]}{a + \dfrac{zf''(z)}{[f'^2}}\right]^2 > 0,\ z \in \mathbb{U}, \quad \textit{for some} \quad a \in \mathbb{R},$$

then,

$$\operatorname{Re} f'(z) > 0,\ z \in \mathbb{U}.$$

Corollary 2.8. *If $f \in \mathcal{A}$, with $f''(0) \neq 0$, and satisfies*

$$\operatorname{Re}\left[\frac{\dfrac{zf'(z)}{f(z)}\left[a + \dfrac{zf'(z)}{f(z)}\left(1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}\right)\right]}{a + \dfrac{f(z)}{zf'(z)}\left(1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}\right)}\right]^2 > 0,\ z \in \mathbb{U}, \quad \textit{for some} \quad a \in \mathbb{R},$$

then,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{U}.$$

Remark 2.1.
(1) For $g \in \mathcal{S}^*$ and $f \in \mathcal{A}$, such that $2f''(0) \neq g''(0)$, setting $p(z) := \dfrac{zf'(z)}{g(z)}$ in the above theorems we will obtain sufficient condition for close-to-convexity.
(2) For $f \in \mathcal{A}$, with $f''(0) \neq 0$, setting $p(z) := \dfrac{f(z)}{z}$ in the above theorems we will obtain sufficient condition for the functions f to satisfy the inequality $\operatorname{Re} \dfrac{f(z)}{z} > 0$, $z \in \mathbb{U}$.

3. Coefficient Bounds

We begin by deriving upper bounds for the general *Taylor-Maclaurin* coefficients $|a_n|$ for $n \geq 3$ of the functions belonging in the class $\mathcal{B}_{\Sigma}^{q,\mu}(\beta, \lambda, h)$, and next we will find estimates for the initial coefficient $|a_2|$.

Using the *Faber polynomial expansion* of functions $f \in \mathcal{S}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see for details

[1] and [2])

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots,a_n)w^n, \tag{3.1}$$

where

$$\begin{aligned}K_{n-1}^{-n}=&\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1}+\frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3\\&+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left[a_5+(-n+2)a_3^2\right]\\&+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)a_3a_4\right]+\sum_{j\geq 7}a_2^{n-j}V_j\end{aligned}$$

such that V_j $(7\leq j\leq n)$ is a homogeneous polynomial in the variables $a_2,a_3,\ldots,a_n$, and the expressions such as (for example) $(-n)!$ are to be interpreted *symbolically* by

$$\begin{aligned}(-n)! &\equiv \Gamma(1-n):=(-n)(-n-1)(-n-2)\ldots,\\ \text{with}\quad n &\in \mathbb{N}_0:=\mathbb{N}\cup\{0\},\ \mathbb{N}:=\{1,2,3,\ldots\}.\end{aligned}$$

In particular, the first three terms of K_{n-1}^{-n} are given by

$$K_1^{-2}=-2a_2,\quad K_2^{-3}=3\left(2a_2^2-a_3\right)\quad\text{and}\quad K_3^{-4}=-4\left(5a_2^3-5a_2a_3+a_4\right).$$

In general, for any p real value the expansion of K_n^p is given below (see for details, [1, 29]; see also [2, p. 349])

$$K_n^p=pa_{n+1}+\frac{p(p-1)}{2}D_n^2+\frac{p!}{(p-3)!3!}D_n^3+\cdots+\frac{p!}{(p-n)!n!}D_n^n, \tag{3.2}$$

where $D_n^p=D_n^p(a_2,a_3,\ldots,a_{n+1})$ (see for details [29]). We also have

$$D_n^m(a_2,a_3,\ldots,a_{n+1})=\sum_{n=1}^{\infty}\frac{m!(a_2)^{\mu_1}\cdot\ldots\cdot(a_{n+1})^{\mu_n}}{\mu_1!\cdot\ldots\cdot\mu_n!}, \tag{3.3}$$

where the sum is taken over all nonnegative integers $\mu_1,\ldots,\mu_n$ satisfying the conditions

$$\begin{cases}\mu_1+\mu_2+\cdots+\mu_n=m\\ \mu_1+2\mu_2+\cdots+n\mu_n=n.\end{cases}$$

It is clear that $D_n^n(a_2,a_3,\ldots,a_{n+1})=a_2^n$.

Theorem 3.1. *Let the function $f\in\mathcal{B}_{\Sigma}^{q,\mu}(\beta,\lambda,h)$ be given by (1.1) with the power expansion of the function h given by*

$$h(z)=1+\sum_{n=1}^{\infty}B_nz^n,\ z\in\mathbb{U}, \tag{3.4}$$

and suppose that $B_1 \neq 0$. *If* $a_k = 0$ *for* $2 \leq k \leq n-1$, *where* $n \geq 3$, *then*

$$|a_n| \leq \frac{|B_1|\cos\beta}{[\lambda+(n-1)]\Theta_n}. \tag{3.5}$$

Proof. For $f \in \mathcal{B}_{\Sigma}^{q,\mu}(\beta,\lambda,h)$ given by (1.1), using the relations (1.6) and (1.7) from [2, page 344] we have

$$e^{i\beta}\left(\frac{z^{1-\lambda}(\Omega_q^{\mu}f(z))'}{[\Omega_q^{\mu}f(z)]^{1-\lambda}}\right) = e^{i\beta}\left(1+\sum_{n=2}^{\infty}\left(1+\frac{n-1}{\lambda}\right)K_{n-1}^{\lambda}(\Theta_2 a_2, \Theta_3 a_3, \ldots, \Theta_n a_n)z^{n-1}\right),$$

and for its inverse map $g = f^{-1}$, according to the expansion formula (1.2) we have

$$e^{i\beta}\left(\frac{w^{1-\lambda}(\Omega_q^{\mu}g(w))'}{[\Omega_q^{\mu}g(w)]^{1-\lambda}}\right) = e^{i\beta}\left(1+\sum_{n=2}^{\infty}\left(1+\frac{n-1}{\lambda}\right)K_{n-1}^{\lambda}(\Theta_2 b_2, \Theta_3 b_3, \ldots, \Theta_n b_n)w^{n-1}\right),$$

where $b_n = \dfrac{1}{n}K_{n-1}^{-n}(a_2, a_3, \ldots, a_n)$, $n = 2, 3, \ldots$ are defined by (3.1).

Furthermore, since $f \in \mathcal{B}_{\Sigma}^{q,\mu}(\beta,\lambda,h)$, from the definition of the subordination there exist two Schwartz functions $u, v : \mathbb{U} \to \mathbb{U}$ of the form $u(z) = \sum\limits_{n=1}^{\infty} p_n z^n$, $v(z) = \sum\limits_{n=1}^{\infty} q_n z^n$, such that

$$e^{i\beta}\left(\frac{z^{1-\lambda}(\Omega_q^{\mu}f(z))'}{[\Omega_q^{\mu}f(z)]^{1-\lambda}}\right) = h(u(z))\cos\beta + i\sin\beta, \tag{3.6}$$

and

$$e^{i\beta}\left(\frac{w^{1-\lambda}(\Omega_q^{\mu}g(w))'}{[\Omega_q^{\mu}g(w)]^{1-\lambda}}\right) = h(v(w))\cos\beta + i\sin\beta. \tag{3.7}$$

Moreover, from (3.3) we have

$$h(u(z)) = 1 + B_1 p_1 z + \left(B_1 p_2 + B_2 p_1^2\right) z^2 + \cdots = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_k D_n^k(p_1, p_2, \ldots, p_n) z^n,$$

and

$$h(v(w)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_k D_n^k(q_1, q_2, \ldots, q_n) w^n.$$

Equating the corresponding coefficients of (3.6) and (3.7) we get, respectively,

$$e^{i\beta}\left(1+\frac{n-1}{\lambda}\right)K_{n-1}^{\lambda}(\Theta_2 a_2, \Theta_3 a_3, \ldots, \Theta_n a_n) = \sum_{k=1}^{n-1} B_k D_{n-1}^k(p_1, p_2, \ldots, p_{n-1})\cos\beta \tag{3.8}$$

and

$$e^{i\beta}\left(1+\frac{n-1}{\lambda}\right)K_{n-1}^{\lambda}(\Theta_2 b_2,\Theta_3 b_3,\ldots,\Theta_n b_n)=\sum_{k=1}^{n-1}B_k D_{n-1}^{k}(q_1,q_2,\ldots,q_{n-1})\cos\beta. \tag{3.9}$$

We observe that if $a_k=0$ for all $2\le k\le n-1$, by the definition of K_n^p we have $b_n=-a_n$, and since $B_1\neq 0$ we have $p_1=\cdots=p_{n-2}=0$ and $q_1=\cdots=q_{n-2}=0$. Hence from (3.8) and (3.9) we obtain, respectively,

$$[\lambda+(n-1)]e^{i\beta}\Theta_n a_n=B_1 p_{n-1}\cos\beta$$

and

$$-[\lambda+(n-1)]e^{i\beta}\Theta_n a_n=B_1 q_{n-1}\cos\beta.$$

Taking the modules of either of the above two equalities and using Lemma 1.2 we obtain our result. □

Theorem 3.2. *Let the function $f\in\mathcal{B}_{\Sigma}^{q,\mu}(\beta,\lambda,h)$ be given by* (1.1). *Then*

$$|a_2|\le\frac{|B_1|\sqrt{2|B_1|}\cos\beta}{\sqrt{|B_1|^2\,|(\lambda-1)(\lambda+2)\Theta_2^2+2(\lambda+2)\Theta_3|\cos\beta+2(|B_1|-|B_2|)(1+\lambda)^2\Theta_2^2}},$$

for those values of all the parameters such that the denominator is not zero.

Proof. If we set $n=2$ and $n=3$ in (3.8) and (3.9), respectively, we obtain

$$e^{i\beta}(1+\lambda)\Theta_2 a_2=B_1p_1\cos\beta \tag{3.10}$$

$$e^{i\beta}\left[\frac{(\lambda-1)(\lambda+2)}{2}\Theta_2^2a_2^2+(\lambda+2)\Theta_3a_3\right]=(B_1p_2+B_2p_1^2)\cos\beta \tag{3.11}$$

$$-e^{i\beta}(1+\lambda)\Theta_2 a_2=B_1q_1\cos\beta \tag{3.12}$$

$$\begin{aligned}&e^{i\beta}\left[\left(\frac{(\lambda-1)(\lambda+2)}{2}\Theta_2^2+2(\lambda+2)\Theta_3\right)a_2^2-(\lambda+2)\Theta_3a_3\right]\\ =\ &(B_1q_2+B_2q_1^2)\cos\beta.\end{aligned} \tag{3.13}$$

From (3.10) and (3.12) we get

$$p_1=-q_1, \tag{3.14}$$

then, adding (3.11) and (3.13) and according to (3.14) we obtain

$$e^{i\beta}[(\lambda-1)(\lambda+2)\Theta_2^2+2(\lambda+2)\Theta_3]a_2^2=B_1\left(p_2+\frac{B_2}{B_1}p_1^2+q_2+\frac{B_2}{B_1}q_1^2\right)\cos\beta.$$

From (3.10), using Lemma 1.3 we have

$$\begin{aligned}&\left|(\lambda-1)(\lambda+2)\Theta_2^2+2(\lambda+2)\Theta_3\right||a_2|^2\le|B_1|\left(\left|p_2+\frac{B_2}{B_1}p_1^2\right|+\left|q_2+\frac{B_2}{B_1}q_1^2\right|\right)\cos\beta\\ &\le 2|B_1|\left(1+\frac{|B_2|-|B_1|}{|B_1|}|p_1^2|\right)\cos\beta=2|B_1|\left[1+\frac{(|B_2|-|B_1|)(1+\lambda)^2\Theta_2^2|a_2^2|}{|B_1|^3\cos^2\beta}\right]\cos\beta.\end{aligned}$$

After some simple computations, from the above inequality we have

$$\left[|B_1|^2\left|(\lambda-1)(\lambda+2)\Theta_2^2+2(\lambda+2)\Theta_3\right|\cos\beta+2(|B_1|-|B_2|)(1+\lambda)^2\Theta_2^2\right]|a_2|^2 \leq 2|B_1|^3\cos^2\beta,$$

which implies our result. □

Remark 3.1. (1) The bound for $|a_2|$ from Theorem 3.2 is smaller than the estimate obtained by Murugusundaramoorthy et al. in [22, Theorem 2.1].

(2) Letting $h(z)=\dfrac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, in Theorem 3.2, we obtain an improvement of the estimate for $|a_2|$ obtained by Murugusundaramoorthy et al. in [22, Theorem 3.1], and it is presented in the next example.

(3) Setting $h(z)=\dfrac{1+(1-2\alpha)z}{1-z}$, $0\leq\alpha<1$, in Theorem 3.2, we obtain an improvement of the estimate for $|a_2|$ obtained by Murugusundaramoorthy et al. in [22, Theorem 4.1], like we will show in Example 3.2.

(4) By setting $\lambda=1$, $\beta=\mu=0$, and $q\to 1^-$ in Theorem 3.2, we get $\Theta_n=1$, hence we obtain an improvement of the estimate for $|a_2|$ obtained by Algahtani in [4, Theorem 2.3].

(5) Taking $\lambda=\beta=\mu=0$ and $q\to 1^-$ in Theorem 3.2, we get $\Theta_n=1$, hence we obtain an improvement of the estimate for $|a_2|$ obtained by Algahtani in [4, Theorem 2.6].

Example 3.1. *Let the function* $f\in\mathcal{B}_\Sigma^{q,\mu}\left(\beta,\lambda,\dfrac{1+Az}{1+Bz}\right)$ *be given by* (1.1), *where* $-1\leq B<A\leq 1$. *If* $a_k=0$ *for* $2\leq k\leq n-1$, *where* $n\geq 3$, *then*

$$|a_n|\leq\frac{(A-B)\cos\beta}{[\lambda+(n-1)]\Theta_n}.$$

Example 3.2. *Let the function* $f\in\mathcal{B}_\Sigma^{q,\mu}\left(\beta,\lambda,\dfrac{1+(1-2\alpha)z}{1-z}\right)$ *be given by* (1.1), *where* $0\leq\alpha<1$. *If* $a_k=0$ *for* $2\leq k\leq n-1$, *where* $n\geq 3$, *then*

$$|a_n|\leq\frac{2(1-\alpha)\cos\beta}{[\lambda+(n-1)]\Theta_n}.$$

4. Conclusion

In the final section, using the Faber polynomial expansion we found upper bounds for $|a_n|$ ($n\geq 3$) coefficients of functions in the class defined by Definition 1.1, and then we obtained an estimate for the initial coefficients $|a_2|$ for the functions of this class. Thus, regarding the proofs of the Theorems 3.1 and 3.2, this technique can be applied for all classes that are defined similarly to the Definition 1.1 in diverse papers enhancing their outcomes (see for example [3, 10, 21, 23, 28] and references therein).

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EQUIVALENCE CONDITIONS OF TWO SYSTEMS OF VECTORS IN THE TAXICAB PLANE AND ITS APPLICATIONS TO TAXICAB POLYGONS

İDRIS ÖREN AND HÜSNÜ ANIL ÇOBAN

Abstract. This study presents the conditions of $M_T(2)$-equivalence for two systems of vectors $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ in R^2_T, where $M_T(2)$ is the group of all isometries of the 2-dimensional taxicab space R^2_T. Firstly a minimal complete system of $M_T(2)$-invariants of $\{x_1, x_2, x_3\}$ is obtained. Then, using the conditions of $M_T(2)$-equivalence, an answer is given to the open problem posed in [10, p.428]. Furthermore, an algorithm is given for constructing taxicab regular polygons in terms of $M_T(2)$-invariants. This algorithm is general and useful to construct the taxicab regular $2n$-gons and gives a tool to solve special cases of the open problem posed in [2, p.32]. Besides, both the conditions of the taxicab regularity of Euclidean regular polygons and Euclidean regularity of taxicab regular polygons are given in terms of $M_T(2)$-invariants.

1. Introduction

Many problems in applied algebra have symmetries or are invariant under certain natural transformations. In particular, all geometric magnitudes and properties are invariant with respect to the underlying transformation group. Properties in Euclidean geometry are invariant under the Euclidean group of rotations, reflections and translations; properties in projective geometry are invariant under the projective transformations, etc. This identification of geometry and invariant theory is expressed in Felix Klein's Erlanger Program (see detailed information in [14, p.14, 193]).

Let R be the field of real numbers. Then the 2-dimensional taxicab space can be introduced by using the metric $d_T(x,y) = |x_1 - y_1| + |x_2 - y_2|$ instead of the well known Euclidean metric $d_E(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, where $x = (x_1, x_2), y = (y_1, y_2) \in R^2$. This space will be denoted by R^2_T which is known as taxicab plane geometry.

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The taxicab metric $d_T(x,y) = \sqrt{p(x-y,x-y)+2\left|(x_1-y_1)(x_2-y_2)\right|}$ is also defined in the paper [3, Definition 3.1, p.302], where $p(x-y,x-y)$ is Euclidean inner product.

Let $M_T(2) = \left\{F : R_T^2 \to R_T^2 : Fx = gx + b, \forall g \in D_4, b \in R_T^2\right\}$ which is known as taxicab group (see [8], [10, p.424]) be the group of all isometries of R_T^2, where the dihedral group D_4 is the (Euclidean) symmetry group of the square.

Let $M_E(2) = \left\{F : R^2 \to R^2 : Fx = gx + b, \forall g \in O(2), b \in R^2\right\}$ which is known Euclidean motion group (see [10, p.424]) be the group of all isometries of the 2-dimensional Euclidean space R^2 where the group $O(2)$ is the orthogonal group.

The complete system of $M_E(n)$-invariants of a system of the vectors $\{x_1, x_2, \dots, x_m\}$ in the n-dimensional Euclidean space is given in [9, Theorem 6] and the complete system of relations between elements of this complete system is given in [9, Theorem 3] where $M_E(n)$ is an n-dimensional Euclidean motion group.

An aim of this study is to present the equivalence conditions of two systems of vectors $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$ and to give a minimal complete system of $M_T(2)$-invariants of the vectors $\{x_1, x_2, x_3\}$ for taxicab plane geometry.

The taxicab geometry play an important role in ecology, optic, fire-spread simulation with square-cell, grid-based maps and nonlinear differential equations. Applications of the taxicab metric in ecology are also well-known. Ecologist have found taxicab metric d_T a useful metric in the measurement of 'niche overlap' and notion of ecological distance between species.(see in papers [13],[11],[16],[4],[12], [7], [1]).

Let us give the well known theorem in the Euclidean geometry as; "If systems $\{x_1, x_2, \dots, x_m\}$ and $\{y_1, y_2, \dots, y_m\}$ of vectors in R^n such that $d_E(x_i, x_j) = d_E(y_i, y_j)$ for all $i, j = 1, 2, \dots, m; i \neq j$, then there exists a unique isometry F of R^n for which $Fx_i = y_i$ for all $i = 1, 2, \dots, m$." [9].

The group of isometries of the taxicab geometry described, and the following open problem is given in [10]: "What (if any) is the taxicab metric analogue of the theorem above for Euclidean isometries?" In this study, an answer is given to this open problem.

Therefore, the following question is one of the fundamental problems of invariant theory (see [14, pp.15]).
"Given a geometric property P, find the corresponding invariants and vice versa. Is there an algorithm for this transition between geometry and algebra?"

Let P be any taxicab regular polygon in the taxicab plane. For P, the following problems are important:

(1) The existence or non-existence of P.
(2) Which Euclidean regular polygons are also the taxicab regular, and which are not?
(3) Find an algorithm to construct taxicab regular polygons.

The above problems for P are geometrically discussed in [2]. Regular polygons in the taxicab plane were studied by means of taxicab circles also in [6]. Some regular polygons in the taxicab 3-space are described in [15]. In papers [2, 6], the following corollaries are obtained:

(i) In [2], the existence of taxicab regular $2n$-gons by means of taxicab circles is proved. Besides, the non-existence of taxicab regular triangle are proved geometrically, and the question "Does there exist any taxicab regular $(2n-1)$-gons?" is posed as an open problem. In [6], the non-existence of taxicab regular triangles and pentagons are proved geometrically by means of taxicab circles.

(ii) In the papers [2, 6], it is proved that all Euclidean squares and some special Euclidean regular octagons are also taxicab regular, and vice versa.

(iii) To construct taxicab regular $2n$-gons, a procedure is given in the proof of Theorem 8 in [2] and a method is demonstrated for any n in [6].

In this study, the solutions of above problems for taxicab regular polygons P in terms of invariants of vectors are investigated. Therefore, an answer is given to the special cases of the open problem posed in [2].

The study is organized as follows. In Section 2, the conditions of G-equivalence of two systems of vectors are given for groups $G = M_T(2)$ and $G = D_4$. The relations between elements of the complete system of $M_T(2)$-invariant functions of two vectors x_1, x_2 is geometrically given. The open problem proposed in [10] is solved for the systems of vectors $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. In Section 3, a minimal complete system of $M_T(2)$-invariants functions of the system of vectors $\{x_1, x_2, x_3\}$ is introduced. In Section 4, both the conditions of the taxicab regularity of Euclidean regular polygons and Euclidean regularity of taxicab regular polygons are given in terms of $M_T(2)$-invariants of the vectors. In Section 5, an algorithm to construct taxicab regular polygons and some corresponding examples are given. In Section 6, in addition to the algorithm a procedure is given to the determine the non-existence of taxicab regular $(2n-1)$-gon having given a line segment as a side for a definite value of n, and some corresponding examples are given.

2. Conditions of G-equivalence of vectors in taxicab geometry

Let G be a group.

Definition 1. *Two systems of vectors $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ in R_T^2 will be called G-equivalent and written by $\{u_1, u_2, u_3\} \overset{G}{\sim} \{v_1, v_2, v_3\}$ if there exists $F \in G$ such that $v_j = Fu_j$ for all $j = 1, 2, 3$.*

Definition 2. *A function $f(u_1, u_2, u_3)$ of vectors u_1, u_2, u_3 in R_T^2 will be called G-invariant if $f(Fu_1, Fu_2, Fu_3) = f(u_1, u_2, u_3)$ for all $F \in G$.*

Example 3. *Let u_1, v_1 be vectors in R_T^2. Since the group D_4 is a subgroup of orthogonal group $O(2)$, we have $p(u_1, v_1)$ is D_4-invariant. That is, since $p(gu_1, gv_1) = p(u_1, v_1)$ for all $g \in D_4$, we obtain that the scalar product $p(u_1, v_1)$ is D_4-invariant. Similarly, the function $p(u_1 - v_1, u_1 - v_1)$ is $M_T(2)$-invariant.*

Example 4. *Let $u_1 = (u_{11}, u_{12}), v_1 = (v_{11}, v_{12})$ be vectors in R_T^2. We define function $q(u_1, v_1) = (u_{11}u_{12})(u_{11}v_{12} + u_{12}v_{11})$. Then $q(u_1, v_1)$ is D_4-invariant. Similarly, the function $q(u_1 - v_1, u_1 - v_1)$ is $M_T(2)$-invariant.*

Theorem 5. *Let $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be two systems of vectors in R_T^2. Then following two conditions are equivalent:*

(*i*) $\{u_1, u_2, u_3\} \overset{M_T(2)}{\sim} \{v_1, v_2, v_3\}$
(*ii*) $\{u_2 - u_1, u_3 - u_1\} \overset{D_4}{\sim} \{v_2 - v_1, v_3 - v_1\}$

Proof. Assume that $\{u_1, u_2, u_3\} \overset{M_T(2)}{\sim} \{v_1, v_2, v_3\}$. Then there exists $F \in M_T(2)$ such that $v_i = Fu_i$ for all $i = 1, 2, 3$, where F has the form $Fu = gu + b, g \in D_4, b \in R_T^2$. These equalities imply that $v_i - v_1 = g(u_i - u_1)$ for all $i = 2, 3$. This means that $\{u_2 - u_1, u_3 - u_1\} \overset{D_4}{\sim} \{v_2 - v_1, v_3 - v_1\}$.
Conversely, assume that$\{u_2 - u_1, u_3 - u_1\} \overset{D_4}{\sim} \{v_2 - v_1, v_3 - v_1\}$. Then there exists $g \in D_4$ such that $v_i - v_1 = g(u_i - u_1)$ for all $i = 2, 3$. Put $b = v_1 - gu_1$. Then $v_i = gu_i + b$ for all $i = 1, 2, 3$. That is, $\{u_1, u_2, u_3\} \overset{M_T(2)}{\sim} \{v_1, v_2, v_3\}$. □

Let $u_1, u_2, \ldots, u_m \in R_T^2$. We denote the matrix $\|p(u_j, u_k)\|_{j,k=1,2,\ldots,m}$ by $Gr\,(u_1, u_2, \ldots, u_m)$ and its determinant by $detGr\,(u_1, u_2, \ldots, u_m)$.
Below we use the following known proposition (see [5, p.192]).

Proposition 6. *Vectors $u_1, u_2, \ldots, u_m \in R_T^2$ are linearly depended if and only if $detGr\,(u_1, u_2, \ldots, u_m) = 0$*

Proof. A proof is given [17, p.75]. □

Example 7. *The rank of the system of vectors $X = \{x_1, x_2\}$ of vectors in R_T^2 is D_4-invariant, but it is not $M_T(2)$-invariant.*

Remark 8. *Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ be two systems of vectors in R_T^2 such that $x_1 \neq 0$ and $y_1 = 0$. Then the systems X and Y are not D_4-equivalent. In the case where $x_1 = y_1 = 0$, the problem of $D_4 -$ equivalence of systems X and Y reduces to the problem of D_4-equivalence of the systems $\{x_2\}$ and $\{y_2\}$. Therefore we will investigate the problem of D_4-equivalence of the systems X and Y such that $x_1 \neq 0$ and $y_1 \neq 0$.*

Theorem 9. *Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ be two systems of vectors in R_T^2 such that $x_1 \neq 0$ and $y_1 \neq 0$. Then following two conditions are equivalent:*

(*i*) $\{x_1, x_2\} \overset{D_4}{\sim} \{y_1, y_2\}$
(*ii*) $p(x_i, x_j) = p(y_i, y_j), q(x_1, x_1) = q(y_1, y_1)$ *and* $q(x_1, x_2) = q(y_1, y_2)$ *for all* $i = 1, 2; i \leq j$.

Proof. Assume that $\{x_1, x_2\} \overset{D_4}{\sim} \{y_1, y_2\}$. Then there exists $g \in D_4$ such that $gx_i = y_i$ for all $i = 1, 2$. Since the functions $p(x_i, x_j)$, $q(x_1, x_1)$ and $q(x_1, x_2)$ are $D_4 - invariants$, that is, $p(x_i, x_j) = p(y_i, y_j)$, $q(x_1, x_1) = q(y_1, y_1)$ and $q(x_1, x_2) =$

$q(y_1, y_2)$ for all $i = 1, 2; i \leq j$.

Conversely, assume that the conditions $p(x_i, x_j) = p(y_i, y_j)$, $q(x_1, x_1) = q(y_1, y_1)$ and $q(x_1, x_2) = q(y_1, y_2)$ for all $i = 1, 2; i \leq j$ are valid.

Denote by $r(X)$ and $r(Y)$ ranks of the systems $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, respectively.

(a) Firstly, consider the case $r(X) = 2$. Then there exist vectors x_1, x_2 which are linearly independent. Let $\|x_1 x_2\|$ be the matrix of column vectors x_1, x_2. Denote by U and V the matrices $\|x_1 x_2\|$ and $\|y_1 y_2\|$ and their transpose matrices by U^T, V^T, respectively. Let $\det U$ be the determinant of U. Linearly independence of x_1, x_2 implies $\det U \neq 0$. $\|p(x_i, x_j)\|_{i,j=1,2}$ is the Gram matrix of vectors x_1, x_2. Then it is easy to see that

$$U^T U = \|p(x_i, x_j)\|_{i,j=1,2} \tag{1}$$

Since $p(x_i, x_j) = p(y_i, y_j)$ for all $i, j = 1, 2$, it is obtained

$$\|p(x_i, x_j)\|_{i,j=1,2} = \|p(y_i, y_j)\|_{i,j=1,2} \tag{2}$$

(1) and (2) imply

$$U^T U = V^T V \tag{3}$$

whence

$$(\det U)^2 = (\det V)^2 \tag{4}$$

Since $\det U \neq 0$, (4) implies that $\det V \neq 0$. That is, the vectors y_1, y_2 are linearly independent. Then there exists a 2×2-matrix g such that $detg \neq 0$ and

$$V = gU \tag{5}$$

(3) and (5) give the equation

$$U^T U = U^T g^T g U \tag{6}$$

Since $\det U \neq 0$, (6) implies $g^T g = I$, where I is the identity matrix. This means that $g \in O(2)$. (5) implies $y_j = g x_j$ for all $j = 1, 2$. Now we prove that $g \in D_4$.

$g \in O(2)$ has the form $g = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $detg = 1$ or $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $detg = -1$.

Consider the matrix $g = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ such that $detg = 1$. Let $x_i = (x_{i1}, x_{i2})$ for all $i = 1, 2$. Since $y_j = g x_j$ for all $j = 1, 2$, it is obtained

$$y_j = (a x_{j1} - b x_{j2}, b x_{j1} + a x_{j2}) \tag{7}$$

for all $j = 1, 2$.

$q(x_1, x_1) = q(y_1, y_1)$ and (7) imply that

$$ab = 0 \tag{8}$$

(8) and $detg = a^2 + b^2 = 1$ give

(i) If $a = 0$, then $b = \mp 1$. So $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(ii) If $b = 0$, then $a = \mp 1$. Therefore $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Similarly, consider the matrix $g = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ such that $detg = -1$. Let $x_i = (x_{i1}, x_{i2})$ for all $i = 1, 2$. Since $y_j = gx_j$ for all $j = 1, 2$, it is obtained

$$y_j = (ax_{j1} + bx_{j2}, bx_{j1} - ax_{j2}) \tag{9}$$

for all $j = 1, 2$.

$q(x_1, x_1) = q(y_1, y_1)$ and (9) imply

$$ab = 0 \tag{10}$$

.

(10) and $detg = a^2 + b^2 = -1$ give

(i) If $a = 0$, then $b = \mp 1$. Therefore $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $g = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

(ii) If $b = 0$, then $a = \mp 1$. Hence $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

So we obtain that $g \in D_4$.

Now, let us prove that there exists $g \in D_4$ such that $gx_i = y_i$ for all $i = 1, 2$. Assume that $gx_1 = y_1$, $hx_2 = y_2$ such that $g, h \in D_4$ and $g \neq h$. Hence the inequality $q(x_1, x_2) \neq q(y_1, y_2)$ is obtained which a contradiction is to the assumption of the theorem. From the equality $q(x_1, x_2) = q(y_1, y_2)$, it is obtained that there exists $g \in D_4$ such that $gx_i = y_i$ for all $i = 1, 2$.

(b) Now, consider the case $r(X) = 1$. The conditions of the theorem and Proposition 6 imply that $r(X) = r(Y)$. Let $\tilde{X}$ and $\tilde{Y}$ denote the linear subspaces of R^2_T spanned by the systems $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, respectively. Then $dim(\tilde{X}) = r(X) = r(Y) = dim(\tilde{Y}) \leq 1$. Since $dim(\tilde{X}) = dim(\tilde{Y}) = 1$, there exist vectors x_2 and y_2 in R^2_T such that $p(x_2, x_2) = 1$, $p(x_1, x_2) = 0$ and $p(y_2, y_2) = 1$, $p(y_1, y_2) = 0$. Consider the systems $\bar{U} = \{x_1, x_2\}$ and $\bar{V} = \{y_1, y_2\}$. Then $r(\bar{U}) = r(\bar{V}) = 2$ and $p(x_i, x_s) = p(y_i, y_s)$ are obtained for all $i, s = 1, 2$. According to the case (a), there exists $g \in O(2)$ such that $\bar{V} = g\bar{U}$. Similarly, the conditions in the theorem and from the case (a), it is obtained that there exists $g \in D_4$ such that $\bar{V} = g\bar{U}$. In particularly, we obtain $y_j = gx_j$ for all $j = 1, 2$.

Hence, from (a) and (b), we have $\{x_1, x_2\} \overset{D_4}{\sim} \{y_1, y_2\}$. □

Corollary 10. *According to Theorem 9, the system*

$$\{p(x_i, x_j), q(x_1, x_1), q(x_1, x_2), 1 \leq i \leq j \leq 2\}$$

is a complete system of D_4-invariants of vectors x_1, x_2.

Using the Theorem 5 and Theorem 9, the following theorem can be obtained.

Theorem 11. *Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be two systems of vectors in R_T^2 such that $x_2 - x_1 \neq 0$ and $y_2 - y_1 \neq 0$. Then following two conditions are equivalent:*

(i) $\{x_1, x_2, x_3\} \overset{M_T(2)}{\sim} \{y_1, y_2, y_3\}$

(ii) $p(x_i - x_1, x_j - x_1) = p(y_i - y_1, y_j - y_1), q(x_2 - x_1, x_2 - x_1) = q(y_2 - y_1, y_2 - y_1)$ *and* $q(x_2 - x_1, x_3 - x_1) = q(y_2 - y_1, y_3 - y_1)$ *for all* $i = 2, 3; i \leq j$.

Corollary 12. *According to Theorem 11, the system*

$$\{p(x_i - x_1, x_j - x_1), q(x_2 - x_1, x_2 - x_1), q(x_2 - x_1, x_3 - x_1), 2 \leq i \leq j \leq 3\}$$

is a complete system of $M_T(2)$-invariants of vectors x_1, x_2, x_3.

Using the Theorem 11, the following theorem gives an answer to the open problem in [10] in terms of $M_T(2)$-invariants.

Theorem 13. *Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be two system of vectors in R_T^2. Then following two conditions are equivalent:*

(i) $\{x_1, x_2, x_3\} \overset{M_T(2)}{\sim} \{y_1, y_2, y_3\}$

(ii) $d_T(x_i, x_j) = d_T(y_i, y_j)$ *and* $d_E(x_i, x_j) = d_E(y_i, y_j)$ *for all* $i \neq j$ *and* $i, j = 1, 2, 3$.

According to Theorem 13, the system $\{d_T(x_i, x_j), d_E(x_i, x_j), i, j = 1, 2, 3; i \neq j\}$ is a complete system of $M_T(2) - invariants$ of vectors x_1, x_2, x_3.

Specially, the system $\{d_T(x_1, x_2), d_E(x_1, x_2)\}$ is a complete system of $M_T(2)$-invariants of vectors x_1, x_2.

Now we investigate relations between elements of the complete system of $M_T(2)$-invariant functions of two vectors x_1, x_2

Theorem 14. *Let x_1 be a fixed point in R_T^2. Then for all points $x_1 \neq x_2$, the following statements are hold:*

(i) *The geometric locus of points x_2 where $d_T(x_1, x_2) = d_E(x_1, x_2)$ are intersection points of taxicab and Euclidean circles with centered x_1. Geometrically, this is a inscribed quadrilateral.*

(ii) *The geometric locus of points x_2 where $d_T(x_1, x_2) = \sqrt{2} d_E(x_1, x_2)$ are tangent points of taxicab and Euclidean circles with centered x_1. Geometrically, this is a circumscribed quadrilateral.*

(iii) *The geometric locus of points x_2 where $d_T(x_1, x_2) < \sqrt{2} d_E(x_1, x_2)$ are intersection points of taxicab and Euclidean circles with centered x_1. The number of the points are only eight.*

Proof. Let $x_1 = (x_{11}, x_{12})$ and $x_2 = (x_{21}, x_{22})$ be two points in R_T^2 such that $x_1 \neq x_2$. Let $d_T(x_1, x_2) = a$ and $d_E(x_1, x_2) = r$, where a and r are positive real numbers. Then, from the equalities $d_E(x_1, x_2) = \sqrt{p(x_1 - x_2, x_1 - x_2)} = r$ and $d_T(x_1, x_2) = \sqrt{p(x_1 - x_2, x_1 - x_2) + 2\left|(x_{11} - x_{21})(x_{12} - x_{22})\right|} = a$, we have

$$p(x_1 - x_2, x_1 - x_2) = r^2 \tag{11}$$

and

$$p(x_1 - x_2, x_1 - x_2) + 2\left|(x_{11} - x_{21})(x_{12} - x_{22})\right| = a^2 \tag{12}$$

The equalities (11) and (12) imply

$$r^2 + 2\left|(x_{11} - x_{21})(x_{12} - x_{22})\right| - a^2 = 0 \tag{13}$$

From (13), following three cases are obtained:

(a) if $x_{11} - x_{21} = 0$ or $x_{12} - x_{22} = 0$, then $r = a$.

Hence the vector $x_1 - x_2$ is parallel to anyone of the lines $x = 0$ or $y = 0$. See Figure 1 for positions of the vector $x_1 - x_2$.

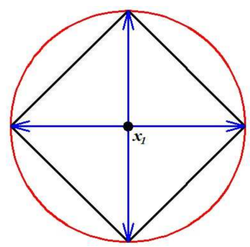

FIGURE 1. The positions of vectors $x_2 - x_1$ with $d_T(x_1, x_2) = d_E(x_1, x_2)$

(b) Let $x_{11} - x_{21} \neq 0$ and $x_{12} - x_{22} \neq 0$. Then there are the following four cases:

$(b.1)$ $x_{11} - x_{21} > 0$ and $x_{12} - x_{22} > 0$,

$(b.2)$ $x_{11} - x_{21} > 0$ and $x_{12} - x_{22} < 0$,

$(b.3)$ $x_{11} - x_{21} < 0$ and $x_{12} - x_{22} > 0$,

$(b.4)$ $x_{11} - x_{21} < 0$ and $x_{12} - x_{22} < 0$.

$(b.1)$ Let $x_{11} - x_{21} > 0$ and $x_{12} - x_{22} > 0$. From (13), it is obtained

$$x_{11} - x_{21} = \frac{a^2 - r^2}{2(x_{12} - x_{22})} \tag{14}$$

Using the equalities (11) and (14), it is obtained

$$x_{12} - x_{22} = \sqrt{\frac{r^2 \pm a\sqrt{2r^2 - a^2}}{2}} \tag{15}$$

From (15), the following cases are obtained:

$(b.1.1)$ if $2r^2 - a^2 = 0$, then we have $x_{11} - x_{21} = x_{12} - x_{22} = \frac{r}{\sqrt{2}}$. Hence the vector $x_1 - x_2$ is parallel to anyone of the lines $y = x$ or $y = -x$. See Figure 2 for positions of the vector $x_1 - x_2$.

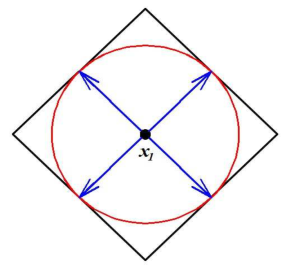

FIGURE 2. The positions of vectors $x_2 - x_1$ with $d_T(x_1, x_2) = \sqrt{2} d_E(x_1, x_2)$

$(b.1.2)$ if $2r^2 - a^2 > 0$, then we have $x_{11} - x_{21} = x_{12} - x_{22} = \sqrt{\frac{r^2 \pm a\sqrt{2r^2-a^2}}{2}}$.

The cases $(b.2)$,$(b.3)$ and $(b.4)$ are similar to $(b.1)$. Then there exist eight intersection points of taxicab circle and Euclidean circle. See Figure 3 for positions of the vector $x_1 - x_2$.

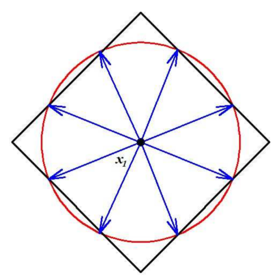

FIGURE 3. The positions of vectors $x_2 - x_1$ with $d_T(x_1, x_2) < \sqrt{2} d_E(x_1, x_2)$

$\square$

Remark 15. *From* (iii) *in Theorem 14, we have*

$$d_E(x_1, x_2) < d_T(x_1, x_2) < \sqrt{2} d_E(x_1, x_2). \tag{16}$$

3. On minimality of the complete system of invariants of vectors

Definition 16. *A system* $\{f_\tau, \tau \in Q\}$ *of* $M_T(2)$ *-invariant functions* $f_\tau(x_1, x_2, x_3)$ *of the systems* $\{x_1, x_2, x_3\}$ *in* R_T^2 *will be called complete if equalities* $f_\tau(x_1, x_2, x_3) = f_\tau(y_1, y_2, y_3)$ *for all* $\tau \in Q$ *imply* $\{x_1, x_2, x_3\} \overset{M_T(2)}{\sim} \{y_1, y_2, y_3\}$.

Denote by $K_{ij}(X)$, $L(X)$ and $M(X)$ by the functions $p(x_i - x_1, x_j - x_1)$ for $2 \leq i \leq j \leq 3$, $q(x_2 - x_1, x_2 - x_1)$ and $q(x_2 - x_1, x_3 - x_1)$, respectively.

According to Theorem 11, the system
$B = \{K_{ij}(X), L(X), M(X), 2 \leq i \leq j \leq 3\}$ is a complete system of $M_T(2)$-invariant functions of vectors x_1, x_2, x_3 in R_T^2.

Definition 17. *A complete system $B = \{f_\tau, \tau \in Q\}$ of $M_T(2)$-invariant functions f_τ of the system $\{x_1, x_2, x_3\}$ in R_T^2 will be called minimal if every proper subset of B is not complete.*

Theorem 18. *The system B is a minimal complete system of $M_T(2)$ -invariants of vectors x_1, x_2, x_3 in R_T^2.*

Proof. A proof follows from the following Lemmas 19-22.

Lemma 19. *The subsystem $B \setminus \{K_{23}(X)\}$ is not a complete system of $M_T(2)$-invariants.*

Proof. Consider the following two systems of vectors in R_T^2 :
$X = \{x_1 = (1,2), x_2 = (3,2), x_3 = (2,4)\}$ and
$Y = \{y_1 = (1,2), y_2 = (3,2), y_3 = (3,3)\}$. Prove the lemma for $i = 2, j = 3$. Then we have $K_{22}(X) = K_{22}(Y) = 4, K_{33}(X) = K_{33}(Y) = 5, L(X) = L(Y) = 0, M(X) = M(Y) = 0$. Since $K_{23}(X)$ and $K_{23}(Y)$ are $M_T(2)$-invariants, $K_{23}(X) = 2$, $K_{23}(Y) = 4$, it is obtained that the systems X and Y are not $M_T(2)$-equivalent. Hence the subsystem $B \setminus \{K_{23}(X)\}$ is not complete. □

Lemma 20. *The subsystem $B \setminus \{K_{ii}(X)\}$ for any $i = 2, 3$ is not a complete system of $M_T(2)$-invariants.*

Proof. Consider the following two systems of vectors in R_T^2 :
$X = \{x_1 = (1,2), x_2 = (3,2), x_3 = (4,3)\}$ and
$Y = \{y_1 = (1,2), y_2 = (3,2), y_3 = (4,4)\}$. Prove the lemma for $i = 3$. Then we have $K_{22}(X) = K_{22}(Y) = 4, K_{23}(X) = K_{23}(Y) = 6, L(X) = L(Y) = 0, M(X) = M(Y) = 0$. Since $K_{33}(X)$ and $K_{33}(Y)$ are $M_T(2)$-invariants, $K_{33}(X) = 10$, $K_{33}(Y) = 13$, it is obtained that the systems X and Y are not $M_T(2)$-equivalent. Hence the subsystem $B \setminus \{K_{33}(X)\}$ is not complete. Similarly, the subsystem $B \setminus \{K_{22}(X)\}$ is not complete. □

Lemma 21. *The subsystem $B \setminus \{L(X)\}$ is not a complete system of $M_T(2) -$ invariants.*

Proof. Consider the following two systems in R_T^2 :
$X = \{x_1 = (1,2), x_2 = (2,3), x_3 = (2,1)\}$ and
$Y = \left\{y_1 = (1,2), y_2 = (1+\sqrt{2}, 2), y_3 = (1, 2-\sqrt{2})\right\}$.
Then we have $K_{22}(X) = K_{22}(Y) = 2, K_{23}(X) = K_{23}(Y) = 0, K_{33}(X) = K_{33}(Y) = 2, M(X) = M(Y) = 0$. Since $L(X)$ and $L(Y)$ are $M_T(2)$-invariants, $L(X) = 2$, $L(Y) = 0$, it is obtained that the systems X and Y are not $M_T(2)$-equivalent. Hence the subsystem $B \setminus \{L(X)\}$ is not complete. □

Lemma 22. *The subsystem $B \setminus \{M(X)\}$ is not a complete system of $M_T(2)-$invariants.*

Proof. Consider the following two systems in R_T^2 :
$X = \{x_1 = (1,2), x_2 = (2,0), x_3 = (5,4)\}$ and
$Y = \{y_1 = (1,2), y_2 = (2,0), y_3 = (-3,0)\}$.
Then we have $K_{22}(X) = K_{22}(Y) = 5, K_{23}(X) = K_{23}(Y) = 0, K_{33}(X) = K_{33}(Y) = 20, L(X) = L(Y) = 4$. Since $M(X)$ and $M(Y)$ are $M_T(2)$-invariants, $M(X) = 12$, $M(Y) = -12$, it is obtained that the systems X and Y are not $M_T(2)$-equivalent. Hence the subsystem $B \setminus \{M(X)\}$ is not complete. □

Lemmas 19-22 imply that the system B is a minimal complete system of $M_T(2)$-invariants. The proof of the theorem is completed. □

4. On the Euclidean regular polygons and taxicab regular polygons

The following definitions about the taxicab polygons are given in [2, p.27-28] "As in the Euclidean plane, a polygon in the taxicab plane consists of three or more coplanar line segments; the line segments (sides) intersect only at endpoints; each endpoint(vertex) belongs to exactly two line segments; no two line segments with a common endpoint are collinear. If the number of sides of a polygon is n for $n \geq 3$ and $n \in N$, then the polygon is called an n-gon. The following definitions for polygons in the taxicab plane are given by means of the taxicab lengths instead of the Euclidean lengths:

Definition 23. *A polygon in the plane is said to be taxicab equilateral if the taxicab lengths of its sides are equal.*

Definition 24. *A polygon in the plane is said to be taxicab equiangular if the measures of its interior angles are equal.*

Definition 25. *A polygon in the plane is said to be taxicab regular if it is both taxicab equilateral and equiangular.*

Definition 24 does not give a new equiangular concept because the taxicab and the Euclidean measure of an angle are the same. That is, every Euclidean equiangular polygon is also the taxicab equiangular, and vice versa. However, since the taxicab plane has a different distance function, Definition 23 and therefore Definition 25 are new concepts."

The following theorem gives us conditions of the taxicab regularity of Euclidean regular polygons in terms of $M_T(2)$-invariants, vice versa.

Theorem 26. *Let $x_1, x_2, \ldots, x_n$ be vertices of an n-sided polygon in the Cartesian plane. Assume that $\{x_{i+1}, x_i\} \overset{M_T(2)}{\sim} \{x_{i+1}, x_{i+2}\}$ and the angle between $x_i - x_{i+1}$ and $x_{i+2} - x_{i+1}$ has measure $\theta = \frac{\pi(n-2)}{n}$ radian for all $1 \leq i \leq n$. Then the n-sided polygon is a taxicab regular n-gon and a Euclidean regular n-gon, where $n = 4$ or $n = 8$.*

Proof. For simplicity, let us consider two vertices $x_1 = (cos\alpha, sin\alpha)$ for $\alpha \in (0, \pi/4)$ and $x_2 = (0, 0)$. Besides, let us start from the vertices. Put $i = 1$ and $\{x_2, x_1\} \overset{M_T(2)}{\sim} \{x_2, x_3\}$. According to Theorem5, we have $\{x_1 - x_2\} \overset{D_4}{\sim} \{x_3 - x_2\}$. Then there exist 8 forms vectors $x_3 - x_2$ such that $x_3 - x_2 = (sin\alpha, cos\alpha)$, $x_3 - x_2 = (-sin\alpha, cos\alpha)$, $x_3 - x_2 = (-cos\alpha, sin\alpha)$,$x_3 - x_2 = (-cos\alpha, -sin\alpha)$, $x_3 - x_2 = (sin\alpha, -cos\alpha)$, $x_3 - x_2 = (-sin\alpha, -cos\alpha)$ and $x_3 - x_2 = (-cos\alpha, -sin\alpha)$.

(i) Let us consider $x_3 - x_2 = (sin\alpha, cos\alpha)$. Then the angle between $x_1 - x_2$ and $x_3 - x_2$ has measure $\theta < \pi/2$. Then $n \leq 3$. Assuming $n \geq 3$, $n = 3$ is obtained . So, $\theta = \pi/3$ and $\alpha = \pi/12$.

Now let us consider $\{x_3, x_2\} \overset{M_T(2)}{\sim} \{x_3, x_4\}$. Since $n = 3$, we obtain that $x_4 = x_1$. Clearly, this is a contradiction. Then $\{x_3, x_2\}$ is not $M_T(2)-equivalent$ to $\{x_3, x_4\}$. That is, $n \neq 3$.

(ii) Let us consider $x_3 - x_2 = (-sin\alpha, cos\alpha)$. Then the angle between $x_1 - x_2$ and $x_3 - x_2$ has measure $\theta = \pi/2$. Then $n = 4$. Since $n = 4$, we have $x_5 = x_1$. Let us consider $\{x_3, x_2\} \overset{M_T(2)}{\sim} \{x_3, x_4\}$ and $\{x_4, x_3\} \overset{M_T(2)}{\sim} \{x_4, x_5\}$. Then the angles between $x_i - x_{i+1}$ and $x_{i+2} - x_{i+1}$ for $i = 1, 2, 3$ have measures $\theta = \pi/2$. Furthermore, we obtain $d_T(x_{i+1}, x_i) = d_T(x_{i+1}, x_{i+2})$ and $d_E(x_{i+1}, x_i) = d_E(x_{i+1}, x_{i+2})$. That is, this is a taxicab square.

(iii) Let us consider $x_3 - x_2 = (-cos\alpha, sin\alpha)$. Then the angle between $x_1 - x_2$ and $x_3 - x_2$ has measure $\theta = \pi - 2\alpha > \pi/2$. Then $n > 4$. Let us consider $g = \begin{pmatrix} -cos2\alpha & -sin2\alpha \\ sin2\alpha & -cos2\alpha \end{pmatrix}$. Since the angle between $x_2 - x_3$ and $x_4 - x_3$ has measure $\theta = \pi - 2\alpha$, we have $g\,(x_2 - x_3) = x_4 - x_3$. This implies $x_4 - x_3 = (-cos3\alpha, sin3\alpha)$. According to Theorem5, $\{x_3, x_2\} \overset{M_T(2)}{\sim} \{x_3, x_4\}$ implies $\{x_2 - x_3\} \overset{D_4}{\sim} \{x_4 - x_3\}$.

So, $\{x_2 - x_3 = (cos\alpha, -sin\alpha)\} \overset{D_4}{\sim} \{x_4 - x_3 = (-cos3\alpha, sin3\alpha)\}$ is obtained.

From Theorem 11, we have $[(cos\alpha)(-sin\alpha)]^2 = [(-cos3\alpha)(sin3\alpha)]^2$. Then this equation implies $\alpha = \pi/8$. That is, $n = 8$. Then the angles between $x_i - x_{i+1}$ and $x_{i+2} - x_{i+1}$ for $i = 3, \ldots, 7$ have measures $\theta = \pi - 2\alpha$ and $g\,(x_i - x_{i+1}) = x_{i+2} - x_{i+1}$. This implies $x_{i+2} - x_{i+1} = (-cos(2i-1)\alpha, sin(2i-1)\alpha)$ and $x_9 = x_1$. This shows that the angles between $x_i - x_{i+1}$ and $x_{i+2} - x_{i+1}$ for $i = 1, 2, \ldots, 7$ have measures $\theta = 3\pi/4$. Furthermore, we obtain $d_T(x_{i+1}, x_i) = d_T(x_{i+1}, x_{i+2})$ and $d_E(x_{i+1}, x_i) = d_E(x_{i+1}, x_{i+2})$. That is, this is a taxicab regular octagon.

If $\alpha = 0$ radians or $\alpha = \pi/2$ radians, the edges of the polygon are parallel to the lines $x = 0$ and $y = 0$. The polygon is a taxicab regular square.

If $\alpha = \pi/4$ radians, the edges of the polygon are parallel to the lines $y = x$ and $y = -x$. The polygon is a taxicab regular square.

Proofs of the cases $x_3-x_2=(-cos\alpha,-sin\alpha)$, $x_3-x_2=(sin\alpha,-cos\alpha)$,$x_3-x_2=(-sin\alpha,-cos\alpha)$ and $x_3-x_2=(-cos\alpha,-sin\alpha)$ are similar to the proof of (i),(ii) and (iii). Hence, we obtain that $n=4$ or $n=8$. $\square$

Corollary 27. (1) *According to Theorem 26, a taxicab regular octagon is a Euclidean regular iff the slopes of sides of a taxicab regular octagon are equal to* $m=\pm tan(\pi/8)$ *or* $m=\pm tan(3\pi/8)$.

(2) *According to Theorem26, a Euclidean regular octagon is a taxicab regular iff the slopes of sides of a Euclidean regular octagon are equal to* $m=\pm tan(\pi/8)$ *or* $m=\pm tan(3\pi/8)$.

(3) *Every taxicab regular square is also Euclidean regular , vice versa.*

Remark 28. (i) *According to Corollary 27, all taxicab regular octagons are not Euclidean regular, vice versa.*
(ii) *According to Corollary 27, all taxicab regular squares are Euclidean regular, vice versa.*

Corollary 27 and Remark 28 that we derived by using $M_T(2)$-invariants, are the same conclusions derived in [2] and [6].

5. The proposed algorithm for taxicab regular polygons

Let x_1 and x_2 be vertices of a side of any polygon and n be the number of sides of polygon in the taxicab plane. Consider a side by $\overline{x_1x_2}$. Since angles in taxicab geometry are measured as in Euclidean geometry, each interior angle of a regular polygon is measured $\theta=\frac{\pi(n-2)}{n}$ radians. Let us introduce the algorithm to construct taxicab regular n-gon having $\overline{x_1x_2}$ as side for a definite value of n, with the following steps:

Step 1 The side $\overline{x_1x_2}$ is rotated through $\beta=\frac{\pi(n+2)}{n}$ radians clockwise about the point x_2 and is obtained a side $\overline{x_2z_3}$ such that $z_3=x_2+g(x_1-x_2)$, where $g=\begin{pmatrix} cos\beta & sin\beta \\ -sin\beta & cos\beta \end{pmatrix}$.
Then $d_E(x_1,x_2)=d_E(z_3,x_2)$ and the angle between vectors x_1-x_2 and z_3-x_2 are equal to θ.

Step 2 For any point x_3 on the line passes points x_2 and z_3, by solving equations $d_T(x_1,x_2)=d_T(x_3,x_2)$ and $p\,(x_1-x_2,x_3-x_2)=d_E(x_1,x_2)d_E(x_2,x_3)cos\theta$, x_3 is obtained.

Step 3 Similarly, for all $i=2,\ldots,n-1$, the side $\overline{x_ix_{i+1}}$ is rotated through $\beta=\frac{\pi(n+2)}{n}$ radians clockwise about the point x_{i+1} and is obtained a side $\overline{x_{i+1}z_{i+2}}$ such that $z_{i+2}=x_{i+1}+g(x_i-x_{i+1})$. Then $d_E(x_i,x_{i+1})=d_E(x_{i+1},z_{i+2})$ and the angle between vectors x_i-x_{i+1} and $z_{i+2}-x_{i+1}$ are equal to θ.

Step 4 For any point x_{i+2} on the line passes points z_{i+2} and x_{i+1}, by solving equations $d_T(x_i,x_{i+1})=d_T(x_{i+2},x_{i+1})$ and $p\,(x_i-x_{i+1},x_{i+2}-x_{i+1})=d_E(x_i,x_{i+1})d_E(x_{i+1},x_{i+2})cos\theta$, x_{i+2} is obtained.
Thus, all vertices $x_3,x_4,\ldots,x_{n+1}$ of the polygon are obtained.

Step 5 If $x_{n+1} = x_1$ and $p(x_n - x_1, x_2 - x_1) = d_E(x_1, x_n)d_E(x_1, x_2)cos\theta$, then this is a taxicab regular n-gon.

Step 6 If $x_{n+1} \neq x_1$, then there is no taxicab regular n-gon having $\overline{x_1x_2}$ as a side.

Remark 29. *According to this algorithm, for definite value of* n*, one can constract taxicab regular* $2n$*-gons, and determine if there exist* $(2n-1)$*-gons, having given a line segment as a side. Clearly, this algorithm is also a tool to give an answer to specal cases of open probleme given in* [2].

5.1. **Illustrations.** In this subsection various examples are given to demonstrate the steps of the proposed algorithm for taxicab regular polygons.

Example 30. *Consider a hexagon with vertices* $x_1 = (1,1), x_2 = (0,0), x_3 = (-1.57735, 0.42265), x_4 = (-2,2), x_5 = (-1,3), x_6 = (0.57735, 2.57735)$. *This polygon is a taxicab regular hexagon(See Figure 4).*

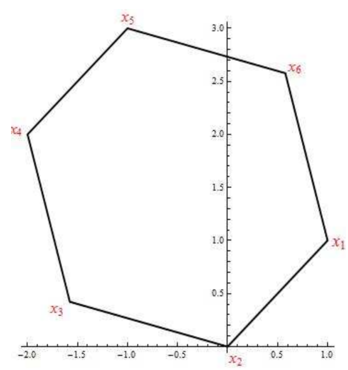

FIGURE 4. The taxicab regular hexagon

Example 31. *Consider a* 10*-gon with vertices*

$$\begin{aligned} x_1 &= (2,3), & x_2 &= (1,2), \\ x_3 &= (-0.726543, 1.72654), & x_4 &= (-2.05146, 2.40162), \\ x_5 &= (-2.72654, 3.72654), & x_6 &= (-2.45309, 5.45309), \\ x_7 &= (-1.45309, 6.45309), & x_8 &= (0.273457, 6.72654), \\ x_9 &= (1.59838, 6.05146), & x_{10} &= (2.27346, 4.72654). \end{aligned}$$

This polygon is a taxicab regular 10-gon.(See Figure 5).

6. TAXICAB REGULARITY OF POLYGONS WITH AN ODD NUMBER OF SIDES

The open problem for $(2n-1)$-gons posed by [2]: "Does there exist any taxicab regular $(2n-1)$-gons? "As the given algorithm in Section 5,the following procedure

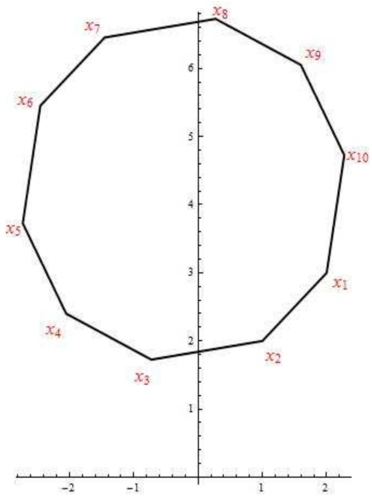

FIGURE 5. The taxicab regular 10-gon

is also a tool to give answer to special cases of open problem given in [2]. That is, for a definite value of n, and given a line segment $\overline{x_1x_2}$, this procedure determines if $(2n-1)$-gon having $\overline{x_1x_2}$ as a side exist or not."

Since angles in taxicab geometry are measured as in Euclidean geometry, it is obtained that each interior angle of a regular polygon has measure $\theta = \frac{\pi(n-2)}{n}$ radians.

Let us consider a Euclidean regular $(2n-1)$-gon with vertices $x_1, \ldots, x_{2n-1}$. For simplicity, let us take a side $\overline{x_1x_2}$ and denote two vertices by $x_1 = y_1$, $x_2 = y_2$. Then there exists a point y_{i+2} on the line parallel to the sides $\overline{x_{i+1}x_{i+2}}$ that passes through the point y_{i+1} such that $d_T(x_1, x_2) = d_T(y_{i+1}, y_{i+2})$ for each $i = 1, \ldots, 2n-3$. Therefore, the angle between sides $\overline{y_iy_{i+1}}$ and $\overline{y_{i+1}y_{i+2}}$ for all $i = 1, \ldots, 2n-3$ equals to $\theta = \frac{\pi(n-2)}{n}$ radians.

But the angle between sides $\overline{y_{2n-2}y_{2n-1}}$ and $\overline{y_{2n-1}y_1}$ is not equal to θ, and the inequality $d_T(y_1, y_2) \neq d_T(y_{2n-1}, y_1)$ holds. If both of these conditions hold at the same time, then $(2n-1)$-gon with vertices $y_1, y_2, ..., y_{2n-1}$ is regular, otherwise it is not.

6.1. **Illustrations.** In this subsection, we give examples related to the procedure introduced above. We have implemented the algoritm proposed in Section 5 in the computer program Mathematica for the examples given in 5.1 Illustrations and 6.1 Illustrations .

Example 32. *Let us consider Euclidean regular triangle with vertices* $x_1 = (2,1), x_2 = (1,1), x_3 = (1.5, 1.86603)$*. Let a side of taxicab regular triangle be* $\overline{x_1x_2}$*. Let us denote vertices* x_1, x_2 *by* y_1, y_2*,respectively.*

Then according to the above procedure, the point $y_3 = (1.36603, 1.63397)$ *on the line parallel to the sides* $\overline{x_2x_3}$ *that passes through the point* y_2 *such that* $d_T(x_1, x_2) = d_T(y_2, y_3)$*is found. Then the triangle with vertices* y_1, y_2 *and* y_3*is not taxicab regular. So there is no taxicab regular triangle with the side* $\overline{y_1y_2}$*. (See Figure 6).*

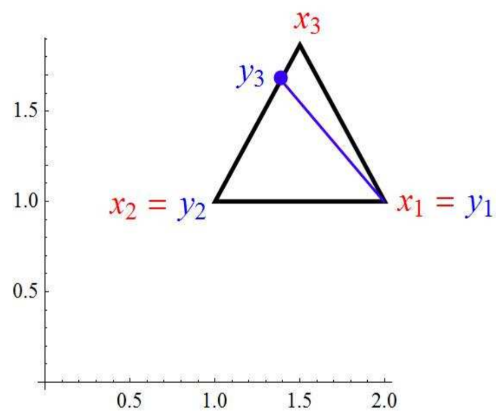

FIGURE 6. While the triangle with vertices x_1, x_2, x_3 is Euclidean regular, the triangle with vertices y_1, y_2, y_3 is not taxicab regular.

Example 33. *Let us consider Euclidean regular pentagon with vertices*

$$\begin{aligned} x_1 &= (2,1), & x_2 &= (1,1), \\ x_3 &= (0.690983, 1.95106), & x_4 &= (1.5, 2.53884), \\ x_5 &= (2.30902, 1.95106). \end{aligned}$$

Let a side of taxicab regular pentagon be $\overline{x_1x_2}$*. Let us denote vertices* x_1, x_2 *by* y_1, y_2*,respectively. Then according to the above procedure, the points*

$$\begin{aligned} y_3 &= (0.754763, 1.75476), & y_4 &= (1.33395, 2.17557), \\ y_5 &= (1.91315, 1.75476) \end{aligned}$$

such that $d_T(x_1, x_2) = d_T(y_2, y_3) = d_T(y_3, y_4) = d_T(y_4, y_5)$ *is found. Clearly, the angle between sides* $\overline{y_4y_5}$ *and* $\overline{y_5y_1}$ *is not equal to* θ*, and the inequalty* $d_T(y_1, y_2) \neq d_T(y_5, y_1)$ *holds. The polygon with vertices* x_1, x_2, x_3, x_4, x_5 *is a Euclidean regular pentagon and but it is not a taxicab regular pentagon with vertices* y_1, y_2, y_3, y_4, y_5*.(See Figure 7).*

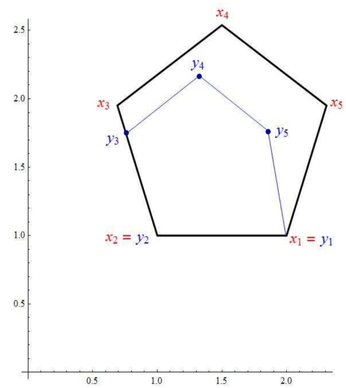

FIGURE 7. While the pentagon with vertices x_1, x_2, x_3, x_4, x_5 is Euclidean regular,the pentagon with vertices y_1, y_2, y_3, y_4, y_5 is not taxicab regular.

7. Conclusions

Finding of an algebraic solution for non-existence of taxicab $(2n-1)$-gon is difficult. In the special case, choosing an initial side in the algorithm, an algebraic solution can be easily found for $n = 3$. Thus, in our paper, the solution of this problem is given numerically. However, the conjecture in [2] still needs to be proven geometrically or algebrically.

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N-SPACES

ATILLA AKPINAR

Abstract. In this paper, we introduce n-spaces constructed over an local ring with the maximal ideal (of non-unit elements). So, we give the example of an octonion n-space. Finally, we give two collineations of quaternion n-space.

1. Introduction and Preliminaries

In the early 1930s, P. Jordan, who is a physicist, has began to study with Jordan algebras. The algebra $\mathbf{H}(\mathbf{O}_3)$ is firstly used by Jordan, to define an octonion plane (over real octonion division algebra) [10]. Freudenthal, in [8], gave the same construction in [10]. Later, Springer, in [12], extended the construction given by Jordan and Freudenthal to the octonion (or Cayley) division algebras defined over a field whose characteristic is different from 2 and 3.

In [3], Bix deals with $\mathbf{J} = \mathbf{H}\ (\mathbf{O}_3, J\gamma)$, the set of 3 by 3 matrices with entries in an octonion algebra $\mathbf{O}$ defined over a local ring R with the maximal ideal I (of non-unit elements), that are symmetric with respect to the canonical involution $J\gamma : X \to \gamma^{-1}\overline{X}^{\mathbf{t}}\gamma$ where the γ_i are elements of $R \setminus I$ and $\gamma :=\text{diag}\{\gamma_1, \gamma_2, \gamma_3\}$. Hence, any element X of $\mathbf{J}$ is of the form

$$X = \begin{pmatrix} \alpha_1 & \gamma_2 a_3 & \gamma_3\overline{a_2} \\ \gamma_1\overline{a_3} & \alpha_2 & \gamma_3 a_1 \\ \gamma_1 a_2 & \gamma_2\overline{a_1} & \alpha_3 \end{pmatrix} \text{ for } \alpha_i \in R \text{ and } a_i \in \mathbf{O}.$$

If it is defined a cubic form N such that $N(X) := \det X$, a quadratic mapping $X \to X^{\sharp}$:=adjoint of X, and a basepoint $C := I_3$ on $\mathbf{J}$ are defined, then the triple $(\mathbf{J}, N, C)$ is a quadratic (exceptional) Jordan algebra under the operator $U_X Y = T(X,Y)X - 2\left(X^{\sharp} \times Y\right)$ [11]. Then, for $X = \begin{pmatrix} \alpha_1 & \gamma_2 a_3 & \gamma_3\overline{a_2} \\ \gamma_1\overline{a_3} & \alpha_2 & \gamma_3 a_1 \\ \gamma_1 a_2 & \gamma_2\overline{a_1} & \alpha_3 \end{pmatrix}$ and $Y =$

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$\begin{pmatrix} \beta_1 & \gamma_2 b_3 & \gamma_3 \overline{b_2} \\ \gamma_1 \overline{b_3} & \beta_2 & \gamma_3 b_1 \\ \gamma_1 b_2 & \gamma_2 \overline{b_1} & \beta_3 \end{pmatrix} \in \mathbf{J}$, we can give the similar results to those given in [11, 3, 7]:

$$N(X) = \alpha_1\alpha_2\alpha_3 - \alpha_1\gamma_2\gamma_3 n(a_1) - \alpha_2\gamma_3\gamma_1 n(a_2) - \alpha_3\gamma_1\gamma_2 n(a_3) + \gamma_1\gamma_2\gamma_3 2t((a_1a_2)a_3),$$

$$X^\sharp = (X_{ij})_{3\times 3} \text{ for } X_{ii} = \alpha_j\alpha_k - \gamma_j\gamma_k n(a_i), x_{ij} = \gamma_i\gamma_k a_i a_j - \gamma_i\alpha_k\overline{a_k} \text{ and } X_{ji} = \overline{X_{ij}},$$

$$X \times Y = (z_{ij})_{3\times 3} \text{ for } \begin{cases} z_{ii} = \frac{1}{2}\left[\alpha_j\beta_k + \beta_j\alpha_k - 2\gamma_j\gamma_k n(a_i, b_i)\right], \\ z_{ij} = \frac{1}{2}\left(\gamma_j\left[\gamma_k\overline{(a_ib_j + b_ia_j)} - (\alpha_k b_k + \beta_k a_k)\right]\right), \ z_{ji} = \overline{z_{ij}} \end{cases},$$

$$T(X,Y) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + 2\gamma_2\gamma_3 n(a_1,b_1) + 2\gamma_3\gamma_1 n(a_2,b_2) + 2\gamma_1\gamma_2 n(a_3,b_3),$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$, n (defined by $n(x) := x\overline{x}$) is the norm (quadratic) form over $\mathbf{O}$, t (defined by $t(x) := \frac{1}{2}(x+\overline{x})$) is the trace (linear) form over $\mathbf{O}$ and finally $n(x,y)$ (defined by $n(x,y) := \frac{1}{2}[n(x+y) - n(x) - n(y)]$) is symmetric bilinear norm w.r.t. n.

Let Π denote the set of elements of rank 1 in $\mathbf{J}$. Then,

$$\Pi = \left\{ X \,\middle|\, X \in \mathbf{J} \setminus I\mathbf{J} \text{ and } X \times X = X^\sharp = 0 \right\}.$$

Note that, if $X \in \Pi$ and α is an element in $R \setminus I$, then $\alpha X \in \Pi$. For $X \in \Pi$, let X_* and X^* be two copies of the set $\{\alpha X \mid \alpha \in R \setminus I\}$.

Now, it is time to give the definition of an octonion plane $\mathbf{P}(\mathbf{J})$ from [3, 6].

Definition 1. *The octonion plane* $\mathbf{P}(\mathbf{J}) = (\mathbf{P}, \mathbf{L}, |, \backsimeq)$ *consists of the incidence structure* $(\mathbf{P}, \mathbf{L}, |)$ *(points, lines, and incidence), and the connection relation is defined as follows:*

$\mathbf{P} = \{X_* \,|X \in \Pi\}$, $\mathbf{L} = \{X^* \,|X \in \Pi\}$,

$X_* |Y^*$, *X_* is on Y^*, if* $V_{Y,X} = 0$, *that is,* $V_{Y,X} =: \{1XY\} = \{X1Y\} = \{XY1\} = X \cdot Y = 0$ *where* $X \cdot Y = \frac{1}{2}(XY + YX)$ *(Jordan multiplication).*

$X_* \backsimeq Y_*$, *X_* is connected to Y_* if* $X \times Y \in I\mathbf{J}$,

$X^* \backsimeq Y^*$, *X^* is connected to Y^* if* $X \times Y \in I\mathbf{J}$,

$X_* \backsimeq Y^*$, *X_* is connected (or near) to Y^* if* $T(X,Y) \in I$.

Now, we recall some informations on projective Klingenberg and Moufang-Klingenberg planes from [2].

Definition 2. *Let* $\mathbb{M} = (\mathbf{P}, \mathbf{L}, \in', \sim')$ *consist of an incidence structure* $(\mathbf{P}, \mathbf{L}, \in')$ *(points, lines, incidence) and an equivalence relation '$\sim'$' (neighbour relation) on* $\mathbf{P}$ *and on* $\mathbf{L}$. *Then* $\mathbb{M}$ *is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:*

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \wedge h$ on both g and h.

(PK3) There is a projective plane $\mathbb{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in')$ *and incidence structure epimorphism* $\Psi : \mathbb{M} \to \mathbb{M}^*$*, such that the conditions*

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim' Q,\ \Psi(g) = \Psi(h) \iff g \sim' h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

A point $P \in' \mathbf{P}$ is called near a line $g \in' \mathbf{L}$ iff there exists a line h such that $P \in' h$ for some line $h \sim' g$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of $\mathbb{M}$.

A Moufang-Klingenberg plane (MK-plane) is a PK-plane $\mathbb{M}$ that generalizes a Moufang plane, and for which $\mathbb{M}^*$ is a Moufang plane (for the details see [2]).

In [9, Chapter III.2, Theorem 1], Jacobson showed that the fact that $(\mathbf{D}_n, J\gamma)$ is a Jordan algebra implies that $\mathbf{D}$ is associative if $n \geq 4$ but alternative with its symmetric elements in the nucleus if $n = 3$. Therefore, in [1], in the case of $n \geq 4$ we were able to study the elements of the quaternion division algebra $\mathbb{Q}$ over a field F, which is associative. For this reason, we could not continue studying by elements of an octonion algebra. But, without the need for Jordan matrix algebras, the obtained results in [1] show the existence of the following two possibilities: either the definition of the octonion plane (octonion 2-space) may be extended to an (octonion) $n-$space or a new geometric structure may be obtained. We need to recall some results in the case $n = 4$ from [1] for better understanding of the construction of the new structure which we call n-space.

Consider $\mathcal{A} := \mathbb{Q} + \mathbb{Q}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1 b_1 + (a_1 b_2 + a_2 b_1)\varepsilon, \quad (a_i, b_i \in \mathbb{Q},\ i = 1, 2)$$

Then $\mathcal{A}$ is a (not commutative) local ring with the maximal ideal $\mathbf{I} = \mathbb{Q}\varepsilon$ of non-units.

$\mathbf{J}' = \mathbf{H}\ (\mathcal{A}_4, J\gamma)$, the set of 4 by 4 matrices, with entries from $\mathcal{A}$, that are symmetric with respect to the canonical involution $J\gamma : X \to \gamma^{-1}\overline{X}^{\mathbf{t}}\gamma$ where the γ_i are non-zero elements of F and $\gamma :=\text{diag}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. Hence, any element X of $\mathbf{J}'$ is of the form

$$X = [x_{ij}] = \begin{pmatrix} \alpha_1 & \gamma_2 a_{12} & \gamma_3 \overline{a_{13}} & \gamma_4 a_{14} \\ \gamma_1 \overline{a_{12}} & \alpha_2 & \gamma_3 a_{23} & \gamma_4 \overline{a_{24}} \\ \gamma_1 a_{13} & \gamma_2 \overline{a_{23}} & \alpha_3 & \gamma_4 a_{34} \\ \gamma_1 \overline{a_{14}} & \gamma_2 a_{24} & \gamma_3 \overline{a_{34}} & \alpha_4 \end{pmatrix} \text{ for } \alpha_i \in F \text{ and } a_i \in \mathcal{A}.$$

If we take a quartic form N such that $N(X) := \det X$, a cubic mapping $X \to X^\sharp$:=adjoint of X, and a basepoint $C := I_4$ on $\mathbf{J}$, then: it is clear that

$$\begin{aligned} T(X,Y) \quad = \quad & \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4 \\ & +2\gamma_1\gamma_2 n(a_{12}, b_{12}) + 2\gamma_1\gamma_3 n(a_{13}, b_{13}) + 2\gamma_1\gamma_4 n(a_{14}, b_{14}) \\ & +2\gamma_2\gamma_3 n(a_{23}, b_{23}) + 2\gamma_2\gamma_4 n(a_{24}, b_{24}) + 2\gamma_3\gamma_4 n(a_{34}, b_{34}), \end{aligned}$$

as $T(X,Y) := T(X\cdot Y) =$trace$(X\cdot Y)$. Moreover, $X\times Y := \frac{1}{6}\left[(X+Y)^{\#} - X^{\#} - Y^{\#}\right]$ because of $X\times X = X^{\#}$.

So, it is obtained the following results for the quaternion 3-space $\mathbf{P}(\mathbf{J}') = (\mathbf{P},\mathbf{L},|\,,\backsimeq)$ where $\mathbf{J}'$ is the 56$-$dimensional special Jordan matrix algebra:

The set of points $\mathbf{P}$ consists of the following four classes (which we call as points of types 1,2,3 and 4, respectively):

$$\left\{P_1 = \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\overline{x_2} & \gamma_1^{-1}\gamma_3\overline{x_3} & \gamma_1^{-1}\gamma_4\overline{x_4} \\ x_2 & \gamma_1^{-1}\gamma_2 n(x_2) & \gamma_1^{-1}\gamma_3 x_2\overline{x_3} & \gamma_1^{-1}\gamma_4 x_2\overline{x_4} \\ x_3 & \gamma_1^{-1}\gamma_2 x_3\overline{x_2} & \gamma_1^{-1}\gamma_3 n(x_3) & \gamma_1^{-1}\gamma_4 x_3\overline{x_4} \\ x_4 & \gamma_1^{-1}\gamma_2 x_4\overline{x_2} & \gamma_1^{-1}\gamma_3 x_4\overline{x_3} & \gamma_1^{-1}\gamma_4 n(x_4) \end{pmatrix} =: \begin{pmatrix} 1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^t \mid x_i\in\mathcal{A}\right\}\cup$$

$$\left\{P_2 = \begin{pmatrix} \gamma_2^{-1}\gamma_1 n(x_1) & x_1 & \gamma_2^{-1}\gamma_3 x_1\overline{x_3} & \gamma_2^{-1}\gamma_4 x_1\overline{x_4} \\ \gamma_2^{-1}\gamma_1\overline{x_1} & 1 & \gamma_2^{-1}\gamma_3\overline{x_3} & \gamma_2^{-1}\gamma_4\overline{x_4} \\ \gamma_2^{-1}\gamma_1 x_3\overline{x_1} & x_3 & \gamma_2^{-1}\gamma_3 n(x_3) & \gamma_2^{-1}\gamma_4 x_3\overline{x_4} \\ \gamma_2^{-1}\gamma_1 x_4\overline{x_1} & x_4 & \gamma_2^{-1}\gamma_3 x_4\overline{x_3} & \gamma_2^{-1}\gamma_4 n(x_4) \end{pmatrix} =: \begin{pmatrix} x_1 \\ 1 \\ x_3 \\ x_4 \end{pmatrix}^t \mid x_1\in\mathbf{I},\ x_3,x_4\in\mathcal{A}\right\}\cup$$

$$\left\{P_3 = \begin{pmatrix} \gamma_3^{-1}\gamma_1 n(x_1) & \gamma_3^{-1}\gamma_2 x_1\overline{x_2} & x_1 & \gamma_3^{-1}\gamma_4 x_1\overline{x_4} \\ \gamma_3^{-1}\gamma_1 x_2\overline{x_1} & \gamma_3^{-1}\gamma_2 n(x_2) & x_2 & \gamma_3^{-1}\gamma_4 x_2\overline{x_4} \\ \gamma_3^{-1}\gamma_1\overline{x_1} & \gamma_3^{-1}\gamma_2\overline{x_2} & 1 & \gamma_3^{-1}\gamma_4\overline{x_4} \\ \gamma_3^{-1}\gamma_1 x_4\overline{x_1} & \gamma_3^{-1}\gamma_2 x_4\overline{x_2} & x_4 & \gamma_3^{-1}\gamma_4 n(x_4) \end{pmatrix} =: \begin{pmatrix} x_1 \\ x_2 \\ 1 \\ x_4 \end{pmatrix}^t \mid x_1,x_2\in\mathbf{I},\ x_4\in\mathcal{A}\right\}\cup$$

$$\left\{P_4 = \begin{pmatrix} \gamma_4^{-1}\gamma_1 n(x_1) & \gamma_4^{-1}\gamma_2 x_1\overline{x_2} & \gamma_4^{-1}\gamma_3 x_1\overline{x_3} & x_1 \\ \gamma_4^{-1}\gamma_1 x_2\overline{x_1} & \gamma_4^{-1}\gamma_2 n(x_2) & \gamma_4^{-1}\gamma_3 x_2\overline{x_3} & x_2 \\ \gamma_4^{-1}\gamma_1 x_3\overline{x_1} & \gamma_4^{-1}\gamma_2 x_3\overline{x_2} & \gamma_4^{-1}\gamma_3 n(x_3) & x_3 \\ \gamma_4^{-1}\gamma_1\overline{x_1} & \gamma_4^{-1}\gamma_2\overline{x_2} & \gamma_4^{-1}\gamma_3\overline{x_3} & 1 \end{pmatrix} =: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix}^t \mid x_i\in\mathbf{I}\right\},$$

the set of lines $\mathbf{L}$ consists of the following four classes (which we call as lines of types 1,2,3 and 4, respectively):

$$\left\{l_1 = \begin{bmatrix} 1 & -m_2 & -m_3 & -m_4 \\ -\gamma_2^{-1}\gamma_1\overline{m_2} & \gamma_2^{-1}\gamma_1 n(m_2) & \gamma_2^{-1}\gamma_1\overline{m_2}m_3 & \gamma_2^{-1}\gamma_1\overline{m_2}m_4 \\ -\gamma_3^{-1}\gamma_1\overline{m_3} & \gamma_3^{-1}\gamma_1\overline{m_3}m_2 & \gamma_3^{-1}\gamma_1 n(m_3) & \gamma_3^{-1}\gamma_1\overline{m_3}m_4 \\ -\gamma_4^{-1}\gamma_1\overline{m_4} & \gamma_4^{-1}\gamma_1\overline{m_4}m_2 & \gamma_4^{-1}\gamma_1\overline{m_4}m_3 & \gamma_4^{-1}\gamma_1 n(m_4) \end{bmatrix} =: \begin{bmatrix} 1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}^t \mid m_i\in\mathbf{I}\right\}\cup$$

$$\left\{l_2 = \begin{bmatrix} \gamma_1^{-1}\gamma_2 n(m_1) & -\gamma_1^{-1}\gamma_2\overline{m_1} & \gamma_1^{-1}\gamma_2\overline{m_1}m_3 & \gamma_1^{-1}\gamma_2\overline{m_1}m_4 \\ -m_1 & 1 & -m_3 & -m_4 \\ \gamma_3^{-1}\gamma_2\overline{m_3}m_1 & -\gamma_3^{-1}\gamma_2\overline{m_3} & \gamma_3^{-1}\gamma_2 n(m_3) & \gamma_3^{-1}\gamma_2\overline{m_3}m_4 \\ \gamma_4^{-1}\gamma_2\overline{m_4}m_1 & -\gamma_4^{-1}\gamma_2\overline{m_4} & \gamma_4^{-1}\gamma_2\overline{m_4}m_3 & \gamma_4^{-1}\gamma_2 n(m_4) \end{bmatrix} =: \begin{bmatrix} m_1 \\ 1 \\ m_3 \\ m_4 \end{bmatrix}^t \mid m_1\in\mathcal{A},\ m_3,m_4\in\mathbf{I}\right\}\cup$$

$$\left\{l_3 = \begin{bmatrix} \gamma_1^{-1}\gamma_3 n(m_1) & \gamma_1^{-1}\gamma_3\overline{m_1}m_2 & -\gamma_1^{-1}\gamma_3\overline{m_1} & \gamma_1^{-1}\gamma_3\overline{m_1}m_4 \\ \gamma_2^{-1}\gamma_3\overline{m_2}m_1 & \gamma_2^{-1}\gamma_3 n(m_2) & -\gamma_2^{-1}\gamma_3\overline{m_2} & \gamma_2^{-1}\gamma_3\overline{m_2}m_4 \\ -m_1 & -m_2 & 1 & -m_4 \\ \gamma_4^{-1}\gamma_3\overline{m_4}m_1 & \gamma_4^{-1}\gamma_3\overline{m_4}m_2 & -\gamma_4^{-1}\gamma_3\overline{m_4} & \gamma_4^{-1}\gamma_3 n(m_4) \end{bmatrix} =: \begin{bmatrix} m_1 \\ m_2 \\ 1 \\ m_4 \end{bmatrix}^t \mid m_1,m_2\in\mathcal{A},\ m_4\in\mathbf{I}\right\}\cup$$

$$\left\{l_4 = \begin{bmatrix} \gamma_1^{-1}\gamma_4 n(m_1) & \gamma_1^{-1}\gamma_4\overline{m_1}m_2 & \gamma_1^{-1}\gamma_4\overline{m_1}m_3 & -\gamma_1^{-1}\gamma_4\overline{m_1} \\ \gamma_2^{-1}\gamma_4\overline{m_2}m_1 & \gamma_2^{-1}\gamma_4 n(m_2) & \gamma_2^{-1}\gamma_4\overline{m_2}m_3 & -\gamma_2^{-1}\gamma_4\overline{m_2} \\ \gamma_3^{-1}\gamma_4\overline{m_3}m_1 & \gamma_3^{-1}\gamma_4\overline{m_3}m_2 & \gamma_3^{-1}\gamma_4 n(m_3) & -\gamma_3^{-1}\gamma_4\overline{m_3} \\ -m_1 & -m_2 & -m_3 & 1 \end{bmatrix} =: \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ 1 \end{bmatrix}^t \mid m_i\in\mathcal{A}\right\}.$$

The incidence relation "$|$", equivalent to $X\cdot Y = 0$, is obtained as follows:

$$\begin{aligned} [1,k_2,k_3,k_4] \quad = \quad & \{(k_2+k_3y_3+k_4y_4,1,y_3,y_4) |\, y_3,y_4\in\mathcal{A}\}\cup \\ & \{(k_2z_2+k_3+k_4z_4,z_2,1,z_4) |\, z_2\in\mathbf{I}, z_4\in\mathcal{A}\}\cup \\ & \{(k_2t_2+k_3t_3+k_4,t_2,t_3,1) |\, t_2,t_3\in\mathbf{I}\}, \end{aligned}$$

$$\begin{aligned} [l_1,1,l_3,l_4] \quad = \quad & \{(1,l_1+l_3x_3+l_4x_4,x_3,x_4) |\ x_3,x_4\in\mathcal{A}\}\cup \\ & \{(z_1,l_1z_1+l_3+l_4z_4,1,z_4) |\, z_1\in\mathbf{I}, z_4\in\mathcal{A}\}\cup \\ & \{(t_1,l_1t_1+l_3t_3+l_4,t_3,1) |\, t_1,t_3\in\mathbf{I}\}, \end{aligned}$$

$$\begin{aligned}
[m_1, m_2, 1, m_4] &= \{(1, x_2, m_1 + m_2x_2 + m_4x_4, x_4) |\, x_2, x_4 \in \mathcal{A}\} \cup \\
&\quad \{(y_1, 1, m_1y_1 + m_2 + m_4y_4, y_4) |\, y_1 \in \mathbf{I}, y_4 \in \mathcal{A}\} \cup \\
&\quad \{(t_1, t_2, m_1t_1 + m_2t_2 + m_4, 1) |\, t_1, t_2 \in \mathbf{I}\}, \\
[n_1, n_2, n_3, 1] &= \{(1, x_2, x_3, n_1 + n_2x_2 + n_3x_3,) |\, x_2, x_3 \in \mathcal{A}\} \cup \\
&\quad \{(y_1, 1, y_3, n_1y_1 + n_2 + n_3y_3,) |\, y_1 \in \mathbf{I}, y_3 \in \mathcal{A}\} \cup \\
&\quad \{(z_1, z_2, 1, n_1z_1 + n_2z_2 + n_3) |\, z_1, z_2 \in \mathbf{I}\}.
\end{aligned}$$

Finally; the connection relation "$\backsimeq$", equivalent to $X \times Y \in I\mathbf{J}$, is obtained as follows:

$$\begin{aligned}
(x_1, x_2, x_3, x_4) &\backsimeq (y_1, y_2, y_3, y_4) \Leftrightarrow x_i - y_i \in \mathbf{I} \text{ for } i = 1, 2, 3, 4, \\
[k_1, k_2, k_3, k_4] &\backsimeq [n_1, n_2, n_3, n_4] \Leftrightarrow k_i - n_i \in \mathbf{I} \text{ for } i = 1, 2, 3, 4.
\end{aligned}$$

Besides, from types of points on lines, it is clear that a point and a line of same type is not connected (near). Moreover, the result is equivalent to $T(X, Y) \notin I = \{0\}$ for a point (or line) X and a line (or point) Y, respectively. In the other cases, we say that they are connected (near).

Now, we are ready to construct the n-space.

2. n-Spaces

Let $\mathbf{R}$ be a local ring with the maximal ideal $\mathbf{I}$ (of non-unit elements). Then $\mathbb{S}_n(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows.

The set of points $\mathbf{P}$ consists of the following $n+1$ points (which we call as points of types 1,2,3,..., $n+1$; respectively):

$$\mathbf{P} = \{P_i = (x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_{n+1}) \mid x_1, ..., x_{i-1} \in \mathbf{I} \text{ and } x_{i+1}, ..., x_{n+1} \in \mathbf{R}\}.$$

The set of lines $\mathbf{L}$ consists of the following $n+1$ lines (which we call as lines of types 1,2,3,..., $n+1$; respectively):

$$\mathbf{L} = \{M_i = [m_1, ..., m_{i-1}, 1, m_{i+1}, ..., m_{n+1}] \mid m_1, ..., m_{i-1} \in \mathbf{R} \text{ and } m_{i+1}, ..., m_n \in \mathbf{I}\}.$$

The incidence relation "$\in$" is defined as follows:

$$\begin{aligned}
M_1 &= [1, m_2, m_3, m_4, m_5, ..., m_{n-1}, m_n, m_{n+1}] \\
&= \{(m_2 + m_3y_3 + \cdots + m_{n+1}y_{n+1}, 1, y_3, ..., y_{n+1}) |\, y_3, ..., y_{n+1} \in \mathbf{R}\} \cup \\
&\quad \left\{ \begin{array}{c} (m_2z_2 + m_3 + m_4z_4 + \cdots + m_{n+1}z_{n+1}, z_2, 1, z_4, ..., z_{n+1}) | \\ z_2 \in \mathbf{I}, z_4, ..., z_{n+1} \in \mathbf{R} \end{array} \right\} \cup \\
&\quad \left\{ \begin{array}{c} (m_2t_2 + m_3t_3 + m_4 + m_5t_5 + \cdots + m_{n+1}t_{n+1}, t_2, t_3, 1, t_5, ..., t_{n+1}) | \\ t_2, t_3 \in \mathbf{I}, t_5, ..., t_{n+1} \in \mathbf{R} \end{array} \right\} \cup \\
&\quad \vdots \\
&\quad \left\{ \begin{array}{c} (m_2k_2 + \cdots + m_{n-1}k_{n-1} + m_n + m_{n+1}k_{n+1}, k_2, k_3, ..., k_{n-1}, 1, k_{n+1}) | \\ k_2, ..., k_{n-1} \in \mathbf{I},\, k_{n+1} \in \mathbf{R} \end{array} \right\} \cup \\
&\quad \{(m_2l_2 + m_3l_3 + \cdots + m_nl_n + m_{n+1}, l_2, l_3, l_4, ..., l_n, 1) |\, l_2, ..., l_n \in \mathbf{I}\},
\end{aligned}$$

$$\begin{aligned}
M_2 &= [m_1, 1, m_3, m_4, m_5, ...m_{n-1}, m_n, m_{n+1}] \\
&= \{(1, m_1 + m_3y_3 + \cdots + m_{n+1}y_{n+1}, y_3, ..., y_{n+1})| \ y_3, ..., y_{n+1} \in \mathbf{R}\} \cup \\
&\quad \left\{ \begin{array}{c} (z_1, m_1z_1 + m_3 + m_4z_4 + \cdots + m_{n+1}z_{n+1}, 1, z_4, ..., z_{n+1})| \\ z_1 \in \mathbf{I}, z_4, ..., z_{n+1} \in \mathbf{R} \end{array} \right\} \cup \\
&\quad \left\{ \begin{array}{c} (t_1, m_1t_1 + m_3t_3 + m_4 + m_5t_5 + \cdots + m_{n+1}t_{n+1}, t_3, 1, t_5, ..., t_{n+1})| \\ t_1, t_3 \in \mathbf{I}, t_5, ..., t_{n+1} \in \mathbf{R} \end{array} \right\} \cup \\
&\quad \vdots \\
&\quad \left\{ \begin{array}{c} (k_1, m_1k_1 + m_3k_3 + \cdots + m_{n-1}k_{n-1} + m_n + m_{n+1}k_{n+1}, k_3, ..., k_{n-1}, 1, k_{n+1})| \\ k_1, k_3, ..., k_{n-1} \in \mathbf{I}, \ k_{n+1} \in \mathbf{R} \end{array} \right\} \cup \\
&\quad \{(l_1, m_1l_1 + m_3l_3 + \cdots + m_nl_n + m_{n+1}, l_3, l_4, ..., l_n, 1)| \ l_1, l_3, ..., l_n \in \mathbf{I}\},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
M_{n+1} &= [m_1, m_2, m_3, m_4, ...m_{n-1}, m_n, 1] \\
&= \{(1, y_2, y_3, ..., y_n, m_1 + m_2y_2 + \cdots + m_ny_n)| \ y_2, ..., y_n \in \mathbf{R}\} \cup \\
&\quad \left\{ \begin{array}{c} (z_1, 1, z_3, z_4, ..., z_n, m_1z_1 + m_2 + m_3z_3 + \cdots + m_nz_n)| \\ z_1 \in \mathbf{I}, z_3, ..., z_n \in \mathbf{R} \end{array} \right\} \cup \\
&\quad \left\{ \begin{array}{c} (t_1, t_2, 1, t_4, ..., t_n, m_1t_1 + m_2t_2 + m_3 + m_4t_4 + \cdots + m_nt_n)| \\ t_1, t_2 \in \mathbf{I}, t_4, ..., t_n \in \mathbf{R} \end{array} \right\} \cup \\
&\quad \vdots \\
&\quad \left\{ \begin{array}{c} (k_1, k_2, ..., k_{n-2}, 1, k_n, m_1k_1 + \cdots + m_{n-2}k_{n-2} + m_{n-1} + m_nk_n)| \\ k_1, k_2, ..., k_{n-2} \in \mathbf{I}, \ k_n \in \mathbf{R} \end{array} \right\} \cup \\
&\quad \{(l_1, l_2, ..., l_{n-1}, 1, m_1l_1 + m_2l_2 + \cdots + m_{n-1}l_{n-1} + m_n)| \ l_1, l_2, ..., l_{n-1} \in \mathbf{I}\}.
\end{aligned}$$

The connection relation "$\sim$" is defined as follows:

$$\begin{aligned}
P &= (x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_{n+1}) \sim (y_1, ..., y_{i-1}, y_i, y_{i+1}, ..., y_{n+1}) = Q \\
&\iff x_i - y_i \in \mathbf{I} \ (1 \leq i \leq n+1), \forall P, Q \in \mathbf{P}; \\
g &= [m_1, ..., m_{i-1}, m_i, m_{i+1}, ..., m_{n+1}] \sim [p_1, ..., p_{i-1}, p_i, p_{i+1}, ..., p_{n+1}] = h \\
&\iff m_i - p_i \in \mathbf{I} \ (1 \leq i \leq n+1), \forall g, h \in \mathbf{L}.
\end{aligned}$$

If we more closely examine the case $n = 2$, then $\mathbb{S}_2(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is obtained as follows:

The set of points $\mathbf{P}$ consists of the following three points (which we call as points of types 1,2,3; respectively):

$$\begin{aligned}
\mathbf{P} &= \{P_1 = (1, x_2, x_3) \mid x_2, x_3 \in \mathbf{R}\} \cup \\
&\quad \{P_2 = (x_1, 1, x_3) \mid x_1 \in \mathbf{I}, x_3 \in \mathbf{R}\} \cup \\
&\quad \{P_3 = (x_1, x_2, 1) \mid x_1, x_2 \in \mathbf{I}\}.
\end{aligned}$$

The set of lines $\mathbf{L}$ consists of the following three lines (which we call as lines of types 1,2,3; respectively):

$$\begin{aligned}\mathbf{L} \quad = \quad & \{ M_1 = [1, m_2, m_3] \mid m_2, m_3 \in \mathbf{I}\} \cup \\ & \{ M_2 = [m_1, 1, m_3,] \mid m_1 \in \mathbf{R}, m_3 \in \mathbf{I}\} \\ & \{ M_3 = [m_1, m_2, 1] \mid m_1, m_2 \in \mathbf{R}\}.\end{aligned}$$

The incidence relation "$\in$" is as follows:

$$M_1 = [1, m_2, m_3] = \{ (m_2 + m_3 y_3, 1, y_3) | \ y_3 \in \mathbf{R}\} \cup \{ (m_2 z_2 + m_3, z_2, 1) | \ z_2 \in \mathbf{I}\},$$

$$M_2 = [m_1, 1, m_3] = \{ (1, m_1 + m_3 y_3, y_3) | \ y_3 \in \mathbf{R}\} \cup \{ (z_1, m_1 z_1 + m_3, 1) | \ z_1 \in \mathbf{I}\},$$

$$M_3 = [m_1, m_2, 1] = \{ (1, y_2, m_1 + m_2 y_2) | \ y_2 \in \mathbf{R}\} \cup \{ (z_1, 1, m_1 z_1 + m_2) | \ z_1 \in \mathbf{I}\}.$$

The connection relation "$\sim$" is as follows:

$$\begin{aligned} P \quad &= \quad (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \iff x_i - y_i \in \mathbf{I} \ (i = 1, 2, 3), \forall P, Q \in \mathbf{P}; \\ g \quad &= \quad [m_1, m_2, m_3] \sim [p_1, p_2, p_3] = h \Leftrightarrow m_i - p_i \in \mathbf{I} \ (i = 1, 2, 3)), \forall g, h \in \mathbf{L}.\end{aligned}$$

So, we have obtained a PK-plane (2-space), isomorphic to the PK-plane given in [2], in the case $n = 2$.

If we take $\mathbf{R} := \mathbb{O} + \mathbb{O}\varepsilon$ where $\mathbb{O}$ is the Cayley division algebra over a field F and $\varepsilon \notin \mathbb{O}$, then $\mathbb{S}_2(\mathbf{R})$ is an octonion plane and also the MK-plane, introduced by Blunck in [4]. Moreover, for $n > 2$, $\mathbb{S}_n(\mathbf{R})$ is the example of n-space (or octonion n-space). Note that the (quaternion) n-space $\mathbb{S}_n(\mathbb{Q} + \mathbb{Q}\varepsilon)$ is a subspace of the (octonion) n-space $\mathbb{S}_n(\mathbb{O}+\mathbb{O}\varepsilon)$. Besides, it is well-known that there is no a projective space constructed over non-associative division rings, and therefore a epimorphism onto an ordinary projective n-space can not exist. This means that the space $\mathbb{S}_n(\mathbb{O} + \mathbb{O}\varepsilon)$ for $n > 2$ is not a PK structure. For this reason, we tend to construct some collineations of the space $\mathbb{S}_n(\mathbb{Q} + \mathbb{Q}\varepsilon)$.

Finally, we would like to complete this paper by giving two collineations of the quaternion n-space $\mathbb{S}_n(\mathbb{Q} + \mathbb{Q}\varepsilon)$.

$$\begin{aligned} \mathrm{T}_{a_2,0,\ldots,0,0} \quad &: \quad \text{for } a_2 \in \mathbb{Q}, \\ (1, x_2, x_3, ..., x_n, x_{n+1}) \quad &\rightarrow \quad (1, x_2 + a_2, x_3 + 0, ..., x_n + 0, x_{n+1} + 0), \\ (x_1, 1, x_3, ..., x_{n+1}) \quad &\rightarrow \quad (x_1, 1, x_3 - (x_3 a_2) x_1, ..., x_{n+1} - (x_{n+1} a_2) x_1), \\ (x_1, x_2, 1, x_4, ..., x_n, x_{n+1}) \quad &\rightarrow \quad (x_1, x_2 + a_2 x_1, 1, x_4, ..., x_n, x_{n+1}), \\ & \quad \vdots \\ (x_1, x_2, x_3, ...x_{n-1}, 1, x_{n+1}) \quad &\rightarrow \quad (x_1, x_2 + a_2 x_1, x_3, ..., x_{n-1}, 1, x_{n+1}),\end{aligned}$$

$$(x_1, x_2, , ...x_{n-1}, x_n, 1) \quad \rightarrow \quad (x_1, x_2 + a_2 x_1, x_3, ..., x_{n-1}, x_n, 1),$$

$$\begin{aligned}
[m_1, m_2, ..., m_n, 1] &\rightarrow [m_1 - m_2 a_2, m_2, ..., m_n, 1] \\
[m_1, m_2, ..., 1, m_{n+1}] &\rightarrow [m_1 - m_2 a_2, m_2, ..., 1, m_{n+1}] \\
&\vdots \\
[m_1, m_2, 1, m_4, ..., m_n, m_{n+1}] &\rightarrow [m_1 - m_2 a_2, m_2, 1, m_4, ..., m_n, m_{n+1}] \\
[m_1, 1, m_3, ..., m_n, m_{n+1}] &\rightarrow [m_1 + a_2, 1, m_3, ..., m_n, m_{n+1}] \\
[1, m_2, m_3, ..., m_n, m_{n+1}] &\rightarrow [1, m_2, m_3, ..., m_n, m_{n+1}]
\end{aligned}$$

Similarly, the transformation $\mathrm{T}_{0,a_3,0,\ldots,0}$ can be defined in the following way: for any $a_3 \in \mathbb{Q}$,

$$\begin{aligned}
(1, x_2, x_3, x_4, ..., x_{n+1}) &\rightarrow (1, x_2 + 0, x_3 + a_3, x_4 + 0, ..., x_{n+1} + 0) \\
(x_1, 1, x_3, x_4, ..., x_{n+1}) &\rightarrow (x_1, 1, x_3 + a_3 x_1, x_4, ..., x_{n+1}), \\
(x_1, x_2, 1, x_4, ..., x_{n+1}) &\rightarrow (x_1, x_2 - (x_2 a_3) x_1, 1, x_4 - (x_4 a_3) x_1, ..., x_{n+1} - (x_{n+1} a_3) x_1), \\
&\vdots \\
(x_1, x_2, x_3, ...x_{n-1}, 1, x_{n+1}) &\rightarrow (x_1, x_2, x_3 + a_3 x_1, x_4, ..., x_{n-1}, 1, x_{n+1}), \\
(x_1, x_2, , ..., x_{n-1}, x_n, 1) &\rightarrow (x_1, x_2, x_3 + a_3 x_1, x_4, ..., x_{n-1}, x_n, 1),
\end{aligned}$$

$$\begin{aligned}
[m_1, m_2, ..., m_n, 1] &\rightarrow [m_1 - m_3 a_3, m_2, ..., m_n, 1], \\
[m_1, m_2, ..., m_{n-1}, 1, m_{n+1}] &\rightarrow [m_1 - m_3 a_3, m_2, ..., m_{n-1}, 1, m_{n+1}], \\
&\vdots \\
[m_1, m_2, 1, m_4, ..., m_n, m_{n+1}] &\rightarrow [m_1 + a_3, m_2, 1, m_4, ..., m_n, m_{n+1}], \\
[m_1, 1, m_3, ..., m_n, m_{n+1}] &\rightarrow [m_1 - m_3 a_3, 1, m_3, ..., m_n, m_{n+1}], \\
[1, m_2, m_3, ..., m_n, m_{n+1}] &\rightarrow [1, m_2, m_3, ..., m_n, m_{n+1}].
\end{aligned}$$

And, continuing on like this, finally, the transformation $\mathrm{T}_{0,0,\ldots,0,a_{n+1}}$ can be defined in the following manner: for any $a_{n+1} \in \mathbb{Q}$,

$$\begin{aligned}
(1, x_2, x_3, ..., x_n, x_{n+1}) &\rightarrow (1, x_2 + 0, x_3 + 0, ..., x_n + 0, x_{n+1} + a_{n+1}) \\
(x_1, 1, x_3, ..., x_n, x_{n+1}) &\rightarrow (x_1, 1, x_3, ..., x_n, x_{n+1} + a_{n+1} x_1), \\
(x_1, x_2, 1, x_4, ..., x_n, x_{n+1}) &\rightarrow (x_1, x_2, 1, x_4, ..., x_n, x_{n+1} + a_{n+1} x_1), \\
&\vdots \\
(x_1, x_2, x_3, ..., x_{n-1}, 1, x_{n+1}) &\rightarrow (x_1, x_2, x_3, ..., x_{n-1}, 1, x_{n+1} + a_{n+1} x_1), \\
(x_1, x_2, , ..., x_n, 1) &\rightarrow (x_1, x_2 - (x_2 a_{n+1}) x_1, ..., x_n - (x_n a_{n+1}) x_1, 1),
\end{aligned}$$

$$[m_1, m_2, ..., m_n, 1] \quad \rightarrow \quad [m_1 + a_{n+1}, m_2, ..., m_n, 1],$$

$$\begin{aligned}
[m_1, m_2, ..., 1, m_{n+1}] &\rightarrow [m_1 - m_{n+1}a_{n+1}, m_2, ..., 1, m_{n+1}], \\
&\vdots \\
[m_1, m_2, 1, m_4, ..., m_n, m_{n+1}] &\rightarrow [m_1 - m_{n+1}a_{n+1}, m_2, 1, m_4, ..., m_n, m_{n+1}], \\
[m_1, 1, m_3, ..., m_n, m_{n+1}] &\rightarrow [m_1 - m_{n+1}a_{n+1}, 1, m_3, ..., m_n, m_{n+1}], \\
[1, m_2, m_3, ..., m_n, m_{n+1}] &\rightarrow [1, m_2, m_3, ..., m_n, m_{n+1}].
\end{aligned}$$

So, in this case, we have the translation transformation $\mathrm{T}_{a_2,a_3,...,a_{n-1},a_n,a_{n+1}}$ of $\mathbb{S}_n(\mathbb{Q}+\mathbb{Q}\varepsilon)$. The other transformation F_a is defined as follows:

$$\begin{aligned}
\mathrm{F}_a &: \text{for } a \notin \mathbb{Q}\varepsilon, \\
(1, x_2, x_3, ..., x_n, x_{n+1}) &\rightarrow (1, ax_2a, x_3a, ..., x_na, x_{n+1}a) \\
(x_1, 1, x_3, ..., x_n, x_{n+1}) &\rightarrow \left(a^{-1}x_1a^{-1}, 1, x_3a^{-1}, ..., x_na^{-1}, x_{n+1}a^{-1}\right) \\
(x_1, x_2, 1, x_4, ..., x_n, x_{n+1}) &\rightarrow \left(a^{-1}x_1, ax_2, 1, x_4, ..., x_n, x_{n+1}\right) \\
&\vdots \\
(x_1, x_2, x_3, ...x_{n-1}, 1, x_{n+1}) &\rightarrow \left(a^{-1}x_1, ax_2, x_3, ...x_{n-1}, 1, x_{n+1}\right) \\
(x_1, x_2, x_3, ..., x_n, 1) &\rightarrow \left(a^{-1}x_1, ax_2, x_3, ..., x_n, 1\right)
\end{aligned}$$

$$\begin{aligned}
[m_1, m_2, m_3, ..., m_n, 1] &\rightarrow \left[m_1a, m_2a^{-1}, m_3, ..., m_n, 1\right] \\
[m_1, m_2, m_3, ..., m_{n-1}, 1, m_{n+1}] &\rightarrow \left[m_1a, m_2a^{-1}, m_3, ..., m_{n-1}, 1, m_{n+1}\right] \\
&\vdots \\
[m_1, m_2, 1, m_4, ..., m_n, m_{n+1}] &\rightarrow \left[m_1a, m_2a^{-1}, 1, m_4, ..., m_n, m_{n+1}\right] \\
[m_1, 1, m_3, ..., m_n, m_{n+1}] &\rightarrow [am_1a, 1, am_3, ..., am_n, am_{n+1}] \\
[1, m_2, m_3, ..., m_n, m_{n+1}] &\rightarrow \left[1, a^{-1}m_2a^{-1}, a^{-1}m_3, ..., a^{-1}m_n, a^{-1}m_{n+1}\right]
\end{aligned}$$

To show that the transformations $\mathrm{T}_{a_2,0,...0,0,0}$, $\mathrm{T}_{0,a_3,...0,0,0}$, $\mathrm{T}_{0,0,...,0,a_{n+1}}$ (and as a result, $\mathrm{T}_{a_2,a_3,...,a_{n-1},a_n,a_{n+1}}$ which is the combination of the all above transformations) and F_a are collineations of $\mathbb{S}_n(\mathbb{Q}+\mathbb{Q}\varepsilon)$, it is basically enough to prove Lemma 3 given in [5]. And also, we will often need the two results that $\mathbb{Q}+\mathbb{Q}\varepsilon$ is associative and that multiplication of any elements in the ideal $\mathbf{I} = \mathbb{Q}\varepsilon$ is equal to zero. Hence, we obtain that it is possible to study in the spaces by means of the collineations, analogous of the collineations given for showing 4-transitivity on the class of MK-plane in [5].

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GREEN FUNCTION AND RESOLVENT OPERATOR OF A SCHRÖDINGER EQUATION WITH GENERAL POINT INTERACTION

EMEL YILDIRIM

Abstract. In this paper, we investigate the time independent Schrödinger equation which has complex valued potential function under the general point interaction. We construct Green function of this problem and we find the resolvent of the problem in terms of Green function.

1. Introduction

In mathematics, a Green's function is used to solve inhomogeneous differential equations under various initial conditions or boundary conditions. The term is also used in physics, quantum mechanics, engineering and quantum field theory to signify various types of correlation functions. It has been shown that should be taken into consideration a perturbation series to an infinite in order to obtain a deep insight not attainable by a finite-order treatment thanks to Green's functions in many quantum-mechanical applications. Therefore, these functions are very important in many disciplines and much work has been done on this theory. Especially, Green's functions have been used in investigating the completeness property of the set of eigenfunctions of self-adjoint operators and in proving the expansion theorem for an arbitrary function in terms of the complete set obtained from a Sturm-Liouville operator [9]. Also, it has been obtained by Sakurai that one of the easiest ways to identify the peculiar feature of a particular quantum mechanics, which contrasts with classical mechanics, is to convert the action integral for a classical motion into the form involving the Green's functions of the Schrödinger equation [15] . As a result of these studies, many researchers have examined of Green's functions of Schrödinger equation with some initial conditions or boundary conditions [20, 16].

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The Green's functions of the Schrödinger equation for the simplest quantum mechanical systems have been investigated in [7]. Sturm Liouville operator with eigenparameter dependent boundary conditions and transmission conditions at a finite number of interior points have been studied and Green's function has been obtained for this problem in [12].

Additionally, differential equations with transmission condition has an important role for many branches of sciences. In particularly, they are used in quantum mechanics and atomic physics. Because some of them are considered to represent many physical systems quantitatively. It is known that there are four parameters point interactions that can be represent as self-adjoint extensions of the nonrelativistic kinetic energy operator in one-dimensional quantum mechanics [10, 11]. For the mathematical theory of differential equations with point interaction, we refer to the monographs[4, 5]. In the literature, many various type of differential equations have been investigated under the point interaction [1, 2, 3, 8, 14, 19]. In particular, dissipative boundary value problems with point interaction have been studied in [6, 17, 18].

In this paper, our aim is to find the Resolvent operator of the Schrödinger equation under general point interaction by constructing the Green function.

2. Some Properties of Solutions of the Schrödinger Equation on Whole Axis

Let us consider the time independent Schrödinger equation

$$-\psi''(x)+q(x)\psi(x)=\lambda^2\psi(x),\quad x\in\mathbb{R}\backslash\{0\} \tag{1}$$

where λ is spectral parameter, $q(x)$ is complex valued function. The equation (1) has bounded solutions $e_{\pm}(x,\lambda)$ which they satisfy following limit conditions.

$$\lim_{x\to\pm\infty}\psi(x)e^{\mp i\lambda x}=1,\quad \lambda\in\overline{\mathbb{C}}_+=\{\lambda\colon \lambda\in\mathbb{C},\operatorname{Im}\lambda\geq 0\}. \tag{2}$$

The solutions $e_{\pm}(x,\lambda)$ are called Jost solution of equation (1) and they have the following representations

$$\begin{aligned} e_-(x,\lambda) &= e^{-i\lambda x}+\int_{-\infty}^{x}K^-(x,t)\,e^{-i\lambda t}dt,\ \lambda\in\overline{\mathbb{C}}_+,\ -\infty<x<0 \\ e_+(x,\lambda) &= e^{i\lambda x}+\int_{x}^{\infty}K^+(x,t)\,e^{i\lambda t}dt,\ \ \lambda\in\overline{\mathbb{C}}_+,\ 0<x<\infty \end{aligned} \tag{3}$$

under the condition

$$\int_{-\infty}^{\infty}(1+|x|)\,|q(x)|\,dx<\infty. \tag{4}$$

$K^{+}(x,t)$ and $K^{-}(x,t)$ are called Kernel functions and they are defined as follows.

$$\begin{aligned} K^{+}(x,t) &= \frac{1}{2}\int\limits_{\frac{x+t}{2}}^{\infty} q(s)ds+\frac{1}{2}\int\limits_{x}^{\frac{x+t}{2}}\int\limits_{x+t-s}^{t-x+s} q(s)K^{+}(s,r)\,dsdr \\ &+\frac{1}{2}\int\limits_{\frac{x+t}{2}}^{\infty}\int\limits_{s}^{t-x+s} q(s)K^{+}(s,r)\,dsdr \\ K^{-}(x,t) &= \frac{1}{2}\int\limits_{-\infty}^{\frac{x+t}{2}} q(s)ds+\frac{1}{2}\int\limits_{x}^{\frac{x+t}{2}}\int\limits_{x+t-s}^{t-x+s} q(s)K^{-}(s,r)\,dsdr \\ &+\frac{1}{2}\int\limits_{-\infty}^{\frac{x+t}{2}}\int\limits_{s}^{t-x+s} q(s)K^{-}(s,r)\,dsdr \end{aligned} \tag{5}$$

Moreover, these functions are continuously differentiable with respect to their arguments and satisfy the following inequalities:

$$\begin{aligned} \left|K^{\pm}(x,t)\right| &\leq c\sigma^{\pm}\left(\frac{x+t}{2}\right) \\ \left|K_{x}^{\pm}(x,t)\pm\frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right|\right| &\leq c\sigma^{\pm}\left(\frac{x+t}{2}\right) \\ \left|K_{t}^{\pm}(x,t)\pm\frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right|\right| &\leq c\sigma^{\pm}\left(\frac{x+t}{2}\right) \end{aligned} \tag{6}$$

where

$$\sigma^{+}(t)=\int\limits_{x}^{\infty}|q(t)|\,dt,\quad \sigma^{-}(t)=\int\limits_{-\infty}^{x}|q(t)|\,dt$$

and $c>0$ is a constant. Furthermore, $\widehat{e}_{\pm}(x,\lambda)$ are unbounded solutions of equation (1) which they satisfy

$$\lim_{x\to\pm\infty}\psi(x)\,e^{\pm i\lambda x}=1 \tag{7}$$

[13].

3. Green Function of the Schrödinger Equation with General Point Interaction

In this section, we shall construct Green function of the Schrödinger equation (1) with the general point interaction

$$\begin{pmatrix} \psi_{+}(0,\lambda) \\ \psi'_{+}(0,\lambda) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_{-}(0,\lambda) \\ \psi'_{-}(0,\lambda) \end{pmatrix} \tag{8}$$

where a, b, c, $d \in \mathbb{R}$ and $ad - bc \neq 0$ and we shall find the Resolvent operator of this problem. Green function of the problem (1)-(8) is a solution to following non-homogeneous differential equation

$$-\psi''(x) + q(x)\psi(x) - \lambda^2\psi(x) = f(x), \quad x \in \mathbb{R}\setminus\{0\} \tag{9}$$

To obtain the general solution of this equation, we need to find solutions of equation (1). The two linear independent solutions of its can be given as:

$$\begin{aligned} U(x,\lambda) &= \begin{cases} U_{+}(x,\lambda), & x > 0 \\ U_{-}(x,\lambda), & x < 0 \end{cases} \\ &= \begin{cases} e_{+}(x,\lambda), & x > 0 \\ \alpha(\lambda)\, e_{-}(x,\lambda) + \beta(\lambda)\, \widehat{e}_{-}(x,\lambda), & x < 0 \end{cases}, \quad \lambda \in \mathbb{C} \end{aligned} \tag{10}$$

$$\begin{aligned} V(x,\lambda) &= \begin{cases} V_{+}(x,\lambda), & x > 0 \\ V_{-}(x,\lambda), & x < 0 \end{cases} \\ &= \begin{cases} \hat{\alpha}(\lambda)\, e_{+}(x,\lambda) + \hat{\beta}(\lambda)\, \widehat{e}_{+}(x,\lambda), & x < 0 \\ e_{-}(x,\lambda), & x < 0 \end{cases}, \quad \lambda \in \mathbb{C} \end{aligned} \tag{11}$$

where $\widehat{e}_{\pm}(x,\lambda)$ are unbounded solutions and $e_{\pm}(x,\lambda)$ are bounded solutions of equation (1). By using the solutions $U(x,\lambda)$ and $V(x,\lambda)$ and the condition (8), we get

$$\begin{aligned} e_{+}(0,\lambda) &= a\alpha(\lambda)\, e_{-}(0,\lambda) + a\beta(\lambda)\, \widehat{e}_{-}(0,\lambda) \\ &\quad + b\alpha(\lambda)\, e'_{-}(0,\lambda) + b\beta(\lambda)\, \widehat{e}'_{-}(0,\lambda) \\ e'_{+}(0,\lambda) &= c\alpha(\lambda)\, e_{-}(0,\lambda) + c\beta(\lambda)\, \widehat{e}_{-}(0,\lambda) \\ &\quad + d\alpha(\lambda)\, e'_{-}(0,\lambda) + d\beta(\lambda)\, \widehat{e}'_{-}(0,\lambda) \\ \widehat{\alpha}(\lambda)\, e_{+}(0,\lambda) + \widehat{\beta}(\lambda)\, \widehat{e}_{+}(0,\lambda) &= ae_{-}(0,\lambda) + be'_{-}(0,\lambda) \\ \widehat{\alpha}(\lambda)\, e'_{+}(0,\lambda) + \widehat{\beta}(\lambda)\, \widehat{e}'_{+}(0,\lambda) &= ce_{-}(0,\lambda) + de'_{-}(0,\lambda). \end{aligned}$$

For these equations to be true, $\alpha(\lambda)$, $\beta(\lambda)$, $\widehat{\alpha}(\lambda)$ and $\widehat{\beta}(\lambda)$ must be as follows, respectively.

$$\alpha(\lambda) = \frac{1}{2i\lambda(ad-bc)}\left(\begin{array}{c} c\widehat{e}_-(0,\lambda)e_+(0,\lambda)+de_+(0,\lambda)\widehat{e}'_-(0,\lambda) \\ -a\widehat{e}_-(0,\lambda)e'_+(0,\lambda)-be'_+(0,\lambda)\widehat{e}'_-(0,\lambda) \end{array}\right)$$

$$\beta(\lambda) = \frac{1}{2i\lambda(ad-bc)}\left(\begin{array}{c} ae_-(0,\lambda)e'_+(0,\lambda)+be'_-(0,\lambda)e'_+(0,\lambda) \\ -ce_-(0,\lambda)-de'_-(0,\lambda)e_+(0,\lambda) \end{array}\right)$$

$$\widehat{\alpha}(\lambda) = \frac{1}{2i\lambda}\left(\begin{array}{c} ae_-(0,\lambda)\widehat{e}'_+(0,\lambda)+be'_-(0,\lambda)\widehat{e}'_+(0,\lambda) \\ -ce_-(0,\lambda)\widehat{e}_+(0,\lambda)-de'_-(0,\lambda)\widehat{e}_+(0,\lambda) \end{array}\right)$$

$$\widehat{\beta}(\lambda) = \frac{1}{2i\lambda}\left(\begin{array}{c} ce_-(0,\lambda)e'_+(0,\lambda)-de'_-(0,\lambda)e_+(0,\lambda) \\ -ae_-(0,\lambda)e'_+(0,\lambda)+be'_-(0,\lambda)e'_+(0,\lambda) \end{array}\right)$$

From these equations, it is seen that $\widehat{\beta}(\lambda)$ can be written in terms of $\beta(\lambda)$ such that

$$\widehat{\beta}(\lambda) = (ad-bc)\beta(\lambda). \tag{12}$$

Moreover, since the linear combination of solutions of a differential equation is the solution of this equation, the general solution of equation (1) can be given by using solutions $U(x,\lambda)$ and $V(x,\lambda)$

$$\begin{aligned} y(x,\lambda) &= \begin{cases} y_-(x,\lambda), & x<0 \\ y_+(x,\lambda), & x>0 \end{cases} \\ &= \begin{cases} c_1U_-(x,\lambda)+c_2V_-(x,\lambda), & x<0 \\ d_1U_+(x,\lambda)+d_2V_+(x,\lambda), & x>0 \end{cases} \end{aligned} \tag{13}$$

where c_1, c_2, d_1 and d_2 are constant. If we use the method of variation of constants, we find the general solution of equation (9) in the form

$$\begin{aligned} \widetilde{y}(x,\lambda) &= \begin{cases} \widetilde{y_-}(x,\lambda), & x<0 \\ \widetilde{y_+}(x,\lambda), & x>0 \end{cases} \\ &= \begin{cases} c_1(x)U_-(x,\lambda)+c_2(x)V_-(x,\lambda), & x<0 \\ d_1(x)U_+(x,\lambda)+d_2(x)V_+(x,\lambda), & x>0 \end{cases} \end{aligned} \tag{14}$$

where $c_1(x), c_2(x)$ and $d_1(x), d_2(x)$ satisfy the following equation systems

$$\begin{aligned} c_1^{'}(x)U_-(x,\lambda)+c_2^{'}(x)V_-(x,\lambda) &= 0 \\ c_1^{'}(x)U_-^{'}(x,\lambda)+c_2^{'}(x)V_-^{'}(x,\lambda) &= -f(x) \end{aligned}$$

and

$$\begin{aligned} d_1^{'}(x)\,U_+(x,\lambda)+d_2^{'}(x)\,V_+(x,\lambda) &= 0 \\ d_1^{'}(x)\,U_+^{'}(x,\lambda)+d_2^{'}(x)\,V_+^{'}(x,\lambda) &= -f(x)\,. \end{aligned}$$

When the Cramer Method's is applied to these equations, the following equations are obtained.

$$c_1^{'}(x)=\frac{V_-(x,\lambda)\,f(x)}{W\left[U_-(x,\lambda)\,,V_-(x,\lambda)\right]}$$

$$c_2^{'}(x)=\frac{-U_-(x,\lambda)\,f(x)}{W\left[U_-(x,\lambda)\,,V_-(x,\lambda)\right]}$$

$$d_1^{'}(x)=\frac{V_+(x,\lambda)\,f(x)}{W\left[U_+(x,\lambda)\,,V_+(x,\lambda)\right]}$$

$$d_2^{'}(x)=\frac{-U_+(x,\lambda)\,f(x)}{W\left[U_+(x,\lambda)\,,V_+(x,\lambda)\right]}$$

Considering that $U_+(x,\lambda)$ and $V_+(x,\lambda)$ are defined on $(0,\infty)$ and $U_-(x,\lambda)$ and $V_-(x,\lambda)$ are defined on $(-\infty,0)$, $c_1(x)\,,c_2(x)\,,d_1(x)$ and $d_2(x)$ can be expressed by the following integral equations;

$$\begin{aligned} c_1(x) &= \alpha_1+\int\limits_{-\infty}^{x}\frac{V_-(t,\lambda)\,f(t)}{W\left[U_-(t,\lambda)\,,V_-(t,\lambda)\right]}dt, \\ c_2(x) &= \alpha_2+\int\limits_{x}^{0}\frac{-U_-(t,\lambda)\,f(t)}{W\left[U_-(t,\lambda)\,,V_-(t,\lambda)\right]}dt\ , \\ d_1(x) &= \alpha_3+\int\limits_{0}^{x}\frac{V_+(t,\lambda)\,f(t)}{W\left[U_+(t,\lambda)\,,V_+(t,\lambda)\right]}dt, \\ d_2(x) &= \alpha_4+\int\limits_{x}^{\infty}\frac{-U_+(t,\lambda)\,f(t)}{W\left[U_+(t,\lambda)\,,V_+(t,\lambda)\right]}dt\ . \end{aligned}$$

If we consider these equation in (14), the general solution of equation (9) can be given as

$$\begin{aligned} \widetilde{y_-}(x,\lambda) &= \alpha_1U_-(x,\lambda)+U_-(x,\lambda)\int\limits_{-\infty}^{x}\frac{V_-(t,\lambda)\,f(t)}{W\left[U_-(t,\lambda)\,,V_-(t,\lambda)\right]}dt \\ &\quad +\alpha_2V_-(x,\lambda)-V_-(x,\lambda)\int\limits_{x}^{0}\frac{U_-(t,\lambda)\,f(t)}{W\left[U_-(t,\lambda)\,,V_-(t,\lambda)\right]}dt. \end{aligned}$$

$$\begin{aligned}\widetilde{y_{+}}(x,\lambda) &= \alpha_3 U_{+}(x,\lambda) + U_{+}(x,\lambda)\int_0^x \frac{V_{+}(t,\lambda) f(t)}{W[U_{+}(t,\lambda), V_{+}(t,\lambda)]} dt \\ &\quad + \alpha_4 V_{+}(x,\lambda) - V_{+}(x,\lambda)\int_x^\infty \frac{U_{+}(t,\lambda) f(t)}{W[U_{+}(t,\lambda), V_{+}(t,\lambda)]} dt.\end{aligned}$$

Since $\widetilde{y_{-}}(x,\lambda) \in L_2(-\infty,0)$ and $\widetilde{y_{+}}(x,\lambda) \in L_2(0,\infty)$, α_1, α_2, α_3 and α_4 must be equal to zero. Therefore the general solution of non-homogeneous equation (9) has the following form:

$$\widetilde{y}(x,\lambda) = \begin{cases} \int\limits_{-\infty}^{x} \frac{V_{-}(t,\lambda) U_{-}(x,\lambda) f(t)}{W[U_{-}(t,\lambda), V_{-}(t,\lambda)]} dt + \int\limits_{x}^{0} \frac{-U_{-}(t,\lambda) V_{-}(x,\lambda) f(t)}{W[U_{-}(t,\lambda), V_{-}(t,\lambda)]} dt, & x<0 \\ \int\limits_{0}^{x} \frac{V_{+}(t,\lambda) U_{+}(x,\lambda) f(t)}{W[U_{+}(t,\lambda), V_{+}(t,\lambda)]} dt + \int\limits_{x}^{\infty} \frac{-U_{+}(t,\lambda) V_{+}(x,\lambda) f(t)}{W[U_{+}(t,\lambda), V_{+}(t,\lambda)]} dt, & x>0 \end{cases}$$

Moreover, by using the representations of $U(x,\lambda)$ and $V(t,\lambda)$, this formula can be rewritten in the form

$$\widetilde{y}(x,\lambda) = U(x,\lambda)\int_{-\infty}^x \frac{V(t,\lambda) f(t)}{W[U(t,\lambda), V(t,\lambda)]} dt - V(x,\lambda)\int_x^\infty \frac{U(t,\lambda) f(t)}{W[U(t,\lambda), V(t,\lambda)]} dt \tag{15}$$

Consequently, we can write the Green function of the problem (1)-(8).

$$G(x,t;\lambda) = \begin{cases} \dfrac{U(x,\lambda) V(t,\lambda)}{W[U(x,\lambda), V(x,\lambda)]}, & -\infty < x \le t, x \neq t \neq 0 \\ \dfrac{V(x,\lambda) U(t,\lambda)}{W[U(x,\lambda), V(x,\lambda)]}, & x \le t < \infty, x \neq t \neq 0 \end{cases} \tag{16}$$

The Wronksian of the solutions $U(x,\lambda)$ and $V(x,\lambda)$ can be calculated easily by using the definiton of them. For x>0,

$$\begin{aligned} W[U(x,\lambda), V(x,\lambda)] &= \begin{vmatrix} e_{+}(x,\lambda) & \widehat{\alpha}(\lambda) e_{+}(x,\lambda) + \widehat{\beta}(\lambda)\widehat{e}_{+}(x,\lambda) \\ e'_{+}(x,\lambda) & \widehat{\alpha}(\lambda) e'_{+}(x,\lambda) + \widehat{\beta}(\lambda)\widehat{e}'_{+}(x,\lambda) \end{vmatrix} \qquad (17) \\ &= \widehat{\alpha}(\lambda) e'_{+}(x,\lambda) e_{+}(x,\lambda) + \widehat{\beta}(\lambda)\widehat{e}'_{+}(x,\lambda) e_{+}(x,\lambda) \\ &\quad -\widehat{\alpha}(\lambda) e_{+}(x,\lambda) e'_{+}(x,\lambda) - \widehat{\beta}(\lambda)\widehat{e}_{+}(x,\lambda) e'_{+}(x,\lambda) \\ &= \widehat{\beta}(\lambda)\left(e_{+}(x,\lambda)\widehat{e}'_{+}(x,\lambda) - \widehat{e}_{+}(x,\lambda) e'_{+}(x,\lambda)\right) \\ &= -2i\lambda\widehat{\beta}(\lambda) \end{aligned}$$

and for x<0,

$$\begin{aligned} W\left[U\left(x,\lambda\right),V\left(x,\lambda\right)\right] &= \begin{vmatrix} \alpha\left(\lambda\right)e_{-}(x,\lambda)+\beta\left(\lambda\right)\widehat{e}_{-}(x,\lambda), & e_{-}(x,\lambda) \\ \alpha\left(\lambda\right)e_{-}'(x,\lambda)+\beta\left(\lambda\right)\widehat{e}_{-}^{\prime}(x,\lambda), & e_{-}'(x,\lambda) \end{vmatrix} \\ &= \alpha\left(\lambda\right)e_{-}(x,\lambda)e_{-}'(x,\lambda)+\beta\left(\lambda\right)\widehat{e}_{-}(x,\lambda)e_{-}'(x,\lambda) \\ &\quad -\alpha\left(\lambda\right)e_{-}'(x,\lambda)e_{-}(x,\lambda)-\beta\left(\lambda\right)\widehat{e}_{-}^{\prime}(x,\lambda)e_{-}(x,\lambda) \\ &= -\beta\left(\lambda\right)\Big(e_{-}(x,\lambda)\widehat{e}_{-}^{\prime}(x,\lambda)-\widehat{e}_{-}(x,\lambda)e_{-}'(x,\lambda)\Big) \\ &= -2i\lambda\beta\left(\lambda\right). \end{aligned}$$

In view of (12) and (17), the Wronskian can be arranged.

$$W\left[U\left(x,\lambda\right),V\left(x,\lambda\right)\right] = \begin{cases} -2i\lambda(ad-bc)\beta\left(\lambda\right), & x>0 \\ \\ -2i\lambda\beta\left(\lambda\right), & x<0 \end{cases} \tag{18}$$

Consequently, the Resolvent operator of this problem can be given as follows.

$$R_{\lambda}\left(f\right)=\int\limits_{-\infty}^{\infty}G(x,t;\lambda)f\left(t\right)dt.$$

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W–LINE CONGRUENCES

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Abstract. By utilizing the Darboux frames, along with a regular surface whose parametric curves are lines of curvature, we analyzed the normal line congruence which preserves the asymptotic curves between its focal surfaces. This allows deriving systems of partial differential equations through which the problem of determining the director surface and the corresponding normal line congruence could be solved. Moreover, a necessary and sufficient condition that the focal surfaces of the normal line congruence are degenerates into curves is derived. As a result the middle focal surface of the normal line congruence is presented as a new surface interrogation tool.

1. Introduction

In Euclidean 3-space, a two-parameter set of lines is called a line congruence. For instance, the normal vector field of a surface constitute such a line congruence but this is not the general situation. Hence, the line congruence of normals forms a special class; which is called normal line congruence. The lines of a line congruence meet a given plane in such a way that through a point of the plane one line, or at most a finite member, pass. Similar results hold if a surface is taken instead of a plane; this surface is called the reference surface or director surface of the line congruence. The lines of the line congruence which pass through a curve on the surface form a one-parameter set of lines i.e. a ruled surface (parameter ruled surface). It is known that on each generator of line congruence, there are two special points, called the focal points. This terminology is justified by the fact that a line congruence can be considered as the set of lines tangents two surfaces, the focal surfaces of the line congruence. Therefore there are two surfaces such that the generating lines of the line congruence are tangents to these surfaces.

There are several different ways that the representation of the line geometry. One of them is the dual vector system; a point on a dual unit sphere corresponds to a straight line in the 3-dimensional Euclidean system. So, the one parameter motion of this point corresponds to a ruled surface, while its two real parameter

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motion corresponds to a line congruence. Nowadays, the differential geometry of the line congruence and the focal surfaces have been widely applied in design and manufacturing, (e.g. Computer Aided Geometric Design/Computer Aided Manufacturing) of products and many other areas such as motion analysis and simulation of rigid bodies via dual number and dual vector systems and model-based object recognition systems [10-13].

This work is organized in the following way: In sec. 2, we present a brief introduction to the basic definitions of the representation of the Darboux frame on a regular surface whose parametric curves are lines of curvature and the normal line congruence. Sec. 3 is dedicated to the main results; we form systems of partial differential equations related to the following properties: the representation preserves, asymptotic curves, and the element area between the focal surfaces. Meanwhile, a necessary and sufficient condition that the focal surfaces of the normal line congruence are degenerates into curves has been derived. Especially, we have been paid pay attention to the director surface to be minimal surface and Weingarten-surface since the focal surfaces have special geometrical properties. Finally, the generalization middle focal surface is presented as a new surface interrogation tool.

2. Line Congruence in Euclidean 3-Space E^3

In the following, we will present some facts about classical results of differential line geometry in order to introduce the notations which will be used through the next sections. These and more recent descriptions about line congruences can be found in the works [1-4,6,8].

Let the vector function $\mathbf{r} = \mathbf{r}(u_1, u_2)$ represent a regular non-spherical and-non developable surface M in Euclidean 3-Space E^3, i.e. $\mathbf{r} : U \subset \mathbb{R}^2 \to E^3$ be a regular parametrized surface and g_{ij} and h_{ij} are the coefficients of the first and second fundamental forms of the surface M, and suppose that the u_1-and u_2 curves of this parametrization are lines of curvature, i.e., the elements g_{12} and h_{12} vanish identically ($g_{12} = h_{12} = 0$). Consider now the unit vectors $\mathbf{e}_1 = \mathbf{e}_1(u_1, u_2)$, $\mathbf{e}_2 = \mathbf{e}_2(u_1, u_2)$, are the tangents of the parametric curves u_2=const., u_1 =const., and the unit vector $\mathbf{e}_3 = \mathbf{e}_3(u_1, u_2)$ of the normal to the surface M at any regular point, then we have:

$$\mathbf{e}_1 = \frac{1}{\sqrt{g_{11}}}\frac{\partial \mathbf{r}}{\partial u_1},\ \mathbf{e}_2 = \frac{1}{\sqrt{g_{22}}}\frac{\partial \mathbf{r}}{\partial u_2},\ \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2, \tag{2.1}$$

which are invariants vector functions on the surface. Using that u_1-and u_2 curves are curvature lines on the surface, we can calculate $ds = \sqrt{g_{11}}du_1$-and $d\overline{s} = \sqrt{g_{22}}du_2$, the arc length parameters of the curves u_2=const., u_1 =const., respectively. The moving frame $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ on the surface M at every regular point is then called the Darboux frame. Hence, by means of the derivatives with respect to the arc length parameter of the curves u_2=const. with tangent $\mathbf{e}_1$ on M, the derivative formula with respect to the Darboux frame, may be stated as [1]:

$$\frac{\partial}{\partial s}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}=\begin{pmatrix}0 & q & k\\ -q & 0 & 0\\ -k & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}, \tag{2.2}$$

where $k=\frac{h_{11}}{g_{11}}=<\frac{\partial\mathbf{e}_1}{\partial s},\mathbf{e}_3>$, and $q=\frac{-(g_{11})_{u_2}}{2g_{11}\sqrt{g_{22}}}=<\frac{\partial\mathbf{e}_1}{\partial s},\mathbf{e}_2>$ are the normal and geodesic curvatures of the curves u_2=const., respectively. Similarly, the derivative formula of the Darboux frame of the curves u_1=const., with tangent $\mathbf{e}_2$ on M is:

$$\frac{\partial}{\partial \overline{s}}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}=\begin{pmatrix}0 & \overline{q} & 0\\ -\overline{q} & 0 & \overline{k}\\ 0 & -\overline{k} & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}, \tag{2.3}$$

also $\overline{k}=\frac{h_{22}}{g_{22}}=<\frac{\partial\mathbf{e}_2}{\partial \overline{s}},\mathbf{e}_3>$, and $\overline{q}=\frac{(g_{22})_{u_1}}{2g_{22}\sqrt{g_{11}}}=-<\frac{\partial\mathbf{e}_2}{\partial \overline{s}},\mathbf{e}_1>$ have the same meaning as in (2.2), for the curves u_1=const. on the surface M. We shall denote $\partial/\partial s$ and $\partial/\partial\overline{s}$ by the suffixes 1 and 2.

Since k, $\overline{k}$, and q, $\overline{q}$ are the invariant quantities of curvature on M, these invariants and their derivatives must fulfill the Gauss and Mainardi-Codazzi equations [1]:

$$\left.\begin{array}{c}-q^2+q_2-k\overline{k}=\overline{q}_1+\overline{q}^2,\\ q(\overline{k}-k)+k_2=0,\\ \overline{q}(\overline{k}-k)+\overline{k}_1=0.\end{array}\right\} \tag{2.4}$$

As stated earlier, given a set of unit vectors $\mathbf{e}_3=\mathbf{e}_3(u_1,u_2)$, the normal line congruence in E^3 is defined in the parameter form:

$$C_N:\mathbf{y}(u_1,u_2,\mu)=\mathbf{r}(u_1,u_2)+\mu\mathbf{e}_3(u_1,u_2),\ \mu\in\mathbb{R}, \tag{2.5}$$

where $\mathbf{r}=\mathbf{r}(u_1,u_2)$ is its director surface and $\mathbf{e}_3=\mathbf{e}_3(u_1,u_2)$ is the unit vector field along the direction of the generating lines of the congruence.

3. Main Results

It is known that the consecutive normals along a line of curvature on M : $\mathbf{r}=\mathbf{r}(u_1,u_2)$ intersect, the points of intersection being the corresponding center of curvature. The locus of the centers of curvature for all points of the surface M is called the surface of centers or centro-surface of M. In general it consists of two sheets, conjugated to the two families of lines of curvature and called focal surfaces of M. The parametric representations of the focal surfaces of C are given by [6,7,14]:

$$\left.\begin{array}{ll}F: & \mathbf{x}(u_1,u_2)=\mathbf{r}(u_1,u_2)+\rho\mathbf{e}_3(u_1,u_2),\ \rho=\frac{1}{k}\neq 0,\\ \overline{F}: & \overline{\mathbf{x}}(u_1,u_2)=\mathbf{r}(u_1,u_2)+\overline{\rho}\mathbf{e}_3(u_1,u_2),\ \overline{\rho}=\frac{1}{\overline{k}}\neq 0.\end{array}\right\} \tag{3.1}$$

Let g^i_{jk}, and h^i_{jk} $(i=1,2)$ are the coefficients of the first and second fundamental forms of the focal surfaces $\mathbf{x}=\mathbf{x}(u_1,u_2)$, and $\overline{\mathbf{x}}=\overline{\mathbf{x}}(u_1,u_2)$, respectively, one can obtain:

$$\left.\begin{array}{l} g^1_{11} = \rho_1^2, \quad g^1_{12} = \rho_1\,\rho_2, \quad g^1_{22} = (1-\frac{\rho}{\overline{\rho}})^2 + \rho_2^2, \\ g^2_{11} = (1-\frac{\overline{\rho}}{\rho})^2 + \overline{\rho}_1^2, \quad g^2_{12} = \overline{\rho}_1\,\overline{\rho}_2, \quad g^2_{22} = \overline{\rho}_2^2, \end{array}\right\} \tag{3.2}$$

and

$$\left.\begin{array}{l} h^1_{11} = -\frac{\rho_1}{\rho}\;, \quad h^1_{12} = 0, \quad h^1_{22} = \frac{\rho\,\overline{\rho}_1}{\overline{\rho}^2}, \\ h^2_{11} = \frac{\overline{\rho}\rho_2}{\rho^2}\;, \quad h^2_{12} = 0, \quad h^2_{22} = -\frac{\overline{\rho}_2}{\overline{\rho}}. \end{array}\right\} \tag{3.3}$$

Hence, the parametric curves on the focal surfaces, which correspond to the lines of curvature on the director surface, are conjugate, but not (generally) lines of curvature. The expression for the Gaussian curvatures of the focal surfaces F, and $\overline{F}$, at the corresponding points, are:

$$\left.\begin{array}{l} K_x = -\frac{\overline{\rho}_1}{\rho_1(\,\overline{\rho}-\rho)^2}, \\ \overline{K}_{\overline{\mathbf{x}}} = -\frac{\rho_2}{\overline{\rho}_2(\,\overline{\rho}-\rho)^2}. \end{array}\right\} \tag{3.4}$$

Moreover, the Mainardi-Codazzi equations may be given as in the following form

$$\left.\begin{array}{l} \frac{\partial}{\partial \overline{s}}[\ln \frac{\sqrt{g_{11}}}{\rho} - \int \frac{d\rho}{\overline{\rho}-\rho}] = 0, \\ \frac{\partial}{\partial s}[\ln \frac{\sqrt{g_{22}}}{f(\rho)} - \int \frac{d\overline{\rho}}{\rho-\overline{\rho}}] = 0. \end{array}\right\} \tag{3.5}$$

The integration of equations (3.5) is reducible to

$$\sqrt{g_{11}} = \rho a(s) e^{\int \frac{d\rho}{\overline{\rho}-\rho}}, \;\; \sqrt{g_{11}} = b(\overline{s}) e^{\int \frac{d\overline{\rho}}{\rho-\overline{\rho}}}. \tag{3.6}$$

Without changing the parametric curves, we may assume that $a(s) = b(\overline{s}) = 1$, then we get:

$$g_{11} = \rho^2 e^{2\int \frac{d\rho}{\overline{\rho}-\rho}}, \; g_{12} = 0, \quad g_{22} = \overline{\rho}^2 e^{2\int \frac{d\overline{\rho}}{\rho-\overline{\rho}}}. \tag{3.7}$$

Therefore

$$h_{11} = \rho e^{-2\int \frac{d\rho}{\overline{\rho}-\rho}}, \quad h_{12} = 0, \qquad h_{22} = \overline{\rho} e^{-2\int \frac{d\overline{\rho}}{\rho-\overline{\rho}}}. \tag{3.8}$$

Thus g_{11}, g_{22} , and h_{11}, h_{22} are expressible as functions of ρ or $\overline{\rho}$, and consequently they are functions of one another. It is clear after simple manipulation that these magnitudes satisfy the Gauss's equation.

3.1. **Weingarten line congruence (W-line congruence).** A line congruence in Euclidean 3-space E^3 is a two-parameter set of straight lines. Such a congruence has a parameterization in the form [14]:

$$L : \mathbf{y}(u, v, \lambda) = \mathbf{p}(u, v) + \lambda \boldsymbol{\xi}(u, v), \|\boldsymbol{\xi}\| = 1, \tag{3.9}$$

where $\mathbf{p}(u, v)$ is its base surface (the surface of reference) and $\boldsymbol{\xi}(u, v)$ is the unit vector giving the direction of the straight lines of the congruence, λ being a parameter on each line. The equations

$$u = u(t), \;\; v = v(t), \quad u'^2 + v'^2 \neq 0, \tag{3.10}$$

define a ruled surface belonging to the line congruence. The ruled surface is called a developable if and only if

$$det[\boldsymbol{\xi}(t), \boldsymbol{\xi}'(t), \mathbf{p}(t)] = 0. \tag{3.11}$$

This is a quadratic equation for u', v'. If it has two real and distinct roots, then the solutions of this equation define two distinct families of developable ruled surfaces. In the generic case, each family consists of the tangent lines to a surface, and these two surfaces M and M^* are called the focal surfaces of the line congruence. The line congruence gives a mapping $f : M \to M^*$ with the property that the line congruence consists of lines which are tangent to both M and M^* and joining $\mathbf{p} \in M$ to $f(\mathbf{p} \in M^*)$. This simple construction plays a fundamental role in the theory of the transformation of surfaces. The classical *Bäckland* theorem studies the transformations of surfaces of constant negative Gaussian curvature in 3-dimensional Euclidean space E^3 by realizing them as the focal surfaces of a pseudo-spherical (p.s.) line congruence. The integrability theorem says that we can construct a new surface in E^3 with constant negative Gaussian curvature from a given one.

We can rephrase this in more current terminology as follows:

Definition 1. *Let L be a line congruence in 3-dimensional Euclidean space E^3 with focal surfaces M, M^* and let $f : M \to M^*$ be the function defined above. The line congruence is called a p.s. line congruence if*
(i) the distance $\|\mathbf{pp}^*\| = r$ is a constant independent of $\mathbf{p}$,
(ii) the angle between the two normals at $\mathbf{p}$ and $\mathbf{p}^*$ is a constant independent of $\mathbf{p}$.

Theorem 1. *(Bäckland 1875): Suppose that L is a p.s. line congruence in E^3 with the focal surfaces M and M^*. Then both focal surfaces have constant negative Gaussian curvature equal to $-sin^2\theta/r^2$ (such surfaces are called p.s. surfaces).*

There is also an integrability theorem:

Theorem 2. *Suppose M is a surface in E^3 of constant negative Gaussian curvature $K = -sin^2\theta/r^2$, where $r > 0$ and $0 < \theta < \pi$ are constants. Given any unit vector $\mathbf{e} \in M_p$, which is not a principal direction, there exists a unique surface M^* and p.s. congruence $f : M \to M^*$ such that if $\mathbf{p}^* = f(\mathbf{p})$, we have $\mathbf{pp}^* = r\mathbf{e}$ and θ is the angle between the normals at $\mathbf{p}$ and $\mathbf{p}^*$.*

Thus one can construct one-parameter family of new surface of constant negative Gaussian curvature from a given one, the results by varying r.

One of the problems of the theory of line congruences is to classify the categories of them which have the property such that this representation preserves the asymptotic curves between the two focal surfaces. This leads to the following definitions for a Weingarten line congruence (W-line congruence):

Definition 2. *A W-line congruence is a line congruence which preserves the asymptotic curves between its focal surfaces.*

Equivalently, for a W-line congruence the second fundamental forms of the two surfaces are proportional.

Corollary 1. *A p.s. line congruence is a W-line congruence.*

Theorem 3. *Consider a line congruence generated by the normals along a regular non-spherical and non-developable surface M in Euclidean 3-Space E^3. If the generators of this congruence are preserving the asymptotic curves on their focal surfaces, then the Gaussian curvatures of the focal surfaces satisfying the relation:*

$$K_x\ \overline{K}_{\overline{\mathbf{x}}} = \frac{1}{(\ \overline{\rho} - \rho)^4}. \tag{3.12}$$

Hence at the corresponding points the curvature is of the same kind.

Proof. Let II_x and $\overline{II}_{\overline{\mathbf{x}}}$ be the second fundamental forms of the focal surfaces $\mathbf{x} = \mathbf{x}(u_1, u_2)$, and $\overline{\mathbf{x}} = \overline{\mathbf{x}}(u_1, u_2)$, respectively. By equations (3.7), we have:

$$\left.\begin{array}{r} II_x = -\frac{\rho_1}{\overline{\rho}} ds^2 + \frac{\rho\ \overline{\rho}_1}{\overline{\rho}^2} d\overline{s}^2, \\ \overline{II}_{\overline{\mathbf{x}}} = \frac{\overline{\rho}\rho_2}{\rho^2}\ ds^2 - \frac{\overline{\rho}_2}{\overline{\rho}} d\overline{s}^2. \end{array}\right\} \tag{3.13}$$

Then the proportionality of the second fundamental forms means $II_x = \lambda\ \overline{II}_{\overline{\mathbf{x}}}$; $\lambda \in \mathbb{R}$, which is equivalent to the following condition on the invariants:

$$\rho_1\overline{\rho}_2 - \rho_2\overline{\rho}_1 = 0 \Rightarrow \frac{\overline{\rho}_1}{\rho_1} = \frac{\overline{\rho}_2}{\rho_2}. \tag{3.14}$$

From this relation, it follows that

$$K_x\ \overline{K}_{\overline{\mathbf{x}}} = \frac{\overline{\rho}_1\rho_2}{\rho_1\ \overline{\rho}_2(\ \overline{\rho} - \rho)^4} = \frac{1}{(\ \overline{\rho} - \rho)^4}, \tag{3.15}$$

as claimed. □

Example 1. *As an example of a p.s. surface, pseudo-sphere can be given as a focal surface of a p.s. line congruence (so a W-line congruence):*

$$\mathbf{x}(u, v) = (sech(u)\cos(v), sech(u)sin(v), u - tanh(u))$$

Let $\left\{\mathbf{e}_1 = \dfrac{\mathbf{x}_u}{\|\ \mathbf{x}_u\|}, \mathbf{e}_2 = \dfrac{\mathbf{x}_v}{\|\ \mathbf{x}_v\|}, \mathbf{e}_3 = \mathbf{x}_u \times \mathbf{x}_v\right\}$ *be an orthonormal frame of the surface* $\mathbf{x}$, *where superscript shows the partial derivatives. Then the W-line congruence is represented by (for simplicity for the equations we can chose)*

$$\mathbf{L}(u, v, \mu) = \mathbf{x}(u, v) + \mu\mathbf{e}_1(u, v),$$

where $\mu \in \mathbb{R}$.

The other focal surface can be given as $\overline{\mathbf{x}} = \mathbf{x} + r\mathbf{e}_1$ *where* $r = \| \overline{\mathbf{x}} - \mathbf{x}\|$. *Therefore, assuming* $r = \frac{\sqrt{2}}{2}$ *the second focal surface is (see Fig.1)*

$$\overline{\mathbf{x}} = \left(\left(\frac{2-\sqrt{2}}{2} \right) sech(u)\cos(v), \left(\frac{2-\sqrt{2}}{2} \right) sech(u) sin(v), u - \left(\frac{2-\sqrt{2}}{2} \right) tanh(u) \right).$$

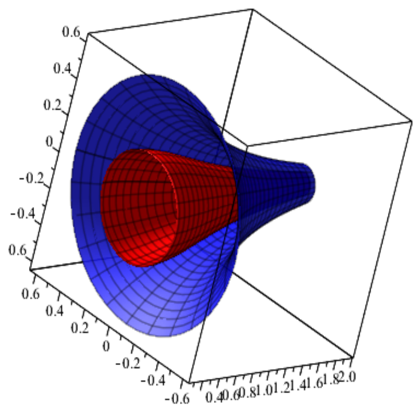

FIGURE 1. Focal surfaces of the line congruence.

3.1.1. *W-surfaces.* From equation (3.14), we see that ρ and $\overline{\rho}$ are connected by functional relation as:

$$f(\rho, \overline{\rho}) = \overline{\rho} - \rho = c, \quad c \in \mathbb{R}. \tag{3.16}$$

Surfaces with this property are called W-surfaces. From this relation it follows that

$$\frac{\overline{\rho}_1}{\rho_1} = \frac{\overline{\rho}_2}{\rho_2} = 1.$$

Therefore, by using the expressions of the Gaussian curvatures in (3.4), we have

$$K_x \ = \overline{K}_{\overline{\mathbf{x}}} = -\frac{1}{c^2}. \tag{3.17}$$

We know that the Gaussian curvature of the focal surfaces of the W-line congruence equal to:

$$\frac{-sin^2\tau}{r^2} = -\frac{1}{c^2} \Rightarrow \overline{\rho} - \rho = |\frac{r}{sin\tau}| \tag{3.18}$$

Surfaces with constant negative Gaussian curvatures are called pseudo-spherical surfaces and they are a result of the sine-Gordon partial differential equation, [8, 12]. Hence, when this functional relation is substituted into (3.7), and (3.8), we obtain:

$$\left.\begin{array}{l} g_{11} = \rho^2 e^{\frac{2\rho}{c}},\ g_{12} = 0, \quad g_{22} = \overline{\rho}^2 e^{\frac{-2\overline{\rho}}{c}}, \\ h_{11} = \rho e^{-\frac{2\rho}{c}}, \quad h_{12} = 0, \qquad h_{22} = \overline{\rho} e^{\frac{-2\overline{\rho}}{c}}. \end{array}\right\} \tag{3.19}$$

Combining the above analysis with the fact that the Gauss and Mainardi-Codazzi equations are the only independent algebraic equations among the fundamental invariants k, $\overline{k}$, and q, $\overline{q}$ and following Bonnet's theorem. Then we may state the following theorem:

Theorem 4. *Among the line congruence in the Euclidean space E^3, the only line congruence whose focal surfaces are pseudo-spherical surfaces, and these surfaces can be geodesically mapped upon the plane, is W-line congruence.*

Now, the second fundamental form of the director surface M is given from the equation:

$$II = \frac{1}{\rho} ds^2 + \frac{1}{\overline{\rho}} d\overline{s}^2, \tag{3.20}$$

If we consider the possibility of the following corresponding $II = \lambda\ II_x;\ \lambda \in \mathbb{R}$, which is equivalent to the following condition on the invariants:

$$\rho_1 \overline{\rho}_1 + \overline{\rho} \rho_1 = 0. \tag{3.21}$$

That is

$$\frac{\partial}{\partial s}(\rho \overline{\rho}) = 0. \tag{3.22}$$

This means that the Gaussian curvature of the director surface M is constant along the lines of curvature $u_2 - const,$. Hence, the following theorem can be given:

Theorem 5. *A necessary and sufficient condition for the Gaussian curvature of the director surface of the congruence C_N is constant along one set of lines of curvature is that the second fundamental forms of the focal surface, conjugate to this set, and the director surface are proportional.*

3.2. **Degenerate focal surfaces.** Now, we proceed to show the case for which the line congruence C_N degenerate into ruled surface. Since one of the families of lines of curvature on a surface are plane curves, they are circular: In this case, either sheet of the centro-surface may degenerate into a curve , i.e. $\mathbf{x}(u_1, c)$, or $\overline{\mathbf{x}}(\overline{c}, u_2)$. In such a case the edge of regression of the developable ruled surface generated by the normals along a line of curvature becomes a single point of that curve. Then, from equations (3.1), the focal surface F is a curve if and only if

$$C:\ \mathbf{x}(u_1, c) = \mathbf{r}(u_1, u_2) + \frac{1}{k}\mathbf{e}_3(u_1, u_2),\ k(u_1, u_2) \neq 0,\ c \in \mathbb{R}, \tag{3.23}$$

or

$$C:\ \mathbf{r}(u_1, u_2) = \mathbf{x}(u_1, c) - \frac{1}{k}\mathbf{e}_3(u_1, u_2),\ k(u_1, u_2) \neq 0. \tag{3.24}$$

Because of $<\mathbf{e}_3, d\mathbf{r}>=0$, then we have

$$<\mathbf{e}_3, (\frac{\partial \mathbf{x}}{\partial s}+\frac{k_1}{k^2}\mathbf{e}_3)>=0, \tag{3.25}$$

and

$$<\mathbf{e}_3, -\frac{\overline{k}}{k}\mathbf{e}_2+\frac{k_2}{k^2}\mathbf{e}_3>=0, \tag{3.26}$$

which is equivalent to

$$-\frac{k_1}{k^2}=<\mathbf{e}_3, \frac{\partial \mathbf{x}}{\partial s}>=\left\|\frac{\partial \mathbf{x}}{\partial s}\right\|\cos\varphi,\ k_2=0, \tag{3.27}$$

where φ is the inclination of the normal to M with the tangent to the curve C. Since then

$$\begin{aligned}\tfrac{\partial}{\partial \overline{s}}(\left\|\tfrac{\partial \mathbf{x}}{\partial s}\right\|\cos\varphi) &= \tfrac{\partial}{\partial \overline{s}}(-\tfrac{k_1}{k^2}),\\ &= \tfrac{\partial}{\partial \overline{s}}(\tfrac{\partial}{\partial s}\tfrac{1}{k}),\\ &= \tfrac{\partial}{\partial s}(\tfrac{\partial}{\partial \overline{s}}\tfrac{1}{k}),\\ &= \tfrac{\partial}{\partial s}(-\tfrac{k_2}{k^2})=0,\end{aligned} \tag{3.28}$$

it follows that $\varphi=\varphi(u_1)$, i.e. is a function of u_1 only. Thus the normals to M, which meet at a point of the curve C, form a right circular cone whose semi-vertical angle φ changes as the point moves along the curve. These intersecting normals emanate from line of curvature ($u_2=const.$) on M , which must then be circular. Thus the surface M has a system of circular lines of curvature. The sphere described with center at the point of concurrence of the normals, and passing through the feet of these normals, will touch M along one of the circular lines of curvature. Thus M is the envelope of a one-parameter family of spheres with centers on the curve C, i.e. M is a canal surface.

By similar argument, we can also have $\overline{k}_1=0$ for the focal surface $\overline{F}$ degenerate into a curve. Hence, both systems of lines of curvature of M are circular lines of curvature ($k_2=0,\ \overline{k}_1=0$), and each sheet of the focal surfaces degenerate to a curve. From the preceding arguments, it follows that each of these curves lies on a one-parameter family of circular cones whose axes are tangents to the other curve. Surfaces of this nature are called Dupin's cyclids. Then as a result:

Theorem 6. *For the line congruence C_N, a necessary and sufficient condition for the focal surfaces degenerate into curves is that the director surface is Dupin cyclide. More explicitly, we have the following:*

$$k_2=0,\ \overline{k}_1=0.$$

3.3. **Generalized middle focal surface.** The Gaussian curvature has the important property of remaining invariant if the surface is subject to an arbitrary bending. A bending is defined as any deformation for which the arc length and angles of all curves on the surface are left invariant. In equation (2.5), as $\mu=\mu(u_1,u_2)$

is a differentiable function with continuos partial derivatives of a certain order the regular surface

$$G:\mathbf{y}(u_1,u_2)=\mathbf{r}(u_1,u_2)+\mu(u_1,u_2)\mathbf{e}_3(u_1,u_2), \tag{3.29}$$

define the graph of the function $\mu=\mu(u_1,u_2)$ on the surface $M:\mathbf{r}=\mathbf{r}(u_1,u_2)$. For each fixed $t\in(\varepsilon,-\varepsilon)$, we define the generalized of the middle focal surface as:

$$\mathbf{y}(u_1,u_2)=\mathbf{r}(u_1,u_2)+t\mu\mathbf{e}_3(u_1,u_2);\ \mu=\frac{(\rho+\overline{\rho})}{2}. \tag{3.30}$$

Thus μ is signed mean distance between the two focal surfaces $\mathbf{x}=\ \mathbf{x}(u_1,u_2)$, and $\overline{\mathbf{x}}=\ \overline{\mathbf{x}}(u_1,u_2)$, and the lines of C_N generate the corresponding between the surfaces M and G.

Two surfaces that can be transformed into each other by bending are called applicable to each other. Equivalently, we will determine whether the generating lines of the congruence C_N establish an area preserving representation between M and G, i.e. it is necessary and sufficient condition for the following condition to be satisfied:

$$|A(G)-A(M)|\rightarrow\min, \tag{3.31}$$

where $A(G)$ and $A(M)$ are the element areas on the surfaces M and G. So, we have to calculate

$$A(G)=\iint\limits_U\sqrt{g_{11}^G\ g_{22}^G-(g_{12}^G)^2}\int du_1du_2, \tag{3.32}$$

where $g_{11}^G,\ g_{22}^G$, and g_{12}^G are the coefficients of the first fundamental form of the surfaces G. By making use of the equations (2.2), (2.3), and (3.26), we obtain

$$\left.\begin{array}{l} g_{11}^G=<\mathbf{y}_1,\mathbf{y}_1>=g_{11}-2t\mu h_{11}+t^2(\mu^2k^2+\mu_1^2),\\ g_{12}^G=<\mathbf{y}_1,\mathbf{y}_2>=t^2\mu_1\mu_2,\\ g_{22}^G=<\mathbf{y}_2,\mathbf{y}_2>=g_{22}-2t\mu h_{22}+t^2(\mu^2\overline{k}^2+\mu_2^2). \end{array}\right\} \tag{3.33}$$

It follows that if ε is sufficiently small subject to the relations $\varepsilon^2=\varepsilon^3=...=0$, then we obtain

$$g_{11}^G g_{22}^G=g_{11}g_{22}(1-4t\mu H), \tag{3.34}$$

or as

$$\sqrt{g_{11}^G\ g_{22}^G}=\sqrt{g_{11}g_{22}}(1-2t\mu H), \tag{3.35}$$

where we have used

$$\sqrt{1-4t\mu H}=1-2t\mu H. \tag{3.36}$$

Substituting equation (3.31) into the element area formula (3.28), then

$$A(G)=A(M)-2t\mu\iint\limits_U Hdsd\overline{s}. \tag{3.37}$$

where H denotes to the mean curvatures of the director surface M. With $\rho+\overline{\rho}\neq0$ on the surface M, it means there is no change of sign of the mean curvature: It exists a real number $m > 0$, with $|\rho + \overline{\rho}| > 0$ for all $\mu \in (u_1, u_2) \in U$. Therefore, the function $(\rho + \overline{\rho})$ is bounded, and the relation (3.27) is hold.

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LCN-TRANSLATION SURFACES IN AFFINE 3-SPACE

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ABSTRACT. We consider translation surfaces in Affine 3- space. Firstly, we give some results of translation surfaces whose mean and Gaussian curvatures vanish [8,16]. Further, we define and investigate LCN-translation surfaces with zero the mean and the Gaussian curvatures in Affine 3-space.

1. INTRODUCTION

A surface that arises when a curve $\alpha(u)$ is translated over another curve $\beta(v)$, is called a translation surface. A translation surface can be defined as the sum of the two generating curves $\alpha(u)$ and $\beta(v)$.Therefore, translation surfaces are made up of quadrilateral, that is, four sided, facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures. A translation surface in an Euclidean 3-space $\mathbb{E}^3$ formed by translating two curves lying in orthogonal planes is the graph of a function $z(u,v) = f(u)+g(v)$, where $f(u)$ and $g(v)$ are smooth functions on some interval of $\mathbb{R}$ [3, 6].

In 1835, H. F. Scherk studied translation surfaces in $\mathbb{E}^3$ defined as graph of the function $z(u,v) = f(u) + g(v)$ and he proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z(u,v) = \frac{1}{a}\log\left|\frac{\cos(au)}{\cos(av)}\right| = \frac{1}{a}\log|\cos(au)| - \frac{1}{a}\log|\cos(av)|,$$

where a is a non-zero constant. These surfaces are now referred as Scherk's minimal surfaces [18].

In mathematics, an Affine space is a geometric structure that generalizes some of the properties of Euclidean spaces in such a way that these are independent of the concepts of distance and measure of angles, keeping only the properties related to parallelism and ratio of lengths for parallel line segments. Affine differential geometry is a type of differential geometry in which the differential invariants are

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invariant under volume-preserving affine transformations. The basic difference between affine and Riemannian differential geometry is that in the affine case we introduce volume forms over a manifold instead of metrics.

In theory of surfaces, there are some special surfaces such as ruled surfaces, minimal surfaces, flat surfaces and surfaces of constant curvature in which differential geometers are interested. Liu described translation surfaces having constant Gaussian and mean curvature in the Euclidean and Minkowski space [12].Goemans studied weingarten translation surfaces Euclidean and Minkowski 3-spaces [6]. In the literature of affine differential geometry, translation surfaces have been also studied previously by many geometers [8, 10, 11, 15, 16, 17]. Manhart gave a complete explicit classification of nondegenerate minimal translation surfaces in Affine space $\mathbb{R}^3$ [11]. Magid and Vrancken showed that the curvatures must be zero and this is equivalent to one of the defining curves being planar. Also, they investigated other, natural, geometric conditions on translation surfaces. In particular, they classified those translation surfaces which are umbilical, affine spheres have trivial normal connection or null mean curvature vectors [10]. Sun classified translation surface with nonzero constant mean curvature in Affine space $\mathbb{R}^3$[15]. Fu and Hou gave a complete classification of nondegenerate affine translation surfaces with constant Gaussian curvature in $\mathbb{R}^3$ [8].Yang, Yu and Liu gave some classification results for nondegenerate linear Weingarten centroaffine translation surfaces in Affine space $\mathbb{R}^3$[16]. Yanga and Fu obtained the complete classification of minimal affine translation surfaces in Affine space [17]. Andrade and Lewiner gave geometric properties of parametric or implicit surfaces, in particular the affine metric, the conormal and normal vectors, and the affine Gaussian and mean curvatures [1, 2]. Huamani studied the surfaces with zero affine mean curvature [7].

The spline surface is composed of quartic Clough-Tocher-type macro elements. Each element is capable of matching boundary data consisting of three points with associated normal vectors. The collection of the macro elements forms a G^1continuous spline surface. Jutler and Sampoli constructed for polynomial spline surfaces with a piecewise linear field of normal vectors [9]. Sampoli, Peternell and Jüttler showed that even the convolution surface of an LN-surface and any rational surface admits rational parametrization [14]. Sampoli showed that for LN spline surfaces (surfaces with a linear field of normal vectors) a closed form representation is available [13].

In this paper, we have pointed out the flat and minimal of the LCN-translation surfaces in Affine 3-space.

2. Preliminaries

In this section we will give some definitions of the main affine structures: the co-normal and normal vectors and the Gaussian and the mean curvatures. The Berwald-Blaschke metric is invariant for Affine transformations and also independent of system of coordinates. This metric is a quadric form. This quadratic form might not be positive definite (non-convex) case. Let $X : \Omega \to \mathbb{R}^3$ be a parametrization of a regular surface locally convex. The first affine fundamental form given by

$$\mathbf{I} = Ldu^2 + 2Mdudv + Ndv^2,$$

where

$$L = [X_u, X_v, X_{uu}], \quad N = [X_u, X_v, X_{vv}], \quad M = [X_u, X_v, X_{uv}].$$

The Berwald-Blaschke metric or the second affine fundamental form given by

$$h = \mathbf{II} = Edu^2 + 2Fdudv + Gdv^2, \tag{1}$$

where

$$E = \frac{L}{|LN - M^2|^{\frac{1}{4}}}, \quad G = \frac{N}{|LN - M^2|^{\frac{1}{4}}}, \quad F = \frac{M}{|LN - M^2|^{\frac{1}{4}}}.$$

From now on, we shall assume that the surface is non-degenerate, that is, $LN - M^2 \neq 0$. Points $LN - M^2$ are negative, zero or positive are called hyperbolic, parabolic or elliptical, respectively [1, 2, 4, 7]. A transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ is affine iff A satisfies $A(u) = C(u) + v_0$, where C is linear and $v_0 \in \mathbb{R}^3$.

Orthonormality relationships are not preserved under an affine transformation A, therefore the Euclidean normal $\mathbf{N}^e$is not an affine covariant vector. However, the direction of the Euclidean normal is covariant ($if \left\langle \mathbf{N}^e, X_u \right\rangle = 0, then \left\langle A^{-T}\mathbf{N}^e, AX_u \right\rangle = 0$ and similarly X_v, $\langle , \rangle$ is Euclidean scalar product). Therefore, a covariant affine normal, called the affine conormal ν can be obtained by scaling the Euclidean normal vector

$$\nu = |\mathbf{K}^e|^{-\frac{1}{4}} \mathbf{N}^e = \frac{X_u \wedge X_v}{[\nu, \nu_u, \nu_v]} = \frac{X_u \wedge X_v}{|LN - M^2|^{\frac{1}{4}}}, \tag{2}$$

where L, N and M are the coefficients of the first affine fundamental form, $\mathbf{K}^e$ and $\mathbf{N}^e$are the Euclidean Gaussian curvature and the Euclidean normal vector, respectively [1, 2, 7].

By definition, it can be seen that $\nu.dX = 0$. Let $d = \pm [\nu, \nu_u, \nu_v] = \pm \left(LN - M^2\right)^{\frac{1}{4}}$, where the signal $\pm$ depends on the point elliptical or hyperbolic. Using this notation, we have

$$\nu = \frac{X_u \wedge X_v}{d}. \tag{3}$$

Since the affine conormal is not in general a unitary vector, it is not orthogonal to its derivatives ν_u, ν_v.But since $[\nu, \nu_u, \nu_v] = d \neq 0$, those derivatives define a proper plane not orthogonal to ν. A contravariant affine vector can then be obtained by looking at a vector orthogonal to that plane and would be the affine equivalent to the Euclidean normal. More precisely, the affine normal vector ξ is defined locally by the relationship:

$$\langle \nu, \xi \rangle = 1, \quad \langle \nu_u, \xi \rangle = 0, \quad \langle \nu_v, \xi \rangle = 0.$$

The affine normal then satisfies:

$$\langle \nu, \xi_u \rangle = \langle \nu, \xi_v \rangle = 0.$$

This last relation shows that a local basis for the embedding space $\mathbb{R}^3$ at a point p of the surface can be obtained by $[X_u, X_v, \xi]$ .This allows to define affine structures from Cartan's moving frames theory. Denote by $\xi = [\xi_1, \xi_2, \xi_3]$ the affine normal vector. Thus we have

$$\xi = \frac{\nu_u \wedge \nu_v}{[\nu, \nu_u, \nu_v]} = \frac{\nu_u \wedge \nu_v}{d} \tag{4}$$

or

$$\xi = \frac{1}{2} \frac{\left|LN - M^2\right|^{\frac{1}{4}}}{\sqrt{LN - M^2}} \left[\frac{\partial}{\partial u} \left(\frac{NX_u - MX_v}{\sqrt{LN - M^2}} \right) + \frac{\partial}{\partial v} \left(\frac{LX_v - MX_u}{\sqrt{LN - M^2}} \right) \right] \tag{5}$$

or

$$\xi = \frac{1}{2} \frac{1}{\sqrt{EG - F^2}} \left[\frac{\partial}{\partial u} \left(\frac{GX_u - FX_v}{\sqrt{EG - F^2}} \right) + \frac{\partial}{\partial v} \left(\frac{EX_v - FX_u}{\sqrt{EG - F^2}} \right) \right], \tag{6}$$

[1, 2, 7].

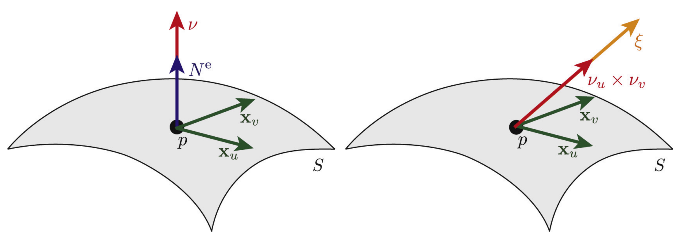

Figure 1

Observe that, the affine normal vector does not belong to the tangent plane to the surfaces S. The curvatures describe the variation of the normal vector. We know that $\nu.\xi_u = 0$, $\nu.\xi_v = 0$.That is , the derivatives ξ_u and ξ_v are orthogonal to ν. In particular ξ_u and $\xi_v \in T_pS$. Therefore, we can define the shape operator S as follows

$$S : T_pS \to T_pS$$

given by $S_p(v) = -D_v\xi$. Since ξ_u and ξ_v are tangents to the surface, we have that there are functions

$$b_{ij} : \Omega \to \mathbb{R}, \ i, j = 1, 2,$$

such that

$$\begin{aligned} \xi_u &= b_{11}X_u + b_{12}X_v, \\ \xi_v &= b_{21}X_u + b_{22}X_v, \end{aligned} \tag{7}$$

where

$$b_{11} = \frac{[\xi_u, X_v, \xi]}{d}, \tag{8}$$

$$
\begin{aligned}
b_{12} &= \frac{[\xi_v, X_v, \xi]}{d}, \\
b_{11} &= \frac{[X_u, \xi_u, \xi]}{d}, \\
b_{11} &= \frac{[X_u, \xi_v, \xi]}{d}.
\end{aligned}
$$

This shows that in the basis $\{X_u, X_v\}$, the Shape Operator $S_p(v) = D_v\xi$ is given by the matrix $B = (b_{ij})$, $i, j = 1, 2$. Notice that this matrix is not necessarily symmetric [1, 2, 7].

Definition 1. *The coefficients b_{ij} form a matrix $B = (b_{ij})$, whose determinant and the half of the trace are the Gaussian and the mean curvatures, respectively. Hence, we have*

$$
\begin{aligned}
\mathbf{K} &= \det B = b_{11}b_{22} - b_{12}b_{21}, \\
\mathbf{H} &= \frac{1}{2}trB = \frac{b_{11} + b_{22}}{2}
\end{aligned}
\tag{9}
$$

[1, 2, 7].

Consider a surface $X^*(u, v)$ in Affine 3-space. This surface is said to be an LCN (linear conormal)-surface, if its conormal vectors admit a linear representation of the form

$$
\nu^* = \overrightarrow{a}u + \overrightarrow{b}v + \overrightarrow{c}
\tag{10}
$$

with certain constant coefficient vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathbb{R}^3$. More precisely, it satisfies the equations

$$
\langle X_u^*, \nu^* \rangle = \langle X_v^*, \nu^* \rangle = 0.
$$

We assume that the three vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are linearly independent. Without loss of generality we may then assume that

$$
\overrightarrow{a} = (1, 0, 0), \overrightarrow{b} = (0, 1, 0), \overrightarrow{c} = (0, 0, 1),
$$

i.e.,$\nu^* = (u, v, 1)$.The tangent planes of an LCN surface have the equations

$$
T(u, v) : z(u, v) + ux + vy + z = 0,
\tag{11}
$$

where $z(u, v)$ is a polynomial or rational function, in case of a polynomial or rational LCN surface, respectively. On the other hand, given a system of tangent planes of the form (11) with a polynomial or rational function $z(u, v)$.The envelope surface $X^* = (x, y, z)$ satisfies the equations

$$
\begin{aligned}
T(u, v) &: z(u, v) + ux + vy + z = 0, \\
T_u(u, v) &: z_u + x = 0, \\
T_v(u, v) &: z_v + y = 0,
\end{aligned}
$$

and the conormal vector evaluates to

$$\nu^* = \left(\frac{z_{uu} z_{vv} - (z_{uv})^2}{d^*} \right) (u, v, 1) . \tag{12}$$

Thus the envelope surface given by

$$X^*(u, v) = (-z_u, -z_v, -z + u z_u + v z_v) . \tag{13}$$

[13, 14].

3. LCN-Translation Surfaces in Affine 3-Space

In this chapter, we define the LCN-translation surfaces in Affine 3-space. Consider a surface in as a the graph of a function $z = r(u, v)$ of two variables, which is itself the sum of two functions f and g of one variable. Here, we restrict our topic to regular surfaces X. Thus, we can express in open form as

$$X: \; z = f(u) + g(v). \tag{14}$$

A surface S defined as the sum of two affine space curves $\alpha(u) = (u, 0, f(u))$ and $\beta(v) = (0, v, g(v))$ is called a translation surface in Affine 3-space. So, a translation surface is defined by a patch

$$X(u, v) = (u, v, f(u) + g(v)) . \tag{15}$$

The coefficients of the first affine fundamental form of the translation surface given by

$$\begin{aligned} L &= f''(u), \; N = g''(v), \; M = 0, \\ d &= \left(LN - M^2\right)^{\frac{1}{4}} = (f'' g'')^{\frac{1}{4}} . \end{aligned} \tag{16}$$

Hence the coefficients of the Berwald–Blaschke metric of the translation surface or the coefficients of the second affine fundamental form of the translation surface are given by

$$E = \frac{f''}{(f'' g'')^{\frac{1}{4}}}, G = \frac{g''}{(f'' g'')^{\frac{1}{4}}}, F = 0. \tag{17}$$

We suppose that the Berwald–Blaschke metric is non-degenerate: $d \neq 0$. Thus, we have the affine conormal and normal vectors are given by

$$\nu = \left(-\frac{f'}{(f'' g'')^{\frac{1}{4}}}, -\frac{g'}{(f'' g'')^{\frac{1}{4}}}, \frac{1}{(f'' g'')^{\frac{1}{4}}} \right) . \tag{18}$$

$$\xi = \begin{pmatrix} -\frac{f''' (f'' g'')^{\frac{1}{4}}}{4 f''^2}, \\ -\frac{g''' (f'' g'')^{\frac{1}{4}}}{4 g''^2}, \\ \frac{(f'' g'')^{\frac{1}{4}} \left(-f' g''^2 f''' + f''^2 \left(4 g''^2 - g' g''' \right) \right)}{4 f''^2 g''^2} \end{pmatrix} . \tag{19}$$

respectively.

Proposition 2. *Let S be a translation surface with non-degenerate in Affine 3-space. Then the Gaussian and the mean curvatures of S can be given by*

$$\begin{aligned}\mathbf{K} &= \frac{f'''^2\left(12g'''^2-7g''g^{(4)}\right)+f''f^{(4)}\left(-7g'''^2+4g''g^{(4)}\right)}{64\left(f''g''\right)^{\frac{5}{2}}}, \qquad (20)\\ \mathbf{H} &= \left(\frac{\left(f''g''\right)^{\frac{1}{4}}\left(7f'''^2-4f''f^{(4)}\right)}{32f''^3}+\frac{\left(f''g''\right)^{\frac{1}{4}}\left(7g'''^2-4g''g^{(4)}\right)}{32g''^3}\right),\end{aligned}$$

respectively [10, 11, 15].

In [8, 15], Fu, Hou and Sun classified vanishing Gaussian curvature and minimal translation surfaces in the Affine 3-space, they proved the following theorems:

Theorem 3. *Let S be a nondegenerate affine translation surface in $\mathbb{R}^3$ with vanishing Gaussian curvature. Then S is affinely equivalent to one of the graph of the following functions*

$$\begin{aligned}z &= u^2+g(v),\\ z &= e^u+v^{\frac{1}{2}},\\ z &= u\ln u\pm v\ln v,\\ z &= \ln u\pm\ln v,\\ z &= u^{\frac{3-2\lambda}{\lambda-1}}\pm v^{\frac{3-2\lambda}{5-3\lambda}},\end{aligned}$$

where $g(v)$ is an arbitrary function and λ is a constant satisfying $\lambda\neq 1,2,\frac{3}{2},\frac{5}{3}$ [8].

Theorem 4. *Let S be a nondegenerate affine minimal translation surface in $\mathbb{R}^3$. Then S is one of the graph of the following functions under affine transformations:*

$$\begin{aligned}z &= u^2\pm v^2,\\ z &= u^{\frac{2}{3}}\pm v^{\frac{2}{3}},\\ z &= u^2\pm v^{\frac{2}{3}}\end{aligned}$$

or

$$\begin{aligned}z &= \ln u-\ln v,\\ z &= \pm\ln u\pm\left(1+\cosh t\right), \quad t+\sinh t=v,\\ z &= \pm\ln u\pm\left(1-\cos t\right), \quad t-\sin t=v,\\ z &= \pm\left(1+\cosh t\right)\pm\left(1+\cosh s\right), \quad t+\sinh t=u, s+\sinh s=v,\\ z &= \pm\left(1+\cosh t\right)\pm\left(1-\cos s\right), \quad t+\sinh t=u, s-\sinh s=v,\\ z &= \pm\left(1-\cos t\right)\pm\left(1-\cos s\right), \quad t-\sin t=u, s-\sin s=v.\end{aligned}$$

[15].

So, using (11) and (13), we can define the LCN- translation surfaces defined by as

$$X^*(u,v) = \left(-f'(u), -g'(v), uf'(u) + vg'(v) - f(u) - g(v)\right). \tag{21}$$

A basis for the tangent vectors is given by

$$\begin{aligned} X_u^* &= \left(-f'', 0, uf''\right), \\ X_v^* &= \left(0, -g'', vg''\right). \end{aligned} \tag{22}$$

The second partial derivatives of $X^*(u,v)$ are given by

$$\begin{aligned} X_{uu}^* &= \left(-f''', 0, f'' + uf'''\right), \\ X_{uv}^* &= (0,0,0) \\ X_{vv}^* &= \left(0, -g''', g''' + vg'''\right). \end{aligned} \tag{23}$$

The coefficients of the first affine fundamental form of the translation surface given by

$$\begin{aligned} L^* &= f''^2 g'',\ N^* = f'' g''^2,\ M^* = 0, \\ d^* &= \left(L^* N^* - {M^*}^2\right)^{\frac{1}{4}} = \left(f''^3 g''^3\right)^{\frac{1}{4}}. \end{aligned} \tag{24}$$

Hence the coefficients of the Berwald–Blaschke metric of the translation surface or the coefficients of the second affine fundamental form of the translation surface are given by

$$E^* = \frac{f''^2 g''}{\left(f''^3 g''^3\right)^{\frac{1}{4}}}, G^* = \frac{f'' g''^2}{\left(f''^3 g''^3\right)^{\frac{1}{4}}}, F^* = 0. \tag{25}$$

We suppose that the Berwald–Blaschke metric is non-degenerate: $d^* \neq 0$. Geometrically $d^* > 0$ means that the Euclidean Gaussian curvature does not vanish, i.e. the LCN translation surface is strongly convex. The affine conormal field of the LCN translation surface given by

$$\nu^* = \frac{f'' g''}{\left(f''^3 g''^3\right)^{\frac{1}{4}}} (u,v,1). \tag{26}$$

Thus, we have the affine normal vector

$$\xi^* = \begin{pmatrix} -\frac{f'' f''' g''^2}{4\left(f''^3 g''^3\right)^{\frac{3}{4}}}, \\ -\frac{f''^2 g'' g'''}{4\left(f''^3 g''^3\right)^{\frac{3}{4}}}, \\ \frac{f'' g''\left(uf''' g'' + f''\left(4g'' - vg'''\right)\right)}{4\left(f''^3 g''^3\right)^{\frac{3}{4}}} \end{pmatrix}. \tag{27}$$

Consequently, the coefficients b_{ij}^* form a matrix $B^* = \left[b_{ij}^*\right]$ are given by

$$
\begin{aligned}
b_{11}^* &= \frac{\left(f''^3 g''^3\right)^{\frac{1}{4}} \left(-5f'''^2 + 4f'' f^{(4)}\right)}{16 f''^4 g''}, b_{12}^* = -\frac{f''' g'' g'''}{16 \left(f''^3 g''^3\right)^{\frac{3}{4}}}, \qquad (28) \\
b_{21}^* &= -\frac{f'' f''' g'''}{16 \left(f''^3 g''^3\right)^{\frac{3}{4}}}, b_{22}^* = \frac{\left(f''^3 g''^3\right)^{\frac{1}{4}} \left(-5g'''^2 + 4g'' g^{(4)}\right)}{16 f'' g''^4}.
\end{aligned}
$$

Proposition 5. *Let S^* be a LCN−translation surface with non-degenerate in Affine 3-space. Then the Gaussian and the mean curvatures of S^* can be given by*

$$
\begin{aligned}
K^* &= \frac{f'' g'' \left(f'''^2 \left(6g'''^2 - 5g'' g^{(4)}\right) + f'' f^{(4)} \left(-5g'''^2 + 4g'' g^{(4)}\right)\right)}{64 \left(f''^3 g''^3\right)^{\frac{3}{2}}}, \qquad (29) \\
H^* &= \frac{1}{2} \left(\frac{\left(f''^3 g''^3\right)^{\frac{1}{4}} \left(-5f'''^2 + 4f'' f^{(4)}\right)}{16 f''^4 g''} + \frac{\left(f''^3 g''^3\right)^{\frac{1}{4}} \left(-5g'''^2 + 4g'' g^{(4)}\right)}{16 f'' g''^4} \right),
\end{aligned}
$$

where $f'' \neq 0$, $g'' \neq 0$, respectively.

We suppose that the LCN-translation surface with non-degenerate given by (29) has zero the Gaussian curvature. Then we obtain

$$
f'''^2 \left(6g'''^2 - 5g'' g^{(4)}\right) + f'' f^{(4)} \left(-5g'''^2 + 4g'' g^{(4)}\right) = 0. \qquad (30)
$$

Here u and v are independent variables, so each side of (30) is equal to a constant $p \in \mathbb{R} \backslash \{0\}$. Hence, the equation (30) is reduced to

$$
-\frac{f'''^2}{f'' f^{(4)}} = p = \frac{-5g'''^2 + 4g'' g^{(4)}}{6g'''^2 - 5g'' g^{(4)}}, \qquad (31)
$$

where $f^{(4)} \neq 0$, $g^{(4)} \neq 0$. By solving (31), we get

$$
\begin{aligned}
f(u) &= c_1 + c_2 u - \frac{c_3 \left(u + pu - pc_4\right)^{2 + \frac{p}{1+p}}}{(1+2p)(2+3p)}, \qquad (32) \\
g(v) &= c_5 + c_6 v - \frac{c_7 \left((1+p) v + c_8 (4+5p)\right)^{-\frac{2+3p}{1+p}}}{(2+3p)(3+4p)},
\end{aligned}
$$

for some constants $c_i \in \mathbb{R}$ and $p \neq \left\{-1, -\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}\right\}$. We draw it as in Figure 2.

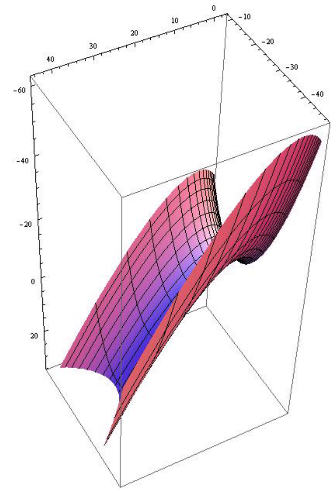

Figure 2

Theorem 6. *Let S^* be a LCN -translation surface with non-degenerate in Affine 3-space. If S^* has zero Gaussian curvature or affine flat then S^* is parametrized as (20) with (32).*

We assume that S^* is affine minimal. Hence, the mean curvature is zero if and only if

$$\frac{\left(f''^3 g''^3\right)^{\frac{1}{4}}\left(-5f'''^2 + 4f''f^{(4)}\right)}{16f''^4 g''} + \frac{\left(f''^3 g''^3\right)^{\frac{1}{4}}\left(-5g'''^2 + 4g''g^{(4)}\right)}{16f''g''^4} = 0. \tag{33}$$

Then, the minimality condition (33) can be separated for the variables

$$\frac{\left(-5f'''^2 + 4f''f^{(4)}\right)}{f''^3} = -\frac{\left(-5g'''^2 + 4g''g^{(4)}\right)}{g''^3}, \tag{34}$$

which implies there exists a constant $p \in \mathbb{R}\backslash\{0\}$ such that

$$\frac{\left(-5f'''^2 + 4f''f^{(4)}\right)}{f''^3} = p = -\frac{\left(-5g'''^2 + 4g''g^{(4)}\right)}{g''^3}, \tag{35}$$

where $f'' \neq 0$, $g'' \neq 0$. Solving this equation for f and g, we get

$$f(u) \quad = \quad c_1 + c_2 u - \frac{2c_3\left(c_4 + u\right)\arctan h\left(\frac{c_3(c_4+u)}{4\sqrt{p}}\right)}{p^{\frac{3}{2}}}, \tag{36}$$

$$g(v) \quad = \quad c_5 + c_6 v - \frac{2c_7\,(c_8+v)\arctan h\left(\frac{c_7(c_8+v)}{4\sqrt{p}}\right)}{p^{\frac{3}{2}}},$$

where $c_i \in \mathbb{R}$ and $p \in \mathbb{R}\backslash\{0\}$. We draw it as in Figure 3.

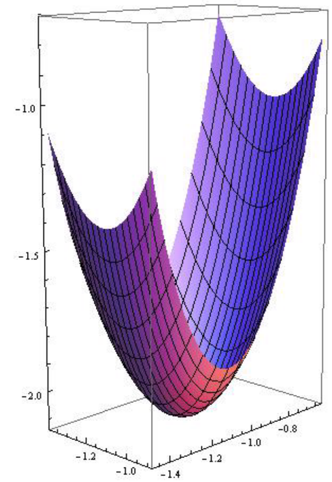

Figure 3

Thus we have following theorem.

Theorem 7. *A LCN-translation surface with non-degenerate S^* is affine minimal in Affine 3-space if and only if it is a part of the surface (20) with (36).*

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SS-SUPPLEMENTED MODULES

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Abstract. A module M is called *ss-supplemented* if every submodule U of M has a supplement V in M such that $U \cap V$ is semisimple. It is shown that a finitely generated module M is ss-supplemented iff it is supplemented and $Rad(M) \subseteq Soc(M)$. A module M is called *strongly local* if it is local and $Rad(M)$ is semisimple. Any direct sum of strongly local modules is ss-supplemented and coatomic. A ring R is semiperfect and $Rad(R) \subseteq Soc({}_RR)$ iff every left R-module is (amply) ss-supplemented iff ${}_RR$ is a finite sum of strongly local submodules.

1. Introduction

Throughout this study, all rings are associative with identity and all modules are unitary left modules. Let R be a ring and M be an R-module. $U \subseteq M$ will mean that U is a submodule of M. $Rad(M)$ and $Soc(M)$ will indicate radical and socle of M. A submodule N of M is called *small* in M, denoted $N << M$, if $M \neq N + K$ for every proper submodule K of M. Let U and V be submodules of M. V is called a *supplement* of U in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V << V$. The module M is called *supplemented* if every submodule of M has a supplement in M. A submodule U of M has *ample supplements* in M if every submodule L of M such that $M = U + L$ contains a supplement of U in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M. For characterizations of supplemented and amply supplemented modules we refer to [7]

A non-zero module M is called *hollow* if every proper submodule of M is small in M and is called *local* if the sum of all proper submodules of M is also a proper submodule of M. Note that local modules are hollow and hollow modules are clearly amply supplemented. A ring R is called *local ring* if ${}_RR$ is a local module.

In [8], Zhou and Zhang generalized the concept of socle of a module M to that of $Soc_s(M)$ by considering the class of all simple submodules of M that are small in M

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in place of the class of all simple submodules of M, that is, $Soc_s(M) = \sum\{N << M \,|\, N \textit{ is simple}\}$. It is clear that $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$.

We call a module M *strongly local* if it is local and $Rad(M)$ is semisimple. We call a ring R *left strongly local ring* if $_RR$ is a strongly local module. Then we have that the following implications on modules:

$$\text{simple} \Longrightarrow \text{strongly local} \Longrightarrow \text{local}$$

Next we mention two examples which show that the above implications are proper. For the local left $\mathbb{Z}$-module $M = \mathbb{Z}_4$, we have $Rad(M) = Soc(M)$. Hence, M is strongly local but not simple. On the other hand, for the local left $\mathbb{Z}$-module $M = \mathbb{Z}_8$, $Soc(M)$ is a proper submodule of $Rad(M)$. Thus M is not a strongly local module.

In section 2 we study on strongly local modules and rings. We show that every left strongly local ring is left perfect and right perfect. A strongly local commutative domain is field.

Let U and V be submodules of a module M. V is called a *Rad-supplement* of U in M if $M = U + V$ and $U \cap V \subseteq Rad(V)$. Since $Soc_s(V) \subseteq Rad(V)$, it is of interest to investigate the analogue of this notion by replacing "$Rad(V)$"with "$Soc_s(V)$". Now, we give the following result playing a key role in our work as a proper generalization of direct summands. Firstly, we need the following well known facts that we include here for completeness.

Lemma 1. *Let M be a module and N be a semisimple submodule of M which is contained in $Rad(M)$. Then $N << M$.*

Proof. Let $N + K = M$ for some submodule K of M. Since N is semisimple, there exists a submodule N' of N such that $N = (N \cap K) \oplus N'$. Hence $M = N + K = [(N \cap K) \oplus N'] + K = N' + K$. Since $N' \cap K = (N' \cap N) \cap K = N' \cap (N \cap K) = 0$, we have $M = N' \oplus K$. It follows from [7, 21.6 (5)] that $Rad(M) = Rad(N') \oplus Rad(K) = Rad(K)$ since $Rad(N') \subseteq Rad(N) = 0$. Then $M = N + K \subseteq Rad(M) + K \subseteq K$. It means that $N << M$. □

Lemma 2. *Let M be a module. Then $Soc_s(M) = Rad(M) \cap Soc(M)$.*

Proof. Let $a \in Rad(M) \cap Soc(M)$. Then Ra is semisimple and so there exist $n \in \mathbb{Z}^+$ and simple submodules S_i of M $(1 \leq i \leq n)$ such that $Ra = S_1 \oplus S_2 \oplus ... \oplus S_n$ by [6, Proposition 3.3]. Since Ra is small in M, it follows from [7, 19.3 (2)] that each S_i is small in M. Thus $a \in Ra \subseteq Soc_s(M)$. □

Lemma 3. *Let M be a module and U, V be submodules of M. Then the following statements are equivalent:*

(1) *$M = U + V$ and $U \cap V \subseteq Soc_s(V)$,*
(2) *$M = U + V$, $U \cap V \subseteq Rad(V)$ and $U \cap V$ is semisimple,*
(3) *$M = U + V$, $U \cap V << V$ and $U \cap V$ is semisimple.*

Proof. (1) $\Longrightarrow$ (2) It follows that $U \cap V \subseteq Soc_s(V) \subseteq Rad(V) \cap Soc(V)$. Hence, we deduce that $U \cap V \subseteq Rad(V)$ and $U \cap V$ is semisimple.

(2) $\Longrightarrow$ (3) It is clear by Lemma 1.

(3) $\Longrightarrow$ (1) It is clear by Lemma 2 □

We say that V an *ss-supplement* of U in M if the equal conditions in the above lemma are satisfied. It is clear that the following implications on submodules of a module hold:

Direct summand $\Longrightarrow$ ss-supplement $\Longrightarrow$ supplement $\Longrightarrow$ Rad-supplement

We call a module M *ss-supplemented* if every submodule of M has an ss-supplement in M. A submodule U of a module M has *ample ss-supplements* in M if every submodule V of M such that $M = U + V$ contains an ss-supplement of U in M. We call a module M *amply ss-supplemented* if every submodule of M has ample ss-supplements in M. It is clear that every ss-supplemented module is supplemented. Of course there exists the same relationship between amply ss-supplemented modules and amply supplemented modules. Later we shall give examples of (amply) supplemented modules which are not (amply) ss-supplemented (see Example 17 and Example 18).

In section 3 we characterize ss-supplemented and amply ss-supplemented modules. For modules with small radical, we give some conditions which are equivalent to being an ss-supplemented module in Theorem 20. It follows that a finitely generated module M is ss-supplemented if and only if it is supplemented and $Rad(M) \subseteq Soc(M)$. Any direct sum of strongly local modules is ss-supplemented and coatomic. A module M is amply ss-supplemented if and only if every submodule of the module M is ss-supplemented. We show that a ring R is semiperfect and $Rad(R) \subseteq Soc({}_RR)$ if and only if every left R-module is (amply) ss-supplemented.

2. Strongly Local Modules and Rings

As we mentioned at introduction, we denote by $Soc_s(M)$ the sum of all simple submodules of a module M that are small in M. Then we have:

Let M be a non-zero module. M is called *indecomposable* if the only direct summands of M are 0 and M.

Lemma 4. *Let M be an indecomposable module. Then M is simple or $Soc(M) \subseteq Rad(M)$.*

Proof. Suppose that M is not simple. Let $M = Soc(M) + X$ for some submodule X of M. Since $Soc(M)$ is semisimple, there exists a submodule Y of $Soc(M)$ such that $Soc(M) = (Soc(M) \cap X) \oplus Y$. Therefore, $M = Soc(M) + X = [(Soc(M) \cap X) \oplus Y] + X = X \oplus Y$. Since M is indecomposable and not simple, it follows that $Y = 0$. It means that $X = M$. Hence $Soc(M) << M$, that is, $Soc(M) \subseteq Rad(M)$. □

Using Lemma 2 and Lemma 4, we have the following result.

Corollary 5. *Let M be a local module which is not simple. Then $Soc_s(M) = Soc(M)$.*

Recall that a module M is called *radical* if M has no maximal submodules, that is, $M = Rad(M)$. Let $P(M)$ be the sum of all radical submodules of M. It is easy to see that $P(M)$ is the largest radical submodule of M. If $P(M) = 0$, M is called *reduced*.

Proposition 6. *Let M be a strongly local module. Then M is reduced.*

Proof. Since M is strongly local, we get $P(M) \subseteq Rad(M) \subseteq Soc(M)$. This implies that $P(M)$ is semisimple and so $P(M) = Rad(P(M)) = 0$. This completes the proof. $\square$

Note that the condition "strongly" in the above proposition is necessary. The following example shows that in general a local module need not be reduced.

Example 7. *Let K be a field. In the polynomial ring $K[x_1, x_2, \ldots]$ with countably many indeterminates x_n, $n \in \mathbb{Z}^+$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \cdots)$ generated by x_1^2 and $x_{n+1}^2 - x_n$ for each $n \in \mathbb{Z}^+$. Then as shown in* [**?**, Example 6.2]*, the quotient ring $R = \frac{K[x_1,x_2,\ldots]}{I}$ is a local ring with the unique maximal ideal $J = \frac{(x_1,x_2,\ldots)}{I} = J^2$. Let M be the left R-module ${}_RR$. Then M is a local module. On the other hand, M is not reduced because $P(M) = Rad(J) = J \neq 0$.*

Proposition 8. *Every factor module of a strongly local module is strongly local.*

Proof. Let M be a strongly local module and N be a submodule of M. Then the factor module $\frac{M}{N}$ is local. Since $Rad(M)$ is the unique maximal submodule of M, it follows from [7, 21.2 (1)] that $Rad(\frac{M}{N}) = \frac{Rad(M)}{N} \subseteq \frac{Soc(M)}{N} = \pi(Soc(M)) \subseteq Soc(\frac{M}{N})$, where $\pi : M \longrightarrow \frac{M}{N}$ is the canonical projection. Hence $\frac{M}{N}$ is strongly local. $\square$

Proposition 9. *Let R be a left strongly local ring. Then $(Rad(R))^2 = 0$. In particular, $Rad(R)$ is nilpotent.*

Proof. Since $Rad(R) \subseteq Soc({}_RR)$, it follows from [7, 21.12 (4)] that $(Rad(R))^2 = 0$. It means that $Rad(R)$ is nilpotent. $\square$

Recall from [7] that an ideal I of a ring R is *right t-nilpotent* if for every sequence $a_1, a_2, ..., a_k$ of elements in I, there is a $k \in \mathbb{Z}^+$ with $a_1a_2...a_k = 0$. Similarly *left t-nilpotent* is defined. Following [7, 43.9], R is called *left perfect* (respectively, *right perfect*) if R is semilocal and $Rad(R)$ is right t-nilpotent (respectively, left t-nilpotent). Here a ring R is *semilocal* if $\frac{R}{Rad(R)}$ is an artinian semisimple ring (see [4]). Note that nilpotent ideals are left and right t-nilpotent. Using this fact, we have the following:

Corollary 10. *Every left strongly local ring is left perfect and right perfect.*

Proof. Let R be a left strongly local ring. Since local rings are semilocal, it follows from Proposition 9 that R is left perfect and right perfect. □

It is well known that an artinian commutative domain is field. We have:

Proposition 11. *A strongly local commutative domain is field.*

Proof. Let R be a strongly local commutative domain and a be any element of R. If $a \in R \backslash Rad(R)$, we can write $Ra = R$ because R is local. Therefore, a is an invertible element of R. Suppose that $a \in Rad(R)$. It follows from Proposition 9 that $a^2 \in (Rad(R))^2 = 0$. By the hypothesis, we get $a = 0$. Hence, R is field. □

3. SS-Supplemented Modules

It is known that a ring R is semiperfect if and only if every finitely generated R-module is (amply) supplemented (see [7, 42.6]). In this section we obtain new characterizations of semiperfect rings via their ss-supplemented modules.

Recall that for a maximal submodule U of a module M, a submodule V of M is a supplement of U in M if and only if $M = U + V$ and V is local (see [7, 41.1 (3)]). Analogous to that we have:

Proposition 12. *Let M be a module and U be a maximal submodule of M. A submodule V of M is an ss-supplement of U in M if and only if $M = U + V$ and V is strongly local.*

Proof. Let V be an ss-supplement of U in M. By [7, 41.1.(3)], V is local and $U \cap V = Rad(V)$ is the unique maximal submodule of V. Since $U \cap V$ is semisimple, we have $Rad(V) \subseteq Soc(V)$. Thus V is strongly local.

Conversely, since V is local and $M = U + V$, we can write $U \cap V \subseteq Rad(V)$. It follows from assumption that $U \cap V$ is semisimple. Hence, V is an ss-supplement of U in M. □

Now, we give examples of (amply) supplemented modules which are not (amply) ss-supplemented. We first need the following facts.

Lemma 13. *Let M be an ss-supplemented module and N be a small submodule of M. Then $N \subseteq Soc_s(M)$.*

Proof. By the assumption, M is the unique ss-supplement of N in M and so $N \cap M = N$ is semisimple. Hence, $N \subseteq Soc_s(M)$ by Lemma 2. □

The following result is a direct consequence of Lemma 13.

Corollary 14. *Let M be an ss-supplemented module and $Rad(M) << M$. Then $Rad(M) \subseteq Soc(M)$.*

It is well known that every local module is amply supplemented. Now we give an analogous characterization of this fact for amply ss-supplemented modules.

Proposition 15. *Every strongly local module is amply ss-supplemented.*

Proof. Let M be a strongly local module. Then, M is local and so it is amply supplemented. Note that M has no supplement submodule except for 0 and M. Since $Rad(M) \subseteq Soc(M)$, M is amply ss-supplemented. □

Proposition 16. *Let R be a ring and M be a hollow R-module. M is (amply) ss-supplemented if and only if it is strongly local.*

Proof. Suppose that M is ss-supplemented. Let $m \in Rad(M)$. Then we get $Rm << M$. Since M is ss-supplemented, it follows from Lemma 13 that $Rm \subseteq Soc_s(M)$. It means that $m \in Soc(M)$ and so $Rad(M) \subseteq Soc(M)$. Suppose that $M = Rad(M)$. Since $M = Rad(M) = Soc(M)$ and the radical of a semisimple module is zero, we have that $M = 0$. This is a contradiction because M is hollow. It means that $M \neq Rad(M)$, that is, M is local by [7, 41.4]. Therefore M is strongly local. The converse follows from Proposition 15. □

Example 17. *For any prime integer p, consider the left $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$. Note that M is a hollow module which is not local. Since hollow modules are (amply) supplemented, M is (amply) supplemented. However, M is not (amply) ss-supplemented module by Proposition 16.*

Every artinian module is supplemented. The next example shows that in general artinian modules need not to be ss-supplemented.

Example 18. *Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{p^k}$, for p is any prime integer and $k \geq 3$. Note that M is artinian. Since $Soc_s(\mathbb{Z}_{p^k}) = Soc(\mathbb{Z}_{p^k}) \cong \mathbb{Z}_p$ and $Rad(M) = p\mathbb{Z}_{p^k}$, M is not strongly local and so it is not ss-supplemented by Proposition 16.*

Lemma 19. *Let M be a supplemented module and $Rad(M) \subseteq Soc(M)$. Then M is ss-supplemented.*

Proof. Let $U \subseteq M$. Since M is supplemented, there exists a submodule V of M such that $M = U + V$ and $U \cap V << V$. Then $U \cap V \subseteq Rad(V) \subseteq Rad(M)$ and so $U \cap V$ is semisimple by the assumption. Hence V is an ss-supplement of U in M. It means that M is ss-supplemented. □

Theorem 20. *Let M be a module with $Rad(M) << M$. Then the following statements are equivalent:*

(1) *M is ss-supplemented,*
(2) *M is supplemented and $Rad(M)$ has an ss-supplement in M,*
(3) *M is supplemented and $Rad(M) \subseteq Soc(M)$.*

Proof. (1) $\Longrightarrow$ (2) It is clear.
(2) $\Longrightarrow$ (3) It follows from Lemma 13.
(3) $\Longrightarrow$ (1) By Lemma 19. □

Since finitely generated modules have small radical, we have the following result.

Corollary 21. *Let M be a finitely generated module. Then M is ss-supplemented if and only if it is supplemented and $Rad(M) \subseteq Soc(M)$.*

Next, in order to prove that every finite sum of ss-supplemented modules is ss-supplemented, we use the following standard lemma (see, [7, 41.2]).

Lemma 22. *Let M be a module and M_1, U be submodules of M with M_1 ss-supplemented. If M_1+U has an ss-supplement in M, U also has an ss-supplement in M.*

Proof. Suppose that X is an ss-supplement of $M_1 + U$ in M and Y is an ss-supplement of $(X+U) \cap M_1$ in M_1. Then $M = X+Y+U$ and $(X+Y) \cap U << X+Y$. Moreover, $X \cap (Y + U)$ is semisimple as a submodule of the semisimple module $X \cap (M_1 + U)$. Note that $Y \cap [(X + U) \cap M_1] = Y \cap (X + U)$ is semisimple. It follows from [3, 8.1.5] that $(X + Y) \cap U$ is semisimple. Hence $X + Y$ is an ss-supplement of U in M. □

Proposition 23. *Let M_1, M_2 be any submodules of a module M such that $M = M_1 + M_2$. Then if M_1 and M_2 are ss-supplemented, M is ss-supplemented.*

Proof. Let U be any submodule of M. The trivial submodule 0 is ss-supplement of $M = M_1 + M_2 + U$ in M. Since M_1 is ss-supplemented, $M_2 + U$ has an ss-supplement in M by Lemma 22. Again applying Lemma 22, we also have that U has an ss-supplement in M. This shows that M is ss-supplemented. □

Using this fact we obtain the following corollary.

Corollary 24. *Every finite sum of ss-supplemented modules is ss-supplemented.*

Now we give an example of an ss-supplemented module which is not strongly local.

Example 25. *The $\mathbb{Z}$-module $M = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is ss-supplemented as a sum of strongly local modules. However, M is not (strongly) local.*

Then we have the following proper implications on modules hold:

$$\begin{tikzcd}
 & \textit{strongly local} \arrow[dl] \arrow[dr] & \\
\textit{local} \arrow[dr] & & ss-\textit{supplemented} \arrow[dl] \\
 & \textit{supplemented} &
\end{tikzcd}$$

Proposition 26. *If M is a (amply) ss-supplemented module, then every factor module of M is (amply) ss-supplemented.*

Proof. Let M be an ss-supplemented module and $\frac{M}{L}$ be a factor module of M. By the assumption, for any submodule U of M which contains L, there exists a submodule V of M such that $M = U + V$, $U \cap V << V$ and $U \cap V$ is semisimple. Let $\pi : M \longrightarrow \frac{M}{L}$ be the canonical projection. Then we have that $\frac{M}{L} = \frac{U}{L} + \frac{V+L}{L}$ and $\frac{U}{L} \cap \frac{V+L}{L} = \frac{(U\cap V)+L}{L} = \pi(U \cap V) << \pi(V) = \frac{V+L}{L}$ by [7, 19.3(4)]. Since $U \cap V$ is semisimple, it follows from [3, 8.1.5] that $\pi(U \cap V) = \frac{(U\cap V)+L}{L} = \frac{U}{L} \cap \frac{V+L}{L}$ is semisimple. That is, $\frac{V+L}{L}$ is an ss-supplement of $\frac{U}{L}$ in $\frac{M}{L}$, as required.

By adapting this argument we can prove similarly that if M is amply ss-supplemented, then so is every factor module of M. $\square$

Recall that a module M is said to be *coatomic* if every proper submodule of M is contained in a maximal submodule of M. It is easy to see that every coatomic module has small radical.

Let p be a prime integer and consider the localization ring $R = \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a,\, b \in \mathbb{Z} \textit{ and } p \nmid b\}$. Note that R is a local ring. Let M be the left R-module $R^{(\mathbb{N})}$. Then M is the direct sum of local submodules but it is not supplemented. Since R is not perfect, $Rad(M)$ is not small in M and so M is not also coatomic. However, any arbitrary direct sum of strongly local modules is ss-supplemented and coatomic, as the next result shows.

Theorem 27. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is a strongly local module. Then, M is ss-supplemented and coatomic.*

Proof. Since M_i is strongly local for every $i \in I$, it is local and $Rad(M_i) \subseteq Soc(M_i)$ and so $Rad(M) = \bigoplus_{i\in I} Rad(M_i) \subseteq \bigoplus_{i\in I} Soc(M_i) = Soc(M)$ by [7, 21.6 (5) and 21.2 (5)]. Applying Lemma 1, we get that $Rad(M)$ is a small submodule of M. Since strongly local modules are local, it follows from [10, Theorem 1.4 (A)] that M is supplemented. Hence, M is ss-supplemented by Theorem 20.

Let U be a proper submodule of M. It follows from [7, 41.1 (6)] that U is contained in a maximal submodule of M, that is, M is coatomic. $\square$

Let M be a module. A module N is called *M-generated* if there exists an epimorphism $f : M^{(I)} \longrightarrow N$ for some index set I.

Corollary 28. *Let M be a strongly local module. Then every M-generated module is ss-supplemented and coatomic.*

Proof. Suppose that N is M-generated. Then, there exists an epimorphism $f : M^{(I)} \longrightarrow N$ for some index set I. By Theorem 27, $M^{(I)}$ is ss-supplemented and coatomic. Hence N is ss-supplemented by Proposition 26 and it is coatomic by [10, Lemma 1.5 (a)]. $\square$

Corollary 29. *Let R be a left strongly local ring. Then every left R-module is ss-supplemented.*

Proof. Since all left R-modules are R-generated, the proof follows from Corollary 28. $\square$

A submodule U of a module M is said to be *cofinite* if M/U is finitely generated (see [1]). Note that maximal submodules of M are cofinite.

Theorem 30. *The following statements are equivalent for a module M:*

(1) *M is the sum of all strongly local submodules,*
(2) *M is ss-supplemented and coatomic,*
(3) *M is coatomic and every cofinite submodule of M has an ss-supplement in M,*
(4) *M is coatomic and every maximal submodule of M has an ss-supplement in M.*

Proof. (1) $\Longrightarrow$ (2) Let $M = \sum_{i\in I} M_i$, where each M_i is strongly local submodules. Put $N = \bigoplus_{i\in I} M_i$. Then, by Theorem 27, N is ss-supplemented and coatomic. Now we consider the epimorphism $f : N \longrightarrow M$ via $f((m_i)_{i\in I}) = \sum_{i\in I} m_i$ for all $(m_i)_{i\in I} \in N$. It follows from Proposition 26 and [10, Lemma 1.5 (a)] that M is ss-supplemented and coatomic.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are clear.

(4) $\Longrightarrow$ (1) Let S be the sum of all strongly local submodules of M. Assume that $S \neq M$. Since M is coatomic, there exists a maximal submodule K of M with $S \subseteq K$. By (4), K has an ss-supplement, say V, in M. It follows from Proposition 12 that V is strongly local. Therefore, $V \subseteq S \subseteq K$, a contradiction. $\square$

The following fact is a direct consequence of Theorem 30.

Corollary 31. *For a coatomic module M, the following statements are equivalent:*

(1) *M is the sum of all strongly local submodules,*
(2) *M is ss-supplemented,*
(3) *Every cofinite (maximal) submodule of M has an ss-supplement in M.*

A ring R is called *left max* if every non-zero left R-module has a maximal submodule. Note that if R is a left max ring, then every left R-module is coatomic. Using this fact and Corollary 31, we obtain the following result.

Corollary 32. *Let R be a left max ring and M be a non-zero left R-module. Then M is the sum of all strongly local submodules of M if and only if it is ss-supplemented.*

Proposition 33. *Let M be a module. If every submodule of M is ss-supplemented, then M is amply ss-supplemented.*

Proof. Let U and V be two submodules of M such that $M = U + V$. Since V is ss-supplemented, there exists a submodule V' of V such that $V = (U \cap V) + V'$, $U \cap V' << V'$ and $U \cap V'$ is semisimple. Note that $M = U + V = U + ((U \cap V) + V') =$

$U + V^{'}$. It means that U has ample ss-supplements in M. Hence M is amply ss-supplemented. □

Lemma 34. *Let M be amply ss-supplemented module and V be an ss-supplement submodule in M. Then V is amply ss-supplemented.*

Proof. Let V be an ss-supplement of a submodule U of M. Let X and Y be submodules of V such that $V = X + Y$. Then $M = (U + X) + Y$. Since M is amply ss-supplemented, $U + X$ has an ss-supplement $Y^{'} \subseteq Y$ in M. It follows that $X + Y^{'} \subseteq V$. By the minimality of V, we have $V = X + Y^{'}$. In addition, $X \cap Y^{'} \subseteq (U + X) \cap Y^{'} << Y^{'}$, that is, $X \cap Y^{'} << Y^{'}$. Since $(U + X) \cap Y^{'}$ is semisimple, $X \cap Y^{'}$ is also semisimple by [3, 8.1.5]. It means that $Y^{'}$ is an ss-supplement of X in V. Finally, V is amply ss-supplemented. □

The next result gives a useful characterization of amply ss-supplemented modules.

Theorem 35. *Let M be a module. Then, M is amply ss-supplemented if and only if every submodule U of M is of the form $U = X + Y$, where X is ss-supplemented and $Y \subseteq Soc_s(M)$.*

Proof. Let U be a submodule of M. Since M is ss-supplemented, U has an ss-supplement V in M. Then $M = U + V$. By the assumption, there exists a submodule X of U such that X is an ss-supplement of V in M. Put $Y = U \cap V$. Since V is an ss-supplement of U in M, we have that $Y \subseteq Soc_s(V) \subseteq Soc_s(M)$. Applying the modular law, we get $U = U \cap M = U \cap (X + V) = X + U \cap V = X + Y$. Note that X is ss-supplemented by Lemma 34.

Conversely, let U be a submodule of M. By the assumption, there exist submodules X and Y of M such that $U = X + Y$, X ss-supplemented and $Y \subseteq Soc_s(M)$. By Proposition 23, U is ss-supplemented. Hence M is amply ss-supplemented from Proposition 33. □

The next result is crucial.

Corollary 36. *For a module M, the following statements are equivalent:*

(1) *M is amply ss-supplemented,*
(2) *Every submodule of M is ss-supplemented,*
(3) *Every submodule of M is amply ss-supplemented.*

Note that it is not in general true that any submodule of an amply supplemented module is (amply) supplemented. Let R be a local Dedekind domain which is not field. Suppose that $M = R^{(\mathbb{N})}$. Then, M is not (amply) supplemented. The group $F = R \times M$ can be converted to a ring by the following operation: $(x, y) \cdot \left(x^{'}, y^{'}\right) = (xx^{'}, xy^{'} + x^{'}y)$ where $x, x^{'} \in R$ and $y, y^{'} \in M$. Then F is a commutative local ring and so F is amply supplemented. Put $L = \{0\} \times M$. Therefore, L is an ideal of F. Hence the submodule L of F is not a (amply) supplemented F-module.

A module M is said to be *π-projective* if whenever U and V are submodules of M such that $M = U + V$, there exists an endomorphism f of M such that $f(M) \subseteq U$ and $(1-f)(M) \subseteq V$. Hollow (local) modules and self-projective modules are π-projective and π-projective supplemented modules are amply supplemented. Similarly, we show that π-projective ss-supplemented modules are amply ss-supplemented. The proof is virtually the same that of [7, 41.15], but we give it for completeness.

Proposition 37. *Let M be a π-projective and ss-supplemented module. Then M is amply ss-supplemented.*

Proof. Let U and V be submodules of M such that $M = U + V$. Since M is π-projective, there exists an endomorphism f of M such that $f(M) \subseteq U$ and $(1-f)(M) \subseteq V$. Note that $(1-f)(U) \subseteq U$. Let $V^{'}$ be an ss-supplement of U in M. Then $M = f(M) + (1-f)(M) = f(M) + (1-f)(U + V^{'}) \subseteq U + (1-f)(V^{'})$, so that $M = U + (1-f)(V^{'})$. Note that $(1-f)(V^{'})$ is a submodule of V. Let $y \in U \cap (1-f)(V^{'})$. Then, $y \in U$ and $y = (1-f)(x) = x - f(x)$ for some $x \in V^{'}$. Next $x = y + f(x) \in U$ so that $y \in (1-f)(U \cap V^{'})$. Since $U \cap V^{'} << V^{'}$, $U \cap (1-f)(V^{'}) = (1-f)(U \cap V^{'}) << (1-f)(V^{'})$ by [7, 19.3(4)]. By [3, 8.1.5], $U \cap (1-f)(V^{'}) = (1-f)(U \cap V^{'})$ is semisimple because $U \cap V^{'}$ is semisimple. Thus $(1-f)(V^{'})$ is an ss-supplement of U in M. Therefore M is amply ss-supplemented module. □

Since every projective module is π-projective, the following result follows from Proposition 37 and Corollary 36.

Corollary 38. *Any submodule of a projective ss-supplemented module is ss-supplemented.*

Now, we characterize the rings whose modules are ss-supplemented. Firstly, we need the following lemmas.

Lemma 39. *Let M be a projective module. Then M is ss-supplemented if and only if it is supplemented and $Rad(M) \subseteq Soc(M)$.*

Proof. Suppose that M is projective supplemented module. Therefore we have $Rad(M) << M$ by [7, 42.5]. Then the proof is obvious from Theorem 20. □

Lemma 40. *Let R be a ring. Then every left R-module is ss-supplemented if and only if every left R-module is the sum of all strongly local submodules.*

Proof. Assume that every left R-module M is ss-supplemented. Then, by [7, 43.9], R is left perfect. This implies that R is a left max ring. Applying Corollary 32, M is the sum of all strongly local submodules of M. The converse follows from Theorem 30. □

Theorem 41. *The following statements are equivalent for a ring R:*

(1) ${}_RR$ *is ss-supplemented,*
(2) R *is semiperfect and* $Rad(R) \subseteq Soc({}_RR)$*,*
(3) R *is semilocal and* $Rad(R) \subseteq Soc({}_RR)$*,*
(4) *Every projective left* R*-module is (amply) ss-supplemented,*
(5) *Every left* R*-module is (amply) ss-supplemented,*
(6) *Every left* R*-module is the sum of all strongly local submodules,*
(7) ${}_RR$ *is a finite sum of strongly local submodules,*
(8) *Every maximal left ideal of* R *has an ss-supplement in* R*.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) By Corollary 21 and [7, 42.6].

(3) $\Longrightarrow$ (4) Let M be a projective R-module. Then, by [7, 21.17 (2)], we can write $Rad(M) = Rad(R)M \subseteq Soc({}_RR)M = Soc(M)$. From [7, 43.9] and Lemma 39, the proof is completed.

(4) $\Longrightarrow$ (5) follows [7, 18.6] and Proposition 26.

(5) $\Longrightarrow$ (6) By Lemma 40.

(6) $\Longrightarrow$ (7) is obvious.

(7) $\Longrightarrow$ (8) By Theorem 30.

(8) $\Longrightarrow$ (1) By Corollary 31. □

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ASYMPTOTIC BEHAVIOUR OF RESONANCE EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH A MATRIX POTENTIAL

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Abstract. We will discuss the asymptotic behaviour of the eigenvalues of a Schrödinger operator with a matrix potential defined by the Neumann boundary condition in $L_2^m(F)$, where F is a d-dimensional rectangle and the potential is an $m \times m$ matrix with $m \geq 2$, $d \geq 2$, when the eigenvalues belong to the resonance domain, roughly speaking they lie near the planes of diffraction.

1. Introduction

In this paper, we consider the Schrödinger operator with a matrix potential $V(x)$ defined by the differential expression

$$L\phi = -\Delta\phi + V\phi \tag{1}$$

and the Neumann boundary condition

$$\frac{\partial \phi}{\partial n}|_{\partial F} \ = 0, \tag{2}$$

in $L_2^m(F)$ where F is the d dimensional rectangle $F = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_d]$, ∂F is the boundary of F, $m \geqslant 2, d \geqslant 2$, $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal of the boundary ∂F, Δ is a diagonal $m \times m$ matrix whose diagonal elements are the scalar Laplace operators $\triangle = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \ldots + \frac{\partial^2}{\partial {x_d}^2}$, $x = (x_1, x_2, \ldots, x_d) \in R^d$, V is a real valued symmetric matrix $V(x) = (v_{ij}(x)), i, j = 1, 2, \ldots, m, v_{ij}(x) \in L_2(F)$, that is, $V^T(x) = V(x)$.

We denote the operator defined by (1)-(2) by $L(V)$, the eigenvalues and the corresponding eigenfunctions of $L(V)$ by Λ_N and Ψ_N, respectively.

The eigenvalues of the operator $L(0)$ which is defined by the differential expression (1) when $V(x) = 0$ and the boundary condition (2) are $|\gamma|^2$, and the

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corresponding eigenspaces are $E_\gamma = span\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \ldots, \Phi_{\gamma,m}(x)\}$, where

$$\begin{aligned} \gamma &= (\gamma^1, \gamma^2, \ldots, \gamma^d) \in \frac{\Gamma^{+0}}{2}, \\ \frac{\Gamma^{+0}}{2} &= \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2} \cdots, \frac{n_d\pi}{a_d}) : n_k \in Z^+ \cup \{0\}, k = 1, 2, \ldots, d\}, \\ \Phi_{\gamma,j}(x) &= (0, \ldots, 0, u_\gamma(x), 0, \ldots, 0), j = 1, 2, \ldots, m, \end{aligned}$$

and the non-zero component of $\Phi_{\gamma,j}(x)$ is $u_\gamma(x) = cos\frac{n_1\pi}{a_1}x_1 cos\frac{n_2\pi}{a_2}x_2 \cdots cos\frac{n_d\pi}{a_d}x_d$, which stands in the jth component. In particular, $u_0(x) = 1$ when $\gamma = (0, 0, \ldots, 0)$.

It can be easily calculated that the norm of $u_\gamma(x)$, $\gamma \in \frac{\Gamma^{+0}}{2}$, in $L_2(F)$ is $\sqrt{\frac{\mu(F)}{|A_\gamma|}}$, where $\mu(F)$ is the measure of the d-dimensional parallelepiped F, $|A_\gamma|$ is the number of vectors in $A_\gamma = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \frac{\Gamma}{2} : \ |\alpha_k| = |\gamma^k|, \ k = 1, 2, \ldots, d\}$, $\frac{\Gamma}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2} \cdots, \frac{n_d\pi}{a_d}) : \ n_k \in Z, \ k = 1, 2, \ldots, d\}$.

From now on, $\langle ., . \rangle$ and $(., .)$ will denote the inner products in $L_2^m(F)$ and $L_2(F)$, respectively.

Since $\{u_\gamma(x)\}_{\gamma \in \frac{\Gamma^{+0}}{2}}$ is a complete system in $L_2(F)$, for any $q(x)$ in $L_2(F)$ we have

$$q(x) = \sum_{\gamma \in \frac{\Gamma^{+0}}{2}} \frac{|A_\gamma|}{\mu(F)} (q, u_\gamma) u_\gamma(x). \tag{3}$$

In our study, it is convenient to use the equivalent decomposition (see [9])

$$q(x) = \sum_{\gamma \in \frac{\Gamma}{2}} q_\gamma u_\gamma(x), \tag{4}$$

where $q_\gamma = \frac{1}{\mu(F)}(q(x), u_\gamma(x))$ for the sake of simplicity. That is, the decomposition (3) and (4) are equivalent for any $d \geq 2$. Thus, according to (4), each matrix element $v_{ij}(x) \in L_2(F)$ of the matrix $V(x)$ can be written in its Fourier series expansion

$$v_{ij}(x) = \sum_{\gamma \in \frac{\Gamma}{2}} v_{ij\gamma} u_\gamma(x), \tag{5}$$

$v_{ij\gamma} = \frac{(v_{ij}, u_\gamma)}{\mu(F)}$, $(v_{ij}, u_\gamma) = \frac{1}{\mu(F)} \int_F v_{ij}(x) u_\gamma(x) dx$ and $v_{ij0} = \frac{1}{\mu(F)} \int_F v_{ij}(x) dx$ $i, j = 1, 2, \ldots, m$.

We assume that $l > \frac{(d+20)(d-1)}{2} + d + 3$ and the Fourier coefficients $v_{ij\gamma}$ of $v_{ij}(x)$ satisfy

$$\sum_{\gamma \in \frac{\Gamma}{2}} |v_{ij\gamma}|^2 (1 + |\gamma|^{2l}) < \infty, \tag{6}$$

for each $i, j = 1, 2, \ldots, m$. Let ρ be a large parameter, $\rho \gg 1$ and α be a positive number with $0 < \alpha < \frac{1}{d+20}$ then for $\Gamma(\rho^\alpha) = \{\gamma \in \frac{\Gamma}{2} : 0 \leq |\gamma| < \rho^\alpha\}$ and $p = l - d$

the condition (6) implies that

$$v_{ij}(x) = \sum_{\gamma \in \Gamma(\rho^{\alpha})} v_{ij\gamma} u_{\gamma}(x) + O(\rho^{-p\alpha}). \tag{7}$$

Here $O(\rho^{-p\alpha})$ is a function in $L_2(F)$ with norm of order $\rho^{-p\alpha}$. Furthermore, by (6), we have

$$M_{ij} \equiv \sum_{\gamma \in \frac{\Gamma}{2}} |v_{ij\gamma}| < \infty, \tag{8}$$

for all $i, j = 1, 2, \ldots, m$.

Notice that, if a function $q(x)$ is sufficiently smooth$\big(q(x) \in W_2^l(F)\big)$ and the support of $\nabla q(x) = \left(\frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}, \ldots, \frac{\partial q}{\partial x_d}\right)$ is contained in the interior of the domain F, then $q(x)$ satisfies condition (6) (See [7]). There is also another class of functions $q(x)$, such that $q(x) \in W_2^l(F)$,

$$q(x) = \sum_{\gamma' \in \Gamma} q_{\gamma'} u_{\gamma'}(x),$$

which is periodic with respect to a lattice

$$\Omega = \{(m_1 a_1, m_2 a_2, \ldots, m_d a_d) : m_k \in \mathbf{Z}, k = 1, 2, \ldots, d\}$$

and thus it also satisfies condition (6).

As in [17]-[22], we divide R^d into two domains: Resonance and Non-resonance domains. In order to define these domains, let us introduce the following sets:

Let $0 < \alpha < \frac{1}{d+20}$, $\alpha_k = 3^k \alpha$, $k = 1, 2, \ldots, d-1$ and

$$\begin{aligned}
V_b(\rho^{\alpha_1}) &\equiv \left\{x \in R^d : \quad \big||x|^2 - |x+b|^2\big| < \rho^{\alpha_1}\right\} \\
E_1(\rho^{\alpha_1}, p) &\equiv \bigcup_{b \in \Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha_1}) \\
U(\rho^{\alpha_1}, p) &\equiv R^d \setminus E_1(\rho^{\alpha_1}, p) \\
E_k(\rho^{\alpha_k}, p) &= \bigcup_{\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p\rho^{\alpha})} \left(\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})\right)
\end{aligned}$$

where $b \neq 0$, $\gamma_i \neq 0$, $i = 1, 2, \ldots, k$ and the intersection $\bigcap\limits_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ in E_k is taken over $\gamma_1, \gamma_2, \ldots, \gamma_k$ which are linearly independent vectors and the length of γ_i is not greater than the length of the other vector in $\Gamma \bigcap \gamma_i R$. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain, and the eigenvalue $|\gamma|^2$ is called a non-resonance eigenvalue if $\gamma \in U(\rho^{\alpha_1}, p)$. The domains $V_b(\rho^{\alpha_1})$, for $b \in \Gamma(p\rho^{\alpha})$ are called resonance domains and the eigenvalue $|\gamma|^2$ is a resonance eigenvalue if $\gamma \in V_b(\rho^{\alpha_1})$.

As noted in [20]-[21], the domain $V_b(\rho^{\alpha_1}) \setminus E_2$, called a single resonance domain, has asymptotically full measure on $V_b(\rho^{\alpha_1})$, that is,

$$\frac{\mu\left((V_b(\rho^{\alpha_1}) \setminus E_2) \bigcap B(q)\right)}{\mu\left(V_b(\rho^{\alpha_1}) \bigcap B(q)\right)} \to 1, \text{ as } \rho \to \infty,$$

where $B(\rho) = \left\{x \in \boldsymbol{R}^d : |x| = \rho\right\}$, if

$$2\alpha_2 - \alpha_1 + (d+3)\alpha < 1, \quad \alpha_2 > 2\alpha_1, \tag{9}$$

hold. Since $0 < \alpha < \frac{1}{d+20}$, the conditions in (9) hold.

In most cases, it is important to know the asymptotic behavior of the eigenvalues of the Schrödinger operator $L(V)$. In this paper, [3] and [8], we construct the asymptotic formulas in the high energy region for eigenvalues of the operator $L(V)$.

In [3], we obtain the asymptotic formulas of arbitrary order for the eigenvalue of $L(V)$ corresponding to the non-resonance eigenvalues $|\gamma|^2$ of $L(0)$ in arbitrary dimension $d \geq 2$.

In [8], we constructed the high energy asymptotics of arbitrary order for the eigenvalue of $L(V)$ corresponding to resonance eigenvalue $|\gamma|^2$ when γ belongs to the special single resonance domains $V_\delta(\rho^{\alpha_1}) \setminus E_2$, where δ is from $\{e_1, e_2, \ldots, e_d\}$ and $e_1 = \left(\frac{\pi}{a_1}, 0, \ldots, 0\right), \ldots, e_d = \left(0, \ldots, \frac{\pi}{a_d}\right)$, $d \geq 2$.

In this paper, we study the case for which $|\gamma|^2$ is a resonance eigenvalue. More precisely, in Theorem (1) and (2) of Section(2), we assume that $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1, 2, \ldots, d-1$ and $\gamma \notin V_{e_k}(\rho^{\alpha_1})$ for $k = 1, 2, \ldots, d$ and prove that the corresponding eigenvalue of $L(V)$ is close to the sum of the eigenvalue of the matrix V_0 and the eigenvalue of the matrix $C = C(\gamma, \gamma_1, \ldots, \gamma_k)$ (See (14)).

In Section(3), this time we assume that $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, $\delta \in \frac{\Gamma}{2} \setminus \{e_1, e_2, \ldots, e_d\}$, that is, γ is in a single resonance domain and we prove the main result Theorem (7) which gives a connection between the eigenvalues of $L(V)$ corresponding to a single resonance domain and the eigenvalues of the Sturm-Liouville operators.

Note that, the case $\delta = e_i$, $i = 1, 2, \ldots, d$, was considered in [8], by a different but simpler method and better formulas were obtained.

2. Asymptotic Formulas for the Eigenvalues in the Resonance Domain

We assume that $\gamma \notin V_{e_k}(\rho^{\alpha_1})$ for $k = 1, 2, \ldots, d$, and $|\gamma|^2$ is a resonance eigenvalue of the operator $L(0)$, that is, $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1, 2, \ldots, d-1$, such that $\mid \gamma \mid \sim \rho$ where $\mid \gamma \mid \sim \rho$ means that $\mid \gamma \mid$ and ρ are asymptotically equal, that is, there exist c_1, c_2 satisfying the inequality $c_1\rho \leq \mid \gamma \mid \leq c_2\rho$, c_i, $i = 1, 2, 3, \ldots$

are positive real constants which do not depend on ρ. To obtain the asymptotic formulas for the eigenvalues of $L(V)$ corresponding to $|\gamma|^2$ we use the binding formula (see (9) in [3])

$$(\Lambda_N - |\gamma|^2)\langle \Psi_N, \Phi_{\gamma,j}\rangle = \langle \Psi_N, V\Phi_{\gamma,j}\rangle. \tag{10}$$

Now, we decompose $V(x)\Phi_{\gamma,j}(x)$ with respect to the basis $\{\Phi_{\gamma',i}(x)\}_{\gamma'\in\frac{\Gamma}{2}, i=1,2,\ldots,m}$. By definition of $\Phi_{\gamma,j}(x)$, it is obvious that

$$V(x)\Phi_{\gamma,j}(x) = (v_{1j}(x)u_\gamma(x),\ldots,v_{mj}(x)u_\gamma(x)). \tag{11}$$

Substituting the decomposition (7) of $v_{ij}(x)$ in (11), we get

$$V(x)\Phi_{\gamma,j}(x) = (\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{1j\gamma'}u_{\gamma'}(x)u_\gamma(x),\ldots,\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{mj\gamma'}u_{\gamma'}(x)u_\gamma(x)) + O(\rho^{-p\alpha}).$$

Since γ does not belong to the domains $V_{e_k}(\rho^{\alpha_1})$, for each $k = 1, 2, ...d$, we may use the following equation

$$\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{ij\gamma'}u_{\gamma'}(x)u_\gamma(x) = \sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{ij\gamma'}u_{\gamma-\gamma'}(x)$$

which is proved in [9] (see equation (18) in [9]), and obtain

$$\begin{aligned} V(x)\Phi_{\gamma,j}(x) &= (\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{1j\gamma'}u_{\gamma-\gamma'}(x),\ldots,\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{mj\gamma'}u_{\gamma-\gamma'}(x)) + O(\rho^{-p\alpha}) \\ &= \sum_{i=1}^{m}\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{ij\gamma'}\Phi_{\gamma-\gamma',i}(x) + O(\rho^{-p\alpha}). \end{aligned} \tag{12}$$

Substituting (12) into (10), we obtain

$$\begin{aligned} <\Psi_N,\Phi_{\gamma,j}> &= \frac{<\Psi_N, V\Phi_{\gamma,j}>}{(\Lambda_N - \mid\gamma\mid^2)} \\ &= \sum_{i=1}^{m}\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{ij\gamma'}\frac{<\Psi_N,\Phi_{\gamma-\gamma',i}>}{(\Lambda_N - \mid\gamma\mid^2)} + O(\rho^{-p\alpha}) \end{aligned} \tag{13}$$

for every vector $\gamma \in \frac{\Gamma}{2}$, satisfying the condition

$$\mid \Lambda_N - \mid\gamma\mid^2\mid > \frac{1}{2}\rho^{\alpha_1}.$$

Letting $p_1 = [\frac{p+1}{2}]$, that is, p_1 is the integer part of $\frac{p+1}{2}$, we define the following sets

$$B_k(\gamma_1,\gamma_2,\ldots,\gamma_k) = \{b : b = \sum_{i=1}^{k} n_i\gamma_i, n_i \in Z, |b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}\},$$

$$B_k(\gamma) = \gamma + B_k(\gamma_1,\gamma_2,\ldots,\gamma_k) = \{\gamma + b : b \in B_k(\gamma_1,\gamma_2,\ldots,\gamma_k)\},$$

$$B_k(\gamma, p_1) = B_k(\gamma) + \Gamma(p_1\rho^\alpha).$$

Let h_τ, $\tau = 1, 2, \ldots, b_k$ denote the vectors of $B_k(\gamma, p_1)$, b_k the number of the vectors in $B_k(\gamma, p_1)$. By its definition, it can easily be obtained that $b_k = O(\rho^{\frac{d}{2}3^d\alpha})$, since $\alpha_k = 3^k\alpha$, $2 \leq k \leq d$. We define the $mb_k \times mb_k$ matrix $C = C(\gamma, \gamma_1, \ldots, \gamma_k)$ by

$$C = \begin{bmatrix} |h_1|^2 I - V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\ V_{h_2-h_1} & |h_2|^2 I - V_0 & \cdots & V_{h_2-h_{b_k}} \\ \vdots & & & \\ V_{h_{b_k}-h_1} & V_{h_{b_k}-h_2} & \cdots & |h_{b_k}|^2 I - V_0 \end{bmatrix}, \tag{14}$$

where $V_{h_\tau-h_\xi}$, $\tau, \xi = 1, 2, \ldots, b_k$ are the $m \times m$ matrices defined by

$$V_{h_\tau-h_\xi} = \begin{bmatrix} v_{11h_\tau-h_\xi} & v_{12h_\tau-h_\xi} & \cdots & v_{1mh_\tau-h_\xi} \\ v_{21h_\tau-h_\xi} & v_{22h_\tau-h_\xi} & \cdots & v_{2mh_\tau-h_\xi} \\ \vdots & & & \\ v_{m1h_\tau-h_\xi} & v_{m2h_\tau-h_\xi} & \cdots & v_{mmh_\tau-h_\xi} \end{bmatrix}. \tag{15}$$

Writing equation (13) for all $h_\tau \in B_k(\gamma, p_1)$, $\tau = 1, 2, \ldots, b_k$ and $j = 1, 2, \ldots, m$, we get

$$(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{h_\tau,j} >= \sum_{i=1}^{m} \sum_{\gamma\prime \in \Gamma(\rho^\alpha)} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau-\gamma\prime,i} > +O(\rho^{-p\alpha}). \tag{16}$$

Similar system of equations for quasi-periodic boundary condition was investigated in [19], [21] and [22]. More recently, in [22], Lemma 2.2.1. states that for $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $h_\tau \in B_k(\gamma, p_1)$ and $\gamma\prime, \gamma_1, \gamma_2, \ldots, \gamma_s \in \Gamma(\rho^\alpha)$, if $h_\tau - \gamma\prime \notin B_k(\gamma, p_1)$ then

$$||\gamma|^2 - |h_\tau - \gamma\prime - \gamma_1 - \ldots - \gamma_s|^2 | > \frac{1}{5}\rho^{\alpha_{k+1}}, \tag{17}$$

for $s = 0, 1, 2, \ldots, p_1 - 1$.

Thus, if an eigenvalue Λ_N of $L(V)$ satisfies

$$|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}, \tag{18}$$

then by (17) and (18), we have

$$|\Lambda_N - |h_\tau - \gamma\prime - \gamma_1 - \ldots - \gamma_s|^2| > \frac{1}{6}\rho^{\alpha_{k+1}}. \tag{19}$$

Now, we prove that if (18) holds then

$$O(\rho^{-p\alpha}) = \sum_{i=1}^{m} \sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau-\gamma\prime,i} > \tag{20}$$

for any $j = 1, 2, \ldots, m$. Here we remark that $\gamma\prime \neq 0$. If it were the case, then we would have from $h_\tau - \gamma\prime \notin B_k(\gamma, p_1)$ that $h_\tau \notin B_k(\gamma, p_1)$ which is a contradiction. So, to prove (20), we argue as Theorem 2.2.2 (a) of [22]: Since Λ_N satisfies the inequality (18), by (19) (for $s = 0$) we have $\mid \Lambda_N - \mid h_\tau - \gamma\prime \mid^2 \mid > \frac{1}{6}\rho^{\alpha_{k+1}}$. Using this, in the equation (13) instead of γ we write $h_\tau - \gamma\prime$ to get

$$< \Psi_N, \Phi_{h_\tau - \gamma\prime, j} >= \sum_{i_1=1}^{m} \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} v_{ij\gamma_1} \frac{< \Psi_N, \Phi_{h_\tau - \gamma\prime - \gamma_1, i_1} >}{(\Lambda_N - \mid h_\tau - \gamma\prime \mid^2)} + O(\rho^{-p\alpha}). \quad (21)$$

Substituting this equation (21) into the right hand side of (20), we obtain

$$\sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau - \gamma\prime, i} > =$$

$$\sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} \frac{v_{ij\gamma\prime}}{\Lambda_N - \mid h_\tau - \gamma\prime \mid^2} \sum_{i_1=1}^{m} \sum_{\substack{\gamma_1 \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{i_1 i \gamma_1} < \Psi_N, \Phi_{h_\tau - \gamma\prime - \gamma_1, i_1} >$$
$$+ O(\rho^{-p\alpha}).$$

In this manner, iterating p_1 times, we get

$$\sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau - \gamma\prime, i} >= \sum_{i_1, i_2, \ldots, i_{p_1}=1}^{m} \sum_{\substack{\gamma\prime, \gamma_1, \gamma_2, \ldots, \gamma_{p_1} \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}}$$

$$\frac{v_{ij\gamma\prime} v_{i_1 i \gamma_1} \ldots v_{i_{p_1} i_{p_1 - 1} \gamma_{p_1}} < \Psi_N, \Phi_{h_\tau - \gamma\prime - \gamma_1 - \cdots - \gamma_{p_1}, i_{p_1}} >}{(\Lambda_N - \mid h_\tau - \gamma\prime \mid^2)(\Lambda_N - \mid h_\tau - \gamma\prime - \gamma_1 \mid^2) \ldots (\Lambda_N - \mid h_\tau - \gamma\prime - \gamma_1 - \cdots - \gamma_{p_1 - 1} \mid^2)}$$
$$+ O(\rho^{-p\alpha}).$$

Taking norm of both sides of the last equality, using (19), the relation (8) and the fact that $p_1 \alpha_{k+1} \geq p_1 \alpha_2 > p\alpha$, we obtain

$$\mid \sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau - \gamma\prime, i} > \mid = O(\rho^{-p\alpha}),$$

which implies (20). Therefore, the equation (16) becomes

$$(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{h_\tau, j} >= \sum_{i=1}^{m} \sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \in B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau - \gamma\prime, i} > + O(\rho^{-p\alpha}). \quad (22)$$

Since $h_\tau - \gamma\prime \in B_k(\gamma, p_1)$, using the notation $h_\xi = h_\tau - \gamma\prime$, the decomposition (22) can be written as

$$(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{h_\tau, j} >= \sum_{i=1}^{m} \sum_{h_\tau - h_\xi \in \Gamma(\rho^\alpha)} v_{ijh_\tau - h_\xi} < \Psi_N, \Phi_{h_\xi, i} > + O(\rho^{-p\alpha}). \quad (23)$$

Isolating the terms where $h_\tau - h_\xi = 0$ in (23), we get

$$\begin{aligned}(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{h_\tau,j} > &= \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{h_\tau,i} > \\ &+ \sum_{i=1}^{m} \sum_{\substack{h_\tau - h_\xi \in \Gamma(\rho^\alpha) \\ h_\tau - h_\xi \neq 0}} v_{ijh_\tau - h_\xi} < \Psi_N, \Phi_{h_\xi,i} > \\ &+ O(\rho^{-p\alpha}). \end{aligned} \tag{24}$$

Writing the equation (24) for all $j = 1, 2, \ldots, m$ and for any $\tau = 1, 2, \ldots, b_k$, , we get the system of equations

$$[(\Lambda_N - |h_\tau|^2)I - V_0]A(N, h_\tau) = \sum_{\substack{\xi=1 \\ \xi \neq \tau}}^{b_k} V_{h_\tau - h_\xi} A(N, h_\xi) + O(\rho^{-p\alpha}), \tag{25}$$

where I is an $m \times m$ identity matrix, $V_{h_\tau - h_\xi}$ is given by (15),

$$O(\rho^{-p\alpha}) = (O(\rho^{-p\alpha}), \ldots, O(\rho^{-p\alpha}))$$

is an $m \times 1$ vector and $A(N, h_\xi)$ is the $m \times 1$ vector

$$A(N, h_\xi) = (< \Psi_N, \Phi_{h_\xi,1} >, < \Psi_N, \Phi_{h_\xi,2} >, \ldots, < \Psi_N, \Phi_{h_\xi,m} >) \tag{26}$$

for any $\xi = 1, 2, \ldots, b_k$. Letting $\lambda_{N,\tau} = \Lambda_N - |h_\tau|^2$, we have

$$\begin{bmatrix} \lambda_{N,1,}I - V_0 & -V_{h_1 - h_2} & \cdots & -V_{h_1 - h_{b_k}} \\ -V_{h_2 - h_1} & \lambda_{N,2}I - V_0 & \cdots & -V_{h_2 - h_{b_k}} \\ \vdots & & & \\ -V_{h_{b_k} - h_1} & -V_{h_{b_k} - h_2} & \cdots & \lambda_{N,b_k}I - V_0 \end{bmatrix} \begin{bmatrix} A(N, h_1) \\ A(N, h_2) \\ \vdots \\ A(N, h_{b_k}) \end{bmatrix} = \begin{bmatrix} O(\rho^{-p\alpha}) \\ O(\rho^{-p\alpha}) \\ \vdots \\ O(\rho^{-p\alpha}), \end{bmatrix}. \tag{27}$$

We may write the system (27) as

$$[\Lambda_N I - C]\mathcal{A}(N, h_1, h_2, \ldots, h_{b_k}) = \mathcal{O}(\rho^{-p\alpha}), \tag{28}$$

where I is an $mb_k \times mb_k$ identity matrix, C is given by (14), $A(N, h_1, h_2, \ldots, h_{b_k})$ is the $mb_k \times 1$ vector

$$\mathcal{A}(N, h_1, h_2, \ldots, h_{b_k}) = (A(N, h_1), A(N, h_2), \ldots, A(N, h_{b_k})) \tag{29}$$

and the right side of the system (28) is the $mb_k \times 1$ vector whose norm is

$$|\mathcal{O}(\rho^{-p\alpha})| = O(\sqrt{b_k}\rho^{-p\alpha}). \tag{30}$$

Theorem 1. *Let* $\mid \gamma \mid^2$ *be a resonance eigenvalue of the operator* $L(0)$*, that is,* $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$*,* $k = 1, 2, \ldots, d-1$ *where* $\mid \gamma \mid\sim \rho$*, and* Λ_N *an eigenvalue* *of the operator $L(V)$ for which* (18) *holds and its corresponding eigenfunction Ψ_N satisfies*

$$|<\Phi_{\gamma,j},\Psi_N>|>c_4\rho^{-c\alpha}. \tag{31}$$

Then there exists an eigenvalue $\eta_s(\gamma)$, $1\le s\le mb_k$ of the matrix C such that

$$\Lambda_N=\eta_s(\gamma)+O(\rho^{-(p-c-\frac{d}{4}3^d)\alpha}).$$

Proof. Since (18) is satisfied, (28) holds. Then multiplying both sides of the equation (28) by $[\Lambda_N I-C]^{-1}$, then taking norm of both sides and by (30), we get

$$|\mathcal{A}(N,h_1,h_2,\dots,h_{b_k})|\le\|[\Lambda_N I-C]^{-1}\|O(\sqrt{b_k}\rho^{-p\alpha}). \tag{32}$$

Using the fact that γ is one of $h_1,h_2,\dots,h_\tau$ (See definition of $B_k(\gamma,p_1)$) and hence by (31) and (32), we obtain

$$c_5\rho^{-c\alpha}<|\mathcal{A}(N,h_1,h_2,\dots,h_{b_k})|\le\|[\Lambda_N I-C]^{-1}\|\sqrt{b_k}c_6\rho^{-p\alpha}.$$

Since $[\Lambda_N I-C]^{-1}$ is symmetric matrix with the eigenvalues $\frac{1}{\Lambda_N-\eta_s(\gamma)}$, $s=1,\dots,mb_k$, we have

$$\max_{s=1,\dots,mb_k}|\Lambda_N-\eta_s(\gamma)|^{-1}=\|[\Lambda_N I-C]^{-1}\|>c_7c_8^{-1}b_k^{-\frac{1}{2}}\rho^{-c\alpha+p\alpha},$$

where $b_k=O(\rho^{\frac{d}{2}3^d\alpha})$, thus

$$\min_{s=1,2,\dots,mb_k}|\Lambda_N-\eta_s(\gamma,\lambda_i)|\le c_9\rho^{-(p-c-\frac{d}{4}3^d)\alpha},$$

and

$$\Lambda_N=\eta_s(\gamma,\lambda_i)+O(\rho^{-(p-c-\frac{d}{4}3^d)\alpha}).$$

□

Theorem 2. *Let $|\gamma|^2$ be a resonance eigenvalue of the operator $L(0)$, that is, $\gamma\in(\bigcap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k}))\setminus E_{k+1}$, $k=1,2,\dots,d-1$ where $|\gamma|\sim\rho$, $\eta_s(\gamma)$ an eigenvalue of the matrix C such that $|\eta_s(\gamma)-|\gamma|^2|<\frac{3}{8}\rho^{\alpha_1}$. Then there is an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N=\eta_s(\gamma)+O(\rho^{-p\alpha+\frac{d}{4}3^d\alpha+\frac{d-1}{2}}). \tag{33}$$

Proof. By the general perturbation theory, there is an eigenvalue Λ_N of the operator $L(V)$ such that $|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{2\alpha_1}$ holds. Thus one can use the system (28) and we prove the theorem for this eigenvalue Λ_N:

Let η_s, $s=1,2,\dots,mb_k$ be an eigenvalue of the matrix C and $\theta_s=(\theta_s^1,\theta_s^2,\dots,\theta_s^{b_k})_{mb_k\times 1}$ the corresponding normalized eigenvector, where $\theta_s^\tau=(\theta_s^{\tau 1},\theta_s^{\tau 2},\dots,\theta_s^{\tau m})_{m\times 1}$, $\tau=1,2,\dots,b_k$. Multiplying the equation (28) by θ_s, since C is symmetric (see (14) and (15)), we get

$$|\Lambda_N-\eta_s||\mathcal{A}(N,h_1,h_2,\dots,h_{b_k})\cdot\theta_s|=|\mathcal{O}(\rho^{-p\alpha})\cdot\theta_s|. \tag{34}$$

By using $b_k = O(\rho^{\frac{d}{2}3^d\alpha})$, (30) and the Cauchy Schwartz Inequality for the right hand side of (34), we have

$$|\Lambda_N - \eta_s||\mathcal{A}(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s| = O(\rho^{-p\alpha + \frac{d}{4}3^d\alpha}). \tag{35}$$

So we need to prove that

$$|\mathcal{A}(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s| > c_{10}\rho^{-\frac{d-1}{2}}, \tag{36}$$

from which the theorem follows.

For this purpose,we first consider the decomposition of the matrix C as $C = A + B$, where

$$A = \begin{bmatrix} |h_1|^2 I & & 0 \\ & \ddots & \\ 0 & & |h_{b_k}|^2 I \end{bmatrix}, \quad B = \begin{bmatrix} V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\ V_{h_2-h_1} & V_0 & \cdots & V_{h_2-h_{b_k}} \\ \vdots & & \ddots & \vdots \\ V_{h_{b_k}-h_1} & V_{h_{b_k}-h_2} & \cdots & V_0 \end{bmatrix}. \tag{37}$$

The eigenvalues and the corresponding eigenspaces of the matrix A are $|h_\tau|^2$ and $E_\tau = span\{e_j : (\tau-1)m+1 \le j \le \tau m\}$, respectively, where

$$\{e_j = (0, \ldots, 0, 1, 0, \ldots, 0)\}_{j=1}^{mb_k}$$

is the standard basis of R^{mb_k}. Now, we use the following notation

$$\theta_s(h_{\tau,j}) \equiv \theta_s \cdot e_j = \theta_s^{\tau j}, \quad \text{if} \quad (\tau-1)m+1 \le j \le \tau m, \tag{38}$$

for $\tau = 1, 2, \cdots, b_k$.

Multiplying $(A+B)\theta_s = \eta_s\theta_s$ by e_j, since A and B are symmetric, we get

$$(\eta_s - |h_\tau|^2)\theta_s(h_{\tau,j}) = \theta_s \cdot Be_j \tag{39}$$

and $(\tau-1)m+1 \le j \le \tau m$, and $\tau = 1, 2, \cdots, b_k$.

On the other hand, if we consider the sum of the elements in the i-th row of the matrix B, by (8)

$$\sum_{\substack{\tau=1 \\ \tau\neq i}}^{b_k} \sum_{j=1}^{m} v_{ijh_i-h_\tau} < \sum_{j=1}^{m} M_{ij}, \tag{40}$$

for all $i = 1, 2, \ldots, m$. Since B is a symmetric matrix and by (40), the sum of elements in each row of B is less then $M = \max_{i=1,2,\ldots,m}\{\sum_{j=1}^{m} M_{ij}\}$, the eigenvalues of B are also less then M from which we have $\| B \| \le M$.

Thus, by (26), (36), (38), we have

$$|\mathcal{A}(N, h_1, \ldots, h_{b_k}) \cdot \theta_s| = |\langle \psi_N, \sum_{\tau=1}^{b_k} \sum_{j=1}^{m} \theta_s(h_{\tau,j})\phi_{h_{\tau,j}} \rangle|, \tag{41}$$

which, together with Parseval's relation, imply

$$\begin{aligned}1 &= \| \sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i})\Phi_{h_\tau,i} \|^2 \\ &= \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &\quad + \sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2. \end{aligned} \tag{42}$$

Now we estimate the first summation in the expression (42):

$$\begin{aligned}&\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &= \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > \\ &+ \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &< 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &+ 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2. \end{aligned} \tag{43}$$

Using Bessel's inequality, Parseval's relation, orthogonality of the functions $\Phi_{h_\tau,i}(x)$, $\tau = 1, 2, \ldots, b_k$, $i = 1, 2, \ldots, m$, the binding formula (39) and $\| B \| \leq M$, we have

$$\begin{aligned}&\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &\leqslant \| \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s(h_{\tau,i})\Phi_{h_\tau,i} \|^2 \\ &= \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}|\theta_s(h_{\tau,i})|^2 \| \Phi_{h_\tau,i} \|^2 \\ &= \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\frac{|\theta_s \cdot Be_i|^2}{|\eta_s-|h_\tau|^2|^2} = O(\rho^{-2\alpha_1}). \end{aligned} \tag{44}$$

The assumption $|\eta_s - |\gamma|^2| < \frac{3}{8}\rho^{\alpha_1}$ of the theorem and $|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}$ imply that $||\gamma|^2 - |h_\tau|^2| < \frac{1}{2}\rho^{\alpha_1}$. So by the well-known formula

$$\frac{1}{\Lambda_N - |h_\tau|^2} = \frac{1}{\Lambda_N - |\gamma|^2}\{\sum_{n=0}^{k}(\frac{|h_\tau|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2})^n + O(\rho^{-(k+1)\alpha_1})\},$$

for $|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}$, and $||\gamma|^2 - |h_\tau|^2| < \frac{1}{2}\rho^{2\alpha_1}$, using (39), we have

$$\begin{aligned}
&\sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\
&\quad = \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) \frac{< \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |h_\tau|^2}|^2 \\
&\quad \leq \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1)|\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |\gamma|^2}|^2 \\
&+ \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1)|\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |\gamma|^2} \frac{|h_\tau|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2}|^2 \\
&\qquad \vdots \\
&+ \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1)|\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |\gamma|^2}[\frac{|h_\tau|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2}]^k|^2 \\
&+ \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1)|\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} > O(\rho^{-(k+1)\alpha_1})|^2.
\end{aligned} \tag{45}$$

To calculate the order of each term in (44), we use Bessel's inequality and the orthogonality of $\Phi_{h_\tau,i}$. So we have

$$\begin{aligned}
& 2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1) \\
& \times \left| \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} > \frac{(|h_\tau|^2 - |\gamma|^2)^r}{(\Lambda_N - |\gamma|^2)^{r+1}} \right|^2 \\
= \quad & 2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \frac{(k+1)}{|\Lambda_N - |\gamma|^2|^{2(r+1)}}
\end{aligned}$$

$$\begin{aligned}
&\times \left| \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} > (|h_\tau|^2-|\gamma|^2)^r \right|^2 \\
\leq\ & c_{11}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \\
&\times \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} \left| < \Psi_N, \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i})(|h_\tau|^2-|\gamma|^2)^r V\Phi_{h_\tau,i} > \right|^2 \\
\leq\ & c_{12}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \left\| \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i})(|h_\tau|^2-|\gamma|^2)^r V\Phi_{h_\tau,i} \right\|^2 \\
\leq\ & c_{13}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \left(\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \left\| \theta_s(h_{\tau,i})(|h_\tau|^2-|\gamma|^2)^r V\Phi_{h_\tau,i} \right\| \right)^2 \\
=\ & c_{14}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \left(\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} |\theta_s(h_{\tau,i})| ||h_\tau|^2-|\gamma|^2|^r \parallel V\Phi_{h_\tau,i} \parallel \right)^2
\end{aligned}$$

$$\leq c_{15}(\rho^{2\alpha_1})^{-2(r+1)}(\frac{1}{2}\rho^{\alpha_1})^{2r}(k+1)(\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \parallel V\Phi_{h_\tau,i} \parallel)^2 = O(\rho^{-2(r+1)\alpha_1}), \tag{46}$$

for $r = 0, 1, 2, \ldots, k$. Now let K be the number of h_τ satisfying $|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}$, then the order of the last summation in (46) is:

$$\begin{aligned}
& \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} (k+1) \\
& \qquad \times \left| \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} > O(\rho^{-(k+1)\alpha_1}) \right|^2 \\
& \leq K \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} (k+1) \\
& \qquad \times \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} |O(\rho^{-(k+1)\alpha_1})|^2 \cdot |\theta_s(h_{\tau,i})|^2 \cdot | < \Psi_N, V\Phi_{h_\tau,i} > |^2 \\
& \leq c_{16} \cdot K \cdot \rho^{-2(k+1)\alpha_1} \cdot \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \parallel V(x)\Phi_{h_\tau,i} \parallel^2 \\
& \leq c_{17} \cdot K^2 \cdot M^2 \cdot \rho^{-2(k+1)\alpha_1} = K^2 \cdot 0(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}),
\end{aligned}$$

since $K = O(\rho^{\frac{d}{2}\alpha_d})$ and we can always choose k in $O(\rho^{-2(k+1)\alpha_1})$ such that

$$K^2 \cdot O(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}), \tag{47}$$

which together with the estimations (44), , (45) and (46) imply

$$O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2.$$

Therefore, from the decomposition (42) we have

$$1 - O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2.$$

Since the number of indexes N satisfying $|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{2\alpha_1}$ is less then ρ^{d-1}, we have

$$1 - O(\rho^{-2\alpha_1}) \leq \rho^{d-1} \max_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{2\alpha_1}} \left\{ |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \right\}$$

which implies together with the relation (41) that

$$|A(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s|^2 \geq \frac{1 - O(\rho^{-2\alpha_1})}{\rho^{d-1}}. \tag{48}$$

It follows from the equation (35) and the estimation (48) that

$$\Lambda_N = \eta_s + \frac{O(\rho^{-p\alpha+\frac{d}{4}3^d\alpha})}{O(\rho^{-\frac{d-1}{2}})},$$

that is, (36) holds. □

3. Asymptotic Formulas for the Eigenvalues in a Single Resonance Domain

Now, we investigate in detail the eigenvalues of $L(V)$ in a single resonance domain. In order the inequalities

$$0 < \alpha < \frac{1}{d+20}, \quad 2\alpha_2 - \alpha_1 + (d+3)\alpha < 1 \tag{49}$$

and

$$\alpha_2 > 2\alpha_1, \tag{50}$$

to be satisfied, we can choose α, α_1 and α_2 as follows

$$\alpha = \frac{1}{d+p}, \;\; \alpha_1 = \frac{p_2}{d+p}, \;\; \alpha_2 = \frac{2p_2+1}{d+p},$$

where $p_2 = [\frac{p-5}{3}] - 1$. Let $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, $\delta \in \frac{\Gamma}{2} \backslash \{e_i\}$, where δ is minimal in its direction. Consider the following sets :

$$\begin{aligned}
&B_1(\delta) = \{b : b = n\delta\,, n \in Z\,,\ |b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}\},\\
&B_1(\gamma) = \gamma + B_1(\delta) = \{\gamma + b : b \in B_1(\delta)\},\\
&B_1(\gamma, p_1) = B_1(\gamma) + \Gamma(p_1\rho^{\alpha}).
\end{aligned}$$

As before, denote by h_τ, $\tau = 1, 2, ..., b_1$ the vectors of $B_1(\gamma, p_1)$, where b_1 is the number of vectors in $B_1(\gamma, p_1)$. Then the matrix $C(\gamma, \delta) = (c_{ij})$, $i, j = 1, 2, \ldots, mb_1$ is defined by

$$C(\gamma,\delta) = \begin{bmatrix} |h_1|^2 I - V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_1}} \\ V_{h_2-h_1} & |h_2|^2 I - V_0 & \cdots & V_{h_2-h_{b_1}} \\ \vdots & & & \\ V_{h_{b_1}-h_1} & V_{h_{b_1}-h_2} & \cdots & |h_{b_1}|^2 I - V_0 \\ & & & \end{bmatrix}, \tag{51}$$

where $V_{h_\tau - h_\xi}$, $\tau, \xi = 1, 2, \ldots, b_1$ are the $m \times m$ matrices defined by (15).
Also we define the matrix $D(\gamma, \delta) = (c_{ij})$ for $i, j = 1, 2, ..., ma_1$, where $h_1, h_2, ..., h_{a_1}$ are the vectors of $B_1(\gamma, p_1) \bigcap \{\gamma + n\delta : n \in Z\}$, and a_1 is the number of vectors in $B_1(\gamma, p_1) \bigcap \{\gamma + n\delta : n \in Z\}$. Clearly $a_1 = O(\rho^{\frac{1}{2}\alpha_2})$.

Lemma 3. *a) If η_{j_s} is an eigenvalue of the matrix $C(\gamma, \delta)$ such that $|\eta_{j_s} - |h_s|^2| < M$ for $s = 1, 2, ..., a_1$, $1 + (s-1)m \leq j_s \leq ms$, then*

$$|\eta_{j_s} - |h_\tau|^2| > \frac{1}{4}\rho^{\alpha_2} \ \ ,\ \ \forall \tau = a_1 + 1, a_1 + 2, ..., b_1.$$

b) If η_{j_s} is an eigenvalue of the matrix $C(\gamma, \delta)$ such that $|\eta_{j_s} - |h_s|^2| < M$ for $s = a_1 + 1, a_1 + 2, ..., b_1$ and $1 + (s-1)m \leq j_s \leq ms$, then

$$|\eta_{j_s} - |h_\tau|^2| > \frac{1}{4}\rho^{\alpha_2} \ \ ,\ \ \forall \tau = 1, 2, ..., a_1.$$

Proof. First we prove

$$||h_\tau|^2 - |h_s|^2| \geq \frac{1}{3}\rho^{\alpha_2}, \qquad \forall s \leq a_1, \qquad \forall \tau > a_1. \tag{52}$$

By definition, if $s \leq a_1$ then $h_s = \gamma + n\delta$, where $|n\delta| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2} + p_1\rho^{\alpha}$. If $\tau > a_1$ then $h_\tau = \gamma + s^{'}\delta + a$, where $|s^{'}\delta| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}$, $a \in \Gamma(p_1\rho^{\alpha}) \setminus \delta R$. Therefore

$$|h_\tau|^2 - |h_s|^2 = 2\gamma \cdot a + 2s^{'}\delta \cdot a + 2s^{'}\gamma \cdot \delta + |s^{'}\delta|^2 + |a|^2 - 2n\gamma \cdot \delta - |n\delta|^2.$$

Since $\gamma \notin V_a(\rho^{\alpha_2})$, $|a| < p_1\rho^{\alpha}$, we have

$$|2\gamma \cdot a| > \rho^{\alpha_2} - c_0\rho^{2\alpha}.$$

The relation $\gamma \in V_\delta(\rho^{\alpha_1})$ and the inequalities for $s^{'}$ and n imply that

$$2s^{'}\gamma \cdot \delta + 2s^{'}\gamma \cdot a + |a|^2 - 2n\gamma \cdot \delta \quad = \quad O(\rho^{\frac{1}{2}\alpha_2 + \alpha_1}),$$

$$||s^{'}\delta|^2 - |n\delta|^2| \quad < \quad \frac{1}{4}\rho^{\alpha_2} + c_0\rho^{\frac{1}{2}\alpha_2+\alpha}.$$

Thus (52) follows from these relations, since $\frac{1}{2}\alpha_2 + \alpha_1 < \alpha_2$ and $\frac{1}{2}\alpha_2 + \alpha < \alpha_2$.

The eigenvalues of $D(\gamma,\delta)$ and $C(\gamma,\delta)$ lay in M-neighborhood of the numbers $|h_k|^2$ for $k = 1,2,...,a_1$ and for $k = 1,2,...,b_1$, respectively. The inequality (52) shows that one can enumerate the eigenvalues η_j $(j = 1,2,\ldots,mb_1)$ of C in the following way:

$$\eta_j \equiv \eta_{j_s}, \quad j_s \le ma_1, \quad 1 + (s-1)m \le j_s \le sm$$

when for $\quad s \le a_1,\ \eta_j$ lay in M-neighborhood of $\ |h_s|^2 \quad$ and

$$\eta_j \equiv \eta_{j_\tau}, \quad j_\tau \ge ma_1, \quad 1 + (\tau-1)m \le j_\tau \le \tau m$$

when for $\tau > a_1$, η_j lay in M-neighborhood $|h_\tau|^2$. Then by (52), we get

$$|\eta_{j_s} - |h_\tau|^2| > \frac{1}{4}\rho^{\alpha_2}, \tag{53}$$

for $s \le a_1$, $\tau > a_1$ and $s > a_1$, $\tau \le a_1$. $\square$

Now, using the notation $h_s = \gamma - (\frac{s}{2})\delta$ if s is even, $h_s = \gamma + (\frac{s-1}{2})\delta$ if s is odd, for $s = 1,2,...,a_1$, (without loss of generality assume that a_1 is even) and using the orthogonal decomposition of $\gamma \in \frac{\Gamma}{2}$, $\gamma = \beta + (l + v(\beta))\delta$, where $\beta \in H_\delta \equiv \{x \in R^d : x\cdot\delta = 0\},\ l \in Z,\ v \in [0,1)$ we can write the matrix $D(\gamma,\delta)$ as

$$D(\gamma,\delta) = |\beta|^2 I + E(\gamma,\delta), \tag{54}$$

where I is a maximal identity matrix and $E(\gamma,\delta)$ is

$$E(\gamma,\delta) = \begin{bmatrix} \left((l+v)^2|\delta|^2\right) I + V_0 & V_\delta & V_{-\delta} & \cdots & V_{\frac{a_1}{2}\delta} \\ V_{-\delta} & \left((l-1+v)^2|\delta|^2\right) I + V_0 & V_{-2\delta} & \cdots & V_{\left(\frac{a_1}{2}-1\right)\delta} \\ V_\delta & V_{2\delta} & \left((l+1+v)^2|\delta|^2\right) I + V_0 & \cdots & V_{\left(\frac{a_1}{2}+1\right)\delta} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{-\frac{a_1}{2}\delta} & . & . & \cdots & \left((l-\frac{a_1}{2}+v)^2|\delta|^2\right) I + V_0 \end{bmatrix}$$

Denote $n_k = -\frac{k}{2}$ if k is even, $n_k = \frac{k-1}{2}$ if k is odd. The system $\{e^{i(n_k+v)t} : k = 1,2,...\}$ is a basis in $L_2^m[0,2\pi]$. Let $T(\gamma,\delta) \equiv T(P(t),\beta)$ be the operator in ℓ_2 corresponding to the Sturm-Liouville operator T, generated by

$$-|\delta|^2 Y''(t) + P(t)Y(t) = \mu Y(t), \tag{55}$$

$$Y(t+2\pi) = e^{i2\pi v(\beta)}Y(t),$$

where $P(t) = (p_{ij}(t)), p_{ij}(t) = \sum\limits_{k=1}^{\infty} v_{ijn_k\delta}e^{in_kt}, v_{ijn_k\delta} = (v_{ij}(x), \frac{1}{|A_{n_k\delta}|}\sum\limits_{\alpha\in A_{n_k\delta}} e^{i(\alpha\cdot x)}),$ $t = x\cdot\delta$. It means that $T(\gamma,\delta)$ is the infinite matrix $(Te^{i(l+n_k+v)t}, e^{i(l+n_m+v)t})$, $k,m = 1,2,\ldots$.

To find the relation between the eigenvalues of $L(V)$ in a single resonance domain and the eigenvalues of the Sturm-Liouville operators defined by (55), we need the following theorems.

Theorem 4. *Let $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$ and $|\gamma| \sim \rho$. Then, for any eigenvalue $\eta_{j_s}(\gamma)$ of the matrix $C(\gamma,\delta)$ satisfying*

$$|\eta_{j_s} - |h_s|^2| < M, \quad 1+(s-1)m \le j_s \le sm, \quad s = 1,2,...,a_1 \tag{56}$$

there exists an eigenvalue $\widetilde{\eta}_{k(j_s)}$ of the matrix $D(\gamma,\delta)$ such that

$$\eta_{j_s} = \widetilde{\eta}_{k(j_s)} + O(\rho^{-\frac{3}{4}\alpha_2}).$$

Proof. Let η_{j_s} be an eigenvalue of the matrix $C(\gamma,\delta)$ satisfying (56) and $\theta_{j_s} = (\theta^1_{j_s}, \theta^2_{j_s}, ..., \theta^{b_1}_{j_s})_{mb_1\times 1}$ be the corresponding normalized eigenvector, $|\theta_{j_s}| = 1$. Now, we consider the decomposition $C = A+B$ and the matrices A, B which are defined in (37). Writing the binding formula (39) for η_{j_s} and using (38), we get

$$(\eta_{j_s} - |h_\tau|^2)\theta_{j_s}(h_{\tau,i}) = \theta_{j_s} \cdot Be_i, \tag{57}$$

$\tau = 1,2,\dots,b_1,\ \ 1+(\tau-1)m \le i \le \tau m.$

For simplicity, we use the following notation in the sequel:

$$e_{\zeta,k} = e_k \quad \text{if} \quad 1+(\zeta-1)m \le k \le \zeta m, \quad \zeta = 1,\dots,b_1,$$

$$Be_i \cdot e_{k_1} = Be_{\tau,i} \cdot e_{\xi,k_1} = b(\tau,i,\xi,k_1).$$

Thus, substituting the orthogonal decomposition

$$Be_i = Be_{\tau,i} = \sum_{\substack{\xi=1,2,\dots,b_1 \\ 1+(m-1)\xi \le k_1 \le m\xi}} b(\tau,i,\xi,k_1) e_{\xi,k_1}$$

into the formula (57), we get

$$\begin{aligned}
(\eta_{j_s} - |h_\tau|^2)\theta_{j_s}(h_{\tau,i}) &= \theta_{j_s} \cdot \sum_{\substack{\xi=1,2,\dots,b_1 \\ 1+(m-1)\xi \le k_1 \le m\xi}} b(\tau,i,\xi,k_1) e_{\xi,k_1} \\
&= \sum_{\substack{\xi=1,2,\dots,b_1 \\ 1+(m-1)\xi \le k_1 \le m\xi}} b(\tau,i,\xi,k_1) \theta_{j_s} \cdot e_{\xi,k_1} \\
&= \sum_{\substack{\xi=1,2,\dots,b_1 \\ 1+(m-1)\xi \le k_1 \le m\xi}} b(\tau,i,\xi,k_1) \theta_{j_s}(h_\xi, k_1).
\end{aligned}$$

It is clear that

$$b(\tau,i,\xi,k_1) = \begin{cases} 0 & \text{if} \quad \xi = \tau, \\ v_{k_1 i h_\xi - h_\tau} & \text{if} \quad \xi \ne \tau, \end{cases}$$

which implies

$$\sum_{\substack{\xi=1,2,\dots,b_1 \\ 1+(m-1)\xi \le k_1 \le m\xi}} b(\tau,i,\xi,k_1) = \sum{}^{\xi=1,2,\dots,b_1}\ v \quad {}_{k_1 i h_\xi - h_\tau}.$$

Thus one has

$$\begin{aligned}(\eta_{j_s}-|h_\tau|^2)\theta_{j_s}(h_\tau,i) &= \sum^{\xi=1,2,\ldots,b_1} v_{k_1ih_\xi-h_\tau}\theta_{j_s}(h_\xi,k_1)\\ &= \sum^{\xi=1,2,\ldots,a_1} v_{k_1ih_\xi-h_\tau}\theta_{j_s}(h_\xi,k_1)\\ &\quad+\sum^{\xi=a_1+1,\ldots,b_1} v_{k_1ih_\xi-h_\tau}\theta_{j_s}(h_\xi,k_1). \end{aligned}\tag{58}$$

Now, writing the equation (58) for all h_τ , $\tau=1,2,...,a_1$, we get the system of linear algebraic equations:

$$\begin{aligned}&(\eta_{j_s}-|h_1|^2)\theta_{j_s}(h_1,i)-\sum^{\xi=1,2,\ldots,a_1} v_{k_1ih_\xi-h_1}\theta_{j_s}(h_\xi,k_1)\\ =\;&\sum^{\xi=a_1+1,\ldots,b_1} v_{k_1ih_\xi-h_1}\theta_{j_s}(h_\xi,k_1)\\[1em] &(\eta_{j_s}-|h_2|^2)\theta_{j_s}(h_2,i)-\sum^{\xi=1,2,\ldots,a_1} v_{k_1ih_\xi-h_2}\theta_{j_s}(h_\xi,k_1)\\ =\;&\sum^{\xi=a_1+1,\ldots,b_1} v_{k_1ih_\xi-h_2}\theta_{j_s}(h_\xi,k_1)\\ &\vdots\\ &(\eta_{j_s}-|h_{a_1}|^2)\theta_{j_s}(h_{a_1},i)-\sum^{\xi=1,2,\ldots,a_1} v_{k_1ih_\xi-h_{a_1}}\theta_{j_s}(h_\xi,k_1)\\ =\;&\sum^{\xi=a_1+1,\ldots,b_1} v_{k_1ih_\xi-h_{a_1}}\theta_{j_s}(h_\xi,k_1)\end{aligned}\tag{59}$$

Using the binding formula (57), the relation (53), and $\| B \| \leq M$, for any $\tau = 1,2,\ldots,a_1$, we find

$$\begin{aligned}\Big|\sum_{\substack{\xi=a_1+1,\ldots,b_1\\ k_1=1,2,\ldots,m\\ \xi\neq\tau}} v_{k_1ih_\xi-h_\tau}\theta_{j_s}(h_\xi,k_1)\Big| &= \Big|\sum_{\substack{\xi=a_1+1,\ldots,b_1\\ k_1=1,2,\ldots,m\\ \xi\neq\tau}} v_{k_1ih_\xi-h_\tau}\frac{\theta_{j_s}\cdot Be_{\xi,k_1}}{(\eta_{j_s}-|h_\xi|^2)}\Big|\\ &\leq \sum_{\substack{\xi=a_1+1,\ldots,b_1\\ k_1=1,2,\ldots,m\\ \xi\neq\tau}} |v_{k_1ih_\xi-h_\tau}|\frac{|\theta_{j_s}|\|B\||e_{\xi,k_1}|}{(\eta_{j_s}-|h_\xi|^2)}\\ &\leq 4\rho^{-\alpha_2}M\sum_{\substack{\xi=a_1+1,\ldots,b_1\\ k_1=1,2,\ldots,m\\ \xi\neq\tau}} |v_{k_1ih_\xi-h_\tau}|\\ &\leq 4\rho^{-\alpha_2}M^2\\ &= O(\rho^{-\alpha_2})\end{aligned}\tag{60}$$

and

$$\begin{aligned}\sum_{\substack{\tau=a_1+1,\ldots,b_1\\ i=1,2,\ldots,m}}|\theta_{j_s}(h_\tau,i)|^2 &= \sum_{\substack{\tau=a_1+1,\ldots,b_1\\ i=1,2,\ldots,m}}\left|\frac{\theta_{j_s}\cdot Be_{\tau,i}}{(\eta_{j_s}-|h_\tau|^2)}\right|^2\\ &= \sum_{\substack{\tau=a_1+1,\ldots,b_1\\ i=1,2,\ldots,m}}\frac{|B\theta_{j_s}\cdot e_{\tau,i}|^2}{(\eta_{j_s}-|h_\tau|^2)^2}\\ &\leq 16M^2\rho^{-2\alpha_2}\\ &= O(\rho^{-2\alpha_2}). \qquad (61)\end{aligned}$$

By (60) and (54), (59) becomes

$$[\theta^1_{j_s},\theta^2_{j_s},\ldots,\theta^{a_1}_{j_s}]^t=(D(\gamma,\delta)-\eta_{j_s}I)^{-1}[O(\rho^{-\alpha_2}),O(\rho^{-\alpha_2}),...,O(\rho^{-\alpha_2})]^t. \qquad (62)$$

By the Parseval's identity and (61), we get

$$\begin{aligned}\sum_{\substack{\tau=1,2,\ldots,a_1\\ i=1,2,\ldots,m}}|\theta_{j_s}(h_\tau,i)|^2 &= \sum_{\substack{\tau=1,2,\ldots,b_1\\ i=1,2,\ldots,m}}|\theta_{j_s}(h_\tau,i)|^2-\sum_{\substack{\tau=a_1+1,\ldots,b_1\\ i=1,2,\ldots,m}}|\theta_{j_s}(h_\tau,i)|^2\\ &\geq 1-O(\rho^{-2\alpha_2}).\end{aligned}$$

Now, taking norm of both sides in (62) and using the above inequality we have

$$\sqrt{1-O(\rho^{-2\alpha_2})}<(\sum_{\substack{\tau=1,2,\ldots,a_1\\ i=1,2,\ldots,m}}|\theta_{j_s}(h_\tau,i)|^2)^{\frac{1}{2}}\leq\|(D(\gamma,\delta)-\eta_{j_s}I)^{-1}\|O(\sqrt{a_1}\rho^{-\alpha_2}).$$

Thus

$$max|\eta_{j_s}-\widetilde{\eta}_{k(j_s)}|^{-1}>\frac{\sqrt{1-O(\rho^{-2\alpha_2})}}{\sqrt{a_1}\rho^{-\alpha_2}},$$

or

$$min|\eta_{j_s}-\widetilde{\eta}_{k(j_s)}|=O(\sqrt{a_1}\rho^{-\alpha_2})=O(\rho^{-\frac{3}{4}\alpha_2}),$$

where the maximum (minimum) is taken over all $\widetilde{\eta}_{k(j_s)}$, $s=1,2,...,a_1$. So the result follows. □

Theorem 5. *For any eigenvalue $\widetilde{\eta}_\tau$ of the matrix $D(\gamma,\delta)$, there exists an eigenvalue $\eta_{j_s(\tau)}$ of the matrix $C(\gamma,\delta)$ such that*

$$\eta_{j_s(\tau)}=\widetilde{\eta}_\tau+O(\rho^{-\frac{1}{2}\alpha_2})$$

Proof. Define the matrix $D' = D'(\gamma, \delta)$ by

$$D' = \begin{bmatrix} |h_1|^2 I - V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{a_1}} & 0 & 0 & \cdots & 0 \\ V_{h_2-h_1} & |h_2|^2 I - V_0 & \cdots & V_{h_2-h_{a_1}} & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ V_{h_{a_1}-h_1} & V_{h_{a_1}-h_2} & \cdots & |h_{a_1}|^2 I - V_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & |h_{a_1+1}|^2 I & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & |h_{b_1-1}|^2 I & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & |h_{b_1}|^2 I \end{bmatrix} \quad (63)$$

So that the spectrum of the matrix D' is

$$\begin{aligned} spec(D') &= spec(D(\gamma,\delta)) \bigcup \{|h_{a_1+1}|^2, |h_{a_1+2}|^2, ..., |h_{b_1}|^2\} \\ &\equiv \{\widetilde{\eta}_1, \widetilde{\eta}_2, ..., \widetilde{\eta}_{ma_1}, |h_{a_1+1}|^2, |h_{a_1+2}|^2, ..., |h_{b_1}|^2\}. \end{aligned}$$

Let us denote by $\Upsilon_\tau = (\Upsilon_\tau^1, \Upsilon_\tau^2, ..., \Upsilon_\tau^{a_1}, 0, ..., 0)_{mb_1\times 1}$, $\Upsilon_\tau^i = (\Upsilon_\tau^{i1}, \Upsilon_\tau^{i2}, \ldots, \Upsilon_\tau^{im})_{m\times 1}$ the normalized eigenvector corresponding to the τ-th eigenvalue of the matrix D', for $\tau = 1, 2, ..., ma_1$ and by $\{e_{k,i}\}_{i=1,2,\cdots,m}$ the eigenvector corresponding to the k-th eigenvalue $|h_k|^2$ of D', for $k = a_1+1, a_1+2, ..., b_1$.

Now, using (62) from the previous theorem, we have

$$\begin{aligned} & (D' - \eta_{j_s} I)[\theta_{j_s}^1, \theta_{j_s}^2, \ldots, \theta_{j_s}^{b_1}]^t \\ = \; & [(D(\gamma,\delta) - \eta_{j_s} I)[\theta_{j_s}^1, \theta_{j_s}^2, \ldots, \theta_{j_s}^{a_1}]^t, (|h_{a_1+1}|^2 - \eta_{j_s})\theta_{j_s}^{a_1+1}, ..., (|h_{b_1}|^2 - \eta_{j_s})\theta_{j_s}^{b_1}] \\ = \; & [O(\rho^{-\alpha_2}), ..., O(\rho^{-\alpha_2}), (|h_{a_1+1}|^2 - \eta_{j_s})\theta_{j_s}^{a_1+1}, ..., (|h_{b_1}|^2 - \eta_{j_s})\theta_{j_s}^{b_1}]. \end{aligned}$$

Taking inner product of both sides of the last equality by Υ_τ for $\tau = 1, 2, ..., ma_1$, using that D' is symmetric and $D'\Upsilon_\tau = \widetilde{\eta}_\tau \Upsilon_\tau$ we have

$$(\eta_{j_s(\tau)} - \widetilde{\eta}_\tau) \sum_{k=1}^{a_1} \theta_{j_s}^k \cdot \Upsilon_\tau^k = \sum_{k=1}^{a_1} O(\rho^{-\alpha_2}) \Upsilon_\tau^k, \quad (64)$$

For the right hand side of the equation (64) using the Cauchy-Schwarz inequality, we get

$$|\sum_{k=1}^{a_1} O(\rho^{-\alpha_2})\Upsilon_\tau^k| \le \sqrt{\sum_{k=1}^{a_1} O(\rho^{-\alpha_2})^2} \sqrt{\sum_{k=1}^{a_1} |\Upsilon_\tau^k|^2} \le \sqrt{a_1(\rho^{-\alpha_2})^2} = O(\sqrt{a_1}\rho^{-\alpha_2}),$$

where $a_1 = O(\rho^{\frac{1}{2}\alpha_2})$. Thus, the equation (64) can be written as

$$(\eta_{j_s(\tau)} - \widetilde{\eta}_\tau) \sum_{k=1}^{a_1} \theta_{j_s}^k \cdot \Upsilon_\tau^k = O(\rho^{-\frac{3}{4}\alpha_2}). \quad (65)$$

In order to get the result, we need to show that for any $\tau = 1, 2, ..., ma_1$ there exists $\theta_{j_s(\tau)}$ such that

$$|\sum_{k=1}^{a_1} \theta^k_{j_s(\tau)} \cdot \Upsilon^k_\tau| = |\theta_{j_s(\tau)} \cdot \Upsilon_\tau| > \sqrt{\frac{1 - O(\rho^{-\frac{3}{2}\alpha_2})}{ma_1}} > c_{18}\rho^{-\frac{1}{4}\alpha_2}. \tag{66}$$

For this, we consider the orthogonal decomposition $\Upsilon_\tau = \sum_{s=1}^{mb_1} (\Upsilon_\tau \cdot \theta_{j_s})\theta_{j_s}$ and the Parseval's identity

$$1 = \sum_{s=1}^{mb_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2 = \sum_{s=1}^{ma_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2 + \sum_{s=ma_1+1}^{mb_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2.$$

First, let us show that

$$\sum_{s=ma_1+1}^{mb_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2 = O(\rho^{-\frac{3}{2}\alpha_2}). \tag{67}$$

Using the decomposition $\Upsilon_\tau = \sum_{\substack{k=1,2...,a_1 \\ i=1,2,...,m}} (\Upsilon_\tau \cdot e_{k,i})e_{k,i}$, the binding formula (57) for $C(\gamma, \delta)$ and A, the relation (53), and the Bessel's inequality we obtain the estimation

$$\begin{aligned}
& \sum_{s=ma_1+1}^{mb_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2 \\
= & \sum_{s=ma_1+1}^{mb_1} |(\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} \Upsilon^{ki}_\tau e_{k,i}) \cdot \theta_{j_s}|^2 \\
= & \sum_{s=ma_1+1}^{mb_1} |\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} \Upsilon^{ki}_\tau (e_{k,i} \cdot \theta_{j_s})|^2 = \sum_{s=ma_1+1}^{mb_1} |\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} \Upsilon^{ki}_\tau \frac{\theta_{j_s} \cdot Be_{k,i}}{(\eta_{j_s} - |h_k|^2)}|^2 \\
\leq & \ 16 \sum_{s=ma_1+1}^{mb_1} \rho^{-2\alpha_2} (\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} |\Upsilon^{ki}_\tau||\theta_{j_s} \cdot Be_{k,i}|)^2 \\
\leq & \sum_{s=ma_1+1}^{mb_1} 16|a_1|m\rho^{-2\alpha_2} \left(\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} |\Upsilon^{ki}_\tau|^2 |\theta_{j_s} \cdot Be_{k,i}|^2 \right) \\
\leq & \ 16\rho^{-2\alpha_2}|a_1|m \sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} |\Upsilon^{ki}_\tau|^2 \sum_{s=ma_1+1}^{mb_1} |\theta_{j_s} Be_{k,i}|^2
\end{aligned}$$

$$\begin{aligned}&\leq \quad 16\rho^{-2\alpha_2}|a_1|m\sum_{\substack{k=1,\dots,a_1\\ i=1,2,\dots,m}}|\Upsilon_\tau^{ki}|^2|Be_{k,i}|^2\leq 16\rho^{-2\alpha_2}|a_1|mM^2\sum_{\substack{k=1,2,\dots,a_1\\ i=1,2,\dots,m}}|\Upsilon_\tau^{ki}|^2\\ &\leq \quad 16|a_1|m\rho^{-2\alpha_2}M^2=O(\rho^{-\frac{3}{2}\alpha_2}).\end{aligned}$$

Therefore one has

$$\sum_{s=1}^{ma_1}|\Upsilon_\tau\cdot\theta_{j_s}|^2=1-O(\rho^{-\frac{3}{2}\alpha_2})$$

from which it follows that there exists an eigenvector $\theta_{j_s(\tau)}$ such that (66) holds. Dividing both sides of (65) by (66) we get the result

$$\eta_{j_s(\tau)}=\widetilde{\eta_\tau}+O(\rho^{-\frac{1}{2}\alpha_2}).$$

□

Theorem 6. *For every eigenvalue ς_s of the Sturm-Liouville operator $T(\gamma,\delta)$, there exists an eigenvalue $\widetilde{\varsigma_s}$ of the matrix $E(\gamma,\delta)$ such that*

$$\varsigma_s=\widetilde{\varsigma_s}+O(\rho^{-\frac{3}{4}\alpha_2}).$$

Proof. Decompose the infinite matrix $T(\gamma,\delta)$ as $T(\gamma,\delta)=\widetilde{A}+\widetilde{B}$ where the matrix $\widetilde{A}$ is defined by

$$\widetilde{A}=\begin{bmatrix}\left((l+v)^2|\delta|^2\right)I+V_0 & & & 0\\ & \left((l-1+v)^2|\delta|^2\right)I+V_0 & & \\ & & \ddots & \\ 0 & & & \left((l-\frac{a_1}{2}+v)^2|\delta|^2\right)I+V_0\\ & & & \end{bmatrix}\tag{68}$$

and $\widetilde{B}=T(\gamma,\delta)-\widetilde{A}$. Let ς_s be an eigenvalue of $T(\gamma,\delta)$, and $\Theta_s=(\Theta_s^1,\Theta_s^2,\Theta_s^3,\ldots)$, $\Theta_s^\tau=(\Theta_s^{\tau1},\ldots,\Theta_s^{\tau m})$ be the corresponding normalized eigenvector, that is, $T\Theta_s=\varsigma_s\Theta_s$. $span\{e_i:(\tau-1)m+1\leq i\leq\tau m\}$ is the eigenspace of the matrix $\tilde{A}$ which corresponds to the eigenvalue $|(\tau'+v)\delta|^2$, where $\tau'=l-\frac{\tau}{2}$ if τ is even, $\tau'=l+\frac{\tau-1}{2}$ if τ is odd, for $\tau=1,2,\ldots$ and $\{e_i\}$ is the standard basis for l_2.
One can easily verify that

$$\left(\varsigma_s-|(\tau'+v)\delta|^2\right)\Theta_s^\tau=\Theta_s\cdot\widetilde{B}e_{\tau,i},\tag{69}$$

where $e_{\tau,i}\equiv e_i$, if $(m-1)\tau+1\leq i\leq m\tau$.

Using the orthogonal decomposition $\widetilde{B}e_{\tau,i}=\sum\limits_{j=1}^{m}\sum\limits_{k=1}^{\infty}(\widetilde{B}e_{\tau,i}\cdot e_{k,j})e_{k,j}$, (69) reduces to

$$\left(\varsigma_s-|(\tau'+v)\delta|^2-|v_{ii0}|^2\right)\Theta_s^{\tau i}=\sum_{j=1}^{m}\sum_{k=1}^{\infty}(\widetilde{B}e_{\tau,i}\cdot e_{k,j})\Theta_s^{kj}$$

and since $\widetilde{B}e_{\tau,i}\cdot e_{k,j}=v_{ji(n_k-n_\tau)\delta}$ for $k\neq\tau$,

$$(\varsigma_s-(\tau'+v)\delta^2)\Theta_s^{\tau i}-\sum_{j=1}^{m}\sum_{k=1}^{a_1}v_{ji(n_k-n_\tau)\delta}\Theta_s^{kj}=\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}v_{ji(n_k-n_\tau)\delta}\Theta_s^{kj}. \quad (70)$$

Now take any eigenvalue ς_s of $T(\gamma,\delta)$, satisfying $\left|\varsigma_s-|(i'+v)\delta|^2\right|<sup|P(t)|$ for $s=1,2,...,\frac{ma_1}{2}$, where $i'=l-\frac{s}{2}$ if s is even, $i'=l+\frac{s-1}{2}$ if s is odd. The relations $\gamma\in V_\delta(\rho^{\alpha_1})$ $(\delta\neq e_i)$ and $\gamma=\beta+(l+v)\delta$, $\beta\cdot\delta=0$ imply

$$|2\gamma\cdot\delta+|\delta|^2|=|(l+v)|\delta|^2+|\delta|^2|<\rho^{\alpha_1},\qquad |l|<c_{19}\rho^{\alpha_1}.$$

Therefore, using the definition of i' and τ', we have

$$|(i'+v)\delta|<\frac{|a_1\delta|}{4}+c_{20}\rho^{\alpha_1}$$

for $s=1,2,...\frac{a_1}{2}$ and

$$|(\tau'+v)\delta|>\frac{|a_1\delta|}{2}-c_{21}\rho^{\alpha_1}$$

for $\tau>a_1$. Since $|a_1|>c_{22}\rho^{\frac{\alpha_2}{2}}$ and $\alpha_2>2\alpha_1$, we have

$$\left||(i'+v)\delta|^2-|(\tau'+v)\delta|^2\right|>c_{23}\rho^{\alpha_2} \quad (71)$$

for $s\leq\frac{a_1}{2}$, $\tau>a_1$, which implies

$$\left|\varsigma_s-|(\tau'+v)|\,\delta^2\right|=||\varsigma_s-|\,|(i'+v)\delta|^2\,|-|\,|(\tau'+v)|\,\delta^2|-|\,|(i'+v)\delta|^2\,||>c_{24}\rho^{\alpha_2}, \quad (72)$$

for $s=1,2,...\frac{a_1}{2}$, $\tau>a_1$.

Since $\widetilde{B}$ corresponds to the operator $P:Y\to P(t)Y$ in $L_2^m[0,2\pi]$, which has norm $sup|P(t)|\leq M$. Using this, equation (69) and (72), we have for the right hand side of (70) that

$$|\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}v_{ij(n_k-n_\tau)\delta}\Theta_s^{kj}|\leqslant\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}|v_{ij(n_k-n_\tau)\delta}|\left|\frac{\Theta_s\cdot\widetilde{B}e_{kj}}{\varsigma_s-|(k'+v)\delta|^2}\right|$$
$$\leq\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}|v_{ij(n_k-n_\tau)\delta}|\frac{\|\Theta_s\|\|\widetilde{B}\|\|e_{kj}\|}{|\varsigma_s-|(k'+v)\delta|^2|}\leq M\rho^{-\alpha_2}\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}|v_{ij(n_k-n_\tau)\delta}|$$
$$\leq c_{25}\rho^{-\alpha_2}, \quad (73)$$

Therefore writing the equation (70) for all $\tau=1,2,...,a_1$, and using (73) we get the following system

$$(E(\gamma,\delta)-\varsigma_sI)[\Theta_s^1,\Theta_s^2,...,\Theta_s^{a_1}]=[O(\rho^{-\alpha_2}),O(\rho^{-\alpha_2}),...,O(\rho^{-\alpha_2})], \quad (74)$$

where I is an $ma_1 \times ma_1$ identity matrix. Using $\Theta_s = \sum_{\tau=1}^{\infty} \Theta_s^{\tau} e_{\tau,i}$, the formula (69) and the inequality (72), we have

$$\sum_{\tau=a_1+1}^{\infty} |\Theta_s^{\tau}|^2 = \sum_{\tau=a_1+1}^{\infty} |\frac{\Theta_s \cdot \widetilde{B} e_{\tau,i}}{\varsigma_s - |(\tau' + v)\delta|^2}|^2 = O(\rho^{-2\alpha_2})$$

and thus

$$\sum_{\tau=1}^{a_1} |\Theta_s^{\tau}|^2 = 1 - O(\rho^{-2\alpha_2}). \tag{75}$$

Multiplying both sides of (74) by $(E(\gamma,\delta) - \varsigma_s I)^{-1}$,

$$[\Theta_s^1, \Theta_s^2, ..., \Theta_s^{a_1}] = (E(\gamma,\delta) - \varsigma_s I)^{-1} [O(\rho^{-\alpha_2}), ..., O(\rho^{-\alpha_2})],$$

then taking norm of both sides and using (75), we get

$$\sqrt{\frac{1 - O(\rho^{-2\alpha_2})}{m}} = \|(E(\gamma,\delta) - \varsigma_s I)^{-1}\| O(\sqrt{a_1}\rho^{-\alpha_2})$$

or

$$\min_{\tau} |\varsigma_s - \widetilde{\varsigma}_{\tau}| = \frac{O(\sqrt{a_1}\rho^{-\alpha_2}) \cdot \sqrt{m}}{\sqrt{1 - O(\rho^{-2\alpha_2})}} = O(\rho^{-\frac{3}{4}\alpha_2}),$$

where the minimum is taken over all eigenvalues $\widetilde{\varsigma}_{\tau}$ of the matrix $E(\gamma,\delta)$. Thus, the result follows. $\square$

Theorem 7. *(Main result) For every $\beta \in H_\delta$, $|\beta| \sim \rho$ and for every eigenvalue $\varsigma_s(v(\beta))$ of the Sturm-Liouville operator $T(\gamma,\delta)$, there is an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = |\beta|^2 + \varsigma_s + O(\rho^{-\frac{1}{2}\alpha_2}).$$

Proof. From Theorem 6 and the definition of $E(\gamma,\delta)$, there exists an eigenvalue $\widetilde{\eta}_{\tau(s)}$ of the matrix $D(\gamma,\delta)$, where γ has a decomposition $\gamma = \beta + (\tau + v(\beta))\delta$, satisfying $\widetilde{\eta}_{\tau(s)} = |\beta|^2 + \varsigma_s + O(\rho^{-\frac{3}{4}\alpha_2})$. Therefore, the result follows from Theorem 5 and Theorem 2. $\square$

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THE RELATIVELY OSCULATING DEVELOPABLE SURFACES OF A SURFACE ALONG A DIRECTION CURVE

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Abstract. We construct a developable surface tangent to a surface along a curve on the surface. We call this surface as relatively osculating developable surface. We choose the curve as the tangent normal direction curve on which the new surface is formed in the Euclidean 3-space. We obtain some results about the existence and uniqueness, and the singularities of relatively osculating developable surfaces. We also give two invariants of curves on a surface which determine these singularities. We present two results for special curves such as asymptotic line and line of curvature which are rulings of the relatively osculating surface.

1. Introduction

One-parameter family of straight lines forms a surface called ruled surface in Euclidean space. It has been an interesting subject that is studied from the end of the 19th century until today. Applications of ruled surfaces have been extensively performed to computer-aided geometric design (CAGD), design of surfaces, technology of manufacture, simulation and rigid bodies [10],[11],[14].

Developable surfaces as a special kind of ruled surfaces are generally characterized by Gaussian curvature, that is, if Gaussian curvature of ruled surfaces becomes vanishing, ruled surfaces can be mapped onto the plane surfaces without distortion of curves: any curve from such a surface drawn onto the flat plane remains the same. Although all developable surfaces are ruled ones, but all ruled surfaces are not developable [11], [17]. Developable surfaces as a kind of ruled surfaces are classified into cylinders, cones or tangent surfaces of space curves [1], [3], [13], [14], [18].

As well known, the inner metric of a surface determines the Gaussian curvature, therefore all the lengths and angles on the surface remain invariant under bending. This feature is what makes developable surfaces important in manufacturing.

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Hence both ruled surfaces and developable surfaces have been paid attention in engineering, architecture, and design, etc. [15], [16].

Based upon a curve in a surface in Euclidean 3-space, a surface has been constructed to be a developable surface tangent to the surface along the curve. This geometric object has been said to be an osculating developable surface along the curve [8]. It has been known that an osculating developable surface is a ruled surface whose rulings are directed by the osculating Darboux vector field along the curve [8].

Singularities of ruled surfaces were studied in the Euclidean 3-space $\mathbb{R}^3$ by Izumiya and Takeuchi [4]. Izumiya and Takeuchi, in their survey of ruled surfaces [5], presented original results about curves in ruled surfaces in the Euclidean 3-space. They studied curves on ruled surfaces by choosing curves as cylindirical helices and Bertrand curves [6]. In their another paper [7], the notions of helices generalized to slant helices and conical geodesic curves were defined in $\mathbb{R}^3$. Also the tangential Darboux developable of a space curve was constructed and its singularities were examined. Interesting results about a geometric invariant of space curve which is closely regarded to singularities of the tangential Darboux developable of the original curve given by Izumiya et al. [7].

The motivation of this study is based on the works of Izumiya and Otani [8], and Hananoi and Izumiya [9]. In [8], the authors constructed osculating developable surface along the curve in the surface by taking a developable surface tangent to a surface forward a curve in the surface into consideration. Then they gave some results such as the uniqueness and the singularities of such a surface.

In [9], Hananoi and Izumiya studied a developable surface which remains normal to a surface along a curve on ruled surface. They had results such as the uniqueness and the singularities of relatively osculating developable surfaces. Recently, Markina and Raffaelli examined the same topic in $\mathbb{R}^{m+1}$. Taking a smooth curve γ in an $m-$dimensional surface M in $\mathbb{R}^{m+1}$, they gave some results about the existence and uniqueness of a flat surface H having the same field of normal vectors as M along γ [12].

The paper is organized as follows: the next two sections present some preliminaries, and introductory relevant notation and terminology. In Sec. 3, new developable surfaces which remain tangent to the base surface are constructed along a tangent normal direction curve and some results such as invariants of M_o characterizing contour generators of M are given. The existence and uniqueness of the surface have been presented for these surfaces. We give two results for special curves such as asymptotic line and line of curvature which are rulings of the relatively osculating surface. Finally, illustrative examples have been given for the base surface and its osculating developable surface.

2. Preliminaries

Some notions, formulas and conclusions for space curves, and ruled surfaces in Euclidean 3-space $\mathbb{R}^3$ are presented in this section, so these basic information are available in the textbooks on differential geometry (See for instance Refs. [5], [7], [14]).

Let M be a regular surface in $\mathbb{R}^3$ and let $\boldsymbol{\alpha} : I \subseteq \mathbb{R} \to M$ be a unit speed curve. At each point on $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, consider the following three vectors: the unit normal vector $\mathbf{e}_3(s)$ to the surface, the unit tangent vector $\mathbf{e}_1 = \mathbf{e}_1(s)$ to the curve and the tangent normal vector $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$. The vector $\mathbf{e}_2$ is tangent to the surface M , but normal to the curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. Then we have an orthonormal frame $\{\mathbf{e}_1(s),\ \mathbf{e}_2(s),\ \mathbf{e}_3(s)\}$ along $\boldsymbol{\alpha}$, which is called the Darboux frame along $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. Darboux equations for this frame are given by:

$$\begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix},$$

or equivaently

$$\mathbf{e}_1' = \boldsymbol{\Omega}_n \times \mathbf{e}_1, \quad \mathbf{e}_2' = \boldsymbol{\Omega}_o \times \mathbf{e}_2, \quad \mathbf{e}_1' = \boldsymbol{\Omega}_r \times \mathbf{e}_3, \tag{1}$$

where

$$\boldsymbol{\Omega}_n = -\kappa_n \mathbf{e}_2 + \kappa_g \mathbf{e}_3, \quad \boldsymbol{\Omega}_r = \tau_g \mathbf{e}_1 + \kappa_g \mathbf{e}_3, \quad \boldsymbol{\Omega}_o = \tau_g \mathbf{e}_1 - \kappa_n \mathbf{e}_2, \tag{2}$$

are said to be the normal, the rectifying, and the osculating Darboux vector fields along $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, respectively [5]. The functions $\kappa_g(s), \kappa_n(s),\ \tau_g(s)$ are entitled as geodesic curvature, normal curvature, and geodesic torsion of $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, respectively [13]. In terms of these quantities, the geodesics, asymptotic lines, and line of curvatures on a smooth surface can be determined, as loci along which $\kappa_g(s) = 0$, $\kappa_n = 0$, and $\tau_g(s) = 0$, respectively. The definitions of the spherical images of each Darboux vector fields are as follows:

$$\left.\begin{aligned} \mathbf{e}_n(\mathbf{s}) &= \frac{\boldsymbol{\Omega}_n}{\|\boldsymbol{\Omega}_n\|} = \frac{-\kappa_n \mathbf{e}_2 + \kappa_g \mathbf{e}_3}{\sqrt{\kappa_g^2 + \kappa_n^2}}, \text{ if } (\kappa_n, \kappa_g) \neq (0,0), \\ \mathbf{e}_r(\mathbf{s}) &= \frac{\boldsymbol{\Omega}_r}{\|\boldsymbol{\Omega}_r\|} = \frac{\tau_g \mathbf{e}_1 + \kappa_g \mathbf{e}_3}{\sqrt{\tau_g^2 + k_g^2}}, \text{ if } (\tau_g, \kappa_g) \neq (0,0), \\ \mathbf{e}_o(\mathbf{s}) &= \frac{\boldsymbol{\Omega}_o}{\|\boldsymbol{\Omega}_o\|} = \frac{\tau_g \mathbf{e}_1 - \kappa_n \mathbf{e}_2}{\sqrt{\tau_g^2 + k_n^2}}, \text{ if } (\tau_g, \kappa_n) \neq (0,0). \end{aligned}\right\} \tag{3}$$

On the other hand, it is known that

$$\kappa(s) = \sqrt{\kappa_g^2 + \kappa_n^2}, \quad \text{and} \quad \tau_g(s) = \frac{\kappa_n \kappa_g' - \kappa_g \kappa_n'}{\kappa_n^2 + \kappa_g^2} + \tau(s), \tag{4}$$

where $\kappa(s)$, and $\tau(s)$ are the curvature and the torsion of $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ as a space curve, respectively [13]. From now on, we shall often not write the parameter s explicitly in our formulae.

2.1. **Ruled and developable surfaces.** A ruled surface in Euclidean 3-space $\mathbb{R}^3$ is a differentiable one-parameter set of straight lines. Such a surface has a parameterization of the form

$$\mathbf{P}(s,v) = \boldsymbol{\alpha}(s) + v\mathbf{e}(s), \;\; v \in \mathbb{R}, \tag{5}$$

where $\boldsymbol{\alpha}(s)$ is the base curve and $\mathbf{e}(s)$ is the unit vector giving the direction of the straight lines of the surface. The unit normal vector of the ruled surface $\mathbf{P}(s,v)$ at each point is defined by

$$\mathbf{n}(s,v) = \frac{\mathbf{P}_s \times \mathbf{P}_v}{\|\mathbf{P}_s \times \mathbf{P}_v\|} = \frac{\boldsymbol{\alpha}' \times \mathbf{e} + v\mathbf{e}' \times \mathbf{e}}{\|\boldsymbol{\alpha}' \times \mathbf{e} + v\mathbf{e}' \times \mathbf{e}\|}. \tag{6}$$

The base curve is not unique, since any curve of the form:

$$\mathbf{C}(s) = \boldsymbol{\alpha}(s) - \eta(s)\mathbf{e}(s) \tag{7}$$

may be used as its base curve, $\eta(s)$ is a smooth function. If there is a common perpendicular vector to two neighboring rulings on $\mathbf{P}(s,v)$, then the foot of the common perpendicular on the main ruling is said to be a central point. The locus of the central points is said to be the striction curve. In Eq. (7) if

$$\eta(s) = \frac{\langle \boldsymbol{\alpha}'(s), \mathbf{e}' \rangle}{\|\mathbf{e}'\|^2}, \tag{8}$$

then $\mathbf{C}(s)$ is named as the striction curve on the ruled surface and it is unique. In the case $\eta = 0$, the base curve is the striction curve. The distribution parameter of $\mathbf{P}(s,v)$ is defined by

$$\lambda(s) = \frac{\det(\boldsymbol{\alpha}', \mathbf{e}, \mathbf{e}')}{\|\mathbf{e}'\|^2}. \tag{9}$$

The parameter of distribution is a real integral invariant of a ruled surface and allows further classification of the ruled surface.

Developable surfaces are briefly introduced as special types of ruled surfaces. If the ruled surface $\mathbf{P}(s,v)$ is a developable one, then we have

$$\lambda(s) = 0 \Leftrightarrow \det(\boldsymbol{\alpha}', \mathbf{e}, \mathbf{e}') = 0. \tag{10}$$

Thus a volume formed by $\boldsymbol{\alpha}', \mathbf{e}$ and $\mathbf{e}'$ is vanishing, i.e, they are linearly dependent. This condition is satisfied provided that there are three non-identically vanishing functions $\eta(s)$, $\xi(s)$ and $\gamma(s)$ satisfying

$$\mu(s)\boldsymbol{\alpha}' + \beta(s)\mathbf{e} + \gamma(s)\mathbf{e}' = \mathbf{0}. \tag{11}$$

We has to analyze the following cases:
Case 1: $\mu = 0$

Since $\langle \mathbf{e}, \mathbf{e}' \rangle = 0$, it follows immediately that Eq. (11) is only satisfied when $\mathbf{e}$ is a constant vector, i.e., $\mathbf{P}(s, v)$ is a part of a cylinder.
Case 2: $\mu \neq 0$ from Eq. (11) it follows:

$$\boldsymbol{\alpha}' = \zeta(s)\mathbf{e} + \upsilon(s)\mathbf{e}', \tag{12}$$

where

$$\zeta(s) = -\frac{\beta}{\mu},\ \upsilon(s) = -\frac{\gamma}{\mu}.$$

Differentiating Eq. (7) and using Eq. (12), we get

$$\mathbf{C}'(s) = (\zeta(s) - \eta'(s))\,\mathbf{e}(s) + (\upsilon(s) - \eta(s))\,\mathbf{e}'. \tag{13}$$

The situation for $\mathbf{C}$ to be striction curve becomes equivalent to that the vectors $\mathbf{C}'$ and $\mathbf{e}'$ are perpendicular to each other. Therefore, we conclude that the ruling becomes parallel to the first differentiation of the striction curve which is also the tangent of the striction curve, i.e.

$$\mathbf{C}' = (\zeta(s) - \eta'(s))\mathbf{e}(s). \tag{14}$$

Thus we have to consider the following sub-case: $\zeta(s) = \eta'(s)$. In this case Eq. (14) yields to that $\mathbf{C} = \mathbf{C}_0$ is a constant vector. So, $\mathbf{P}(s, v)$ becomes a part of a cone as follows:

$$\mathbf{P}(s, v) = \mathbf{C}_0 + (\eta(s) + v)\mathbf{e}(s),\ v \in \mathbb{R}. \tag{15}$$

We now define the concept "contour generators". Let M be an orientable surface and $\mathbf{n}$ a unit normal vector field on M. For a unit vector $\mathbf{x}$ in the unit sphere $\mathbb{S}^2 = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1 \right\}$, the normal contour generator of the orthogonal projection with the direction $\mathbf{x}$ is defined to be

$$\{\mathbf{p} \in M \mid < \mathbf{n}, \mathbf{x} >= 0\}. \tag{16}$$

Moreover, for a fixed point $\mathbf{c} \in \mathbb{R}^3$, the normal contour generator of the central projection with the center $\mathbf{c}$ is defined to be

$$\{\mathbf{p} \in M \mid < \mathbf{n}, \mathbf{p} - \mathbf{c} >= 0\}. \tag{17}$$

3. The relatively osculating developable surfaces

In this section, we present a relatively osculating developable surface along the $\mathbf{e}_2(s)$-direction curve

$$\boldsymbol{\beta}(s) = \int_0^s \mathbf{e}_2(s) ds$$

as follows:

$$M_o : \widetilde{\mathbf{P}}(s, v) = \boldsymbol{\beta}(s) + v\mathbf{e}_o(s), \tag{18}$$

where $v \in \mathbb{R}$, and

$$\mathbf{e}_o(s) = \frac{\tau_g \mathbf{e}_1 - \kappa_n \mathbf{e}_2}{\sqrt{\tau_g^2 + k_n^2}}$$

under the assumption $(\tau_g,\kappa_n) \neq (0,0)$. Firstly differentiating $\mathbf{e}_o$, we find

$$\mathbf{e}_o' = \left(\kappa_g + \frac{\kappa_n\tau_g' - \tau_g\kappa_n'}{\tau_g^2 + \kappa_n^2}\right)\left(\frac{\kappa_n\mathbf{e}_1 + \tau_g\mathbf{e}_2}{\sqrt{\tau_g^2 + \kappa_g^2}}\right), \tag{19}$$

and thus $\lambda(s) = 0$. This results that M_o is a developable surface. Furthermore, we propose two invariants $\delta_o(s)$, and $\sigma_o(s)$ of M_o as follows:

$$\delta_o = \kappa_g + \frac{\kappa_n\tau_g' - \tau_g\kappa_n'}{\tau_g^2 + \kappa_n^2}, \text{ and } \quad \sigma_o = -\left[\frac{\kappa_n}{\sqrt{\tau_g^2 + \kappa_n^2}} + \left(\frac{\tau_g}{\delta_o\sqrt{\tau_g^2 + \kappa_g^2}}\right)'\right], \tag{20}$$

where $\delta_o \neq 0$. We can also calculate that

$$\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v = -\left(v\delta_o + \frac{\tau_g}{\sqrt{\tau_g^2 + \kappa_n^2}}\right)\mathbf{e}_3. \tag{21}$$

Hence, the normal vector of M_o is in the same direction to the normal vector of M. This is the reason why we name M_o the relatively osculating developable surface of M along $\boldsymbol{\beta}(s)$.

On the other hand, the invariants $\delta_o(s)$, and $\sigma_o(s)$ of M_o describe contour generators of M as follows:

Theorem 1. Let M_o be the relatively osculating developable surface of M expressed by Eq. (18). Then we have the following:

(A) The following are equivalent:

(1) M_o is a cylinder,

(2) $\delta_o(s) = 0$,

(3) $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ is a contour generator with respect to an orthogonal projection.

(B) If $\delta_o(s) \neq 0$, then the following are equivalent:

(1) M_o is a cone,

(2) $\sigma_o(s) = 0$,

(3) $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ is a contour generator with respect to a central projection.

Proof. (A) From Eq. (18), it is obvious that M_o is a cylinder if and only if $\mathbf{e}_o(s)$ is constant, i.e. $\delta_o(s) = 0$. Therefore, the condition (1) becomes equivalent to the situation (2). Suppose that the condition (3) holds. Then there exists a fixed vector $\mathbf{x} \in \mathbb{S}^2$ such that $\langle \mathbf{e}_3, \mathbf{x}\rangle = 0$. So there are a, $b \in \mathbb{R}$ such that $\mathbf{x} = a\mathbf{e}_1 + b\mathbf{e}_2$. Since $\langle \mathbf{e}_3', \mathbf{x}\rangle = 0$, we have $-a\kappa_n - b\tau_g = 0$, so that we have $\mathbf{x} = \pm\mathbf{e}_o(s)$. Namely, the situation (1) holds. Suppose that $\mathbf{e}_o(s)$ is constant. Then we choose $\mathbf{x} = \mathbf{e}_o(s) \in \mathbb{S}^2$. By the definition of $\mathbf{e}_o(s)$, we have $\langle \mathbf{x}, \mathbf{e}_3\rangle = 0$. Hence the condition (1) entails the situation (3).

(B) The situation (1) determines that the singular value set of M_n is a constant vector. Thus, in view of Eqs. (8), (9), and from Eq. (19), we have

$$\mathbf{C}'(s) = -\left[\frac{\kappa_n}{\sqrt{\tau_g^2+\kappa_n^2}} + \left(\frac{\tau_g}{\delta_o\sqrt{\tau_g^2+\kappa_g^2}}\right)'\right]\mathbf{e}_o(s) = -\sigma_o(s)\mathbf{e}_o(s).$$

Then M_o is a cone if and only if $\sigma_o(s)=0$. It follows that the situations (1) and (2) are equivalent. By the definition of the the central projection means that there is a fixed point $\mathbf{c}\in\mathbb{R}^3$ such that $\langle \mathbf{e}_3, \boldsymbol{\beta}-\mathbf{c}\rangle = 0$. If the condition (1) holds, then $\mathbf{C}(s)$ is constant. For the constant point $\mathbf{c}=\mathbf{C}(s)$, we have

$$\langle \mathbf{e}_3, \boldsymbol{\beta}-\mathbf{c}\rangle = \langle \mathbf{e}_3, \boldsymbol{\beta}-\mathbf{C}\rangle = \left\langle \mathbf{e}_3, \frac{\langle \boldsymbol{\beta}', \mathbf{e}_o'\rangle}{\|\mathbf{e}_o'\|^2}\ \mathbf{e}_o\right\rangle = \langle \mathbf{e}_3, \mathbf{e}_o\rangle = 0. \tag{22}$$

This implies that (3) holds. On the contrary, by (3), there is a fixed point $\mathbf{c}\in\mathbb{R}^3$ such that $\langle \mathbf{e}_3\ , \boldsymbol{\beta}-\mathbf{c}\rangle = 0$. Differentiating both side of Eq. (22), we have

$$0 = \langle \mathbf{e}_3, \boldsymbol{\beta}-\mathbf{c}\rangle' = \langle -\kappa_n\mathbf{e}_1 - \tau_g\mathbf{e}_2\ , \boldsymbol{\beta}-\mathbf{c}\rangle, \tag{23}$$

so we may write $\boldsymbol{\beta}-\mathbf{c} = f(s)\mathbf{e}_o(s)$, where $f(s)$ is a differentiable function. Differentiating Eq. (23) again, we have:

$$0 = \langle \mathbf{e}_3, \boldsymbol{\beta}-\mathbf{c}\rangle'' = \langle -\kappa_n\mathbf{e}_1 - \tau_g\mathbf{e}_2, \mathbf{e}_2\rangle + \left\langle -\left(\kappa_n\mathbf{e}_1+\tau_g\mathbf{e}_2\right)', \boldsymbol{\beta}-\mathbf{c}\right\rangle,$$

or equivaently,

$$0 = \langle \mathbf{e}_3, \boldsymbol{\beta}-\mathbf{c}\rangle'' = -\tau_g + f\delta_o\sqrt{\tau_g^2+\kappa_n^2}.$$

It follows that

$$\mathbf{c} = \boldsymbol{\beta}(\mathbf{s}) - \frac{\tau_g}{\delta_o\sqrt{\tau_g^2+\kappa_n^2}}\mathbf{e}_o(s) = \boldsymbol{\beta} - \frac{\langle\ \boldsymbol{\beta}', \mathbf{e}_o'\rangle}{\|\ \mathbf{e}_o'\|^2}\mathbf{e}_o(s) = \mathbf{C}(s).$$

Therefore, $\mathbf{C}(s)$ is constant, so that (1) holds $\boxdot$.

Theorem 2 (Existence and uniqueness). Let $M\subset\mathbb{R}^3$ be a regular surface and $\boldsymbol{\beta}: I\to M\subset\mathbb{R}^3$ be a unit-speed curve given by $\boldsymbol{\beta} = \int \mathbf{e}_2(s)ds$ with $\kappa_n^2+\tau_g^2\neq 0$. Then there exists uniquely a relatively osculating developable surface represented by Eq. (18).

Proof. For the existence, we have the relatively osculating developable surface along $\boldsymbol{\beta}=\boldsymbol{\beta}(s)$ represented by Eq. (18). On the other hand, since M_o is a ruled surface, we suppose that

$$M_o : \widetilde{\mathbf{P}}(s,v) = \boldsymbol{\beta}(s) + v\boldsymbol{\zeta}(s), \tag{24}$$

where $v\in\mathbb{R}$, with $(\tau_g,\kappa_n)\neq(0,0)$, and

$$\boldsymbol{\zeta}(s) = \zeta_1(s)\mathbf{e}_1 + \zeta_2(s)\mathbf{e}_2 + \zeta_3(s)\mathbf{e}_3,\ \boldsymbol{\zeta}'(s)\neq\mathbf{0}.$$

It can be immediately seen that M_o is developable if and only if

$$\det(\boldsymbol{\beta}^{'},\boldsymbol{\zeta},\boldsymbol{\zeta}^{'})=0\Leftrightarrow\zeta_3\zeta_1^{'}-\zeta_1\zeta_3^{'}-\zeta_2\left(\zeta_1\tau_g+\zeta_3\kappa_g\right)+\kappa_n\left(\zeta_3^2+\zeta_1^2\right)=0. \tag{25}$$

Conversely, since M_o is a relatively osculating developable surface along $\boldsymbol{\beta}=\boldsymbol{\beta}(s)$, we have

$$\left(\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v\right)(s,v)=\psi\left(s,v\right)\mathbf{e}_3. \tag{26}$$

Also, the normal vector $\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v$ at the point $(s,0)$ is

$$\left(\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v\right)(s,0)=\zeta_3\mathbf{e}_1-\zeta_1\mathbf{e}_3. \tag{27}$$

By means of Eqs. (26) and (27) we find:

$$\zeta_3=0,\text{ and }\zeta_1=-\psi\left(s,0\right), \tag{28}$$

which follows from Eq. (25) that

$$-\zeta_1\left(\zeta_1\kappa_n+\zeta_2\tau_g\right)=0. \tag{29}$$

If $(s,0)$ is a regular point (i.e., $\psi\left(s,0\right)\neq 0$), then $\zeta_1(s)\neq 0$. Thus, we have

$$\zeta_2=-\frac{\kappa_n}{\tau_g}\zeta_1,\text{ with }\tau_g\neq 0. \tag{30}$$

Therefore, we obtain

$$\boldsymbol{\zeta}(s)=\zeta_1\mathbf{e}_1-\frac{\kappa_n}{\tau_g}\zeta_1\mathbf{e}_2=\frac{\zeta_1}{\cos\varphi}\mathbf{e}_o(s), \tag{31}$$

where $(\tau_g,\kappa_n)\neq(0,0)$, and $\varphi\neq\dfrac{\pi}{2}$. It follows that $\boldsymbol{\zeta}(s)$ becomes equal to the direction of $\mathbf{e}_o(s)$. If $\tau_g\neq 0$ (i.e., $\varphi\neq\dfrac{\pi}{2}$), we have the same result as the above case.

On the other hand, suppose that M_o has a singular point at $(s_0,0)$. Then $\psi\left(s_0,0\right)=\zeta_1(s_0)=\zeta_3(s_0)=0$, and we have $\boldsymbol{\zeta}(s_0)=\zeta_2(s_0)\mathbf{e}_2(s_0)$. If the singular point $\boldsymbol{\beta}(s_0)$ is in the closure of the set of points where the relatively osculating developable surface along $\boldsymbol{\beta}(s)$ is regular, then there is a point $\boldsymbol{\beta}(s)$ in any neighborhood of $\boldsymbol{\beta}(s_0)$ such that the uniqueness of the relatively osculating developable surface is satisfied at $\boldsymbol{\beta}(s)$. Passing to the limit $s\to s_0$, uniqueness of the relatively osculating surface holds at s_0. Assume that there is an open interval $J\subseteq I$ such that M_o is singular at $\boldsymbol{\beta}(s)$ for any $s\in J$. Then $\mathbf{P}(s,v)=\boldsymbol{\beta}(s)+v\zeta_2(s)\ \mathbf{e}_2(s)$ for any $s\in J$. This means that $\zeta_1(s)=\zeta_3(s)=0$ for $s\in J$. It follows that

$$\left(\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v\right)(s,v)=-v\zeta_2^2\left(\tau_g\mathbf{e}_1+\kappa_g\mathbf{e}_3\right)\ . \tag{32}$$

Thus the above vector is directed to $\mathbf{e}_3$, i.e. $\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v\parallel\mathbf{e}_3(s)$ if and only if $\kappa_g\neq 0$ and $\tau_g=0$ for any $s\in J$. In this case, $\mathbf{e}_0(s)=\pm\mathbf{e}_3$. This determines that uniqueness holds $\boxdot$.

Proposition 1. Let M_o be the relatively osculating developable surface expressed by Eq. (18) with $(\tau_g,\kappa_n) \neq (0,0)$. If there are two osculating developable surfaces along $\boldsymbol{\beta}(s)$, then $\boldsymbol{\beta}(s)$ is a straight line.

Proof. Assume that $(\tau_g,\kappa_n) \neq (0,0)$, the relatively osculating developable surface along the direction curve $\boldsymbol{\beta}(s)$ is unique by Theorem 2. If $\kappa_n = \tau_g = 0$, then $\boldsymbol{\beta}(s)$ is a plane curve. In this case, a plane Π at $\boldsymbol{\beta}(s_0)$ is a relatively osculating developable surface along $\boldsymbol{\beta}(s)$. If there is another relatively osculating developable surface M_o along $\boldsymbol{\beta}(s)$, then M_o is tangent to Π along $\boldsymbol{\beta}(s)$. By definition, Π is tangent to M_o along a ruling of M_o, which is $\boldsymbol{\beta}(s)$. Thus $\boldsymbol{\beta}(s)$ is a line. If $\kappa_n = \tau_g = 0$ at an isolated point $s_0 \in I$ except at s_0, then there is a point $s \in I$ in any neighborhood of s_0 such that the uniqueness of the relatively osculating developable surface is satisfied at $s \in I$. Passing to the limit $s \to s_0$, uniqueness of the relatively osculating developable surface is satisfied at $s_0 \in I$ $\boxdot$.

Proposition 2. Let M_o be the relatively osculating developable surface expressed by Eq. (18) with $(\tau_g,\kappa_n) \neq (0,0)$. Then $\kappa_n = \tau_g = 0$ if and only if $\boldsymbol{\beta}(s)$ is a ruling of M_o.

Proof. In general, the torsion of the curve $\boldsymbol{\beta}(s)$ as a space curve is given by

$$\tau_{\boldsymbol{\beta}}(s) := \frac{\det(\boldsymbol{\beta}',\ \boldsymbol{\beta}'',\boldsymbol{\beta}''')}{\left\|\boldsymbol{\beta}' \times \boldsymbol{\beta}''\right\|^2} = -\kappa_n + \frac{\kappa_g\tau_g' - \tau_g\kappa_g'}{\tau_g^2+\kappa_g^2}\ . \tag{33}$$

Assuming that $\kappa_n = \tau_g = 0$, the torsion $\tau_{\boldsymbol{\beta}}$ becomes constantly equal to 0. Thus, $\boldsymbol{\beta}(s)$ becomes a plane curve. Moreover, we have $\mathbf{e}_3' = -\kappa_n\,\mathbf{e}_1 - \tau_g\mathbf{e}_2 = 0$. The assumption that M_o is an osculating developable surface implies that M_o is a plane generated by $\boldsymbol{\beta}(s)$. Thus $\boldsymbol{\beta}(s)$ is a line. For the converse, we assume that $\boldsymbol{\beta}(s) = \int_0^s \mathbf{e}_2(s)ds$ is a ruling of the osculating developable M_o. Since $\boldsymbol{\beta}(s)$ is a ruling in $\mathbb{R}^3$; $\mathbf{e}_2$ is a constant vector. The supposition that M_o is a developable surface determines that $\mathbf{e}_3' = \mathbf{0}$. Thus, by the Darboux equations we have $\kappa_n = \tau_g = 0$ $\boxdot$.

Therefore, we can give the following corollaries:

Corollary 1. The relatively osculating developable surface M_o represented by Eq. (18) is a non-cylindrical if and only if $\delta_o(s) \neq 0$.

Proof. It is a straighforward result from the definition of non-cylindirical ruled surface.

Corollary 2. The relatively osculating developable surface M_o represented by Eq. (18) is a tangential developable if and only if $\delta_o(s) \neq 0$, and $\sigma_o(s) \neq 0$.

Proof. According to the proof of Theorem 1, when $\delta_o(s) \neq 0$, and $\sigma_o(s) \neq 0$, we have $\mathbf{e}_o' \neq \mathbf{0}$, and $\mathbf{C}' \neq \mathbf{0}$. Since $\det(\boldsymbol{\beta}', \mathbf{e}_o, \mathbf{e}_o') = 0$, $< \mathbf{C}', \mathbf{e}_o' >= 0$ and $< \mathbf{e}_o, \mathbf{e}_o' >= 0$, we find $\mathbf{C}' \| \mathbf{e}_o$. This determines that the surface M_o is a tangent surface $\boxdot$.

3.1. **Special curves on a surface.** Based on the Theorem 3.3 of Ref. [8], we divide the singularities of relatively osculating developable surfaces M_o forward special curves by using the two invariants δ_o, and σ_o in the following:

(A). If $\kappa_n = 0$, then α is an asymptotic line on M, and

$$M_o : \widetilde{\mathbf{P}}(s,v) = \boldsymbol{\beta}(s) + v\mathbf{e}_1(s), v \in \mathbb{R}. \tag{34}$$

In this case, we obtain the invariants as follows:

$$\delta_o = \kappa_g, \quad \text{and} \quad \sigma_o = -\left(\frac{\tau_g}{\delta_o\sqrt{\tau_g^2 + \kappa_g^2}}\right)'.$$

Corollary 3. Let M_o be the relatively osculating developable surface expressed by Eq. (34). Then we have the following:
(1) M_o is non-singular at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v_0 \neq 0$.
(2) M_o is locally diffeomorphic to Cuspidal edge CE at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v_0 = -\kappa_g^{-1}(s_0) \neq 0$, and $\kappa_g'(s_0) \neq 0$.
(3) M_o is locally diffeomorphic to Swallowtail SW at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v = -\kappa_g^{-1}(s_0) \neq 0$, $\kappa_g'(s_0) = 0$, and $\left(\kappa_g^{-1}\right)''(s_0) \neq 0$.
Here,

$$\left.\begin{array}{l} CE = \left\{(x_1, x_2, x_3) | x_1 = u,\ x_2 = v^2\ ,\ x_3 = v^3\right\},\ \text{(see Fig. 1)}. \\ SW = \left\{(x_1, x_2, x_3) | x_1{=}u,\ x_2{=}3v^2{+}uv^2,\ x_3{=}4v^3{+}2uv\right\},\ \text{(see Fig. 2)} \end{array}\right\}$$

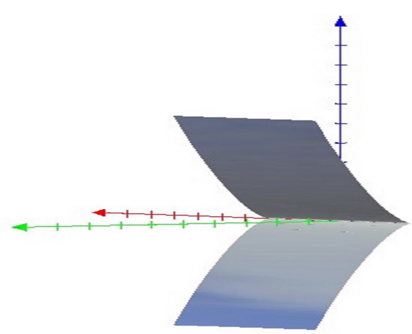

FIGURE 1. Cuspidal edge.

Proof. Singularities of the relatively osculating developable surface expressed by Eq. (34) are

$$\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v = -\left(v\kappa_g + 1\right)\mathbf{e}_3. \tag{35}$$

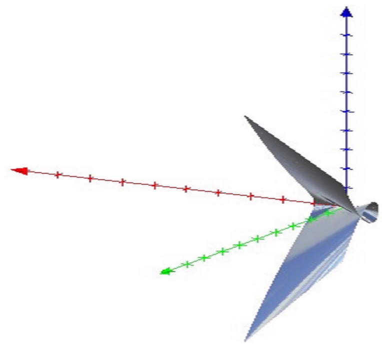

Figure 2. Swallowtail.

Therefore, $\widetilde{\mathbf{P}}(s_0, v_0)$ is non-singular if and only if $\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v \neq \mathbf{0}$. This condition is equivalent to $v_0 = -\kappa_g^{-1}(s_0)$. This completes the proof of assertion (1). If there is a parameter s_0 such that

$$v_0 = -\kappa_g^{-1}(s_0), \quad \text{and} \quad v_0' = \frac{\kappa_g'(s_0)}{\kappa_g^2(s_0)} \neq 0 \ \left(\text{i.e. } \kappa_g' \neq 0 \right),$$

then M_o is locally diffeomorphic to CE at $\widetilde{\mathbf{P}}(s_0, v_0)$. This completes the proof of assertion (2). We also have, if there is a parameter s_0 such that

$$v_0 = -\kappa_g^{-1}(s_0), \quad v_0' = \frac{\kappa_g'(s_0)}{\kappa_g^2(s_0)} = 0, \quad \text{and} \quad \left(\kappa_g^{-1}\right)''(s_0) \neq 0,$$

then M_o is locally diffeomorphic to SW at points $\widetilde{\mathbf{P}}(s_0, v_0)$. This concludes the proof of affirmation (3).

(B). If $\tau_g = 0$, then $\boldsymbol{\alpha}$ becomes a line of curvature on M, and

$$M_o : \widetilde{\mathbf{P}}(s, v) = \boldsymbol{\beta}(s) - v\mathbf{e}_2(s), v \in \mathbb{R}, \tag{36}$$

which is recognized as the tangent surface of $\boldsymbol{\beta}(s)$. In this case we obtain the invariants as $\delta_o = \kappa_g$, and $\sigma_o = -1$.

Corollary 4. Let M_o be the relatively osculating developable surface expressed by Eq. (36). Then we have the following:
(1) M_o is non-singular at points $\mathbf{P}(s_0, v_0)$ if and only if $v_0 \neq 0$.
(2) M_o is locally diffeomorphic to CE at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v_0 = -\kappa_g^{-1}(s_0) \neq 0$, and $\kappa_g'(s_0) \neq 0$.
(3) M_o is locally diffeomorphic to SW at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v = -\kappa_g^{-1}(s_0) \neq 0$, $\kappa_g'(s_0) = 0$, and $\left(\kappa_g^{-1}\right)''(s_0) \neq 0$.

Proof. Singularities of the relatively osculating developable surface expressed by Eq. (36) are

$$\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v = -v\kappa_g \mathbf{e}_3. \tag{37}$$

Therefore, $\widetilde{\mathbf{P}}(s_0, v_0)$ is non-singular if and only if $\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v \neq \mathbf{0}$. This condition is equivalent to $v_0 = -c\kappa_g^{-1}(s_0)$, $c \neq 0$. This completes the proof of assertion (1). If there is a parameter s_0 such that

$$v_0 = -c\kappa_n^{-1}(s_0),\ c \neq 0, \quad \text{and} \quad v_0' = \frac{c\kappa_n'(s_0)}{\kappa_n^2(s_0)} \neq 0,\ (\text{i.e. } \kappa_n' \neq 0)\,,$$

then M_o is locally diffeomorphic to CE at $\widetilde{\mathbf{P}}(s_0, v_0)$. This finishes the proof of affirmation (2). Again, if there exists a parameter s_0 such that

$$v_0 = -c\kappa_g^{-1}(s_0),\ c \neq 0, \quad v_0' = \frac{c\kappa_g'(s_0)}{\kappa_g^2(s_0)} = 0, \quad \text{and} \quad \left(\kappa_g^{-1}\right)''(s_0) \neq 0,$$

then M_o is locally diffeomorphic to SW at pointgs $\widetilde{\mathbf{P}}(s_0, v_0)$. This finishes the proof of affirmation (3) $\boxdot$.

3.1.1. *Curves on the unit sphere.* We now deal with the case when M is the unit sphere $\mathbb{S}^2 = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\|^2 = 1\right\}$. Let $\boldsymbol{\alpha} : I \subseteq \mathbb{R} \to \mathbb{S}^2$ be a unit speed curve. In this case, we have $\mathbf{t}(s) = \boldsymbol{\alpha}'$, $\mathbf{g}(s) = \boldsymbol{\alpha} \times \mathbf{t}$, and since s is a natural parameter of $\boldsymbol{\alpha}(s)$, it follows that $\|\mathbf{t}\| = 1$, and the frame $\{\boldsymbol{\alpha} = \boldsymbol{\alpha}(s), \mathbf{t}(s), \mathbf{g}(s)\}$ forms a moving orthonormal frame fitted to each point of the spherical curve $\boldsymbol{\alpha}(s)$. This frame is said to be the Darboux frame relative to $\mathbf{x}(s)$. By construction, the Darboux formula is

$$\begin{bmatrix} \boldsymbol{\alpha}' \\ \mathbf{t}' \\ \mathbf{g}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \mathbf{t} \\ \mathbf{g} \end{bmatrix}, \tag{38}$$

where $\gamma = \gamma(s)$ is the geodesic curvature of $\boldsymbol{\alpha}(s)$. It follows that $\delta_o = \gamma$, $\sigma_o = \pm 1$, $\mathbf{e}_0 = \pm\mathbf{g}(s)$, and $\boldsymbol{\beta}(s) = \int_0^s \mathbf{g}(s)ds$. Thus, we have:

$$M_o : \widetilde{\mathbf{P}}(s, v) = \boldsymbol{\beta}(s) + v\mathbf{g}(s),\ v \in \mathbb{R}, \tag{39}$$

which is recognized as the tangent developable surface of $\boldsymbol{\beta}(s)$. Then we have the following lemma as a result of Corollary 4.

Lemma 1. Let M_o be the tangent developable expressed by Eq. (39). Then we have the following:
(1) M_o is non-singular at points $\beta(s_0)$ if and only if $v_0 \neq 0$.
(2) M_o is locally diffeomorphic to CE at points $\mathbf{P}(s_0, v_0)$ if and only if $v_0 = -\gamma^{-1}(s_0) \neq 0$, and $\gamma'(s_0) \neq 0$.
(3) M_o is locally diffeomorphic to SW at points $\mathbf{P}(s_0, v_0)$ if and only if $v = -\gamma^{-1}(s_0) \neq 0$, $\gamma'(s_0) = 0$, and $\left(\gamma^{-1}\right)''(s_0) \neq 0$.

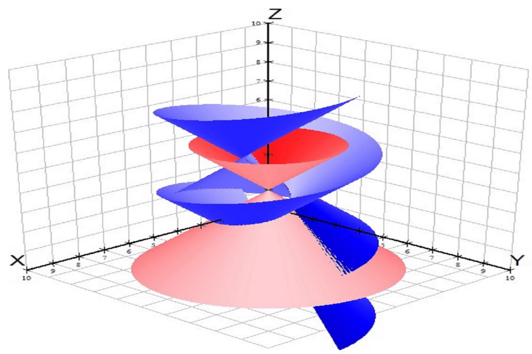

FIGURE 3.

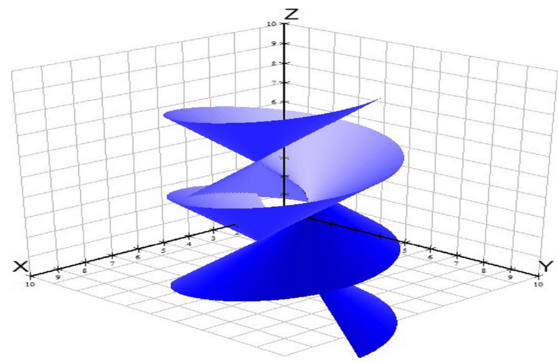

FIGURE 4.

4.

Proposition 2. THE RELATIVELY OSCULATING DEVELOPABLE SURFACE M_o REPRESENTED BY EQ. (39) IS A CYLINDRICAL IF $\boldsymbol{\alpha}(s)$ IS A GREAT CIRCLE.

Proof. ASSUME THAT $\boldsymbol{\alpha}(s)$ BECOMES A GREAT CIRCLE. THEN $\gamma(s) = 0$, AND $\mathbf{g}(s)$ IS CONSTANT. THEREFORE, M_o IS A CIRCULAR CYLINDER.

4.1. **Examples.** We close this section with some examples:

Example 1. Let the base surface M be given as the following parameterization:

$$\mathbf{P}(s,v) = (\cos s - \frac{1}{\sqrt{2}} v \cos s, \sin s - \frac{1}{\sqrt{2}} v \sin s, \frac{v}{\sqrt{2}}). \tag{40}$$

The directrix curve $\boldsymbol{\beta}$ of the relatively osculating developable surface is $\boldsymbol{\beta} = (-\frac{1}{\sqrt{2}} \sin s, \frac{1}{\sqrt{2}} \cos s, \frac{s}{\sqrt{2}})$. The normal curvature and geodesic torsion of the base

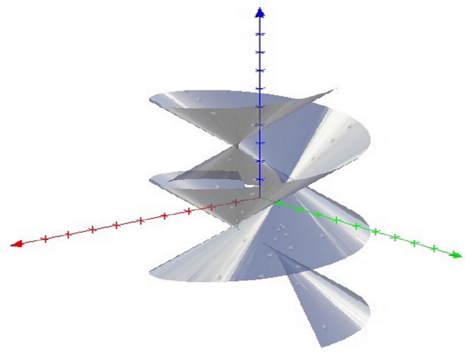

FIGURE 5.

curve are, respectively, computed as $\kappa_n = -\frac{1}{\sqrt{2}}$, and $\tau_g = 0$. Then the ruling line $\mathbf{e}_o$ of the relatively osculating developable surface is obtained as $\mathbf{e}_o = (\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\sin s, -\frac{1}{\sqrt{2}})$. As a result, the relatively osculating developable surface M_o is given with the parameterization:

$$\widetilde{\mathbf{P}}(s,v)=(-\frac{1}{\sqrt{2}}\sin s+\frac{1}{\sqrt{2}}v\cos s, \frac{1}{\sqrt{2}}\cos s+\frac{1}{\sqrt{2}}v\sin s, \frac{s}{\sqrt{2}}-\frac{v}{\sqrt{2}}). \tag{41}$$

The base surface given by (40) and the relatively osculating developable surface given by (41) have been together plotted in Fig. 3. The relatively osculating developable surface given by (41) has been alone illustrated in Fig 4. The relatively osculating developable surface has been illustrated by reflecting surface in Fig. 5.

Example 2. Given the base surface M as follows:

$$\mathbf{P}(s,v)=(\cos\frac{s}{\sqrt{2}}-\frac{1}{\sqrt{2}}v\sin\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}+\frac{1}{\sqrt{2}}v\cos\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}+\frac{v}{\sqrt{2}}). \tag{42}$$

The directrix curve $\boldsymbol{\beta}$ of the relatively osculating developable surface is $\boldsymbol{\beta} = (\sqrt{2}\sin\frac{s}{\sqrt{2}}, -\sqrt{2}\cos\frac{s}{\sqrt{2}}, 0)$. The normal curvature and geodesic torsion of the base curve are, respectively, computed as $\kappa_n = 0$, and $\tau_g = \frac{1}{2}$. Then the ruling line $\mathbf{e}_o$ of the relatively osculating developable surface is obtained as $\mathbf{e}_o = (-\frac{1}{2}\sin\frac{s}{\sqrt{2}}, \frac{1}{2}\cos\frac{s}{\sqrt{2}}, \frac{1}{2})$. As a result, the relatively osculating developable surface M_o is given with the below parameterization:

$$\widetilde{\mathbf{P}}(s,v) = (\sqrt{2}\sin\frac{s}{\sqrt{2}} - \frac{v}{2}\sin\frac{s}{\sqrt{2}}, -\sqrt{2}\cos\frac{s}{\sqrt{2}} + \frac{v}{2}\cos\frac{s}{\sqrt{2}}, \frac{v}{2}). \tag{43}$$

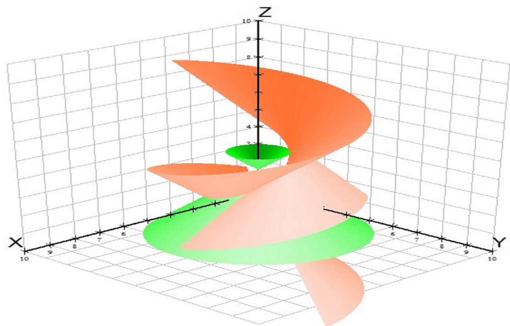

FIGURE 6.

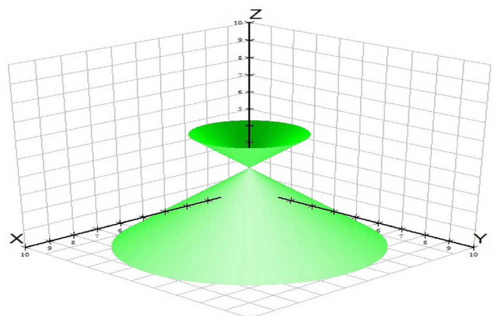

FIGURE 7.

The base surface given by (42) and the relatively osculating developable surface given by (43) have been together plotted in Fig. 6. The relatively osculating developable surface given by (42) has been alone illustrated in Fig. 7. The relatively osculating developable surface has been illustrated by reflecting surface in Fig. 8.

5. CONCLUSION

In this work, we have constructed a developable surface tangent to a surface forward a curve in the surface which we defined it as relatively osculating developable surface. We have chosen the curve as the tangent normal direction curve on which the new surface is formed in Euclidean space. We have obtained some results about the existence and uniqueness, and the singularities of such developable surfaces. We have also given two invariants of curves on a surface which describe these singularities. We have given two results for special curves such as asymptotic line and line of curvature which are rulings of the relatively osculating developable surface.

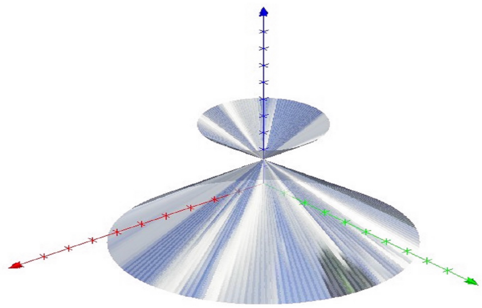

FIGURE 8.

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LIE IDEALS AND JORDAN TRIPLE $(\alpha, \beta)-$DERIVATIONS IN RINGS

NADEEM UR REHMAN AND EMINE KOÇ SÖGÜTCÜ

Abstract. In this paper we prove that on a 2-torsion free semiprime ring R every Jordan triple (α, β)-derivation (resp. generalized Jordan triple (α, β)-derivation) on Lie ideal L is an (α, β)-derivation on L (resp. generalized (α, β)-derivation on L)

1. Introduction

Throughout the present paper R will denote an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0$; $x \in R$, implies $x = 0$. For any $x, y \in R$, we denote the commutator $[x, y] = xy - yx$. Recall that R is prime if for $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. A Lie ideal L is said to be square-closed if $a^2 \in L$ for all $a \in L$. Recall that a derivation of a ring R is an additive map $\delta : R \longrightarrow R$ such that $(xy)^{\delta} = (x)^{\delta} y + x(y)^{\delta}$ holds for all $x, y \in R$. On the other hand, $\delta : R \longrightarrow R$ an additive mapping is called a Jordan derivation if $\left(x^2\right)^{\delta} = (x)^{\delta} x + x(x)^{\delta}$ holds for all $x \in R$. A famous result due to Herstein [11, Theorem 3.3] shows that a Jordan derivation of a prime ring of characteristic not 2 must be a derivation. This result was extended to 2-torsion free semiprime rings by Cusack [10] and subsequently, by Bresar [7]. Following [6]. an additive mapping $\delta : R \rightarrow R$ is called a Jordan triple derivation if $(xyx)^{\delta} = (x)^{\delta} yx + x(y)^{\delta} x + xy(x)^{\delta}$ holds for all $x, y \in R$. One can easily prove that any Jordan derivation on an 2-torsion free ring is a Jordan triple derivation (see [11, Lemma 3.5]). Bresar has proved the following result.

Theorem 1.1. *([6, Theorem 4.3]) Let R be a 2-torsion free semiprime ring and $\delta : R \rightarrow R$ be a Jordan triple derivation. In this case δ is a derivation.*

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To understand our results it is better to review some generalizations of the notion of derivation. An additive mapping $F : R \to R$ is said to be generalized derivation (resp. a generalized Jordan derivation) on R if there exists a derivation $\delta : R \to R$ such that $(xy)^F = (x)^F y + x(y)^\delta$ (resp. $(x^2)^F = (x)^F x + x(x)^\delta$) holds for all $x, y \in R$. An additive mapping $F : R \to R$ is said to be generalized Jordan triple derivation on R if there exists a Jordan triple derivation $\delta : R \to R$ such that $(xyx)^F = (x)^F yx + x(y)^\delta x + xy(x)^\delta$ holds for all $x, y \in R$. In 2003, Jing and Lu [14, Theorem 3.5] proved that every generalized Jordan triple derivation on a 2-torsion free prime rings R is a generalized derivation. Recently, Vukman [20] extended Jing and Lu result for 2-torsion free semiprime rings.

If $\delta : R \longrightarrow R$ is a additive and if α and β are endomorphisms of R, then δ is said to be an (α, β)-derivation of R when for all $x, y \in R$, $(xy)^\delta = (x)^\delta \alpha(y) + \beta(x)(y)^\delta$. Note that for I, the identity map on R, an (I, I)-derivation is just a derivation. An example of (α, β)-derivation when R has a nontrivial central idempotent e is to let $\delta(x) = ex$, $\alpha(x) = (1 - e)x$, and $\beta = I$ (or δ) (formally). Here, δ is not a derivation because $(ee)^\delta = eee \neq 2eee = (ee)e + e(ee) = (e)^\delta e + e(e)^\delta$. In any ring with endomorphism α, if we let $d = I - \alpha$, then d is an (α, I)-derivation, but not a derivation when R is semiprime, unless $\alpha = I$. An additive mapping $\delta : R \to R$ is called Jordan triple (α, β)-derivation if $(xyx)^\delta = (x)^\delta \alpha(yx) + \beta(x)(y)^\delta \alpha(x) + \alpha(xy)(x)^\delta$ for all $x, y \in R$. Obviously, every (α, β)-derivation on a 2-torsion free ring is a Jordan triple (α, β)-derivation, but converse need not be true in general. In 2007, Liu and Shiue [15, Theorem 2] show that the converse is true for 2-torsion free semiprime rings R and probed the following result:

Theorem 1.2. *Let R be a 2-torsion free semiprime rings and let α, β be automorphisms of R. If $\delta : R \to R$ is a Jordan triple (α, β)-derivation, then δ is an (α, β)-derivation.*

An additive map $F : R \longrightarrow R$ is called a generalized (α, β)-derivation, for α and β endomorphisms of R, if there exists an (α, β)-derivation $\delta : R \longrightarrow R$ such that $(xy)^F = (x)^F \alpha(y) + \beta(x)(y)^\delta$ holds for all $x, y \in R$. Clearly, this notion include those of (α, β)-derivation when $F = \delta$, of derivation when $F = \delta$ and $\alpha = \beta = I$, and of generalized derivation, which is the case when $\alpha = \beta = I$. Maps of the form $(x)^F = ax + xb$ for $a, b \in R$ with $(x)^\delta = xb - bx$ and $\alpha = \beta = I$ are generalized derivations, and more generally, maps $(x)^\delta = a\alpha(x) + \beta(x)b$ are generalized (α, β)-derivation. To see this observe that $(xy)^F = a\alpha(x)\alpha(y) + \beta(x)\beta(y)b = (a\alpha(x) + \beta(x)b)\alpha(x) + \beta(x)(\beta(y)b - b\alpha(y))$, and as we have just seen above, $(x)^\delta = b\alpha(x) - \beta(x)b$ is an (α, β)-derivation of R. As for derivation, a generalized Jordan (α, β)-derivation F assumes $x = y$ in the definition above; that is, we assume only that $(x^2)^F = (x)^F \alpha(x) + \beta(x)(x)^\delta$, holds for all $x \in$. An additive map $F : R \longrightarrow R$ is called generalized Jordan triple (α, β)-derivation, for α and β endomorphisms of R, if there exists a Jordan triple (α, β)-derivation $\delta : R \longrightarrow R$ such that $(xyx)^F = (x)^F \alpha(yx) + \beta(x)(y)^\delta \alpha(x) + \beta(xy)(x)^\delta$, holds for all $x, y \in R$.

Clearly, this notion includes those of triple (α, β)-derivation when $F = \delta$, of triple derivation when $F = \delta$ and $\alpha = \beta = I$, and of generalized triple derivation which is the case $\alpha = \beta = I$. In 2007, Liu and Shiue [15, Theorem 3] proved the following generalization of all above results:

Theorem 1.3. *Let R be a 2-torsion free semiprime rings and α, β be automorphisms of R. If $F : R \to R$ is a generalized Jordan triple (α, β)-derivation, then F is a generalized (α, β)-derivation.*

The present paper is motivated by the previous results and we here continue this line of investigation to generalize Theorem 1.2 and Theorem 1.3 on Lie ideal of R.

2. Jordan Triple Derivations

It is obvious to see that every derivation is a Jordan triple derivation, but the converse need not to be true in general. In [6], Bresar proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Motivated by the result due to Bresar, in the present section it is shown that on a 2-torsion free semiprime ring R every Jordan triple $(\alpha, \beta)-$derivation on Lie ideal L is an $(\alpha, \beta)-$derivation on L. More precisely, we prove the following:

Theorem 2.1. *Let R be a 2-torsion free semiprime ring, α, β be automorphisms of R and $L \nsubseteq Z(R)$ be a nonzero square-closed Lie ideal of R. If $\delta : R \longrightarrow L$ satisfying*

$$(aba)^\delta = a^\delta \alpha(ba) + \beta(a) b^\delta \alpha(a) + \beta(ab) a^\delta \text{ for all } a, b \in L$$

and $a^\delta, \beta(a) \in L$, then δ is a $(\alpha, \beta)-$derivation on L.

Corollary 2.1. *Let R be a 2-torsion free semiprime ring, α, β be automorphisms of R and $L \nsubseteq Z(R)$ be a nonzero square-closed Lie ideal of R. If $\delta : R \longrightarrow L$ satisfying*

$$(a^2)^\delta = a^\delta \alpha(a) + \beta(a) a^\delta \text{ for all } a \in L$$

and $a^\delta, \beta(a) \in L$, then δ is a $(\alpha, \beta)-$derivation on L.

To facilitate our discussion, we shall begin with the following lemmas:

Lemma 2.1 ([4], Lemma 4)**.** *If $L \nsubseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aLb = \{0\}$, then $a = 0$ or $b = 0$.*

Lemma 2.2 ([19], Lemma 2.4)**.** *Let R be a 2-torsion free semiprime ring, L be a Lie ideal of R and $a \in L$ such that $L \nsubseteq Z(R)$. If $aLa = 0$, then $a^2 = 0$ and there exists a nonzero ideal $K = R[L, L]R$ of R generated by $[L, L]$ such that $[K, R] \subseteq L$ and $Ka = aK = 0$.*

Corollary 2.2 ([12], Corollary 2.1)**.** *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \nsubseteq Z(R)$ and let $a, b \in L$.*

(1) *if* $aLa = 0$, *then* $a = 0$.
(2) *If* $aL = 0$(*or* $La = 0$), *then* $a = 0$
(3) *If* L *is square-closed and* $aLb = 0$, *then* $ab = 0$ *and* $ba = 0$.

Lemma 2.3. *Let R be a 2-torsion free semiprime ring, L be a noncentral Lie ideal of R, β be a homomorphisms of R and $a, b \in L$. If $aub + \beta(bu)a = 0$, for all $u \in L$ then $aub = 0$.*

Proof. If

$$aub + \beta(bu)a = 0, \text{ for all } u \in L. \tag{2.1}$$

Then replacing u by ubv in (2.1), we get

$$a(ubv)b + \beta(bu)\beta(bv)a = 0. \tag{2.2}$$

Now application of (2.1), yields that

$$-\beta(bu)avb + \beta(bu)\beta(bv)a = 0. \tag{2.3}$$

Again, by (2.1), we obtain $-\beta(bu)avb - \beta(bu)avb = 0$ that is $\beta(bu)avb = 0$. Again by (2.1) $aubvb = 0$. Hence $aubLb = 0$, so $aub = 0$ for all $u \in L$.

Lemma 2.4 ([19], Lemma 2.7)**.** *Let $G_1, G_2, \cdots, G_n$ be additive groups and R be a 2-torsion free semiprime ring and $L \nsubseteq Z(R)$ is a Lie ideal of R. Suppose that mappings $S : G_1 \times G_2 \times \cdots \times G_n \longrightarrow R$ and $T : G_1 \times G_2 \times \cdots \times G_n \longrightarrow R$ are additive in each argument. If $S(a_1, a_2, \cdots, a_n)xT(a_1, a_2, \cdots, a_n) = 0$ for all $x \in L$, $a_i \in G_i$ $i = 1, 2, \cdots n$, then $S(a_1, a_2, \cdots, a_n)xT(b_1, b_2, \cdots, b_n) = 0$ for all $x \in L$, $a_i, b_i \in G_i$ $i = 1, 2, \cdots n$.*

Lemma 2.5. *Let R be a ring, L be a Lie ideal of R and $\delta : R \to R$ be a Jordan triple $(1, \beta)-$derivation. For arbitrary $a, b, c \in L$, we have*

$$(abc + cba)^\delta = a^\delta(bc) + \beta(a)b^\delta(c) + \beta(ab)c^\delta + c^\delta(ba) + \beta(c)b^\delta(a) + \beta(cb)a^\delta.$$

Proof. We have

$$(aba)^\delta = a^\delta(ba) + \beta(a)b^\delta(a) + \beta(ab)a^\delta, \quad \text{for all } a, b \in L. \tag{2.4}$$

We compute, $W = ((a + c)b(a + c))^\delta$ in two different ways. On one hand, we find that $W = (a + c)^\delta b(a + c) + \beta(a + c)b^\delta(a + c) + \beta((a + c)b)(a + c)^\delta$, and on the other hand $W = (aba)^\delta + (abc + cba)^\delta + (cbc)^\delta$. Comparing two expressions we obtain the required result.

Remark 2.1. It is easy to see that every Jordan $(1, \beta)-$derivation of a 2-torsion free ring satisfies (2.4) (see [1] for reference).

For the purpose of this section we shall write; $\Delta(a, b, c) = (abc)^\delta - a^\delta(bc) - \beta(a)b^\delta(c) - \beta(ab)c^\delta$, and $\Lambda(a, b, c) = abc - cba$. We list a few elementary properties of δ and Λ:

(i) $\Delta(a, b, c) + \Delta(c, b, a) = 0$

(ii) $\Delta((a+b),c,d) = \Delta(a,c,d) + \Delta(b,c,d)$ and $\Lambda((a+b),c,d) = \Lambda(a,c,d) + \Lambda(b,c,d)$

(iii) $\Delta(a,(b+c),d) = \Delta(a,b,d) + \Delta(a,c,d)$ and $\Lambda(a,(b+c),d) = \Lambda(a,b,d) + \Lambda(a,c,d)$

(iv) $\Delta(a,b,(c+d)) = \Delta(a,b,c) + \Delta(a,b,d)$ and $\Lambda(a,b,(c+d)) = \Lambda(a,b,c) + \Lambda(a,b,d)$.

Proposition 2.1. *Let R be a semiprime ring and $L \not\subseteq Z(R)$ be a square-closed Lie ideal of R. If $\Delta(a,b,c) = 0$ holds for all $a,b,c \in L$, then δ is an $(1,\beta)-$derivation of L.*

Proof. We have $\Delta(a,b,c) = 0$ for all $a,b,c \in L$, that is,

$$(abc)^\delta = a^\delta(bc) + \beta(a)b^\delta(c) + \beta(ab)c^\delta.$$

Let $M = abxab$. We have

$$\begin{aligned} M^\delta = (a(bxa)b)^\delta &= a^\delta(bxab) + \beta(a)b^\delta(xab) + \beta(ab)x^\delta(ab) \\ &+ \beta(abx)a^\delta(b) + \beta(abxa)b^\delta \text{ for all } x,a,b \in L. \end{aligned} \tag{2.5}$$

On the other hand,

$$M^\delta = ((ab)x(ab))^\delta = (ab)^\delta(xab) + \beta(ab)x^\delta(ab) + \beta(abx)(ab)^\delta. \tag{2.6}$$

Comparing (2.5) with (2.6) we get

$$\{(ab)^\delta - a^\delta(b) - \beta(a)b^\delta\}(xab) + \beta(abx)\{(ab)^\delta - a^\delta(b) - \beta(a)b^\delta\} = 0$$

that is, $a^b(xab) + \beta(abx)a^b = 0$, where a^b stands for $(ab)^\delta - a^\delta(b) - \beta(a)b^\delta$. Thus by Lemma 2.3 we find that $a^b(xab) = 0$, for all $a,b,x \in L$. Now by Lemma 2.4, we get $a^b(xcd) = 0$, for all $a,b,c,d,x \in L$. Hence, by using Corollary 2.2, we obtain $a^b = 0$ for all $a,b \in L$ that is δ is a $(1,\beta)-$derivation on L.

Lemma 2.6. *Let R be a ring and L be a Lie ideal of R. For any $a,b,c,x \in L$, we have*

$$\Delta(a,b,c)x\Lambda(a,b,c) + \beta(\Lambda(a,b,c))\beta(x)\Delta(a,b,c) = 0.$$

Proof. For any $a,b,c,x \in L$, suppose that $N = abcxcba + cbaxabc$. Now we find

$$\begin{aligned} N^\delta &= (a(bcxcb)a + c(baxab)c)^\delta = (a(bcxcb)a)^\delta + (c(baxab)c)^\delta \\ &= a^\delta(bcxcba) + \beta(a)b^\delta(cxcba) + \beta(ab)c^\delta(xcba) \\ &\quad +\beta(abc)x^\delta(cba) + \beta(abcx)c^\delta(ba) + \beta(abcxc)b^\delta(a) \\ &\quad +\beta(abcxcb)a^\delta + c^\delta(baxabc) + \beta(c)b^\delta(axabc) \\ &\quad +\beta(cb)a^\delta(xabc) + \beta(cba)x^\delta(abc) + \beta(cbax)a^\delta(bc) \\ &\quad +\beta(cbaxa)b^\delta(c) + \beta(cbaxab)c^\delta. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} N^\delta &= ((abc)x(cba) + (cba)x(abc))^\delta \\ &= (abc)^\delta(xcba) + \beta(abc)x^\delta(cba) + \beta(abcx)(cba)^\delta \\ &\quad +(cba)^\delta(xabc) + \beta(cba)x^\delta(abc) + \beta(cbax)(abc)^\delta. \end{aligned}$$

On comparing last two expressions we get

$$-\Delta(c,b,a)(xcba)+\Delta(c,b,a)(xabc)+\beta(abcx)\Delta(c,b,a)-\beta(cbax)\Delta(c,b,a)=0.$$

This implies that $\Delta(a,b,c)x\Lambda(a,b,c)+\beta(\Lambda(a,b,c))\beta(x)\Delta(a,b,c)=0$ for all $a,b,c\in L$.

Lemma 2.7. *Let R be a semiprime ring and $L\nsubseteq Z(R)$ be a square-closed Lie ideal of R. Then $\Delta(a,b,c)x\Lambda(r,s,t)=0$ holds for all $a,b,c,r,s,t,x\in L$.*

Proof. By Lemma 2.6, we have $\Delta(a,b,c)x\Lambda(a,b,c)+\beta(\Lambda(a,b,c))\beta(x)\Delta(a,b,c)=0$ for all $a,b,c\in L$. Thus we get $\Delta(a,b,c)x\Lambda(a,b,c)=0$ by Lemma 2.3. Now by Lemma 2.4 we find that$\Delta(a,b,c)x\Lambda(r,s,t)=0$, for all $a,b,c,r,s,t\in L$.

For an arbitrary ring R, we set $S=\{a\in C(L)\mid aL\subseteq C(L)\}$, where $C(L)$ is center of L.

Lemma 2.8. *Let R be a semiprime ring, L be a square-closed Lie ideal of R and $a\in L$. If $axy=yxa$ holds for all $x,y\in L$, then $a\in S$.*

Proof: Let $x,y,z,w\in L$. We get

$$a(wz)yx=yx(wz)a=ya(wz)x=y(awz)x=yzwax=(yzwa)x=awyzx.$$

This implies that

$$aw(zy-yz)x=0,\text{ for all }x,y,z,w\in L.$$

That is,

$$aw\,[z,y]\,Law\,[z,y]=0,\text{ for all }y,z,w\in L.$$

By Corollary 2.2, we have

$$aw\,[z,y]=0,\text{ for all }y,z,w\in L.$$

Replacing z by a in this equation, we get

$$aw\,[a,y]=0,\text{ for all }y,w\in L.$$

Hence $ayw\,[a,y]=0=yaw\,[a,y]$ for all $y,w\in L$, and so $[a,y]\;L\,[a,y]=0$, for all $y\in L$. By Corollary 2.2, we have $[a,y]=0$, for all $y\in L$. Therefore, $axy=yxa=yax$ for all $x,y\in L$. That is $aL\subseteq C(L)$. Thus, $a\in S$.

Lemma 2.9. *Let R be a semiprime ring, L be a square-closed Lie ideal of R, $a\in C(L)$, $c\in L$, β be a homomorphisms of R and $\beta(L)\subseteq L$. If $(\beta(ab)-ab)c=0$ holds for all $b\in L$, then $a(\beta(b)-b)c=0$.*

Proof: Replacing b by bx, $x \in L$ in the hypothesis and using $a \in C(L)$, we have

$$\begin{aligned} 0 &= (\beta(abx) - abx)c = \beta(ab)\beta(x)c - abxc \\ &= \beta(ba)\beta(x)c - abxc = \beta(b)\beta(ax)c - abxc \\ &= \beta(b)axc - abxc = a\beta(b)xc - abxc \\ &= a(\beta(b) - b)xc. \end{aligned}$$

That is,

$$a(\beta(b) - b)xc = 0, \text{for all } b, x \in L.$$

Using $\beta(L) \subseteq L$ and replacing x by $cxa(\beta(b) - b)$, we obtain that

$$a(\beta(b) - b)cxa(\beta(b) - b)c = 0, \text{for all } b, x \in L.$$

This implies that

$$a(\beta(b) - b)cLa(\beta(b) - b)c = 0, \text{for all } b \in L.$$

By Corollary 2.2, we have

$$a(\beta(b) - b)c = 0, \text{for all } b \in L.$$

Lemma 2.10. *Let R be a 2-torsion free semiprime ring and L be a square-closed Lie ideal of R. If $\Lambda(a, b, c) = 0$ for all $a, b, c \in L$, then $L \subseteq Z(R)$.*

Proof. Assume that $L \nsubseteq Z(R)$. We have $\Lambda(a, b, c) = 0$ for all $a, b, c \in L$ that is, $abc = cba$. Replacing b by $2tb$, we get $2atbc = 2ctba$ for all $a, b, c, t \in L$. Again replacing t by $2tw$ and using the fact that R is 2-torsion free to get, $atwbc = ctwba$ and hence $a(tw)bc = bc(tw)a = ba(tw)c = awtbc$. Thus we find that $a[t, w]bc = 0$ for all $a, b, c, t, w \in L$. By Corollary 2.2, we get $[t, w] = 0$ for all $t, w \in L$, that is L is a commutative Lie ideal of R. And so, we have $[a, [a, t]] = 0$ for all $t \in R$ and hence by Sublemma on page 5 of [11], $a \in Z(R)$. Hence $L \subseteq Z(R)$, a contradiction. This completes the proof of the theorem.

Proof of Theorem 2.1. Since $\alpha^{-1}\delta$ is a Jordan triple $(1, \alpha^{-1}\beta)$-derivation, replacing δ by $\alpha^{-1}\delta$ we may assume that δ is a Jordan triple $(1, \beta)$-derivation. Then, our goal will be to show that δ is a $(1, \beta)-$derivation of associative triple systems. We have

$$\begin{aligned} \Lambda(\Delta(a, b, c), r, s)x\Lambda(\Delta(a, b, c), r, s) &= (\Delta(a, b, c)rs - sr\Delta(a, b, c))x\Lambda(\Delta(a, b, c), r, s) \\ &= \Delta(a, b, c)rsx\Lambda(\Delta(a, b, c), r, s) \\ &\quad -sr\Delta(a, b, c)x\Lambda(\Delta(a, b, c), r, s). \end{aligned}$$

By Lemma 2.7, the above relation reduces to

$$\Lambda(\Delta(a, b, c), r, s)L\Lambda(\Delta(a, b, c), r, s) = 0, \text{ for all } a, b, c, r, s \in L.$$

By Corollary 2.2, we have

$$\Lambda(\Delta(a, b, c), r, s) = 0, \text{ for all } a, b, c, r, s \in L.$$

We obtain that

$$\Delta(a,b,c)rs - sr\Delta(a,b,c) = 0, \text{ for all } a,b,c,r,s \in L.$$

Using $\Delta(a,b,c), r, s \in L$ and Lemma 2.8, we have $\Delta(a,b,c) \in S$. This implies that

$$rs\Delta(a,b,c) - sr\Delta(a,b,c) = 0, \text{ for all } a,b,c,r,s \in L.$$

That is,

$$[r,s]\Delta(a,b,c) = 0, \text{ for all } a,b,c,r,s \in L. \tag{2.7}$$

Similarly, we have

$$\Delta(a,b,c)[r,s] = 0, \text{ for all } a,b,c,r,s \in L. \tag{2.8}$$

Let $a \in S$ and $b,c \in L$. Thus, $a, ab, ac, abc \in C(L)$ and $abc = cba$. Consider $N = abcxcba$. We have

$$\begin{array}{rcl} N^{\delta} & = & (a(bcxcb)a)^{\delta} \\ & = & a^{\delta}(bcxcba) + \beta(a)b^{\delta}(cxcba) + \beta(ab)c^{\delta}(xcba) \\ & & +\beta(abc)x^{\delta}(cba) + \beta(abcx)c^{\delta}(ba) + \beta(abcxc)b^{\delta}(a) \\ & & +\beta(abcxcb)a^{\delta}. \end{array}$$

On the other hand, we have

$$\begin{array}{rcl} N^{\delta} & = & ((abc)x(cba))^{\delta} = ((abc)x(abc))^{\delta} \\ & = & (abc)^{\delta}(xabc) + \beta(abc)x^{\delta}(abc) + \beta(abcx)(abc)^{\delta} \end{array}$$

Comparing the last two equations and using $abc = cba$, we have

$$\Delta(a,b,c)xabc + \beta(abc)\beta(x)\Delta(c,b,a) = 0.$$

Using $\Delta(a,b,c) = -\Delta(c,b,a)$, we have

$$\Delta(a,b,c)xabc - \beta(abc)\beta(x)\Delta(a,b,c) = 0.$$

Since $abc \in C(L)$, we find that

$$-\Delta(a,b,c)abcx + \beta(abc)\beta(x)\Delta(a,b,c) = 0.$$

Using $abcx \in C(L)$, we have

$$-(abc)x\Delta(a,b,c) + \beta(abc)\beta(x)\Delta(a,b,c) = 0.$$

This implies that

$$(\beta(abc)\beta(x) - (abc)x)\Delta(a,b,c) = 0.$$

By Lemma 2.9, we have

$$(abc)(\beta(x) - x)\Delta(a,b,c) = 0, \text{ for all } a,b,c,x \in L.$$

Multiplying y form the right hand side, using $abc \in C(L)$ and $\Delta(a,b,c) \in S$, we have

$$(\beta(x) - x)(abc)y\Delta(a,b,c) = 0, \text{for all } a,b,c,x,y \in L.$$

By Lemma 2.4, we have

$$(\beta(x)-x)(srt)y\Delta(a,b,c)=0, \text{ for all } a,s\in S \text{ and } x,r,t,b,c,y\in L.$$

Using $\Delta(a,b,c)\in S$, we have

$$(\beta(x)-x)\Delta(a,b,c)^2L(\beta(x)-x)\Delta(a,b,c)^2=0, \text{ for all } a\in S \text{ and } x,b,c\in L.$$

By Corollary 2.2 and using $abc=cba$, for all $b,c\in L$, we have

$$(\beta(x)-x)\Delta(a,b,c)^2=0, \text{ for all } a\in S \text{ and } x,b,c\in L.$$

Using $\Delta(a,b,c)\in S$, we get

$$\Delta(a,b,c)^2(\beta(x)-x)=0, \text{ for all } a\in S \text{ and } x,b,c\in L. \tag{2.9}$$

Using equations (2.8) and (2.9), we have

$$\begin{aligned}
2\Delta(a,b,c)^3 &= \Delta(a,b,c)^2\Delta(a,b,c)+\Delta(a,b,c)^2\Delta(a,b,c)\\
&= \Delta(a,b,c)^2\Delta(a,b,c)-\Delta(a,b,c)^2\Delta(c,b,a)\\
&= \Delta(a,b,c)^2(\Delta(a,b,c)-\Delta(c,b,a))\\
&= \Delta(a,b,c)^2((abc)^\delta-a^\delta(bc)-\beta(a)b^\delta c-\beta(ab)c^\delta\\
&\quad -(cba)^\delta+c^\delta(ba)+\beta(c)b^\delta(a)+\beta(cb)a^\delta)\\
&= \Delta(a,b,c)^2(-a^\delta(bc)-\beta(a)b^\delta c-\beta(ab)c^\delta+c^\delta(ba)\\
&\quad +\beta(c)b^\delta(a)+\beta(cb)a^\delta)\\
&= \Delta(a,b,c)^2(-a^\delta(bc)-\beta(a)b^\delta c-\beta(ab)c^\delta+c^\delta(ba)\\
&\quad +\beta(c)b^\delta(a)+\beta(cb)a^\delta\\
&\quad +a^\delta\beta(bc)-a^\delta\beta(bc)+a^\delta\beta(cb)-a^\delta\beta(cb)+ab^\delta c-ab^\delta c)\\
&= \Delta(a,b,c)^2(a^\delta(\beta(bc)-bc)-a^\delta(\beta(bc)-\beta(cb))+(\beta(cb)a^\delta-a^\delta\beta(cb))\\
&\quad -(\beta(a)-a)b^\delta c+(\beta(c)-c)b^\delta a+(ab-\beta(ab))c^\delta)\\
&= \Delta(a,b,c)^2(a^\delta(\beta(bc)-bc)-a^\delta[\beta(b),\beta(c)]\\
&\quad +[\beta(cb),a^\delta]-(\beta(a)-a)b^\delta c+(\beta(c)-c)b^\delta a+(ab-\beta(ab))c^\delta)\\
&= 0.
\end{aligned}$$

We have, $2\Delta(a,b,c)^3=0$. Since R is 2-torsion free, we have $\Delta(a,b,c)^3=0$. Using $\Delta(a,b,c)\in S$, we have $\Delta(a,b,c)^2x\Delta(a,b,c)^2=0$, for all $x\in L$. By Corollary 2.2, we have $\Delta(a,b,c)^2=0$. Similarly, we get $\Delta(a,b,c)=0$, for all $a\in S$ and $b,c\in L$. Also, if $a\in S$, then $aL\subseteq C(L)$ and $\beta(a),\beta^{-1}(a)\in S$. Let $a\in S$ and $x,y,b,c\in L$. Using the last equation, we have

$$\begin{aligned}
(ayxbc)^\delta &= ((ayx)bc)^\delta=(ayx)^\delta(bc)+\beta(ayx)b^\delta c+\beta((ayx)b)c^\delta\\
&= (a^\delta(yx)+\beta(a)y^\delta x+\beta(ay)x^\delta)(bc)+\beta(ayx)b^\delta c+\beta((ayx)b)c^\delta.
\end{aligned}$$

On the other hand,

$$(ayxbc)^{\delta} = a^{\delta}(yxbc) + \beta(a)y^{\delta}xbc + \beta(ay)(xbc)^{\delta}.$$

Comparing the last two equations, we have

$$ay\beta^{-1}(\Delta(x,b,c)) = 0, \text{ for all } a \in S \text{ and } x,b,c \in L.$$

Replacing a by $\beta^{-1}(\Delta(x,b,c))$, we have

$$\beta^{-1}(\Delta(x,b,c))L\beta^{-1}(\Delta(x,b,c)) = 0, \text{ for all } x,b,c \in L.$$

Corollary 2.2, we find that

$$\Delta(x,b,c) = 0, \text{ for all } x,b,c \in L.$$

By Proposition 2.1, we conclude that δ is an $(1,\beta)-$derivation of L. This completes the proof of the theorem.

Example 2.1. *Let S be any ring and let* $R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a,b \in S \right\}$ *and* $L = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b \in S \right\}$. *Define* $d : R \to R$ *by* $d \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *and* $\beta : R \to R$ *by* $\beta \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -a & 0 & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *It is easy to check that R is a ring, L is a Lie ideal of R, β is an one to one, onto and d is a Jordan triple $(1,\beta)$-derivation on L but not an $(1,\beta)$-derivation.*

3. Generalized Jordan Triple $(\alpha,\beta)-$Derivations

An additive mapping $\mu : R \longrightarrow R$ is said to be a Jordan triple left centralizer on L if $(aba)^{\mu} = a^{\mu}ba$ for all $a,b \in L$ and called a Jordan left centralizer on L if $(a^2)^{\mu} = a^{\mu}a$.

To facilitate our discussion, we shall begin with the following lemma:

Lemma 3.1 ([12], Theorem 3.1)**.** *Let R be a 2-torsion free semiprime ring and $L \nsubseteq Z(R)$ be a square-closed Lie ideal. If $\mu : R \to R$ is Jordan triple left centralizer on L, then μ is a Jordan left centralizer on L.*

Theorem 3.1. *Let R be a $2-$torsion free semiprime ring, α,β be automorphisms of R and $L \nsubseteq Z(R)$ be a square-closed Lie ideal. If $F : R \to R$ is generalized Jordan triple $(\alpha,\beta)-$derivation on L such that $a^{\delta}, \beta(a) \in L$, then F is a generalized $(\alpha,\beta)-$derivation on L.*

Proof. We are given that F is a generalized Jordan triple (α, β)-derivation on L. Therefore we have

$$(aba)^F = a^F \alpha(ba) + \beta(a) b^\delta \alpha(a) + \beta(ab) a^\delta \text{ for all } a, b \in L. \tag{3.1}$$

In (3.1), we take δ is a Jordan triple (α, β)-derivation on L. Since R is a $2-$torsion free semiprime ring, so in view of Theorem 2.1, δ is (α, β)-derivation on L. Now we write $\Gamma = F - \delta$. Then

$$\begin{aligned} \Gamma(aba) &= (aba)^{F-\delta} \\ &= (aba)^F - (aba)^\delta \\ &= (a^F - a^\delta)\alpha(ba) \text{ for all } a, b \in L. \end{aligned}$$

Then we have $\Gamma(aba) = \Gamma(a)\alpha(ba)$ for all $a, b \in L$. So, $\alpha^{-1}\Gamma$ becomes a Jordan triple left centralizer. In other words $\alpha^{-1}\Gamma$ is a Jordan triple left centralizer on L. Since R is a $2-$torsion free semiprime ring one can conclude that $\alpha^{-1}\Gamma$ is a Jordan left centralizer by Lemma 3.1. Hence

$$\alpha^{-1}\Gamma(ab) = \alpha^{-1}\Gamma(a)b \text{ for all } a, b \in L.$$

That is, $\Gamma(ab) = \Gamma(a)\alpha(b)$ and hence F is of the form $F = \Gamma + \delta$, where δ is an $(\alpha, \beta)-$ derivation and $\Gamma(ab) = \Gamma(a)\alpha(b)$. Therefore, F is a generalized Jordan $(\alpha, \beta)-$derivation on L.

Since every generalized (α, β)-derivation is also a generalized Jordan Triple (α, β) derivation, we immediately obtain

Corollary 3.1. *Let R be a $2-$torsion free semiprime ring, α, β be automorphisms of R and $L \nsubseteq Z(R)$ be a square-closed Lie ideal. If $F : R \to R$ is generalized Jordan $(\alpha, \beta)-$derivation on L such that $a^\delta, \beta(a) \in L$, then F is a generalized $(\alpha, \beta)-$derivation on L.*

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ON THE K_a-CONTINUITY OF REAL FUNCTIONS

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ABSTRACT. The aim of the present paper is to define K_a-continuity which is associated to the number sequence $a = (a_n)$ and to give some new results.

1. Introduction and Preliminaries

Robbins proposed a problem and he asked readers to show that a function $f : \mathbb{R} \to \mathbb{R}$ with the following property has to be linear:

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} f(x_k) = f(x_0) \text{ whenever } \lim_n \frac{1}{n} \sum_{k=1}^{n} x_k = x_0, \ x_0 \in \mathbb{R},$$

in 1946 ([9]). Solution by R. C. Buck [10] was published in 1948 (the problem was also solved by five others). Since then, different type continuities defined and studied by authors. Antoni and Salat [3] defined the concept of $A-$continuity for real functions based on $A-$summability. After that the notion of $F-$continuity based on almost convergence ($F-$convergence) was introduced in the paper [11] by Öztürk. This method studied by Borsik and Salat [4] and they remark that almost convergence and $A-$summability are not equivalent. Also some authors studied different concepts of continuity [2, 10, 12, 13].

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_n)$ be a number sequence. The sequence $(A(x)_n)$ where $A(x)_n = \sum_{k=1}^{\infty} a_{nk} x_k$ is called the $A-$transform of x whenever the series converges for $n = 1, 2, 3, ...$ The sequence x is said to be $A-$summable to l if the sequence $(A(x)_n)$ converges to l and we write $A-\lim_n x_n = l$. A is called regular if $\lim_n x_n = l$ implies $A - \lim_n x_n = l$ ([5, 6]).

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A sequence (x_n) of real numbers is said to be almost convergent ($F-$convergent) to number l if

$$\lim_p \frac{1}{p}\sum_{k=1}^{p} x_{n+k} = l$$

holds uniformly in $n = 1, 2, 3, ...$ and we write $F - \lim_n x_n = l$[8].

Definition 1. *Let $A = (a_{nk})$ be a regular matrix of real numbers and (x_n) be a number sequence. A function $f : \mathbb{R} \to \mathbb{R}$ is $A-$continuous at a point $x_0 \in \mathbb{R}$ if $A - \lim_n f(x_n) = f(x_0)$ whenever $A - \lim_n x_n = x_0$([2, 3]).*

Definition 2. *A function $f : \mathbb{R} \to \mathbb{R}$ is $F-$continuous at a point $x_0 \in \mathbb{R}$ if $F - \lim_n f(x_n) = f(x_0)$ whenever $F - \lim_n x_n = x_0$.*

In the present paper, we study the concept of K_a-continuity based on K_a-convergence, was defined by Lazic and Jovovic [7]. It is now natural to ask: Is the K_a-continuity a special case of $A-$continuity or do K_a-continuity and $F-$continuity contain each other? In general the answer is no. Simple examples show that these continuity methods do not contain each other. Namely, these methods are overlap.

We now recall some definitions and properties:

The notion of K_a-convergence was defined by Lazic and Jovovic [7] in 1993, which is obviously associated to the matrix $A = (a_{nk})$,

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 & ... \\ a_2 & a_1 & 0 & 0 & \\ a_3 & a_2 & a_1 & 0 & \\ . & & & & \\ . & & & & \end{pmatrix}.$$

Let $a = (a_n)$ and (x_n) be number sequences, set $y_n = \sum_{i=1}^{n} a_{n-i+1} x_i \ (n = 1, 2, 3, ...)$, then we say that (y_n) is the K_a-*transformation* of the (x_n).

Definition 3. [7] *The sequence (x_n) of real numbers is said to be K_a-convergent to the number l if, its K_a-transformation (y_n) converges to the number l, i.e. $\lim_n y_n = l$. This limit is denoted by $K_a - \lim_n x_n = l$.*

Proposition 4. [7] *Let $a = (a_n)$ be a number sequence and the series $\sum a_n$ be absolutely convergent, i.e.*

$$\sum_{n=1}^{\infty} |a_n| < \infty. \tag{1}$$

(i) If (x_n) is convergent, $\lim_n x_n = l$ and the condition (1) is satisfied then,

$$K_a - \lim_n x_n = l \sum_{n=1}^{\infty} a_n.$$

(ii) The convergence method K_a is regular if and only if the condition (1) and

$$\sum_{n=1}^{\infty} a_n = 1 \tag{2}$$

are valid (for more properties and details, see also [7]*).*

Now, we will give examples which show that K_a-convergence and almost convergence do not imply each other.

Example 5. *Let $a = (a_n) = (2, 2, -2, 0, 0, ...)$ and let*

$$x = (x_i) = (1, 0, 1, -1, 2, -3, 5, -8, ...) \left[x_i = x_{i-2} - x_{i-1} \text{ for } i \geq 3\right].$$

Then,

$$(y_k) = \left(\sum_{i=1}^{k} a_{k-i+1} x_i\right) = (2, 2, 0, 0, ...).$$

Therefore $K_a - \lim_n x_n = 0$. However, $F - \lim_n x_n$ does not exist. Also, observe that $\sum_{n=1}^{\infty} a_n = 2$ and K_a is not regular.

Example 6. *Let $a = (a_n) = (1, 0, 1, 0, 0, ...)$ and let*

$$(x_i) = \left(1, 1, \frac{1}{2^3}, \frac{1}{2^4}, 1, 1, \frac{1}{2^7}, \frac{1}{2^8}, 1, 1, ...\right).$$

Then,

$$(y_n) = \left(\sum_{i=1}^{n} a_{n-i+1} x_i\right) = \left(1, 1, 1 + \frac{1}{2^3}, 1 + \frac{1}{2^4}, \frac{1}{2^3} + 1, \frac{1}{2^4} + 1, ...\right).$$

Hence $K_a - \lim_n x_n = 1$. However, $F - \lim_n x_n \neq 1$. Also, observe that $\sum_{n=1}^{\infty} a_n = 2$ and K_a is not regular.

Now, we introduce the notion of K_a-continuity.

Definition 7. *Let $a = (a_n)$ and (x_n) be number sequences. The function $f : \mathbb{R} \to \mathbb{R}$ is K_a-continuous at a point $x_0 \in \mathbb{R}$ if $K_a - \lim_n f(x_n) = f(x_0)$ whenever $K_a - \lim_n x_n = x_0$.*

Lemma 8. *If (f_n) is a sequence of K_a-continuous functions defined on a subset D of $\mathbb{R}$, $\sum\limits_{n=1}^{\infty}|a_n| = M \neq 0$ and (f_n) is uniformly convergent to a function f, then f is K_a-continuous on D.*

Proof. Let (x_n) be a K_a-convergent sequence and $\varepsilon > 0$. Since (f_n) is uniformly convergent, then there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2(M+1)}$ for all $x \in D$, whenever $n \geq N$. As f_N is K_a-continuous, there exists a positive integer N_1, greater than N, such that $\left|\sum\limits_{i=1}^{n} a_{n-i+1} f_N(x_i) - f_N(x_0)\right| < \frac{\varepsilon}{2}$ for $n \geq N_1(\varepsilon)$. Then, for all $n \geq N_1$, we get

$$\begin{aligned}\left|\sum_{i=1}^{n} a_{n-i+1} f(x_i) - f(x_0)\right| &\leq \left|\sum_{i=1}^{n} a_{n-i+1}\left(f(x_i) - f_N(x_i)\right)\right| \\ &\quad + \left|\sum_{i=1}^{n} a_{n-i+1} f_N(x_i) - f_N(x_0)\right| \\ &\quad + |f_N(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{2(M+1)}M + \frac{\varepsilon}{2} + \frac{\varepsilon}{2(M+1)} = \varepsilon.\end{aligned}$$

This completes the proof. $\square$

2. Main Results

In this section we prove our main theorems.

Theorem 9. *Let $a = (a_n)$ be a number sequence and $\sum\limits_{n=1}^{\infty}|a_n| < \infty$. If $f : \mathbb{R} \to \mathbb{R}$ is K_a-continuous at a point $x_0 \in \mathbb{R}$, then f is a linear function.*

Proof. Let $\sum\limits_{n=1}^{\infty} a_n = N$ and $N \neq 0$. First, we can assume that $g : \mathbb{R} \to \mathbb{R}$ is K_a-continuous at a point 0 and $g(0) = 0$ as a special case.

Let $x = (b, c, d, b, c, d, ...)$ such that $b, c, d \in \mathbb{R}$ and $b + c + d = 0$ and let $a = (a_n) = (1, 1, 1, 0, 0, ...)$. Then the sequence K_a-convergent to 0. Indeed,

$$\left(\sum_{i=1}^{n} a_{n-i+1} x_i\right) = (b, b + c, 0, 0, ...).$$

This means $K_a - \lim\limits_n x_n = 0$. According to assumption, we have $K_a - \lim\limits_n g(x_n) = g(0) = 0$, i.e., the sequence $(g(x_n)) = (g(b), g(c), g(d), ...)$ is K_a-convergent to

0. Also, by a direct calculation, we can see that

$$\begin{aligned} & \left(\sum_{i=1}^{n} a_{n-i+1} g\left(x_i\right)\right) \\ = \quad & \left(g\left(b\right), g\left(b\right)+g\left(c\right), g\left(b\right)+g\left(c\right)+g\left(d\right), g\left(c\right)+g\left(d\right)+g\left(b\right), \ldots\right)., \end{aligned}$$

$K_a-\lim_n g\left(x_n\right)=g\left(b\right)+g\left(c\right)+g\left(d\right)$. Hence

$$g\left(b\right)+g\left(c\right)+g\left(d\right)=0 \tag{3}$$

Since $d=-b-c$, we get $g\left(-b-c\right)=-g\left(b\right)-g\left(c\right)$. Putting $c=0$ we have

$$g\left(-b\right)=-g\left(b\right) \quad \left(b\in\mathbb{R}\right) \tag{4}$$

Let $x,y\in\mathbb{R}$ arbitrary. Put $d=x+y$, $b=-x$, $c=-y$ then $b+c+d=0$ and according to (3) and (4), we get

$$g\left(x+y\right)=-g\left(-x\right)-g\left(-y\right)=g\left(x\right)+g\left(y\right),\ g\left(nx\right)=ng\left(x\right).$$

If a sequence (x_n) is K_a-convergent to zero, so that $\lim_n \sum_{i=1}^{n} a_{n-i+1}x_i=0$, then it can be seen that

$$\lim_n \sum_{i=1}^{n} a_{n-i+1} g\left(x_i\right)=\lim_n g\left(\sum_{i=1}^{n} a_{n-i+1}x_i\right)=0.$$

Hence g is continuous in the usual sense at zero. On the basis of well known knowledge on Cauchy equation we get $g\left(x\right)=Cx$ for $x\in\mathbb{R}$, C being a constant (p. 44-45, [1]).

Now, we shall discuss the general case. Let $f:\mathbb{R}\to\mathbb{R}$ be K_a-continuous at a point $x_0\in\mathbb{R}$. We write new coordinates $x'=x-x_0$, $y'=Ny-f\left(x_0\right)$. Put $g\left(x'\right)=Nf\left(x\right)-f\left(x_0\right)$. It is easy to see that from the K_a-continuity of f at x_0 the K_a-continuity of g at 0 follows. Hence, g has the form $g\left(x'\right)=C'x'$, i.e., $Nf\left(x\right)-f\left(x_0\right)=C'x'=C'\left(x-x_0\right)=C'x-C'x_0$, $f\left(x\right)=\frac{C'}{N}x+\frac{-C'x_0+f\left(x_0\right)}{N}=Cx+B$ where $C=\frac{C'}{N}$ and $B=\frac{-C'x_0+f\left(x_0\right)}{N}$. The proof is finished. □

Theorem 10. *Let $a=(a_n)$ be a number sequence, $\sum_{n=1}^{\infty}|a_n|<\infty$ and $f:\mathbb{R}\to\mathbb{R}$ have the following property:*

there exists such a point $x_0\in\mathbb{R}$ that the following implication

$$K_a-\lim_n x_n=x_0\Rightarrow\lim_n f\left(x_n\right)=\frac{f\left(x_0\right)}{N}, \tag{5}$$

where $N=\sum_{n=1}^{\infty}a_n\left(N\neq 0\right)$, is valid. Then f is a constant function.

Proof. From (5) and Proposition 4, we have

$$K_a - \lim_n x_n = x_0 \Rightarrow K_a - \lim_n f(x_n) = f(x_0).$$

Hence f is K_a−continuous at a point $x_0 \in \mathbb{R}$. The Theorem 9 says that f is linear. Put $b = x_0 - 1$, $c = x_0 + 1$ and $a = (a_n) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, ...\right)$. Then the sequence $(x_n) = (b, c, b, c, ...)$ is K_a−convergent to x_0, i.e.,

$$\left(\sum_{i=1}^{n} a_{n-i+1} x_i\right) = \left(\frac{x_0 - 1}{2}, x_0, x_0, ...\right),$$

$K_a - \lim_n x_n = x_0$. It follows from (5) that

$$(f(x_n)) = (f(b), f(c), f(b), f(c), ...)$$

converges. The last statement yields

$$f(b) = f(c). \tag{6}$$

Since f is a linear function it follows from (6) that f is a constant function. □

We note that if $\sum_{n=1}^{\infty} a_n = 1$ then the matrix $A = (a_{nk})$ given via the sequence $a = (a_n)$ is regular. In that case, the K_a−continuity is a special case of A−continuity. But, here $\sum_{n=1}^{\infty} |a_n| < \infty$ and therefore our main theorems Theorem 9 and Theorem 10 are not a consequence of the results concerning the A−continuity.

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DECOMPOSITION OF SOFT CONTINUITY VIA SOFT LOCALLY b-CLOSED SET

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ABSTRACT. In this paper, we introduce soft locally b-closed sets in soft topological spaces which are defined over an initial universe with a fixed set of parameters and study some of their properties. We investigate their relationships with different types of subsets of soft topological spaces with the help of counterexamples. Also, the concept of soft locally b-continuous functions is presented. Finally, a decomposition of soft continuity is obtained.

1. INTRODUCTION

Molodtsov [18, 19] initiated and applied soft set theory, while modelling the problems in the field of science including engineering physics, computer science, economics, social sciences and medical sciences, to deal with uncertain data and not clear objects without complete information. Then many researchers [9, 16, 17, 20, 24] presented some new definitions and results and also discussed in detail the application of soft set theory in decision making problems.

In [20], Shabir and Naz introduced the primary concepts of soft topological spaces. Later, Aygünoğlu and Aygün [7], Hussain and Ahmad [11], Yuksel et al. [23], Zorlutuna et al. [24] continued to study many basic concepts and properties of soft topological spaces. Kharal and Ahmad [13] and Zorlutuna et al. [24] discussed the mappings of soft classes and their properties in soft topological spaces. Recently, different forms of soft open sets [1, 2, 3, 6, 8, 12, 14, 15, 21] were studied.

In the present paper, we introduce soft locally b-closed set in soft topological spaces which are defined over an initial universe with a fixed set of parameters. Also, we give the notion of soft locally b-continuous function and obtain another decomposition of soft continuity.

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2. Preliminaries

Here, we present the basic definitions and results of soft sets and soft topological spaces which have already given in earlier studies. Let X be an initial universe set and E be the set of all possible parameters with respect to X. Let $P(X)$ denote the power set of X. Then a soft set over X is defined as follows.

Definition 1. [18] *A pair (F, A) is called a soft set over X where $A \subseteq E$ and $F : A \to P(X)$ is a set valued mapping. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\forall \varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon-$approximate elements of the soft set (F, A). It is worth noting that $F(\varepsilon)$ may be arbitrary. Some of them may be empty, and some may have nonempty intersection.*

The set of all soft sets over X is denoted by $SS(X)_E$. For null soft set (Φ), absolute soft set $(\widetilde{X})$, soft subset $(\sqsubseteq)$, soft union $(\sqcup)$, soft intersection $(\sqcap)$, soft relative complement, their properties and the relations to each other; the interested reader is refer to [5, 9, 17, 20, 24].

Definition 2. [24] *The soft set $(F, E) \in SS(X)_E$ is called a soft point in $\widetilde{X}$, denoted by e_F, if for the element $e \in E$, $F(e) \neq \emptyset$ and $F(e^{'}) = \emptyset$ for all $e^{'} \in E \backslash \{e\}$. The soft point e_F is said to be in the soft set (G, E), denoted by $e_F \in (G, E)$, if for the element $e \in E$ and $F(e) \subseteq G(e)$.*

Definition 3. [20] *Let $\widetilde{\tau}$ be the collection of soft sets over X, then $\widetilde{\tau}$ is said to be a soft topology on X if*

(1) Φ, $\widetilde{X}$ belong to $\widetilde{\tau}$,
(2) the union of any number of soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,
(3) the intersection of any two soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triplet $(X, \widetilde{\tau}, E)$ is called a soft topological space over X. The members of $\widetilde{\tau}$ are said to be soft open sets in X. A soft set (F, E) over X is said to be a soft closed set in X , if its relative complement $(F, E)^c$ belongs to $\widetilde{\tau}$. We will denote the family of all soft open sets (resp., soft closed sets) of a soft topological space $(X, \widetilde{\tau}, E)$ by SOS(X) (resp., SCS(X)).

Throughout the paper, the spaces X and Y stand for soft topological spaces with $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\upsilon}, K)$ assumed unless otherwise stated.

Definition 4. *Let X be a soft topological space and (F, E) be a soft set over X.*

(1) [20] The soft closure of (F, E) is the soft set in X defined as:
$cl(F, E) = \sqcap\{(G, E) :\ (G, E)$ is soft closed and $(F, E) \sqsubseteq (G, E)\}$.
(2) [24] The soft interior of (F, E) is the soft set in X defined as:
$int(F, E) = \sqcup\{(H, E) : (H, E)$ is soft open and $(H, E) \sqsubseteq (F, E)\}$.

Clearly, $cl(F, E)$ is the smallest soft closed set over X which contains (F, E) and $int(F, E)$ is the largest soft open set over X which is contained in (F, E).

Definition 5. *Let X be a soft topological space. A soft set (F,E) is called*

(1) a soft semiopen set [8] in X if $(F,E) \sqsubseteq cl(int(F,E))$,
(2) a soft preopen set [6] in X if $(F,E) \sqsubseteq int(cl(F,E))$,
(3) a soft α-open set [1] in X if $(F,E) \sqsubseteq int(cl(int(F,E)))$,
(4) a soft β-open set [12] in X if $(F,E) \sqsubseteq cl(int(cl(F,E)))$,
(5) a soft regular open set [22] in X if $(F,E) = int(cl(F,E))$,
(6) a soft A-set [21] in X if $(F,E) = (G,E)\backslash(H,E)$, where (G,E) is a soft open set and (H,E) is a soft regular open set in X,
(7) a soft t-set [21] in X if $int(cl(F,E)) = int(F,E)$,
(8) a soft B-set [21] in X if $(F,E) = (G,E) \sqcap (H,E)$, where (G,E) is a soft open set and (H,E) is a soft t-set in X,
(9) a soft b-open (briefly; sb-open) set [2] in X if $(F,E) \sqsubseteq int(cl(F,E)) \sqcup cl(int(F,E))$,
(10) a soft locally closed set (briefly; soft LC-set) [14] in X if $(F,E) = (G,E) \sqcap (H,E)$, where (G,E) is soft open and (H,E) is soft closed in X.

The relative complement of a soft semiopen (soft preopen, soft α-open, soft β-open, soft regular open, soft b-open) set is called a soft semiclosed (soft preclosed, soft α-closed, soft β-closed, soft regular closed, soft b-closed) set. We will denote the family of all soft semiopen sets (resp., soft preopen sets, soft α-open sets, soft β-open sets, soft regular open sets, soft A-sets, soft B-sets, soft b-open sets and soft locally closed sets) of a soft topological space X by SSOS(X) (resp., SPOS(X), SαOS(X), SβOS(X), SROS(X), SAS(X), SBS(X), SbOS(X) and SLCS(X)).

Definition 6. *Let X be a soft topological space and (F,E) be a soft set over X.*

(1) The soft semiclosure [8] of (F,E) is the soft set in X defined as:
$scl(F,E) = \sqcap\{(G,E) :\ (G,E)$ is soft semiclosed and $(F,E) \sqsubseteq (G,E)\}$.
(2) The soft semiinterior [8] of (F,E) is the soft set in X defined as:
$sint(F,E) = \sqcup\{(H,E) : (H,E)$ is soft semiopen and $(H,E) \sqsubseteq (F,E)\}$.
(3) The soft preclosure [3] of (F,E) is the soft set in X defined as:
$pcl(F,E) = \sqcap\{(G,E) :\ (G,E)$ is soft preclosed and $(F,E) \sqsubseteq (G,E)\}$.
(4) The soft preinterior [3] of (F,E) is the soft set in X defined as:
$pint(F,E) = \sqcup\{(H,E) : (H,E)$ is soft preopen and $(H,E) \sqsubseteq (F,E)\}$.
(5) The soft b-closure [2] of (F,E) is the soft set in X defined as:
$bcl(F,E) = \sqcap\{(G,E) :\ (G,E)$ is soft b-closed and $(F,E) \sqsubseteq (G,E)\}$.
(6) The soft b-interior [2] of (F,E) is the soft set in X defined as:
$bint(F,E) = \sqcup\{(H,E) : (H,E)$ is soft b-open and $(H,E) \sqsubseteq (F,E)\}$.

Theorem 7. [21] *Let X be a soft topological space. A soft set (F,E) over X is soft open if and only if it is both a soft preopen set and a soft B-set.*

Proof. Necessity is trivial, we prove the sufficiency. Since (F,E) is a soft B-set, we have $(F,E) = (G,E) \sqcap (H,E)$, where (G,E) is a soft open set and $int(cl(H,E)) =$

$int(H,E)$. Since (F,E) is soft preopen, we have

$$\begin{aligned}(F,E) &\sqsubseteq int(cl(F,E)) = int(cl((G,E)\sqcap(H,E)))\\ &\sqsubseteq int(cl(G,E)\sqcap cl(H,E)) = int(cl(G,E))\sqcap int(cl(H,E))\\ &= int(cl(G,E))\sqcap int(H,E).\end{aligned}$$

Hence

$$\begin{aligned}(F,E) &= (G,E)\sqcap(H,E) = ((G,E)\sqcap(H,E))\sqcap(G,E)\\ &\sqsubseteq (int(cl(G,E))\sqcap int(H,E))\sqcap(G,E)\\ &= (int(cl(G,E))\sqcap(G,E))\sqcap int(H,E) = (G,E)\sqcap int(H,E).\end{aligned}$$

Notice $(F,E) = (G,E)\sqcap(H,E) \sqsupseteq (G,E)\sqcap int(H,E)$, we have $(F,E) = (G,E)\sqcap int(H,E)$. Thus we obtain (F,E) is soft open. □

Theorem 8. [12] *Let X be a soft topological space. A soft set (F,E) over X is soft α-open if and only if it is both a soft preopen set and a soft semiopen set.*

Proof. Necessity. Since (F,E) is a soft α-open set, we have $(F,E)\sqsubseteq int(cl(int(F,E)))$. Hence $(F,E)\sqsubseteq cl(int(F,E))$ and $(F,E)\sqsubseteq int(cl(F,E))$. Thus (F,E) is both soft preopen and soft semiopen.

Sufficiency. Since (F,E) is both a soft preopen set and a soft semiopen set, then $(F,E)\sqsubseteq cl(int(F,E))$ and $(F,E)\sqsubseteq int(cl(F,E))$. Thus

$$(F,E)\sqsubseteq int(cl(cl(int(F,E)))) = int(cl(int(F,E))).$$

It follows that (F,E) is soft α-open. □

3. Soft Locally b-Closed Sets

In this section, we introduce soft locally b-closed sets in soft topological spaces and study some of their properties.

Definition 9. *A soft set (F,E) in a soft topological space X is called a soft locally b-closed set (briefly; soft LbC-set) if $(F,E) = (G,E)\sqcap(K,E)$ where (G,E) is soft open and (K,E) is soft b-closed.*

Remark 10. *The following examples show that a soft locally b-closed set need not be soft open and a soft locally b-closed set need not be soft b-closed.*

Example 11. *Let $X = \{x_1,x_2,x_3,x_4\}$, $E = \{e_1,e_2,e_3\}$ and*

$$\widetilde{\tau} = \{\Phi, \widetilde{X}, (F_1,E), (F_2,E), ..., (F_{15},E)\}$$

where $\Phi, \widetilde{X}, (F_1,E), (F_2,E), ..., (F_{15},E)$ are soft sets over X, defined as follows:

$$\begin{aligned}(F_1,E) &= \{(e_1,\{x_1\}),\ (e_2,\{x_2,x_3\}),(e_3,\{x_1,x_4\})\},\\ (F_2,E) &= \{(e_1,\{x_2,x_4\}),\ (e_2,\{x_1,x_3,x_4\}),(e_3,\{x_1,x_2,x_4\})\},\end{aligned}$$

$$\begin{aligned}
(F_3,E) &= \{(e_1,\emptyset),(e_2,\{x_3\}),(e_3,\{x_1\})\},\\
(F_4,E) &= \{(e_1,\{x_1,x_2,x_4\}),\ (e_2,X),(e_3,X)\},\\
(F_5,E) &= \{(e_1,\{x_1,x_3\}),\ (e_2,\{x_2,x_4\}),(e_3,\{x_2\})\},\\
(F_6,E) &= \{(e_1,\{x_1\}),\ (e_2,\{x_2\}),(e_3,\emptyset)\}\\
(F_7,E) &= \{(e_1,\{x_1,x_3\}),\ (e_2,\{x_2,x_3,x_4\}),(e_3,\{x_1,x_2,x_4\})\},\\
(F_8,E) &= \{(e_1,\emptyset),(e_2,\{x_4\}),(e_3,\{x_2\})\},\\
(F_9,E) &= \{(e_1,X),\ (e_2,X),(e_3,\{x_1,x_2,x_3\})\},\\
(F_{10},E) &= \{(e_1,\{x_1,x_3\}),\ (e_2,\{x_2,x_3,x_4\}),(e_3,\{x_1,x_2\})\},\\
(F_{11},E) &= \{(e_1,\{x_2,x_3,x_4\}),\ (e_2,X),(e_3,\{x_1,x_2,x_3\})\},\\
(F_{12},E) &= \{(e_1,\{x_1\}),\ (e_2,\{x_2,x_3,x_4\}),(e_3,\{x_1,x_2,x_4\})\},\\
(F_{13},E) &= \{(e_1,\{x_1\}),\ (e_2,\{x_2,x_4\}),(e_3,\{x_2\})\},\\
(F_{14},E) &= \{(e_1,\{x_3,x_4\}),\ (e_2,\{x_1,x_2\}),(e_3,\emptyset)\},\\
(F_{15},E) &= \{(e_1,\{x_1\}),\ (e_2,\{x_2,x_3\}),(e_3,\{x_1\})\}.
\end{aligned}$$

Then $\widetilde{\tau}$ defines a soft topology on X and thus $(X,\widetilde{\tau},E)$ is a soft topological space over X in [2]. Let (H,E) be a soft b-closed set over X such that $(H,E) = \{(e_1,\{x_1,x_2,x_3\}),\ (e_2,\{x_4\}),(e_3,\{x_1,x_3\})\}$. Then $(F_8,E)\sqcap(H,E) = (F,E) = \{(e_1,\emptyset),(e_2,\{x_4\}),(e_3,\emptyset)\}$ is a soft locally b-closed set in X, but (F,E) is not soft open.

Example 12. *Let $X=\{x_1,x_2,x_3,x_4\}$, $E=\{e_1,e_2\}$ and*

$$\widetilde{\tau} = \{\Phi,\widetilde{X},(F_1,E),(F_2,E),(F_3,E)\}$$

where $\Phi,\widetilde{X},(F_1,E),(F_2,E),(F_3,E)$ are soft sets over X, defined as follows:

$$\begin{aligned}
(F_1,E) &= \{(e_1,\{x_4\}),\ (e_2,\{x_2\})\},\\
(F_2,E) &= \{(e_1,\{x_2,x_4\}),\ (e_2,\{x_2,x_3,x_4\})\},\\
(F_3,E) &= \{(e_1,\{x_1,x_2,x_4\}),\ (e_2,X)\}.
\end{aligned}$$

Then $\widetilde{\tau}$ defines a soft topology on X and thus $(X,\widetilde{\tau},E)$ is a soft topological space over X in [10]. Clearly, $(F_2,E)=\{(e_1,\{x_2,x_4\}),\ (e_2,\{x_2,x_3,x_4\})\}$ is a soft locally b-closed set in X but not soft b-closed.

Proposition 13. *Let (F,E) be any soft set in a soft topological space X. (F,E) is soft locally b-closed if and only if there exists a soft open set (G,E) such that $(F,E)=(G,E)\sqcap bcl(F,E)$.*

Proof. Necessity. Since (F,E) is soft locally b-closed, $(F,E)=(G,E)\sqcap(K,E)$ where (G,E) is soft open and (K,E) is soft b-closed. Hence $(F,E)\sqsubseteq(G,E)$ and $(F,E)\sqsubseteq(K,E)$ then $(F,E)\sqsubseteq bcl(F,E)\sqsubseteq bcl(K,E)=(K,E)$. Therefore

$(F,E) \sqsubseteq (G,E) \sqcap bcl(F,E) \sqsubseteq (G,E) \sqcap bcl(K,E) = (G,E) \sqcap (K,E) = (F,E)$. Hence, $(H,E) = (G,E) \sqcap bcl(H,E)$.

Sufficiency. Since $bcl(F,E)$ is soft b-closed and $(F,E) = (G,E) \sqcap bcl(F,E)$, then (F,E) is soft locally b-closed. □

Theorem 14. [2] *In a soft topological space X, every soft closed set is soft b-closed.*

Now we are ready to give the relation between soft locally b-closed set and soft locally closed set.

Theorem 15. *In a soft topological space X, every soft locally closed set is a soft locally b-closed set.*

The following example shows that the converse implication does not hold.

Example 16. *Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and*

$$\widetilde{\tau} = \{\Phi, \widetilde{X}, (F_1,E), (F_2,E), (F_3,E)\}$$

where $\Phi, \widetilde{X}, (F_1,E), (F_2,E), (F_3,E)$ are soft sets over X, defined as follows:

$$\begin{aligned} (F_1,E) &= \{(e_1,\{x_1\}),\ (e_2,\{x_1\})\}, \\ (F_2,E) &= \{(e_1,\{x_2\}),\ (e_2,\{x_2\})\}, \\ (F_3,E) &= \{(e_1,\{x_1,x_2\}),\ (e_2,\{x_1,x_2\})\}. \end{aligned}$$

Then $\widetilde{\tau}$ defines a soft topology on X and thus $(X, \widetilde{\tau}, E)$ is a soft topological space over X. Clearly, $(F,E) = \{(e_1,\{x_3\}),\ (e_2,\{x_1,x_3\})\}$ is a soft locally b-closed set in X but not soft locally closed.

Theorem 17. [2] *In a soft topological space X,*
(1) An arbitrary union of soft b-open sets is a soft b-open set.
(2) The intersection of a soft open set and a soft b-open set is a soft b-open set.

From the Theorem 17, we have the following.

Corollary 18. *The intersection of a soft locally b-closed set and a soft locally closed set is soft locally b-closed.*

Proposition 19. *Let X be a soft topological space. If (F,E) is soft locally b-closed in X then*

(1) $bcl(F,E) \backslash (F,E)$ is a soft b-closed set.

(2) $[(F,E) \sqcup (\widetilde{X} \backslash bcl(F,E))]$ is soft b-open.

(3) $(F,E) \sqsubseteq bint((F,E) \sqcup (\widetilde{X} \backslash bcl(F,E)))$.

Proof. (1) If (F,E) is soft locally b-closed, there exists a soft open set (G,E) such that $(F,E) = (G,E) \sqcap bcl(F,E)$. Then

$$\begin{aligned} bcl(F,E)\backslash(F,E) &= bcl(F,E)\backslash[(G,E) \sqcap bcl(F,E)] \\ &= bcl(F,E) \sqcap [\widetilde{X}\backslash((G,E) \sqcap bcl(F,E))] \\ &= bcl(F,E) \sqcap [(\widetilde{X}\backslash(G,E)) \sqcup (\widetilde{X}\backslash bcl(F,E))] \\ &= [bcl(F,E) \sqcap (\widetilde{X}\backslash(G,E))] \sqcup [bcl(F,E) \sqcap (\widetilde{X}\backslash bcl(F,E))] \\ &= bcl(F,E) \sqcap (\widetilde{X}\backslash(G,E)) \end{aligned}$$

which is soft b-closed by Theorem 17.

(2) Since $bcl(F,E)\backslash(F,E)$ is soft b-closed, then $[\widetilde{X}\backslash(bcl(F,E)\backslash(F,E))]$ is soft b-open and

$$\widetilde{X}\backslash(bcl(F,E)\backslash(F,E)) = \widetilde{X}\backslash(bcl(F,E) \sqcap (\widetilde{X}\backslash(F,E))) = (F,E) \sqcup (\widetilde{X}\backslash bcl(F,E)).$$

(3) It is obvious that

$$(F,E) \sqsubseteq (F,E) \sqcup (\widetilde{X}\backslash bcl(F,E)) = bint[(F,E) \sqcup (\widetilde{X}\backslash bcl(F,E))].$$

□

Theorem 20. *Let X be closed under finite unions of soft b-closed sets. Then the following relation hold:*

$$bcl(F,E) \sqcup bcl(G,E) = bcl((F,E) \sqcup (G,E)).$$

Proof. We have $(F,E) \sqsubseteq (F,E) \sqcup (G,E)$ and $(G,E) \sqsubseteq (F,E) \sqcup (G,E)$. Since $bcl(F,E) \sqsubseteq bcl((F,E) \sqcup (G,E))$ and $bcl(G,E) \sqsubseteq bcl((F,E) \sqcup (G,E))$ we have $bcl(F,E) \sqcup bcl(G,E) \sqsubseteq bcl((F,E) \sqcup (G,E))$ [2].

Now, $bcl(F,E)$ and $bcl(G,E)$ are soft b-closed sets. Then we have $(F,E) \sqcup (G,E) \sqsubseteq bcl(F,E) \sqcup bcl(G,E)$ since $(F,E) \sqsubseteq bcl(F,E)$ and $(G,E) \sqsubseteq bcl(G,E)$. That is, $bcl(F,E) \sqcup bcl(G,E)$ is a soft b-closed set containing $(F,E) \sqcup (G,E)$. But $bcl((F,E) \sqcup (G,E))$ is the smallest soft b-closed set containing $(F,E) \sqcup (G,E)$. Hence $bcl((F,E) \sqcup (G,E)) \sqsubseteq bcl(F,E) \sqcup bcl(G,E)$. So, we obtain $bcl((F,E) \sqcup (G,E)) = bcl(F,E) \sqcup bcl(G,E)$. □

The union of two soft locally b-closed sets is generally not soft locally b-closed. To define the union of two soft locally b-closed sets, we need to give the following concept:

Definition 21. [23] *Let X be a soft topological space and (F,E), (G,E) are soft sets over X. (F,E) and (G,E) are said to be soft separated sets if $(F,E) \sqcap cl(G,E) = \Phi$ and $(G,E) \sqcap cl(F,E) = \Phi$.*

Theorem 22. *Suppose X is closed under finite unions of soft b-closed sets. Let (F,E) and (G,E) be soft locally b-closed. If (F,E) and (G,E) are soft separated, then $(F,E) \sqcup (G,E)$ is soft locally b-closed.*

Proof. Since (F,E) and (G,E) are soft locally b-closed, $(F,E) = (S,E) \sqcap bcl(F,E)$ and $(G,E) = (T,E) \sqcap bcl(G,E)$, where (S,E) and (T,E) are soft open in X. Put $(H,E) = (S,E) \sqcap (\widetilde{X} \backslash cl(G,E))$ and $(K,E) = (T,E) \sqcap (\widetilde{X} \backslash cl(F,E))$. Then $(H,E) \sqcap bcl(F,E) = ((S,E) \sqcap (\widetilde{X} \backslash cl(G,E))) \sqcap bcl(F,E) = (F,E) \sqcap (\widetilde{X} \backslash cl(G,E)) = (F,E)$, since $(F,E) \sqsubseteq (\widetilde{X} \backslash cl(G,E))$. Similarly, $(K,E) \sqcap bcl(G,E) = (G,E)$. And $(H,E) \sqcap bcl(G,E) \sqsubseteq (H,E) \sqcap cl(G,E) = \Phi$ and $(K,E) \sqcap bcl(F,E) \sqsubseteq (K,E) \sqcap cl(F,E) = \Phi$. Since (H,E) and (K,E) are soft open, $((H,E) \sqcup (K,E)) \sqcap bcl((F,E) \sqcup (G,E)) = ((H,E) \sqcup (K,E)) \sqcap (bcl(F,E) \sqcup bcl(G,E)) = ((H,E) \sqcap bcl(F,E)) \sqcup ((H,E) \sqcap bcl(G,E)) \sqcup ((K,E) \sqcap bcl(F,E)) \sqcup ((K,E) \sqcap bcl(G,E)) = (F,E) \sqcup (G,E)$. We obtain $(F,E) \sqcup (G,E)$ is soft locally b-closed. □

Lemma 23. [12] *Let X be a soft topological space, (F,E) a soft set over X and $e_K \in SS(X)_E$. Then $e_K \in pcl(F,E)$ if and only if $(F,E) \sqcap (G,E) \neq \Phi$ for every $(G,E) \in SPOS(X)$.*

Lemma 24. *Let X be a soft topological space. If (F,E) is a soft semiopen set, then $pcl(F,E) = cl(F,E)$.*

Proof. We have $pcl(F,E) \sqsubseteq cl(F,E)$ for every soft set (F,E) over X. We show that $cl(F,E) \sqsubseteq pcl(F,E)$ if $(F,E) \in SSOS(X)$. Let $e_K \in cl(F,E)$ and $e_K \in (G,E) \in SPOS(X)$, then $e_K \in (G,E) \sqsubseteq int(cl(G,E))$ and hence $(F,E) \sqcap int(cl(G,E)) \neq \Phi$. Since $(F,E) \in SSOS(X)$, $(F,E) \sqcap int(cl(G,E)) \sqsubseteq cl(int(F,E)) \sqcap int(cl(G,E)) \sqsubseteq cl(int(F,E) \sqcap cl(G,E)) \sqsubseteq cl((F,E) \sqcap (G,E))$. Therefore, we obtain $cl((F,E) \sqcap (G,E)) \neq \Phi$ and so $(F,E) \sqcap (G,E) \neq \Phi$. By Lemma 23, $e_K \in pcl(F,E)$ and hence $cl(F,E) \sqsubseteq pcl(F,E)$. □

Definition 25. *A soft set (F,E) in a soft topological space X is called a soft Ψ- set if $(F,E) = (G,E) \sqcap (K,E)$ where (G,E) is soft open and $int(cl(K,E)) \sqsubseteq (K,E)$.*

It is clear that every soft Ψ- set is a soft B-set.

Theorem 26. *Let X be a soft topological space. A soft set (F,E) over X is soft Ψ- set if and only if there exists a soft open set (G,E) such that $(F,E) = (G,E) \sqcap scl(F,E)$.*

Proof. Let (F,E) be a soft Ψ- set. Then there exists a soft open set (G,E) and a soft semiclosed set (H,E) such that $(F,E) = (G,E) \sqcap (H,E)$. We have $(F,E) \sqsubseteq (G,E)$, $(F,E) \sqsubseteq (H,E)$, $(F,E) \sqsubseteq scl(H,E)$, $(F,E) \sqsubseteq (G,E) \sqcap scl(F,E) \sqsubseteq (G,E) \sqcap (H,E) = (F,E)$. Hence $(F,E) = (G,E) \sqcap scl(F,E)$.

The converse is obvious since $scl(F,E)$ is soft semiclosed. □

Remark 27. *Soft semiopen sets and soft locally b-closed sets are independent from each other as shown in the following examples.*

Example 28. *Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3\}$. Let us take the soft topology $\widetilde{\tau}$ on X and the soft set $(F, E) = \{(e_1, \emptyset),\ (e_2, \{x_4\}),\ (e_3, \emptyset)\}$ in Example 11. Clearly, (F, E) is a soft locally b-closed set in X but not soft semiopen.*

Example 29. *Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2\}$. Let us take the soft topology $\widetilde{\tau}$ on X and the soft set $(G, E) = \{(e_1, \{x_2, x_3, x_4\}), (e_2, \{x_2, x_3, x_4\})\}$ in Example 12. Clearly, (G, E) is a soft semiopen set in X but not soft locally b-closed.*

Theorem 30. *Let X be a soft topological space. A soft set (F, E) over X is a soft B-set if it is soft locally b-closed and soft semiopen.*

Proof. Let (F, E) be soft locally b-closed and soft semiopen. Then by Proposition 13, there exists a soft open set (G, E) such that $(F, E) = (G, E) \sqcap bcl(F, E) = (G, E) \sqcap [scl(F, E) \sqcap pcl(F, E)]$. By Lemma 24, we have $(F, E) = (G, E) \sqcap [scl(F, E) \sqcap cl(F, E)] = (G, E) \sqcap scl(F, E)$. Hence (F, E) is a soft Ψ- set by Theorem 26, so (F, E) is a soft B-set. □

Remark 31. *Soft α-open sets and soft locally b-closed sets are independent from each other as shown in the following examples.*

Example 32. *Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3\}$. Let us take the soft topology $\widetilde{\tau}$ on X and the soft set $(F, E) = \{(e_1, \emptyset),\ (e_2, \{x_4\}),\ (e_3, \emptyset)\}$ in Example 11. Clearly, (F, E) is a soft locally b-closed set in X but not soft α-open.*

Example 33. *Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2\}$. Let us take the soft topology $\widetilde{\tau}$ on X and the soft set $(G, E) = \{(e_1, \{x_2, x_3, x_4\}), (e_2, \{x_2, x_3, x_4\})\}$ in Example 12. Clearly, (G, E) is a soft α-open set in X but not soft locally b-closed.*

Theorem 34. *Let X be a soft topological space. A soft set (F, E) over X is soft open if and only if it is both a soft α-open set and a soft locally b-closed set.*

Proof. It is immediate from Theorem 7, Theorem 8 and Theorem 30. □

Definition 35. [4] *Let X be a soft topological space. A soft set (F, E) over X is called a soft generalized b-closed set (briefly; soft gb-closed set) if $bcl(F, E) \sqsubseteq (G, E)$, whenever $(F, E) \sqsubseteq (G, E)$ and (G, E) is soft open.*

Theorem 36. *Let X be a soft topological space. A soft set (F, E) over X is soft b-closed if and only if it is both a soft gb-closed set and a soft locally b-closed set.*

Proof. Necessity. Let (F, E) be soft b-closed. $(F, E) = (F, E) \sqcap \widetilde{X}$, then (F, E) soft locally b-closed. Also, if $(F, E) \sqsubseteq (G, E)$ where (G, E) is soft open, then $bcl(F, E) = (F, E) \sqsubseteq (G, E)$. Hence (F, E) is soft gb-closed.

Sufficiency. If (F, E) is soft locally b-closed, then there exists a soft open set (G, E) such that $(F, E) = (G, E) \sqcap bcl(F, E)$. Since $(F, E) \sqsubseteq (G, E)$ and (F, E) is soft gb-closed then $bcl(F, E) \sqsubseteq (G, E)$. Therefore $bcl(F, E) \sqsubseteq (G, E) \sqcap bcl(F, E) = (F, E)$. Hence (F, E) is soft b-closed. □

4. Decompositions of Soft Continuity

In this section, we introduce soft locally b-continuous functions and give a decomposition of soft continuity via the notion of soft locally b-closed set.

Definition 37. [13] *Let $SS(X)_E$ and $SS(Y)_K$ be families of soft sets, $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Then the mapping $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ is defined as:*

(1) Let $(F, E) \in SS(X)_E$. The image of (F, E) under f_{pu}, written as $f_{pu}(F, E) = (f_{pu}(F), p(E))$, is a soft set in $SS(Y)_K$ such that

$$f_{pu}(F)(y) = \begin{cases} \cup_{x \in p^{-1}(y) \cap A} u(F(x)) & , p^{-1}(y) \cap A \neq \emptyset \\ \emptyset & , otherwise \end{cases}$$

for all $y \in K$.

(2) Let $(G, K) \in SS(Y)_K$. The inverse image of (G, K) under f_{pu}, written as $f_{pu}^{-1}(G, K) = (f_{pu}^{-1}(G), p^{-1}(K))$, is a soft set in $SS(X)_E$ such that

$$f_{pu}^{-1}(G)(x) = \begin{cases} u^{-1}(G(p(x))) & , p(x) \in K \\ \emptyset & , otherwise \end{cases}$$

for all $x \in E$.

Definition 38. *Let X and Y be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then f_{pu} is called*

(1) soft continuous [24] if for each $(G, K) \in$SOS(Y), $f^{-1}(G, K) \in$SOS(X).
(2) soft semicontinuous [15] if for each $(G, K) \in$SOS(Y), $f_{pu}^{-1}(G, K) \in$SSOS(X).
(3) soft α-continuous [1] if for each $(G, K) \in$SOS(Y), $f_{pu}^{-1}(G, K) \in$SαOS(X).
(4) soft B-continuous [21] if for each $(G, K) \in$SOS(Y), $f_{pu}^{-1}(G, K) \in$SBS(X).
(5) soft LC-continuous [14] if for each $(G, K) \in$SOS(Y), $f_{pu}^{-1}(G, K) \in$SLCS(X).
(6) soft b-continuous [2] if for each $(G, K) \in$SOS(Y), $f_{pu}^{-1}(G, K) \in$SbOS(X).

Definition 39. *Let X and Y be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then f_{pu} is called soft locally b-continuous if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft locally b-closed set in X.*

Remark 40. *Soft α-continuous functions and soft locally b-continuous functions are independent from each other as shown in the following examples.*

Example 41. *Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, $E = \{e_1, e_2\}$, $K = \{k_1, k_2\}$, $\widetilde{\tau} = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ in Example 16 and $\widetilde{\upsilon} = \{\Phi, \widetilde{Y}, (H, K)\}$ such that*

$$(H, K) = \{(k_1, \{y_1\}), (k_2, \{y_1, y_3\})\}.$$

Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\upsilon}, K)$ be soft topological spaces. Define $u : X \longrightarrow Y$, $p : E \longrightarrow K$ as

$u(x_1) = \{y_3\},\ u(x_2) = \{y_2\},\ u(x_3) = \{y_1\}$ and $p(e_1) = \{k_1\},\ p(e_2) = \{k_2\}$.

Let $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then (H,K) is soft open in Y and $f_{pu}^{-1}(H,K) = \{(e_1, \{x_3\}), (e_2, \{x_1, x_3\})\}$ is a soft locally b-closed set. Therefore, f_{pu} is soft locally b-continuous. But, $f_{pu}^{-1}(H,K)$ is not a soft α-open set and so f_{pu} is not soft α-continuous.

Example 42. *Let $X = \{x_1, x_2, x_3, x_4\},\ Y = \{y_1, y_2, y_3, y_4\},\ E = \{e_1, e_2\}$ and $K = \{k_1, k_2\}$ $\widetilde{\tau} = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ in Example 12 and $\widetilde{\upsilon} = \{\Phi, \widetilde{Y}, (G, K)\}$ such that*

$$(G,K) = \{(k_1, \{y_1, y_2, y_3\}), (k_2, \{y_1, y_2, y_3\})\}.$$

Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\upsilon}, K)$ be soft topological spaces. Define $u : X \longrightarrow Y,\ p : E \longrightarrow K$ as

$$u(x_1) = \{y_4\},\ u(x_2) = \{y_1\},\ u(x_3) = \{y_2\},\ u(x_4) = \{y_3\} \text{ and}$$
$$p(e_1) = \{k_1\},\ p(e_2) = \{k_2\}.$$

Let $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then (G,K) is soft open in Y and $f_{pu}^{-1}(G,K) = \{(e_1, \{x_2, x_3, x_4\}), (e_2, \{x_2, x_3, x_4\})\}$ is a soft α-open set. Therefore, f_{pu} is soft α-continuous. But, $f_{pu}^{-1}(G,K)$ is not a soft locally b-closed set and so f_{pu} is not soft locally b-continuous.

Theorem 43. *Let X and Y be soft topological spaces. Then $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ is soft continuous if and only if it is soft α-continuous and soft locally b-continuous.*

Proof. This is an immediate consequence of Theorem 34. □

Definition 44. *Let X and Y be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then f_{pu} is called soft gb-continuous if for each (G,K) ∈SOS(Y), $f_{pu}^{-1}(G,K)$ is a soft gb-closed set in X.*

Theorem 45. *Let X and Y be soft topological spaces. Then $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ is soft b-continuous if and only if it is soft gb-continuous and soft locally b-continuous.*

Proof. This is an immediate consequence of Theorem 36. □

5. Conclusion

In the present study, we have introduced soft locally b-closed sets in soft topological spaces which are defined over an initial universe with a fixed set of parameters and we have studied some of their properties. We have investigated their relationships with different types of subsets of soft topological spaces with the help of counterexamples. Also, the concept of soft locally b-continuous functions have been presented. Finally, a decomposition of soft continuity has been obtained.

In future, these findings may be extended to new types of soft sets such as soft α-locally closed and soft pre-locally closed sets in soft topological spaces. We expect that results in this paper will be helpfull for further studies in soft topological spaces.

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ESTIMATION OF POPULATION MEAN UNDER DIFFERENT STRATIFIED RANKED SET SAMPLING DESIGNS WITH SIMULATION STUDY APPLICATION TO BMI DATA

ARZU ECE CETIN AND NURSEL KOYUNCU

Abstract. In this article, we have compared the performance of ratio-type estimators in some stratified ranked set sampling methods. These sampling methods are stratified random sampling, stratified ranked set sampling, stratified double ranked set sampling and stratified median ranked set sampling. In these methods, the ratio type estimators using auxiliary variable information such as coefficient of variation and kurtosis are examined. We have used a real data set to see the performance of estimators. We use the data concerning body mass index (BMI) as a study variable and the age and the weight as auxiliary variables for 800 people in Turkey in 2014. We stratified the data set using gender. A simulation study is carried out to see performance of the proposed ratio type estimators in these stratified ranked set sampling designs. The performances of these estimators are compared in terms of mean squared error (MSE) and percent relative efficiency (PRE). The importance of this study is to compare these stratified sampling designs with those in the sampling literature by performing a detailed simulation using a real data set.

1. Introduction

Ranked set sampling (RSS) technique was first introduced by Mclntyre [11], and Dell and Clutter [3] showed that the mean of the RSS is an unbiased estimator of the population mean, whether or not there are errors in ranking. Following, stratified ranked set sampling (SRSS) was suggested by Samawi and Muttlak [13] to obtain a more efficient estimator for a population mean. Samawi [14] proposed an efficient estimator in stratified ranked set sampling. Al- Saleh and Al-Kaddiri [1] introduced the concept of double-ranked set sampling (DRSS) and showed that the DRSS estimator is more efficient than the usual RSS estimator in estimating the finite population mean. Using SRSS, the performances of the combined and

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separate ratio estimates were obtained by Samawi and Siam [15]. Muttlak [12] has suggested the median ranked set sampling (MRSS) method for estimating the population mean. Ibrahim at al. [5] suggested estimating the population mean using stratified median ranked set sampling (SMRSS). Al-Omari [2] suggested ratio estimation of the population mean using auxiliary information in simple random sampling (SRS) and median ranked set sampling (MRSS). Following Kadilar and Cingi [6], Mandowara and Mehta [10] used the idea of SRSS instead of stratified simple random sampling (SSRS) and obtained more efficient ratio type estimators. Koyuncu [7] has proposed ratio and exponential type estimators in MRSS. Khan et al. [9] improved ratio-type estimators using stratified double-ranked set sampling (SDRSS). Khan et al. [8] introduced efficient classes of ratio-type estimators of population mean under stratified median ranked set sampling. In this article we compares the performance ratio-type estimators given by Mandowara and Mehta [10], ratio-type estimators using stratified double-ranked set sampling (SDRSS) given by Khan et al. [9] and efficient classes of ratio-type estimators of population mean under stratified median ranked set sampling given by Khan et al [8]. The aim of this study is to make a performance comparison of the proposed ratio-type estimators in these stratified sampling designs as a result of a simulation study using a real data set. The remainder of the paper is organized as follows. In Section 2, the designs of stratified sampling methods are explained and their ratio-type estimators are given and MSE equations are offered. The results of simulation are reported in Section 3. Finally, we arrive at a conclusion from these results in the last section.

2. Stratified Sampling Methods

2.1. **Stratified Simple Random Sampling.** In stratified sampling the population of N units is first divided into L subpopulations of $N_1, N_2, \ldots, N_L$ units. These subpopulations, or also known as strata, are not overlapping and when combined together they form the whole population, i.e. $N_1 + N_2 + \ldots + N_L = N$ To obtain the full benefit from stratification, the values of the N_h, $h = 1, 2, \ldots, L$ must be known. After the strata have been determined, a sample is drawn from each stratum. The sample sizes within the strata are denoted by $n_1, n_2, \ldots, n_L$ respectively and $n = \sum_{h=1}^{L} n_h$ simple random sample is taken in each stratum, the whole procedure is described as stratified simple random sampling (SSRS). Let x_{hi} and y_{hi} show the observed values of the variable of interest and the auxiliary variable for h^{th} stratum, respectively. When the relationship between the X and Y variables is positive, Hansen et al. [4] proposed compound and separate proportional predictors, respectively,

$$\bar{y}_{RC} = \frac{\bar{y}_{st}}{\bar{x}_{st}}\bar{X} \tag{1}$$

$$\bar{y}_{RS} = \frac{\bar{y}_h}{\bar{x}_h}\bar{X}_h \tag{2}$$

where $\bar{y}_{st} = \sum_{h=1}^{L} W_h \overline{y}_h$, $\bar{x}_{st} = \sum_{h=1}^{L} W_h \overline{x}_h$ are the unbiased estimators of population means of the variable of interest and the auxiliary variable in SSRS and $W_h = N_h/N$ is the stratum weight.To the first degree of approximation the mean square errors (MSE) of the estimators $\overline{y}_{RC}$ and $\overline{y}_{RS}$ respectively are given as follows

$$MSE(\overline{y}_{RS}) = \sum_{h=1}^{L} W_h^2 \psi_h \overline{Y}_h^2 (C_{yh}^2 + C_{xh}^2 - 2C_{yxh}) \tag{3}$$

where $\psi_h = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)$ correction term for h^{th} stratum, C_{xh} and C_{yh} are the population coefficients of variation of auxiliary and study variables for h^{th} stratum, respectively. $C_{xyh} = \rho C_{yh} C_{xh}$ and ρ is the population correlation coefficient between the auxiliary and the study variables.

$$MSE(\overline{y}_{RC}) = \sum_{h=1}^{L} W_h^2 \psi_h (S_{yh}^2 + RS_{xh}^2 - 2RS_{yxh}) \tag{4}$$

where $S_{xh}^2 = \frac{1}{(N_h - 1)} \sum_{i=1}^{N_h} (X_{hi} - \overline{X}_h)^2$ and $S_{yh}^2 = \frac{1}{(N_h - 1)} \sum_{i=1}^{N_h} (Y_{hi} - \overline{Y}_h)^2$ are the population variances of the auxiliary and the study variables for h^{th} stratum, respectively. $S_{yxh} = \frac{1}{(N_h - 1)} \sum_{i=1}^{N_h} (Y_{hi} - \overline{Y}_h)(X_{hi} - \overline{X}_h)$ is the population covariance between auxiliary variate and variate of interest in stratum h, and $R = \frac{\overline{Y}}{\overline{X}}$ is the population ratio.

2.2. **Stratified Ranked Set Sampling.** In ranked set sampling, r independent random sets, each of size r and each unit in the set being selected with equal probability and without replacement, are selected from the population. The members of each random set are ranked with respect to the characteristic of the study variable or auxiliary variable. Then, the smallest unit is selected from the first ordered set and the second smallest unit is selected from the second ordered set. By this way, this procedure is continued until the unit with the largest rank is chosen from the r^{th} set. This cycle may be repeated m times, so $n = mr$ units have been measured during this process. In stratified ranked set sampling, for the h^{th} stratum of the population, first choose r_h independent samples each of size r_h, $h = 1, 2, ..., L$. Rank each sample, and use RSS scheme to obtain L independent RSS samples of size r_h, one from each stratum. Let $r_1 + r_2 + \ldots + r_L = r$. This complete one cycle of stratified ranked set sample. The cycle may be repeated m times until $n = mr$ elements have been obtained. A modification of the above procedure is suggested here to be used for the estimation of the ratio using stratified ranked set sample. For the h^{th} stratum, first choose r_h independent samples each of size r_h of independent bivariate elements from the h^{th} subpopulation (stratum) $h = 1, 2, ..., L$. Rank each sample with respect to one of the variables say Y or X. Then use the RSS sampling scheme to obtain L independent RSS samples of size r_h one from each stratum. This complete one cycle of stratified ranked set sample. Sampling units for stratified ranked set sample can be ranking is on the variable X or Y. When the ranking is on the variable Y, for the k^{th} cycle and the stratum h^{th}, the SRSS is denoted by $\{(Y_{h(1)k}, X_{h[1]k}), ..., (Y_{h(r_h)k}, X_{h[r_h]k}) : k = 1, 2, ..., m : h = 1, 2, ..., L\}$, where $Y_{h(i)k}$ is the i^{th} judgement ordering in the i^{th} set for the study variable and $X_{h[i]k}$ is the i^{th} order statistic in the i^{th} set for the auxiliary variable. When ranking in terms of Y and X variables, the formulas are the same, but the variable ordered is represented by the index () and the other variable is represented by the index []. The compound and separate ratio estimators of population mean respectively given by Samawi and Siam [15], using stratified ranked set sampling is defined as

$$\bar{y}_{SS(c)} = \frac{\bar{y}_{(SRSS)}}{\bar{x}_{[SRSS]}} \bar{X} \tag{5}$$

$$\bar{y}_{SS(s)} = \frac{\bar{y}_{(SRSS)}}{\bar{x}_{[SRSS]}} \bar{X}_h \tag{6}$$

where $\bar{y}_{(SRSS)} = \sum_{h=1}^{L} W_h \overline{y}_{h(r_h)}$ and $\bar{x}_{[SRSS]} = \sum_{h=1}^{L} W_h \overline{x}_{h[r_h]}$ are the unbiased estimators of population means $\overline{Y}$ and $\overline{X}$ in SRSS.

The MSE of the estimator $\bar{y}_{SS(c)}$ and $\bar{y}_{SS(s)}$ to the first degree of approximation are respectively given by

$$MSE(\overline{y}_{SS(c)}) \cong \sum_{h=1}^{L} \frac{W_h^2}{n_h} \overline{Y}_h^2 \left(\frac{S_{xh}^2}{\bar{X}^2} + \frac{S_{yh}^2}{\bar{Y}^2} - 2\frac{S_{xhyh}}{\bar{X}\bar{Y}} - \frac{m}{n_h} \sum_{i=1}^{r_h} (D_{xh[i]} - D_{yh(i)})^2 \right) \tag{7}$$

where $D_{yh(i)} = \dfrac{\mu_{yh(i)} - \mu_{yh}}{\bar{Y}}$ and $D_{xh[i]} = \dfrac{\mu_{xh[i]} - \mu_{xh}}{\bar{X}}$.

$$MSE(\overline{y}_{SS(s)}) \cong \sum_{h=1}^{L} \frac{W_h^2}{n_h} \left(S_{yh}^2 + R_h^2 S_{xh}^2 - 2R_h S_{xhyh} - \frac{m}{n_h} \sum_{i=1}^{r_h} (M_{xh[i]} - M_{yh(i)})^2 \right) \tag{8}$$

where $M_{yh(i)}^2 = \dfrac{(\mu_{yh(i)} - \mu_{yh})^2}{\bar{Y}_h^2}$, $M_{xh[i]}^2 = \dfrac{(\mu_{xh[i]} - \mu_{xh})^2}{\bar{X}_h^2}$ and

$M_{xh[i]yh(i)} = \dfrac{(\mu_{xh[i]} - \mu_{xh})(\mu_{yh(i)} - \mu_{yh})}{\bar{Y}_h \bar{X}_h}$.

Following Samawi and Siam[15], Mandowara and Mehta[10] suggested a modified ratio-type estimator for population mean ($\bar{Y}$) using SRSS, when the population coefficient of variation of the auxiliary variable for the h^{th} stratum C_{xh} and $\beta_{2h(x)}$ the coefficient of kurtosis of the auxiliary variable X in the h^{th} stratum, are known as

$$\bar{y}_{MM1} = \bar{y}_{(SRSS)} \frac{\bar{X} + C_{st}}{\bar{x}_{[SRSS]} + C_{st}} \tag{9}$$

$$\bar{y}_{MM2} = \bar{y}_{(SRSS)} \frac{\bar{X} + \beta_{st}}{\bar{x}_{[SRSS]} + \beta_{st}} \tag{10}$$

$$\bar{y}_{MM3} = \bar{y}_{(SRSS)} \frac{\bar{X}\beta_{st} + C_{st}}{\bar{x}_{[SRSS]}\beta_{st} + C_{st}} \tag{11}$$

$$\bar{y}_{MM4} = \bar{y}_{(SRSS)} \frac{\bar{X}C_{st} + \beta_{st}}{\bar{x}_{[SRSS]}C_{st} + \beta_{st}} \tag{12}$$

where $C_{st} = \sum_{h=1}^{L} W_h C_{xh}$ and $\beta_{st} = \sum_{h=1}^{L} W_h \beta_{2h(x)}$

The MSE of the estimators $\bar{y}_{MM1}$,$\bar{y}_{MM2}$,$\bar{y}_{MM3}$ and $\bar{y}_{MM4}$ to the first degree of approximation are respectively given by

$$MSE(\bar{y}_{MMj}) = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \left\{ S_{yh}^2 + R^2\lambda_j^2 S_{xh}^2 - 2R\lambda_j S_{xhyh} - \bar{Y}^2 \frac{m}{n_h} \sum_{i=1}^{r_h} (D_{yh(i)} - \lambda_j D_{xh[i]})^2 \right\} \tag{13}$$

where $\lambda_1 = \dfrac{\bar{X}}{\bar{X} + C_{st}}$, $\lambda_2 = \dfrac{\bar{X}}{\bar{X} + \beta_{st}}$, $\lambda_3 = \dfrac{\bar{X}\beta_{st}}{\bar{X}\beta_{st} + C_{st}}$ and $\lambda_4 = \dfrac{\bar{X}C_{st}}{\bar{X}C_{st} + \beta_{st}}$.

2.3. **Stratified Double Ranked Set Sampling.** In stratified double-ranked set sampling, for the h^{th} stratum of the population, first choose r_h^3 independent random samples $(h = 1, 2, ..., L)$. Arrange these selected units randomly into r_h sets, each of size r_h^2 . The procedure of RSS is then applied on each of the sets to obtain the r_h sets of ranked set samples each of size r_h . These ranked set samples are collected together to form r_h sets of observations each of size r_h . The RSS procedure is then applied again on this set to obtain L independent DRSS samples each of size r_h, to get $r_1 + r_2 + ... + r_L = r$ observations. This completes one cycle of $SDRSS$. The whole process is repeated m times to get the desired sample size $n = mr$. Following Samawi and Siam [15], Khan and et al. [9] propose combined ratio-type estimator of population mean $\overline{Y}$ using $SDRSS$ and is defined as

$$\bar{y}_{R(S_tDRSS)SS} = \bar{y}_{(S_tDRSS)} \left(\frac{\bar{X}}{\bar{x}_{[S_tDRSS]}} \right) \tag{14}$$

where $\bar{y}_{(S_tDRSS)} = \sum_{h=1}^{L} W_h \bar{y}_{h(DRSS)}$ and $\bar{x}_{[S_tDRSS]} = \sum_{h=1}^{L} W_h \bar{x}_{h[DRSS]}$ are the unbiased estimators of population means $\bar{Y}$ and $\bar{X}$ respectively in $SDRSS$.

The MSE of the estimators $\bar{y}_{R(S_tDRSS)SS}$ to the first degree of approximation is given by

$$\begin{aligned} \bar{y}_{R(S_tDRSS)SS} \cong & \left[\sum_{h=1}^{L} \frac{W_h^2}{mr_h} (C_{yh}^2 + C_{xh}^2 - 2\rho_{yxh} C_{xh} C_{yh}) \right] \\ & - \bar{Y}^2 \left[\sum_{h=1}^{L} \frac{W_h^2}{mr_h^2} \sum_{i=1}^{r_h} (W_{yh(i:rh)}^{(i:rh)} - W_{xh[i:rh]}^{[i:rh]})^2 \right] \end{aligned} \tag{15}$$

where

$$W^{2[i:rh]}_{yh[i:rh]} = \frac{(\mu^{(i:rh)}_{yh(i:rh)} - \bar{Y}_h)^2}{\bar{Y}^2}, \qquad W^{2[i:rh]}_{xh[i:rh]} = \frac{(\mu^{[i:rh]}_{xh[i:rh]} - \bar{X}_h)^2}{\bar{X}^2},$$

$$W^{[i:rh]}_{yh[i:rh]} = \frac{\mu^{(i:rh)}_{yh(i:rh)} - \bar{Y}_h}{\bar{Y}}, \qquad W^{[i:rh]}_{xh[i:rh]} = \frac{\mu^{[i:rh]}_{xh[i:rh]} - \bar{X}_h}{\bar{X}},$$

$$W^{[i:rh]}_{xh[i:rh]} W^{(i:rh)}_{yh(i:rh)} = \frac{(\mu^{(i:rh)}_{yh(i:rh)} - \bar{Y}_h)(\mu^{[i:rh]}_{xh[i:rh]} - \bar{X}_h)}{\bar{Y}\bar{X}}$$

Khan et al. [9] suggested efficient classes of ratio-type estimators of population mean under stratified median ranked set sampling and is defined as

$$\bar{y}_{R(S_tDRSS)SD} = \bar{y}_{(S_tDRSS)} \frac{\sum_{h=1}^{L} W_h(\bar{X}_h + C_{xh})}{\sum_{h=1}^{L} W_h(\bar{x}_{h[DRSS]} + C_{xh})} \tag{16}$$

$$\bar{y}_{R(S_tDRSS)KC} = \bar{y}_{(S_tDRSS)} \frac{\sum_{h=1}^{L} W_h(\bar{X}_h + C_{xh})}{\sum_{h=1}^{L} W_h(\bar{x}_{h[DRSS]} + C_{xh})} \tag{17}$$

$$\bar{y}_{R(S_tDRSS)US1} = \bar{y}_{(S_tDRSS)} \frac{\sum_{h=1}^{L} W_h(\bar{X}_h + \beta_{2(xh)})}{\sum_{h=1}^{L} W_h(\bar{x}_{h[DRSS]} + \beta_{2(xh)})} \tag{18}$$

$$\bar{y}_{R(S_tDRSS)US2} = \bar{y}_{(S_tDRSS)} \frac{\sum_{h=1}^{L} W_h(\bar{X}_h C_{xh} + \beta_{2h(x)})}{\sum_{h=1}^{L} W_h(\bar{x}_{h[DRSS]} C_{xh} + \beta_{2h(x)})} \tag{19}$$

The MSE of the estimators $\bar{y}_{R(S_tDRSS)SD}$, $\bar{y}_{R(S_tDRSS)KC}$, $\bar{y}_{R(S_tDRSS)US1}$, $\bar{y}_{R(S_tDRSS)US2}$, to the first degree of approximation are respectively given by

$$\begin{aligned} MSE(\bar{y}_{R(S_tDRSS)F}) =& \bar{Y}^2 \left[\sum_{h=1}^{L} \frac{W_h^2}{mr_h} (C_{yh}^2 + \lambda_i^2 C_{xh}^2 - 2\lambda_i \rho_{yxh} C_{xh} C_{yh}) \right] \\ & - \bar{Y} \left[\sum_{h=1}^{L} \frac{W_h^2}{mr_h^2} \sum_{i=1}^{r_h} \left(W^{(i:rh)}_{yh(i:rh)} - \lambda_i W^{[i:rh]}_{xh[i:rh]} \right)^2 \right] \end{aligned} \tag{20}$$

where $i = 1, 2, 3, 4$, $F = SD, KC, US1, US2$, $\lambda_1 = \dfrac{\bar{X}}{\bar{X} + C_{st}}$, $\lambda_2 = \dfrac{\bar{X}}{\bar{X} + \beta_{st}}$, $\lambda_3 = \dfrac{\bar{X}\beta_{st}}{\bar{X}\beta_{st} + C_{st}}$ and $\lambda_4 = \dfrac{\bar{X}C_{st}}{\bar{X}C_{st} + \beta_{st}}$.

2.4. **Stratified Median Ranked Set Sampling.** The MRSS procedure as proposed by Muttlak [12] can be formed by selecting r random samples of size n units from the population and rank the units within each sample with respect to a variable of interest. If the sample size r_h is odd, then from each sample select for the measurement the $\left(\dfrac{r_h + 1}{2}\right) th$ smallest ranked unit,i.e., the median of the sample. If the sample size n is even, then select for the measurement from the first

$\dfrac{r_h}{2}$ samples the $\left(\dfrac{r_h}{2}\right) th$ smallest ranked unit and from the second $\dfrac{r_h}{2}$ samples the $\left(\dfrac{r_h}{2}+1\right) th$ smallest ranked. The cycle can be repeated m times if needed to get a sample of size mr units. If the MRSS is performed in each stratum instead of SRSS described, the method is known as stratified median ranked set sampling $SMRSS$. To illustrate the method, let us consider the following two cases, if the subpopulations involve odd number of elements in each set, and the second example if the subpopulations involve even number of elements in each set. Note that the number of subpopulations (strata) is immaterial, either odd or even. Following Ibrahim et al. [5], Khan and et al. [8] propose two efficient classes of ratio-type estimators for estimating the finite population mean under stratified median ranked set sampling using the known auxiliary information . Khan and et al. [8] propose the following class of estimators in $SMRSS$, given by

$$\bar{y}_{(S_tMRSSk)p} = \bar{y}_{(S_tMRSSk)} \frac{\sum_{h=1}^{L} W_h(a_h\bar{X}_h + b_h)}{\sum_{h=1}^{L} W_h(a_h\bar{x}_{h[MRSS]} + b_h)} \tag{21}$$

where $\bar{y}_{(S_tMRSS)} = \sum_{h=1}^{L} W_h\bar{y}_{h(MRSS)}$ and $\bar{x}_{[S_tMRSS]} = \sum_{h=1}^{L} W_h\bar{x}_{h[MRSS]}$ are the unbiased estimators of population means $\bar{Y}$ and $\bar{X}$ respectively in $SMRSS$. Also, a_h and b_h are known population parameters, which can be coefficient of variation, coefficient of skewness, coefficient of kurtosis and coefficient of quartiles of the auxiliary variable and $k = O, E$ denote the sample size odd and even respectively.

The MSE of the estimators for odd and even sample sizes are respectively, given by

$$MSE(\bar{y}_{(S_tMRSSO)p}) \cong$$
$$\bar{Y}^2 \sum_{i=1}^{L} \frac{W_h^2}{r_h} \left[\frac{\sigma^2_{y_h\left(\frac{r_h+1}{2}\right)}}{\bar{Y}_h^2} + \lambda^2 \frac{\sigma^2_{x_h\left[\frac{r_h+1}{2}\right]}}{\bar{X}_h^2} - 2\lambda \frac{\sigma^2_{yx_h\left(\frac{r_h+1}{2}\right)}}{\bar{Y}_h\bar{X}_h} \right] \tag{22}$$

$$MSE(\bar{y}_{R(S_tMRSSE)p}) \cong$$
$$\bar{Y}^2 \sum_{i=1}^{L} \frac{W_h^2}{2r_h} \left[\left(\frac{\sigma^2_{y_h\left(\frac{r_h}{2}\right)} + \sigma^2_{y_h\left(\frac{r_h+2}{2}\right)}}{\bar{Y}_h^2} \right) + \lambda^2 \left(\frac{\sigma^2_{x_h\left[\frac{r_h}{2}\right]} + \sigma^2_{x_h\left[\frac{r_h+2}{2}\right]}}{\bar{X}_h^2} \right) \right]$$
$$-\lambda\bar{Y}^2 \sum_{h=1}^{L} \frac{W_h^2}{r_h} \left(\frac{\sigma_{yx_h\left(\frac{r_h}{2}\right)} + \sigma_{yx_h\left(\frac{r_h+2}{2}\right)}}{\bar{Y}_h\bar{X}_h} \right) \tag{23}$$

where $\lambda = \dfrac{\sum_{h=1}^{L} W_h a_h \bar{X}_h}{\sum_{h=1}^{L} W_h (a_h \bar{X}_h + b_h)}$

Khan and et al. [8] proposed $\bar{y}_{R(S_tMRSSk)p}$ a_h and b_h are known population parameters; coefficient of variation, coefficient of skewness, coefficient of kurtosis and coefficient of quartiles of the auxiliary variable,

$$a_h = 1, b_h = 0 \quad \bar{y}_{(S_tMRSSk)0} = \bar{y}_{(S_tMRSSk)} \left(\frac{\bar{X}}{\bar{x}_{[S_tMRSSk]}} \right) \tag{24}$$

$$a_h = 1, b_h = C_{xh} \quad \bar{y}_{(S_tMRSSk)1} = \bar{y}_{(S_tMRSSk)} \frac{\sum_{h=1}^{L} W_h (\bar{X}_h + C_{xh})}{\sum_{h=1}^{L} W_h (\bar{x}_{h[MRSS]} + C_{xh})} \tag{25}$$

$$a_h = 1, b_h = \beta_{2(xh)} \quad \bar{y}_{(S_tMRSSk)2} = \bar{y}_{(S_tMRSSk)} \frac{\sum_{h=1}^{L} W_h (\bar{X}_h + \beta_{2(xh)})}{\sum_{h=1}^{L} W_h (\bar{x}_{h[MRSS]} + \beta_{2(xh)})} \tag{26}$$

$$a_h = \beta_{2(xh)}, b_h = C_{xh} \quad \bar{y}_{(S_tMRSSk)3} = \bar{y}_{(S_tMRSSk)} \frac{\sum_{h=1}^{L} W_h (\bar{X}_h \beta_{2(xh)} + C_{xh})}{\sum_{h=1}^{L} W_h (\bar{x}_{h[MRSS]} \beta_{2(xh)} + C_{xh})} \tag{27}$$

$$a_h = C_{xh}, b_h = \beta_{2(xh)} \quad \bar{y}_{(S_tMRSSk)4} = \bar{y}_{(S_tMRSSk)} \frac{\sum_{h=1}^{L} W_h (\bar{X}_h C_{xh} + \beta_{2(xh)})}{\sum_{h=1}^{L} W_h (\bar{x}_{h[MRSS]} C_{xh} + \beta_{2(xh)})} \tag{28}$$

The MSE of $\bar{y}_{R(S_tMRSSk)1}$, $\bar{y}_{R(S_tMRSS2)p}$, $\bar{y}_{R(S_tMRSS3)p}$, $\bar{y}_{R(S_tMRSS4)p}$ and $\bar{y}_{R(S_tMRSS5)p}$ for odd and even sample sizes are respectively, given by

$$MSE(\bar{y}_{R(S_tMRSSO)p}) \cong \bar{Y}^2 \sum_{i=1}^{L} \frac{W_h^2}{r_h} \left[\frac{\sigma^2_{y_h\left(\frac{r_h+1}{2}\right)}}{\bar{Y}_h^2} + \lambda_i^2 \frac{\sigma^2_{x_h\left[\frac{r_h+1}{2}\right]}}{\bar{X}_h^2} - 2\lambda_i \frac{\sigma_{yx_h\left(\frac{r_h+1}{2}\right)}}{\bar{Y}_h \bar{X}_h} \right] \tag{29}$$

$$\begin{aligned} MSE(\bar{y}_{R(S_tMRSSE)p}) \cong {} & \bar{Y}^2 \sum_{i=1}^{L} \frac{W_h^2}{2r_h} \left[\left(\frac{\sigma^2_{y_h\left(\frac{r_h}{2}\right)} + \sigma^2_{y_h\left(\frac{r_h+2}{2}\right)}}{\bar{Y}_h^2} \right) + \lambda_i^2 \left(\frac{\sigma^2_{x_h\left[\frac{r_h}{2}\right]} + \sigma^2_{x_h\left[\frac{r_h+2}{2}\right]}}{\bar{X}_h^2} \right) \right] \\ & - \lambda_i \bar{Y}^2 \sum_{h=1}^{L} \frac{W_h^2}{r_h} \left(\frac{\sigma_{yx_h\left(\frac{r_h}{2}\right)} + \sigma_{yx_h\left(\frac{r_h+2}{2}\right)}}{\bar{Y}_h \bar{X}_h} \right) \end{aligned} \tag{30}$$

$p = 1,2,3,4,5$ $i = 0,1,2,3,4$ where $\lambda_0 = \dfrac{\bar{X}}{\bar{X}} = 1, \lambda_1 = \dfrac{\bar{X}}{\bar{X}+C_{st}}$, $\lambda_2 = \dfrac{\bar{X}}{\bar{X}+\beta_{st}}$, $\lambda_3 = \dfrac{\bar{X}\beta_{st}}{\bar{X}\beta_{st}+C_{st}}$ and $\lambda_4 = \dfrac{\bar{X}C_{st}}{\bar{X}C_{st}+\beta_{st}}$

Khan et al. [8] proposed an other class of ratio-type estimators in $SMRSS$, given by

$$MSE(\bar{y}_{(S_tMRSSk)G}) = \bar{y}_{(S_tMRSSk)}\left[\omega\frac{\sum_{h=1}^{L}W_h(\bar{X}_h+q_{[1h]})}{\sum_{h=1}^{L}W_h(\bar{x}_{h[MRSS]}+q_{[1h]})} + (1-\omega)\frac{\sum_{h=1}^{L}W_h(\bar{X}_h+q_{[3h]})}{\sum_{h=1}^{L}W_h(\bar{x}_{h[MRSS]}+q_{[3h]})}\right] \tag{31}$$

where ω is scalar quantity $q_{[1h]}$ and $q_{[3h]}$ are the first and third quartiles of auxiliary variable in the h^{th} stratum respectively.

The MSEs of $\bar{y}_{(S_tMRSSk)G}$, upto first order of approximation, for odd and even sample sizes are respectively, given by

$$\begin{aligned} MSE(\bar{y}_{(S_tMRSSO)G}) \cong & \bar{Y}^2\sum_{i=1}^{L}\frac{W_h^2}{r_h}\left[\frac{\sigma^2_{y_h(\frac{r_h+1}{2})}}{\bar{Y}_h^2} + (\eta_2 + k(\eta_1-\eta_2))^2\frac{\sigma^2_{x_h[\frac{r_h+1}{2}]}}{\bar{X}_h^2}\right] \\ & - 2\bar{Y}^2\left\{\eta_2+\omega(\eta_1-\eta_2)\right\}\sum_{h=1}^{L}\frac{W_h^2}{r_h}\left(\frac{\sigma^2_{yxh\left(\frac{r_h+1}{2}\right)}}{\bar{Y}_h\bar{X}_h}\right) \end{aligned} \tag{32}$$

$$\begin{aligned} & MSE(\bar{y}_{R(S_tMRSSE)G}) \\ & \cong \bar{Y}^2\sum_{i=1}^{L}\frac{W_h^2}{2r_h}\left[\frac{\sigma^2_{y_h(\frac{r_h+2}{2})}+\sigma^2_{y_h(\frac{r_h}{2})}}{\bar{Y}_h^2} + (\eta_2+\omega(\eta_1-\eta_2))^2\frac{\sigma^2_{x_h[\frac{r_h+2}{2}]}+\sigma^2_{x_h[\frac{r_h}{2}]}}{\bar{X}_h^2}\right] \\ & \quad - \left\{\eta_2+\omega(\eta_1-\eta_2)\right\}\bar{Y}^2\sum_{h=1}^{L}\frac{W_h^2}{r_h}\left(\frac{\sigma_{yx_h\left(\frac{r_h}{2}\right)}+\sigma_{yx_h\left(\frac{r_h+2}{2}\right)}}{\bar{Y}_h\bar{X}_h}\right) \end{aligned} \tag{33}$$

where $\eta_1 = \dfrac{\sum_{h=1}^{L}W_h\bar{X}_h}{\sum_{h=1}^{L}W_h\left(\bar{X}_h+q_{[1h]}\right)}$ and $\eta_2 = \dfrac{\sum_{h=1}^{L}W_h\bar{X}_h}{\sum_{h=1}^{L}W_h\left(\bar{X}_h+q_{[3h]}\right)}$.

Estimators proposed by Khan et al. [8] using the following values for the scalar number ω in the estimator $\bar{y}_{R(S_tMRSSk)G}$,

$$\omega = 1, \quad \bar{y}_{(S_tMRSSk)6} = \bar{y}_{(S_tMRSSk)} \frac{\sum_{h=1}^{L} W_h(\bar{X}_h + q_{[1h]})}{\sum_{h=1}^{L} W_h(\bar{x}_{h[MRSS]} + q_{[1h]})} \tag{34}$$

$$\omega = 0, \quad \bar{y}_{(S_tMRSSk)7} = \bar{y}_{(S_tMRSSk)} \frac{\sum_{h=1}^{L} W_h(\bar{X}_h + q_{[3h]})}{\sum_{h=1}^{L} W_h(\bar{x}_{h[MRSS]} + q_{[3h]})} \tag{35}$$

The MSE of $\bar{y}_{R(S_tMRSSk)6}$ and $\bar{y}_{R(S_tMRSSk)7}$ for odd and even sample sizes are respectively, given by

$$MSE(\bar{y}_{R(S_tMRSSO)G})) \cong \bar{Y}^2 \sum_{i=1}^{L} \frac{W_h^2}{r_h} \left[\frac{\sigma^2_{y_h\left(\frac{r_h+1}{2}\right)}}{\bar{Y}_h^2} + \eta_i^2 \frac{\sigma^2_{x_h\left[\frac{r_h+1}{2}\right]}}{\bar{X}_h^2} - 2\eta_i \frac{\sigma_{yx_h\left(\frac{r_h+1}{2}\right)}}{\bar{Y}_h \bar{X}_h} \right] \tag{36}$$

$$\begin{aligned} &MSE(\bar{y}_{R(S_tMRSSE)G}) \\ &\cong \bar{Y}^2 \sum_{i=1}^{L} \frac{W_h^2}{2r_h} \left[\left(\frac{\sigma^2_{y_h\left(\frac{r_h}{2}\right)} + \sigma^2_{y_h\left(\frac{r_h+2}{2}\right)}}{\bar{Y}_h^2} \right) + \eta_i^2 \left(\frac{\sigma^2_{x_h\left[\frac{r_h}{2}\right]} + \sigma^2_{x_h\left[\frac{r_h+2}{2}\right]}}{\bar{X}_h^2} \right) \right] \\ &\quad - \eta_i \bar{Y}^2 \sum_{h=1}^{L} \frac{W_h^2}{r_h} \left(\frac{\sigma_{yx_h\left(\frac{r_h}{2}\right)} + \sigma_{yx_h\left(\frac{r_h+2}{2}\right)}}{\bar{Y}_h \bar{X}_h} \right) \end{aligned} \tag{37}$$

where $i = 1, 2$ $G = 6, 7$

3. Simulation Study

In this section a simulation study is conducted to investigate the performance of SSRS, SRSS, $SDRSS$ and $SMRSS$ in ratio type estimators the population mean. To observe performances of the estimators, we use the real data concerning body mass index (BMI) as a study variable and the age and weight as auxiliary variable for 800 people in Turkey in 2014. We have investigated correlation quantity between study variable Y and auxiliary variable X for odd or even sample sizes. Also, we considered on both variable Y and X. The simulation study was performed first by using BMI with age variables and second by using BMI with weight variables. The correlation coefficient of BMI with age was 0.60 and the correlation coefficient with weight was 0.86. Thus, sampling methods were compared in different correlations. For both cases, 10000 samples of size $r_h = 4, 5, 6, 7$ were selected from $N = 800$ units using SSRS, SRSS, S_tDRSS and S_tMRSS methods. Also, we stratified the data set using gender ($h = 1, 2$). Estimators are compared in terms of mean squared errors (MSEs) and percent relative efficiencies (PREs). We used the following expressions to obtain the MSEs and PREs, respectively

$$MSE(\bar{y}_{(\alpha)}) = \frac{\sum_{k=1}^{10000} \left[\bar{y}_{(\alpha)} - \bar{Y}\right]^2}{10000} \tag{38}$$

$$PRE(\bar{y}_{(\alpha)}) = \frac{MSE(\bar{y}_{st})}{MSE(\bar{y}_{(\alpha)})} * 100 \tag{39}$$

$\alpha = st, SRSSk$, S_tMRSSk, $R(S_tDRSS)$, $RC, RS, SS(c)$, $MM1$, $MM2$, $MM3$, $MM4$, $R(S_tMRSS)1$, $R(S_tMRSS)2$, $R(S_tMRSS)3$, $R(S_tMRSS)4$, $R(S_tMRSS)5$, $R(S_tMRSS)6$, $R(S_tMRSS)7$, $R(S_tDRSS)SS$, $R(S_tDRSS)SD$, $R(S_tDRSS)KC$, $R(S_tDRSS)US1$, $R(S_tDRSS)US2$.

In this study, the PRE values of the average estimators in other sampling methods were calculated based on the classical mean estimator in the SSRS method. In Table 1, statistical summary of population information about BMI, age and weight variables are given. In Table 2, statistical summary of population stratified information about BMI, age and weight variables are given. Simulation results obtained when the auxiliary variable is taken as age are given in Table 3 and Table 4. The results obtained when the weight is taken as auxiliary variables are given in Table 5 and Table 6. Since the efficiency of the estimators changes according to the sample size being odd and even, the sample size is considered as 4, 5, 6 and 7 in the simulation study and the results are given in all tables. When ranking on variable X and Y, the results obtained by using the MSE and PRE formulas of the estimators calculated by using the population stratified information of the Body Mass Index (Y) and Age (X_1) variables of the SSRS, $SDRSS$, SRSS and $SMRSS$ methods are given in the following Table 3 and Table 4, respectively. When ranking on variable X and Y, the results obtained by using the MSE and PRE formulas of the estimators calculated by using the population stratified information of the Body Mass Index (Y) and Weight (X_2) variables of the SSRS, $SDRSS$, SRSS and $SMRSS$ methods are given in the following Table 5 and Table 6, respectively.

From Table 3, it can be easily seen that when the sample size was both odd and even and ranking on age, the lowest predictive value of MSE and the highest predictive value of PRE were found to be $\overline{y}_{(S_tMRSSk)7}$ the estimator proposed by Khan et al. [8].

From Table 4, it have seen that when the sample size was both odd and even and ranking on BMI, the lowest predictive value of MSE and the highest predictive value of PRE were found to be the $\overline{y}^*_{(S_tMRSSk)7}$ estimator proposed by Khan et al.[8].

From Table 5, it have seen that when the sample size was both odd and even and ranking on weight, the lowest predictive value of MSE and the highest predictive value of PRE were found to be the $\overline{y}_{(S_tMRSSk)6}$ estimator proposed by Khan et al. [8].

From Table 6, it can be easily seen that when the sample size was both odd and even and ranking on BMI, the lowest predictive value of MSE and the highest predictive value of PRE were found to be the $\overline{y}^*_{(S_t MRSSk)7}$ estimator proposed by Khan et al. [8].

According to the results of the ranking of X and Y variables, it was seen that the PRE value was highest in the ranking on Y and the MSE value was lowest in the ranking according to Y and the weight auxiliary variable gave the better results in the ranking according to Y.

Table 1. Population Information about Body Mass Index (Y), Age (X_1) and Weight (X_2) Variables

N=800	$\bar{X}_1$=30.12	$\bar{X}_2$=67.55
$\bar{Y}_1$=23.77	S^2_{x1}=121.84	S^2_{x2}=191.53
S^2_y=17.6	$\beta_{2(x_1h)} = 0.78$	$\beta_{2(x_2h)} = -0.58$
C_y=0.17	R_1=0.78	R_2=0.35
C_{x1}=0.36	R^2_1=0.62	R^2_2=0.12
C_{x2}=0.2	ρ_{x_1y}=0.6	ρ_{x_2y}=0.86
S_{x_1y}=28.24	S_{x_2y}=50.36	

Table 2. Population Stratified Information about Body Mass Index (Y), Age (X_1) and Weight (X_2) Variables

Age		**Weight**	
Stratum1	**Stratum2**	**Stratum1**	**Stratum2**
N_1=477	N_2=323	N_1=477	N_2=323
r_1=9	r_2=8	r_1=9	r_2=8
m_1=6	m_2=5	m_1=6	m_2=5
W_1=0.59	W_2=0.4	W_1=0.59	W_2=0.4
$\bar{X}_1$=27.68	$\bar{X}_2$=33.73	$\bar{X}_1$=59.99	$\bar{X}_2$=78.72
$\bar{Y}_1$=22.36	$\bar{Y}_2$=25.85	$\bar{Y}_1$=22.36	$\bar{Y}_2$=25.85
$R_{x1[1]}$=0.8	$R_{x1[2]}$=0.76	$R_{x2[1]}$=0.37	$R_{x2[2]}$=0.32
$S_{x1[1]}$=10.19	$S_{x1[2]}$=11.26	$S_{x2[1]}$=10.47	$S_{x2[2]}$=3.58
$S_{y1(1)}$=3.99	$S_{y1(2)}$=10.16	$S_{y2(1)}$=3.99	$S_{y2(2)}$=10.16
$S_{xy1(1)}$=25.37	$S_{xy1(2)}$=19.96	$S_{xy2(1)}$=37.7	$S_{xy2(2)}$=30.2
$C_{x1[1]}$=0.36	$C_{x1[2]}$=0.33	$C_{x2[1]}$=0.17	$C_{x2[2]}$=0.04
$C_{y1(1)}$=0.17	$C_{y1(2)}$=10.16	$C_{y2(1)}$=3.99	$C_{y2(2)}$=10.16
$\beta_{2[x_1(1)]}$=2.72	$\beta_{2[x_1(2)]}$=-0.27	$\beta_{2[x_2(1)]}$=2.72	$\beta_{2[x_1(2)]}$=-0.27
$\rho_{xy1(1)}$=0.62	$\rho_{xy1(2)}$=0.49	$\rho_{xy2(1)}$=0.9	$\rho_{xy2(2)}$=0.82

TABLE 3. MSE and PRE values of estimators according to even and odd of sample size when ranking on variable Age (X_1)

		r_h=4		r_h=5		r_h=6		r_h=7	
		MSE	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
SSRS	$\bar{y}_{RC}$	5.99	31.69	4.88	31.67	4.03	31.11	3.39	31.99
	$\bar{y}_{RS}$	6.09	31.14	4.96	31.15	4.11	30.54	3.44	31.53
	$\bar{y}_{R(S_tDRSS)SS}$	7.37	25.75	10.56	14.63	8.67	14.49	11.68	9.30
SDRSS	$\bar{y}_{R(S_tDRSS)SD}$	7.21	26.33	10.37	14.91	8.46	14.85	11.44	9.50
	$\bar{y}_{R(S_tDRSS)KC}$	6.73	28.20	9.78	15.81	7.85	16.00	10.74	10.12
	$\bar{y}_{R(S_tDRSS)US1}$	21.16	8.97	24.77	6.24	23.84	5.27	27.20	3.99
	$\bar{y}_{R(S_tDRSS)US2}$	6.12	30.99	8.97	17.24	7.03	17.86	9.75	11.15
	$\bar{y}_{SS(s)}$	3.76	50.43	2.72	56.67	2.07	60.45	1.66	65.24
	$\bar{y}_{SS(c)}$	3.67	51.66	2.66	57.97	2.03	61.69	1.63	66.50
SRSS	$\bar{y}_{MM1}$	3.59	52.78	2.61	59.16	1.99	62.88	1.60	67.72
	$\bar{y}_{MM2}$	3.37	56.31	2.45	62.88	1.88	66.60	1.51	71.50
	$\bar{y}_{MM3}$	7.67	24.75	7.17	21.55	6.70	18.74	6.46	16.80
	$\bar{y}_{MM4}$	2.89	65.64	2.13	72.37	1.65	76.02	1.34	80.85
	$\bar{y}_{R(S_tMRSSk)1}$	3.92	48.36	5.07	30.47	3.96	31.66	4.94	21.98
	$\bar{y}_{R(S_tMRSSk)2}$	3.81	49.80	4.93	31.36	3.84	32.67	4.79	22.66
	$\bar{y}_{R(S_tMRSSk)3}$	3.49	54.41	4.520	34.21	3.49	35.95	4.37	24.85
SMRSS	$\bar{y}_{R(S_tMRSSk)4}$	3.18	59.67	2.54	60.69	1.65	75.68	1.52	71.10
	$\bar{y}_{R(S_tMRSSk)5}$	2.64	71.85	3.44	44.93	2.53	49.56	3.23	33.60
	$\bar{y}_{R(S_tMRSSk)6}$	1.14	166.00	1.57	98.32	0.95	131.71	1.32	82.24
	$\bar{y}_{R(S_tMRSSk)7}$	0.84	223.81**	1.21	127.81	0.63	197.09	0.94	115.02

**Shows most efficient estimator.

TABLE 4. MSE and PRE values of estimators according to even and odd of sample size when ranking on variable Y

		r_h=4		r_h=5		r_h=6		r_h=7	
		MSE	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
SSRS	$\bar{y}^*_{RC}$	5.99	31.69	4.88	31.67	4.03	31.11	3.39	31.99
	$\bar{y}^*_{RS}$	6.09	31.14	4.96	31.15	4.11	30.54	3.44	31.53
	$\bar{y}^*_{R(S_tDRSS)SS}$	6.15	30.87	8.30	18.62	4.85	25.89	7.45	14.58
	$\bar{y}^*_{R(S_tDRSS)SD}$	6.01	31.59	8.11	19.06	4.72	26.77	7.28	14.92
SDRSS	$\bar{y}^*_{R(S_tDRSS)KC}$	5.60	33.89	7.55	20.46	4.37	28.74	6.78	16.03
	$\bar{y}^*_{R(S_tDRSS)US1}$	13.18	14.40	14.87	10.40	13.94	9.01	15.49	7.02
	$\bar{y}^*_{R(S_tDRSS)US2}$	4.85	39.12	6.50	23.78	3.80	33.03	5.88	18.45
	$\bar{y}^*_{SS(s)}$	6.14	30.88	4.83	31.95	4.10	30.57	3.45	31.48
	$\bar{y}^*_{SS(c)}$	6.00	31.61	4.72	32.73	4.01	31.29	3.37	32.21
	$\bar{y}^*_{MM1}$	5.85	32.43	4.60	33.57	3.91	32.08	3.29	33.01
SRSS	$\bar{y}^*_{MM2}$	5.41	35.05	4.26	36.24	3.62	34.62	3.05	35.59
	$\bar{y}^*_{MM3}$	8.98	21.13	8.40	18.40	7.92	15.85	7.48	14.51
	$\bar{y}^*_{MM4}$	4.48	42.37	3.54	43.57	3.01	41.61	2.54	42.66
	$\bar{y}^*_{R(S_tMRSSk)1}$	3.27	58.05	5.14	30.08	2.47	50.67	3.93	27.58
	$\bar{y}^*_{R(S_tMRSSk)2}$	3.17	59.74	4.99	30.97	2.40	52.26	3.82	28.44
	$\bar{y}^*_{R(S_tMRSSk)3}$	2.91	65.81	4.56	33.87	2.18	57.43	3.48	31.22
SMRSS	$\bar{y}^*_{R(S_tMRSSk)4}$	5.53	34.32	5.94	25.99	4.20	29.87	4.62	23.51
	$\bar{y}^*_{R(S_tMRSSk)5}$	2.29	82.65	3.58	43.10	1.64	76.22	2.66	40.80
	$\bar{y}^*_{R(S_tMRSSk)6}$	0.93	202.23	1.38	111.80	0.61	205.75	0.97	111.97
	$\bar{y}^*_{R(S_tMRSSk)7}$	0.67	281.13	0.94	164.36	0.41	304.56**	0.63	170.89

*Shows ranking on variable Y.

**Shows most efficient estimator.

TABLE 5. MSE and PRE values of estimators according to even and odd of sample size when ranking on variable Weight (X_2)

		$r_h=4$		$r_h=5$		$r_h=6$		$r_h=7$	
		MSE	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
SSRS	$\bar{y}_{RC}$	0.44	429.60	0.34	452.29	0.29	432.77	0.25	429.82
	$\bar{y}_{RS}$	0.43	434.29	0.34	453.64	0.28	434.55	0.25	432.36
SDRSS	$\bar{y}_{R(S_tDRSS)SS}$	0.58	325.97	3.45	44.71	0.37	330.87	3.28	33.12
	$\bar{y}_{R(S_tDRSS)SD}$	0.58	325.16	3.45	44.82	0.38	329.49	3.27	33.19
	$\bar{y}_{R(S_tDRSS)KC}$	0.58	323.70	3.43	45.01	0.38	327.03	3.26	33.33
	$\bar{y}_{R(S_tDRSS)US1}$	0.43	438.72	2.27	68.07	0.22	556.98	2.11	51.52
	$\bar{y}_{R(S_tDRSS)US2}$	0.43	433.89	1.98	77.73	0.22	558.13	1.83	59.22
SRSS	$\bar{y}_{SS(s)}$	0.42	446.16	0.34	451.14	0.28	446.11	0.24	448.94
	$\bar{y}_{SS(c)}$	0.42	442.22	0.34	447.59	0.28	443.17	0.24	444.91
	$\bar{y}_{MM1}$	0.42	442.38	0.34	447.70	0.28	443.29	0.24	444.98
	$\bar{y}_{MM2}$	0.42	442.65	0.34	447.89	0.28	443.47	0.24	445.08
	$\bar{y}_{MM3}$	0.54	349.21	0.46	331.78	0.41	305.38	0.36	294.93
	$\bar{y}_{MM4}$	0.60	316.23	0.52	294.96	0.47	266.77	0.42	254.00
SMRSS	$\bar{y}_{R(S_tMRSSk)1}$	0.21	870.08	0.34	448.64	0.15	790.65	0.25	430.97
	$\bar{y}_{R(S_tMRSSk)2}$	0.21	870.72	0.34	448.65	0.15	790.28	0.25	430.72
	$\bar{y}_{R(S_tMRSSk)3}$	0.21	871.80	0.34	448.67	0.15	789.58	0.25	430.26
	$\bar{y}_{R(S_tMRSSk)4}$	0.42	450.22	0.56	272.47	0.38	324.80	0.48	225.29
	$\bar{y}_{R(S_tMRSSk)5}$	0.50	377.12	0.65	234.84	0.48	262.16	0.57	188.02
	$\bar{y}_{R(S_tMRSSk)6}$	0.31	604.43**	0.45	337.81	0.25	495.68	0.36	299.67
	$\bar{y}_{R(S_tMRSSk)7}$	0.33	563.24	0.48	321.52	0.27	463.49	0.38	284.58

**Shows most efficient estimator.

TABLE 6. MSE and PRE values of estimators according to even and odd of sample size when ranking on variable Y

		$r_h=4$		$r_h=5$		$r_h=6$		$r_h=7$	
		MSE	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
SSRS	$\bar{y}^{*}_{RC}$	0.44	429.60	0.34	452.29	0.29	432.77	0.25	429.82
	$\bar{y}_{RS}$	0.43	434.29	0.34	453.64	0.28	434.55	0.25	432.36
SDRSS	$\bar{y}^{*}_{R(S_tDRSS)SS}$	0.55	345.06	3.81	40.51	0.49	251.27	3.92	27.67
	$\bar{y}^{*}_{R(S_tDRSS)SD}$	0.55	345.05	3.80	40.63	0.50	250.76	3.91	27.74
	$\bar{y}^{*}_{R(S_tDRSS)KC}$	0.55	345.02	3.78	40.84	0.50	249.84	3.89	27.88
	$\bar{y}^{*}_{R(S_tDRSS)US1}$	0.39	480.77	2.53	61.06	0.26	481.15	2.59	41.96
	$\bar{y}^{*}_{R(S_tDRSS)US2}$	0.37	502.21	2.19	70.44	0.23	531.01	2.24	48.49
SRSS	$\bar{y}^{*}_{SS(s)}$	0.41	461.43	0.33	466.77	0.27	454.34	0.23	457.12
	$\bar{y}^{*}_{SS(c)}$	0.40	463.41	0.32	470.36	0.27	457.29	0.23	459.81
	$\bar{y}^{*}_{MM1}$	0.40	464.69	0.32	471.78	0.27	458.77	0.23	461.35
	$\bar{y}^{*}_{MM2}$	0.40	466.98	0.32	474.31	0.27	461.41	0.23	464.09
	$\bar{y}^{*}_{MM3}$	0.51	365.59	0.44	347.36	0.39	320.71	0.35	308.85
	$\bar{y}^{*}_{MM4}$	0.56	337.79	0.49	315.16	0.43	286.48	0.39	271.87
SMRSS	$\bar{y}^{*}_{R(S_tMRSSk)1}$	0.21	896.95	0.33	466.16	0.15	825.55	0.24	448..20
	$\bar{y}^{*}_{R(S_tMRSSk)2}$	0.21	898.90	0.33	467.58	0.15	826.65	0.24	449.31
	$\bar{y}^{*}_{R(S_tMRSSk)3}$	0.21	902.32	0.32	470.09	0.15	828.56	0.24	451.26
	$\bar{y}^{*}_{R(S_tMRSSk)4}$	0.42	447.73	0.55	278.78	0.39	319.83	0.48	223.98
	$\bar{y}^{*}_{R(S_tMRSSk)5}$	0.50	376.82	0.63	244.27	0.48	259.43	0.57	189.46
	$\bar{y}^{*}_{R(S_tMRSSk)6}$	0.26	721.73**	0.29	518.47	0.20	604.62	0.24	445.07
	$\bar{y}^{*}_{R(S_tMRSSk)7}$	0.28	669.91	0.31	494.40	0.20	562.40	0.25	422.55

*Shows ranking on variable Y.
**Shows most efficient estimator.

4. Conclusion

In this article, it is aimed to compare the performances of the population mean estimators of various stratified sampling methods in the literature. These sampling methods are SSRS, SRSS, $SDRSS$ and $SMRSS$. In these methods, the ratio type estimators using auxiliary variable information such as coefficient of variation and kurtosis are examined. MSE and PRE values of these estimators are shown on a numerical sample and their performance is evaluated. Firstly, general information about these methods and estimators is given and introduced. Then, the MSE and PRE values of these estimators were found and the results of the simulation were interpreted. To observe performances of the estimators, we use the real data. The simulation study was performed first by using BMI with age variables and second by using BMI with weight variables. In the simulation study, when the sample size is odd and even and sorted by X and Y variables and different correlations are calculated by using different auxiliary variables, performance evaluation is performed. The aim here is to compare the same sampling methods in different correlations. According to the results obtained from the simulation, the best sampling method was found to be the $SMRSS$ method.

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A STUDY ON COMPARISONS OF BAYESIAN AND CLASSICAL PARAMETER ESTIMATION METHODS FOR THE TWO-PARAMETER WEIBULL DISTRIBUTION

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Abstract. The main objective of this paper is to determine the best estimators of the shape and scale parameters of the two parameter Weibull distribution. Therefore, both classical and Bayesian approximation methods are considered. For parameter estimation of classical approximation methods maximum likelihood estimators (MLEs), modified maximum likelihood estimators-I (MMLEs-I), modified maximum likelihood estimators -II (MMLEs-II), least square estimators (LSEs), weighted least square estimators (WLSEs), percentile estimators (PEs), moment estimators (MEs), L-moment estimators (LMEs) and TL- moment estimators (TLMEs) are used. Since the Bayesian estimators don't have the explicit form. There are Bayes estimators are obtained by using Lindley's and Tierney Kadane's approximation methods in this study. In Bayesian approximation, the choice of loss function and prior distribution is very important. Hence, Bayes estimators are given based on both the non- informative and informative prior distribution. Moreover, these estimators have been calculated under different symmetric and asymmetric loss functions. The performance of classical and Bayesian estimators are compared with respect to their biases and MSEs through a simulation study. Finally, a real data set taken from Turkish State Meteorological Service is analysed for better understanding of methods presented in this paper.

1. Introduction

Weibull distribution is one of the most popular among life-time distributions. The Weibull distribution was first proposed by W. Weibull who used it to model the distribution of the breaking strength of materials. The distribution has played major role in the reliability theory, see for example, [1] and [2]. Also, the distribution has found wide applications in many areas of environmental sciences, and renewable energy [3],[4],[5]and [6] . In addition to these application areas, Weibull distribution

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is now being used in a wide range of fields in medical, biological, and earth sciences. For details, see [7],[8] and [9] .

It is crucial to determine the best parameter estimation method for any probability function. There are various different estimation methods in the literature for estimating the parameters of the Weibull distribution. Notable among them are given as follows: In terms of classical parameter estimation methods, Trustrum and Jayatilaka [10] investigated the moment estimator, maximum likelihood estimator and least squares method based on the Monte Carlo simulation.Hung [11] and Lu et al.[12] discussed the properties of the weighted least square estimators and showed that weighted least squares estimators performed better than least squares estimators. Pobocikova and Sedliackova [13] compared the maximum likelihood estimators, moment estimators, least squares estimators and weighted least square estimators. Teimouri et al. [14] presented the maximum likelihood estimators, method of logarithm moment, percentile estimator, L- moment estimator, method of moment. Alizadeh et al. [15] considered estimation of the probability density function and cumulative density function.

In terms of Bayesian parameter estimation methods, Al Omari and Ibrahim [16] conducted a study on Bayesian survival estimator for Weibull distribution with censored data. Also, Guure et al. [17] provided the Bayesian estimation of two parameter Weibull distribution under three loss functions using extension of Jeffey's prior information. Pandey et. al [18] compared Bayesion estimator and maximum likelihood estimation of the scale parameter of the Weilbull distribution under linex loss function, with the assumption that the shape parameter is kwown. Similar work can be seen in [19],[20] .

The maximum likelihood estimators (MLEs) and the moment estimators (MEs) are the most well-known among parameter estimation methods. In this article, the least square error estimators (LSEs) and the weighted square error estimators (WLSEs), the percentile estimators (PEs),the L-moment estimators (LMEs),the TL-moment estimators (TLMEs), modified maximum likelihood estimators (MMLE-I) are considered besides these methods. Moreover, we propose the modified maximum likelihood estimators-II (MMLE-II). Further, we compute Bayes estimators of the unknown parameters with informative prior and non-informative prior under squared error loss function (SELF), general entropy loss function (GELF), weighted square loss function (WSELF) and precautionary loss function (PLF). It is clear that Bayesian estimators cannot be found in explicit form. Therefore, in this paper, we consider the Lindley's and Tierney Kadane's procedures.

There are numerous studies for Weibull distribution in literature. But, as far as we know this, this is the first study which compares all these aforementioned estimation methods for choosing the best estimation method for the two- parameter Weibull distribution. The objective of this study is to estimate the parameters of the model from both classical and Bayesian viewpoint. Finally, a better estimation method is given for the distribution parameters. In the recent past, many researchers have compared various parameter estimation methods for estimating the parameters of the different distribution. See, for example, [21] for the generalized Rayleigh distribution, [22] for the Fréchet distribution, [23] for two parameter Maxwell distribution,[24] for generalized logistic distribution.

The rest of the paper is organized as follows: Weibull distribution is described in section 2. In section 3, some classical estimation methods are given to estimate the unknown parameters. In section 4, Bayes estimators of the unknown parameters are obtained by using Lindley's and Tierney Kadane's approximations. In Section 5, a simulation study is presented to evaluate the performances of the estimators with respect to their biases and mean square errors (MSE). Finally, a real life example taken from Turkish State Meteorological Service is given.

2. Weibull Distribution

The popularity of the Weibull distribution is attributable to the fact that it is commonly used to model different data types, such as wind speed, geothermal energy and finance.

The probability density function (PDF) and the cumulative density function (CDF) of the two-parameter Weibull distribution with the shape parameter α and the scale parameter β are given by:

$$F(x;\alpha,\beta) = 1 - exp\left\{ -\left(\frac{x}{\beta}\right)^{\alpha} \right\} \ 0 < x < \infty;\ \alpha > 0,\ \beta > 0 \tag{1}$$

and

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} exp\left\{ -\left(\frac{x}{\beta}\right)^{\alpha} \right\},\ 0 < x < \infty. \tag{2}$$

The mean and variance of the Weibull distribution are defined as follows:

$$E(x) = \beta\Gamma\left(1 + \frac{1}{\alpha}\right) \ and\ V(x) = \beta^2 \left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right)\right]$$

respectively. Here, Γ is the gamma function.

3. The Methods for Parameter Estimation

In this section, we presented the methods of classical estimation for the Weibull distribution used in this study.

3.1 Moment Estimators. The MEs are found by equating theoretical moments to corresponding sample moments as shown below:

$$\beta\Gamma\left(1 + \frac{1}{\alpha}\right) = \bar{X} \ and\ \beta^2\Gamma\left(1 + \frac{2}{\alpha}\right) = \frac{\sum_{i=1}^{n} X_i^2}{n}. \tag{3}$$

Then, by solving equation 3 the MEs of α and β are found as

$$\hat{\beta} = \frac{\bar{X}}{\Gamma\left(1+\frac{1}{\alpha}\right)} \; and \; \frac{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1+\frac{2}{\hat{\alpha}}\right)} = \frac{\sum_{i=1}^{n} X_i^2}{n\bar{X}^2} \tag{4}$$

respectively.

3.2 Maximum Likelihood Estimators. Let $X_1, X_2, ..., X_n$ be a random sample from Weibull distribution. The log-likelihood function is given by:

$$lnL = nln\alpha - n\alpha ln\beta + (\alpha - 1)\sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}. \tag{5}$$

By taking the partial derivative of 5 with respect to α and β, and equating them to zero, we obtain the following log-likelihood equations:

$$\frac{\partial lnL}{\alpha} = \frac{n}{\alpha} - nln\beta + \sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{a} ln\frac{x_i}{\beta} = 0 \tag{6}$$

and

$$\frac{\partial lnL}{\partial\beta} = \frac{n\alpha}{\beta} + \frac{\alpha\sum_{i=1}^{n} x_i^{\alpha}}{\beta^{\alpha+1}} = 0. \tag{7}$$

Solutions of these likelihood equations are called as the MLEs of shape parameter α and scale parameter β , see for example [25] , [26] . However, they do not give closed form expressions since they include nonlinear terms $g_1(x) = lnx$ and $g_2(x) = x_i^{\alpha}$ in 6 and 7. Therefore, numerical methods are applied to solve the required equations. In this study, we apply the well-known Newton Rapson method to solve these equations.

3.3 Least Squares and Weighted Least Squares Estimators. The LSEs and WLSEs were originally suggested by Swain et al.[27] to estimate the parameters of beta distributions. See, for example, Kundu and Ragab [21] and Alkasabeh and Ragab [24] .

Let $X_1, ..., X_n$ is a random sample of size n from a distribution function $G(.)$ and $X_{i:n}; i = 1, 2, ..., n$ denotes the ordered sample. The expected value and variance of $G(X_{i:n})$ are easily obtained from the relation between the Beta and uniform distribution as

$$E(G(X_{i:n})) = \frac{i}{n+1} \; and \; Var(G(X_{i:n})) = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

Since $E(G(X_{i:n})) = \frac{i}{n+1}$, $i = 1, 2, ..., n$, a regression model can be written as follows:

$$G(X_{i:n}) = \frac{i}{n+1} + \varepsilon_i, i = 1, 2, ..., n.$$

Then the LSEs of the unknown parameters can be obtained by minimizing the sum of squares of errors

$$\sum_{i=1}^{n}(G(X_{i:n}) - \frac{i}{n+1})^2 \tag{8}$$

with respect to unknown parameters. Therefore, the LSEs of the unknown parameters of Weibull distribution are found by minimizing

$$\sum_{i=1}^{n}(1 - \exp(-(x_{i:n}/\beta)^{\alpha}))^2 \tag{9}$$

with respect to α and β . Since the variances of errors depend on i, the heteroscedasticity problem arises. This problem adversely affects the performance of the estimators. To overcome this problem, we use the method of weighted least squares. The weighted least squares estimators of the unknown parameters can be obtained by minimizing

$$\sum_{i=1}^{n}W_i(G(X_{i:n}) - \frac{i}{n+1})^2 \tag{10}$$

with respect to the unknown parameters. Therefore, the WLSEs of the unknown parameters of the two-parameter Weibull distribution are obtained by minimizing

$$\sum_{i=1}^{n}w_i(1 - \exp\{-(x_{i:n}/\beta)^{\alpha}\})^2 \tag{11}$$

with respect to α and β. Where $W_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$.

3.4 The Percentile Estimators. The Percentile estimators (PEs) of α and β are obtained by minimizing the function given below:

$$\sum_{i=1}^{n}\left\{X_{i:n} - F^{-1}(\frac{i}{n+1})\right\}^2 \tag{12}$$

with respect to unknown parameters [28], [29] . Here, F^{-1} is the inverse distribution function and $X_{i:n}$ is ordered observations i.e. $X_{1:n} < X_{2:n} < ... < X_{n:n}$.

Then the PEs of the shape and scale parameters of the Weibull distribution are obtained by minimizing function

$$\sum_{i=1}^{n}\left(X_{i:n} - \beta ln(\frac{n+1}{n+1-i})^{\frac{1}{\alpha}}\right)^2 \tag{13}$$

with respect to α and β.

3.5 L- Moment Estimators. The L- moment estimators (LMEs) was introduced by Hosking [30]. These estimators have an estimation method based on linear combination of order statistics. The LMEs have lower sample variances and they are more robust outliers in data. In recently, a few authors have studied the L-moment estimator for the Weibull distribution [14]-[31] .

Let $X_1, X_2, ..., X_n$ be a random sample of size n and $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order random variables. Then the population L-moments and sample L-moments are given as follows:

$$L_k = k^{-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} E(X_{k-j:k}), k = 1, 2, 3, ..., \tag{14}$$

$$l_k = \frac{1}{k\binom{n}{k}} \sum_{i=1}^{n} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \binom{i-1}{k-j-1} \binom{n-i}{j} X_{i:n}, k = 1, 2, 3, ... \tag{15}$$

respectively. Here, k is the number of the unknown parameters, $E(X_{i:n})$ are the expected values of the order statistics and n is sample size.
By using equations 14, the population L-moments of two-parameter Weibull distribution derived as

$$L_1 = \beta\Gamma(1 + \frac{1}{\alpha}) \ and \ L_2 = \beta\Gamma(1 + \frac{1}{\alpha}) - \frac{\beta\Gamma(1 + \frac{1}{\alpha})}{2^{\frac{1}{\alpha}}}. \tag{16}$$

The idea lying under L moment estimators are the same as in the moment estimators. In other words, on equating the first two population moments to corresponding sample moments, the estimating equations are

$$\beta\Gamma(1 + \frac{1}{\alpha}) = l_1 \ and \ \beta\Gamma(1 + \frac{1}{\alpha}) - \frac{\beta\Gamma(1 + \frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} = l_2. \tag{17}$$

Then the LMEs of the parameters follow from 17 as

$$\hat{\alpha} = \frac{ln2}{ln(\frac{l_1}{l_1 - l_2})} \ and \ \hat{\beta} = \frac{l_1}{\Gamma(1 + \frac{1}{\hat{\alpha}})}, \tag{18}$$

respectively, where, $l_1 = \bar{x}$ and $l_2 = \frac{1}{n(n-1)} \sum (2j - n - 1) X_{j:n}$.
3.6 Trimmed L-Moments Estimators. Elamir and Seheult [32] proposed TL-moments as a robust generalization of L-moments. The TL-moments always exist even if the mean of the distribution does not exist, for example, the TL-moments exist for Cauchy distribution.

Let $X_1, X_2, ..., X_n$ be a random sample of size n and $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the corresponding order statistics. Elamir and Seheult [32] defined the kth the population and sample TL-moments

$$\lambda_k^{(s,t)} = k^{-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} E(X_{k+s-j:k+s+t}), k = 1, 2, 3, ..., s, t = 0, 1, 2, ... \tag{19}$$

and

$$l_k^{(s,t)} = \frac{1}{k\binom{n}{k+s+t}} \sum_{j=s}^{n-t} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \binom{j-1}{k+s-i-1} \binom{n-j}{t+i} X_{j:n} \ k = 1, 2, 3... \tag{20}$$

respectively. It should be noted that TL-moments reduce to the L-moments when $s = t = 0$. In this study, we focus on asymmetric cases where $s = 0, t = 1$. By putting $s = 0$ and $t = 1$ in equations 19 and 20 , we have

$$\lambda_k^{(0,1)} = k^{-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} E(X_{k-j:k+1}) \tag{21}$$

and

$$l_k^{(0,1)} = \frac{1}{k\binom{n}{k+t}} \sum_{j=0}^{n-1} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \binom{j-1}{k-i-1} \binom{n-j}{i+1} X_{j:n}. \tag{22}$$

The population TL-moments of the two-parameter Weibull distribution can be obtained from 21 as

$$\lambda_1^{(0,1)} = \frac{\beta\Gamma(1+\frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} \ and \ \lambda_2^{(0,1)} = \frac{3\beta\Gamma(1+\frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} - \frac{2\beta\Gamma(1+\frac{1}{\alpha})}{3^{\frac{1}{\alpha}}} \tag{23}$$

The TLMEs are obtained by equating the first two sample TL-moments to the corresponding population TL-moments. Hence, the estimating equations are

$$\frac{\beta\Gamma(1+\frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} = l_1^{(0,1)} \ and \ \frac{3\beta\Gamma(1+\frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} - \frac{2\beta\Gamma(1+\frac{1}{\alpha})}{3^{\frac{1}{\alpha}}} = l_2^{(0,1)}. \tag{24}$$

The solutions of these equations are the following TLMEs:

$$\hat{\alpha} = \frac{log(\frac{2}{3})}{log(\frac{3l_1^{(0,1)}-2l_2^{(0,1)}}{3l_1^{(0,1)}})} \ and \ \hat{\beta} = \frac{2^{1/\alpha} l_1^{(0,1)}}{\Gamma(1+\frac{1}{\hat{\alpha}})}$$

where

$$l_1^{(0,1)} = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} (n-j) X_{j:n} \ and \ l_2^{(0,1)} = \frac{3}{2n(n-1)(n-2)} \sum_{j=1}^{n} (n-j)(3j-n-1)) X_{j:n}.$$

3.7 Modified Maximum Likelihood Estimators-I. Cohen and Whitten [33] recommend modifications of the MLEs for estimating the shape and scale parameters of the Weibull distribution. The MMLE-I of the shape parameter α and scale parameter β , say $\hat{\alpha}_{MMLE-I}$ and $\hat{\beta}_{MMLE-I}$ respectively, of the Weibull distribution is obtained by solving the following equations:

$$\frac{-nX_{1:n}^{\alpha}}{ln(\frac{n}{n+1})} = \sum_{i=1}^{n} X_{i:n}^{\alpha} \ and \ \hat{\beta}_{MMLE-I} = \frac{1}{n}(\sum_{i=1}^{n} X_i^{\hat{\alpha}_{MMLE}})^{\frac{1}{\hat{\alpha}_{MMLE-I}}}. \tag{25}$$

3.8 Modified Maximum Likelihood Estimators-II. We proposed modifications of the MLEs for estimating the unknown parameters of the Weibull distribution. Then, MMLE of the shape parameter α, say $\hat{\alpha}_{MMLE-II}$, is estimated by solving the following equation:

$$\frac{\gamma + \frac{ln\sum_{i=1}^{n} x_i^{\alpha}}{n}}{\alpha} = \frac{\sum_{i=1}^{n} lnx_i}{n}, \tag{26}$$

where $\gamma \cong 0.57722$ is Euler constant.
Here, by inserting $\hat{\alpha}_{MMLE-II}$ instead of $\hat{\alpha}$ into equation 7 , MMLE of the scale parameter β , say $\hat{\beta}_{MMLE-II}$ is obtained as

$$\hat{\beta}_{MMLE-II} = (\frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{\alpha}_{MMLE-II}})^{\frac{1}{\hat{\alpha}_{MMLE-II}}}. \tag{27}$$

4. Bayesian Analysis

In this section, we consider the Bayesian estimation by using Lindley's and Tierney-Kadane's approximations under different loss function for estimating the unknown parameters of Weibull distribution. Bayesian analysis has many applications in statistical theory and analysis[34]. In Bayesian analysis the role of two factors are crucial. These are (i) the choice of the loss function (LF) and (ii) the choice of the prior distribution. For more details about the priors and loss functions, see [35],[36].
In this study, GELF, PLF,WSELF and SELF are considered and described as follows:
The SELF was proposed by Legendre [37] and Gauss [38] to developed least square theory. This loss function is commonly used and defined as

$$L_{SELF} = (\hat{\theta} - \theta)^2, \tag{28}$$

where θ is the parameter to be estimated by an estimator $\hat{\theta}$. The Bayes estimator under equation 28 is the posterior mean given by

$$\hat{\theta}_{SELF} = E(\theta|x). \tag{29}$$

This loss function is symmetrical in nature. It gives equal weight to both underestimation and over estimation. However, from a practical point of view, this is not always appropriate and realistic, see for example [39]. Hence, asymmetric loss functions would be more useful to develop Bayesian procedures.
Calabria and Pulcini [40] proposed general entropy loss function. It is one of the most popular asymmetrical loss functions.
The GELF is given by

$$L_{GELF} = (\frac{\hat{\theta}}{\theta})^k - klog(\frac{\hat{\theta}}{\theta}) - 1, k \neq 0, \tag{30}$$

where $\hat{\theta}$ is the estimator of θ. k reflects the magnitude and degree of symmetry. The Bayes estimator under equation 30 is given by

$$\hat{\theta}_{GELF} = [E(\theta^{-k}|x)]^{-\frac{1}{k}}, \tag{31}$$

provided $E_\theta(\theta^{-k}|x)$.
The PLF, which is proposed by Norstrom [41] , is one of the asymmetric loss functions. This loss function approach is useful to derive conservative estimators since it approaches infinity near the origin and prevents underestimation. It is

very useful when underestimation may lead to significant results [42]. The PLF is defined as

$$L_{PLF} = \frac{(\theta - \hat{\theta})^2}{\hat{\theta}}, \tag{32}$$

where $\hat{\theta}$ is the estimator of θ. The Bayes estimator of under equation 32 is given by

$$\hat{\theta}_{PLF} = \sqrt{E(\theta^2|x)}, \tag{33}$$

provided $\sqrt{E(\theta^2|x)}$ exists and is finite.
WSELF is another useful asymmetric loss function. This function is a weighted version of SELF. More detail about this loss function can be found in [35]and [43]. The WSELF is defined as:

$$L_{WSELF}(\hat{\theta}, \theta) = \frac{(\theta - \hat{\theta})^2}{\hat{\theta}}. \tag{34}$$

The Bayes estimator under WSELF is given by

$$\hat{\theta}_{WSELF} = [E(\theta^{-1}|x)]^{-1}. \tag{35}$$

provided $E(\theta^{-1}|x)^{-1}$ exists and is finite.
The prior distribution summarizes the information about unknown parameter before the data is available. The prior distribution is then synthesized with the information in the data procedure the posterior distribution. In other words, analytically, combining the prior distribution and likelihood function results in the posterior distribution. The posterior distribution expresses what is known after seeing data. In the Bayesian analysis, all inferences are made from the posterior distribution [44].
The prior distribution has two forms: these are (i) "non-informative prior" and (ii) "informative prior" [45].
Here we assume that α and β have two independent gamma prior distributions i.e. $\alpha \sim gamma(a, b)$ and $\beta \sim gamma(c, d)$ respectively. The gamma prior is very flexible and suitable. Thus, this paper considers two special cases of the gamma prior corresponding to $a = b = c = d = 0$ and $a, b, c, d \geq 0$ (a, b, c, d are the hyper-parameters of the prior distribution). It should be mentioned that for $a = b = c = d = 0$ the prior distribution is non-informative prior (NP) distribution. For $a, b, c, d \geq 0$, the prior distribution is referred to as the gamma prior (GP) distribution. Thus, the proposed prior for α and β may be considered as

$$v_1(\alpha) \propto \alpha^{a-1} e^{-b\alpha}, \alpha > 0 \text{ and } v_2(\beta) \propto \beta^{c-1} e^{-d\beta}, \beta > 0. \tag{36}$$

The joint prior distribution α and β is given as

$$v(\alpha, \beta) \propto \alpha^{a-1} \beta^{c-1} e^{-d\beta - b\alpha},\ \alpha, \beta, a, b, c, d \geq 0. \tag{37}$$

Based on the observations, the likelihood function becomes

$$L(\alpha,\beta)=\alpha^{n}\beta^{-n\alpha}\prod_{i=1}^{n}X_i^{(\alpha-1)}e^{-\sum(\frac{X_i}{\beta})^{\alpha}}. \tag{38}$$

Combining 37 with 38 and using Bayes theorem, the joint posterior density of α and β is

$$p(\alpha,\beta|x)=K^{-1}\alpha^{n+a-1}\beta^{-n\alpha+c-1}exp(-d\beta-b\alpha)\prod_{i=1}^{n}x_i^{\alpha-1}e^{\sum_{i=1}^{n}(\frac{x_i}{\beta})^{\alpha}}. \tag{39}$$

Here $K=\int_0^\infty\int_0^\infty \alpha^{n+a-1}\beta^{-n\alpha+c-1}exp(-d\beta-b\alpha)\prod_{i=1}^{n}x_i^{\alpha-1}e^{\sum_{i=1}^{n}(\frac{x_i}{\beta})^{\alpha}}d\alpha d\beta$.
It can be seen that the analytical solution of the Bayes estimators are not obtained. Hence, we use the Lindley's and Tierney-Kadane's approximation. These methods are described below.

4.1 Lindley's procedure. Lindley's [46] introduced an approximation method for the evaluation of the ratio of the two integrals. This procedure can be applied to compute the posterior expectation of the arbitrary function $u(\theta)$ as given by

$$E(u(\theta)|x)=\frac{\int u(\theta)e^{L(\theta)+G(\theta)}d\theta}{\int e^{L(\theta)+G(\theta)}d\theta},$$

where
$u(\theta)=$ a function of θ only,
$L(\theta)=$ Log-likelihood function,
$G(\theta)=$ Log of joint prior density function.
According to Lindley's approximation, the ratio of integral $E\{u(\theta)|x\}$ can be approximated asymptotically given below:

$$E(u(\theta)|x)\approx[u+\frac{1}{2}\sum_i\sum_j(u_{ij}+2u_i\rho_i)\sigma_{ij}+\frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l]+O(1/n^2). \tag{40}$$

Here, $i;j;k;l=1,2,...,n;\theta=(\theta_1,\theta_2,...,\theta_m)$, $u_i=\frac{\partial u(\theta)}{\partial\theta_i}$, $u_{ij}=\frac{\partial u(\theta)}{\partial\theta_i\partial\theta_j}$, $L_{ijk}=\frac{\partial L(\theta)}{\partial\theta_i\partial\theta_j\partial\theta_k}$, $\rho_j=\frac{\partial G(\theta)}{\partial\theta_j}$ and σ_{ij} are elements of the covariance matrix.
For the two-parameter Weibull distribution, equation 40 reduces to

$$\begin{aligned}E(u(\alpha,\beta)|x)&=u+\frac{1}{2}(u_{11}\sigma_{11}+u_{22}\sigma_{22})+u_{12}\sigma_{12}+u_1(\sigma_{11}\rho_1+\sigma_{21}\rho_2)+u_2(\sigma_{12}\rho_1+\sigma_{22}\rho_2)\\&+0.5[L_{111}(u_1\sigma_{11}^2+u_2\sigma_{11}\sigma_{12})+L_{112}(3u_1\sigma_{11}\sigma_{12}+u_2(\sigma_{11}\sigma_{22}+2\sigma_{12}^2))\\&+L_{122}(u_1(\sigma_{11}\sigma_{22}+2\sigma_{12}^2)+3u_2\sigma_{12}\sigma_{22})+L_{222}(u_1\sigma_{12}\sigma_{22}+u_2\sigma_{22}^2)]_{\hat{\alpha},\hat{\beta},}.\end{aligned} \tag{41}$$

Here, the $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of α and β, respectively.
All other quantities appearing in the above expression of $E(u(\alpha,\beta)|x)$ for Weibull

distribution is given by

$$\hat{\rho}_\alpha = \frac{a-1}{\hat{\alpha}} - b, \hat{\rho}_\beta = \frac{c-1}{\hat{\beta}} - d,$$

$$\hat{L}_{111} = \frac{2n}{\hat{\alpha}^3} - \sum_{i=1}^{n}((ln(\frac{x_i}{\hat{\beta}}))^3(\frac{x_i}{\hat{\beta}})^{\hat{\alpha}}),$$

$$\hat{L}_{112} = \sum_{i=1}^{n} log(\frac{x_i}{\hat{\beta}})(\frac{x_i}{\hat{\beta}})^{\hat{\alpha}}[\frac{1}{\hat{\beta}}(2+\hat{\alpha}.log(\frac{x_i}{\hat{\beta}}))],$$

$$\hat{L}_{122} = \sum_{i=1}^{n}((\frac{x_i}{\hat{\beta}})^{\hat{\alpha}}(\frac{1}{\hat{\beta}^2}(\hat{\alpha}+1)(\hat{\alpha}+ln(\frac{x_i}{\hat{\beta}})+1)+\hat{\alpha})) + \frac{n}{\hat{\beta}}^2$$

$$\hat{L}_{222} = \frac{-2n\hat{\alpha}}{\hat{\beta}^3} + \frac{\hat{\alpha}(\hat{\alpha}+1)(\hat{\alpha}+2)}{\hat{\beta}^3}\sum_{i=1}^{n}(\frac{x_i}{\hat{\beta}})^{\hat{\alpha}},$$

and

$$\sigma_{ij} = \begin{bmatrix} Var\{\hat{\alpha}\} & Cov\{\hat{\alpha},\hat{\beta}\} \\ Cov\{\hat{\alpha},\hat{\beta}\} & Var\{\hat{\beta}\} \end{bmatrix} = \frac{1}{n}\begin{bmatrix} 0.6080\alpha^2 & 0.2570\beta \\ 0.2570\beta & 1.1087\frac{\beta^2}{\alpha^2} \end{bmatrix}.$$

All constant are evaluated at $(\hat{\alpha}, \hat{\beta})$.
Then, by using Lindley's method the Bayesion estimators of the parameter α under SELF is obtained as
If $u(\alpha,\beta) = \alpha, u_1 = 1, u_2 = u_{22} = u_{12} = u_{21} = u_{11} = 0$, then

$$\begin{aligned}\hat{\alpha}_{SELF} =& \hat{\alpha} + (\hat{\sigma}_{11}\hat{\rho}_1 + \hat{\sigma}_{21}\hat{\rho}_2) \\ &+ 0.5[\hat{L}_{111}\sigma_{11}^2 + 3\hat{L}_{112}\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + \hat{L}_{222}\hat{\sigma}_{12}\hat{\sigma}_{22}].\end{aligned}$$

So, the Bayes estimator of β under SELF is obtained as,
If $u(\alpha,\beta) = \beta$, $u_2 = 1, u_{22} = u_{12} = u_{21} = u_{11} = 0$, then

$$\begin{aligned}\hat{\beta}_{SELF} =& \hat{\beta} + (\hat{\sigma}_{12}\hat{\rho}_1 + \hat{\sigma}_{22}\hat{\rho}_2) \\ &+ 0.5[\hat{L}_{111}\sigma_{11}\sigma_{12} + \hat{L}_{112}(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + 3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22} + \hat{L}_{222}\hat{\sigma}_{22}^2].\end{aligned}$$

Bayes estimator of α under the GELF is defined as
If $u(\alpha,\beta) = \alpha^{-k}, u_1 = -k\alpha^{-(k+1)}, u_{11} = k(k+1)\alpha^{-(k+2)}, u_2 = u_{22} = u_{12} = u_{21} = 0$, then

$$\begin{aligned}&E(\alpha^{-k}|x) = \hat{\alpha}^{-k} + 0.5(\hat{u}_{11}\hat{\sigma}_{11}) + \hat{u}_1(\hat{\sigma}_{11}\hat{\rho}_1 + \hat{\sigma}_{21}\hat{\rho}_2) + \\ &0.5[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2 + 3\hat{L}_{112}\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}\hat{u}_1(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + \hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}].\end{aligned}$$

Therefore, $\hat{\alpha}_{GELF} = E[\alpha^{-k}|x]^{-1/k}$.
Bayes estimator of β under the general entropy loss function is given by
If $u(\alpha,\beta) = \beta^{-k}, u_2 = -k\beta^{-(k+1)}$, $u_{22} = k(k+1)\beta^{-(k+2)}$, $u_1 = u_{11} = u_{12} = u_{21} = 0$, then

$$E(\beta^{-k}|x) = \hat{\beta}^{-k} + 0.5(\hat{u}_{22}\hat{\sigma}_{22}) + \hat{u}_2(\hat{\sigma}_{12}\hat{\rho}_1 + \hat{\sigma}_{22}\hat{\rho}_2) +$$

$$0.5[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{112}\hat{u}_2(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\sigma_{12}^2) + 3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22} + \hat{L}_{222}\hat{u}_2\hat{\sigma}_{22}^2].$$

Hence, $\hat{\beta}_{GELF} = E[\beta^{-k}|x]^{-1/k}$.
Bayes estimator of α under the WSELF is as follows
If $u(\alpha,\beta) = \alpha^{-1}, u_1 = -\alpha^{-2}, u_{11} = 2\alpha^{-3}, u_2 = u_{22} = u_{12} = 0$, then

$$E(\alpha^{-1}|x) = \hat{\alpha}^{-1} + 0.5(\hat{u}_{11}\hat{\sigma}_{11}) + \hat{u}_1(\hat{\sigma}_{11}\hat{\rho}_1 + \hat{\sigma}_{21}\hat{\rho}_2) +$$
$$0.5[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2 + 3\hat{L}_{112}\hat{u}_1(\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}\hat{u}_1(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + \hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}].$$

So, $\hat{\alpha}_{WSELF} = [E(\alpha^{-1}|x)]^{-1}$.
The Bayes estimator of β under the WSELF is given in following form
If $u(\alpha,\beta) = \beta^{-1}, u_2 = -\beta^{-2}, u_{22} = 2\beta^{-3}, u_1 = u_{11} = u_{12} = 0$, then

$$E(\beta^{-1}|x) = \hat{\beta}^{-1} + 0.5(\hat{u}_{22}\hat{\sigma}_{22}) + \hat{u}_2(\hat{\sigma}_{12}\hat{\rho}_1 + \hat{\sigma}_{22}\hat{\rho}_2) +$$
$$0.5[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{112}\hat{u}_2(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + 3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22} + \hat{L}_{222}\hat{u}_2\hat{\sigma}_{22}^2].$$

So, the Bayes estimator of β is $\hat{\beta}_{WSELF} = [E(\beta^{-1}|x)]^{-1}$.
Finally, the Bayes estimator of α under PLF is
If $u(\alpha,\beta) = \alpha^2, u_1 = 2\alpha, u_{11} = 2, u_2 = u_{22} = u_{12} = 0$, then

$$E(\alpha^2|x) = \hat{\alpha}^2 + 0.5(\hat{u}_{11}\hat{\sigma}_{11}) + \hat{u}_1(\hat{\sigma}_{11}\hat{\rho}_1 + \hat{\sigma}_{21}\hat{\rho}_2) +$$
$$0.5[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2 + 3\hat{L}_{112}\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}\hat{u}_1(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + \hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}].$$

Hence,the Bayes estimator of α is as follows
$\hat{\alpha}_{PLF} = \sqrt{E(\alpha^2|x)}$.
Bayes estimator of β under PLF is given by
If $u(\alpha,\beta) = \beta^2, u_2 = 2\beta, u_{22} = 2, u_1 = u_{11} = u_{12} = 0$, then

$$E(\beta|x) = \hat{\beta} + 0.5(\hat{u}_{22}\hat{\sigma}_{22}) + \hat{u}_2(\hat{\sigma}_{12}\hat{\rho}_1 + \hat{\sigma}_{22}\hat{\rho}_2) +$$
$$0.5[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{112}\hat{u}_2(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + 3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22} + \hat{L}_{222}\hat{u}_2\hat{\sigma}_{22}^2].$$

So, $\hat{\beta}_{PLF} = \sqrt{E(\beta^2|x)}$.

4.2 Tierney Kadane's Procedure. Lindley's procedure seems to be become more and more complex in p- parameter case ($p > 2$). Therefore, in multi-parameter case, Tierney Kadane's (T-K) procedure is used as an alternative to Lindley's procedure [47],[48].
According to this procedure, posterior expectation for multi-parameter case can be approximated by:

$$E(u(\theta)|x) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}} exp[n(L_1^*(\hat{\theta}^*) - L_1(\hat{\theta}))]. \tag{42}$$

Here, $\hat{\theta}^*$ and $\hat{\theta}$ maximize L_1^* and L_1,respectively,

$$L_1 = \frac{[L(\theta) + log(v(\theta))]}{n}, L_1^* = L_1 + \frac{[log(u(\theta))]}{n},$$

where
$v(\theta) =$ joint prior distribution of θ,
$L(\theta) =$ Log-likelihood function of θ,
$u(\theta) =$ loss function of θ.
In equation 42, $\sum^*$ and $\sum$ are elements of the negative of the inverse of the matrices of the second derivatives of L_1^* and L_1 at the point $\hat{\theta}^*$ and θ, respectively.
For the two parameter case, $\theta = (\alpha, \beta)$, equation 42 becomes:

$$E(u(\alpha,\beta)|x) = \sqrt{\frac{|\sum^*|}{|\sum|}} exp[n(L_1^*(\hat{\alpha}^*, \hat{\beta}^*) - L_1(\hat{\alpha}, \hat{\beta}))]. \tag{43}$$

Here, $(\hat{\beta}, \hat{\alpha})$ and $(\hat{\beta}^*, \hat{\alpha}^*)$ maximize $L_1(\alpha, \beta)$ and $L_1^*(\alpha, \beta)$, respectively. $\sum$ and $\sum^*$ are given below:

$$\sum^* = \begin{bmatrix} -\frac{\partial^2 L_1^*}{\partial \alpha^2} & -\frac{\partial^2 L_1^*}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 L_1^*}{\partial \alpha \partial \beta} & -\frac{\partial^2 L_1^*}{\partial \beta} \end{bmatrix}^{-1}_{(\hat{\alpha}^*, \hat{\beta}^*)} \quad and \quad \sum = \begin{bmatrix} -\frac{\partial^2 L_1}{\partial \alpha^2} & -\frac{\partial^2 L_1}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 L_1}{\partial \alpha \partial \alpha} & -\frac{\partial^2 L_1}{\partial \beta^2} \end{bmatrix}^{-1}_{(\hat{\alpha}, \hat{\beta})}. \tag{44}$$

All other quantities appearing in the above expression of $E(u(\alpha, \beta|x))$ for Weibull distribution can be obtained as

$$L_1(\alpha,\beta) = \frac{1}{n}[nln\alpha - n\alpha ln\beta + (\alpha-1)\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} (\frac{x_i}{\beta})^{\alpha} + (\alpha-1)ln\alpha + (c-1)ln\beta - (b\alpha + d\beta)]. \tag{45}$$

Thus the Bayes estimator of α under SELF is given in the following form:
If $u(\alpha, \beta) = \alpha$ and $L_1^* = \frac{1}{n}log\alpha + L_1(\alpha, \beta)$, then

$$\hat{\alpha}_{SELF} = [\sqrt{\frac{|\sum^*|}{|\sum|}} exp[n(\frac{1}{n}log\hat{\alpha} + L_1(\hat{\alpha}^*, \hat{\beta}^*) - L_1(\hat{\alpha}, \hat{\beta}))]].$$

Also, the Bayes estimator of β under SELF using this procedure is defined as:
If $u(\alpha, \beta) = \beta$ and $L_1^* = \frac{1}{n}log\beta + L_1(\alpha, \beta)$, then

$$\hat{\beta}_{SELF} = [\sqrt{\frac{|\sum^*|}{|\sum|}} exp[n(\frac{1}{n}log\hat{\beta} + L_1(\hat{\alpha}^*, \hat{\beta}^*) - L_1(\hat{\alpha}, \hat{\beta}))]].$$

Bayes estimator of α under GELF is given by:
If $u(\alpha, \beta) = \alpha^{-k}$ and $L_1^* = \frac{1}{n}log(\alpha^{-k}) + L_1(\alpha, \beta)$, then

$$\hat{\alpha}_{GELF} = [\sqrt{\frac{|\sum^*|}{|\sum|}} exp[n(\frac{1}{n}log(\hat{\alpha}^{-k}) + L_1^*(\hat{\alpha}^*, \hat{\beta}^*) - L_1(\hat{\alpha}, \hat{\beta}))]]^{-1/k}.$$

Bayes estimator of β under GELF is given by:
If $u(\alpha,\beta)=\beta^{-k}$ and $L_1^*=\frac{1}{n}log(\beta^{-k})+L_1(\alpha,\beta)$, then

$$\hat{\beta}_{GELF}=[\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log(\hat{\beta}^{-k})+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]]^{-1/k}.$$

Bayes estimator of α under WSELF is as follows:
If $u(\alpha,\beta)=\alpha^{-1}$ and $L_1^*=\frac{1}{n}log(\alpha^{-1})+L_1(\alpha,\beta)$, then

$$\hat{\alpha}_{WSELF}=[\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log(\hat{\alpha^{-1}})+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]]^{-1}.$$

Bayes estimator of β under WSELF is as follows:
If $u(\alpha,\beta)=\beta^{-1}$ and $L_1^*=\frac{1}{n}log(\beta)^{-1}+L_1(\alpha,\beta)$, then

$$\hat{\beta}_{WSELF}=[\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log\hat{(}\beta^{-1})+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]]^{-1}.$$

Bayes estimator of α under PLF is
If $u(\alpha,\beta)=\alpha^2$ and $L_1^*=\frac{1}{n}log(\alpha)^2+L_1(\alpha,\beta)$, then

$$\hat{\alpha}_{PLF}=\sqrt{\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log(\hat{\alpha}^2)+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]}.$$

Bayes estimator of β under PLF is
If $u(\alpha,\beta)=\beta^2$ and $L_1^*=\frac{1}{n}log(\beta)^2+L_1(\alpha,\beta)$, then

$$\hat{\beta}_{PLF}=\sqrt{\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log(\hat{\beta}^2)+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]}.$$

5. Simulation study

In this section, an extensive Monte Carlo simulation study was carried out to compare the performances of the Bayesian and classical estimators with respect to the biases and mean squared errors (MSEs) for different sample sizes and parameter values. All The computations were performed in Matlab R. 2013. over 10.000 replications for different cases. We consider the sample sizes $n = 10(10)100$, the shape parameter values $\alpha = 0.5, 1.5$ and the scale parameter β was taken to be 1 throughout the study. The bias and MSE values of the classical estimators are given in Table 1.

For Bayesian estimators, we know that the Gamma prior provides flexible approach in both informative and non-informative cases [48]. In case of the non-informative prior (NP), we chose hyper-parameter values as $a = b = c = d = 0$. In case of the GP, we chose hyper-parameter values as $a = 0.4, 1, 1.5, 3$, $b = 0.2, 1$, $c = 0.4, 1, 1.5, 3$ and $d = 0.2, 1$. In both cases i.e. informative and non-informative, we considered as $k = \pm 1.5$ for GELF. Because of the large number of tables and results, only results for $a = c = 0.4$, $b = d = 0.2$ and $k = 1.5$ were reported. Moreover, Lindley's and T-K methods were used to obtain the Bayes estimator of the unknown parameters. The results of simulation for these approximation methods were summarized in Table 2-3.

Table 1. The simulated, means and MSEs values for the classical different parameter estimators of α and β

			$\hat{\alpha}$		$\hat{\beta}$	
n	α	Estimator	Mean	MSE	Mean	MSE
		MLE	0.5382	0.0121	1.0704	0.2607
		LME	0.5290	0.0152	0.9837	0.2598
		TLME	0.5079	0.0138	0.9579	0.2395
		MMLE-II	0.5330	0.0123	1.0687	0.2610
20	0.5	MMLE-I	0.4739	0.0220	0.9874	0.2616
		ME	0.6511	0.0403	1.3510	0.5322
		LSE	0.5008	0.0147	1.1271	0.3244
		WLSE	0.5052	0.0131	1.1114	0.2973
		PE	0.4887	0.0336	1.1229	0.2395
		MLE	0.5243	0.0068	1.0544	0.1694
		LME	0.5210	0.0101	0.9960	0.1752
		TLME	0.5039	0.0081	0.9807	0.1622
		MMLE-II	0.5226	0.0069	1.0541	0.1700
30	0.5	MMLE-I	0.4702	0.0172	0.9816	0.1779
		ME	0.6151	0.0263	1.2863	0.3425
		LSE	0.4984	0.0087	1.0932	0.2037
		WLSE	0.5034	0.0076	1.0795	0.1878
		PE	0.4800	0.0276	1.0577	0.2268
		MLE	0.5147	0.0037	1.0276	0.0951
		LME	0.5125	0.0059	0.9891	0.1019
		TLME	0.5032	0.0045	0.9829	0.0945
		MMLE-II	0.5139	0.0038	1.0280	0.0959
50	0.5	MMLE-I	0.4689	0.0139	0.9654	0.1154
		ME	0.5782	0.0149	1.1955	0.1874
		LSE	0.5000	0.0048	1.0490	0.1094
		WLSE	0.5039	0.0041	1.0401	0.1014
		PE	0.4664	0.0210	0.9622	0.3263

Table 1. Continued

			$\hat{\alpha}$		$\hat{\beta}$	
n	α	Estimator	Mean	MSE	Mean	MSE
		MLE	0.5068	0.0016	1.0112	0.0458
		LME	0.5068	0.0029	0.9931	0.0506
		TLME	0.5012	0.0022	0.9894	0.0472
		MMLE-II	0.5063	0.0017	1.0111	0.0461
100	0.5	MMLE-I	0.4697	0.0106	0.9586	0.0661
		ME	0.5487	0.0076	1.1264	0.0959
		LSE	0.4994	0.0023	1.0223	0.0516
		WLSE	0.5020	0.0019	1.0170	0.0481
		PE	0.4653	0.0142	0.9208	0.1755
		MLE	1.6090	0.1041	1.0010	0.0245
		LME	1.5292	0.0878	0.9943	0.0247
		TLME	1.5333	0.1304	0.9968	0.0270
		MMLE-II	1.5989	0.1055	1.0008	0.0246
20	1.5	MMLE-I	1.4225	0.1959	0.9712	0.0281
		ME	1.6194	0.1051	1.0016	0.0245
		LSE	1.4949	0.1262	1.0165	0.0276
		WLSE	1.5087	0.1116	1.0129	0.0262
		PE	1.4362	0.0968	1.0172	0.0734
		MLE	1.5740	0.0625	0.9999	0.0166
		LME	1.5225	0.0563	0.9956	0.0167
		TLME	1.5218	0.0793	0.9977	0.0179
		MMLE-II	1.5689	0.0634	0.9998	0.0167
30	1.5	MMLE-I	1.4112	0.1556	0.9732	0.0202
		ME	1.5826	0.0651	1.0003	0.0167
		LSE	1.4981	0.0787	1.0104	0.0183
		WLSE	1.5125	0.0683	1.0071	0.0174
		PE	1.4471	0.0667	1.0107	0.0021
		MLE	1.5438	0.0331	0.9995	0.0099
		LME	1.5137	0.0319	0.9969	0.0100
		TLME	1.5133	0.0445	0.9982	0.0108
		MMLE-II	1.5413	0.0344	0.9994	0.0100
50	1.5	MMLE-I	1.4103	0.1258	0.9763	0.0133
		ME	1.5496	0.0353	0.9997	0.0100
		LSE	1.4989	0.0450	1.0058	0.0110
		WLSE	1.5107	0.0381	1.0034	0.0104
		PE	1.4549	0.0398	1.0057	0.1723
		MLE	1.5213	0.0151	1.0000	0.0050
		LME	1.5065	0.0152	0.9988	0.0050
		TLME	1.5067	0.0206	0.9993	0.0054
		MMLE-II	1.5199	0.0158	1.0000	0.0050
100	1.5	MMLE-I	1.4063	0.0976	0.9790	0.0081
		ME	1.5244	0.0165	1.0001	0.0050
		LSE	1.4998	0.0213	1.003	0.0055
		WLSE	1.5075	0.0175	1.0017	0.0052
		PE	1.4658	0.0205	1.0025	0.0735

Table 2.The simulated, means and MSEs values under different loss function for the Lindley approximation of α and β

			Lindley's approximation							
			$\hat{\alpha}$				$\hat{\beta}$			
			NP		GP		NP		GP	
n	α	LF	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
20	0.5	SELF	0.5178	0.0102	0.5240	0.0100	1.1895	0.3391	1.2313	0.3487
		GELF	0.5008	0.0092	0.5060	0.0089	0.9441	0.2152	0.9746	0.2060
		WSELF	0.5038	0.0094	0.5092	0.0090	0.9825	0.2258	1.0182	0.2177
		PLF	0.5258	0.0108	0.5321	0.0108	1.2758	0.4192	1.3101	0.4347
30	0.5	SELF	0.5110	0.0061	0.5154	0.0062	1.1349	0.2053	1.1635	0.2104
		GELF	0.4993	0.0057	0.5032	0.0057	0.9640	0.1468	0.9868	0.1429
		WSELF	0.5015	0.0057	0.5054	0.0058	0.9930	0.1524	1.0186	0.1494
		PLF	0.5163	0.0063	0.5207	0.0065	1.1976	0.2441	1.2220	0.2518
50	0.5	SELF	0.5068	0.0034	0.5097	0.0035	1.0759	0.1071	1.0974	0.1128
		GELF	0.4996	0.0033	0.5023	0.0033	0.9709	0.0875	0.9897	0.0880
		WSELF	0.5009	0.0033	0.5037	0.0033	0.9899	0.0894	1.0099	0.0906
		PLF	0.5099	0.0035	0.5129	0.0036	1.1157	0.1209	1.1356	0.1284
100	0.5	SELF	0.5029	0.0016	0.5031	0.0016	1.0354	0.0487	1.0418	0.0507
		GELF	0.4992	0.0015	0.4994	0.0016	0.9816	0.0441	0.9876	0.0453
		WSELF	0.4999	0.0015	0.5002	0.0016	0.9918	0.0445	0.9979	0.0459
		PLF	0.5045	0.0016	0.5047	0.0016	1.0564	0.0521	1.0628	0.0545
20	1.5	SELF	1.5480	0.0877	1.5585	0.0894	1.0161	0.0250	1.0177	0.0250
		GELF	1.4974	0.0799	1.5069	0.0807	0.9874	0.0245	0.9889	0.0243
		WSELF	1.5061	0.0809	1.5158	0.0818	0.9930	0.0245	0.9946	0.0243
		PLF	1.5720	0.0932	1.5828	0.0955	1.0274	0.0258	1.0287	0.0258
30	1.5	SELF	1.5340	0.0553	1.5387	0.0548	1.0102	0.0169	1.0122	0.0163
		GELF	1.4989	0.0517	1.5031	0.0509	0.9906	0.0166	0.9927	0.0160
		WSELF	1.5053	0.0521	1.5096	0.0514	0.9945	0.0166	0.9965	0.0160
		PLF	1.5498	0.0577	1.5545	0.0574	1.0179	0.0172	1.0199	0.0166
50	1.5	SELF	1.5202	0.0306	1.5229	0.0300	1.0057	0.0100	1.0090	0.0100
		GELF	1.4984	0.0294	1.5010	0.0287	0.9938	0.0099	0.9970	0.0098
		WSELF	1.5025	0.0295	1.5051	0.0288	0.9961	0.0099	0.9994	0.0098
		PLF	1.5295	0.0315	1.5322	0.0309	1.0104	0.0101	1.0137	0.0101
100	1.5	SELF	1.5096	0.0145	1.5119	0.0150	1.0032	0.0050	1.0037	0.0049
		GELF	1.4985	0.0142	1.5008	0.0146	0.9971	0.0050	0.9976	0.0049
		WSELF	1.5006	0.0142	1.5029	0.0147	0.9983	0.0050	0.9988	0.0049
		PLF	1.5142	0.0147	1.5166	0.0152	1.0056	0.0051	1.0061	0.0049

In all cases, the biases and MSEs of the estimators decrease as the sample size n increases. It indicates that all the estimators are asymptotically unbiased and

Table 3.The simulated, means and MSEs values under different loss function for Tierney Kadaneâ€^{TM}s approximation parameter estimators of α and β

			Tierney-Kadane's approximation							
			$\hat{\alpha}$				$\hat{\beta}$			
			NP		GP		NP		GP	
n	α	LF	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
20	0.5	SELF	0.5288	0.0107	0.5348	0.0107	1.2014	0.3498	1.2423	0.3448
		GELF	0.5081	0.0091	0.5143	0.0089	0.9001	0.3049	0.9501	0.1962
		WSELF	0.5123	0.0094	0.5185	0.0092	0.9549	0.2193	1.0046	0.2093
		PLF	0.5369	0.0115	0.5428	0.0116	1.3497	0.4496	1.3823	0.4838
30	0.5	SELF	0.5175	0.0063	0.5218	0.0065	1.1420	0.2066	1.1656	0.2083
		GELF	0.5040	0.0057	0.5084	0.0058	0.9430	0.1433	0.9714	0.1393
		WSELF	0.5068	0.0058	0.5111	0.0059	0.9809	0.1485	1.0084	0.1460
		PLF	0.5229	0.0066	0.5271	0.0069	1.2317	0.2645	1.2527	0.2662
50	0.5	SELF	0.5103	0.0034	0.5132	0.0036	1.0765	0.1096	1.0982	0.1124
		GELF	0.5024	0.0032	0.5053	0.0033	0.9609	0.0888	0.9836	0.0871
		WSELF	0.5040	0.0033	0.5069	0.0034	0.9833	0.0906	1.0058	0.0898
		PLF	0.5134	0.0036	0.5163	0.0037	1.1258	0.1273	1.1470	0.1318
100	0.5	SELF	0.5045	0.0016	0.5047	0.0016	1.0534	0.0551	1.0486	0.0505
		GELF	0.5006	0.0015	0.5008	0.0016	0.9966	0.0481	0.9923	0.0443
		WSELF	0.5014	0.0016	0.5016	0.0016	1.0078	0.0490	1.0034	0.0450
		PLF	0.5060	0.0016	0.5063	0.0017	1.0769	0.0599	1.0718	0.0549
20	1.5	SELF	1.5464	0.0339	1.4050	0.0321	1.0161	0.0250	1.0184	0.0250
		GELF	1.4859	0.1329	1.3065	0.1373	0.9819	0.0246	0.9849	0.0243
		WSELF	1.4801	0.1009	1.3183	0.1119	0.9887	0.0245	0.9915	0.0243
		PLF	1.4260	0.0342	1.4348	0.0329	1.0299	0.0259	1.0320	0.0260
30	1.5	SELF	1.5436	0.0277	1.4485	0.0274	1.0101	0.0169	1.0126	0.0163
		GELF	1.4844	0.0379	1.400	0.0317	0.9882	0.0167	0.9908	0.0160
		WSELF	1.4848	0.0325	1.4114	0.0279	0.9926	0.0166	0.9952	0.0160
		PLF	1.4608	0.0285	1.4654	0.0283	1.019	0.0172	1.0213	0.0167
50	1.5	SELF	1.5229	0.0211	1.4757	0.0205	1.0057	0.0100	1.0091	0.0100
		GELF	1.4894	0.0212	1.4523	0.0206	0.9929	0.0099	0.9964	0.0098
		WSELF	1.4841	0.0211	1.4571	0.0205	0.9955	0.0099	0.9989	0.0098
		PLF	1.4822	0.0212	1.4849	0.0207	1.0108	0.0101	1.0143	0.0101
100	1.5	SELF	1.4886	0.0122	1.4908	0.0126	1.0032	0.0050	1.0037	0.0049
		GELF	1.4873	0.0124	1.4795	0.0126	0.9969	0.0050	0.9974	0.0049
		WSELF	1.4895	0.0123	1.4818	0.0126	0.9982	0.0050	0.9987	0.0049
		PLF	1.4931	0.0123	1.4953	0.0126	1.0057	0.0051	1.0062	0.0049

consistent for the parameters α and β. When the classical methods are compared with each other, for the shape parameter α, as far as bias is concerned, LSE, WLSE and TLME work the best for all sample sizes. With respect to the MSEs, for $\alpha < 1$, MMLE-II performs better than the other estimators for small sample sizes $(n < 20)$ and otherwise MLE outperforms the rest. For $\alpha \geq 1$, LME works the best for small sample sizes $(n \leq 20)$. For large sample sizes $(n \geq 50)$, MLE and MMLE-II both works very well.
Similarly, if we compare the classical estimators for β , comparing the biases, for $\alpha < 1$, it is observed that LME and TLME work the best for particularly small sample sizes and for large sample sizes $(n > 50)$, the performances of the LME and TLME are close to that of the MLE and MMLE-II. When $\alpha \geq 1$, LME and TLME work the better than the other estimators for small sample sizes $(n \leq 20)$ and otherwise MLE and MMLE-II outperform the rest.
Then, if we compare the performance of Bayes estimators obtained by Lindley's method, it is clear that as far as MSE and bias are concerned, Bayes estimators under GELF and WSELF work the best in all cases. Similarly, comparing the performance of Bayes estimators obtained by Tierney Kadane's approximation, it is observed that, if $\alpha \leq 1$, Bayes estimators obtained under GELF works the best in all cases for estimating α parameter, followed by Bayes estimation under the WSELF. When $\alpha > 1$, for estimating parameter, Bayesian estimations under SELF and PLF work very well.
For estimating β parameter, Bayes estimation under WSELF performs better than the other estimators for small sample sizes $(n \leq 20)$ and otherwise Bayesian estimations under WSELF and GELF give the same result.
When we compare the Bayesian and classical methods for estimating the α and β parameters, it is clear that as far as bias and MSE are concerned; Bayesian methods outperform the classical methods. Furthermore, Lindley's method works well than the Tierney-Kadane's method in the most of the cases. Also, the GP gives better estimators than the NP for all loss functions.

6. Application

In this section, an actual data set is used to illustrate the estimation procedure developed in section 3-4. The data set measured from Sivas, Turkey during 2017 was used. There were 6011 observations recorded. The data was taken from the Turkish State Meteorological Service. All measurements were made at the heights of 10m in hourly basis.
In this paper, the performance of the Weibull distribution (WD) was compared with the Gamma distribution (GD), log- normal distribution (LND) and inverse Gauss distribution (IGD). These distributions for wind speed data were analyzed seasonally and annually. To determine the distribution providing better fit to wind speed data, we computed the root mean square error (RMSE), the coefficient of

determination (R^2) and Akaike information criteria(AIC) values for each distribution, as shown in Table 4. The formulas for model selection criteria were given in Table 5. In addition to these statistical criteria, the cumulative density function of the WD, GD, IGD and LND were presented in Figure 1 for seasonal and annual wind speed data.

Table 4. RMSE, R^2 and AIC values for distributions

n		Criteria	WD	IGD	LN	GD
		RMSE	0.0215	0.0746	0.0502	0.0301
6011	Annual	R2	0.9946	0.9214	0.9664	0.989
		AIC	2.5736	2.7704	2.7488	2.6058
		RMSE	0.0215	0.0821	0.0571	0.0353
1560	Winter	R2	0.9945	0.9053	0.9564	0.9847
		AIC	6.6889	7.2235	6.9984	6.7801
		RMSE	0.025	0.0627	0.0419	0.0261
1710	Spring	R2	0.9926	0.9453	0.977	0.9918
		AIC	7.2749	7.7628	8.3423	7.3414
		RMSE	0.0236	0.0796	0.0538	0.0338
1429	Summer	R2	0.9936	0.9084	0.9606	0.986
		AIC	6.1904	6.7319	6.8202	6.298
		RMSE	0.0231	0.0721	0.0482	0.0287
1312	Autumn	R2	0.9936	0.9276	0.9693	0.99
		AIC	5.5252	5.8949	5.6003	5.5729

Table 5. The formulas of criteria for model evaluation

Criteria	Formulas
RMSE	$2k - 2\ln\alpha$
R^2	$1 - \left(\sum_{i=1}^{n}\hat{F}\left(X_{(i)}\right) - \frac{i}{n+1}\right)^2 / \left(\sum_{i=1}^{n}\hat{F}\left(X_i\right) - \bar{\hat{F}}\left(X_i\right)\right)^2$
AIC	$\left[\sum_{i=1}^{n}\left(\hat{F}\left(X_{(i)}\right) - \frac{i}{n+1}\right)^2 / n\right]^{1/2}$

According to Table 4, Weibull distribution has the smallest RMSE, AIC values and the highest R^2 values. In Table 5, k is the number of the unknown parameters, In L is the value of log-likelihood function for each distribution, $\hat{F}$ is the estimated cumulative density function, X_i is $i-th$ order statistics, n is sample size and $\bar{\hat{F}} = \sum_{i=1}^{n}\hat{F}_i/n$.

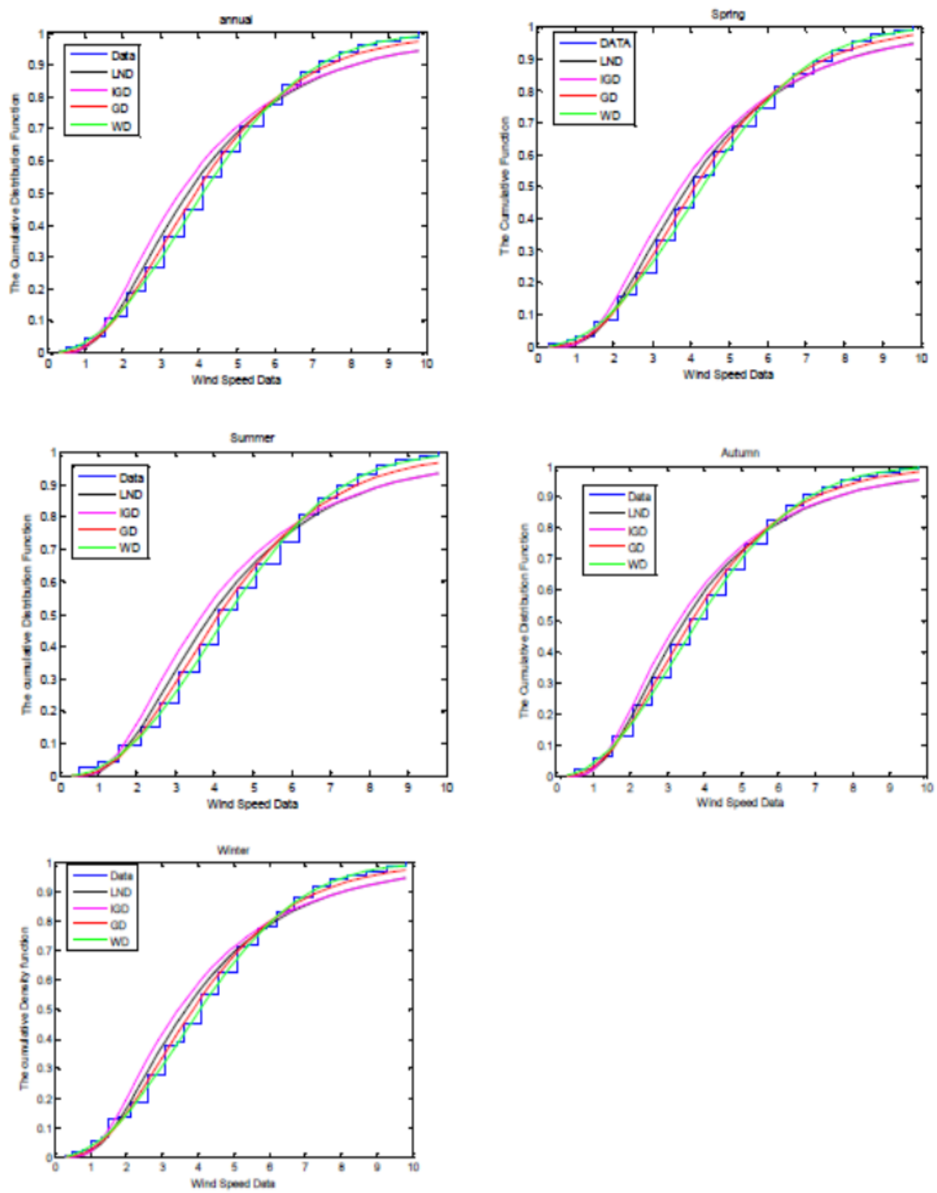

Figure 1. The cumulative density function for annual and seasonal wind speed data (m/s) for Sivas.

Table 6. Classical parameter estimations for the wind speed data.

n		Estimator	$\hat{\alpha}$	$\hat{\beta}$
6011	Annual	MLE	2.1520	4.9177
		LME	2.1422	4.9202
		TLME	2.1256	4.9326
		MMLE-I	2.1318	4.9088
		MMLE-II	3.0342	5.2794
		ME	2.1551	4.9203
		LSE	2.1083	4.9360
		WLSE	2.1571	4.9221
		PE	2.1876	4.9300
1560	Winter	MLE	2.1052	4.8413
		LME	2.1020	4.8463
		TLME	2.0534	4.8832
		MMLE-I	2.0780	4.8291
		MMLE-II	2.6032	5.0557
		ME	2.1139	4.8465
		LSE	2.1089	4.8786
		WLSE	2.1168	4.8610
		PE	2.1452	4.8600
1710	Spring	MLE	2.2734	5.0726
		LME	2.2605	5.0705
		TLME	2.3060	5.0389
		MMLE-I	2.2730	5.0724
		MMLE-II	2.6083	5.2082
		ME	2.2674	5.0704
		LSE	2.1956	5.0535
		WLSE	2.2540	5.0615
		PE	2.2746	5.0723
1429	Summer	MLE	2.2483	5.1749
		LME	2.2290	5.1816
		TLME	2.1682	5.2274
		MMLE-I	2.2058	5.1573
		MMLE-II	2.5317	5.2892
		ME	2.2541	5.1812
		LSE	2.1641	5.2318
		WLSE	2.2420	5.1998
		PE	2.3019	5.1970

Table 6. Continued.

n		Estimator	$\hat{\alpha}$	$\hat{\beta}$
1312	Autumn	MLE	2.0056	4.5226
		LME	2.0020	4.5226
		TLME	2.0117	4.5155
		MMLE-I	1.9993	4.5196
		MMLE-II	2.5919	4.7879
		ME	2.0060	4.5228
		LSE	1.9794	4.5225
		WLSE	2.0062	4.5257
		PE	2.0140	4.5285

Table 7.Lindley's and Tierney Kadane's parameter estimations under NP for the wind speed data

			Lindley's		Tierney-Kadane's	
n		LF	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
6011	Annual	SELF	2.1517	4.9178	2.1494	4.9178
		GELF	2.1515	4.9176	2.1493	4.9177
		WSELF	2.1515	4.9176	2.1492	4.9176
		PLF	2.1518	4.9179	2.1496	4.9179
1560	Winter	SELF	2.1041	4.8419	2.0963	4.8418
		GELF	2.1034	4.8412	2.0956	4.8414
		WSELF	2.1033	4.8411	2.0954	4.8413
		PLF	2.1046	4.8423	2.0967	4.8423
1710	Spring	SELF	2.2724	5.0731	2.2619	5.0731
		GELF	2.2717	5.0725	2.2611	5.0726
		WSELF	2.2716	5.0724	2.2611	5.0724
		PLF	2.2728	5.0734	2.2624	5.0734
1429	Summer	SELF	2.2471	5.1755	2.2349	5.1753
		GELF	2.2463	5.1748	2.2340	5.1749
		WSELF	2.2462	5.1747	2.2338	5.1748
		PLF	2.2476	5.1759	2.2355	5.1758
1312	Autumn	SELF	2.0044	4.5232	1.9973	4.5233
		GELF	2.0036	4.5224	1.9965	4.5223
		WSELF	2.0035	4.5223	1.9963	4.5222
		PLF	2.0049	4.5237	1.9978	4.5237

Table 8. Lindley's and Tierney Kadane's parameter estimations under GP for the wind speed

			Lindley's		Tierney-Kadane's	
n		LF	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
6011	Annual	SELF	2.1517	4.9186	2.1489	4.918
		GELF	2.1515	4.9184	2.1487	4.9131
		WSELF	2.1515	4.9184	2.1487	4.9131
		PLF	2.1518	4.9187	2.1490	4.9187
1560	Winter	SELF	2.1039	4.8365	2.0939	4.8365
		GELF	2.1034	4.8358	2.0953	4.8358
		WSELF	2.1033	4.8357	2.0951	4.8357
		PLF	2.1046	4.8369	2.0965	4.8369
1710	Spring	SELF	2.2727	5.0731	2.2623	5.0731
		GELF	2.2720	5.0725	2.2615	5.0725
		WSELF	2.2719	5.0725	2.2614	5.0725
		PLF	2.2731	5.0734	2.2627	5.0734
1429	Summer	SELF	2.2494	5.1776	2.2371	5.1777
		GELF	2.2486	5.1769	2.2361	5.1770
		WSELF	2.2484	5.1768	2.2360	5.1769
		PLF	2.2499	5.1780	2.2376	5.1781
1312	Autumn	SELF	2.0059	4.5218	1.9984	4.5218
		GELF	2.0047	4.5210	1.9976	4.5210
		WSELF	2.0046	4.5209	1.9975	4.5209
		PLF	2.0060	4.5223	1.9989	4.5223

It is clear that the results in Figure 1 are consistent with Table 4. Thus, in terms of all criteria, WD performed better than GD, IGD and LND for the seasonal and the annual wind speed data. Therefore, the two-parameter Weibull distribution was used for modelling the wind speed data. The estimators of the α and β obtained by using Bayesian and classical methods are given in Table 6-8. In light of the aforementioned information, we recommend the Bayesian estimations under WSELF and GELF for estimating the unknown parameters of Weibull distribution.

7. Conclusion

In this paper, we obtained different methods of estimation of the unknown parameters both with Bayesian and classical approximation. In classical method, the parameters α and β were estimated by using nine different method. In Bayesian method, we computed the Bayesian estimators of unknown parameters based on symmetric and asymmetric loss functions. The Bayes estimators do not have explicit form. Hence, we used the Lindley and Tierney Kadane's techniques under the assumption of Gamma priors. We also compare the performances of the estimators via simulation study. It is clear from the simulation results given in Table 1-3 that Lindley approximation under GELF and WSELF are more preferable than the other estimators according to the MSE and bias criteria in both scenarios i.e. informative prior and non-informative prior (especially for sample size $n > 50$).

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T_0 CONVERGENCE APPROACH SPACES

MUHAMMAD QASIM AND MEHMET BARAN

Abstract. In previous papers, several T_0-objects in set-based topological category have been introduced and compared. In this paper, we give the characterization of general $\overline{T_0}$ (resp. T_0, and T_0') convergence approach spaces as well as show how these notions are linked to each other.

1. Introduction

In 1989, Colebunders and Lowen [16] introduced convergence approach space to satisfy the categorical properties such as Cartesian closedness which are failed in approach space [17].

Classical T_0 separation of topology plays a vital role not only in mathematics such as to get an alternative characterization of locally semi-simple coverings in terms of light morphisms in algebraic topology [13] but also in computer science where this concept correspond to access the values through observations [26]. In addition to that, T_0 axiom has been used to build topological models in denotational semantics of programming language and lambda calculus where Hausdorff topologies fail to build such models [24, 25]. Furthermore, it has been used to characterize digital line in digital topology and to construct cellular complex in image processing and computer graphs [10, 14, 15].

Due to huge importance of T_0 separation, this concept has been extended to topological categories by several mathematicians such as Brümmer [8] in 1971, Marny [21] in 1973, Hoffmann [11] in 1974, Harvey [9] in 1977 and Baran [2] in 1991. Moreover, in 1991, Weck-Schwarz [27] and in 1995, Baran [3] analyzed the relationship among these various generalization of T_0 objects. One of the main reason to extend T_0 separation was to define T_2 objects in arbitrary topological categories [5].

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The main object of this paper is to characterize each of T_0, $\overline{T_0}$ and T_0' convergence approach spaces and show how these are related to each other.

2. Preliminaries

Let $\mathcal{E}$ and $\mathcal{B}$ be two categories. The functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is called topological functor if (i) $\mathcal{U}$ is concrete (i.e., faithful and amnestic) (ii) $\mathcal{U}$ consists of small fibers and (iii) every $\mathcal{U}$-source has a unique initial lift [1, 22, 23].

Note that topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is called normalized if subterminals have a unique structure.

Let X be a set, $A \subseteq X$, $F(X)$ be the set of all filters and $\mathcal{A}$ be collection of subsets of X. The stack of $\mathcal{A}$ and the indicator map $\theta_A : X \to [0,\infty]$ are defined by $[\mathcal{A}] = \{B \subseteq X | \exists A \in \mathcal{A} : A \subseteq B\}$ and

$$\theta_A(x) = \begin{cases} 0, & x \in A \\ \infty, & x \notin A \end{cases}$$

respectively.

Definition 1. *(cf.* [16, 18, 20]*) A map $\lambda : F(X) \longrightarrow [0,\infty]^X$ is called a convergence approach structure on X if it satisfies the followings:*

(i) $\forall x \in X : \lambda[x](x) = 0$,
(ii) $\forall \alpha, \beta \in F(X) : \alpha \subset \beta \Rightarrow \lambda\beta \leq \lambda\alpha$,
(iii) $\forall \alpha, \beta \in F(X) : \lambda(\alpha \cap \beta) = \sup\{\lambda(\alpha), \lambda(\beta)\}$.

The pair (X, λ) is called a convergence approach space.

Definition 2. *(cf.* [16, 18, 20]*) Let (X, λ) and (X', λ') be convergence approach spaces. The map $f : (X, \lambda) \longrightarrow (X', \lambda')$ is called a contraction map if it satisfies for all $\alpha \in F(X) : \lambda'(f(\alpha)) \circ f \leq \lambda\alpha$.*

The category whose objects are convergence approach spaces and morphisms are contraction maps is denoted by **CApp** and it is a Cartesian closed topological category over **Set** [16, 18, 20].

Definition 3. *(cf.* [16, 18, 20]*) Let X be a non-empty set and (X_i, λ_i) be the class of convergence approach spaces.*

(i) *A source $\{f_i : X \to (X_i, \lambda_i)\}$ in **CApp** has initial lift if and only if for all $\alpha \in F(X)$, $\lambda\alpha = \sup_{i \in I} \lambda_i(f_i(\alpha)) \circ f_i$, where $f_i(\alpha)$ is a filter generated by $\{f_i(A_i), i \in I\}$, i.e., $f_i(\alpha) = \{A_i \subset X_i : \exists B \in \alpha$ such that $f_i(B) \subset A_i\}$.*
(ii) *A sink $\{f_i : (X_i, \lambda_i) \to X\}$ in **CApp** has final lift if and only if for all $\alpha \in F(X)$ and $x \in X$,*

$$\lambda(\alpha)(x) = \begin{cases} 0, & \alpha = [x] \\ \inf\limits_{i \in I} \inf\limits_{y \in f_i^{-1}(x)} \inf\limits_{\substack{\beta \in F(X_i) \\ \subset \alpha}} \lambda_i(\beta)(y), & \alpha \neq [x] \end{cases}$$

(iii) *The discrete structure* (X, λ_{dis}) *on* X *in* **CApp** *is defined by for all* $\alpha \in F(X)$ *and* $x \in X$,

$$\lambda_{dis}(\alpha) = \begin{cases} \theta_{\{x\}}, & \alpha = [x] \\ \infty, & \alpha \neq [x] \end{cases}$$

(iv) *The indiscrete structure* (X, λ_{ind}) *on* X *in* **CApp** *is defined by for all* $\alpha \in F(X)$ *and* $x \in X$,

$$\lambda_{ind}(\alpha)(x) = 0$$

3. T_0 Convergence Approach Spaces

Let B be a nonempty set, $B^2 \coprod B^2$ be the coproduct of B^2 and $B^2 \vee_\triangle B^2$ be two distinct copies of B^2 identified along the diagonal [2]. Let $q : B^2 \coprod B^2 \to B^2 \vee_\triangle B^2$ be the quotient map. A point (x, y) in $B^2 \vee_\triangle B^2$ is denoted by $(x, y)_1$ (resp. $(x, y)_2$) if (x, y) is in the first (resp. second) component of $B^2 \vee_\triangle B^2$. Note that $(x, x)_1 = (x, x)_2 = (x, x)$.

Definition 4. *(cf.* [2]*) A map* $A : B^2 \vee_\triangle B^2 \to B^3$ *is called a principle axis map if*

$$A((x, y)_i) = \begin{cases} (x, y, x), & i = 1 \\ (x, x, y), & i = 2 \end{cases}$$

Definition 5. *(cf.* [2]*) A map* $\nabla : B^2 \vee_\triangle B^2 \to B^2$ *is called a folding map if* $\nabla((x, y)_i) = (x, y)$ *for* $i = 1, 2$.

Definition 6. *(cf.* [2, 21]*) Let* $\mathcal{U} : \mathcal{E} \to$ **Set** *be topological in the sense of* [1, 22] *and* X *be an object in* $\mathcal{E}$ *with* $\mathcal{U}(X) = B$.

(i) X *is* $\overline{T_0}$ *iff initial lift of the* $\mathcal{U}$*-source* $\{A : B^2 \vee_\triangle B^2 \to \mathcal{U}(X^3) = B^3$ *and* $\nabla : B^2 \vee_\triangle B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ *is discrete, where* $\mathcal{D}$ *is a discrete functor which is left adjoint to* $\mathcal{U}$.

(ii) X *is* T_0' *iff initial lift of the* $\mathcal{U}$*-source* $\{id : B^2 \vee_\triangle B^2 \to \mathcal{U}(B^2 \vee_\triangle B^2)'^2 \vee_\triangle B^2$ *and* $\nabla : B^2 \vee_\triangle B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ *is discrete, where* $(B^2 \vee_\triangle B^2)'$ *is the final lift of* $\mathcal{U}$*-sink* $\{q \circ i_1, q \circ i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\triangle B^2\}$ *and* $i_k : B^2 \to B^2 \coprod B^2$ *are the canonical injections for* $k = 1, 2$.

(iii) X *is* T_0 *iff* X *doesn't contain an indiscrete subspace with (at least) two points.*

Theorem 7. *A convergence approach space* (X, λ) *is* $\overline{T_0}$ *iff for all* $x, y \in X$ *with* $x \neq y$, $\lambda([x])(y) = \infty$ *or* $\lambda([y])(x) = \infty$.

Proof. Let (X, λ) be $\overline{T_0}$ for all $x, y \in X$ with $x \neq y$. Note that $[(x, y)_1] \in F(X^2 \vee_\triangle X^2)$, $(x, y)_2 \in X^2 \vee_\triangle X^2$ and

$$\lambda_{dis}([\nabla(x, y)_1])(\nabla(x, y)_2) = \lambda_{dis}([(x, y)])(x, y) = 0,$$

$$\lambda([\pi_1 A(x, y)_1](\pi_1 A(x, y)_2) = \lambda([x])(x) = 0,$$

$$\lambda([\pi_2 A(x,y)_1](\pi_2 A(x,y)_2) = \lambda([y])(x)$$

and

$$\lambda([\pi_3 A(x,y)_1](\pi_3 A(x,y)_2) = \lambda([x])(y),$$

where $\pi_i : X^3 \to X$ are the projection maps, $i = 1,2,3$. Since (X,λ) is $\overline{T_0}$, by Definition 3 (i),

$$\begin{aligned}\infty &= \sup\{\lambda_{dis}([\nabla(x,y)_1])(\nabla(x,y)_2), \lambda([\pi_1 A(x,y)_1])(\pi_1 A(x,y)_2),\\ &\quad \lambda([\pi_2 A(x,y)_1])(\pi_2 A(x,y)_2), \lambda([\pi_3 A(x,y)_1])(\pi_3 A(x,y)_2)\}\\ &= \sup\{0, \lambda([x])(y), \lambda([y])(x)\} = \sup\{\lambda([x])(y), \lambda([y])(x)\}\end{aligned}$$

and consequently, $\lambda([x])(y) = \infty$ or $\lambda([y])(x) = \infty$.

Conversely, let $\overline{\lambda}$ be an initial convergence approach structure on $X^2 \vee_\triangle X^2$ induced by $A : X^2 \vee_\triangle X^2 \to (X^3, \lambda^3)$ and $\nabla : X^2 \vee_\triangle X^2 \to (X^2, \lambda_{dis})$, where λ_{dis} is discrete convergence approach structure on X^2 and λ^3 is the product convergence approach structure on X^3 induced by $\pi_i : X^3 \to X$ the projection maps for $i = 1,2,3$. Suppose $\alpha \in F(X^2 \vee_\triangle X^2)$ and $v \in X^2 \vee_\triangle X^2$ with $\nabla v = (x,y)$. By Definition 1, we show that

$$\overline{\lambda}(\alpha) = \begin{cases}\theta_{\{v\}}, & \alpha = [v]\\ \infty, & \alpha \neq [v]\end{cases}$$

where $\theta_{\{v\}}$ is the indicator of $\{v\}$. Let w be any point in $X^2 \vee_\triangle X^2$. Note that

$$\lambda_{dis}(\nabla\alpha)(\nabla w) = \begin{cases}\theta_{\{(x,y)\}}\nabla w, & \nabla\alpha = [(x,y)]\\ \infty, & \nabla\alpha \neq [(x,y)]\end{cases}$$

$$= \begin{cases}0, & \nabla\alpha = [(x,y)] \text{ and } \nabla w = (x,y)\\ \infty, & \nabla\alpha = [(x,y)] \text{ and } \nabla w \neq (x,y)\\ \infty, & \nabla\alpha \neq [(x,y)] \text{ and } \nabla w \neq (x,y)\end{cases}$$

Case I: If $x = y$, then $\nabla w = (x,x)$ implies $w = (x,x)_1 = (x,x)_2 = v$ and $\nabla\alpha = [(x,x)]$ implies $\alpha = [(x,x)_i] = [(x,x)]$ for $i = 1,2$. By Definition 3 (i), $\overline{\lambda}(\nabla\alpha)(\nabla w) = \overline{\lambda}([(x,x)])(x,x) = 0$ since $\overline{\lambda}$ is a convergence approach structure on $X^2 \vee_\triangle X^2$.

Suppose that $x \neq y$. $\nabla w = (x,y)$ implies $w = (x,y)_1$ or $u = (x,y)_2$ and $\nabla\alpha = [(x,y)]$ implies $\alpha = [(x,y)_1]$, $[(x,y)_2]$, $[\{(x,y)_1,(x,y)_2\}]$ or $\alpha \supset [\{(x,y)_1,(x,y)_2\}]$. Firstly, we show that the case $\alpha \supset [\{(x,y)_1,(x,y)_2\}]$ with $\alpha \neq [\emptyset]$ and $\alpha \neq [\{(x,y)_1,(x,y)_2\}]$ cannot occur. To end this, if $[\emptyset] \neq \alpha \neq [\{(x,y)_1,(x,y)_2\}]$, then $\alpha \supset [\{(x,y)_1,(x,y)_2\}]$ iff $\alpha = [(x,y)_1]$ or $\alpha = [(x,y)_2]$. Clearly, if $\alpha = [(x,y)_1]$ or $[(x,y)_2]$, then $\alpha \supset [\{(x,y)_1,(x,y)_2\}]$. Conversely, if $\alpha \supset [\{(x,y)_1,(x,y)_2\}]$ with $[\emptyset] \neq \alpha \neq [\{(x,y)_1,(x,y)_2\}]$, then there exists $V \in \alpha$ such that $V \neq \{(x,y)_1,(x,y)_2\}$ and $V \neq \emptyset$. Since V and $\{(x,y)_1,(x,y)_2\}$ are in α and α is a filter, it follows that

$V \cap \{(x,y)_1,(x,y)_2\} = \{(x,y)_1\}$ or $\{(x,y)_2\}$ is in α, i.e., $\alpha = [(x,y)_1]$ or $[(x,y)_2]$. Hence, we must have $\alpha = [(x,y)_1]$, $[(x,y)_2]$ or $[\{(x,y)_1,(x,y)_2\}]$.

If $\alpha = [(x,y)_i]$ and $w = (x,y)_i$, $i = 1,2$, then $\overline{\lambda}([(x,y)_i])((x,y)_i) = 0$ since $\overline{\lambda}$ is a convergence approach structure on $X^2 \vee_\triangle X^2$.

If $\alpha = [(x,y)_2]$ and $w = (x,y)_1$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}(\nabla[(x,y)_2])(\nabla(x,y)_1) = \lambda_{dis}([(x,y)])(x,y) = 0,$$

$$\lambda(\pi_1 A\alpha)(\pi_1 Aw) = \lambda([\pi_1 A(x,y)_2])(\pi_1 A(x,y)_1)) = \lambda([x])(x) = 0,$$

$$\lambda(\pi_2 A\alpha)(\pi_2 Aw) = \lambda([\pi_2 A(x,y)_2])(\pi_2 A(x,y)_1) = \lambda([x])(y)$$

and

$$\lambda(\pi_3 A\alpha)(\pi_3 Aw) = \lambda([\pi_3 A(x,y)_2])(\pi_3 A(x,y)_1) = \lambda([y])(x),$$

by Definition 3 (i),

$$\begin{aligned}
\overline{\lambda}(\alpha)(w) &= \overline{\lambda}([(x,y)_2])((x,y)_1) \\
&= \sup\{\lambda_{dis}([\nabla(x,y)_2])(\nabla(x,y)_1), \lambda([\pi_1 A(x,y)_2])(\pi_1 A(x,y)_1), \\
&\qquad \lambda([\pi_2 A(x,y)_2])(\pi_2 A(x,y)_1), \lambda([\pi_3 A(x,y)_2])(\pi_3 A(x,y)_1)\} \\
&= \sup\{0, \lambda([y])(x), \lambda([x])(y)\} = \sup\{\lambda([y])(x), \lambda([x])(y)\} = \infty
\end{aligned}$$

since by the assumption $\lambda([y])(x) = \infty$ or $\lambda([x])(y) = \infty$.

If $\alpha = [\{(x,y)_1,(x,y)_2\}]$ and $w = (x,y)_1$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}(\nabla[\{(x,y)_1,(x,y)_2\}])(\nabla(x,y)_1) = \lambda_{dis}([x])(x) = 0,$$

$$\lambda(\pi_1 A\alpha)(\pi_1 Aw) = \lambda([\{\pi_1 A(x,y)_1, \pi_1 A(x,y)_2\}])(\pi_1 A(x,y)_1) = \lambda([x])(x) = 0,$$

$$\lambda(\pi_2 A\alpha)(\pi_2 Aw) = \lambda([\{\pi_2 A(x,y)_1, \pi_2 A(x,y)_2\}])(\pi_2 A(x,y)_1) = \lambda([\{x,y\}])(y)$$

and

$$\lambda(\pi_3 A\alpha)(\pi_3 Aw) = \lambda([\{\pi_3 A(x,y)_1, \pi_3 A(x,y)_2\}])(\pi_3 A(x,y)_1) = \lambda([\{x,y\}])(x).$$

Note that $[\{x,y\}] \subset [y]$ and $[\{x,y\}] \subset [x]$. Since λ is a convergence approach structure, we get $\lambda([y])(x) \leq \lambda([\{x,y\}])(x)$ and $\lambda([x])(y) \leq \lambda([\{x,y\}])(y)$. The assumption $\lambda([y])(x) = \infty$ (resp. $\lambda([x])(y) = \infty$) implies $\lambda([\{x,y\}])(x) = \infty$ (resp. $\lambda([\{x,y\}])(y) = \infty$).

By Definition 3 (i),

$$\begin{aligned}
\overline{\lambda}(\alpha)(w) &= \overline{\lambda}([\{(x,y)_1,(x,y)_2\}])((x,y)_1) \\
&= \sup\{\lambda_{dis}([\{\nabla(x,y)_1, \nabla(x,y)_2\}])(\nabla(x,y)_1), \lambda([\{\pi_1 A(x,y)_1, \pi_1 A(x,y)_2\}]) \\
&\qquad (\pi_1 A(x,y)_1), \lambda([\{\pi_2 A(x,y)_1, \pi_2 A(x,y)_2\}])(\pi_2 A(x,y)_1), \lambda([\{\pi_3 A(x,y)_1, \\
&\qquad \pi_3 A(x,y)_2\}])(\pi_3 A(x,y)_1)\} = \sup\{0,\infty\} = \infty.
\end{aligned}$$

For the cases $\alpha = [(x,y)_1]$ or $[\{(x,y)_1,(x,y)_2\}]$ and $w = (x,y)_2$, it can be done analogously to the above argument.

Case II: Let $(z,z) = \nabla w \neq (x,y)$ for some $z \in X$ and $\nabla\alpha = [(x,y)]$. It follows that $w = (z,z)_1 = (z,z)_2$ and $\alpha = [(x,y)_1]$, $[(x,y)_2]$ or $[\{(x,y)_1,(x,y)_2\}]$.

If $\alpha = [(x,y)_i]$ or $[\{(x,y)_1,(x,y)_2\}]$ for $i = 1,2$ and $w = (z,z)_1 = (z,z)_2$, then $\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}([(x,y)])(z,z) = \infty$ since λ_{dis} is a discrete convergence approach structure and $(x,y) \neq (z,z)$. It follows that

$$\begin{aligned}\overline{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla w), \lambda(\pi_1 A\alpha)(\pi_1 Aw), \lambda(\pi_2 A\alpha)(\pi_2 Aw), \lambda(\pi_3 A\alpha)(\pi_3 Aw)\} \\ &= \sup\{\infty, \lambda(\pi_1 A\alpha)(z,z), \lambda(\pi_2 A\alpha)(z,z), \lambda(\pi_3 A\alpha)(z,z)\} = \infty.\end{aligned}$$

Case III: Suppose $\nabla w \neq (x,y)$ and $\nabla\alpha \neq [(x,y)]$, then $\lambda_{dis}(\nabla\alpha)(\nabla w) = \infty$ since λ_{dis} is a discrete convergence approach structure, and consequently

$$\begin{aligned}\overline{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla w), \lambda(\pi_1 A\alpha)(\pi_1 Aw), \lambda(\pi_2 A\alpha)(\pi_2 Aw), \lambda(\pi_3 A\alpha)(\pi_3 Aw)\} \\ &= \sup\{\infty, \lambda(\pi_1 A\alpha)(\pi_1 Aw), \lambda(\pi_2 A\alpha)(\pi_2 Aw), \lambda(\pi_3 A\alpha)(\pi_3 Aw)\} = \infty.\end{aligned}$$

Therefore, for all $\alpha \in F(X^2 \vee_\triangle X^2)$ and $\forall v \in X^2 \vee_\triangle X^2$, we get

$$\overline{\lambda}(\alpha) = \begin{cases}\theta_{\{v\}}, & \alpha = [v] \\ \infty, & \alpha \neq [v]\end{cases}$$

i.e., by Definition 3 (iii), $\overline{\lambda}$ is discrete convergence approach structure on $X^2 \vee_\triangle X^2$ and by Definition 6 (i), (X,λ) is $\overline{T_0}$. $\square$

Let X be a non-empty set and $\alpha, \beta \in F(X)$. We denote by $\alpha \cup \beta$ the smallest filter containing both α and β, i.e., $\alpha \cup \beta$ is the filter generated by the set $\{V \cap W : V \in \alpha, W \in \beta\}$.

Lemma 8. *Let $(X_j, \lambda_j)_{j\in I}$ be a class of* **CApp** *objects and $X = \coprod_{j\in I} X_j$, the coproduct of $\{X_j\}_{j\in I}$. The coproduct convergence approach structure λ on X with respect to the family of canonical injections $i_j : (X_j, \lambda_j) \to X = \coprod_{j\in I} X_j$ is defined by*

$$\lambda(\alpha)(x_k) = \begin{cases}0, & \text{if } \alpha = [x_k] \\ \lambda_k(\alpha \cup [X_k])(x_k), & \text{if } i_k(\beta) \subset \alpha \text{ for some } k \in I \text{ and } \beta_k \in F(X_k) \\ \infty, & \text{if } i_k(\beta) \not\subset \alpha \text{ for all } k \in I \text{ and } \beta_k \in F(X_k)\end{cases}$$

Proof. Let $\alpha \in F(X)$ with $\alpha \neq [x]$ for all $x \in X = \coprod_{j\in I} X_j$. By definition 3 (iii), $\lambda(\alpha)(x_k) = \inf\{\lambda_k(\beta_k)(x_k) : \beta_k \in F(X_k)$ for some $k \in I$ such that $i_k(\beta_k) \subset \alpha\}$. If $i_k(\beta_k) \subset \alpha$ for some $k \in I$ and $\beta_k \in F(X_k)$, then such k can be at most one and for this k, $\alpha \cup [X_k]$ is the greatest element $\beta_k \in F(X_k)$ such that $i_k(\beta_k) \subset \alpha$, i.e., $i_k(\alpha \cup [X_k]) = \alpha$. Hence, $\lambda(\alpha)(x_k) = \lambda_k(\alpha \cup [X_k])(x_k)$. $\square$

Theorem 9. *Every convergence approach space is T_0'.*

Proof. Let (X,λ) be a convergence approach space. We show that (X,λ) is T_0'. Let $\overline{\lambda}$ be an initial convergence approach structure on $X^2 \vee_\triangle X^2$ induced by $\nabla : X^2 \vee_\triangle X^2 \to (X^2, \lambda_{dis})$ and $id : X^2 \vee_\triangle X^2 \to (X^2 \vee_\triangle X^2, \lambda^*)$, where λ_{dis} is discrete convergence approach structure on X^2 and λ^* is the final convergence approach structure on $X^2 \vee_\triangle X^2$ induced by $q \circ i_k : X^2 \to X^2 \vee_\triangle X^2$ for $k = 1, 2$ and let $v \in X^2 \vee_\triangle X^2$ with $\nabla v = (x, y)$. Suppose $\alpha \in F(X^2 \vee_\triangle X^2)$ and $w \in X^2 \vee_\triangle X^2$. Note that

$$\begin{aligned} \lambda_{dis}(\nabla\alpha)(\nabla w) &= \begin{cases} \theta_{\{(x,y)\}}\nabla w, & \nabla\alpha = [(x,y)] \\ \infty, & \nabla\alpha \neq [(x,y)] \end{cases} \\ &= \begin{cases} 0, & \nabla\alpha = [(x,y)] \text{ and } \nabla w = (x,y) \\ \infty, & \nabla\alpha = [(x,y)] \text{ and } \nabla w \neq (x,y) \\ \infty, & \nabla\alpha \neq [(x,y)] \text{ and } \nabla w \neq (x,y) \end{cases} \end{aligned}$$

Case I: If $x = y$, then $\nabla w = (x, x)$ implies $w = (x, x)_1 = (x, x)_2 = (x, x) = v$ and $\nabla\alpha = [(x, x)]$ implies $\alpha = [(x, x)_i] = [(x, x)]$ for $i = 1, 2$. By Definition 3 (i), $\overline{\lambda}(\alpha)(w) = \overline{\lambda}([(x, x)_i])(x, x)_i = 0$ since $\overline{\lambda}$ is a convergence approach structure on $X^2 \vee_\triangle X^2$.

Let $x \neq y$. $\nabla\alpha = [(x, y)]$ implies $\alpha = [(x, y)_1]$, $[(x, y)_2]$, $[\{(x, y)_1, (x, y)_2\}]$ or $\alpha \supset [\{(x, y)_1, (x, y)_2\}]$ and $\nabla w = (x, y)$ implies $w = (x, y)_1$ or $w = (x, y)_2$. By using the similar argument given in the proof of Theorem 7, we must have $\alpha = [(x, y)_1]$, $[(x, y)_2]$ or $[\{(x, y)_1, (x, y)_2\}]$.

If $\alpha = [(x, y)_j]$ and $w = (x, y)_j$ for $j = 1, 2$, then $\overline{\lambda}([(x, y)_j])((x, y)_j) = 0$ since $\overline{\lambda}$ is a convergence approach structure on $X^2 \vee_\triangle X^2$.

If $\alpha = [(x, y)_1]$ and $w = (x, y)_2$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}(\nabla[(x, y)_1])(\nabla(x, y)_2) = \lambda_{dis}([(x, y)])(x, y) = 0,$$
$$\lambda^*(id\alpha)(idw) = \lambda^*(\alpha)(w) = \lambda^*([(x, y)_1])((x, y)_2).$$

Since $i_2\beta \not\subset \alpha = [(x, y)_1]$ for all $\beta \in F(X^2)$, by Lemma 8,

$$\lambda^*(\alpha)(w) = \lambda^*([(x, y)_1])((x, y)_2) = \infty.$$

Hence, by Definition 3 (i),

$$\begin{aligned} \overline{\lambda}(\alpha)(w) &= \overline{\lambda}([(x, y)_1])((x, y)_2) \\ &= \sup\{\lambda_{dis}([\nabla(x, y)_1])(\nabla(x, y)_2), \lambda^*(id[(x, y)_1])(id(x, y)_2)\} \\ &= \sup\{0, \infty\} = \infty. \end{aligned}$$

Suppose $\alpha = [\{(x, y)_1, (x, y)_2\}]$ and $w = (x, y)_2$.

In particular,

$$\lambda^*(id\alpha)(idw) = \lambda^*(\alpha)(w) = \lambda^*([\{(x, y)_1, (x, y)_2\}])((x, y)_2).$$

Since λ^* is a final convergence approach structure on $X^2 \vee_\triangle X^2$ and $[\{(x, y)_1, (x, y)_2\}] \subset [(x, y)_1]$, we get $\lambda^*([(x, y)_1])((x, y)_2) \leq \lambda^*([\{(x, y)_1, (x, y)_2\}])((x, y)_2)$. By the same statement used above, $\lambda^*([(x, y)_1])((x, y)_2) = \infty$, and consequently,

$$\lambda^*([\{(x, y)_1, (x, y)_2\}])((x, y)_2) = \infty.$$

By Definition 3 (i),

$\overline{\lambda}(\alpha)(w) = \overline{\lambda}([\{(x, y)_1, (x, y)_2\}])((x, y)_1)$

$= \sup\{\lambda_{dis}(\nabla[\{(x,y)_1,(x,y)_2\}])(\nabla(x,y)_2), \lambda^*(id[\{(x,y)_1,(x,y)_2\}]), (id(x,y)_2)\}$
$= \sup\{0,\infty\} = \infty.$

For the cases $\alpha = [(x,y)_2]$ (resp. $[\{(x,y)_1,(x,y)_2\}]$) and $w = (x,y)_1$, by Lemma 8 and the argument used above, we get $\overline{\lambda}(\alpha)(w) = \infty$.

Case II: Let $(z,z) = \nabla w \neq (x,y)$ for some $z \in X$ and $\nabla\alpha = [(x,y)]$. It follows that $w = (z,z)_1 = (z,z)_2$ and $\alpha = [(x,y)_1]$, $[(x,y)_2]$ or $[\{(x,y)_1,(x,y)_2\}]$.

If $\alpha = [(x,y)_i]$ (resp. $[\{(x,y)_i,(x,y)_j\}]$) for $i,j = 1,2$ with $i \neq j$ and $w = (z,z)_1 = (z,z)_2 = (z,z)$, then $\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}([(x,y)])(z,z) = \infty$ since λ_{dis} is a discrete convergence approach structure and $(x,y) \neq (z,z) = \nabla w$. It follows that

$$\begin{aligned}\overline{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla w), \lambda^*(id\alpha)(idw)\}\\ &= \sup\{\infty, \lambda^*(\alpha)(w)\} = \infty.\end{aligned}$$

Case III: Suppose $\nabla w \neq (x,y)$ and $\nabla\alpha \neq [(x,y)]$, then $\lambda_{dis}(\nabla\alpha)(\nabla w) = \infty$ since λ_{dis} is a discrete convergence approach structure, and consequently

$$\begin{aligned}\overline{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla w), \lambda^*(id\alpha)(idw)\}\\ &= \sup\{\infty, \lambda^*(\alpha)(w)\} = \infty.\end{aligned}$$

Therefore, for all $\alpha \in F(X^2 \vee_\triangle X^2)$,

$$\overline{\lambda}(\alpha) = \begin{cases}\theta_{\{v\}}, & \alpha = [v]\\ \infty, & \alpha \neq [v]\end{cases}$$

, i.e., by Definition 3 (iii), $\overline{\lambda}(\alpha)$ is discrete convergence approach structure over $X^2 \vee_\triangle X^2$. By Definition 6 (ii), (X,λ) is T_0'. □

Theorem 10. *A convergence approach space (X,λ) is T_0 iff for all $x,y \in X$ with $x \neq y$, $\lambda([y])(x) > 0$ or $\lambda([x])(y) > 0$.*

Proof. The proof is the same as the proof of [12, 19]. □

Example 11. *Let X be a set with $|X| \geq 2$. By Theorems 7, 9 and 10, every indiscrete convergence approach space, i.e., for all $\alpha \in F(X)$ and for all $x \in X$, $\lambda(\alpha)(x) = 0$ is T_0' but neither $\overline{T_0}$ nor T_0.*

Example 12. *Let X be a non-empty set, $F(X)$ be the set of all filters and $\lambda : F(X) \to [0,\infty]^X$ be a map defined as follows: For all $\alpha \in F(X)$ and $u \in X$,*

$$\lambda(\alpha)(u) = \begin{cases}0, & \alpha = [u]\\ 1, & \alpha \neq [u]\end{cases}$$

Clearly, (X,λ) is a convergence approach space. By Theorems 7, 9 and 10, (X,λ) is T_0 (resp. T_0') but not $\overline{T_0}$.

Remark 13. (I) *In* **Top** *(category of topological spaces and continuous maps),* $\overline{T_0}$*,* T_0' *and* T_0 *are equivalent and reduce to classical* T_0 *axiom (i.e., for each distinct points* x *and* y*, there exists a neighborhood of* x *doesn't contain* y *or vice versa)* [4].

(II) *For any arbitrary topological category,*

(i) $\overline{T_0}$ *implies* T_0' *but converse is not true in general* [3].

(ii) *There is no relation between* T_0 *and each of* $\overline{T_0}$ *and* T_0' [3].

(a) $\overline{T_0}$ *could be only discrete objects such as in* ∞**pqsMet** *(extended pseudo-quasi-semi metric spaces and non-expansive maps)* [7].

(b) $\overline{T_0}$ *could be all objects, e.g., in* **Born** *(bornological spaces and bounded maps)* [3].

(c) *In category* **Born**, $T_0 \implies \overline{T_0} = T_0'$ [3].

(d) *In category* **Lim** *of limit spaces and filter convergence maps,* $\overline{T_0} = T_0 \implies T_0'$ [3].

(e) *In category* **SUConv** *of semi-uniform convergence spaces and uniformly continuous maps,* $\overline{T_0} \implies T_0 \implies T_0'$ [6].

(III) *In convergence approach space* (X, λ)*, by Theorems 7, 9 and 10,* $\overline{T_0} \implies T_0 \implies T_0'$ *but converse of each implication is not true in general by Examples 11 and 12.*

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A GENERALIZED NONLINEAR ITERATIVE ALGORITHM FOR THE EXPLICIT MIDPOINT RULE OF NONEXPANSIVE SEMIGROUP

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Abstract. In this paper, we introduce a new iterative midpoint rule for finding a solution of fixed point problem for a nonexpansive semigroup in real Hilbert spaces. We establish a strong convergence theorem for the sequences generated by our proposed iterative scheme. Furthermore, we provide application to Fredholm integral equations. A numerical example is presented to illustrate the convergence result. Our results improve and extend the corresponding results in the literature.

1. Introduction

Let $\mathbb{R}$ denote the set of all real numbers, H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ and C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be contraction if there exists a constant $\alpha \in (0,1)$ such that $\|T(x) - T(y)\| \leq \alpha\|x - y\|$, for all $x, y \in C$. If $\alpha = 1$, T is called nonexpansive on C.

The fixed point problem (FPP) for a nonexpansive mapping T is: To find $x \in C$ such that $x \in Fix(T)$, where $Fix(T)$ is the fixed point set of the nonexpansive mapping T.

The explicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, we refer to [2, 3, 4, 5, 9, 10, 11, 16, 19, 20, 21, 22, 23, 25, 27, 28] and the references cited therein. For instance, consider the initial value problem for the differential equation $y^{'}(t) = f(y(t))$ with the initial condition $y(0) = y_0$, where f is a continuous function from $\mathbb{R}^d$ to $\mathbb{R}^d$. The explicit midpoint rule in which a

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sequence $\{y_n\}$ is generated by the following the recurrence relation

$$\frac{1}{h}(y_{n+1} - y_n) = f(\frac{y_{n+1} - y_n}{2}).$$

In 2015, Xu et al. [30] extended and generalized the results of Alghamdi et al. [1] and applied the viscosity method on the midpoint rule for nonexpansive mappings and they gave the generalized viscosity explicit method:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}).$$

In 2016, Rizvi [24] introduced the following iterative method for the explicit midpoint rule of nonexpansive mappings:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n B)T(\frac{x_n + x_{n+1}}{2}).$$

A family $S := \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(1) $T(0)x = x$ for all $x \in C$
(2) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$
(3) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$
(4) For all $x \in C, s \to T(s)x$ is continuous.

Plubtieng and Punpaeng [18] introduced and studied the following iterative method to prove a strong convergence theorem for FPP in a real Hilbert space:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)x_n ds, \qquad \forall n \in \mathbb{N}.$$

where f is a contraction mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$ and $\{s_n\}$ is a positive real divergent sequence.
Kang et al. [12] considerd an iterative algorithm in a Hilbert space as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds.$$

Under the certain conditions, the sequence $\{x_n\}$ strongly converges to a unique solution of the variational inequality $\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0,\ \forall x \in Fix(T)$.
Motivated and inspired by the results mentioned and related literature in [1, 12, 24, 30], we propose an iterative midpoint algorithm based on the viscosity method for finding a common element of the set of solutions of nonexpansive semigroup in Hilbert spaces. Then we prove strong convergence theorems that extend and improve the corresponding results of Rizvi [24], Xu [30], and others. Finally, we give an example and numerical result to illustrate our main result.

The rest of paper is organized as follows. The next section presents some preliminary results. Section 3 is devoted to introduce midpoint algorithm for solving it. The last section presents a numerical example to demonstrate the proposed algorithms.

2. Preliminaries

For each point $x \in H$, there exists a unique nearest point of C, denote by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C y\rangle \leq 0 \tag{1}$$

Further, it is well known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leq (\frac{1}{2})\|(T(x) - x) - (T(y) - y)\|^2, \tag{2}$$

and therefore, we get, for all $(x, y) \in H \times Fix(T)$,

$$\langle (x - T(x)), (y - T(y))\rangle \leq (\frac{1}{2})\|(T(x) - x)\|^2, \tag{3}$$

see, e.g. [8].

It is also known that H satisfies Opial's condition [17], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{4}$$

holds for every $y \in H$ with $y \neq x$.

Lemma 1. [6] *The following inequality holds in real space H:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \qquad \forall x, y \in H.$$

Definition 2. *A mapping $M : C \to H$ is said to be monotone, if*

$$\langle Mx - My, x - y\rangle \geq 0, \qquad \forall x, y \in C.$$

M is called α-inverse-strongly-monotone if there exist a positive real number α such that

$$\langle Mx - My, x - y\rangle \geq \alpha\|Mx - My\|^2, \qquad \forall x, y \in C.$$

Definition 3. *A mapping $B : H \to H$ is said to be strongly positive linear bounded operator, if there exists a constant $\bar{\gamma} > 0$ such that $\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2$, $\forall x \in H$.*

Lemma 4. [15] *Assume that B is a strong positive linear bounded self adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 5. [26] *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. For each $x \in C$ and $t > 0$. Then, for any $0 \leq h < \infty$,*

$$\lim_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)xds - T(h)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 6. [29] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 0$ where α_n is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that*
(i) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$; *(ii)* $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\Sigma_{n=1}^{\infty}\delta_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Nonlinear midpoint algorithm

In this section, we prove a strong convergence theorem based on the explicit iterative for fixed point of nonexpansive semigroup. We firstly present the following unified algorithm.
Let $S = \{T(s) : s \in [0,+\infty)\}$ be a nonexpansive semigroup on C such that $Fix(S) \neq \emptyset$. Also $f : C \to H$ be a α-contraction mapping and B, D be strongly positive bounded linear self adjoint operators on H with coefficients $\bar{\gamma}_1, \bar{\gamma}_2 > 0$ such that $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$, $\bar{\gamma}_1 \leq \|B\| \leq 1$ and $\|D\| = \bar{\gamma}_2 \leq 1$.

Algorithm 7. *For given $x_0 \in C$ arbitrary, let the sequence $\{x_n\}$ be generated by:*

$$x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n Dx_n + ((1-\epsilon_n)I - \beta_n D - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds. \tag{5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$ are the sequence in $(0,1)$ such that $\epsilon_n \leq \alpha_n$ and $\{s_n\} \subset [s,\infty)$ with $s > 0$ satisfying conditions:
(C1)

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\epsilon_n = 0,\ \Sigma_{n=1}^{\infty}\alpha_n = \Sigma_{n=1}^{\infty}\beta_n = \infty;$$

(C2)

$$\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty \quad or \quad \lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1;$$

$$\sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty \quad or \quad \lim_{n\to\infty}\frac{\beta_{n+1}}{\beta_n} = 1;$$

$$\sum_{n=1}^{\infty}|\epsilon_n - \epsilon_{n-1}| < \infty \quad or \quad \lim_{n\to\infty}\frac{\epsilon_{n+1}}{\epsilon_n} = 1;$$

(C3)

$$\lim_{n\to\infty}s_n = \infty,\ \sup_{n\in\mathbb{N}}|s_{n+1} - s_n| \text{ is bounded.}$$

Lemma 8. *For any $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$, there exists a unique fixed point for sequence $\{x_n\}$.*

Proof. As a matter of fact, for fixed $x \in C$, we can define the sequence $\{P_n : H \to H\}$ as follows:

$$P_n(x) := \alpha_n\gamma f(x) + \beta_n Dx + ((1-\epsilon_n)I - \beta_n D - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)x\ ds,\ \forall x \in H.$$

We may assume without loss of generality that $\alpha_n \leq (1-\epsilon_n-\beta_n\|D\|)\|B\|^{-1}$. Since B and D are linear bounded self adjoint operators, we have
$\|B\| = \sup\{|\langle Bx, x\rangle| : x \in H, \|x\| = 1\}$,
$\|D\| = \sup\{|\langle Dx, x\rangle| : x \in H, \|x\| = 1\}$
and observe that

$$\begin{aligned}\langle((1-\epsilon_n)I - \beta_n D - \alpha_n B)x, x\rangle &= (1-\epsilon_n)\langle x, x\rangle - \beta_n\langle Dx, x\rangle - \alpha_n\langle Bx, x\rangle \\ &\geq 1-\epsilon_n - \beta_n\|D\| - \alpha_n\|B\| \geq 0.\end{aligned}$$

Therefore, $(1-\epsilon_n)I - \beta_n D - \alpha_n B$ is positive. Then, by strong positivity of B and D, we get

$$\begin{aligned}\|(1-\epsilon_n)I - \beta_n D - \alpha_n B\| &= \sup\{\langle((1-\epsilon_n)I - \beta_n D - \alpha_n B)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{(1-\epsilon_n)\langle x, x\rangle - \beta_n\langle Dx, x\rangle \\ &\quad -\alpha_n\langle Bx, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1-\epsilon_n - \beta_n\bar{\gamma}_2 - \alpha_n\bar{\gamma}_1 \\ &\leq 1 - \beta_n\bar{\gamma}_2 - \alpha_n\bar{\gamma}_1.\end{aligned} \tag{6}$$

For any $x, y \in C$

$$\begin{aligned}\|P_n x - P_n y\| &\leq \alpha_n\gamma\|f(x) - f(y)\| + \beta_n\|D\|\|x-y\| \\ &\quad +\|(1-\epsilon_n)I - \beta_n D - \alpha_n B\|\frac{1}{s_n}\int_0^{s_n}\|T(s)x - T(s)y\|ds \\ &\leq \alpha_n\gamma\alpha\|x-y\| + \beta_n\bar{\gamma}_2\|x-y\| + (1-\beta_n\bar{\gamma}_2 - \alpha_n\gamma_1)\|x-y\| \\ &= (1-(\gamma_1-\gamma\alpha)\alpha_n)\|x-y\|.\end{aligned}$$

Therefore, Banach contraction principle guarantees that P_n has a unique fixed point in H, and so the iteration (5) is well defined. □

Lemma 9. *Let $p \in Fix(S)$. Then the sequence $\{x_n\}$ generated by Algorithm 7 is bounded.*

Proof. Let $p \in Fix(S)$, we obtain

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + \beta_n Dx_n \\ &\quad +((1-\epsilon_n)I - \beta_n D - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds - p\| \\ &\leq \alpha_n\|\gamma f(x_n) - Bp\| + \beta_n\|Dx_n - Dp\| + \epsilon_n\|p\| \\ &\quad +\|((1-\epsilon_n)I - \beta_n D - \alpha_n B)\|\|\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2}) - T(s)p\|ds \\ &\leq \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Bp\|) + \beta_n\|Dx_n - Dp\| + \epsilon_n\|p\| \\ &\quad +(1-\beta_n\bar{\gamma}_2 - \alpha_n\bar{\gamma}_1)\|\frac{x_n + x_{n+1}}{2} - p\| \\ &\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| + \beta_n\bar{\gamma}_2\|x_n - p\| + \alpha_n\|p\| \\ &\quad +\frac{(1-\beta_n\bar{\gamma}_2 - \alpha_n\bar{\gamma}_1)}{2}(\|x_n - p\| + \|x_{n+1} - p\|).\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1+\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1}{2}\|x_{n+1}-p\| &\leq (\alpha_n\gamma\alpha+\frac{1+\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1}{2})\|x_n-p\| \\ &\quad +\alpha_n(\|\gamma f(p)-Bp\|+\|p\|).\end{aligned}$$

Then

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\frac{2(\bar{\gamma}_1-\gamma\alpha)\alpha_n}{1+\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1})\|x_n-p\|+\frac{2\alpha_n(\bar{\gamma}_1-\gamma\alpha)}{1+\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1}\frac{\|\gamma f(p)-Bp\|+\|p\|}{\bar{\gamma}_1-\gamma\alpha} \\ &\leq \max\{\|x_n-p\|,\frac{\|\gamma f(p)-Bp\|+\|p\|}{\bar{\gamma}_1-\gamma\alpha}\} \qquad (7) \\ &\quad \vdots \\ &\leq \max\{\|x_0-p\|,\frac{\|\gamma f(p)-Bp\|+\|p\|}{\bar{\gamma}_1-\gamma\alpha}\}.\end{aligned}$$

Hence $\{x_n\}$ is bounded. □

Now, set $t_n := \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds$. Then $\{t_n\}$ and $\{f(x_n)\}$ are bounded.

Lemma 10. *The following properties are satisfying for the Algorithm 7*

P1. $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.
P2. $\lim_{n\to\infty}\|x_n-t_n\|=0$.
P3. $\lim_{n\to\infty}\|T(s)t_n-t_n\|=0$.

Lemma 11. *In order to prove P1, one can write*

$$\begin{aligned}\|t_{n+1}-t_n\| &= \|\frac{1}{s_{n+1}}\int_0^{s_{n+1}}T(s)(\frac{x_{n+1}+x_{n+2}}{2})ds-\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\ &= \|\frac{1}{s_{n+1}}\int_0^{s_{n+1}}(T(s)(\frac{x_{n+1}+x_{n+2}}{2})-T(s)(\frac{x_n+x_{n+1}}{2}))ds \\ &\quad +(\frac{1}{s_{n+1}}-\frac{1}{s_n})\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds \\ &\quad +\frac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}}T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\ &= \|\frac{1}{s_{n+1}}\int_0^{s_{n+1}}(T(s)(\frac{x_{n+1}+x_{n+2}}{2})-T(s)(\frac{x_n+x_{n+1}}{2}))ds \\ &\quad +(\frac{1}{s_{n+1}}-\frac{1}{s_n})\int_0^{s_n}(T(s)(\frac{x_n+x_{n+1}}{2})-T(s)p)ds \\ &\quad +\frac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}}(T(s)(\frac{x_n+x_{n+1}}{2})-T(s)p)ds\| \\ &\leq \|\frac{x_{n+1}+x_{n+2}}{2}-\frac{x_n+x_{n+1}}{2}\|+\frac{|s_{n+1}-s_n|s_n}{s_{n+1}s_n}\|\frac{x_n+x_{n+1}}{2}-p\|\end{aligned}$$

$$
\begin{aligned}
& +\frac{|s_{n+1}-s_n|}{s_{n+1}}\|\frac{x_n+x_{n+1}}{2}-p\|\frac{1}{2}(\|x_{n+1}-x_n\|+\|x_{n+2}-x_{n+1}\|) \\
\leq \quad & +\frac{|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\|+\|x_{n+1}-p\|). \qquad (8)
\end{aligned}
$$

Next, we show that the sequence $\{x_n\}$ *is asymptotically regular, i.e.,* $\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\|=0$. *By (8) we estimate that*

$$
\begin{aligned}
&\|x_{n+2}-x_{n+1}\| = \|(\alpha_{n+1}\gamma f(x_{n+1})+\beta_{n+1}Dx_{n+1} \\
&\quad +((1-\epsilon_{n+1})I-\beta_{n+1}D-\alpha_{n+1}B)\frac{1}{s_{n+1}}\int_0^{s_{n+1}}T(s)(\frac{x_{n+1}+x_{n+2}}{2})ds) \\
&\quad -(\alpha_n\gamma f(x_n)+\beta_nDx_n+((1-\epsilon_n)I-\beta_nD-\alpha_nB)\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds)\| \\
&= \|((1-\epsilon_{n+1})I-\beta_{n+1}D-\alpha_{n+1}B)(\frac{1}{s_{n+1}}\int_0^{s_{n+1}}T(s)(\frac{x_{n+1}+x_{n+2}}{2})ds \\
&\quad -\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds)+((\epsilon_n+\beta_nD+\alpha_nB) \\
&\quad -(\epsilon_{n+1}+\beta_{n+1}D+\alpha_{n+1}B))\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds+(\alpha_{n+1}-\alpha_n)\gamma f(x_n) \\
&\quad +\alpha_{n+1}(\gamma f(x_{n+1})-\gamma f(x_n))+(\beta_{n+1}-\beta_n)Dx_n+\beta_{n+1}(Dx_{n+1}-Dx_n)\| \\
&\leq (1-\beta_{n+1}\bar{\gamma}_2-\alpha_{n+1}\bar{\gamma}_1)\|t_{n+1}-t_n\|+|\epsilon_{n+1}-\epsilon_n|\|\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +M|\alpha_n-\alpha_{n+1}|+N|\beta_n-\beta_{n+1}|+\alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\| \\
&\leq (1-\beta_{n+1}\bar{\gamma}_2-\alpha_{n+1}\bar{\gamma}_1)\|t_{n+1}-t_n\|+|\epsilon_{n+1}-\epsilon_n|\|\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +M|\alpha_n-\alpha_{n+1}|+N|\beta_n-\beta_{n+1}|+\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\| \\
&\leq \frac{1-\beta_{n+1}\bar{\gamma}_2-\alpha_{n+1}\bar{\gamma}_1}{2}(\|x_{n+1}-x_n\|+\|x_{n+2}-x_{n+1}\|) \\
&\quad +(1-\beta_{n+1}\bar{\gamma}_2-\alpha_{n+1}\bar{\gamma}_1)\frac{|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\|+\|x_{n+1}-p\|)+|\epsilon_{n+1}-\epsilon_n|\|t_n\| \\
&\quad +M|\alpha_n-\alpha_{n+1}|+N|\beta_n-\beta_{n+1}|+\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|,
\end{aligned}
$$

where $M := \sup\{\|\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds\|+\|f(x_n)\|\}$ *and* $N := \sup\{\|\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds\|+\|x_n\|\}$. *Then*

$$
\begin{aligned}
(1+\alpha_{n+1}\bar{\gamma}_1+\beta_{n+1}\bar{\gamma}_2)\|x_{n+2}-x_{n+1}\| &\leq (1-\beta_{n+1}\bar{\gamma}_2+(2\alpha\gamma-\bar{\gamma}_1)\alpha_{n+1})\|x_{n+1}-x_n\| \\
&\quad +(1-\beta_{n+1}\bar{\gamma}_2-\alpha_{n+1}\bar{\gamma}_1)\frac{2|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\| \\
&\quad +\|x_{n+1}-p\|)+2|\epsilon_{n+1}-\epsilon_n|\|t_n\| \\
&\quad +2M|\alpha_n-\alpha_{n+1}|+2N|\beta_n-\beta_{n+1}|.
\end{aligned}
$$

Therefore

$$\|x_{n+2}-x_{n+1}\| \leq (1-\frac{2(\beta_{n+1}\bar{\gamma}_2+(\bar{\gamma}_1-\alpha\gamma)\alpha_{n+1})}{1+\alpha_{n+1}\bar{\gamma}_1+\beta_{n+1}\bar{\gamma}_2})\|x_{n+1}-x_n\|$$
$$+(\frac{1-\beta_{n+1}\bar{\gamma}_2-\alpha_{n+1}\bar{\gamma}_1}{1+\alpha_{n+1}\bar{\gamma}_1+\beta_{n+1}\bar{\gamma}_2})(\frac{2|s_{n+1}-s_n|}{s_{n+1}})(\|x_n-p\|+\|x_{n+1}-p\|)$$
$$+\frac{2}{1+\alpha_{n+1}\bar{\gamma}_1+\beta_{n+1}\bar{\gamma}_2}|\epsilon_{n+1}-\epsilon_n|\|t_n\|+\frac{2M}{1+\alpha_{n+1}\bar{\gamma}_1+\beta_{n+1}\bar{\gamma}_2}|\alpha_n-\alpha_{n+1}|$$
$$+\frac{2N}{1+\alpha_{n+1}\bar{\gamma}_1+\beta_{n+1}\bar{\gamma}_2}|\beta_n-\beta_{n+1}|.$$

Lemma 6 and (C1)-(C2) implies

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{9}$$

And similarly, we have

$$\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\|=0. \tag{10}$$

Also by (8), (9),(10) and (C3) we have $\lim_{n\to\infty}\|t_{n+1}-t_n\|=0$*.*

In order to prove P2, one can write

$$\begin{aligned}\|x_n-t_n\| &\leq \|x_{n+1}-x_n\| \\ &\quad +\|\alpha_n\gamma f(x_n)+\beta_n Dx_n+((1-\epsilon_n)I-\beta_n D-\alpha_n B)t_n-t_n\| \\ &\leq \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-Bt_n\|+\beta_n\bar{\gamma}_2\|x_n-t_n\|+\epsilon_n\|t_n\|.\end{aligned}$$

Then

$$(1-\beta_n\bar{\gamma}_2)\|x_n-t_n\| \leq \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-Bt_n\|+\epsilon_n\|t_n\|.$$

By (C1) and (9), we obtain

$$\lim_{n\to\infty}\|x_n-t_n\|=0. \tag{11}$$

In order to prove P3, set $K:=\{w\in C:\|w-p\|\leq\|x_0-p\|+\frac{1}{\bar{\gamma}_1-\gamma\alpha}(\|\gamma f(p)-Bp\|+\|p\|)\}$*. Then* K *is a nonempty bounded closed convex subset of* C *which is* $T(s)$*-invariant for each* $s\in[0,+\infty)$ *and contains* $\{x_n\}$*. So, without loss of generality, we may assume that* $S:=\{T(s):s\in[0,+\infty)\}$ *is a nonexpansive semigroup on* K*.*

$$\begin{aligned}\|T(s)x_n-x_n\| &= \|T(s)x_n-T(s)\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds \\ &\quad +T(s)\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds \\ &\quad -\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds+\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds-x_n\| \\ &\leq \|T(s)x_n-T(s)\frac{1}{s_n}\int_0^{s_n}T(s)(\frac{x_n+x_{n+1}}{2})ds\|\end{aligned}$$

$$
\begin{aligned}
&+\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&+\|\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
\leq\ & \|x_n - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&+\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&+\|\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
=\ & 2\|\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
&+\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\|
\end{aligned}
$$

Since $\frac{x_n+x_{n+1}}{2} \in C$*, from (11) and Lemma 5, we obtain* $\lim_{n\to\infty} \|T(s)x_n - x_n\| = 0$. *Therefore*

$$
\begin{aligned}
\|T(s)t_n - t_n\| &\leq \|T(s)t_n - T(s)x_n\| + \|T(s)x_n - x_n\| + \|x_n - t_n\| \\
&\leq \|t_n - x_n\| + \|T(s)x_n - x_n\| + \|x_n - t_n\|.
\end{aligned}
$$

Then we have $\lim_{n\to\infty} \|T(s)t_n - t_n\| = 0$.

4. Convergence algorithm

Theorem 12. *The Algorithm (5) converges strongly* $z \in Fix(S)$*, which is a unique solution of the variational inequality* $\langle(\gamma f - B)z, y - z\rangle \leq 0$*, for all* $y \in Fix(S)$.

Proof. Let $q = P_{Fix(S)}$. We get

$$
\begin{aligned}
\|q(I-B+\gamma f)(x) - q(I-B+\gamma f)(y)\| &\leq \|(I-B+\gamma f)(x) - (I-B+\gamma f)(y)\| \\
&\leq \|I-B\|\|x-y\| + \gamma\|f(x) - f(y)\| \\
&\leq (1-\bar{\gamma}_1)\|x-y\| + \gamma\alpha\|x-y\| \\
&= (1-(\bar{\gamma}_1 - \gamma\alpha))\|x-y\|.
\end{aligned}
$$

Then $q(I-B+\gamma f)$ is a contraction mapping from H into itself. Therefore by Banach contraction principle, there exists $z \in H$ such that $z = q(I-B+\gamma f)z = P_{Fix(S)}(I-B+\gamma f)z$.

We show that $\langle(\gamma f - B)z, x_n - z\rangle \leq 0$. To show this inequality, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - B)z, t_n - z\rangle = \lim_{i\to\infty}\langle(\gamma f - B)z, t_{n_i} - z\rangle. \tag{12}$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\} \subseteq K$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that

$t_{n_i} \rightharpoonup w$. Now, we prove that $w \in Fix(S)$. Assume that $w \notin Fix(S)$. Since $t_{n_i} \rightharpoonup w$ and $T(s)w \neq w$, from Opial's conditions (4) and Lemma 10 (P3), we have

$$\begin{aligned}
\liminf_{i\to\infty} \|t_{n_i} - w\| &< \liminf_{i\to\infty} \|t_{n_i} - T(s)w\| \\
&\leq \liminf_{i\to\infty} (\|t_{n_i} - T(s)t_{n_i}\| + \|T(s)t_{n_i} - T(s)w\|) \\
&\leq \liminf_{i\to\infty} \|t_{n_i} - w\|,
\end{aligned}$$

which is a contradiction. Thus, we obtain $w \in Fix(S)$. Now from (1), we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle (\gamma f - B)z, x_n - z\rangle &= \limsup_{n\to\infty} \langle (\gamma f - B)z, t_n - z\rangle \\
&\leq \limsup_{i\to\infty} \langle (\gamma f - B)z, t_{n_i} - z\rangle \qquad (13) \\
&= \langle (\gamma f - B)z, w - z\rangle \leq 0.
\end{aligned}$$

Now we prove that x_n is strongly convergence to z.

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \alpha_n \langle \gamma f(x_n) - Bz, x_{n+1} - z\rangle + \beta_n \langle Dx_n - Dz, x_{n+1} - z\rangle \\
&\quad -\epsilon_n \langle z, x_{n+1} - z\rangle + \langle ((1-\epsilon_n)I - \beta_n D - \alpha_n B)(t_n - z), x_{n+1} - z\rangle \\
&\leq \alpha_n (\gamma \langle f(x_n) - f(z), x_{n+1} - z\rangle + \langle \gamma f(z) - Bz, x_{n+1} - z\rangle) \\
&\quad +\beta_n \|D\| \|x_n - z\| \|x_{n+1} - z\| - \epsilon_n \|z\| \|x_{n+1} - z\| \\
&\quad +\|(1-\epsilon_n)I - \beta_n D - \alpha_n B\| \|t_n - z\| \|x_{n+1} - z\| \\
&\leq \alpha_n \alpha\gamma \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad +\beta_n \bar{\gamma}_2 \|x_n - z\| \|x_{n+1} - z\| - \epsilon_n \|z\| \|x_{n+1} - z\| \\
&\quad +(1 - \beta_n \bar{\gamma}_2 - \alpha_n \bar{\gamma}_1) \|\frac{x_n + x_{n+1}}{2} - z\| \|x_{n+1} - z\| \\
&\leq \alpha_n \alpha\gamma \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad +\beta_n \bar{\gamma}_2 \|x_n - z\| \|x_{n+1} - z\| - \epsilon_n \|z\| \|x_{n+1} - z\| \\
&\quad +\frac{1 - \beta_n \bar{\gamma}_2 - \alpha_n \bar{\gamma}_1}{2} (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - z\| \\
&= \frac{1 + \beta_n \bar{\gamma}_2 - \alpha_n (\bar{\gamma}_1 - 2\alpha\gamma)}{2} \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad -\epsilon_n \|z\| \|x_{n+1} - z\| + \frac{1 - \beta_n \bar{\gamma}_2 - \alpha_n \bar{\gamma}_1}{2} \|x_{n+1} - z\|^2 \\
&\leq \frac{1 + \beta_n \bar{\gamma}_2 - \alpha_n (\bar{\gamma}_1 - 2\alpha\gamma)}{4} (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad +\alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle - \epsilon_n \|z\| \|x_{n+1} - z\| \\
&\quad +\frac{1 - \beta_n \bar{\gamma}_2 - \alpha_n \bar{\gamma}_1}{2} \|x_{n+1} - z\|^2 \\
&\leq \frac{1 + \beta_n \bar{\gamma}_2 - \alpha_n (\bar{\gamma}_1 - 2\alpha\gamma)}{4} \|x_n - z\|^2
\end{aligned}$$

$$\begin{aligned}
&+\frac{3-\beta_n\bar{\gamma}_2-\alpha_n(3\bar{\gamma}_1-2\alpha\gamma)}{4}\|x_{n+1}-z\|^2\\
&+\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle-\epsilon_n\|z\|\|x_{n+1}-z\|\\
\leq\ &\frac{1+\beta_n\bar{\gamma}_2-\alpha_n(\bar{\gamma}_1-2\alpha\gamma)}{4}\|x_n-z\|^2+\frac{3}{4}\|x_{n+1}-z\|^2\\
&+\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle-\epsilon_n\|z\|\|x_{n+1}-z\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
4\|x_{n+1}-z\|^2 \quad\leq\quad &(1+\beta_n\bar{\gamma}_2-\alpha_n(\bar{\gamma}_1-2\alpha\gamma))\|x_n-z\|^2+3\|x_{n+1}-z\|^2\\
&+4\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle+4\epsilon_n\|z\|\|x_{n+1}-z\|.
\end{aligned}$$

Then

$$\begin{aligned}
\|x_{n+1}-z\|^2 \quad\leq\quad &(1-(\alpha_n(\bar{\gamma}_1-2\alpha\gamma)-\beta_n\bar{\gamma}_2))\|x_n-z\|^2\\
&+4\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle+4\epsilon_n\|z\|\|x_{n+1}-z\|\\
= \quad &(1-k_n)\|x_n-z\|^2+4\alpha_n l_n, \qquad (14)
\end{aligned}$$

where $k_n=\alpha_n(\bar{\gamma}_1-2\alpha\gamma)+\beta_n\bar{\gamma}_2$ and $l_n=\langle\gamma f(z)-Bz, x_{n+1}-z\rangle-\|z\|\|x_{n+1}-z\|$. Since $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$ and $\Sigma_{n=0}^{\infty}\alpha_n=\Sigma_{n=0}^{\infty}\beta_n=\infty$, it is easy to see that $\lim_{n\to\infty}k_n=0$, $\Sigma_{n=0}^{\infty}k_n=\infty$ and $\limsup_{n\to\infty}l_n\leq 0$. Hence, from (13) and (14) and Lemma 6, we deduce that $x_n\to z$, where $z=P_{Fix(S)}(I-B+\gamma f)z$. $\square$

5. Numerical examples

In this section, we give some examples and numerical results for supporting our main theorem. All the numerical results have been produced in Matlab 2017 on a Linux workstation with a 3.8 GHZ Intel annex processor and 8 Gb of memory

Example 13. *Consider a Fredholm integral equation of the following form*

$$x(t)=g(t)+\int_0^t F(t,k,x(k))\quad dk, \qquad t\in[0,1], \tag{15}$$

where g is a continuous function on $[0,1]$ and $F:[0,1]\times[0,1]\times R\to R$ is continuous. Note that if F satisfies the Lipschitz continuity condition, i.e.,

$$|F(t,k,x)-F(t,k,y)|\leq|x-y|, \qquad \forall t,k\in[0,1], \quad x,y\in\mathbb{R},$$

then equation (15) has at least one solution in $L^2[0,1]$ (see [13]*).*
Define a mapping $T(s):L^2[0,1]\to L^2[0,1]$ by

$$(T(s)x)(t)=e^{-3s}(g(t)+\int_0^t F(t,k,x(k))\ dk), \qquad t\in[0,1].$$

It is easy to observe that $S=\{T(s):s\in[0,+\infty)\}$ is a nonexpansive semigroup. In fact, we have, for $x,y\in L^2[0,1]$,

$$\|T(s)x-T(s)y\|^2 \quad = \quad \int_0^1|(T(s)x)(t)-(T(s)y)(t)|^2\ dt$$

$$\begin{aligned}
&= \int_0^1 |e^{-3s}\int_0^1 (F(t,k,x(k))-F(t,k,y(k)))\ dk|^2\ dt\\
&\leq \int_0^1 (\int_0^1 |x(k)-y(k)|^2\ dk)\ dt\\
&= \int_0^1 |x(k)-y(k)|^2\ dk\\
&= \|x-y\|^2.
\end{aligned}$$

This means that to find the solution of integral equation (15) is reduced to find a fixed point of the nonexpansive semigroup S in $L^2[0,1]$. For any given function $x_0 \in L^2[0,1]$, define a sequence of functions x_n in $L^2[0,1]$ by

$$x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n Dx_n + ((1-\epsilon_n)I - \beta_n D - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds$$

satisfying the conditions of Algorithm 7. Then the sequence $\{x_n\}$ converges strongly in $L^2[0,1]$ to the solution of integral equation (15) which is also a solution of the following variational inequality

$$\langle(\gamma f - B)z, y - z\rangle \leq 0, \quad \forall y \in Fix(S).$$

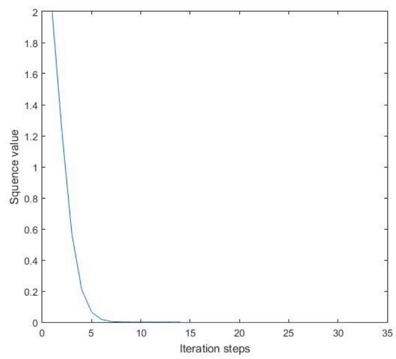
Example 14. *Let $H = R$, the set of all real numbers, with the inner product defined by $\langle x,y\rangle = xy$, $\forall x,y \in R$, and induced usual norm $|\ .\ |$. Let $C = [-1,3]$; Let $f(x) = \frac{1}{9}x$, $B(x) = \frac{1}{4}x$, $D(x) = x$ and let, for each $x \in C$, $T(s)x = \frac{1}{1+2s}x$. Then there exists a unique sequence $\{x_n\} \subset R$ generated by the iterative scheme*

$$\begin{aligned}
x_{n+1} &= (\frac{1}{9\sqrt{n}} + \frac{1}{2n})x_n \qquad (16)\\
&\quad +((1-\frac{1}{(n+1)^2})I - \frac{1}{2n}D - \frac{1}{\sqrt{n}}B)\frac{1}{s_n}\int_0^{s_n}\frac{1}{1+2s}(\frac{x_n+x_{n+1}}{2})ds
\end{aligned}$$

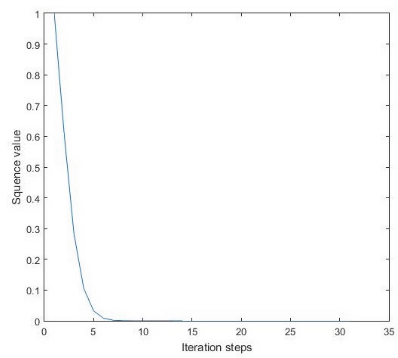
where $\alpha_n = \frac{1}{\sqrt{n}}$, $\beta_n = \frac{1}{2n}$, $\epsilon_n = \frac{1}{(n+1)^2}$ and $s_n = n$. Then $\{x_n\}$ converges to $\{0\} \in Fix(S)$. f is contraction mapping with constant $\alpha = \frac{1}{6}$ and B, D are strongly positive bounded linear operators with constant $\bar{\gamma}_1 = \frac{1}{5}$ on C. Therefore, we can choose $\gamma = 1$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $Fix(S) = \{0\} \neq \emptyset$. After simplification, scheme (16) reduce to

$$x_{n+1} = \frac{\frac{1}{9\sqrt{n}} + \frac{1}{2n} + \frac{1}{4n}(1 - \frac{1}{(n+1)^2} - \frac{1}{2n} - \frac{1}{4\sqrt{n}})\ln(1+2n)}{1 - \frac{1}{4n}(1 - \frac{1}{(n+1)^2} - \frac{1}{2n} - \frac{1}{4\sqrt{n}})\ln(1+2n)}x_n.$$

Following the proof of Theorem 12, we obtain that $\{x_n\}$ converges strongly to $w = \{0\} \in Fix(S)$.



The graph of x(n) with initial value x(1)=2



The graph of x(n) with initial value x(1)=1

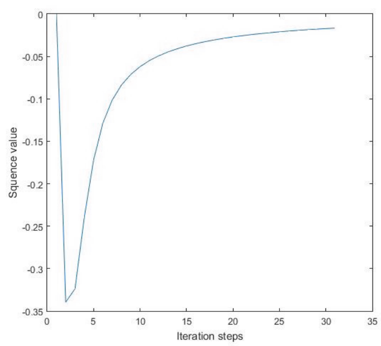
Let $H = R$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, $\forall x, y \in R$, and induced usual norm $|\ .\ |$. Let $C = [0, 4]$; Let $f(x) = \frac{1}{10}(x-3)$, $B(x) = \frac{1}{3}x$, $D(x) = \frac{1}{2}x$ and let, for each $x \in C$, $T(s)x = e^{-2s}x$. Then there exists a unique sequence $\{x_n\} \subset R$ generated by the iterative scheme

$$x_{n+1} = \frac{3}{20n+5}(x_n-3) + \frac{1}{2\sqrt{n+2}}x_n + ((1-\frac{1}{n^2})I - \frac{1}{\sqrt{n+2}}D - \frac{3}{4n+1}B)\frac{1}{s_n}\int_0^{s_n} e^{-2s}(\frac{x_n+x_{n+1}}{2})ds \tag{17}$$

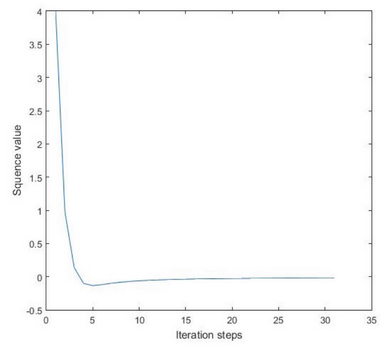
where $\alpha_n = \frac{3}{4n+1}$, $\beta_n = \frac{1}{\sqrt{n+2}}$, $\epsilon_n = \frac{1}{n^2}$ and $s_n = 2n$. Then $\{x_n\}$ converges to $\{0\} \in Fix(S)$. f is contraction mapping with constant $\alpha = \frac{1}{9}$ and B, D are strongly positive bounded linear operators with constant $\bar{\gamma}_1 = \frac{1}{4}$ on C. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $Fix(S) = \{0\} \neq \emptyset$. After simplification, scheme (17) reduce to

$$x_{n+1} = \frac{(\frac{3}{20n+5} + \frac{1}{2\sqrt{n+2}} - \frac{1}{8n}(e^{-4n}-1)(1-\frac{1}{n^2}-\frac{1}{2\sqrt{n+2}}-\frac{1}{4n+1}))x_n - \frac{9}{20n+5}}{1+\frac{1}{8n}(e^{-4n}-1)(1-\frac{1}{n^2}-\frac{1}{2\sqrt{n+2}}-\frac{1}{4n+1})}.$$

Following the proof of Theorem 12, we obtain that $\{x_n\}$ converges strongly to $w = \{0\} \in Fix(S)$.



The graph of x(n) with initial value x(1)=0



The graph of x(n) with initial value x(1)=4

6. Conculsion

In this paper, we present a viscosity nonlinear midpoint algorithm for solving equilibrium problems in real Hilbert spaces. The methods propose a theoretical generalization of some existing results in the literature and primary numerical experiments also demonstrate the potential applicability of these methods. We establish the algorithm's strong convergence under mild and standard assumptions. This work open the doors for many promising research directions such as obtaining error bound and convergence rate of our algorithms as well as extensions to Banach spaces.

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ON GENERALIZED CHEEGER-GROMOLL METRIC AND HARMONICITY

R. KADA BEN OTMANE, A. ZAGANE, AND M. DJAA

Abstract. In this paper, we introduce the Generalized Cheeger-Gromoll metric on the tangent bundle TM, as a natural metric on TM. We establish a necessary and sufficient conditions under which a vector field is harmonic with respect to the Generalized Cheeger-Gromoll metric. We also construct some examples of harmonic vector fields.

1. Introduction

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the energy functional is defined by

$$E(\phi) = \int_K e(\phi) dv_g \tag{1}$$

or over any compact subset $K \subset M$.

$$e(\phi) = \frac{1}{2} trace_g(\phi^* h) = \frac{1}{2} trace_g h(d\phi, d\phi) \tag{2}$$

is the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d}{dt}\phi_t\Big|_{t=0}$, we have

$$\frac{d}{dt} E(\phi_t)\Big|_{t=0} = -\int_K h(\tau(\phi), V) dv_g \tag{3}$$

where

$$\tau(\phi) = trace_g \nabla d\phi \tag{4}$$

is the tension field of ϕ. Then ϕ is harmonic if and only if $\tau(\phi) = 0$.

One can refer to [14], [15], [22] for background on harmonic maps and [9], [12] for background on generalized harmonic maps.

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The geometry of the tangent bundle TM equipped with the Sasaki metric has been studied by many authors such as Sasaki [25], K.Yano and S. Ishihara [27], P.Dombrowski [13], A. Salimov, A. Gezer and N. Cengiz [26],[23], etc.... The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM. J. Cheeger and D. Gromoll has introduced the notion of Cheeger-Gromoll metric [5], this metric has been studied also by many authors (see [1], [2], [16], [17], [24], [26]).

The existence and explicit construction of harmonic mappings between two given Riemannian manifolds (M, g) and (N, h) are two of the most fundamental problems of the theory of harmonic mappings. If M is compact N has non positive sectional curvature, then any smooth map from M to N can be deformed into a harmonic map using the heat flow method [Eells and Sampson 1964]. However, there is no general existence theory of harmonic mappings if the target manifold does not satisfy the non positive curvature condition. This fact makes it interesting to find harmonic maps defined by vector fields as a maps from Riemannian manifold (M, g) to its tangent bundle TM.

The main idea in this note consists in the modification of the Sasaki metric. First we introduce a natural metric called Generalized Cheeger-Gromoll metric on the tangent bundle TM, originally defined by M. Anastasiei [2]. Afterward we establish necessary and sufficient conditions under which a vector field is harmonic with respect to the Generalized Cheeger-Gromoll metric (Theorem 10 and Theorem 11). We also construct some examples of harmonic vector fields and we give a formula for the construction of non trivial examples of vector fields (Theorem 16 and Corollary 17). After that we study the harmonicity of the map $\sigma : (M, g) \longrightarrow (TN, \widetilde{h})$ (Theorem 21, Theorem 22) and the map $\phi : (TM, \widetilde{g}) \longrightarrow (N, h)$ (Theorem 24, Theorem 25).

1.1. **Basic Notion and Definition on TM.** Let (M, g) be an m-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by:

$$\mathcal{V}_{(x,u)} = \ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)}; a^i \in \mathbb{R}\},$$

$$\mathcal{H}_{(x,u)} = \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)}; a^i \in \mathbb{R}\},$$

where $(x,u) \in TM$,such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$. Let $X = X^i \dfrac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by:

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{5}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{ \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \} \tag{6}$$

For consequences, we have $(\dfrac{\partial}{\partial x^i})^H = \dfrac{\delta}{\delta x^i}$ and $(\dfrac{\partial}{\partial x^i})^V = \dfrac{\partial}{\partial y^i}$, then $(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

If $w = w^i \dfrac{\partial}{\partial x^i} + \overline{w}^j \dfrac{\partial}{\partial y^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}$$

$$w^v = \{ \overline{w}^j + w^i u^j \Gamma^k_{ij} \} \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}$$

Proposition 1. [27]*Let (M,g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X, Y \in \Gamma(TM)$ and $p = (x,u) \in TM$ we have:*
1. $[X^H, Y^H]_p = [X,Y]^H_p - (R_x(X,Y)u)^V$,
2. $[X^H, Y^V]_p = (\nabla_X Y)^V_p$,
3. $[X^V, Y^V]_p = 0$.

2. Generalized Cheeger-Gromoll metric

2.1. **Generalized Cheeger-Gromoll metric.**

Definition 2. *Let (M,g) be a Riemannian manifold and $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$, $\alpha \neq 0$ are smooth functions. On the tangent bundle TM, we define a Generalized Cheeger-Gromoll metric noted $\widetilde{g}$ by:*
1. $\widetilde{g}(X^H, Y^H)_p = g_x(X,Y)$,
2. $\widetilde{g}(X^H, Y^V)_p = 0$,
3. $\widetilde{g}(X^V, Y^V)_p = \alpha(r) g(X,Y) + \beta(r) g(X,u) g(Y,u)$,

where $X, Y \in \Gamma(TM)$, $p = (x,u) \in TM$ and $r = g(u,u)$.

For more details see [2].

Remark 3. *1) If $\alpha = 1$ and $\beta = 0$, then $\widetilde{g}$ is the Sasaki metric* [25],
2) If $\alpha = \beta = \dfrac{1}{r+1}$, then $\widetilde{g}$ is the Cheeger-Gromoll metric [5], [17].

Lemma 4. [1], [16]*Let (M,g) be a Riemannian manifold and $f : \mathbb{R} \to \mathbb{R}$ a smooth function. For all $X, Y \in \Gamma(TM)$, $p = (x,u) \in TM$, $u = u^i \frac{\partial}{\partial x^i} \in T_x M$ and $r = g(u,u)$, we have:*

(1) $X^H(f(r))_p = 0$,
(2) $X^V(f(r))_p = 2f'(r)g(X,u)_x$
(3) $X^H(g(Y,u))_p = g(\nabla_X Y, u)_x$,
(4) $X^V(g(Y,u))_p = g(X,Y)_x$.

Proof. Locally, the statement is a direct consequence of formulas (**??**) and (5). □

Lemma 5. *Let (M,g) be a Riemannian manifold and $(TM,\widetilde{g})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric, we have:*

$$\begin{aligned}
1)\ X^H\widetilde{g}(Y^V,Z^V) &= \widetilde{g}((\nabla_X Y)^V, Z^V) + \widetilde{g}(Y^V,(\nabla_X Z)^V),\\
2)\ X^V\widetilde{g}(Y^V,Z^V) &= 2\alpha' g(X,u)g(Y,Z) + 2\beta' g(X,u)g(Y,u)g(Z,u)\\
&\quad +\beta[g(Z,u)g(X,Y) + g(Y,u)g(X,Z)].
\end{aligned}$$

for all $X,Y,Z \in \Gamma(TM)$.

Proof. Using Lemma 4, we obtain:

$$\begin{aligned}
1)\ X^H\widetilde{g}(Y^V,Z^V) &= X^H\big[\alpha g(X,Y) + \beta g(X,u)g(Y,u)\big]\\
&= \alpha X^H g(Y,Z) + \beta X^H(g(Y,u)g(Z,u))]\\
&= \alpha[g(\nabla_X Y, Z) + g(Y,\nabla_X Z)]\\
&\quad +\beta[g(\nabla_X Y,u)g(Z,u) + g(Y,u)g(\nabla_X Z,u)]\\
&= \widetilde{g}((\nabla_X Y)^V, Z^V) + \widetilde{g}(Y^V,(\nabla_X Z)^V).
\end{aligned}$$

$$\begin{aligned}
2)\ X^V\widetilde{g}(Y^V,Z^V) &= X^V\big[\alpha g(X,Y) + \beta g(X,u)g(Y,u))\big]\\
&= X^V(\alpha)g(Y,Z) + X^V(\beta)g(Y,u)g(Z,u) + \beta X^V(g(Y,u)g(Z,u))\\
&= 2\alpha' g(X,u)g(Y,Z) + 2\beta' g(X,u)g(Y,u)g(Z,u)\\
&\quad +\beta g(Z,u)g(X,Y) + \beta g(Y,u)g(X,Z).
\end{aligned}$$

□

2.2. Levi-Civita connection of the Generalized Cheeger-Gromoll metric.

Lemma 6. *Let (M,g) be a Riemannian manifold and $(TM,\widetilde{g})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. If ∇ (resp. $\widetilde{\nabla}$) denote the Levi-Civita connection of (M,g) (resp. $(TM,\widetilde{g})$), then we have:*

1) $\widetilde{g}(\widetilde{\nabla}_{X^H}Y^H, Z^H) = \widetilde{g}((\nabla_X Y)^H, Z^H)$,

2) $\widetilde{g}(\widetilde{\nabla}_{X^H}Y^H, Z^V) = -\dfrac{1}{2}\widetilde{g}((R(X,Y)u)^V, Z^V)$,

3) $\widetilde{g}(\widetilde{\nabla}_{X^H}Y^V, Z^H) = \dfrac{\alpha(r)}{2}\widetilde{g}((R(u,Y)X)^H, Z^H)$,

4) $\widetilde{g}(\widetilde{\nabla}_{X^H}Y^V, Z^V) = \widetilde{g}((\nabla_X Y)^V, Z^V)$,

5) $\widetilde{g}(\widetilde{\nabla}_{X^V}Y^H, Z^H) = \dfrac{\alpha(r)}{2}\widetilde{g}((R(u,X)Y)^H, Z^H)$,

6) $\widetilde{g}(\widetilde{\nabla}_{X^V}Y^H, Z^V) = 0,$

7) $\widetilde{g}(\widetilde{\nabla}_{X^V}Y^V, Z^H) = 0,$

8) $$\widetilde{g}(\widetilde{\nabla}_{X^V}Y^V, Z^V) = \widetilde{g}\Big(\frac{\alpha'}{\alpha}\big[g(X,u)Y^V + g(Y,u)X^V\big] + \big[\frac{\beta-\alpha'}{\alpha+r\beta}g(X,Y) + \frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+r\beta)}g(X,u)g(Y,u)\big]U^V, Z^V\Big).$$

for all $X, Y, U \in \Gamma(TM)$, $U_x = u = u^i \frac{\partial}{\partial x^i} \in T_xM$ and $(x,u) \in TM$.

Proof. Using Lemma 4, Lemma 5 and Kozul formula, we obtain:

1) $$\begin{aligned}2\widetilde{g}(\widetilde{\nabla}_{X^H}Y^H, Z^H) =& X^H\widetilde{g}(Y^H, Z^H) + Y^H\widetilde{g}(Z^H, X^H) - Z^H\widetilde{g}(X^H, Y^H)\\ &+ \widetilde{g}(Z^H, [X^H, Y^H]) + \widetilde{g}(Y^H, [Z^H, X^H]) - \widetilde{g}(X^H, [Y^H, Z^H])\\ =& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y])\\ &+ g(Y,[Z,X]) - g(X,[Y,Z])\\ =& 2g(\nabla_X Y.Z)\\ =& 2\widetilde{g}((\nabla_X Y)^H, Z^H).\end{aligned}$$

2) $$\begin{aligned}2\widetilde{g}(\widetilde{\nabla}_{X^H}Y^H, Z^V) =& X^H\widetilde{g}(Y^H, Z^V) + Y^H\widetilde{g}(Z^V, X^H) - Z^V\widetilde{g}(X^H, Y^H)\\ &+ \widetilde{g}(Z^V, [X^H, Y^H]) + \widetilde{g}(Y^H, [Z^V, X^H]) - \widetilde{g}(X^H, [Y^H, Z^V])\\ =& \widetilde{g}(Z^V, [X^H, Y^H])\\ =& -\widetilde{g}((R(X,Y)u)^V, Z^V).\end{aligned}$$

3) $$\begin{aligned}2\widetilde{g}(\widetilde{\nabla}_{X^H}Y^V, Z^H) =& X^H\widetilde{g}(Y^V, Z^H) + Y^V\widetilde{g}(Z^H, X^H) - Z^H\widetilde{g}(X^H, Y^V)\\ &+ \widetilde{g}(Z^H, [X^H, Y^V]) + \widetilde{g}(Y^V, [Z^H, X^H]) - \widetilde{g}(X^H, [Y^V, Z^H])\\ =& -\widetilde{g}((R(Z,X)u)^V, Y^V)\\ =& -\alpha g(R(Z,X)u, Y) - \beta g(Y,u)g(R(Z,X)u,u)]\\ =& \alpha g(R(u,Y)X, Z)\\ =& \alpha\widetilde{g}((R(u,Y)X)^H, Z^H).\end{aligned}$$

4) $$\begin{aligned}2\widetilde{g}(\widetilde{\nabla}_{X^H}Y^V, Z^V) =& X^H\widetilde{g}(Y^V, Z^V) + Y^V\widetilde{g}(Z^V, X^H) - Z^V\widetilde{g}(X^H, Y^V)\\ &+ \widetilde{g}(Z^V, [X^H, Y^V]) + \widetilde{g}(Y^V, [Z^V, X^H]) - \widetilde{g}(X^H, [Y^V, Z^V])\\ =& X^H\widetilde{g}(Y^V, Z^V) + \widetilde{g}(Z^V, [X^H, Y^V]) + \widetilde{g}(Y^V, [Z^V, X^H])\\ =& \widetilde{g}((\nabla_X Y)^V, Z^V) + \widetilde{g}(Y^V, (\nabla_X Z)^V)\\ &+ \widetilde{g}(Z^V, (\nabla_X Y)^V) - \widetilde{g}(Y^V, (\nabla_X Z)^V)\\ =& 2\widetilde{g}((\nabla_X Y)^V, Z^V).\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Theorem 7. [16], [21]*Let (M,g) be a Riemannian manifold and $(TM,\widetilde{g})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. If ∇ (resp. $\widetilde{\nabla}$) denote the Levi-Civita connection of (M,g) (resp. $(TM,\widetilde{g})$), then we have:*

$$\begin{aligned}
1)\,(\widetilde{\nabla}_{X^H}Y^H)_p &= (\nabla_X Y)^H_p - \frac{1}{2}(R_x(X,Y)u)^V,\\
2)\,(\widetilde{\nabla}_{X^H}Y^V)_p &= (\nabla_X Y)^V_p + \frac{\alpha}{2}(R_x(u,Y)X)^H,\\
3)\,(\widetilde{\nabla}_{X^V}Y^H)_p &= \frac{\alpha}{2}(R_x(u,X)Y)^H,\\
4)\,(\widetilde{\nabla}_{X^V}Y^V)_p &= \frac{\alpha'}{\alpha}\big[g_x(X,u)Y^V_p + g_x(Y,u)X^V_p\big]\\
&\quad +\big[\frac{\beta-\alpha'}{\alpha+r\beta}g_x(X,Y) + \frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+r\beta)}g_x(X,u)g_x(Y,u)\big]U^V_p.
\end{aligned}$$

for all $X,Y,U\in\Gamma(TM)$, $U_x = u = u^i\frac{\partial}{\partial x^i}\in T_xM$ and $p=(x,u)\in TM$. where R denote the curvature tensor of (M,g).

Proof. The statement is a direct consequence of Lemma 6. □

3. Generalized Cheeger-Gromoll metric and Harmonicity

3.1. Harmonicity of a vector field $X:(M,g)\longrightarrow(TM,\widetilde{g})$.

Lemma 8. *Let (M,g) be a Riemannian manifold. If $X,Y\in\Gamma(TM)$ are vector fields on M and $(x,u)\in TM$ such that $Y_x=u$, then we have:*

$$d_xY(X_x) = X^H_{(x,u)} + (\nabla_X Y)^V_{(x,u)}.$$

Proof. Let (U,x^i) be a local chart on M in $x\in M$ and $\pi^{(-1)}(U),x^i,y^j)$ be the induced chart on TM, if $X_x = X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x = Y^i(x)\frac{\partial}{\partial x^i}|_x = u$, then

$$d_xY(X_x) = X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)} + X^i(x)\frac{\partial Y^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,u)},$$

thus the horizontal part is given by:

$$\begin{aligned}
(d_xY(X_x))^h &= X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)} - X^i(x)Y^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,u)}\\
&= X^H_{(x,u)},
\end{aligned}$$

and the vertical part is given by:

$$\begin{aligned}
(d_xY(X_x))^v &= \{X^j(x)\frac{\partial Y^k}{\partial x^i}(x) + X^i(x)Y^j(x)\Gamma^k_{ij}(x)\}\frac{\partial}{\partial y^k}|_{(x,u)}\\
&= (\nabla_X Y)^V_{(x,u)}.
\end{aligned}$$

□

Lemma 9. *Let (M^m, g) be a Riemannian m-dimensional manifold and $(TM, \widetilde{g})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$, then the energy density associated to X is given by:*

$$e(X) = \frac{m}{2} + \frac{1}{2} trace_g \big[\alpha g(\nabla X, \nabla X) + \beta g(\nabla X, u)^2\big].$$

Proof. Let $X \in \Gamma(TM)$ and $(E_1, \cdots, E_m)$ be a local orthonormal frame on M, then:

$$e(X) = \frac{1}{2}\sum_{i=1}^{m} \widetilde{g}(dX(E_i), dX(E_i))$$

Using Lemma 8, we obtain:

$$\begin{aligned}
e(X) &= \frac{1}{2}\sum_{i=1}^{m} \widetilde{g}(E_i^H + (\nabla_{E_i} X)^V, E_i^H + (\nabla_{E_i} X)^V) \\
&= \frac{1}{2}\sum_{i=1}^{m} \big[\widetilde{g}(E_i^H, E_i^H) + \widetilde{g}((\nabla_{E_i} X)^V, (\nabla_{E_i} X)^V))\big] \\
&= \frac{1}{2}\sum_{i=1}^{m} \big[g(E_i, E_i) + \alpha g(\nabla_{E_i} X, \nabla_{E_i} X) + \beta g(\nabla_{E_i} X, u)^2\big] \\
&= \frac{m}{2} + \frac{1}{2} trace_g \big[\alpha g(\nabla X, \nabla X) + \beta g(\nabla X, u)^2\big].
\end{aligned}$$

□

Theorem 10. *Let (M^m, g) be a Riemannian m-dimensional manifold and $(TM, \widetilde{g})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$, then the tension field associated to X is given by:*

$$\tau(X) = \big[trace_g(\alpha R(X, \nabla X)*)\big]^H + \big[trace_g A(X)\big]^V.$$

where $A(X)$ is a bilinear map defined by:

$$\begin{aligned}
A(X) &= \nabla^2 X + \frac{2\alpha'}{\alpha} g(\nabla X, X)\nabla X + \big[\frac{\beta - \alpha'}{\alpha + \|X\|^2\beta} g(\nabla X, \nabla X) \\
&\quad + \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha + \|X\|^2\beta)} g(\nabla X, X)^2\big] X,
\end{aligned}$$

and $\|X\|^2 = g(X, X)$.

Proof. Let $x \in M$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla^M_{E_i} E_i)_x = 0$ and $X_x = u$, then:

$$\tau(X)_x = \sum_{i=1}^{m} \{\nabla^X_{E_i} dX(E_i) - dX(\nabla^M_{E_i} E_i)\}_x$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\{\widetilde{\nabla}_{dX(E_i)}dX(E_i)\}_{(x,u)} \\
&= \sum_{i=1}^{m}\{\widetilde{\nabla}_{[E_i^H+(\nabla_{E_i}X)^V]}[E_i^H+(\nabla_{E_i}X)^V]\}_{(x,u)} \\
&= \sum_{i=1}^{m}\{\widetilde{\nabla}_{E_i^H}E_i^H+\widetilde{\nabla}_{E_i^H}(\nabla_{E_i}X)^V+\widetilde{\nabla}_{(\nabla_{E_i}X)^V}(E_i)^H+\widetilde{\nabla}_{(\nabla_{E_i}X)^V}(\nabla_{E_i}X)^V\}_{(x,u)},
\end{aligned}
$$

Using Theorem 7, we obtain:

$$
\begin{aligned}
\tau(X) &= \sum_{i=1}^{m}\Big[(\nabla_{E_i}E_i)^H-\frac{1}{2}(R(E_i,E_i)X)^V+(\nabla_{E_i}\nabla_{E_i}X)^V+\frac{\alpha}{2}(R(X,\nabla_{E_i}X)E_i)^H \\
&\quad +\frac{\alpha}{2}(R(X,\nabla_{E_i}X)E_i)^H+\frac{\alpha'}{\alpha}\big[g(\nabla_{E_i}X,X)(\nabla_{E_i}X)^V+g(\nabla_{E_i}X,X)(\nabla_{E_i}X)^V\big] \\
&\quad +\big[\frac{\beta-\alpha'}{\alpha+\|X\|^2\beta}g(\nabla_{E_i}X,\nabla_{E_i}X)+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+\|X\|^2\beta)}g(\nabla_{E_i}X,X)g(\nabla_{E_i}X,X)\big]X^V\Big] \\
&= \sum_{i=1}^{m}\Big[\alpha(R(X,\nabla_{E_i}X)E_i)^H+(\nabla_{E_i}\nabla_{E_i}X)^V+\frac{2\alpha'}{\alpha}g(\nabla_{E_i}X,X)(\nabla_{E_i}X)^V \\
&\quad +\big[\frac{\beta-\alpha'}{\alpha+\|X\|^2\beta}g(\nabla_{E_i}X,\nabla_{E_i}X)+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+\|X\|^2\beta)}g(\nabla_{E_i}X,X)^2\big]X^V\Big] \\
&= \Big[trace_g\Big(\alpha R(X,\nabla X)*\Big)\Big]^H+\Big[trace_g\Big(\nabla^2X+\frac{2\alpha'}{\alpha}g(\nabla X,X)\nabla X \\
&\quad +\big[\frac{\beta-\alpha'}{\alpha+\|X\|^2\beta}g(\nabla X,\nabla X)+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+\|X\|^2\beta)}g(\nabla X,X)^2\big]X\Big)\Big]^V.
\end{aligned}
$$

□

Theorem 11. *Let (M^m,g) be a Riemannian m-dimensional manifold and $(TM,\widetilde{g})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. If $X\in\Gamma(TM)$, then X is harmonic if and only the following conditions are verified*

$$trace_g\Big(R(X,\nabla X)*\Big)=0$$

and

$$
\begin{aligned}
&trace_g\Big(\nabla^2X+\frac{2\alpha'}{\alpha}g(\nabla X,X)\nabla X \\
&\qquad +\big[\frac{\beta-\alpha'}{\alpha+\|X\|^2\beta}g(\nabla X,\nabla X)+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+\|X\|^2\beta)}g(\nabla X,X)^2\big]X\Big)=0 \qquad (7)
\end{aligned}
$$

Proof. The statement is a direct consequence of Theorem 10. □

Corollary 12. *Let (M^m, g) be a Riemannian m-dimensional manifold and $(TM, \widetilde{g})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$ is a parallel (i.e $\nabla X = 0$), then X is harmonic.*

Theorem 13. *Let (M^m, g) be a Riemannian compact m-dimensional manifold and $(TM, \widetilde{g})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$, then X is harmonic if and only if X is parallel (i.e $\nabla X = 0$).*

Proof. If X is parallel from Corollary 12, we deduce that X is harmonic vector field.
Inversely, let φ_t be a compactly supported variation of X defined by:

$$\begin{array}{rcl} \varphi : \mathbb{R} \times M & \longrightarrow & T_x M \\ (t, x) & \longmapsto & \varphi(t, x) = \varphi_t(x) = (t+1)X_x \end{array}$$

From Lemma 9 we have:

$$e(\varphi_t) = \frac{m}{2} + \frac{(t+1)^2}{2} trace_g \big[\alpha g(\nabla X, \nabla X) + \beta g(\nabla X, X)^2\big]$$

$$E(\varphi_t) = \frac{m}{2} Vol(M) + \frac{(t+1)^2}{2} \int_M trace_g \big[\alpha g(\nabla X, \nabla X) + \beta g(\nabla X, X)^2\big] dv_g$$

If X is a critical point of the energy functional, then we have :

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} E(\varphi_t)|_{t=0} \\ &= \frac{\partial}{\partial t}\Big[\frac{m}{2} Vol(M) + \frac{(t+1)^2}{2} \int_M trace_g \big[\alpha g(\nabla X, \nabla X) + \beta g(\nabla X, X)^2\big] dv_g\Big]_{t=0} \\ &= \int_M trace_g \big[\alpha g(\nabla X, \nabla X) + \beta g(\nabla X, X)^2\big] dv_g \end{aligned}$$

then $g(\nabla X, \nabla X) + g(\nabla X, X)^2 = 0$,
hence $\nabla X = 0$. □

Example 14. *Let $\mathbb{R}^n$ equipped with the canonical metric (flat manifold and non compact) and the vector field :*

$$\begin{array}{rcl} X : \mathbb{R}^n & \longrightarrow & T\mathbb{R}^n \\ x = (x_1, \cdots, x_n) & \longmapsto & X_x = (X_x^1, \cdots, X_x^n) \end{array},$$

we have:

$$\tau(X) = \sum_{i=1}^{n} \Big(\frac{\partial^2 X^1}{\partial x_i^2}, ..., \frac{\partial^2 X^n}{\partial x_i^2}\Big)$$

1) If X is constant, then X is harmonic.
2) If $X^i = a_i x_i$ and $a_i \neq 0$, then X is harmonic ($\tau(X) = 0$) but

$$\nabla X = \sum_i a_i \frac{\partial}{\partial x_i} \otimes dx_i \neq 0.$$

indeed

$$\nabla X(\partial x_j) = \nabla_{\partial x_j} X = \sum_i a_i \nabla_{\partial x_j}(x_i \frac{\partial}{\partial x_i}) = \sum_i a_i \delta_{ij} \frac{\partial}{\partial x_i} = a_j \frac{\partial}{\partial x_j}.$$

Example 15. *Let $\mathbb{S}^1$ equipped with the metric:*

$$g_{\mathbb{S}^1} = \frac{4}{(1+x^2)^2} dx^2$$

as $\mathbb{S}^1$ is compact then. The vector field $X = a(x)\frac{\partial}{\partial x}$, $a \in \mathcal{C}^\infty(\mathbb{S}^1)$ is harmonic if and only if X is parallel, i.e

$$\begin{aligned} \nabla X = 0 \quad &\Leftrightarrow \quad \nabla_{\frac{\partial}{\partial x}} a \frac{\partial}{\partial x} = 0 \\ &\Leftrightarrow \quad \frac{\partial a}{\partial x} + a\Gamma = 0 \\ &\Leftrightarrow \quad \frac{\partial a}{\partial x} - \frac{2x}{1+x^2} a = 0 \\ &\Leftrightarrow \quad a(x) = k(1+x^2)\ , \ k \in \mathbb{R} \\ &\Leftrightarrow \quad X = k(1+x^2)\frac{\partial}{\partial x}\ , \ k \in \mathbb{R} \end{aligned}$$

Theorem 16. *Let $(\mathbb{R}^m, g_0)$ the real euclidean space and $(T\mathbb{R}^m, \widetilde{g_0})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. If $X = (X^1, \cdots, X^m) \in \Gamma(T\mathbb{R}^m)$, then X is harmonic if and only if X verifies the following system of equations*

$$\begin{aligned} \sum_{i=1}^m \frac{\partial^2 X^k}{\partial (x^i)^2} + \sum_{i,j=1}^m \Big(\frac{2\alpha'}{\alpha} X^j \frac{\partial X^j}{\partial x^i} \frac{\partial X^k}{\partial x^i} + \frac{\beta - \alpha'}{\alpha + \|X\|^2 \beta} X^k \big(\frac{\partial X^j}{\partial x^i}\big)^2 \Big) \\ + \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha + \|X\|^2\beta)} X^k \sum_{i=1}^m \Big(\sum_{j=1}^m X^j \frac{\partial X^j}{\partial x^i} \Big)^2 = 0. \end{aligned} \tag{8}$$

for all $k = \overline{1,m}$.

Proof. Let $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1,m}}$ be a canonical frame on $\mathbb{R}^m$. Using Theorem 11, we have: $\tau(X) = 0$ equivalent the following equations (**??**) and (**??**) are verified.
As $\mathbb{R}^m$ is flat, then the equation (**??**) is obvious.
Hence,

$$\begin{aligned} &\tau(X) = 0 \Leftrightarrow (\mathbf{??}) \\ &\Leftrightarrow trace_g \Big[\nabla^2 X + \frac{2\alpha'}{\alpha} g(\nabla X, X) \nabla X \\ &+ \big[\frac{\beta - \alpha'}{\alpha + \|X\|^2\beta} g(\nabla X, \nabla X) + \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha + \|X\|^2\beta)} g(\nabla X, X)^2 \big] X \Big] = 0 \end{aligned}$$

$$
\begin{aligned}
&\sum_{i=1}^{m}\Big[\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^i}}X+\frac{2\alpha'}{\alpha}g(\nabla_{\frac{\partial}{\partial x^i}}X,X)(\nabla_{\frac{\partial}{\partial x^i}}X)\\
&\quad+\big[\frac{\beta-\alpha'}{\alpha+\|X\|^2\beta}g(\nabla_{\frac{\partial}{\partial x^i}}X,\nabla_{\frac{\partial}{\partial x^i}}X)+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+\|X\|^2\beta)}g(\nabla_{\frac{\partial}{\partial x^i}}X,X)^2\big]X\Big]=0\\
&\Leftrightarrow\sum_{i=1}^{m}\Big[\sum_{k=1}^{m}\big(\frac{\partial^2X^k}{\partial(x^i)^2}\frac{\partial}{\partial x^k}\big)+\frac{2\alpha'}{\alpha}\sum_{j=1}^{m}\big(\frac{\partial X^j}{\partial x^i}X^j\big)\sum_{k=1}^{m}\big(\frac{\partial X^k}{\partial x^i}\frac{\partial}{\partial x^k}\big)\\
&\quad+\big[\frac{\beta-\alpha'}{\alpha+\|X\|^2\beta}\sum_{j=1}^{m}\big(\frac{\partial X^j}{\partial x^i}\big)^2+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+\|X\|^2\beta)}\big(\sum_{j=1}^{m}X^j\frac{\partial X^j}{\partial x^i}\big)^2\big]\sum_{i=1}^{k}\big(X^k\frac{\partial}{\partial x^k}\big)\Big]=0\\
&\Leftrightarrow\sum_{i=1}^{m}\frac{\partial^2X^k}{\partial(x^i)^2}+\sum_{i,j=1}^{m}\Big(\frac{2\alpha'}{\alpha}X^j\frac{\partial X^j}{\partial x^i}\frac{\partial X^k}{\partial x^i}+\frac{\beta-\alpha'}{\alpha+\|X\|^2\beta}X^k\big(\frac{\partial X^j}{\partial x^i}\big)^2\Big)\\
&\qquad+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+\|X\|^2\beta)}X^k\sum_{i=1}^{m}\Big(\sum_{j=1}^{m}X^j\frac{\partial X^j}{\partial x^i}\Big)^2=0.
\end{aligned}
$$

for all $k=\overline{1,m}$. □

Corollary 17. *Let $(\mathbb{R}^m, g_0)$ the real euclidean space and $(T\mathbb{R}^m, \widetilde{g_0})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric and $X=(X^1,\cdots,X^m)\in\Gamma(T\mathbb{R}^m)$. If α and β are constant functions, then X is a harmonic if and only if for all $k=\overline{1,m}$:*

$$\sum_{i=1}^{m}\frac{\partial^2X^k}{\partial(x^i)^2}+\frac{\beta}{\alpha+\|X\|^2\beta}X^k\sum_{i,j=1}^{m}\big(\frac{\partial X^j}{\partial x^i}\big)^2=0.$$

Remark 18. *Using Corollary 17, we can construct many examples of non trivial harmonic vector fields.*

Example 19. *If $\mathbb{R}^n$ is endowed with the canonical metric and $T\mathbb{R}^m$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. From corollary* 17, *we deduce that:*

1) *If $X=(y(x_1),0,\cdots,0)\in\Gamma(T\mathbb{R}^m)$ is a harmonic vector field if and only the function y is solution of differential equation:*

$$y''+\frac{\beta y'y}{\alpha+\beta y^2}y'=0.$$

2) *If $X=(y(x_1,x_2),0,\cdots,0)\in\Gamma(T\mathbb{R}^m)$ is a harmonic vector field if and only the function y is the solution of the partial derivative equation:*

$$\frac{\partial^2y}{\partial x_1^2}+\frac{\partial^2y}{\partial x_2^2}+\frac{\beta y}{\alpha+\beta y^2}\big(\frac{\partial y}{\partial x_1}+\frac{\partial y}{\partial x_2}\big)=0.$$

where $\alpha,\beta\in\mathbb{R}^+$.

3.2. **Harmonicity of the map** $\sigma : (M,g) \longrightarrow (TN, \widetilde{h})$**.**

Lemma 20. *Let* (M^m, g), (N^n, h) *two Riemannian manifolds and* $f : N \to]0, +\infty[$ *a smooth function. Let* $(TN, \widetilde{h})$ *the tangent bundle of* N *equipped with the Generalized Cheeger-Gromoll metric. If*

$$\begin{array}{rcl} \sigma : (M,g) & \longrightarrow & (TN, \widetilde{h}) \\ x & \longmapsto & (\varphi(x), v) \end{array}$$

a map, such that $\varphi = \pi_{TN} \circ \sigma$ *and* $v = Y_{\varphi(x)} \in T_{\varphi(x)}N$ *where* $Y \in \Gamma(TN)$, $\pi_{TN} : TN \to N$ *is the canonical projection, then:*

$$d\sigma(X) = (d\varphi(X))^H + (\nabla^\varphi_X \sigma)^V$$

for all $X \in \Gamma(TM)$.

Proof. Using Lemma 8, we obtain:

$$\begin{aligned} d_x\sigma(X_x) &= d_x(Y \circ \varphi)(X_x) \\ &= d_{\varphi(x)}Y(d\varphi(X_x)) \\ &= (d\varphi(X))^H_{(\varphi(x),v)} + (\nabla_{d\varphi(X)}Y)^V_{(\varphi(x),v)} \\ &= (d\varphi(X))^H_{(\varphi(x),v)} + (\nabla^\varphi_X \sigma)^V_{(\varphi(x),v)} \end{aligned}$$

where $Y_{\varphi(x)} = v \in T_{\varphi(x)}N$ $\square$

Theorem 21. *Let* (M^m, g), (N^n, h) *two Riemannian manifolds and* $f : N \to]0, +\infty[$ *a smooth function. Let* $(TN, \widetilde{h})$ *the tangent bundle of* N *equipped with the Generalized Cheeger-Gromoll metric. The tension field of the map*

$$\begin{array}{rcl} \sigma : (M,g) & \longrightarrow & (TN, h^f) \\ x & \longmapsto & (\varphi(x), v) \end{array}$$

such that $\varphi = \pi_{TN} \circ \sigma$*, is given by:*

$$\tau(\sigma) = \big[\tau(\varphi) + trace_g(\alpha R^N(\sigma, \nabla^\varphi \sigma) d\varphi(*))\big]^H + \big[trace_g A(\sigma)\big]^V$$

where $A(\sigma)$ *is a bilinear map defined by:*

$$\begin{aligned} A(\sigma) \quad = \quad & (\nabla^\varphi)^2\sigma + \frac{2\alpha'}{\alpha} h(\nabla^\varphi \sigma, \sigma)\nabla^\varphi \sigma \\ & + \big[\frac{\beta - \alpha'}{\alpha + \|\sigma\|^2\beta} h(\nabla^\varphi \sigma, \nabla^\varphi \sigma) + \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha + \|\sigma\|^2\beta)} h(\nabla^\varphi \sigma, \sigma)^2\big]\sigma \end{aligned}$$

and $\|\sigma\|^2 = h(\sigma, \sigma) = r$.

Proof. Let $x \in M$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla^M_{E_i} E_i)_x = 0$ and $\sigma(x) = U_{\varphi(x)} = v$. From the Lemma 20 and theorem 7, we obtain:

$$\tau(\sigma)_x \quad = \quad trace_g(\nabla d\sigma)_x$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\{\widetilde{\nabla}_{d\sigma(E_i)}d\sigma(E_i)\}_{(\varphi(x),v)}\\
&= \sum_{i=1}^{m}\{\widetilde{\nabla}_{[(d\varphi(E_i))^H+(\nabla^{\varphi}_{E_i}\sigma)^V]}[(d\varphi(E_i))^H+(\nabla^{\varphi}_{E_i}\sigma)^V]\}_{(\varphi(x),v)}\\
&= \sum_{i=1}^{m}\{\widetilde{\nabla}_{(d\varphi(E_i))^H}(d\varphi(E_i))^H+\widetilde{\nabla}_{(d\varphi(E_i))^H}(\nabla^{\varphi}_{E_i}\sigma)^V+\widetilde{\nabla}_{(\nabla^{\varphi}_{E_i}\sigma)^V}(d\varphi(E_i))^H\\
&\quad+\widetilde{\nabla}_{(\nabla^{\varphi}_{E_i}\sigma)^V}(\nabla^{\varphi}_{E_i}\sigma)^V\}_{(\varphi(x),v)}\\
&= \sum_{i=1}^{m}\Big[(\nabla^N_{d\varphi(E_i)}d\varphi(E_i))^H-\frac{1}{2}(R^N(d\varphi(E_i),d\varphi(E_i))\sigma)^V\\
&\quad+(\nabla^N_{d\varphi(E_i)}\nabla^{\varphi}_{E_i}\sigma)^V+\frac{\alpha}{2}(R^N(v,\nabla^{\varphi}_{E_i}\sigma)d\varphi(E_i))^H\\
&\quad+\frac{\alpha}{2}(R^N(v,\nabla^{\varphi}_{E_i}\sigma)d\varphi(E_i))^H\\
&\quad+\frac{\alpha'}{\alpha}\big[h(\nabla^{\varphi}_{E_i}\sigma,v)(\nabla^{\varphi}_{E_i}\sigma)^V+h(\nabla^{\varphi}_{E_i}\sigma,v)(\nabla^{\varphi}_{E_i}\sigma)^V\big]\\
&\quad+\Big(\frac{\beta-\alpha'}{\alpha+r\beta}h(\nabla^{\varphi}_{E_i}\sigma,\nabla^{\varphi}_{E_i}\sigma)+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+r\beta)}h(\nabla^{\varphi}_{E_i}\sigma,v)^2\Big)U^V\Big]_{(\varphi(x),v)}
\end{aligned}
$$

After offsetting the values of $r=h(\sigma,\sigma)=\|\sigma\|^2$ and $\sigma(x)=U_{\varphi(x)}=v$, we have:

$$
\begin{aligned}
\tau(\sigma) &= \sum_{i=1}^{m}\Big[(\nabla^{\varphi}_{E_i}d\varphi(E_i))^H+\alpha(R^N(\sigma,\nabla^{\varphi}_{E_i}\sigma)d\varphi(E_i))^H\\
&\quad+(\nabla^{\varphi}_{E_i}\nabla^{\varphi}_{E_i}\sigma)^V+\frac{2\alpha'}{\alpha}h(\nabla^{\varphi}_{E_i}\sigma,\sigma)(\nabla^{\varphi}_{E_i}\sigma)^V\\
&\quad+\Big(\frac{\beta-\alpha'}{\alpha+\|\sigma\|^2\beta}h(\nabla^{\varphi}_{E_i}\sigma,\nabla^{\varphi}_{E_i}\sigma)+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+\|\sigma\|^2\beta)}h(\nabla^{\varphi}_{E_i}\sigma,\sigma)^2\Big)\sigma^V\Big]\\
\tau(\sigma) &= \Big[\tau(\varphi)+trace_g\big(\alpha R^N(\sigma,\nabla^{\varphi}\sigma)d\varphi(*)\big)\Big]^H\\
&\quad+\Big[trace_g\big[(\nabla^{\varphi})^2\sigma+\frac{2\alpha'}{\alpha}h(\nabla^{\varphi}\sigma,\sigma)\nabla^{\varphi}\sigma\\
&\quad+\big(\frac{\beta-\alpha'}{\alpha+\|\sigma\|^2\beta}h(\nabla^{\varphi}_{E_i}\sigma,\nabla^{\varphi}_{E_i}\sigma)+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+\|\sigma\|^2\beta)}h(\nabla^{\varphi}\sigma,\sigma)^2\big)\sigma\big]\Big]^V
\end{aligned}
$$

□

Theorem 22. *Let (M^m,g), (N^n,h) two Riemannian manifolds and $f: N \to]0,+\infty[$ a smooth function. Let (TN,h^f) the tangent bundle of N equipped with the Generalized Cheeger-Gromoll metric. The map*

$$\sigma:(M,g) \longrightarrow (TN,h^f)$$

$$x \longmapsto (\varphi(x), v)$$

such that $\varphi = \pi_{TN} \circ \sigma$, is a harmonic if and only if the following conditions are verified

$$\tau(\varphi) = -trace_g\big(\alpha R^N(\sigma, \nabla^\varphi \sigma) d\varphi(*)\big)$$

and

$$\begin{aligned} 0 &= trace_g\big[(\nabla^\varphi)^2\sigma + \frac{2\alpha'}{\alpha} h(\nabla^\varphi \sigma, \sigma)\nabla^\varphi \sigma \\ &\quad +\big(\frac{\beta - \alpha'}{\alpha + \|\sigma\|^2\beta} h(\nabla^\varphi_{E_i}\sigma, \nabla^\varphi_{E_i}\sigma) + \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha + \|\sigma\|^2\beta)} h(\nabla^\varphi \sigma, \sigma)^2\big)\sigma\big] \end{aligned} \tag{9}$$

3.3. Harmonicity of the map $\phi : (TM, \widetilde{g}) \longrightarrow (N, h)$.

Lemma 23. *Let (M^m, g) be a Riemannian m-dimensional manifold and $(TM, \widetilde{g})$ its tangent bundle equipped with the Generalized Cheeger-Gromoll metric. The canonical projection*

$$\begin{aligned} \pi : (TM, \widetilde{g}) &\longrightarrow (M, g) \\ (x, u) &\longmapsto x \end{aligned}$$

is harmonic: i.e. $\tau(\pi) = 0$.

Proof. Let $(x, u) \in TM$ and $\{E_i\}_{i=\overline{1,m}}$ such that $E_1 = \frac{u}{\|u\|}$ is an orthonormal basis on M in x, then

$$\{E_i^H, \frac{1}{\sqrt{\alpha + r\beta}} E_1^V, \frac{1}{\sqrt{\alpha}} E_j^V\}_{i=\overline{1,m}, j=\overline{2,m}}$$

is an orthonormal basis on TM in (x, u).

$$\begin{aligned} \tau(\pi) &= trace_{\widetilde{g}} \nabla d\pi \\ &= \sum_{i=1}^m \Big\{\nabla^\pi_{E_i^H} d\pi(E_i^H) - d\pi(\nabla^{TM}_{E_i^H} E_i^H)\Big\} \\ &\quad +\nabla^\pi_{(\frac{1}{\sqrt{\alpha+r\beta}}E_1^V)} d\pi(\frac{1}{\sqrt{\alpha + r\beta}} E_1^V) - d\pi(\nabla^{TM}_{(\frac{1}{\sqrt{\alpha+r\beta}}E_1^V)}(\frac{1}{\sqrt{\alpha + r\beta}} E_1^V)) \\ &\quad + \sum_{j=2}^m \Big\{\nabla^\pi_{(\frac{1}{\sqrt{\alpha}}E_j^V)} d\pi(\frac{1}{\sqrt{\alpha}} E_j^V) - d\pi(\nabla^{TM}_{(\frac{1}{\sqrt{\alpha}}E_j^V)}(\frac{1}{\sqrt{\alpha}} E_j^V))\Big\} \end{aligned}$$

as $d\pi(E_i^V) = 0$ and $d\pi(E_i^H) = E_i \circ \pi$ then:

$$\begin{aligned} \tau(\pi) &= \sum_{i=1}^m \Big\{(\nabla^M_{E_i} E_i) \circ \pi - d\pi(\nabla^M_{E_i} E_i)^H\Big\} \\ &\quad -\frac{1}{\sqrt{\alpha + r\beta}} d\pi\big[E_1^V(\frac{1}{\sqrt{\alpha + r\beta}}) E_1^V + \frac{1}{\sqrt{\alpha + r\beta}} \nabla^{TM}_{E_1^V} E_1^V\big] \end{aligned}$$

$$
\begin{aligned}
& -\sum_{j=2}^{m}\Big\{\frac{1}{\sqrt{\alpha}}d\pi\big[E_j^V(\frac{1}{\sqrt{\alpha}})E_j^V+\frac{1}{\sqrt{\alpha}}\nabla^{TM}_{E_j^V}E_j^V\big]\Big\}\\
= \quad & -\frac{1}{\alpha+r\beta}d\pi(\nabla^{TM}_{E_1^V}E_1^V)-\sum_{j=2}^{m}\Big\{\frac{1}{\alpha}d\pi(\nabla^{TM}_{E_j^V}E_j^V)\Big\}\\
= \quad & 0
\end{aligned}
$$

□

Theorem 24. *Let (M^m,g), (N^n,h) two Riemannian manifolds and $f : M \to]0,+\infty[$ a smooth function. Let $(TM,\widetilde{g})$ the tangent bundle of M equipped with the Generalized Cheeger-Gromoll metric.*
Let $\varphi : (M,g) \longrightarrow (N,h)$ a smooth map. The tension field of the map:

$$
\begin{array}{rcl}
\phi : (TM,\widetilde{g}) & \longrightarrow & (N,h)\\
(x,y) & \longmapsto & \varphi(x)
\end{array}
$$

is given by:

$$\tau(\phi)=\big[\tau(\varphi)\big]\circ\pi.$$

Proof. Let $(x,u)\in TM$ and $\{E_i\}_{i=\overline{1,m}}$ such that $E_1=\frac{u}{\|u\|}$ is an orthonormal basis on M in x, then $\{E_i^H,\frac{1}{\sqrt{\alpha+r\beta}}E_1^V,\frac{1}{\sqrt{\alpha}}E_j^V\}_{i=\overline{1,m},j=\overline{2,m}}$ is an orthonormal basis on TM in (x,u).
as the ϕ is defined by:

$$
\begin{array}{rcl}
\phi : (TM,\widetilde{g}) & \xrightarrow{\pi} & (M,g) \xrightarrow{\varphi} (N,h)\\
(x,y) & \longmapsto & x \longmapsto \varphi(x)
\end{array}
$$

i.e. $\phi=\varphi\circ\pi$, we have:

$$
\begin{aligned}
\tau(\phi) &= \tau(\varphi\circ\pi)\\
&= d\varphi(\tau(\pi))+trace_{\widetilde{g}}\nabla d\varphi(d\pi,d\pi)
\end{aligned}
$$

$$
\begin{aligned}
trace_{\widetilde{g}}\nabla d\varphi(d\pi,d\pi) &= \sum_{i=1}^{m}\Big\{\nabla^{\varphi}_{d\pi(E_i^H)}d\varphi(d\pi(E_i^H))-d\varphi(\nabla^M_{d\pi(E_i^H)}d\pi(E_i^H))\Big\}\\
&+\sum_{j=2}^{m}\Big\{\nabla^{\varphi}_{d\pi(\frac{1}{\sqrt{\alpha}}E_j^V)}d\varphi(d\pi(\frac{1}{\sqrt{\alpha}}E_j^V))-d\varphi(\nabla^M_{d\pi(\frac{1}{\sqrt{\alpha}}E_j^V)}d\pi(\frac{1}{\sqrt{\alpha}}E_j^V))\Big\}\\
&+\nabla^{\varphi}_{d\pi(\frac{1}{\sqrt{\alpha+r\beta}}E_1^V)}d\varphi(d\pi(\frac{1}{\sqrt{\alpha+r\beta}}E_1^V))-d\varphi(\nabla^M_{d\pi(\frac{1}{\sqrt{\alpha+r\beta}}E_1^V)}d\pi(\frac{1}{\sqrt{\alpha+r\beta}}E_1^V))\\
&=\sum_{i=1}^{m}\Big\{(\nabla^{\varphi}_{E_i}d\varphi(E_i))\circ\pi-d\varphi(\nabla^M_{E_i}E_i)\circ\pi\Big\}\\
&=\sum_{i=1}^{m}\Big\{\nabla^{\varphi}_{E_i}d\varphi(E_i)-d\varphi(\nabla^M_{E_i}E_i)\Big\}\circ\pi
\end{aligned}
$$

$$= \tau(\varphi) \circ \pi$$

Using Lemma 23, we obtain:

$$\tau(\phi) = \tau(\varphi) \circ \pi$$

$\square$

Theorem 25. *Let (M^m, g), (N^n, h) two Riemannian manifolds and $f : M \to]0, +\infty[$ a smooth function. Let $(TM, \widetilde{g})$ the tangent bundle of M equipped with the Generalized Cheeger-Gromoll metric. The map*

$$\begin{array}{rcl} \phi : (TM, \widetilde{g}) & \longrightarrow & (N, h) \\ (x, y) & \longmapsto & \varphi(x) \end{array}$$

is harmonic if and only if φ is a harmonic.

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RESULTS ON QUASI-STATISTICAL LIMIT AND QUASI-STATISTICAL CLUSTER POINTS

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Abstract. In this paper we introduce the concepts of quasi-statistical limit point and quasi-statistical cluster point of a sequence. We give some inclusion results concerning these concepts. We also give the relationship between the Knopp core and quasi-statistical core of a sequence. Finally we state some theorems which deal with quasi-summability and quasi-statistical convergence of a sequence under some assumptions.

1. Introduction

The convergence of sequences has many generalizations with the aim of providing deeper insights into summability theory. One of the most important generalizations is statistical convergence [1], [2], [12], [14]. It is quite effective especially when the classical limit does not exist since it is broader than ordinary convergence. Therefore concept of convergence has been studied by many authors [6], [7], [8], [9], [13]. It has also been used in number theory, trigonometric series and approximation theory [15], [16]. In [10] Ganichev and Kadets have introduced the concept of quasi-statistical filter. Then by using the filter Özgüç and Yurdakadim have defined the quasi-statistical convergence and have studied the relationship between statistical convergence and quasi-statistical convergence in [11]. The statistical analogues of limit points, limit superior, limit inferior and core of a sequence have been obtained by Fridy and Orhan [3], [4], [5].

In this study we introduce the concepts of quasi-statistical limit point and quasi-statistical cluster point of a sequence. We give some inclusion results concerning these concepts. We also give the relationship between the Knopp core and quasi-statistical core of a sequence. Finally we state some theorems which deal with quasi-summability and quasi-statistical convergence of a sequence under some assumptions.

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Now let us recall the basic notations and definitions which we need throughout the paper.

If K is a set of positive integers, $|K|$ will denote the cardinality of K. The natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : \ k \in K\}\right|,$$

if it exists.

The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $K_\varepsilon = \{k \in \mathbb{N} : \ |x_k - L| \ge \varepsilon\}$ has natural density zero. In this case we write $st - \lim x = L$.

Throughout the paper we assume that $c := (c_n)$ is a sequence of positive real numbers such that

$$\lim_n c_n = \infty \text{ and } \limsup_n \frac{c_n}{n} < \infty. \tag{1}$$

We define the quasi-density of $E \subset \mathbb{N}$ corresponding to the sequence (c_n) by

$$\delta_c(E) := \lim_n \frac{1}{c_n} \left|\{k \le n : \ k \in E\}\right|$$

if it exists.

The sequence $x = (x_k)$ is called quasi-statistically convergent to L provided that for every $\varepsilon > 0$ the set $E_\varepsilon = \{k \in \mathbb{N} : \ |x_k - L| \ge \varepsilon\}$ has quasi-density zero. In this case we write $st_q - \lim x = L$ or $x_k \to L \ (st_q)$.

The next result has been obtained in [11] to present the relationship between quasi-statistical convergence and statistical convergence.

Lemma 1. *If $x = (x_k)$ is quasi-statistically convergent to L then it is statistically convergent to L.*

An example has been given in order to show that the converse of Lemma 1 does not hold (see [11]).

The following result has also been given to relate the statistical convergence and quasi-statistical convergence.

Under the assumptions (1) and

$$d := \inf_n \frac{c_n}{n} > 0 \tag{2}$$

we immediately obtain that

"$x = (x_k)$ is statistically convergent to L if and only if x is quasi-statistically convergent to L."

By S_q, we denote the set of all quasi-statistically convergent sequences.

It is easy to see that every convergent sequence is quasi-statistically convergent, i.e., $c \subset S_q$ where c is the set of all convergent sequences.

If x is a sequence we write $\{x_k : k \in \mathbb{N}\}$ to denote the range of x. If $\{x_{k(j)} : j \in \mathbb{N}\}$ is a subsequence of x and $K = \{k(j) : j \in \mathbb{N}\}$, then we abbreviate $\{x_{k(j)}\}$ by $\{x\}_K$.

In case $\delta_c(K) = 0$, $\{x\}_K$ is called a subsequence of quasi-density zero or a thin subsequence. On the other hand $\{x\}_K$ is called a nonthin subsequence of x if K does not have quasi density zero. Note that $\{x\}_K$ is a nonthin subsequence if either $\delta_c(K)$ is a positive number or does not exist.

The number L is an ordinary limit point of a sequence x if there is a subsequence of x that converges to L.

Definition 2. *The number λ is a quasi-statistical limit point of the sequence x if there is a nonthin subsequence which converges to λ.*

Note that we will denote by Λ_x^c, L_x, the set of quasi-statistical limit points of x, and the set of ordinary limit points of x, respectively. It is clear that $\Lambda_x^c \subseteq L_x$ for any sequence x.

Proposition 3. *$\Lambda_x \subseteq \Lambda_x^c$ holds where Λ_x denotes the set of statistical limit points of x.*

Proof. Let $\lambda \in \Lambda_x$. Then there exists a subset M such that $\delta(M) \neq 0$ and $\{x\}_M$ converges to λ. One can write that

$$\frac{1}{n}\left|\{k \leq n : k \in M\}\right| \leq H\frac{1}{c_n}\left|\{k \leq n : k \in M\}\right|$$

where $H := \sup\limits_n \frac{c_n}{n}$ and it follows that $\delta_c(M) \neq 0$. This completes the proof. □

It is known that under the conditions (1) and (2) quasi-statistical convergence coincides with statistical convergence. If we assume that the sequence $c = (c_n)$ satisfies the conditions (1) and (2), we have $\Lambda_x^c = \Lambda_x$.

Example 4. *Define x by*

$$x_k = \begin{cases} r_n & ; \quad k = n^2, n = 0, 1, 2, .. \\ k & ; \quad \textit{otherwise} \end{cases}$$

where $\{r_n\}_{n=1}^{\infty}$ is a sequence whose range is the set of all rational numbers. It is known that $\Lambda_x = \varnothing$, $L_x = \mathbb{R}$ (see Example 2 of [3]*). Since $\Lambda_x^c \subseteq \Lambda_x$, we get that $\Lambda_x^c = \varnothing$.*

Definition 5. *The number γ is called a quasi-statistical cluster point of the sequence x if the set $\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$ does not have quasi-density zero for every $\varepsilon > 0$.*

We will denote the set of all quasi-statistical cluster points of x by Γ_x^c. It is clear that $\Gamma_x^c \subseteq L_x$ for every sequence x.

Proposition 6. *$\Gamma_x \subseteq \Gamma_x^c$ holds where Γ_x denotes the set of all statistical cluster points of x.*

Proof. Let $\gamma \in \Gamma_x$. Then $\delta(M := \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) \neq 0$ for every $\varepsilon > 0$. One can write that

$$0 \neq \delta(\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) = \frac{1}{n} |\{k \leq n : k \in M\}| \leq H \frac{1}{c_n} |\{k \leq n : k \in M\}|$$

and it follows that $\delta_c(M) \neq 0$. This completes the proof. □

Under the conditions (1) and (2), $\Gamma_x^c = \Gamma_x$ holds.

Following result presents the inclusion relationship between Γ_x^c and Λ_x^c.

Theorem 7. *For every sequence x, $\Lambda_x^c \subseteq \Gamma_x^c$ holds.*

Proof. Let $\gamma \in \Lambda_x^c$. Then $\lim\limits_j x_{k(j)} = \gamma$ and $\limsup\limits_n \frac{1}{c_n} |\{k(j) \leq n\}| = r > 0$ hold. Also the set $\{j : |x_{k(j)} - \gamma| \geq \varepsilon\}$ is finite for every $\varepsilon > 0$ so

$$\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\} \supseteq \{k(j) : j \in \mathbb{N}\} - \text{finite set}.$$

Therefore

$$\frac{1}{c_n} |\{k \leq n : |x_k - \gamma| < \varepsilon\}| \geq \frac{1}{c_n} |\{k(j) : j \in \mathbb{N}\}| - \frac{1}{c_n} O(1) \geq \frac{r}{2}$$

for infinitely many n. Hence $\delta_c(\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) \neq 0$ for every $\varepsilon > 0$ which completes the proof. □

It is known that Λ_x does not need to be closed but Γ_x and L_x are closed sets. In a similar proof to given by Fridy [3], one can also show the following.

Proposition 8. *For any sequence x, the set Γ_x^c is closed.*

Theorem 9. *If $\delta_c(\{k : x_k \neq y_k\}) = 0$ then $\Lambda_x^c = \Lambda_y^c$ and $\Gamma_x^c = \Gamma_y^c$.*

Proof. Assume that $\delta_c(\{k : x_k \neq y_k\}) = 0$ and let $\lambda \in \Lambda_x^c$, the nonthin sequence $\{x\}_K$ converges to λ. Note that $\delta_c(\{k : x_k = y_k\}) \neq 0$. Therefore the latter set yields a nonthin subsequence $\{y\}_{K'}$ of $\{y\}_K$ which converges to λ. Hence $\Lambda_x^c \subseteq \Lambda_y^c$. By symmetry one can also get $\Lambda_x^c \supseteq \Lambda_y^c$. The second assertion can be proved in a similar way. □

The following theorem is easy to prove by using the same technique in Theorem 2 of [3]. Therefore we omit it.

Theorem 10. *If x is a number sequence then there exists a sequence y such that $L_y = \Gamma_x^c$ and $\delta_c(\{k : x_k \neq y_k\}) = 0$. Moreover, the range of y is a subset of the range of x.*

Another noteworthy and useful result concerning the quasi-statistical cluster points is as follows.

Theorem 11. *If x is a number sequence that has a bounded nonthin subsequence, then x has a quasi-statistical cluster point.*

Proof. For such x, the above theorem ensures that there exists a sequence y such that $L_y = \Gamma_x^c$ and $\delta_c(\{k : x_k \neq y_k\}) = 0$. Then y must have a bounded nonthin subsequence, so by the Bolzano-Weierstrass Theorem $L_y \neq \varnothing$ which implies $\Gamma_x^c \neq \varnothing$. □

Now we immediately get the following corollary.

Corollary 12. *If x is a bounded sequence, then x has a quasi-statistical cluster point.*

Theorem 13. *If x is a bounded sequence then it has a thin subsequence $\{x\}_K$ such that $\{x_k : k \in \mathbb{N} - K\} \cup \Gamma_x^c$ is a compact set.*

Proof. Again by the above results one can choose a bounded sequence y such that $L_y = \Gamma_x^c$, $\{y_k : k \in \mathbb{N}\} \subseteq \{x_k : k \in \mathbb{N}\}$, and $\delta_c(K) = 0$ where $K := \{k \in \mathbb{N} : x_k \neq y_k\}$. This implies

$$\{x_k : k \in \mathbb{N} - K\} \cup \Gamma_x^c = \{y_k : k \in \mathbb{N}\} \cup L_y$$

and the right-hand set is compact. □

2. Quasi-Statistical Limit Superior and Inferior

The aim of this section is to present quasi-statistical limit superior and inferior to obtain some quasi-statistical analogues of ordinary limit superior, inferior and statistical limit superior, inferior as in [4]. We also introduce quasi-statistical core of a sequence and prove some results.

Definition 14. *If x is a real number sequence then the quasi-statistical limit superior and inferior of x are defined by*

$$st_q - \limsup x = \begin{cases} \sup B_x^c & , \quad \text{if } B_x^c \neq \emptyset \\ -\infty & , \quad \text{if } B_x^c = \emptyset \end{cases},$$

$$st_q - \liminf x = \begin{cases} \inf A_x^c & , \quad \text{if } A_x^c \neq \emptyset \\ +\infty & , \quad \text{if } A_x^c = \emptyset \end{cases}$$

where $B_x^c = \{b \in \mathbb{R} : \delta_c(\{k : x_k > b\}) \neq 0\}$, $A_x^c = \{a \in \mathbb{R} : \delta_c(\{k : x_k < a\}) \neq 0\}$. We now give a simple example to understand the concepts just defined.

Example 15. *Let $c := (c_n)$ be the sequence of positive real numbers such that $\lim_n c_n = \infty$, and $\lim_n \frac{\sqrt{n}}{c_n} = \infty$. We can choose a subsequence $\{c_{n_p}\}$ such that $c_{n_p} > 1$ for each $p \in \mathbb{N}$.*

Consider the sequence $x = (x_k)$ defined by

$$x_k := \begin{cases} c_k & , \quad k \text{ is square and } c_k \in \{c_{n_p} : p \in \mathbb{N}\} \\ 2 & , \quad k \text{ is square and } c_k \notin \{c_{n_p} : p \in \mathbb{N}\} \\ 1 & , \quad k \text{ is odd and } k \text{ is not square} \\ 0 & , \quad k \text{ is even and } k \text{ is not square} \end{cases}.$$

One can easily see that $st_q - \limsup x = 1$, $st_q - \liminf x = 0$.

Definition 16. *The real number sequence x is said to be quasi-statistically bounded if there is a number M such that $\delta_c(\{k \in \mathbb{N} : |x_k| > M\}) = 0$.*

The sequence x in the above example is not quasi-statistically convergent but quasi-statistically bounded. Also note that quasi-statistical boundedness implies that $st_q - \limsup$, $st_q - \liminf$ are finite and if the sequence is quasi-statistically bounded then $st_q - \limsup x$ is the greatest element of the set of quasi-statistical cluster points and $st_q - \liminf x$ is the least element of this set.

Now we investigate the relationship between $st_q - \limsup x$ and $st - \limsup x$ and also the relationship between $st_q - \liminf x$ and $st - \liminf x$.

Remark 17. *Let $H < \infty$. Then*

$$st_q - \liminf x \leq st - \liminf x \leq \ st - \limsup x \leq st_q - \limsup x$$

holds for any real sequence.

Proof. Let $\alpha_2 = st_q - \liminf x$ and $\alpha_1 = st - \liminf x$. Then $\delta(\{k : x_k < \alpha_1 + \varepsilon\}) \neq 0$ and $\delta(\{k : x_k < \alpha_1 - \varepsilon\}) = 0$ holds for every $\varepsilon > 0$. Since $H < \infty$, we have that $\delta(\{k : x_k < \alpha_1 + \varepsilon\}) \leq \delta_c(\{k : x_k < \alpha_1 + \varepsilon\})$ and this implies $\delta_c(\{k : x_k < \alpha_1 + \varepsilon\}) \neq 0$. Then

$$\alpha_1 + \varepsilon \in A_x^c = \{a \in \mathbb{R} : \delta_c(\{k : x_k < a\}) \neq 0\}$$

and it is known that $\inf A_x^c = \alpha_2$ which implies $\alpha_2 \leq \alpha_1 + \varepsilon$ for every $\varepsilon > 0$. ε is arbitrary and we obtain that $\alpha_1 = st - \liminf x \geq \alpha_2 = st_q - \liminf x$. Now let $\beta_2 = st_q - \limsup x$, $\beta_1 = st - \limsup x$. Then $\delta(\{k : x_k > \beta_1 - \varepsilon\}) \neq 0$ and $\delta(\{k : x_k > \beta_1 + \varepsilon\}) = 0$ holds for every $\varepsilon > 0$. Since $H < \infty$ we have that $\delta(\{k : x_k > \beta_1 - \varepsilon\}) \leq \delta_c(\{k : x_k > \beta_1 - \varepsilon\})$ and this implies $\delta_c(\{k : x_k > \beta_1 - \varepsilon\}) \neq 0$. Then

$$\beta_1 - \varepsilon \in B_x^c = \{b \in \mathbb{R} : \delta(\{k : x_k > b\}) \neq 0\}$$

and it is known that $\sup B_x^c = \beta_2$ which implies $\beta_1 - \varepsilon \leq \beta_2$ for every $\varepsilon > 0$. ε is arbitrary and we obtain that $\beta_1 = st - \limsup x \leq \beta_2 = st_q - \limsup x$ which completes the proof. □

Knopp has introduced the concept of the core of a sequence and has proved the well known core theorem. In order to produce natural analogues of Knopp core and statistical core of a sequence, we can replace limit points and statistical cluster points with quasi-statistical cluster points in [4], [5].

Definition 18. *If x is a quasi-statistically bounded real sequence then the quasi-statistical core of x is the closed interval $st_q{-}core\,\{x\} = [st_q - \liminf x, st_q - \limsup x]$. In case x is not quasi-statistically bounded, $st_q - core\,\{x\}$ is defined accordingly as either $[st_q - \liminf x, \infty)$, $(-\infty, \infty)$ or $(-\infty, st_q - \limsup x]$.*

One can easily see from Remark 1 that

$$st - core\{x\} \subseteq st_q - core\,\{x\} \subseteq K - core\{x\}.$$

Recall that the sequence $x = (x_k)$ is said to be strongly quasi-summable to L if

$$\lim_n \frac{1}{c_n} \sum_{k=1}^{n} |x_k - L| = 0.$$

The space of all strongly quasi-summable sequences is denoted by N_q.

$$N_q := \left\{ x : \text{for some } L,\ \lim_n \frac{1}{c_n} \sum_{k=1}^{n} |x_k - L| = 0 \right\}.$$

Also the sequence $x = (x_k)$ is said to be quasi-summable to L if

$$\lim_n \frac{1}{c_n} \sum_{k=1}^{n} x_k = L.$$

Now we can give a result concerning with the quasi-summability and quasi-statistical limit superior.

Theorem 19. *Let the sequence x is bounded above, $\ell = \delta_c(\mathbb{N})$ and $\beta l = st_q - \limsup x$. If the sequence x is quasi-summable to $\beta\ell^2$ then x is quasi-statistically convergent to $\beta\ell$.*

Proof. Suppose that x is not quasi-statistically convergent to $\beta\ell$. Then $st_q - \liminf x < \beta l$, therefore there is a number $\mu < \beta l$ such that $\delta_c(\{k : x_k < \mu\}) \neq 0$. Let $K^{'} := \{k : x_k < \mu\}$. Then by the definition of quasi-statistical limit superior, $\delta_c(\{k : x_k > \beta l + \varepsilon\}) = 0$, for every $\varepsilon > 0$. Now define

$$K^{''} := \{k : \mu \leq x_k \leq \beta l + \varepsilon\},\ K^{'''} := \{k : x_k > \beta l + \varepsilon\},\ B := \sup x < \infty.$$

Since $\delta_c(K^{'}) \neq 0$, there are infinitely many n such that

$$\frac{1}{c_n} \left| K_n^{'} \right| \geq r > 0,$$

for each such n we have

$$\begin{aligned}
\frac{1}{c_n} \sum_{k=1}^{n} x_k &= \frac{1}{c_n} \sum_{k=1,\ k \in K_n^{'}}^{n} x_k + \frac{1}{c_n} \sum_{k=1, k \in K_n^{''}}^{n} x_k + \frac{1}{c_n} \sum_{k=1, k \in K_n^{'''}}^{n} x_k \\
&< \frac{\mu}{c_n} \left| K_n^{'} \right| + \frac{\beta\ell + \varepsilon}{c_n} \left| K_n^{''} \right| + \frac{B}{c_n} \left| K_n^{'''} \right| \\
&= \frac{\mu}{c_n} \left| K_n^{'} \right| + (\beta\ell + \varepsilon)(\ell - \frac{\left| K_n^{'} \right|}{c_n}) + o(1) \\
&\leq (\mu - \beta\ell) \frac{\left| K_n^{'} \right|}{c_n} + \beta\ell^2 + \varepsilon(\ell - \frac{\left| K_n^{'} \right|}{c_n}) \\
&\leq \beta\ell^2 - (\beta\ell - \mu) \frac{\left| K_n^{'} \right|}{c_n} + \varepsilon(\ell - r) + o(1)
\end{aligned}$$

$$\leq \beta\ell^2 - (\beta\ell - \mu)r + \varepsilon(\ell - r) + o(1)$$

Since ε is arbitrary it follows that $\liminf \frac{1}{c_n}\sum_{k=1}^{n} x_k < \beta\ell^2$. Hence x is not quasi-summable to $\beta\ell^2$ and this completes the proof. □

This theorem is a generalization of Theorem 5 in [3]. Using the symmetry, we also have the following for lower bounds.

Corollary 20. *Let the sequence x is bounded below, $\ell = \delta_c(\mathbb{N})$ and $\alpha l = st_q - \liminf x$. If the sequence x is quasi-summable to $\alpha\ell^2$ then x is quasi-statistically convergent to $\alpha\ell$.*

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META FUZZY INDEX FUNCTIONS

NIHAT TAK

Abstract. Meta-analysis was introduced to aggregate the findings of different primary studies in statistical aspects. However, in the proposed study, the term "meta" is used to aggregate different models for a specific topic with the help of fuzzy c-means clustering method. One of the motivations of the proposed method is based on the concept of indices. In the literature, there are numerous proposed indices under different conditions for a specific purpose. Our assumption is that each index has some information for a given dataset. Therefore, meta fuzzy index functions, which include each index in each function with a certain degree of membership value, are introduced in the proposed method. Currency crisis and process capability indices are chosen as applications in order to show that the proposed method can be useful tool in terms of indices.

1. Introduction

Meta-analysis was defined as a method of combining the results of multiple independent studies based on statistical methods on a given subject by Glass [16] in 1976. His aim was to aggregate 375 different psychotherapy outcome studies to reduce the confusion among different outcomes. Besides, DerSimonian and Laird [12] defined in their paper that meta-analysis is a collection of analytic results for integrating the findings to get more reliable results. Aforementioned advantages, the studies based on meta-analysis have become more popular in the last few decades. However, rather than aggregating different studies outcomes, the outcomes of different selected methods for a given dataset are aggregated in the study. Because there are numerous proposed indexes for a purpose, indexes are the focus of the paper.

Indexes represent the proportional changes of a simple or compound event in time or space. The expression of changes in percentage rather than absolute figures is preferred in terms of interpretation and understanding of events. In other

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words, a function of numerous indicators is defined as an index in statistics. Indices widely used for judging the pulse economy. Although they used to measure the effect of changes in prices in the beginning, today we use indices for industrial production, cost of living, agricultural production, currency crisis, process capability, etc. There are multiple proposed methods in literature for each title given above. Another motivation of Meta Fuzzy Index Functions (MFIFs) is the assumption that each method for an index has some information for a given dataset. Thus, our aim is to aggregate these methods in functions with the help of fuzzy c-means (FCM) clustering algorithm and call it MFIFs. Process Capability Indices (PCI) and Currency Crisis Indices (CCI) are selected as applications.

The PCI, which is defined as a statistical measure of process capability, tends to determine whether a production process is able to produce items within specification tolerance. There are several conditions, some of which are assumed to be normally distributed and to have a large sample size. Some of well-known PCIs are Cp, Cpk, and Cpm, which are introduced by [26, 19, 7]. For these PCIs, it is assumed that measured characteristic is normally distributed, and it is not, very often, possible to maintain the mean of the process on the center of the tolerance interval. Some of the well-known PCIs will be discussed in detail in Section 2.1.

A Currency crisis is defined as the crisis that consists of reduction in the reserves of an economy that uses fixed exchanged regime. The reduction in the reserves occurs when market actors change their national assets to international assets at the instant and, in parallel, as the consequence of the action of Central Bank's reserves. To forecast the crisis there are numerous currency crisis indices in literature, some of which are introduced by Kaminsky et al. [18] , Corsetti et al. [11] etc. Kaminsky's [18] index is composed of changes in nominal exchange rate and changes in foreign reserves while Corsetti's [11] index is composed of changes in nominal exchange rate and changes in international reserves. In short, there are different point of views for currency crisis' indicators for different researchers. Thus, the currency crisis indices are suitable for our purpose in the paper. The currency crisis indices will be given in detail in Section 2.2.

The assumption of the proposed method is that each index has some information about the process or the situation. Thus, we tried to aggregate indices in functions with the help of FCM algorithm. The first step of the MFIFs is the clustering the indices using FCM. Degree of membership values obtained from FCM for each cluster give coefficients of the indices in each function. Finally, the function that explains the process or the situation best is chosen as the meta-fuzzy index function. The MFIFs will be discussed in detail in Section 4.

2. Preliminaries

2.1. **Process Capability Indices (PCI).** Process capability analysis is conducted to examine whether the products produced during the production process have the desired tolerances or not. In other words, it is the measure whether a process is capable of producing an item within specification limits. There are numerous PCIs in literature to measure the process potential and performance. At the beginning, it is assumed that the dataset of a process is normally distributed, and its tolerance limits are symmetric. Later, researchers proposed different PCIs under different conditions, such as processes with symmetric tolerance limits and normally distributed, asymmetric tolerance limits and normally distributed, and symmetric tolerance limits and asymmetric distribution. Some of the PCIs proposed are given below. The concerned univariate measurements will be denoted with the corresponding random variable by X. The expectation and standard deviation of X will be denoted by μ and σ respectively. When it is assumed that the measured characteristics of a process is approximately normally distributed and its tolerance limits are symmetric, following PCIs are commonly used by manufacturers. The C_p index introduced by Sullivan [26] is given in Equation 1,

$$PCI_1 = C_p = \frac{USL - ASL}{6\sigma} \tag{1}$$

and used when $\mu = M$ where $M = (USL + LSL)/6\sigma$. USL and LSL stand for upper specification and lower specification limits respectively. It is not always possible to maintain the process on the center of tolerance interval $[LSL, USL]$. In this case, C_{pk} and C_{pm} indices might be useful and given in Equation 2,3,

$$PCI_2 = C_{pk} = \frac{min(USL - \mu, \mu - LSL)}{3\sigma} \tag{2}$$

and

$$PCI_3 = C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - \tau)^2}} \tag{3}$$

When it is assumed that the measured characteristics of a process is approximately normally distributed and its tolerance limits are asymmetric, following PCIs are used by researchers. Some of the studies conducted for asymmetric tolerance and normal distributed of a process was proposed by Kane [19] , Boyles [5], Pearn and Chen [23], Chan et al. [7], and Chen et al. [8].

$$PCI_4 = C^{\star}_{pm} = \frac{d'}{3\sqrt{\sigma^2 + (\mu - T)}} \tag{4}$$

$$d' = \frac{(USL - T) + (T - LSL)}{2} \tag{5}$$

where T is the target value.

$$PCI_5 = C'_{pm} = \frac{d^{\star}}{3\sqrt{\sigma^2 + (A^{\star})^2}} \tag{6}$$

$$A^{\star} = max\left\{\frac{d^{\star}(\mu - T))}{D_u}, \frac{d^{\star}(T - \mu))}{D_l}\right\}, \tag{7}$$

$$D_u = USL - T, D_l = T - LSL \tag{8}$$

When the measured characteristics of a process has an asymmetric distribution and symmetric tolerance limits, some of the PCAs are introduced by Clements [9], Pearn and Kotz [25], Pearn and Chen [24], and Wright [28].

$$PCI_6 = \widehat{C_p} = \frac{USL - LSL}{U_p - U_l} \tag{9}$$

$$PCI_7 = \widehat{C}_{pk} = min\left\{\frac{USL - m}{m - L_p}, \frac{m - LSL}{U_p - m}\right\} \tag{10}$$

$$PCI_8 = \widehat{C}_{pmk} = min\left\{\frac{USL - m}{3\sqrt{[\frac{U_p - m}{3}]^2 + (m-T)^2}}, \frac{m - LSL}{3\sqrt{[\frac{m - L_p}{3}]^2 + (m-T)^2}}\right\} \tag{11}$$

$$U_p = \%99.865, L_p = \%0.135 \tag{12}$$

$$PCI_9 = \hat{C}'_{pk} = min\left\{\frac{USL - m}{[U_p - L_p]/2}, \frac{m - LSL}{[U_p - L_p]/2}\right\} \tag{13}$$

$$PCI_{10} = \hat{C}'_{pm} = \frac{USL - LSL}{6\sqrt{\left[\frac{U_p - L_p}{6}\right]^2 + (m-T)^2}} \tag{14}$$

$$PCI_{11} = \hat{C}'_{pmk} = min\left\{\frac{USL - m}{3\sqrt{\left[\frac{U_p - L_p}{6}\right]^2 + (m-T)^2}}, \frac{m - LSL}{3\sqrt{\left[\frac{U_p - L_p}{6}\right]^2 + (m-T)^2}}\right\} \tag{15}$$

$$PCI_{12} = C_s = \frac{(d - |\mu - T|)/\sigma}{3\sqrt{1 + [(\mu - T)/\sigma]^2 + |k_1|}} \tag{16}$$

Large value implies a better process for each index given above. The PCAs which are used in the application are given above.

2.2. **Currency Crisis Indices (CCI).** Although there is no consensus on defining currency crisis, there is more or less consensus on the indicators of a currency crisis. Usually, exchange rate, interest rate, and international reserves are considered as the indicators of the currency crisis. The definitions of CCIs are obtained using three or different pairwise combination of these indicators. The CCIs which will be used in the study are proposed by Eichengreen [15], Kaminsky [18], Corsetti et al. [10], Krkoska [21], Von Hagen and Ho [29], Bussierre [4], Yiu et al. [31], Alvarez-Plata and Schrooten [1], Bunda and Co-Zorri [6], and Johansen [17]. The definitions of CCIs given by the researchers introduced above are given in Equation 17-26 and cited from Ari and Cergibozan [2].

$$CCI1_t = \frac{1}{\sigma_{RER}}\Delta NER_{TR,t} - \frac{1}{\sigma_{RES}}\Delta RES_{TR,t} + \frac{1}{\sigma_{NIR}}\Delta(NIR_{TR,t} - NIR_{US,t}) \tag{17}$$

where NER is the nominal exchange rate, RES is international reserves, NIR is the nominal interest rate.

$$CCI2_t = ln\left(\frac{NER_t}{NER_{t-3}}\right) - \frac{\sigma^2_{NER}}{\sigma^2_{RES}} ln\left(\frac{RES_t}{RES_{t-3}}\right) \tag{18}$$

$$CCI3_t = \frac{1}{\sigma^2_{RER}}\Delta RER_t - \frac{1}{\sigma^2_{RES}}\Delta RES_t + \frac{1}{\sigma^2_{RIR}}(RIR_t - RIR_{t-1}) \tag{19}$$

where RER is the reel exchange rate, RIR is the reel exchange rate, σ^2_{RER}, σ^2_{RES}, and σ^2_{RIR} the standard deviations of the reel exchange rate, reserves and the reel interest rate respectively.

$$\begin{aligned} CCI4_t \quad = \quad & \frac{1}{\sigma_{NER}}\Delta NER_{TR_j} - \frac{1}{\sigma_{RES}}(\Delta RES_{TR_j} - \Delta RES_{US_j}) \\ & + \frac{1}{\sigma_{NIR}}(\Delta NIR_{TR_j} - \Delta NIR_{US_j}) \end{aligned} \tag{20}$$

where NIR_{TR} and NIR_{US} are the nominal interest rates, RES_{TR} and RES_{US} are the international reserves excluding gold, σ_{NER} is the standard deviation of nominal exchange rate, σ_{RES} is standard deviation of the change in the international reserves gap between TR and the US, and σ_{NIR} indicates the standard deviation of the difference between the nominal interest rate of TR and the US.

$$CCI5_t = \frac{1}{\sigma_{RER}}\Delta RER_t - \frac{1}{\sigma_{RES}}\Delta RES_t + \frac{1}{\sigma_{NIR}}(NIR_t - NIR_{t-1}) \tag{21}$$

$$CCI6_t = \triangle RER_t \tag{22}$$

$$CCI7_t = \Delta NER_t - \frac{\sigma_{NER}}{\sigma_{RES}}\Delta RES_t \tag{23}$$

$$CCI8_t = \Delta NER_t - \Delta RES_t + (NIR_t - NIR_{t-1}) \tag{24}$$

$$CCI9_t = \Delta NER_t - \frac{\sigma_{NER}}{\sigma_{RES}}\Delta RES_t + \frac{\sigma_{NER}}{\sigma_{NIR}}(NIR_t - NIR_{t-1}) \tag{25}$$

$$CCI10_t = 0.75\Delta NER_t - 0.25\Delta RES_t \tag{26}$$

2.3. **Fuzzy C-means (FCM).** Fuzzy set theory was introduced by Zadeh [32] in 1965. Fuzzy logic has been commonly studied topic by then. Fuzzy sets were used in many fields, engineering, health science, economics, statistics, etc. Fuzzy logic lets an object to become a member of different classes or clusters with a degree of membership value. In this sense, fuzzy c-means clustering algorithm is developed by Dunn [13] in 1973 and improved by Bezdek [3] in 1981. FCM is also very useful tool in many fields. The detailed steps of FCM algorithm are given in Step 3 in Section 3.

3. Proposed Method

"Which index should we use for a given time series", "What are the characteristics of the dataset" lead us to come up with MFIFs. Thus, the aim of MFIFs is to propose a method that gives a way to aggregate the information of indices for a specific topic in functions by using FCM and get better outcomes. Thus, the inputs are MFIFs are the outputs of different indexes for a given dataset. Clustering the inputs by using FCM, the functions are obtained. In this case, a cluster represents a function in MFIFs. Thus, there are as many functions as the number of clusters.

Although there are many fuzzy clustering techniques like[20, 22], FCM[3] is used in the proposed method because of its easy-to-use structure and fame. The detailed steps of the proposed method are given below step by step.

- Step 1. Each index is calculated for the given dataset. When the scale of each index values is not the same, input matrix is standardized. Thus, the input matrix consists of the output of the indices.
- Step 2. Let c be the number of fuzzy sets (number of functions).
- Step 3. Using FCM algorithm, the degrees of membership values for each observation (indices) are calculated. In other words, the coefficients of indices in a function will be determined with the help of FCM.

  FCM Algorithm
  - Step 3.1. Initialize $\mu = \left[\mu_{ij}\right]$ matrix, determine the number of clusters and initial cluster centers.
  - Step 3.2. Calculate the membership value μ with the formula given in Equation 27.

$$\mu_{ik} = \left[\sum_{j=1}^{c}\left(\frac{d(z_k, v_i)}{d(z_k, v_j)}\right)^{\frac{2}{f_i-1}}\right]^{-1}, i = 1, 2, .., c; k = 1, 2, .., n \tag{27}$$

    under the constraint; $\sum_{i=1}^{c}\mu_{ik} = 1, if \mu_{ik} < \alpha - cut$, then μ_{ik} value will be taken as zero. where Z is the input matrix, v is the cluster centers, $d(.)$ stands for Euclidean distance function, c is the number of clusters, and f_i is the fuzzy index value.
  - Step 3.3. Calculate the new cluster centers.

$$v_i = \frac{\sum_{k=1}^{n}\mu_{ik}^{f_i} z_k}{\sum_{k=1}^{n}\mu_{ik}^{f_i}} \tag{28}$$

  - Step 3.4. Repeat Step 2 and Step 3 until the difference of clusters between two iterations drops under some threshold or the number of iterations is reached.

- Step 4. Using the degrees of membership values, new vales of meta fuzzy index functions are calculated with the formula given in Equation 29.

$$MFIF_i = \frac{Z_i^T \mu_i}{\sum_{i=1}^{c} \mu_i}, i = 1, 2, ..c \tag{29}$$

where $MFIF$ stands for the index values of the proposed method and c stands for the number of clusters.

- Step 5. Step 4 is repeated for the number of cluster times.
- Step 6. The cluster that explains the purpose best is selected as the meta-fuzzy index function.

4. Applications

4.1. **Application of CCIs.** The first application of the proposed method is CCIs. CCIs are chosen because determining a currency crisis in a certain year is commonly studied topic by researchers and there are numerous proposed CCIs in literature. Each CCI has its own characteristic. However, our assumption is that each CCI has information about a dataset. Therefore, CCIs are suitable for MFIFs.

The data that were monthly observed were obtained from the Central Bank of Turkey. [27] The elements of the input matrix, which are observed from January of 1990 to June of 2014, are calculated using the given indices above. Summary of the input matrix is given in Table 1.

Table 1. Some observations of the input matrix

no	CCI1	CCI2	CCI3	CCI4	CCI5	CCI6	CCI7	CCI8	CCI9	CCI10	CCI11
1	-0.38	-0.08	-24.13	-0.65	-1.52	-0.12	-0.045	-0.09	-5.33	0.03	-0.06
2	-0.24	-0.04	-17.72	-0.54	-1.08	-0.01	-0.01	-0.03	-5.73	0.03	-0.02
3	0.16	0.04	6.62	1.73	0.56	0.03	0.08	0.09	-6.23	0.01	0.08
4	-0.08	0.06	-28.19	-0.17	-1.474	0.04	-0.01	-0.02	-21.13	0.02	-0.01
5	0.02	0.09	-1.19	0.6	-0.07	-0.02	0.02	0.03	-0.28	0.02	0.02
6	-0.28	-0.01	-14.01	-0.76	-1.07	-0.15	-0.06	-0.07	6.93	0.04	-0.06
.	.	.	.	.	.	.	.	.	.	.	.

Because the values of the indices are not in the same scale, the input matrix is standardized. The centers of the clusters are initialized randomly, and the number of clusters (functions) is taken as three. The $\alpha - cut$ value is taken as 0.1. The degree of membership matrix is given in Table 2.

Table 2 shows that which index belongs to which function with what degree of membership values. Using Equation 29, the following functions and graphs are obtained.

Using the first column of degree of memberships matrix and the CCIs, the first function is obtained as below.

Table 2. Degree of membership values of indices

Index	1	2	3
CCI1	0.12	0.02	0.86
CCI2	0.30	0.10	0.60
CCI3	0.01	0.00	0.99
CCI4	0.01	0.00	0.98
CCI5	0.00	0.00	1.00
CCI6	0.00	1.00	0.00
CCI7	0.08	0.01	0.90
CCI8	0.01	0.00	0.98
CCI9	0.96	0.01	0.04
CCI10	0.98	0.01	0.02
CCI11	0.00	0.00	1.00

$$MFIF_1 = (\mu_{11}CCI1 + \mu_{12}CCI2 + ... + \mu_{1n}CCIn)/\sum_{i=1}^{n}\mu_{1i} \tag{30}$$

$$MFIF_1 = (0.12*CC1+0.3*CCI2+...+0.98*CCI10)/(0.12+0.3+...+0.98) \tag{31}$$

Looking at $MFIF_1$, it is obvious that most contribution is made by $CCI9$ and $CCI10$, which means $CCI9$ and $CCI10$ reacts similar given the dataset. The graph of the first function is given in Figure 1. The time period of the currency crisis is determined if the points in a graph are 2 standard deviation away from the margin. In this case, the first function is capable of detecting three crisis that occurred in 1994, 2001, and 2008.

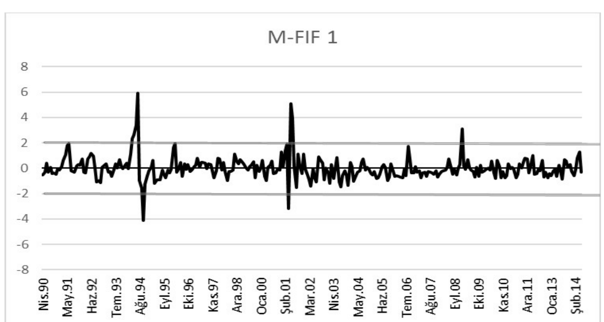

Figure 1. The first M-FIF for the crisis in Turkey

For the second function ($MFIF_2$), $CCI6$ dominates the other indices. The second $MFIF$ looks like as follows.

$$MFIF_2 = (\mu_{21}CCI1 + \mu_{22}CCI2 + ... + \mu_{2n}CCIn)/\sum_{i=1}^{n}\mu_{2i} \tag{32}$$

$$MFIF_2 = 1 * CCI6 \tag{33}$$

Figure 2 indicates 4 crises in Turkey, that occurred in 1994, 2001, 2006, and 2008.

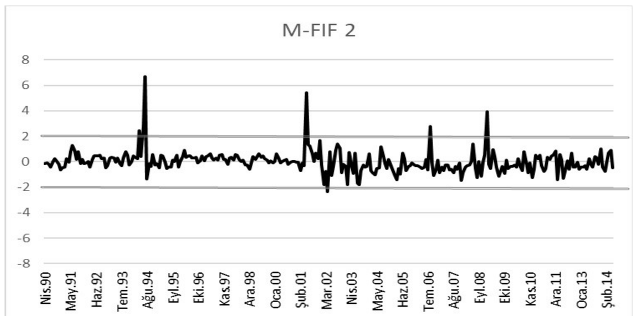

Figure 2. The second M-FIF for the crisis in Turkey

The third M-FIF contains the rest of the indices with higher degree of membership values.

$$MFIF_3 = (\mu_{31}CCI1 + \mu_{32}CCI2 + ... + \mu_{3n}CCIn)/\sum_{i=1}^{n}\mu_{3i} \tag{34}$$

$$MFIF_3 = (0.86 * CCI1 + 0.60 * CCI2 + ... + 1 * CCI11)/\sum_{i=1}^{n}\mu_{3i} \tag{35}$$

The third function indicates 2 crises that occurred in 1994 and 2001. The graph of M-FIF3 is given in Figure 3.

Implementing the proposed method to the currency crisis data set of Turkey, three different functions are obtained. The question arises as which function will be more useful for CCIs. In this case, the function which can detect more crisis is chosen as the M-FIF, which is the second function.

Overall, in the application of CCIs, we obtain three M-FIFs. Each M-FIF has different information about the crises in Turkey. The outcomes of MFIFs might be investigated by an economist in details. Our belief is that the M-FIFs is a useful tool in terms of CCIs.

FIGURE 3. The third M-FIF for the crisis in Turkey.

4.2. **Application of PCIs.** PCIs are chosen as an application for the proposed method because there are numerous PCIs given under different conditions in literature. Some of them are explained in Chapter 3. Our belief is that each PCI has some information about a dataset no matter what distribution it has. Therefore, we tried to aggregate these indices in different functions. While some of the functions have more information about the process, some others have less information. Because there will be as many index functions as the number of clusters, we are looking for the best function in which indexes explain the process best. In this case, the best function that gives the highest value is selected the M-FIF. For the application, elastomer bearing sliding valve shaft dataset is chosen. The data set is obtained from [30].

The dataset is non-normally distributed, and its tolerance limits are asymmetric. At the beginning of the application, the bootstrap sampling method is used to determine how the indices react to the dataset. The bootstrap method, which was introduced by Efron [14], is a resampling technique with replacement used to estimate statistics/indexes on a population. Because there is just one value that is obtained from the indexes, we used bootstrap method to be able to learn the characteristics of an index. Ten bootstrap samples are obtained from the dataset and the index values of each sample are given in Table 3.

10 bootstrap samples of 11 indices are clustered using FCM. The degree of membership values of each index are given in Table 4.

Using Equation 26, we obtain the results in Table 5. Table 5 shows the M-FIF values for each bootstrap sample.

Table 5 gives the M-FIFs. Looking at Table 4, it is obvious that only $PCI12$ belongs to the first function with 1 degree of membership value. The function looks

TABLE 3. Values of indices and bootstrap samples

Index	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10
PCI1	0.81	0.84	0.93	0.80	0.77	0.84	0.90	0.88	0.91	0.81
PCI2	0.69	0.73	0.81	0.73	0.71	0.77	0.78	0.77	0.83	0.71
PCI3	0.76	0.80	0.87	0.78	0.76	0.82	0.85	0.83	0.88	0.78
PCI4	0.64	0.66	0.73	0.62	0.59	0.65	0.71	0.70	0.70	0.64
PCI5	0.64	0.67	0.74	0.63	0.60	0.67	0.72	0.70	0.71	0.64
PCI6	1.10	1.04	1.15	1.01	1.05	1.13	1.06	1.04	1.10	1.02
PCI7	0.92	0.84	0.93	0.79	0.79	0.92	0.85	0.84	0.85	0.81
PCI8	0.91	0.84	0.93	0.77	0.76	0.92	0.85	0.84	0.84	0.80
PCI9	0.82	0.89	0.97	0.92	0.98	0.92	0.90	0.89	0.95	0.84
PCI10	1.10	1.04	1.14	0.99	1.02	1.13	1.05	1.04	1.10	1.02
PCI11	0.82	0.88	0.96	0.89	0.94	0.92	0.89	0.88	0.94	0.84
PCI12	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 4. Degree of membership values of indices

Index	1	2	3
PCI1	0.004	0.086	0.910
PCI2	0.003	0.024	0.973
PCI3	0.001	0.012	0.987
PCI4	0.034	0.105	0.861
PCI5	0.029	0.097	0.874
PCI6	0.001	0.984	0.015
PCI7	0.006	0.132	0.861
PCI8	0.005	0.103	0.891
PCI9	0.010	0.449	0.540
PCI10	0.001	0.988	0.011
PCI11	0.010	0.361	0.629
PCI12	1.000	0.000	0.000

like as below.

$$MFIF_1 = (\mu_{11}PCI1 + \mu_{12}PCI2 + ... + \mu_{1n}PCIn)/\sum_{i=1}^{n}\mu_{1i} \tag{36}$$

$$MFIF_1 = 1 * PCI12 \tag{37}$$

The second $M-FIF$ includes mainly two indices mainly but a few more with lower degree of membership values. $MFIF_2$ is given in Equation 38.

$$MFIF_2 = (\mu_{21}PCI1 + \mu_{22}PCI2 + ... + \mu_{2n}PCIn)/\sum_{i=1}^{n}\mu_{2i} \tag{38}$$

Table 5. Index values of M-FIFs

Samples	1	2	3
1	0.070	0.978	0.776
2	0.072	0.956	0.788
3	0.079	1.051	0.869
4	0.068	0.930	0.760
5	0.067	0.962	0.751
6	0.073	1.027	0.818
7	0.075	0.975	0.824
8	0.074	0.962	0.811
9	0.076	1.015	0.841
10	0.069	0.931	0.759

$$MFIF_2 = (0 * PCI1 + ... + 0.105 * PCI4 + ... + 0 * PCI12) / \sum_{i=1}^{12} \mu_{2i} \tag{39}$$

The third function includes the most of the indices more than 0.90 degree of membership values. The $MFIF_3$ looks like as below.

$$MFIF_3 = (\mu_{31} PCI1 + \mu_{32} PCI2 + ... + \mu_{3n} PCIn) / \sum_{i=1}^{n} \mu_{3i} \tag{40}$$

$$MFIF_2 = (0.91 * PCI1 + 0.973 * PCI2 + ... + 0 * PCI12) / \sum_{i=1}^{12} \mu_{3i} \tag{41}$$

Because we have three M-FIFs, the one that explains the process best is chosen as the M-FIF. Deciding which function will be MFIF is selected by looking at the mean of each cluster. The one which has the highest value is chosen as the M-FIF. In this case, the second function is chosen as the M-FIF for the process capability.

5. Conclusions

A new approach in terms of indices is proposed in the study. The need of the proposed method has arisen the question that which method we should prefer for a given dataset (i.e. normally distributed, linear, non-linear, semi-linear, etc.). Therefore, the methods that we might use for a given dataset are clustered with FCM algorithm. While the methods that perform better for a dataset are clustered in a function with higher degree of membership values, the methods that perform worse are clustered in a different function with higher degree of membership values. The advantages of the proposed method are discussed below.

- Because there are numerous methods in literature for a specific problem, there is also confusion which method will be chosen. With the help of the proposed method, the best method or methods are clustered in the same

function with certain degree of membership values. Thus, we are able to say that which method/methods are capable of dealing with a specific dataset.
- Because of the first advantage, we do not need to look for the assumptions of methods.
- MFIFs are able to aggregate the information of each index into functions.
- The M-FIFs approach is the first method in literature that aggregates the indexes into functions.

However, the major defect of the proposed method is the determination of the number and centers of the clusters. In the applications given above, the number of clusters is chosen as three and the centers are randomly initialized. Specifying the cluster centers with expert opinion might give better classification of the methods. Besides, increasing the number of clusters might give even better outcomes. This part of the proposed method is left for future work.

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OSCILLATION RESULTS FOR SECOND ORDER HALF-LINEAR FUNCTIONAL DYNAMIC EQUATIONS WITH UNBOUNDED NEUTRAL COEFFICIENTS ON TIME SCALES

ORHAN ÖZDEMIR

Abstract. This study aims to present some new sufficient conditions for the oscillatory behavior of solutions to a class of second order half-linear functional dynamic equations with mixed neutral term i.e., the neutral term contains both retarded and advanced arguments. The results obtained are applicable in the case where the studied equation has unbounded neutral coefficients and they are new even for the linear case. Illustrative examples are also provided.

1. Introduction

In this study, we are concerned with the oscillation of second order half-linear mixed neutral dynamic equations of the form

$$\Big(r(t)\big(y^{\Delta}(t)\big)^{\beta}\Big)^{\Delta}+\sum_{i=1}^{n}q_i(t)x^{\beta}\big(h_i(t)\big)=0,\quad t\in[t_0,\infty)_{\mathbb{T}} \tag{1.1}$$

where $n\geq 1$ is an integer, β is a ratio of positive odd integers, $\mathbb{T}$ is a time scale unbounded above with $t_0\in\mathbb{T}$, and

$$y(t):=x(t)+p_1(t)x\big(\tau_1(t)\big)+p_2(t)x\big(\tau_2(t)\big). \tag{1.2}$$

For some basic facts on time scale calculus and dynamic equations on time scales, one may consult the excellent texts by Bohner and Peterson [8, 9]. Throughout this study it is assumed that the reader is familiar with time scale calculus, and the following conditions are always satisfied:

(i) $q_i:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ are nonnegative rd-continuous functions such that not all of the $q_i(t)$ vanish in a neighborhood of infinity for $i=1,2,...,n$ and $r:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ is a positive rd-continuous function with

$$\int_{t_0}^{\infty}r^{-1/\beta}(s)\Delta s=\infty;$$

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(ii) $\tau_1, \tau_2, h_i : [t_0, \infty)_{\mathbb{T}} \to \mathbb{T}$ are rd-continuous functions such that τ_1 and τ_2 are strictly increasing, $\tau_1(t) < t$, $\tau_2(t) > t$ and $\lim_{t\to\infty} \tau_1(t) = \lim_{t\to\infty} h_i(t) = \infty$, for $i = 1, 2, ..., n$;

and, either

(iii$_\mathfrak{a}$) $p_1, p_2 : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ are rd-continuous functions with $p_1(t) \geq 0$, $p_2(t) \geq 1$ and $p_2(t) \not\equiv 1$ eventually;

or

(iii$_\mathfrak{b}$) $p_1, p_2 : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ are rd-continuous functions with $p_2(t) \geq 0$, $p_1(t) \geq 1$ and $p_1(t) \not\equiv 1$ eventually.

Wherever we write "$t \geq t_n$" we mean "$t \in [t_n, \infty)_{\mathbb{T}}$".

By a solution of equation (1.1) we mean a function $x \in C_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ which has the properties $y \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ and $r\left(y^{\Delta}\right)^{\beta} \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$, and satisfies equation (1.1) on $[T_x, \infty)_{\mathbb{T}}$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[T_x, \infty)_{\mathbb{T}}$, for some $T_x \geq t_0$, and

$$\sup\{|x(t)| : t \geq T_1\} > 0 \quad \text{for all } T_1 \geq T_x.$$

We make the standing hypothesis that (1.1) admits such solutions. Such a solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros on $[T_x, \infty)_{\mathbb{T}}$ and otherwise it is called *nonoscillatory*. Equation (1.1) is said to be *oscillatory* if all its solutions are *oscillatory*.

The study of oscillation of the solutions of neutral differential and difference equations presents a strong theoretical interest. One reason for this is that they arise in several areas of applied mathematics including circuit theory, bifurcation analysis, population dynamics, stability theory, the dynamics of delayed network systems and others. Besides, these equations are used in the analysis of computer networks containing lossless transmission lines, as in high speed networks where lossless transmission lines serve to connect switching circuits in the network. Also, second-order neutral delay differential equations are of great interest in biology in explaining self-balancing of the human body and in robotics in constructing biped robots [11, 13, 15, 25]. Interested readers can refer to the books by Hale [15] and, Kolmanovskii and Myshkis [22] for more applications in science and technology.

Note that equation (1.1) with $\mathbb{T} = \mathbb{R}$, $n = 2$, $h_1(t) < t$ and $h_2(t) > t$ were encountered in the study of vibrating masses attached to an elastic bar [15, 23]. Since it has some direct applications in science, the oscillatory behavior of equation (1.1) and it's special and more general forms have been studied by numerous authors utilizing different methods. In reviewing the related literature, most of such results are concerned with the cases where the functions $p_j(t)$ are constant or bounded functions, for $j = 1, 2$; see for example [6, 10, 12, 16, 19, 23, 28, 30, 36, 37] and the references cited therein. However, to the best of our knowledge, there does not appear to be any oscillation results for second order mixed neutral dynamic equations in the case where the neutral term includes unbounded neutral coefficients.

Motivated by the papers mentioned above, the purpose of this study is to establish some new oscillation criteria that can be applied in the case where equation (1.1) has unbounded neutral coefficients, that is, $p_1(t) \to \infty$ and/or $p_2(t) \to \infty$ as $t \to \infty$. Moreover, it should be also pointed out that the results obtained here are new even for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, for the linear case when $\beta = 1$, for $r(t) = 1$, for $n = 1$ and for discrete deviating arguments such as $\tau_1(t) = t - a$, $\tau_2(t) = t + b$ and $h_i(t) = t \mp c$ with $a, b, c > 0$ and $i = 1, 2, ..., n$.

For convenience, we will use the following notations:

$$\gamma = \frac{\beta+1}{\beta}, \qquad \theta_+(t) := \max\{0, \theta(t)\} \qquad \text{and} \quad A(t, t_1) := \int_{t_1}^{t} r^{-1/\beta}(s)\Delta s,$$

for any rd-continuous function θ.

The following lemma is required in our main results. Since the proof is standard we omit the details here.

Lemma 1.1. *Assume that conditions* (i)-(iii$_\mathfrak{a}$) (*or* (i), (ii), (iii$_\mathfrak{b}$)) *hold and* $x(t)$ *is an eventually positive solution of* (1.1). *Then* $y(t)$ *satisfies* $y(t) > 0$, $y^{\Delta}(t) > 0$ *and* $\left(r(t)\left(y^{\Delta}(t)\right)^{\beta}\right)^{\Delta} < 0$, *for all* t *large enough.*

Lemma 1.2. [17] *If* X *and* Y *are nonnegative and* $\lambda > 1$, *then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda - 1) Y^{\lambda},$$

where equality holds if and only if $X = Y$.

2. Oscillation results when (iii$_\mathfrak{a}$) holds

In this section, we establish some new criteria for the oscillation of (1.1) in the cases where $h_i(t) \le \tau_2(t)$ and $h_i(t) \ge \tau_2(t)$ for $i = 1, 2, ..., n$, respectively. For notational purposes, we let

$$\psi(t) := \frac{1}{p_2(\tau_2^{-1}(t))}\left[1 - \frac{1}{p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} - \frac{p_1(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}\right]$$

where τ_2^{-1} denotes the inverse function of τ_2, and throughout this section we assume that $\psi(t) > 0$ for all sufficiently large t.

Theorem 2.1. *Assume that conditions* (i)-(iii$_\mathfrak{a}$) *hold and* $h_i(t) \le \tau_2(t)$ *for* $i = 1, 2, ..., n$. *If there exists a positive function* $\eta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left(\Psi_1(s) - \frac{r(s)\left(\eta^{\Delta}(s)\right)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)}\right)\Delta s = \infty \tag{2.1}$$

for $T > t_1$ *where* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *sufficiently large and*

$$\Psi_1(t) = \eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^{\beta}\big(h_i(t)\big)\frac{A^{\beta}\big(\tau_2^{-1}(h_i(t)), t_1\big)}{A^{\beta}\big(t, t_1\big)}\right),$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ and $x(h_i(t)) > 0$ for $t \geq t_1$ and $i = 1, 2, ..., n$. The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. In view of Lemma 1.1, it is obvious that

$$y(t) > 0,\ y^{\Delta}(t) > 0 \text{ and } \left(r(t)\big(y^{\Delta}(t)\big)^{\beta}\right)^{\Delta} < 0 \quad \text{for } t \geq t_1.$$

Since $r(t)\big(y^{\Delta}(t)\big)^{\beta}$ is decreasing, we have

$$\begin{aligned} y(t) &= y(t_1) + \int_{t_1}^{t} \frac{\left(r(s)\left(y^{\Delta}(s)\right)^{\beta}\right)^{1/\beta}}{r^{1/\beta}(s)}\Delta s \\ &\geq \left(r(t)\left(y^{\Delta}(t)\right)^{\beta}\right)^{1/\beta} A(t, t_1) \quad \text{for } t \geq t_1. \end{aligned} \tag{2.2}$$

Thus

$$\left(\frac{y(t)}{A(t, t_1)}\right)^{\Delta} \leq 0, \tag{2.3}$$

i.e., $y(t)/A(t, t_1)$ is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$. From the definition of $y(t)$, we get

$$\begin{aligned} x(t) &= \frac{1}{p_2(\tau_2^{-1}(t))}\left(y(\tau_2^{-1}(t)) - x(\tau_2^{-1}(t)) - p_1(\tau_2^{-1}(t))x(\tau_1(\tau_2^{-1}(t)))\right) \\ &= \frac{y(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))} - \frac{y(\tau_2^{-1}(\tau_2^{-1}(t))) - x(\tau_2^{-1}(\tau_2^{-1}(t)))}{p_2(\tau_2^{-1}(t))p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} \\ &\quad + \frac{p_1(\tau_2^{-1}(\tau_2^{-1}(t)))x(\tau_1(\tau_2^{-1}(\tau_2^{-1}(t))))}{p_2(\tau_2^{-1}(t))p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} \\ &\quad - \frac{p_1(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))}\left[\frac{y(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))) - x(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}\right. \\ &\quad \left. - \frac{p_1(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))x(\tau_1(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}\right] \\ &\geq \frac{y(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))} - \frac{y(\tau_2^{-1}(\tau_2^{-1}(t)))}{p_2(\tau_2^{-1}(t))p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} - \frac{p_1(\tau_2^{-1}(t))y(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}{p_2(\tau_2^{-1}(t))p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}. \end{aligned} \tag{2.4}$$

Using the fact that the functions y, τ_1 and τ_2 are strictly increasing, and noting that $\tau_1(t) < t < \tau_2(t)$, we get

$$y(\tau_2^{-1}(\tau_2^{-1}(t))) < y(\tau_2^{-1}(t)) \tag{2.5}$$

and

$$y(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))) < y(\tau_2^{-1}(t)). \tag{2.6}$$

Using (2.5) and (2.6) in (2.4) gives

$$x(t) \geq \frac{1}{p_2(\tau_2^{-1}(t))}\left[1 - \frac{1}{p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} - \frac{p_1(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}\right] y(\tau_2^{-1}(t)) \tag{2.7}$$

for $t \geq t_1$. Since $\lim_{t\to\infty} h_i(t) = \infty$, we can choose $t_2 \geq t_1$ such that all $h_i(t) \geq t_1$ for $t \geq t_2$, where $i = 1, 2, ..., n$. Thus, from (2.7) we obtain

$$x(h_i(t)) \geq \psi(h_i(t)) y\left(\tau_2^{-1}(h_i(t))\right), \qquad i = 1, 2, ..., n \tag{2.8}$$

for $t \geq t_2$. Using (2.8) in (1.1) gives

$$\left(r(t)\big(y^{\Delta}(t)\big)^{\beta}\right)^{\Delta} + \sum_{i=1}^{n} q_i(t)\psi^{\beta}\big(h_i(t)\big)y^{\beta}\big(\tau_2^{-1}(h_i(t))\big) \leq 0. \tag{2.9}$$

Define the function $\omega(t)$ by the Riccati substitution

$$\omega(t) := \eta(t)\frac{r(t)\big(y^{\Delta}(t)\big)^{\beta}}{y^{\beta}(t)} \quad \text{for } t \geq t_2. \tag{2.10}$$

Clearly $\omega(t) > 0$ and from (2.9) we see that

$$\begin{aligned}
\omega^{\Delta}(t) &= \frac{\eta(t)}{y^{\beta}(t)}\left(r(t)\big(y^{\Delta}(t)\big)^{\beta}\right)^{\Delta} + \left(\frac{\eta(t)}{y^{\beta}(t)}\right)^{\Delta} r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta} \\
&\leq \eta^{\Delta}(t)\,\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}}{y^{\beta}\big(\sigma(t)\big)} - \eta(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}\big(y^{\beta}(t)\big)^{\Delta}}{y^{\beta}(t)y^{\beta}\big(\sigma(t)\big)} \\
&\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^{\beta}\big(h_i(t)\big)\frac{y^{\beta}\big(\tau_2^{-1}(h_i(t))\big)}{y^{\beta}(t)}\right) \quad \text{for } t \geq t_2,
\end{aligned} \tag{2.11}$$

where $\sigma(t)$ is the forward jump operator on time scale $\mathbb{T}$. Using the fact $y(t)/A(t,t_1)$ is nonincreasing, and noting that $h_i(t) \leq \tau_2(t)$ implies $\tau_2^{-1}(h_i(t)) \leq t$, we obtain

$$\frac{y\big(\tau_2^{-1}(h_i(t))\big)}{y(t)} \geq \frac{A\big(\tau_2^{-1}(h_i(t)), t_1\big)}{A\big(t, t_1\big)}. \tag{2.12}$$

for $i = 1, 2, ..., n$. Substituting (2.12) into (2.11) gives

$$\begin{aligned}
\omega^{\Delta}(t) &\leq \eta^{\Delta}(t)\,\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}}{y^{\beta}\big(\sigma(t)\big)} - \eta(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}\big(y^{\beta}(t)\big)^{\Delta}}{y^{\beta}(t)y^{\beta}\big(\sigma(t)\big)} \\
&\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^{\beta}\big(h_i(t)\big)\frac{A^{\beta}\big(\tau_2^{-1}(h_i(t)), t_1\big)}{A^{\beta}\big(t, t_1\big)}\right) \quad \text{for } t \geq t_2.
\end{aligned} \tag{2.13}$$

From [8, Theorem 1.90], we obtain

$$\big(y^{\beta}(t)\big)^{\Delta} \geq \begin{cases} \beta y^{\beta-1}\big(\sigma(t)\big)y^{\Delta}(t), & \text{if } 0<\beta\leq 1, \\ \beta y^{\beta-1}(t)y^{\Delta}(t), & \text{if } \beta>1. \end{cases} \tag{2.14}$$

If $0<\beta\leq 1$, then we have from (2.13) and (2.14) that

$$\begin{aligned} \omega^{\Delta}(t) \quad &\leq \quad \eta^{\Delta}(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}}{y^{\beta}\big(\sigma(t)\big)} - \beta\eta(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}y^{\Delta}(t)}{y^{\beta}(t)y\big(\sigma(t)\big)} \\ &\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^{\beta}\big(h_i(t)\big)\frac{A^{\beta}\big(\tau_2^{-1}(h_i(t)),t_1\big)}{A^{\beta}\big(t,t_1\big)}\right). \end{aligned} \tag{2.15}$$

If $\beta>1$, then we have from (2.13) and (2.14) that

$$\begin{aligned} \omega^{\Delta}(t) \quad &\leq \quad \eta^{\Delta}(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}}{y^{\beta}\big(\sigma(t)\big)} - \beta\eta(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}y^{\Delta}(t)}{y(t)y^{\beta}\big(\sigma(t)\big)} \\ &\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^{\beta}\big(h_i(t)\big)\frac{A^{\beta}\big(\tau_2^{-1}(h_i(t)),t_1\big)}{A^{\beta}\big(t,t_1\big)}\right). \end{aligned} \tag{2.16}$$

Using the fact that $y(t)$ is increasing and $r(t)\big(y^{\Delta}(t)\big)^{\beta}$ is decreasing, we get $y(t)\leq y\big(\sigma(t)\big)$ and $y^{\Delta}(t)\geq r^{1/\beta}(\sigma(t))y^{\Delta}\big(\sigma(t)\big)/r^{1/\beta}(t)$, respectively. Thus, combining (2.15) and (2.16) we obtain for $\beta>0$ and $t\geq t_2$,

$$\begin{aligned} \omega^{\Delta}(t) \quad &\leq \quad \eta^{\Delta}(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}}{y^{\beta}\big(\sigma(t)\big)} - \beta\eta(t)\frac{r^{\gamma}\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta+1}}{r^{1/\beta}(t)y^{\beta+1}\big(\sigma(t)\big)} \\ &\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^{\beta}\big(h_i(t)\big)\frac{A^{\beta}\big(\tau_2^{-1}(h_i(t)),t_1\big)}{A^{\beta}\big(t,t_1\big)}\right) \\ &= \quad \eta^{\Delta}(t)\frac{\omega\big(\sigma(t)\big)}{\eta\big(\sigma(t)\big)} - \beta\frac{\eta(t)}{r^{1/\beta}(t)}\frac{\omega^{\gamma}\big(\sigma(t)\big)}{\eta^{\gamma}\big(\sigma(t)\big)} \\ &\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^{\beta}\big(h_i(t)\big)\frac{A^{\beta}\big(\tau_2^{-1}(h_i(t)),t_1\big)}{A^{\beta}\big(t,t_1\big)}\right). \end{aligned} \tag{2.17}$$

If we apply Lemma 1.2 with

$$X=\frac{[\beta\eta(t)]^{1/\gamma}}{\big[r^{1/\beta}(t)\eta^{\gamma}(\sigma(t))\big]^{1/\gamma}}\omega(\sigma(t)) \quad \text{and} \quad Y=\left[\frac{\beta}{\beta+1}\frac{\big[r^{1/\beta}(t)\eta^{\gamma}(\sigma(t))\big]^{1/\gamma}}{[\beta\eta(t)]^{1/\gamma}}\frac{\eta^{\Delta}(t)}{\eta(\sigma(t))}\right]^{\beta}$$

we see that

$$\eta^{\Delta}(t)\frac{\omega\big(\sigma(t)\big)}{\eta\big(\sigma(t)\big)} - \beta\frac{\eta(t)}{r^{1/\beta}(t)}\frac{\omega^{\gamma}\big(\sigma(t)\big)}{\eta^{\gamma}\big(\sigma(t)\big)} \leq \frac{1}{(\beta+1)^{\beta+1}}\frac{r(t)\big(\eta^{\Delta}(t)\big)^{\beta+1}}{\eta^{\beta}(t)}. \tag{2.18}$$

Using (2.18) in (2.17) gives

$$\omega^{\Delta}(t) \leq \frac{r(t)\left(\eta^{\Delta}(t)\right)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(t)} - \eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^{\beta}\left(h_i(t)\right)\frac{A^{\beta}\left(\tau_2^{-1}(h_i(t)),t_1\right)}{A^{\beta}\left(t,t_1\right)}\right).$$

Integrating the latter inequality from t_2 to t yields

$$\int_{t_2}^{t}\left(\eta(s)\left(\sum_{i=1}^{n} q_i(s)\psi^{\beta}\left(h_i(s)\right)\frac{A^{\beta}\left(\tau_2^{-1}(h_i(s)),t_1\right)}{A^{\beta}\left(s,t_1\right)}\right) - \frac{r(s)\left(\eta^{\Delta}(s)\right)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)}\right)\Delta s \leq \omega\left(t_2\right)$$

which contradicts (2.1) and completes the proof. □

Following, we give an oscillation criterion for (1.1) by using the integral averaging technique due to Philos [27]. First we need to introduce, the function class $\mathcal{P}$. Let $D_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \geq t_0\}$, $D \equiv \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$ and $H, h \in C_{rd}(D,\mathbb{R})$. The function $H \in C_{rd}(D,\mathbb{R})$ is said to belongs to the class $\mathcal{P}$ if

- ($\mathcal{P}_1$) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ on D_0,
- ($\mathcal{P}_2$) H has a nonpositive rd-continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on D_0 with respect to second variable and satisfies

$$H^{\Delta_s}(t,s) + H(t,s)\frac{\eta^{\Delta}(s)}{\eta(\sigma(s))} = \frac{h(t,s)}{\eta(\sigma(s))}H^{1/\gamma}(t,s),$$

where the function η is as in Theorem 2.1.

Theorem 2.2. *Assume that conditions* (i)-(iii$_{\mathfrak{a}}$) *hold and* $h_i(t) \leq \tau_2(t)$ *for* $i = 1,2,...,n$. *Suppose also that there exist functions* $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ *and* $H, h \in C_{rd}(D,\mathbb{R})$ *with* H *belongs to the class* $\mathcal{P}$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}\left[H(t,s)\,\Psi_1(s) - \frac{r(s)\left(h_+(t,s)\right)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)}\right]\Delta s = \infty, \tag{2.19}$$

where $\Psi_1(t)$ *is as in Theorem 2.1 and* $t_* > t_1$ *for sufficiently large* $t_1 \in [t_0,\infty)_{\mathbb{T}}$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ and $x(h_i(t)) > 0$ for $t \geq t_1$ and $i = 1,2,...,n$. Proceeding as in the proof of Theorem 2.1, we again arrive at (2.17) for $t \geq t_2$. In view of ($\mathcal{P}_1$) and ($\mathcal{P}_2$), it follows from (2.17) that

$$\int_{t_2}^{t} H(t,s)\eta(s)\left(\sum_{i=1}^{n} q_i(s)\psi^{\beta}\left(h_i(s)\right)\frac{A^{\beta}\left(\tau_2^{-1}(h_i(s)),t_1\right)}{A^{\beta}\left(s,t_1\right)}\right)\Delta s \leq -\int_{t_2}^{t} H(t,s)\omega^{\Delta}(s)\Delta s$$
$$+\int_{t_2}^{t} H(t,s)\eta^{\Delta}(s)\frac{\omega\left(\sigma(s)\right)}{\eta\left(\sigma(s)\right)}\Delta s - \int_{t_2}^{t} H(t,s)\beta\frac{\eta(s)}{r^{1/\beta}(s)}\frac{\omega^{\gamma}\left(\sigma(s)\right)}{\eta^{\gamma}\left(\sigma(s)\right)}\Delta s.$$

Using the integration by parts formula on time scales, we obtain

$$\int_{t_2}^{t} H(t,s)\,\Psi_1(s)\Delta s \le H(t,t_2)\omega(t_2) + \int_{t_2}^{t} H^{\Delta_s}(t,s)\omega\big(\sigma(s)\big)\Delta s$$

$$+\int_{t_2}^{t} H(t,s)\eta^{\Delta}(s)\frac{\omega\big(\sigma(s)\big)}{\eta\big(\sigma(s)\big)}\Delta s - \int_{t_2}^{t} H(t,s)\beta\frac{\eta(s)}{r^{1/\beta}(s)}\frac{\omega^{\gamma}\big(\sigma(s)\big)}{\eta^{\gamma}\big(\sigma(s)\big)}\Delta s$$

$$\le H(t,t_2)\omega(t_2) + \int_{t_2}^{t}\frac{h_+(t,s)}{\eta\big(\sigma(s)\big)}H^{1/\gamma}(t,s)\omega\big(\sigma(s)\big)\Delta s$$

$$-\int_{t_2}^{t} H(t,s)\beta\frac{\eta(s)}{r^{1/\beta}(s)}\frac{\omega^{\gamma}\big(\sigma(s)\big)}{\eta^{\gamma}\big(\sigma(s)\big)}\Delta s. \qquad (2.20)$$

Applying Lemma 1.2 with

$$X = \frac{[H(t,s)\beta\eta(s)]^{1/\gamma}}{\big[r^{1/\beta}(s)\eta^{\gamma}(\sigma(s))\big]^{1/\gamma}}\omega\big(\sigma(s)\big) \text{ and } Y = \left[\frac{\beta}{\beta+1}\frac{\big[r^{1/\beta}(s)\eta^{\gamma}(\sigma(s))\big]^{1/\gamma}}{[\beta\eta(s)]^{1/\gamma}}\frac{h_+(t,s)}{\eta(\sigma(s))}\right]^{\beta}$$

we obtain,

$$\frac{h_+(t,s)}{\eta\big(\sigma(s)\big)}H^{1/\gamma}(t,s)\omega\big(\sigma(s)\big) - H(t,s)\beta\frac{\eta(s)}{r^{1/\beta}(s)}\frac{\omega^{\gamma}\big(\sigma(s)\big)}{\eta^{\gamma}\big(\sigma(s)\big)} \le \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)}.$$

Substituting the latter inequality into (2.20), we conclude that

$$\int_{t_2}^{t}\left[H(t,s)\,\Psi_1(s) - \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)}\right]\Delta s \le H(t,t_2)\omega(t_2), \qquad (2.21)$$

which contradicts (2.19). This proves the theorem. □

From Theorem 2.2, we immediately have the following oscillation criterion.

Corollary 2.1. *Suppose that all conditions of Theorem 2.2 are satisfied with* (2.19) *replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t} H(t,s)\,\eta(s)\left(\sum_{i=1}^{n} q_i(s)\psi^{\beta}\big(h_i(s)\big)\frac{A^{\beta}\big(\tau_2^{-1}(h_i(s)),t_1\big)}{A^{\beta}\big(s,t_1\big)}\right)\Delta s = \infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}\frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{\eta^{\beta}(s)}\Delta s < \infty,$$

then equation (1.1) *is oscillatory.*

Theorem 2.3. *Assume that conditions* (i)-(iii$_\mathfrak{a}$) *hold and* $h_i(t) \ge \tau_2(t)$ *for* $i = 1,2,...,n$. *If there exists a positive function* $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{T}^{t}\left(\eta(s)\left(\sum_{i=1}^{n} q_i(s)\psi^{\beta}\big(h_i(s)\big)\right) - \frac{r(s)\big(\eta^{\Delta}(s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)}\right)\Delta s = \infty, \qquad (2.22)$$

for $T > t_1$ with $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ and $x(h_i(t)) > 0$ for $t \geq t_1$ and $i = 1, 2, ..., n$. Proceeding as in the proof of Theorem 2.1, we again arrive at (2.11) for $t \geq t_2$. Using the fact that $\tau_2(t)$ is strictly increasing and noting that $h_i(t) \geq \tau_2(t)$, we have

$$\tau_2^{-1}(h_i(t)) \geq t$$

for $i = 1, 2, ..., n$. So, from the fact that y is increasing, we obtain

$$\frac{y\left(\tau_2^{-1}(h_i(t))\right)}{y(t)} \geq 1, \qquad i = 1, 2, ..., n. \tag{2.23}$$

Using (2.23) in (2.11) yields

$$\begin{aligned} \omega^{\Delta}(t) \quad &\leq \quad \eta^{\Delta}(t) \frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}}{y^{\beta}\big(\sigma(t)\big)} - \eta(t) \frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}\big(y^{\beta}(t)\big)^{\Delta}}{y^{\beta}(t) y^{\beta}\big(\sigma(t)\big)} \\ &\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^{\beta}\big(h_i(t)\big)\right) \quad \text{for } t \geq t_2. \end{aligned} \tag{2.24}$$

The rest of proof is similar to that of Theorem 2.1, and so we omit the details. □

Theorem 2.4. *Assume that conditions* (i)-(iii$_{\mathfrak{a}}$) *hold and $h_i(t) \geq \tau_2(t)$ for $i = 1, 2, ..., n$. Suppose also that there exist functions $\eta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $H, h \in C_{rd}(D, \mathbb{R})$ with H belongs to the class $\mathcal{P}$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_*)} \int_{t_*}^{t} \left[H(t,s)\eta(s) \left(\sum_{i=1}^{n} q_i(s)\psi^{\beta}\big(h_i(s)\big) \right) - \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)} \right] \Delta s = \infty, \tag{2.25}$$

where $t_* > t_1$ for sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from (2.23), (2.24) and Theorem 2.2, and so we omit the details. □

3. Oscillation results when (iii$_{\mathfrak{b}}$) holds

In this section, we establish some new criteria for the oscillation of (1.1) in the cases where $h_i(t) \leq \tau_1(t)$ and $h_i(t) \geq \tau_1(t)$ for $i = 1, 2, ..., n$, respectively. For notational purposes, we let

$$\begin{aligned} \phi(t) := \frac{1}{p_1(\tau_1^{-1}(t))} \Bigg[1 &- \frac{1}{p_1(\tau_1^{-1}(\tau_1^{-1}(t)))} \frac{A\big(\tau_1^{-1}(\tau_1^{-1}(t)), t_1\big)}{A\big(\tau_1^{-1}(t), t_1\big)} \\ &- \frac{p_2(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))} \frac{A\big(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))), t_1\big)}{A\big(\tau_1^{-1}(t), t_1\big)} \Bigg] \end{aligned}$$

where τ_1^{-1} denotes the inverse function of τ_1, and throughout this section we assume that $\phi(t) > 0$ for all sufficiently large t.

Theorem 3.1. *Suppose that conditions* (i), (ii), (iii$_\mathfrak{b}$) *hold and* $h_i(t) \leq \tau_1(t)$ *for* $i = 1, 2, ..., n$. *If there exists a positive function* $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left(\Psi_2(s) - \frac{r(s)\big(\eta^{\Delta}(s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)} \right) \Delta s = \infty \tag{3.1}$$

for $T > t_1$ *where* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *sufficiently large and*

$$\Psi_2(t) = \eta(t) \left(\sum_{i=1}^n q_i(t)\phi^{\beta}\big(h_i(t)\big) \frac{A^{\beta}\big(\tau_1^{-1}(h_i(t)), t_1\big)}{A^{\beta}\big(t, t_1\big)} \right),$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ and $x(h_i(t)) > 0$ for $t \geq t_1$ and $i = 1, 2, ..., n$. Following a similar procedure to the proof of Theorem 2.1, we see that (2.2), (2.3) hold again, and from the definition of $y(t)$, we obtain

$$x(t) \geq \frac{y(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))} - \frac{y(\tau_1^{-1}(\tau_1^{-1}(t)))}{p_1(\tau_1^{-1}(t))p_1(\tau_1^{-1}(\tau_1^{-1}(t)))} - \frac{p_2(\tau_1^{-1}(t))y(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))}{p_1(\tau_1^{-1}(t))p_1(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))}. \tag{3.2}$$

Using the fact that τ_1 and τ_2 are strictly increasing, and noting that $\tau_1(t) < t < \tau_2(t)$, we get

$$\tau_1^{-1}(\tau_1^{-1}(t)) > \tau_1^{-1}(t) \tag{3.3}$$

and

$$\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))) > \tau_1^{-1}(t). \tag{3.4}$$

Taking $y(t)/A(t, t_1)$ is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$ into account that, we deduce from (3.3) and (3.4) that

$$\frac{A\big(\tau_1^{-1}(\tau_1^{-1}(t)), t_1\big)y(\tau_1^{-1}(t))}{A\big(\tau_1^{-1}(t), t_1\big)} \geq y(\tau_1^{-1}(\tau_1^{-1}(t))) \quad \text{for } t \geq t_1, \tag{3.5}$$

and

$$\frac{A\big(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))), t_1\big)y(\tau_1^{-1}(t))}{A\big(\tau_1^{-1}(t), t_1\big)} \geq y(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t)))) \quad \text{for } t \geq t_1, \tag{3.6}$$

respectively. Using (3.5) and (3.6) in (3.4) yields

$$x(t) \quad \geq \quad \frac{1}{p_1(\tau_1^{-1}(t))} \left[1 - \frac{1}{p_1(\tau_1^{-1}(\tau_1^{-1}(t)))} \frac{A\big(\tau_1^{-1}(\tau_1^{-1}(t)), t_1\big)}{A\big(\tau_1^{-1}(t), t_1\big)} \right.$$

$$-\frac{p_2(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))}\frac{A\big(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))),t_1\big)}{A\big(\tau_1^{-1}(t),t_1\big)}\Bigg]y(\tau_1^{-1}(t)) \qquad (3.7)$$

for $t \geq t_1$. Since $\lim_{t\to\infty} h_i(t) = \infty$, we can choose $t_2 \geq t_1$ such that all $h_i(t) \geq t_1$ for $t \geq t_2$, where $i = 1, 2, ..., n$. Thus, from (3.7) we obtain

$$x(h_i(t)) \geq \phi(h_i(t))y\left(\tau_1^{-1}(h_i(t))\right), \qquad i = 1, 2, ..., n \qquad (3.8)$$

for $t \geq t_2$. Using (3.8) in (1.1) gives

$$\Big(r(t)\big(y^{\Delta}(t)\big)^{\beta}\Big)^{\Delta} + \sum_{i=1}^{n} q_i(t)\phi^{\beta}\big(h_i(t)\big)y^{\beta}\big(\tau_1^{-1}(h_i(t))\big) \leq 0. \qquad (3.9)$$

Define again Riccati substitution ω by (2.10). Then, it follows from (2.10) and (3.9) that

$$\begin{aligned}\omega^{\Delta}(t) \quad &\leq \quad \eta^{\Delta}(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}}{y^{\beta}\big(\sigma(t)\big)} - \eta(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}\big(y^{\beta}(t)\big)^{\Delta}}{y^{\beta}(t)y^{\beta}\big(\sigma(t)\big)} \\ &\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\phi^{\beta}\big(h_i(t)\big)\frac{y^{\beta}\big(\tau_1^{-1}(h_i(t))\big)}{y^{\beta}(t)}\right) \quad \text{for } t \geq t_2. \qquad (3.10)\end{aligned}$$

Using the fact $y(t)/A(t, t_1)$ is nonincreasing, and noting that $h_i(t) \leq \tau_1(t)$ implies $\tau_1^{-1}(h_i(t)) \leq t$, we obtain

$$\frac{y\big(\tau_1^{-1}(h_i(t))\big)}{y(t)} \geq \frac{A\big(\tau_1^{-1}(h_i(t)), t_1\big)}{A\big(t, t_1\big)}. \qquad (3.11)$$

for $i = 1, 2, ..., n$. Substituting (3.11) into (3.10) gives

$$\begin{aligned}\omega^{\Delta}(t) \quad &\leq \quad \eta^{\Delta}(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}}{y^{\beta}\big(\sigma(t)\big)} - \eta(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}\big(y^{\beta}(t)\big)^{\Delta}}{y^{\beta}(t)y^{\beta}\big(\sigma(t)\big)} \\ &\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\phi^{\beta}\big(h_i(t)\big)\frac{A^{\beta}\big(\tau_1^{-1}(h_i(t)), t_1\big)}{A^{\beta}(t, t_1)}\right) \quad \text{for } t \geq t_2. \quad (3.12)\end{aligned}$$

The remainder of the proof is similar to that of Theorem 2.1, and so the details are omitted. □

Theorem 3.2. *Suppose that conditions* (i), (ii), ($\mathrm{iii}_{\mathfrak{b}}$) *hold and* $h_i(t) \leq \tau_1(t)$ *for* $i = 1, 2, ..., n$. *Suppose also that there exist functions* $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ *and* $H, h \in C_{rd}(D, \mathbb{R})$ *with* H *belongs to the class* $\mathcal{P}$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}\left[H(t,s)\,\mathit{\Psi}_2(s) - \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)}\right]\Delta s = \infty, \qquad (3.13)$$

where $\mathit{\Psi}_2(t)$ *is as in Theorem 3.1 and* $t_* > t_1$ *for sufficiently large* $t_1 \in [t_0,\infty)_{\mathbb{T}}$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof follows from (3.11), (3.12) and Theorem 2.2, and so we omit the details. □

From Theorem 3.3, we immediately have the following oscillation criterion.

Corollary 3.1. *Suppose that all conditions of Theorem 3.2 are satisfied with* (3.13) *replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t} H\left(t,s\right)\eta(s)\left(\sum_{i=1}^{n} q_i(s)\phi^{\beta}\big(h_i(s)\big)\frac{A^{\beta}\big(\tau_1^{-1}(h_i(s)),t_1\big)}{A^{\beta}\big(s,t_1\big)}\right)\Delta s=\infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}\frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{\eta^{\beta}(s)}\Delta s<\infty,$$

then equation (1.1) *is oscillatory.*

Theorem 3.3. *Assume that conditions* (i), (ii), ($\mathrm{iii}_{\mathfrak{b}}$) *hold and $h_i(t)\ge\tau_1(t)$ for $i=1,2,...,n$. If there exists a positive function $\eta\in C_{rd}^{1}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}\right)$ such that*

$$\limsup_{t\to\infty}\int_{T}^{t}\left(\eta(s)\left(\sum_{i=1}^{n} q_i(s)\phi^{\beta}\big(h_i(s)\big)\right)-\frac{r(s)\big(\eta^{\Delta}(s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)}\right)\Delta s=\infty, \tag{3.14}$$

for $T>t_1$ with $t_1\in[t_0,\infty)_{\mathbb{T}}$ sufficiently large, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(\tau_1(t))>0$, $x(\tau_2(t))>0$ and $x(h_i(t))>0$ for $t\ge t_1$ and $i=1,2,...,n$. Proceeding as in the proof of Theorem 3.1, we again arrive at (3.10) for $t\ge t_2$. Using the fact that $\tau_1(t)$ is strictly increasing and noting that $h_i(t)\ge\tau_1(t)$, we have

$$\tau_1^{-1}(h_i(t))\ge t$$

for $i=1,2,...,n$. So, from the fact that y is increasing, we obtain

$$\frac{y\left(\tau_1^{-1}(h_i(t))\right)}{y(t)}\ge 1,\qquad i=1,2,...,n. \tag{3.15}$$

Using (3.15) in (3.10) yields

$$\begin{aligned}\omega^{\Delta}\left(t\right)\quad\le\quad&\eta^{\Delta}\left(t\right)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}}{y^{\beta}\big(\sigma(t)\big)}-\eta(t)\frac{r\big(\sigma(t)\big)\big(y^{\Delta}\big(\sigma(t)\big)\big)^{\beta}\big(y^{\beta}(t)\big)^{\Delta}}{y^{\beta}(t)y^{\beta}\big(\sigma(t)\big)}\\&-\eta(t)\left(\sum_{i=1}^{n} q_i(t)\phi^{\beta}\big(h_i(t)\big)\right)\quad\text{for } t\ge t_2.\end{aligned} \tag{3.16}$$

The rest of proof is similar to the proof of Theorem 2.1, and so the details are omitted. □

Theorem 3.4. *Suppose that conditions* (i)*,* (ii)*,* ($\mathfrak{iii}_{\mathfrak{b}}$) *hold and* $h_i(t) \geq \tau_1(t)$ *for* $i = 1, 2, ..., n$*. Suppose also that there exist functions* $\eta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ *and* $H, h \in C_{rd}(D, \mathbb{R})$ *with* H *belongs to the class* $\mathcal{P}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_*)} \int_{t_*}^{t} \left[H(t,s)\eta(s) \left(\sum_{i=1}^{n} q_i(s)\phi^{\beta}\big(h_i(s)\big) \right) - \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)} \right] \Delta s = \infty, \tag{3.17}$$

where $t_* > t_1$ *for sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$*. Then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof follows from (3.15), (3.16) and Theorem 2.2, and so we omit the details. □

4. Examples and Remarks

Example 4.1. Consider the neutral differential equation

$$\left[\left(\left(x(t) + x(\frac{t}{2}) + tx(2t) \right)' \right)^3 \right]'^4 x^3(t-1) + t^5 x^3(2t-1) = 0, \tag{4.1}$$

for $t \geq 13$. Here we have $\mathbb{T} = \mathbb{R}$, $\beta = 3$, $n = 2$, $r(t) = p_1(t) = 1$, $p_2(t) = t$, $q_1(t) = t^4$, $q_2(t) = t^5$, $\tau_1(t) = t/2$, $\tau_2(t) = 2t$, $h_1(t) = t - 1$ and $h(t) = 2t - 1$. It is clear that conditions (i)-($\mathfrak{iii}_{\mathfrak{a}}$) hold, $\tau_2(t) > h_2(t) > t > h_1(t)$ and

$$\psi(t) = \frac{1}{t/2}\left[1 - \frac{1}{t/4} - \frac{1}{t/8}\right] = \frac{2t-24}{t^2} > 0, \quad \text{for } t \geq 13. \tag{4.2}$$

On the other hand, we see that

$$A(t,t_1) = \int_{t_1}^{t} \frac{1}{r^{1/\beta}(s)} ds = \int_{13}^{t} ds = t - 13.$$

With $\eta(t) = t$, condition (2.1) is satisfied due to

$$\limsup_{t\to\infty} \int_T^t \left[s^5 \left(\frac{2s-26}{(s-1)^2} \right)^3 \left(\frac{s-27}{2s-26} \right)^3 + s^6 \left(\frac{4s-26}{(2s-1)^2} \right)^3 \left(\frac{s-27/2}{s-13} \right)^3 \right] ds = \infty$$

and

$$\limsup_{t\to\infty} \int_T^t \frac{1}{4^4 s^3} ds < \infty.$$

where $T > 13$. Hence, by Theorem 2.1, every solution of (4.1) is oscillatory.

Example 4.2. Let $\mathbb{T} = \mathbb{Z}$ and consider the neutral difference equation

$$\Delta \left[\frac{1}{t^{1/3}} \Big(\Delta\big(x(t) + 3x(t-3) + 7x(t+1)\big) \Big)^{1/3} \right] + \sum_{i=1}^{n} t^i x^{1/3}(t+i+1) = 0, \tag{4.3}$$

for $t \geq 2$. Here we have $\beta = 1/3$, $r(t) = 1/t^{1/3}$, $p_1(t) = 3$, $p_2(t) = 7$, $\tau_1(t) = t - 3$, $\tau_2(t) = t + 1$, $q_i(t) = t^i$ and $h_i(t) = t + i + 1$ for $i = 1, 2, ..., n$. It is clear that conditions (i)-($\mathrm{iii}_{\mathfrak{a}}$) hold, $t < \tau_2(t) < h_i(t)$ for $i = 1, 2, ..., n$ and

$$\psi(t) = \frac{1}{7}\left[1 - \frac{1}{7} - \frac{3}{7}\right] = \frac{3}{49} > 0. \tag{4.4}$$

It is easy to see that condition (2.22) holds with $\eta(t) = c > 0$. So, by Theorem 2.3, all solutions of equation (4.3) are oscillatory.

Example 4.3. Consider the second order neutral dynamic equation

$$\left(\left(\left(x(t) + 5tx(\frac{t}{2}) + tx(4t)\right)^{\Delta}\right)^{3}\right)^{\Delta} + t^2x^3\left(\frac{t}{4}\right) + t^3x^3\left(\frac{t}{8}\right) = 0, \tag{4.5}$$

for $t \geq 2$ and $\mathbb{T} := \overline{2^{\mathbb{Z}}} = \{2^k : k \in \ \mathbb{Z}\} \cup \{0\}$. Here we have $\beta = 3$, $n = 2$, $r(t) = 1$, $p_1(t) = 5t$, $p_2(t) = t$, $\tau_1(t) = t/2$, $\tau_2(t) = 4t$, $q_1(t) = t^2$, $q_2(t) = t^3$, $h_1(t) = t/4$ and $h_2(t) = t/8$. It is clear that conditions (i), (ii), ($\mathrm{iii}_{\mathfrak{b}}$) hold and $\tau_1(t) \geq h_i(t)$ for $i = 1, 2$. Also, it obvious that $A(t, t_1) = t - 2$ for any time scale $\mathbb{T}$, and

$$\phi(t) = \frac{1}{10t}\left[1 - \frac{1}{20t}\frac{4t-2}{2t-2} - \frac{2t}{80t}\frac{16t-2}{2t-2}\right] = \frac{32t^2 - 43t + 2}{400t^3 - 400t^2} > 0 \tag{4.6}$$

for $t \geq 2$. Then, with $\eta(t) = t^2$, we see that condition (3.1) holds for $T > 2$. Hence, by Theorem 3.1, every solution of second order mixed neutral q-difference equation (4.5) is oscillatory.

Example 4.4. Consider the neutral differential equation

$$\Big(x(t) + tx(t - 2\pi) + x(t + \pi)\Big)'' + 2x\left(t - \frac{\pi}{2}\right) + tx(t + 2\pi) = 0, \quad t \geq 5. \tag{4.7}$$

Here we have $\mathbb{T} = \mathbb{R}$, $\beta = 1$, $n = 2$, $r(t) = p_2(t) = 1$, $p_1(t) = t$, $\tau_1(t) = t - 2\pi$, $\tau_2(t) = t + \pi$, $q_1(t) = 2$, $q_2(t) = t$, $h_1(t) = t - \frac{\pi}{2}$ and $h_2(t) = t + 2\pi$. It is clear that conditions (i), (ii), ($\mathrm{iii}_{\mathfrak{b}}$) hold, and $\tau_1(t) < h_1(t) < h_2(t)$. On the other hand, we see that $A(t, t_1) = t - 5$, and $\phi(t) > 0$. With $\eta(t) = 1$, it is easy to see that condition (3.14) holds. Hence, by Theorem 3.3, every solution of equation (4.7) is oscillatory. In fact, $x(t) = \sin t$ is such a solution.

Remark 4.1. Note that oscillation results presented in [6, 10, 12, 16, 19, 23, 28, 30, 36, 37] fail to apply to the equations (4.1), (4.3), (4.5) and (4.7).

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CURVES AND RULED SURFACES ACCORDING TO ALTERNATIVE FRAME IN DUAL SPACE

SÜLEYMAN ŞENYURT AND ABDUSSAMET ÇALIŞKAN

Abstract. In this paper, the vectorial moments of the alternative vectors are expressed in terms of altrenative frame. According to the new versions of these vectorial moments, the parametric equations of the closed ruled surfaces corresponding to the $(\widehat{N}), (\widehat{C}), (\widehat{W})$ dual curves are given. The integral invariants of these surfaces are computed and illustrated by presenting with examples.

1. Introduction

There are many studies on the classical differential geometry of curve and surface theories and are still being studied. A ruled surface in IR^3 is a surface which contains at least one 1-parameter family of straight lines. Thus a ruled surface has a parametrization in the form

$$\vec{\varphi}(s,v) = \vec{\alpha}(s) + v\vec{x}(s) \tag{1}$$

where we call α the anchor curve and the generator vector x as ruled surface. When the above ruled surface satisfies $\varphi(s+2\pi, v) = \varphi(s,v)$ it is called closed ruled surface. The properties of the ruled surface obtained according to the condition of the anchor curve or the generator vector are available in the books of differential geometry, [1, 2, 13]. Bertrand offsets, Mannheim offsets and involute-evolute offsets are obtained when special curves such as Bertrand, Mannheim and involute-evolute are taken as base curves. The geometric properties of these curves and surfaces are available in some references, [8, 9, 10, 12, 14, 15, 16].

If the vectorial moment of the x vector is denoted by x^*, then $x^* = \alpha \wedge x$. If X has the norm $\|X\| = 1$, then it is dual point on the dual unit sphere. According to E.Study theorem, there exists a one-to-one transformation between the dual points on the unit dual sphere and the oriented lines in IR^3. A one-parameter set of points (a dual curve) on dual unit sphere corresponds to a one-parameter family

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of oriented lines in E^3, which defines a ruled surface. This dual curve is called the dual spherical image of the ruled surface, [5, 7].

The dual expression of a ruled surface in (1) is

$$\varphi(s,u) = \vec{x}(s) \wedge \vec{x}^*(s) + u\vec{x}(s) \tag{2}$$

where the $\vec{x}(s) \wedge \vec{x}^*(s)$ is the anchor curve. s is not the arc-parameter of this curve [5, 7]. The dual angle of pitch of the closed ruled surface in (2) is defined by [5]

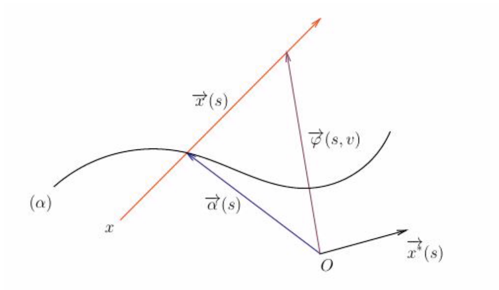

FIGURE 1. The dual expression of a ruled surface.

$$\Lambda_X = -\langle D, X\rangle = \lambda_x - \varepsilon L_x \tag{3}$$

Here, λ_x and L_x are real integral invariants [5].

Osman Gürsoy's study showed that the dual integral invariant of a closed ruled surface, the dual angle of pitch, corresponds to the dual spherical surface area described by the dual spherical indicatrix of the closed ruled surface. Further, geometric interpretations of the real angle of pitch and the real pitch of a closed ruled surface were given [6]. In [4], the pitch, the angle of pitch and the dual angle of pitch of closed ruled surface corresponding to a closed curve on dual unit sphere were investigated. In [17], a differential equation characterizing the dual spherical curves and an explicit solution of this differential equation was given. By investigating one parameter spherical motion in with two different kinds of dual indicatrice curves, Yaylı and Saraçoğlu obtained the ruled surfaces that correspond to tangent, principal normal and binormal indicatrices of the dual curve were developable, [20].

2. Preliminaries

In E^3, standard inner product is given by

$$\langle x, x\rangle = x_1^2 + x_2^2 + x_3^2 \tag{4}$$

where $x = (x_1, x_2, x_3) \in E^3$. Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame. $T(s)$ is the tangent vector field, $N(s)$ is the principal normal vector field and $B(s)$ is the binormal vector field of curve $\alpha(s)$, respectively. The Frenet formulas are given by [2]

$$T'(s) = \kappa(s)N(s),\ N'(s) = -\kappa(s)T(s) + \tau(s)B(s),\ B'(s) = -\tau(s)N(s). \tag{5}$$

Here curvature and torsion of the curve $\alpha(s)$ are defined with [2]

$$\kappa(s) = \|\alpha''(s)\|,\ \tau(s) = \frac{\langle \alpha'(s) \wedge \alpha''(s), \alpha'''(s)\rangle}{\|\alpha'(s) \wedge \alpha''^2}. \tag{6}$$

The vector W is called unit Darboux vector and defined by [3]

$$W = \frac{w}{\|w\|} = \frac{1}{\sqrt{\kappa^2 + \tau^2}}(\tau T + \kappa B) \tag{7}$$

It is obvious that the Darboux vector is perpendicular to the principal normal vector field N. If C is taken as $C = W \wedge N$, then $\{N, C, W\}$ are another orthonormal moving frame along the curve α. This frame is called an alternative frame. The derivative formulae of the alternative frame is given by

$$\begin{bmatrix} N' \\ C' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & \beta & 0 \\ \beta & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix} \tag{8}$$

where $\beta = \sqrt{\kappa^2 + \tau^2}$ and $\gamma = \frac{\kappa^2}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)'$.The relationship between the Frenet frame and alternative frame are

$$\begin{cases} C = -\overline{\kappa}T + \overline{\tau}B \\ W = \overline{\tau}T + \overline{\kappa}B \end{cases} \quad \text{or} \quad \begin{cases} T = -\overline{\kappa}C + \overline{\tau}W \\ B = \overline{\tau}C + \overline{\kappa}W \end{cases} \tag{9}$$

where principal normal vector N is same in both frames, $\overline{\kappa} = \frac{\kappa}{\beta}$ and $\overline{\tau} = \frac{\tau}{\beta}$, [11, 19].

Let $f(s)$, $g(s)$ and $h(s)$ be al least C^3- functions. $\alpha(s)$ can be written in the form of

$$\alpha(s) = f(s)T(s) + g(s)N(s) + h(s)B(s) \tag{10}$$

as a linear combination of the Frenet vectors $\{T, N, B\}$, [18]. By differentiating both side of (10), it is obtained [18]

$$f'(s) - g(s)\kappa(s) = 1,\ h'(s) + g(s)\tau(s) = 0,\ g'(s) + f(s)\kappa(s) - h(s)\tau(s) = 0. \tag{11}$$

3. Curves and Ruled Surfaces According to Alternative Frame in Dual Space

The geometric location of $\widehat{N} = N + \varepsilon N^*$, $\widehat{C} = C + \varepsilon C^*$ and $\widehat{W} = W + \varepsilon W^*$ vectors draws closed curves on the dual sphere. These closed curves are shown as $(\widehat{N}), (\widehat{C})$ and $(\widehat{W})$ respectively. According to Study's theorem these closed curves correspond to closed ruled surfaces. The dual expressions of the closed ruled surfaces corresponding to $(\widehat{N}), (\widehat{C})$ and $(\widehat{W})$ dual curves are

$$\begin{aligned} \psi_{\widehat{N}}(s,v) &= \beta_N(s) + vN(s),\ \beta_N(s) = N(s) \wedge N^*(s), \\ \psi_{\widehat{C}}(s,v) &= \beta_C(s) + vC(s),\ \beta_C(s) = C(s) \wedge C^*(s), \end{aligned} \tag{12}$$

$$\psi_{\widehat{W}}(s,v) \;=\; \beta_W(s)+vW(s),\ \beta_W(s)=W(s)\wedge W^*(s).$$

It is known that the curve α is written in the form of Frenet vectors. Using the equations (9) and (10), we can write as the linear combination of alternative vectors as follows:

$$\begin{aligned}\alpha(s) &= f(-\overline{\kappa}C+\overline{\tau}W)+gN+h(\overline{\tau}C+\overline{\kappa}W)\\ &= gN+\frac{g'}{\beta}C+\left(\frac{f\tau+h\kappa}{\beta}\right)W.\end{aligned} \tag{13}$$

Considering the above equation, vectorial moments of N, C, W are given respectively

$$\begin{aligned}N^* &= \alpha\wedge N=\Big(gN+\frac{g'}{\beta}C+\Big(\frac{f\tau+h\kappa}{\beta}\Big)W\Big)\wedge N\\ &= -\frac{g'}{\beta}W+\Big(\frac{f\tau+h\kappa}{\beta}\Big)C,\\ C^* &= \alpha\wedge C=\Big(gN+\frac{g'}{\beta}C+\Big(\frac{f\tau+h\kappa}{\beta}\Big)W\Big)\wedge C\\ &= gW-\Big(\frac{f\tau+h\kappa}{\beta}\Big)N,\\ W^* &= \alpha\wedge W=\Big(gN+\frac{g'}{\beta}C+\Big(\frac{f\tau+h\kappa}{\beta}\Big)W\Big)\wedge W\\ &= \frac{g'}{\beta}N-gC.\end{aligned} \tag{14}$$

Using (14) in (12), The dual expressions of the closed ruled surfaces corresponding to $(\widehat{N}),(\widehat{C})$ and $(\widehat{W})$ dual curves are given by

$$\begin{aligned}\psi_{\widehat{N}}(s,v) &= N\wedge N^*+vN=N\wedge\left(\frac{1}{\beta}\big(\left(f\tau+h\kappa\right)C-g'W\big)\right)+vN\\ &= \frac{1}{\beta}\big(g'C+\left(f\tau+h\kappa\right)W\big)+vN,\\ \psi_{\widehat{C}}(s,v) &= C(s)\wedge C^*(s)+vC(s)=C\wedge\left(-\frac{1}{\beta}(f\tau+h\kappa)N+gW\right)+vC\\ &= \frac{1}{\beta}(f\tau+h\kappa)W+gN+vC,\\ \varphi_{\widehat{W}}(s,v) &= W(s)\wedge W^*(s)+vW(s)=W\wedge\left(\frac{g'}{\beta}N-gC\right)+vW(s)\\ &= gN+\frac{g'}{\beta}C+vW(s).\end{aligned} \tag{15}$$

Theorem 1. *Distribution parameters of the closed ruled surfaces corresponding to the* $(\widehat{N}), (\widehat{C}), (\widehat{W})$ *dual curves are given by*

$$
\begin{aligned}
P_{\widehat{N}} &= \frac{1}{\beta}\left(-\left(\frac{f\tau+h\kappa}{\beta}\right)' - \frac{\gamma}{\beta}g'\right),\\
P_{\widehat{C}} &= \frac{g'\gamma+\beta\left(\frac{1}{\beta}(f\tau+h\kappa)\right)'}{\beta^2+\gamma^2},\\
P_{\widehat{W}} &= 0.
\end{aligned} \tag{16}
$$

Proof. We know that the distribution parameter of the closed ruled surface corresponding to the $(\widehat{N})$ dual curve is calculated by

$$P_{\widehat{N}} = \frac{\det((N\wedge N^*)', N, N')}{\|N'^2}. \tag{17}$$

It can be written that

$$
\begin{aligned}
(N\wedge N^*)' &= \left(\frac{1}{\beta}\right)'\Big(g'C+(f\tau+h\kappa)W\Big)+\frac{1}{\beta}\Big(g''C+g'(-\beta N+\gamma W)\\
&\quad +(f\tau+h\kappa)'W-\gamma(f\tau+h\kappa)\Big)C\\
&= -g'N+\left((\frac{g'}{\beta})'-\frac{\gamma}{\beta}(f\tau+h\kappa)\right)C+\left((\frac{f\tau+h\kappa}{\beta})'+\frac{\gamma}{\beta}g'\right)W.
\end{aligned}
$$

If this value is substituted into (17), the following result is obtained

$$
\begin{aligned}
P_{\widehat{N}} &= \frac{1}{\beta^2}\begin{vmatrix} -g' & \left(\frac{g'}{\beta}\right)'-\frac{\gamma}{\beta}(f\tau+h\kappa) & -\left(\frac{f\tau+h\kappa}{\beta}\right)'-\frac{\gamma}{\beta}g' \\ 1 & 0 & 0 \\ 0 & \beta & 0 \end{vmatrix}\\
&= \frac{1}{\beta}\left(-\left(\frac{f\tau+h\kappa}{\beta}\right)'-\frac{\gamma}{\beta}g'\right).
\end{aligned} \tag{18}
$$

Similarly, with the values of

$$
\begin{aligned}
(C\wedge C^*)' &= g'N+gN'+\left(\frac{f\tau+h\kappa}{\beta}\right)'W+\frac{1}{\beta}(f\tau+h\kappa)W'\\
&= g'N+\left(g\beta-\frac{\gamma}{\beta}(f\tau+h\kappa)\right)C+\left(\frac{1}{\beta}(f\tau+h\kappa)\right)'W,\\
(W\wedge W^*)' &= g'N+gN'+\left(\frac{g'}{\beta}\right)'C+\frac{g'}{\beta}C'
\end{aligned}
$$

$$= \left(g\beta + \left(\frac{g'}{\beta}\right)'\right) C + \frac{\gamma}{\beta} g' W,$$

distribution parameters of the closed ruled surfaces corresponding to the $(\widehat{C})$ and $(\widehat{W})$ dual curves are

$$P_{\widehat{C}} = \frac{g'\gamma + \beta\left(\frac{1}{\beta}(f\tau + h\kappa)\right)'}{\beta^2 + \gamma^2}, \quad P_{\widehat{W}} = 0. \tag{19}$$

□

Theorem 2. *Gauss curvatures of closed ruled surfaces corresponding to $(\widehat{N}), (\widehat{C})$ and $(\widehat{W})$ dual curves are*

$$K_{\widehat{N}}(P) = -\left(\frac{\left(\frac{\gamma}{\beta}g' + \left(\frac{f\tau+h\kappa}{\beta}\right)'\right)\beta}{\sqrt{\left(\left(\frac{g'}{\beta}\right)' - \frac{\gamma(f\tau+h\kappa)}{\beta} + v\beta\right)^2 + \left(\frac{\gamma}{\beta}g' + \left(\frac{f\tau+h\kappa}{\beta}\right)'\right)^2}}\right)^2,$$

$$K_{\widehat{C}}(P) = -\left(\frac{\beta\left(\frac{f\tau+h\kappa}{\beta}\right)' + \gamma g'}{\sqrt{(g' - v\beta)^2 + \left[\left(\frac{f\tau+h\kappa}{\beta}\right)' + v\gamma\right]^2}}\right)^2,$$

$$K_{\widehat{W}}(P) = 0.$$

Proof. For the closed ruled surface $\psi_{\widehat{N}}(s,v)$, the partial derivative is taken according to s and v, it is found

$$\begin{aligned}
\psi_{\widehat{N}_v}(s,v) &= N, \\
\psi_{\widehat{N}_s}(s,v) &= \left(\frac{g'}{\beta}C + \left(\frac{f\tau+h\kappa}{\beta}\right)W\right)' + vN'(s) \\
&= -g'N + \left(\left(\frac{g'}{\beta}\right)' - \frac{\gamma}{\beta}(f\tau+h\kappa) + v\beta\right)C \\
&\quad + \left(\frac{\gamma}{\beta}g' + \left(\frac{f\tau+h\kappa}{\beta}\right)'\right)W.
\end{aligned}$$

Taking into account that inner product, we compute

$$\langle \psi_{\widehat{N}_v}(s,v), \psi_{\widehat{N}_s}(s,v)\rangle = -g' \neq 0. \tag{20}$$

Using the Gram-Schmidt process, it can be seen that

$$\begin{aligned}
y_1 &= x_1 = N,\\
y_2 &= -\frac{\langle y_1, x_2\rangle}{\langle y_1, y_1\rangle} y_1 + x_2\\
&= \left(\left(\frac{g'}{\beta}\right)' - \frac{\gamma(f\tau + h\kappa)}{\beta} + v\beta\right) C + \left(\frac{\gamma}{\beta} g' + \left(\frac{f\tau + h\kappa}{\beta}\right)'\right) W,
\end{aligned}$$

$$\begin{aligned}
E_1 &= \frac{y_1}{\|y_1\|} = N,\\
E_2 &= \frac{y_2}{\|y_2\|} = \frac{\left(\left(\frac{g'}{\beta}\right)' - \frac{\gamma(f\tau + h\kappa)}{\beta} + v\beta\right) C + \left(\frac{\gamma}{\beta} g' + \left(\frac{f\tau + h\kappa}{\beta}\right)'\right) W}{\sqrt{\left(\left(\frac{g'}{\beta}\right)' - \frac{\gamma(f\tau + h\kappa)}{\beta} + v\beta\right)^2 + \left(\frac{\gamma}{\beta} g' + \left(\frac{f\tau + h\kappa}{\beta}\right)'\right)^2}}.
\end{aligned}$$

For a closed ruled surface with parametrization $\psi_{\widehat{N}}(s, v)$, the normal vector is given by

$$\begin{aligned}
N_{\widehat{N}} &= E_1 \wedge E_2\\
&= \frac{-\left(\frac{\gamma}{\beta} g' + \left(\frac{f\tau + h\kappa}{\beta}\right)'\right) C + \left(\left(\frac{g'}{\beta}\right)' - \frac{\gamma(f\tau + h\kappa)}{\beta} + v\beta\right) W}{\sqrt{\left(\left(\frac{g'}{\beta}\right)' - \frac{\gamma(f\tau + h\kappa)}{\beta} + v\beta\right)^2 + \left(\frac{\gamma}{\beta} g' + \left(\frac{f\tau + h\kappa}{\beta}\right)'\right)^2}}.
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
S_{\widehat{N}}(E_2) &= D_{E_2} N_{\widehat{N}} = D_{\frac{y_2}{\|y_2\|}} N_{\widehat{N}} = \frac{1}{\|y_2\|} D_{\lambda y_1 + x_2} N_{\widehat{N}} = \frac{1}{\|y_2\|}\left(\lambda \frac{\partial N_{\widehat{N}}}{\partial v} + \frac{\partial N_{\widehat{N}}}{\partial s}\right)\\
&\Rightarrow \ \langle S_{\widehat{N}}(E_2), E_1\rangle = \left(\frac{\left(\frac{\gamma}{\beta} g' + \left(\frac{f\tau + h\kappa}{\beta}\right)'\right)\beta}{\sqrt{\left(\left(\frac{g'}{\beta}\right)' - \frac{\gamma(f\tau + h\kappa)}{\beta} + v\beta\right)^2 + \left(\frac{\gamma}{\beta} g' + \left(\frac{f\tau + h\kappa}{\beta}\right)'\right)^2}}\right),
\end{aligned} \tag{21}$$

where $\lambda = -\frac{\langle y_1, x_2\rangle}{\langle y_1, y_1\rangle}$. Since shape operator is self-adjoint, we can write $\langle S(E_2), E_1\rangle = \langle S(E_1), E_2\rangle$. If the main direction of the surface is the asymptotic direction, the shape operator is $\langle S(E_1), E_1\rangle = 0$, [1]. Then, Gauss curvature of closed ruled surface $\psi_{\widehat{N}}(s, v)$ is

$$K_{\widehat{N}}(P) = det(S_P) = \begin{bmatrix} \langle S(E_1), E_1\rangle & \langle S(E_1), E_2\rangle \\ \langle S(E_2), E_1\rangle & \langle S(E_2), E_2\rangle \end{bmatrix}$$

$$= -\left(\frac{\left(\frac{\gamma}{\beta}g' + \left(\frac{f\tau+h\kappa}{\beta}\right)'\right)\beta}{\sqrt{\left(\left(\frac{g'}{\beta}\right)' - \frac{\gamma(f\tau+h\kappa)}{\beta} + v\beta\right)^2 + \left(\frac{\gamma}{\beta}g' + \left(\frac{f\tau+h\kappa}{\beta}\right)'\right)^2}}\right)^2.$$

Likewise, Gauss curvatures of closed ruled surfaces $\psi_{\widehat{C}}(s,v)$ and $\psi_{\widehat{W}}(s,v)$ are

$$\begin{aligned} K_{\widehat{C}}(P) &= -\left(\frac{\beta\left(\frac{f\tau+h\kappa}{\beta}\right)' + \gamma g'}{\sqrt{(g'-v\beta)^2 + \left[\left(\frac{f\tau+h\kappa}{\beta}\right)' + v\gamma\right]^2}}\right)^2, \\ K_{\widehat{W}}(P) &= 0. \end{aligned}$$

□

Theorem 3. *The instantaneous dual Pfaffian vector and the dual Steiner vector are given by respectively*

$$\widehat{w} = \tau T + \kappa B + \varepsilon(g\kappa T + g'N - g\tau B) \tag{22}$$

and

$$D = d + \varepsilon d^* = T\oint\tau + B\oint\kappa + \varepsilon(T\oint g\kappa + N\oint g' - B\oint g\tau). \tag{23}$$

Proof. The instantaneous dual Pfaffian vector has the same role with the dual Darboux vector. w and w^* are respectively the Darboux vector and vectorial moment of the Darboux vector.
The instantaneous dual Pfaffian vector is

$$\widehat{\omega} = \omega + \varepsilon\omega^*. \tag{24}$$

Vectorial moment of Darboux vector is given by

$$\begin{aligned} \omega^* &= \alpha \wedge w \\ &= gN\wedge(\tau T+\kappa B) + \frac{g'}{\beta}C\wedge(\tau T+\kappa B) + \left(\frac{f\tau+h\kappa}{\beta}\right)W\wedge(\tau T+\kappa B) \\ &= g\kappa T + g'N - g\tau B \end{aligned}$$

If this statement is substituted in (24), the instantaneous dual Pfaffian vector is

$$\widehat{\omega} = \tau T + \kappa B + \varepsilon(g\kappa T + g'N - g\tau B). \tag{25}$$

Also, by taking definition of dual Steiner vector,[5], we can write

$$D = \oint\widehat{\omega} = T\oint\tau + B\oint\kappa + \varepsilon(T\oint g\kappa + N\oint g' - B\oint g\tau). \tag{26}$$

□

Theorem 4. *The dual angles of pitch of closed ruled surfaces corresponding to* $(\widehat{N}), (\widehat{C}), (\widehat{W})$ *dual curves are*

$$\begin{aligned}
\Lambda_{\widehat{N}} &= \varepsilon\Big(\frac{\kappa(f\tau+h\kappa)+g'\beta\tau}{\beta^2}\lambda_T - \frac{\tau(f\tau+h\kappa)-g'\beta\kappa}{\beta^2}\lambda_B - \oint g'\Big),\\
\Lambda_{\widehat{C}} &= \frac{1}{\beta}\Big(\kappa\lambda_T - \tau\lambda_B\Big) - \varepsilon\Big(\frac{1}{\beta}\Big(g\tau\lambda_T + g\kappa\lambda_B - \tau\oint g\tau - \kappa\oint g\kappa\Big)\Big),\\
\Lambda_{\widehat{W}} &= -\frac{1}{\beta}\Big(\tau\lambda_T + \kappa\lambda_B\Big) - \varepsilon\Big(\frac{1}{\beta}\Big(g\kappa\lambda_T - g\tau\lambda_B + \tau\oint g\kappa - \kappa\oint g\tau\Big)\Big).
\end{aligned} \tag{27}$$

Here λ_T *and* λ_B *are the angles of pitch of closed ruled surfaces drawn by* T *and* B*, respectively.*

Proof. If take into account equations (3) and (23), the dual angle of pitch of closed ruled surface corresponding to the $(\widehat{N})$ dual curve is

$$\begin{aligned}
\Lambda_{\widehat{N}} &= -\langle D, \widehat{N}\rangle = -\langle d+\varepsilon d^*, N+\varepsilon N^*\rangle\\
&= -\langle T\oint\tau + B\oint\kappa, N\rangle - \varepsilon\big[\langle T\oint\tau + B\oint\kappa, \frac{1}{\beta}\big((f\tau+h\kappa)C - g'W\big)\rangle\\
&\quad +\langle T\oint g\kappa + N\oint g' - B\oint g\tau, N\rangle\big]\\
&= \varepsilon\Big(\frac{\kappa(f\tau+h\kappa)+g'\beta\tau}{\beta^2}\oint\tau - \frac{\tau(f\tau+h\kappa)-g'\beta\kappa}{\beta^2}\oint\kappa - \oint g'\Big)\\
&= \varepsilon\Big(\frac{\kappa(f\tau+h\kappa)+g'\beta\tau}{\beta^2}\lambda_T - \frac{\tau(f\tau+h\kappa)-g'\beta\kappa}{\beta^2}\lambda_B - \oint g'\Big).
\end{aligned}$$

Similarly, the dual angles of pitch of closed ruled surfaces corresponding to the $(\widehat{C})$ and $(\widehat{W})$ dual curves are

$$\begin{aligned}
\Lambda_{\widehat{C}} &= -\langle D, \widehat{C}\rangle = -\langle d+\varepsilon d^*, C+\varepsilon C^*\rangle\\
&= -\langle T\oint\tau + B\oint\kappa, C\rangle - \varepsilon\big[\langle T\oint\tau + B\oint\kappa, -\frac{1}{\beta}\big((f\tau+h\kappa)N + gW\big)\rangle\\
&\quad +\langle T\oint g\kappa + N\oint g' - B\oint g\tau, C\rangle\big]\\
&= \frac{1}{\beta}\Big(\kappa\oint\tau - \tau\oint\kappa\Big) - \varepsilon\Big(\frac{1}{\beta}\Big(g\tau\oint\tau - \tau\oint g\tau + g\kappa\oint\kappa - \kappa\oint g\kappa\Big)\Big)\\
&= \frac{1}{\beta}\Big(\kappa\lambda_T - \tau\lambda_B\Big) - \varepsilon\Big(\frac{1}{\beta}\Big(g\tau\lambda_T + g\kappa\lambda_B - \tau\oint g\tau - \kappa\oint g\kappa\Big)\Big),
\end{aligned}$$

$$
\begin{aligned}
\Lambda_{\widehat{W}} &= -\langle D, \widehat{W}\rangle = -\langle d+\varepsilon d^*, W+\varepsilon W^*\rangle \\
&= -\langle T\oint\tau + B\oint\kappa, W\rangle - \varepsilon\big[\langle T\oint\tau + B\oint\kappa, \frac{g'}{\beta}N - gC\rangle \\
&\quad +\langle T\oint g\kappa + N\oint g' - B\oint g\tau, W\rangle\big] \\
&= -\frac{1}{\beta}\Big(\tau\oint\tau + \kappa\oint\kappa\Big) - \varepsilon\Big(\frac{1}{\beta}\Big(g\kappa\oint\tau - g\tau\oint\kappa + \tau\oint g\kappa - \kappa\oint g\tau\Big)\Big) \\
&= -\frac{1}{\beta}\Big(\tau\lambda_T + \kappa\lambda_B\Big) - \varepsilon\Big(\frac{1}{\beta}\Big(g\kappa\lambda_T - g\tau\lambda_B + \tau\oint g\kappa - \kappa\oint g\tau\Big)\Big).
\end{aligned}
$$

□

Example 1. Let $\alpha(s) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, s)$ be a circular helix curve. Then, it is easy to show that

$$
\begin{aligned}
N(s) &= (\cos s, \sin s, 0),\ C(s) = (-\sin s, \cos s, 0), \\
W(s) &= (0,0,1), \\
\kappa(s) &= \frac{1}{\sqrt{2}},\ \tau(s) = \frac{1}{\sqrt{2}}.
\end{aligned}
$$

Considering equation (2), we obtain closed ruled surfaces corresponding to the $(\widehat{N}), (\widehat{C}), (\widehat{W})$ dual curves as

$$
\begin{aligned}
\psi_{\widehat{N}}(s,v) &= N\wedge N^* + vN \\
&= \Big(v\cos s, v\sin s, \frac{1}{\sqrt{2}}s\Big), \\
\psi_{\widehat{C}}(s,v) &= C\wedge C^* + vC \\
&= \Big(-\frac{1}{\sqrt{2}}\cos s - v\sin s, \frac{1}{\sqrt{2}}\sin s + v\cos s, \frac{s}{\sqrt{2}}\Big), \\
\psi_{\widehat{W}}(s,v) &= W\wedge W^* + vW \\
&= \big(-\frac{1}{\sqrt{2}}\sin s, -\frac{1}{\sqrt{2}}\cos s, v\big).
\end{aligned}
$$

These closed ruled surfaces are shown in Fig.2. Let us find the functions $f(s), g(s), h(s)$. From the equation (13), we can write

$$
\begin{aligned}
-\frac{1}{\sqrt{2}}\cos s &= g(s)\cos s - g'(s)\sin s, \\
-\frac{1}{\sqrt{2}}\sin s &= g(s)\sin s + g'(s)\cos s, \\
\frac{s}{\sqrt{2}} &= f(s) + h(s).
\end{aligned}
$$

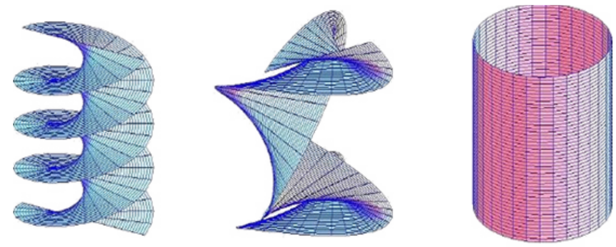

FIGURE 2. The figures (i), (ii) and (iii) show closed ruled surfaces corresponding to the $(\widehat{N})$, $(\widehat{C})$ and $(\widehat{W})$ dual curves, respectively. Anchor curves of these surfaces are helix curve.

Using the equation (11), the solutions of $f(s), g(s), h(s)$ are given by

$$f(s)=\frac{s}{2},\ g(s)=-\frac{1}{\sqrt{2}},\ h(s)=\frac{s}{2}\cdot \tag{28}$$

By taking into account the above equation and the equation (27), the dual angles of pitch of closed ruled surfaces corresponding to $(\widehat{N}), (\widehat{C}), (\widehat{W})$ dual curves are

$$\begin{aligned}
\Lambda_{\widehat{N}} &= \varepsilon\Big(\frac{s}{2}(\lambda_T-\lambda_B)-c\Big),\\
\Lambda_{\widehat{C}} &= \frac{1}{\sqrt{2}}(\lambda_T-\lambda_B)-\varepsilon\Big(-\frac{1}{2}(\lambda_T+\lambda_B)+\frac{1}{\sqrt{2}}\oint ds\Big)\\
&= \frac{1}{\sqrt{2}}(\lambda_T-\lambda_B)-\varepsilon\Big(-\frac{1}{2}(\lambda_T+\lambda_B)+\frac{1}{\sqrt{2}}L_T\Big),\\
\Lambda_{\widehat{W}} &= -\frac{1}{\sqrt{2}}(\lambda_T+\lambda_B)-\frac{\varepsilon}{2}(\lambda_B-\lambda_T).
\end{aligned}$$

Here L_T is the pitch of closed ruled surface drawn by the T and c is an arbitrary constant known as the integration constant.

Example 2. Let $\alpha(s)=(\frac{4}{5}\cos s, 1-\sin s, -\frac{3}{5}\cos s)$ be a curve. Then, it is easy to show that

$$\begin{aligned}
N(s) &= \Big(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\Big),\\
C(s) &= \Big(\frac{4}{5}\sin s, \cos s, -\frac{3}{5}\sin s\Big),\\
W(s) &= \Big(-\frac{3}{5}, 0, -\frac{4}{5}\Big),\\
\kappa(s) &= 1,\ \tau(s)=0.
\end{aligned}$$

Considering equation (2), we obtain closed ruled surfaces corresponding to $(\widehat{N}), (\widehat{C}), (\widehat{W})$ dual curves as

$$\begin{aligned}
\psi_{\widehat{N}}(s,v) &= N \wedge N^{*} + vN \\
&= \Big(\frac{4}{5}\cos s \sin s - v\frac{4}{5}\cos s, \cos^{2} s + v\sin s, -\frac{3}{5}\cos s\sin s + v\frac{3}{5}\cos s\Big), \\
\psi_{\widehat{C}}(s,v) &= C \wedge C^{*} + vC \\
&= v\Big(\frac{4}{5}\sin s, \cos s, -\frac{3}{5}\sin s\Big), \\
\psi_{\widehat{W}}(s,v) &= W \wedge W^{*} + vW \\
&= \Big(\frac{4}{5}\cos s - v\frac{3}{5}, 1-\sin s, -\frac{3}{5}\cos s - v\frac{4}{5}\Big).
\end{aligned}$$

These closed ruled surfaces are shown in Fig.3. Let us find the functions $f(s), g(s), h(s)$. From the equation (13), we can write

$$\begin{aligned}
4\cos s &= -4f(s)\sin s - 4g(s)\cos s - 3h(s), \\
1-\sin s &= -f(s)\cos s + g(s)\sin s, \\
-3\cos s &= 3f(s)\sin s + 3g(s)\cos s - 4h(s).
\end{aligned}$$

Applying the (11), the solutions of $f(s), g(s), h(s)$ are given by

$$f(s) = -\cos s,\ \ g(s) = \sin s - 1,\ \ h(s) = 0. \tag{29}$$

By taking into account the above equation and the equation (27), the dual angles of pitch of closed ruled surfaces corresponding to $(\widehat{N}), (\widehat{C}), (\widehat{W})$ are

$$\begin{aligned}
\Lambda_{\widehat{N}} &= \varepsilon\Bigg(\cos s\lambda_B - \oint \cos s ds\Bigg), \\
\Lambda_{\widehat{C}} &= \lambda_T - \varepsilon\Bigg((\sin s - 1)\lambda_B - \oint(\sin s - 1)ds\Bigg), \\
\Lambda_{\widehat{W}} &= -\lambda_B - \varepsilon(g\lambda_T - c).
\end{aligned}$$

where c is an arbitrary constant known as the integration constant.

Example 3. The Viviani's curve is formed by the intersection of a cylinder and a sphere. It is parametrized by

$$\alpha(t) = \left(a(1+\cos t), a\sin t, 2a\sin\frac{t}{2}\right).$$

Here, 2a is radius of sphere. The expression for the alternative invariants of Viviani's curve are given by

$$N(s) \quad = \quad \Bigg(\frac{-3-12\cos t - \cos 2t}{\sqrt{88\cos t + 162 + 6\cos 2t}}, \frac{-12\sin t - \sin 2t}{\sqrt{88\cos t + 162 + 6\cos 2t}},$$

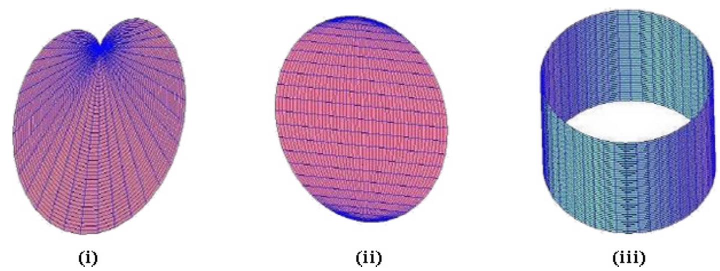

FIGURE 3. The figures (i), (ii) and (iii) show closed ruled surfaces corresponding to the $(\widehat{N})$, $(\widehat{C})$ and $(\widehat{W})$ dual curves, respectively.

$$\left.\frac{2\sqrt{2}\sin\frac{t}{2}}{\sqrt{81+44\cos t+3\cos 2t}}\right),$$

$$\begin{aligned}
C(s) &= W\wedge N,\\
W(s) &= \frac{\begin{bmatrix}\dfrac{\sqrt{3\cos t+13}}{a\,(\cos t+3)^{3/2}}\left(\dfrac{3\sin(t/2)+\sin(3t/2)}{\sqrt{26+6\cos t}}\right)+\dfrac{-6\sqrt{2}\cos(t/2)\sin t}{a(3\cos t+13)\sqrt{3+\cos t}}\,,\\ \dfrac{\sqrt{3\cos t+13}}{a\,(\cos t+3)^{3/2}}\left(\dfrac{-2\sqrt{2}\cos^3(t/2)}{\sqrt{13+3\cos t}}\right)+\dfrac{6\cos(t/2)}{a(3\cos t+13)}\left(\dfrac{\sqrt{2}\cos t}{\sqrt{3+\cos t}}\right),\\ \dfrac{\sqrt{3\cos t+13}}{a\,(\cos t+3)^{3/2}}\left(\dfrac{2\sqrt{2}}{\sqrt{13+3\cos t}}\right)+\dfrac{6\cos(t/2)}{a(3\cos t+13)}\left(\dfrac{\sqrt{2}\cos(t/2)}{\sqrt{3+\cos t}}\right)\end{bmatrix}}{\left((3\cos t+13)a^{-1}(\cos t+3)^{-3}+36\cos(t/2)^2a^{-2}(3\cos t+13)^{-2}\right)},\\
\kappa(s) &= \frac{\sqrt{3\cos t+13}}{a(\cos t+3)^{\frac{3}{2}}},\ \tau(s)=\frac{6\cos\frac{t}{2}}{3a\cos t+13a}.
\end{aligned}$$

Considering the equation(15), closed ruled surfaces $\psi_{\widehat{N}}(s,v),\psi_{\widehat{C}}(s,v)$ and $\psi_{\widehat{W}}(s,v)$ corresponding to $(\widehat{N}),(\widehat{C})$ and $(\widehat{W})$ dual curves is plotted by using Maple program (Fig. 4). Herein, associated calculations of these surfaces are computed by Maple program.

4. Conclusion

In this study, the vectorial moments of the alternative vectors are written using the data in equation (13). The dual expressions of the closed ruled surfaces which corresponds to the dual curves drawn by the $\widehat{N},\widehat{C}$ and $\widehat{W}$ on the dual sphere are expressed in terms of alternative vectors. The distribution parameters and Gauss curvatures of closed ruled surfaces $\psi_{\widehat{N}}(s,v)$, $\psi_{\widehat{C}}(s,v)$ and $\psi_{\widehat{W}}(s,v)$, which are obtained using the equation (13), are calculated. Applying (13), likewise, it is shown that the closed ruled surface corresponding to the $(\widehat{W})$ is developable. The dual angles of pitch of these surfaces obtained using the equation (13) are expressed

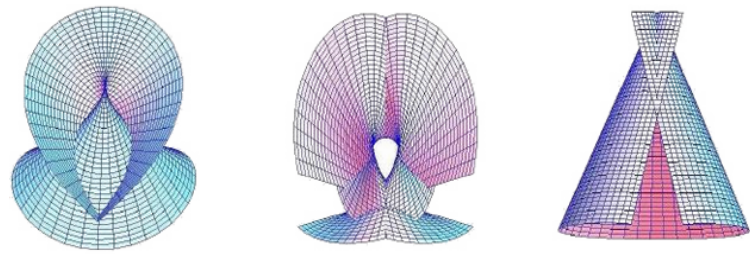

FIGURE 4. The figures (i), (ii) and (iii) show closed ruled surfaces corresponding to the $(\widehat{N}), (\widehat{C})$ and $(\widehat{W})$ dual curves, respectively. Anchor curves of these surfaces are Viviani's curve.

in terms of the angles of pitch of closed ruled surfaces drawn by T and B. Upon inspection of helix and Viviani's curves, the related ruled surfaces are generated.

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ANALYSIS OF JOINT RELIABILITY IMPORTANCE IN LINEAR m-CONSECUTIVE-k,l-OUT-OF-n:F SYSTEM

CIHANGIR KAN AND MURAT OZKUT

Abstract. Combinatorial techniques have an important role to compute the joint reliability importance (JRI) of some coherent systems. We obtain combinatorial formula for calculation of the JRI of two components in a generalised version of consecutive type systems consisting of n linearly ordered components such that system fails if and only if (iff) there are at least m l-overlapping runs of k consecutive failed components ($n \geq m(k-l)+l, l < k$). Overlapping runs mean having common elements which is denoted by l. We concentrate on both s-independent & identical components and exchangeable components. Explicit combinatorial formulae are provided for computing the JRI of the above mentioned cases. For both cases, we also compare the results with linear m-consecutive-k-out-of-n:F system (nonoverlapping case when $l = 0$). In addition, some numerical and illustrative examples are presented.

Acronyms and Notations

MRI	Marginal Reliability Importance
JRI	Joint Reliability Importance
$\mathrm{Lin}/m/\mathrm{Con}/k/l/n : F$	Linear m-consecutive-k,l-out-of-n:F
n	number of components
X_i	the state of component i, $i = 1, \ldots, n$ ($X_i = 1$ if the ith component fails, and $X_i = 0$ if the ith component works)
E	the event that the system works
k_ϕ	minimum number of failed components that may cause system failure
z_ϕ	maximum number of failed components such that a system can still work successfully

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1. Introduction

The marginal reliability importance (MRI) of a component measures the change in system reliability with respect to the change in component reliability([5, 6, 24, 35, 36, 37]). MRI is very useful in engineering fields such as design and improvement of a system. If $MRI_{E+j}(i)$ ($MRI_{E-j}(i)$) denotes MRI of ith component when jth component is functioning (failed), then $JRI(i,j) = MRI_{E+j}(i) - MRI_{E-j}(i)$ where the JRI is a measure of the interaction of the components in their contribution to system reliability (see [3, 14, 19, 20, 21]). Type and degree of interactions between two components are represented not only by the sign but also by the value of the JRI of two components in a coherent system. If the sign of the JRI of two components is nonnegative (nonpositive), it is called reliability complements (substitutes) ([19]). Moreover, if $JRI > 0$ ($JRI < 0$), then one component becomes more (less) important when the other is functioning which is also considered as synergy (diminishing returns). For $JRI = 0$, then one component's importance is not affected by the functioning of the other component ([3]). In literature, there are many studies on computation and analysis of JRI. Hong, Koo, and Lie [22] obtained a closed-form equation for the JRI of two components in a k-out-of- $n : G$ system, and examined its properties with respect to component reliability, and system parameters k and n. Gao, Cui, and Li [14] deeply analyzed JRI of three components in a k-out-of-$n : G$ system with independent components. Gertsbakh and Shpungin [17] combinatorially computed the JRI of two components. Rani, Jain, and Dewan [34] presented conditional marginal and conditional JRI in series–parallel systems. Eryilmaz [10] presented JRI in linear m-consecutive-k-out-of-$n : F$ systems. Mahmoud and Eryilmaz [28] studied exchangeable dependent components which is generalization of some results in Hong, Koo, and Lie [22] and Gao, Cui, and Li [14]. Zhu, Mahmoud, and Mohamed [38] presented JRI in m-consecutive-k-out-of-$n : F$ system that consists of Markov dependent components. Zhu, Mahmoud, and Mohamed [39] computed the JRI in consecutive-k-within-m-out-of-$n : F$ system with Markov dependent components. Eryilmaz and Mahmoud [8] firstly proposed and studied the m-consecutive-k, l-out-of-$n : F$ system. Zhu et al. [40] derived closed-form formulas for the reliability of the m-consecutive-k, l-out-of-$n : F$ and G systems, and computed JRI of this system when the components are non-homogenous Markov-dependent. One can see an extensive review of reliability importance measures in Kuo and Zhu [24] and Kuo, Way, and Zuo [25].

Eryilmaz, Oruc, and Oger [12] obtained general formula for computing the joint reliability importance of two components for a binary coherent system that consists of exchangeable dependent components. In that study, the joint reliability importance can be easily calculated if the path sets of the system are known. On the other hand, achieving the full list of path sets for the computation of JRI of any coherent system is not an easy task. Hence, only combinatorial formula for a series-parallel system is given in the study of Eryilmaz, Oruc, and Oger [12]. From this point of view, combinatorial techniques have an important role to compute JRI of some coherent systems. In this paper, combinatorial method has been used for computing the JRI of two components in Lin/m/Con/k/l/$n : F$ systems consisting of n linearly ordered components such that the system fails iff there are at least m l-overlapping runs of k consecutive failed components ($n \geq m(k-l)+l, l<k$). Unlike the study done by Zhu et al. [40], we concentrate on both s-independent & identical components and exchangeable components. For both cases, we also compare the results with linear m-consecutive-k-out-of-n:F system (nonoverlapping case when $l=0$). Eryilmaz [11] mentioned Birnbaum importance of a component when the system consists of exchangeable dependent components which is distinguished from our paper. We give explicit formula for calculation of JRI of two components under these two cases. And finally, some numerical and illustrative examples are presented.

2. Lin/m/Con/k/l/$n : F$ System

The Lin/m/Con/k/l/$n : F$ is a system that consists of n linearly ordered components such that the system fails iff there are at least m l-overlapping runs of k consecutive failed components ($n \geq m(k-l)+l, l<k$). Overlapping runs mean having common elements. For instance, 1111 is a sequence which contains two overlapping runs of length three and three overlapping runs of length two. Now, consider the states of a system with 16 components be 1110011100110100. For $m=5, k=2$ and $l=0$, this system is functioning when $l=1$ it is failed. When $l=0$, the Lin/m/Con/k/l/$n : F$ system becomes the non-overlapping Lin/m/Con/k/$n : F$ system which is introduced by Griffith [18] and Papastavridis [33]. When $l=k-1$, it reduces to the overlapping Lin/m/Con/k/$n : F$ systems. When $m=1$, the Lin/m/Con/k/l/$n : F$ system reduces to the Lin/Con/k/$n : F$ system. Also when $k=1$, the Lin/m/Con/k/l/$n : F$ system becomes m-out-of-$n : F$ system. This advanced system model with addition of the new parameter l creates diversity for real life applications in quality control, statistics and probability. Recently, there are many discussions on Lin/m/Con/k/l/$n : F$ system. For instance, Agarwal and Mohan [1] computed reliability of the system with the help of graphical evaluation and review technique under assumptions of i.i.d. components and $(k-1)$-step Markov dependent components. Some recent contributions on Lin/m/Con/k/l/$n : F$ system are the works of Gera [16], Levitin and Dai [27], Cui, Lin, and Du [7] and Zhu et al. [40].

The reliability of Lin/m/Con/k/l/$n : F$ system is closely related with the run statistics $N_{n,k,l}^{L(1:n)}$, which denotes the total number of $l-$overlapping runs of failures of length k in a linearly ordered sequence of binary trials $X_1, X_2, \ldots, X_n$. The distribution of the random variable $N_{n,k,l}^{L(1:n)}$ has been named the binomial distribution of order k for $l-$overlapping runs of length k, and introduced and studied by Aki and Hirano [2]. The reliability of Lin/m/Con/k/l/$n : F$ system can be expressed as $P\{E\} = P\{N_{n,k,l}^{L(1:n)} < m\}$. Some recent discussions on this topic are Eryilmaz [9], Eryilmaz and Mahmoud [8], Levitin [26], Makri and Psillakis [29, 30] and Makri and Psillakis [31]. For extensive reviews of the runs related literature, we refer to Balakrishnan and Koutras [4], Fu and Lou [13], and Koutras [23].

3. The Reliability of Lin/m/Con/k/l/n : F system

Eryilmaz [9] computed that the reliability of Lin/m/Con/k/l/n : F system consisting of s-independent components with common working probability $P\{X_i = 0\} = p$, and $r_i(n)$ denotes the total number of path sets of this structure including i working components,

$$\sum_{i=n-z_\phi}^{n} r_i(n)p^i(1-p)^{n-i}$$

where

$$z_\phi = n - 1 - \left[\frac{n - m(k-l) - l}{k}\right]$$

$n \geq m(k-l)+l$, and $[x]$ denotes the integer part of x and for the derivation of $r_i(n)$, see, Theorems 2.1 of Makri, Philippou, and Psillakis [32] and Eq. (1) of Eryilmaz and Mahmoud [8]). For simplicity of calculation, throughout this paper, we will denote $N_{i,a,k,l,s,n}$ as $N_{n,k,l}^{L(1:n)}$, hence

$$r_i(n) = \sum_{s=0}^{m-1} \sum_{a} \binom{i+1}{a} N_{n,k,l}^{L(1:n)}.$$

In an explicit way, by using Theorem 1 of Eryilmaz and Mahmoud [8] it can be written as

$$\begin{aligned} r_i(n) \quad &= \quad C(n-i; i+1, 0; k-1; k-1) \\ &\quad + \sum_{s=1}^{m-1} \sum_{a=1}^{\min(i+1,s)} \binom{i+1}{a}\binom{s-1}{a-1} C(n-i-al-s(k-l); a, i-a+1; k-l-1, k-1) \end{aligned}$$

where the quantities $C(\beta; a, r-a; m_1-1, m_2-1)$ can be calculated via the following formula (see, e.g. Makri, Philippou, and Psillakis [32]):

$$C(\beta; a, r-a; m_1-1, m_2-1) = \sum_{j_1=0}^{\left[\frac{\beta}{m_1}\right]} \sum_{j_2=0}^{\left[\frac{\beta-m_1j_1}{m_2}\right]} (-1)^{j_1+j_2}\binom{a}{j_1}\binom{r-a}{j_2}\binom{\beta-m_1j_1-m_2j_2+r-1}{r-1}$$

4. The JRI of Lin/m/Con/k/l/n : F system

Consider Lin/m/Con/k/l/n : F system consists of n binary components. Let X_i denote the state of ith component ($X_i = 1$ if the ith component fails, and $X_i = 0$ if it works, $i = 1, 2, \ldots, n$.) and E be the event that system functions. Then the JRI of components i and j can be defined as (see Kuo and Zhu [2012])

$$\begin{aligned} JRI(i,j) \quad &= \quad P\{E|X_i=1, X_j=1\} - P\{E|X_i=1, X_j=0\} - P\{E|X_i=0, X_j=1\} \\ &\quad + P\{E|X_i=0, X_j=0\}. \end{aligned} \tag{1}$$

Eryilmaz [10] expressed (1) by using the law of total probability as follows

$$JRI(i,j) = \frac{P\{E\} - P\{E, X_j=0\} - P\{E, X_i=0\} + P\{E, X_i=0, X_j=0\}}{1 - P\{X_j=0\} - P\{X_i=0\} + P\{X_i=0, X_j=0\}}$$

$$- \frac{P\{E, X_j = 0\} - P\{E, X_i = 0, X_j = 0\}}{P\{X_j = 0\} - P\{X_i = 0, X_j = 0\}}$$
$$- \frac{P\{E, X_i = 0\} - P\{E, X_i = 0, X_j = 0\}}{P\{X_i = 0\} - P\{X_i = 0, X_j = 0\}} + \frac{P\{E, X_i = 0, X_j = 0\}}{P\{X_i = 0, X_j = 0\}}.$$

So we need to calculate $P\{E, X_i = 0\}$ and $P\{E, X_i = 0, X_j = 0\}$ for the computation of JRI. It can easily be written as

$$\begin{aligned} &P\{E, X_i = 0\} \\ &\quad = P\{N^{L(1:n)}_{n,k,l} < m, X_i = 0\} \\ &\quad = P\{N^{L(1:i-1)}_{i-1,k,l} + N^{L(i+1:n)}_{n-i,k,l} < m, X_i = 0\} \\ &\quad = \sum_{s_1+s_2<m}\sum P\{N^{L(1:i-1)}_{i-1,k,l} = s_1, N^{L(i+1:n)}_{n-i,k,l} = s_2, X_i = 0\} \end{aligned} \tag{2}$$

and

$$\begin{aligned} &P\{E, X_i = 0, X_j = 0\} \\ &\quad = P\{N^{L(1:i-1)}_{i-1,k,l} + N^{L(i+1:j-1)}_{j-i-1,k,l} + N^{L(j+1:n)}_{n-j,k,l} < m, X_i = 0, X_j = 0\} \\ &\quad = \sum_{s_1+s_2+s_3<m}\sum\sum P\{N^{L(1:i-1)}_{i-1,k,l} = s_1, N^{L(i+1:j-1)}_{j-i-1,k,l} = s_2, N^{L(j+1:n)}_{n-j,k,l} = s_3, X_i = 0, X_j = 0\}. \end{aligned} \tag{3}$$

In the following subsections, we will obtain combinatorial formulas for the JRI of Lin/m/Con/$k/l/n : F$ systems consisting of

i. s-independent and identical components (common working probability $P\{X_i = 0\} = p$),
ii. exchangeable s-dependent components.

4.1. **S-Independent and Identical Components.** Consider a Lin/m/Con/$k/l/n$: F system when the components are s-independent with same working probability $P\{X_i = 0\} = p$. Assume that in the sequence of the first $i-1$ components there are m_1 working ones and in the sequence of the last $n-i$ components there are m_2 working components, and let $S^{(a:b)}_{b-a+1}$ denote the number of working components among the components $a, a+1, \ldots, b$, for $a < b$. That is $S^{(a:b)}_{b-a+1} = \sum_{i=a}^{b}(1-X_i)$, then by conditioning on s_1, s_2 and working components (2) can be rewritten as

$$\begin{aligned} P\{E, X_i = 0\} = \sum_{s_1+s_2<m}\sum \sum_{m_1=0}^{i-1}\sum_{m_2=0}^{n-i} P\Big\{&N^{L(1:i-1)}_{i-1,k,l} = s_1, N^{L(i+1:n)}_{n-i,k,l} = s_2, \\ &S^{(1:i-1)}_{i-1} = m_1, S^{(i+1:n)}_{n-i} = m_2, X_i = 0\Big\}. \end{aligned}$$

For the simplicity of calculation, above equation can be written as a sum of 4 terms (4,5,6 and 7) as follows

$$\begin{aligned} P\{E, X_i = 0\} = \sum_{m_1=0}^{i-1}\sum_{m_2=0}^{n-i} P\Big\{&N^{L(1:i-1)}_{i-1,k,l} = 0, N^{L(i+1:n)}_{n-i,k,l} = 0, \\ &S^{(1:i-1)}_{i-1} = m_1, S^{(i+1:n)}_{n-i} = m_2, X_i = 0\Big\} \end{aligned} \tag{4}$$

$$+\sum_{s_2=1}^{\min\left(m-1,\left[\frac{n-i-l}{k-l}\right]\right)}\sum_{m_1=0}^{i-1}\sum_{m_2=low_{m_2}}^{up_{m_2}} P\left\{N^{L(1:i-1)}_{i-1,k,l}=0, N^{L(i+1:n)}_{n-i,k,l}=s_2,\right.$$
$$\left. S^{(1:i-1)}_{i-1}=m_1, S^{(i+1:n)}_{n-i}=m_2, X_i=0\right\} \tag{5}$$
$$+\sum_{s_1=1}^{\min\left(m-1,\left[\frac{i-1-l}{k-l}\right]\right)}\sum_{m_1=low_{m_1}}^{up_{m_1}}\sum_{m_2=0}^{n-i} P\left\{N^{L(1:i-1)}_{i-1,k,l}=s_1, N^{L(i+1:n)}_{n-i,k,l}=0,\right.$$
$$\left. S^{(1:i-1)}_{i-1}=m_1, S^{(i+1:n)}_{n-i}=m_2, X_i=0\right\} \tag{6}$$
$$+\sum_{s_1=1}^{\min\left(m-2,\left[\frac{i-1-l}{k-l}\right]\right)}\sum_{s_2=1}^{\min\left(m-1-s_1,\left[\frac{n-i-l}{k-l}\right]\right)}\sum_{m_1=low_{m_1}}^{up_{m_1}}\sum_{m_2=low_{m_2}}^{up_{m_2}} P\left\{N^{L(1:i-1)}_{i-1,k,l}=s_1,\right.$$
$$\left. N^{L(i+1:n)}_{n-i,k,l}=s_2, S^{(1:i-1)}_{i-1}=m_1, S^{(i+1:n)}_{n-i}=m_2, X_i=0\right\} \tag{7}$$

where

$$up_{m_1}=i-1-k-(s_1-1)(k-l),$$
$$low_{m_1}=\left|\begin{array}{ll} 1+\left[\frac{i-2-s_1(k-l)-l}{k}\right] & \text{if } \frac{i-1-l}{k-l}<m-1 \\ 0 & \text{otherwise}\end{array}\right. ,$$
$$up_{m_2}=n-i-k-(s_2-1)(k-l),$$
$$low_{m_2}=\left|\begin{array}{ll} 1+\left[\frac{n-i-1-s_2(k-l)-l}{k}\right] & \text{if } \frac{n-i-l}{k-l}<m-1 \\ 0 & \text{otherwise}\end{array}\right. .$$

For better understanding of the terms 4,5,6 and 7, an explanation is given at Appendix.

Now, consider the probability $P\{E, X_i=0, X_j=0\}$. From (3)

$$P\{E, X_i=0, X_j=0\}$$
$$=\sum\sum\sum_{s_1+s_2+s_3<m}\sum_{m_1=0}^{i-1}\sum_{m_2=0}^{j-i-1}\sum_{m_3=0}^{n-j} P\left\{N^{L(1:i-1)}_{i-1,k,l}=s_1, S^{(1:i-1)}_{i-1}=m_1, N^{L(i+1:j-1)}_{j-i-1,k,l}=s_2,\right.$$
$$\left. S^{(i+1:j-1)}_{j-i-1}=m_2, N^{L(j+1:n)}_{n-j,k,l}=s_3, S^{(j+1:n)}_{n-j}=m_3, X_i=0, X_j=0\right\}$$

By using the independence of components, we can write,

$$P\left\{N^{L(1:i-1)}_{i-1,k,l}=s_1, N^{L(i+1:j-1)}_{j-i-1,k,l}=s_2, N^{L(j+1:n)}_{n-j,k,l}=s_3,\right.$$
$$\left. S^{(1:i-1)}_{i-1}=m_1, S^{(i+1:j-1)}_{j-i-1}=m_2, S^{(j+1:n)}_{n-j}=m_3, X_i=0, X_j=0\right\}$$
$$=P\left\{N^{L(1:i-1)}_{i-1,k,l}=s_1, S^{(1:i-1)}_{i-1}=m_1\right\}P\left\{N^{L(i+1:j-1)}_{j-i-1,k,l}=s_2, S^{(i+1:j-1)}_{j-i-1}=m_2\right\}$$
$$\times P\left\{N^{L(j+1:n)}_{n-j,k,l}=s_3, S^{(j+1:n)}_{n-j}=m_3\right\}P\left\{X_i=0\right\}P\left\{X_j=0\right\}$$

So, the cardinality of $N^{L(i:j)}_{j-i-1,k,l}$ and $S^{(i:j)}_{j-i-1}$ denotes the number of having s l-overlapping failure runs of length k in a linear binary sequence of length $j-i-1(i<j)$ with M number of working component(s) which is given as

$$\{N^{L(i:j)}_{j-i-1,k,l}=s, S^{(i:j)}_{j-i-1}=M\}=$$

$$\begin{cases} C(j-i-1-M; M+1, 0; k-1, k-1), & \text{for } s=0, \\ \sum_a \binom{M+1}{a}\binom{s-1}{a-1} C(j-i-M-al-s(k-l); a, M+1-a; k-l-1, k-1), & \text{for } s>0. \end{cases} \tag{8}$$

The quantity $C(\beta; a, r-a; m_1-1, m_2-1)$ can be calculated via the following formula (see, e.g. Makri, Philippou, and Psillakis [32]):

$$C(\beta; a, r-a; m_1-1, m_2-1) = \sum_{j_1=0}^{\left[\frac{\beta}{m_1}\right]} \sum_{j_2=0}^{\left[\frac{\beta-m_1 j_1}{m_2}\right]} (-1)^{j_1+j_2} \binom{a}{j_1}\binom{r-a}{j_2}\binom{\beta-m_1 j_1-m_2 j_2+r-1}{r-1}.$$

$P\{E, X_i=0, X_j=0\}$ can be rewritten explicitly which was shown at Appendix.

4.1.1. *Numerical Studies and Illustrations.* In this subsection, we present illustrative computational results for the JRI of components in a Lin/m/Con/k/l/n : F system when the components are s-independent with same working probability $P\{X_i=0\}=p$. In Figure 1., we compare graph of JRI(2,5) considered as a function of component reliability p for $m=3, k=2, n=20$ and $l=0,1$.

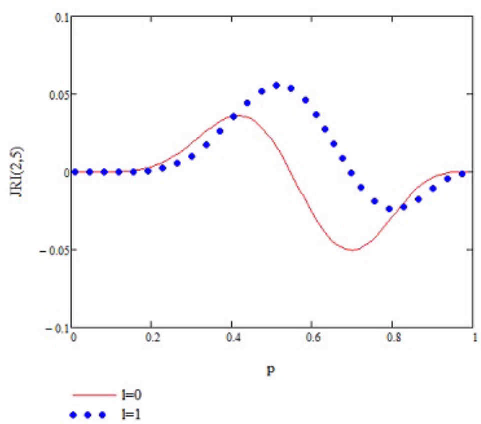

FIGURE 1. JRI(2,5) in a Lin/3/Con/2/l/20:F system as a function of p for $l=0$ and 1.

It can be easily seen that for Lin/3/Con/2/0/20 : F system, the sign of JRI(2,5) changes around at the point $p=0.55$. On the other hand, for Lin/3/Con/2/1/20 : F system this point shifts around $p=0.7$. As a result for the values $i=2$ and $j=5$, the graph of Lin/3/Con/2/1/20 : F system can be considered as a graph of Lin/3/Con/2/0/20 : F system shifted to the right. The sign of JRI(2,5) may not change for some values of n, m, k as seen in Figure 2.

In Table 1., we present all pairwise JRI values of the Lin/2/Con/2/1/5 : F system for different values of p.

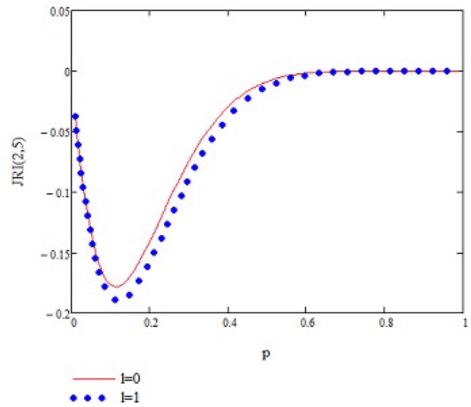

FIGURE 2. JRI(2,5) in a Lin/2/Con/5/l/12:F system as a function of p for $l = 0$ and

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
JRI(1,2)	−0.171	−0.288	−0.357	−0.384	−0.375	−0.336	−0.273	−0.192	−0.099
JRI(1,3)	0.639	0.352	0.133	−0.024	−0.125	−0.176	−0.183	−0.152	−0.089
JRI(1,4)	0.729	0.512	0.343	0.216	0.125	0.064	0.027	0.008	0.001
JRI(1,5)	−0.081	−0.128	−0.147	−0.144	−0.125	−0.096	−0.063	−0.032	−0.009
JRI(2,3)	0.549	−0.192	−0.073	−0.264	−0.375	−0.416	−0.393	−0.312	−0.179
JRI(2,4)	0.639	0.352	0.133	−0.024	−0.125	−0.176	−0.183	−0.152	−0.089

Table 1. All Pairwise JRI Values for the Lin/2/Con/2/1/5 : F System

Note that the Lin/m/Con/k/l/n : F system is symmetric, more precisely $JRI(i,j) = JRI(n-i+1, n-j+1)$, By Table 1, $JRI(1,2) < JRI(1,5) < 0$ for all $0 < p < 1$. which means component 2 should be more reliable than component 5 to decrease the diminishing return effect of component 1. while $JRI(1,4) > 0$ for all $0 < p < 1$. which means that component 1 and 4 have complementary synergy. On the other hand, one can see that components 1 and 3, and 2 and 4 are reliability complements for $p < 0.4$, while they are reliability substitutes for $p \geq 0.4$. The components 2 and 3 are reliability substitutes for $p \geq 0.2$.

In Table 2., we show the sign of JRI between component 1 and others for different values of m, k, l and n with different component reliability p when Lin/m/Con/k/l/n : F system contains s-independent and identical components.

p	n	m	k	l	$JRI(1,2)$	$JRI(1,3)$	$JRI(1,4)$	$JRI(1,5)$	$JRI(1,6)$	$JRI(1,7)$	$JRI(1,j), j>7$
0.85	20	3	2	1	−	−	−	−	−	−	−
0.9	20	3	2	1	−	−	−	−	−	−	−
0.9	30	3	2	1	−	−	−	−	−	−	−
0.9	30	3	3	1	−	−	−	+	−	−	−
0.9	30	3	3	2	−	−	−	−	+	−	−
0.9	30	4	3	1	−	−	−	+	−	+	−
0.9	30	4	3	2	−	−	−	−	−	+	−

Table 2. The sign of $JRI(1, j),\ j > 1$ for different Lin/m/Con/$k/l/n : F$ Systems

This table shows the effect of the system values n, m, k, l and p on the JRI. Also we observe that components 1 and 5 are reliability complements in Lin/3/Con/3/1/30 : F system when the components are s-independent with same working probability $p = 0.9$. However, they are reliability substitutes in Lin/3/Con/3/2/30 : F system with same component reliability. For Lin/3/Con/3/l/30 : F systems, when we change the value of l from 1 to 2, the reliability complementary components 1 and 5 turns into reliability substitutive components, while the reliability substitutive components 1 and 6 turns into reliability complementary components. As a result, the sign of JRI may change as the value of n, m, k, l and p change.

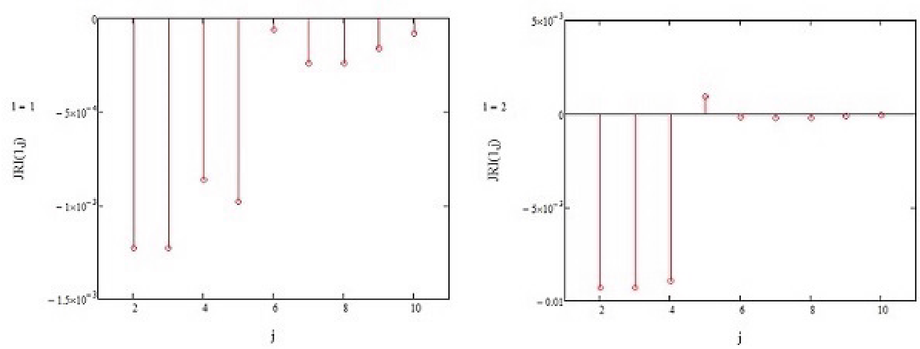

FIGURE 3. JRI(1, j), $j = 2, ..., 10$, of Lin/2/Con/3/l/10:F system with common working probability$p = 0.9$ for $l = 1$ and $l = 2$.

In Figure 3 we present $JRI(1, j)$, $j = 2, \ldots, 10$, of Lin/2/Con/3/l/10 : F system consisting of independent and identical components with working probability $p = 0.9$ for the two cases: when $l = 1$ and $l = 2$. From this figure, we observe that the sign of the $JRI(1, 5)$ for these two cases are different. When $JRI(1, j) < 0$ ($j = 2, \ldots, 10$), increasing in l causes diminishing on the value of $JRI(1, j)$ in Lin/2/Con/3/l/10 : F system, generally.

4.2. **Exchangeable S-Dependent Components.** In this section, JRI formula is obtained for Lin/m/Con/$k/l/n : F$ system consisting of exchangeable components.

A sequence of components $X_1, X_2, \ldots, X_n$ is exchangeable if for each n,

$$P\{X_{\pi_1} = x_1, \ldots, X_{\pi_n} = x_n\} = P\{X_1 = x_1, \ldots, X_n = x_n\}$$

for any permutation $(\pi_1, \ldots, \pi_n)$ of the indices in $(1, \ldots, n)$, i.e. the joint distribution of $X_1, X_2, \ldots, X_n$ is symmetric in $x_1, x_2, \ldots, x_n$. The exchangeability means that the components have identical distribution, but they affect one other within the system. That means, the joint distribution of $X_1, X_2, \ldots, X_n$ is invariant under permutation of its arguments. From George and Bowman (1995), any sequence with a 0 s and $n - a$ 1 s has probability

$$\begin{aligned} g(n,a) &= P\{X_1 = 0, \ldots, X_a = 0, X_{a+1} = 1, \ldots, X_n = 1\} \\ &= \sum_{i=0}^{n-a} (-1)^i \binom{n-a}{i} \lambda_{a+i} \\ &= \sum_{i=0}^{a} (-1)^i \binom{a}{i} \theta_{n-a+i} \end{aligned}$$

where $\lambda_a = P\{X_1 = 0, \ldots, X_a = 0\}$ and $\theta_a = P\{X_1 = 1, \ldots, X_a = 1\}$ with $\lambda_0 = 1, \theta_0 = 1$.

Since (2) can be obtained by the sum of (9), (10), (11) and (12), for exchangeable components $p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1}$ can be replaced by $g(n, m_1 + m_2 + 1)$ in (2) [see Eryilmaz [10]]. Similarly, $g(n, m_1 + m_2 + m_3 + 2)$ can be substituted in (3).

4.2.1. *Numerical Studies and Illustrations.* In this subsection, we consider Lin/m/Con/k/l/n : F system consisting of exchangeable components and present illustrative computational results for the JRI of components. Suppose p have a Beta distribution with parameters α and β. Hence for exchangeable random variables $X_1, \ldots, X_n$,

$$\begin{aligned} \lambda_a &= P\{X_1 = 0, \ldots, X_a = 0\} \\ &= \int_0^1 p^a \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} dp \\ &= \frac{\Gamma(a+\alpha)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(a+\alpha+\beta)} \end{aligned}$$

when $a \geq 1$.

In Figure 4, we present $JRI(1, j)$ of Lin/m/Con/3/l/10 : F system consisting of exchangeable components with parameters $\alpha = 9$ and $\beta = 1$ for the two cases: when $m = 2$ and $m = 3$. Clearly, when $m = 2$, the sign $JRI(1, j)$, $j = 2, \ldots, 10$, are same for the values $l = 1$ and $l = 2$ but the sign of $JRI(1, 5)$ is different for another case, $m = 3$, when $l = 1$ and $l = 2$ we observe that the sign of the $JRI(1, 5)$ for these two cases are opposite. Similar to s-independent and identical case, increasing in l causes diminishing on the value of JRI between the first and the other components in Lin/m/Con/k/l/n : F system consisting of exchangeable components, for most cases.Since systems in Figure 3 containing s-independent components and systems in Figure 4 containing exchangeable components with parameters $\alpha = 9$ and $\beta = 1$

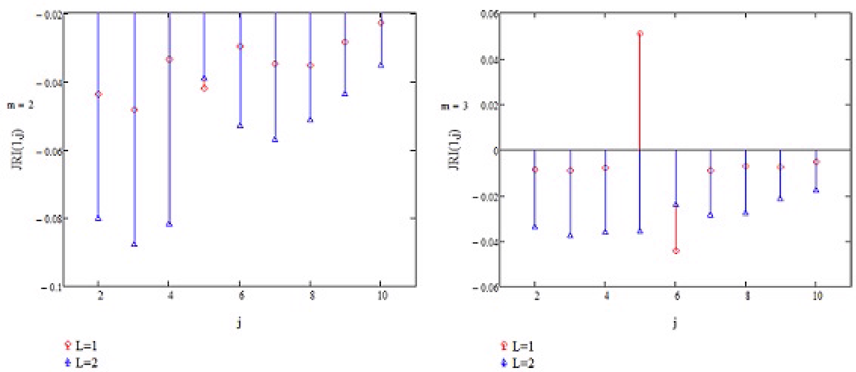

FIGURE 4. JRI$(1, j)$, $j = 2, ..., 10$, of Lin/m/Con/3/l/10:F system with exchangeable components with parameters $\alpha = 9$ and $\beta = 1$ for the two cases: when $m = 2$ ($l = 1, 2$) and $m = 3$ ($l = 1, 2$).

have the same working component reliability $p = 0.9$, one can easily compare $JRI(1, j)$ of those common systems.
In Table 3, the sign of $JRI(1, j)$, $j = 2, \ldots, 10$, of various Lin/m/Con/k/l/$n : F$ systems consisting of exchangeable components with parameters $\alpha = 9$ and $\beta = 1$ are given.

α	β	p	n	m	k	l	$JRI(1,2)$	$JRI(1,3)$	$JRI(1,4)$	$JRI(1,j)$, $j > 4$
4	6	0.4	20	3	2	1	+	+	+	+
10	6	0.625	20	3	2	1	−	+	+	+
10	4	0.714	20	3	2	1	−	−	−	−
4	6	0.4	10	3	2	1	−	+	+	+
4	6	0.4	20	2	2	1	+	+	+	+
4	6	0.4	20	2	3	1	−	−	+	+
4	6	0.4	20	2	3	2	−	−	+	+

Table 3. The sign of $JRI(1, j)$, $j > 1$ for different Lin/m/Con/k/l/$n : F$ systems consisting of exchangeable components with parameters $\alpha = 9$ and $\beta = 1$.

From Table 2 and 3 we can see that the dependency may effect the sign of the JRI between the first and the other components. In addition, the sign of JRI between the first and the other components in Lin/m/Con/k/l/$n : F$ systems consisting of exchangeable components highly depend on the values of $n, m.k$, and l. However, from Table 3 one can say that increasing component reliability p will change reliability complement components into reliability substitute components, i.e. increasing p from 0.4 to 0.714 when $n = 20, m = 3, k = 2$, and $l = 1$.

5. Conclusions

We have studied on Lin/m/Con/k/l/n : F system which is the generalization of consecutive k-out-of-n : F system. A Lin/m/Con/k/l/n : F system becomes a non-overlapping Lin/m/Con/k/n : F system, an overlapping Lin/m/Con/k/n : F system, a Lin/Con/k/n : F system and a m-out-of-n : F system for $l = 0, l = k-1, m = 1$ and $k = 1$ respectively. We have derived combinatorial formula for the computation of JRI of two components in Lin/m/Con/k/l/n : F system when components are s-independent & identical components and exchangeable. One possible future effort can be carried on the computation of JRI in an arbitrary dependent case or by changing the type of the system from linear form into circular form.

6. Appendix

For better understanding of the terms 4,5,6 and 7, consider a binary sequence in the following form

$$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components}} \quad \overset{\text{i}^{th} \text{ working component}}{0} \quad \underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components}}$$

For the operation of system, s is the total number of l-overlapping failure runs of length k must be less than m. We can denote this by $s = s_1 + s_2 (s < m)$ where s_1 and s_2 denote the l-overlapping runs of length k in the first sequence $i-1$ components and in the last sequence $n-i$ components, respectively. Hence we have four possible cases for operation of system.

Case 1 (4)	$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components with } s_1=0} \quad \overset{\text{i}^{th} \text{ working component}}{0} \quad \underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components with } s_2=0}$
Case 2 (5)	$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components with } s_1=0} \quad \overset{\text{i}^{th} \text{ working component}}{0} \quad \underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components with } 0<s_2<m}$
Case 3 (6)	$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components with } 0<s_1<m} \quad \overset{\text{i}^{th} \text{ working component}}{0} \quad \underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components with } s_2=0}$
Case 4 (7)	$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components with } 0<s_1<m-1} \quad \overset{\text{i}^{th} \text{ working component}}{0} \quad \underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components with } 0<s_2<m-1}$

Now let us consider terms of the sum 4,5,6 and 7 one by one. For term 4, where $s_1 = s_2 = 0$,

$$\sum_{m_1=0}^{i-1} \sum_{m_2=0}^{n-i} P\{N_{i-1,k,l}^{L(1:i-1)} = 0, N_{n-i,k,l}^{L(i+1:n)} = 0, S_{i-1}^{(1:i-1)} = m_1, S_{n-i}^{(i+1:n)} = m_2, X_i = 0\}$$

$$= \sum_{m_1=0}^{i-1} \sum_{m_2=0}^{n-i} C(i-1-m_1; m_1+1, 0; k-1, k-1) \times C(n-i-m_2; m_2+1, 0; k-1, k-1)$$

$$\times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1}$$

$$= \sum_{m_1=0}^{i-1} \sum_{m_2=0}^{n-i} \sum_{j=0}^{\min(m_1+1, \left[\frac{i-1-m_1}{k}\right])} (-1)^j \binom{m_1+1}{j} \binom{i-1-kj}{m_1}$$

$$\times \sum_{j=0}^{\min(m_2+1,\left[\frac{n-i-m_2}{k}\right])} (-1)^j \binom{m_2+1}{j}\binom{n-i-kj}{m_2} \times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1} \quad (9)$$

In term 5, where $s_1 = 0$ and $0 < s_2 < m$,

$$\begin{aligned}
&\sum_{s_2=1}^{\min\left(m-1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} P\left\{N_{i-1,k,l}^{L(1:i-1)} = 0, N_{n-i,k,l}^{L(i+1:n)} = s_2,\right.\\
&\qquad\qquad \left. S_{i-1}^{(1:i-1)} = m_1, S_{n-i}^{(i+1:n)} = m_2, X_i = 0\right\}\\
&= \sum_{s_2=1}^{\min\left(m-1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} C(i-1-m_1; m_1+1, 0; k-1, k-1)\\
&\quad \times \sum_{a=1}^{u} \binom{m_2+1}{a}\binom{s_2-1}{a-1} C(n-i-m_2-al-(k-l)s_2; a, m_2+1-a; k-l-1, k-1)\\
&\qquad \times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1}\\
&= \sum_{s_2=1}^{\min\left(m-1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{j=0}^{\min(m_1+1,\left[\frac{i-1-m_1}{k}\right])} (-1)^j \binom{m_1+1}{j}\binom{i-1-kj}{m_1}\\
&\times \sum_{a=1}^{u} \binom{m_2+1}{a}\binom{s_2-1}{a-1} \sum_{j_1=0}^{\min(a,\left[\frac{n-i-al-(k-l)s_2-m_2}{k-l}\right])} \sum_{j_2=0}^{\min(m_2+1-a,\left[\frac{n-i-al-(k-l)(s_2+j_1)-m_2}{k}\right])}\\
&\qquad \times (-1)^{j_1+j_2}\binom{a}{j_1}\binom{m_2+1-a}{j_2}\binom{n-i-al-(k-l)s_2-(k-l)j_1-kj_2}{m_2}\\
&\qquad \times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1} \qquad (10)
\end{aligned}$$

where $u = \left| \begin{array}{ll} \min(m_2+1, s_2) & \text{for } l = 0 \\ \min(m_2+1, s_2, \left[\frac{n-i-s_2(k-l)-m_2}{l}\right]) & \text{otherwise} \end{array}\right.$.

In term 6, where $s_2 = 0$ and $0 < s_1 < m$,

$$\begin{aligned}
&\sum_{s_1=1}^{\min\left(m-1,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{n-i} P\left\{N_{i-1,k,l}^{L(1:i-1)} = s_1, N_{n-i,k,l}^{L(i+1:n)} = 0,\right.\\
&\qquad\qquad \left. S_{i-1}^{(1:i-1)} = m_1, S_{n-i}^{(i+1:n)} = m_2, X_i = 0\right\}\\
&= \sum_{s_1=1}^{\min\left(m-1,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{n-i} \sum_{a=1}^{v} \binom{m_1+1}{a}\binom{s_1-1}{a-1}\\
&\qquad \times C(i-1-m_1-al-s_1(k-l); a, m_1+1-a; k-l-1, k-1)\\
&\qquad \times C(n-i-m_2; m_2+1, 0; k-1, k-1) \times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1}\\
&= \sum_{s_1=1}^{\min\left(m-1,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{n-i} \sum_{a=1}^{v} \binom{m_1+1}{a}\binom{s_1-1}{a-1}
\end{aligned}$$

$$\times \sum_{j_1=0}^{\min(a,\left[\frac{i-1-al-s_1(k-l)-m_1}{k-l}\right])} \sum_{j_2=0}^{\min(m_1+1-a,\left[\frac{i-1-al-(k-l)(s_1+j_1)-m_1}{k}\right])} \left\{(-1)^{j_1+j_2}\binom{a}{j_1}\right.$$

$$\left.\times\binom{m_1+1-a}{j_2}\binom{i-1-al-s_1(k-l)-(k-l)j_1-kj_2}{m_1}\right\}$$

$$\times \sum_{j=0}^{\min(m_2+1,\left[\frac{n-i-m_2}{k}\right])} (-1)^j\binom{m_2+1}{j}\binom{n-i-kj}{m_2}\times p^{m_1+m_2+1}\times(1-p)^{n-m_1-m_2-1} \quad (11)$$

where $v = \begin{vmatrix} \min(m_1+1,s_1) & \text{for } l=0 \\ \min(m_1+1,s_1,\left[\frac{i-1-s_1(k-l)-m_1}{l}\right]) & \text{otherwise} \end{vmatrix}$.

In term 7, that is $s_1 \geq 1$, $s_2 \geq 1$, and $s_1+s_2<m$,

$$\sum_{s_1=1}^{\min\left(m-2,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{s_2=1}^{\min\left(m-1-s_1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=low_{m_2}}^{up_{m_2}}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)}=s_1, N_{n-i,k,l}^{L(i+1:n)}=s_2, S_{i-1}^{(1:i-1)}=m_1, S_{n-i}^{(i+1:n)}=m_2, X_i=0\}$$

$$= \sum_{s_1=1}^{\min\left(m-2,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{s_2=1}^{\min\left(m-1-s_1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=low_{m_2}}^{up_{m_2}}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)}=s_1, S_{i-1}^{(1:i-1)}=m_1\}\times P\{N_{n-i,k,l}^{L(i+1:n)}=s_2, S_{n-i}^{(i+1:n)}=m_2\}\times P\{X_i=0\} \quad (12)$$

where

$$P\{N_{i-1,k,l}^{L(1:i-1)}=s_1, S_{i-1}^{(1:i-1)}=m_1\} = \sum_{a=1}^{v}\binom{m_1+1}{a}\binom{s_1-1}{a-1}$$
$$\times C(i-1-m_1-al-s_1(k-l);a,m_1+1-a;k-l-1,k-1)$$
$$\times p^{m_1}\times(1-p)^{i-1-m_1},$$

$$P\{N_{n-i,k,l}^{L(i+1:n)}=s_2, S_{n-i}^{(i+1:n)}=m_2\} = \sum_{a=1}^{u}\binom{m_2+1}{a}\binom{s_2-1}{a-1}$$
$$\times C(n-i-m_2-al-(k-l)s_2;a,m_2+1-a;k-l-1,k-1)$$
$$\times p^{m_2}\times(1-p)^{n-i-m_2}.$$

Explanation for $P\{E,X_i=0,X_j=0\}$:

$P\{E,X_i=0,X_j=0\}$ can be rewritten explicitly as follows

$$P\{E,X_i=0,X_j=0\} =$$

$$\sum_{m_1=0}^{i-1}\sum_{m_2=0}^{j-i-1}\sum_{m_3=0}^{n-j} P\{N_{i-1,k,l}^{L(1:i-1)}=0, S_{i-1}^{(1:i-1)}=m_1\}P\{N_{j-i-1,k,l}^{L(i+1:j-1)}=0, S_{j-i-1}^{(i+1:j-1)}=m_2\}$$

$$\times P\{N_{n-j,k,l}^{L(j+1:n)}=0, S_{n-j}^{(j+1:n)}=m_3\}P\{X_i=0\}P\{X_j=0\}$$

$$+ \sum_{s_1=1}^{\min(m-1,\left[\frac{i-1-l}{k-l}\right])} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{j-i-1} \sum_{m_3=0}^{n-j} P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, S_{i-1}^{(1:i-1)} = m_1\}$$

$$\times P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = 0, S_{j-i-1}^{(i+1:j-1)} = m_2\} P\{N_{n-j,k,l}^{L(j+1:n)} = 0, S_{n-j}^{(j+1:n)} = m_3\}$$

$$\times P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_2=1}^{\min(m-1,\left[\frac{j-i-1-l}{k-l}\right])} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{m_3=0}^{n-j} P\{N_{i-1,k,l}^{L(1:i-1)} = 0, S_{i-1}^{(1:i-1)} = m_1\}$$

$$\times P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = s_2, S_{j-i-1}^{(i+1:j-1)} = m_2\} P\{N_{n-j,k,l}^{L(j+1:n)} = 0, S_{n-j}^{(j+1:n)} = m_3\}$$

$$\times P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_3=1}^{\min(m-1,\left[\frac{n-j-l}{k-l}\right])} \sum_{m_1=0}^{i-1} \sum_{m_2=0}^{j-i-1} \sum_{m_3=low_{m_3}}^{up_{m_3}} P\{N_{i-1,k,l}^{L(1:i-1)} = 0, S_{i-1}^{(1:i-1)} = m_1\}$$

$$\times P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = 0, S_{j-i-1}^{(i+1:j-1)} = m_2\} P\{N_{n-j,k,l}^{L(j+1:n)} = s_3, S_{n-j}^{(j+1:n)} = m_3\}$$

$$\times P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_1=1}^{\min(m-2,\left[\frac{i-1-l}{k-l}\right])} \sum_{s_2=1}^{\min\left(m-1-s_1,\left[\frac{j-i-1-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{m_3=0}^{n-j}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, S_{i-1}^{(1:i-1)} = m_1\} P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = s_2, S_{j-i-1}^{(i+1:j-1)} = m_2\}$$

$$\times P\{N_{n-j,k,l}^{L(j+1:n)} = 0, S_{n-j}^{(j+1:n)} = m_3\} P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_1=1}^{\min(m-2,\left[\frac{i-1-l}{k-l}\right])} \sum_{s_3=1}^{\min\left(m-1-s_1,\left[\frac{n-j-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{j-i-1} \sum_{m_3=low_{m_3}}^{up_{m_3}}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, S_{i-1}^{(1:i-1)} = m_1\} P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = 0, S_{j-i-1}^{(i+1:j-1)} = m_2\}$$

$$\times P\{N_{n-j,k,l}^{L(j+1:n)} = s_3, S_{n-j}^{(j+1:n)} = m_3\} P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_2=1}^{\min(m-2,\left[\frac{j-i-1-l}{k-l}\right])} \sum_{s_3=1}^{\min\left(m-1-s_2,\left[\frac{n-j-l}{k-l}\right]\right)} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{m_3=low_{m_3}}^{up_{m_3}}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)} = 0, S_{i-1}^{(1:i-1)} = m_1\} P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = s_2, S_{j-i-1}^{(i+1:j-1)} = m_2\}$$

$$\times P\{N_{n-j,k,l}^{L(j+1:n)} = s_3, S_{n-j}^{(j+1:n)} = m_3\} P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_1=1}^{\min(m-3,\left[\frac{i-1-l}{k-l}\right])} \sum_{s_2=1}^{\min(m-2-s_1,\left[\frac{j-i-1-l}{k-l}\right])} \sum_{s_3=1}^{\min(m-1-s_1-s_2,\left[\frac{n-j-l}{k-l}\right])} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{m_3=low_{m_3}}^{up_{m_3}}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, S_{i-1}^{(1:i-1)} = m_1\} P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = s_2, S_{j-i-1}^{(i+1:j-1)} = m_2\}$$
$$\times P\{N_{n-j,k,l}^{L(j+1:n)} = s_3, S_{n-j}^{(j+1:n)} = m_3\} P\{X_i = 0\} P\{X_j = 0\}.$$

where

$$up_{m_1} = i - 1 - k - (s_1 - 1)(k - l),$$
$$low_{m_1} = \left| \begin{array}{ll} 1 + \left[\frac{i-2-s_1(k-l)-l}{k}\right] & \text{if } \frac{i-1-l}{k-l} < m - 1 \\ 0 & \text{otherwise} \end{array} \right. ,$$
$$up_{m_2} = j - i - 1 - k - (s_2 - 1)(k - l),$$
$$low_{m_2} = \left| \begin{array}{ll} 1 + \left[\frac{j-i-2-s_2(k-l)-l}{k}\right] & \text{if } \frac{j-i-1-l}{k-l} < m - 1 \\ 0 & \text{otherwise} \end{array} \right. ,$$
$$up_{m_3} = n - j - k - (s_3 - 1)(k - l),$$
$$low_{m_3} = \left| \begin{array}{ll} 1 + \left[\frac{n-j-1-s_3(k-l)-l}{k}\right] & \text{if } \frac{n-j-l}{k-l} < m - 1 \\ 0 & \text{otherwise} \end{array} \right. .$$

By substituting the equation (8) in (3) one can obtain explicitly.

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AN EVALUATION OF SOME METHODS USED FOR DETERMINATION OF HOMOGENOUS STRUCTURAL BREAK POINT IN MEAN OF PANEL DATA

SELIM DAĞLIOĞLU AND M. AKIF BAKIR

Abstract. In this study, performances of correct break point estimation of Simple Mean Shift Model Method, Fluctuation Test, Wald Statistic Test and Kim Test methods used to investigate presence of structural break and determine the date of break in a panel data consisting of N time series, each of T length, belonging to N cross-section have been investigated. In this context, 108 Monte Carlo simulations with each 3000 repeats have been carried out for 3, 3, 4 and 3 levels of factors, respectively number of cross-section units, length of series, size of break and proportion of break, to evaluate the performance of these tests used for determination of structural break in panel data. According to the Monte Carlo simulations it is concluded that Simple Mean Shift Model approach has better performance of break point estimation than other methods. Moreover, while Wald Test puts forth its best performance in the case where the breaks in series are at the half of the series, Fluctuation and Kim Tests showed their best performances in the case that the breaks are at the third quarter of series. Generally, correct break point estimation performances of tests decrease as the number of cross-section or length of series increases, even if it is limited. The changes at the levels of the proportion of break factor also lead to high accuracy estimation performance of different methods. Moreover, increases at the size of break usually decreases rates of correct estimation of methods and they approach to zero while means of the series changed 40% and over after break.

1. Introduction

Structural break(s) is(are) permanent change(s) in the structure of variables, due to permanent effects of economic or financial shocks, policy changes, cultural and technological changes, etc., on the distribution of variables. Changes in the behaviour of economic time series such as employment, growth and unemployment can occur in the long run due to policy changes and various economic events.

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However, when the models used in examining the data for such variables are estimated, it is usually assumed that the model parameters do not change over the sampling periods. This assumption makes the analysis relatively simple. However, the assumption that a time series is not subject to a change throughout the sample becomes more difficult to achieve as the length of the series increases. In the case of structural breaks in series, continuing analysis without considering this structural change can lead to incorrect estimations of model parameters. A typical example of this is that the investigation of the presence of unit root in Nelson and Plosser data; Nelson and Plosser [1], Perron [2], Zivot and Andrews [3] and Lumsdaine and Papell [4] have achieved different results. Despite the use of the same data set in these studies, the results differ depending on whether structural breaks are taken into account and whether structural breaks are included in the model.

The time series consists of observations obtained over a single cross-sectional unit at different times. Policy or technology changes often lead to permanent changes in the structure of the time series. For this reason, structural breaks are often encountered in time series. However, some difficulties arise when estimating the break point in the time series. If a structural break occurs at any time point k_0 of time series y_t, the break point k_0 can not be consistently predicted, regardless of how large the sample is, and the estimator $\hat{k}$ of the break point k_0 is not consistent. Therefore, it is usually attempted to estimate the break fraction instead of estimating the k_0's in which the structural change occurs in the time series. The effectiveness of the approach using a single time series in determining structural break depends on two assumptions: First, the magnitude of structural break (the difference between pre-break mean and post-break mean) is large enough. The second is that the true point of break point k_0 is far enough from the beginning and end of the sample. In a single series it is impossible to identify break point when the regime has a single observation [5], [6]. In the study of both single and multiple structural break points in time series, asymptotic framework is used in which the magnitudes of change(s) asymptotically converge to zero as the sample size increases in order to obtain critical statistics [7]. In other words, obtaining the limit distribution of the test statistics requires the assumption that the size of the structural break decreases as the sample size increases [8]. In the structural break literature this assumption is called the shrinking magnitude of structural break assumption. According to this assumption, as the sample size increases in the time series, the break point can be determined [9]. Both the break point inconsistency and the necessity of reduced break are related to the problem of defining the break point in time series models. The main reason for these two situations to emerge is that time series can not carry enough information. Additional information is needed in order to determine the actual break point in the time series. This information is tried to be obtained by increasing the sample size. When examining structural break in panel data, the additional information carried by the cross-sectional dimension of panel data eliminate the necessity of artificially increasing the number of observations using the reduced shrinking magnitude of structural break assumption. In addition, panel data can be used to derive asymptotics around the actual break date, since it has the cross-sectional dimension as well as the time dimension [9].

Although methods using panel data have significant advantages when compared to methods using only time series or only cross-sectional data, methods using panel data are much more complex. In this context, different methods for determining structural break point in panel data have been developed in the structural break literature. It has become widespread that structural break problem has been examined in panel data in recent years. The studies on the structural break problem in the panel data are generally considered in two directions. The first is to investigate the existence of unit root in panel data in the presence of structural break. The second is to determine the existence and date of structural break point. Two approaches have been adopted in panel studies in relation to the assumptions made about the position of structural break data. While the first considers the assumption that structural breaks in all series of the panel have emerged in a common date, in the second approach break point is assumed to be random in which break point occur on a different date for each series depending on the distribution of the random variable. The methods assuming the random break point are more complicated than the methods considering the common break point hypothesis.

The assumption of the common break point has been used in the studies by Han and Park [10], Joseph and Wolfson [11], Bai [12], Bai et al. [13], Emerson and Kao [14], Bai and Perron [15], Kao et. al. [16], Feng et. al. [9], Kim [17], Horváth and Hušková [18], Chan et. al. [19] and Li et. al. [20]. On the other hand, the assumption of random break point is considered in studies such as Joseph and Wolfson [11], Joseph and Wolfson [21], Joseph, Vandal and Wolfson [22], Joseph at al. [23], Joseph at al. [24] and Liao [6].

While there have been various methods developed in the literature on structural breaks in panel data, no study has been found on the comparison of the performance of these methods in the context of determining break point [25]. The contribution of this study is to compare the correct break point estimation performance of some methods used to determine the structural break point under the assumption of the common break point, according to the factors the number of cross sections, time series dimension, break size and break fraction. In this context, with the aid of Monte Carlo simulations, the Simple Mean Shift Model Method proposed in Bai [5], the Fluctuation Test and the Wald Statistic Test proposed in Emerson and Kao [14] and the Kim Test proposed in Kim [17] performance are evaluated.

In the next section of the study, the performances of the considered methods estimating the breakpoint are discussed. In the third section of the study, the data generating process and the issues considered in determination of factor levels and the assumption of Monte Carlo simulation are explained. In the fourth part of the study, the results obtained by Monte Carlo simulations are given. In the fifth and last part, the results obtained in the study are discussed and some suggestions are made.

2. Methods for Determination of Break point

Bai [5] considers the following simple mean shift model:

$$\begin{aligned} y_{it} &= \mu_{i1} + u_{it} \qquad t = 1,2,...,k_0 \\ y_{it} &= \mu_{i2} + u_{it} \qquad t = k_0+1,...,T. \end{aligned} \tag{1}$$

where $E(u_{it}) = 0$ for all i and t. In this model, each series has a break point at k_0, where k_0 is unknown. The μ_{i1} and μ_{i2} are pre-break mean and post-break mean of y_{it}, respectively. For the simple mean shift model, he proposes the OLS estimator of k_0 as in Equation 2:

$$\hat{k} = \underset{1\le k\le T-1}{\arg\min}\, SSR(k). \tag{2}$$

where sum of square of residuals $SSR_{iT}(k)$ is

$$SSR_{iT}(k) = \begin{cases} \sum\limits_{t=1}^{k}(y_{it}-\bar{y}_{i1})^2 + \sum\limits_{t=k+1}^{T}(y_{it}-\bar{y}_{i2})^2 & , \quad k = 1,2,...,T-1 \\ \sum\limits_{t=1}^{T}(y_{it}-\bar{y}_{i})^2 & , \quad k = T \end{cases} \tag{3}$$

for each $k = 1,2,...,T$. Also $\bar{y}_i$ is the average of all the observations of cross-section unit defined by,

$$\begin{aligned} \bar{y}_{i1} &= \frac{1}{k}\sum_{t=1}^{k} y_{it} \\ \bar{y}_{i2} &= \frac{1}{T-k}\sum_{t=k+1}^{T} y_{it} \end{aligned} \tag{4}$$

and sum of residual squares over all equations is as in Equation 5:

$$SSR(k) = \sum_{i=1}^{N} SSR_{iT}(k). \tag{5}$$

Emerson and Kao [14] consider the one-way random effect panel regression model with the deterministic time trend given in Equation 6:

$$\begin{aligned} y_{it} &= \alpha + \beta_t X_t + v_{it} \\ v_{it} &= \mu_i + u_{it}. \end{aligned} \tag{6}$$

where β is the slope parameter, $X_t = \frac{t}{T}$, unobservable individual effects are $\mu_i \sim iid\left(0, \sigma_\mu^2\right)$ and disturbance term of AR(1) is $u_{it} = \rho u_{it-1} + \varepsilon_{it}, \varepsilon \sim iid\left(0, \sigma_\varepsilon^2\right)$. They propose two different methods for testing the following null hypothesis

$$H_0 : \beta_t = \beta; \forall t \in [1, T] \tag{7}$$

meaning that there is no change in the model against the following alternative hypothesis

$$H_1 : \beta_t = \begin{cases} \beta_1 & , \quad t = 1, 2, ..., k \\ \beta_2 & , \quad t = k+1, ..., T \end{cases} \tag{8}$$

meaning that there exists a change in the $k-$point. They proposed to estimate the break point according to these two methods, The first is based on the fluctuation test of Ploberger, Kramer and Kontrus [26], while the second one is based on the mean statistics of Andrew and Ploberger [27] and exponential Wald statistic and the Wald statistic of Andrew [28]. In testing null hypothesis with fluctuation test, if the difference

$$\max_{i=1,\ldots,k} \left| \hat{\beta}_k - \hat{\beta}_T \right| \tag{9}$$

is big enough, that is when $\hat{\beta}_k$ is too much fluctuating, the null hypothesis is rejected. In other words, there is a structural break at this point and $\left| \hat{\beta}_k - \hat{\beta}_T \right|$ is the estimate of the break point. In Equation (9), $\hat{\beta}_T$ denotes the estimate of the slope parameter over all panel data estimated by OLS method, and $\hat{\beta}_k$, which is estimated with recursive OLS, is

$$\hat{\beta}_k = \frac{\sum\limits_{i=1}^{N} \left[\sum\limits_{t=1}^{k} \left(X_t - \bar{X}_k \right) y_{it} \right]}{\sum\limits_{i=1}^{N} \sum\limits_{t=1}^{k} \left(X_t - \bar{X}_k \right)^2} \tag{10}$$

where

$$\bar{X}_k = \frac{1}{k} \sum_{t=1}^{k} X_t.$$

In the Wald statistic test, the break point is estimated to be

$$\hat{k} = \underset{[Tr^+ \leq k \leq T - [Tr^+]]}{\arg\min} W_1(k). \tag{11}$$

Here,

$$\tilde{\sigma}_u^2 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (v_{it} - \bar{v}_i)^2 \tag{12}$$

and the estimation of σ_0^2 is

$$\sigma_0^2 = \frac{\sigma_\varepsilon^2}{(1-\rho)^2}. \tag{13}$$

Thus,

$$W_1(k) = \frac{\tilde{\sigma}_u^2}{3\sigma_0^2} W(k). \tag{14}$$

In addition,

$$\hat{\beta}_{1k} = \frac{\sum\limits_{i=1}^{N}\left[\sum\limits_{t=1}^{k}\left(X_t - \bar{X}_{1k}\right) y_{it}\right]}{\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{k}\left(X_t - \bar{X}_{1k}\right)^2} \tag{15}$$

$$\hat{\beta}_{2k} = \frac{\sum\limits_{i=1}^{N}\left[\sum\limits_{t=k+1}^{T}\left(X_t - \bar{X}_{2k}\right) y_{it}\right]}{\sum\limits_{i=1}^{N}\sum\limits_{t=k+1}^{T}\left(X_t - \bar{X}_{2k}\right)^2} \tag{16}$$

$$\bar{X}_{1k} = \frac{1}{k}\sum_{t=1}^{k} X_t$$

and

$$\bar{X}_{2k} = \frac{1}{T-k}\sum_{t=k+1}^{T} X_t.$$

Then $W(k)$ is calculated as follows:

$$W(k) = \frac{1}{\hat{\sigma}_u^2}\frac{\left(\hat{\beta}_{1k} - \hat{\beta}_{2k}\right)^2}{\left[\left(\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{k}\left(X_t - \bar{X}_{1k}\right)^2\right)^{-1} + \left(\sum\limits_{i=1}^{N}\sum\limits_{t=k+1}^{T}\left(X_t - \bar{X}_{2k}\right)^2\right)^{-1}\right]}. \tag{17}$$

Kim [17] considers the model with the deterministic trend and the disturbance component given in Equation 18:

$$y_{it} = d_{it} + u_{it}\ , \quad i = 1, ..., N \text{ and } t = 1, ..., T. \tag{18}$$

The deterministic component d_{it} can be considered in three different ways to be

$$d_{it} = \begin{cases} \mu_i + \beta_i t + \gamma_i B_t & , \textit{ Model I (Joint broken trend)} \\ \mu_i + \beta_i t + \theta_i C_t + \gamma_i B_t & , \textit{ Model II (Locally broken trend)} \\ \mu_i + \beta_i t + \theta_i C_t & , \textit{ Model III (Mean shift)} \end{cases} \tag{19}$$

where

$$C_t = \begin{cases} 0 & , \quad t \leq k_0 \\ 1 & , \quad t > k_0 \end{cases} \tag{20}$$

and

$$B_t = (t - k_0)\, C_t. \tag{21}$$

Here, Equation (20) can be rewritten for all of three models, if $t \leq k_0$, then $d_{it} = \mu_i + \beta_i t$ and if $t > k_0$, then

$$d_{it} = \begin{cases} \mu_i - k_0\gamma_i + (\beta_i + \gamma_i)\, t & , \textit{ Model I (Joint broken trend)} \\ \mu_i - k_0\gamma_i + \theta_i + (\beta_i + \gamma_i)\, t & , \textit{ Model II (Locally broken trend)} \\ \mu_i + \beta_i t + \theta_i & , \textit{ Model III (Mean shift)} \end{cases}$$

Models I and II are extended form of the panel data models reviewed by Perron and Zhu [29] for the univariate case. Model III, on the other hand, is an extended form so as to include a deterministic trend of the mean shift model examined in Bai [5].

The regression coefficients in the model are not restricted to be common for each section. For this reason, instead of estimating the regression coefficients jointly by pooling the cross-section data, the regression coefficients can be estimated separately for each equation using the OLS method. Thus, in the Kim Test, the individual OLS estimators of the regression coefficients for each equations are used for each cross section unit [17].

The Kim test assumes that the actual break point is unknown and the break fraction defined to be $\lambda_1 = k_0/T$; $\lambda_1 \in [\pi, 1-\pi]$, $\pi \in (0, 1/2)$ is constant for every T. It is also assumed that the break point k_0 is common to all equations and that the break fraction λ_1 remains constant as the sample size grows.

Using the deterministic time trend definitions given in Equation (19), the model in Equation (18) can be rewritten with matrix notation for each equation as

$$\underset{(T\times 1)}{Y_i} = \underset{(T\times 3 \text{ or } T\times 4)}{X_{k_0}} \underset{(3\times 1 \text{ or } 4\times 1)}{\Pi_i} \underset{(T\times 1)}{U_i} \tag{22}$$

where $Y_i = (y_{i1}, ..., y_{iT})'$ and $U_i = (u_{i1}, ..., u_{iT})'$. The variables and coefficients of Equation (22) are defined as follows:

$$X_{k_0} = \begin{cases} [\iota, \tau, B] & , \; \textit{Model I} \\ [\iota, \tau, C, B] & , \; \textit{Model II} \\ [\iota, \tau, C] & , \; \textit{Model III} \end{cases}$$

$$\Pi_i = \begin{cases} (\mu_i, \beta_i, \gamma_i)' & , \; \textit{Model I} \\ (\mu_i, \beta_i, \theta_i, \gamma_i)' & , \; \textit{Model II} \\ (\mu_i, \beta_i, \theta_i)' & , \; \textit{Model III} \end{cases}$$

where $\iota = (1, ..., 1)', \tau = (1, ..., T)', C = (C_1, ..., C_T)', B = (B_1, ..., B_T)', X_{k_0}$ is the collection of all dependent variables, and Π_i is the regression coefficient for the corresponding equation.

Then, the whole N equation system can be written as

$$Y = X_{k_0}\Pi + U \tag{23}$$

where $Y = [Y_1, ..., Y_N], \Pi = [\Pi_1, ..., \Pi_N]$ and $U = [U_1, ..., U_N]$. Also the row vectors are defined as $\mu = (\mu_1, ..., \mu_N), \beta = (\beta_1, ..., \beta_N), \theta = (\theta_1, ..., \theta_N)$ and$\gamma = (\gamma_1, ..., \gamma_N)$. Then, an alternative expression for Π is $[\mu', \beta', \gamma']'$, $[\mu', \beta', \theta', \gamma']'$ and $[\mu', \beta', \theta']'$ for Model I, II and III, respectively.

A general break point and a general break fraction are denoted by k, and $\alpha = k/T$, respectively, and X_k is defined similarly to X_{k_0}. Then, the sum of residual squares for each k, can be defined as follows:

$$SSR(k) = tr\left[Y'(I - P_k)Y\right] \tag{24}$$

where $P_k = X_k \left(X_k' X_k\right)^{-1} X_k'$ and $tr[.]$ is trace operator. Thus, estimated break point is the one minimizing the sum of residual squares such as

$$\hat{k} = \arg\min_k SSR\left(k\right) \tag{25}$$

and

$$\hat{\lambda} = \frac{\hat{k}}{T}. \tag{26}$$

3. Data generation and Monte-Carlo simulations

In this section, we evaluate the correct estimation performance of the Simple Mean Shift Model Method (hereafter referred to as Bai Test) proposed by Bai [5], the Fluctuation Test, Wald Statistic Test (hereafter referred to as Wald test) and Kim Test proposed in Kim [17], for the break date with Monte Carlo simulations.

The panel data to which the tests are applied are generated in accordance with the following model, also given in Equation (1):

$$\begin{aligned} y_{it} &= \mu_{i1} + u_{it},\ t = 1, 2, ..., k_0 \\ y_{it} &= \mu_{i2} + u_{it},\ t = k_0 + 1, ..., T \end{aligned}$$

where, $i = 1, 2, ..., N$, $t = 1, 2, ..., T$, y_{it} is the observation value of the ith section unit at time t, μ_{i1} is the pre-break mean which is mean of the panel data before the break date, μ_{i2} is the post-break mean which is mean of the panel data after the break date, k_0 is the common break point and u_{it} indicates the disturbance terms. In the simulations, the disturbance terms are generated from independent and identically distributed $u_{it} \overset{iid}{\sim} N\left(0;1\right)$, and, μ_{i1} and μ_{i2} are from $\mu_{i1} \overset{iid}{\sim} N\left(3;0,24\right)$ and $\mu_{i2} \overset{iid}{\sim} N\left(3 \times \gamma;0,24\right)$ where γ denotes the break ratio.

The number of repetitions are decided by taking into account the difference between asymptotics of the estimated break points in sequential run of simulations. In the study, the number of repetitions was determined as 3000 runs with the difference 0,001 between the average values of the break points predicted in each repetition. In total, Monte Carlo simulations are repeated as many times as the number of factor combinations depending on the level of the four factors under investigation.

Various issues have been taken into account to determine the factor levels. These issues can be summarized as follows: When examining the effects of time and cross-section length on break point estimation performance, the levels of these factors are defined as small, medium and large. The levels are chosen as 12, 32 and 120 for both time dimension T and cross-sectional dimension N.

If the break point k_0 is defined as a set of fixed values, the marginal effect of the break point can not be observed due to the coexistence of changes in the break point at different time dimension and the effects of changes in time dimension. For this reason, instead of taking the break point k_0 as a member of a fixed value set in simulations, k_0 is defined as an integer between 1 and T, $k_0 = [T\lambda]$, $\lambda \in (0, 1)$.

Thus, in the simulations, breaks are allowed to occur in the first, second and third quarter of the panel data, respectively, taking into account $\lambda \in \{0,25; 0,50; 0,75\}$ to define the break fraction.

The final factor by which the effect on the break point estimation performance is investigated is the magnitude of the break $(\mu_{i2} - \mu_{i1})$. When the magnitude of break factor levels are determined, the post-break mean is defined as

$$\mu_{i2} = \mu_{i1} \times \gamma$$

where γ is the break ratio. Then, the magnitude of the break is constant and written in the following form:

$$(\mu_{i2} - \mu_{i1}) = \gamma \times \mu_{i1} - \mu_{i1} = (\gamma - 1)\,\mu_{i1}$$

Thus, the magnitude of the break is defined as the ratio of the pre-break mean. Expression of the magnitude of break in this way allows it to be fixed for different factor levels and to define the post-break mean to be smaller than the pre-break mean. For this reason, when examining the effect of magnitude of break on the performance of the tests, the break ratio factor, γ, is strictly defined as the pre-break mean is used. The levels of the break ratio are defined as $\gamma \in \{1,1; 0,8; 1,4; 1,9\}$ so as to include the case where the post-break panel mean is smaller than the pre-break panel mean.

Simulation is performed at a total of 108 points of the experimental design for the factors time dimension, cross-section dimension, break fraction and break ratio, with the levels 3, 3, 3 and 4, respectively.

4. Simulation Results

In this section, the simulation results obtained via the simulation design described in the third section about correct break point estimation performance of the Bai, Fluctuation, Wald, and Kim tests are given. After generation of panel data, indicator variable is generated by using break point estimates of Bai, Fluctuation, Wald and Kim Tests to estimate correct estimation rates. Indicator variable shows whether estimated break point is equal actual break point or not. Indicator variable defined as dummy variable is given below:

$$D_j = \begin{cases} 1 & , \quad \hat{k}_j = k_0 \\ 0 & , \quad \hat{k}_j \neq k_0 \end{cases}$$

where j $(j = 1, ..., 4)$ shows the method used for estimating break point. This variable can take two values as 0 or 1. Since the mean of the indicator variable is on the [0, 1] interval, it shows the correct estimation rate of tests under the certain factor assumptions. These rates have been used for evaluating the performance for correct estimation of tests. Figure 1 shows the the effects of the changes in the cross-section dimension on the correct estimation rates of the tests for different time dimension in the case that the breakpoint occurs at the first quarter of the series of the panel data and mean of the series decreases by %20 after break. When

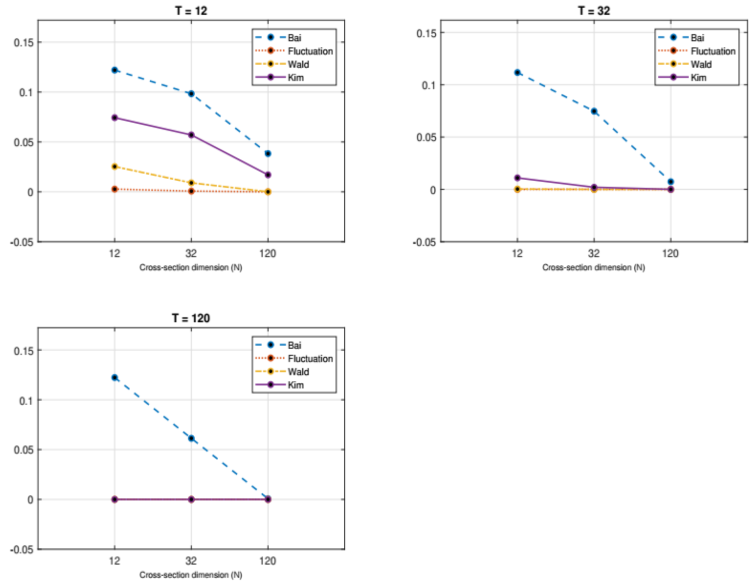

FIGURE 1. Simulation results for correct break point estimation rates ($\gamma = 0,8$ and $\lambda = 0,25$)

the break fraction and the break ratio factor are fixed, correct estimation rates of test convergences to zero by decreasing as both time and cross-section dimensions increase. Nevertheless, it is seen that decreasing at the rate of correct break point estimation are small. Moreover, while in the case where $T = 12$, correct estimation rates of the Bai and Kim tests are different from zero, they converge to zero as the time and cross-section dimensions increase. Correct estimation rates of the tests, except the Bai's, are zero for the bigger factor levels of T.If a break occurs in the first quarter of the series and the mean of the series increases by 40% after break, the correct break point estimation rates of all the tests decrease as the cross-sectional dimension increases under different time dimensions. Compared to the case where the mean of the series is reduced by 20%, the correct estimation rates of the tests show a similar tendency. Nevertheless, in the current case, the correct break points estimation rates of the tests are generally lower for all levels of the cross-sectional dimension. In other words, an increase in the rate of break ratio causes to a decrease in the correct estimation rates of the methods. In both cases, the highest correct estimation rates are achieved with the Bai Test. In the case where a break in the first quarter of the series and an 10% increase in the mean of the series after break, the correct break point estimation rates of all the tests, except rates of Bai Test, decreases as the cross-section size increases for different time dimensions. In panel data, the Bai Test correct estimation rates increase (Figure 3). However, when

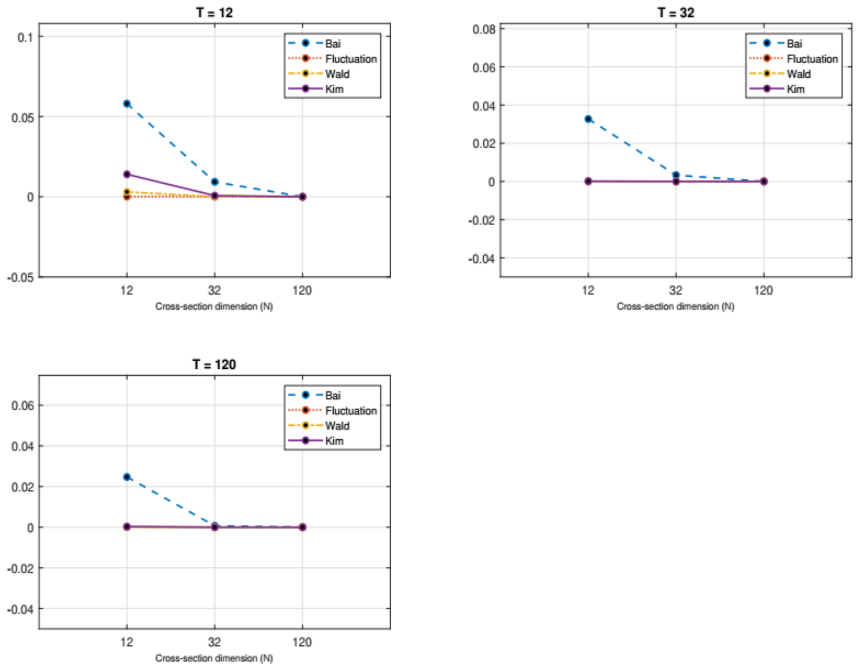

FIGURE 2. Simulation results for correct break point estimation rates ($\gamma = 1,4$ and $\lambda = 0,25$)

Figures 1 and 3 are evaluated jointly, the correct break point estimation rates of the test converges each other for both break ratios of 10% and 20%.

Figures 1, 2 and 3 show that increments in the cross-section dimensions have an effects towards decreasing the correct estimation rates of break points. A similar situation is observed for time dimension and the break ratio. In other words, according to the results obtained in Figures 1, 2 and 3, it can be said that the increases in the cross section and time dimension and the increase in the break ratio in the series have negative effect on the correct break points estimation rates of the tests. Figure 4 shows the effects of the break fraction on the correct break point estimation rates of Bai, Fructuation, Wald and Kim Tests when the time dimension and cross section size are fixed at 12. From the Figure 4, it is seen that the Wald Test has the highest correct estimation rate in the case that the breaks occur at the midpoint of the series. In addition, if the breaks occur in the later periods of the series, the correct break point estimation rates of the Fluctuation test increase. The Fluctuation Test shows highest correct estimation rates in the panels where break occurs in the second half of the series and the break ratio is small. Compared to other methods, the Bai Test has a generally high correct estimation

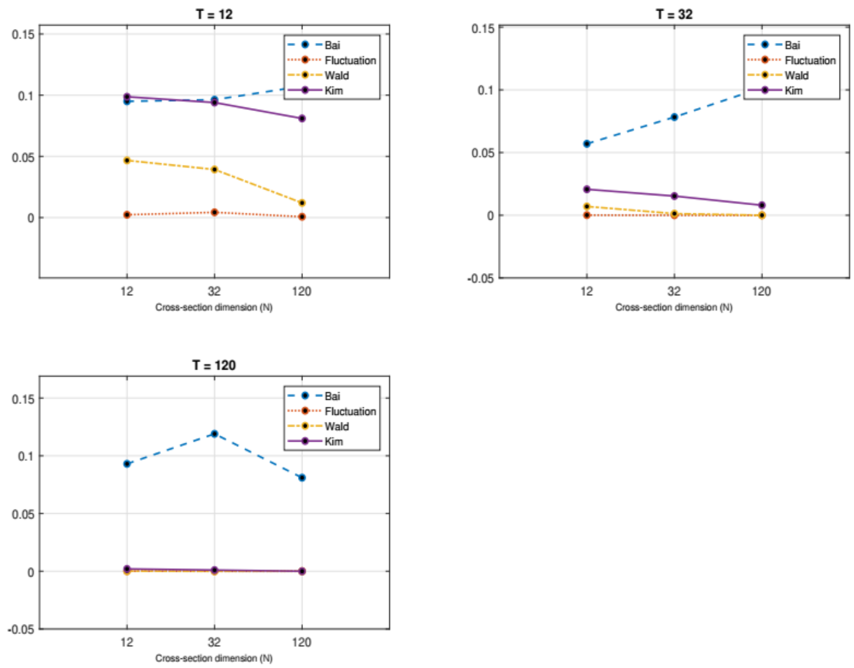

FIGURE 3. Simulation results for correct break point estimation rates ($\gamma = 1,1$ and $\lambda = 0,25$)

rate and they are less influenced by break fraction. In addition, Kim Test provides higher correct break point estimation rates if the break occurs in the later stages of the series similar to Fluctuation test.

An important finding emerging from Figure 4 is that the changes in the break fraction have effects on the correct estimation rates of the break point of the tests at different ways. Moreover, changes in break fraction has limited effects on correct break point estimation rates of the tests. Figure 5 shows the effects of the changes in the break ratio on the correct break point estimation rates of tests when the break occurred in first quarter, half or the third quarter of panel data formed by $N = 32$ and $T = 12$. In general, the correct break point estimation rate of the methods decreases as the break ratio increases.

In addition, if the post-break mean is 40% or more bigger than the pre-break mean, methods with some exceptions can not accurately estimate the break point depending on the break fraction in general. According to the region where the breaks occur in the series, the test having the highest correct break point estimation rate varies. In the case where the break occurs in the first, second and third quarter of the series, the Bai, Wald and Kim tests have the highest correct estimation rates,

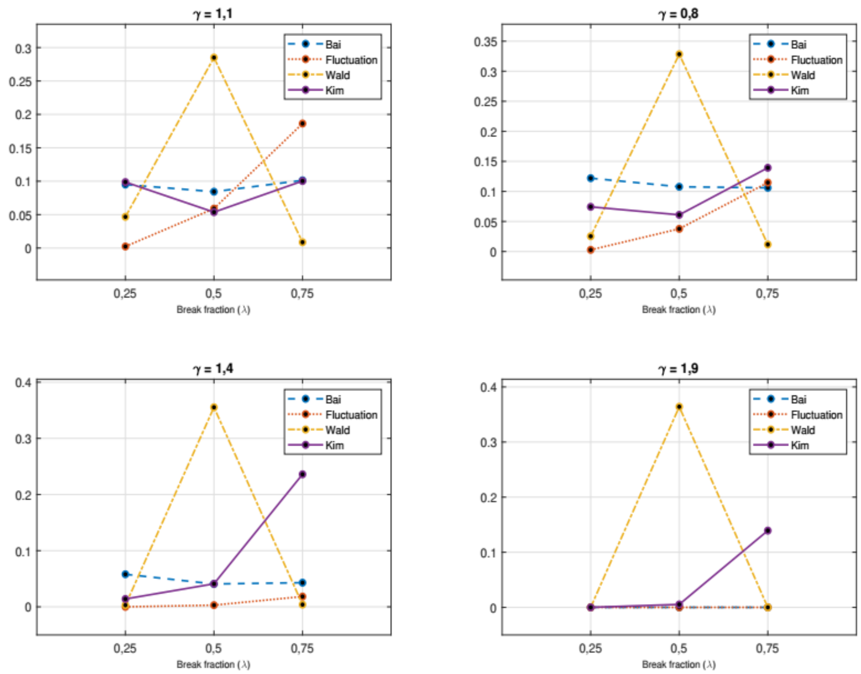

FIGURE 4. Simulation results for correct break point estimation rates ($N = 12$ and $T = 12$)

in order. Nevertheless, in the case where the break occurs in the third quarter of the series and the break ratio is small, the Fluctuation Test has the highest correct break point estimation rate. While the changes in the break fraction have a limited effect on the correct break point estimation rates of the tests except Wald test, they lead to change the tests having the highest correct break point estimation rates based on the occurrence of break in different regions of the series. Figure 6 shows the effects of changes in panel time dimension on the correct break point estimation rates of Bai, Fluctuation, Wald and Kim Tests where the mean of the series is reduced by 20% compared to the pre-break mean in the panel data consisting of 32 cross-sectional units. From Figure 6, it is seen that Bai Test mostly has higher correct estimation rates than others. Furthermore, the change in the break fraction leads to significant changes in the correct estimation rates of the tests. When the break occurs in the middle of the series, the Wald Test has higher correct estimation rates than others for $T = 12$, whereas in the third quarter of the series, the Kim Test has higher correct estimation rate for the smallest time dimension. As the time dimension increases at all levels of the break fraction factor, a decrease in

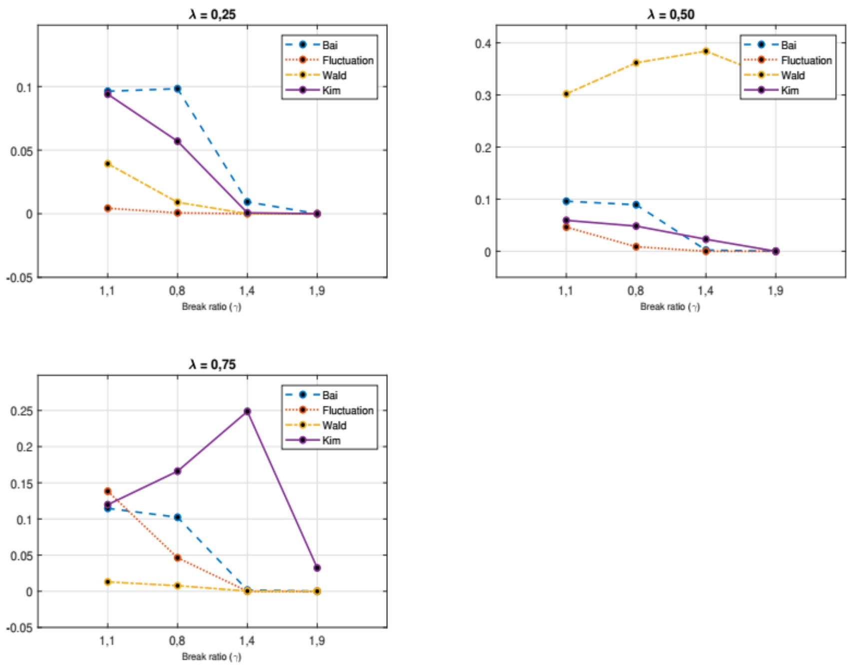

FIGURE 5. Simulation results for correct break point estimation rates ($N = 32$ and $T = 12$)

the correct estimation performance of the tests occurs and correct estimation rates approach to zero.

Nevertheless, the Bai test is the one affected least against time dimension. Thus, for medium and large time dimension, Bai Test has the highest correct break point estimation rate. Figure 7 shows the effects of the cross-section dimension on the correct break point estimation rates under the conditions that the break occurs in the middle of the series, and the mean of the series increases by 10% in post-break. it is seen from Figure 7 that the increase in cross-section dimension have effect on the correct break point estimation rates of the methods on different ways. However, the length of the series forming the panel has a limited impact on the correct estimation rates for Bai Test, while other methods lead to a reduction in the correct estimates. Accordingly, the highest accurate estimation rates for the panels with short time series at these levels of the fraction section and fraction rate factors are reached with the Wald Test, while the highest rates of the other methods are reached with Bai Test as the time series length increases. Figure 8 shows the effects of the cross-section size under the conditions the break point in the third quarter of the series and the mean of the series increases by 10% after

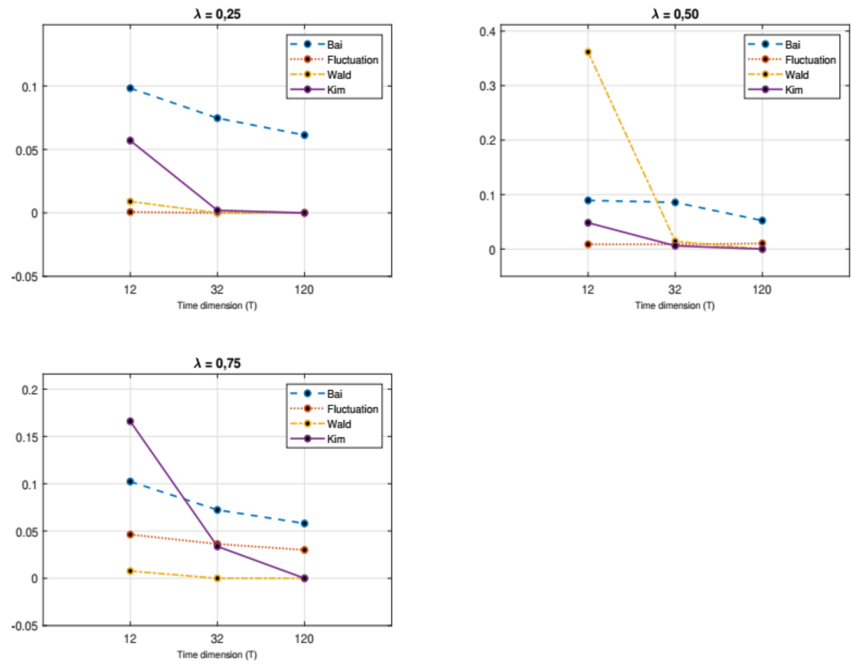

FIGURE 6. Simulation results for correct break point estimation rates ($\gamma = 0,8$ and $N = 32$)

break. It is seen in Figure 8 that increase in the cross section dimension have a different effects on the correct break point estimation rate of the methods. While the increase in cross-section size leads to a decrease in the correct break point estimation rate of the Fluctuation test, it leads to a slight increase in the correct estimation rate of Bai Test. However, the length of time has a different effect on the correct estimation rates of the methods and the correct estimation rates of the tests vary under different time dimension values. While changes in time dimension have a limited effect on the correct estimation rates of Bai Test, it leads to decrease in the correct estimation rates of the other methods. Accordingly, while the highest correct estimation rates in the panels with short time series at this level of the break fraction and the break ratio factors are reached with the Fluctuation Test, as the time lengths increase the highest correct estimation rates are reached with the Bai Test. Moreover, when Figures 3, 7 and 8 are evaluated altogether, it can be concluded that Bai Test has the highest correct estimation rates in the case where break occurs in the first quarter of the series, Wald Test has the highest correct estimation rates in the case where break occurs in the second quarter of the series

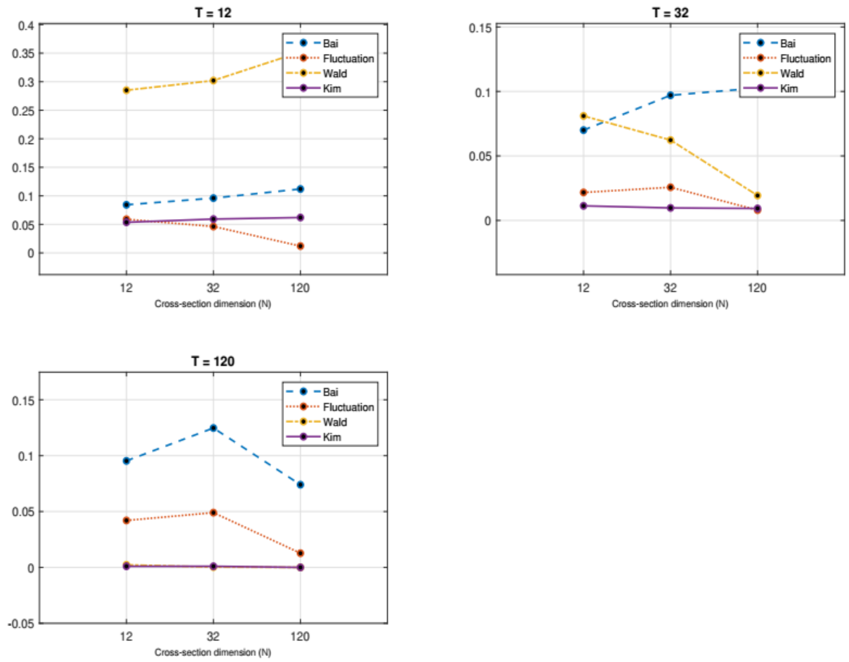

FIGURE 7. Simulation results for correct break point estimation rates ($\gamma = 1,1$ and $\lambda = 0,5$)

and Fluctuation Test has the highest rate in the case where break occurs in the third quarter of the series.

However, while the level of the time dimension factor is 12, the Kim Test reveals the correct estimation rates similar to that of the Bai Test. Figure 9 concludes the effects of break ratio where a break occurs in the third quarter of the series by fixing time dimension at 32. From the Figure 9 it is seen that the correct estimation performance of the tests generally decreases as the break ratio increases. All methods can no longer accurately estimate the actual break point when post-break means of the series are bigger at the rate of %40 or greater than pre-break means of the series. In addition, as the cross-section dimension increases, there is a limited decrease in the correct estimation performance of the tests. The Figure 10 shows that in the panel data with 32 section units and 12 time points, the ratio of the pre-break mean to the post-break mean and the part in which the break occurs are observed to have a significant effect on the correct estimation rates of the methods. While the highest correct break point estimation rates are reached with the Bai Test if the break occurs in the first quarter of the series, the Wald Test has the highest correct estimation rates in case the break occurs in the middle of

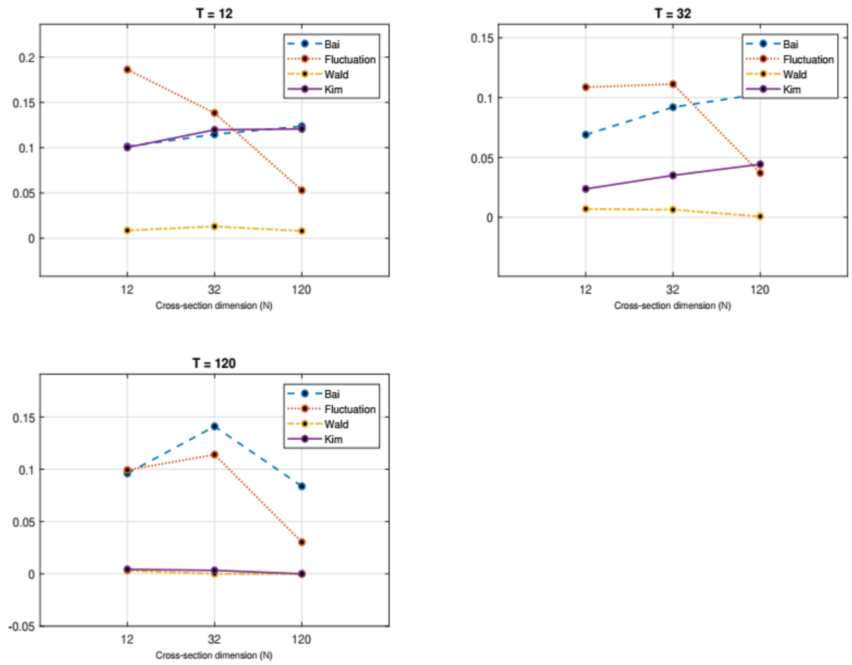

FIGURE 8. Simulation results for correct break point estimation rates ($\gamma = 1,1$ and $\lambda = 0,75$)

the series. When the break occurs in the third quarter of the series, the Fluctuation Test has higher correct estimation rates if the change in the mean of the series is small. Nevertheless, the Kim Test reveals higher correct estimation rates if the change in the mean of the series is large. The Bai Test is the method of which the correct estimation performance is least affected by the changes of break fraction.

5. Conclusion and Suggestions

The correct estimation performance of the Bai, Fluctuation, Wald Statistics and Kim Tests, which are used to determine the structural break date in panel data, are examined via Monte Carlo simulations for the factors time dimension, cross-section dimension, break fraction and break ratio. The results can be concluded as follows:

- Bai Test has higher correct break point estimation rate than the other test methods except for some specific factor levels. If the break occurs in the first quarter of the series, the Bai Test shows a higher correct estimation performance. The Bai Test is the method that is least affected by the changes in the time dimension and the place on where the break point is.

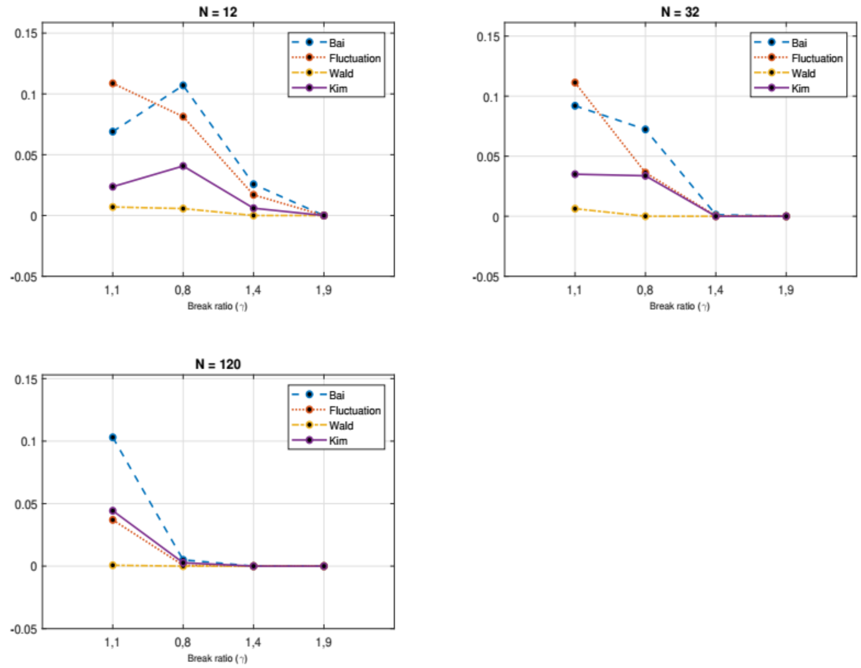

Figure 9. Simulation results for correct break point estimation rates ($T = 32$ and $\lambda = 0,75$)

- Wald Test has the highest correct estimation performance if the break occurs in the middle of the series. Nevertheless, the Bai Test shows a higher correct estimation performance if the break occurs in the middle of the series in panel data with large time dimensions since the changes in the time dimension have effects on the correct break point estimation rate of Bai and Wald Test on different ways.
- Correct estimation performance of the Fluctuation Test increases as the distance between the starting point of the series forming the panel and the break point increases. The Fluctuation Test shows a higher correct estimation performance when the break is in the third quarter of the series.
- The Kim Test shows a lower correct estimation performance than in other cases when the break is in the middle of the series. If the break occurs in the third quarter of the series, the Kim Test reveals a higher correct estimation performance. The Kim Test shows a higher correct estimation performance than the Fluctuation Test in the case when the break occurs in the third quarter of the series and where the time dimension of the panel is medium or large.

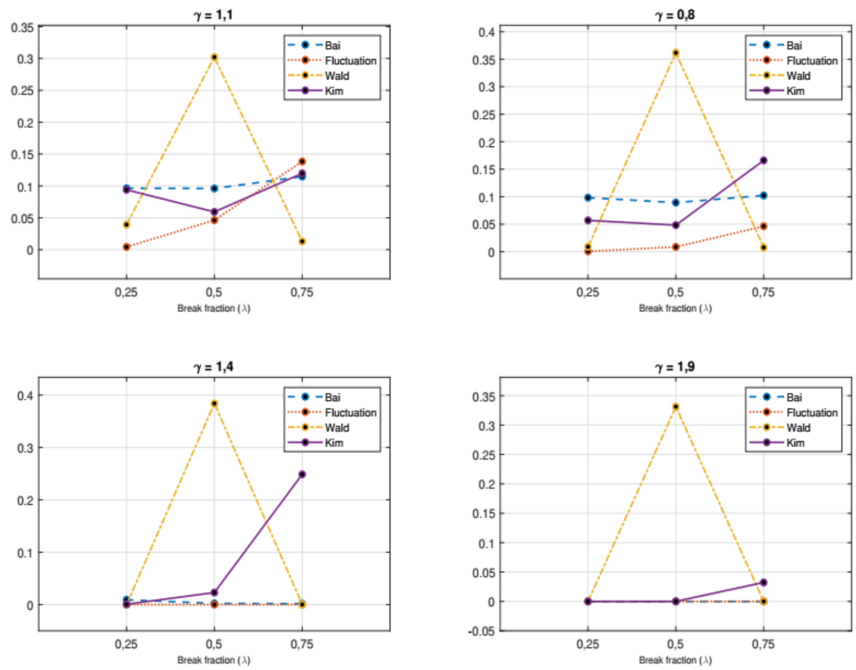

Figure 10. Simulation results for correct break point estimation rates ($N = 32$ and $T = 12$)

- Mostly, increases in the cross-sectional dimension and/or time dimension have the effect on the correct estimation performance of the methods, which is slight but on decreasing direction. Nevertheless, the time dimension changes have a small effect on the Bai Test.
- Changes in the break ratio generally have a negative effect on the correct break point estimation performance of Bai, Fluctuation, Wald and Kim Tests. The methods can no longer correct estimate the actual break point when post-break mean is bigger 40% or more than the pre-break mean of the series.
- While the highest correct estimation rates are reached with the Bai Test in the case when the break occurs in the first quarter of the series, the highest correct estimation rates can be reached by the Wald Test if the break occurs in the middle of the series. In the case when the break occurs in the third quarter of the series, if the change in the mean of the series is small, the Fluctuation Test shows higher correct estimation rates, whereas in the case of a large change in the mean of the series, the Kim Test reveals

higher correct estimation rates. The method of which correct estimation performance is less affected by the changes in the break fraction is Bai Test.

When evaluating the results of the tests concerning the correct break point estimation performance, it is seen that the correct estimation rates of the tests are usually adversely affected by the increase in the factor levels such as increase in the time or the cross-sectional dimensions. This may be due to a larger range of estimation values when the time dimension increases, or the fact that, when the cross-section dimension increases, the estimation of the break point estimates the same value as the standard error decreases. The results so far are the results obtained by evaluating test performances without considering the magnitude of the difference between the actual date of break and the break point estimate. However, it should be taken into consideration that the performance of the tests can be seen adverse since they do not estimate actual break date correctly at all, although they steadily lead very close estimation to the break point. On the other hand, we can only conclude that the tests have a good estimation performance since they have estimated the true break date only once although they generally produce very distant estimations to the actual break point.

6. Concluding Remarks

This paper investigates the performance of methods determining structural break point in a panel data with only one common break point at the time dimension. It is assumed that there is no serial correlation and/or cross section dependency. Also, the performance evaluation is performed with a consideration that there does not exist cross sectional heterogeneity. Although this study is limited for evaluation of performance of the methods assuming time and/or cross sectional dependency, it would be extended of the performance evaluation of these methods in terms of sensitivity or robustness for panel data set with time and/or cross sectional dependency.

The Monte Carlo simulations are based on the Equation 1 of the study which is the basic model of Bai (2010). Therefore, it could be expected that the Bai method or the methods which are compatible to this methodology such as Kim (2011) Model III has better performances more than others in this study. Since correct estimation performance of the methods have been investigated instead of comparing the performance of the considered tests in testing the null hypothesis, the data set is generated by considering only the model based on Equation 1 and the correct estimation rates of the methods were compared. Therefore, although data are produced according to the model proposed by Bai (2010), other methods have executed higher correct estimation performance than Bai (2010) and Kim (2011) at some factor levels. Thus, it has been possible to compare which test is more likely to correct estimate the break point according to the factor levels of the break point estimation methods considered in the study. Nevertheless, in the future studies, it may be useful to compare the correct estimation perspectives of the tests according to factor levels by considering the data generation processes based on other models.

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THE GOMPERTZ EXTENDED GENERALIZED EXPONENTIAL DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract. In this article, a new class of distribution of the exponential family of distributions called the Gompertz extended generalized exponential (G-EGE) distribution for life time processes is proposed. The mathematical properties of the G-EGE distribution such as reliability, hazard rate function, reversed hazard, cumulative, odd functions, quantiles function, kurtosis, skewness and order statistics were derived. The parameters of the G-EGE distribution were estimated using the maximum likelihood method. The efficiency and flexibility of the G-EGE distribution were examined using a simulation study and a real life data application. The results revealed that the G-EGE distribution outperformed some existing distributions in terms of their test statistics.

1. Introduction

Modeling lifetime processes has received several attentions in recent years. However, the lifetime processes rely on the phenomena of distribution. Thus, developing a flexible distributions depends on how the researcher compound one or more distributions to form a more flexible distribution [1]. One of such distributions in modeling lifetime processes is exponential distribution. The exponential distribution is used to describing the time between events with a Poisson processes. Thus, the exponential distribution has been used to model processes with continuous memoryless random processes and constant failure rate. However, the occurrence of constant failure rate is almost impossible in real life. Hence, to account for this shortcoming in distribution theory, [2] modeled lifetime processes with inverted exponential (IE) distribution. The inverted exponential distribution was extensively studied in [3]; who applied it to various data from the field of engineering and medicine. [4] proposed the transmuted inverse exponential distribution and studied its statistical

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properties using data from medicine and engineering. [5] also examined the statistical properties of the exponentiated generalized inverted exponential distribution. [6] proposed the Kumaraswamy inverse exponential distribution. More so, [7] proposed the extended generalized exponential distribution. [8] proposed the Harris extended exponential distribution. [9] proposed the extended Poisson exponential distribution. [10] proposed fractional beta exponential distribution. [11] proposed the exponentiated generalized extended exponential distribution. [12] proposed the moments of the alpha power transformed generalized exponential distribution. [13] proposed the extended weighted exponential distribution. [14] proposed the type I general exponential class of distribution. [15] proposed the Gompertz alpha power inverted exponential distribution. [16] proposed extended new generalized exponential distribution. [17] proposed the alpha power Gompertz distribution. [18] proposed the odd exponentiated half logistic-G family of distribution. [19] proposed a new distribution using the tangent function. [20] proposed generalized exponential distribution. [21] proposed the alpha power inverted exponential distribution. [22] proposed the alpha power Weibull distribution. [23] proposed a new extension of generalized exponential distribution. [24] proposed transmuted exponentiated generalized-G family of distributions. [25] proposed exponentiated generalized-G Poisson distribution. [26] proposed exponentiated generalized class of distributions. [27] proposed a new method for generating distributions with an application to exponential distribution. [28] proposed a method for estimating the generalized inverted exponential distribution.

The cumulative distribution function (cdf) of the extended generalized distribution is given as

$$G(x;\gamma,\beta)=\frac{(\beta-e^{-x})^{\gamma}-(\beta-1)^{\gamma}}{(\beta^{\gamma}-(\beta-1)^{\gamma})}\ (\beta^{\gamma}-(\beta-1)^{\gamma})\neq 0\ x>0,\ \gamma>1\ ,\beta>1. \tag{1}$$

The corresponding probability density function (pdf) to Equation (1) is given as

$$g(x;\gamma,\beta)=\frac{\gamma(\beta-e^{-x})^{\gamma-1}e^{-x}}{(\beta^{\gamma}-(\beta-1)^{\gamma})}\ (\beta^{\gamma}-(\beta-1)^{\gamma})\neq 0\ x>0,\ \gamma>1\ ,\beta>1, \tag{2}$$

where γ is shape parameter and β is the scale parameter.

Also, the Gompertz distribution is a continuous distribution used to describe the lifespan of stochastic processes. Hence, there exist a relationship between the exponential and the Gompertz distributions. A lot of researchers have developed different compound distributions using the exponential and Gompertz distributions. However, no knowledge of Gompertz extended generalized exponential distribution was found in existing literature. Hence, this study is motivated to bridge the gap in existing literature by proposing a lifetime distribution called Gompertz extended generalized exponential (G-EGE) distribution using the Gompertz-G characterization. This distribution is further applied to glass fibre to examine its efficiency and flexibility.

Let $G(x;\tau)$ and $g(x;\tau)$ be the baseline model with parameter vector τ. Then, the cdf of Gompertz-G family proposed in [29] is given as

$$F(x) = \int_0^{B[G(x;\tau)]} u(t)dt, \tag{3}$$

where $u(t)$ is the probability density function of the Gompertz distribution and $B[G(x;\tau)] = -log[1 - G(x;\tau)]$ is the link function.

The cumulative distribution function in Equation (3) can be expressed as

$$F(x) = \int_0^{-\log[1-G(x;\tau)]} \theta\, e^{\lambda t - \frac{\theta}{\lambda}(e^{\lambda t}-1)} dt = 1 - e^{\frac{\theta}{\lambda}(1-(1-G(x,\tau))^{-\lambda})}\ for\ \theta\ >\ 0\ \lambda\ >\ 0, \tag{4}$$

where λ and θ are additional two shape parameters.

The pdf that corresponds to the G-family of distribution is given as

$$f(x) = [\frac{d}{dx}B[G(x;\tau)]]u[B[G(x;\tau)]] = \theta g(x;\tau)[1 - G(x;\tau)]^{-\lambda-1} e^{\frac{\theta}{\lambda}(1-(1-G(x;\tau))^{-\lambda})}. \tag{5}$$

A random variable X with pdf in Equation (5) is denoted by $X \sim Gompertz - G(\theta, \lambda, \tau)$.

The aim of this study is to propose a G-EGE class of the family of the exponential distribution and examining its statistical characteristics extensively.

This paper is unfolded as follows. In Section 2, we define the G-EGE distribution and a plot for its pdf, cdf and hazard rate function (hrf). Useful mixture representation of the pdf is derived in Section 3. In Section 4 derives some mathematical properties of the newly proposed class of distribution. In Section 5, the order statistics is obtained. The maximum likelihood estimates (MLEs) of the newly proposed class of distribution and simulation are performed in Section 6. The viability of the new class of distribution is examined in Section 7 by means of real life data sets. Section 7 is the concluding remarks.

2. The Gompertz Extended Generalized Exponential Distribution

In this section, we shall establish the pdf and the cdf of the newly proposed continuous distribution. Let X be a continuous random variable. Then, X follows an G-EGE distribution if its pdf is given as

$$\begin{aligned} f_{(G-EGE)}(x;\theta,\lambda,\beta,\gamma) = \theta\ \gamma\ \exp\big(-x\big)\ \Big(\beta^{\gamma} - \big(\beta-1\big)^{\gamma}\Big)^{\lambda}\Big(\beta - \exp\big(-x\big)\Big)^{\gamma-1} \\ \times \Big[\beta^{\gamma} - \big(\beta - \exp\big(-x\big)\big)^{\gamma}\Big]^{-(\lambda+1)} \\ \times \exp\left(\frac{\theta}{\lambda}\left\{1 - \Big(\frac{\beta^{\gamma} - \big(\beta-1\big)^{\gamma}}{\beta^{\gamma} - \big(\beta - \exp(-x)\big)^{\gamma}}\Big)^{\lambda}\right\}\right) \end{aligned} \tag{6}$$

$$for\ \theta\ >\ 0\ \lambda\ >\ 0\ x>0,\ \gamma>1\ ,\beta>1.$$

The cdf that corresponds to the pdf is given as

$$F_{(G-EGE)}(x)=1-\exp\left(\frac{\theta}{\lambda}\left\{1-\Big(\frac{\beta^{\gamma}-\big(\beta-1\big)^{\gamma}}{\beta^{\gamma}-\big(\beta-\exp(-x)\big)^{\gamma}}\Big)^{\lambda}\right\}\right) \tag{7}$$

$$for\ \theta\ >\ 0\ \lambda\ >\ 0\ x>0,\ \gamma>1\ ,\beta>1,$$

where γ is shape parameter and β is the scale parameter; λ and θ are additional two shape parameters.

Figure 1 shows the plots of the G-EGE density for some selected values of the parameters γ, β, λ and θ. The pdf plots indicate that the G-EGE distribution can be unimodal, left skewed, increasing and decreasing.

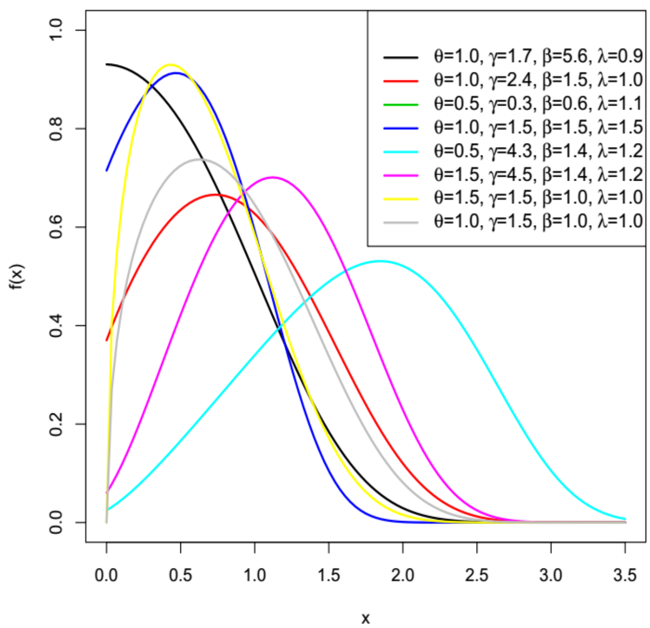


FIGURE 1. The plots of the G-EGE pdf for some parameter values.

The Hazard Rate function (hrf), reliability function (rf) and cumulative hazard rate function (chrf) of the random variable X are given respectively as

$$hrf(x)=\frac{f_{(G-EGE)}(x)}{1-F_{(G-EGE)}(x)}=\theta\ \gamma\ \exp\big(-x\big)\ \Big(\beta^{\gamma}-\Big(\beta-1\Big)^{\gamma}\Big)^{\lambda}\Big(\beta-\exp\big(-x\big)\Big)^{\gamma-1}$$
$$\times\left[\beta^{\gamma}-\big(\beta-\exp\big(-x\big)\big)^{\gamma}\right]^{-(\lambda+1)}. \tag{8}$$

Figure 2 shows the plots for the hazard rate function of the G-EGE distribution. The plots shows that the G-EGE density is increasing and bathtub depending on the values of the parameters γ, β, λ, and θ.

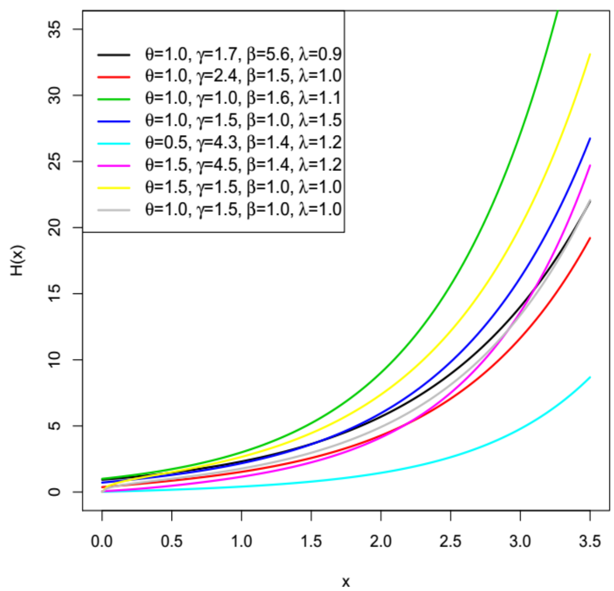


FIGURE 2. The plots of the G-EGE hrf for some parameter values.

$$R(x) = 1 - F_{(G-EGE)}(x) = \exp\left(\frac{\theta}{\lambda}\left\{1 - \left(\frac{\beta^{\gamma} - \left(\beta - 1\right)^{\gamma}}{\beta^{\gamma} - \left(\beta - \exp(-x)\right)^{\gamma}}\right)^{\lambda}\right\}\right). \tag{9}$$

$$H(x) = -InR_{(G-EGE)}(x) = \left\{\frac{\theta}{\lambda}\left(\frac{\beta^{\gamma} - \left(\beta - 1\right)^{\gamma}}{\beta^{\gamma} - \left(\beta - \exp(-x)\right)^{\gamma}}\right)^{\lambda}\right\} - \frac{\theta}{\lambda}. \tag{10}$$

3. MIXTURE REPRESENTATION

The quantity $\left(\beta - \exp(-x)\right)^{\gamma}$ can be expressed as

$$\sum_{k=0}^{\gamma}(-1)^k\binom{\gamma}{k}\beta^{\gamma-k}\exp(-xk).$$

More so, the quantity $\Big(\beta^{\gamma}-\big(\beta-\exp(-x)\big)^{\gamma}\Big)^{\lambda+1}$ can be expressed as

$$\sum_{k=0}^{\gamma}\sum_{p=0}^{\lambda+1}(-1)^{p(k+1)}\binom{\lambda+1}{p}\binom{\gamma}{k}^{p}\beta^{\lambda(\gamma+1)+p(\gamma-\lambda-k)}\exp(-xkp).$$

Thus, inserting these expressions into Equation (6) and after some algebraic simplification we expanded Equation (6) as

$$\begin{aligned} f(x) \quad = \quad & \sum_{k=0}^{\gamma}\sum_{p=0}^{\lambda+1}\sum_{i=0}^{\gamma-1}\frac{(\gamma-1)!}{(\gamma-i-1)!i!}\theta\ \gamma\ \exp\big(-x\big)\ \Bigg(\beta^{\gamma}-\Big(\beta-1\Big)^{\gamma}\Bigg)^{\lambda}\Bigg(-1\Bigg)^{i-p(k+1)} \\ & \times\exp\big(-xi\big)a^{m+j}\beta^{\Big(\gamma-i-1\Big)-\Big(\lambda(\gamma+1)+p(\gamma-\lambda-k)\Big)}\exp\big(xkp\big) \\ & \times\exp\Bigg(\frac{\theta}{\lambda}\Bigg\{1-\Big(\frac{\beta^{\gamma}-\big(\beta-1\big)^{\gamma}}{\beta^{\gamma}-\big(\beta-\exp(-x)\big)^{\gamma}}\Big)^{\lambda}\Bigg\}\Bigg), \qquad (11) \end{aligned}$$

where

$$a^{j}=\Big[\frac{(\lambda-p+1)!p!}{(\lambda+1)!}\Big]\ and\ a^{m}=\Big[\frac{(\gamma-k)!k!}{\gamma!}\Big]^{p}.$$

Expanding the binomial terms, we have

$$f(x)=\sum_{k=0}^{\gamma}\sum_{p=0}^{\lambda+1}\sum_{i=0}^{\gamma-1}\upsilon_{i,k,p}\ \exp\Bigg(-x\ D_{ikp}-m\Bigg(\beta^{\gamma}-\big(\beta-e^{-x}\big)^{\gamma}\Bigg)^{-\lambda}\Bigg), \qquad (12)$$

where

$$\begin{aligned} \upsilon_{i,k,p} &= \frac{(\gamma-1)!}{(\gamma-i-1)!i!}\theta\ \gamma\ \Bigg(\beta^{\gamma}-\Big(\beta-1\Big)^{\gamma}\Bigg)^{\lambda}\Bigg(-1\Bigg)^{i-p(k+1)}a^{m+j} \\ & \times\beta^{-\Big(\lambda(\gamma+1)+p(\gamma-\lambda-k)\Big)+\Big(\gamma-i-1\Big)}\ \exp\big(\frac{\theta}{\lambda}\big). \\ D_{ikp} &= \Big(i-kp+1\Big), \\ m &= \frac{\theta}{\lambda}\Big(\frac{1}{\beta^{\gamma}-(\beta-1)^{\gamma}}\Big)^{-\lambda}. \end{aligned}$$

4. Mathematical Properties

This section investigates some statistical properties of the G-EGE distribution. This includes quantile and random number generation, Skewness, Kurtosis and order statistics. These structural properties of the G-EGE distribution can be computed efficiently by using programming softwares like R, Mathematical, Maple and Matlab.

4.1. **Quantile function and random number generation.** Let X be a random variable such that $X \sim G-EGE(\theta,\beta,\gamma,\lambda)$. Then, the quantile function of X for $p \in (0,1)$ is obtained by inverting Equation (7) as

$$x_p = -\log\Big[\beta - \Big(\beta^{\gamma} - \big(\beta^{\gamma} - (\beta-1)^{\gamma}\big)\big(1 - \frac{\lambda}{\theta}\log(1-p)\big)^{-\frac{1}{\lambda}}\Big)^{\frac{1}{\gamma}}\Big]. \qquad (13)$$

Setting $p = 0.5$ in Equation (13) gives the median M of X as

$$x_{0.5} = -\log\Big[\beta - \Big(\beta^{\gamma} - \big(\beta^{\gamma} - (\beta-1)^{\gamma}\big)\big(1 - \frac{\lambda}{\theta}\log(0.5)\big)^{-\frac{1}{\lambda}}\Big)^{\frac{1}{\gamma}}\Big] \; 0 < p < 1. \qquad (14)$$

Simulating the G-EGE random variable is flexible. If U is a uniform variates on the interval $(0,1)$, then the random variable $X = x_p$ at $p = U$ follows the $x_p \sim G-EGE(\theta,\beta,\gamma,\lambda)$ of Equation (6).

However, the 25^{th} and 75^{th} percentile for the random variable X are obtained as

$$x_{0.25} = -\log\Big[\beta - \Big(\beta^{\gamma} - \big(\beta^{\gamma} - (\beta-1)^{\gamma}\big)\big(1 - \frac{\lambda}{\theta}\log(0.75)\big)^{-\frac{1}{\lambda}}\Big)^{\frac{1}{\gamma}}\Big], \qquad (15)$$

$$x_{0.75} = -\log\Big[\beta - \Big(\beta^{\gamma} - \big(\beta^{\gamma} - (\beta-1)^{\gamma}\big)\big(1 - \frac{\lambda}{\theta}\log(0.25)\big)^{-\frac{1}{\lambda}}\Big)^{\frac{1}{\gamma}}\Big]. \qquad (16)$$

4.2. **Skewness and Kurtosis.** The Bowleys formula for coefficient of skewness is given as

$$Sk = \frac{x_{0.75} - 2x_{0.5} + x_{0.25}}{x_{0.75} - x_{0.25}}.$$

However, the Moors formula for coefficient of kurtosis is given as

$$Ks = \frac{x_{0.875} - x_{0.625} - x_{0.375} + x_{0.125}}{x_{0.75} - x_{0.25}}.$$

4.3. **Order statistics.** Let $X_1, X_2, \cdots, X_n$ be a random sample of size n of the $f_{APEGE}(x)$ distribution and $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the corresponding order statistics. Then, probability density function of the ith order statistics X_k, say $f_k(x)$ is expressed as

$$g_k(y_k) = \frac{n!}{\Big(k-1\Big)!\Big(nk\Big)!}\Big[F_{G-EGE}(y_k)\Big]^{k-1} f_{G-EGE}(y_k)\Big[1 - F_{G-EGE}(y_k)\Big]^{k-1}. \qquad (17)$$

We can write

$$\begin{aligned} g_k(y_k) = &\frac{n!}{\Big(k-1\Big)!\Big(nk\Big)!} \\ &\times \Bigg[1-\exp\Bigg(\frac{\theta}{\lambda}\Bigg\{1-\Big(\frac{\beta^\gamma-\big(\beta-1\big)^\gamma}{\beta^\gamma-\big(\beta-\exp(-x)\big)^\gamma}\Big)^\lambda\Bigg\}\Bigg)\Bigg]^{k-1} \\ &\times \sum_{k=0}^{\gamma}\sum_{p=0}^{\lambda+1}\sum_{i=0}^{\gamma-1} \upsilon_{i,k,p}\ \exp\Bigg(-x\Big(i-kp+1\Big)+\frac{\theta}{\lambda}\Bigg\{1-\Big(\frac{\beta^\gamma-\big(\beta-1\big)^\gamma}{\beta^\gamma-\big(\beta-\exp(-x)\big)^\gamma}\Big)^\lambda\Bigg\}\Bigg) \\ &\times \Bigg[\exp\Bigg(\frac{\theta}{\lambda}\Bigg\{1-\Big(\frac{\beta^\gamma-\big(\beta-1\big)^\gamma}{\beta^\gamma-\big(\beta-\exp(-x)\big)^\gamma}\Big)^\lambda\Bigg\}\Bigg)\Big)\Bigg]^{n-k}. \end{aligned} \tag{18}$$

The order statistics for the G-EGE distribution can be obtained as follows:

- The minimum order statistics is obtained for $k = 1$.
- The median is obtained when $k = m = 1$, given n is odd expressed as $n = 2m + 1$.
- The maximum order statistics is obtained for $k = n$ for even n expressed as $n = 2m$.

5. Parameter Estimation

Several approaches have been employed for parameter estimation in literature. In this article, the maximum likelihood method was adopted to obtain the parameters of the G-EGE. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a random sample of the G-EGE model with unknown parameter vector $\theta = (\theta, \beta, \gamma, \lambda)^T$. Then, the log-likelihood function ℓ of the G-EGE distribution can be expressed as

$$\ell = n\log\theta + n\log\gamma - \sum_{i=1}^{n} x_i + n\lambda\log z - \Big(\lambda+1\Big)\sum_{i=1}^{n} s_i + \sum_{i=1}^{n}\frac{\theta}{\lambda}\Bigg\{1-\Big(\frac{z}{s_i}\Big)^\lambda\Bigg\}, \tag{19}$$

where

$$z = \beta^\gamma - \Big(\beta-1\Big)^\gamma \ \textit{and}\ s = \beta^\gamma - \Big(\beta-\exp(-x)\Big)^\gamma$$

However, the partial derivative of the ℓ with respect to each parameter is given as

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \frac{1}{\lambda}\sum_{i=1}^{n}\Bigg\{1-\Big(\frac{z}{s_i}\Big)^\lambda\Bigg\}, \tag{20}$$

$$\frac{\partial\ell}{\partial\gamma} = \frac{n}{\gamma} + \frac{n\lambda z'_\gamma}{z} - \Big(\lambda+1\Big)\sum_{i=1}^{n} s'_\gamma - \sum_{i=1}^{n}\theta\Bigg\{z^{\lambda-1}z'_\gamma s_i^{-\lambda} - z^\lambda s_i^{-\lambda-1}s'_\gamma\Bigg\}, \tag{21}$$

$$\frac{\partial \ell}{\partial \lambda} = n \log z - \sum_{i=1}^{n} s_i - \frac{n\theta}{\lambda^2} + \sum_{i=1}^{n} \left\{ \frac{\theta}{\lambda^2} \left(\frac{z}{s_i} \right)^{\lambda} - \frac{\theta}{\lambda} \left(\frac{z}{s_i} \right)^{\lambda} \ln \left(\frac{z}{s_i} \right) \right\}, \tag{22}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} = & \frac{n\lambda\gamma \Big(\beta^{\gamma-1} - \big(\beta - 1\big)^{\gamma-1} \Big)}{\beta^{\gamma} - \big(\beta-1\big)^{\gamma}} - \gamma\big(\lambda+1\big) \sum_{i=1}^{n} \Big(\beta^{\gamma-1} - \big(\beta - \exp(-x)\big)^{\gamma-1} \Big) \\ & - \sum_{i=1}^{n} \gamma\theta \bigg(z^{\lambda-1} s_i^{-\lambda} \Big(\beta^{\gamma-1} - (\beta-1)^{\gamma-1} \Big) - z^{\lambda} s_i^{(\lambda+1)} \Big(\beta^{\gamma} - \big(\beta - \exp(-x)\big)^{\gamma-1} \Big) \bigg), \end{aligned} \tag{23}$$

where

$$z'_{\gamma} = \frac{\partial z}{\partial \gamma}; s'_{\gamma} = \frac{\partial s}{\partial \gamma}.$$

The solution to the vector is obtained analytically using Newton-Raphson algorithm. Software like MATLAB, R, MAPLE, and so on could be used to obtain the estimates.

5.1. **Simulations study.** A simulation is carried out to test the flexibility and efficiency of the G-EGE distribution. Table 1 shows the simulation for different values of parameters for the G-EGE distribution. The simulation is performed as follows:

- Data are generated using

$$x = -\log \Big[\beta - \Big(\beta^{\gamma} - \big(\beta^{\gamma} - \big(\beta - 1\big)^{\gamma} \big) \big(1 - \frac{\lambda}{\theta} \log\big(1-p\big) \big)^{-\frac{1}{\lambda}} \Big)^{\frac{1}{\gamma}} \Big], \ 0 < p < 1$$

- The values of the parameters are set as follows: $\gamma = 1.5$, $\theta = 1.3$, $\lambda = 1.5$, and $\beta = 2.0$
- The sample sizes are taken as $n = 50, 100, 150, 250$ and 350.
- Each sample size is replicated 1000 times.

The bias is calculated by (for $S = \hat{a}, \hat{b}, \hat{\alpha}, \hat{\lambda},$)

$$\hat{B}ias_S = \frac{1}{1000} \sum_{i=1}^{1000} \Big(\hat{S}_i - S \Big).$$

Also, the MSE is obtained as

$$\hat{M}SE_S = \frac{1}{1000} \sum_{i=1}^{1000} \Big(\hat{S}_i - S \Big)^2.$$

The simulation study investigates the average estimates (MEs), biases, variance, means squared errors and roots means squared errors. The results are shown in Table 1. The results of the Monte Carlo study show that the MSEs and RMSEs decay towards zero as the sample size increases. This corroborates the first-order asymptotic theory. The mean estimates of the parameters tend to the true parameter

Table 1. A simulation Study of the G-EGE Distribution

Sample size	Parameter	Average estimate	Bias	Variance	MSE	RMSE
50	γ	1.4859	-0.0141	0.1470	0.1472	0.3836
	θ	1.2792	-0.0208	0.0299	0.0303	0.1741
	λ	1.5913	0.0913	0.1078	0.1162	0.3408
	β	2.1554	0.1554	0.3791	0.4032	0.6350
100	γ	1.5168	0.0168	0.0739	0.0742	0.2723
	θ	1.3097	0.0097	0.0146	0.0147	0.1211
	λ	1.5657	0.0657	0.0775	0.0818	0.2860
	β	2.0793	0.0793	0.1920	0.1983	0.4453
150	γ	1.5143	0.0143	0.0549	0.0551	0.2348
	θ	1.3153	0.0153	0.0105	0.0107	0.1036
	λ	1.5752	0.0752	0.0716	0.0773	0.2779
	β	2.0492	0.0492	0.1212	0.1237	0.3517
250	γ	1.5187	0.0187	0.0340	0.0343	0.1853
	θ	1.3325	0.0325	0.0065	0.0076	0.0869
	λ	1.5665	0.0665	0.0491	0.0535	0.2314
	β	2.0381	0.0381	0.0729	0.0743	0.2726
350	γ	1.5135	0.0135	0.0228	0.0229	0.1515
	θ	1.3317	0.0317	0.0040	0.0050	0.0706
	λ	1.5792	0.0792	0.0416	0.0478	0.2187
	β	2.0248	0.0248	0.0501	0.0508	0.2253

values as the sample size increases. This corroborates the fact that the asymptotic normal distribution provides an adequate approximation of the estimates.

6. Data Analysis

In this section, the flexibility of the newly developed G-EGE model is proven by means of a real life datasets. The fits of G-EGE model is compared with Weibull Frechét (WFr), extended generalized exponential (EGE), Weibull alpha power inverted exponential (WAPIE), Kumaraswamy Frechét (KFr), transmuted Frechét (TFr), transmuted Marshall-Olkin Frechét (TMOFr), Kumaraswamy alpha power inverted exponential (KAPIE), Kumaraswamy inverted exponential (KIE), beta Lomax (BL), alpha power inverted exponential (APIE) and exponential(E) distributions. However, these models were chosen base on their relationship that enables us make effective and efficient conclusion about their test statistics.

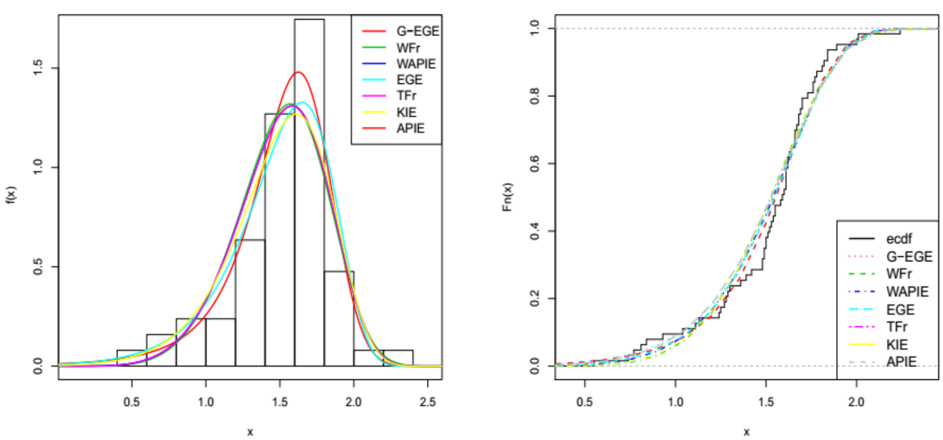
The following criteria were used to determine the best fit: Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), and Hannan and Quinn Information Criteria (HQIC). The test statistics are given as follows: $AIC = -2\hat{\ell} + 2k$, $BIC = -2\hat{\ell} + k\log(n)$,

$CAIC = -2\hat{\ell} + \frac{2kn}{n-k-1}$, $HQIC = -2\hat{\ell} + 2k\log(\log(n))$, where n is the sample size, k is the number of model parameters and $\hat{\ell}$ is minus twice the maximized log-likelihood. The model with the lowest values test statistics is chosen as the best model to fit the datasets.

The first set of data on 1.5 cm strengths of glass fibres were obtained by workers at the UK National Physical Laboratory was used to compare the performance of the G-EGE distribution as used by [30], [31], [32], [33], [34], [35] and [36].

The performance of a model is determined by the value that corresponds to the lowest Akaike Information Criteria (AIC) as the *best* model. In the real life cases considered in Table 2, the *G-EGE* distribution has the lowest AIC value with 37.6.

Figure 3 shows the plots of the estimated densities together with the estimated cdfs of the models under consideration. These plots show that the G-EGE distribution produces a better fit than others models.



(a) Empirical estimated densities

(b) Empirical estimated cdfs

FIGURE 3. The plots of empirical estimated pdfs and cdfs of the G-EGE model

7. Conclusion

The G-EGE distribution has been successfully derived. The basic statistical properties of the G-EGE distribution such as the order statistics, cumulative hazard function, reversed hazard function, quantile, median, hazard function, odds function have been successfully established. The G-EGE distribution was also explicitly expressed as a linear function of the exponential distribution. The order statistics of the proposed distribution was also derived. A simulation study of the proposed model was also illustrated. The simulation shows that the shape of the proposed

Table 2. Performance rating of the G-EGE distribution with glass fibers dataset

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
G-EGE	$\hat{\theta} = 0.0085$ $\hat{\lambda} = 3.5696$ $\hat{\beta} = 8.6251$ $\hat{\gamma} = 0.1765$	**37.6**	**38.3**	**46.2**	**41.0**	**0.14**	**0.84**
Weibull Frechét	$\hat{\alpha} = 0.0207$ $\hat{\beta} = 10.0442$ $\hat{a} = 0.4430$ $\hat{b} = 0.3690$	39.3	39.7	47.6	42.4	0.26	1.42
Weibull Alpha Power Inverted Exponential	$\hat{\alpha} = 0.0058$ $\hat{\beta} = 4.9797$ $\hat{\lambda} = 0.3655$ $\hat{\gamma} = 2.0357$	39.6	40.2	48.1	42.9	0.27	1.46
Kumaraswamy Frechét	$\hat{\alpha} = 2.1160$ $\hat{\beta} = 0.7401$ $\hat{a} = 5.5043$ $\hat{b} = 857.3434$	47.6	48.3	56.2	51.0	0.26	1.42
Kumaraswamy Alpha Power Inverted Exponential	$\hat{a} = 1.04428$ $\hat{b} = 19.3039$ $\hat{c} = 7.4277$ $\hat{\alpha} = 0.0021$	52.7	53.4	61.3	56.1	0.51	2.77
Kumaraswamy Inverted Exponential	$\hat{\alpha} = 3.0232$ $\hat{\lambda} = 163.2152$ $\hat{\beta} = 2.6961$	53.4	53.8	59.9	56.0	0.51	2.83
Transmuted Marshall-Olkin Frechét	$\hat{\alpha} = 0.6524$ $\hat{\beta} = 6.8744$ $\hat{\lambda} = 376.2684$ $\hat{\gamma} = 0.1499$	56.5	57.2	65.1	59.9	2.50	3.10
Beta Lomax	$\hat{\alpha} = 18.1737$ $\hat{\beta} = 26.7645$ $\hat{a} = 10.8769$ $\hat{b} = 0.0329$	56.8	57.5	65.4	60.2	2.54	3.20
Transmuted Frechét	$\hat{\alpha} = 1.3068$ $\hat{\beta} = 2.7898$ $\hat{\lambda} = 0.1298$	100.1	100.5	106.6	102.7	0.99	4.28
G	$\hat{\alpha} = 0.5128$ $\hat{\beta} = 0.5009$	141.4	141.6	145.6	143.1	2.02	3.42
Extended Generalized Exponential	$\hat{\alpha} = 144.0791$ $\hat{\beta} = 0.0550$ $\hat{\lambda} = 137.8711$ $\hat{\gamma} = 7.994$	145.3	145.9	153.8	148.6	0.99	4.25
Exponential	$\hat{\lambda} = 0.6637$	179.6	181.8	185.9	179.7	1.00	4.29
Alpha Power Inverted Exponential	$\hat{\alpha} = 53.5634$ $\hat{\lambda} = 0.3509$	196.3	196.5	200.6	198.0	0.78	4.24

distribution could be inverted bathtub or decreasing (depending on the value of the parameters). The new distribution was applied to a real life data. It shows that the G-EGE distribution performed better than some existing models in literature.

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A COMPARATIVE STUDY OF CLASSIFIERS FOR EARLY DIAGNOSIS OF GESTATIONAL DIABETES MELLITUS

PRIYA SHIRLEY MULLER AND M. NIRMALA

Abstract. Gestational Diabetes Mellitus (GDM), usually found deploying a medical test called the Oral Glucose Tolerance Test (OGTT), is a prevalent complication during pregnancy. Early detection of GDM and identifying the most influential risk factors of GDM pose to be a challenging problem and is found to be crucial as GDM has dreadful health indications for both mother and the baby. The performances of computational techniques like Radial Basis Function (RBF) neural network and Multilayer Perceptron Network (MLP) were collated with that of the statistical technique Discriminant Analysis (DA) on real time GDM datasets for diagnosis of GDM in multigravida pregnant women, specifically women who have been pregnant more than once, without even a visit to the hospital. The most influential risk factors were identified using DA while the overall performance of MLP beyond doubt established itself to be the most effective technique for early diagnosis of GDM in women during pregnancy.

1. Introduction

Diabetes Mellitus is causing havoc and concern amongst the health experts as it is greatly instrumental in the increasing burden of diseases which are non-communicable. Sadly, India is no different. According to the World Health Organization (WHO), existence of diabetes mellitus (DM) in adults showed a rise of more than 120% from 135 million in the year 1995 to a staggering 300 million in 2025 [1]. In a survey conducted, the percentage of pregnant woman who was diagnosed with GDM in the urban population of Chennai was found to be 16.2% [2].

GDM is defined as intolerance of carbohydrate levels of differing severity with onset or foremost identification during pregnancy [3]. The birth of a child from GDM mother is susceptible in getting affected by obesity while growing up and

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possibly with DM type ii during subsequent stages of their lifespan [4]. Moreover, their offspring are more prone to an added threat of health issues like jaundice, hypoglycemia and fetal macrosomia. Delivery complications like Caesarean section, pre-eclampsia and an extended danger of having type 2 diabetes or even Type1 after delivery are more incident amongst ladies with GDM. However, gestational diabetes is a treatable condition. The WHO has recommended using a 2 hour 75 g OGTT to systemize the diagnosis of GDM, which is generally performed between 24 and 28 weeks [5]. Thus a pregnant woman who may be prone to gestational diabetes shall undergo the conventional medical blood tests only in the period of six to seven months of her pregnancy. Discerning individuals who are at danger of developing GDM is the growing need of the hour. Various studies have put on record that early detection of gestational diabetes actually lowered mortality of mother and child and also helped improve the woman's well being in terms of health [6][7][8]. More importantly, as the rate of babies who are born dead is relatively high in India and gestational diabetes mellitus is undoubtedly one of the causes, early diagnosis and awareness of GDM is an utmost priority in the society today [9].

2. Literature Review

Nanda et al.[10] used an analysis on predicting complications during pregnancy in the early stage to build a methodology for forecasting gestational diabetes using biochemical markers, characteristics of the pregnant women. The classification power of the models for detection of GDM in pregnant women who were prone to developing GDM was collated by Tran et al.[11] using a few diagnostic norms on the basis of 75-g oral glucose tolerance test and finally summarized for screening of GDM selectively in places like Vietnam, an ordinary prognostic model using Body Mass Index (BMI) and age at booking was adequate. Okeh et al.[12] applied a semi-parametric linear mixed model to determine the effect of covariates on the precision of the results of diagnostic tests by deriving a general cut off estimate for selecting patients to perform glucose tests during pregnancy explained implementing gestational diabetes data. Fuzzy integral was used by Zhang et al.[13] to develop the classification model of GDM. Training of BPN was done to obtain the Sugeno measure and the BP neural network was optimized using the algorithm of simulated annealing to acquire an estimated global solution which was optimal. A universal screening program to detect GDM was extremely cost-effective in Israel and India concluded Lohse et al.[14] by examining whether selection process of pregnant women for diagnosing GDM was economical and used published core diabetes model to estimate the long-term impact of screening through their study.

The above survey infers that while taking into consideration the facts and figures needed to be collected for the analysis, there is certainly a minimum of one data for which the pregnant woman is in need of help of a medical staff from the hospital. By providing newly designed input variables, the article aims to diagnose GDM in an early stage among pregnant women without performing a blood test. The article utilizes Artificial Neural Networks namely a supervised MLP network using Back propagation algorithm and RBF Network and the statistical technique Discriminant Analysis for classification of GDM and compares the efficiency of these diagnostic models.

3. Methodology

3.1. **Artificial Neural Network.** A computational arrangement which bears a strong resemblance to the biological networks consisting of neurons in the human brain basically explains an artificial neural network. Because of their ability to adapt easily, a salient feature of these networks, these networks go a long way in solving problems in diagnosis of diseases. Neural networks are known for recognizing the patterns which are hidden between predictor variables and dependent variables and are commonly applied to model complex relationships between them.

3.1.1. *Fundamentals of Multilayer Perceptron Network.* Using hidden layers, the separation of the relationship between the inputs and the output into a sequence of stages which are linearly separable is the most important essence of neural networks [15]. The diagnostic system comprises of three varied modules. The input module which receives data from the patient is the first module. It then transfers it to the second module, which classifies the given input patient's case record. The classification system output is displayed by the third module which is an output module. For an input pattern z_p, with an only pass forward, the MLP Network's return is evaluated. For every output unit o_k , the output is given by

$$O_{k,p} = f_{o_k}\Big(\sum_{J=1}^{J+1} w_{kj} f_{y_j}\big(\sum_{i=1}^{I+1} v_{ji} z_{i,p}\big)\Big) \tag{1}$$

where the activation function for o_k is f_{ok} and the activation function for y_j, a hidden unit is f_{yj} ; the weight linking hidden unit y_j and output unit o_k is w_{kj}; the estimate of z_i of input pattern z_p is $z_{i,p}$; in the following layer the neurons' threshold estimates are indicated by the bias units.
Back propagation Training Algorithm. The most powerful tool for training ANN is probably the hugely popular Back propagation algorithm. It coaches a Multilayer Perceptron network for a group of values of input whose outputs are already known. The network inspects the response of its output values to the given input values weighing up with the target output values for every entry of the sample set that is submitted and the error value is determined. Till the value of the error is brought to a minimum, these sample patterns are continuously handed over to the MLP network [16].

3.1.2. *Fundamentals of Radial Basis Function Network.* RBF is one of the frequently implemented algorithms of neural networks in various medical and engineering domains because of their faster learning speed, more compact topology and universal approximation. These networks have been independently proposed by

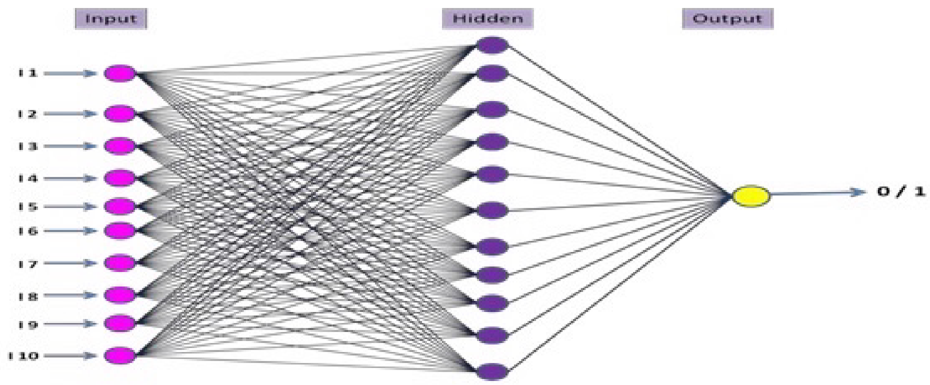

FIGURE 1. Architecture Design of MLP Network

numerous researchers [17] [18][19][20] and are a popular alternative to the Multilayer Perceptrons. It is a Feed Forward Neural Network (FFNN) containing 3 different modules which are the input layer, the hidden layer and the output layer. A parameter vector in the hidden module called center exists in every neuron. By evaluating distance between the inputs of the network and centers of the hidden module, the outputs of the first module are determined. The outputs of the linear hidden layer are the weighted forms of the returns of the first module. The general expression of the RBF network is [21]:

$$y_j^l = \sum_{i=1}^{I} w_{ij} \varnothing\big(||x - c_i||\big) + \beta_j \tag{2}$$

The Euclidean distance is taken to be the norm while the most frequently used Gaussian function is assumed to be the radial basis function as it has well known mathematical features, is highly nonlinear and provides good locality as a local RBF [22] and is defined by:

$$\phi(r) = e^{(-\alpha_i ||x - c_i||^2)} \tag{3}$$

I denotes neuron count in the middle layer $i \in \{1, 2, ...I\}$
J denotes neuron count in the middle layer $J \in \{1, 2, ...J\}$
c_i denotes centre vector of the i^{th} neuron
x denotes input data vector
w_{ij} denotes connecting value of the i^{th} neuron and j^{th} output
y_j^l denotes output of the j^{th} neuron Network
ϕ denotes radial basis function
α_i denotes spread parameter of the i^{th} neuron
β_j denotes value of the bias of the output j^{th} neuron.

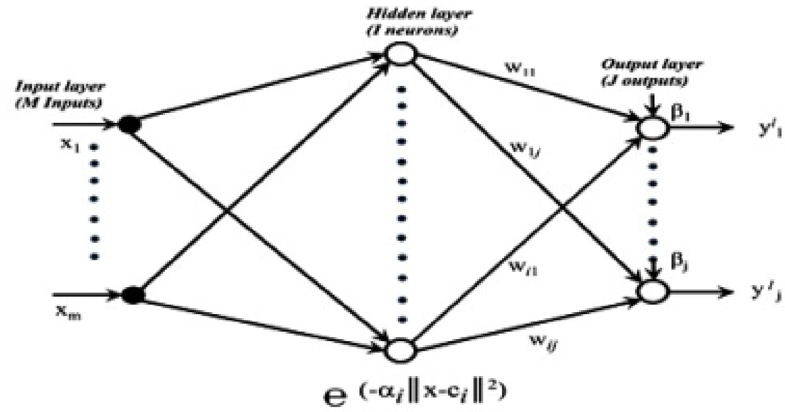

FIGURE 2. Architecture Design of RBF network

The structure of a radial basis function neural network is depicted in Fig. 2. The inputs of m dimensions $(x_1, ..., x_m)$ situated in the input module are first passed on to the hidden module, which comprises of I neurons. The Euclidean distance connecting the centers and inputs are evaluated by each neuron which contains the basis function, which is an activation function. To shape the curve $(\alpha_1, ..., \alpha_i)$ the RBF contains a spread parameter and is very often taken to be the Gaussian function. The hidden layer's weighted outputs denoted by $(w_{11}, ..., w_{ij})$ are then broadcasted to the last module. Here the dimensions of the middle layer are given by I where $i \in \{1, 2, ...I\}$ which depicts the number of neurons in the layer while the dimension of the output is denoted by J where $j \in \{1, 2, ...J\}$ and bias parameters by $(\beta_1, ..., \beta_j)$. The linear combination of the bias parameters and returns of the second module are evaluated by the last layer. The results of the radial basis network are then eventually acquired $(y_1^l, ..., y_j^l)$. During the training period, the parameters of the RBF network are modulated in such a way that the data used for training is made to fit the network model in best possible way [23].

3.2. **Discriminant Analysis.** In biomedicine models, one of the most commonly accepted statistical techniques extensively implemented is Discriminant Analysis [24]. It is basically a multivariate method which segregates different sets of observation values and assigns fresh observation values to already defined sets[25]. Based on the population size, the statistical problem is to build a classification function. The score of the discriminant function can be generated with unstandardized discriminant function scores and raw scores. To maximize the differences between the two groups, the discriminant function coefficients are chosen, whose mean is equal to zero and standard deviation is one. For every group the mean discriminant function coefficient known as centroids can be found which are generated by the discriminant function brought down from the starting independent variables.

The dimensions along which the groups differ are shown by differences in the location of these centroids. Through their capacity to exactly discriminate every data point to their derived groups, the utility of these functions can be examined. When the classification functions are ascertained groups are then differentiated. In order to achieve this purpose, from the linear discriminant functions, the classification functions are acquired.

The classification function coefficient C_j for the j^{th} group, $j = 1, ..., k$ whose sample sizes are all equal is given by:

$$C_j = c_{j0} + c_{j1}x_1 + c_{j2}x_2 + ... + c_{jp}x_p \tag{4}$$

where c_{j0} is a constant and x stands for the raw scores of each predictor. If M denotes mean column matrix for group j and W denotes within-group variance-covariance matrix, $c_{j0} = (-1/2)C_jM_j$. When the size of the sample is unequal in every group, if in group j, size is denoted by n_j and N denotes the entire size of the sample, then C_j is as follows:

$$C_j = c_{j0} + \sum_{i=1}^{p} c_{ij}x_i + \ln\left(\frac{n_j}{N}\right) \tag{5}$$

4. Data Analysis

The variables used in the study were selected based on the various characteristics which are relevant medically for a woman who is pregnant to have gestational diabetes on consultation with gynecologists. The real time data sets of 336 records of which 188 were of multigravida patients, every set containing ten variables, were collected from the records of outgoing patients in a Chennai multi-specialty hospital located in India during the period January to May 2013.

Table 1. The variables for the study

S.No	**Study Variable**	**Classification Network Variable**
1	History of stillbirth	Y or N [character]
2	Pre pregnancy body mass index	Integer [continuous]
3	Abnormal baby in previous pregnancy	Y or N [character]
4	History of miscarriage	Y or N [character]
5	Delivery of a large infant	Y or N [character]
6	Age	Integer [continuous]
7	History of GDM	Y or N [character]
8	History of polycystic ovary syndrome	Y or N [character]
9	Family history of diabetes	Y or N [character]
10	Infections (Urinary, Skin, Vaginal)	Y or N [character]

Table 1 shows the variables chosen for the study. Of the ten parameters, three include common details like BMI and age of the patient and history of diabetes in family amongst relatives of first degree. Details on previous pregnancy namely child born weighing above 3.8kg, presence or absence of GDM, the demise of a child within 5 months, a baby's birth which has flaws in major organs like the heart or brain, the birth of an infant that has died in the womb strictly after having survived through at least 5 months of pregnancy are included in five other variables. Particulars on history of infections and syndrome of polycystic ovaries are revealed in the remaining two variables[26].

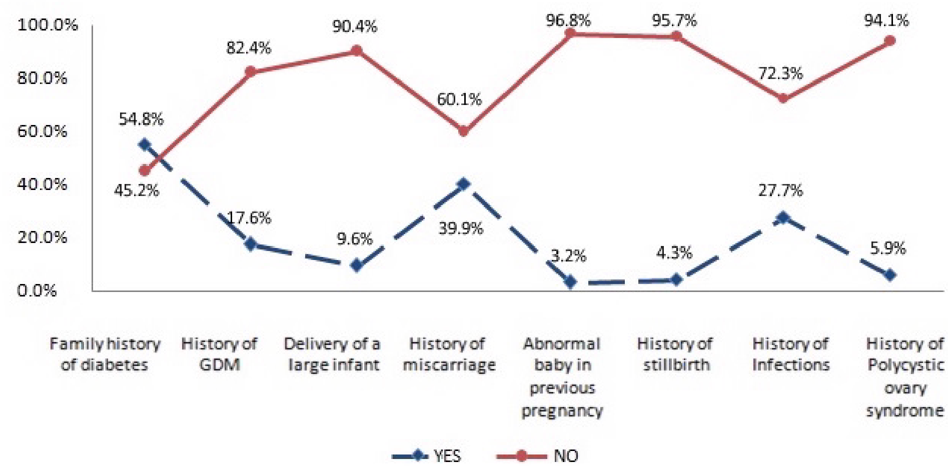

FIGURE 3. Graph showing the patients' history summary statistics

The information on the statistics of the records containing history of the patients is shown in Figure 3 by means of a graph. It was observed that the age of the pregnant ladies on an average was 32.8 years while average BMI of the patients was 26.4. The prevalence rate of GDM was found to be an alarming 34.04% in this study.

5. RESULTS

The results of the three diagnostic models are discussed below.

Results of MLP Model

MATLAB R2014a, a toolbox of Neural Network was implemented to construct the diagnostic models for both MLP and RBF. A typical FFNN using back- propagation was implemented to develop a classification system. Ten input neurons constituted the input layer, fifteen hidden neurons were used in the middle layer while the output layer comprised of a single neuron. 1 or 0 were the only possible outputs of the model as diagnosing GDM was considered as a binary classification problem i.e. Output 1 was regarded as "GDM patient" and a value of 0 was interpreted as "non-GDM patient". As the optimal neuron count lying in the middle layer cannot be predetermined, stopping criteria, the neurons in the second module and the network layer count was determined through trial and error procedure. Hence the neurons in the hidden layer were kept altering and tests were carried out on various architectures through which it was found that the architecture with hidden layer consisting of 15 neurons produced the best classification results. 70% of the data set was selected for training, 15% of them were chosen for validation while the remaining 15% was allotted for testing. The learning rate for network training was set to 0.28 and the momentum was set to 0.8. Until an average squared error of minimum less than 0.045 was reached, the model was executed.

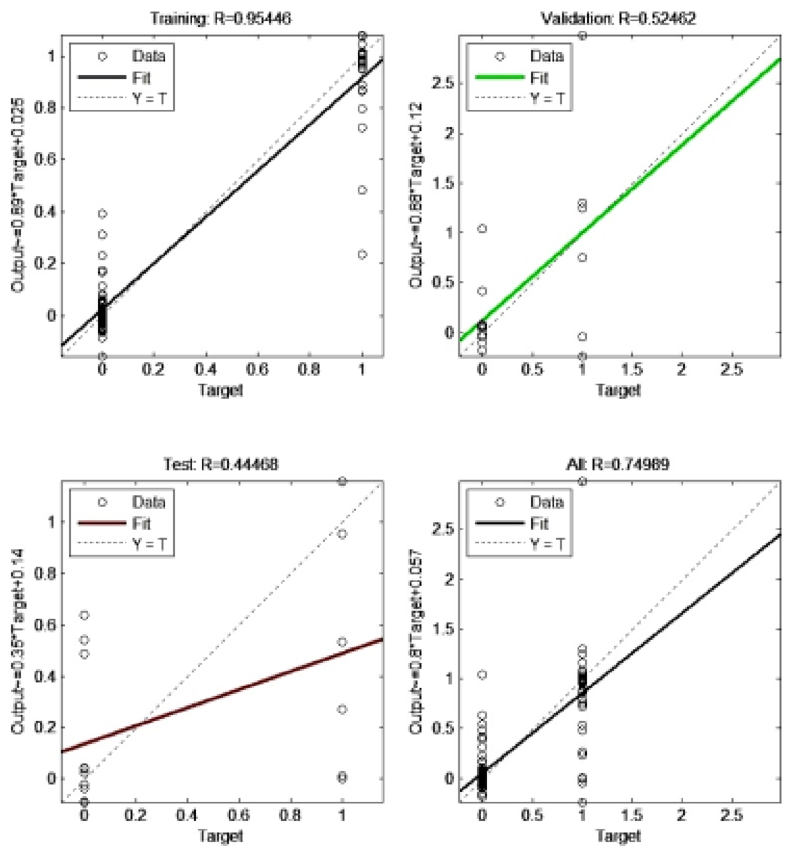

FIGURE 4. Regression Testing

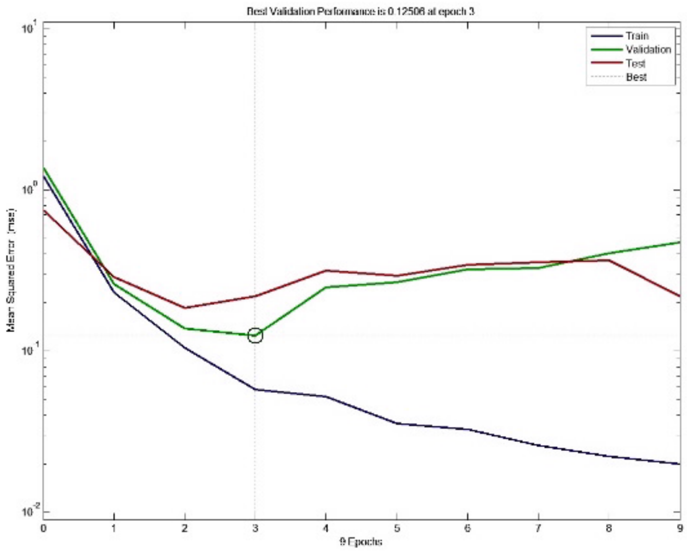

FIGURE 5. Performance Analysis

The regression testing outcomes performed on the MLP architecture for training, testing and validation and an amalgamation of all of them is depicted in Fig. 4. The performances of the MLP generated for training, validation and testing with respect to the mean square error is shown in Fig. 5. The mean square value was found to be 0.12506 and the performance of best validation was reached in the 3rd generation. As the generation proceeded, it was seen gradient descent learning algorithm minimized the error. The global minimum of mean square error was 0.075309 at the ninth generation as depicted in Fig. 6. A surge in the gradient value was noted right after ninth generation. Fig. 7 depicts the linear separability of the chosen data set classified into 2 distinguished groups namely GDM pregnant women with output 1 and non GDM patients with output 0. An astonishing 92.86 % of the given data was classified correctly while only the remaining 7.14 % were classified incorrectly. These results of the MLP model proved that the system was trained effectively and may very possibly be implemented for discerning women who are pregnant having high or low risk of gestational diabetes.

Results of RBF Model

The datasets were divided equally for training and testing. The outputs in the model were either 1 or 0 as detection of GDM was considered as a binary classification problem.

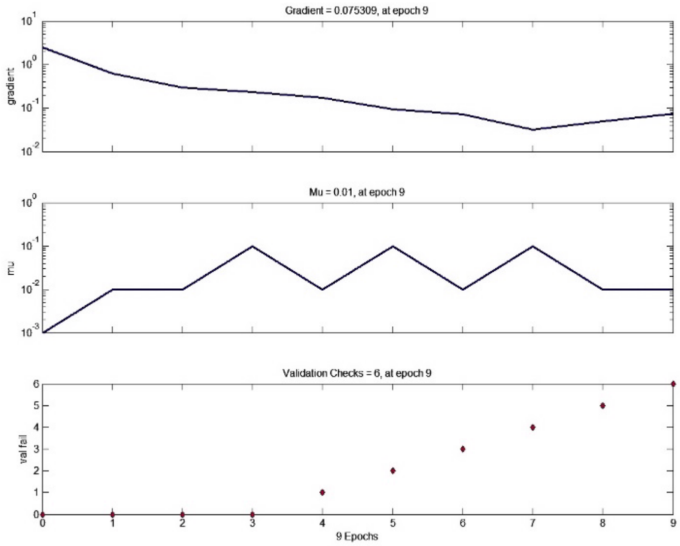

FIGURE 6. Validation Performance

The graphs generated for trained dataset and tested dataset are shown above in fig.8 and fig.9. The performance of RBF neural networks was considered best at nine centers while 16 centers were maximum tried. Using the best centers, 0.1213 was found to be the root mean square error. Execution time of RBF network was lesser than MLP. The classification accuracy of a model is used to analyze its discriminatory power. The measures of accuracy namely the sensitivity and specificity brief about the test accuracy. The true positive rate or sensitivity of a model is the capacity to accurately discern the patients with GDM while the true negative rate or specificity of the model is the capacity to accurately discern patients without gestational diabetes. The total of the number of true negative and true positive values divided by the overall size of the sample gives the overall accuracy of the model.

TABLE 2. Classification Table of RBF Model

Observed		**Predicted**		
		Output GDM		**Percentage Correct**
		No	Yes	
Output GDM	No	23	46	33.33
	Yes	01	24	96.00
Overall Percentage				50.00

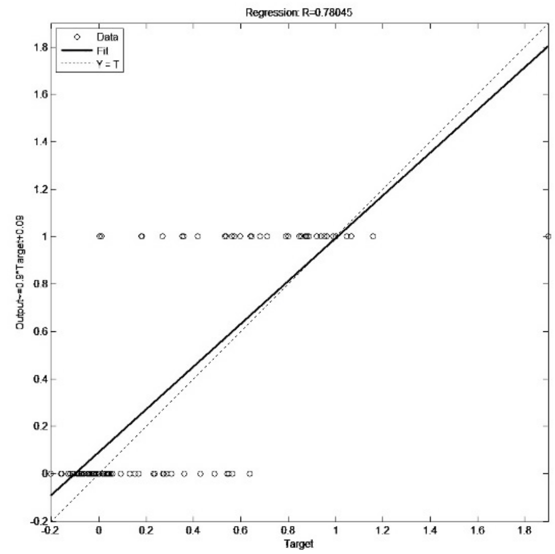

Figure 7. Classification of GDM

The classification results using RBF is shown in Table 2. 50% of the records were used for testing. Sensitivity was found to be 96.00% in the RBF neural network model and specificity was 33.33%. The overall accuracy was calculated to be a modest 50.00% for the model.

Results of Discriminant Analysis Model

To detect GDM and non GDM patients and also to determine most significant parameters of GDM, Discriminant Analysis model was implemented using version 20 of SPSS, namely the Statistical Package for Social Sciences for Windows. In DA, Wilks' lambda is applied by the mean differences ANOVA F test. Lambda value lies between 0 and 1, wherein 0 indicates that the group means differ and a value of 1 indicate that all means of the group are equal. Hence an independent variable will contribute more to the discriminant function as the lambda value gets smaller for the variable. Thus the significance of the contributions of the variables is revealed through the Wilks' lambda's F test. Corresponding to each discriminant function, the Pearsonian correlations of all the variables are depicted by the structure matrix table in SPSS, which are known as discriminant loadings or correlations or structure coefficients.

The significance of discriminant analysis was indicated using Wilks' Lambda test. From the table, it is inferred that pre pregnancy body mass index, diabetes history in family and presence or absence of GDM history were the variables which were the most influential with GDM occurrence since they had the least p values.

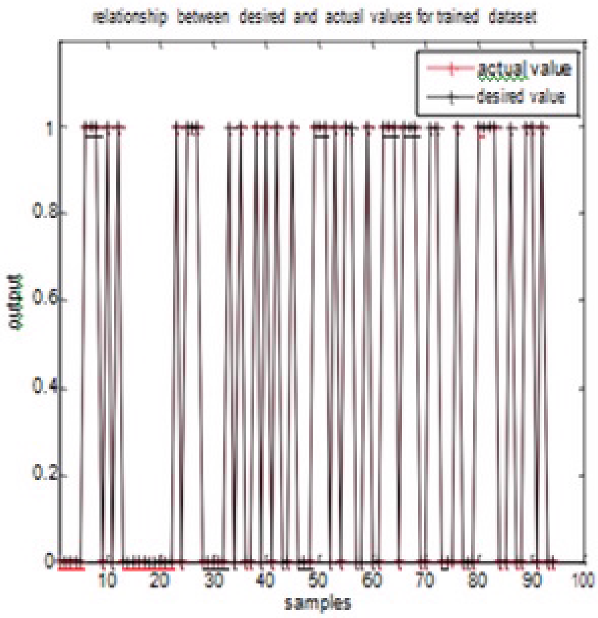

FIGURE 8. Relationship between desired and actual values for training dataset

TABLE 3. Testing Equality of Group Means

Study Variable	**F Value**	**Wilks' Lambda**	**P Value**
Pre pregnancy body mass index	16.130	0.920	<0.001**
Abnormal baby in previous pregnancy	2.953	0.984	0.087
Infections (Urinary, Skin, Vaginal)	6.455	0.966	0.012*
Delivery of a large infant	6.657	0.965	0.011*
Age	5.850	0.970	0.017*
History of miscarriage	7.283	0.962	0.008**
History of GDM	95.894	0.660	<0.001**
History of polycystic ovary syndrome	2.190	0.988	0.141
History of stillbirth	3.030	0.984	0.083
Family history of diabetes	27.594	0.871	<0.001**

Note: * stands for 5% level of significancenificance
** stands for 1% level of sig

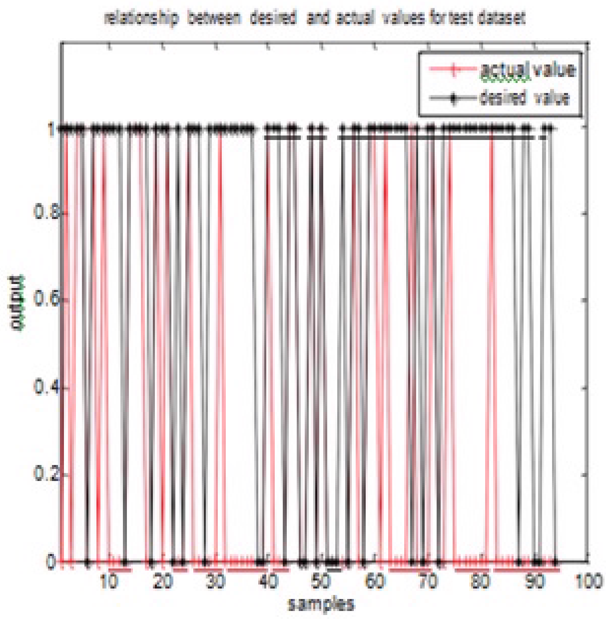

FIGURE 9. Relationship between desired and actual values for test dataset

Moreover, large infant delivery, age and infections in the past were the variables with 5% level of significance whereas the variable history of miscarriage had 1% level of significance[27]. Using structure matrix and the standardized coefficients, discriminant functions are well explained. In each discriminant function, standardized beta coefficients are given for every variable. The contribution of a variable to the discrimination between GDM and non GDM patients will be less if the value of the standardized coefficient is less and vice-versa. It is concluded from table 4 that the most vital part in discriminating the two groups was contributed by history of GDM while a few other variables like infections history, history of diabetes in family and miscarriage history also played crucial roles. 64 of the 188 pregnant women in the study had GDM in current pregnancy. Table 5 shows that using the discriminant analysis model, 45 of the 64 pregnant women with GDM were correctly identified while 112 pregnant women of the 124 patients who did not have GDM were correctly identified.

TABLE 4. Canonical Discriminant Function Coefficients

Study Variable	**Standardized Coefficients**	**Unstandardized Coefficients**
Pre pregnancy body mass index	0.053	0.017
Abnormal baby in previous pregnancy	0.145	0.830
Infections (Urinary, Skin, Vaginal)	0.313	0.707
Delivery of a large infant	0.244	0.841
Age	0.156	0.041
History of miscarriage	0.304	0.629
History of GDM	0.800	2.577
History of polycystic ovary syndrome	0.234	0.997
History of stillbirth	-0.094	-0.469
Family history of diabetes	0.472	1.011
Constant		-3.199

TABLE 5. Classification Table

Observed		**Predicted**		
		Output GDM		**Percentage Correct**
		Yes	No	
Output GDM	Yes	45	19	70.31
	No	12	112	90.32
Overall Percentage				83.51

6. DISCUSSION

To determine the most efficient model and the model with the best discriminatory power, the measures of accuracy of the three diagnostic models were compared and analyzed. Another measure which exhibits information on the classification accuracy of the test namely Youden's index is calculated using the specificity and sensitivity values of the model and is defined as follows:

$$\text{Youden's index} = \text{Specificity} + \text{Sensitivity} - 1. \tag{6}$$

This index lies between -1 and 1. The test is considered flawless if there are no false negatives or false positives thereby yielding a value of 1. Thus, the accuracy of the model is higher when Youden's index value of the model is larger.

For all the three classification methods, table 6 displays a comparison of the measures namely accuracy, specificity, sensitivity and Youden's index. All models had specificity, sensitivity, accuracy and Youden's index range between 33.33-94.74%, 70.31-96.00%, 50.00-92.86% and 0.29-0.84 respectively. The sensitivity was more

Table 6. Comparative Predictions of the three Diagnostic Models

Model	**Sensitivity (%)**	**Specificity (%)**	**Accuracy (%)**	**Youden's Index**
MLP	88.89	94.74	92.86	0.84
RBF	96.00	33.33	50.00	0.29
Discriminant Analysis	70.31	90.32	83.51	0.61

than 70% in each model of which RBF had the highest (96.00%). In this study, the MLP model had the highest specificity (94.74%), the best classification accuracy (92.86%) and the highest Youden's index (0.84). Based on the above comparison analysis carried out, the MLP model was found to be the best classification method and has clearly outperformed RBF and discriminant analysis models.

7. Conclusion

GDM is a public health concern. Only women who have the traditional risk factors like obesity or family history of GDM are usually screened earlier on in pregnancy. Unfortunately, women who do not have these common risk factors and develop GDM often remain undiagnosed until the second trimester and a delay in diagnosis often leads to therapies for GDM becoming less effective. Hence, there is a growing need for early detection of gestational diabetes. Nearly three-fourth of the population in India exists in rural environment and basic amenity for even diagnosis of DM is inadequate. Performing OGTT to diagnose GDM is burdensome and unfavorable in this current setting. Furthermore, the amount involved is exorbitant to undergo three medical tests. Therefore, the necessity is also for an inexpensive and uncomplicated procedure to detect gestational diabetes. To address these needs, the methods identified in this study offer every pregnant woman the opportunity to know her risk early on without a visit to the hospital because of which the costs for the various blood tests are saved and hence would prove immensely favorable for all pregnant women. In conclusion, with a staggering 92.86% overall accuracy, MLP neural network with back propagation algorithm significantly outperformed RBF and discriminant analysis models. Moreover, through discriminant analysis, it was found that the variables, diabetes history in family, pre pregnancy BMI and GDM history of the patient are the significant factors which play the most crucial role in diagnosing gestational diabetes, which will assist pregnant women to be mindful of in an early stage and take precautionary measures like actively participate in physical exercise and make changes in dietary behavior so that gestational diabetes can be successfully warded off.

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COMMUNICATIONS
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GRAND LORENTZ SEQUENCE SPACE AND ITS MULTIPLICATION OPERATOR

OĞUZ OĞUR

Abstract. In this paper, we introduce the grand Lorentz sequence spaces $\ell^{\theta}_{p,q)}$ and study on some topological properties. Also, we characterize some properties of the multiplication operator, such as compactness, Fredholmness etc., defined on $\ell^{\theta}_{p,q)}$.

1. Introduction

Let (X,S,μ) be a $\sigma-$finite measure space and let g be a complex-valued measurable function defined on X. The *non-increasing rearrangement* g^* of g is defined by

$$g^*(s)=\inf\{t>0:F_\mu(t)\leq s\},\ \ s\geq 0,$$

where $F_\mu(t)=\mu\{x\in X:|g(x)|>t\}$, $t\geq 0$, is the *distribution function* of g. If μ is counting measure on $S=2^{\mathbb{N}}$, then we can write the *distribution function* and *the non-increasing rearrangement of a complex-valued sequence* (x_n), respectively, as follows;

$$F_\mu(t)=\mu\{n\in\mathbb{N}:|x_n|>t\},t\geq 0$$

and

$$x_{\phi(n)}=\inf\{t>0:F_\mu(t)\leq n-1\}$$

if $n-1\leq t<n$ with $F_\mu(t)<\infty$. By the definition of non-increasing rearrangement, we can interpret that $\left(x_{\phi(n)}\right)$ can be obtained by permuting $(|x_n|)_{n\in R}$, where $R=\{n\in\mathbb{N}:x_n\neq 0\}$, in the decreasing order. Here, $x_{\phi(n)}=0$ for $n>\mu(R)$ if $\mu(R)<\infty$ [2].

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Lorentz introduced the classical Lorentz space $\Lambda_{q,w}$, $0<q<\infty$, which the space of all measurable functions f defined on $(0,1)$ with

$$\|f\|_{\Lambda_{q,w}} = \left(\int_0^1 (f^*(x))^q w(x)dx \right)^{\frac{1}{q}},$$

where f^* is the non-increasing rearrangement of f and w is a weight function [12], [13]. The space $\Lambda_{q,w}$ and its special case $L^{p,q}$, $0<q,p\leq\infty$, have been widely studied by many authors. For more details see [3], [5], [7].

The Lorentz sequence spaces $\ell_{p,q}$ is the space of all complex-valued sequences $x=(x_n)$ such that

$$\|x\|_{p,q} = \begin{cases} \left(\sum\limits_{n=1}^{\infty} n^{\frac{q}{p}-1} (x_{\phi(n)})^q \right)^{\frac{1}{q}}, & 1\leq p\leq\infty,\ 1\leq q<\infty \\ \sup_n n^{\frac{1}{p}} x_{\phi(n)}, & 1\leq p<\infty,\ q=\infty \end{cases}$$

is finite, where $\left(x_{\phi(n)}\right)$ is non-increasing rearrangement of x. The spaces $\ell_{p,q}$ have been used to introduce and investigate some classes of operators, like $(p,q)-$nuclear, $(p,q;r)-$absolutely summing operator [14]. Kato [11] characterized the dual space of $\ell_{p,q}\{E\}$, where E is a Banach space. See also [2], [10], [15].

The idea of grand spaces was raised by Iwaniec and Sbordone [8]. They introduced the grand Lebesgue spaces $L^{p)}$ for $1<p<\infty$. Samko and Umarkhadzhiev [17] studied some properties of grand Lebesgue spaces on sets of infinite measure. Jain and Kumari [9] introduced the grand Lorentz spaces $\Lambda_{q),w}$, $0<q<\infty$ and studied on its basic properties. Also, they characterized boundedness of maximal operator on the space $\Lambda_{q),w}$. Later, Rafeiro and others [16] introduced the grand Lebesgue sequence space $\ell^{p),\theta}=\ell^{p),\theta}(X)$ by the norm

$$\|x\|_{\ell^{p),\theta}(X)} = \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k\in X} |x_k|^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{p(1+\varepsilon)}} \|x\|_{\ell^{p(1+\varepsilon)}(X)}$$

where X is one of the sets $\mathbb{Z}^n$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ for $1\leq p<\infty$, $\theta>0$. They studied various operators of harmonic analysis, e. g. maximal, convolution, Hardy etc.

In this paper, we are inspired by this work and introduce the grand Lorentz sequence spaces $\ell^{\theta}_{p,q)}$ as follows; let $\theta>0$. The grand Lorentz sequence space $\ell^{\theta}_{p,q)}$ is the set of all sequences $a=(a_n)$ such that $\|a\|_{p,q),\theta}<\infty$, where $\|a\|_{p,q),\theta}$ is defined by

$$\begin{cases} \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum\limits_{n=1}^{\infty} \left(n^{\frac{1}{p(1+\varepsilon)}} a_{\phi(n)} \right)^{q(1+\varepsilon)} n^{-1} \right)^{\frac{1}{q(1+\varepsilon)}}, & 1\leq p\leq\infty, 1\leq q<\infty \\ \sup_{n\geq 1} n^{\frac{1}{p}} a_{\phi(n)}, & 1\leq p<\infty, q=\infty \end{cases}$$

where $\left(a_{\phi(n)}\right)$ is the non-increasing rearrangement of the sequence $a=(a_n)$. In case $p=q$, the grand Lorentz sequence space $\ell^{\theta}_{p,q)}$ coincides with the grand Lebesgue space $\ell^{p),\theta}(\mathbb{N})$. In this work, we study on some topological properties and inclusion theorems of the space $\ell^{\theta}_{p,q)}$. Also, we characterize some properties of multiplication operator on the $\ell^{\theta}_{p,q)}$.

We will need the following lemma:

Lemma 1. *(Hardy, Littlewood and Polya) Let (r_n^*) and $({}^*r_n)$ be the non-increasing and non-decreasing rearrangements of a finite sequence (r_n) of positive numbers. Then, we have for any two sequences (a_n) and (b_n) of positive numbers such that*

$$\sum_n a_n^{**} b_n \leq \sum_n a_n b_n \leq \sum_n a_n^* b_n^*$$

[6].

2. Main Results

2.1. **Grand Lorentz Sequence Space.**

Theorem 2. *The grand Lorentz sequence space $\ell^{\theta}_{p,q)}$ is a normed space for $1 \leq q \leq p \leq \infty$ and a quasi-normed space for $1 \leq p < q \leq \infty$.*

Proof. By definition of the norm of $\ell^{\theta}_{p,q)}$, we can write

$$\|a\|_{p,q),\theta} = \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \|a\|_{p,q(1+\varepsilon)} . \tag{1}$$

Let $1 \leq q < p \leq \infty$. For any $a, b \in \ell^{\theta}_{p,q)}$, since $n^{\frac{q}{p}-1}$ is decreasing sequence of positive numbers and so by Lemma 1, we have

$$\begin{aligned}
\|a+b\|_{p,q),\theta} &= \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(a_{\vartheta(n)} + b_{\vartheta(n)} \right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&= \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} \left(n^{\left(\frac{q}{p}-1\right)\frac{1}{q(1+\varepsilon)}} \left(a_{\vartheta(n)} + b_{\vartheta(n)} \right) \right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&\leq \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(a_{\vartheta(n)} \right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&\quad + \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(b_{\vartheta(n)} \right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&\leq \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(a_{\phi(n)} \right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&\quad + \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(b_{\psi(n)} \right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}}
\end{aligned}$$

$$= \quad \|a\|_{p,q),\theta} + \|b\|_{p,q),\theta}$$

where $\left(a_{\vartheta(n)} + b_{\vartheta(n)}\right)$, $\left(a_{\phi(n)}\right)$ and $\left(b_{\psi(n)}\right)$ are the non-increasing rearrangements of $(a_n + b_n)$, (a_n) and (b_n), respectively.

Let $1 \le p < q < \infty$. Then, we have $p < q(1+\varepsilon)$ for $\varepsilon > 0$ and hence $\|a\|_{p,q(1+\varepsilon)}$ is a quasi-norm. Thus, we get

$$\begin{aligned} \|a+b\|_{p,q),\theta} &= \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \|a+b\|_{p,q(1+\varepsilon)} \\ &\le \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \left(2^{\frac{1}{p}} \left(\|a\|_{p,q(1+\varepsilon)} + \|b\|_{p,q(1+\varepsilon)}\right)\right) \\ &\le 2^{\frac{1}{p}} \left(\|a\|_{p,q),\theta} + \|b\|_{p,q),\theta}\right). \end{aligned}$$

For $1 \le p < \infty$ and $q = \infty$, we have $\|a\|_{p,\infty),\theta} = \|a\|_{p,\infty}$. The proof is completed. □

Remark 3. *Let $\alpha > 0$ and let us take the sequence*

$$(a_n) = \left(n^{\frac{-1}{p}} (\ln(n+1))^{-\alpha}\right)$$

as in [16]*. It is easy to see that the sequence (a_n) is decreasing and thus the non-increasing rearrangement of (a_n) is itself. Therefore, we have*

$$\sum_{n=1}^{\infty} \left(n^{\frac{1}{p}} n^{\frac{-1}{p}} (\ln(n+1))^{-\alpha}\right)^q n^{-1} = \sum_{n=1}^{\infty} n^{-1} (\ln(n+1))^{-\alpha q}.$$

If $\alpha > \frac{1}{q}$, then $(a_n) \in \ell_{p,q}$. Using similar technique as in [16]*, we get $(a_n) \in \ell^{\theta}_{p,q)}$ if and only if $\alpha \ge \frac{1-\theta}{q}$. Thus, we get $(a_n) \in \ell^{\theta}_{p,q)}$ and $(a_n) \notin \ell_{p,q}$ whenever $\frac{1-\theta}{q} \le \alpha \le \frac{1}{q}$.*

Definition 4. *The vanishing grand Lorentz sequence space $\mathring{\ell}^{\theta}_{p,q)}$, $1 \le p \le \infty$, $1 \le q < \infty$, consists of all sequences $(a_n) \in \ell^{\theta}_{p,q)}$ such that*

$$\lim_{\varepsilon\to 0} \varepsilon^{\theta} \sum_{n=1}^{\infty} \left(n^{\frac{1}{p(1+\varepsilon)}} a_{\phi(n)}\right)^{q(1+\varepsilon)} n^{-1} = 0.$$

Lemma 5. *The space $\mathring{\ell}^{\theta}_{p,q)}$ is a closed subspace of the space $\ell^{\theta}_{p,q)}$.*

Proof. The proof can be obtained by using similar technique as in [16]. □

Remark 6. *It is enough to take the supremum in (1) on the finite interval for ε, which means*

$$\|a\|_{p,q),\theta} = \sup_{0<\varepsilon<\frac{1}{W(1/e)}} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \|a\|_{p,q(1+\varepsilon)}$$

where $W(t)$ is the Lambert function. Note that $\frac{1}{W(1/e)} \approx 3.59$ (see [4], [16]*).*

Lemma 7. *Let $a = (a_n) \in \ell^{\theta}_{p,q)}$, $1 \leq p, q < \infty$ and $\theta > 0$. Then, we have the following inequalities for all $n \in \mathbb{N}$:*

$$a_{\phi(n)} \leq h\left(\frac{1}{W(e^{-1})}\right)^{\frac{-\theta}{q}} \left(\frac{p}{q}R(\varepsilon_0)\right)^{\frac{-1}{q}} n^{\frac{-1}{p}} \|a\|_{p,q),\theta}$$

if $1 \leq p \leq q < \infty$ and

$$a_{\phi(n)} \leq h\left(\frac{1}{W(e^{-1})}\right)^{-\frac{\theta}{q}} n^{\frac{1}{q}-\frac{1}{p}} \|a\|_{p,q),\theta}$$

if $1 \leq q < p \leq \infty$, where $h(x) = x^{\frac{1}{1+x}}$, $R(x) = (1+x)^{-\frac{1}{1+x}}$ and $\varepsilon_0 \approx 1,7182$.

Proof. Let $a = (a_n) \in \ell^{\theta}_{p,q)}$ and let $1 \leq p \leq q < \infty$. Since $p \leq q(1+\varepsilon)$, we have by Lemma 2 in [11] that

$$\begin{aligned}
\|a\|_{p,q),\theta} &= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p,q(1+\varepsilon)} \\
&\geq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(n^{\frac{1}{p}} \left(\frac{p}{q(1+\varepsilon)}\right)^{\frac{1}{q(1+\varepsilon)}} a_{\phi(n)}\right) \\
&\geq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}} (1+\varepsilon)^{-\frac{1}{q(1+\varepsilon)}} n^{\frac{1}{p}} a_{\phi(n)} \\
&= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}} (R(\varepsilon))^{\frac{1}{q}} n^{\frac{1}{p}} a_{\phi(n)}. \\
&\geq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}} (R(\varepsilon_0))^{\frac{1}{q}} n^{\frac{1}{p}} a_{\phi(n)}. \\
&= h\left(\frac{1}{W(e^{-1})}\right)^{\frac{\theta}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}} (R(\varepsilon_0))^{\frac{1}{q}} n^{\frac{1}{p}} a_{\phi(n)}.
\end{aligned}$$

Here $R(x) = (1+x)^{-\frac{1}{1+x}}$ attains the minimum at the point $\varepsilon_0 \approx 1,7182$.
Let $1 \leq q < p < \infty$. Then, since $n^{\frac{q}{p}-1}$ is decreasing, we have

$$\begin{aligned}
\|a\|_{p,q),\theta} &= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p,q(1+\varepsilon)} \\
&\geq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\sum_{n=1}^{k} \left(n^{\frac{1}{p(1+\varepsilon)}} a_{\phi(n)}\right)^{q(1+\varepsilon)} n^{-1}\right)^{\frac{1}{q(1+\varepsilon)}} \\
&\geq a_{\phi(k)} \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\sum_{n=1}^{k} n^{\frac{q}{p}-1}\right)^{\frac{1}{q(1+\varepsilon)}}
\end{aligned}$$

$$\begin{aligned} &\geq a_{\phi(k)} \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(k^{\frac{q}{p}-1}\right)^{\frac{1}{q(1+\varepsilon)}} \\ &\geq h\left(\frac{1}{W(e^{-1})}\right)^{\frac{\theta}{q}} n^{\frac{1}{p}-\frac{1}{q}} a_{\phi(k)}. \end{aligned}$$

□

Theorem 8. *The space $\ell^{\theta}_{p,q)}$ is complete for $1 \leq p, q \leq \infty$.*

Proof. Let $a^{(s)} = \left(a_n^{(s)}\right) \in \ell^{\theta}_{p,q)}$ such that

$$\lim_{s,t\to\infty} \left\|a^{(s)} - a^{(t)}\right\|_{p,q),\theta} = 0.$$

For $q = \infty$, the proof is clear. Let $q < \infty$. Then, there exists a natural number s_0 such that

$$\left\|a^{(s)} - a^{(t)}\right\|_{p,q),\theta} < \eta$$

whenever $s, t \geq s_0$. By Lemma 3, we have

$$\begin{aligned} \left|a_k^{(s)} - a_k^{(t)}\right| &\leq h\left(\frac{1}{W(e^{-1})}\right)^{-\frac{\theta}{q}} \begin{cases} k^{\frac{1}{q}-\frac{1}{p}} \left\|a^{(s)} - a^{(t)}\right\|_{p,q),\theta}, & q < p \\ \left(\frac{p}{q} R(\varepsilon_0)\right)^{-\frac{1}{q}} k^{-\frac{1}{p}} \left\|a^{(s)} - a^{(t)}\right\|_{p,q),\theta}, & p \leq q \end{cases} \\ &< h\left(\frac{1}{W(e^{-1})}\right)^{-\frac{\theta}{q}} \begin{cases} k^{\frac{1}{q}-\frac{1}{p}} \eta, & q < p \\ \left(\frac{p}{q} R(\varepsilon_0)\right)^{-\frac{1}{q}} k^{-\frac{1}{p}} \eta, & p \leq q \end{cases} \end{aligned}$$

where $h(x) = x^{\frac{1}{1+x}}$, $R(x) = (1+x)^{-\frac{1}{1+x}}$. This shows that $\left(a_k^{(s)}\right)$ is a Cauchy sequence in $\mathbb{C}$. Thus, we have $(a_k) \in \mathbb{C}$ such that $\lim_{s\to\infty} \left|a_k^{(s)} - a_k\right| = 0$. By using the equality (1) with classical method, we get $\ell^{\theta}_{p,q)}$is a complete space. □

Lemma 9. *Let $1 \leq p < \infty,\ 1 \leq q < q_1 \leq \infty$. Then, we have the following*

$$\ell^{\theta}_{p,q)} \subset \ell^{\theta}_{p,q_1)}.$$

Proof. Let $a = (a_n) \in \ell^{\theta}_{p,q)}$ and $p < q$. Then, we have by Proposition 2 in [11] that

$$\begin{aligned} \|a\|_{p,q_1),\theta} &= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q_1}} \|a\|_{p,q_1(1+\varepsilon)} \\ &\leq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q_1}} \left(\frac{q(1+\varepsilon)}{p}\right)^{\frac{1}{q(1+\varepsilon)}-\frac{1}{q_1}} \|a\|_{p,q(1+\varepsilon)} \\ &\leq \left(\frac{q}{p}\left(1 + \frac{1}{W(e^{-1})}\right)\right)^{\frac{1}{q}-\frac{1}{q_1}} \|a\|_{p,q),\theta} \\ &< \infty. \end{aligned}$$

where $h(x) = x^{\frac{1}{1+x}}$. The inclusion can be obtained by similar way for $p \geq q$ with Lemma 3. $\square$

Theorem 10. *Let either* $1 \leq p < p_1 \leq \infty,\ 1 \leq q < \infty$ *or* $1 \leq p < p_1 < \infty,\ q = \infty$. *Then, the inclusion*

$$\ell^{\theta}_{p,q)} \subset \ell^{\theta}_{p_1,q)}$$

holds.

Proof. Let $a \in \ell^{\theta}_{p,q)}$. Then, we have

$$\begin{aligned} \|a\|_{p_1,q),\theta} &= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p_1,q(1+\varepsilon)} \\ &\leq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p,q(1+\varepsilon)} \\ &= \|a\|_{p,q),\theta} \\ &< \infty \end{aligned}$$

which shows $a \in \ell^{\theta}_{p_1,q)}$. $\square$

Corollary 11. *Let* $1 \leq p_1 < p \leq q < q_1 \leq \infty$. *Then, the inclusions*

$$\ell^{p_1),\theta} \subset \ell^{\theta}_{p,q)} \subset \ell^{q_1),\theta}$$

hold.

Theorem 12. *The grand Lorentz sequence space* $\ell^{\theta}_{p,q)}$ *is strictly convex for* $1 < p < \infty$ *and* $1 < q < \infty$.

Proof. Let $a, b \in \ell^{\theta}_{p,q)}$ such that $\|a\|_{p,q),\theta} = \|b\|_{p,q),\theta} = 1$ and $\left\|\frac{a+b}{2}\right\|_{p,q),\theta} = 1$. Then, we have by using similar technique as in [1] that

$$\begin{aligned} 1 &= \left\|\frac{a+b}{2}\right\|_{p,q),\theta} = \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \left\|\frac{a+b}{2}\right\|_{p,q(1+\varepsilon)} \\ &\leq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \left(\frac{\|a\|_{p,q(1+\varepsilon)} + \|b\|_{p,q(1+\varepsilon)}}{2}\right) \\ &\leq \left(\frac{\|a\|_{p,q),\theta} + \|b\|_{p,q),\theta}}{2}\right) \\ &= 1 \end{aligned}$$

which shows $a = b$. $\square$

2.2. **Multiplication Operator.** In this section, we characterize some properties of the multiplication operators on $\ell^{\theta}_{p,q)}$. Let $v=(v_n)$ be a complex-valued sequence and let us define the linear transformation M_v on the sequence space X into the linear space of all complex-valued sequences by

$$M_v(x)=vx=(v_n x_n).$$

If the linear transformation M_v is bounded with range in X, then it is called *multiplication operator* on X.

Theorem 13. *Let $v=(v_n)$ be a complex-valued sequence. Then, M_v is a multiplication operator on $\ell^{\theta}_{p,q)}$, $1\le p,q\le\infty$ if and only if v is a bounded sequence.*

Proof. Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$ and let $q<\infty$. Then, there exists a positive number $K>0$ such that

$$\|M_v(a)\|_{p,q),\theta}\le K\|a\|_{p,q),\theta}$$

for all $a\in\ell^{\theta}_{p,q)}$. Let us define

$$e^{(k)}_n=\begin{cases} s^{-\frac{\theta}{p}}, & k=n\\ 0, & k\neq n\end{cases}$$

where $s=\left(\frac{1}{W(e^{-1})}\right)^{\frac{W(e^{-1})}{1+W(e^{-1})}}$ for all $n\in\mathbb{N}$. Then, the non-increasing rearrangement of $\left(e^{(k)}_n\right)$ is

$$e^{(k)}_{\phi(n)}=\begin{cases} s^{-\frac{\theta}{p}} & ,n=1\\ 0 & ,n\neq 1\end{cases}.$$

Then, we have $\left(e^{(k)}_n\right)\in\ell^{\theta}_{p,q)}$ with $\left\|e^{(k)}\right\|_{p,q),\theta}=1$. By the boundedness of M_v, it can be written $\left\|M_v e^{(k)}\right\|_{p,q),\theta}\le K\left\|e^{(k)}\right\|_{p,q),\theta}=K$. Thus, we get

$$\begin{aligned}\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{n=1}^{\infty}\left(n^{\frac{1}{p(1+\varepsilon)}}v_{\psi(n)}e^{(k)}_{\psi(n)}\right)^{q(1+\varepsilon)}n^{-1}\right)^{\frac{1}{q(1+\varepsilon)}} &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\left(v_{\psi(1)}e^{(k)}_{\psi(1)}\right)^{q(1+\varepsilon)}\right)^{\frac{1}{q(1+\varepsilon)}}\\ &= s^{-\frac{\theta}{p}}\sup_{\varepsilon>0}\left(\varepsilon^{\frac{\theta}{q(1+\varepsilon)}}v_{\psi(1)}\right)\\ &\le K\end{aligned}$$

which gives that $v_{\psi(1)}\le K.s^{-\frac{\theta}{q}+\frac{\theta}{p}}$. This shows that v is bounded. If $q=\infty$, the proof is similar as was used in the classical Lorentz sequence spaces.

Conversely, let v be a bounded sequence. Then, there exists $T>0$ such that $|v_k|\le T$ for all $k\in\mathbb{N}$. Thus, we get

$$\|M_v a\|_{p,q),\theta} = \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=1}^{\infty}\left(k^{\frac{1}{p(1+\varepsilon)}}v_{\psi(k)}a_{\psi(k)}\right)^{q(1+\varepsilon)}k^{-1}\right)^{\frac{1}{q(1+\varepsilon)}}$$

$$\begin{aligned}
&\leq \quad T \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k=1}^{\infty} \left(k^{\frac{1}{p(1+\varepsilon)}} a_{\psi(k)} \right)^{q(1+\varepsilon)} k^{-1} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&= \quad T \left\| a \right\|_{p,q),\theta}
\end{aligned}$$

for $q < \infty$. If $q = \infty$, then

$$\sup_{k\in\mathbb{N}} k^{\frac{1}{p}} v_{\psi(k)} a_{\psi(k)} \leq T \left\| a \right\|_{p,q),\theta}.$$

□

Theorem 14. *Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$, $1 \leq p, q \leq \infty$. Then, M_v is invertible if and only if there exists $\mu > 0$ such that $|v_n| \geq \mu . s^{-\frac{\theta}{q}+\frac{2\theta}{p}}$, where $s = \left(\frac{1}{W(e^{-1})}\right)^{\frac{W(e^{-1})}{1+W(e^{-1})}}$ for all $n \in \mathbb{N}$.*

Proof. Let M_v be invertible operator on $\ell^{\theta}_{p,q)}$, $1 \leq p, q \leq \infty$. Then, there exists $\rho > 0$ such that

$$\left\| M_v a \right\|_{p,q),\theta} \geq \mu \left\| a \right\|_{p,q),\theta}$$

for all $a \in \ell^{\theta}_{p,q)}$. Thus, for $\left(e^{(k)}_n\right) \in \ell^{\theta}_{p,q)}$, we get

$$\left\| M_v e^{(k)} \right\|_{p,q),\theta} = s^{\frac{\theta}{q}-\frac{\theta}{p}} \left| v_k \right| \geq \mu s^{\frac{\theta}{p}}$$

which gives $|v_k| \geq s^{-\frac{\theta}{q}+\frac{2\theta}{p}} \mu$. Conversely, let define $z_k = (v_k)^{-1}$. By using Theorem 5, the proof can be obtained. □

Theorem 15. *Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$, $1 \leq p, q \leq \infty$. Then, a necessary and sufficient condition for M_v to have closed range is that for some $\varrho > 0$*

$$|v_n| \geq \varrho$$

for each $n \in R = \{n \in \mathbb{N} : v_n \neq 0\}$.

Proof. Assume that $|v_n| \geq \varrho$ for $\varrho > 0$ and for all $n \in R$. Let $q < \infty$ and let $g^{(k)}, g \in \ell^{\theta}_{p,q)}$ such that $M_v g^{(k)} \to g$ as $k \to \infty$. Then, we write

$$\lim_{m,n\to\infty} \left\| M_v g^{(m)} - M_v g^{(n)} \right\|_{p,q),\theta} = 0.$$

Put $x^{(mn)} = g^{(m)} - g^{(n)}$. Thus, we have

$$\left\{ l \in \mathbb{N} : \left| x^{(mn)}_l \right| > \frac{r}{\varrho} \right\} \subseteq \left\{ l \in \mathbb{N} : \left| v_l x^{(mn)}_l \right| > r \right\}$$

for each $r > 0$ and so $\varrho x^{(mn)}_{\phi(l)} \leq v_{\psi(l)} x^{(mn)}_{\psi(l)}$, where $x^{(mn)}_{\phi(l)}$ and $v_{\psi(l)} x^{(mn)}_{\psi(l)}$ are the non-increasing rearrangement of the sequences $\left(x^{(mn)}_l\right)$ and $\left(v_l x^{(mn)}_l\right)$, respectively.

Thus, we have

$$\begin{aligned}\left\|vx^{(mn)}\right\|_{p,q),\theta} &= \left\|M_v g^{(m)} - M_v g^{(n)}\right\|_{p,q),\theta}\\ &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{l\in R}\left(l^{\frac{1}{p(1+\varepsilon)}}v_{\psi(l)}x^{(mn)}_{\psi(l)}\right)^{q(1+\varepsilon)}l^{-1}\right)^{\frac{1}{q(1+\varepsilon)}}\\ &\geq \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{l\in R}\varrho^{q(1+\varepsilon)}\left(l^{\frac{1}{p(1+\varepsilon)}}x^{(mn)}_{\phi(l)}\right)^{q(1+\varepsilon)}l^{-1}\right)^{\frac{1}{q(1+\varepsilon)}}\\ &= \varrho\left\|x^{(mn)}\right\|_{p,q),\theta}.\end{aligned}$$

Since $\left\|vx^{(mn)}\right\|_{p,q),\theta}\to 0$ as $m,n\to\infty$, we have $x^{(mn)}\to 0$ as $m,n\to\infty$. This means that $g^{(m)}$ is a Cauchy sequence in $\ell^{\theta}_{p,q)}|_R$, where $\ell^{\theta}_{p,q)}|_R = \left\{a=(a_k)\in\ell^{\theta}_{p,q)} : a_k = 0 \text{ if } k\in\mathbb{N}\backslash R\right\}$ is a closed subspace of $\ell^{\theta}_{p,q)}$. Thus, we get $f\in\ell^{\theta}_{p,q)}|_R$ such that $g^{(m)}\to f$ as $m\to\infty$. Since M_v is bounded linear operator, we can write $M_v g^{(m)}\to M_v f$. This gives $M_v f = g$. Because of $Ker\,(M_v) = \ell^{\theta}_{p,q)}|_{\mathbb{N}\backslash R}$, M_v has closed range.

Conversely, assume that M_v has closed range and there exists $(l_n)\in R$ such that $|v_{l_n}| < \frac{1}{n}$. Let

$$e^{(l_n)}_m = \begin{cases} s^{-\frac{\theta}{p}}, & m = l_n\\ 0, & m\neq l_n\end{cases}$$

where $s = \left(\frac{1}{W(e^{-1})}\right)^{\frac{W(e^{-1})}{1+W(e^{-1})}}$ and let $q<\infty$. Then, $\left\|e^{(l_n)}\right\|_{p,q),\theta} = 1$. Thus, we get

$$\begin{aligned}\left\|M_v e^{(l_n)}\right\|_{p,q),\theta} &= \left\|ve^{(l_n)}\right\|_{p,q),\theta}\\ &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{m=1}^{\infty}\left(m^{\frac{1}{p(1+\varepsilon)}}v_{\psi(m)}e^{(l_n)}_{\psi(m)}\right)^{q(1+\varepsilon)}m^{-1}\right)^{\frac{1}{q(1+\varepsilon)}}\\ &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\left(v_{\psi(1)}e^{(l_n)}_{\psi(1)}\right)^{q(1+\varepsilon)}\right)^{\frac{1}{q(1+\varepsilon)}}\\ &= s^{\frac{\theta}{p}-\frac{\theta}{q}}v_{l_n}\\ &< \frac{1}{n}s^{\frac{\theta}{p}-\frac{\theta}{q}}\left\|e^{(l_n)}\right\|_{p,q),\theta}\end{aligned}$$

which means M_v is not bounded different from zero. Thus, $|v_n|\geq\varrho$ for some $\varrho>0$ and all $n\in R$. For the case $q=\infty$ the proof can be obtained by similar way. □

Theorem 16. *Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$. Then, M_v is compact if and only if $|v_n|\to 0$ as $n\to\infty$.*

Proof. The proof can be obtained by the similar way used in the classical Lorentz sequence space. $\square$

Corollary 17. *Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$. Then, M_v is Fredholm if and only if the set $\mathbb{N}\backslash R$ has finite elements and there exists $\rho > 0$ such that*

$$|v_n| \geq \varrho$$

for all $n \in \mathbb{N}$, where $R = \{n \in \mathbb{N} : v_n \neq 0\}$.

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GENERALIZED FUZZY SUBHYPERSPACES BASED ON FUZZY POINTS

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Abstract. We define $(\in, \in \vee q^{\delta})$-fuzzy subhyperspaces and $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspaces, as a generalization of fuzzy subhyperspaces, $(\in, \in \vee q)$-fuzzy subhyperspaces and $(\in, \in \vee q_k)$-fuzzy subhyperspaces. In this way, we show that $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspaces are the largest family of generalized fuzzy subhyperspaces based on concepts of belongingness and quasi-coincidence. Moreover, we study some properties and investigate the difference of generalized fuzzy subhyperspaces, supported by examples.

1. Introduction

The theory of fuzzy set was initiated by Zadeh [23] in 1965. It was extended to algebra by Rosenfeld [17] with defining fuzzy subgroups. Then other fuzzy algebraic structures have been investigated, such as fuzzy semigroups, fuzzy ideals, fuzzy vector spaces and so on. For more information about fuzzy algebraic structures refer to [15] and [16].

Algebraic hyperstructures was introduced by Marty [14] in 1934, when he defined hypergroups. Similarly fuzzy algebraic hyperstructures were investigated in many branches ([8]). Ameri [1] introduced fuzzy subhyperspaces of hypervector spaces in the sense of Scafatti-Tallini [18]. Fuzzy subhyperspaces were studied more in [2], [3] and [13].

After defining the concept of $(\in, \in \vee q)$-fuzzy subgroups by Bhakat and Das [4] as an important generalization of Rosenfeld's fuzzy subgroups, this notion and its another type, $(\in, \in \vee q_k)$-fuzzy subgroups, were studied on many algebraic structures (see [9]). In context of hyperstructures theory, as an extension of fuzzy subhyperstructures, Davvaz and Corsini defined $(\in, \in \vee q)$-fuzzy subhyperquasigroups in [5]. Furthermore, semihypergroups were characterized by $(\in, \in \vee q_k)$-fuzzy hyperideals and $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy hyperideals in [19] and [20], respectively. Moreover,

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$(\in, \in \vee q)$-fuzzy n-ary subhypergroups in [12] and [22], $(\in, \in \vee q)$-fuzzy and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy n-ary subpolygroups in [11] and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroups in [21], had been studied. Also, this concept and related topics were investigated on H_v-rings in [6], hypermodules in [24] and (m,n)-ary hypermodules in [7].

A new generalization of $(\in, \in \vee q)$-fuzzy subgroups was defined by Jun et al. ([10]) which called $(\in, \in \vee q^\delta)$-fuzzy subgroups. Now, in this paper, we introduce new generalizations of a fuzzy subhyperspace. In this regards, $(\in, \in \vee q^\delta)$-fuzzy subhyperspaces and $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspaces as generalizations of fuzzy subhyperspaces, $(\in, \in \vee q)$-fuzzy subhyperspaces and also $(\in, \in \vee q_k)$-fuzzy subhyperspaces are defined. It is shown that these notions construct a bigger family for generalized fuzzy subhyperspaces and also indicated that subhyperspaces are characterized by them. Moreover, connections and differences of them are studied, supported by illustrative examples.

2. Preliminaries

In this section we present some definitions and properties of hypervector spaces and fuzzy subhyperspaces that we shall use in later.

Definition 1. [18] *Let K be a field, $(V,+)$ be an Abelian group and $P_*(V)$ be the set of all non-empty subsets of V. We define a hypervector space over K to be the quadruplet $(V,+,\circ,K)$, where "$\circ$" is an external hyperoperation*

$$\circ : K \times V \longrightarrow P_*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

- (H_1) $a \circ (x+y) \subseteq a \circ x + a \circ y$, *right distributive law,*
- (H_2) $(a+b) \circ x \subseteq a \circ x + b \circ x$, *left distributive law,*
- (H_3) $a \circ (b \circ x) = (ab) \circ x$,
- (H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
- (H_5) $x \in 1 \circ x$,

where in (H_1), $a \circ x + a \circ y = \{p + q : p \in a \circ x, q \in a \circ y\}$. Similarly it is in ($H_2$). Also in ($H_3$), $a \circ (b \circ x) = \bigcup_{t \in b \circ x} a \circ t$.

V is called strongly right distributive, if we have equality in (H_1). In a similar way we define the strongly left distributive hypervector spaces. V is called strongly distributive, if it is strongly right and left distributive.

A non-empty subset W of V is called a subhyperspace of V if W is itself a hypervector space with the external hyperoperation on V, i.e. for all $a \in K$ and $x, y \in W$, $x - y \in W$ and $a \circ x \subseteq W$.

In the sequel of this paper, V denotes a hypervector space over the field K, unless otherwise is specified.

Example 2. [2] *In classical vector space* $(\mathbb{R}^3, +, ., \mathbb{R})$ *we define:*

$$\begin{cases} \circ: \quad \mathbb{R} \times \mathbb{R}^3 \longrightarrow P_*(\mathbb{R}^3) \\ \qquad a \circ (x_0, y_0, z_0) = L, \end{cases}$$

where L *is a line with the parametric equations:*

$$L: \begin{cases} x = ax_0, \\ y = ay_0, \\ z = t. \end{cases}$$

Then $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ *is a strongly left distributive hypervector space.*

Definition 3. [1] *A fuzzy subset* μ *of* V *is called a fuzzy subhyperspace of* V*, if for all* $a \in K$ *and* $x, y \in V$*, the following conditions are satisfied:*

1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,
2) $\bigwedge_{t \in a \circ x} \mu(t) \geq \mu(x)$.

Example 4. *(modified example 2.16, of* [3]*) Consider the hypervector space* $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ *in Example 2. Define a fuzzy subset* μ *of* V *by the following:*

$$\mu(x, y, z) = \begin{cases} t_3 & (x, y, z) \in \{0\} \times \{0\} \times \mathbb{R}, \\ t_2 & (x, y, z) \in \mathbb{R} \times \{0\} \times \mathbb{R} \backslash \{0\} \times \{0\} \times \mathbb{R}, \\ t_1 & otherwise, \end{cases}$$

where $0 \leq t_1 < t_2 < t_3 \leq 1$*. Then* μ *is a fuzzy subhyperspace of* V*.*

3. (α, β)-Fuzzy Subhyperspaces

A fuzzy subset μ of a hypervector space V defined by

$$\mu(y) = \begin{cases} t \ (\neq 0), & \text{if } y = x \\ 0, & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point* with the *support* x and the *value* t and is denoted by x_t. For a fuzzy point x_t and the fuzzy subset μ we write
(1) $x_t \in \mu \Leftrightarrow \mu(x) \geq t$.
(2) $x_t q \mu \Leftrightarrow \mu(x) + t > 1$.
(3) $x_t q_k \mu \Leftrightarrow \mu(x) + t + k > 1$, for $k \in [0, 1)$.
(4) $x_t q^\delta \mu \Leftrightarrow \mu(x) + t > \delta$, for $\delta \in (0, 1]$.
(5) $x_t q_k^\delta \mu \Leftrightarrow \mu(x) + t + k > \delta$, for $(k, \delta) \in [0, 1) \times (0, 1]$.
In case (1) we say that x_t is belong to μ and in (2) x_t is quasi-coincident with the fuzzy subset μ. For a fuzzy point x_t, we write $x_t \in \vee q\mu$ ($x_t \in \wedge q\mu$) if $x_t \in \mu$ or $x_t q\mu$ ($x_t \in \mu$ and $x_t q\mu$). Similarly, we have $x_t \in \vee q_k\mu$ and $x_t \in \wedge q_k\mu$. Also, for $\alpha \in \{\in, q, q_k, \in\vee q, \in\wedge q, \ldots\}$, the notation $x_t \overline{\alpha} \mu$ means that $x_t \alpha \mu$ does not hold.

Definition 5. *A fuzzy subset* μ *of* V *is called an* $(\in, \in\vee q_k^\delta)$*-fuzzy subhyperspace of* V*, if for all* $t, r \in [0, 1)$*,* $x, y \in V$ *and* $a \in K$*:*

$k\delta 1)$ $x_t \in \mu$ *and* $y_r \in \mu$ *imply that* $(x - y)_{t \wedge r} \in \vee q_k^\delta \mu$*;*

$k\delta 2)$ *$x_t \in \mu$ implies that $z_t \in \vee q_k^\delta \mu$, for all $z \in a \circ x$.*

μ is called an $(\in, \in \vee q)$-fuzzy subhyperspace of V, if $\delta = 1$ and $k = 0$. It is called an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V, if $\delta = 1$. Also, μ is called an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V, if $k = 0$.

Theorem 6. *A fuzzy subset μ of V is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V if and only if for all $x, y \in V$ and $a \in K$ the following conditions hold:*

$k\acute{\delta}1)$ $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$;

$k\acute{\delta}2)$ $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{\delta-k}{2}$.

Proof. Let μ be an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V. Assume that $(k\acute{\delta}1)$ is not valid, i.e. there exist $x, y \in V$ such that $\mu(x-y) < \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$. Then $\mu(x-y) < t \leq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$, for some $t \in (0,1]$. Thus $t \leq \mu(x)$ and $t \leq \mu(y)$, and so $x_t, y_t \in \mu$. Hence $(x-y)_t \in \vee q_k^\delta \mu$. But $\mu(x-y) < t$ and also $\mu(x-y) + t < t + t \leq \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$. It follows that $\mu(x-y) + t + k \leq \delta$. Therefore, $(x-y)_t \overline{\in \vee q_k^\delta} \mu$, which is a contradiction. Consequently, $(k\acute{\delta}1)$ is valid.

Now if there exist some $x, z \in V$ and $a \in K$, such that $z \in a \circ x$ and $\mu(z) < \mu(x) \wedge \frac{\delta-k}{2}$, then $\mu(z) < t \leq \mu(x) \wedge \frac{\delta-k}{2}$, for some $t \in (0,1]$. Thus $t \leq \mu(x)$ and so $x_t \in \mu$. Hence $z_t \in \vee q_k^\delta \mu$. But $\mu(z) < t$ and also $\mu(z) + t < t + t \leq \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$. Thus $\mu(z) + t + k \leq \delta$. Therefore $z_t \overline{\in \vee q_k^\delta} \mu$, which is a contradiction. Consequently, $(k\acute{\delta}2)$ is valid.

Conversely, let $x_t \in \mu$ and $y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. Thus by $(k\acute{\delta}1)$, $\mu(x-y) \geq t \wedge r \wedge \frac{\delta-k}{2}$. If $t \wedge r \leq \frac{\delta-k}{2}$, then $\mu(x-y) \geq t \wedge r$ and so $(x-y)_{t\wedge r} \in \mu$. If $t \wedge r > \frac{\delta-k}{2}$, then $\mu(x-y) \geq \frac{\delta-k}{2}$ and $\mu(x-y) + (t \wedge r) > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$. Thus $\mu(x-y) + (t \wedge r) + k > \delta$. Hence $(x-y)_{t\wedge r} q_k^\delta \mu$. Therefore $(x-y)_{t\wedge r} \in \vee q_k^\delta \mu$. Similarly, $(k\acute{\delta}2)$ implies $(k\delta 2)$. □

Corollary 7. *Let $\mu \in FS(V)$, i.e. μ is a fuzzy subset of V. Then μ is*

1) *an $(\in, \in \vee q)$-fuzzy subhyperspace of V if and only if for all $x, y \in V$ and $a \in K$, $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$ and $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge 0.5$;*
2) *an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V if and only if for all $x, y \in V$ and $a \in K$, $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{1-k}{2}$ and $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{1-k}{2}$;*
3) *an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V if and only if for all $x, y \in V$ and $a \in K$, $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta}{2}$ and $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{\delta}{2}$.*

Example 8. *Consider the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 2. Define a fuzzy subset μ of V by the following:*

$$\mu(x,y,z) = \begin{cases} \frac{1}{2} & (x,y,z) = (0,0,0), \\ \frac{1}{3} & (x,y,z) \in \mathbb{R} \times \{0\} \times \{0\} \setminus (0,0,0), \\ \frac{1}{5} & o.w. \end{cases}$$

Then μ is an $(\in, \in \vee q_{0.5}^{0.7})$-fuzzy subhyperspace of V, but it is not an $(\in, \in \vee q_{0.3}^{0.8})$-fuzzy subhyperspace of V, since the condition $(k\acute{\delta}2)$ is not valid (if $x = (0,0,0)$ and $z = (0,0,2)$, then for all $a \in \mathbb{R}$, $z \in a \circ x$, so $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{\delta-k}{2}$).

In the next example one can see that an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace is not an $(\in, \in \vee q)$-fuzzy subhyperspace or $(\in, \in \vee q_k)$-fuzzy subhyperspace or $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V, in general.

Example 9. *Consider the $(\in, \in \vee q_{0.5}^{0.7})$-fuzzy subhyperspace μ of $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 8. Then by Corollary 7, it follows that:*

1) *μ is not an $(\in, \in \vee q)$-fuzzy subhyperspace of V, because if $x = (0,0,0)$ and $z = (0,0,3)$, then for all $a \in \mathbb{R}$ and $z \in a \circ x$, $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge 0.5$;*
2) *μ is not an $(\in, \in \vee q_{0.5})$-fuzzy subhyperspace of V, because if $x = (0,0,0)$ and $z = (0,0,2)$, then for all $a \in \mathbb{R}$ and $z \in a \circ x$, $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{1-k}{2}$;*
3) *μ is not an $(\in, \in \vee q^{0.7})$-fuzzy subhyperspace of V, because if $x = (1,0,0)$, $a = 2$ and $z = (2,0,5)$, then $z \in a \circ x$, $\mu(x) = \frac{1}{3}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{\delta}{2}$.*

Theorem 10. *Let $\mu \in FS(V)$, $\delta \in (0,1]$ and $k \in [0,1)$. Then μ is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V if and only if $\mu_t (\neq \emptyset)$ is a subhyperspace of V, for all $t \in (0, \frac{\delta-k}{2}]$.*

Proof. Suppose μ is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V, $t \in (0, \frac{\delta-k}{2}]$ and $x, y \in \mu_t$. Then by Theorem 6,

$$\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2} \geq t \wedge t \wedge \frac{\delta-k}{2} = t.$$

Thus $x - y \in \mu_t$. Moreover, for all $a \in K$, $z \in a \circ x$ and $x \in \mu_t$, we have $\mu(z) \geq \mu(x) \wedge \frac{\delta-k}{2} \geq t$, which means that $a \circ x \subseteq \mu_t$. Hence μ_t is a subhyperspace of V, for all $t \in (0, \frac{\delta-k}{2}]$.

Conversely, let μ_t be a subhyperspace of V, for all $t \in (0, \frac{\delta-k}{2}]$ and let $(k\acute{\delta}1)$ is not valid. Then there exist $x, y \in V$ and $t \in (0,1)$ such that

$$\mu(x-y) < t < \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}.$$

Thus $x, y \in \mu_t$ for some $0 < t \leq \frac{\delta-k}{2}$, but $x - y \notin \mu_t$, which is a contradiction. Hence $(k\acute{\delta}1)$ is valid. Similarly, we can show $(k\acute{\delta}2)$ is valid. Therefore, by Theorem 6, μ is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V. □

Corollary 11. *Let $\mu \in FS(V)$. Then μ is*

1) *an $(\in, \in \vee q)$-fuzzy subhyperspace of V if and only if $\mu_t (\neq \emptyset)$ is a subhyperspace of V, for all $t \in (0, 0.5]$;*

2) *an* $(\in, \in \vee q_k)$*-fuzzy subhyperspace of* V *if and only if* $\mu_t(\neq \emptyset)$ *is a subhyperspace of* V*, for all* $t \in (0, \frac{1-k}{2}]$*;*
3) *an* $(\in, \in \vee q^{\delta})$*-fuzzy subhyperspace of* V *if and only if* $\mu_t(\neq \emptyset)$ *is a subhyperspace of* V*, for all* $t \in (0, \frac{\delta}{2}]$.

Theorem 12. *Let* $\mu \in FS(V)$*. Then* $\mu_t(\neq \emptyset)$ *is a subhyperspace of* V*, for all* $t \in (\frac{\delta-k}{2}, 1]$*, if and only if*

(*i*) $\mu(x-y) \vee \frac{\delta-k}{2} \geq \mu(x) \wedge \mu(y)$*, for all* $x, y \in V$*;*
(*ii*) $\bigwedge\limits_{z \in a \circ x} \mu(z) \vee \frac{\delta-k}{2} \geq \mu(x)$*, for all* $x \in V$ *and* $a \in K$*.*

Proof. Let $\mu_t(\neq \emptyset)$ be a subhyperspace of V, for all $t \in (\frac{\delta-k}{2}, 1]$. If there exist $x, y \in V$ such that

$$\mu(x-y) \vee \frac{\delta-k}{2} < \mu(x) \wedge \mu(y),$$

then $t_0 = \mu(x) \wedge \mu(y) \in (\frac{\delta-k}{2}, 1]$ and $x, y \in \mu_{t_0}$. Thus $x-y \in \mu_{t_0}$ and so $\mu(x-y) \geq t_0$, which is a contradiction with $\mu(x-y) \vee \frac{\delta-k}{2} < t_0$. Hence (i) holds. Similarly, condition (ii) will be obtained.

Conversely, assume that $t \in (\frac{\delta-k}{2}, 1]$ and $x, y \in \mu_t$. Then

$$\mu(x-y) \vee \frac{\delta-k}{2} < \mu(x) \wedge \mu(y) \geq t > \frac{\delta-k}{2}.$$

Thus $\mu(x-y) \geq t$ and so $x-y \in \mu_t$. Now let $a \in K$, $x \in \mu_t$ and $z \in a \circ x$. Then

$$\mu(z) \vee \frac{\delta-k}{2} \geq \bigwedge_{z \in a \circ x} \mu(z) \vee \frac{\delta-k}{2} \geq \mu(x) \geq t,$$

which implies that $\mu(z) \geq t$, for all $z \in a \circ x$. Hence $a \circ x \subseteq \mu_t$, for all $t \in (\frac{\delta-k}{2}, 1]$. Therefore, μ_t is a subhyperspace of V. $\square$

Corollary 13. *Let* $\mu \in FS(V)$*. Then*

1) $\mu_t(\neq \emptyset)$ *is a subhyperspace of* V *for all* $t \in (0.5, 1]$ *if and only if for all* $x, y \in V$ *and* $a \in K$*,* $\mu(x-y) \vee 0.5 \geq \mu(x) \wedge \mu(y)$ *and* $\bigwedge\limits_{z \in a \circ x} \mu(z) \vee 0.5 \geq \mu(x)$*;*
2) $\mu_t(\neq \emptyset)$ *is a subhyperspace of* V *for all* $t \in (\frac{1-k}{2}, 1]$ *if and only if for all* $x, y \in V$ *and* $a \in K$*,* $\mu(x-y) \vee \frac{1-k}{2} \geq \mu(x) \wedge \mu(y)$ *and* $\bigwedge\limits_{z \in a \circ x} \mu(z) \vee \frac{1-k}{2} \geq \mu(x)$*;*
3) $\mu_t(\neq \emptyset)$ *is a subhyperspace of* V *for all* $t \in (\frac{\delta}{2}, 1]$ *if and only if for all* $x, y \in V$ *and* $a \in K$*,* $\mu(x-y) \vee \frac{\delta}{2} \geq \mu(x) \wedge \mu(y)$ *and* $\bigwedge\limits_{z \in a \circ x} \mu(z) \vee \frac{\delta}{2} \geq \mu(x)$.

Theorem 14. *A non-empty subset* S *of* V *is a subhyperspace of* V *if and only if* χ_S *is an* $(\in, \in \vee q_k^{\delta})$*-fuzzy subhyperspace of* V*.*

Proof. Let S be a subhyperspace of V and $t \in (0, \frac{\delta-k}{2}]$. If $x, y \in \chi_{S_t}$, then $\chi_S(x), \chi_S(y) \geq t$. Thus $\chi_S(x) = \chi_S(y) = 1$ and so $x - y \in S$. Hence $\chi_S(x-y) = 1 \geq t$, i.e. $x - y \in \chi_{S_t}$. Similarly, $a \circ x \subseteq \chi_{S_t}$, for all $a \in K$ and $x \in \chi_{S_t}$. Consequently, χ_{S_t} is a subhyperspace of V, for all $t \in (0, \frac{\delta-k}{2}]$. Therefore, by Theorem 10, χ_S is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V.

Conversely, let χ_S be an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V, $a \in K$ and $x \in S$. Then by Theorem 6, for all $z \in a \circ x$, $\bigwedge\limits_{z \in a \circ x} \chi_S(z) \geq \chi_S(x) \wedge \frac{\delta-k}{2} = 1 \wedge \frac{\delta-k}{2} = \frac{\delta-k}{2}$. Since $\delta \in (0,1]$ and $k \in [0,1)$, so $\chi_S(z) = 1$, for all $z \in a \circ x$. Thus $a \circ x \subseteq S$. Similarly, $x + y \in S$, for all $x, y \in S$. Therefore, S is a subhyperspace of V. □

It is well-known that the characterization function of any subhyperspace is a fuzzy subhyperspace. Hence the following corollary is obtained from Theorem 14:

Corollary 15. *A non-empty subset S of V is a subhyperspace of V if and only if χ_S is an $(\in, \in \vee q)$-fuzzy subhyperspace of V if and only if χ_S is an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V if and only if χ_S is an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V.*

Proposition 16. *Let $\delta \in (0,1]$, $k \in [0,1)$. Then*

1) *Every $(\in, \in \vee q_k)$-fuzzy subhyperspace of V is an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V, if $\delta + k < 1$;*
2) *Every $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V is an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V, if $\delta + k > 1$;*
3) *For $\delta + k = 1$, μ is an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V if and only if it is an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V;*
4) *Every $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V is an $(\in, \in \vee q)$-fuzzy subhyperspace of V, if $\delta = k + 1$;*
5) *Every $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V is a fuzzy subhyperspace of V, if $\delta = k$.*

Proof. 1) Let μ be an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V. Then by Corollary 7(2), for all $x, y \in V$ and $a \in K$, it follows that:

$$\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{1-k}{2} \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta}{2},$$

and

$$\bigwedge_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{1-k}{2} \geq \mu(x) \wedge \frac{\delta}{2}.$$

Thus by Corollary 7(3), μ is an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V.

2) The proof is completed by Corollary 7, similarly.

3) One can conclude by Corollary 7(2) and Corollary 7(3).

4) It is straightforward by Theorem 6 and Corollary 7(1).

5) The proof is obtained by Theorem 6 and Definition 3. □

The following example shows that the converse of assertions (1) and (2) of Proposition 16 are not generally true.

Example 17. *Consider the fuzzy subset μ of $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 8. Then by Corollary 7, it follows that:*

1) *μ is an $(\in, \in \vee q^{0.2})$-fuzzy subhyperspace of V, but it is not an $(\in, \in \vee q_{0.4})$-fuzzy subhyperspace of V, because if $x = (2, 0, 0)$, $a = 4$ and $z = (8, 0, 3)$, then $z \in a \circ x$, $\mu(x) = \frac{1}{3}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{1-k}{2}$ $(\delta = 0.2,\ k = 0.4)$.*
2) *μ is an $(\in, \in \vee q_{0.7})$-fuzzy subhyperspace of V, but it is not an $(\in, \in \vee q^{0.5})$-fuzzy subhyperspace of V, because if $x = (3, 0, 0)$, $a = 1$ and $z = (3, 0, 2)$, then $z \in a \circ x$, $\mu(x) = \frac{1}{3}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{\delta}{2}$ $(\delta = 0.5,\ k = 0.7)$.*

Theorem 18. *If μ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V, such that $\mu(x) \leq \frac{\delta-k}{2}$, for all $x \in V$, then μ is a fuzzy subhyperspace of V.*

Proof. By Theorem 6, for all $x, y \in V$ and $a \in K$, $\mu(x - y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2} = \mu(x) \wedge \mu(y)$ and $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{\delta-k}{2} = \mu(x)$. So μ is a fuzzy subhyperspace of V. □

Note that in the $(\in, \in \vee q_{0.5}^{0.7})$-fuzzy subhyperspace μ of $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 8, $\mu(x) \nleq \frac{\delta-k}{2}$, for some $x \in V$ and μ is not a fuzzy subhyperspace of V.

Corollary 19. *Let $\mu \in FS(V)$. Then*

1) *If μ is an $(\in, \in \vee q)$-fuzzy subhyperspace of V, such that $\mu(x) < 0.5$, for all $x \in V$, then μ is a fuzzy subhyperspace of V;*
2) *If μ is an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V, such that $\mu(x) < \frac{1-k}{2}$, for all $x \in V$, then μ is a fuzzy subhyperspace of V;*
3) *If μ is an $(\in, \in \vee q^{\delta})$-fuzzy subhyperspace of V, such that $\mu(x) < \frac{\delta}{2}$, for all $x \in V$, then μ is a fuzzy subhyperspace of V.*

Proposition 20. *Let $0 < \delta_2 \leq \delta_1 \leq 1$ and $\mu \in FS(V)$. If μ is an $(\in, \in \vee q_k^{\delta_1})$-fuzzy subhyperspace of V, then it is an $(\in, \in \vee q_k^{\delta_2})$-fuzzy subhyperspace of V.*

Proof. By Theorem 6, $\mu(x - y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta_1-k}{2}$ and $\mu(z) \geq \mu(x) \wedge \frac{\delta_1-k}{2}$, for all $x, y \in V$, $a \in K$ and $z \in a \circ x$. Since $\delta_1 \geq \delta_2$, thus $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta_2-k}{2}$ and $\mu(z) \geq \mu(x) \wedge \frac{\delta_2-k}{2}$. Hence the proof is completed by Theorem 6. □

In the following example it can be seen that the converse of Proposition 20, is not generally valid.

Example 21. *Consider the fuzzy subset μ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then μ is an $(\in, \in \vee q_{0.5}^{0.8})$-fuzzy subhyperspace of V and μ is not an $(\in, \in \vee q_{0.5}^{0.95})$-fuzzy subhyperspace of V, while $\delta_2 = 0.8 \leq \delta_1 = 0.95$.*

Corollary 22. *Let $0 < \delta_2 \leq \delta_1 \leq 1$ and $\mu \in FS(V)$. If μ is an $(\in, \in \vee q^{\delta_1})$-fuzzy subhyperspace of V, then it is an $(\in, \in \vee q^{\delta_2})$-fuzzy subhyperspace of V.*

Proof. It is straightforward by Corollary 7 and Proposition 20. □

Next example shows that the converse of Corollary 22, is not valid, in general.

Example 23. *Consider the fuzzy subset μ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then μ is an $(\in, \in \vee q^{0.3})$-fuzzy subhyperspace of V and μ is not an $(\in, \in \vee q^{0.5})$-fuzzy subhyperspace of V, while $\delta_2 = 0.3 \leq \delta_1 = 0.5$.*

Proposition 24. *Let $0 \leq k_1 \leq k_2 < 1$ and $\mu \in FS(V)$. If μ is an $(\in, \in \vee q^{\delta}_{k_1})$-fuzzy subhyperspace of V, then it is an $(\in, \in \vee q^{\delta}_{k_2})$-fuzzy subhyperspace of V.*

Proof. It is completed by a similar manner of the proof of Proposition 20. $\square$

The converse of Proposition 24, is not valid in general. See the following example:

Example 25. *Consider the fuzzy subset μ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then μ is an $(\in, \in \vee q^{0.7}_{0.4})$-fuzzy subhyperspace of V and μ is not an $(\in, \in \vee q^{0.7}_{0.2})$-fuzzy subhyperspace of V, while $k_1 = 0.2 \leq k_2 = 0.4$.*

The following corollary is immediately concluded by Corollary 7 and Proposition 24.

Corollary 26. *Let $0 \leq k_1 \leq k_2 < 1$ and $\mu \in FS(V)$. If μ is an $(\in, \in \vee q_{k_1})$-fuzzy subhyperspace of V, then it is an $(\in, \in \vee q_{k_2})$-fuzzy subhyperspace of V.*

The converse of Corollary 26, is not valid in general. See the next example:

Example 27. *Consider the fuzzy subset μ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then μ is an $(\in, \in \vee q_{0.7})$-fuzzy subhyperspace of V and μ is not an $(\in, \in \vee q_{0.5})$-fuzzy subhyperspace of V, while $k_1 = 0.5 \leq k_2 = 0.7$.*

Theorem 28. *Let S be a subhyperspace of V. Then for every $t \in (0, \frac{\delta-k}{2}]$, there exists an $(\in, \in \vee q^{\delta}_{k})$-fuzzy subhyperspace μ of V such that $\mu_t = S$.*

Proof. Let $t \in (0, \frac{\delta-k}{2}]$ and define a fuzzy subset μ of V as

$$\mu(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $\mu_t = S$. Now if there exist $x, y \in V$ such that $\mu(x-y) < \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$, then $\mu(x-y) = 0$ and $\mu(x) = \mu(y) = t$, which is a contradiction. Thus $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$, for all $x, y \in V$. Similarly, $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{\delta-k}{2}$, for all $a \in K$. Hence μ is an $(\in, \in \vee q^{\delta}_{k})$-fuzzy subhyperspace of V, by Theorem 6. $\square$

Corollary 29. *Let S be a subhyperspace of V. Then*

1) *For every $t \in (0, 0.5]$, there exists an $(\in, \in \vee q)$-fuzzy subhyperspace μ of V, such that $\mu_t = S$;*
2) *For every $t \in (0, \frac{1-k}{2}]$, there exists an $(\in, \in \vee q_k)$-fuzzy subhyperspace μ of V, such that $\mu_t = S$;*
3) *For every $t \in (0, \frac{\delta}{2}]$, there exists an $(\in, \in \vee q^{\delta})$-fuzzy subhyperspace μ of V, such that $\mu_t = S$.*

Theorem 30. *If μ_i is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V, for all $i \in I$, then $\mu = \cap_{i\in I}\mu_i$ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V.*

Proof. Let $x_t, y_r \in \mu$, for $x, y \in V$ and $t, r \in (0,1)$ and $(x-y)_{t\wedge r} \overline{\in \vee q_k^{\delta}} \mu$. Then $\mu(x-y) < t \wedge r$ and $\mu(x-y) + t \wedge r \leq \delta - k$, which imply that $\mu(x-y) < \frac{\delta-k}{2}$. Now, put $I_1 = \{i \in I \mid (x-y)_{t\wedge r} \in \mu_i\}$ and $I_2 = \{i \in I \mid (x-y)_{t\wedge r} q_k^{\delta} \mu_i\} \cap \{j \in I \mid (x-y)_{t\wedge r} \overline{\in} \mu_j\}$. Then $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. If $I_2 = \emptyset$, then $(x-y)_{t\wedge r} \in \mu_i$, for all $i \in I$, which implies that $\mu(x-y) \geq t \wedge r$, that is a contradiction. Hence $I_2 \neq \emptyset$, and so for every $i \in I_2$, $\mu_i(x-y) < t \wedge r$ and $\mu_i(x-y) + t \wedge r > \delta - k$, that is $t \wedge r > \frac{\delta-k}{2}$. Thus from $x_t, y_r \in \mu$, we can obtain $\mu_i(x) \wedge \mu_i(y) > \mu(x) \wedge \mu(y) > t \wedge r > \frac{\delta-k}{2}$. Now, set $\alpha = \mu_i(x-y) < \frac{\delta-k}{2}$ and take $\alpha < \beta < \frac{\delta-k}{2}$. Then $x_\alpha, y_\beta \in \mu_i$ but $\mu_i(x-y) = \alpha < \beta$ and $\mu_i(x-y) + \beta < \delta - k$. This contradicts that μ_i is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V. Thus $\mu_i(x-y) \geq \frac{\delta-k}{2}$, which is a contradiction. Hence $(x-y)_{t\wedge r} \in \vee q_k^{\delta} \mu$. Similarly, the condition $(k\delta 2)$ will be proven. Therefore, $\mu = \cap_{i\in I}\mu_i$ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V. □

For any fuzzy set μ of V and $t \in (0,1]$, we denote

$$(\mu)_t = \{x \in V \mid x_t \ q_k^{\delta} \ \mu\} \quad \text{and} \quad [\mu]_t = \{x \in V \mid x_t \ \in \vee q_k^{\delta} \ \mu\}.$$

Obviously, $[\mu]_t = \mu_t \cup (\mu)_t$.

Theorem 31. *Let $\mu \in FS(V)$. Then μ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V if and only if $[\mu]_t$ is a subhyperspace of V, for all $t \in (0,1]$.*

Proof. let μ be an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V and $x, y \in [\mu]_t$, for $t \in (0,1]$. Then $\mu(x) \geq t$ or $\mu(x) + t > \delta - k$, and $\mu(y) \geq t$ or $\mu(y) + t > \delta - k$. Using Theorem 6,

1) for $\mu(x), \mu(y) \geq t$, $\mu(x-y) \geq t \wedge \frac{\delta-k}{2}$. If $t > \frac{\delta-k}{2}$, then $\mu(x-y) + t > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$, and so $(x-y)_t q_k^{\delta} \mu$. If $t \leq \frac{\delta-k}{2}$, then $\mu(x-y) \geq t$ and so $(x-y)_t \in \mu$. Thus $x - y \in [\mu]_t$. Similarly, we can show $a \circ x \subseteq [\mu]_t$, for all $a \in K$, in this case.

2) for $\mu(x) \geq t$ and $\mu(y) + t > \delta - k$, $\mu(x-y) \geq t \wedge \delta - k - t \wedge \frac{\delta-k}{2}$. If $t > \frac{\delta-k}{2}$, then $\mu(x-y) > \delta - k - t$, and so $(x-y)_t q_k^{\delta} \mu$. If $t \leq \frac{\delta-k}{2}$, then $\mu(x-y) \geq t$ and so $(x-y)_t \in \mu$. Thus $x - y \in [\mu]_t$. Similarly, we can show $a \circ x \subseteq [\mu]_t$, for all $a \in K$, in this case.

3) for $\mu(x) + t > \delta - k$ and $\mu(y) \geq t$, we can prove similar to the case (2).

4) for $\mu(x) + t > \delta - k$ and $\mu(y) + t > \delta - k$, if $t > \frac{\delta-k}{2}$, then $\mu(x-y) > \delta - k - t$, and if $t \leq \frac{\delta-k}{2}$, then $\mu(x-y) \geq t$. Thus $(x-y)_t \in \vee q_k^{\delta} \mu$ and so $x - y \in [\mu]_t$. Similarly, $a \circ x \subseteq [\mu]_t$, for all $a \in K$.

Therefore, $[\mu]_t$ is a subhyperspace of V.

Conversely, let μ be a fuzzy subset of V and there exist $x, y \in V$ such that $\mu(x-y) < \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$, for $\delta \in (0,1]$ and $k \in [0,1)$. Then $\mu(x-y) < t \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$, for some $t \in (0,1)$. Thus $x, y \in [\mu]_t$ and so $x - y \in [\mu]_t$. But

$\mu(x-y) < t$ and $\mu(x-y)+t \geq \delta - k$, which is a contradiction. Hence (1) of Theorem 6, and similarly the assertion (2) of Theorem 6, are valid. Therefore, μ is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V. □

4. Conclusion

We define $\lambda(x)+t > \delta$ and $\lambda(x)+t+k > \delta$ as new connections between a fuzzy point and a fuzzy subset on a hypervector space to generalize the concept of fuzzy subhyperspaces. These new connections help us to find new generalizations for fuzzy subhyperspaces and specially the largest family of them based on the concepts of belongingness and quasi-coincidence. This study can be extended to other algebraic structures and hyperstructures, in future. The following figure shows how we extend the family of generalized fuzzy subhyperspaces:

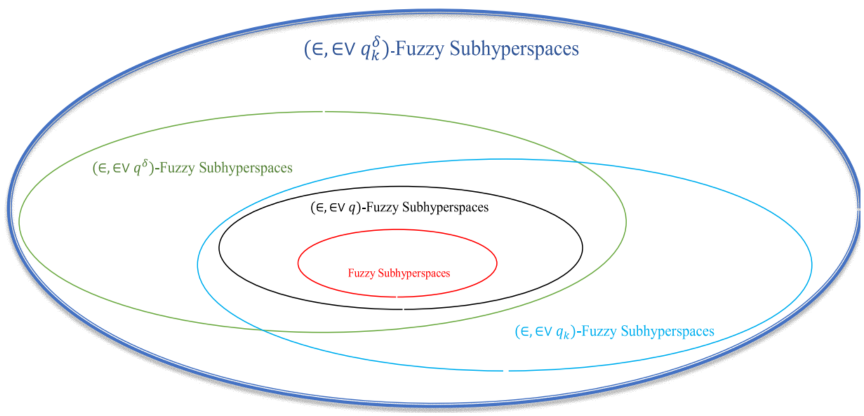

Figure 1. Generalizations of fuzzy subhyperspces

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THE COMPARISON OF DIFFERENT ESTIMATION METHODS FOR THE PARAMETERS OF FLEXIBLE WEIBULL DISTRIBUTION

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Abstract. This article presents different parameter estimation methods for flexible Weibull distribution introduced by Bebbington et al. (Reliability Engineering and System Safety 92:719-726, 2007), which is a modified version of the Weibull distribution and is suitable to model different shapes of the hazard rate. We consider both frequentist and Bayesian estimation methods and present a comprehensive comparison of them. For frequentist estimation, we consider the maximum likelihood estimators, least squares estimators, weighted least squares estimators, percentile estimators, the maximum product spacing estimators, the minimum spacing absolute distance estimators, the minimum spacing absolute log-distance estimators, Cramér von Mises estimators, Anderson Darling estimators, and right tailed Anderson Darling estimators, and compare them using a comprehensive simulation study. We also consider Bayesian estimation by assuming gamma priors for both shape and scale parameters. We use a Markov Chain Monte Carlo algorithm to compute the posterior summaries. A real data example is also a part of this work.

1. Introduction

Weibull distribution is one of the most widely used distributions in reliability, and has a monotonic hazard rate, which may be increasing or decreasing. In many reliability applications, however, the failure rate often non-monotonic, which motivated [1] to introduce a new extension of the Weibull distribution having bathtub-shaped failure rate. To define it, let X have the flexible Weibull (FW for short) distribution, say $X \sim \mathrm{FW}(\alpha, \lambda)$. [1] defined the cumulative distribution function (cdf) of

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X as

$$G(x) = 1 - \exp\bigl(-\exp(\alpha t - \lambda/t)\bigr), \tag{1}$$

where α and λ are the shape parameters. The exponential distribution is obtained by $\lambda = 0$ and $\alpha = \log(\theta)$. The probability density function (pdf) corresponding to (1) is given by

$$g(x) = (\alpha + \lambda/x^2)\exp\bigl(\alpha t - \lambda/x\bigr)\exp\bigl(-\exp(\alpha t - \lambda/t)\bigr), \qquad x > 0 \tag{2}$$

[1] pointed out that as λ decreases, the failure rate function becomes more bathtub-like while it becomes shallower as α increases.

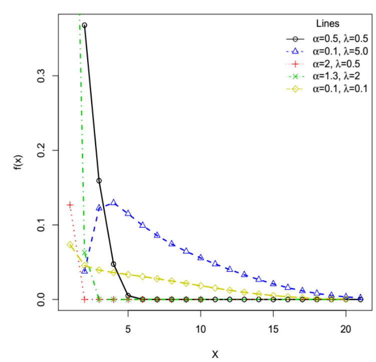


FIGURE 1. Density Plot of flexible Weibull for some selected parameter values.

Note that the FW distribution has the closed-form density, hazard and survival functions. In Figure-1, we have depicted the density of FW distribution for various combinations of parameters. It is clear from the figure that the distribution is very flexible and adopts various shapes for different combinations of parameters.

In the literature, [2] developed a R Package 'reliaR' to generate random numbers from FW to estimate its parameters and study other reliability characteristics. [3] discussed Bayesian estimation and prediction for FW under type-II censoring scheme. [4] discussed parameter estimation of the flexible Weibull distribution for type I censored data. [5] proposed a new extension of FW distribution using the odd generalized exponential generator. [6] proposed a generalized class of FW distribution for repairable systems. [7] proposed a generalized class of FW distribution. [8] discussed estimation and prediction for type-II hybrid censored data assuming FW distribution. [9] studied the penalized maximum likelihood estimation for the modified extended Weibull distribution. [10] discussed the reliability properties of the proportional hazard reverse transformation using FW distribution. [11] presented estimation and prediction for FW based on progressive type-II censored data. [12] proposed exponentiated additive Weibull distribution where FW is a special case of the proposed distribution.

The aim of this article is to compare different parameter estimation methods, including both classical and Bayesian. In particular, we compare the maximum likelihood, the maximum and the minimum spacing distances (minimum spacing absolute distance and minimum spacing absolute-log distance), ordinary and weighted least squares, percentiles, the minimum distance methods including Cramér-von-Mises, Anderson-Darling and right-tail Anderson-Darling. Further, we also compute the parameter estimates of FW by using the Bayesian method, where we use the Markov Chain Monte Carlo (MCMC) to obtain the posterior summaries. Several authors have used different methods of estimations for different distributions, for example, [13, 14, 15, 16, 17, 18, 19, 20].

The rest of the article is organized as follows: Section 2 discusses some new properties of the FW distribution. Section 3 deals with different methods of estimation of the model parameters. Section 4 presents simulation study while a real life example to show the practical application is presented in Section 5. Finally, some concluding remarks are given in Section 6.

2. New properties

This section discusses some statistical properties.

2.1. **Moments, Skewness and Kurtosis.** We calculate the mean, variance, skewness and kurtosis numerically and depict in Figure-2. It is clear from the figure that as λ increases, the mean and variance also increase. However, the skewness and kurtosis decrease by increasing λ. It is also noticed that a small value of α results into large value of mean, variance, skewness and kurtosis.

2.2. **Quantile function.** To generate random variable from FW, we invert Equation-1 as follows $X = F^{-1}(u)$, where $u \sim Uniform(0,1)$. The simplified form is

$$X = F^{-1}(u) = \frac{1}{2\alpha}\left(\log(-\log u) + \sqrt{\{\log(-\log u)\}^2 + 4\alpha\lambda}\right) \tag{3}$$

The skewness and kurtosis measures can be investigated using the quantile function. For example, the Bowley skewness [21] based on quantiles is given by

$$B = \frac{F^{-1}(3/4) + F^{-1}(1/4) - 2\,F^{-1}(2/4)}{F^{-1}(3/4) - F^{-1}(1/4)}.$$

Similarly, the Moors' kurtosis [22] is

$$M = \frac{F^{-1}(3/8) - F^{-1}(1/8) + F^{-1}(7/8) - F^{-1}(5/8)}{F^{-1}(6/8) - F^{-1}(2/8)}.$$

2.3. **Reliability properties of FW distribution.** A key property to characterize the distribution is log-concave, i.e., the density is log-concave if $d^2/dx^2 \log f < 0$, otherwise convex. The hazard would be decreasing if density is log-concave. For the FW, it is observed that the density is log-concave for $\lambda > \alpha$.

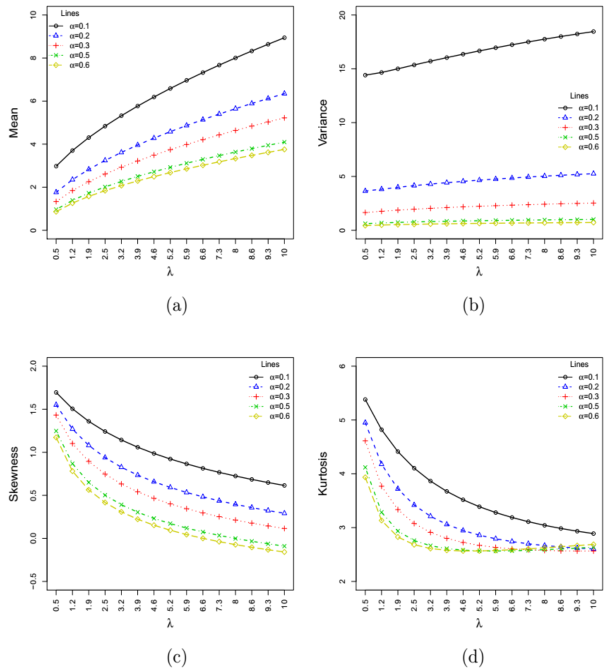


FIGURE 2. Plots of the FW (a) Mean (b) Variance (c) Skewness, and (d) Kurtosis for some selected parameter values.

2.4. **Stochastic ordering.** Stochastic ordering is an important tool in reliability theory and finance to assess comparative behavior. Let X_1 and X_2 be two random variables having cdfs, sfs and pdfs $F_1(x)$, $F_2(x)$, $\bar{F}_1(x) = 1 - F_1(x)$, $\bar{F}_2(x) = 1 - F_2(x)$, $f_1(x)$, and $f_2(x)$, respectively. The random variable X_1 is said to be smaller than X_2 in the following ordering as:

(i) stochastic order (denoted by $X_1 \leq_{st} X_2$) if $\bar{F}_1(x) \leq \bar{F}_2(x)$ for all x;
(ii) likelihood ratio order (denoted by $X_1 \leq_{lr} X_1$) if $f_1(x)/f_2(x)$ is decreasing in $x \geq 0$;
(iii) hazard rate order (denoted by $X_1 \leq_{hr} X_2$) if $\bar{F}_1(x)/\bar{F}_2(x)$ is decreasing in $x \geq 0$;
(iv) reversed hazard rate order (denoted by $X_1 \leq_{rhr} X_2$) if $F_1(x)/F_2(x)$ is decreasing in $x \geq 0$.

All these four stochastic orders defined in (i)–(iv) are related to each other [23] and the following implications hold:

$$(X_1 \leq_{rhr} X_2) \Leftarrow (X_1 \leq_{lr} X_2) \Rightarrow (X_1 \leq_{hr} X_2) \Rightarrow (X_1 \leq_{st} X_2). \tag{4}$$

The following theorem shows that the FW distribution has likelihood ratio ordering when appropriate assumptions are satisfied.

Theorem 2.1. *Let $X_1 \sim FW(\alpha_1, \lambda_1)$ and $X_2 \sim FW(\alpha_2, \lambda_2)$. If $\alpha_1 < \alpha_2$ for fixed $\lambda_1 = \lambda_2 = \lambda$, and $\lambda_2 > \lambda_2$, for $\alpha_1 = \alpha_2 = \alpha$ then $X_1 \leq_{lr} X_2$.*

Proof. It is not difficult to show that $\frac{d}{dx} \log \frac{f_1(x;\alpha_1,\lambda_1)}{f_2(x;\alpha_2,\lambda_1)} < 0$ for the following conditions:

- $\alpha_1 < \alpha_2$ for fixed $\lambda_1 = \lambda_2 = \lambda$,
- $\lambda_2 > \lambda_2$ and $\alpha_1 = \alpha_2 = \alpha$.

Thus, likelihood ratio ordering holds and $X_1 \leq_{lr} X_2$. □

2.5. **Stress and Strength Analysis.** Stress-Strength reliability is defined as $G = Pr(X_1 > X_2) = \int_0^\infty f_1(x)F_2(x)dx$, $X_1 \sim FW(\alpha_1, \lambda_1)$ and $X_2 \sim FW(\alpha_2, \lambda_2)$, whereas the $f_1(x)$ is the pdf of X_1 and $F_2(x)$ cdf of X_2.

$$G = Pr(X_1 > X_2) = 1 - \int_0^\infty (\alpha_1 + \lambda_1/x^2) \exp\big(\alpha_1 x - \lambda_1/x\big) \times \exp\Big(-\exp\big(\alpha_1 x - \lambda_1/x\big) - \exp\big(\alpha_2 x - \lambda_2/x\big)\Big) dx \tag{5}$$

The above equation can be solved numerically.

3. Parameters estimation methods

This section describes ten different methods of estimation to obtain the estimators of the parameters α and λ of the FW distribution.

3.1. **Maximum likelihood estimators.** Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from Equation (2). Then, the log-likelihood function is given by

$$\ell(\alpha, \lambda) = \sum_{i=1}^n \log\big(\alpha + \lambda/x_i^2\big) + \alpha \sum_{i=1}^n x_i - \sum_{i=1}^n (\lambda/x_i) - \sum_{i=1}^n \exp\big(\alpha x_i - \lambda/x_i\big) \tag{6}$$

The resulting partial derivatives of the log-likelihood function are

$$\frac{\partial\, \ell(\alpha, \lambda)}{\partial \alpha} = \sum_{i=1}^n \frac{1}{\alpha + \lambda/x_i^2} + \sum_{i=1}^n x_i - \sum_{i=1}^n x_i \exp\big(\alpha x_i - \lambda/x_i\big) \tag{7}$$

$$\frac{\partial\, \ell(\alpha, \lambda)}{\partial \lambda} = \sum_{i=1}^n \frac{1}{\alpha x_i^2 + \lambda} + \sum_{i=1}^n x_i^{-1} - \sum_{i=1}^n x_i^{-1} \exp\big(\alpha x_i - \lambda/x_i\big) \tag{8}$$

Equating these partial derivatives to zero do not yield closed-form solutions for the MLEs and thus a numerical method, like Newton Raphson, is used for solving these equations simultaneously.

3.2. **Least Squares Estimators.** The least squares and weighted least squares estimators were proposed by [24] to estimate the parameters of beta distributions. To define these, suppose $F(X_{(j)})$ denote the distribution function of the ordered random variables $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ where $\{X_1, X_2, \cdots, X_n\}$ is a random sample of size n from the distribution function $F(\cdot)$. Then, the least squares estimators of α and λ, say $\hat{\alpha}_{LSE}$ and $\hat{\lambda}_{LSE}$ can be obtained by minimizing

$$S(\alpha, \lambda) = \sum_{i=1}^{n} \left[F\left(x_{i:n} | \alpha, \lambda\right) - \frac{i}{n+1} \right]^2$$

with respect to α and λ, where $F(\cdot)$ is the cdf (1). Equivalently, the estimators can be obtained by solving:

$$\sum_{i=1}^{n} \left[F\left(x_{i:n} \mid \alpha, \lambda\right) - \frac{i}{n+1} \right] \eta_1\left(x_{i:n} \mid \alpha, \lambda\right) = 0,$$
$$\sum_{i=1}^{n} \left[F\left(x_{i:n} \mid \alpha, \lambda\right) - \frac{i}{n+1} \right] \eta_2\left(x_{i:n} \mid \alpha, \lambda\right) = 0,$$

where

$$\eta_1\left(x_{i:n} \mid \alpha, \lambda\right) = \alpha \exp\Big(\big(\alpha x - \lambda/x\big) - \exp\big(\alpha x - \lambda/x\big) \Big), \tag{9}$$

and

$$\eta_2\left(x_{i:n} \mid \alpha, \lambda\right) = \frac{\lambda}{x^2} \exp\Big(\big(\alpha x - \lambda/x\big) - \exp\big(\alpha x - \lambda/x\big) \Big). \tag{10}$$

The weighted least squares estimators, $\widehat{\alpha}_{WLSE}$ and $\widehat{\lambda}_{WLSE}$, can be obtained by minimizing

$$W\left(\alpha, \lambda\right) = \sum_{i=1}^{n} \frac{\left(n+1\right)^2 \left(n+2\right)}{i\left(n-i+1\right)} \left[F\left(x_{i:n} \mid \alpha, \lambda\right) - \frac{i}{n+1} \right]^2.$$

These estimators can be obtained by solving:

$$\sum_{i=1}^{n} \frac{\left(n+1\right)^2 \left(n+2\right)}{i\left(n-i+1\right)} \left[F\left(x_{i:n} \mid \alpha, \lambda\right) - \frac{i}{n+1} \right] \eta_1\left(x_{i:n} \mid \alpha, \lambda\right) = 0,$$
$$\sum_{i=1}^{n} \frac{\left(n+1\right)^2 \left(n+2\right)}{i\left(n-i+1\right)} \left[F\left(x_{i:n} \mid \alpha, \lambda\right) - \frac{i}{n+1} \right] \eta_2\left(x_{i:n} \mid \alpha, \lambda\right) = 0.$$

3.3. **Percentile Estimators.** If the data come from a distribution function which has a closed form, then the unknown parameters can be estimated by fitting straight line to the theoretical points obtained from the distribution function and the sample percentile points. This method was originally suggested by [25, 26] and it has been used for Weibull distribution and for generalized exponential distribution. In this paper, we apply the same technique for the two-parameter FW distribution. Let $X_{(j)}$ be the jth order statistic, i.e, $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. If p_j denote some estimate of $F(x_{(j)};\alpha,\lambda)$, then the estimate of α and λ can be obtained by minimizing

$$\sum_{j=1}^{n}\left(x_{(j)}-\frac{1}{2\alpha}\left(\log(-\log p_j)+\sqrt{\{\log(-\log p_j)\}^2+4\alpha\lambda}\right)\right)^2,$$

with respect to α and λ. Several type of estimators for p_j can be used [27] and this paper considers $p_j=\frac{j}{n+1}$.

3.4. **Maximum and Minimum Product of Spacings Estimators.** The maximum product spacing (MPS) method was introduced by [28, 29] as an alternative to MLE for the estimation of the unknown parameters of continuous univariate distributions. The MPS method was also derived independently by [30] as an approximation to the Kullback-Leibler measure of information. To motivate our choice, [29] proved that this method is as efficient as the MLE estimators and consistent under more general conditions.

We define the uniform spacings of a random sample from the FW distribution as:

$$D_i(\alpha,\lambda)=F\left(x_{i:n}\mid\alpha,\lambda\right)-F\left(x_{i-1:n}\mid\alpha,\lambda\right),\qquad i=1,2,\ldots,n,$$

where $F(x_{0:n}\mid\alpha,\lambda)=0$ and $F(x_{n+1:n}\mid\alpha,\lambda)=1$. Clearly $\sum_{i=1}^{n+1}D_i(\alpha,\lambda)=1$.

The maximum product of spacings estimators $\widehat{\alpha}_{MPS}$ and $\widehat{\lambda}_{MPS}$, of the parameters α and λ are obtained by maximizing the geometric mean of the spacings with respect to α and λ

$$G\left(\alpha,\lambda\right)=\left[\prod_{i=1}^{n+1}D_i(\alpha,\lambda)\right]^{\frac{1}{n+1}},\tag{11}$$

or, equivalently, by maximizing the function

$$H\left(\alpha,\lambda\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\alpha,\lambda).\tag{12}$$

The estimators $\widehat{\alpha}_{MPS}$ and $\widehat{\lambda}_{MPS}$ of the parameters α and λ can be obtained by solving the nonlinear equations

$$\begin{aligned}\frac{\partial}{\partial\alpha}H\left(\alpha,\lambda\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\lambda)}\left[\eta_1(x_{i:n}|\alpha,\lambda)-\eta_1(x_{i-1:n}|\alpha,\lambda)\right]&=0,\\ \frac{\partial}{\partial\lambda}H\left(\alpha,\lambda\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\lambda)}\left[\eta_2(x_{i:n}|\alpha,\lambda)-\eta_2(x_{i-1:n}|\alpha,\lambda)\right]&=0,\end{aligned}$$

where $\eta_1\left(\cdot\mid\alpha,\lambda\right)$ and $\eta_2\left(\cdot\mid\alpha,\lambda\right)$ are given by (9) and (10), respectively.

Similarly, the minimum spacing distance estimators of $\widehat{\alpha}_{MSADE}$ and $\widehat{\lambda}_{MSADE}$ of α and λ are obtained by minimizing

$$T\left(\alpha,\lambda\right)=\sum_{i=1}^{n+1}h\left(D_i\left(\alpha,\lambda\right),\frac{1}{n+1}\right), \tag{13}$$

where $h(x,y)$ is an appropriate distance. Some choices of $h(x,y)$ are the absolute distance $|x-y|$ and the absolute-log distance $|\log x-\log y|$. These estimators are called the "minimum spacing absolute distance estimator" (MSADE) and the "minimum spacing absolute-log distance estimator" (MSALDE). The MSADE and MSALDE of parameters α and λ can be obtained by minimizing

$$T\left(\alpha,\lambda\right)=\sum_{i=1}^{n+1}\left|(D_i(\alpha,\lambda)-\frac{1}{n+1}\right| \tag{14}$$

and

$$T\left(\alpha,\lambda\right)=\sum_{i=1}^{n+1}\left|\log D_i(\alpha,\lambda)-\log\frac{1}{n+1}\right|, \tag{15}$$

with respect to α and λ, respectively.

The estimators $\hat{\alpha}_{MSADE}$ and $\hat{\lambda}_{MSADE}$ of α and λ can be obtained by solving the following nonlinear equations

$$\frac{\partial}{\partial\alpha}T(\alpha,\lambda)=\sum_{i=1}^{n+1}\frac{D_i(\alpha,\lambda)-\frac{1}{n+1}}{\left|D_i(\alpha,\lambda)-\frac{1}{n+1}\right|}\left[\eta_1\left(x_{i:n}\mid\alpha,\lambda\right)-\eta_1\left(x_{i-1:n}\mid\alpha,\lambda\right)\right]=0$$

$$\frac{\partial}{\partial\lambda}T(\alpha,\lambda)=\sum_{i=1}^{n+1}\frac{D_i(\alpha,\lambda)-\frac{1}{n+1}}{\left|D_i(\alpha,\lambda)-\frac{1}{n+1}\right|}\left[\eta_2\left(x_{i:n}\mid\alpha,\lambda\right)-\eta_2\left(x_{i-1:n}\mid\alpha,\lambda\right)\right]=0,$$

where $D_i(\alpha,\lambda)\neq\frac{1}{n+1}$.

The estimators $\hat{\alpha}_{MSALDE}$, and $\hat{\lambda}_{MSALDE}$ of α and λ can be obtained by solving the nonlinear equations

$$\begin{aligned}\frac{\partial}{\partial\alpha}T(\alpha,\lambda)=\sum_{i=1}^{n+1}&\frac{\log D_i(\alpha,\lambda)-\log\frac{1}{n+1}}{\left|\log D_i(\alpha,\lambda)-\log\frac{1}{n+1}\right|}\frac{1}{D_i(\alpha,\lambda)}\\&\times\left[\eta_1\left(x_{i:n}\mid\alpha,\lambda\right)-\eta_1\left(x_{i-1:n}\mid\alpha,\lambda\right)\right]=0\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\lambda}T(\alpha,\lambda)=\sum_{i=1}^{n+1}&\frac{\log D_i(\alpha,\lambda)-\log\frac{1}{n+1}}{\left|\log D_i(\alpha,\lambda)-\log\frac{1}{n+1}\right|}\frac{1}{D_i(\alpha,\lambda)}\\&\times\left[\eta_2\left(x_{i:n}\mid\alpha,\lambda\right)-\eta_2\left(x_{i-1:n}\mid\alpha,\lambda\right)\right]=0,\end{aligned}$$

where $\log D_i(\alpha,\lambda)\neq\log\frac{1}{n+1}$.

3.5. **Minimum Distances Estimators.** This section presents three estimation methods for α and λ based on the minimization of the goodness-of-fit statistics with respect to α and λ. This class of statistics is based on the difference between the estimate of the cumulative distribution function and the empirical distribution function.

3.5.1. *Cramér-von-Mises Estimators.* To motivate our choice of Cramér-von-Mises type minimum distance estimators, [31] provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. Thus, the Cramér-von Mises estimators $\widehat{\alpha}_{CME}$ and $\widehat{\lambda}_{CME}$ of the parameters α and λ are obtained by minimizing the following function.

$$C(\alpha,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F\left(x_{i:n}\mid\alpha,\lambda\right)-\frac{2i-1}{2n}\right)^{2}. \tag{16}$$

These estimators can be obtained by solving the following non-linear equations

$$\begin{aligned}\sum_{i=1}^{n}\left(F\left(x_{i:n}\mid\alpha,\lambda\right)-\frac{2i-1}{2n}\right)\eta_1\left(x_{i:n}\mid\alpha,\lambda\right) &= 0,\\ \sum_{i=1}^{n}\left(F\left(x_{i:n}\mid\alpha,\lambda\right)-\frac{2i-1}{2n}\right)\eta_2\left(x_{i:n}\mid\alpha,\lambda\right) &= 0,\end{aligned}$$

where $\eta_1\left(\cdot\mid\alpha,\lambda\right)$ and $\eta_2\left(\cdot\mid\alpha,\lambda\right)$ are given by (9) and (10) respectively.

3.5.2. *Anderson-Darling and Right-tail Anderson-Darling Estimators.* The Anderson-Darling (AD) test [32] is an alternative method to detect sample distribution departure from the assumed distribution. Specifically, the AD test converge very quickly towards the asymptote [33, 34, 35]. The Anderson-Darling estimators $\widehat{\alpha}_{ADE}$ and $\widehat{\lambda}_{ADE}$ of the parameters α and λ are obtained by minimizing the following function with respect to the parameters.

$$A(\alpha,\lambda)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log F\left(x_{i:n}\mid\alpha,\lambda\right)+\log\overline{F}\left(x_{n+1-i:n}\mid\alpha,\lambda\right)\right\}. \tag{17}$$

These estimators can be obtained by solving the following non-linear equations:

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)\left[\frac{\eta_1\left(x_{i:n}\mid\alpha,\lambda\right)}{F\left(x_{i:n}\mid\alpha,\lambda\right)}-\frac{\eta_1\left(x_{n+1-i:n}\mid\alpha,\lambda\right)}{\overline{F}\left(x_{n+1-i:n}\mid\alpha,\lambda\right)}\right] &= 0,\\ \sum_{i=1}^{n}(2i-1)\left[\frac{\eta_2\left(x_{i:n}\mid\alpha,\lambda\right)}{F\left(x_{i:n}\mid\alpha,\lambda\right)}-\frac{\eta_2\left(x_{n+1-i:n}\mid\alpha,\lambda\right)}{\overline{F}\left(x_{n+1-i:n}\mid\alpha,\lambda\right)}\right] &= 0,\end{aligned}$$

where $\eta_1\left(\cdot\mid\alpha,\lambda\right)$ and $\eta_2\left(\cdot\mid\alpha,\lambda\right)$ are given by (9) and (10), respectively.

The Right-tail Anderson-Darling estimators $\widehat{\alpha}_{RTADE}$ and $\widehat{\lambda}_{RTADE}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$R(\alpha,\lambda)=\frac{n}{2}-2\sum_{i=1}^{n}F\left(x_{i:n}\mid\alpha,\lambda\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\overline{F}\left(x_{n+1-i:n}\mid\alpha,\lambda\right). \quad (18)$$

Equivalently

$$\begin{aligned}-2\sum_{i=1}^{n}\eta_1\left(x_{i:n}\mid\alpha,\lambda\right)+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\eta_1\left(x_{n+1-i:n}\mid\alpha,\lambda\right)}{\overline{F}\left(x_{n+1-i:n}\mid\alpha,\lambda\right)} &= 0,\\ -2\sum_{i=1}^{n}\eta_2\left(x_{i:n}\mid\alpha,\lambda\right)+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\eta_2\left(x_{n+1-i:n}\mid\alpha,\lambda\right)}{\overline{F}\left(x_{n+1-i:n}\mid\alpha,\lambda\right)} &= 0,\end{aligned}$$

where $\eta_1\left(\cdot\mid\alpha,\lambda\right)$ and $\eta_2\left(\cdot\mid\alpha,\lambda\right)$ are given by (9) and (10), respectively.

4. Bayesian analysis

This section discusses the Bayesian estimation of the FW distribution. To this end, the likelihood function can be written as

$$L(\alpha,\lambda|\boldsymbol{x})=\exp\left(\sum_{i=1}^{n}\log\left(\alpha+\lambda/x_i^2\right)\right)\exp(\alpha\sum_{i=1}^{n}x_i-\lambda\sum_{i=1}^{n}x_i^{-1})\exp\left(-\sum_{i=1}^{n}\exp\left(\alpha x_i-\lambda/x_i\right)\right)$$

Next assuming $\alpha\sim Gamma(a,b)$, i.e., $f(\alpha)=\frac{b^a}{\Gamma(a)}\alpha^{a-1}\exp(-b\alpha)$, and $\lambda\sim Gamma(c,d)$, the joint posterior of α and λ can be written as

$$\begin{aligned}P(\alpha,\lambda|\boldsymbol{x})\propto\quad &\alpha^{a-1}\exp(-\alpha(b-\sum_{i=1}^{n}x_i))\lambda^{c-1}\exp(-\lambda(d+\sum_{i=1}^{n}x_i^{-1}))\\ &\times\exp\left(\sum_{i=1}^{n}\log\left(\alpha+\lambda/x_i^2\right)-\sum_{i=1}^{n}\exp\left(\alpha x_i-\lambda/x_i\right)\right) \quad (19)\end{aligned}$$

The marginal distribution of λ is $P(\lambda|\boldsymbol{x})\sim Gamma(c,d+\sum_{i=1}^{n}x_i^{-1})$ while $P(\alpha|\lambda,\boldsymbol{x})\sim\alpha^{a-1}\exp(-\alpha(b-\sum_{i=1}^{n}x_i))\exp\left(\sum_{i=1}^{n}\log\left(\alpha+\lambda/x_i^2\right)-\sum_{i=1}^{n}\exp\left(\alpha x_i-\lambda/x_i\right)\right)$ for α.

To generate marginal of α, we propose the adaptive rejection sampling. To this end, it is not difficult to show that $P(\alpha|\lambda,\boldsymbol{x})$ is log-concave and thus, the idea of [36] can be used. For Metropolis Hastings (MH) sampling, we assume the gamma density as transition kernel $q(\alpha^{(i)}|\alpha^{(*)})$ for sampling value of α. The choice of gamma distribution has been done purely for illustration purpose, and other suitable distributions can be considered. After generating the marginal densities, the next step is to calculate the posterior summaries, $\mathbb{E}(\boldsymbol{\theta}|\boldsymbol{x})=\int_{\boldsymbol{\theta}}\boldsymbol{\theta}\mathbb{P}(\boldsymbol{\theta}|\boldsymbol{x})$. The steps to calculate the Bayes estimates are as follow:

MH Algorithm-Step 1: Generate λ from the Gamma distribution.

(1) To generate the α, evaluate the acceptance probability by $k(\alpha^{(i)}, \alpha^{(*)}) = \min\left(1, \frac{P(\alpha^{(*)}|\boldsymbol{x})q(\alpha^{(i)}|\alpha^{(*)})}{P(\alpha^{(i)}|\boldsymbol{x})q(\alpha^{(*)}|\alpha^{(i)})}\right)$, where $P(\alpha|\boldsymbol{x}, \lambda)$ has been defined above.

(2) Generate a random u from $Uniform(0, 1)$

(3) If $k(\alpha^{(i)}, \alpha^{(*)}) \geq u$, $\alpha^{(i+1)} = \alpha^{(*)}$, otherwise $\alpha^{(i+1)} = \alpha^{(i)}$.

Step 2: Suppose at the i-th step, α and λ take the values α_i and λ_i and we can generate $\mathbb{P}(\lambda_{i+1}|\boldsymbol{x})$, and $\mathbb{P}(\alpha_{i+1}|\lambda_i, \boldsymbol{x})$;

Step 3: Repeat the above step N times;

Step 4: Calculate the Bayes estimator of $g(\alpha, \lambda)$ by $\frac{1}{N-M}\sum_{i=M+1}^{N} g(\alpha_i, \lambda_i)$, where M denotes the burn-in sample.

In the next section, a simulation study is done to assess the performance of different estimation methods.

5. Simulation Study

This section presents Monte Carlo simulation studies to assess the performance of the frequentist estimators derived in the previous section. In particular, we use bias, the root mean squared error, the average absolute difference between the theoretical and the empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and empirical distribution functions as the performance assessment criteria. For comparison, we considered the following sample sizes: n = 20, 40, 60, 80, 100. Ten thousand independent samples of the aforementioned sizes were generated from EW distribution with parameters $(\alpha, \lambda) = \{(0.5, 0.5), (1.5, 0.5), (1.5, 2.0), (3.0, 2.0)\}$. It is noticed that 10,000 repetitions are sufficiently large to have stable results. For all the methods considered in this study, first we estimated the parameters using the method of maximum likelihood and then these estimates are used as the initial values. Since the MLE are not in closed form, we used the 'fitdist' function of R package fitdistrplus, which optimized the logarithm of the likelihood function numerically, to estimate the parameters. The results of the simulation studies are tabulated in Tables 1-4.

For each estimate, we calculated the bias, the root mean-squared error (RMSE), the average absolute difference between the theoretical and the empirical estimate of the distribution functions (D_{abs}), and the maximum absolute difference between the theoretical and the empirical distribution functions (D_{max}). The statistics are obtained using the following formulae:

$$\text{Bias}(\hat{\alpha}) = \frac{1}{K}\sum_{i=1}^{K}(\hat{\alpha}_i - \alpha), \quad \text{Bias}(\hat{\lambda}) = \frac{1}{K}\sum_{i=1}^{K}(\hat{\lambda}_i - \lambda) \tag{20}$$

$$\text{RMSE}(\hat{\alpha}) = \sqrt{\frac{1}{K}\sum_{i=1}^{K}(\hat{\alpha}_i - \alpha)^2}, \quad \text{RMSE}(\hat{\lambda}) = \sqrt{\frac{1}{K}\sum_{i=1}^{K}(\hat{\lambda}_i - \lambda)^2} \tag{21}$$

$$D_{\text{abs}}(\hat{\alpha}) = \frac{1}{(nK)} \sum_{i=1}^{K} \sum_{j=1}^{n} \left| F(x_{ij}|\alpha, \lambda) - F(x_{ij}|\hat{\alpha}, \hat{\lambda}) \right| \tag{22}$$

$$D_{\max}(\hat{\alpha}) = \frac{1}{nK} \sum_{i=1}^{K} \max_{j} \left| F(x_{ij}|\alpha, \lambda) - F(x_{ij}|\hat{\alpha}, \hat{\lambda}) \right| \tag{23}$$

where n denotes the sample size and K is the number of iterations. Simulated bias, RMSE, D_{abs}, $D_{\max}$ for the estimates are given in Tables 1-4. The row with label $\sum$Ranks shows the partial sum of the ranks and superscript indicates the rank of each of the estimators among all the estimators for that metric. For example, Table-1 shows the bias of MLE($\hat{\alpha}$) as 1.731^8 for $n = 20$. This indicates, bias of $\hat{\alpha}$ obtained using the method of maximum likelihood ranks 8^{th} among all other estimators.
The following observations can be drawn from the Tables 1-4.

1. All the estimators show the property of consistency, i.e., the RMSE decreases as the sample size increases, except in the case of PCE and MSALDE for $\alpha = 0.5$. However, assuming $\alpha > 1$, the RMSE of MSALDE decreases by increasing the sample size. Furthermore, assuming $\alpha = 1.5, \lambda = 0.5$,the RMSE of assuming α increases with the sample size for the MLE.
2. The bias of $\hat{\alpha}$ and $\hat{\lambda}$ decreases with increasing n for all the method of estimations.
3. It is noticed that the MLE and PCE performed the worst than the rest methods. The MSALDE performs the best when $\alpha, \lambda > 1$. The CVM and AD are suggested only when $\alpha > 1$.
4. D_{abs} is smaller than $D_{\max}$ for all the estimation techniques. Again, the statistics gets smaller with the increase of sample size.
5. In terms of performance of the methods of estimation, the MSADE and AD estimators uniformly produces the least biases of the estimates with the least RMSE, see the ranking of $\sum Ranks$ rows in the tables, for the most configurations considered in our studies.
6. It is also observed that for the estimation of λ, PCE performed the worst, as the RMSE is the highest as compared to the other methods.

For the Bayesian analysis, we generated $12,000$ samples of α and λ, and the Bayes estimates with other posterior summaries, like MCMC error, median, 95% Bayesian intervals have been tabulated in Table-5. For the parameter combinations mentioned above to compute the posterior summaries, hyperparameters are selected in such a way that the mean of the priors equal to the parameters' nominal values with large variances. Moreover, we used $M = 2,000$ as a burn-in period for our calculations. From the table, it is clear that as the sample size increases, the Bayes estimates approaches to the nominal values and the Bayesian intervals become more smaller for large sample sizes. Furthermore, the MCMC error decreases with the increase of sample size.

TABLE 1. Simulation results for $\alpha = \lambda = 0.5$.

n	Est.	MLE	LSE	WLS	PCE	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\alpha}$)	1.731^{8}	1.680^{7}	12.813^{10}	-0.381^{2}	1.409^{5}	-0.385^{3}	-0.311^{1}	1.893^{9}	1.101^{4}	1.629^{6}
	RMSE($\hat{\alpha}$)	1.820^{7}	1.925^{8}	14.131^{10}	0.381^{2}	1.492^{5}	0.389^{3}	0.373^{1}	2.159^{9}	1.184^{4}	1.724^{6}
	Bias($\hat{\lambda}$)	-0.365^{4}	-0.367^{5}	0.000^{1}	29.943^{10}	-0.377^{7}	3.425^{8}	12.480^{9}	-0.360^{2}	-0.374^{6}	-0.363^{3}
	RMSE($\hat{\lambda}$)	0.367^{4}	0.369^{5}	0.003^{1}	32.900^{10}	0.378^{6}	3.742^{8}	13.973^{9}	0.362^{2}	0.380^{7}	0.367^{3}
	D_{abs}	0.363^{10}	0.359^{6}	0.327^{5}	0.137^{3}	0.360^{7}	0.136^{1}	0.137^{2}	0.361^{8}	0.315^{4}	0.361^{9}
	$D_{\max}$	0.528^{8}	0.517^{6}	0.777^{10}	0.495^{4}	0.500^{5}	0.483^{2}	0.494^{3}	0.533^{9}	0.442^{1}	0.517^{7}
	$\sum$Ranks	41^{10}	$37^{7.5}$	$37^{7.5}$	31^{4}	35^{6}	$25^{1.5}$	$25^{1.5}$	39^{9}	26^{3}	34^{5}
40	Bias($\hat{\alpha}$)	1.608^{8}	1.565^{7}	11.733^{10}	-0.498^{3}	1.419^{5}	-0.398^{2}	-0.298^{1}	1.665^{9}	1.064^{4}	1.559^{6}
	RMSE($\hat{\alpha}$)	1.643^{7}	1.649^{8}	12.976^{10}	0.498^{2}	1.453^{5}	0.399^{1}	0.518^{3}	1.753^{9}	1.102^{4}	1.599^{6}
	Bias($\hat{\lambda}$)	-0.370^{4}	-0.372^{5}	0.000^{1}	36.321^{10}	-0.378^{6}	4.038^{8}	12.135^{9}	-0.368^{2}	-0.379^{7}	-0.370^{3}
	RMSE($\hat{\lambda}$)	0.371^{4}	0.373^{5}	0.000^{1}	38.809^{10}	0.379^{6}	4.232^{8}	13.750^{9}	0.369^{2}	0.381^{7}	0.371^{3}
	D_{abs}	0.363^{10}	0.361^{6}	0.321^{5}	0.137^{3}	0.363^{9}	0.137^{1}	0.137^{2}	0.362^{7}	0.315^{4}	0.362^{8}
	$D_{\max}$	0.519^{5}	0.513^{3}	0.780^{10}	0.539^{9}	0.503^{2}	0.533^{7}	0.538^{8}	0.522^{6}	0.439^{1}	0.514^{4}
	$\sum$Ranks	38^{10}	34^{6}	$37^{8.5}$	$37^{8.5}$	33^{5}	$27^{1.5}$	32^{4}	35^{7}	$27^{1.5}$	30^{3}
60	Bias($\hat{\alpha}$)	1.568^{8}	1.533^{6}	11.149^{10}	-0.399^{2}	1.429^{5}	-0.399^{3}	-0.268^{1}	1.599^{9}	1.051^{4}	1.537^{7}
	RMSE($\hat{\alpha}$)	1.589^{8}	1.582^{7}	12.279^{10}	0.399^{1}	1.449^{5}	0.399^{2}	0.586^{3}	1.650^{9}	1.075^{4}	1.562^{6}
	Bias($\hat{\lambda}$)	-0.372^{4}	-0.373^{5}	0.000^{1}	50.076^{10}	-0.378^{6}	4.431^{8}	11.772^{9}	-0.371^{2}	-0.380^{7}	-0.372^{4}
	RMSE($\hat{\lambda}$)	0.372^{3}	0.374^{5}	0.000^{1}	54.888^{10}	0.378^{6}	4.565^{8}	13.445^{9}	0.371^{2}	0.382^{7}	0.372^{4}
	D_{abs}	0.363^{9}	0.362^{6}	0.318^{5}	0.137^{3}	0.363^{10}	0.137^{1}	0.137^{2}	0.362^{8}	0.314^{4}	0.362^{7}
	$D_{\max}$	0.515^{5}	0.511^{3}	0.778^{10}	0.560^{9}	0.503^{2}	0.555^{7}	0.558^{8}	0.517^{6}	0.437^{1}	0.512^{4}
	$\sum$Ranks	$37^{9.5}$	$32^{4.5}$	$37^{9.5}$	35^{7}	34^{6}	29^{2}	$32^{4.5}$	36^{8}	27^{1}	31^{3}
80	Bias($\hat{\alpha}$)	1.550^{8}	1.520^{6}	10.711^{10}	-0.403^{3}	1.439^{5}	-0.398^{2}	-0.231^{1}	1.570^{9}	1.045^{4}	1.527^{7}
	RMSE($\hat{\alpha}$)	1.566^{8}	1.554^{7}	11.731^{10}	0.403^{2}	1.454^{5}	0.399^{1}	0.788^{3}	1.604^{9}	1.063^{4}	1.546^{6}
	Bias($\hat{\lambda}$)	-0.373^{4}	-0.374^{5}	0.000^{1}	58.682^{10}	-0.378^{6}	4.719^{8}	11.443^{9}	-0.372^{2}	-0.381^{7}	-0.373^{3}
	RMSE($\hat{\lambda}$)	0.373^{3}	0.374^{5}	0.000^{1}	64.961^{10}	0.378^{6}	4.847^{8}	13.138^{9}	0.373^{2}	0.382^{7}	0.373^{4}
	D_{abs}	0.363^{9}	0.362^{6}	0.315^{5}	0.137^{3}	0.363^{10}	0.137^{1}	0.137^{2}	0.362^{8}	0.314^{4}	0.362^{7}
	$D_{\max}$	0.514^{5}	0.511^{3}	0.775^{10}	0.574^{9}	0.504^{2}	0.568^{7}	0.571^{8}	0.515^{6}	0.437^{1}	0.511^{4}
	$\sum$Ranks	37^{9}	$32^{4.5}$	37^{9}	37^{9}	34^{6}	$27^{1.5}$	$32^{4.5}$	36^{7}	$27^{1.5}$	31^{3}
100	Bias($\hat{\alpha}$)	1.540^{8}	1.515^{6}	10.478^{10}	-0.405^{3}	1.445^{5}	-0.394^{2}	-0.191^{1}	1.555^{9}	1.042^{4}	1.522^{7}
	RMSE($\hat{\alpha}$)	1.552^{8}	1.541^{7}	11.406^{10}	0.405^{2}	1.457^{5}	0.400^{1}	0.898^{3}	1.582^{9}	1.056^{4}	1.537^{6}
	Bias($\hat{\lambda}$)	-0.373^{4}	-0.374^{5}	0.000^{1}	57.713^{10}	-0.378^{6}	4.881^{8}	11.130^{9}	-0.373^{2}	-0.382^{7}	-0.373^{3}
	RMSE($\hat{\lambda}$)	0.374^{3}	0.375^{5}	0.000^{1}	60.414^{10}	0.378^{6}	5.045^{8}	12.823^{9}	0.373^{2}	0.383^{7}	0.374^{4}
	D_{abs}	0.363^{9}	0.362^{7}	0.314^{4}	0.137^{3}	0.363^{10}	0.137^{1}	0.137^{2}	0.363^{8}	0.314^{5}	0.362^{6}
	$D_{\max}$	0.513^{5}	0.510^{3}	0.774^{10}	0.583^{9}	0.505^{2}	0.573^{7}	0.580^{8}	0.514^{6}	0.436^{1}	0.511^{4}
	$\sum$Ranks	$37^{9.5}$	33^{5}	$36^{7.5}$	$37^{9.5}$	34^{6}	27^{1}	32^{4}	$36^{7.5}$	28^{2}	30^{3}

6. Data Analysis

This section shows empirically that the FW distribution can be used as an alternative to some well-known two-parameter models like gamma, log-normal, Weibull, exponentiated exponential (EE), Nadarajah and Haghighi (NH) [37], Birnbaum-Saunders (BS), and inverse Gaussian (IG) distributions. For model comparison, we consider three well-known statistics and three model selection criteria. These measures and selection criteria are: Anderson-Darling (A^*), Cramér–von Mises (W^*) and Kolmogorov-Smirnov (K-S) measures, Akaike information criterion (AIC),Bayesian information criterion (BIC), and loglikelihood. The least value of these measures and selection criteria may indicate better fit. The cdfs of the EE, NH, BS and pdf

TABLE 2. Simulation results for $\alpha = 1.5, \lambda = 0.5$.

n	Est.	MLE	LSE	WLS	PCE	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\alpha}$)	-0.770^{4}	-0.824^{6}	-0.817^{5}	-1.487^{8}	-0.857^{7}	2.513^{9}	2.546^{10}	-0.758^{3}	-0.335^{2}	-0.321^{1}
	RMSE($\hat{\alpha}$)	0.785^{3}	0.847^{6}	0.837^{5}	1.487^{8}	0.897^{7}	3.236^{9}	3.468^{10}	0.789^{4}	0.390^{2}	0.380^{1}
	Bias($\hat{\lambda}$)	0.715^{8}	0.642^{6}	0.650^{7}	121.751^{10}	0.565^{5}	-0.225^{2}	0.038^{1}	0.723^{9}	0.245^{3}	0.271^{4}
	RMSE($\hat{\lambda}$)	0.773^{6}	0.717^{4}	0.719^{5}	175.811^{10}	0.622^{3}	1.735^{9}	1.435^{8}	0.803^{7}	0.312^{1}	0.359^{2}
	D_{abs}	0.332^{7}	0.332^{9}	0.332^{8}	0.831^{10}	0.325^{5}	0.162^{4}	0.110^{1}	0.331^{6}	0.154^{2}	0.156^{3}
	$D_{\max}$	0.497^{8}	0.486^{5}	0.487^{6}	1.000^{10}	0.470^{4}	0.581^{9}	0.356^{3}	0.497^{7}	0.218^{1}	0.225^{2}
	$\sum$Ranks	$36^{6.5}$	$36^{6.5}$	$36^{6.5}$	56^{10}	31^{3}	42^{9}	33^{4}	$36^{6.5}$	11^{1}	13^{2}
40	Bias($\hat{\alpha}$)	-0.803^{4}	-0.831^{7}	-0.825^{6}	-1.488^{8}	-0.819^{5}	3.290^{10}	1.953^{9}	-0.800^{3}	-0.174^{2}	-0.168^{1}
	RMSE($\hat{\alpha}$)	0.809^{3}	0.840^{6}	0.833^{5}	1.488^{8}	1.102^{7}	3.891^{10}	2.712^{9}	0.810^{4}	0.228^{2}	0.221^{1}
	Bias($\hat{\lambda}$)	0.667^{7}	0.631^{5}	0.638^{6}	143.170^{10}	0.578^{4}	-0.340^{3}	0.957^{9}	0.669^{8}	0.115^{1}	0.125^{2}
	RMSE($\hat{\lambda}$)	0.695^{6}	0.666^{4}	0.670^{5}	207.450^{10}	0.607^{3}	1.084^{8}	4.460^{9}	0.705^{7}	0.159^{1}	0.181^{2}
	D_{abs}	0.332^{7}	0.332^{8}	0.332^{9}	0.832^{10}	0.327^{5}	0.164^{4}	0.143^{3}	0.332^{6}	0.082^{1}	0.083^{2}
	$D_{\max}$	0.494^{8}	0.488^{5}	0.489^{6}	1.000^{10}	0.476^{4}	0.657^{9}	0.392^{3}	0.493^{7}	0.117^{1}	0.121^{2}
	$\sum$Ranks	35^{5}	35^{5}	37^{7}	56^{10}	28^{3}	44^{9}	42^{8}	35^{5}	8^{1}	10^{2}
60	Bias($\hat{\alpha}$)	-0.814^{6}	-0.832^{8}	-0.827^{7}	-1.488^{9}	-0.740^{4}	3.846^{10}	0.287^{3}	-0.812^{5}	-0.086^{2}	-0.082^{1}
	RMSE($\hat{\alpha}$)	0.817^{3}	0.838^{6}	0.832^{5}	1.488^{7}	1.551^{8}	4.443^{10}	1.988^{9}	0.818^{4}	0.151^{2}	0.146^{1}
	Bias($\hat{\lambda}$)	0.653^{7}	0.628^{5}	0.635^{6}	159.279^{10}	0.586^{4}	-0.321^{3}	9.004^{9}	0.653^{8}	0.057^{1}	0.063^{2}
	RMSE($\hat{\lambda}$)	0.671^{6}	0.651^{4}	0.655^{5}	227.698^{10}	0.616^{3}	1.257^{8}	13.300^{9}	0.677^{7}	0.099^{1}	0.115^{2}
	D_{abs}	0.332^{8}	0.332^{6}	0.332^{7}	0.832^{10}	0.326^{4}	0.167^{3}	0.441^{9}	0.332^{5}	0.047^{1}	0.049^{2}
	$D_{\max}$	0.493^{7}	0.488^{4}	0.490^{5}	1.000^{10}	0.478^{3}	0.696^{9}	0.655^{8}	0.492^{6}	0.070^{1}	0.074^{2}
	$\sum$Ranks	37^{7}	33^{4}	$35^{5.5}$	56^{10}	26^{3}	43^{8}	47^{9}	$35^{5.5}$	8^{1}	10^{2}
80	Bias($\hat{\alpha}$)	-0.819^{6}	-0.833^{8}	-0.829^{7}	-1.489^{9}	-0.663^{4}	4.236^{10}	0.156^{3}	-0.818^{5}	-0.089^{2}	-0.086^{1}
	RMSE($\hat{\alpha}$)	0.821^{3}	0.837^{6}	0.832^{5}	1.489^{7}	1.874^{8}	4.842^{10}	1.874^{9}	0.822^{4}	0.139^{2}	0.134^{1}
	Bias($\hat{\lambda}$)	0.645^{7}	0.626^{5}	0.632^{6}	177.803^{10}	0.593^{4}	-0.307^{3}	9.562^{9}	0.645^{8}	0.055^{1}	0.060^{2}
	RMSE($\hat{\lambda}$)	0.658^{6}	0.644^{4}	0.648^{5}	259.177^{10}	0.632^{3}	1.370^{8}	13.698^{9}	0.662^{7}	0.089^{1}	0.102^{2}
	D_{abs}	0.332^{7}	0.332^{6}	0.332^{8}	0.832^{10}	0.325^{4}	0.168^{3}	0.463^{9}	0.332^{5}	0.044^{1}	0.046^{2}
	$D_{\max}$	0.492^{7}	0.488^{4}	0.489^{5}	1.000^{10}	0.479^{3}	0.719^{9}	0.675^{8}	0.491^{6}	0.065^{1}	0.068^{2}
	$\sum$Ranks	$36^{6.5}$	33^{4}	$36^{6.5}$	56^{10}	26^{3}	43^{8}	47^{9}	35^{5}	8^{1}	10^{2}
100	Bias($\hat{\alpha}$)	-0.822^{6}	-0.833^{8}	-0.829^{7}	-1.489^{9}	-0.589^{4}	4.437^{10}	0.314^{3}	-0.821^{5}	-0.091^{2}	-0.089^{1}
	RMSE($\hat{\alpha}$)	0.824^{3}	0.836^{6}	0.832^{5}	1.489^{7}	2.171^{9}	5.080^{10}	1.938^{8}	0.824^{4}	0.131^{2}	0.127^{1}
	Bias($\hat{\lambda}$)	0.640^{7}	0.626^{5}	0.631^{6}	190.293^{10}	0.589^{4}	-0.281^{3}	8.652^{9}	0.641^{8}	0.054^{1}	0.058^{2}
	RMSE($\hat{\lambda}$)	0.651^{6}	0.640^{4}	0.643^{5}	271.665^{10}	0.625^{3}	1.539^{8}	13.037^{9}	0.654^{7}	0.083^{1}	0.093^{2}
	D_{abs}	0.332^{7}	0.332^{6}	0.332^{8}	0.832^{10}	0.323^{4}	0.168^{3}	0.429^{9}	0.332^{5}	0.043^{1}	0.044^{2}
	$D_{\max}$	0.491^{7}	0.489^{4}	0.489^{5}	1.000^{10}	0.479^{3}	0.729^{9}	0.646^{8}	0.491^{6}	0.063^{1}	0.065^{2}
	$\sum$Ranks	$36^{6.5}$	33^{4}	$36^{6.5}$	56^{10}	27^{3}	43^{8}	46^{9}	35^{5}	8^{1}	10^{2}

of the IG distributions are, respectively, given by

$$
\begin{aligned}
F_{EE}(x;\alpha,\lambda) &= \left(1-\mathrm{e}^{-\lambda\,x}\right)^{\alpha}, \quad x,\theta>0,\\
F_{NH}(x;\alpha,\lambda) &= 1-\mathrm{e}^{1-(1+\lambda\,x)^{\alpha}}, \quad x,\alpha,\lambda>0,\\
F_{BS}(x;\alpha,\beta) &= \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{x}{\beta}\right)^{1/2}-\left(\frac{\beta}{x}\right)^{1/2}\right\}\right], \quad x,\alpha,>0,\\
f_{IG}(x;\mu,\lambda) &= \sqrt{\frac{\lambda}{2\,\pi x^{3}}}\exp\left[-\lambda(x-\mu)^{2}/(2x\mu^{2})\right], \quad x,\mu,\lambda>0.
\end{aligned}
$$

6.1. **Strength of glass fibres.** This data set corresponds to the strengths of 15 cm fibres and taken from [38]. The data are: 0.37, 0.40, 0.70, 0.75, 0.80 ,0.81 ,0.83, 0.86, 0.92, 0.92, 0.94, 0.95, 0.98, 1.03, 1.06, 1.06, 1.08, 1.09, 1.10, 1.10, 1.13, 1.14, 1.15, 1.17, 1.20, 1.20, 1.21, 1.22, 1.25, 1.28, 1.28, 1.29, 1.29, 1.30, 1.35, 1.35, 1.37, 1.37, 1.38, 1.40, 1.40, 1.42, 1.43, 1.51, 1.53, 1.61. A summary of these data is: n = 46, $\bar{x}$

TABLE 3. Simulation results for $\alpha = 1.5, \lambda = 2.0$.

n	Est.	MLE	LSE	WLS	PCE	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\alpha}$)	-0.779^{7}	-0.833^{9}	-0.826^{8}	-1.467^{10}	-0.523^{5}	0.253^{3}	0.163^{1}	-0.774^{6}	0.242^{2}	0.264^{4}
	RMSE($\hat{\alpha}$)	0.791^{5}	0.849^{8}	0.840^{7}	1.468^{9}	2.258^{10}	0.775^{4}	0.408^{3}	0.795^{6}	0.335^{1}	0.364^{2}
	Bias($\hat{\lambda}$)	2.839^{8}	2.514^{6}	2.554^{7}	297.882^{10}	2.245^{5}	1.455^{4}	-0.102^{1}	2.855^{9}	-0.337^{3}	-0.291^{2}
	RMSE($\hat{\lambda}$)	3.024^{7}	2.764^{5}	2.779^{6}	416.378^{10}	2.515^{4}	5.565^{9}	0.382^{1}	3.123^{8}	0.512^{2}	0.536^{3}
	D_{abs}	0.456^{9}	0.456^{8}	0.455^{7}	0.955^{10}	0.437^{5}	0.110^{2}	0.012^{1}	0.455^{6}	0.146^{3}	0.146^{4}
	$D_{\max}$	0.796^{9}	0.778^{6}	0.781^{7}	1.000^{10}	0.751^{5}	0.260^{4}	0.088^{1}	0.793^{8}	0.218^{2}	0.219^{3}
	$\sum$Ranks	45^{9}	$42^{6.5}$	$42^{6.5}$	59^{10}	34^{5}	26^{4}	8^{1}	43^{8}	13^{2}	18^{3}
40	Bias($\hat{\alpha}$)	-0.807^{7}	-0.835^{9}	-0.829^{8}	-1.470^{10}	0.323^{5}	0.271^{4}	0.074^{1}	-0.807^{6}	0.226^{2}	0.235^{3}
	RMSE($\hat{\alpha}$)	0.812^{4}	0.842^{8}	0.835^{7}	1.470^{9}	4.616^{10}	0.829^{6}	0.272^{1}	0.814^{5}	0.274^{2}	0.285^{3}
	Bias($\hat{\lambda}$)	2.661^{9}	2.497^{6}	2.532^{7}	359.246^{10}	2.394^{5}	1.856^{4}	-0.106^{1}	2.659^{8}	-0.360^{3}	-0.340^{2}
	RMSE($\hat{\lambda}$)	2.748^{7}	2.612^{5}	2.633^{6}	489.784^{10}	2.609^{4}	6.308^{9}	0.397^{1}	2.778^{8}	0.446^{2}	0.450^{3}
	D_{abs}	0.455^{9}	0.455^{7}	0.455^{8}	0.955^{10}	0.426^{5}	0.137^{2}	0.010^{1}	0.455^{6}	0.145^{4}	0.145^{3}
	$D_{\max}$	0.798^{9}	0.789^{6}	0.791^{7}	1.000^{10}	0.765^{5}	0.339^{4}	0.077^{1}	0.797^{8}	0.212^{2}	0.213^{3}
	$\sum$Ranks	45^{9}	$41^{6.5}$	43^{8}	59^{10}	34^{5}	29^{4}	6^{1}	$41^{6.5}$	15^{2}	17^{3}
60	Bias($\hat{\alpha}$)	-0.817^{6}	-0.835^{8}	-0.830^{7}	-1.470^{10}	1.062^{9}	0.464^{3}	0.037^{1}	-0.816^{5}	0.460^{2}	0.467^{4}
	RMSE($\hat{\alpha}$)	0.819^{4}	0.839^{7}	0.833^{6}	1.470^{9}	5.858^{10}	0.861^{8}	0.227^{1}	0.821^{5}	0.482^{2}	0.489^{3}
	Bias($\hat{\lambda}$)	2.605^{9}	2.496^{6}	2.526^{7}	378.613^{10}	2.483^{5}	0.597^{2}	-0.079^{1}	2.602^{8}	-0.710^{4}	-0.700^{3}
	RMSE($\hat{\lambda}$)	2.661^{6}	2.572^{4}	2.592^{5}	514.546^{10}	2.739^{8}	4.407^{9}	0.757^{3}	2.679^{7}	0.730^{2}	0.725^{1}
	D_{abs}	0.456^{9}	0.455^{7}	0.455^{8}	0.955^{10}	0.412^{5}	0.081^{2}	0.009^{1}	0.455^{6}	0.269^{4}	0.269^{3}
	$D_{\max}$	0.799^{9}	0.792^{6}	0.794^{7}	1.000^{10}	0.767^{5}	0.348^{2}	0.068^{1}	0.798^{8}	0.399^{3}	0.400^{4}
	$\sum$Ranks	43^{9}	38^{5}	40^{7}	59^{10}	42^{8}	26^{4}	8^{1}	39^{6}	17^{2}	18^{3}
80	Bias($\hat{\alpha}$)	-0.821^{6}	-0.835^{8}	-0.830^{7}	-1.471^{9}	1.735^{10}	0.575^{4}	0.024^{1}	-0.821^{5}	0.457^{2}	0.463^{3}
	RMSE($\hat{\alpha}$)	0.823^{4}	0.838^{7}	0.833^{6}	1.471^{9}	6.779^{10}	0.894^{8}	0.199^{1}	0.824^{5}	0.473^{2}	0.479^{3}
	Bias($\hat{\lambda}$)	2.575^{9}	2.493^{5}	2.519^{6}	393.765^{10}	2.548^{7}	-0.044^{1}	-0.086^{2}	2.572^{8}	-0.713^{4}	-0.706^{3}
	RMSE($\hat{\lambda}$)	2.618^{6}	2.550^{4}	2.569^{5}	535.269^{10}	2.868^{8}	3.119^{9}	0.621^{1}	2.630^{7}	0.728^{3}	0.724^{2}
	D_{abs}	0.455^{9}	0.455^{7}	0.455^{8}	0.955^{10}	0.397^{5}	0.056^{2}	0.008^{1}	0.455^{6}	0.269^{4}	0.269^{3}
	$D_{\max}$	0.798^{9}	0.794^{6}	0.795^{7}	1.000^{10}	0.770^{5}	0.368^{2}	0.062^{1}	0.798^{8}	0.399^{3}	0.399^{4}
	$\sum$Ranks	43^{8}	37^{5}	$39^{6.5}$	58^{10}	45^{9}	26^{4}	7^{1}	$39^{6.5}$	$18^{2.5}$	$18^{2.5}$
100	Bias($\hat{\alpha}$)	-0.824^{5}	-0.835^{8}	-0.831^{7}	-1.471^{9}	2.478^{10}	0.636^{4}	0.014^{1}	-0.824^{6}	0.456^{2}	0.460^{3}
	RMSE($\hat{\alpha}$)	0.825^{4}	0.837^{7}	0.833^{6}	1.471^{9}	7.672^{10}	0.937^{8}	0.182^{1}	0.826^{5}	0.469^{2}	0.473^{3}
	Bias($\hat{\lambda}$)	2.559^{8}	2.493^{5}	2.516^{6}	410.757^{10}	2.619^{9}	-0.355^{2}	-0.069^{1}	2.556^{7}	-0.715^{4}	-0.709^{3}
	RMSE($\hat{\lambda}$)	2.593^{7}	2.538^{5}	2.555^{6}	546.167^{10}	2.987^{9}	2.385^{4}	0.710^{1}	2.602^{8}	0.727^{3}	0.724^{2}
	D_{abs}	0.455^{9}	0.455^{7}	0.455^{8}	0.955^{10}	0.382^{5}	0.045^{2}	0.008^{1}	0.455^{6}	0.269^{4}	0.269^{3}
	$D_{\max}$	0.798^{9}	0.795^{6}	0.796^{7}	1.000^{10}	0.771^{5}	0.393^{2}	0.056^{1}	0.798^{8}	0.399^{3}	0.399^{4}
	$\sum$Ranks	42^{8}	38^{5}	$40^{6.5}$	58^{10}	48^{9}	22^{4}	6^{1}	$40^{6.5}$	$18^{2.5}$	$18^{2.5}$

= 1.13, s = 0.2713669, skewness = −0.79359, kurtosis = 0.59954. The boxplot of these observations displayed in Figure 3(a) indicates that the distribution is right-skewed. The TTT plot [39] of these data is shown in Figure 3(b). The TTT plot suggests an increasing failure rate and thus, the FW distribution could in principle be appropriate for modeling the current data. Table **??** provides the MLEs of the parameters and the values of A^*, W^*, K-S, AIC, BIC, and loglikelihood for each model. On the basis of results listed in the table, we conclude that the FW distribution provides the best fit with the lowest values of model selection criteria. This indicates that the FW distribution has the ability to fit left-skewed data with increasing failure rate. For a visual comparison, we provide QQ-plots for all fitted models in Figure 4. Clearly, the FW model provides the closest fit to the data.

TABLE 4. Simulation results for $\alpha = 2, \lambda = 3$.

n	Est.	MLE	LSE	WLS	PCE	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\alpha}$)	-2.474^{9}	0.51^{4}	0.51^{2}	-2.904^{10}	0.511^{7}	0.51^{5}	0.51^{6}	0.51^{3}	-1.360^{8}	0.264^{1}
	RMSE($\hat{\alpha}$)	2.474^{9}	0.51^{4}	0.51^{2}	2.904^{10}	0.536^{7}	0.51^{5}	0.51^{6}	0.51^{3}	1.365^{8}	0.364^{1}
	Bias($\hat{\lambda}$)	2.862^{8}	0.01^{3}	0.01^{5}	85.223^{10}	0.011^{6}	0.01^{2}	0.01^{1}	0.01^{4}	4.050^{9}	-0.291^{7}
	RMSE($\hat{\lambda}$)	2.862^{8}	0.01^{2}	0.01^{4}	114.487^{10}	0.119^{6}	0.01^{1}	0.01^{5}	0.01^{3}	4.237^{9}	0.536^{7}
	D_{abs}	0.013^{7}	0.00^{3}	0.00^{1}	1.000^{10}	0.000^{6}	0.00^{3}	0.00^{5}	0.00^{3}	0.495^{9}	0.146^{8}
	$D_{\max}$	0.172^{7}	0.00^{3}	0.00^{1}	1.000^{10}	0.000^{6}	0.00^{3}	0.00^{5}	0.00^{3}	0.923^{9}	0.219^{8}
	$\sum$Ranks	48^{8}	19^{3}	15^{1}	60^{10}	38^{7}	19^{3}	28^{5}	19^{3}	52^{9}	32^{6}
40	Bias($\hat{\alpha}$)	-2.473^{9}	0.51^{6}	0.51^{4}	-2.91^{10}	0.509^{3}	0.51^{7}	0.51^{8}	0.51^{5}	0.226^{1}	0.235^{2}
	RMSE($\hat{\alpha}$)	2.474^{9}	0.51^{5}	0.51^{3}	2.91^{10}	0.512^{8}	0.51^{6}	0.51^{7}	0.51^{4}	0.274^{1}	0.285^{2}
	Bias($\hat{\lambda}$)	2.862^{9}	0.01^{3}	0.01^{5}	100.15^{10}	0.014^{6}	0.01^{2}	0.01^{1}	0.01^{4}	-0.360^{8}	-0.340^{7}
	RMSE($\hat{\lambda}$)	2.862^{9}	0.01^{2}	0.01^{4}	129.73^{10}	0.214^{6}	0.01^{1}	0.01^{5}	0.01^{3}	0.446^{7}	0.450^{8}
	D_{abs}	0.013^{7}	0.00^{3}	0.00^{1}	1.00^{10}	0.000^{6}	0.00^{3}	0.00^{5}	0.00^{3}	0.145^{9}	0.145^{8}
	$D_{\max}$	0.260^{9}	0.00^{3}	0.00^{1}	1.00^{10}	0.000^{6}	0.00^{3}	0.00^{5}	0.00^{3}	0.212^{7}	0.213^{8}
	$\sum$Ranks	52^{9}	22^{3}	18^{1}	60^{10}	$35^{7.5}$	22^{3}	31^{5}	22^{3}	$33^{6.5}$	$35^{7.5}$
60	Bias($\hat{\alpha}$)	-2.473^{9}	0.51^{6}	0.51^{4}	-2.912^{10}	0.509^{3}	0.51^{7}	0.51^{8}	0.51^{5}	0.460^{1}	0.467^{2}
	RMSE($\hat{\alpha}$)	2.474^{9}	0.51^{5}	0.51^{3}	2.912^{10}	0.518^{8}	0.51^{6}	0.51^{7}	0.51^{4}	0.482^{1}	0.489^{2}
	Bias($\hat{\lambda}$)	2.862^{9}	0.01^{3}	0.01^{5}	106.919^{10}	0.017^{6}	0.01^{2}	0.01^{1}	0.01^{4}	-0.710^{8}	-0.700^{7}
	RMSE($\hat{\lambda}$)	2.862^{9}	0.01^{2}	0.01^{4}	137.635^{10}	0.303^{6}	0.01^{1}	0.01^{5}	0.01^{3}	0.730^{8}	0.725^{7}
	D_{abs}	0.013^{7}	0.00^{3}	0.00^{1}	1.000^{10}	0.000^{6}	0.00^{4}	0.00^{5}	0.00^{2}	0.269^{9}	0.269^{8}
	$D_{\max}$	0.318^{7}	0.00^{3}	0.00^{1}	1.000^{10}	0.001^{6}	0.00^{4}	0.00^{5}	0.00^{2}	0.399^{8}	0.400^{9}
	$\sum$Ranks	50^{9}	22^{3}	18^{1}	60^{10}	35^{7}	24^{4}	31^{5}	20^{2}	35^{7}	35^{7}
80	Bias($\hat{\alpha}$)	-2.473^{9}	0.51^{6}	0.510^{4}	-2.913^{10}	0.508^{3}	0.51^{7}	0.51^{8}	0.51^{5}	0.457^{1}	0.463^{2}
	RMSE($\hat{\alpha}$)	2.474^{9}	0.51^{4}	0.511^{7}	2.913^{10}	0.516^{8}	0.51^{5}	0.51^{6}	0.51^{3}	0.473^{1}	0.479^{2}
	Bias($\hat{\lambda}$)	2.862^{9}	0.01^{3}	0.011^{5}	112.751^{10}	0.019^{6}	0.01^{2}	0.01^{1}	0.01^{4}	-0.713^{8}	-0.706^{7}
	RMSE($\hat{\lambda}$)	2.862^{9}	0.01^{1}	0.143^{5}	144.653^{10}	0.350^{6}	0.01^{4}	0.01^{3}	0.01^{2}	0.728^{8}	0.724^{7}
	D_{abs}	0.013^{7}	0.00^{2}	0.000^{5}	1.000^{10}	0.000^{6}	0.00^{3}	0.00^{4}	0.00^{1}	0.269^{9}	0.269^{8}
	$D_{\max}$	0.362^{7}	0.00^{2}	0.000^{5}	1.000^{10}	0.001^{6}	0.00^{3}	0.00^{4}	0.00^{1}	0.399^{8}	0.399^{9}
	$\sum$Ranks	50^{9}	18^{2}	31^{5}	60^{10}	35^{7}	24^{3}	26^{4}	16^{1}	35^{7}	35^{7}
100	Bias($\hat{\alpha}$)	-2.473^{9}	0.51^{6}	0.510^{4}	-2.914^{10}	0.508^{3}	0.51^{7}	0.51^{8}	0.51^{5}	0.456^{1}	0.460^{2}
	RMSE($\hat{\alpha}$)	2.474^{9}	0.51^{4}	0.511^{7}	2.914^{10}	0.515^{8}	0.51^{5}	0.51^{6}	0.51^{3}	0.469^{1}	0.473^{2}
	Bias($\hat{\lambda}$)	2.862^{9}	0.01^{3}	0.011^{5}	115.576^{10}	0.020^{6}	0.01^{2}	0.01^{1}	0.01^{4}	-0.715^{8}	-0.709^{7}
	RMSE($\hat{\lambda}$)	2.862^{9}	0.01^{1}	0.142^{5}	149.390^{10}	0.372^{6}	0.01^{4}	0.01^{3}	0.01^{2}	0.727^{8}	0.724^{7}
	D_{abs}	0.013^{7}	0.00^{2}	0.000^{5}	1.000^{10}	0.000^{6}	0.00^{3}	0.00^{4}	0.00^{1}	0.269^{9}	0.269^{8}
	$D_{\max}$	0.395^{7}	0.00^{2}	0.000^{5}	1.000^{10}	0.001^{6}	0.00^{3}	0.00^{4}	0.00^{1}	0.399^{8}	0.399^{9}
	$\sum$Ranks	50^{9}	18^{2}	31^{5}	60^{10}	35^{7}	24^{3}	26^{4}	16^{1}	35^{7}	35^{7}

7. CONCLUDING REMARKS

This article studied the performance of different estimation methods for flexible Weibull distribution. The distribution parameters are estimated by eleven different methods of estimation, namely, the maximum likelihood estimators, least squares and weighted least squares estimators, the maximum product of spacings estimators, the minimum spacing absolute distance estimators, the minimum spacing absolute-log distance estimators, Cramér-von-Mises estimators, Anderson-Darling, right-tail Anderson-Darling, and the Bayes estimators. The results of the simulation study showed that among the frequentist estimators, Cramér-von-Mises estimators and Anderson-Darling perform better than their counterparts. Contrary to frequentist methods, Bayesian method outperformed the rest estimation methods. In the future, different estimation methods can be compared using censored and record data. Furthermore, different confidence intervals, like approximate, bootstrap, and

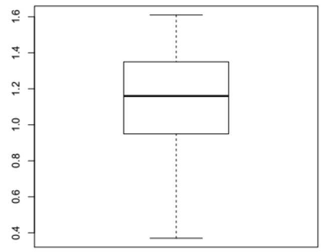
TABLE 5. Monte Carlo Markov Chain results for Bayesian analysis.

Parameter	n	Estimate	SD	MC error	95% CI	Median
$\alpha = 0.5$	20	0.4990	0.5096	0.0051	(0.0125,1.912)	0.3418
	40	0.4995	0.5045	0.0035	(0.0122,1.872)	0.3443
	60	0.4992	0.4999	0.0029	(0.0124,1.865)	0.3455
	80	0.4998	0.4977	0.0023	(0.0128,1.845)	0.346
	100	0.4982	0.4950	0.0021	(0.0127,1.851)	0.3452
$\lambda = 0.5$	20	0.4968	0.5039	0.0051	(0.0119,1.866)	0.3404
	40	0.4976	0.4969	0.0033	(0.0128,1.86)	0.3462
	60	0.4969	0.4968	0.0026	(0.0121,1.845)	0.3454
	80	0.4964	0.4958	0.0023	(0.0122,1.841)	0.3447
	100	0.4965	0.4948	0.0022	(0.0122,1.837)	0.3452
$\alpha = 1.5$	20	1.489	0.8649	0.0089	(0.3095,3.635)	1.318
	40	1.488	0.855	0.0058	(0.3104,3.561)	1.326
	60	1.493	0.835	0.0051	(0.3065,3.559)	1.325
	80	1.497	0.8246	0.0043	(0.3091,3.563)	1.326
	100	1.498	0.8157	0.0038	(0.3078,3.58)	1.33
$\lambda = 0.5$	20	0.4934	0.4977	0.0046	(0.0132,1.836)	0.3365
	40	0.4976	0.4975	0.0034	(0.0123,1.845)	0.3459
	60	0.4991	0.4963	0.0028	(0.0119,1.838)	0.3463
	80	0.5008	0.4927	0.0027	(0.0127,1.847)	0.3463
	100	0.4996	0.4905	0.0023	(0.0127,1.846)	0.3456
$\alpha = 1.5$	20	1.501	0.8659	0.0093	(0.3163,3.629)	1.336
	40	1.501	0.8657	0.0065	(0.3155,3.611)	1.335
	60	1.497	0.8653	0.0050	(0.3118,3.606)	1.335
	80	1.498	0.8649	0.0046	(0.3125,3.607)	1.334
	100	1.499	0.8645	0.0039	(0.3119,3.604)	1.331
$\lambda = 2$	20	1.97	0.9931	0.0099	(0.5345,4.331)	1.811
	40	1.977	0.9874	0.0069	(0.5475,4.289)	1.818
	60	1.977	0.9822	0.0058	(0.5536,4.297)	1.82
	80	1.980	0.9820	0.0048	(0.5502,4.301)	1.823
	100	1.989	0.9814	0.0045	(0.5475,4.308)	1.822
$\alpha = 1.5$	20	2.982	1.222	0.0116	(1.098,5.859)	2.813
	40	2.979	1.220	0.0080	(1.103,5.849)	2.807
	60	2.976	1.218	0.0066	(1.101,5.842)	2.806
	80	2.99	1.188	0.0028	(1.093,5.816)	2.822
	100	2.989	1.176	0.0026	(1.094,5.813)	2.822
$\lambda = 2$	20	1.98	0.991	0.0099	(0.5403,4.333)	1.816
	40	1.986	0.9893	0.0075	(0.5383,4.324)	1.825
	60	1.988	0.9891	0.0056	(0.5407,4.331)	1.829
	80	1.989	0.9925	0.0028	(0.5408,4.361)	1.824
	100	1.989	0.9913	0.0024	(0.5377,4.368)	1.824

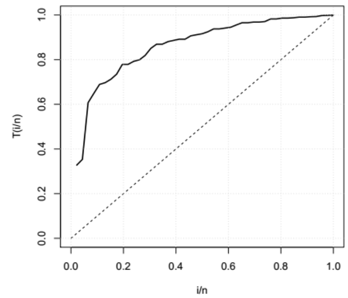
Bayesian can also be compared. Also, bias-corrected estimators can be studied for the flexible Weibull distribution.

8. CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.



(a)



(b)

FIGURE 3. (a) Histogram (b) TTT plot for the strengths of glass fibres data.

TABLE 6. MLEs, their standard errors (in parentheses) and goodness-of-fit measures of the strengths of glass fibres data.

Distribution	Estimates		A^*	W^*	K-S	AIC	BIC	Loglikelihood
FW(α, λ)	1.9908	2.96114	0.4157	0.0622	0.0605	10.6989	14.3562	3.34946
	(0.2304)	(0.3925)						
Gamma(α,θ)	11.6769	0.0979	1.3219	0.1920	0.1324	26.3742	30.0315	11.1871
	(3.6130)	(0.0313)						
Weibull(c, λ)	2.79206	0.0490	0.5254	0.0661	0.0921	13.2132	16.8705	4.6065
	(0.2133)	(0.0138)						
Log-normal(μ, σ)	0.0850	0.2964	1.8996	0.2838	0.1596	30.5075	34.1648	13.2538
	(0.0437)	(0.0309)						
NH(α, λ)	35.5990	0.0193	0.8102	0.1129	0.4296	79.2266	82.8838	37.6133
	(27.5059)	(0.0150)						
EE(α, λ)	20.4136	3.1137	2.0367	0.3076	0.1601	33.2085	36.8658	14.6043
	(6.6018)	(0.3384)						
BS(α, β)	0.3042	1.0797	2.0263	0.3029	0.1714	31.9066	35.5639	13.9533
	(0.0317)	(0.0478)						
IG(μ, λ)	1.1312	311.8473	2.0538	0.3075	0.1712	32.2376	35.8949	14.1188
	(0.0516)	(2.4703)						

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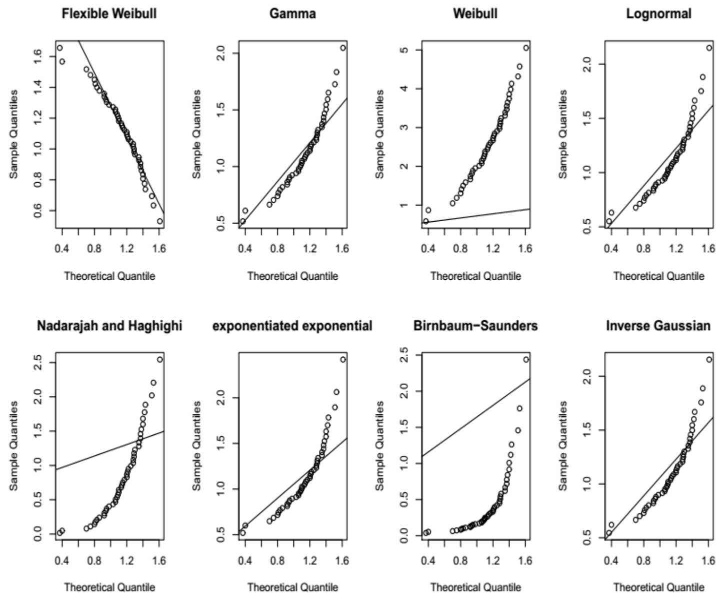

FIGURE 4. Q-Q plots for the strengths of glass fibre data.

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INTERVAL OSCILLATION CRITERIA FOR IMPULSIVE CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we derive new interval oscillation criteria for impulsive conformable fractional differential equations having fixed moments of impulse actions. The results are extended to a more general class of nonlinear impulsive conformable fractional differential equations. Examples are also given to illustrate the relevance of the result.

1. Introduction

In recent years fractional differential equations are recognized as an excellent source of knowledge in modelling dynamical processes in self similar and porous structures, electrical networks, probability and statistics, visco elasticity, electro chemistry of corrosion, electro dynamics of complex medium, polymer rheology, industrial robotics, economics, biotechnology etc. For the theory and applications of fractional differential equations we refer the monographs [10, 18]. But the most commonly used definitions are based on the integration with singular kernel and which are nonlocal: Riemann-Liouville derivative and Caputo derivative. Moreover for this type of derivative useful product rule and chain rule are not applicable. But in 2014 Khalil et. al [9] introduced a new fractional derivative called the conformable derivative which is closely similar to classical derivative.

The oscillation of fractional differential equations as a new research field has received significant attention and some interesting results have already been obtained. We refer to [2, 3, 4, 5, 6, 14, 22, 24] and the references quoted therein.

The oscillation theory of impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied

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sciences. For further details and applications one can refer the monographs [1, 12] and reference cited therein.

In [13], Q.L. Li and W.S. Cheng considered the following interval oscillation criteria for second order forced delay differential equation under impulses effects of the form

$$\begin{aligned}(p(t)x'(t))' + q(t)x(t-\tau) + \sum_{i=1}^{n} q_i(t)\Phi_{\alpha_i}(x(t-\tau)) &= f(t), \quad t \neq t_k\\ x(t_k^+) = a_k x(t_k), \quad x'(t_k^+) &= b_k x'(t_k), \quad k = 1, 2, \cdots\end{aligned}$$

where $0 \le t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $p, q, q_i, f \in PLC[t_0, \infty)$. By using the Riccati technique, some interesting oscillation results were obtained.

In the last decades, interval oscillation of impulsive differential equations was arousing the interest of many researchers, see [7, 8, 11, 15, 16, 19, 20, 21, 23, 25] and the references cited therein. Most of the existing literature concentrated on interval oscillation criteria for case of without delay and only very few papers appeared for case of with delay. As far as author knowledge, it seems that there has been no paper dealing with interval oscillation criteria for impulsive conformable fractional differential equations.

Motivated by this gap, we propose to initiate the following model of the form

$$\left.\begin{aligned}&T_\alpha\left(r(t)g\left(T_\alpha x(t)\right)\right) + q(t)x(t-\rho) + \sum_{i=1}^{n} q_i(t)f_i(x(t-\rho)) = f(t), \quad t \neq t_k\\ &x(t_k^+) = a_k x(t_k), \quad T_\alpha(x(t_k^+)) = b_k T_\alpha(x(t_k)), \quad k = 1, 2, \cdots,\end{aligned}\right\} \tag{1}$$

where T_α denotes the conformable fractional derivative of order $0 < \alpha \le 1$.

In the sequel, we assume that the following hypotheses (H) hold:

(H_1) $r(t) \in C^\alpha([t_0, \infty), (0, \infty))$, $q(t), q_i(t), f(t) \in PLC([t_0, \infty), \mathbb{R})$, $i = 1, 2, \cdots, n$, where PLC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1, 2, \cdots$, and left continuous at $t = t_k, a_k, b_k$ are real-valued sequences satisfying $a_k > -1, a_k \le b_k$, $k = 1, 2, \cdots$, $t - \rho < t$, $\lim_{t\to\infty} t - \rho = \infty$, $0 < t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$.

(H_2) $f_i, g \in C(\mathbb{R}, \mathbb{R})$ are convex in $[0, \infty)$ with $xf_i(x) > 0$ and $\frac{f_i(x)}{x} \ge \epsilon_i > 0$ for $x \neq 0$, $i = 1, 2, \cdots, n$, $xg(x) > 0$, $g(x) \le \eta x$ for $x \neq 0$, $g^{-1} \in C(\mathbb{R}, \mathbb{R})$ are continuous functions with $xg^{-1}(x) > 0$ for $x \neq 0$ and there exist positive constant ζ such that $g^{-1}(xy) \le \zeta g^{-1}(x)g^{-1}(y)$ for $xy \neq 0$ and

$$\int_{t_0}^{\infty} s^{\alpha-1} g^{-1}\left(\frac{1}{r(s)}\right) ds = \infty.$$

(H_3) For any $T \ge 0$ there exists intervals $[c_1, d_1]$ and $[c_2, d_2]$ contained in $[T, \infty)$ such that $c_1 < d_1 \le d_1 + \rho \le c_2 < d_2$, $c_j, d_j \notin \{t_k\}$, $j = 1, 2$, $k = 1, 2, \cdots$

and $r(t) > 0$, $q(t) \geq 0$, $q_i(t) \geq 0$, $i = 1, 2, \cdots, n$ for $t \in [c_1 - \rho, d_1] \cup [c_2 - \rho, d_2]$ and $f(t)$ has different signs in $[c_1 - \rho, d_1]$ and $[c_2 - \rho, d_2]$, for instance, let

$$f(t) \leq 0 \quad \text{for} \quad t \in [c_1 - \rho, d_1] \quad \text{and} \quad f(t) \geq 0 \quad \text{for} \quad t \in [c_2 - \rho, d_2].$$

Denote

$$J(s) := \max\{j : \ t_0 < t_j < s\}, \quad r_j := \max\{r(t) : \ t \in [c_j, d_j]\}, \quad j = 1, 2$$
$$J_p(c_j, d_j) = \{p \in C^\alpha[c_j, d_j], \ \ p(t) \neq 0, \ p(c_j) = p(d_j) = 0, \ j = 1, 2\}.$$

For two constants $c, d \notin \{t_k\}$ with $c < d$ and a function $\varphi \in C([c, d], \mathbb{R})$, we define an operator $\Phi : C([c, d], \mathbb{R}) \to \mathbb{R}$ by

$$\Phi_c^d[\varphi] = \begin{cases} 0, & J(c) = J(d) \\ \varphi(t_{J(c)+1})\tau(c) + \sum_{k=J(c)+2}^{J(d)} \varphi(t_i)\sigma(t_i), & J(c) < J(d), \end{cases}$$

where

$$\tau(c) = t_{I(c)+1}^{1-\alpha} \frac{a_{J(c)+1} - b_{J(c)+1}}{a_{J(c)+1}(t_{J(c_1)+1}^\alpha - c_1^\alpha)}$$

and

$$\sigma(t) = t_j^{1-\alpha} \frac{a_j - b_j}{a_j(t_j^\alpha - t_{j-1}^\alpha)}.$$

This paper is organized as follows: In Section 2, we present some definitions and results that will be needed in the sequel. The main results are given in Section 3. In Section 4, some examples is considered to illustrate the main results.

2. Preliminaries

In this section, we recall some definitions and results which will be used in our main results.

Definition 1. *A solution of equation* (1) *is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Equation* (1) *is called oscillatory if all its solutions are oscillatory.*

We use the following definition introduced by R.R. Khalil et al. [9].

Definition 2. *Given* $f : [0, \infty) \to \mathbb{R}$. *Then the conformable fractional derivative of* f *of order* α *is defined by*

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0, \alpha \in (0, 1]$.

If f is α-differentiable in some $(0, a), a > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

Definition 3. *$I^a_\alpha(f)(t) = I^a_1(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$.*

Conformable fractional derivative has the following properties :

Theorem 4. *Let $\alpha \in (0,1]$ and f, g be α- differentiable at a point $t > 0$. Then*

(i) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.
(ii) $T_\alpha(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$.
(iii) $T_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.
(iv) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
(v) $T_\alpha\left(\frac{f}{g}\right) = \dfrac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
(vi) *If f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\dfrac{df}{dt}(t)$.

3. Main Results

In this section, we established some new interval oscillation criteria for the equation (1) by using Riccati transformation.

Theorem 5. *Assume that conditions $(H_1)-(H_3)$ hold, furthermore for any $T \geq 0$ there exist c_j, d_j satisfying with $T \leq c_1 < d_1$, $T \leq c_2 < d_2$ and $p(t) \in J_p(c_1, d_1)$ such that*

$$\int_{c_j}^{d_j} \left[(p'(t))^2 t^{2-2\alpha}\eta r(t) + w(t)p^2(t)(1-\alpha)t^{-\alpha}\right] dt - \int_{c_j}^{t_{J(c_j)+1}} Q(t)p^2(t)M^j_{J(c_j)}(t)dt$$
$$- \sum_{k=J(c_j)+1}^{J(d_j)-1} \int_{t_k}^{t_{k+1}} Q(t)p^2(t)M^j_{J(c_j)}(t)dt - \int_{t_{J(d_j)}}^{d_j} Q(t)p^2(t)M^j_{J(d_j)}(t)dt \leq \Lambda(p, c_j, d_j) \tag{2}$$

where $Q(t) = q(t) + \sum_{i=1}^{n} \epsilon_i q_i(t)$, $\Lambda(p, c_j, d_j) = 0$ for $J(c_j) = J(d_j)$ and

$$\Lambda(p, c_j, d_j) = r_j \left\{ p^2(t_{J(c_j)+1})t^{1-\alpha}_{J(c_j)+1} \frac{a_{J(c_j)+1} - b_{J(c_j)+1}}{a_{J(c_j)+1}(t^\alpha_{J(c_j)+1} - c^\alpha_j)} \right.$$
$$\left. + \sum_{k=J(c_j)+2}^{J(d_j)} p^2(t_k)t^{1-\alpha}_k \frac{a_k - b_k}{a_k(t^\alpha_k - t^\alpha_{k-1})} \right\}$$

for $J(c_j) < J(d_j)$, $j = 1, 2$

$$M^j_k(t) = \begin{cases} \dfrac{\rho\alpha}{\rho\alpha a_k + b_k(t^\alpha - t^\alpha_k)} \dfrac{(t-\rho)^\alpha - (t_k - \rho)^\alpha}{t^\alpha_k - (t_k - \rho)^\alpha}, & t \in (t_k, t_k + \rho) \\ \dfrac{(t-\rho)^\alpha - t^\alpha_k}{t^\alpha - t^\alpha_k}, & t \in [t_k + \rho, t_{k+1}), \end{cases}$$

then every solution of problem (1) is oscillatory.

Proof. Assume to the contrary that $x(t)$ is a non-oscillatory solution of (1). Without loss of generality we may assume that $x(t)$ is an eventually positive solution of (1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Therefore it follows from (1) that

$$T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] = f(t) - q(t)x(t-\rho) - \sum_{i=1}^{n} q_i(t)f_i(x(t-\rho)) \quad \text{for} \quad t \in [t_1, \infty).$$

Thus $T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] \geq 0$ or $T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] < 0$, $t \geq t_1$ for some $t_1 \geq t_0$. We now claim that

$$T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] \geq 0 \quad \text{for} \quad t \geq t_1. \tag{3}$$

Suppose not, then $T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] < 0$ and there exists $t_2 \in [t_1, \infty)$ such that $T_\alpha\left[r(t_2)g(T_\alpha(x(t_2)))\right] < 0$. Since $r(t)g(T_\alpha(x(t)))$ is strictly decreasing on $[t_1, \infty)$. It is clear that

$$r(t)g(T_\alpha(x(t))) < r(t_2)g(T_\alpha(x(t_2))) := -\mu$$

where $\mu > 0$ is a constant for $t \in [t_2, \infty)$, we have

$$\begin{aligned}
&r(t)g(T_\alpha(x(t))) < -\mu\\
&\qquad T_\alpha(x(t)) < g^{-1}\left(\frac{-\mu}{r(t)}\right)\\
&\qquad T_\alpha(x(t)) \leq -\zeta_1 g^{-1}\left(\frac{1}{r(t)}\right), \quad \text{where} \quad \zeta_1 = \zeta g^{-1}(\mu) \text{ for } t \in [t_2, \infty).
\end{aligned}$$

Integrating the above inequality from t_2 to t, we have

$$x(t) \leq x(t_2) - \zeta_1 \int_{t_2}^{t} s^{\alpha-1} g^{-1}\left(\frac{1}{r(s)}\right) ds.$$

Letting $t \to \infty$, we get $\lim\limits_{t\to+\infty} x(t) = -\infty$ which contradiction proves that (3) holds. Define the Riccati transformation

$$w(t) := \frac{r(t)g(T_\alpha(x(t)))}{x(t)}.$$

It follows from (1) that $w(t)$ satisfies

$$T_\alpha(w(t)) \leq \frac{f(t)}{x(t)} - \left[q(t) + \sum_{i=1}^{n} \epsilon_i q_i(t)\right] \frac{x(t-\rho)}{x(t)} - \frac{w^2(t)}{\eta r(t)}.$$

By the assumption, we can choose $c_1, d_1 \geq t_0$ such that $r(t) > 0$, $q(t) \geq 0$ and $q_i(t) \geq 0$ for $t \in [c_1 - \rho, d_1]$, $i = 1, 2, \cdots, n$ and $f(t) \leq 0$ for $t \in [c_1 - \rho, d_1]$ from (1) we can easily to see that

$$t^{1-\alpha} w'(t) \leq -\frac{w^2(t)}{\eta r(t)} - Q(t)\frac{x(t-\rho)}{x(t)}. \tag{4}$$

For $t = t_k,\ k = 1, 2, \cdots$, one has

$$w(t_k^+) = \frac{r(t_k^+)g(T_\alpha(x(t_k^+)))}{x(t_k^+)} \leq \frac{b_k}{a_k} w(t_k).$$

At first, we consider the case in which $J(c_1) < J(d_1)$. In this case, all the impulsive moments in $[c_1, d_1]$ are $t_{J(c_1)+1},\ t_{J(c_1)+2}, \cdots,\ t_{J(d_1)}$. Choose an $p(t) \in J_p(c_1, d_1)$ and multiplying by $p^2(t)$ on both sides on (4), integrating it from c_1 to d_1, we obtain

$$\begin{aligned}
&\int_{c_1}^{t_{J(c_1)+1}} p^2(t)t^{1-\alpha}w'(t)dt + \int_{t_{J(c_1)+1}}^{t_{J(c_1)+2}} p^2(t)t^{1-\alpha}w'(t)dt \\
&+ \cdots + \int_{t_{J(d_1)}}^{d_1} p^2(t)t^{1-\alpha}w'(t)dt \\
&\leq -\int_{c_1}^{t_{J(c_1)+1}} p^2(t)\frac{w^2(t)}{\eta r(t)}dt - \int_{t_{J(c_1)+1}}^{t_{J(c_1)+2}} p^2(t)\frac{w^2(t)}{\eta r(t)}dt - \cdots - \int_{t_{J(d_1)}}^{d_1} p^2(t)\frac{w^2(t)}{\eta r(t)}dt \\
&- \int_{c_1}^{t_{J(c_1)+1}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt - \int_{t_{J(c_1)+1}}^{t_{J(c_1)+1}+\rho} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt \\
&- \int_{t_{J(c_1)+1}+\rho}^{t_{J(c_1)+2}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt - \cdots - \int_{t_{J(c_1)-1}+\rho}^{t_{J(d_1)}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt \\
&- \int_{t_{J(d_1)}}^{d_1} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt.
\end{aligned}$$

Using the integration by parts on the left-hand side, and noting that the condition $p(c_1) = p(d_1) = 0$, we get

$$\begin{aligned}
&\sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k)t_k^{1-\alpha}\left[w(t_k) - w(t_k^+)\right] \leq \int_{c_1}^{d_1}\left[p'(t)t^{1-\alpha}\sqrt{\eta r(t)} - \frac{p(t)w(t)}{\sqrt{\eta r(t)}}\right]^2 dt \\
&- \int_{c_1}^{t_{J(c_1)+1}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt \\
&- \sum_{k=J(c_1)+1}^{J(d_1)-1}\left[\int_{t_k}^{t_k+\rho} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt + \int_{t_k+\rho}^{t_{k+1}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt\right] \\
&- \int_{t_{J(d_1)}}^{d_1} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt + \int_{c_1}^{d_1} t^{2-2\alpha}\eta r(t)(p'(t))^2dt \\
&+ \int_{c_1}^{d_1}(1-\alpha)t^{-\alpha}p^2(t)w(t)dt. \qquad (5)
\end{aligned}$$

There are several cases to consider to estimate $\frac{x(t-\rho)}{x(t)}$.

Case 1: For $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$. If $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$, since $t_{k+1} - t_k > \rho$,

we consider two sub cases:
Case 1.1: If $t \in [t_k + \rho, t_{k+1}]$, then $t - \rho \in [t_k, t_{k+1} - \rho]$ and there are no impulsive moments in $(t - \rho, t)$, then for any $t \in [t_k + \rho, t_{k+1}]$ one has

$$x(t) - x(t_k^+) = T_\alpha(x(\xi)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right), \quad \xi \in (t_k, t).$$

Since $r(t)g(T_\alpha(x(t)))$ is non-increasing

$$x(t) \geq T_\alpha(x(\xi)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right) > \frac{r(t)g(T_\alpha(x(t)))}{r(\xi)} \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right).$$

From the fact that $r(t)$ is nondecreasing, we get

$$\frac{r(t)g(T_\alpha(x(t)))}{x(t)} < \frac{\alpha r(\xi)}{t^\alpha - t_k^\alpha} < \frac{\alpha r(t)}{t^\alpha - t_k^\alpha}.$$

We obtain

$$\frac{T_\alpha(x(t))}{x(t)} < \frac{\alpha}{t^\alpha - t_k^\alpha}.$$

Integrating it from $t - \rho$ to t, we have

$$\frac{x(t-\rho)}{x(t)} > \frac{(t-\rho)^\alpha - t_k^\alpha}{t^\alpha - t_k^\alpha}.$$

Case 1.2: If $t \in (t_k, t_k + \rho)$ then $t - \rho \in (t_k - \rho, t_k)$ and there is an impulsive moment t_k in $(t - \rho, t)$. Similar to Case 1.1, we obtain

$$x(t) - x(t_k - \rho) = T_\alpha(x(\xi_1)) \left(\frac{t^\alpha - (t_k - \rho)^\alpha}{\alpha} \right), \quad \xi_1 \in (t_k - \rho, t_k]$$

or

$$\frac{T_\alpha(x(t))}{x(t)} < \frac{\alpha}{t^\alpha - (t_k - \rho)^\alpha}.$$

Integrating it from $t - \rho$ to t, we get

$$\frac{x(t-\rho)}{x(t_k)} > \frac{(t-\rho)^\alpha - (t_k - \rho)^\alpha}{t_k^\alpha - (t_k - \rho)^\alpha} > 0, \quad t \in (t_k, t_k + \rho). \tag{6}$$

For any $t \in (t_k, t_k + \rho)$, we have

$$x(t) - x(t_k^+) < T_\alpha(x(t_k^+)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right), \quad \xi_2 \in (t_k, t).$$

Using the impulsive conditions in equation (1), we get

$$x(t) - a_k x(t_k) < b_k T_\alpha(x(t_k)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right)$$

$$\frac{x(t)}{x(t_k)} < b_k \frac{T_\alpha(x(t_k))}{x(t_k)} \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right) + a_k.$$

Using $\frac{T_\alpha(x(t_k))}{x(t_k)} < \frac{1}{\rho}$, we obtain

$$\frac{x(t)}{x(t_k)} < a_k + \frac{b_k}{\rho}\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right).$$

That is,

$$\frac{x(t_k)}{x(t)} > \frac{\rho\alpha}{\rho\alpha a_k + b_k(t^\alpha - t_k^\alpha)}. \tag{7}$$

From (6) and (7), we get

$$\frac{x(t-\rho)}{x(t)} > \frac{\rho\alpha}{\rho\alpha a_k + b_k(t^\alpha - t_k^\alpha)} \frac{(t-\rho)^\alpha - (t_k-\rho)^\alpha}{t_k^\alpha - (t_k-\rho)^\alpha} \geq 0.$$

Case 2: If $t \in [c_1, t_{J(c_1)+1}]$, we consider three sub cases:
Case 2.1: If $t_{J(c_1)} > c_1-\rho$ and $t \in [t_{J(c_1)}+\rho, t_{J(c_1)+1}]$ then $t-\rho \in [t_{J(c_1)}, t_{J(c_1)+1}-\rho]$ and there are no impulsive moments in $(t-\rho, t)$. Making a similar analysis of the Case 1.1 and using Mean-value Theorem on $(t_{J(c_1)}, t_{J(c_1)+1}]$, we get

$$\frac{x(t-\rho)}{x(t)} > \frac{(t-\rho)^\alpha - t_{J(c_1)}^\alpha}{t^\alpha - t_{J(c_1)}^\alpha} > 0, \quad t \in [t_{J(c_1)}+\rho, t_{J(c_1)+1}].$$

Case 2.2: If $t_{J(c_1)} > c_1-\rho$ and $t \in [c_1, t_{J(c_1)}+\rho)$, then $t-\rho \in [c_1-\rho, t_{J(c_1)})$ and there is an impulsive moments $t_{J(c_1)}$ in $(t-\rho, t)$. Making a similar analysis of the Case 1.2, we have

$$\begin{aligned}\frac{x(t-\rho)}{x(t)} &> \frac{\rho\alpha}{\rho\alpha a_{J(c_1)} + b_{J(c_1)}(t^\alpha - t_{J(c_1)}^\alpha)} \frac{(t-\rho)^\alpha - (t_{J(c_1)}-\rho)^\alpha}{t_{J(c_1)}^\alpha - (t_{J(c_1)}-\rho)^\alpha} \\ &\geq 0, \quad t \in (c_1, t_{J(c_1)}+\rho).\end{aligned}$$

Case 2.3: If $t_{J(c_1)} < c_1-\rho$, then for any $t \in [c_1, t_{J(c_1)+1}]$, $t-\rho \in [c_1-\rho, t_{J(c_1)+1}-\rho]$ and there are no impulsive moments in $(t-\rho, t)$. Making a similar analysis of the Case 1.1, we obtain

$$\frac{x(t-\rho)}{x(t)} > \frac{(t-\rho)^\alpha - t_{J(c_1)}^\alpha}{t^\alpha - t_{J(c_1)}^\alpha} > 0, \quad t \in [c_1, t_{J(c_1)+1}].$$

Case 3: For $t \in (t_{J(d_1)}, d_1]$, there are three sub cases:
Case 3.1: If $t_{J(d_1)}+\rho < d_1$ and $t \in [t_{J(d_1)}+\rho, d_1]$ then $t-\rho \in [t_{J(d_1)}, d_1-\rho]$ and there are no impulsive moments in $(t-\rho, t)$. Making a similar analysis of the Case 2.1, we have

$$\frac{x(t-\rho)}{x(t)} > \frac{(t-\rho)^\alpha - t_{J(d_1)}^\alpha}{t^\alpha - t_{J(d_1)}^\alpha} > 0, \quad t \in [t_{J(d_1)}+\rho, d_1].$$

Case 3.2: If $t_{J(d_1)}+\rho < d_1$ and $t \in [t_{J(d_1)}, t_{J(d_1)}+\rho)$, then $t-\rho \in [t_{J(d_1)}-\rho, t_{J(d_1)})$ and there is an impulsive moments $t_{J(d_1)}$ in $(t-\rho, t)$. Making a similar analysis of the Case 2.2, we obtain

$$\frac{x(t-\rho)}{x(t)} > \frac{\rho\alpha}{\rho\alpha a_{J(d_1)} + b_{J(d_1)}(t^\alpha - t^\alpha_{J(d_1)})} \frac{(t-\rho)^\alpha - (t_{J(d_1)} - \rho)^\alpha}{t^\alpha_{J(d_1)} - (t_{J(d_1)} - \rho)^\alpha} \geq 0.$$

Case 3.3: If $t_{J(d_1)} + \rho \geq d_1$, then for any $t \in (t_{J(d_1)}, d_1]$, we get $t - \rho \in (t_{J(d_1)} - \rho, d_1 - \rho]$ and there is an impulsive moments $t_{J(d_1)}$ in $(t-\rho, t)$. Making a similar analysis of the Case 3.2, we get

$$\frac{x(t-\rho)}{x(t)} > \frac{\rho\alpha}{\rho\alpha a_{J(d_1)} + b_{J(d_1)}(t^\alpha - t^\alpha_{J(d_1)})} \frac{(t-\rho)^\alpha - (t_{J(d_1)} - \rho)^\alpha}{t^\alpha_{J(d_1)} - (t_{J(d_1)} - \rho)^\alpha} \geq 0.$$

Combining all these cases, we have

$$\frac{x(t-\rho)}{x(t)} > \begin{cases} M^1_{J(c_1)}(t) & \text{for} \quad t \in [c_1, t_{J(c_1)+1}] \\ M^1_k(t) & \text{for} \quad t \in (t_k, t_{k+1}], \quad k = J(c_1)+1, \cdots, J(d_1)-1 \\ M^1_{J(d_1)}(t) & \text{for} \quad t \in (t_{J(d_1)+1}, d_1]. \end{cases}$$

Hence by (5), we have

$$\begin{aligned} &\sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} \left[w(t_k) - w(t_k^+)\right] \\ &\qquad \leq \int_{c_1}^{t_{J(c_1)+1}} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M^1_{J(c_1)}(t)\right] dt \\ &\qquad\quad + \sum_{k=J(c_1)+1}^{J(d_1)-1} \int_{t_k}^{t_{k+1}} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M^1_k(t)\right] dt \\ &\qquad\quad + \int_{t_{J(d_1)}}^{d_1} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M^1_{J(d_1)}(t)\right] dt \\ &\qquad\quad + \int_{c_1}^{d_1} (1-\alpha) t^{-\alpha} p^2(t) w(t) dt \end{aligned} \tag{8}$$

Since $r(t)g(T_\alpha(x(t)))$ is non-increasing in $(c_1, t_{J(c_1)+1}]$. Thus

$$\begin{aligned} x(t) > x(t) - x(c_1) &= T_\alpha(x(\xi_3)) \left(\frac{t^\alpha - c_1^\alpha}{\alpha}\right) \\ &\geq \frac{r(t) g(T_\alpha(x(t)))}{r(\xi_3)} \left(\frac{t^\alpha - c_1^\alpha}{\alpha}\right), \quad \xi_3 \in (c_1, t) \end{aligned}$$

Letting $t \to t^-_{J(c_1)+1}$, it follows that

$$w(t_{J(c_1)+1}) < \frac{r_1}{t^\alpha_{J(c_1)+1} - c_1^\alpha}. \tag{9}$$

Similarly we can prove that on $(t_{k-1}, t_k]$, $k = J(c_1)+2, \cdots, J(d_1)$,

$$w(t_k) < \frac{r_1}{t_k^\alpha - t_{k-1}^\alpha}. \tag{10}$$

Hence (9) and (10), we have

$$\begin{aligned}
&\sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k)t_k^{1-\alpha}w(t_k)\left[\frac{a_k-b_k}{a_k}\right] \\
&\qquad \geq r_1\Bigg[p^2(t_{J(c_1)+1})t_{J(c_1)+1}^{1-\alpha}\frac{a_{J(c_1)+1}-b_{J(c_1)+1}}{a_{J(c_1)+1}}\frac{1}{t_{J(c_1)+1}^\alpha - c_1^\alpha} \\
&\qquad + \sum_{k=J(c_1)+2}^{J(d_1)} p^2(t_k)t_k^{1-\alpha}\frac{a_k-b_k}{a_k}\frac{1}{t_k^\alpha - t_{k-1}^\alpha}\Bigg] \\
&\qquad \geq \Lambda(p, c_1, d_1).
\end{aligned}$$

Thus we have

$$\sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k)t_k^{1-\alpha}w(t_k)\left[\frac{a_k-b_k}{a_k}\right] \geq \Lambda(p, c_1, d_1).$$

Therefore (8), we get

$$\begin{aligned}
&\int_{c_1}^{t_{J(c_1)+1}}\left[(p'(t))^2t^{2-2\alpha}\eta r(t) - p^2(t)Q(t)M^1_{J(c_1)}(t)\right]dt \\
&+\sum_{k=J(c_1)+1}^{J(d_1)-1}\int_{t_k}^{t_{k+1}}\left[(p'(t))^2t^{2-2\alpha}\eta r(t) - p^2(t)Q(t)M^1_k(t)\right]dt \\
&+\int_{t_{J(d_1)}}^{d_1}\left[(p'(t))^2t^{2-2\alpha}\eta r(t) - p^2(t)Q(t)M^1_{J(d_1)}(t)\right]dt + \int_{c_1}^{d_1}(1-\alpha)t^{-\alpha}p^2(t)w(t)dt \\
&> \Lambda(p, c_1, d_1)
\end{aligned}$$

which contradicts (2).

If $J(c_1) = J(d_1)$ then $\Lambda(p, c_1, d_1) = 0$ and there are no impulsive moments in $[c_1, d_1]$. Similar to the proof of (8), we obtain

$$\int_{c_1}^{d_1}\left[(p'(t))^2t^{2-2\alpha}\eta r(t) - p^2(t)Q(t)M^1_{J(c_1)}(t) + p^2(t)(1-\alpha)t^{-\alpha}w(t)\right]dt > 0$$

This again contradicts our assumption. Finally if $x(t)$ is eventually negative, we can consider $[c_2, d_2]$ and reach similar contradiction. The proof of theorem is complete. □

Following Kong [11] and Philos [17], we introduce a class of functions: Let $D = \{(t,s) : t_0 \le s \le t\}$, then a function $H_1, H_2 \in C(D, \mathbb{R})$ is said to belong to the class $\mathcal{H}$ if

(H_4) $H_1(t,t) = H_2(t,t) = 0$, $H_1(t,s) > 0, H_2(t,s) > 0$ for $t > s$ and

(H_5) H_1 and H_2 have partial derivatives $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial s}$ on D such that

$$\frac{\partial H_1}{\partial t} = h_1(t,s)H_1(t,s), \quad \frac{\partial H_2}{\partial s} = -h_2(t,s)H_2(t,s),$$

where $h_1, h_2 \in L_{loc}(D, \mathbb{R})$.

$$\begin{aligned}\Omega_{1,j} = &\int_{c_j}^{t_{J(c_j)+1}} H_1(t,c_j)Q(t)M^j_{J(c_j)}(t)dt + \sum_{k=J(c_i)+1}^{J(\lambda_j)-1} \int_{t_k}^{t_{k+1}} H_1(t,c_j)Q(t)M^j_k(t)dt \\ &+ \int_{t_{J(\lambda_j)}}^{\lambda_j} H_1(t,c_j)Q(t)M^j_{J(d_j)}(t)dt \\ &+ \int_{c_j}^{\lambda_j} H_1(t,c_j)\left[\frac{w^2(t)}{\eta r(t)} - w(t)t^{1-\alpha}h_1(t,c_j) - (1-\alpha)t^{-\alpha}w(t)\right]dt\end{aligned}$$

and

$$\begin{aligned}\Omega_{2,j} = &\int_{\lambda_j}^{t_{J(\lambda_j)+1}} H_2(d_j,t)Q(t)M^j_{J(\lambda_j)}(t)dt + \sum_{k=J(\lambda_j)+1}^{J(d_j)-1} \int_{t_k}^{t_{k+1}} H_2(d_j,t)Q(t)M^j_k(t)dt \\ &+ \int_{t_{J(d_j)}}^{d_j} H_2(d_j,t)Q(t)M^j_{J(d_j)}(t)dt \\ &+ \int_{\lambda_j}^{d_j} H_2(d_j,t)\left[\frac{w^2(t)}{\eta r(t)} + w(t)t^{1-\alpha}h_2(d_j,t) - (1-\alpha)t^{-\alpha}w(t)\right]dt.\end{aligned}$$

Theorem 6. *Assume that conditions $(H_1) - (H_3)$ hold, furthermore for any $T \ge 0$ there exist c_j, d_j satisfying $(H_4), (H_5)$ with $c_1 < \lambda_1 < d_1 \le c_2 < \lambda_2 < d_2$. If there exists $H_1, H_2 \in \mathcal{H}$ such that*

$$\frac{1}{H_1(\lambda_1,c_1)}\Omega_{1,1} + \frac{1}{H_2(d_1,\lambda_1)}\Omega_{2,1} > \Lambda(H_1,H_2;c_j,d_j), \tag{11}$$

where

$$\Lambda(H_1,H_2;c_j,d_j) = -\left\{\frac{r_j}{H_1(\lambda_j,c_j)}\Phi^{\lambda_j}_{c_j}[H_1(.,c_j)] + \frac{r_j}{H_2(d_j,\lambda_j)}\Phi^{d_j}_{\lambda_j}[H_2(d_j,.)]\right\},$$

then every solution of problem (1) is oscillatory.

Proof. Suppose to the contrary that there is a non-oscillatory solution $x(t)$ of problem (1). Notice whether or not there are impulsive moments in $[c_1, \lambda_1]$ and $[\lambda_1, d_1]$, we should consider the following cases $J(c_1) < J(\lambda_1) < J(d_1)$, $J(c_1) = J(\lambda_1) <$

$J(d_1)$, $J(c_1) < J(\lambda_1) = J(d_1)$ and $J(c_1) = J(\lambda_1) = J(d_1)$. Moreover, the impulsive moments of $x(t-\rho)$ having following two cases, $t_{J(\lambda_s)} + \rho > \lambda_s$ and $t_{J(\lambda_s)} + \rho \leq \lambda_s$. Consider the case $J(c_1) < J(\lambda_1) < J(d_1)$, with $t_{J(\lambda_s)} + \rho > \lambda_s$. For this case, the impulsive moments are $t_{J(\lambda_1)+1},\ t_{J(\lambda_1)+2}, \cdots,\ t_{J(d_1)}$ in $[\lambda_1, d_1]$.

Multiplying by $H_1(t, c_1)$ on both sides on (4), integrating it from c_1 to λ_1, we obtain

$$\int_{c_1}^{\lambda_1} H_1(t,c_1)t^{1-\alpha}w'(t)dt \leq -\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x(t-\rho)}{x(t)}dt - \int_{c_1}^{\lambda_1} H_1(t,c_1)\frac{w^2(t)}{\eta r(t)}dt.$$

Applying integration by parts formula on the L.H.S, we get,

$$\begin{aligned}
&\sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[w(t_k) - w(t_k^+)\right] - H_1(\lambda_1,c_1)\lambda_1^{1-\alpha}w(\lambda_1) \\
&\quad - \int_{c_1}^{\lambda_1} w(t)\left[h_1(t,c_1)H_1(t,c_1)t^{1-\alpha} + H_1(t,c_1)(1-\alpha)t^{-\alpha}\right]dt \\
&\leq -\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x(t-\rho)}{x(t)}dt - \int_{c_1}^{\lambda_1} \frac{w^2(t)}{\eta r(t)}H_1(t,c_1)dt
\end{aligned} \tag{12}$$

By Theorem 5, we divide the interval $[c_1, \lambda_1]$ into several and calculating the function $\dfrac{x(t-\rho)}{x(t)}$, we obtain

$$\begin{aligned}
\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x(t-\rho)}{x(t)}dt \geq{}& \int_{c_1}^{t_{J(c_1)+1}} H_1(t,c_1)Q(t)M^1_{J(c_1)}(t)dt \\
&+ \sum_{k=J(c_1)+1}^{J(\lambda_1)-1}\int_{t_k}^{t_{k+1}} H_1(t,c_1)Q(t)M^1_{k(t)}dt \\
&+ \int_{t_{J(\lambda_1)}}^{\lambda_1} H_1(t,c_1)Q(t)M^1_{J(\lambda_1)}(t)dt.
\end{aligned} \tag{13}$$

From (12) and (13), we obtain

$$\begin{aligned}
&\int_{c_1}^{t_{J(c_1)+1}} H_1(t,c_1)Q(t)M^1_{J(c_1)}(t)dt + \sum_{k=J(c_1)+1}^{J(\lambda_1)-1}\int_{t_k}^{t_{k+1}} H_1(t,c_1)Q(t)M^1_{k(t)}dt \\
&+ \int_{t_{J(\lambda_1)}}^{\lambda_1} H_1(t,c_1)Q(t)M^1_{J(\lambda_1)}(t)dt \\
&+ \int_{c_1}^{\lambda_1} H_1(t,c_1)w(t)\left[\frac{w(t)}{\eta r(t)} - t^{1-\alpha}h_1(t,c_1) - (1-\alpha)t^{-\alpha}\right]dt \\
&\leq -\sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[\frac{a_k - b_k}{a_k}\right]w(t_k) - H_1(\lambda_1,c_1)\lambda_1^{1-\alpha}w(\lambda_1).
\end{aligned} \tag{14}$$

On the other hand multiplying both sides of (4) by $H_2(d_1,t)$ and integrating from λ_1 to d_1 and using the similar of above, we get

$$\begin{aligned}
&\int_{\lambda_1}^{t_{J(\lambda_1)+1}} H_2(d_1,t)Q(t)M^1_{J(\lambda_1)}(t)dt + \sum_{k=J(\lambda_1)+1}^{J(d_1)-1}\int_{t_k}^{t_{k+1}} H_2(d_1,t_k)Q(t)M^1_{k(t)}dt\\
&+\int_{t_{J(d_1)}}^{d_1} H_2(d_1,t)Q(t)M^1_{J(d_1)}(t)dt\\
&+\int_{\lambda_1}^{d_1} H_2(d_1,t)w(t)\left[\frac{w(t)}{\eta r(t)} + t^{1-\alpha}h_2(d_1,t) - (1-\alpha)t^{-\alpha}\right]dt\\
&\le -\sum_{k=J(\lambda_1)+1}^{J(d_1)} H_2(d_1,t_k)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k) + H_2(d_1,\lambda_1)\lambda_1^{1-\alpha}w(\lambda_1). \qquad (15)
\end{aligned}$$

Dividing (14) and (15) by $H_1(\lambda_1,c_1)$ and $H_2(d_1,\lambda_1)$ respectively and adding them, we get

$$\begin{aligned}
&\frac{1}{H_1(\lambda_1,c_1)}\Omega_{1,1} + \frac{1}{H_2(d_1,\lambda_1)}\Omega_{2,1}\\
&\qquad \le -\left[\frac{1}{H_1(\lambda_1,c_1)}\sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k)\right.\\
&\qquad\quad \left.+\frac{1}{H_2(d_1,\lambda_1)}\sum_{k=J(\lambda_1)+1}^{J(d_1)} H_2(d_1,t_k)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k)\right]. \qquad (16)
\end{aligned}$$

Using the method as in (9) , we obtain

$$\left.\begin{aligned}
-\sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k) &\le -r_1\Phi_{c_1}^{\lambda_1}[H_1(\cdot,c_1)]\\
-\sum_{k=J(\lambda_1)+1}^{J(d_1)} H_2(d_1,t_k)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k) &\le -r_1\Phi_{\lambda_1}^{d_1}[H_2(d_1,\cdot)].
\end{aligned}\right\} \qquad (17)$$

From (16) and (17), we obtain

$$\begin{aligned}
&\frac{1}{H_1(\lambda_1,c_1)}\Omega_{1,1} + \frac{1}{H_2(d_1,\lambda_1)}\Omega_{2,1}\\
&\qquad \le -\left\{\frac{r_1}{H_1(\lambda_1,c_1)}\Phi_{c_1}^{\lambda_1}[H_1(\cdot,c_1)] + \frac{r_1}{H_2(d_1,\lambda_1)}\Phi_{\lambda_1}^{d_1}[H_2(d_1,\cdot)]\right\}\\
&\qquad \le \Lambda(H_1,H_2;c_1,d_1) \qquad (18)
\end{aligned}$$

which is contradiction to the condition (11). Suppose $x(t) < 0$, we take interval $[c_2,d_2]$ for equation (1). The proof is similar and hence omitted. The proof is complete. □

4. Examples

In this section, we present some examples to illustrate our results established in Section 3.

Example 7. *Consider the following impulsive conformable fractional differential equations*

$$\left.\begin{array}{ll} T_{\frac{1}{2}}\left(2\left(T_{\frac{1}{2}}(x(t))\right)\right)+mx(t-\frac{\pi}{8})+2mx(t-\frac{\pi}{8})=f(t), & t\neq 2k\pi\pm\frac{\pi}{4}, \\ x(t_k^+)=\frac{1}{3}x(t_k), \quad T_{\frac{1}{2}}(x(t_k^+))=\frac{2}{3}T_{\frac{1}{2}}(x(t_k)), & k=1,2,\cdots. \end{array}\right\} \tag{19}$$

Here $\alpha=\frac{1}{2}$, $a_k=\frac{1}{3}$, $b_k=\frac{2}{3}$, $r(t)=2$, $q(t)=m$, $q_1(t)=2m$, $g(x)=x$, $f_1(x)=x$, $\eta=1$, $f(t)=\cos t-2t\sin t+3m\sin(t-\frac{\pi}{8})$ *and* m *is a positive constant. Also* $\rho=\frac{\pi}{8}$, $t_{k+1}-t_k=\pi/2>\pi/8$. *For any* $T>0$, *we choose* k *large enough such that* $T<c_1=4k\pi-\frac{\pi}{2}<d_1=4k\pi$ *and* $c_2=4k\pi+\frac{\pi}{8}<d_2=4k\pi+\frac{\pi}{2}$, $k=1,2,\cdots$. *Then there is an impulsive movement* $t_k=4k\pi-\frac{\pi}{4}$ *in* $[c_1,d_1]$ *and an impulsive moment* $t_{k+1}=4k\pi+\frac{\pi}{4}$ *in* $[c_2,d_2]$. *For* $\epsilon_1=1$, *we have* $Q(t)=3m$, *and we take* $p(t)=\sin 8t\in J_p(c_j,d_j)$, $j=1,2$, $t_{J(c_1)}=4k\pi-\frac{7\pi}{4}$, $t_{J(d_1)}=4k\pi-\frac{\pi}{4}$, *then by using simple calculation, the left side of equation* (2) *is the following :*

$$\begin{aligned}
&\int_{c_j}^{d_j}\left[(p'(t))^2t^{2-2\alpha}\eta r(t)+w(t)p^2(t)(1-\alpha)t^{-\alpha}\right]dt-\int_{c_j}^{t_{J(c_j)+1}}Q(t)p^2(t)M^j_{J(c_j)}(t)dt \\
&-\sum_{k=J(c_j)+1}^{J(d_j)-1}\int_{t_k}^{t_{k+1}}Q(t)p^2(t)M^j_{J(c_j)}(t)dt-\int_{t_{J(d_j)}}^{d_j}Q(t)p^2(t)M^j_{J(d_j)}(t)dt \\
&\leq\int_{4k\pi-\frac{\pi}{2}}^{4k\pi}\left(2t(8\cos 8t)^2+\frac{t^{\frac{1}{2}}\cos t}{\sin t}t^{-\frac{1}{2}}\sin^2(8t)\right)dt \\
&-3m\Bigg\{\int_{4k\pi-\frac{\pi}{2}}^{4k\pi-\frac{\pi}{4}}\sin^2(8t)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{2}}-\left(4k\pi-\frac{7\pi}{4}\right)^{\frac{1}{2}}}{t^{\frac{1}{2}}-\left(4k\pi-\frac{7\pi}{4}\right)^{\frac{1}{2}}}\right)dt \\
&\quad+\int_{4k\pi-\frac{\pi}{4}}^{4k\pi-\frac{\pi}{8}}\sin^2(8t)\left(\frac{\frac{\pi}{16}}{\frac{\pi}{48}+\frac{2}{3}\left(t^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}\right)^{\frac{1}{2}}\right)}\right) \\
&\qquad\qquad\times\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}-\frac{\pi}{8}\right)^{\frac{1}{2}}}{\left(4k\pi-\frac{\pi}{4}\right)^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}-\frac{\pi}{8}\right)^{\frac{1}{2}}}\right) \\
&\qquad+\int_{4k\pi-\frac{\pi}{8}}^{4k\pi}\sin^2(8t)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}\right)^{\frac{1}{2}}}{t^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}\right)^{\frac{1}{2}}}\right)dt\Bigg\} \\
&\simeq 1182.67634-m(1.94487).
\end{aligned}$$

for m large enough. On the other hand, note that $J(c_1) = k-1$, $J(d_1) = k$, $r_1 = 2$, we have $\Lambda(p, c_i, d_i) = 0$. Therefore the condition (2) is satisfied in $[c_1, d_1]$. Similarly, we can prove that for $t \in [c_2, d_2]$. Hence by Theorem 5, every solution of (19) is oscillatory. In fact $x(t) = \sin t$ is one such solution of problem (19).

Example 8. *Consider the following impulsive conformable fractional differential equations*

$$\left.\begin{array}{l} T_{\frac{1}{3}}\left(3\left(T_{\frac{1}{3}}(x(t))\right)\right) + \frac{m}{2}x(t-\frac{\pi}{8}) + mx(t-\frac{\pi}{8}) = f(t), \quad t \neq 2k\pi \pm \frac{\pi}{4}, \\ x(t_k^+) = 4x(t_k), \quad T_{\frac{1}{3}}(x(t_k^+)) = 5T_{\frac{1}{3}}(x(t_k)), \qquad k = 1, 2, \cdots. \end{array}\right\} \tag{20}$$

Here $\alpha = \frac{1}{3}$, $a_k = 4$, $b_k = 5$, $r(t) = 3$, $q(t) = \frac{m}{2}$, $q_1(t) = m$, $g(x) = 2x$, $f_1(x) = x$, $\eta = 2$, $f(t) = -4t^{\frac{1}{3}}\sin t - 6t^{\frac{4}{3}}\cos t + \frac{3m}{2}\cos(t-\frac{\pi}{8})$ and m is a positive constant. Also $\rho = \frac{\pi}{8}$, $t_{k+1} - t_k = \pi/2 > \pi/8$. For any $T > 0$, we choose k large enough such that $T < c_1 = 4k\pi - \frac{\pi}{2} < d_1 = 4k\pi$ and $c_2 = 4k\pi + \frac{\pi}{8} < d_2 = 4k\pi + \frac{\pi}{2}$, $k = 1, 2, \cdots$. Then there is an impulsive movement $t_k = 4k\pi - \frac{\pi}{4}$ in $[c_1, d_1]$ and an impulsive moment $t_{k+1} = 4k\pi + \frac{\pi}{4}$ in $[c_2, d_2]$. For $\epsilon_1 = 1$, we have $Q(t) = \frac{3m}{2}$, and we take $p(t) = \sin 16t \in J_p(c_j, d_j)$, $j = 1, 2$, $t_{J(c_1)} = 4k\pi - \frac{7\pi}{4}$, $t_{J(d_1)} = 4k\pi - \frac{\pi}{4}$, then by using simple calculation, the left side of Equation (2) is the following :

$$\begin{aligned}
&\int_{c_j}^{d_j} \left[(p'(t))^2 t^{2-2\alpha}\eta r(t) + w(t)p^2(t)(1-\alpha)t^{-\alpha}\right] dt - \int_{c_j}^{t_{J(c_j)+1}} Q(t)p^2(t)M^j_{J(c_j)}(t)dt \\
&- \sum_{k=J(c_j)+1}^{J(d_j)-1} \int_{t_k}^{t_{k+1}} Q(t)p^2(t)M^j_{J(c_j)}(t)dt - \int_{t_{J(d_j)}}^{d_j} Q(t)p^2(t)M^j_{J(d_j)}(t)dt \\
&\leq \int_{4k\pi-\frac{\pi}{2}}^{4k\pi} \left(6t^{\frac{4}{3}}(16\cos 16t)^2 - \frac{4t^{\frac{1}{3}}\sin t}{\cos t}\sin^2(16t)\right) dt \\
&- \frac{3m}{2}\Bigg\{ \int_{4k\pi-\frac{\pi}{2}}^{4k\pi-\frac{\pi}{4}} \sin^2(16t)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{3}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{3}}}{t^{\frac{1}{3}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{3}}}\right) dt \\
&\quad + \int_{4k\pi-\frac{\pi}{4}}^{4k\pi-\frac{\pi}{8}} \sin^2(16t)\left(\frac{\frac{\pi}{24}}{\frac{2\pi}{6} + 5\left(t^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{3}}\right)}\right)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4} - \frac{\pi}{8}\right)^{\frac{1}{3}}}{\left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4} - \frac{\pi}{8}\right)^{\frac{1}{3}}}\right) \\
&\quad + \int_{4k\pi-\frac{\pi}{8}}^{4k\pi} \sin^2(16t)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{3}}}{t^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{3}}}\right) dt\Bigg\} \\
&\qquad\qquad \simeq 32367.58257 - m(0.66712).
\end{aligned}$$

for m large enough. On the other hand, note that $J(c_1) = k-1$, $J(d_1) = k$, $r_1 = 3$, we have $\Lambda(p, c_i, d_i) = 0$. Therefore the condition (2) is satisfied in $[c_1, d_1]$. Similarly, we can prove that for $t \in [c_2, d_2]$. Hence by Theorem 5, every solution of (20) is oscillatory. In fact $x(t) = \cos t$ is one such solution of problems(20).

Remark 9. *In this paper, some new oscillation results are obtained, generalizing the results of* [13] *to impulsive conformable fractional differential equations. The improvement factors impulses, delay and forcing term that affect the interval qualitative properties of solution in the sequence of subintervals in* $[0, \infty)$*, were taken into account together. Our newly obtained results in this paper have improved and extended some of the results already prevailing in the existing literature.*

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RELATIVE SUBCOPURE-INJECTIVE MODULES

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Abstract. In this paper, copure-injective modules are examined from an alternative perspective. For two modules A and B, A is called B-subcopure-injective if for every copure monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. The class $\underline{\mathfrak{CPI}}^{-1}(A) = \{B : A$ is B-subcopure-injective$\}$ is called the subcopure-injectivity domain of A. We obtain characterizations of copure-injective modules, right CDS rings and right V-rings with the help of subcopure-injectivity domains. Since subcopure-injectivity domains clearly contains all copure-injective modules, studying the notion of modules which are subcopure-injective only with respect to the class of copure-injective modules is reasonable. We refer to these modules as sc-indigent. We studied the properties of subcopure-injectivity domains and of sc-indigent modules and investigated these modules over some certain rings.

1. Introduction and preliminaries

Throughout this paper, R will denote an associative ring with identity, and modules will be unital right R-modules, unless otherwise stated. As usual, the category of right R-modules is denoted by $Mod - R$.

Some new studies in module theory have focused on to approach to the injectivity from the point of relative notions. The injectivity domain $\mathfrak{In}^{-1}(A)$ for a module A, is the class of all modules B such that A is B-injective [1]. Given A and B modules, A is called B-subinjective if for every monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. Instead of using the injectivity domain, in latest articles, authors have proposed to consider an alternative sight so-called subinjectivity domain $\underline{\mathfrak{In}}^{-1}(A)$, contains of modules B such that A is B-subinjective ([2]). It is clear that injectivity of A is equivalent to that $\underline{\mathfrak{In}}^{-1}(A) = Mod{-}R$. If B is injective, then A is exactly B-subinjective. So by [2, Proposition 2.3], the class of injective modules is the smallest

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possible subinjectivity domain. The recent studies of non-injective modules have been made to figure out the notion of modules that are subinjective only with respect to the class of injective modules. This kind of non-injective modules are called indigent in [2]. So far, it is not known whether the existence of indigent modules for an arbitrary ring, but a positive answer is known for some rings, such as Noetherian rings ([3, Proposition 3.4]).

A submodule A of a right R-module B is said to be pure if for every left R-module K the natural induced map $i \otimes 1_K : A \otimes K \to B \otimes K$ is a monomorphism. Recall that a module A is said to be B-pure-injective if for every pure monomorphism $f : C \to B$ and every homomorphism $g : C \to A$, there exists a homomorphism $h : B \to A$ such that $hf = g$. A module A is said to be pure-injective if it is B-pure-injective for every module B. As an analogue to the injectivity profile of [12], the pure-injectivity profile of a ring is introduced in [5]. The pure-injectivity domain $\mathfrak{PI}^{-1}(A)$ of a module A, consists of those modules B such that A is B-pure-injective. Inspired by the notion of subinjectivity, the notion of pure-subinjectivity introduced in [11]. A module A is called B-pure-subinjective if for every pure monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. The pure-subinjectivity domain of a module A is the class $\underline{\mathfrak{PI}}^{-1}(A) = \{B : A \text{ is } B\text{-pure-subinjective}\}$. If B is pure-injective, then A is exactly B-pure-subinjective. So by [11, Theorem 2.4], for a module A, the class $\underline{\mathfrak{PI}}^{-1}(A)$ must contain the class of pure-injective modules at least. In [11], modules whose pure-subinjectivity domain consists of only pure-injective modules is called pure-subinjectively poor (ps-poor for short).

An R-module A is said to be finitely embedded (or cofinitely generated) if $E(A) = E(S_1) \oplus E(S_2) \oplus ... \oplus E(S_n)$, where $S_1, S_2, ..., S_n$ are simple R-modules (see [16]). If an R-module A is isomorphic to $\prod\{E(S_\alpha) | S_\alpha \text{is a simple right } R\text{-module}, \alpha \in I\}$, where I is some index set, then A is called a cofree module (see [6]). A right R-module A is said to be cofinitely related if there is an exact sequence $0 \to A \to B \to C \to 0$ of R-modules with B finitely embedded, cofree and C finitely embedded (see [6]). As a dual notion of purity, by using cofinitely related modules, the notion of copurity is introduced in [7]. An exact sequence of R-modules $0 \to A \to B \to C \to 0$ is called a copure exact sequence if every cofinitely related right R-module is injective relative to this sequence.

Following idea on pure-injectivity profile of [5], in [15], the copure-injectivity profile of a ring is introduced. For two modules A and B, A is called B-copure-injective if for every copure monomorphism $f : C \to B$ and a homomorphism $g : C \to A$, there exists a homomorphism $h : B \to A$ such that $hf = g$. A is copure-injective if it is injective with respect to every copure exact sequences (see [8]). The copure-injectivity domain $\mathfrak{CPI}^{-1}(A)$ of A is the class of modules B such that A is B-copure-injective. In [15], copure-injectively-poor (shortly copi-poor) modules introduced as modules with minimal copure-injectivity domain and studied properties of copi-poor modules. The existence of copi-poor modules are studied and investigated over some certain rings, but we do not know whether copi-poor modules exist over arbitrary rings (see [15]).

Inspired by the notion of pure-subinjectivity from [11], in this paper we initiate the study of an alternative perspective on the analysis of the copure-injectivity of a module, as we introduce the notions of relative subcopure-injectivity and assign to every module its subcopure-injectivity domain. The aim of this paper is to investigate the viability of obtaining valuable information about a ring R from the perspective of subcopure-injectivity domain.

In Section 2, relative subcopure-injectivity and subcopure-injectivity domains of modules introduced. We investigate the properties of the notion of subcopure-injectivity and we compare subcopure-injectivity domains with (copure-)injectivity domains. We obtain characterizations of copure-injective modules, right CDS rings and right V-rings with the help of subcopure-injectivity domains.

In section 3, we introduced and studied the concept of cc-injective modules in terms of relative subcopure-injective modules. We give examples of cc-injective modules and compare cc-injective modules with cotorsion modules in Example 19. We prove that R is a right V-ring if and only if every cc-injective right R-module is injective. We investigate when the class of B-subcopure-injective modules is closed under extensions.

An R-module is copure-injective if and only if its subcopure-injectivity domain consists of $Mod-R$. Since subcopure-injectivity domains clearly contain all copure-injective modules, it is reasonable to investigate modules which are subcopure-injective only with respect to the class of copure-injective modules. It is thus to keep in line with [11], we refer to these modules as sc-indigent. In Section 4 of this paper, we studied and investigated sc-indigent modules over some certain rings. We compared sc-indigent modules with indigent modules and ps-poor modules.

2. Relative subcopure-injective modules

In this section, we study the B-subcopure-injective modules for a module B and examine its fundamental properties.

Definition 1. *For two modules A and B, A is called B-subcopure-injective if for every copure monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. The class $\underline{\mathfrak{CPI}}^{-1}(A) = \{B : A \text{ is } B\text{-subcopure-injective}\}$ is called the subcopure-injectivity domain of A.*

Hiremath proved in [8, Theorem 7] that every module can be embedded as a copure submodule in a direct product of cofinitely related modules. By [8, Proposition 3], every cofinitely related module is copure-injective and every direct product of copure-injective modules is copure-injective. This gives the below result that we use frequently in the sequel.

Lemma 2. *For every module A, there exists a copure monomorphism $\alpha : A \to C$ with C is copure-injective.*

Our next Lemma gives a characterization of the B-subcopure-injective modules for a module B.

Lemma 3. *Let A and B be two modules. The following conditions are equivalent:*

(1) *A is B-subcopure-injective.*
(2) *For every homomorphism $g : B \to A$ and every copure monomorphism $\alpha : B \to C$ with C copure-injective, there exists $h : C \to A$ such that $h\alpha = g$.*
(3) *For every homomorphism $g : B \to A$ and every copure monomorphism $\alpha : B \to C$ with C direct product of cofinitely related modules, there exists $h : C \to A$ such that $h\alpha = g$.*
(4) *For every $g : B \to A$ there exist a copure monomorphism $\alpha : B \to C$ with C copure-injective and $h : C \to A$ such that $h\alpha = g$.*

Proof. (1) $\Rightarrow$ (2) Obvious. (2) $\Rightarrow$ (3) It follows from [8, Proposition 3].

(3) $\Rightarrow$ (4) Let $g : B \to A$ be a homomorphism. By Lemma 2, there exists a copure monomorphism $\alpha : B \to C$ with C copure-injective, whence C is a direct summand of F where $F = \prod_{i \in I} F_i$ with each F_i cofinitely related by [8, Theorem 8]. So $i\alpha : B \to F$ is copure monomorphism where $i : C \to F$. By (3), there exists $h : F \to A$ such that $(hi)\alpha = h(i\alpha) = g$,where $i\alpha : B \to F$.

(4) $\Rightarrow$ (1) Let $g : B \to A$ be a homomorphism and $\bar{\alpha} : B \to D$ a copure monomorphism. By (4), there exists a monic copure map $\alpha : B \to C$ with C copure-injective and a homomorphism $h : C \to A$ such that $h\alpha = g$. So by the copure-injectivity of C, there exists a homomorphism $\bar{h} : D \to C$ such that $\alpha = \bar{h}\bar{\alpha}$. Then $h\bar{h} : D \to A$ and $h\bar{h}\bar{\alpha} = h\alpha = g$. Hence, A is B-subcopure-injective. □

Proposition 4. *Let A be an R-module. The following conditions are equivalent:*

(1) *A is copure-injective.*
(2) $\underline{\mathfrak{CPI}}^{-1}(A) = Mod - R$.
(3) *A is A-subcopure-injective.*

Proof. (1) $\Rightarrow$ (2) For any R-module B and any copure-injective module A, every copure monomorphism $\alpha : B \to D$ and a homomorphism $g : B \to A$, there exists a homomorphism $h : D \to A$ such that $h\alpha = g$. Hence, A is B-subcopure-injective and so $B \in \underline{\mathfrak{CPI}}^{-1}(A)$. Consequently, $\underline{\mathfrak{CPI}}^{-1}(A) = Mod - R$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Assume that A is A-subcopure-injective. For any copure monomorphism $\alpha : A \to B$ with B copure-injective and $1_A : A \to A$, there exists a homomorphism $g : B \to A$ such that $g\alpha = 1_A$. Thus α splits. This means that A is copure-injective. □

The next result asserts that subcopure-injectivity domain $\underline{\mathfrak{CPI}}^{-1}(A)$ of A how small can be. It should contain the copure-injective modules at least.

Proposition 5. $\bigcap_{A \in Mod - R} \underline{\mathfrak{CPI}}^{-1}(A) = \{C \in Mod - R \mid C \text{ is copure-injective}\}$.

Proof. Suppose that each R-module is B-subcopure-injective for an R-module B. Then, by Proposition 4, B is copure-injective. Conversely, let A be any R-module and B a copure-injective module. Let $g : B \to A$ be a homomorphism and $\alpha : B \to C$ a copure monomorphism. Since B is copure-injective, the splitting map $\alpha : B \to C$ gives the homomorphism $\beta : C \to B$ such that $\beta\alpha = 1_B$. So $\beta(\alpha g) = (\beta\alpha)g = g$. Hence $B \in \underline{\mathfrak{CPI}}^{-1}(A)$ for any R-module A. □

Clearly, $\underline{\mathfrak{CPI}}^{-1}(A)$ contains $\underline{\mathfrak{In}}^{-1}(A)$ for any module A. The following example shows that equality need not hold.

Example 6. *Let $G = Z(n)$ be a cyclic group of order n. Since G is finite it is cofinitely related and so it is copure-injective $\mathbb{Z}$-module* [8, Proposition 3]. *So $G \in \underline{\mathfrak{CPI}}^{-1}(G)$ by Proposition 4. But $G \notin \underline{\mathfrak{In}}^{-1}(G)$, otherwise G would be an injective $\mathbb{Z}$-module.*

It is natural to investigate conditions to get the coincidence of the injectivity, and subcopure-injectivity domains, either for a certain class of modules or all the modules in $Mod - R$. We start by proving that, for all modules, subcopure-injectivity domains are the same as their subinjectivity domains over a right V-ring. Recall that a ring R is a right V-ring if and only if all exact sequences in $Mod - R$ are copure if and only if all copure-injective modules are injective (see [8, Proposition 5]).

Corollary 7. *Let R be a ring. The following conditions are equivalent:*

(1) *R is a right V-ring.*
(2) *$\underline{\mathfrak{CPI}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}(A)$ for each R-module A.*
(3) *$\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \underline{\mathfrak{In}}^{-1}(A)$ for each R-module A.*

Proof. (1) ⇒ (2) It is easy since for any module A, over a right V-ring its extension is copure.
(2) ⇒ (3) It is obvious.
(3) ⇒ (1) For a copure injective right R-module A, by Proposition 4, $A \in \underline{\mathfrak{CPI}}^{-1}(A)$. By (3), $A \in \underline{\mathfrak{In}}^{-1}(A)$. This says that A is injective, and so R is a right V-ring by [8, Proposition 5].

□

Proposition 8. *Let A be a module. The following conditions are equivalent:*

(1) *A is copure-injective.*
(2) *$\underline{\mathfrak{CPI}}^{-1}(A)$ is closed under copure submodules.*
(3) *$\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{CPI}^{-1}(A)$.*
(4) *$\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{CPI}^{-1}(A)$.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) are clear since $\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{CPI}^{-1}(A) = Mod - R$.

(2) $\Rightarrow$ (1) For a copure-injective extension C of A, $C \in \underline{\mathfrak{CPI}}^{-1}(A)$, so A is also in $\underline{\mathfrak{CPI}}^{-1}(A)$ by (2). Then by Proposition 4, A is copure-injective.

(3) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) For a copure-injective extension C of A, $C \in \underline{\mathfrak{CPI}}^{-1}(A)$. This implies that A is C-copure-injective i.e. $C = A \oplus B$ for some submodule B of A, whence A is copure-injective. □

The rings for which every right R-module is copure-injective are called right CDS, [8, Corollary 18]. As a result of Proposition 8, we get the following Corollary.

Corollary 9. *Let R be a ring. The following conditions are equivalent:*

(1) *R is right CDS.*
(2) *$\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{CPI}^{-1}(A)$ for each R-module A.*
(3) *$\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{CPI}^{-1}(A)$ for each R-module A.*

Proof. (2) $\Rightarrow$ (3) It is clear.

(1) $\Rightarrow$ (2) Let A be an R-module. Since R is a right CDS ring, A is copure-injective. The rest follows from Proposition 8.

(3) $\Rightarrow$ (1) For any right R-module A, $\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{CPI}^{-1}(A)$ by the hypothesis. Thus every right R-module A is copure-injective by Proposition 8, whence R is right CDS. □

Remark 10. *If A is R-subcopure-injective, for a ring R and a module A, then $\underline{\mathfrak{CPI}}^{-1}(A)$ and $Mod-R$ need not be equal. For example if R is copure-injective ring that is not CDS, then for every module A, A is R-subcopure-injective by Proposition 5. But by the definition of right CDS ring, we can find a module A that is not copure-injective.*

Proposition 11. *Let A be a module. The following conditions are equivalent:*

(1) *A is injective.*
(2) *$\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{In}^{-1}(A)$.*
(3) *$\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) By the copure-injectivity of $E(A)$, $E(A) \in \underline{\mathfrak{CPI}}^{-1}(A)$. By (3), $E(A) \in \mathfrak{In}^{-1}(A)$, and hence A is injective. □

Corollary 12. *Let R be a ring. The following conditions are equivalent:*

(1) *R is semisimple.*
(2) *$\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{In}^{-1}(A)$ for each R-module A.*
(3) *$\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$ for each R-module A.*

Proof. (2) $\Rightarrow$ (3) It is clear.

(1) $\Rightarrow$ (2) Let A be an R-module. Since R is semisimple, A is injective. The rest follows from Proposition 11.

(3) $\Rightarrow$ (1) For any right R-module A, $\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$ by the hypothesis. Thus every right R-module A is injective by Proposition 11, whence R is semisimple. $\square$

In general, factors of copure-injective modules need not be copure-injective (see, [8, Remark 24]). But if R is a Dedekind domain, every copure factor of copure-injective module is copure-injective by [8, Corollary 28]. Hence, by the following Proposition, $\underline{\mathfrak{CPI}}^{-1}(A)$ is closed under copure homomorphic images over Dedekind domains for a module A.

Proposition 13. *$\underline{\mathfrak{CPI}}^{-1}(A)$ is closed under copure quotients for any module A if and only if every copure homomorphic image of a copure-injective module is copure-injective.*

Proof. Let B be a copure submodule of copure-injective module A. Since $A \in \underline{\mathfrak{CPI}}^{-1}(\frac{A}{B})$, by the hypothesis $\frac{A}{B} \in \underline{\mathfrak{CPI}}^{-1}(\frac{A}{B})$, and so $\frac{A}{B}$ is copure-injective. Conversely, let A be a module and C a copure submodule of B with $B \in \underline{\mathfrak{CPI}}^{-1}(A)$. By Lemma 2, there exists a copure monomorphism $\alpha : B \to D$ with D copure-injective. Let $f : \frac{B}{C} \to A$ be any homomorphism. Consider the following pushout diagram:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\alpha"] \arrow[d, "\pi"] & D \arrow[r] \arrow[d, "\pi'"] & \frac{D}{B} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \frac{B}{C} \arrow[r, "\alpha'"] \arrow[d, "f"] & E \arrow[r] & \frac{D}{B} \arrow[r] & 0 \\
 & A & & &
\end{tikzcd}$$

where $\pi : B \to \frac{B}{C}$ is the natural epimorphism. By commutativity of the following diagram:

$$\begin{tikzcd}
B \arrow[r, "\alpha"] \arrow[d, "\pi"] & D \arrow[d, "\pi''"] \\
\frac{B}{C} \arrow[r, "\alpha''"] & \frac{D}{C}
\end{tikzcd}$$

and the pushout diagram property, there exists a map $\phi : E \to \frac{D}{C}$ such that $\phi\pi' = \pi''$ and $\phi\alpha' = \alpha''$. Since A is B-subcopure-injective, there exists a homomorphism $\varphi : D \to A$ such that $\varphi\alpha = f\pi$. Then, $\varphi(C) = \varphi\alpha(C) = f\pi(C) = f(0) = 0$. Hence, $Ker(\phi\pi') \subseteq Ker\varphi$, and so there exists $\psi : \frac{D}{C} \to A$ such that $\psi\pi'' = \varphi$. For every $x \in B$, $\psi(x + C) = \psi\pi''(x) = \varphi(x) = f\pi(x) = f(x + C)$. Thus ψ extends f. Then by the hypothesis, $\frac{D}{C}$ is copure-injective, so by Lemma 3, $\frac{B}{C} \in \underline{\mathfrak{CPI}}^{-1}(A)$. $\square$

Proposition 14. *$\underline{\mathfrak{CPI}}^{-1}(\prod_{i\in I} A_i) = \bigcap_{i\in I} \underline{\mathfrak{CPI}}^{-1}(A_i)$ for any set of modules $\{A_i\}_{i\in I}$.*

Proof. Let $B \in \underline{\mathfrak{CPI}}^{-1}(\prod_{i\in I} A_i)$, $i \in I$ and $f : B \to A_i$ be a homomorphism. Then there exists a homomorphism $g : C \to \prod_{i\in I} A_i$ such that $g\alpha = i_{A_i} f$, where $\alpha : B \to C$ is the monic map with C copure-injective and $i_{A_i} : A_i \to \prod_{i\in I} A_i$ is the inclusion map. Let $\pi_{A_i} : \prod_{i\in I} A_i \to A_i$ denote the natural projection. Since $\pi_{A_i} g\alpha = \pi_{A_i} i_{A_i} f = f$, f is extended to $\pi_{A_i} g$. Therefore $B \in \underline{\mathfrak{CPI}}^{-1}(A_i)$ for any $i \in I$. Conversely , let $B \in \underline{\mathfrak{CPI}}^{-1}(A_i)$ for all $i \in I$ and $f : B \to \prod_{i\in I} A_i$. Hence for each $i \in I$, there exists $g_i : C \to A_i$ with $g_i\alpha = \pi_{A_i} f$. Now define $g : C \to \prod_{i\in I} A_i$ by $x \mapsto g_i(x)$. Since $g\alpha = f$, g extends f. Thus, $B \in \underline{\mathfrak{CPI}}^{-1}(\prod_{i\in I} A_i)$. □

Corollary 15. *Let B be a module. Then B-subcopure-injective modules are closed under direct summands and finite direct sums.*

Proof. Let A be a module with decomposition $A = \oplus_{i=1}^{n} A_i$. By Proposition 14, $B \in \underline{\mathfrak{CPI}}^{-1}(A)$ if and only if $B \in \bigcap_{i=1}^{n} \underline{\mathfrak{CPI}}^{-1}(A_i)$. Now the result follows. □

The following shows that Proposition 14 do not hold for infinite direct sums.

Example 16. *Let $K_i = \mathbb{Z}_{p_i}$ and $G = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{p_i}$ where p_i is a prime integer for all $i \in \mathbb{N}$. Since every $\mathbb{Z}_{p_i}$ is pure-injective, every $\mathbb{Z}_{p_i}$ is copure-injective by* [8, Proposition 9]*. So $G \in \underline{\mathfrak{CPI}}^{-1}(\mathbb{Z}_{p_i})$ for all $i \in \mathbb{N}$. But $G \notin \underline{\mathfrak{CPI}}^{-1}(G)$ since G is not copure-injective by* [8, Examples-(ii)].

Proposition 17. *If $B \in \underline{\mathfrak{CPI}}^{-1}(A)$, then every direct summand of B is in $\underline{\mathfrak{CPI}}^{-1}(A)$.*

Proof. Suppose C is a direct summand of B, and let $f : C \to A$ be a homomorphism. By Lemma 2, there exist copure monomorphisms $i : B \to D$ and $j : C \to E$ with D and E copure-injective. Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "i_C"] \arrow[d, "j"] & B \arrow[d, "i"] \\ & E & D \end{tikzcd}$$

where $i_C : C \to B$ the inclusion map. Since D is copure-injective, there exists $h : E \to D$ such that $hj = ii_C$. Let $\pi_C : B \to C$ be the projection map. Since A is B-subcopure-injective, there exists a homomorphism $g : D \to A$ such that $gi = f\pi_C$. Then, $(gh)j = g(hj) = gii_C = f\pi_C i_C = f$, and so by Lemma 3, A is C-subcopure-injective. □

3. CC-INJECTIVE MODULES

In this section, we introduced and studied the concept of cc-injective modules in terms of relative subcopure-injective modules.

A module C is said to be co-absolutely co-pure (c.c. in short) if every exact sequence of modules ending with C is copure, equivalently $Ext^1_R(C, A) = 0$ for every co-finitely related module A. Clearly every projective module is c.c. But the converse need not be true, for instance, the additive group $\mathbb{Q}$ is a c.c. $\mathbb{Z}$-module but $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module (see, [9, Example on page 290]).

Definition 18. *A right module A is called cc-injective if $Ext^1_R(B, A) = 0$ for any c.c. module B.*

Recall that a module A is called cotorsion if $Ext^1_R(B, A) = 0$ for every flat module B. A module A is called linearly compact if any family of cosets having the finite intersection property has a nonempty intersection. A commutative ring is called classical if the injective hull $E(S)$ of all simple modules S are linearly compact (see [17, §3]).

Example 19. *(1) By definition, any cofinitely related module is cc-injective.*
(2) By [9, Remark 15]*, c.c. modules need not be flat in general. By* [9, Corollary 14] *c.c. modules are flat over a commutative ring. So, in this case every cotorsion module is cc-injective.*
(3) By [9, Remark 12]*, flat modules need not be c.c. Over a commutative classical ring flat modules are c.c. by* [9, Proposition 11]*. So, in this case every cc-injective module is cotorsion.*

Remark 20. *Over a commutative ring R every simple R-module is cotorsion by* [13, Lemma 2.14]*. So by Example 19(2), every simple R-module is cc-injective.*

Lemma 21. *Every copure-injective module is cc-injective.*

Proof. Let A be a copure-injective module and B a c.c. module. By [9, Proposition 5], there exists a copure exact sequence $0 \to D \to P \to B \to 0$ with P projective. If we apply $Hom(-, A)$ to this sequence, we have $Hom(P, A) \to Hom(D, A) \to Ext^1_R(B, A) \to Ext^1_R(P, A) = 0$. Since A is copure-injective, $Hom(P, A) \to Hom(D, A)$ is epic, and so $Ext^1_R(B, A) = 0$ for any c.c. module B. Hence A is cc-injective. □

Proposition 22. *For a ring R, the following conditions are equivalent:*

1. *R is a right V-ring.*
2. *Every copure-injective right R-module is injective.*
3. *Every cc-injective right R-module is injective.*

Proof. (1) $\Leftrightarrow$ (2) It follows by [8, Proposition 5].

(3) $\Rightarrow$ (2) It immediately from Lemma 21.

(1) $\Rightarrow$ (3) Let A be a cc-injective R-module and B any R-module. Since R is right V, B is a c.c. module by [9, Proposition 4]. Thus $Ext^1_R(B, A) = 0$ for any R-module B, and so A is injective. □

Proposition 23. *Let B be an R-module and $\alpha : B \to C$ a copure monomorphism with C copure-injective. If $C/im(\alpha)$ is c.c., then every cc-injective module is B-subcopure-injective.*

Proof. Let A be a cc-injective module and $C/im(\alpha)$ a c.c. module. Applying functor $Hom(-, A)$ to the exact sequence $0 \to B \to C \to C/im(\alpha) \to 0$, we have $Hom(C, A) \to Hom(B, A) \to Ext^1_R(C/im(\alpha), A)$. Since $C/im(\alpha)$ is c.c., $Ext^1_R(C/im(\alpha), A) = 0$ and so $Hom(C, A) \to Hom(B, A)$ is epic. Hence A is B-subcopure-injective by Lemma 3. □

Theorem 24. *Let A and B be two modules. Consider the following conditions:*

(1) *A is B-subcopure-injective.*
(2) *For every homomorphism $g : B \to A$, there exist a monomorphism $\alpha : B \to C$ with C copure-injective and a homomorphism $h : C \to A$ such that $h\alpha = g$.*
(3) *For every homomorphism $g : B \to A$, there exist a monomorphism $\alpha : B \to C$ with C cc-injective and a homomorphism $h : C \to A$ such that $h\alpha = g$.*
(4) *For every homomorphism $g : B \to A$ and for any extension $\alpha : B \hookrightarrow C$ with C/B is c.c., there exists $h : C \to A$ such that $h\alpha = g$.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Also, if $D/im(\alpha)$ is c.c. for a copure monomorphism $\alpha : B \to D$ with D copure-injective, then (4) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Obvious by Lemma 3.

(2) $\Rightarrow$ (3) It follows from Lemma 21, since every copure-injective module is cc-injective.

(2) $\Rightarrow$ (1) Let $\alpha : B \to C$ be a copure-monomorphism and $g : B \to A$ a homomorphism. By (2), exists a monomorphism $\beta : B \to D$ with D copure-injective and a homomorphism $h : D \to A$ such that $h\beta = g$. Since D is copure-injective, there exists a homomorphism $f : C \to D$ such that $f\alpha = \beta$. Hence, $(hf)\alpha = h\beta = g$, and so (1) follows.

(3) $\Rightarrow$ (4) Let C be an extension of B with C/B is c.c. and $g : B \to A$ a homomorphism. So, $0 \to B \xrightarrow{\alpha} C \to C/B \to 0$ is copure exact. Then consider the exact sequence with E cc-injective:

$$0 \to Hom_R(C/B, E) \to Hom_R(C, E) \xrightarrow{\alpha^*} Hom_R(B, E) \to Ext^1_R(C/B, E) = 0$$

Since, α^* is surjective, by (3), there exists a monomorphism $f : B \to E$ and a homomorphism $h : E \to A$ such that $hf = g$. Since α^* is surjective, there exists a homomorphism $\beta : C \to E$ such that $\beta\alpha = f$. Hence, $h(\beta\alpha) = hf = g$, and so (4) follows.

$(4) \Rightarrow (1)$: Let $\alpha : B \to D$ be a copure monomorphism with D copure-injective and $D/im(\alpha)$ is c.c. So, by (4), for any homomorphism $g : B \to A$ there exists $h : D \to A$ such that $h\alpha = g$. Thus A is B-subcopure-injective by Lemma 3. □

Now we investigate when the class of B-subcopure-injective modules is closed under extensions.

Proposition 25. *Let B be an R-module and $\alpha : B \to C$ a copure monomorphism with C copure-injective. The class of B-subcopure-injective modules is closed under extensions if and only if for every exact sequence $0 \to A' \to A \to C \to 0$ with A' B-subcopure-injective, A is B-subcopure-injective.*

Proof. Let $0 \to A' \to A \to C \to 0$ be an exact sequence with A' B-subcopure-injective. Since C is copure-injective, it is B-subcopure-injective. By the hypothesis, A is B-subcopure-injective. Conversely, let $0 \to A' \to A \xrightarrow{\pi} A'' \to 0$ be an exact sequence with A' and A'' B-subcopure-injective. Then by Lemma 3, for every map $g : B \to A$, there exists a map $h : C \to A''$ such that $\pi g = h\alpha$ where $\alpha : B \to C$ is the copure monomorphism with C copure-injective. If we consider the pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, equal] & D \arrow[r, "\beta"] \arrow[d, "f"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r, "\pi"] & A'' \arrow[r] & 0
\end{tikzcd}$$

there exists a homomorphism $\gamma : B \to D$ such that $f\gamma = g$ and $\beta\gamma = \alpha$. By hypothesis, D is B-subcopure-injective, so by Lemma 3, there exists a homomorphism $h' : C \to D$ such that $h'\alpha = \gamma$. Thus, $fh'\alpha = f\gamma = g$ and so, A is B-subcopure-injective by Lemma 3. □

A ring R is said to be right co-noetherian if every homomorphic image of a finitely embedded R-module is finitely embedded, equivalently for each simple right R-module S the injective hull $E(S)$ is Artinian (see [10, Theorem]). Over a commutative noetherian ring, the injective hull of each simple right R-module is Artinian by [14, Exercise 4.17]. Thus every commutative Noetherian ring is co-noetherian. In the following, for an ideal I, we deal with an R-module structure of an R/I-module.

Proposition 26. *Let R be a right co-noetherian ring and $f : R \to S$ a ring epimorphism. If A is cc-injective S-module, then A is cc-injective R-module.*

Proof. Let A be a cc-injective S-module. Since $f : R \to S$ is a ring epimorphism, $S \cong R/I$ for some ideal I of R and so A can be considered as R/I-module. Let C be an extension of A by a c.c. module F as R-modules. Since F is c.c., the exact sequence $0 \to A \to C \to F \to 0$ is copure. Then $A \cap CI = AI$ for each right ideal I by [7, proposition 16]. Since A is an R/I-module, $A \cap CI = AI = 0$, and so $\frac{A+CI}{CI} \cong A$. Thus we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{A+CI}{CI} & \longrightarrow & \frac{C}{CI} & \longrightarrow & \frac{C}{A+CI} & \longrightarrow & 0 \end{array}$$

Since $\frac{C}{A} \otimes \frac{R}{I} \cong \frac{C}{A+CI}$ is c.c. as an R/I-module, so the second exact sequence splits and so does the first. Hence $Ext^1_R(F, A) = 0$, and A is cc-injective R-module. $\square$

4. SC-INDIGENT MODULES

Indigent (resp. ps-poor) modules were introduced and some results about them were obtained in [2] (resp. [11]). Proposition 5 says that subcopure-injectivity domain of any module A contains all copure-injective modules, so studying the notion of modules which are subcopure-injective only with respect to the class of copure-injective modules is reasonable. It is thus to keep in line with [2], we refer to these modules as subcopure-injectively indigent (sc-indigent for short). In this section, sc-indigent modules investigated over certain rings and compared these modules with indigent modules and ps-poor modules.

Definition 27. *A module A is said to be subcopure-injectively indigent (sc-indigent for short), if $\underline{\mathfrak{CPI}}^{-1}(A)$ consists of only copure-injective modules.*

Remark 28. *Let A be a module with decomposition $A = B \oplus C$. If B is sc-indigent, then so is A, by Proposition 14.*

Proposition 29. *For a ring R, the following conditions are equivalent:*

(1) *R is right CDS.*
(2) *Every R-module is sc-indigent.*
(3) *There exists a copure-injective sc-indigent R-module.*
(4) *0 is an sc-indigent R-module.*
(5) *R has an sc-indigent module and every sc-indigent R-module is copure-injective.*
(6) *R has an sc-indigent module and every factor of an sc-indigent R-module is sc-indigent.*
(7) *R has an sc-indigent module and every summand of an sc-indigent R-module is sc-indigent.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (5) are clear since every R-module is copure-injective.
The implications (2) ⇒ (4) and (2) ⇒ (6) ⇒ (7) are clear.
(4) ⇒ (2) It immediately from Remark 28.
(2) ⇒ (3) The copure-injective extension C of any module A is sc-indigent.
(3) ⇒ (1) Let C be a copure-injective sc-indigent module and A a module. Since C is A-subcopure-injective, A is copure-injective. Then R is a right CDS ring.

(5) $\Rightarrow$ (1) By (5), there exist an sc-indigent module B. Then $A \oplus B$ is also sc-indigent for any module A by Remark 28. So A is copure-injective by (5). Also A is copure-injective. Thus R is a right CDS ring.
(7) $\Rightarrow$ (2) Let A be an R-module. Then $A \oplus B$ is an sc-indigent module for some sc-indigent module B. Hence, A is sc-indigent by the hypothesis. $\square$

Remark 30. *Over a commutative uniserial ring R, every R-module is sc-indigent since such rings are CDS by* [4, Theorem 10.4]*.*

Remark 31. *An sc-indigent module need not be indigent. Consider the ring $R = \mathbb{Z}/p^2\mathbb{Z}$, for some prime integer p. R is an artinian principal ideal ring. Hence it is a CDS-ring by* [4, Theorem 10.4]*. So every R-module is sc-indigent. Since $\mathbb{Z}/p^2\mathbb{Z}$ is injective $\mathbb{Z}/p^2\mathbb{Z}$-module, $\underline{\mathfrak{In}}^{-1}(\mathbb{Z}/p^2\mathbb{Z}) = Mod-R$. But since R is not a semisimple ring, $\mathbb{Z}/p^2\mathbb{Z}$ is not an indigent R-module.*

Remark 32. *An indigent module need not be sc-indigent. Let R be a commutative Noetherian ring which is not CDS and Γ a complete set of representatives of finitely presented right R-modules. Set $F := \bigoplus_{S_i \in \Gamma} S_i$. Thus the character module F^+ of F is a pure-injective indigent R-module by* [3, Proposition 3.4]*. Since R is commutative, F^+ is copure-injective by* [8, Proposition 9]*, and so $\underline{\mathfrak{CPI}}^{-1}(F^+) = Mod - R$. But since R is not a CDS-ring, F^+ is not an sc-indigent R-module.*

Proposition 33. *Indigent modules and sc-indigent modules coincide over a right V-ring R.*

Proof. Let R be a right V-ring. Then by Corollary 7, $\underline{\mathfrak{CPI}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}(A)$ for any R-module A. Hence A is indigent if and only if A is sc-indigent by [8, Proposition 5]. $\square$

Proposition 34. *A module A is sc-indigent if and only if $\prod_{i \in I} A_i$ is sc-indigent where $A_i = A$ for all $i \in I$.*

Proof. Clear by Proposition 14. $\square$

By Remark 28 and Proposition 34, sc-indigent rings are characterized as follows:

Corollary 35. *For a ring R, the following are equivalent:*

1. *R_R is sc-indigent.*
2. *Any direct product of copies of R is sc-indigent.*
3. *Every free R-module is sc-indigent.*
4. *There exists a cyclic projective sc-indigent R-module.*

Theorem 36. *Let R be a ring, B an R-module and A an R/I-module for any ideal I of R. If $B/BI \in \underline{\mathfrak{CPI}}^{-1}(A_{R/I})$, then $B \in \underline{\mathfrak{CPI}}^{-1}(A_R)$.*

Proof. Let $B/BI \in \underline{\mathfrak{CPI}}^{-1}(A_{R/I})$, and C be a copure extension of B and $g : B \to A$ an R-homomorphism. Since copure short exact sequences of R-modules form a proper class by [7, Proposition 8], B/BI can be embedded in C/CI as a copure submodule via $f : B/BI \to C/CI$ defined by $f(b + BI) = b + CI$ for any $b \in B$. Since $BI \subseteq Ker(g)$, there exists a homomorphism $h : B/BI \to A$ such that $h\pi_B = g$ where $\pi_B : B \to B/BI$. By assumption, there exists an R/I-homomorphism $\bar{h} : C/CI \to A$ such that $\bar{h}f = g$. Since h is also an R-homomorphism and $\bar{h}\pi_C i_B = g$ where $\pi_C : C \to C/CI$ and $i_B : B \to C$ is the inclusion. Thus $B \in \underline{\mathfrak{CPI}}^{-1}(A_R)$. $\square$

Corollary 37. *Let I be an ideal of a ring R and A and B be R/I-modules. Then the following statements hold:*

(1) *$B \in \underline{\mathfrak{CPI}}^{-1}(A_R)$ if and only if $B \in \underline{\mathfrak{CPI}}^{-1}(A_{R/I})$.*
(2) *A is a copure-injective R-module if and only if A is a copure-injective R/I-module.*
(3) *A is an sc-indigent R-module if and only if A is an sc-indigent R/I-module.*

Proof. (1) If A_R is B-subcopure-injective, then clearly it is a B-subcopure-injective R/I-module. The converse follows by Theorem 36.

(2) By using Proposition 4, (2) follows from (1).

(3) Clear by (1) and (2). $\square$

Recall [11] that a module A is called ps-poor if pure-subinjectivity domain of A consists of only pure-injective modules. Over a commutative classical ring R, by [8, Corollary 17], pure-injective modules and copure-injective modules coincide. Hence, the following result is immediate.

Proposition 38. *Let R be a commutative classical ring. Then an R-module A is sc-indigent if and only if A is ps-poor.*

Since by [16, Theorem 2] and [17, Proposition 4.1], every commutative (co-)noetherian ring is classical, we have the following result.

Corollary 39. *Let R be a commutative (co-)noetherian ring. Then an R-module A is sc-indigent if and only if A is ps-poor.*

Remark 40. *ps-poor abelian groups and sc-indigent abelian groups coincide by Corollary 39.*

Corollary 41. *Every finitely embedded $\mathbb{Z}$-module is copure-injective but not sc-indigent.*

Proof. Let A be a finitely embedded $\mathbb{Z}$-module. Then A is cofinitely related by [6, Proposition 17]. So A is copure-injective by [8, Proposition 3]. Since $\mathbb{Z}$ is not a CDS ring, by Proposition 29, A is not an sc-indigent module. $\square$

Proposition 42. *If a ring R has an sc-indigent cc-injective module B, then every module with its copure injective extension has c.c cokernel is copure-injective.*

Proof. Let A be an R-module with the exact sequence $0 \to A \to C \to C/A \to 0$, where $A \to C$ is a copure extension of A with C is copure-injective. Consider the sequence $0 \to Hom(C/A, B) \to Hom(C, B) \to Hom(A, B) \to Ext^1(C/A, B)$. Since C/A is c.c., $Ext^1(C/A, B) = 0$. So by Lemma 3, $A \in \underline{\mathfrak{CPI}}^{-1}(B)$, that is A is copure-injective. □

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FOURIER-BESSEL TRANSFORMS OF DINI-LIPSCHITZ FUNCTIONS ON LEBESGUE SPACES $L_{p,\gamma}(\mathbb{R}^n_+)$

ISMAIL EKINCIOGLU, ESRA KAYA, AND S. ELIFNUR EKINCIOGLU

Abstract. In this paper, we prove a generalization of Titchmarsh's theorem for the Laplace-Bessel differential operator in the space $L_{p,\gamma}(\mathbb{R}^n_+)$ for functions satisfying the (ψ, p)-Laplace-Bessel Lipschitz condition for $1 < p \leq 2$ and $\gamma > 0$.

1. Introduction

Integral transforms and their inverse transforms are widely used to solve various problems in calculus, fourier analysis, mechanics, mathematical physics, and computational mathematics. Fourier transform is one of the most important integral transforms. Since it was introducted by Fourier in the early 1880s, it has become an important mathematical concept that is at the centre of the highly developed branch of mathematics called Fourier Analysis. It has many application areas. The Fourier transform of the kernel of singular integral operator is very important in applications of singular integral operator theory. The properties of the Fourier transform of the kernel give information about the existence of the solution of singular integral equations. Since singular integrals are convolution type operators, their Fourier transforms are the product of the Fourier transforms of two functions.

As it is well known that if Lipschitz conditions are applied on a function $f(x)$, then these conditions greatly affect the absolute convergence of the Fourier-Bessel series and behaviour of $F_\gamma f$ Fourier-Bessel transforms of f. In general, if $f(x)$ belongs to a certain function class, then the Lipschitz conditions have bearing as to the dual space to which the Fourier coefficients and Fourier-Bessel transforms of $f(x)$ belong. Younis (see [12]) worked the same phenomena for the wider Dini Lipschitz class for some classes of functions. Daher, El Quadih, Daher and El Hamma proved an analog Younis (see [12, Theorem 2.5]) in for the Fourier-Bessel transform for functions satisfies the Fourier-Bessel Dini Lipschitz condition in the

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Lebesgue space $L^2_{\alpha,n}$ (see [10]). El Hamma and Daher proved a generalization of Titchmarsh's theorem for the Bessel transform in the space $L_{2,\gamma}(\mathbb{R}^n_+)$ (see [1]) .

In this paper we prove a generalization of Titchmarsh's theorem for the Laplace-Bessel transform in the space $L_{p,\gamma}(\mathbb{R}^n_+)$, where $1 < p \leq 2$ and $\gamma > 0$.

2. Preliminaries

Let $\mathbb{R}^n_+$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, ..., x_n)$, defined by the inequality $x_n > 0$. We write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}_+$. S^n_+ denote the unit sphere on $\mathbb{R}^n_+$, which can be defined as $S^n_+ = \{x \in \mathbb{R}^n_+ : \ |x| = 1\}$. $\mathbb{S}_+ = \mathbb{S}(\mathbb{R}^n_+)$ be the space of functions which are the restrictions to $\mathbb{R}^n_+$ of the test functions of the Schwartz that are even with respect to x_n, decreasing sufficiently rapidly at infinity, together with all derivatives of the form

$$D^\alpha_\gamma = D^{\alpha'}_{x'} B^{\alpha_n}_n = D^{\alpha_1}_1 ... D^{\alpha_{n-1}}_{n-1} B^{\alpha_n}_n = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \ldots \cdot \frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}} B^{\alpha_n}_n,$$

i.e., for all $\varphi \in \mathbb{S}_+$, $\sup\limits_{x \in \mathbb{R}^n_+} \left| x^\beta D^\alpha_\gamma \varphi \right| < \infty$, where $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ are multi-indexes, and $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$ and $B_n = \dfrac{\partial^2}{\partial x_n^2} + \dfrac{\gamma}{x_n} \dfrac{\partial}{\partial x_n}$ is the Bessel differential expansion. For $\gamma \geq 0$, we introduce the Bessel normalized function of the first kind j_γ defined by

$$j_\gamma(z) = \Gamma(\gamma + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \gamma + 1)} \left(\frac{z}{2} \right)^{2n}, \tag{1}$$

where Γ is the gamma-function (see [9]). Moreover, from (1) we see that

$$\lim_{z \to 0} \frac{j_{\frac{\gamma-1}{2}}(z) - 1}{z^2} \neq 0$$

by consequence, there exist $C > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Rightarrow \left| j_{\frac{\gamma-1}{2}}(z) - 1 \right| \geq C\, |z|^2. \tag{2}$$

The function $u = j_{\frac{\gamma-1}{2}}(z)$ satisfies the differential equation

$$B_{x_n} u(x, y) = B_{y_n} u(x, y)$$

with the initial conditions $u(x, 0) = f(x)$ and $u_y(x, 0) = 0$ is function infinitely differentiable, even, and, moreover entire analytic.

The Fourier-Bessel transformation and its inverse on $\mathbb{S}_+$ are defined by

$$F_\gamma f(x) = \int_{\mathbb{R}^n} f(y)\, e^{-i(x'y')} j_{\frac{\gamma-1}{2}}(x_n y_n)\ y_n^\gamma dy,$$
$$F^{-1}_\gamma f(x) = C_{n,\gamma} F_\gamma f(-x', x_n),$$

where $(x', y') = x_1y_1 + \ldots + x_{n-1}y_{n-1}$, j_γ, $\gamma > 0$, is the normalized Bessel function, and

$$C_{n,\gamma} = (2\pi)^{n-1}2^{\gamma-1}\Gamma^2((\gamma+1)/2),$$

(see [4, 9, 11]). This transform is associated to the Laplace-Bessel differential operator

$$\Delta_\gamma = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}, \ \gamma > 0. \tag{3}$$

The expression (3) is a hybrid of the Hankel transform.

For a fixed parameter $\gamma > 0$, let $L_{p,\gamma} = L_{p,\gamma}(\mathbb{R}^n_+)$ be the space of measurable functions with a finite norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\gamma dx\right)^{1/p}, \quad 1 \le p < \infty.$$

The space of the essentially bounded measurable function on $\mathbb{R}^n_+$ is denoted by $L_{\infty,\gamma}(\mathbb{R}^n_+)$. For for $f \in L_{p,\gamma}$, I.A. Kipriyanov (for $n = 1$ B.M. Levitan [7, 8]) investigated the generalized convolution (Δ_γ-convolution)

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_+} f(y)\, T^y g(x)\, y_n^\gamma dy,$$

associated with the Laplace-Bessel differential operator, where T^y is the generalized shift operator (Δ_γ-shift) defined by

$$T^y f(x) = C_\gamma \int_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_ny_n\cos\theta + y_n^2}\right) \sin^{\gamma-1}\theta d\theta,$$

being $C_\gamma = \pi^{-\frac{1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)[\Gamma\left(\frac{\gamma}{2}\right)]^{-1}$ (see [5, 6, 7, 8]). We note that this convolution satisfies the property $(f \otimes g)(x) = (g \otimes f)(x)$ (see [2, 3]). The following relation connect the generalized shift operator and the Fourier-Bessel transform, we have

$$F_\gamma[T^y f(x)] = j_{\frac{\gamma-1}{2}}(x_ny_n)F_\gamma[f(x)]. \tag{4}$$

Given $1 < p \le 2$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $f \in L_{p,\gamma}$, we have the Hausdorff-Young inequality

$$\|F_\gamma f\|_{q,\gamma} \le C_q \, \|f\|_{p,\gamma}, \tag{5}$$

where and C_q is a positive constant.

3. Fourier-Bessel Transforms of Dini-Lipschitz Functions

In this section we give the main result of this paper. We need first to define (ψ, p)-Laplace Bessel Lipschitz class.

Definition 1. *A function $f \in L_{p,\gamma}(\mathbb{R}^n_+)$ is said to be in the (ψ, p)-Laplace Bessel Lipschitz class, denoted by Lip(ψ, γ, p), if*

$$\|T^y f(x) - f(x)\|_{p,\gamma} = O(\psi(y)) \quad as \quad y \to 0,$$

where $\psi(x)$ is a continuous increasing function on $\mathbb{R}^n_+$, $\psi(0) = 0$, and $\psi(xs) = \psi(x)\psi(s)$ for all $x, s \in \mathbb{R}^n_+$.

Theorem 2. *Let $f(x)$ belong to $Lip(\psi, \gamma, p)$. Then*

$$\int_{|\xi|\geq\tau} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi = O(\psi(\tau^{-q})), \quad as \quad \tau \to +\infty.$$

Proof. Let $f \in Lip(\psi, \gamma, p)$. Then we have

$$\|T^y f(x) - f(x)\|_{p,\gamma} = O(\psi(y)) \quad \text{as} \quad y \to 0.$$

Now we consider Fourier-Bessel transform of generalized shift operator. We get

$$\begin{aligned}
F_\gamma[T^y f(x)](\xi) &= \int_{\mathbb{R}^n_+} T^y f(x) j_{\frac{\gamma-1}{2}}(x_n\xi_n) x_n^\gamma dx \\
&= \int_{\mathbb{R}^n_+} T^y \left[j_{\frac{\gamma-1}{2}}(x_n\xi_n)\right] f(x) x_n^\gamma dx \\
&= \int_{\mathbb{R}^n_+} j_{\frac{\gamma-1}{2}}(x_n\xi_n) j_{\frac{\gamma-1}{2}}(y_n\xi_n) f(x) x_n^\gamma dx \\
&= j_{\frac{\gamma-1}{2}}(y_n\xi_n) \int_{\mathbb{R}_n^+} f(x) j_{\frac{\gamma-1}{2}}(x_n\xi_n) x_n^\gamma dx \\
&= j_{\frac{\gamma-1}{2}}(y_n\xi_n) F_\gamma(f)(\xi),
\end{aligned}$$

where $T^y(j_p(\sqrt{\lambda}x)) = j_p(\sqrt{\lambda}y)j_p(\sqrt{\lambda}x)$. From formulas (4) and (5), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n_+} F_\gamma |T^y f(x) - f(x)|^q x_n^\gamma dx &= \int_{\mathbb{R}^n_+} |F_\gamma T^y f(x) - F_\gamma f(x)|^q x_n^\gamma dx \\
&= \int_{\mathbb{R}^n_+} |j_{\frac{\gamma-1}{2}}(\xi y) F_\gamma f(\xi) - F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \\
&= \int_{\mathbb{R}^n_+} |F_\gamma f(\xi)[1 - j_{\frac{\gamma-1}{2}}(\xi y)]|^q \xi_n^\gamma d\xi \\
&= \int_{\mathbb{R}^n_+} |1 - j_{\frac{\gamma-1}{2}}(\xi y)|^q |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \\
&\leq C_q \int_{\mathbb{R}^n_+} |T^y f(x) - f(x)|^q \xi_n^\gamma d\xi \\
&\leq C_q \|T^y f(x) - f(x)\|^q_{p,\gamma}.
\end{aligned}$$

From (2), we have

$$\begin{aligned}
\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi &= C_q \int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |1 - j_{\frac{\gamma-1}{2}}(\xi h)|^q |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \\
&\geq C_q |h|^{-1} \int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi,
\end{aligned}$$

$0 < h \leq 1$. It follows from the above consideration that there exists a positive constant C such that

$$\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \leq C\psi^q(h) = C\psi(h^q).$$

Therefore, we get

$$\int_{\tau\leq|\xi|\leq 2\tau} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \leq C\psi(\tau^{-q}).$$

In fact, we have

$$\begin{aligned}\int_{\tau\leq|\xi|<\infty} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi &= \sum_{k=1}^{\infty} \int_{2^{k-1}\tau\leq|\xi|<2^k\tau} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \\ &\leq C_q\psi(\tau^{-q}) + C_q\psi\left((2\tau)^{-q}\right) + C_q\psi\left((2^2\tau)^{-q}\right) + \ldots \\ &\leq C_q\psi(\tau^{-q})\left(1+\psi(2^{-q})+\psi^2(2^{-q})+\psi^3(2^{-q})+\ldots\right).\end{aligned}$$

Thus, we can write

$$\int_{\tau\leq|\xi|<\infty} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \leq C_1\psi(\tau^{-q}),$$

where $C_1 = C_q\left(1-\psi(2^{-q})\right)^{-1}$ since $2^{-q} < 1$. Finally, we get

$$\int_{|\xi|\geq\tau} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi = O(\psi(\tau^{-q})) \quad \text{as} \quad \tau\to\infty.$$

Thus, the proof of theorem is completed. □

We can give the following result which is used for many the theorem given above. It is well known that

$$F_\gamma\left(B_n^{\alpha_n} f\right)(x) = (-x_n^2)^{\alpha_n} F_\gamma f(x), \tag{6}$$

$$F_\gamma\left(D_i^{2\alpha_i} f\right)(x) = (-x_i^2)^{\alpha_i} F_\gamma f(x), \quad i = 1,\ldots,n-1, \tag{7}$$

$$F_\gamma\left(\Delta_\gamma f\right)(x) = -|x|^2 F_\gamma f(x) \quad \text{and} \quad F_\gamma(f\otimes g) = F_\gamma f\, F_\gamma g, \tag{8}$$

$$F_\gamma\left(D_{x'}^{2\alpha'} B_n^{\alpha_n} f\right)(x) = (-1)^{|\alpha|} x^{2\alpha} F_\gamma f(x) \tag{9}$$

We can use the mathematical induction method for $k = 1$, we get

$$\begin{aligned}F_\gamma\big(\Delta_\gamma f\big)(x) &= C_{n,\gamma}\int_{\mathbb{R}_+^n} \Delta_\gamma f(y) e^{-ix'y'} j_{\frac{\gamma-1}{2}}(x_n y_n) y_n^\gamma dy \\ &= C_{n,\gamma}\int_{\mathbb{R}_+^n}\left(\sum_{k=1}^{n}\frac{\partial^2 f(y)}{\partial y_k^2} + \frac{\gamma}{y_n}\frac{\partial f(y)}{\partial y_n}\right) e^{-ix'y'} j_{\frac{\gamma-1}{2}}(x_n y_n) y_n^\gamma dy\end{aligned}$$

$$= C_{n,\gamma} \int_{\mathbb{R}^n_+} \left(\sum_{k=1}^{n} \frac{\partial^2 f(y)}{\partial y_k^2} e^{-ix'y'} j_{\frac{\gamma-1}{2}}(x_n y_n) y_n^\gamma dy \right.$$
$$\left. - C_{n,\gamma} \int_{\mathbb{R}^n_+} \left(\sum_{k=1}^{n} \frac{\gamma}{y_n} \frac{\partial f(y)}{\partial y_n} \right) e^{-ix'y'} j_{\frac{\gamma-1}{2}}(x_n y_n) y_n^\gamma dy = I_1 + I_2. \right.$$

If we apply partial integration to the second term of I_1 and I_2, then we have

$$F_\gamma\big(\Delta_\gamma u\big)(x) = C_{n,\gamma} \int_{\mathbb{R}^n_+} f(y) e^{-ix'y'} \big(\Delta_\gamma j_{\frac{\gamma-1}{2}}(x_n y_n)\big) y_n^\gamma dy.$$

Here, if we use the following equality [8],

$$\int_0^\infty f(y) \Delta_\gamma j_{\frac{\gamma-1}{2}}(xy) y^\gamma dy = -|x|^2 \int_0^\infty f(y) j_{\frac{\gamma-1}{2}}(xy) y^\gamma dy$$

then we have

$$F_\gamma\left(\Delta_\gamma f\right)(x) = -|x|^2 F_\gamma f(x).$$

Since $f \in Lip(\psi, \gamma, p)$, it is clear that

$$||F_\gamma\left(\Delta_\gamma f\right)||_{L_{q,\gamma}(|\xi| \geq \tau)} \leq C_{n,\gamma} O(\psi(\tau^{-q}))$$

as $\tau \to +\infty$.

There are many examples. Here is one of them and a simple method to produce many more: $f(x) = |x|^{\frac{1}{p}}$ for $1 < p < \infty$, where $f(0) = 0$ is understood. These functions are uniformly continuous on all of $\mathbb{R}^n_+$. If $p = 2$, f belongs to the Lipschitz class at $\mathbb{R}_+$.

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WIJSMAN ASYMPTOTICAL $\mathcal{I}_2$-STATISTICALLY EQUIVALENT DOUBLE SET SEQUENCES OF ORDER η

UĞUR ULUSU AND ESRA GÜLLE

Abstract. In this study, we present notions of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence of order η, Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence of order η and Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalence of order η for double set sequences where $0 < \eta \leq 1$. Also, we investigate some properties of these notions and some relationships between them.

1. Introduction

Pringshiem [1] introduced the notion of convergence for double sequences. Then, Mursaleen and Edely [2] studied the notion of statistical convergence. After that, Das et al. [3] studied the notion of $\mathcal{I}$-convergence for double sequences. Recently, Bhunia et al. [4], Çolak and Altın [5], Savaş [6] and Altın et al. [7] presented various type of convergence of order α for double sequences.

Patterson [8] introduced the notion of asymptotical equivalence for double sequences. After that, the notions of asymptotical Cesàro equivalence, asymptotical $\mathcal{I}$-equivalence and asymptotical statistically equivalence for double sequences were studied by Kavita et al. [9], Hazarika and Kumar [10] and Esi and Açıkgöz [11], respectively.

To date, a variety of convergence types for set sequences have been studied by several authors. In this study, the notion of Wijsman convergence which is one of these types is handled (see, [12, 13, 14]). Several authors extended the notion of Wijsman convergence to the new notions for double set sequences via using the notions of statistical convergence, $\mathcal{I}$-convergence and Cesàro summability (see, [15, 16, 17, 18, 19, 20]).

The notions of asymptotical equivalence in Wijsman sense for double set sequences were presented by Nuray et al. [21]. Also, the notions of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence and Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence

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for double set sequences were introduced in [22] and [23], respectively. Lately, new notions of asymptotical equivalence of order α for double set sequences were studied by Gülle [24].

More study on the concepts of convergence or asymptotical equivalence for real sequences or set sequences can be found in [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

2. Definitions and Notations

The fundamental definitions and notations required for this study are following. (see, [1, 3, 8, 12, 13, 14, 21, 22, 23, 25]).

A double sequence (x_{ij}) is convergent to L if for $\varepsilon > 0$, there exists a number $N_\varepsilon \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ for $i, j > N_\varepsilon$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be ideal if

1) $\emptyset \in \mathcal{I}$, 2) For $E, F \in \mathcal{I}$, $E \cup F \in \mathcal{I}$, 3) For $E \in \mathcal{I}$ and $F \subseteq E$, $F \in \mathcal{I}$.

An ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be non trivial if $\mathbb{N} \notin \mathcal{I}$ and a non trivial ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be admissible if $\{j\} \in \mathcal{I}$ for $j \in \mathbb{N}$.

A non trivial ideal $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be strongly admissible if $\{j\} \times \mathbb{N}$ and $\mathbb{N} \times \{j\}$ belong to $\mathcal{I}_2$ for $j \in \mathbb{N}$.

Obviously any strongly admissible ideal is admissible.

Throughout the study, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ will be taken as strongly admissible ideal.

Two non negative double sequences (x_{ij}) and (y_{ij}) are said to be asymptotical equivalent if

$$\lim_{i,j\to\infty} \frac{x_{ij}}{y_{ij}} = 1.$$

Let X be any non empty set. A function $f : \mathbb{N} \to 2^X$ is defined by $f(n) = U_n \in 2^X$ for each $n \in \mathbb{N}$, where 2^X is power set of X. The sequence $\{U_n\} = (U_1, U_2, ...)$, which is the range's elements of f, is said to be set sequences.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non empty subset U of X, distance from x to U is defined by

$$\mu(x, U) = \inf_{u\in U} \rho(x, u).$$

A double sequence $\{U_{ij}\}$ is Wijsman convergent to U if for each $x \in X$,

$$\lim_{i,j\to\infty} \mu(x, U_{ij}) = \mu(x, U).$$

Throughout the study, we will take (X, ρ) as metric space and U_{ij}, V_{ij} as any non empty closed subsets of X.

The term $\mu_x(U_{ij}, V_{ij})$ is defined as follows:

$$\mu_x(U_{ij}, V_{ij}) = \begin{cases} \dfrac{\mu(x, U_{ij})}{\mu(x, V_{ij})} & , \quad x \notin U_{ij} \cup V_{ij} \\ \\ L & , \quad x \in U_{ij} \cup V_{ij}. \end{cases}$$

Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical equivalent if for each $x \in X$,

$$\lim_{i,j\to\infty} \mu_x(U_{ij}, V_{ij}) = 1.$$

Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-equivalent of multiple L if for each $x \in X$ and $\varepsilon > 0$,

$$\Big\{(i,j) \in \mathbb{N}\times\mathbb{N} : |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\Big\} \in \mathcal{I}_2.$$

Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent of multiple L if for each $x \in X$ and ε, $\delta > 0$,

$$\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\Big|\Big\{i \leq m, j \leq n : |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\Big\}\Big| \geq \delta\Big\} \in \mathcal{I}_2.$$

The set of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent double sequences is denoted by $S(\mathcal{I}^L_{W_2})$.

Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent of multiple L if for each $x \in X$ and $\varepsilon > 0$,

$$\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n} |\mu_x(U_{ij}, V_{ij}) - L|^p \geq \varepsilon\Big\} \in \mathcal{I}_2$$

where $0 < p < \infty$.

The set of Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent double sequences is denoted by $C[\mathcal{I}^L_{W_2}]^p$.

3. New Notions

In this section, we present notions of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence of order η, Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence of order η and Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalence of order η for double set sequences.

Definition 1. *Let $0 < \eta \leq 1$. Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple L of order η if for each $x \in X$ and ε, $\delta > 0$,*

$$\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(mn)^\eta}\Big|\Big\{(i,j) : i \leq m, j \leq n,\ |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\Big\}\Big| \geq \delta\Big\} \in \mathcal{I}_2$$

and we write $U_{ij} \overset{\mathcal{I}_2^W(S_L^\eta)}{\sim} V_{ij}$, and simply Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent of order η if $L = 1$.

The class of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple L of order η double sequences will be denoted by $\mathcal{I}_2^W(S_L^\eta)$.

Example 2. *Let $X = \mathbb{R}^2$ and double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ be defined as following:*

$$U_{ij} := \begin{cases} \big\{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - \frac{ij}{2})^2 = \frac{(ij)^2}{4}\big\} & , \text{ if } ij = c^2 \text{ and } c \in \mathbb{N} \\ \big\{(0,1)\big\} & , \text{ if not.} \end{cases}$$

and

$$V_{ij} := \begin{cases} \big\{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 + \frac{ij}{2})^2 = \frac{(ij)^2}{4}\big\} & , \text{ if } ij = c^2 \text{ and } c \in \mathbb{N} \\ \big\{(0,1)\big\} & , \text{ if not.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, ($\mathcal{I}_2^f$ is the class of finite subsets of $\mathbb{N} \times \mathbb{N}$), then the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent of order η.

Remark 3. *For $\eta = 1$, the notion of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence to multiple L of order η coincides with the notion of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence of multiple L for double set sequences in* [22].

Definition 4. *Let $0 < \eta \leq 1$. Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalent to multiple L of order η if for each $x \in X$ and $\varepsilon > 0$,*

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left| \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \mu_x(U_{ij}, V_{ij}) - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2$$

and we write $U_{ij} \overset{\mathcal{I}_2^W(C_L^\eta)}{\sim} V_{ij}$, and simply Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalent of order η if $L = 1$.

Definition 5. *Let $0 < \eta \leq 1$ and $0 < p < \infty$. Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalent to multiple L of order η if for each $x \in X$ and $\varepsilon > 0$,*

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p \geq \varepsilon \right\} \in \mathcal{I}_2$$

and we write $U_{ij} \overset{\mathcal{I}_2^W[C_L^\eta]^p}{\sim} V_{ij}$, and simply Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalent of order η if $L = 1$.

The class of Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalent to multiple L of order η double sequences will be denoted by ${\mathcal{I}_2}^W[C_L^\eta]^p$.

If $p = 1$, then the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalent to multiple L of order η and we write $U_{ij} \overset{\mathcal{I}_2^W[C_L^\eta]}{\sim} V_{ij}$, and simply Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalent of order η if $L = 1$.

Example 6. *Let $X = \mathbb{R}^2$ and double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ be defined as following:*

$$U_{ij} := \begin{cases} \{(x_1, x_2) \in \mathbb{R}^2 : (x_1+2)^2 + x_2^2 = \frac{1}{ij}\} & , \text{ if } ij = c^2 \text{ and } c \in \mathbb{N} \\ \{(-1, 1)\} & , \text{ if not.} \end{cases}$$

and

$$V_{ij} := \begin{cases} \{(x_1, x_2) \in \mathbb{R}^2 : (x_1-2)^2 + x_2^2 = \frac{1}{ij}\} & , \text{ if } ij = c^2 \text{ and } c \in \mathbb{N} \\ \{(-1, 1)\} & , \text{ if not.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, then the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalent of order η.

Remark 7. *For $\eta = 1$, the notions of Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence to multiple L of order η and Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalence to multiple L of order η coincide with the notions of Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence of multiple L and Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalence of multiple L for double set sequences in* [23], *respectively.*

4. Inclusions Theorems

In this section, we investigate some properties of the new asymptotical equivalence notions that introduced in Section 3 and some relationships between them.

Theorem 8. *If $0 < \eta \leq \gamma \leq 1$, then $\mathcal{I}_2^W(S_L^\eta) \subseteq \mathcal{I}_2^W(S_L^\gamma)$.*

Proof. Suppose that $0 < \eta \leq \gamma \leq 1$ and $U_{ij} \overset{\mathcal{I}_2^W(S_L^\eta)}{\sim} V_{ij}$. For each $x \in X$ and $\varepsilon > 0$,

$$\frac{1}{(mn)^\gamma}\Big|\big\{(i,j) : i \leq m, j \leq n, \, |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\big\}\Big|$$
$$\leq \frac{1}{(mn)^\eta}\Big|\big\{(i,j) : i \leq m, j \leq n, \, |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\big\}\Big|$$

and so for $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\gamma}\Big|\big\{(i,j) : i \leq m, j \leq n, \, |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\big\}\Big| \geq \delta\right\}$$
$$\subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\eta}\Big|\big\{(i,j) : i \leq m, j \leq n, \, |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\big\}\Big| \geq \delta\right\}.$$

Consequently, by our assumption, we get $\mathcal{I}_2^W(S_L^\eta) \subseteq \mathcal{I}_2^W(S_L^\gamma)$. □

If we take $\gamma = 1$ in Theorem 8, we obtain the following:

Corollary 9. *If double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple L of order η, then the double sequences are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent of multiple L, i.e., $\mathcal{I}_2^W(S_L^\eta) \subseteq S(\mathcal{I}_{W_2}^L)$.*

Theorem 10. *If $0 < \eta \leq \gamma \leq 1$ and $0 < p < \infty$, then $\mathcal{I}_2^W[C_L^\eta]^p \subseteq \mathcal{I}_2^W[C_L^\gamma]^p$.*

Proof. Suppose that $0 < \eta \leq \gamma \leq 1$ and $U_{ij} \overset{\mathcal{I}_2^W[C_L^\eta]^p}{\sim} V_{ij}$. For each $x \in X$,

$$\frac{1}{(mn)^\gamma} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p \leq \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p$$

and so for $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\gamma} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p \geq \varepsilon\right\}$$
$$\subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p \geq \varepsilon\right\}.$$

Consequently, by our assumption, we get $\mathcal{I}_2^W[C_L^\eta]^p \subseteq \mathcal{I}_2^W[C_L^\gamma]^p$. □

If we take $\gamma = 1$ in Theorem 10, we obtain the following:

Corollary 11. *If double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalent to multiple L of order η, then the double sequences are Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalent of multiple L, i.e., $\mathcal{I}_2^W[C_L^\eta]^p \subseteq C[\mathcal{I}_{W_2}^L]^p$.*

Now, we shall give a theorem that gives a relation between $\mathcal{I}_2^W[C_L^\eta]^p$ and $\mathcal{I}_2^W[C_L^\eta]^q$ where $0 < \eta \leq 1$ and $0 < p < q < \infty$.

Theorem 12. *If $0 < \eta \leq 1$ and $0 < p < q < \infty$, then $\mathcal{I}_2^W[C_L^\eta]^q \subset \mathcal{I}_2^W[C_L^\eta]^p$.*

Proof. Assume that $0 < p < q < \infty$ and $U_{ij} \overset{\mathcal{I}_2^W[C_L^\eta]^q}{\sim} V_{ij}$. For each $x \in X$,

$$\frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p < \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^q$$

and so for $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p \geq \varepsilon\right\}$$
$$\subset \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^q \geq \varepsilon\right\}.$$

Hence, by our assumption, we get $U_{ij} \overset{\mathcal{I}_2^W[C_L^\eta]^p}{\sim} V_{ij}$. Consequently, $\mathcal{I}_2^W[C_L^\eta]^q \subset \mathcal{I}_2^W[C_L^\eta]^p$. □

Theorem 13. *If double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent to multiple L of order η, then the double sequences are Wijsman asymptotical $\mathcal{I}_2$-statistically to multiple L of order γ where $0<\eta\leq\gamma\leq 1$ and $0<p<\infty$.*

Proof. Assume that $0<\eta\leq\gamma\leq 1$ and the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent to multiple L of order η. For each $x\in X$ and $\varepsilon>0$,

$$\begin{aligned}\sum_{i,j=1,1}^{m,n}\left|\mu_x(U_{ij},V_{ij})-L\right|^p &\geq \sum_{\substack{i,j=1,1\\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon}}^{m,n}\left|\mu_x(U_{ij},V_{ij})-L\right|^p\\ &\geq \varepsilon^p\left|\left\{(i,j):\ i\leq m, j\leq n,\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon\right\}\right|\end{aligned}$$

and so

$$\begin{aligned}&\frac{1}{\varepsilon^p\,(mn)^\eta}\sum_{i,j=1,1}^{m,n}\left|\mu_x(U_{ij},V_{ij})-L\right|^p\\ &\qquad\geq\frac{1}{(mn)^\eta}\left|\left\{(i,j):\ i\leq m, j\leq n,\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon\right\}\right|\\ &\qquad\geq\frac{1}{(mn)^\gamma}\left|\left\{(i,j):\ i\leq m, j\leq n,\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon\right\}\right|.\end{aligned}$$

Then for $\delta>0$,

$$\begin{aligned}&\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{(mn)^\gamma}\left|\left\{(i,j):\ i\leq m, j\leq n,\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon\right\}\right|\geq\delta\right\}\\ &\qquad\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{(mn)^\eta}\sum_{i,j=1,1}^{m,n}|\mu_x(U_{ij},V_{ij})-L|^p\geq\varepsilon^p\,\delta\right\}.\end{aligned}$$

Consequently, by our assumption, we get that the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple L of order γ. □

If we take $\gamma=\eta$ in Theorem 13, we obtain the following:

Corollary 14. *If double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent to multiple L of order η, then the double sequences are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple L of order η where $0<\eta\leq 1$ and $0<p<\infty$.*

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GENERALIZED HERMITE-HADAMARD TYPE INEQUALITIES FOR PRODUCTS OF CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper, we think products of two co-ordinated convex functions for the Hermite-Hadamard type inequalities. Using these functions we obtained Hermite-Hadamard type inequalities which are generalizations of some results given in earlier works.

1. Introduction

The following inequality discovered by C. Hermite and J. Hadamard for convex functions is well known in the literature as the Hermite–Hadamard inequality (see, e.g., [13]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$.

Hermite–Hadamard inequality provides a lower and an upper estimation for the integral average of any convex function defined on a compact interval. This inequality has a notable place in mathematical analysis, optimization and so on. However, many studies have been established to demonstrate its new proofs, refinements, extensions and generalizations. A few of these studies are ([4], [9]-[11], [13]-[17], [24]-[27], [29], [34], [35], [37]) referenced works and also the references included there.

On the other hand, Hermite-Hadamard inequality is considered for convex functions on the co-ordinates in [12], [18]. If we look at the convexity of the co-ordinates, there are a lot of definitions of co-ordinated convex function. They may be stated as follows [12]:

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Definition 1. *Let us consider a bidimensional interval* $\Delta := [a,b] \times [c,d]$ *in* $\mathbb{R}^2$ *with* $a < b$ *and* $c < d$. *A function* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *is said to be convex on* Δ *if the following inequality satisfies*

$$f(tx + (1-t)\ z, ty + (1-t)\ w) \leq t\ f(x,y) + (1-t)\ f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ *and* $t \in [0,1]$.

A modification of definition of co-ordinated convex function was defined by Dragomir [12] as follows:

Definition 2. *A function* $f : \Delta \to \mathbb{R}$ *is said to be convex on the co-ordinates on* Δ *if the partial mappings* $f_y : [a,b] \to \mathbb{R}$, $f_y(x) = f(x,y)$ *and* $f_x : [c,d] \to \mathbb{R}$, $f_x(y) = f(x,y)$ *are convex where defined for all* $x \in [a,b]$ *and* $y \in [c,d]$.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 3. *A function* $f : \Delta \to \mathbb{R}$ *is called co-ordinated convex on* Δ *if the following inequality satisfies*

$$\begin{aligned} & f(tx + (1-t)\ y, su + (1-s)\ v) \\ \leq \quad & ts\ f(x,u) + t(1-s)f(x,v) + s(1-t)f(y,u) + (1-t)(1-s)f(y,v) \end{aligned} \tag{1}$$

for all $(x,u),(y,v) \in \Delta$ *and* $t,s \in [0,1]$.

The following Hermite-Hadamard type inequalities for co-ordinated convex functions were obtained by Dragomir in [12]:

Theorem 4. *Suppose that* $f : \Delta \to \mathbb{R}$ *is co-ordinated convex, then we have the following inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ & \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ dydx \qquad (2) \\ & \leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ & \quad \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ & \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \end{aligned}$$

The above inequalities are sharp.

The following Hermite-Hadamard type inequalities for products of two co-ordinated convex functions were given by Latif and Alomari in [18]:

Theorem 5. *Let $f, g : \Delta \rightarrow [0, \infty)$ be co-ordinated convex functions on Δ, then we have the following Hermite-Hadamard type inequalities*

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) g(x,y) dx dy \qquad (3) \\ \leq \quad & \frac{1}{9} K(a,b,c,d) + \frac{1}{18} \left[L(a,b,c,d) + M(a,b,c,d) \right] + \frac{1}{36} N(a,b,c,d) \end{aligned}$$

and

$$\begin{aligned} & 4 f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) g\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \qquad (4) \\ \leq \quad & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) g(x,y) dx dy + \frac{5}{36} K(a,b,c,d) \\ & + \frac{7}{36} \left[L(a,b,c,d) + M(a,b,c,d) \right] + \frac{2}{9} N(a,b,c,d) \end{aligned}$$

where

$$\begin{aligned} K(a,b,c,d) &= f(a,c)g(a,c) + f(b,c)g(b,c) + f(a,d)g(a,d) + f(b,d)g(b,d), \\ L(a,b,c,d) &= f(a,c)g(b,c) + f(b,c)g(a,c) + f(a,d)g(b,d) + f(b,d)g(a,d), \\ M(a,b,c,d) &= f(a,c)g(a,d) + f(b,c)g(b,d) + f(a,d)g(a,c) + f(b,d)g(b,c) \end{aligned}$$

and

$$N(a,b,c,d) = f(a,c)g(b,d) + f(b,c)g(a,d) + f(a,d)g(b,c) + f(b,d)g(a,c).$$

Now, we give the definitions of Riemann-Liouville fractional integrals for two variable functions:

Definition 6. [28] *Let $f \in L_1([a,b] \times [c,d])$. The Riemann-Liouville fractional integrals $J^{\alpha,\beta}_{a+,c+}$, $J^{\alpha,\beta}_{a+,d-}$, $J^{\alpha,\beta}_{b-,c+}$ and $J^{\alpha,\beta}_{b-,d-}$ are defined by*

$$J^{\alpha,\beta}_{a+,c+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t,s) ds dt, \quad x > a, \ y > c,$$

$$J^{\alpha,\beta}_{a+,d-} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t,s) ds dt, \quad x > a, \ y < d,$$

$$J^{\alpha,\beta}_{b-,c+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t,s) ds dt, \quad x < b, \ y > c$$

and

$$J_{b-,d-}^{\alpha,\beta}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt,\quad x<b,\ y<d.$$

The following Hermite-Hadamard type inequality utilizing co-ordinated convex functions was proved by Sarikaya in [28]:

Theorem 7. *Let $f,g:\Delta:=[a,b]\times[c,d]\to[0,\infty)$ be two co-ordinated convex on Δ with $0\le a<b$ and $0\le c<d$ and $f\in L(\Delta)$. Then for $\alpha,\beta>0$ we have the following Hermite-Hadamard type inequality*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (5)\\
\le\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\
& \times\Big[J_{a+,c+}^{\alpha,\beta}f(b,d)g(b,d)+J_{a+,d-}^{\alpha,\beta}f(b,c)g(b,c)\\
& +J_{b-,c+}^{\alpha,\beta}f(a,d)g(a,d)+J_{b-,d-}^{\alpha,\beta}f(a,c)g(a,c)\Big]\\
\le\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

Now, let's give the notations $A_k(x;m,n)$ and $B_k(x;m,n)$ used throughout the study:

$$A_k(x;m,n)=\int_m^n (n-x)^2 w_k(x)dx,\qquad B_k(x;m,n)=\int_m^n (n-x)(x-m)w_k(x)dx$$

for $k=1,2$.

In [7], Budak gave the following inequalities which are used the main results:

Theorem 8. *Suppose that $w_1:[a,b]\to\mathbb{R}$ is non-negative, integrable and symmetric about $x=\frac{a+b}{2}$ (i.e. $w_1(x)=w_1(a+b-x)$). If $f,g:I\to\mathbb{R}$ are two real-valued, non-negative and convex functions on I, then for any $a,b\in I$, we have*

$$\int_a^b f(x)g(x)w_1(x)dx\le\frac{M(a,b)}{(b-a)^2}A_1(x;a,b)+\frac{N(a,b)}{(b-a)^2}B_1(x;a,b)\qquad (6)$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b)\ \text{and}\ N(a,b)=f(a)g(b)+f(b)g(a).$$

Theorem 9. *Suppose that conditions of Theorem 8 hold, then we have the following inequality*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_a^b w_1(x)dx \tag{7}$$

$$\leq \int_a^b f(x)g(x)w_1(x)dx+\frac{M(a,b)}{(b-a)^2}B_1(x;a,b)+\frac{N(a,b)}{(b-a)^2}A_1(x;a,b)$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 8.

Many convexity is defined on co-ordinates and several inequalities are done by using these definitions. For example, Alomari and Darus proved Hadamard type inequalities for the $s-$convex functions and $\log-$convex functions on the co-ordinates in a rectangle from the plane $\mathbb{R}^2$ in [2] and [3] respectively. In [23] Ozdemir *et al.* gave Hadamard type inequalities for $h-$convex functions on the co-ordinates. For the others, please refer to ([1]-[3], [5]-[8], [12], [18]-[23],[28],[30]-[33], [36]).

The aim of this paper is to establish Hermite-Hadamard type inequalities for product of co-ordinated convex functions. The results presented in this paper provide extensions of those given in [6] and [18]

2. Main Results

Theorem 10. *Let $f,g:\Delta\subset\mathbb{R}^2\to[0,\infty)$ be co-ordinated convex functions on Δ. Also, $w_1:[a,b]\to\mathbb{R}$ is non-negative, integrable and symmetric about $x=\frac{a+b}{2}$ (i.e. $w_1(x)=w_1(a+b-x)$) and $w_2:[c,d]\to\mathbb{R}$ is non-negative, integrable and symmetric about $y=\frac{c+d}{2}$ (i.e. $w_2(y)=w_2(c+d-y)$). Then, we have the following Hermite-Hadamard type inequality*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)w_1(x)w_2(y)dydx$$

$$\leq \frac{A_2(y;c,d)}{(b-a)^3(d-c)^3}\left[K(a,b,c,d)A_1(x;a,b)+L(a,b,c,d)B_1(x;a,b)\right]$$

$$+\frac{B_2(y;c,d)}{(b-a)^3(d-c)^3}\left[M(a,b,c,d)A_1(x;a,b)+N(a,b,c,d)B_1(x;a,b)\right]$$

where $K(a,b,c,d)$, $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ defined by as in Theorem 5.

Proof. Since f and g are co-ordinated convex functions on Δ, the functions f_x and g_x are convex on $[c,d]$. If the inequality (6) is applied for the functions f_x and g_x, then we obtain

$$\frac{1}{d-c}\int_{c}^{d} f_x(y)g_x(y)w_2(y)dy \leq \frac{A_2(y;c,d)}{(d-c)^3}\left[f_x(c)g_x(c)+f_x(d)g_x(d)\right] \tag{8}$$

$$+\frac{B_2(y;c,d)}{(d-c)^3}\left[f_x(c)g_x(d)+f_x(d)g_x(c)\right].$$

That is,

$$\frac{1}{d-c}\int_{c}^{d} f(x,y)g(x,y)w_2(y)dy \leq \frac{A_2(y;c,d)}{(d-c)^3}\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right] \tag{9}$$

$$+\frac{B_2(y;c,d)}{(d-c)^3}\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right].$$

Multiplying the inequality (9) by $\frac{w_1(x)}{(b-a)}$ and then integrating respect to x from a to b, we get

$$\frac{1}{(b-a)(d-c)}\int_{a}^{b}\int_{c}^{d} f(x,y)g(x,y)w_1(x)w_2(y)dydx \tag{10}$$

$$\leq \frac{A_2(y;c,d)}{(b-a)(d-c)^3}\int_{a}^{b}\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right]w_1(x)dx$$

$$+\frac{B_2(y;c,d)}{(b-a)(d-c)^3}\int_{a}^{b}\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right]w_1(x)dx.$$

Applying the inequality (6) to each integrals in (10), we have

$$\int_{a}^{b} f(x,c)g(x,c)w_1(x)dx \leq \frac{A_1(x;a,b)}{(b-a)^2}\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right] \tag{11}$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right],$$

$$\int_{a}^{b} f(x,d)g(x,d)w_1(x)dx \leq \frac{A_1(x;a,b)}{(b-a)^2}\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right] \tag{12}$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right],$$

$$\int_a^b f(x,c)g(x,d)w_1(x)dx \leq \frac{A_1(x;a,b)}{(b-a)^2}\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right] \tag{13}$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]$$

and

$$\int_a^b f(x,d)g(x,c)w_1(x)dx \leq \frac{A_1(x;a,b)}{(b-a)^2}\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right] \tag{14}$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right].$$

Substituting the inequalities (11)-(14) in the inequality (10) and then arranging the result obtained, we get desired result. On the other hand, the same result is obtained by using the convexity of functions f_y and g_y. □

Theorem 11. *Let $f,g:\Delta\subset\mathbb{R}^2\to[0,\infty)$ be co-ordinated convex functions on Δ with $a<b, c<d$. Also, $w_1:[a,b]\to\mathbb{R}$ is non-negative, integrable and symmetric about $x=\frac{a+b}{2}$ (i.e. $w_1(x)=w_1(a+b-x)$) and $w_2:[c,d]\to\mathbb{R}$ is non-negative, integrable and symmetric about $y=\frac{c+d}{2}$ (i.e. $w_2(y)=w_2(c+d-y)$). Then, we have the following Hermite-Hadamard type inequality*

$$
\begin{aligned}
&4\int_a^b\int_c^d f\left(\frac{a+b}{2},\frac{c+d}{2}\right) g\left(\frac{a+b}{2},\frac{c+d}{2}\right) w_1(x)w_2(y)dydx \\
\leq\ & \int_a^b\int_c^d f\left(x,y\right)g\left(x,y\right)w_1(x)w_2(y)dydx \\
&+\frac{K(a,b,c,d)}{(b-a)^2(d-c)^2}\left[B_1(x;a,b)A_2(y;c,d)+B_2(y;c,d)A_1(x;a,b)+B_1(x;a,b)B_2(y;c,d)\right] \\
&+\frac{L(a,b,c,d)}{(b-a)^2(d-c)^2}\left[B_2(y;c,d)B_1(x;a,b)+A_2(y;c,d)A_1(x;a,b)+A_1(x;a,b)B_2(y;c,d)\right] \\
&+\frac{M(a,b,c,d)}{(b-a)^2(d-c)^2}\left[B_2(y;c,d)B_1(x;a,b)+A_2(y;c,d)A_1(x;a,b)+B_1(x;a,b)A_2(y;c,d)\right]
\end{aligned}
$$

$$+\frac{N(a,b,c,d)}{(b-a)^2(d-c)^2}\left[A_1(x;a,b)B_2(y;c,d)+A_2(y;c,d)B_1(x;a,b)+A_2(y;c,d)A_1(x;a,b)\right].$$

Proof. Since f and g are co-ordinated convex functions on Δ, the functions f_x, g_x, f_y and g_y are convex. Applying the inequality (7) for the functions $f_{\frac{c+d}{2}}$ and $g_{\frac{c+d}{2}}$ with $y=\frac{c+d}{2}$ and then multiplying both sides of the result obtained by $2\int_c^d w_2(y)dy$, we get

$$4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d w_1(x)w_2(y)dydx \tag{15}$$
$$\leq 2\int_a^b\int_c^d f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)w_1(x)w_2(y)dydx$$
$$+\left\{2\int_c^d\left[\frac{f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)}{(b-a)^2}\right]w_2(y)dy\right\}B_1(x;a,b)$$
$$+\left\{2\int_c^d\left[\frac{f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)}{(b-a)^2}\right]w_2(y)dy\right\}A_1(x;a,b).$$

Similarly, if we apply the inequality (7) for the functions $f_{\frac{a+b}{2}}$ and $g_{\frac{a+b}{2}}$ with $x=\frac{a+b}{2}$ and then multiply both sides of the result obtained by $2\int_a^b w_1(x)dx$, we get

$$4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d w_1(x)w_2(y)dydx \tag{16}$$
$$\leq 2\int_a^b\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)w_1(x)w_2(y)dydx$$
$$+\left\{2\int_a^b\left[\frac{f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)}{(d-c)^2}\right]w_1(x)dx\right\}B_2(y;c,d)$$
$$+\left\{2\int_a^b\left[\frac{f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)}{(d-c)^2}\right]w_1(x)dx\right\}A_2(y;c,d).$$

Using the inequality (7) for each integrals in inequalities (15) and (16), we have

$$2f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\int\limits_c^d w_2(y)dy \leq \int\limits_c^d f(a,y)g(a,y)w_2(y)dy \tag{17}$$

$$+\left[\frac{f(a,c)g(a,c)+f(a,d)g(a,d)}{(d-c)^2}\right]B_1(y;c,d)$$

$$+\left[\frac{f(a,c)g(a,d)+f(a,d)g(a,c)}{(d-c)^2}\right]A_1(y;c,d),$$

$$2f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\int\limits_c^d w_2(y)dy \leq \int\limits_c^d f(b,y)g(b,y)w_2(y)dy \tag{18}$$

$$+\left[\frac{f(b,c)g(b,c)+f(b,d)g(b,d)}{(d-c)^2}\right]B_1(y;c,d)$$

$$+\left[\frac{f(b,c)g(b,d)+f(b,d)g(b,c)}{(d-c)^2}\right]A_1(y;c,d),$$

$$2f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\int\limits_c^d w_2(y)dy \leq \int\limits_c^d f(a,y)g(b,y)w_2(y)dy \tag{19}$$

$$+\left[\frac{f(a,c)g(b,c)+f(a,d)g(b,d)}{(d-c)^2}\right]B_1(y;c,d)$$

$$+\left[\frac{f(a,c)g(b,d)+f(a,d)g(b,c)}{(d-c)^2}\right]A_1(y;c,d),$$

$$2f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\int\limits_c^d w_2(y)dy \leq \int\limits_c^d f(b,y)g(a,y)w_2(y)dy \tag{20}$$
$$+\left[\frac{f(b,c)g(a,c)+f(b,d)g(a,d)}{(d-c)^2}\right]B_1(y;c,d)$$
$$+\left[\frac{f(b,c)g(a,d)+f(b,d)g(a,c)}{(d-c)^2}\right]A_1(y;c,d),$$

$$2f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)\int\limits_a^b w_1(x)dx \leq \int\limits_a^b f(x,c)g(x,c)w_1(x)dx \tag{21}$$
$$+\left[\frac{f(a,c)g(a,c)+f(b,c)g(b,c)}{(b-a)^2}\right]B_1(x;a,b)$$
$$+\left[\frac{f(a,c)g(b,c)+f(b,c)g(a,c)}{(b-a)^2}\right]A_1(x;a,b),$$

$$2f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\int\limits_a^b w_1(x)dx \leq \int\limits_a^b f(x,d)g(x,d)w_1(x)dx \tag{22}$$
$$+\left[\frac{f(a,d)g(a,d)+f(b,d)g(b,d)}{(b-a)^2}\right]B_1(x;a,b)$$
$$+\left[\frac{f(a,d)g(b,d)+f(b,d)g(a,d)}{(b-a)^2}\right]A_1(x;a,b),$$

$$2f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\int\limits_a^b w_1(x)dx \le \int\limits_a^b f(x,d)g(x,d)w_1(x)dx \tag{23}$$

$$+\left[\frac{f(a,d)g(a,d)+f(b,d)g(b,d)}{(b-a)^2}\right]B_1(x;a,b)$$

$$+\left[\frac{f(a,d)g(b,d)+f(b,d)g(a,d)}{(b-a)^2}\right]A_1(x;a,b),$$

$$2f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)\int\limits_a^b w_1(x)dx \le \int\limits_a^b f(x,c)g(x,d)w_1(x)dx \tag{24}$$

$$+\left[\frac{f(a,c)g(a,d)+f(b,c)g(b,d)}{(b-a)^2}\right]B_1(x;a,b)$$

$$+\left[\frac{f(a,c)g(b,d)+f(b,c)g(a,d)}{(b-a)^2}\right]A_1(x;a,b)$$

and

$$2f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\int\limits_a^b w_1(x)dx \le \int\limits_a^b f(x,d)g(x,c)w_1(x)dx \tag{25}$$

$$+\left[\frac{f(a,d)g(a,c)+f(b,d)g(b,c)}{(b-a)^2}\right]B_1(x;a,b)$$

$$+\left[\frac{f(a,d)g(b,c)+f(b,d)g(a,c)}{(b-a)^2}\right]A_1(x;a,b).$$

When the inequalities (17)-(25) is written in (15) and (16) and then the results obtained are added side by side and rearranged, we obtain

$$\begin{aligned}
& 8f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d w_1(x)w_2(y)dydx \qquad (26) \\
\leq\; & 2\int_a^b\int_c^d f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)w_1(x)w_2(y)dydx \\
& +2\int_a^b\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)w_1(x)w_2(y)dydx \\
& +\frac{B_1(x;a,b)}{(b-a)^2}\int_c^d\left[f(a,y)g(a,y)+f(b,y)g(b,y)\right]w_2(y)dy \\
& +\frac{A_1(x;a,b)}{(b-a)^2}\int_c^d\left[f(a,y)g(b,y)+f(b,y)g(a,y)\right]w_2(y)dy \\
& +\frac{B_2(y;c,d)}{(d-c)^2}\int_a^b\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right]w_1(x)dx \\
& +\frac{A_2(y;c,d)}{(d-c)^2}\int_a^b\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right]w_1(x)dx \\
& +\frac{2K(a,b,c,d)}{(b-a)^2(d-c)^2}B_1(x;a,b)B_2(y;c,d) \\
& +\frac{2L(a,b,c,d)}{(b-a)^2(d-c)^2}A_1(x;a,b)B_2(y;c,d) \\
& +\frac{2M(a,b,c,d)}{(b-a)^2(d-c)^2}B_1(x;a,b)A_2(y;c,d) \\
& +\frac{2N(a,b,c,d)}{(b-a)^2(d-c)^2}A_1(x;a,b)A_2(y;c,d).
\end{aligned}$$

The inequality (7) is applied to $f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)$ and then the result is multiplied by $w_1(x)$ and integrated over $[a,b]$,we get

$$2\int_a^b\int_c^d f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)w_1(x)w_2(y)dydx \qquad (27)$$

$$\leq \int_a^b \int_c^d f(x,y)g(x,y)w_1(x)w_2(y)dydx$$

$$+\frac{B_2(y;c,d)}{(d-c)^2}\int_a^b [f(x,c)g(x,c)+f(x,d)g(x,d)]\,w_1(x)dx$$

$$+\frac{A_2(y;c,d)}{(d-c)^2}\int_a^b [f(x,c)g(x,d)+f(x,d)g(x,c)]\,w_1(x)dx.$$

Similarly, if we apply the inequality (7) to $f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)$ and then the result is multiplied by $w_2(y)$ and integrated over $[c,d]$,we get

$$2\int_a^b \int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)w_1(x)w_2(y)dydx \tag{28}$$

$$\leq \int_a^b \int_c^d f(x,y)g(x,y)w_1(x)w_2(y)dydx$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\int_c^d [f(a,y)g(a,y)+f(b,y)g(b,y)]\,w_2(y)dy$$

$$+\frac{A_1(x;a,b)}{(b-a)^2}\int_c^d [f(a,y)g(b,y)+f(b,y)g(a,y)]\,w_2(y)dy.$$

Substituting the inequalities (27) and (28) in the inequality (26) and reordering the results obtained, we have

$$8f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b \int_c^d w_1(x)w_2(y)dydx \tag{29}$$

$$\leq 2\int_a^b \int_c^d f(x,y)g(x,y)w_1(x)w_2(y)dydx$$

$$+\frac{2B_1(x;a,b)}{(b-a)^2}\int_c^d [f(a,y)g(a,y)+f(b,y)g(b,y)]\,w_2(y)dy$$

$$
\begin{aligned}
&+\frac{2A_1(x;a,b)}{(b-a)^2}\int\limits_c^d [f(a,y)g(b,y)+f(b,y)g(a,y)]\,w_2(y)dy \\
&+\frac{2B_2(y;c,d)}{(d-c)^2}\int\limits_a^b [f(x,c)g(x,c)+f(x,d)g(x,d)]\,w_1(x)dx \\
&+\frac{2A_2(y;c,d)}{(d-c)^2}\int\limits_a^b [f(x,c)g(x,d)+f(x,d)g(x,c)]\,w_1(x)dx \\
&+\frac{2K(a,b,c,d)}{(b-a)^2(d-c)^2}B_1(x;a,b)B_2(y;c,d) \\
&+\frac{2L(a,b,c,d)}{(b-a)^2(d-c)^2}A_1(x;a,b)B_2(y;c,d) \\
&+\frac{2M(a,b,c,d)}{(b-a)^2(d-c)^2}B_1(x;a,b)A_2(y;c,d) \\
&+\frac{2N(a,b,c,d)}{(b-a)^2(d-c)^2}A_1(x;a,b)A_2(y;c,d).
\end{aligned}
$$

By applying the inequality (6) to each integral in (29) and later rearranging the results obtained, we obtain desired inequality. □

Remark 12. *If we choose $w_1(x)=1$ and $w_2(y)=1$ in Theorem 10 and Theorem 11, we get (3) and (4) respectively.*

Remark 13. *If we choose $w_1(x)=\frac{\alpha}{(b-a)^{\alpha-1}}\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]$ with $\alpha>0$ and $w_2(y)=\frac{\beta}{(d-c)^{\beta-1}}\left[(d-y)^{\beta-1}+(y-c)^{\beta-1}\right]$ with $\beta>0$ in Theorem 10 and Theorem 11, we get*

$$
\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\
&\times\Big[J^{\alpha,\beta}_{a+,c+}f(b,d)g(b,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)g(b,c) \\
&\quad +J^{\alpha,\beta}_{b-,c+}f(a,d)g(a,d)+J^{\alpha,\beta}_{b-,d-}f(a,c)g(a,c)\Big] \\
\leq\ &\left[\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]K(a,b,c,d)
\end{aligned}
$$

$$+\left[\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]L(a,b,c,d)$$
$$+\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]M(a,b,c,d)$$
$$+\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]N(a,b,c,d)$$

and

$$\begin{aligned}
&4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\ &\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\
&\times\Big[J^{\alpha,\beta}_{a+,c+}f(b,d)g(b,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)g(b,c)\\
&\quad+J^{\alpha,\beta}_{b-,c+}f(a,d)g(a,d)+J^{\alpha,\beta}_{b-,d-}f(a,c)g(a,c)\Big]\\
&+\left\{\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\right\}K(a,b,c,d)\\
&+\left\{\frac{1}{2}\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]+\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\right\}L(a,b,c,d)\\
&+\left\{\frac{1}{2}\left[\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right]+\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\right\}M(a,b,c,d)\\
&+\left\{\frac{1}{4}-\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\right\}N(a,b,c,d)
\end{aligned}$$

which is proved by Budak and Sarikaya [6].

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A NOTE ON HYPERBOLIC (p,q)-FIBONACCI QUATERNIONS

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Abstract. In this paper, we introduce a new quaternion sequence called hyperbolic (p,q)-Fibonacci quaternions. This new quaternion sequence includes hyperbolic Fibonacci, hyperbolic k-Fibonacci, hyperbolic Pell, hyperbolic k-Pell, hyperbolic Jacobsthal, hyperbolic k-Jacobsthal quaternions. We give generating function and Binet's formula for these quaternions. We also obtain some identities such as d'Ocagne's, Catalan's and Cassini's identities involving hyperbolic (p,q)-Fibonacci quaternions.

1. Introduction

Fibonacci numbers have been applied in different scientific areas such as engineering, and architecture. Recently, Fibonacci numbers have been studied and generalized by many authors in many ways. For example, one of the generalization of Fibonacci numbers is (p,q)-Fibonacci numbers [15,17].

For positive real numbers p and q, the sequence of (p,q)-Fibonacci numbers, denoted by $\{\mathcal{F}_n\}_{n\geq 0}$, is defined by the recurrence relation

$$\mathcal{F}_n = p\mathcal{F}_{n-1} + q\mathcal{F}_{n-2}, \quad n \geq 2$$

with initial conditions $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$ [17].

The nth term of the sequence $\{\mathcal{F}_n\}_{n\geq 0}$ is given by

$$\mathcal{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1}$$

where $\alpha = \dfrac{p+\sqrt{p^2+4q}}{2}$, $\beta = \dfrac{p-\sqrt{p^2+4q}}{2}$ are the roots of the characteristic equation $t^2 - pt - q = 0$ [17].

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It must be note that $\alpha+\beta=p$, $\alpha-\beta=\sqrt{p^2+4q}$ and $\alpha\beta=-q$. Moreover, the generating function for the sequence $\{\mathcal{F}_n\}_{n\geq 0}$ [29] is given by

$$f_{p,q}(t)=\frac{t}{1-pt-qt^2}.$$

The (p,q)-Fibonacci sequence is the generalization of the familiar second-order recurrent sequences, that is, for special values of p and q, are defined as follows:

- If $p=q=1$, then $\{\mathcal{F}_n\}_{n\geq 0}$ is the (classical) Fibonacci sequence $\{F_n\}_{n\geq 0}$ [24].
- If $p=k$, $q=1$, then $\{\mathcal{F}_n\}_{n\geq 0}$ is the k-Fibonacci sequence $\{F_{k,n}\}_{n\geq 0}$ [9].
- If $p=2$, $q=1$, then $\{\mathcal{F}_n\}_{n\geq 0}$ is the Pell sequence $\{P_n\}_{n\geq 0}$ [18].
- If $p=2$, $q=k$ then $\{\mathcal{F}_n\}_{n\geq 0}$ is the k-Pell sequence $\{P_{k,n}\}_{n\geq 0}$ [7].
- If $p=1$, $q=2$, then $\{\mathcal{F}_n\}_{n\geq 0}$ is the Jacobsthal sequence $\{J_n\}_{n\geq 0}$ [19].
- If $p=k$, $q=2$, then $\{\mathcal{F}_n\}_{n\geq 0}$ is the k-Jacobsthal sequence $\{J_{k,n}\}_{n\geq 0}$ [22].

Quaternions (real quaternions), introduced by Sir William Rowan Hamilton in the mid nineteenth century, are four-dimensional hypercomplex numbers. Quaternions are widely used in high-tech areas such as computer graphics, signal processing, and robotics, see for example [1,8,10,11], among others.

Quaternions form a four-dimensional non-commutative associative algebra over the real numbers, are defined as follows:

$$\mathbf{H}=\{q=q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}\quad|\quad q_0,q_1,q_2,q_3\in\mathbb{R}\},$$

where $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ is a basis of $\mathbf{H}$, and the imaginary units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy the following equalities

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1,\quad \mathbf{ij}=\mathbf{k}=-\mathbf{ji},\quad \mathbf{jk}=\mathbf{i}=-\mathbf{kj},\quad \mathbf{ki}=\mathbf{j}=-\mathbf{ik}.$$

For more details on quaternions, one can see, for example [14,32].

Horadam [16] defined the Fibonacci quaternions as

$$QF_n=F_n+F_{n+1}\mathbf{i}+F_{n+2}\mathbf{j}+F_{n+3}\mathbf{k},$$

where F_n is the nth Fibonacci number.

Fibonacci quaternions have been studied and generalized by many authors, some of which can be found in [2-6,12,13,20,21,26-28,30,31], among others. One of the generalization for Fibonacci quaternions is done by Ipek. In [20], Ipek introduced the (p,q)-Fibonacci quaternions as

$$Q\mathcal{F}_n=\mathcal{F}_n+\mathcal{F}_{n+1}\mathbf{i}+\mathcal{F}_{n+2}\mathbf{j}+\mathcal{F}_{n+3}\mathbf{k},$$

where $\mathcal{F}_n$ is the nth (p,q)-Fibonacci number.

The author also defined the (p,q)-Fibonacci quaternions recursively by the relation

$$Q\mathcal{F}_n=pQ\mathcal{F}_{n-1}+qQ\mathcal{F}_{n-2},\quad n\geq 2.$$

Moreover, Patel and Ray [26] investigated some properties of (p,q)-Fibonacci and (p,q)-Lucas quaternions.

Alexander Mac-Farlane first described hyperbolic quaternions in 1891, and these numbers are not associative. Kurt Godel used the name of these quaternions in 1949, but the author actually implied split quaternions in his definition. Hyperbolic quaternions [25], just like real quaternions, are a generalization of complex numbers by four real numbers. Moreover, just like real quaternions, hyperbolic quaternions are not commutative. But hyperbolic quaternions have zero divisors.

In [23], Kosal studied on hyperbolic quaternions and their algebraic properties. In [5], Aydin defined the hyperbolic k-Fibonacci quaternions. The author also investigated some algebraic properties of the hyperbolic k-Fibonacci quaternions.

Hyperbolic quaternions are defined as

$$\mathbf{K} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \quad | \quad q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1, \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

Let $p = p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3$ and $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ be two hyperbolic quaternions. Then the addition and subtraction of two hyperbolic quaternions are defined as

$$p \pm q = (p_0 \pm q_0) + \mathbf{i}(p_1 \pm q_1) + \mathbf{j}(p_2 \pm q_2) + \mathbf{k}(p_3 \pm q_3).$$

The multiplication of a hyperbolic quaternion by a real scalar λ is defined as

$$\lambda p = \lambda p_0 + \mathbf{i}\lambda p_1 + \mathbf{j}\lambda p_2 + \mathbf{k}\lambda p_3.$$

The multiplication of two hyperbolic quaternions is defined as

$$\begin{aligned} pq = {} & (p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3) + \mathbf{i}(p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2) \\ & + \mathbf{j}(p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1) + \mathbf{k}(p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0). \end{aligned}$$

The conjugate of a hyperbolic quaternion q is denoted by $\overline{q}$ and defined by

$$\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3.$$

Moreover, the norm of the hyperbolic quaternion q is

$$N(q) = q\overline{q} = {q_0}^2 - {q_1}^2 - {q_2}^2 - {q_3}^2.$$

The main objective of this paper is to introduce hyperbolic (p,q)-Fibonacci quaternions. We then give the generating function and Binet's formula for the hyperbolic (p,q)-Fibonacci quaternions. In addition, we obtain some well-known identities involving these quaternions.

2. The Hyperbolic (p,q)-Fibonacci Quaternions

In this section, we first give the definition of the hyperbolic (p,q)-Fibonacci quaternions. We then investigate some properties of these quaternions.

Definition 1. *For positive real numbers p and q, hyperbolic (p,q)-Fibonacci quaternions are defined by the relation*

$$HQ\mathcal{F}_n = \mathcal{F}_n + \mathbf{i}\mathcal{F}_{n+1} + \mathbf{j}\mathcal{F}_{n+2} + \mathbf{k}\mathcal{F}_{n+3}, \tag{2}$$

where $\mathcal{F}_n$ is the nth (p,q)-Fibonacci number, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *satisfy the equalities*

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1, \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}. \tag{3}$$

Let $HQ\mathcal{F}_n$ be the nth (p,q)-Fibonacci number. Then, after some necessary calculations, one can obtain the following recurrence relation:

$$HQ\mathcal{F}_n = pHQ\mathcal{F}_{n-1} + qHQ\mathcal{F}_{n-2}, \quad n \geq 2, \tag{4}$$

with initial conditions

$$HQ\mathcal{F}_0 = \mathbf{i} + \mathbf{j}p + \mathbf{k}(p^2 + q), \tag{5}$$

$$HQ\mathcal{F}_1 = 1 + \mathbf{i}p + \mathbf{j}(p^2 + q) + \mathbf{k}(p^3 + 2pq). \tag{6}$$

Particular cases of Definition 1 are

- Hyperbolic Fibonacci quaternions are

$$HQF_n = F_n + \mathbf{i}F_{n+1} + \mathbf{j}F_{n+2} + \mathbf{k}F_{n+3},$$

where F_n is the nth Fibonacci number, with initial conditions

$$HQF_0 = \mathbf{i} + \mathbf{j} + \mathbf{k}2,$$
$$HQF_1 = 1 + \mathbf{i} + \mathbf{j}2 + \mathbf{k}3.$$

- Hyperbolic k-Fibonacci quaternions [5] are

$$HQF_{k,n} = F_{k,n} + \mathbf{i}F_{k,n+1} + \mathbf{j}F_{k,n+2} + \mathbf{k}F_{k,n+3},$$

where $F_{k,n}$ is the nth k-Fibonacci number, with initial conditions

$$HQF_{k,0} = \mathbf{i} + \mathbf{j}k + \mathbf{k}(k^2 + 1),$$
$$HQF_{k,1} = 1 + \mathbf{i}k + \mathbf{j}(k^2 + 1) + \mathbf{k}(k^3 + 2k).$$

- Hyperbolic Pell quaternions are

$$HQP_n = P_n + \mathbf{i}P_{n+1} + \mathbf{j}P_{n+2} + \mathbf{k}P_{n+3},$$

where P_n is the nth Pell number, with initial conditions

$$HQP_0 = \mathbf{i} + 2\mathbf{j} + \mathbf{k}5,$$
$$HQP_1 = 1 + \mathbf{i}2 + \mathbf{j}5 + \mathbf{k}12.$$

- Hyperbolic k-Pell quaternions are

$$HQP_{k,n} = P_{k,n} + \mathbf{i}P_{k,n+1} + \mathbf{j}P_{k,n+2} + \mathbf{k}P_{k,n+3},$$

where $P_{k,n}$ is the nth k-Pell number, with initial conditions

$$\begin{aligned} HQP_{k,0} &= \mathbf{i} + \mathbf{j}2 + \mathbf{k}(4+k), \\ HQP_{k,1} &= 1 + \mathbf{i}2 + \mathbf{j}(4+k) + \mathbf{k}(8+4k). \end{aligned}$$

- Hyperbolic Jacobsthal quaternions are

$$HQJ_n = J_n + \mathbf{i}J_{n+1} + \mathbf{j}J_{n+2} + \mathbf{k}J_{n+3},$$

where J_n is the nth Jacobsthal number, with initial conditions

$$\begin{aligned} HQJ_0 &= \mathbf{i} + \mathbf{j} + \mathbf{k}3, \\ HQJ_1 &= 1 + \mathbf{i} + \mathbf{j}3 + \mathbf{k}5. \end{aligned}$$

- Hyperbolic k-Jacobsthal quaternions are

$$HQJ_{k,n} = J_{k,n} + \mathbf{i}J_{k,n+1} + \mathbf{j}J_{k,n+2} + \mathbf{k}J_{k,n+3},$$

where $J_{k,n}$ is the nth k-Jacobsthal number, with initial conditions

$$\begin{aligned} HQJ_{k,0} &= \mathbf{i} + \mathbf{j}k + \mathbf{k}(k^2+2), \\ HQJ_{k,1} &= 1 + \mathbf{i}k + \mathbf{j}(k^2+2) + \mathbf{k}(k^3+4k). \end{aligned}$$

Let $HQ\mathcal{F}_n = \mathcal{F}_n + \mathbf{i}\mathcal{F}_{n+1} + \mathbf{j}\mathcal{F}_{n+2} + \mathbf{k}\mathcal{F}_{n+3}$ and $HQ\mathcal{F}_m = \mathcal{F}_m + \mathbf{i}\mathcal{F}_{m+1} + \mathbf{j}\mathcal{F}_{m+2} + \mathbf{k}\mathcal{F}_{m+3}$ be two hyperbolic (p,q)-Fibonacci quaternions. Then the addition and subtraction of two hyperbolic (p,q)-Fibonacci quaternions are defined by

$$\begin{aligned} HQ\mathcal{F}_n \pm HQ\mathcal{F}_m = (\mathcal{F}_n \pm \mathcal{F}_m) &+ \mathbf{i}(\mathcal{F}_{n+1} \pm \mathcal{F}_{m+1}) + \mathbf{j}(\mathcal{F}_{n+2} \pm \mathcal{F}_{m+2}) \\ &+ \mathbf{k}(\mathcal{F}_{n+3} \pm \mathcal{F}_{m+3}). \end{aligned} \tag{7}$$

The multiplication of a hyperbolic (p,q)-Fibonacci quaternion by a real scalar λ is defined by

$$\lambda HQ\mathcal{F}_n = \lambda\mathcal{F}_n + \mathbf{i}\lambda\mathcal{F}_{n+1} + \mathbf{j}\lambda\mathcal{F}_{n+2} + \mathbf{k}\lambda\mathcal{F}_{n+3}. \tag{8}$$

The multiplication of two hyperbolic (p,q)-Fibonacci quaternions is defined by

$$\begin{aligned} &HQ\mathcal{F}_n \times HQ\mathcal{F}_m \\ &\qquad = (\mathcal{F}_n\mathcal{F}_m + \mathcal{F}_{n+1}\mathcal{F}_{m+1} + \mathcal{F}_{n+2}\mathcal{F}_{m+2} + \mathcal{F}_{n+3}\mathcal{F}_{m+3}) \\ &\qquad\quad + \mathbf{i}(\mathcal{F}_n\mathcal{F}_{m+1} + \mathcal{F}_{n+1}\mathcal{F}_m + \mathcal{F}_{n+2}\mathcal{F}_{m+3} - \mathcal{F}_{n+3}\mathcal{F}_{m+2}) \\ &\qquad\quad + \mathbf{j}(\mathcal{F}_n\mathcal{F}_{m+2} - \mathcal{F}_{n+1}\mathcal{F}_{m+3} + \mathcal{F}_{n+2}\mathcal{F}_m + \mathcal{F}_{n+3}\mathcal{F}_{m+1}) \\ &\qquad\quad + \mathbf{k}(\mathcal{F}_n\mathcal{F}_{m+3} + \mathcal{F}_{n+1}\mathcal{F}_{m+2} - \mathcal{F}_{n+2}\mathcal{F}_{m+1} + \mathcal{F}_{n+3}\mathcal{F}_m). \end{aligned} \tag{9}$$

The generating function for the hyperbolic (p,q)-Fibonacci quaternions is given in the following theorem.

Theorem 2. *The generating function for the hyperbolic (p,q)-Fibonacci quaternions is given by*

$$G_{p,q}(t) = \frac{t + \mathbf{i}(1+t-pt) + \mathbf{j}(1+2t-pt) + \mathbf{k}(2+3t-2pt)}{1-pt-qt^2}.$$

Proof. Let $G_{p,q}(t)$ be the generating function for the hyperbolic (p,q)-Fibonacci quaternions. Then we write

$$G_{p,q}(t) = \sum_{n=0}^{\infty} HQ\mathcal{F}_n t^n = HQ\mathcal{F}_0 + HQ\mathcal{F}_1 t + ... + HQ\mathcal{F}_n t^n + ... \quad . \tag{10}$$

Multiplying the Eq. (10) with pt and qt^2 respectively, we get

$$ptG_{p,q}(t) = pHQ\mathcal{F}_0 t + pHQ\mathcal{F}_1 t^2 + ... + pHQ\mathcal{F}_{n-1} t^n + ...$$

and

$$qt^2 G_{p,q}(t) = qHQ\mathcal{F}_0 t^2 + qHQ\mathcal{F}_1 t^3 + ... + qHQ\mathcal{F}_{n-2} t^n + ... \quad .$$

Then we have

$$\begin{aligned}(1-pt-qt^2)G_{p,q}(t) &= HQ\mathcal{F}_0 + (HQ\mathcal{F}_1 - pHQ\mathcal{F}_0)t \\ &\quad + \sum_{n=2}^{\infty}(HQ\mathcal{F}_n - pHQ\mathcal{F}_{n-1} - qHQ\mathcal{F}_{n-2})t^n \\ &= HQ\mathcal{F}_0 + (HQ\mathcal{F}_1 - pHQ\mathcal{F}_0)t.\end{aligned}$$

By the Eqs. (5) and (6), we get

$$(1-pt-qt^2)G_{p,q}(t) = t + \mathbf{i}(1+t-pt) + \mathbf{j}(1+2t-pt) + \mathbf{k}(2+3t-2pt)$$

which is the desired result. □

Particular cases of Theorem 2 are

- The generating function of the hyperbolic (classical) Fibonacci quaternions is
$$f(t) = \frac{t + \mathbf{i} + \mathbf{j}(1+t) + \mathbf{k}(2+t)}{1-t-t^2}.$$
- The generating function of the hyperbolic k-Fibonacci quaternions is
$$f_k(t) = \frac{t + \mathbf{i}(1+t(1-k)) + \mathbf{j}(1+t(2-k)) + \mathbf{k}(2+t(3-2k))}{1-kt-t^2}.$$
- The generating function of the hyperbolic Pell quaternions is
$$g(t) = \frac{t + \mathbf{i}(1-t) + \mathbf{j} + \mathbf{k}(2-t)}{1-2t-t^2}.$$
- The generating function of the hyperbolic k-Pell quaternions is
$$g_k(t) = \frac{t + \mathbf{i}(1-t) + \mathbf{j} + \mathbf{k}(2-t)}{1-2t-kt^2}.$$

- The generating function of the hyperbolic Jacobsthal quaternions is

$$h(t)=\frac{t+\mathbf{i}+\mathbf{j}(1+t)+\mathbf{k}(2+t)}{1-t-2t^2}.$$

- The generating function of the hyperbolic k-Jacobsthal quaternions is

$$h_k(t)=\frac{t+\mathbf{i}(1+t(1-k))+\mathbf{j}(1+t(2-k))+\mathbf{k}(2+t(3-2k))}{1-kt-2t^2}.$$

The following theorem gives the Binet's formula for the hyperbolic (p,q)-Fibonacci quaternions.

Theorem 3. *The nth term of the hyperbolic (p,q)-Fibonacci quaternion is given by*

$$HQ\mathcal{F}_n=\frac{\alpha^*\alpha^n-\beta^*\beta^n}{\alpha-\beta},$$

where $\alpha^*=1+\mathbf{i}\alpha+\mathbf{j}\alpha^2+\mathbf{k}\alpha^3$, $\alpha=\frac{p+\sqrt{p^2+4q}}{2}$ *and* $\beta^*=1+\mathbf{i}\beta+\mathbf{j}\beta^2+\mathbf{k}\beta^3$, $\beta=\frac{p-\sqrt{p^2+4q}}{2}$.

Proof. Using the definition of the hyperbolic (p,q)-Fibonacci quaternions and the Binet's formula of the (p,q)-Fibonacci numbers, we have

$$\begin{aligned}HQ\mathcal{F}_n&=\mathcal{F}_n+\mathbf{i}\mathcal{F}_{n+1}+\mathbf{j}\mathcal{F}_{n+2}+\mathbf{k}\mathcal{F}_{n+3}\\&=\frac{\alpha^n-\beta^n}{\alpha-\beta}+\mathbf{i}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}+\mathbf{j}\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}+\mathbf{k}\frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}\\&=\frac{\alpha^n(1+\mathbf{i}\alpha+\mathbf{j}\alpha^2+\mathbf{k}\alpha^3)-\beta^n(1+\mathbf{i}\beta+\mathbf{j}\beta^2+\mathbf{k}\beta^3)}{\alpha-\beta}.\end{aligned}$$

If we take $\alpha^*=1+\mathbf{i}\alpha+\mathbf{j}\alpha^2+\mathbf{k}\alpha^3$ and $\beta^*=1+\mathbf{i}\beta+\mathbf{j}\beta^2+\mathbf{k}\beta^3$, we obtain the desired result. □

Particular cases of Therorem 3 are

- The Binet's formula of the nth hyperbolic (classical) Fibonacci quaternion is

$$HQF_n=\frac{1}{\sqrt{5}}(\alpha^*\alpha^n-\beta^*\beta^n),$$

  where $\alpha^*=1+\mathbf{i}\alpha+\mathbf{j}\alpha^2+\mathbf{k}\alpha^3$, $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta^*=1+\mathbf{i}\beta+\mathbf{j}\beta^2+\mathbf{k}\beta^3$, $\beta=\frac{1-\sqrt{5}}{2}$.
- The Binet's formula of the nth hyperbolic k-Fibonacci quaternion [5] is

$$HQF_{k,n}=\frac{1}{\sqrt{k^2+4}}({r_1}^*{r_1}^n-{r_2}^*{r_2}^n),$$

where ${r_1}^* = 1+\mathbf{i}r_1+\mathbf{j}{r_1}^2+\mathbf{k}{r_1}^3$, $r_1 = \frac{k+\sqrt{k^2+4}}{2}$ and ${r_2}^* = 1+\mathbf{i}r_2+\mathbf{j}{r_2}^2+\mathbf{k}{r_2}^3$, $r_2 = \frac{k-\sqrt{k^2+4}}{2}$.

- The Binet's formula of the nth hyperbolic Pell quaternion is

$$HQP_n = \frac{1}{2\sqrt{2}}(x_1^* {x_1}^n - {x_2}^* x_2^n),$$

where ${x_1}^* = 1+\mathbf{i}x_1+\mathbf{j}{x_1}^2+\mathbf{k}{x_1}^3$, $x_1 = 1+\sqrt{2}$ and ${x_2}^* = 1+\mathbf{i}x_2+\mathbf{j}{x_2}^2+\mathbf{k}{x_2}^3$, $x_2 = 1-\sqrt{2}$.

- The Binet's formula of the nth hyperbolic k-Pell quaternion is

$$HQP_{k,n} = \frac{1}{2\sqrt{1+k}}(y_1^* {y_1}^n - {y_2}^* y_2^n),$$

where ${y_1}^* = 1+\mathbf{i}y_1+\mathbf{j}{y_1}^2+\mathbf{k}{y_1}^3$, $y_1 = 1+\sqrt{1+k}$ and ${y_2}^* = 1+\mathbf{i}y_2+\mathbf{j}{y_2}^2+\mathbf{k}{y_2}^3$, $y_2 = 1-\sqrt{1+k}$.

- The Binet's formula of the nth hyperbolic Jacobsthal quaternion is

$$HQJ_n = \frac{2^* 2^n - (-1)^* (-1)^n}{3},$$

where $2^* = 1+\mathbf{i}2+\mathbf{j}4+\mathbf{k}8$ and $(-1)^* = 1-\mathbf{i}+\mathbf{j}-\mathbf{k}$.

- The Binet's formula of the nth hyperbolic k-Jacobsthal quaternion is

$$HQJ_{k,n} = \frac{1}{\sqrt{k^2+8}}(w_1^* {w_1}^n - {w_2}^* w_2^n),$$

where ${w_1}^* = 1+\mathbf{i}w_1+\mathbf{j}{w_1}^2+\mathbf{k}{w_1}^3$, $w_1 = \frac{k+\sqrt{k^2+8}}{2}$ and ${w_2}^* = 1+\mathbf{i}w_2+\mathbf{j}{w_2}^2+\mathbf{k}{w_2}^3$, $w_2 = \frac{k-\sqrt{k^2+8}}{2}$.

The d'Ocagne's identity involving the hyperbolic (p, q)-Fibonacci quaternions is given in the following theorem.

Theorem 4. *Let m and n be two positive integers, such that $n \leq m$. Then we have*

$$HQ\mathcal{F}_m \times HQ\mathcal{F}_{n+1} - HQ\mathcal{F}_{m+1} \times HQ\mathcal{F}_n = \frac{(-q)^n}{\sqrt{p^2+4q}}(\alpha^* \beta^* \alpha^{m-n} - \beta^* \alpha^* \beta^{m-n}).$$

Proof. Using the Binet's formula of the hyperbolic (p, q)-Fibonacci quaternions, we have

$$\begin{aligned}
&HQ\mathcal{F}_m \times HQ\mathcal{F}_{n+1} - HQ\mathcal{F}_{m+1} \times HQ\mathcal{F}_n \\
&= \frac{\alpha^*\alpha^m - \beta^*\beta^m}{\alpha-\beta} \times \frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta} - \frac{\alpha^*\alpha^{m+1} - \beta^*\beta^{m+1}}{\alpha-\beta} \times \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta} \\
&= \frac{1}{(\alpha-\beta)^2}(\alpha^*\beta^*(\alpha^{m+1}\beta^n - \alpha^m\beta^{n+1}) + \beta^*\alpha^*(\alpha^n\beta^{m+1} - \alpha^{n+1}\beta^m))
\end{aligned}$$

$$= \frac{1}{(\alpha-\beta)}(\alpha^*\beta^*\alpha^m\beta^n - \beta^*\alpha^*\alpha^n\beta^m)$$
$$= \frac{1}{(\alpha-\beta)}(\alpha\beta)^n(\alpha^*\beta^*\alpha^{m-n} - \beta^*\alpha^*\beta^{m-n}).$$

Since $\alpha - \beta = \sqrt{p^2+4q}$ and $\alpha\beta = -q$, we obtain the desired result. □

Note that, if we take $p = k$, $q = 1$ as a special case in Theorem 4, we obtain the equivalent result for d'Ocagne's identity involving the hyperbolic k-Fibonacci quaternions given in [5].

The following theorem gives the Catalan's identity for the hyperbolic (p, q)-Fibonacci quaternions.

Theorem 5. *Let n and r be two positive integers. Then we have*

$$HQ\mathcal{F}_{n-r} \times HQ\mathcal{F}_{n+r} - HQ\mathcal{F}_n^2 = \frac{(-q)^{n-r}}{p^2+4q}(\alpha^*\beta^*\beta^r - \beta^*\alpha^*\alpha^r)(\alpha^r - \beta^r).$$

Proof. Using the Binet's formula of the hyperbolic (p, q)-Fibonacci quaternions, we have

$$HQ\mathcal{F}_{n-r} \times HQ\mathcal{F}_{n+r} - HQ\mathcal{F}_n \times HQ\mathcal{F}_n$$
$$= \frac{\alpha^*\alpha^{n-r} - \beta^*\beta^{n-r}}{\alpha-\beta} \times \frac{\alpha^*\alpha^{n+r} - \beta^*\beta^{n+r}}{\alpha-\beta} - \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta} \times \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta}$$
$$= \frac{1}{(\alpha-\beta)^2}(\alpha^*\beta^*(\alpha\beta)^{n-r}(\alpha^r\beta^r - \beta^{2r}) + \beta^*\alpha^*(\alpha\beta)^{n-r}(\alpha^r\beta^r - \alpha^{2r}))$$
$$= \frac{1}{(\alpha-\beta)^2}(\alpha\beta)^{n-r}(\alpha^*\beta^*\beta^r - \beta^*\alpha^*\alpha^r)(\alpha^r - \beta^r).$$

Since $\alpha - \beta = \sqrt{p^2+4q}$ and $\alpha\beta = -q$, we obtain the desired result. □

Note that, if we take $p = k$, $q = 1$ as a special case in Theorem 5, we obtain the equivalent result for Catalan's identity involving the hyperbolic k-Fibonacci quaternions given in [5].

If we take $r = 1$ in Theorem 5, we obtain the Cassini's identity involving the hyperbolic (p, q)-Fibonacci quaternions as

$$HQ\mathcal{F}_{n-1} \times HQ\mathcal{F}_{n+1} - HQ\mathcal{F}_n^2 = \frac{(-q)^{n-1}}{\sqrt{p^2+4q}}(\alpha^*\beta^*\beta - \beta^*\alpha^*\alpha).$$

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ON THE INTUITIONISTIC FUZZY PROJECTIVE MENELAUS AND CEVA'S CONDITIONS

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Abstract. In this work, the intuitionistic fuzzy versions of Menelaus and Ceva's theorems in intuitionistic fuzzy projective plane are defined and the conditions to the intuitionistic fuzzy versions of Menelaus and Ceva 6-figures are determined.

1. Introduction

Ceva's and Menelaus theorems are two classic theorems in plane geometry. The main question of these theorems is to determine conditions under which three points are collinear and conditions under which three lines are concurrent. Ceva's theorem characterizes the concurrency of lines and Menelaus's theorem characterizes the collinearity of points. Kelly B. Funk [9] gave Menelaus and Ceva theorems in projective planes $\mathcal{P}_2(F)$ where F is the field of characteristic not equal to two. The definitions of the original Menelaus and Ceva $6-$figures are given in [3, 9].

After the introduction of Fuzzy set theory by Zadeh [15] several researches were conducted on generalizations of fuzzy theory.

A model of fuzzy projective geometries was introduced by Kuijken and Van Maldeghem [14]. This provided a link between the fuzzy versions of classical theories that are very closely related some basic results on fuzzy projective geometries are published in [1, 2, 5, 8]. Fiber geometry that is a particular kind of fuzzy geometries is introduced by Kuijken and Van Maldeghem. In these geometry, the points and lines of the base geometry mostly have multiple degrees of membership. The fibered version of Menelaus and Ceva's 6-figures was studied in [6].

Intuitionistic fuzzy set theory was firstly published by Atanassov [4]. A model of intuitionistic fuzzy projective geometry and the link between fibered and intuitionistic fuzzy projective geometry were given by Ghassan E. Arif [10].

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In the present paper, intuitionistic fuzzy projective Menelaus and Ceva's conditions in the intuitionistic fuzzy projective plane with base plane that is projective plane are given.

2. Preliminaries

We firstly recall the basic notions from the theory of projective geometry, fuzzy projective geometry and intuitionistic fuzzy projective geometry. We assume that the reader is familiar with the basic notions of fuzzy mathematics, although this is not strictly necessary as the paper is self-contained in this respect.

We denote by $\wedge$ and $\vee$, minimum and maximum operators respectively.

Definition 1. *Let $\mathcal{P} = (P, B, \sim)$ be any projective plane with point set P and line set B, i.e., P and B are two disjoint sets endowed with a symmetric relation $\sim$ (called the* incidence relation*) such that the graph $(P \cup B, \sim)$ is a bipartite graph with classes P and B, and such that two distinct points p, q in $\mathcal{P}$ are incident with exactly one line (denoted by $\langle pq \rangle$), every two distinct lines L, M are incident with exactly one point (denoted by $L \cap M$), and every line is incident with at least three points. A set S of* collinear *points is a subset of P each member of which is incident with a common line L. Dually, one defines a set of* concurrent *lines* [5].

Definition 2. (see [15])*A fuzzy set λ of a set X is a function $\lambda : X \to [0, 1]$.*

Definition 3. (see [4]) *Let X be a nonempty fixed set. An intuitionistic fuzzy set A on X is an object having the form*

$$A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$$

where the function $\lambda : X \to I$ and $\mu : X \to I$ denote the degree of membership (namely, $\lambda(x)$) and the degree of nonmembership (namely, $\mu(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \lambda(x) + \mu(x) \leq 1$ for each $x \in X$. An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ can be written in the $A = \{\langle x, \lambda, \mu\rangle : x \in X\}$, or simply $A = \langle \lambda, \mu \rangle$.

Definition 4. *(see [10]) An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ on n -dimensional projective space S is an intuitionistic fuzzy $n-$dimensional projective space on S if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$ and $\mu(p) \leq \mu(q) \vee \mu(r)$, for any three collinear points p, q, r of A we denoted $[A, S]$.*

The projective space S is called the base projective space of $[A, S]$ if $[A, S]$ is an intuitionistic fuzzy point, line, plane, ... ,we use base point, base line, base plane,... , respectively.

Definition 5. *(see [10]) Consider the projective plane $\mathcal{P} = (P, B, \sim)$. Suppose $a \in P$ and $\alpha, \alpha' \in [0, 1]$. The $\mathcal{IF}$-point (a, α, α') is the following intuitionistic fuzzy set on the point set P of $\mathcal{P}$:*

$$(a, \alpha, \alpha') : P \to [0, 1] : \begin{cases} a \to \alpha, \ a \to \alpha' & \\ x \to 0 & \text{if } x \in P \backslash \{a\}. \end{cases}$$

The point a is called the base point of the $\mathcal{IF}$-point (a,α,α'). An $\mathcal{IF}$-line (L,α,α') with base line L is defined in a similar way.

The $\mathcal{IF}$-lines (L,α,α') and (M,β,β') intersect in the unique $\mathcal{IF}$- point $(L\cap M,\alpha\wedge\beta,\alpha'\vee\beta')$. The $\mathcal{IF}$-points (a,α,α') and (b,β,β') span the unique $\mathcal{IF}$-line $(\langle a,b\rangle,\alpha\wedge\beta,\alpha'\vee\beta')$.

Definition 6. *(see [10]) Suppose $\mathcal{P}$ is a projective plane $\mathcal{P}=(P,B,\sim)$. The intuitionistic fuzzy set $Z=\langle\lambda,\mu\rangle$ on $P\cup B$ is an intuitionistic fuzzy projective plane on $\mathcal{P}$ denoted by $\mathcal{IFP}$ if*

1) $\lambda(L)\geq\lambda(p)\wedge\lambda(q)$ *and* $\mu(L)\leq\mu(p)\vee\mu(q)$, $\forall p,q$*:* $\langle p,q\rangle=L$
2) $\lambda(p)\geq\lambda(L)\wedge\lambda(M)$ *and* $\mu(p)\leq\mu(L)\vee\mu(M)$, $\forall L,M$*:* $L\cap M=p$.

Theorem 7. *(see [7]) Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$ that is Desarguesian. Choose three $\mathcal{IF}$-points $(a_i,\alpha_i,\alpha'_i), i\in\{1,2,3\}$ with noncollinear base points, and three other points (b_i,β_i,β'_i), $i\in\{1,2,3\}$ with noncollinear base points, such that the f-lines $(\langle a_i,b_i\rangle,\alpha_i\wedge\beta_i,\alpha'_i\vee\beta'_i)$, for $i\in\{1,2,3\}$, meet in an IF-point (p,γ,η) of $\mathcal{IFP}$, with $a_i\neq b_i\neq p\neq a_i$. Then the three $\mathcal{IF}$-points $(c_{\{i,j\}},\gamma_{\{i,j\}},\gamma'_{\{i,j\}})$ obtained by intersecting $(\langle a_i,a_j\rangle,\alpha_i\wedge\alpha_j,\alpha'_i\vee\alpha'_j)$ and $(\langle b_i,b_j\rangle,\beta_i\wedge\beta_j,\beta'_i\vee\beta'_j)$, for $i\neq j$ and $i,j\in\{1,2,3\}$), are collinear.*

Theorem 8. *(see [7]) Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with Pappian base plane $\mathcal{P}$. Choose two different lines L_1 and L_2 in $\mathcal{P}$. Choose two triples of $\mathcal{IF}-$points (a_i,α_i,α'_i) and (b_i,β_i,β'_i) with a_i on L_1 and b_i on L_2, $i=1,2,3$ and such that no three of the base points a_1, a_2, b_1, b_2 are collinear. Then the three intersection $\mathcal{IF}-$points $(c_1,\gamma_1,\gamma'_1)=(a_2b_3\cap a_3b_2,\alpha_2\wedge\alpha_3\wedge\beta_2\wedge\beta_3,\alpha'_2\vee\alpha'_3\vee\beta'_2\vee\beta'_3)$, $(c_2,\gamma_2,\gamma'_2)=(a_1b_3\cap a_3b_1,\alpha_1\wedge\alpha_3\wedge\beta_1\wedge\beta_3,\alpha'_1\vee\alpha'_3\vee\beta'_1\vee\beta'_3)$ and $(c_3,\gamma_3,\gamma'_3)=(a_1b_2\cap a_2b_1,\ \alpha_1\wedge\alpha_2\wedge\beta_1\wedge\beta_2,\ \alpha'_1\vee\alpha'_2\vee\beta'_1\vee\beta'_2)$ are collinear.*

Definition 9. *(see [11]) Let $\mathcal{P}$ be a projective plane. A $6-$figure in $\mathcal{P}$ is a sequence of six distinct points $(a_1a_2a_3\ ,\ b_1b_2b_3)$ such that $a_1a_2a_3$ constitutes a nondegenerate triangle with $b_1\in\langle a_2,a_3\rangle$, $b_2\in\langle a_1,a_3\rangle$, $b_3\in\langle a_1,a_2\rangle$. The points a_1,a_2,a_3,b_1,b_2,b_3 are called vertices of this $6-$figures. Such a configuration is said to be a Menelaus $6-$figure or a Ceva $6-$figure if b_1,b_2 and b_3 are collinear or if $\langle a_1,b_1\rangle$, $\langle a_2,b_2\rangle$, $\langle a_3,b_3\rangle$ are concurrent, respectively.*

Definition 10. *(see [6]) Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three $f-$points (a_i,α_i), $i\in\{1,2,3\}$ in $\mathcal{FP}$ with non collinear base points and the other three $f-$points (b_k,β_k), $k\in\{1,2,3\}$ with $(b_k,\beta_k)\in(\langle a_i,a_j\rangle,\alpha_i\wedge\alpha_j)$ for $i\neq j\neq k$, $\{i,j,k\}=\{1,2,3\}$. If the $f-$points (b_k,β_k) are $f-$collinear, the configuration that consists of these six $f-$points is called an $f-$Menelaus $6-$figure. It is called $f-$Menelaus line spanned with $f-$points (b_k,β_k) for $k=\{1,2,3\}$.*

Theorem 11. *(see [6]) Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three $f-$points (a_i,α_i), $i\in\{1,2,3\}$ in $\mathcal{FP}$ with non collinear base points and the other three f-points (b_k,β_k), $k\in\{1,2,3\}$ with $(b_k,\beta_k)\in(\langle a_i,a_j\rangle,\alpha_i\wedge\alpha_j)$ for*

$i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$. The configuration that consists of these six $f-$points is Menelaus $6-$figure if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.

Corollary 12. *(see [6]) Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three $f-$points (a_i, α_i), $i \in \{1,2,3\}$ in $\mathcal{FP}$ with non collinear base points and the other three f-points (b_k, β_k), $k \in \{1,2,3\}$ with $(b_k, \beta_k) \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$. The configuration that consists of these six $f-$points is Ceva $6-$figure iff $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.*

We view Menalaus and Ceva's theorem in projective plane and extend them $\mathcal{IFP}$, intuitionistic fuzzy projective plane.

Theorem 13. *Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$. Let a_1, a_2, a_3 be three non-collinear points in $\mathcal{P}$ and be*

$$(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$$

be three points of $\mathcal{IFP}$. Suppose that the point b_3 on $\langle a_1, a_2 \rangle$ is obtained by intersecting $\langle a_1, a_2 \rangle$ with the join of two chosen points b_1 and b_2 where b_1 on $\langle a_2, a_3 \rangle$ and b_2 on $\langle a_1, a_3 \rangle$. Then the point (b_3, β_3, β_3') obtained by intersecting $(\langle a_1, a_2 \rangle, \alpha_1 \wedge \alpha_2, \alpha_1' \vee \alpha_2')$ with the join of the two points (b_1, β_1, β_1') and (b_2, β_2, β_2'), where $(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$ and $(b_2, \beta_2, \beta_2'), (a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$ are collinear, is independent of the chosen points (b_1, β_1, β_1') and (b_2, β_2, β_2').

Proof. In $\mathcal{IFP}$, since the three points $(b_2, \beta_2, \beta_2'), (a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$ and the three points

$$(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$$

are collinear,

$$\begin{gathered} \alpha_1 \wedge \alpha_3 = \alpha_1 \wedge \beta_2 = \alpha_3 \wedge \beta_2 \\ \alpha_1' \vee \alpha_3' = \alpha_1' \vee \beta_2' = \alpha_3' \vee \beta_2' \end{gathered}$$

and

$$\begin{gathered} \alpha_2 \wedge \alpha_3 = \alpha_2 \wedge \beta_1 = \alpha_3 \wedge \beta_1 \\ \alpha_2' \vee \alpha_3' = \alpha_2' \vee \beta_1' = \alpha_3' \vee \beta_1'. \end{gathered}$$

One can easily calculate that $\beta_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\beta_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3^{2\prime}$. It is seen that the point (b_3, β_3, β_3') is independent of the choice of the points (b_1, β_1, β_1') and (b_2, β_2, β_2'). □

Theorem 14. *Let an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$ be $\mathcal{IFP}$. Let three points in this plane no three base points of which are collinear be $(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$. If the point (b_3, β_3, β_3') be obtain by intersecting of the lines $(\langle a_1, a_2 \rangle, \alpha_1 \wedge \alpha_2, \alpha_1' \vee \alpha_2')$ and $(\langle b_1, b_2 \rangle, \beta_1 \wedge \beta_2, \beta_1' \vee \beta_2')$, where $(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$ and $(b_2, \beta_2, \beta_2'), (a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$ are collinear, then the configuration that consists of the six points*

$$(a_i, \alpha_i, \alpha_i'), (b_i, \beta_i, \beta_i'), \quad i \in \{1,2,3\}$$

is an intuitionistic fuzzy Menelaus $6-$figure.

Proof. Since the three points (b_i, β_i, β_i'), $i \in \{1,2,3\}$ are collinear, from Definition 9 the configuration that consists of the six points $(a_i, \alpha_i, \alpha_i'), (b_i, \beta_i, \beta_i')$, $i \in \{1,2,3\}$ is an intuitionistic fuzzy Menelaus 6−figure. □

Theorem 15. *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and with $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k\rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$. If b_1, b_2 and b_3 in $\mathcal{P}$ are collinear, then the three points (b_j, β_j, β_j'), $j \in \{1,2,3\}$ are collinear if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.*

Proof. A configuration is picked such that three points

$$(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$$

and

$$(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k\rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k') \text{ for } i \neq j \neq k, \quad \{i,j,k\} = \{1,2,3\}.$$

Suppose the three points (b_j, β_j, β_j'), $j \in \{1,2,3\}$ be collinear. Since three points (b_j, β_j, β_j') are collinear and the three points $(a_i, \alpha_i, \alpha_i')$, $(a_j, \alpha_j, \alpha_j')$ and (b_k, β_k, β_k'), for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$ are collinear $\beta_i \wedge \beta_j = \beta_i \wedge \beta_k$, $\beta_i' \vee \beta_j' = \beta_i' \vee \beta_k'$ and $\alpha_i \wedge \alpha_j = \alpha_i \wedge \beta_k = \alpha_j \wedge \beta_k$, $\alpha_i' \vee \alpha_j' = \alpha_i' \vee \beta_k' = \alpha_j' \vee \beta_k'$. Then it is seen that $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.

Conversely, if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^3$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$ are satisfied, $\beta_1 \wedge \beta_2 = \beta_1 \wedge \beta_3 = \beta_2 \wedge \beta_3$ and $\beta_1' \vee \beta_2' = \beta_1' \vee \beta_3' = \beta_2' \vee \beta_3'$. Then three points (b_i, β_i, β_i'), $i \in \{1,2,3\}$ are collinear. □

Corollary 16. ***The intuitionistic fuzzy projective Menelaus condition ($\mathcal{IFPMC}$):*** *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2')$ and $(a_3, \alpha_3, \alpha_3')$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$with $(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j\rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$. The configuration that consists of these six points is Menelaus 6−figure if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.*

Definition 17. *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$ with*

$$(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j\rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j') \textit{ for } i \neq j \neq k, \{i,j,k\} = \{1,2,3\}.$$

If the lines $(\langle a_i, b_i\rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i'), i = 1,2,3$ are concurrent, the configuration that consists of these six points is called an intuitionistic fuzzy Ceva 6−figure. The intersection point of the lines $(\langle a_i, b_i\rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$, $i = 1,2,3$ is called intuitionistic fuzzy Ceva point.

Theorem 18. *Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$. Let a_1, a_2, a_3 be three non-collinear points in $\mathcal{P}$ and be*

$$(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2') \text{ and } (a_3, \alpha_3, \alpha_3')$$

be three points of $\mathcal{IFP}$. Let points $b_1$and b_2 be chosen such that b_1 on $\langle a_2, a_3\rangle$ and b_2 on $\langle a_1, a_3\rangle$. Suppose that the point b_3 on $\langle a_1, a_2\rangle$ is obtained by intersecting $\langle a_1, a_2\rangle$ with the join $(\langle a_1, b_1\rangle \cap \langle a_2, b_2\rangle)$ and a_3. Then the point (b_3, β_3, β_3') obtained by intersecting $(\langle a_1, a_2\rangle, \alpha_1 \wedge \alpha_2, \alpha_1' \vee \alpha_2')$ with the join of the two points $(\langle a_1, b_1\rangle, \alpha_1 \wedge \beta_1, \alpha_1' \vee \beta_1') \cap (\langle a_2, b_2\rangle, \alpha_2 \wedge \beta_2, \alpha_2' \vee \beta_2')$ and $(a_3, \alpha_3, \alpha_3')$, where $(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$ and $(b_2, \beta_2, \beta_2'), (a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$ are collinear, and independent of the chosen points (b_1, β_1, β_1') and (b_2, β_2, β_2').

Proof. Since three points$(a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$,$(b_2, \beta_2, \beta_2')$ and three points

$$(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$$

are collinear,

$$\begin{aligned} &\alpha_1 \wedge \alpha_3 = \alpha_1 \wedge \beta_2 = \alpha_3 \wedge \beta_2 \text{ and } \alpha_2 \wedge \alpha_3 = \alpha_2 \wedge \beta_1 = \alpha_3 \wedge \beta_1 \\ &\alpha_1' \vee \alpha_3' = \alpha_1' \vee \beta_2' = \alpha_3' \vee \beta_2' \text{ and } \alpha_2' \vee \alpha_3' = \alpha_2' \vee \beta_1' = \alpha_3' \vee \beta_1'. \end{aligned}$$

One calculates that $\beta_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\beta_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3^{2\prime}$ hence the point (b_3, β_3, β_3') is independent of the chosen of the points (b_1, β_1, β_1') and (b_2, β_2, β_2'). □

Theorem 19. *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1, 2, 3\}$ in $\mathcal{IFP}$ with non collinear base points and with $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k\rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. If the lines $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle$ and $\langle a_3, b_3\rangle$ in $\mathcal{P}$ are concurrent, then three lines $(\langle a_i, b_i\rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$, for $i \in \{1, 2, 3\}$ are concurrent if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.*

Proof. A configuration is chosen such that three points$(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2')$ and $(a_3, \alpha_3, \alpha_3')$ and $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k\rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. Suppose three lines $(\langle a_i, b_i\rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$, for $i \in \{1, 2, 3\}$ are concurrent. Then three membership degree pairs in concurrent point $\alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$ and $\alpha_i' \vee \alpha_j' \vee \beta_i' \vee \beta_j'$, $i \neq j$, $\{i, j\} = \{1, 2, 3\}$ are equal. Since three points $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k\rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$, $\alpha_i \wedge \alpha_j = \alpha_i \wedge \beta_k = \alpha_j \wedge \beta_k$ and $\alpha_i' \vee \alpha_j' = \alpha_i' \vee \beta_k' = \alpha_j' \vee \beta_k'$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$ are valid. One can easily get $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.

Conversely, by using points (b_j, β_j, β_j'), $(a_i, \alpha_i, \alpha_i')$ and $(a_i, \alpha_k, \alpha_k')$ are collinear for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$ in $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$ it is shown that three pair of values $\alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$ and $\alpha_i' \vee \alpha_j' \vee \beta_i' \vee \beta_j'$, $i \neq j$, $\{i, j\} = \{1, 2, 3\}$ are equal. □

Corollary 20. ***(The intuitionistic fuzzy projective Ceva condition($\mathcal{IFPCC}$))*** *Let $\mathcal{FP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$ with $(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$. The configuration that consists of these six points is Ceva 6−figure iff $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.*

The following theorem show that intuitionistic fuzzy Ceva 6−figures can be obtained as a corollary of intuitionistic fuzzy Menelaus 6−figures.

Theorem 21. *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$ with $(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$, three lines $\langle a_i, b_i \rangle$ are concurrent in $\mathcal{P}$. If the configuration that consists of these six points is intuitionistic fuzzy Menelaus 6−figure, it is intuitionistic fuzzy Ceva 6−figure.*

Proof. Let the configuration chosen such that three points $(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2')$ and $(a_3, \alpha_3, \alpha_3')$ and $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$ be intuitionistic fuzzy Menelaus 6−figure. Three membership degree pairs in intersection point of three lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$ are $\alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$ and $\alpha_i' \vee \alpha_j' \vee \beta_i' \vee \beta_j'$, $i \neq j$, $\{i,j\} = \{1,2,3\}$. It is easily seen that these are equal. So three lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$ for $i \in \{1,2,3\}$ are concurrent. □

The reverse of this theorem isn't true in $\mathcal{IFP}$.

Fano projective plane, denoted by $PG(2,2)$, consists seven points and seven lines. Fano projective plane is only example that is both Menelaus 6−figure and Ceva 6−figure. Even if the base plane $\mathcal{P}$ of $\mathcal{IFP}$ is Fano plane, the reverse of the process is not always valid in $\mathcal{IFP}$.

Theorem 22. *Let $\wedge$ and $\vee$be a triangular norm and conorm, respectively. Let $\mathcal{IFP}$ be any nontrivial intuitionistic fuzzy projective plane with base plane $\mathcal{P}$ that is Fano plane. Let three points be $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$ with $(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$, three lines $\langle a_i, b_i \rangle$ are concurrent in $\mathcal{P}$. If the configuration that consists of these six points is intuitionistic fuzzy Ceva 6−figure, it can not be intuitionistic fuzzy Menalaus 6−figure.*

Proof. The configuration picked such that points

$$(a_1, 0.5, 0.5), (a_2, 0.5, 0.5), (a_3, 0, 5, 0.5) \text{ and } (b_1, 0.6, 0.4), (b_2, 0.7, 0.3), (b_3, 0.8, 0.2)$$

is Ceva 6−figure in $\mathcal{IFP}$. But, using the minimum and maximum operators for $\wedge$ and $\vee$, it is easily seen that the points $(b_1, 0.6, 0.4), (b_2, 0.7, 0.3)$ and $(b_3, 0.8, 0.2)$ are not collinear. □

Conclusion 23. *In this study, the intuitionistic fuzzy versions of Menelaus and Ceva 6-figures in intuitionistic fuzzy projective plane are given. So, the obtained conditions and results for the intuitionistic fuzzy versions of Menelaus and Ceva will contribute to the intuitionistic fuzzy projective geometry. While the fibered and fuzzy versions of some classical results in projective planes by using t-norm are given, the intuitionistic fuzzy versions of these theorems include both t-norm and conorm. It seen that the triangular norms and conorms have important role in the intuitionistic fuzzy versions of theorems related to theory.*

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ALTERNATIVE PARTNER CURVES IN THE EUCLIDEAN 3−SPACE

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Abstract. In the present paper, a new type of special curve couple which are called WC^*-partner curves are introduced according to alternative moving frame $\{N, C, W\}$. The distance function between the corresponding points of reference curve and its partner curve is obtained. Besides, the angle function between the vector fields of alternative frame of the curves is expressed by means of alternative curvatures f and g. In addition to these, various characterizations are obtained related to these curves.

1. Introduction

The curves are the fundamental structure of differential geometry. Numerous studies of curves are carried out in 3−dimensional Euclidean space. Two curves which have some special properties at their corresponding points are called curve pairs. Hence, curve pairs are attracted the attention of many researchers [1, 2, 3, 13]. The most famous types of curve pairs are Bertrand partner curves. The Bertrand curves were firstly described by Bertrand Russell in 1850. These curves have the common principal normal vector. The classic characterization for Bertrand curves is that a regular curve α in $\mathbb{E}^3$ is the Bertrand curve if and only if $a\kappa(s)+b\tau(s)=1$ holds [7]. The other famous curve pair are the Mannheim partner curves. These curves are defined by Mannheim with the equality $\kappa^2+\tau^2=w^2=$constant. Another characterization can be made as two curves α and β in $\mathbb{E}^3$ which are called Manneim partner curves if the principal normal vector fields of α coincide with the binormal vector fields of β at the corresponding points of curves [5, 6, 12, 14].

This paper is expected to define a new kind of curve pairs which are called WC^*-partner curves and give various characterization of these curves. For this purpose, an alternative frame on original curve is used and another curve is defined using this frame. First of all, a brief summary of curve theory and alternative frame

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are presented. Afterwards, the definition and main characterizations corporated to distance function and angle function of WC^*-partner curves are introduced.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a regular unit speed curve in the Euclidean 3−space where s measures its arc length. Also, let $T = \alpha^{'}$ be its unit tangent vector, $N = \dfrac{T^{'}}{\|T'\|}$ be its principal normal vector and $B = T \times N$ be its binormal vector. The triple $\{T, N, B\}$ be the Frenet frame of the curve α. Then the Frenet formula of the curve is given by

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are curvature and torsion of α, respectively [10]. Also, the geodesic curvature of spherical image of principal normal indicatrix of a space curve α is given

$$\sigma = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} (\frac{\tau}{\kappa})'.$$

If we reconstruct the above equation via the harmonic curvature function H which is introduced by Özdamar in [8], we can easily see that

$$\sigma = \frac{H'}{\kappa(1 + H^2)^{3/2}}, \quad H = \frac{\tau}{\kappa}.$$

From the equation (2.1), the unit Darboux vector W of α is as follows

$$W = \frac{1}{\sqrt{\kappa^2 + \tau^2}} (\tau T + \kappa B). \tag{2.2}$$

It is obvious that the Darboux vector is vertical to the principal normal vector field N from equation (2.2). With the help of the vector fields W and N, along $\alpha(s)$, $C = W \times N$ unit vector field is defined. These three orthogonal vectors creates a new frame defined by Uzunoğlu et al. in [11]. This frame is designation by $\{N, C, W\}$ and alternative frame to curve rather than the Frenet frame $\{T, N, B\}$. The alternative frame and derivative formula of the alternative frame are given by

$$\begin{pmatrix} N \\ C \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-\kappa}{\sqrt{\kappa^2 + \tau^2}} & 0 & \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \\ \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} & 0 & \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.3}$$

and

$$\begin{pmatrix} N' \\ C' \\ W' \end{pmatrix} = \begin{pmatrix} 0 & f & 0 \\ -f & 0 & g \\ 0 & -g & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}, \tag{2.4}$$

where

$$f=\kappa\sqrt{1+H^2},\ g=\frac{H^{'}}{1+H^2}.$$

Since the principal normal vector N is common in both frames, following equations are available from the equations (2.1), (2.2) and (2.4),

$$\begin{aligned} C &= -\bar{\kappa}T+\bar{\tau}B \\ W &= \bar{\tau}T+\bar{\kappa}B \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} T &= -\bar{\kappa}C+\bar{\tau}W \\ B &= \bar{\tau}C+\bar{\kappa}W \end{aligned} \tag{2.6}$$

where $\bar{\kappa}=\dfrac{\kappa}{f}$ and $\bar{\tau}=\dfrac{\tau}{f}$.

A regular curve α is called a helix if the tangent lines of the curve makes a constant angle with a fixed direction. This curve is characterized by the property that $\dfrac{\tau}{\kappa}$ is constant [4]. If the principal normal lines of the curve makes a constant angle with a fixed direction, then the curve is called a slant helix and characterized by the equally

$$\frac{g}{f}=\frac{H^{'}}{\kappa(1+H^2)^{3/2}}=\sigma$$

is constant [11]. Then the characterization of a slant helix according to alternative frame is given as follows.

Remark 1. *A regular curve $\alpha(s)$ according to alternative frame $\{N,C,W\}$ with alternative curvatures f and g is a slant helix if and only if $\dfrac{g(s)}{f(s)}=$ constant* [11].

3. ALTERNATIVE PARTNER CURVES IN THE EUCLIDEAN 3−SPACE

This section aims to define a new type of partner curves by considering alternative frame and find some characterizations for these curves corporated to distance function between the corresponding points of the curves, curvatures of the curves and angle function.

Definition 1. *Let $\alpha=\alpha(s)$ and $\alpha^*=\alpha^*(s^*)$ be two regular space curves parameterized by its arc length s and s^* with Frenet frames $\{T,N,B\}$, $\{T^*,N^*,B^*\}$, curvatures κ,κ^* and torsions τ,τ^* respectively in the Euclidean $3-$space. Also, let the alternative moving frames and alternative curvatures of curves be $\{N,C,W\}$, f,g and $\{N^*,C^*,W^*\}$, f^*,g^*, respectively. The curves α and α^* are called WC^*-partner curves if the vector fields W and C^* coincide i.e., $W=C^*$ holds at the corresponding points of the curves.*

From Definition 1, we can easily write the parametric representation of $\alpha^*(s^*)$ as follows

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)W(s) \tag{3.1}$$

where $\lambda = \lambda(s)$ is the distance function between corresponding points of the curves α and α^*. Because the vector fields W and C^* are the equal, we can represent the relationship between the alternative frames of α and α^*. If $\theta = \theta(s)$ is the angle function between vector fields N and W^*, the following equations are obtained thanks to axis rotation equations.

$$\begin{pmatrix} N^* \\ C^* \\ W^* \end{pmatrix} = \begin{pmatrix} \cos(90-\theta) & \sin(90-\theta) & 0 \\ 0 & 0 & 1 \\ -\sin(90-\theta) & \cos(90-\theta) & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}$$

$$\begin{aligned} N^* &= \sin\theta N + \cos\theta C \\ W^* &= -\cos\theta N + \sin\theta C \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} N &= \sin\theta N^* - \cos\theta W^* \\ C &= \cos\theta N^* + \sin\theta W^*. \end{aligned} \tag{3.3}$$

Theorem 1. *Let $\{\alpha, \alpha^*\}$ be WC^*-partner curves according to alternative frame in Euclidean $3-$space. The distance function $\lambda = \lambda(s)$ between the corresponding points of the α and α^* is as follows,*

$$\lambda(s) = -\frac{\kappa}{fg}.$$

Proof. If we take derivative of the equation (3.1) according to s, we get

$$T^*\frac{ds^*}{ds} = T + \lambda' W + \lambda W'.$$

Using the equations (2.6) and (3.3), we obtain that

$$(-\bar{\kappa}^* C^* + \bar{\tau}^* W^*)\frac{ds^*}{ds} = -(\bar{\kappa} + \lambda g)\cos\theta N^* + (\bar{\tau} + \lambda')C^* - (\bar{\kappa} + \lambda g)\sin\theta W^*.$$

If we consider the above equalities, we can easily see that

$$\lambda(s) = -\frac{\kappa}{fg}.$$

□

Theorem 2. *Let $\{\alpha, \alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. $\{N, C, W, f, g\}$ and $\{N^*, C^*, W^*, f^*, g^*\}$ are the alternative frame elements of the curves α and α^*, respectively. Then the following relation exists among curvatures.*

$$\frac{g^*}{f^*} = -\tan\theta = \text{constant} \quad \text{and} \quad (f^*)^2 + (g^*)^2 = g^2$$

Proof. Since $\{\alpha,\alpha^*\}$ is the WC^*-partner curves, $W=C^*$ and their derivatives are equal.

$$\begin{aligned}(C^*)' &= -f^*N^*+g^*W^*,\\ W' &= -gC.\end{aligned}$$

From the last equation and equation (3.3), we have

$$-f^*N^*+g^*W^*=-g(\cos\theta N^*+\sin\theta W^*)$$

$$\begin{aligned}f^* &= g\cos\theta,\\ g^* &= -g\sin\theta.\end{aligned}$$

So, we obtain that

$$\frac{g^*}{f^*}=-\tan\theta$$

and

$$(f^*)^2+(g^*)^2=g^2.$$

□

Theorem 3. *Let $\{\alpha,\alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. $\theta=\theta(s)$ be the angle function between vector fields N and W^*. Then the following relation exists.*

$$\theta=\int_0^s f ds,\quad s=\int_0^{s^*}\frac{f^*}{g\cos\theta}ds^*$$

Proof. From the equation (3.2), we have

$$N^*=\sin\theta N+\cos\theta C$$

If we take the derivative of each side of the above equation according to s, we obtain

$$\frac{dN^*}{ds^*}\frac{ds^*}{ds}=\cos\theta\frac{d\theta}{ds}N+\sin\theta N'-\sin\theta\frac{d\theta}{ds}C+\cos\theta C'$$

$$f^*C^*\frac{ds^*}{ds}=\cos\theta\frac{d\theta}{ds}N+\sin\theta(fC)-\sin\theta\frac{d\theta}{ds}C+\cos\theta(-fN+gW)$$

Because $\{\alpha,\alpha^*\}$ is the WC^*-partner curves, we have

$$f^*W\frac{ds^*}{ds}=(\cos\theta\frac{d\theta}{ds}-f\cos\theta)N+(f\sin\theta-\sin\theta\frac{d\theta}{ds})C+g\cos\theta W$$

$$f^*\frac{ds^*}{ds}=g\cos\theta\ \text{ and } s=\int_0^{s^*}\frac{f^*}{g\cos\theta}ds^*.$$

Also, from $f\sin\theta - \sin\theta\dfrac{d\theta}{ds} = 0$ and $\cos\theta\dfrac{d\theta}{ds} - f\cos\theta = 0$, we get $f = \dfrac{d\theta}{ds}$ and

$$\theta = \int_0^s f ds.$$

□

Theorem 4. *Let $\{\alpha, \alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. α^* is a helix if and only if*

$$\frac{(\bar{\kappa} + \lambda g)\sin\theta}{(\bar{\tau} + \lambda')}$$

is constant.

Proof. If we take the derivative of the equation (3.1) according to parameter s, we have

$$T^*\frac{ds^*}{ds} = T + \lambda^{'} W + \lambda W'$$

and if we use the equation (2.6) and the alternative frame formulas, we get

$$(-\bar{\kappa}^* C^* + \bar{\tau}^* W^*)\frac{ds^*}{ds} = -\bar{\kappa}C + \bar{\tau}W + \lambda^{'} W - \lambda(gC).$$

From equation (3.2) and $W = C^*$,

$$(-\bar{\kappa}^* W + \bar{\tau}^* W^*)\frac{ds^*}{ds} = -\bar{\kappa}(\cos\theta N^* + \sin\theta W^*) + \bar{\tau}W + \lambda^{'} W - \lambda g(\cos\theta N^* + \sin\theta W^*)$$

$$\begin{aligned} \bar{\kappa}^*\frac{ds^*}{ds} &= -(\bar{\tau} + \lambda') \\ \bar{\tau}^*\frac{ds^*}{ds} &= -(\bar{\kappa} + \lambda g)\sin\theta \end{aligned}$$

and

$$\frac{\bar{\tau}^*}{\bar{\kappa}^*} = \frac{(\bar{\kappa} + \lambda g)\sin\theta}{(\bar{\tau} + \lambda')}. \tag{3.4}$$

Because of $\bar{\tau}^* = \dfrac{\tau^*}{f^*}$ and $\bar{\kappa}^* = \dfrac{\kappa^*}{f^*}$, we know that $\dfrac{\bar{\tau}^*}{\bar{\kappa}^*} = \dfrac{\tau^*}{\kappa^*}$. So, from equation (3.4), α^* is a helix if and only if

$$\frac{(\bar{\kappa} + \lambda g)\sin\theta}{(\bar{\tau} + \lambda')}$$

is constant. □

Theorem 5. *Let $\{\alpha, \alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. α^* is a slant helix if and only if*

$$\frac{g^*}{f^*} = \textit{constant}$$

Proof. If we use the derivative of the alternative frame, we have

$$\frac{dN^*}{ds^*} = f^* C^*$$

and

$$\frac{dW^*}{ds^*} = -g^* C^*.$$

Using the above two equations, we obtain that

$$\frac{g^*}{f^*} = -\frac{\frac{dW^*}{ds^*}}{\frac{dN^*}{ds^*}}.$$

Also if we take the derivative of the first equality of equation (3.2) according to s, we get

$$\begin{aligned} \frac{dN^*}{ds^*}\frac{ds^*}{ds} &= \cos\theta\frac{d\theta}{ds}N + \sin\theta N' - \sin\theta\frac{d\theta}{ds}C + \cos\theta C', \\ &= \cos\theta\frac{d\theta}{ds}N + \sin\theta(fC) - \sin\theta\frac{d\theta}{ds}C + \cos\theta(-fN + gW), \\ &= \left(\cos\theta\frac{d\theta}{ds} - f\cos\theta\right)N + \left(f\sin\theta - \sin\theta\frac{d\theta}{ds}\right)C + g\cos\theta W \end{aligned} \tag{3.5}$$

From the proof of the Theorem 3, we know that

$$f = \frac{d\theta}{ds}. \tag{3.6}$$

If we use the above equation in (3.5), we obtain that

$$\frac{dN^*}{ds^*}\frac{ds^*}{ds} = g\cos\theta W. \tag{3.7}$$

Similarly if we take the derivative of the second equality of equation (3.2) according to parameter s, we can easily see that

$$\begin{aligned} \frac{dW^*}{ds^*}\frac{ds^*}{ds} &= \sin\theta\frac{d\theta}{ds}N - \cos\theta N' + \cos\theta\frac{d\theta}{ds}C + \sin\theta C', \\ &= \sin\theta\frac{d\theta}{ds}N - \cos\theta(fC) + \cos\theta\frac{d\theta}{ds}C + \sin\theta(-fN + gW), \\ &= \left(\sin\theta\frac{d\theta}{ds} - f\sin\theta\right)N + \left(-f\cos\theta + \cos\theta\frac{d\theta}{ds}\right)C + g\sin\theta W \end{aligned} \tag{3.8}$$

If we use the equation (3.6) in (3.8), we have

$$\frac{dW^*}{ds^*}\frac{ds^*}{ds} = g\sin\theta W. \tag{3.9}$$

By proportioning the equations (3.7) and (3.9), we get

$$\frac{g^*}{f^*} = -\tan\theta = \text{constant}$$

□

Theorem 6. *Let $\{\alpha, \alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. Then the following relation exists*

$$\frac{g}{f} = \frac{f^*}{f\cos\theta}\frac{ds^*}{ds}$$

Proof. Using alternative frame $\{N, C, W\}$, we have

$$N' = fC \text{ and } W' = -gC.$$

If we calculate the ratio of these two equations, we obtain

$$\frac{g}{f} = -\frac{W'}{N'}. \tag{3.10}$$

From the following equations

$$\frac{dC^*}{ds^*} = -f^*N^* + g^*W^* \quad \text{and} \quad W = C^*,$$

we can see that

$$\frac{dW}{ds^*}\frac{ds^*}{ds} = (-f^*N^* + g^*W^*)\frac{ds^*}{ds}.$$

Also, from equation (3.2) ,

$$\begin{aligned} \frac{dW}{ds^*}\frac{ds^*}{ds} &= [-f^*(\sin\theta N + \cos\theta C) + g^*(-\cos\theta N + \sin\theta C)]\frac{ds^*}{ds} \\ &= [(-f^*\sin\theta - g^*\cos\theta)N + (-f^*\cos\theta + g^*\sin\theta)C]\frac{ds^*}{ds} \end{aligned} \tag{3.11}$$

If we use the equations $f^* = g\cos\theta$ and $g^* = -g\sin\theta$ in Theorem 2, we obtain

$$g^* = -\frac{f^*\sin\theta}{\cos\theta}$$

and if we write this equation in (3.11) , we get

$$\frac{dW}{ds^*}\frac{ds^*}{ds} = \left[\left(-f^*\sin\theta + \frac{f^*\sin\theta}{\cos\theta}\cos\theta\right)N + \left(-f^*\cos\theta - \frac{f^*\sin\theta}{\cos\theta}\sin\theta\right)C\right]\frac{ds^*}{ds}.$$

$$W' = -\frac{f^*}{\cos\theta}\frac{ds^*}{ds}C.$$

Also from the equation (3.10), we have

$$\frac{g}{f} = -\frac{W'}{N'} = -\frac{-\frac{f^*}{\cos\theta}\frac{ds^*}{ds}C}{fC}$$

$$\frac{g}{f} = \frac{f^*}{f\cos\theta}\frac{ds^*}{ds}.$$

So this completes the proof. □

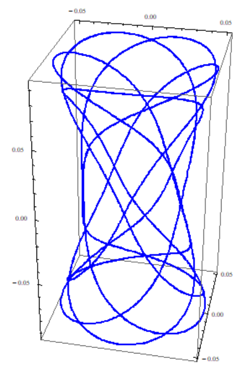

FIGURE 1. The curve α

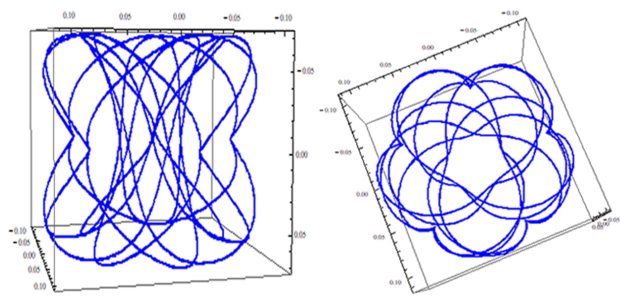

FIGURE 2. $WC^{*}-$partner curve of α

Example 1. *Let α be spatial curve given by the parametrization ([9])*

$$\alpha(s) = (\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s, \frac{-9}{208}\cos 16s + \frac{1}{117}\cos 36s, \frac{6}{65}\sin 10s).$$

If the necessary arrangements are made, we obtain the curvatures of α as follows

$$\kappa(s) = -24\sin 10s,\ \ \tau(s) = 24\cos 10s,\ \ f(s) = 24,\ \ g(s) = 10.$$

From the Theorem 1, the distance function is obtained as $\lambda(s) = \dfrac{\sin 10s}{10}$. Then the $WC^{*}-$partner curve α^{*} of α is obtained as

$$\alpha^{*}(s^{*}) = \alpha(s) + \lambda(s)W(s)$$

$$\begin{aligned} \alpha^{*}(s) \ \ = \ \ & (\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{9}{130}\cos 6s\sin 10s - \frac{4}{130}\cos 46s\sin 10s, \\ & \frac{-9}{208}\cos 16s + \frac{1}{117}\cos 36s + \frac{9}{130}\sin 6s\sin 10s - \frac{4}{130}\sin 46s\sin 10s, \end{aligned}$$

$$\frac{6}{65}\sin 10s+\frac{12}{130}\cos 20s\sin 10s).$$

Figure 1 shows the graph of the curve α and Figure 2 shows the $WC^{*}-$partner curve α^{*}.

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FORECASTING MORTALITY RATES WITH A GENERAL STOCHASTIC MORTALITY TREND MODEL

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ABSTRACT. This paper presents a model, which can closely predict the future mortality rates whose efficiency is performed through the comparisons with respect to Lee-Carter and mortality trend models. This general model estimates the logit function of death rate in terms of general tendency of the mortality evolution independent of age, the mortality steepness, additional effects of childhood, youth and old age. Generalized linear model (GLM) is used to estimate the parameters. Moreover, the weighted least square (WLS) and random walk with drift (RWWD) methods are employed to project the future values of the parameters. In order to ensure the stability of the outputs and construct the confidence intervals, Monte Carlo simulation is used. The impact of the proposed model is implemented on USA, France, Italy, Japan and Israel mortality rates for both genders based on their ageing structure. A detailed comparison study is performed to illustrate modified mortality rates on the net single premiums over mortality trend model and Lee-Carter model.

1. INTODUCTION

Having well-constructed mortality tables and accurate future mortality rate projections are important in many areas. In the case of under or over estimation of the rates, unpredictable losses can be experienced, especially, by the life and pension companies. For this reason, modeling mortality rates accurately has gained importance in recent decades, and been used especially to measure the longevity risk. The future values of the mortality rates are projected by many methods. The main idea of these studies is to model or systematize mortality rates from the past to the future so that the actuarial calculations become proper for both present and the future. In the literature, Lee and Li (2005) proposed a multi-population mortality modeling as an extension of Lee-Carter method which is accounted as a stochastic model [10]. Also, Jarner and Kryger (2011) studied a multi-population mortality model which includes long-term trend and short- to mid-term deviations

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using time series model [8]. An application of Canada and US female mortality has been conducted by Li and Hardy (2011) with respect to basis risk in longevity index hedges [9] incorporating four extensions to the Lee-Carter model. Börger (2010) proposed one-year period longevity risk by considering the adequacy of Solvency II scenarios [1]. The adequacy of longevity shock has been analyzed by comparing the resulting capital requirement to the Value-at-Risk (VaR) based on a stochastic mortality model. On the other hand, Plat (2011) modeled the changes of long-term mortality trend from the aspects of mortality and longevity risk proposing one-year VaR measure which aims at covering the risk of the variation in the projection year as well as the risk of changes in the best estimate projection for future years [11]. Another remarkable study was conducted by Richards et al. (2014) using different methods like Lee-Carter and Cairns-Blake-Dowd in order to determine one-year period longevity risk [12]. Börger et al. (2014) proposed a new mortality trend model, which contributed to a better quantification of mortality and longevity risk over time, under modern solvency regimes [2]. In their work, mortality trend model represents young and old age effects more precisely: They use three variables separately for each group of age. The outputs of this model are employed to compare the capital requirements with respect to Solvency II standard formula.

Constructing a model that includes additional effects of specific age groups is crucial for many pension systems, since different age groups have different effects on the trend. If the parameters of specific age effects are not used, the model becomes less sensitive to inner trends of each age groups. As the proportion of a specific age group whose inner trend is not well represented increase, the gap between installments and the compensations increase, too. For this reason, in this paper, we aim (i) to create a mortality trend model which includes both stochastic and deterministic terms to project future mortality rates accurately; (ii) to propose a modified trend model which includes the impact of young ages which is crucial, especially for populations having higher proportions at younger ages; (iii) to incorporate more stochastic structure to capture the stylized facts of mortality trend by modifying the model proposed by Börger (2014). We show the effect of these models on the valuation of net single premiums. The inclusion of the childhood effect parameter as modification to the linear model having the impact of old, center and young ages on mortality rates is expected to give more sensitive estimation of mortality rates. In this aspect, to our best knowledge, this study contributes to the literature of the quantification of influence of childhood effect by determining the threshold age to describe the childhood based on population dynamics. Validation of modified mortality trend model is performed using Mean Absolute Percentage Error (MAPE), R-Squared and applied to mortality tables from selected nationalities: USA, France, Italy, Japan and Israel. The choice of these countries is made according to their ageing structure such as young, middle and elderly populations. The results of the mortality trend model, the modified mortality trend model and Lee-Carter model are compared to demonstrate the efficiency on predictions.

This paper is organized as follows: Trend and modified trend models are presented in Section 2 along with the outline of the proposed model. Section 3 includes the application of the modified model on the mortality tables of the selected countries. Parameter estimation, projection of future mortality rates and comparison of the methods are also performed. The impact of the models on net single premium valuations is determined in Section 4. Last section concludes the paper.

2. Trend Models

The popularity of the trend models has been increased during the last decades with the Lee-Carter model [3]. There are also other trend models such as Heligman-Pollard [6], Cairns-Blake-Dowd (CBD) [14] which have significant contributions in this field. In this study, Lee-Carter is taken as benchmark model for the comparison, as it is the most commonly used method in the literature and practices.

2.1. **Lee-Carter (LC).** An extended version of the LC mortality rate model proposed by Girosi & King (2007) is [4]:

$$\log[q_{xt}] = \alpha_x + \beta_x \kappa_t + \varepsilon_{xt}, \tag{1}$$

where $q_{x,t}$ is the central rate of mortality at age x and in year t, α_x is the general tendency in the trend of mortality rates which depends on age, β_x explains the rate of decline in response, κ_t is a level of mortality index and the random error ε_{xt} follows a Normal distribution with mean 0 and variance σ_ε^2 with the following restrictions

$$\sum \beta_x = 1 \quad \text{and} \quad \sum \kappa_t = 0. \tag{2}$$

Here, x and t are age and time components, respectively. Singular Value Decomposition (SVD) is used in order to estimate β_x and κ_t parameters.

2.2. **Mortality trend model (MTM).** The mortality rate, $q_{x,t}$, is expressed as [2]:

$$\operatorname{logit} q_{x,t} = \alpha_x + \kappa_t^{(1)} + \kappa_t^{(2)}(x - x_{center}) + \kappa_t^{(3)}(x_{young} - x)^+ + \kappa_t^{(4)}(x - x_{old})^+ + \gamma_{t-x} \tag{3}$$

where $\operatorname{logit}(q_{x,t}) = \ln(\frac{q_{x,t}}{1-q_{x,t}})$ and $x^+ = \max\{x, 0\}$. Here, α_x is the location parameter, $\kappa^{(1)}$, $\kappa^{(2)}$, $\kappa^{(3)}$ and $\kappa^{(4)}$ are time dependent parameters, which represent the trend of ages, and γ is a normally distributed random error.

Börger (2014) states that including ages smaller than 20 disturbs the general trend of the mortality rate evolution and it leads to an increase in the empirical errors of the model, since mortality rates decreases over years at the childhood ages contrary to the rest of the ages [2]. Hence, small ages are generally not taken into account in the most of the similar studies. In order to increase the sensitivity to younger ages, we propose adding another additional effect parameter: childhood effect. Therefore, modified mortality trend model (M-MTM) should include the ages smaller than 20 in the analysis preventing such disturbance.

2.3. **Modified mortality trend model (M-MTM).** The model with childhood effect parameter is given as:

$$\begin{aligned}\operatorname{logit} q_{x,t} &= \alpha_x + \kappa_t^{(1)} + \kappa_t^{(2)}(x - x_{center}) + \kappa_t^{(3)}(x_{child} - x)\mathbb{I}_{(x<x_{child})} \\ &+\kappa_t^{(4)}(x_{young} - x)\mathbb{I}_{(x_{child}<x<x_{young})} + \kappa_t^{(5)}(x - x_{old})\mathbb{I}_{(x>x_{old})} + \gamma_{t-x} \qquad (4)\end{aligned}$$

M-MTM includes another time dependent parameter $\kappa^{(5)}$ at which, indicator functions, $\mathbb{I}$, are used to define valid ranges for $\kappa^{(3)}$, $\kappa^{(4)}$, and $\kappa^{(5)}$. Having these parameters specified for the age ranges is crucial to express the special effects of the age intervals. This model is expected to incorporate the impact of young ages into the estimation and projection of the mortality rates with more precision especially for the countries having high rate of birth and young population. Moreover, since the age boundaries such as x_{child}, x_{young}, x_{center} and x_{old} can be rearrangeable based on mortality breaks, the model is also applicable for the countries having dominancy on elderly people which is shown by Hasgul (2015) [5].

The M-MTM requires certain steps to estimate the parameters based on time-varying structured data. Following algorithm illustrates that the parameter estimation is the first performed by conditional GLM with respect to the age constraints on parameters $\kappa^{(3)}$, $\kappa^{(4)}$, and $\kappa^{(5)}$. After estimation of $\kappa^{(\cdot)}$ parameters, the future values of the parameters are projected using WLS and RWWD methods. The future mortality rates are obtained by substituting the projected $\kappa^{(\cdot)}$ parameters into the mortality trend model. Finally, a Monte Carlo algorithm is employed to resemble the predictions.

Let the mortality rates be defined as $q_{x,t}$ at age x and in year t. The steps of the algorithm:

(1) Apply the logit transformation to $q_{x,t}$ as $\log(q_{x,t}/(1 - q_{x,t}))$
(2) Apply GLM to the transformed mortality rates in given intervals as in Eqn[4]
(3) Test if WLS model for $\kappa^{(1)}$ is significant
(4) Test if $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$, $\kappa^{(5)}$ are stationary
(5) Apply WLS model for prediction of $\kappa^{(1)}$
(6) Apply appropriate time series model for forecasting $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$, $\kappa^{(5)}$
(7) Find future prediction of mortality rates via future parameters
(8) Resemble the predictions of m-simulations by MC

The parameters in Eqn.(4), $\kappa^{(1)}$, $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ represent general tendency, mortality steepness, additional effects of childhood, young and old ages, respectively.

The estimation of constant parameter, α_x for a fixed x defined by

$$\alpha_x = \frac{1}{t_{max} - t_{min} + 1} \sum_{t=t_{min}}^{t_{max}} \operatorname{logit} q_{x,t}, \qquad (5)$$

where t_{min} and t_{max} are starting and ending year of the time span of the data, respectively.

Estimation of $\kappa^{(\cdot)}$ is performed through a GLM, which enables more flexible fitting and compatibility with non-normal distribution of errors. By employing GLM, $\kappa_t^{(\cdot)}$ are estimated in the interval $[t_{min}, t_{max}]$. The GLM equation of the M-MTM is given as follows

$$\begin{pmatrix} logit(q_{x_{min},t}) - \alpha_{x_{min}} \\ logit(q_{x_{min}+1,t}) - \alpha_{x_{min}+1} \\ \vdots \\ logit(q_{x_{max},t}) - \alpha_{x_{max}} \end{pmatrix} \approx M \begin{pmatrix} \kappa_t^{(1)} \\ \vdots \\ \kappa_t^{(5)} \end{pmatrix},$$

where M is the coefficient matrix of the mortality trend model.

After the estimation process the next step is the projection of the future model parameters, coefficients and the future mortality rates.

The mortality tendency $\kappa^{(1)}$ is forecasted by employing WLS method which advantageously reflects the behaviour of the random errors and can be used with either linear or non-linear functions.

The first step of projection process is fitting a weighted least square model to $\kappa_t^{(1)}$. The weights (w_t) are given as:

$$w_t = \left(1 + \frac{1}{h}\right)^{t - t_{max}}; \quad \text{for } h > 0. \tag{6}$$

Note that the weights are chosen in such a way that the last years of the data has more contribution to the model than the earlier years in the projected model with the help of weight factor, h [2]. As t gets close to t_{max}, w_t gets larger with an increasing momentum for all h greater than zero.

We add a stochastic term to the best fitted regression line $l_{t_{max}}$, where $t \in [t_{min}, t_{max}]$. Hence, the forecast for $\kappa_t^{(1)}$ is obtained as follows:

$$\kappa_t^{(1)} = l_{t-1}(t) + \varepsilon_t^{(1)}(\sigma^{(1)} + \bar{\sigma}^{(1)}), \tag{7}$$

where $\varepsilon_t^{(1)} \sim^{iid} \mathcal{N}(0,1)$ for all $t \in [t_{min}, t_{max}]$. The volatility, $\sigma^{(1)}$, is the standard deviation of the empirical errors $(\kappa_t^{(1)} - l_{t-1}(t))$ for $[t_{min}+2, t_{max}]$ obtained from the WLS estimation and the term $\bar{\sigma}^{(1)}$ is optional volatility which is assumed as zero [2]. The projection of $\kappa_t^{(1)}$ over time is done iteratively by including each new element of projection in the following forecast.

As experienced from the literature, $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ parameters are generally non-stationary and have trends over time. Thus, RWWD is proposed to capture this stochastic pattern. In RWWD, the mean and standard deviation differences between $\kappa_t^{(\cdot)}$ and $\kappa_{t-1}^{(\cdot)}$ are assumed to be the drift, $\mu_t^{(\cdot)}$, and the volatility,

$\sigma^{(\cdot)}$, respectively. The dynamics of each parameter are given as follows:

$$\kappa_t^{(i)} = \kappa_{t-1}^{(i)} + \mu_t^{(i)} + \epsilon_t^{(i)}; \qquad \text{for } i = 2,3,4,5 \tag{8}$$
$$\text{with } \epsilon_t^{(\cdot)} \sim \mathcal{N}(0, \sigma^{(\cdot)}).$$

Finally, the future mortality rate projections are determined by Monte Carlo simulations. In addition, standard deviation of the generated samples are used in the construction of the confidence intervals (CI) for projections.

3. Implementation of the Model

MTM, M-MTM and LC Model are applied to the mortality rates of selected countries over years and the results are compared in order to test whether proposed model illustrates significancy with respect to age, time and country specific characteristics.

The reliability of the model should be examined through a validation process in order to apply the model to the selected data. Throughout the validation process, mortality rates corresponding to the last 8 years of each country are employed as in-sample justification and compared with projections. MAPE and R-Squared statistics are used in the comparison of the observed rates versus projections, and 95% confidence interval is constructed. Depending on the accuracy of in-sample forecast process, 10 years of projections are made on gender base for the selected countries.

The projections are made with the help of 'demography' package of R-code which uses the time series forecasts.

3.1. **Description of the data.** Five countries are selected with respect to their ageing structure. The main indicator for the classification of age structure is the median age. Italy (IT) and Japan (JPN) are supposed to have elderly populations with median ages 44.5 and 46, respectively. France (FR) and the United States of America (USA) are supposed as middle-aged countries having median age around 39, whereas Israel represents a young age population with median age of 30. Mortality rates are retrieved from Human Mortality Database [7]. The longest possible common (joint) time span (1960-2012) is selected. However, since Israel (ISR) mortality rates on the data sources are found to be available only after 1983, this country's mortality is studied within the years 1983-2014. Mortality rates for the ages 10 and 100 are taken into account in the study.

3.2. **Parameter estimation.** GLM has the advantage on relaxing normality assumption on dependent variable. Having two outcomes as being alive or dead in defined period allows us to assume that the distribution of mortality rates are Bernoulli distributed with the probabilities p_x and q_x, respectively. Thus, the logit function can be used as a link in order to transform response variable, $q_{x,t}$, into normal distribution which enables us to employ GLM and the parameters of the trend model in Eqn (4) are estimated. The dependence between mortality rates and

Table 1. The median ages and the selected time frame

Country	Group	Years	Child*	Young*	Center*	Old*	Median
Italy	Old	1960-2012	30; 30	55; 55	60; 60	85; 85	44.5
Japan	Old	1960-2012	30; 30	55; 55	65; 65	85; 85	46.1
France	Middle	1960-2012	35; 42	60; 55	67; 65	85; 85	40.9
USA	Middle	1960-2012	27; 25	55; 55	60; 60	85; 85	37.6
Israel	Young	1983-2014	40; 40	55; 55	60; 60	85; 85	29.9

*Age boundaries for genders (Male; Female)

its regressors are quantified (Table 2) and the correlations indicate that the general mortality trend, $\kappa^{(1)}$, has strong positive correlation to the corresponding mortality rates for all countries. On the other hand, old age effect, $\kappa^{(5)}$, expose negative correlations for all countries. Japan female (F) and France (F) cases yield the most significant correlations with the response for each parameter (old age parameter is in negative direction) compared to the others.

Table 2. Correlation coefficients between response variable and regressors in M-MTM

MALE	**IT**	**JPN**	**USA**	**FR**	**ISR**
general ($\kappa^{(1)}$)	0.9835	0.9967	0.9944	0.9903	0.9845
center ($\kappa^{(2)}$)	-0.4786	0.8148	0.6722	0.686	0.4841
child ($\kappa^{(3)}$)	0.2978	0.8523	0.8041	0.9749	0.6438
young ($\kappa^{(4)}$)	-0.6998	0.1016	-0.5707	0.0994	-0.6241
old ($\kappa^{(5)}$)	-0.7027	-0.8527	-0.9246	-0.8354	-0.6139

FEMALE	**IT**	**JPN**	**USA**	**FR**	**ISR**
general ($\kappa^{(1)}$)	0.9916	0.9979	0.9961	0.9868	0.9906
center ($\kappa^{(2)}$)	0.7347	0.9616	0.1592	0.9437	0.5068
child ($\kappa^{(3)}$)	0.8278	0.897	0.8884	0.979	0.6371
young ($\kappa^{(4)}$)	0.1404	0.9689	0.3562	0.6889	-0.245
old ($\kappa^{(5)}$)	-0.8931	-0.9455	-0.8584	-0.9526	-0.7567

The parameter estimates are obtained according to the proposed algorithm. Findings for some selected years are illustrated for USA male (M) data for a slice of the time span in Table 3. We see that the change in parameters over years is recognizable, especially in $\kappa^{(3)}$, $\kappa^{(4)}$ and κ^5 which correspond to childhood, young and old ages, respectively. Even if the changes in $\kappa^{(3)}$, $\kappa^{(4)}$ and κ^5 are greater than in $\kappa^{(1)}$ proportionally, it is important to note that $\kappa^{(1)}$ is the leading variable which has major effect on the mortality rate and makes it dramatically decrease over time

TABLE 3. Estimated $\kappa^{(\cdot)}$ parameters for USA (M) mortality rates

Parameter	**1960**	**1961**	**1962**	**.**	**2010**	**2011**	**2012**
$\kappa^{(1)}$	-3.7819	-3.8041	-3.7866	.	-4.4662	-4.4751	-4.4866
$\kappa^{(2)}$	0.0851	0.0852	0.0851	.	0.0805	0.0804	0.0799
$\kappa^{(3)}$	0.0053	0.0044	0.0028	.	-0.0221	-0.0212	-0.0244
$\kappa^{(4)}$	-0.0064	-0.0064	-0.0068	.	0.0024	0.0030	0.0032
$\kappa^{(5)}$	-0.0280	-0.0304	-0.0234	.	0.0422	0.0419	0.0452

(Figure 1). In other words, mortality rates generally decrease with respect to the major effect of $\kappa^{(1)}$.

In order to compare the parameters of different populations with each other, some transformations on the parameters should be conducted. Hence, some adjustments on $\kappa^{(1)}$ and $\kappa^{(2)}$ are done as follows [2] :

$$\kappa_t^{(2)} \Longleftarrow \kappa_t^{(2)} + \varphi_1,$$

$$\alpha_x \Longleftarrow \alpha_x - \varphi_1(x - x_{center}),$$

$$\varphi_2 = \alpha_{center},$$

$$\alpha_x \Longleftarrow \alpha_x - \varphi_2,$$

$$\kappa_t^{(1)} \Longleftarrow \kappa_t^{(1)} + \varphi_2,$$

where φ is the slope of the fitted regression line of α_x for $x \in \{x_{child}, ..., x_{old}\}$. These adjustments do not detort the results. However, they bring the parameters onto a comparable scale so that the explanation on the parameters become legitimate. $\kappa^{(\cdot)}$ estimates for the selected countries (M, F: 1960-2012) are illustrated in Figures 1-5. These graphs depict that the mortality rates of all countries in the study decrease over time and females generally tend to live longer compared to males which can be inferred from general tendency parameter ($\kappa^{(1)}$). While childhood parameter ($\kappa^{(3)}$) decreases over time, the old age parameter ($\kappa^{(5)}$) increases. In other words, these graphs shows us that the mortality rates of people aged below the boundary of childhood parameter ($\kappa^{(3)}$) decrease more than the amount the general tendecy ($\kappa^{(1)}$) proposes. However, the mortality rate of people aged above the boundary of old age parameter ($\kappa^{(5)}$) decreases less than the amount general tendency proposes.

This illustrates even though the mortality trend model has a decreasing pattern, the contribution of each age class may differ in the amount of the decay keeping up the pace of the decay based on mortality structure.

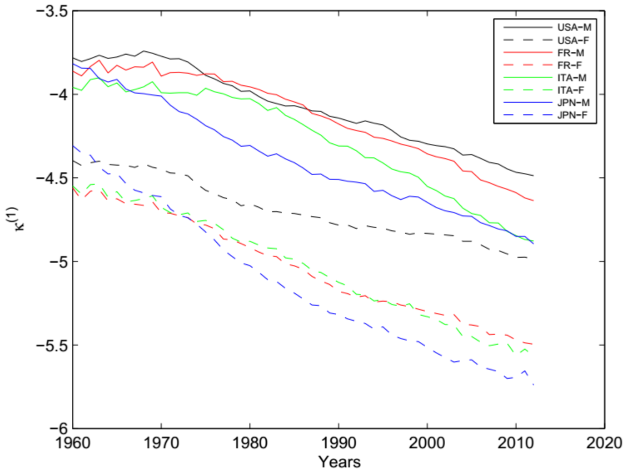

FIGURE 1. Estimation of general tendency parameter ($\kappa^{(1)}$)

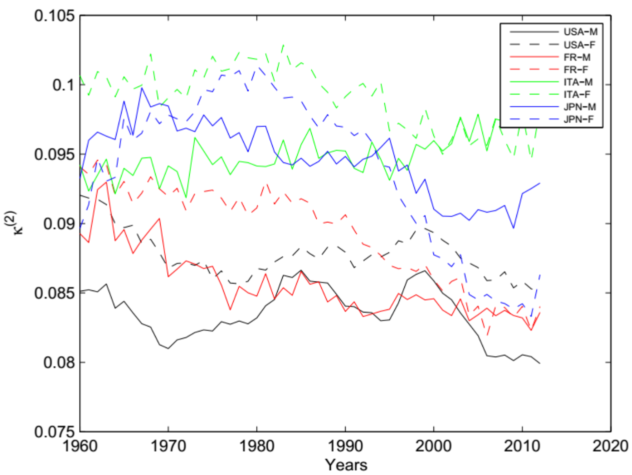

FIGURE 2. Estimation of slope in the logit parameter ($\kappa^{(2)}$)

3.3. **Projection of future mortality rates.** A future time frame of 2013-2022 is achieved by estimating $\kappa^{(\cdot)}$. A validation period 2005-2012 is taken into account to determine the estimation power of M-MTM. Projection of $\kappa^{(1)}$ is performed based on WLS as presented in earlier sections. Whereas, for the projections of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$, a stochastic modeling approach, RWWD is employed.

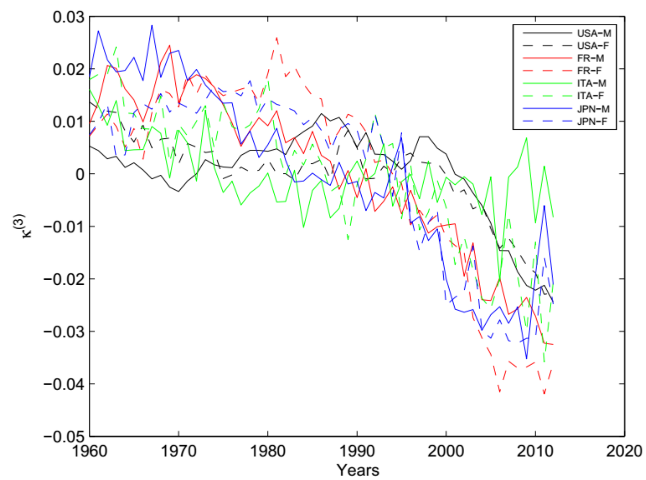

FIGURE 3. Estimation of childhood parameter ($\kappa^{(3)}$)

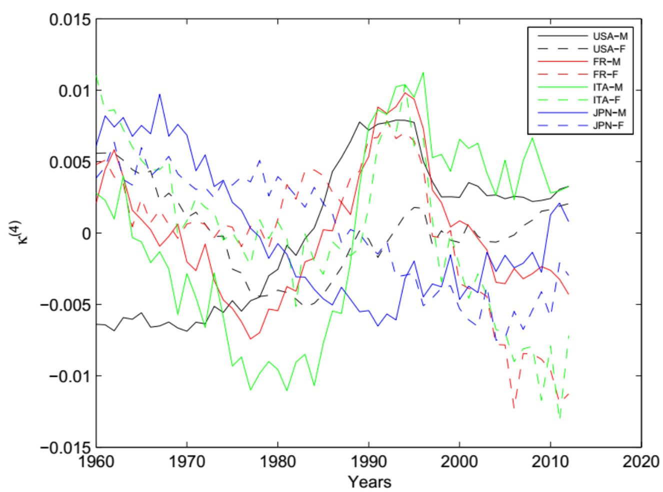

FIGURE 4. Estimation of young age parameter ($\kappa^{(4)}$)

WLS method is applied with five different values of weight factor h in order to detect the best choice. $\kappa^{(1)}$ projection values with the weight factors $h = 1$, $h = 2$, $h = 5$, $h = 10$ and $h = 20$ are shown in Figure 6. Small h values leads to less constribution of previous years. From Figure 6, we see that the projections with $h = 5$, $h = 10$ and $h = 20$ appear to have more stable paths than the projections

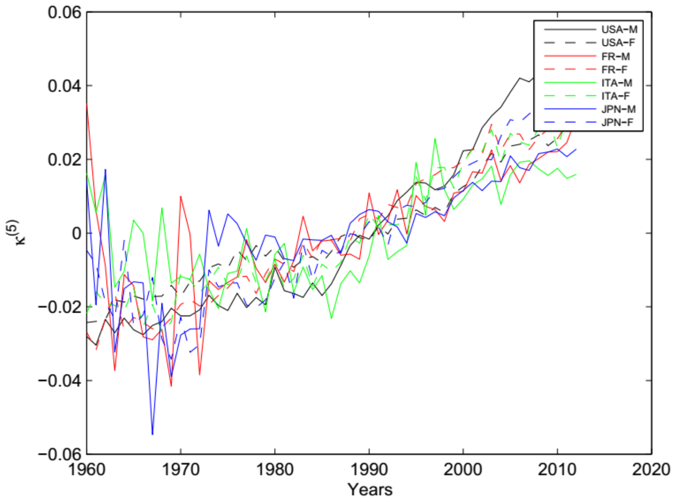

FIGURE 5. Estimation of old age parameter ($\kappa^{(5)}$)

with $h = 1$ and $h = 2$ considering the estimated $\kappa^{(1)}$ values. Hence, it can be inferred that the contribution of past years are important.

The linear model of years and $\kappa^{(1)}$ is significant (p-value < 0.001) and has $R^2 =$ 98%. Residuals also follow normal distribution (p-value of 0.9933).

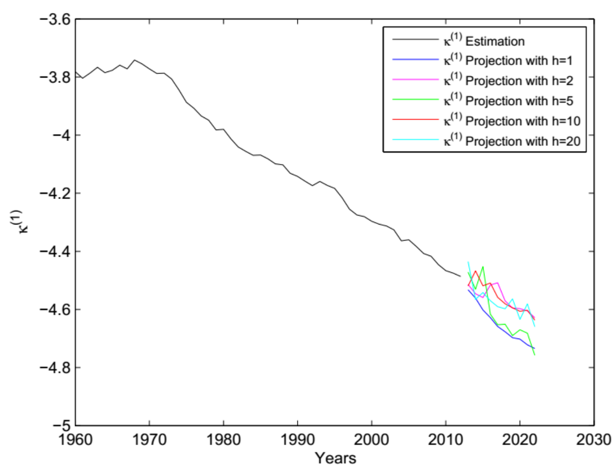

FIGURE 6. Effect of choice of h on $\kappa^{(1)}$ trend

The sequential estimation of $\kappa^{(1)}$ requires the following steps:

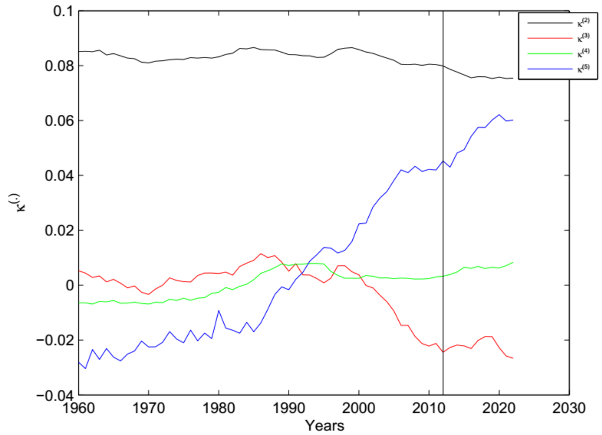

FIGURE 7. Projections of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ for the test and the forecast periods

(i) Apply WLS to n-estimated $\kappa^{(1)}$'s to predict $(n+1)^{th}$ value of $\kappa^{(1)}$,
(ii) Apply WLS to $(n+1)$-estimated $\kappa^{(1)}$'s to predict $(n+2)^{th}$ value of $\kappa^{(1)}$,
(iii) Continue until k projected ($n+k$ in total) $\kappa^{(1)}$ values are obtained.

Since the data of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ are non-stationary processes and have drifts, RWWD method is used to model future values of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$. Average and standard deviation of differences between n^{th} and $(n+1)^{th}$ values in each series of $\kappa^{(\cdot)}$'s correspond to drift and volatility, respectively, given by the following equation.

$$\hat{\kappa}_{t+1}^{(\cdot)} = \kappa_t^{(\cdot)} + \mu^{(\cdot)} + \epsilon_t^{(\cdot)}, \tag{9}$$

where $\mu^{(\cdot)}$ is drift, $\sigma^{(\cdot)}$ is standard deviation and it is assumed that $\epsilon_t \sim \mathcal{N}(0, \sigma^{(\cdot)})$.

The trends of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ for a time interval between 1960-2022 are shown in Figure 7. While interpreting the projections of $\kappa^{(4)}$ (additional effect of young ages) and $\kappa^{(5)}$ (additional effect of old ages) parameters, the sustainable decrease in $\kappa^{(1)}$ parameter should be considered as well. Although, an increase in the projections of parameters is observed, it does not necessarily indicate the increase in the mortality rates of specific age groups, such as young and old ages. Since $\kappa^{(1)}$ has a linear decrease as shown in Figure 6, we infer that the rate of reduction in mortality rates of young and old ages decreases over time as values of $\kappa^{(4)}$ and $\kappa^{(5)}$ increase (Figure 7).

Based on the forecasts of $\kappa^{(1)}$, $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$, mortality rates corresponding to the period of 8 years are projected by employing Monte Carlo simulation with $m = 10^4$ trials for the random components for USA (M) case. To justify the precision and accuracy of the model, MAPE, R^2-values and 95% confidence interval of projected mortality rates are determined. Table 4 shows that MAPE values are smaller than 10%. The range of error is between 4.7% and 8% and the average of all errors is found to be 6.4%. Moreover, R^2-values are considerably high which indicate the accuracy of the proposed M-MTM.

TABLE 4. MAPE and R^2-values for USA (M) between 2005-2012

%	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**Average**
MAPE	4.75	5.43	5.60	5.63	7.47	8.00	6.89	7.65	6.43
R^2	99.99	99.98	99.95	99.97	99.93	99.94	99.98	99.94	

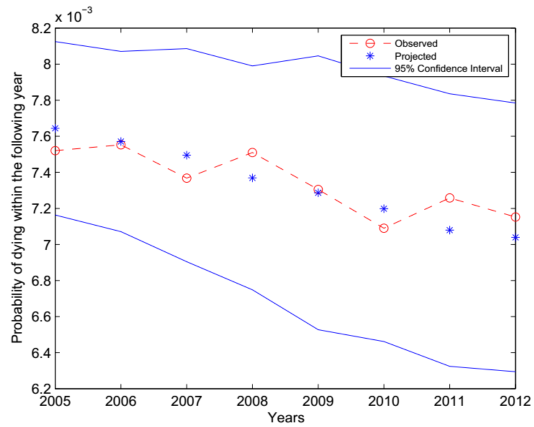

FIGURE 8. In-sample estimates of USA (M) mortality rates (α = 5%, M, x=55)

For the illustration purposes and space limitations, the predicted mortality rates of age 55 for USA (M) are exibited with 95% confidence interval based on the mean and standard deviation of estimates obtained from Monte Carlo simulation for years 2005-2012. As it is seen in Figure 8, all the observed mortality rates remain within the confidence interval of in-sample estimates.

The high values of R^2 (>99%) are also the indication to a systematic risk which is presumed to arise from the high proportion of variance in the dependent variable explained by independent variables in the M-MTM.

Henceforth, we predict future 10-years (2013-2022) mortality rates with 95% confidence band using M-MTM whose outcomes are plotted in Figure 9 and 10 for M and F, respectively. We see that the pattern of mortality rates in USA at age 55 is consistent for both genders.

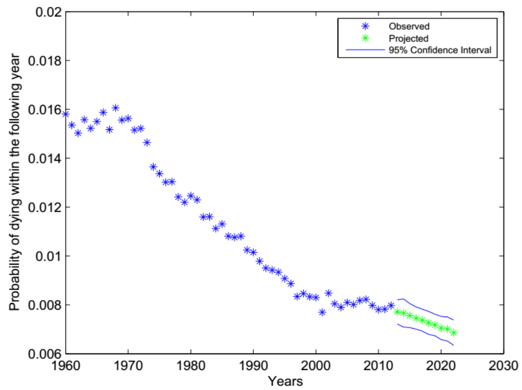

FIGURE 9. Projection for USA mortality rates and 95% CI (x=55, M)

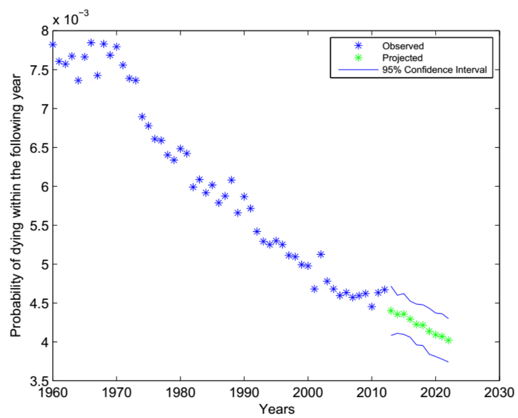

FIGURE 10. Projection for USA mortality rates and 95% CI (x=55, F)

3.4. **Comparison of models.** The comparison of three models, MTM, M-MTM and LC, is carried out for each country on gender base for the years 2005-2012 except for Israel whose time frame is taken as 2007-2014. MAPE values are shown in Table 5. The average of the values are taken as indicators of performance of the models. The MAPE values for years between 2005 and 2012 yield an average of value ranging between 4.69% and 20.78% for old and middle age countries. However, MAPE averages for young age country (Israel) are around 17% and 20%. The best performance is marked by bold font for each case in Table 5. It is observed that the minimum average error is achieved by M-MTM, except Italy (M) where LC model outperforms compare to the others. It is also interesting to note that Italy (F) results do not show any superiority in any model. It can be concluded that M-MTM estimation carries almost the majority in good performance compared to the other two. To determine if the MAPE values of each model are statistically

Table 5. MAPE values for old and middle-aged countries (in %)

GROUP	DATA	GND	MODEL	2005	2006	2007	2008	2009	2010	2011	2012	Mean
			MTM	16.56	19.59	18.33	17.37	20.54	26.00	23.24	27.38	21.13
OLD	**ITA**	**M**	**M-MTM**	15.18	17.33	16.96	14.29	19.65	24.76	22.68	25.83	19.59
			LC	8.36	9.21	10.14	9.40	13.37	15.45	16.36	18.45	**12.59**
			MTM	8.42	8.42	8.84	8.45	8.54	9.74	8.06	7.66	8.52
OLD	**ITA**	**F**	**M-MTM**	7.48	7.14	8.58	7.63	7.61	9.75	8.23	9.25	**8.21**
			LC	8.64	5.64	8.38	8.45	9.62	10.34	10.94	11.68	9.21
			MTM	11.04	9.03	9.28	10.04	9.53	12.09	17.24	11.98	11.28
OLD	**JPN**	**M**	**M-MTM**	8.01	5.70	6.13	7.15	6.92	9.44	14.01	8.99	**8.30**
			LC	9.13	7.54	8.24	8.67	9.13	10.81	14.34	11.06	9.87
			MTM	13.21	14.06	14.12	16.42	17.15	19.25	29.18	21.90	18.16
OLD	**JPN**	**F**	**M-MTM**	10.77	11.59	11.79	13.85	14.83	16.70	27.15	19.32	**15.75**
			LC	16.49	17.39	17.37	19.10	20.19	21.45	30.63	23.62	20.78
			MTM	8.85	10.92	12.86	13.50	12.47	13.82	15.57	18.50	13.31
MIDDLE	**FR**	**M**	**M-MTM**	8.59	11.15	12.92	13.10	12.27	13.53	14.63	17.53	12.96
			LC	9.10	9.72	11.48	12.28	12.29	13.85	14.74	17.52	12.62
			MTM	8.51	7.53	6.51	7.85	8.88	8.65	7.58	10.16	**8.21**
MIDDLE	**FR**	**F**	**M-MTM**	6.72	6.93	7.14	7.52	8.31	9.94	9.23	11.64	**8.43**
			LC	10.30	10.86	11.18	11.59	11.39	13.05	13.15	14.32	11.98
			MTM	5.43	6.37	6.96	7.88	9.41	10.78	9.60	10.01	8.31
MIDDLE	**USA**	**M**	**M-MTM**	4.76	5.47	5.50	5.99	7.49	8.48	7.35	7.74	**6.60**
			LC	6.04	6.85	7.43	7.54	9.01	9.55	9.86	10.29	8.32
			MTM	7.42	7.61	7.22	8.06	7.88	8.56	9.07	9.37	8.15
MIDDLE	**USA**	**F**	**M-MTM**	4.70	3.69	3.78	4.29	5.08	5.03	5.30	5.69	**4.69**
			LC	6.60	7.62	7.60	8.76	9.54	10.59	11.07	11.37	9.14
				2007	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	
			MTM	15.69	15.26	19.21	19.10	22.13	26.42	28.15	24.69	21.33
YOUNG	**ISR**	**M**	**MMTM**	14.55	13.07	14.70	16.08	18.05	22.73	23.60	20.01	**17.85**
			LC	16.48	15.86	17.45	17.83	23.60	28.49	25.03	24.39	21.14
			MTM	14.86	18.50	17.51	17.44	17.91	17.49	22.07	18.78	**18.07**
YOUNG	**ISR**	**F**	**MMTM**	13.72	18.88	17.06	19.13	20.06	20.53	24.92	21.93	19.53
			LC	16.44	19.29	19.48	19.26	18.45	17.88	25.52	20.38	19.59

different from each other, t-test is performed for each population considered (Table 6). The results of comparison tests indicate that the M-MTM generally has a better precision than MTM and LC. For Italy (M) which is in old group, LC has a better precision compared with M-MTM and MTM. It can be generalized that M-MTM performs better than Lee Carter for the rest of the cases. MTM is found to be preferable for some cases such as Israel (F) mortality rates. This can be distinguished in pairwise comparisons which are also differentiated by colors (Table 6). As blue color refers to M-MTM is preferable, green to MTM and red color stands for superiority of LC to the others. Black color shows no favoration on the model choice.

Table 6. MAPE comparisons via p-values of t-tests

DATA	GND	MMTM&MTM	MMTM&LC	MTM&LC
USA	**M**	0,00012*	0.00005*	0.94370
	F	<0.00001*	0.00002*	0.03590*
FR	**M**	0.05530	0.25270	0.01760*
	F	0.62840	<0.00001*	0.00005*
ITA	**M**	0.00091	<0.00001*	<0.00001*
	F	0.37610	0.08410	0.37480
JPN	**M**	<0.00001*	0.00020*	0.00100*
	F	<0.00001*	<0.00001*	0.00002*
ISR	**M**	0.00016*	0.00018*	0.11270
	F	0.00310*	0.8754	0.02430*

*statistically significant at 5% level

4. Performance of the Models on the Net Single Premium Calculations

In order to illustrate the impact of the models on the valuation of NSP, we assume a hypothetical term life insurance scenario with the following assumptions: (i) a constant annual interest rate of 10%, (ii) an 8-year term life insurance for ages 25, 35, 45 and 55. The risk premium corresponding to one unit benefit of life insurance payment for aged x, which covers next n years [13].

$$A^{1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x q_{x+k}; \tag{10}$$

where v denotes discount factor, ${}_kp_x$ is the probability of living k years at age x and q_{x+k} stands for the probability of death in one year between age $(x+k)$ and $(x+k+1)$.

Table 7 demonstrates the values of NSP under three models and the original mortality rates (between 2005-2012) and depicts the best model yielding the closest NSP to the one obtained with respect to the original. Moreover, Table 7 shows that MTM gives the closest values to original NSP for age 25, whereas, M-MTM outperforms the other two models for the ages, 35, 45, 55 for the term life insurance of 8 years.

5. Conclusion

The prediction of future mortality rates using proposed model (M-MTM) incorporates the childhood effect into the mortality trend model (MTM) [2] and the stochastic approach to estimate its parameters are found to yield remarkable results. Since the mortality trend of young people has a different slope than the rest of the population, forecasts including younger ages, such as 5-20 and 10-20, have

TABLE 7. The impact of models on NSP of 8-year Term Life insurance

Age	Country	Gender	M-MTM	MTM	LC	Original	Best
25-33	**USA**	**M**	0.008287	**0.008302**	0.008182	0.008561	MTM
		F	**0.003286**	0.002875	0.003235	0.003530	**M-MTM**
	FR	**M**	0.005655	**0.005223**	0.006684	0.005179	MTM
		F	0.001987	**0.001790**	0.002190	0.001855	MTM
	ITA	**M**	0.004739	**0.003997**	0.004964	0.003912	MTM
		F	0.001407	**0.001330**	0.001400	0.001280	MTM
	JPN	**M**	0.003565	**0.003613**	0.002969	0.003874	MTM
		F	0.001551	**0.001616**	0.001173	0.001996	MTM
	ISR	**M**	**0.003750**	0.004143	0.004262	0.003676	**M-MTM**
		F	0.001554	**0.001414**	0.001533	0.001399	MTM
35-43	**USA**	**M**	**0.012007**	0.012344	0.012515	0.011400	**M-MTM**
		F	**0.006408**	0.006022	0.006347	0.006580	**M-MTM**
	FR	**M**	0.010960	**0.009764**	0.010964	0.009125	MTM
		F	**0.004422**	0.004099	0.004697	0.004385	**M-MTM**
	ITA	**M**	0.007123	0.006334	**0.006067**	0.005757	LC
		F	0.003176	**0.003003**	0.002924	0.003065	MTM
	JPN	**M**	**0.006319**	0.005884	0.005558	0.006762	**M-MTM**
		F	0.003098	**0.003271**	0.002572	0.003709	MTM
	ISR	**M**	**0.005950**	0.005785	0.007240	0.005901	**M-MTM**
		F	0.003413	0.003030	**0.003306**	0.003256	LC
45-53	**USA**	**M**	**0.026204**	0.027077	0.023213	0.026604	**M-MTM**
		F	0.015079	**0.015142**	0.014813	0.016409	MTM
	FR	**M**	0.026190	0.026482	**0.025168**	0.024169	LC
		F	**0.011161**	0.010772	0.010458	0.011906	**M-MTM**
	ITA	**M**	0.016626	0.017518	**0.013139**	0.014356	LC
		F	0.008301	**0.008529**	0.007575	0.008677	MTM
	JPN	**M**	0.015239	0.014135	**0.015901**	0.016220	LC
		F	**0.007166**	0.006875	0.006604	0.008543	**M-MTM**
	ISR	**M**	0.015268	**0.015095**	0.013805	0.014885	MTM
		F	**0.008291**	0.008404	0.008285	0.008333	**M-MTM**
55-63	**USA**	**M**	**0.060004**	0.061893	0.051921	0.055702	**M-MTM**
		F	**0.033970**	0.035285	0.037536	0.032919	**M-MTM**
	FR	**M**	**0.057042**	0.061561	0.049459	0.053340	**M-MTM**
		F	0.020789	**0.022280**	0.019778	0.023075	MTM
	ITA	**M**	0.045992	0.050541	**0.038009**	0.037771	LC
		F	**0.020042**	0.021421	0.018474	0.019912	**M-MTM**
	JPN	**M**	0.037035	0.034616	**0.038211**	0.040878	LC
		F	**0.015273**	0.013961	0.013958	0.017944	**M-MTM**
	ISR	**M**	**0.042078**	0.043441	0.032790	0.040081	**M-MTM**
		F	**0.021520**	0.023886	0.020389	0.021706	**M-MTM**

always been challenging in the mortality trend modeling. However, we show that M-MTM handles this problem with a better accuracy in predictions. The implementation and illustration of the proposed model are done on the mortality rates of 5 countries in order to determine the effect of demographic structure (old, middle and young age). Monte Carlo simulation is used to generate possible projections and construct confidence interval for these projections. Comparison of the proposed model is done with respect to MTM and LC model where the performance is examined via efficiency indicators. In most of the cases, our proposed model performs more accurate results than the other two models. In other words, in the case that these three models are conducted in a wider range of ages including childhood ages, modified model projects the future mortality rates with less margin of errors. Additionally, projections performed by M-MTM generally have narrower confidence intervals and more precise forecasts compared to MTM and LC model. For this reason, the future mortality projected by M-MTM would be more contributing. For a term life insurance, net single premium (NSP) estimation by M-MTM for the ages over 35 generally gives closest results to realized NSP compared to the estimations by other two models. The outcomes of this study show that M-MTM is advantageous in mortality modeling since the opportunity of changing age boundaries including the childhood makes M-MTM applicable for all types of different age-level populations.

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FIRST ORDER MAXIMAL DISSIPATIVE SINGULAR DIFFERENTIAL OPERATORS

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ABSTRACT. In this paper, using the Calkin-Gorbachuk method, the general form of all maximal dissipative extensions of the minimal operator generated by first order linear multipoint symmetric singular differential-operator expression in the direct sum of Hilbert space of vector-functions has been found. Later on, the structure of spectrum of these extensions is researched. Finally, the results are supported by an application.

1. INTRODUCTION

Operator theory is important to understand the nature of the spectral properties of an operator associated with a boundary value problem acting on a Hilbert space. To obtain such an information as is well known that the corresponding inner product is useful. A linear closed densely defined operator $T : D(T) \subset X \to X$ in a Hilbert space X is called to be dissipative if and only if

$$Im(T\psi, \psi)_X \geq 0, \ \psi \in D(T),$$

where $Im(\cdot , \cdot)$ and $D(T)$ denote the imaginary part of the inner product and the domain of the operator T, respectively (see [3]). If a dissipative operator has no any proper dissipative extension, then it is called maximal dissipative [3]. A direct result on dissipative operators is that their spectrum lies in the closed upper half-plane. Therefore, open lower half-plane does not belong to the spectrum of T. Maximal dissipative operators play a very important role in mathematics and physics. In physics, there are many interesting applications of the dissipative operators in areas like hydrodynamic, laser and nuclear scattering theories.

Remember that the general theory of self-adjoint extensions of linear densely-defined closed symmetric operators in any Hilbert space was mentioned in the well-known work of Neumann [9]. The complete informations of Vishik's and Birman's

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investigations on the all non-negative selfadjoint extensions of a positive closed symmetric operator have been given by Fischbacher in [2].

The functional model theory of Nagy and Foias [6] is a basic method for investigation the spectral properties of dissipative operators. The maximal dissipative extensions and their spectral analysis of the minimal operator having equal deficiency indices generated by formally symmetric differential-operator expression in one finite or infinite interval case in the Hilbert space of vector-functions have been researched by Gorbachuk [3]. This method has been generalized in terms of boundary values by Rofe-Beketov, Kholkin in [8].

In the present study, in Section 3, using the Calkin-Gorbachuk method, the representation of all maximal dissipative extensions of the minimal operator generated by the first order linear symmetric differential-operator expression in the direct sum of Hilbert spaces of vector-functions in two infinite interval case is obtained. Later on, in Section 4, we also investigate the structure of spectrum of these dissipative extensions.

2. Statement of the problem

Let X be a separable Hilbert space and $a_1, a_2 \in \mathbb{R}$ such that $a_1 < a_2$. In the Hilbert spaces

$$\mathcal{X} = L^2(X, (-\infty, a_1)) \oplus L^2(X, (a_2, \infty))$$

of vector-functions on $(-\infty, a_1) \cup (a_2, \infty)$, consider the following linear multipoint differential operator expression for first order of the form

$$l(\nu) = (l_1(\nu_1), l_2(\nu_2)),\ \nu = (\nu_1, \nu_2),$$

where

$$\begin{aligned} l_1(\nu_1) &= i\nu_1' + \Omega_1\nu_1, \\ l_2(\nu_2) &= i\nu_2' + \Omega_2\nu_2, \end{aligned}$$

where $\Omega_m : D(\Omega_m) \subset X \to X,\ m = 1, 2$ are linear selfadjoint operators.

The minimal Υ_0^1 and Υ_0^2 operators corresponding to differential operator expression $l_1(\cdot)$ and $l_2(\cdot)$ in $L^2(X, (-\infty, a_1))$ and $L^2(X, (a_2, \infty))$ can be constructed by using the same technique in [4], respectively. The operators $\Upsilon^1 = (\Upsilon_0^1)^*,\ \Upsilon^2 = (\Upsilon_0^2)^*$ are maximal operators corresponding to $l_1(\cdot)$ and $l_2(\cdot)$ in $L^2(X, (-\infty, a_1))$ and $L^2(X, (a_2, \infty))$, respectively. In this case, the operators

$$\Upsilon_0 = \Upsilon_0^1 \oplus \Upsilon_0^2 \text{ and } \Upsilon = \Upsilon^1 \oplus \Upsilon^2$$

in the Hilbert space $\mathcal{X}$ are called minimal and maximal operators corresponding to differential operator expression $l(\cdot)$, respectively.

We have that the domains of the operators Υ and Υ_0 are of the form

$$\begin{aligned} D(\Upsilon) &= \{\nu \in \mathcal{X} : l(\nu) \in \mathcal{X}\}, \\ D(\Upsilon_0) &= \{\nu \in D(\Upsilon) : \nu_1(a_1) = \nu_2(a_2) = 0\}. \end{aligned}$$

Our aim in this paper is to obtain all maximal dissipative extensions of the minimal operator Υ_0 in $\mathcal{X}$ in terms of boundary values and investigate the spectrum of them. Then, we give an application of obtained results to the concrete model.

3. Representation of maximal dissipative extensions

In this section, we will study the abstract representation of all maximal dissipative extensions of Υ_0 in terms of boundary values using the Calkin-Gorbachuk method.

Firstly, let us define the deficiency indices of any symmetric operator in a Hilbert space.

Definition 1. [7] *Let T be a symmetric operator, λ be an arbitrary non-real number and $\mathfrak{X}$ be a Hilbert space. We denote by $\mathcal{R}_{\overline{\lambda}}$ and $\mathcal{R}_\lambda$ the ranges of the operator $\left(T-\overline{\lambda}I\right)$ and $(T-\lambda I)$, respectively, where I is identity operator on $\mathfrak{X}$. Clearly, $\mathcal{R}_{\overline{\lambda}}$ and $\mathcal{R}_\lambda$ are subspaces of $\mathfrak{X}$, which need not necessarily be closed. We call $\left(\mathfrak{X}-\mathcal{R}_{\overline{\lambda}}\right)$ and $(\mathfrak{X}-\mathcal{R}_\lambda)$, which are their orthogonal complements, the deficiency spaces of the operator T and we denote them by $\mathcal{N}_{\overline{\lambda}}$ and $\mathcal{N}_\lambda$, respectively: thus*

$$\mathcal{N}_{\overline{\lambda}}=\mathfrak{X}-\mathcal{R}_{\overline{\lambda}},\quad \mathcal{N}_\lambda=\mathfrak{X}-\mathcal{R}_\lambda.$$

The numbers

$$n_{\overline{\lambda}}=dim\mathcal{N}_{\overline{\lambda}},\quad n_\lambda=dim\mathcal{N}_\lambda$$

are called deficiency indices of the operator T.

Let us prove the following auxiliary result we will need:

Lemma 2. *The deficiency indices of Υ_0 are of the form*

$$(n_+(\Upsilon_0),n_-(\Upsilon_0))=(dimX,dimX).$$

Proof. Here, without loss generality it will be assumed that $\Omega_1=\Omega_2=0$. The general solution of the differential equations can be given as follows:

$$i\nu'_{1\pm}(\xi)=\mp i\nu_{1\pm}(\xi),\ \xi<a_1,$$
$$i\nu'_{2\pm}(\xi)=\mp i\nu_{2\pm}(\xi),\ \xi>a_2$$

where

$$\begin{aligned}\nu_{1\pm}(\xi)&=exp\left(\mp(\xi-a_1)\right)\kappa_1,\ \kappa_1\in X,\ \xi<a_1,\\ \nu_{2\pm}(\xi)&=exp\left(\mp(\xi-a_2)\right)\kappa_2,\ \kappa_2\in X,\ \xi>a_2,\end{aligned}$$

respectively. Hence, we have

$$\begin{aligned}n_+(\Upsilon_0^1)&=dimKer(\Upsilon^1+iI)=0,\\ n_-(\Upsilon_0^1)&=dimKer(\Upsilon^1-iI)=dimX,\\ n_+(\Upsilon_0^2)&=dimKer(\Upsilon^2+iI)=dimX,\\ n_-(\Upsilon_0^2)&=dimKer(\Upsilon^2-iI)=0,\end{aligned}$$

where I is identity operator in the corresponding space. Therefore, we get

$$\begin{aligned} n_+(\Upsilon_0) &= n_+(\Upsilon_0^1) + n_+(\Upsilon_0^2) = dimX, \\ n_-(\Upsilon_0) &= n_-(\Upsilon_0^1) + n_-(\Upsilon_0^2) = dimX. \end{aligned}$$

□

Consequently, the operator Υ_0 has a maximal dissipative extension (see [3]). In order to describe all maximal dissipative extensions of Υ_0, it is necessary to construct a space of boundary values for it.

Definition 3. [3] *Let $\mathfrak{X}$ be any Hilbert space and $S : D(S) \subset \mathfrak{X} \to \mathfrak{X}$ be a closed densely defined symmetric operator on the Hilbert space having equal finite or infinite deficiency indices. A triplet $(\mathbf{X}, \beta_1, \beta_2)$, where $\mathbf{X}$ is a Hilbert space, β_1 and β_2 are linear mappings from $D(S^*)$ into $\mathbf{X}$, is called a space of boundary values for the operator S, if for any $\eta, \kappa \in D(S^*)$*

$$(S^*\eta, \kappa)_{\mathfrak{X}} - (\eta, S^*\kappa)_{\mathfrak{X}} = (\beta_1(\eta), \beta_2(\kappa))_{\mathbf{X}} - (\beta_2(\eta), \beta_1(\kappa))_{\mathbf{X}}$$

while for any $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{X}$, there exists an element $\eta \in D(S^*)$ such that $\beta_1(\eta) = \mathcal{G}_1$ and $\beta_2(\eta) = \mathcal{G}_2$.

Lemma 4. *The triplet (X, β_1, β_2), where*

$$\begin{aligned} \beta_1 : D(\Upsilon) \to X,\ \beta_1(\nu) &= \frac{1}{\sqrt{2}} (\nu_1(a_1) - \nu_2(a_2)) \quad \text{and} \\ \beta_2 : D(\Upsilon) \to X,\ \beta_2(\nu) &= \frac{1}{i\sqrt{2}} (\nu_1(a_1) + \nu_2(a_2)),\ \nu = (\nu_1, \nu_2) \in D(\Upsilon) \end{aligned}$$

is a space of boundary values of the minimal operator Υ_0 in $\mathcal{X}$.

Proof. For any $\nu = (\nu_1, \nu_2), \vartheta = (\vartheta_1, \vartheta_2)$ from $D(\Upsilon)$, one can easily check that

$$\begin{aligned} (\Upsilon\nu, \vartheta)_{\mathcal{X}} - (\nu, \Upsilon\vartheta)_{\mathcal{X}} &= (\Upsilon^1\nu_1, \vartheta_1)_{L^2(X,(-\infty,a_1))} + (\Upsilon^2\nu_2, \vartheta_2)_{L^2(X,(a_2,\infty))} \\ &- (\nu_1, \Upsilon^1\vartheta_1)_{L^2(X,(-\infty,a_1))} - (\nu_2, \Upsilon^2\vartheta_2)_{L^2(X,(a_2,\infty))} \\ &= \left[(i\nu_1' + \Omega_1\nu_1, \vartheta_1)_{L^2(X,(-\infty,a_1))} - (\nu_1, i\vartheta_1' + \Omega_1 v_1)_{L^2(X,(-\infty,a_1))}\right] \\ &+ \left[(i\nu_2' + \Omega_2\nu_2, \vartheta_2)_{L^2(X,(a_2,\infty))} - (\nu_2, i\vartheta_2' + \Omega_2\vartheta_2)_{L^2(X,(a_2,\infty))}\right] \\ &= \left[(i\nu_1', \vartheta_1)_{L^2(X,(-\infty,a_1))} - (\nu_1, i\vartheta_1')_{L^2(X,(-\infty,a_1))}\right] \\ &+ \left[(i\nu_2', \vartheta_2)_{L^2(X,(a_2,\infty))} - (\nu_2, i\vartheta_2')_{L^2(X,(a_2,\infty))}\right] \\ &= i\left[(\nu_1', \vartheta_1)_{L^2(X,(-\infty,a_1))} + (\nu_1, \vartheta_1')_{L^2(X,(-\infty,a_1))}\right] \\ &+ i\left[(\nu_2', \vartheta_2)_{L^2(X,(a_2,\infty))} + (\nu_2, \vartheta_2')_{L^2(X,(a_2,\infty))}\right] \\ &= i\left[(\nu_1(a_1), \vartheta_1(a_1))_X - (\nu_2(a_2), \vartheta_2(a_2))_X\right] \\ &= (\beta_1(\nu), \beta_2(\vartheta))_X - (\beta_2(\nu), \beta_1(\vartheta))_X. \end{aligned}$$

Now let $f_1,\ f_2 \in X$. Let us find the function $\nu = (\nu_1, \nu_2) \in D(\Upsilon)$ such that

$$\beta_1(\nu) = \frac{1}{\sqrt{2}}\,(\nu_1(a_1) - \nu_2(a_2)) = f_1$$

and

$$\beta_2(\nu) = \frac{1}{i\sqrt{2}}\,(\nu_1(a_1) + \nu_2(a_2)) = f_2.$$

Hence, we can obtain

$$(\nu_1)(a_1) = (if_2 + f_1)/\sqrt{2},\quad (\nu_2)(a_2) = (if_2 - f_1)/\sqrt{2}.$$

If we choose the functions $\nu_1(\ \cdot\ )$ and $\nu_2(\ \cdot\ )$ as

$$\begin{array}{rcl} \nu_1(\tau) & = & e^{\tau - a_1}(if_2 + f_1)/\sqrt{2},\ \tau < a_1 \text{ and} \\ \nu_2(\tau) & = & e^{a_2 - \tau}(if_2 - f_1)/\sqrt{2},\ \tau > a_2, \end{array}$$

then we have $\nu = (\nu_1, \nu_2) \in D(\Upsilon)$ and $\beta_1(\nu) = f_1,\ \beta_2(\nu) = f_2$. □

With the use of the Calkin-Gorbachuk method [3], we obtain the following:

Theorem 5. *If $\widetilde{\Upsilon}$ is a maximal dissipative extension of Υ_0 in $\mathcal{X}$, then it is generated by the differential operator expression $l(\ \cdot\ )$ and the boundary condition*

$$\nu_2(a_2) = K\nu_1(a_1),$$

where $K : X \to X$ is a contraction operator. Moreover, the contraction operator K in X is uniquely determined by the extension $\widetilde{\Upsilon}$, i.e. $\widetilde{\Upsilon} = \Upsilon_K$, and vice versa.

Proof. Each maximal dissipative extension $\widetilde{\Upsilon}$ of Υ_0 is described by the differential operator expression $l(\ \cdot\ )$ with the boundary condition

$$(C - I)\beta_1(\nu) + i(C + I)\beta_2(\nu) = 0,\ \nu \in D(\Upsilon),$$

where $C : X \to X$ is a contraction operator and I is identity operator in corresponding space. Therefore, from Lemma 4, we obtain

$$(C - E)\,(\nu_1(a_1) - \nu_2(a_2)) + (C + E)\,(\nu_1(a_1) + \nu_2(a_2)) = 0,\ \nu = (\nu_1, \nu_2) \in D(\widetilde{\Upsilon}).$$

Hence it is obtained that

$$\nu_2(a_2) = -C\nu_1(a_1).$$

Choosing $K = -C$ in the last boundary condition we have

$$\nu_2(a_2) = K\nu_1(a_1).$$

□

4. The spectrum of the maximal dissipative extensions

In this section, we will investigate the structure of the spectrum of the maximal dissipative extensions Υ_K of the minimal operator Υ_0 in $\mathcal{X}$.

Theorem 6. *The point spectrum $\sigma_p(\Upsilon_K)$ of any maximal dissipative extension Υ_K is of the form:*
(1) If $KerK \neq \{0\}$, then $\sigma_p(\Upsilon_K) \supset H_+$, where $H_+ = \{\lambda \in \mathbb{C} : Im\lambda > 0\}$;
(2) If $KerK = \{0\}$, then $\sigma_p(\Upsilon_K) = \emptyset$.

Proof. Let us consider the following eigenvalue problem defined by

$$l(\nu) = \lambda\nu,\ \lambda = \lambda_r + i\lambda_i,\ \nu \in \mathcal{X}, \lambda \in H_+,$$

with the boundary condition

$$\nu_2(a_2) = K\nu_1(a_1).$$

Then, we have

$$\begin{aligned} &\nu_1'(\xi) = i(\Omega_1 - \lambda)\nu_1,\ \xi < a_1,\\ &\nu_2'(\xi) = i(\Omega_2 - \lambda)\nu_2,\ \xi > a_2,\\ &\nu_2(a_2) = K\nu_1(a_1). \end{aligned}$$

The general solutions of these differential equations are as follows:

$$\begin{aligned} \nu_1(\xi;\lambda) &= exp\left(i(\Omega_1 - \lambda)(\xi - a_1)\right) f_1,\ \xi < a_1, f_1 \in X,\\ \nu_2(\xi;\lambda) &= exp\left(i(\Omega_2 - \lambda)(\xi - a_2)\right) f_2,\ \xi > a_2, f_2 \in X \end{aligned}$$

with the boundary condition

$$\nu_2(a_2;\lambda) = K\nu_1(a_1;\lambda).$$

Moreover, $f_1 = \nu_1(a_1;\lambda),\ f_2 = \nu_2(a_2;\lambda)$.
It is clear that for any $f_1 \in X$, we can write

$$\nu_1(\xi;\lambda) = exp\left(i(\Omega_1 - \lambda_r)(\xi - a_1)\right) exp\left(\lambda_i(\xi - a_1)\right) f_1 \in L^2(X, (-\infty, a_1))$$

and for $f_2 \in X$ such that $f_2 \neq 0$, we get

$$\nu_2(\xi;\lambda) = exp\left(i(\Omega_2 - \lambda_r)(\xi - a_2)\right) exp\left(\lambda_i(\xi - a_2)\right) f_2 \notin L^2(X, (a_2, \infty)).$$

(1) If we choose the function $\nu \in \mathcal{X}$ of the following special form

$$\nu^*(\xi;\lambda) = \left(exp\left(i(\Omega_1 - \lambda_r)(\xi - a_1)\right) exp\left(\lambda_i(\xi - a_1)\right) f, 0\right),\ f \in KerK,$$

then we obtain $\Upsilon_K\nu^*(\xi;\lambda) = \lambda\nu^*(\xi;\lambda)$ and $\nu_2^*(a_2;\lambda) = K\nu_1^*(a_1;\lambda)$, for any $\lambda \in H_+$.
(2) If $KerK = \{0\}$, then from the boundary condition $0 = K\nu_1(a_1;\lambda)$ we have $\nu_1(a_1;\lambda) = f_1 = 0$. Hence, the boundary value problem $\Upsilon_K\nu = \lambda\nu,\ \lambda \in H_+,\ \nu \in \mathcal{X}$ have a zero solution once.

Now, let us consider the eigenvalue problem defined by

$$\Upsilon_K\nu = \lambda\nu,\ \nu \in \mathcal{X},\ \lambda \in \mathbb{R}.$$

Then we have

$$\begin{aligned}
&\nu_1'(\xi) = i(\Omega_1 - \lambda)\nu_1,\ \xi < a_1,\\
&\nu_2'(\xi) = i(\Omega_2 - \lambda)\nu_2,\ \xi > a_2,\\
&\nu_2(a_2) = K\nu_1(a_1).
\end{aligned}$$

The general solutions of these differential equations are as follows:

$$\begin{aligned}
\nu_1(\xi;\lambda) &= exp\,(i(\Omega_1 - \lambda)(\xi - a_1))\, f_1 \notin L^2(X, (-\infty, a_1)),\ f_1 \in X\\
\nu_2(\xi;\lambda) &= exp\,(i(\Omega_2 - \lambda)(\xi - a_2))\, f_2 \notin L^2(X, (a_2, \infty)),\ f_2 \in X.
\end{aligned}$$

Consequently, for $KerK \neq \{0\}$ we have

$$\sigma_p(\Upsilon_K) \supset H_+$$

and for $KerK = \{0\}$ we get

$$\sigma_p(\Upsilon_K) = \emptyset.$$

□

Theorem 7. *The residual spectrum $\sigma_r(\Upsilon_K)$ of any maximal dissipative extension Υ_K is empty, i.e.*

$$\sigma_r(\Upsilon_K) = \emptyset.$$

Proof. From Theorem 6 we get $\sigma_r(\Upsilon_K) \subset \mathbb{R}$ for $KerK \neq \{0\}$, and $\sigma_r(\Upsilon_K) \subset \mathbb{R} \cap H_+$ for $KerK = \{0\}$. In order to prove this theorem we will investigate the point spectrum of the adjoint operator Υ_K^* of Υ_K in $\mathcal{X}$. Let us consider the eigenvalue problem defined by

$$\Upsilon_K^*\vartheta = \lambda\vartheta,\ \lambda \in \mathbb{R},\ \vartheta = (\vartheta_1, \vartheta_2) \in \mathcal{X}.$$

In this case, we have

$$\begin{aligned}
i\vartheta_1'(\xi) + \Omega_1\vartheta_1(\xi) &= \lambda\vartheta_1(\xi),\ \xi < a_1,\\
i\vartheta_2'(\xi) + \Omega_2\vartheta_2(\xi) &= \lambda\vartheta_2(\xi),\ \xi > a_2
\end{aligned}$$

with the boundary condition

$$\vartheta_1(a_1) = K^*\vartheta_2(a_2).$$

Hence, it is obtained

$$\begin{aligned}
\vartheta_1(\xi;\lambda) &= exp\,(i(\Omega_1 - \lambda)(\xi - a_1))\, g_1,\ \xi < a_1\\
\vartheta_2(\xi;\lambda) &= exp\,(i(\Omega_2 - \lambda)(\xi - a_2))\, g_2,\ \xi > a_2,\ g_1, g_2 \in X.
\end{aligned}$$

Therefore for any $g_1, g_2 \in X$ and for each $\lambda \in \mathbb{R}$, we get

$$\begin{aligned}
\vartheta_1(\,\cdot\,;\lambda) &\notin L^2(X, (-\infty, a_1)),\\
\vartheta_2(\,\cdot\,;\lambda) &\notin L^2(X, (a_2, \infty)).
\end{aligned}$$

Now, let us consider the residual spectrum of Υ_K, namely,

$$\Upsilon_K^*\vartheta = \lambda\vartheta,\ \lambda \in \mathbb{C},\ \lambda_i = Im\lambda > 0,\ \vartheta = (\vartheta_1, \vartheta_2) \in \mathcal{X}.$$

We have

$$\begin{aligned}\vartheta_1(\xi;\lambda) &= exp\left((i\Omega_1 - i\lambda_r + \lambda_i)(\xi - a_1)\right) g_1,\ \xi < a_1\\ \vartheta_2(\xi;\lambda) &= exp\left((i\Omega_2 - i\lambda_r + \lambda_i)(\xi - a_2)\right) g_2,\ \xi > a_2.\end{aligned}$$

As a result, we get $\vartheta_1(\,\cdot\,;\lambda) \in L^2(X,(-\infty,a_1))$ and $\nu_2(\,\cdot\,;\lambda) \notin L^2(X,(a_2,\infty))$ for any $g_2 = \vartheta_2(a_2) \neq 0$.

The necessary and sufficient condition for $\vartheta_2(\,\cdot\,;\lambda) \in L^2(X,(a_2,\infty))$ is $g_2 = \vartheta_2(a_2) = 0$. From the boundary condition we get

$$\vartheta_1(a_1) = K^*\vartheta_2(a_2)$$

which implies $\vartheta_1(a_1) = 0$. Then, $Ker(\Upsilon_K^*) = \{0\}$.
Consequently, we have $\lambda \notin \sigma_r(\Upsilon_K)$ for any $\lambda \in \mathbb{C}$ with $Im\lambda > 0$. □

By the general theory of linear closed operators in a Hilbert spaces and Theorem 6-Theorem 7, one can immediately obtain the following:

Theorem 8. *If $KerK \neq \{0\}$, then the continuous spectrum $\sigma_c(\Upsilon_K)$ of any maximal dissipative extension Υ_K in $\mathcal{X}$ coincides with $\mathbb{R}$, i.e.*

$$\sigma_c(\Upsilon_K) = \mathbb{R}.$$

Moreover, $\sigma(\Upsilon_K) = \{\lambda \in \mathbb{C} : Im\lambda \geq 0\}$.

With the use of Theorem 6-Theorem 8, the following result can be obtained.

Corollary 9. *If $KerK \neq \{0\}$, then the point spectrum $\sigma_p(\Upsilon_K)$ of any maximal dissipative extension Υ_K in $\mathcal{X}$ is of the form $\sigma_p(\Upsilon_K) = \{\lambda \in \mathbb{C} : Im\lambda > 0\}$.*

Theorem 10. *If $KerK = \{0\}$, then the spectrum of any maximal dissipative extension Υ_K in $\mathcal{X}$ is of the form*

$$\sigma(\Upsilon_K) = \sigma_c(\Upsilon_K) = \mathbb{R}.$$

Proof. Let us consider the following spectrum problem defined by

$$\Upsilon_K\nu = \lambda\nu + f,\ \lambda \in \mathbb{C},\ Im\lambda = \lambda_i > 0,\ \nu = (\nu_1,\nu_2),\ f = (f_1,f_2) \in \mathcal{X}.$$

Then, we have

$$\begin{aligned}i\nu_1'(\xi) + \Omega_1\nu_1(\xi) &= \lambda\nu_1(\xi) + f_1(\xi),\ \xi < a_1,\\ i\nu_2'(\xi) + \Omega_2\nu_2(\xi) &= \lambda\nu_2(\xi) + f_2(\xi),\ \xi > a_2,\\ \nu_2(a_2) &= K\nu_1(a_1).\end{aligned}$$

Hence, the general solutions of the following differential equations

$$\begin{aligned}\nu_1'(\xi) &= i(\Omega_1 - \lambda E)\nu_1(\xi) - if_1(\xi),\ \xi < a_1,\\ \nu_2'(\xi) &= i(\Omega_2 - \lambda E)\nu_2(\xi) - if_2(\xi),\ \xi > a_2\end{aligned}$$

are of the forms

$$\nu_1(\xi;\lambda) = exp\left(i(\Omega_1 - \lambda E)(\xi - a_1)\right) f_\lambda + i\int\limits_{\xi}^{a_1} exp\left(i(\Omega_1 - \lambda E)(\xi - \tau)\right) f_1(\tau)d\tau,$$

$$\xi < a_1,\ f_\lambda \in X,$$

$$\nu_2(\xi;\lambda) = i\int\limits_{\xi}^{\infty} exp\left(i(\Omega_2 - \lambda E)(\xi - \tau)\right) f_2(\tau)d\tau,\ \xi > a_2.$$

Additionally, from the boundary condition we have

$$\int\limits_{a_2}^{\infty} exp\left(i(\Omega_2 - \lambda E)(a_2 - \tau)\right) f_2(\tau)d\tau = Kf_\lambda.$$

Consequently, the solution of above considered spectrum problem can be expressed by

$$\begin{aligned}\nu_1(\xi;\lambda) &= exp\left(i(\Omega_1 - \lambda E)(\xi - a_1)\right)\left(K^{-1}\int\limits_{a_2}^{\infty} exp\left(i(\Omega_2 - \lambda E)(a_2 - \tau)\right) f_2(\tau)d\tau\right)\\ &\quad + i\int\limits_{\xi}^{a_1} exp\left(i(\Omega_1 - \lambda E)(\xi - \tau)\right) f_1(\tau)d\tau,\ \xi < a_1,\\ \nu_2(\xi;\lambda) &= i\int\limits_{\xi}^{\infty} exp\left(i(\Omega_2 - \lambda E)(\xi - \tau)\right) f_2(\tau)d\tau,\ \xi > a_2\end{aligned}$$

in the spaces $L^2(X, (-\infty, a_1))$ and $L^2(X, (a_2, \infty))$, respectively.
As a result, we have $H_+ \subset \rho(\Upsilon_K)$. Since for $\lambda \in \mathbb{R}$ the problem

$$\Upsilon_K \nu = \lambda\nu,\ \nu \in \mathcal{X}$$

has zero solution once, $\sigma_p(\Upsilon_K) = \emptyset$ in case that $KerK = \{0\}$.

For $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0$ and $f = (f_1, f_2) \in \mathcal{X}$ the resolvent operator $R_\lambda(\Upsilon_K)$ in $\mathcal{X}$ can be written in the form

$$\|R_\lambda(\Upsilon_K))f(\xi)\|^2_{\mathcal{X}} \geq \|i\int\limits_{\xi}^{\infty} \exp\left(i(\Omega_2 - \lambda E)(\xi - \tau)\right) f_2(\tau)d\tau\|^2_{L^2(X,(a_2,\infty))}.$$

The vector functions $f^*(\xi;\lambda)$ have the form $f^*(\xi,\lambda) = \left(0, exp\left(i(\Omega_2 - \lambda E)\xi\right) f\right)$, $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0,\ f \in X$ belong to $\mathcal{X}$. Indeed,

$$\|f^*(\xi,\lambda)\|^2_{L^2(X,(a_2,\infty))} \quad = \quad \int\limits_{a_2}^{\infty} \|exp\left(i(\Omega_2 - \lambda E)\xi\right) f\|^2_X d\xi$$

$$
\begin{aligned}
&= \int_{a_2}^{\infty} exp\left(-2\lambda_i \xi\right) d\xi \|f\|_X^2 \\
&= \frac{1}{2\lambda_i} exp\left(-2\lambda_i a_2\right) \|f\|_X^2 < \infty .
\end{aligned}
$$

For the such functions $f^*(\lambda; \cdot)$, we have

$$
\begin{aligned}
\|R_\lambda(\Upsilon_K) f^*(\cdot\,; \lambda)\|_{\mathcal{X}}^2 &\geq \|i \int_{\xi}^{\infty} exp\left(i(\Omega_2 - \lambda)(\xi - \tau) - i(\lambda - \Omega_2)\tau\right) f d\tau\|_{L^2(X,(a_2,\infty))}^2 \\
&= \| \int_{\xi}^{\infty} exp\left(-i\lambda\xi\right) exp\left(-2\lambda_i \tau\right) exp\left(i\Omega_2 \xi\right) f d\tau\|_{L^2(X,(a_2,\infty))}^2 \\
&= \|exp\left(-i\lambda\xi\right) exp\left(i\Omega_2\xi\right) \int_{\xi}^{\infty} exp\left(-2\lambda_i\tau\right) f d\tau\|_{L^2(X,(a_2,\infty))}^2 \\
&= \|exp\left(-i\lambda\xi\right) \int_{\xi}^{\infty} exp\left(-2\lambda_i\tau\right) d\tau\|_{L^2(X,(a_2,\infty))}^2 \|f\|_X^2 \\
&= \frac{1}{4\lambda_i^2} \int_{a_2}^{\infty} exp\left(-2\lambda_i\tau\right) d\tau \|f\|_X^2 \\
&= \frac{1}{8\lambda_i^3} exp\left(-2\lambda_i a_2\right) \|f\|_X^2 .
\end{aligned}
$$

Using the above inequality we get

$$
\|R_\lambda(\Upsilon_K) f^*(\cdot\,; \lambda)\|_{\mathcal{X}} \geq \frac{exp\left(\lambda_i a_2\right)}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}} \|f\|_X^2 = \frac{1}{2\lambda_i} \|f^*(\xi; \lambda)\|_{L^2(X,(a_2,\infty))},
$$

i.e., for $\lambda_i = Im\lambda > 0$ and $f \neq 0$ we can write

$$
\frac{\|R_\lambda(\Upsilon_K) f^*(\lambda; \cdot)\|_{\mathcal{X}}}{\|f^*(\lambda; \xi)\|_{\mathcal{X}}} \geq \frac{1}{2\lambda_i}
$$

and it is also obvious that

$$
\|R_\lambda(\Upsilon_K)\| \geq \frac{\|R_\lambda(\Upsilon_K) f^*(\cdot\,; \lambda)\|_{\mathcal{X}}}{\|f^*(\xi; \lambda)\|_{\mathcal{X}}}, \ f \neq 0.
$$

As a consequence, we get

$$
\|R_\lambda(\Upsilon_K)\| \geq \frac{1}{2\lambda_i} \text{ for } \lambda \in \mathbb{C}, \ \lambda_i = Im\lambda > 0,
$$

which shows that every $\lambda \in \mathbb{R}$ belongs to the continuous spectrum of the extension Υ_K. □

5. Examples

Example 11. *Let us consider the following linear multipoint differential operator expression for first order of the form*

$$l((\nu,\vartheta)) = \left(i\nu'(\tau,\varsigma)+\varsigma\nu(\tau,\varsigma), i\vartheta'(\tau,\varsigma)+\varsigma\vartheta(\tau,\varsigma)\right)$$

in the Hilbert space

$$\mathcal{X} = L^2\left((-\infty,-1)\times\mathbb{R}\right)\oplus L^2\left((1,\infty)\times\mathbb{R}\right).$$

Let $\widetilde{L}$ be a maximal dissipative extension of the minimal operator generated by above differential expression. Then, $\widetilde{L}$ is generated by the differential operator expression $l(\,\cdot\,)$ and the following boundary condition

$$\vartheta(1,\varsigma) = \nu(-1,\varsigma)$$

in $\mathcal{X}$.

By Corollary 9, Theorem 8 and Theorem 7, the point, continuous and residual spectrum of the maximal dissipative extension $\widetilde{L}$ in $\mathcal{X}$ are of the forms

$$\sigma_p\left(\widetilde{L}\right) = \{\lambda\in\mathbb{C} : Im\lambda > 0\},$$
$$\sigma_c\left(\widetilde{L}\right) = \mathbb{R},$$
$$\sigma_r\left(\widetilde{L}\right) = \emptyset,$$

respectively.

Consequently, the spectrum of the maximal dissipative extension $\widetilde{L}$ in $\mathcal{X}$ is of the form

$$\sigma\left(\widetilde{L}\right) = \{\lambda\in\mathbb{C} : Im\lambda \geq 0\}.$$

Remark 12. *In special case the representation of selfadjoint extensions of corresponding mentioned above minimal operator and their spectral analysis have been surveyed in* [1] *and* [5].

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GENERALIZED DIFFERENCE SEQUENCE SPACES OF FRACTIONAL ORDER DEFINED BY ORLICZ FUNCTIONS

NAZLIM DENIZ ARAL AND MIKAIL ET

Abstract. The main purpose of this paper is to introduce the concepts of $\Delta^{\alpha}-$lacunary statistical convergence of order β $(0 < \beta \leq 1)$ with the fractional order of α and $\Delta^{\alpha}-$lacunary strongly convergence of order β $(0 < \beta \leq 1)$ with the fractional order of α. We establish some connections between $\Delta^{\alpha}-$lacunary strongly convergence of order β and $\Delta^{\alpha}-$lacunary statistical convergence of order β.

1. Introduction

The idea of statistical convergence was given by Zygmund [45] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [42] and Fast [20] and later reintroduced by Schoenberg [38]. Over the years and under different names statistical convergence was discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Çakallı et al. ([7],[8],[9]). Caserta et al. [10], Çınar et al. [11], Connor [13], Et et al. ([15],[17]), Fridy [22], Fridy and Orhan [23], Mursaleen [33], Salat [41], Mohiuddine et al. ([5], [31]) and many others.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k), \text{provided that the limit exists.}$$

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0.$$

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Recently, Çolak [12] generalized the statistical convergence by ordering the interval $(0,1]$ and defined the statistical convergence of order β and strong $p-$Cesàro summability of order β, where $0<\beta\leq 1$ and p is a positive real number. Şengül and Et ([19],[39]) generalized the concepts such as lacunary statistical convergence of order β and lacunary strong $p-$Cesàro summability of order β for sequences of real numbers.

Difference sequence spaces was defined by Kızmaz [27] and the concept was generalized by Et et al. ([14],[18]) as follows:

$$\Delta^m(X)=\{x=(x_k):(\Delta^m x_k)\in X\},$$

where X is any sequence space, $m\in\mathbb{N}$, $\Delta^0 x=(x_k)$, $\Delta x=(x_k-x_{k+1})$, $\Delta^m x=(\Delta^m x_k)=\left(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1}\right)$ and so $\Delta^m x_k=\sum_{v=0}^{m}(-1)^v\binom{m}{v}x_{k+v}$.

If $x\in\Delta^m(X)$ then there exists one and only one sequence $y=(y_k)\in X$ such that $y_k=\Delta^m x_k$ and

$$x_k=\sum_{v=1}^{k-m}(-1)^m\binom{k-v-1}{m-1}y_v=\sum_{v=1}^{k}(-1)^m\binom{k+m-v-1}{m-1}y_{v-m},\qquad(1)$$

$$y_{1-m}=y_{2-m}=\cdots=y_0=0$$

for sufficiently large k, for instance $k>2m$. After then some properties of difference sequence spaces have been studied in ([1],[2],[16],[18], [25], [26], [32], [36]).

By $\Gamma(r)$, we denote the Gamma function of a real number r and $r\notin\{0,-1,-2,-3,...\}$. By the definition, it can be expressed as an improper integral as:

$$\Gamma(r)=\int_0^\infty e^{-t}t^{r-1}dt.$$

From the definition, it is observed that:

(i) For any natural number n, $\Gamma(n+1)=n!$,

(ii) For any real number n and $n\notin\{0,-1,-2,-3,...\}$, $\Gamma(n+1)=n\Gamma(n)$,

(iii) For particular cases, we have $\Gamma(1)=\Gamma(2)=1,\Gamma(3)=2!,\Gamma(4)=3!,....$

For a proper fraction α, we define a fractional difference operator $\Delta^\alpha:w\to w$ defined by

$$\Delta^\alpha(x_k)=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}x_{k+i}.\qquad(2)$$

In particular, we have

$$\begin{aligned}
\Delta^{\frac{1}{2}}x_k &= x_k-\frac{1}{2}x_{k+1}-\frac{1}{8}x_{k+2}-\frac{1}{16}x_{k+3}-\frac{5}{128}x_{k+4}-\frac{7}{256}x_{k+5}-\frac{21}{1024}x_{k+6}\cdots\\
\Delta^{-\frac{1}{2}}x_k &= x_k+\frac{1}{2}x_{k+1}+\frac{3}{8}x_{k+2}+\frac{5}{16}x_{k+3}+\frac{35}{128}x_{k+4}+\frac{63}{256}x_{k+5}+\frac{231}{1024}x_{k+6}\cdots\\
\Delta^{\frac{1}{3}}x_k &= x_k-\frac{1}{3}x_{k+1}-\frac{1}{9}x_{k+2}-\frac{5}{81}x_{k+3}-\frac{10}{243}x_{k+4}-\frac{22}{729}x_{k+5}-\frac{154}{6561}x_{k+6}\cdots\\
\Delta^{\frac{2}{3}}x_k &= x_k-\frac{2}{3}x_{k+1}-\frac{1}{9}x_{k+2}-\frac{4}{81}x_{k+3}-\frac{7}{243}x_{k+4}-\frac{14}{729}x_{k+5}-\frac{91}{6561}x_{k+6}\cdots
\end{aligned}$$

Without loss of generality, we assume throughout that the series defined in (2) is convergent. Moreover, if α is a positive integer, then the infinite sum defined in (2) reduces to a finite sum i.e., $\sum_{i=0}^{\alpha}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}$. In fact, this operator generalized the difference operator introduced by Et and Çolak [14].

Recently, using fractional operator Δ^{α} (fractional order of α, $\alpha \in \mathbb{R}$) Baliarsingh et al. ([3],[4],[35]) defined the sequence space $\Delta^{\alpha}(X)$ such as: $\Delta^{\alpha}(X) = \{x = (x_k) : (\Delta^{\alpha} x_k) \in X\}$, where X is any sequence space.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in ([7],[8],[9],[21],[23], [24], [40]).

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg [29] got interested in Orlicz sequence spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or ℓ_p $(0 \le p < \infty)$. Subsequently, Lindenstrauss and Tzafriri [30] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \quad \text{for some} \quad \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

becomes a Banach space, called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = |x|^p$ for $1 \le p < \infty$. Lindenstrauss and Tzafriri [30] proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to l_p $(1 \le p < \infty)$. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [28].

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Recently, Orlicz sequence spaces were studied by Bhardwaj and Singh [6],Mursaleen et al. ([16],[34]), Savaş and Rhoades [37], Tripathy et al. [43] and many others.

2. Main Results and proofs

Definition 1. *Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$ and α be a proper fraction. The sequence $x = (x_k)$ is said to be $\Delta^\alpha-$lacunary statistically convergent of order β of fractional order of α (or $\Delta^\alpha(S_\theta^\beta)-$convergent to L) to the number L, if there is a real number L such that*

$$\lim_{r\to\infty} \frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}| = 0$$

for all $\varepsilon > 0$. In this case, we write $x_k \to L(\Delta^\alpha(S_\theta^\beta)$.

The set of all $\Delta^\alpha(S_\theta^\beta)-$convergent sequences will be denoted by $\Delta^\alpha(S_\theta^\beta)$. If $\theta = (2^r)$, then we write $\Delta^\alpha(S^\beta)$ instead of $\Delta^\alpha(S_\theta^\beta)$. In the special cases $\theta = (2^r)$ and $\beta = 1$, we write $\Delta^\alpha(S)$ instead of $\Delta^\alpha(S_\theta^\beta)$.

In particular, $\Delta^\alpha(S_\theta^\beta)-$convergence includes many special cases; for example, in case of $\alpha = m \in \mathbb{N}, \beta = 1, \Delta^\alpha-$lacunary statistical convergence of order β reduces to the Δ^m-lacunary statistical convergence which was defined and studied by Tripathy and Et [44].

Definition 2. *Let M be an Orlicz function, $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1], \alpha$ be a proper fraction and $p = (p_k)$ be a sequence of strictly positive real numbers. The sequence $x = (x_k)$ is said to be strongly $\Delta^\alpha(N_\theta^\beta,(p))-$summable to L with respect to the Orlicz function M (or strongly $\Delta^\alpha(N_\theta^\beta, M,(p))-$summable to L), if there is a real number L such that*

$$\lim_{r\to\infty} \frac{1}{h_r^\beta} \sum_{k\in I_r} \left[M\left(\frac{|\Delta^\alpha x_k - L|}{\rho}\right)\right]^{p_k} = 0,$$

for all $\varepsilon > 0$ and some $\rho > 0$. In this case, we write $x_k \to L(\Delta^\alpha(N_\theta^\beta, M,(p)))$.

The set of all $\Delta^\alpha(N_\theta^\beta, M,(p))-$summable sequences will be denoted by $\Delta^\alpha(N_\theta^\beta, M,(p))$. In the special cases $M(x) = x, p_k = p$ for each $k \in \mathbb{N}$, we obtain the set $\Delta^\alpha(N_\theta^\beta, p)$. If $\theta = (2^r)$, $M(x) = x, p_k = 1$ for each $k \in \mathbb{N}$ and $\beta = 1$, then we write $\Delta^\alpha(|\sigma_1|)$ instead of $\Delta^\alpha(N_\theta^\beta, p)$ and say that $x = (x_k)$ is strongly $\Delta^\alpha-$Cesàro summable to L.

The proof the following theorems are straightforward, so we choose to state these results without proof.

Theorem 3. *Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$, α be a proper fraction and $x = (x_k), y = (y_k)$ are sequences of real numbers, then*

i) If $x_k \to L(\Delta^\alpha(S_\theta^\beta))$ and $c \in \mathbb{C}$, then $cx_k \to cL(\Delta^\alpha(S_\theta^\beta))$.

ii) If $x_k \to L_1(\Delta^\alpha(S_\theta^\beta))$ and $y_k \to L_2(\Delta^\alpha(S_\theta^\beta))$, then $(x_k + y_k) \to (L_1 + L_2)(\Delta^\alpha(S_\theta^\beta))$.

Theorem 4. *Let the sequence (p_k) be bounded, then the sequence space $\Delta^\alpha(N_\theta^\beta, M, (p))$ is a linear space over the set of complex numbers.*

Theorem 5. *If a $\Delta^\alpha-$bounded sequence (that is $x \in \Delta^\alpha(\ell_\infty)$) is $\Delta^\alpha-$statistically convergent to L then it is strongly $\Delta^\alpha-$Cesàro summable to L.*

Proof. Suppose that $x \in \Delta^\alpha(\ell_\infty) \cap \Delta^\alpha(S)$ with $x_k \to L(\Delta^\alpha(S)$. Without loss of generality we may assume that $L = 0$. Set $K = \|\Delta^\alpha x\|_\infty$. Let $\varepsilon > 0$ be given and choose N_ε such that $\frac{1}{n}|\{k \leq n : |\Delta^\alpha x_k| \geq \frac{\varepsilon}{2}\}| < \frac{\varepsilon}{2K}$ for all $n > N_\varepsilon$. Now, we get

$$\frac{1}{n}\sum_{k=1}^{n}|\Delta^\alpha x_k| = \frac{1}{n}\sum_{\substack{1\leq k\leq n \\ |\Delta^\alpha x_k|\geq\frac{\varepsilon}{2}}}|\Delta^\alpha x_k| + \frac{1}{n}\sum_{\substack{1\leq k\leq n \\ |\Delta^\alpha x_k|<\frac{\varepsilon}{2}}}|\Delta^\alpha x_k| \leq \frac{1}{n}\frac{n\varepsilon}{2K}K + \frac{n}{n}\frac{\varepsilon}{2} = \varepsilon.$$

for all $n > N_\varepsilon$. Thus $\lim \frac{1}{n}\sum_{k=1}^{n}|\Delta^\alpha x_k| = 0$ which means that $x \in \Delta^\alpha(|\sigma_1|)$. □

Converse of Theorem 5 does not holds. For this choose $\alpha = 1$, then the sequence $x = (0, -1, -1, -2, -2, -3-, 3, -4, -4, ...)$ belongs to $\Delta(|\sigma_1|)$ and does not belong to $\Delta(S)$.

Theorem 6. *$\Delta^\alpha(N_\theta^\beta, p)$ is a Banach space normed by*

$$\|x\|_{\Delta_1^\alpha} = \sum_{i=1}^{\infty}|x_i| + \sup_r\left(\frac{1}{h_r^\beta}\sum_{k\in I_r}|\Delta^\alpha x_k|^p\right)^{1/p}, 1 \leq p < \infty \tag{3}$$

and a complete p-normed space for $0 < p < 1$ by

$$\|x\|_{\Delta_2^\alpha} = \sum_{i=1}^{\infty}|x_i| + \sup_r\frac{1}{h_r^\beta}\sum_{k\in I_r}|\Delta^\alpha x_k|^p = 0 \tag{4}$$

Proof. Proof follows from Theorem 3 [4] and Theorem 2.4 [39]. □

Theorem 7. *$\Delta^\alpha(N_\theta^\beta, p)$ is a $BK-$space normed by* (3).

Proof. Omitted. □

Theorem 8. *Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0, 1]$, α be a proper fraction and p be a fixed positive real number, then*

i) If $x_k \to L(\Delta^\alpha(N_\theta^\beta, p))$, then $x_k \to L(\Delta^\alpha(S_\theta^\beta))$ and the inclusion is strict,

ii) ([24]) If $x \in \Delta^\alpha(\ell_\infty)$ and $x_k \to L(\Delta^\alpha(S_\theta))$, then $x_k \to L(\Delta^\alpha(N_\theta, p))$.

Proof. The inclusion part of the proof is easy. In order to establish "*the inclusion is strict*", let θ be given, choose $\alpha = m$, $\beta = 1, p = 1$ and define a sequence $x = (x_k)$ by $\Delta^m x$ to be $1, 2, ..., \left[\sqrt{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $\Delta^m x_k = 0$ otherwise (5)

It is clear that x is not Δ^m-bounded. Since

$$\frac{1}{h_r}\left|\{k \in I_r : |\Delta^m x_k - 0| \geq \varepsilon\}\right| = \frac{\left[\sqrt{h_r}\right]}{h_r} \to 0, \text{as } r \to \infty$$

and

$$\frac{1}{h_r}\sum_{k \in I_r} |\Delta^m x_k - 0| = \frac{\left[\sqrt{h_r}\right]\left(\left[\sqrt{h_r}\right]+1\right)}{2h_r} \to \frac{1}{2}, \text{as } r \to \infty.$$

From (1) we have $x \in \Delta^m(S_\theta)$, $x_k \notin \Delta^m(N_\theta)$. □

Theorem 9. *Let* $0 < \alpha \leq 1$ *and* $\theta = (k_r)$ *be a lacunary sequence. If* $\liminf_r q_r > 1$, *then* $\Delta^\alpha\left(S^\beta\right) \subset \Delta^\alpha\left(S_\theta^\beta\right)$.

Proof. Suppose that $\liminf_r q_r > 1$; then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \Longrightarrow \left(\frac{h_r}{k_r}\right)^\beta \geq \left(\frac{\delta}{1+\delta}\right)^\beta \Longrightarrow \frac{1}{k_r^\beta} \geq \frac{\delta^\beta}{(1+\delta)^\beta}\frac{1}{h_r^\beta}.$$

If $x_k \to L\left(\Delta^\alpha\left(S^\beta\right)\right)$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r^\beta}\left|\{k \leq k_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}\right| &\geq \frac{1}{k_r^\beta}\left|\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}\right| \\ &\geq \frac{\delta^\beta}{(1+\delta)^\beta}\frac{1}{h_r^\beta}\left|\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}\right|;\end{aligned}$$

So $x \in \Delta^\alpha\left(S_\theta^\beta\right)$. □

Theorem 10. *Let* $0 < \alpha \leq 1$ *and* $\theta = (k_r)$ *be a lacunary sequence. If* $\limsup_r q_r < \infty$, *then* $\Delta^\alpha\left(S_\theta^\beta\right) \subset \Delta^\alpha\left(S^\beta\right)$.

Proof. Omitted. □

In the following theorems, assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$.

Theorem 11. *Let* $\beta, \gamma \in (0, 1]$ *be real numbers such that* $\beta \leq \gamma$, M *be an Orlicz function and* $\theta = (k_r)$ *be a lacunary sequence, then* $\Delta^\alpha(N_\theta^\beta(M, (p)) \subset \Delta^\alpha(S_\theta^\gamma)$.

Proof. Let $x \in \Delta^{\alpha}(N_{\theta}^{\beta}(M,(p))$, $\varepsilon > 0$ be given and $\sum_1$ and $\sum_2$ denote the sums over $k \in I_r, |\Delta^{\alpha}x_k - L| \geq \varepsilon$ and $|\Delta^{\alpha}x_k - L| < \varepsilon$ respectively. As $h_r^{\beta} \leq h_r^{\gamma}$ for each r, we have

$$\begin{aligned}
\frac{1}{h_r^{\beta}}\sum_{k\in I_r}\left[M\left(\frac{|\Delta^{\alpha}x_k - L|}{\rho}\right)\right]^{p_k} &\geq \frac{1}{h_r^{\gamma}}\left[\begin{array}{c} \sum_1\left[M\left(\frac{|\Delta^{\alpha}x_k - L|}{\rho}\right)\right]^{p_k} \\ +\sum_2\left[M\left(\frac{|\Delta^{\alpha}x_k - L|}{\rho}\right)\right]^{p_k}\end{array}\right] \\
&\geq \frac{1}{h_r^{\gamma}}\left[\sum_1 M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k} \\
&\geq \frac{1}{h_r^{\gamma}}\sum_1 \min\left([M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\right), \quad \varepsilon_1 = \frac{\varepsilon}{\rho} \\
&\geq \frac{1}{h_r^{\gamma}}|\{k \in I_r : |\Delta^{\alpha}x_k - L| \geq \varepsilon\}| \\
&\quad \times \min\left([M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\right).
\end{aligned}$$

As $x \in \Delta^{\alpha}(N_{\theta}^{\beta}(M,(p))$, the left hand side of the above inequality tends to zero as $r \to \infty$. Therefore, the right hand side of the above inequality tends to zero as $r \to \infty$, hence $x \in \Delta^{\alpha}(S_{\theta}^{\gamma})$. □

Corollary 12. *Let $0 < \beta \leq 1$, M be an Orlicz function and $\theta = (k_r)$ be a lacunary sequence, then $\Delta^{\alpha}(N_{\theta}^{\beta}(M,(p)) \subset \Delta^{\alpha}(S_{\theta}^{\beta})$.*

Theorem 13. *Let M be an Orlicz function, $x = (x_k)$ be a $\Delta^{\alpha}-$bounded sequence and $\theta = (k_r)$ be a lacunary sequence. If $\lim_{r\to\infty}\dfrac{h_r}{h_r^{\beta}} = 1$, then $x \in \Delta^{\alpha}(S_{\theta}^{\beta}) \Rightarrow x \in \Delta^{\alpha}(N_{\theta}^{\beta}(M,(p))$.*

Proof. Suppose that $x = (x_k)$ be a $\Delta^{\alpha}-$bounded sequence, that is $x \in \Delta^{\alpha}(\ell_{\infty})$ and $x_k \to L(\Delta^{\alpha}(S_{\theta}^{\beta}))$. As $x \in \Delta^{\alpha}(\ell_{\infty})$, then there is a constant $T > 0$ such that $|\Delta^{\alpha}x_k| \leq T$. Given $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{h_r^{\beta}}\sum_{k\in I_r}\left[M\left(\frac{|\Delta^{\alpha}x_k - L|}{\rho}\right)\right]^{p_k} &= \frac{1}{h_r^{\beta}}\sum_1\left[M\left(\frac{|\Delta^{\alpha}x_k - L|}{\rho}\right)\right]^{p_k} \\
&\quad + \frac{1}{h_r^{\beta}}\sum_2\left[M\left(\frac{|\Delta^{\alpha}x_k - L|}{\rho}\right)\right]^{p_k} \\
&\leq \frac{1}{h_r^{\beta}}\sum_1 \max\left\{\left[\left[M\left(\frac{T}{\rho}\right)\right]^h, \left[M\left(\frac{T}{\rho}\right)\right]^H\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
& & & +\frac{1}{h_r^\beta}\sum_{2}\max\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k} \\
& & \leq & \max\left\{[M(K)]^h, [M(K)]^H\right\} \\
& & & \times\frac{1}{h_r^\beta}\left|\{k\in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}\right| \\
& & & +\frac{h_r}{h_r^\beta}\max\left\{[M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\right\}, \\
& \frac{T}{\rho} & = & K, \quad \frac{\varepsilon}{\rho} = \varepsilon_1.
\end{aligned}
$$

Hence $x \in \Delta^\alpha(N_\theta^\beta(M,(p))$. □

Theorem 14. *If* $\lim p_k > 0$ *and* $x = (x_k)$ *is strongly* $\Delta^\alpha(N_\theta^\beta(M,(p))-$*summable to* L *with respect to the Orlicz function* M, *then that limit* L *is unique.*

Proof. Let $\lim p_k = s > 0$. Suppose that $x_k \to L(\Delta^\alpha(N_\theta^\beta, p))$ and $x_k \to L_1(\Delta^\alpha(N_\theta^\beta, p))$. Then we have

$$\lim_{r\to\infty}\frac{1}{h_r^\beta}\sum_{k\in I_r}\left[M\left(\frac{|\Delta^\alpha x_k - L|}{\rho_1}\right)\right]^{p_k} = 0, \quad \text{for some} \quad \rho_1 > 0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r^\beta}\sum_{k\in I_r}\left[M\left(\frac{|\Delta^\alpha x_k - L_1|}{\rho_2}\right)\right]^{p_k} = 0, \quad \text{for some} \quad \rho_2 > 0.$$

We define the $\rho = \max(2\rho_1, 2\rho_2)$. As M is nondecreasing and convex, we have

$$
\begin{aligned}
\frac{1}{h_r^\beta}\sum_{k\in I_r}\left[M\left(\frac{L-L_1}{\rho}\right)\right]^{p_k} & \leq \frac{D}{h_r^\beta}\sum_{k\in I_r}\frac{1}{2^{p_k}} \\
& \quad \times\left(\left[M\left(\frac{|\Delta^\alpha x_k - L|}{\rho_1}\right)\right]^{p_k} + \left[M\left(\frac{|\Delta^\alpha x_k - L_1|}{\rho_2}\right)\right]^{p_k}\right) \\
& \leq \frac{D}{h_r^\beta}\sum_{k\in I_r}\left[M\left(\frac{|\Delta^\alpha x_k - L|}{\rho_1}\right)\right]^{p_k} \\
& \quad +\frac{D}{h_r^\beta}\sum_{k\in I_r}\left[M\left(\frac{|\Delta^\alpha x_k - L_1|}{\rho_2}\right)\right]^{p_k} \\
& \to 0, \quad (r\to\infty),
\end{aligned}
$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. Hence,

$$\lim_{r\to\infty}\frac{1}{h_r^\beta}\sum_{k\in I_r}\left[M\left(\frac{L-L_1}{\rho}\right)\right]^{p_k} = 0.$$

As $\lim_{k\to\infty} p_k = s$, we have

$$\lim_{k\to\infty}\left[M\left(\frac{|L-L_1|}{\rho}\right)\right]^{p_k}=\left[M\left(\frac{|L-L_1|}{\rho}\right)\right]^{s}$$

and so $L = L_1$. Thus, the limit is unique. $\square$

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SOME GENERAL INTEGRAL INEQUALITIES FOR LIPSCHITZIAN FUNCTIONS VIA CONFORMABLE FRACTIONAL INTEGRAL

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Abstract. In this paper, the author establishes some Hadamard-type and Bullen-type inequalities for Lipschitzian functions via Riemann Liouville fractional integral.

1. Introduction

Hermite-Hadamard Inequality. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions (see [7]). Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f.

Ostrowski's Inequality. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$, then we the following inequality holds

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]$$

for all $x \in [a, b]$ (see [1]).

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Simpson's Inequality. Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4$$

(see [3, 11] and therein).

Bullen's inequality. Suppose that $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. Then we have the inequalities:

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]$$

(see [5] and [16]). In what follows we recall the following definition.

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called an M -Lipschitzian function on the interval I of real numbers with $M \ge 0$, if*

$$|f(x) - f(y)| \le M\,|x-y|$$

for all $x, y \in I$.

For some recent results are connected with Hermite-Hadamard type integral inequalities for Lipschitzian functions, see [4, 8, 9, 17, 18]. In [17], Tseng et al. established some Hadamard-type and Bullen-type inequalities for Lipschitzian functions as follows:

Theorem 2. *Let I be an interval in $\mathbb{R}$, $a \le A \le B \le b$ in I, $V = (1-\alpha)a + \alpha b$, $\alpha \in [0,1]$ and let $f : I \to \mathbb{R}$ be an L -Lipschitzian function with $L \ge 0$. Then we have the inequality*

$$\left|\alpha f(A) + (1-\alpha)f(B) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{LV_\alpha(A,B)}{2\,(b-a)}, \tag{2}$$

where

$$V_\alpha(A,B) = \left\{\begin{array}{c} (A-a)^2 - (A-V)^2 + (B-V)^2 + (b-B)^2\,, \\ a \le V \le A \le B \le b, \\ (A-a)^2 + (V-A)^2 + (B-V)^2 + (b-B)^2\,, \\ a \le A \le V \le B \le b, \\ (A-a)^2 + (V-A)^2 + (b-B)^2 - (V-B)^2\,, \\ a \le A \le B \le V \le b \end{array}\right. .$$

Theorem 3. *Let I be an interval in $\mathbb{R}$, $a \leq A \leq B \leq C \leq b$ in I, $V_1 = (1-\alpha)a+\alpha b$, $V_2 = \gamma a + (\alpha + \beta) b$, $\alpha, \beta, \gamma \in [0,1]$, $\alpha + \beta + \gamma = 1$, and let $f : I \to \mathbb{R}$ be an L-Lipschitzian function with $L \geq 0$. Then we have the inequality*

$$\left| \alpha f(A) + \beta f(B) + \gamma f(C) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{L V_{\alpha,\beta,\gamma}(A,B,C)}{2(b-a)}, \tag{3}$$

where $V_{\alpha,\beta,\gamma}$ is defined as in [17, Section 3].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 4. *Let $f \in L[a,b]$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$ (see [13]*).*

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities, see [2, 10, 14, 15, 19]. In [15], Sarıkaya et. al. represented Hermite–Hadamard's inequalities in fractional integral forms as follows:

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \leq \frac{f(a)+f(b)}{2} \tag{4}$$

with $\alpha > 0$.

Definition 6. *Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$ and set $\beta = \alpha - n$. Then the left conformable factional integral of any order $\alpha > 0$ is defined by*

$$\left(I_{\alpha}^{a} f\right)(x) = \frac{1}{n!} \int_a^x (x-t)^n (t-a)^{\beta-1} f(t)dt,$$

and analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$\left({}^{b}I_{\alpha}f\right)(x) = \frac{1}{n!}\int_{x}^{b}(t-x)^{n}(b-t)^{\beta-1}f(t)dt.$$

Notice that, if $\alpha = n+1$ then $\beta = \alpha - n = 1$ and hence $\left(I_{\alpha}^{a}f\right)(x) = J_{a+}^{n+1}f(x)$ and $\left({}^{b}I_{\alpha}f\right)(x) = J_{b-}^{n+1}f(x)$. Also, if $n = 0$ and $\alpha = 1$ then $\beta = 1$ and hence $\left(I_{\alpha}^{a}f\right)(b) = \left({}^{b}I_{\alpha}f\right)(a) = \int_{a}^{b} f(t)dt$.

The Beta function defined as follows:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_{0}^{1} t^{a-1}(1-t)^{b-1}dt,\ a,b>0.$$

The Incomplete Beta function is defined by

$$B_{x}(a,b) = \int_{0}^{x} t^{a-1}(1-t)^{b-1}dt,\ x\in[0,1],a,b>0,$$

for $x = 1$, the incomplete beta function coincides with the complete beta function. In [12], Set et. al. represented Hermite–Hadamard's inequalities for conformable fractional integrals as follows:

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for conformable fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\left(I_{\alpha}^{a}f\right)(b) + \left({}^{b}I_{\alpha}f\right)(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{5}$$

The aim of this paper is to indicate generalizations of some integral inequalities for Lipschitzian functions via conformable fractional integral. The results are obtained in this study is a generalization of the results which are obtained in Theorem 2 and Theorem 3 by using conformable fractional integrals.

2. A generalization of Hadamard and Ostrowski type inequalities for Lipschitzian functions via fractional integrals

Throughout this section, let I be an interval in $\mathbb{R}$, $a \le x \le y \le b$ in I and let $f : I \to \mathbb{R}$ be an M-Lipschitzian function. In the next theorem, let $\lambda \in [0,1]$, $A = (1-\lambda)a + \lambda b$, and $A_{\alpha,\beta,n}$, $\alpha > 0$, $n = 0,1,2$, $\beta = \alpha - n$, as follows:

(1) If $a \le A \le x \le y \le b$, then

$$A_{\alpha,\beta,n}(x,y,A) = K_{\alpha,\beta,n}(x,y,A) + L_{\alpha,\beta,n}^{*}(x,y,A).$$

(2) If $a \leq x \leq A \leq y \leq b$, then

$$A_{\alpha,\beta,n}(x,y,A) = K^*_{\alpha,\beta,n}(x,y,A) + L^*_{\alpha,\beta,n}(x,y,A).$$

(3) If $a \leq x \leq y \leq A \leq b$, then

$$A_{\alpha,\beta,n}(x,y,A) = K^*_{\alpha,\beta,n}(x,y,A) + L_{\alpha,\beta,n}(x,y,A).$$

where

$$\begin{aligned}
K_{\alpha,\beta,n}(x,y,A) &= (A-a)^{\alpha}\left[(x-a)\,B(\beta,n+1) - (A-a)\,B(\beta+1,n+1))\right],\\
K^*_{\alpha,\beta,n}(x,y,A) &= (A-a)^{\alpha}\Big\{(x-a)\Big[2B_{\frac{x-a}{A-a}}(\beta,n+1) - B(\beta,n+1)\Big])\\
&\quad + (A-a)\Big[B(\beta+1,n+1) - 2B_{\frac{x-a}{A-a}}(\beta+1,n+1)\Big]\Big\},\ A \neq a,\\
K^*_{\alpha,\beta,n}(x,y,a) &= 0,\\
L_{\alpha,\beta,n}(x,y,A) &= (b-A)^{\alpha}\left[(A-y)\,B(n+1,\beta) + (b-A)B(n+2,\beta)\right],\\
L^*_{\alpha,\beta,n}(x,y,A) &= (b-A)^{\alpha}\Big\{(y-A)\Big[2B_{\frac{y-A}{b-A}}(n+1,\beta) - B(n+1,\beta)\Big])\\
&\quad + (b-A)\Big[B(n+2,\beta) - 2B_{\frac{y-A}{b-A}}(n+2,\beta)\Big]\Big\},\ A \neq b,\\
L^*_{\alpha,\beta,n}(x,y,b) &= 0.
\end{aligned}$$

Theorem 8. *Let $x, y, \alpha, \lambda, A, A_{\alpha,\beta,n}$ and the function f be defined as above. Then we have the inequality for fractional integrals*

$$\begin{aligned}
&\left|\lambda^{\alpha} f(x) + (1-\lambda)^{\alpha} f(y) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\left(I^{a}_{\alpha} f\right)(A) + \left({}^{b}I_{\alpha} f\right)(A)\right]\right|\\
&\leq \frac{\Gamma(\alpha+1)A_{\alpha,\beta,n}(x,y,A)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)} M.
\end{aligned} \tag{6}$$

Proof. Using the hypothesis of f, we have the following inequality

$$\begin{aligned}
&\left|\lambda^{\alpha} f(x) + (1-\lambda)^{\alpha} f(y) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\left(I^{a}_{\alpha} f\right)(A) + \left({}^{b}I_{\alpha} f\right)(A)\right]\right|\\
&= \frac{\Gamma(\alpha+1)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left|\int_a^A [f(x)-f(t)]\,(A-t)^n\,(t-a)^{\beta-1}\,dt\right.\\
&\quad \left. + \int_A^b [f(y)-f(t)]\,(t-A)^n\,(b-t)^{\beta-1}\,dt\right|\\
&\leq \frac{\Gamma(\alpha+1)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\int_a^A |f(x)-f(t)|\,(A-t)^n\,(t-a)^{\beta-1}\,dt\right.
\end{aligned}$$

$$+\int_A^b |f(y)-f(t)|\,(t-A)^n\,(b-t)^{\beta-1}\,dt\Bigg]$$

$$\le \frac{\Gamma(\alpha+1)M}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt\right.$$

$$\left.+\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt\right]. \tag{7}$$

Now using simple calculations, we obtain the following identities

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt \quad\text{and}\quad \int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt.$$

1. If $a\le A\le x\le y\le b$, then

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt$$

$$= (A-a)^{\alpha}\left[(x-a)\,B(\beta,n+1)-(A-a)\,B(\beta+1,n+1))\right]$$
$$= K_{\alpha,\beta,n}(x,y,A).$$

and

$$\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt$$

$$= (b-A)^{\alpha}\Big\{(y-A)\Big[2B_{\frac{y-A}{b-A}}(n+1,\beta)-B(n+1,\beta)\Big])$$
$$+(b-A)\Big[B(n+2,\beta)-2B_{\frac{y-A}{b-A}}(n+2,\beta)\Big]\Big\}$$
$$= L^{*}_{\alpha,\beta,n}(x,y,A).$$

2. If $a\le x\le A\le y\le b$, then

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt$$

$$= (A-a)^{\alpha}\Big\{(x-a)\Big[2B_{\frac{x-a}{A-a}}(\beta,n+1)-B(\beta,n+1)\Big])$$
$$+(A-a)\Big[B(\beta+1,n+1)-2B_{\frac{x-a}{A-a}}(\beta+1,n+1)\Big]\Big\}$$
$$= K^{*}_{\alpha,\beta,n}(x,y,A).$$

and

$$\begin{aligned}&\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt\\ = \quad & (b-A)^\alpha \Big\{(y-A)\Big[2B_{\frac{y-A}{b-A}}(n+1,\beta)-B(n+1,\beta)\Big])\\ & +(b-A)\Big[B(n+2,\beta)-2B_{\frac{y-A}{b-A}}(n+2,\beta)\Big]\Big\}\\ = \quad & L^*_{\alpha,\beta,n}(x,y,A).\end{aligned}$$

3. If $a \le x \le y \le A \le b$, then

$$\begin{aligned}&\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt\\ &= (A-a)^\alpha \Big\{(x-a)\Big[2B_{\frac{x-a}{A-a}}(\beta,n+1)-B(\beta,n+1)\Big])\\ &\quad +(A-a)\Big[B(\beta+1,n+1)-2B_{\frac{x-a}{A-a}}(\beta+1,n+1)\Big]\Big\}\\ &= K^*_{\alpha,\beta,n}(x,y,A),\end{aligned}$$

and

$$\begin{aligned}&\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt\\ &= (b-A)^\alpha\left[(A-y)\,B(n+1,\beta)+(b-A)B(n+2,\beta)\right]=L_{\alpha,\beta,n}(x,y,A).\end{aligned}$$

Using the inequality (7) and the above identities $\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt$ and $\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt$, we derive the inequality (6). This completes the proof. □

Under the assumptions of Theorem 8, we have the following corollaries and remarks as follows:

Remark 9. *In Theorem 8, if we take $\alpha=\beta=1$ and $n=0$, then the inequality (6) reduces the inequality (2) in Theorem 2 under the appropriate symbols.*

Corollary 10. *In Theorem 8, let $\delta \in \left[\frac{1}{2},1\right]$, $x=\delta a+(1-\delta)b$ and $y=(1-\delta)a+\delta b$. Then, we have the inequality*

$$\begin{aligned}&\Big|\lambda^\alpha f(\delta a+(1-\delta)b)+(1-\lambda)^\alpha f((1-\delta)a+\delta b)\\ &-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\,\Gamma(\alpha-n)}\left[\left(I^a_\alpha f\right)(A)+\left({}^bI_\alpha f\right)(A)\right]\Big|\\ \le\quad & \frac{\Gamma(\alpha+1)A_{\alpha,\beta,n}(\delta a+(1-\delta)b,(1-\delta)a+\delta b,A)}{n!\,(b-a)^\alpha\,\Gamma(\alpha-n)}M. \qquad (8)\end{aligned}$$

Specially if we choose, if we take $x = y = A$, then we have Ostrowski-type inequality as follows:

$$\begin{aligned} & \left| \left[\lambda^{\alpha} + (1-\lambda)^{\alpha}\right] f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)} \left[\left(I_{\alpha}^{a} f\right)(A) + \left({}^{b}I_{\alpha} f\right)(A) \right] \right| \\ \leq \quad & \frac{\Gamma(\alpha+1) A_{\alpha,\beta,n}(x,y,A)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)} M, \end{aligned} \tag{9}$$

where

$$A_{\alpha,\beta,n}(x,y,A) = (x-a)^{\alpha+1} \left(B(\beta, n+1) - B(\beta+1, n+1) \right) + (b-x)^{\alpha+1} B(n+2, \beta).$$

Remark 11. *In the inequality (9), if we take $\alpha = n+1$, then the inequality (9) reduces the inequality (2.4) obtained via Riemann-Liouville fractional integrals in* [10, Corollary 2.1].

Corollary 12. *We have the following weighted Hadamard-type inequalities for Lipschitzian functions via conformable fractional integrals as follows:*

In the inequality (8), if we take $\delta = 1$, then we have

$$\begin{aligned} & \left| \lambda^{\alpha} f(a) + (1-\lambda)^{\alpha} f(b) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)} \left[\left(I_{\alpha}^{a} f\right)(A) + \left({}^{b}I_{\alpha} f\right)(A) \right] \right| \\ \leq \quad & \frac{\Gamma(\alpha+1) A_{\alpha,\beta,n}(a,b,A)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)} M, \end{aligned}$$

where

$$\begin{aligned} A_{\alpha,\beta,n}(a,b,A) \quad = \quad & (A-a)^{\alpha+1} \left[B(\beta+1, n+1) - B(\beta, n+1) \right] \\ & + (b-A)^{\alpha+1} \left[B(n+1, \beta) - B(n+2, \beta) \right], \end{aligned}$$

in this inequality, specially if we choose $\lambda = \frac{x-a}{b-a}$ for $x \in [a,b]$, then

$$\begin{aligned} & \left| \frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{(b-a)^{\alpha}} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)} \left[\left(I_{\alpha}^{a} f\right)(x) + \left({}^{b}I_{\alpha} f\right)(x) \right] \right| \\ \leq \quad & \frac{\Gamma(\alpha+1) A_{\alpha,\beta,n}(a,b,x)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)} M, \end{aligned}$$

Corollary 13. *In the inequality (9),*

(i) if we choose $\lambda = \frac{1}{2}$, then

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)} \left[\left(I_{\alpha}^{a} f\right)\left(\frac{a+b}{2}\right) + \left({}^{b}I_{\alpha} f\right)\left(\frac{a+b}{2}\right) \right] \right| \\ \leq \; & \frac{2^{\alpha-1}\Gamma(\alpha+1) A_{\alpha,\beta,n}(\frac{a+b}{2}, \frac{a+b}{2}, \frac{a+b}{2})}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)} M, \end{aligned}$$

where

$$A_{\alpha,\beta,n}(\frac{a+b}{2}, \frac{a+b}{2}, \frac{a+b}{2})$$

$$= \left(\frac{b-a}{2}\right)^{\alpha+1}\left[B(\beta,n+1)-B(\beta+1,n+1)+B(n+2,\beta)\right].$$

(ii) In the inequality (9), if we take $\lambda=\frac{1}{2}$ and $\delta=\frac{3}{4}$ then

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\left[(I_{\alpha}^{a}f)\left(\frac{a+b}{2}\right)+({}^{b}I_{\alpha}f)\left(\frac{a+b}{2}\right)\right]\right|\\&\le\frac{2^{\alpha-1}\Gamma(\alpha+1)A_{\alpha,\beta,n}(\frac{3a+b}{4},\frac{a+3b}{4},\frac{a+b}{2})}{n!\,(b-a)^{\alpha}\Gamma(\alpha-n)}M,\end{aligned}$$

where

$$\begin{aligned}&A_{\alpha,\beta,n}(\frac{3a+b}{4},\frac{a+3b}{4},\frac{a+b}{2})\\&=\left(\frac{b-a}{2}\right)^{\alpha+1}\big[B_{1/2}(\beta,n+1)+B_{1/2}(n+1,\beta)-2B_{1/2}(\beta+1,n+1)\\&-2B_{1/2}(n+2,\beta)+B(\beta+1,n+1)+B(n+2,\beta)-B(\beta,n+1)\big].\end{aligned}$$

3. A generalization of Bullen and Simpson type inequalities for Lipschitzian functions via fractional integrals

Throughout this section, let I be an interval in $\mathbb{R}$, $a\le x\le y\le z\le b$ in I and $f:I\to\mathbb{R}$ be an M-lipschitzian function. In the next theorem, let $\lambda+\eta+\mu=1$, $\lambda,\eta,\mu\in[0,1]$, $A=(1-\lambda)a+\lambda b$, $C=\mu a+(\lambda+\eta)\,b$, and define $I_{\alpha,\lambda,\eta,\mu}$, $\alpha>0$, as follows:

(1) If $A\le C\le x\le y\le z$ or $A\le x\le C\le y\le z$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M_{\alpha,\lambda,\eta,\mu}(x,y,z)+N_{\alpha,\lambda,\eta,\mu}(x,y,z)+O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(2) If $A\le x\le y\le C\le z$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M_{\alpha,\lambda,\eta,\mu}(x,y,z)+N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(3) If $A\le x\le y\le z\le C$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M_{\alpha,\lambda,\eta,\mu}(x,y,z)+N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(4) If $x\le A\le C\le y\le z$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+N_{\alpha,\lambda,\eta,\mu}(x,y,z)+O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(5) If $x\le A\le y\le C\le z$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(6) If $x\le A\le y\le z\le C$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(7) If $x \le y \le A \le C \le z$, then

$I_{\alpha,\lambda,\eta,\mu}(x,y,z) = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z) - N_{\alpha,\lambda,\eta,\mu}(x,y,z) + O^*_{\alpha,\lambda,\eta,\mu}(x,y,z)$.

(8) If $x \le y \le A \le z \le C$ or $x \le y \le z \le A \le C$, then

$I_{\alpha,\lambda,\eta,\mu}(x,y,z) = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z) - N_{\alpha,\lambda,\eta,\mu}(x,y,z) + O_{\alpha,\lambda,\eta,\mu}(x,y,z)$.

Where

$$\begin{aligned}
M_{\alpha,\lambda,\eta,\mu}(x,y,z) &= (A-a)^\alpha \left[(x-a)\,B(\beta,n+1) - (A-a)\,B(\beta+1,n+1)\right],\\
N_{\alpha,\lambda,\eta,\mu}(x,y,z) &= (C-A)^\alpha \left[(y-A)\,B(n+1,\beta) - (C-A)\,B(n+2,\beta)\right],\\
O_{\alpha,\lambda,\eta,\mu}(x,y,z) &= (b-C)^\alpha \left[(C-z)\,B(n+1,\beta) + (b-C)\,B(n+2,\beta)\right],
\end{aligned}$$

$$\begin{aligned}
&M^*_{\alpha,\lambda,\eta,\mu}(x,y,z) = (A-a)^\alpha \Big\{(x-a)\Big[2B_{\frac{x-a}{A-a}}(\beta,n+1) - B(\beta,n+1)\Big])\\
&+ (A-a)\Big[B(\beta+1,n+1) - 2B_{\frac{x-a}{A-a}}(\beta+1,n+1)\Big]\Big\},\ A \ne a\ (or\ \lambda \ne 0),\\
&M^*_{\alpha,0,\eta,\mu}(x,y,z) = 0,\\
&N^*_{\alpha,\lambda,\eta,\mu}(x,y,z) = (C-A)^\alpha \Big\{(y-A)\Big[2B_{\frac{y-A}{C-A}}(n+1,\beta) - B(n+1,\beta)\Big])\\
&+ (C-A)\Big[B(n+2,\beta) - 2B_{\frac{y-A}{C-A}}(n+2,\beta)\Big]\Big\},\ A \ne C\ (or\ \eta \ne 0),\\
&N^*_{\alpha,\lambda,0,\mu}(x,y,z) = 0,\\
&O^*_{\alpha,\lambda,\eta,\mu}(x,y,z) = (b-C)^\alpha \Big\{(z-C)\Big[2B_{\frac{z-C}{b-C}}(n+1,\beta) - B(n+1,\beta)\Big])\\
&+ (b-C)\Big[B(n+2,\beta) - 2B_{\frac{z-C}{b-C}}(n+2,\beta)\Big]\Big\},\ C \ne b\ (or\ \mu \ne 0),\\
&O^*_{\alpha,\lambda,\eta,0}(x,y,z) = 0.
\end{aligned}$$

Theorem 14. *Let $x, y, z, \lambda, \eta, \mu, A_1, A_2, A_{\alpha,\lambda,\eta,\mu}$ and the function f be defined as above. Then we have the inequality*

$$\begin{aligned}
&|\lambda^\alpha f(x) + \eta^\alpha f(y) + \mu^\alpha f(z)\\
&- \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\,\Gamma(\alpha-n)} \left[(I^a_\alpha f)(A) + \left({}^C I_\alpha f\right)(A) + \left({}^b I_\alpha f\right)(C)\right]\Big|\\
&\le \frac{\Gamma(\alpha+1) I_{\alpha,\lambda,\eta,\mu}(x,y,z)}{n!\,(b-a)^\alpha\,\Gamma(\alpha-n)} M.
\end{aligned} \tag{10}$$

Proof. Using the hypothesis of f, we have the inequality

$$\begin{aligned}
&|\lambda^\alpha f(x) + \eta^\alpha f(y) + \mu^\alpha f(z)\\
&- \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\,\Gamma(\alpha-n)} \left[(I^a_\alpha f)(A) + \left({}^C I_\alpha f\right)(A) + \left({}^b I_\alpha f\right)(C)\right]\Big|\\
&= \frac{\Gamma(\alpha+1)}{n!\,(b-a)^\alpha\,\Gamma(\alpha-n)} \left|\int_a^A [f(x)-f(t)]\,(A-t)^n\,(t-a)^{\beta-1}\,dt\right.
\end{aligned}$$

$$+\int_{A}^{C}[f(y)-f(t)](t-A)^{n}(C-t)^{\beta-1}dt+\int_{C}^{b}[f(z)-f(t)](t-C)^{n}(b-t)^{\beta-1}dt\Bigg|$$

$$\leq\frac{\Gamma(\alpha+1)}{n!(b-a)^{\alpha}\Gamma(\alpha-n)}\Bigg|\int_{a}^{A}|f(x)-f(t)|(A-t)^{n}(t-a)^{\beta-1}dt$$

$$+\int_{A}^{C}|f(y)-f(t)|(t-A)^{n}(C-t)^{\beta-1}dt+\int_{C}^{b}|f(z)-f(t)|(t-C)^{n}(b-t)^{\beta-1}dt\Bigg|$$

$$\leq\frac{\Gamma(\alpha+1)M}{n!(b-a)^{\alpha}\Gamma(\alpha-n)}\Bigg|\int_{a}^{A}|x-t|(A-t)^{n}(t-a)^{\beta-1}dt \tag{11}$$

$$+\int_{A}^{C}|y-t|(t-A)^{n}(C-t)^{\beta-1}dt+\int_{C}^{b}|z-t|(t-C)^{n}(b-t)^{\beta-1}dt\Bigg|.$$

Now, using simple calculations, we obtain the following identities $\int_a^A |x-t|(A-t)^n(t-a)^{\beta-1}dt$, $\int_A^C |y-t|(t-A)^n(C-t)^{\beta-1}dt$ and $\int_C^b |z-t|(t-C)^n(b-t)^{\beta-1}dt$.

(1) If $A\leq C\leq x\leq y\leq z$ or $A\leq x\leq C\leq y\leq z$, then we have

$$\begin{aligned}&\int_{a}^{A}|x-t|(A-t)^{n}(t-a)^{\beta-1}dt\\&=(A-a)^{\alpha}[(x-a)B(\beta,n+1)-(A-a)B(\beta+1,n+1)]\\&=M_{\alpha,\lambda,\eta,\mu}(x,y,z),\\&\int_{A}^{C}|y-t|(t-A)^{n}(C-t)^{\beta-1}dt\\&=(C-A)^{\alpha}[(y-A)B(n+1,\beta)-(C-A)B(n+2,\beta)]\\&=N_{\alpha,\lambda,\eta,\mu}(x,y,z),\end{aligned}$$

and

$$\begin{aligned}&\int_{C}^{b}|z-t|(t-C)^{n}(b-t)^{\beta-1}dt\\&=(b-C)^{\alpha}\Big\{(z-C)\Big[2B_{\frac{z-C}{b-C}}(n+1,\beta)-B(n+1,\beta)\Big])\\&+(b-C)\Big[B(n+2,\beta)-2B_{\frac{z-C}{b-C}}(n+2,\beta)\Big]\Big\}\end{aligned}$$

$$= O^*_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(2) If $A \le x \le y \le C \le z$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt$$
$$= (C-A)^\alpha \left\{(y-A)\left[2B_{\frac{y-A}{C-A}}(n+1,\beta) - B(n+1,\beta)\right]\right)$$
$$+ (C-A)\left[B(n+2,\beta) - 2B_{\frac{y-A}{C-A}}(n+2,\beta)\right]\right\}$$
$$= N^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt = O^*_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(3) If $A \le x \le y \le z \le C$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt = N^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt$$
$$= (b-C)^\alpha\left[(C-z)\,B(n+1,\beta) + (b-C)\,B(n+2,\beta)\right]$$
$$= O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(4) If $x \le A \le C \le y \le z$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt = N_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_{C}^{b} |z-t| (t-C)^{n} (b-t)^{\beta-1} dt = O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(5) If $x \leq A \leq y \leq C \leq z$, then we have

$$\int_{a}^{A} |x-t| (A-t)^{n} (t-a)^{\beta-1} dt = M^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_{A}^{C} |y-t| (t-A)^{n} (C-t)^{\beta-1} dt = N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_{C}^{b} |z-t| (t-C)^{n} (b-t)^{\beta-1} dt = O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(6) If $x \leq A \leq y \leq z \leq C$, then we have

$$\int_{a}^{A} |x-t| (A-t)^{n} (t-a)^{\beta-1} dt = M^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_{A}^{C} |y-t| (t-A)^{n} (C-t)^{\beta-1} dt = N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_{C}^{b} |z-t| (t-C)^{n} (b-t)^{\beta-1} dt = O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(7) If $x \leq y \leq A \leq C \leq z$, then we have

$$\int_{a}^{A} |x-t| (A-t)^{n} (t-a)^{\beta-1} dt = M^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_{A}^{C} |y-t| (t-A)^{n} (C-t)^{\beta-1} dt = -N_{\alpha,\lambda,\eta,\mu}(x,y,z)$$

and

$$\int_{C}^{b} |z-t| (t-C)^{n} (b-t)^{\beta-1} dt = O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(8) If $x \leq y \leq A \leq z \leq C$ or $x \leq y \leq z \leq A \leq C$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt = -N_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt = O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

Using the inequality (11) and the above identities $\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt$, $\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt$ and $\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt$, we derive the inequality (10). This completes the proof. □

Under the assumptions of Theorem 14, we have the following corollaries and remarks as follows:

Remark 15. *In Theorem 14, if we take $\alpha = \beta = 1$ and $n = 0$, then then the inequality (10) reduces the inequality (3) in Theorem 3 under the appropriate symbols.*

Corollary 16. *In Theorem 14, let $\delta \in \left[\frac{1}{2}, 1\right]$, $x = \delta a + (1-\delta)b$, $y = \frac{a+b}{2}$ and $z = (1-\delta)a + \delta b$. Then, we have the inequality*

$$\begin{aligned} &\Bigg|\lambda^\alpha f(\delta a + (1-\delta)b) + \eta^\alpha f(\frac{a+b}{2}) + \mu^\alpha f((1-\delta)a + \delta b) \\ &- \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\,\Gamma(\alpha-n)} \left[\left(I^a_\alpha f\right)(A) + \left({}^C I_\alpha f\right)(A) + \left({}^b I_\alpha f\right)(C)\right]\Bigg| \\ \leq\ & \frac{\Gamma(\alpha+1) I_{\alpha,\lambda,\eta,\mu}(\delta a + (1-\delta)b, \frac{a+b}{2}, (1-\delta)a + \delta b)}{n!\,(b-a)^\alpha\,\Gamma(\alpha-n)} M. \end{aligned}$$

Corollary 17. *In Corollary 16, if we take $\delta = 1$, $\lambda = \mu = \frac{\theta}{2}$ and $\eta = 1-\theta$ with $\theta \in [0,1]$, then we have the following weighted Bullen-type inequality for M-Lipschitzian functions via fractional integrals*

$$\begin{aligned} &\Bigg|\left(\frac{\theta}{2}\right)^\alpha (f(a) + f(b)) + (1-\theta)^\alpha f\left(\frac{a+b}{2}\right) \\ &- \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\,\Gamma(\alpha-n)} \left[\left(I^a_\alpha f\right)(A) + \left({}^C I_\alpha f\right)(A) + \left({}^b I_\alpha f\right)(C)\right]\Bigg| \\ \leq\ & \frac{\Gamma(\alpha+1) I_{\alpha,\frac{\theta}{2},1-\theta,\frac{\theta}{2}}(a, \frac{a+b}{2}, b)}{n!\,(b-a)^\alpha\,\Gamma(\alpha-n)} M, \end{aligned} \tag{12}$$

where

$$\begin{aligned} & I_{\alpha,\frac{\theta}{2},1-\theta,\frac{\theta}{2}}(a,\frac{a+b}{2},b) \\ = & (b-a)^{\alpha+1}\left\{ \begin{array}{c} \left(\frac{\theta}{2}\right)^{\alpha+1}\left[B\left(\beta+1,n+1\right)+B\left(n+1,\beta\right)-B\left(n+2,\beta\right)\right] \\ +\left(1-\theta\right)^{\alpha+1}\left[\begin{array}{c} B\left(n+2,\beta\right)-\frac{1}{2}B\left(n+1,\beta\right)+B_{1/2}\left(n+1,\beta\right) \\ -2B_{1/2}\left(n+2,\beta\right) \end{array}\right] \end{array} \right\}. \end{aligned}$$

Specially, in the inequality (12), if we take $n=0$ and $\alpha=\beta=1$, then the inequality (12) reduces to the following general inequality for M-Lipschitzian functions

$$\begin{aligned} & \left|\left(\frac{\theta}{2}\right)\left(f(a)+f(b)\right)+\left(1-\theta\right)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq & \frac{M}{4}\left(b-a\right)\left[2\theta^2+\left(1-\theta\right)^2\right]. \end{aligned} \tag{13}$$

Remark 18. *In the inequality (12), if we take $\alpha=n+1$, then the inequality (12) reduces the inequality obtained via Riemann-Liouville fractional integrals in* [10, Corollary 3.2].

Remark 19. *In the inequality (13), if we take $\theta=\frac{1}{3}$, then the inequality (13) reduces to the following Simpson-type inequality for M-Lipschitzian functions*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{6}\left(b-a\right).$$

Remark 20. *In the inequality (13), if we take $\theta=\frac{1}{2}$, then the inequality (13) reduces to the following Bullen type inequality for M-Lipschitzian functions*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{3M}{16}\left(b-a\right).$$

Remark 21. *In the inequality (13), if we take $\theta=0$, then the inequality (13) reduces to the following Midpoint type inequality for M -Lipschitzian functions*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{4}\left(b-a\right).$$

Remark 22. *In the inequality (13), if we take $\theta=1$, then the inequality (13) reduces to the following Trapezoid type inequality for M -Lipschitzian functions*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{2}\left(b-a\right).$$

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APPROXIMATION BY SAMPLING TYPE DISCRETE OPERATORS

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Abstract. In this paper, we deal with discrete operators of sampling type. It is known that this type of operators are related to generalized sampling series and they have important applications. In this work, using bounded and uniformly continuous functions we get general estimations under usual supremum norm with the help of summability method. We also study the degree of approximation with respect to suitable Lipschitz class of continuous functions. Finally, we give specific kernels which verify our kernel assumptions.

1. Introduction

Sampling type discrete operators have significant applications in speech processing, medicine, economic forecasting, geophysics and etc. (see [2, 11, 12, 13, 14, 15, 16, 25]). In this paper, we mainly inspired from the paper [1], where Angeloni and Vinti had some convergence results using discrete operators. The authors utilized from convergence in φ-variation to get some convergence results in that work. Now, our aim is to get some approximations under usual supremum norm by generalizing them using Bell-type summability method. In this process, we use bounded and uniformly continuous functions on $\mathbb{R}$. Furthermore, we study the rate of approximation for our main theorem using suitable Lipschitz class. Then, taking some appropriate kernels we also get more general case of generalized sampling series. Finally, we illustrate the kernels $l_{k,w}$ which satisfy our kernel assumptions.

Some notations and definitions are given below.

- $\|\cdot\|_{l^1}$ denotes the l^1 norm, i.e., for a given $u_k : \mathbb{Z} \to \mathbb{R}$, $\|u_k\|_{l^1} = \sum_{k\in\mathbb{Z}} |u_k|$.
- By $\|\cdot\|$, we mean the usual supremum norm on $\mathbb{R}$.
- The space of bounded and uniformly continuous functions on $\mathbb{R}$ is shown by $BUC(\mathbb{R})$.
- Let $\mathcal{A} = \{A^{\upsilon}\}_{\upsilon\in\mathbb{N}} = \{[a^{\upsilon}_{nw}]\}_{\upsilon\in\mathbb{N}}$ $(n, w \in \mathbb{N})$ be a family of infinite matrices of real or complex numbers. Then, for a given sequence $x = (x_k)$ the

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following double sequence $(\mathcal{A}x)_n^{\upsilon}$

$$(\mathcal{A}x)_n^{\upsilon} := \left\{ \sum_{w=1}^{\infty} a_{nw}^{\upsilon} x_w \right\} \quad (n, \upsilon \in \mathbb{N})$$

is called by $\mathcal{A}$-transform of x, if the series is convergent for all $n, \upsilon \in \mathbb{N}$. Moreover, if

$$\lim_{n\to\infty} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} x_w = L \text{ uniformly in } \upsilon$$

holds, we call "x is $\mathcal{A}$-summable to L" and denote by

$$\mathcal{A} - \lim x = L$$

(see [9]).

- $\mathcal{A}$ is called regular if for any $\lim_k x_k = L$ implies that $\mathcal{A}-\lim x = L$ ([9, 10]).
- A characterization for the regularity of the given method $\mathcal{A}$ is found by Bell in [10] such that

$$\mathcal{A} \text{ is regular } \Leftrightarrow \begin{array}{l} \cdot \text{ for each } w \in \mathbb{N},\ \lim_{n\to\infty} a_{nw}^{\upsilon} = 0 \text{ (uniformly in } \upsilon), \\ \cdot\ \lim_{n\to\infty} \sum\limits_{w=1}^{\infty} a_{nw}^{\upsilon} = 1 \text{ (uniformly in } \upsilon), \\ \cdot \text{ for all } n, \upsilon \in \mathbb{N},\ \sum\limits_{w=1}^{\infty} |a_{nw}^{\upsilon}| < \infty \text{ and there exist integers} \\ \quad N \text{ and } M \text{ such that } \sup\limits_{n \geq N, \upsilon \in \mathbb{N}} \sum\limits_{w=1}^{\infty} |a_{nw}^{\upsilon}| \leq M. \end{array}$$

- Throughout the paper, we will assume that $\mathcal{A}$ is regular with nonnegative real entries.

We should note that Bell-type summability method consists many well-known methods such as Cesàro summability [18], almost convergence [23], order summability [19, 20] and etc. It also allows us to increase the speed of convergence [21, 27, 29]. Some applications of Bell-type summability method are given in [3, 4, 5, 6, 7, 8, 17, 22, 24, 28].

Now, we can define our operator as follows:

$$\mathcal{T}_{n,\upsilon}(f; x) = \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} f\left(x - \tfrac{k}{w}\right) l_{k,w} \quad (x \in \mathbb{R},\ n, \upsilon \in \mathbb{N}) \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is bounded and $l_{k,w} \in l^1(\mathbb{Z})$ is a family of discrete kernels for all $w \in \mathbb{N}$.

Our aim is to prove the following general convergence result

$$\lim_{n\to\infty} \|\mathcal{T}_{n,\upsilon}(f) - f\| = 0 \text{ (uniformly in } \upsilon \in \mathbb{N})$$

for all $f \in BUC(\mathbb{R})$. It is not hard to see that operator (1.1) coincides with the following operator

$$T_w(f; x) = \sum_{k\in\mathbb{Z}} f\left(x - \tfrac{k}{w}\right) l_{k,w}$$

when $\mathcal{A} = \{A^{\upsilon}\} = \{I\}$ (identity matrix). Furthermore, we will indicate that operator (1.1) contains the $\mathcal{A}$-transform of generalized sampling series, defined by

$$\mathcal{S}_{n,\upsilon}(f;x) = \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} f\left(\tfrac{k}{w}\right) \chi(wx-k) \quad (x\in\mathbb{R},\ n,\upsilon \in \mathbb{N}) \tag{1.2}$$

where $f, \chi : \mathbb{R} \to \mathbb{R}$ and generalized sampling series

$$S_w(f;x) = \sum_{k\in\mathbb{Z}} f\left(\tfrac{k}{w}\right) \chi(wx-k)$$

is a special case of (1.2).

2. Approximation in Usual Supremum Norm

In this section, we will prove our main approximation theorem. For this, we need the following conditions on the kernel of the corresponding operator.

(l_1) There exists a constant $A > 0$ such that $\sup\limits_{n,\upsilon\in\mathbb{N}} \sum\limits_{w=1}^{\infty} a_{nw}^{\upsilon} \| l_{k,w}\|_{l^1} = A < \infty$,

(l_2) $\mathcal{A} - \lim \left(\sum\limits_{k\in\mathbb{Z}} l_{k,w} \right) = 1$,

(l_3) there exists $r > 0$ such that $\mathcal{A} - \lim \left(\sum\limits_{|k|\geq r} |l_{k,w}| \right) = 0$.

Here, when $\mathcal{A}$ is taken the identity matrix, conditions $(l_1) - (l_3)$ reduce to the approximate identities given in [1].

The following lemma shows that (1.1) is well defined for all bounded functions.

Lemma 2.1. *If f is bounded on $\mathbb{R}$ and (l_1) holds, then $\|\mathcal{T}_{n,\upsilon}(f)\| < \infty$ for every $n,\upsilon \in \mathbb{N}$. Moreover, if $f \in L^1(\mathbb{R})$, then $\mathcal{T}_{n,\upsilon}(f) \in L^1(\mathbb{R})$.*

Proof. Since f is bounded, there exists a positive number M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Considering this with (l_1), we get

$$\begin{aligned} |\mathcal{T}_{n,\upsilon}(f;x)| &\leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} \left| f\left(x - \tfrac{k}{w}\right)\right| |l_{k,w}| \\ &\leq MA \end{aligned}$$

and having supremum over $x\in\mathbb{R}$, we have

$$\|\mathcal{T}_{n,\upsilon}(f)\| \leq MA < \infty$$

for all $n,\upsilon \in \mathbb{N}$, which shows that $\mathcal{T}_{n,\upsilon}$ maps from the space of bounded functions into itself.

For the second part of the theorem, assume that $f \in L^1(\mathbb{R})$. Then, it is possible to write that

$$\int_{\mathbb{R}} |\mathcal{T}_{n,\upsilon}(f;x)|\, dx \leq \int_{\mathbb{R}} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left| f\left(x - \tfrac{k}{w}\right)\right| dx$$

and from a theorem of integration by series (see [26]),

$$\int_{\mathbb{R}} |\mathcal{T}_{n,\upsilon}(f;x)|\,dx \leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right)\right\|_{L^1}$$

holds for all $n,\upsilon \in \mathbb{N}$. Since $\left\| f\left(\cdot - \frac{k}{w}\right)\right\|_{L^1} = \|f\|_{L^1}$, then

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{T}_{n,\upsilon}(f;x)|\,dx &\leq \|f\|_{L^1} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \\ &\leq A\|f\|_{L^1}\end{aligned}$$

is obtained, where $\|f\|_{L^1}$ is the classical L^1 norm, i.e., $\|f\|_{L^1} = \int_{\mathbb{R}} |f(x)|\,dx$. □

Lemma 2.2. *Assume that* (l_1) *holds. If* $f \in BUC(\mathbb{R})$, *then* $\mathcal{T}_{n,\upsilon}(f) \in BUC(\mathbb{R})$ *for all* $n,\upsilon \in \mathbb{N}$.

Proof. By the previous lemma it is clear that if f is bounded, then $\mathcal{T}_{n,\upsilon}(f)$ is too. Now, let $\varepsilon > 0$ be given and let $|x-y| < \delta$ where δ corresponds to given ε and f. Then,

$$|\mathcal{T}_{n,\upsilon}(f;x) - \mathcal{T}_{n,\upsilon}(f;y)| \leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left| f\left(x - \tfrac{k}{w}\right) - f\left(y - \tfrac{k}{w}\right)\right|$$

holds. Since $\left|x - \frac{k}{w} - \left(y - \frac{k}{w}\right)\right| = |x-y| < \delta$, from (l_1)

$$|\mathcal{T}_{n,\upsilon}(f;x) - \mathcal{T}_{n,\upsilon}(f;y)| \leq A\varepsilon$$

for all $n,\upsilon \in \mathbb{N}$. □

The main approximation theorem is given below.

Theorem 2.3. *Assume that* $(l_1)-(l_3)$ *hold. Then, for all* $f \in BUC(\mathbb{R})$ *we have*

$$\lim_{n\to\infty} \|\mathcal{T}_{n,\upsilon}(f) - f\| = 0 \text{ uniformly in } \upsilon.$$

Proof. From triangle inequality, it is possible to write that

$$\begin{aligned}|\mathcal{T}_{n,\upsilon}(f;x) - f(x)| &= \left| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} \left(f\left(x - \tfrac{k}{w}\right) - f(x)\right)\right. \\ &\quad \left. + f(x)\left(\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} - 1\right)\right| \\ &\leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right\| \\ &\quad + \|f\| \left| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| \\ &:= A_1 + A_2\end{aligned}$$

holds. In A_1, we concentrate on the continuity of f. Since f is uniformly continuous, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \tag{2.1}$$

whenever $|x-y| < \delta$. Then, for a fixed $\bar{r}$ it is easy to find a number w_1 satisfying

$$\left|\tfrac{\bar{r}}{w}\right| < \delta$$

for all $w > w_1$.Now, if we divide A_1 as follows

$$\begin{aligned} A_1 = {} & \sum_{w=1}^{w_1} a_{nw}^{\upsilon} \sum_{|k|<\bar{r}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot) \right\| \\ & + \sum_{w=w_1+1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<\bar{r}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot) \right\| \\ & + \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|\geq\bar{r}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot) \right\| \\ & := A_1^1 + A_1^2 + A_1^3 \end{aligned}$$

from (2.1) and (l_1)

$$A_1^2 \leq A\varepsilon$$

holds, since $\left|x - \frac{k}{w} - x\right| = \left|\frac{k}{w}\right| < \frac{\bar{r}}{w} < \delta$.

For A_1^1, from the regularity of $\mathcal{A}$, one can find a number $n_1 = n_1(\varepsilon)$ such that

$$A_1^1 < D' w_1 \varepsilon$$

where $D' := \max_{1\leq w\leq w_1} \left\{ \sum_{|k|<\bar{r}} |l_{k,w}| \left\| f\left(\cdot - \frac{k}{w}\right) - f(\cdot) \right\| \right\}$. And from (l_3), we see that

$$A_1^3 < 2\|f\|\varepsilon$$

for sufficiently large $n \in \mathbb{N}$.

Finally, it follows from (l_2)

$$A_2 < \|f\|\varepsilon$$

yields for sufficiently large $n \in \mathbb{N}$. Hence, having supremum over $x\in\mathbb{R}$ in the first inequality, we complete the proof. □

3. Rate of Convergence

In this section we investigate the rate of approximation, and therefore we need the following Lipschitz class.

For any given $\alpha > 0$, define $Lip(\alpha)$ as follows:

$$Lip(\alpha) = \{ f \in BUC(\mathbb{R}) : \|f(\cdot - t) - f(\cdot)\| = O(|t|^{\alpha}) \text{ as } t \to 0 \}$$

where $f(t) = O(g(t))$ as $t \to 0$ means that, there exist $\delta, N > 0$ such that $|f(t)| \leq N|g(t)|$ for $|t| < \delta$. Let Ψ be family of all functions $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, such that $\xi(0) = 0$, $\xi(t) > 0$ for $t > 0$ and ξ be continuous at $t = 0$. Now, for any fixed $\alpha > 0$ and $\xi \in \Psi$, consider the following conditions:

$$\left(\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} - 1\right) = O\left(\xi\left(1/n\right)\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon), \tag{3.1}$$

there exists a constant $r_0 > 0$ such that

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<r_0} \frac{|l_{k,w}|}{w^{\alpha}} = O\left(\xi\left(1/n\right)\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon), \tag{3.2}$$

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|\geq r_0} |l_{k,w}| = O\left(\xi\left(1/n\right)\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon) \tag{3.3}$$

and for a given $\mathcal{A} = \{[a_{nw}^{\upsilon}]\}_{\upsilon\in\mathbb{N}}$

$$\text{for each } w \in \mathbb{N},\ a_{nw}^{\upsilon} = O\left(\xi\left(1/n\right)\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon). \tag{3.4}$$

We obtain the following rates of approximations.

Theorem 3.1. *Suppose that for any fixed $\xi \in \Psi$ and $\alpha > 0$, (3.1)-(3.4) and (l_1) hold. Then, for all $f \in Lip\left(\alpha\right)$*

$$\left\|\mathcal{T}_{n,\upsilon}\left(f\right) - f\right\| = O\left(\xi\left(1/n\right)\right) \textit{ as } n \to \infty \textit{ (uniformly in } \upsilon).$$

Proof. From the proof of Theorem 2.3, we observe that

$$\begin{aligned}\left\|\mathcal{T}_{n,\upsilon}\left(f\right) - f\right\| &\leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \\ &+ \|f\| \left|\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| \\ &:= B_1 + B_2\end{aligned}$$

holds. In B_1 for some fixed $r_0 > 0$, we can find a number w_2 such that for all $w > w_2$, $\left|x - \frac{k}{w} - x\right| = \left|\frac{k}{w}\right| < \frac{r_0}{w} < \delta$ and since $f \in Lip\left(\alpha\right)$, there exists a constant $N > 0$ such that

$$\left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \leq N \left|\tfrac{k}{w}\right|^{\alpha}$$

hold. Then, we get

$$\begin{aligned}B_1 &= \sum_{w=1}^{w_2} a_{nw}^{\upsilon} \sum_{|k|<r_0} |l_{k,w}| \left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \\ &+ \sum_{w=w_2+1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<r_0} |l_{k,w}| \left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \\ &+ \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|\geq r_0} |l_{k,w}| \left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \\ &\leq D'' w_2 \max_{1\leq w\leq w_2} a_{nw}^{\upsilon} \\ &+ N \sum_{w=w_2+1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<r_0} |l_{k,w}| \left(\tfrac{r_0}{w}\right)^{\alpha}\end{aligned}$$

$$+2\|f\|\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{|k|\geq r_0}|l_{k,w}|$$
$$:=B_1^1+B_1^2+B_1^3$$

where $D'':=\max_{1\leq w\leq w_2}\left\{\sum_{|k|<r_0}|l_{k,w}|\left\|f\left(\cdot-\frac{k}{w}\right)-f(\cdot)\right\|\right\}$. From (3.4), (3.2) and (3.3) it is clear that

$$B_1^1,B_1^2,B_1^3=O\left(\xi\left(1/n\right)\right)\ \text{ as } n\to\infty \text{ (uniformly in } \upsilon),$$

yields.

Finally, from (3.1) we conclude that

$$B_2=O\left(\xi\left(1/n\right)\right)\ \text{ as } n\to\infty \text{ (uniformly in } \upsilon).$$

□

Notice that, it is possible to find regular methods such that (3.4) is satisfied, for instance, $\{C_1\}$ (Cesàro Matrix) and $\mathcal{F}$ (almost convergence matrix) which are given in Corollary 4.3.

4. Conclusions and Applications

In the present section, we give some applications of the operators of type (1.1).

Let $f:\mathbb{R}\to\mathbb{R}$ be given, and suppose that $l_{k,w}\equiv\chi(k)$, that is, $l_{k,w}$ is not depending on w where $\chi:\mathbb{R}\to\mathbb{R}$. Then, (1.1) reduces to

$$\bar{\mathcal{T}}_{n,\upsilon}(f;x)=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}f\left(x-\frac{k}{w}\right)\chi(k),\ x\in\mathbb{R}$$

which is in some cases equal to $\mathcal{A}-$transform of generalized sampling series, namely

$$\mathcal{S}_{n,\upsilon}(f;x)=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}f\left(\frac{k}{w}\right)\chi(wx-k),\ x\in\mathbb{R}.$$

In this case (l_1) and (l_2) coincide with the following assumptions

(l_1') $\chi\in l^1(\mathbb{Z})$
(l_2') $\sum_{k\in\mathbb{Z}}\chi(k)=1$

where on the other hand, (l_3) is clearly not satisfied. But these two conditions are still enough to verify the following approximations (see also [1]).

Theorem 4.1. *Let $f\in BUC(\mathbb{R})$. If (l_1'), (l_2') hold, then*

$$\lim_{n\to\infty}\left\|\bar{\mathcal{T}}_{n,\upsilon}(f)-f\right\|=0 \text{ (uniformly in } \upsilon\in\mathbb{N}).$$

Proof. Considering (l_2'), by the proof of the Theorem 2.3, we obtain the following inequalities

$$\left\|\bar{\mathcal{T}}_{n,\upsilon}(f)-f\right\|\leq\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}|\chi(k)|\left\|f\left(\cdot-\frac{k}{w}\right)-f(\cdot)\right\|+\|f\|\left|\sum_{w=1}^{\infty}a_{nw}^{\upsilon}-1\right|.$$

Since $\sum_{k\in\mathbb{Z}} |\chi(k)| < \infty$ from (l_1'), for all $\varepsilon > 0$ there exists a number $\breve{r} > 0$ such that

$$\sum_{|k|\geq \breve{r}} |\chi(k)| < \varepsilon$$

and hence, for sufficiently large $n \in \mathbb{N}$

$$\begin{aligned}\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|\geq \breve{r}} |\chi(k)| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right\| &< 2\|f\| \sum_{w=1}^{\infty} a_{nw}^{\upsilon}\varepsilon \\ &\leq 2M\|f\|\varepsilon\end{aligned}$$

holds where M comes from the regularity of $\mathcal{A}$. In a similar way with the proof of Theorem 2.3, it is possible to show

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<\breve{r}} |\chi(k)| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right\| < \varepsilon\left(\bar{D}\bar{w}_1 + \bar{A}M\right)$$

for sufficiently large $n \in \mathbb{N}$, where $\bar{D} := \max_{1\leq w\leq \bar{w}_1} \left\{\sum_{|k|<\breve{r}} |\chi(k)| \left\| f\left(\cdot - \frac{k}{w}\right) - f(\cdot)\right\|\right\}$. Finally, by the regularity of $\mathcal{A}$

$$\|f\| \left|\sum_{w=1}^{\infty} a_{nw}^{\upsilon} - 1\right| < \|f\|\varepsilon$$

for sufficiently large $n \in \mathbb{N}$. Since ε is arbitrary, the proof is completed. □

Although $\bar{\mathcal{T}}$ and $\mathcal{S}$ are similar, they are different in general. However, in some cases, they coincide (see [1]).

Corollary 4.2. *Let $f \in B_{\pi w}^1(\mathbb{R})$ (the Paley-Wiener Space $B_{\pi w}^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : |f(z)| \leq \exp(\pi w|z|)\|f\|$ for every $z \in \mathbb{C}\}$) for some $w > 0$ and $\chi \in B_\pi^\infty(\mathbb{R})$. If (l_1') and (l_2') hold, then*

$$\lim_{n\to\infty} \|\mathcal{S}_{n,\upsilon}(f) - f\| = 0 \text{ (uniformly in } \upsilon \in \mathbb{N}).$$

Proof. It is proved in [1] that $B_{\pi w}^1(\mathbb{R}) \subset Lip(\mathbb{R})$, and therefore bounded elements of $B_{\pi w}^1(\mathbb{R})$ are also elements of $BUC(\mathbb{R})$. On the other hand, using similar arguments in Lemma 4.2 in [1] we get

$$\mathcal{S}_{n,\upsilon}(f) = \bar{\mathcal{T}}_{n,\upsilon}(f)$$

for all $n, \upsilon \in \mathbb{N}$ and $f \in B_{\pi w}^1(\mathbb{R})$. Consequently, by the Theorem 4.1, the proof completes. □

Remark 4.1. *It may clearly be seen that, Corollary 4.2 holds for $f \in B_{\pi w}^p(\mathbb{R})$ where $1 \leq p \leq 2$. In this case, we need to assume $\chi \in B_\pi^q(\mathbb{R})$ to apply Lemma 4.2 in* [1] *where $1/p + 1/q = 1$. For some examples of χ which satisfy $\chi \in B_\pi^\infty(\mathbb{R})$, (l_1') and (l_2'), we refer to Example 4.5 in* [1].

It is clear that operator (1.1) can be written as

$$\mathcal{T}_{n,\upsilon}(f;x)=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}T_{w}(f;x) \tag{4.1}$$

where T_w is given by

$$T_w(f;x)=\sum_{k\in\mathbb{Z}}f\left(x-\tfrac{k}{w}\right)l_{k,w}\ \ (x\in\mathbb{R},\ w\in\mathbb{N}). \tag{4.2}$$

Considering (4.1) and (4.2), we get the following corollary.

Corollary 4.3. *Taking specific regular matrices, we observe the following estimations:*

- *Assume that $\mathcal{A}=\mathcal{F}=\{F^{\upsilon}\}=\{[a_{nw}^{\upsilon}]\}$ where $a_{nw}^{\upsilon}=1/n$ if $\upsilon\leq w\leq n+\upsilon-1$; $a_{nw}^{\upsilon}=0$ if otherwise. Assume further that $(l_1)-(l_3)$ hold for $\mathcal{A}=\mathcal{F}$ (almost convergence matrix). Then, for all $f\in BUC(\mathbb{R})$,*

$$\lim_{n\to\infty}\left\|\frac{T_{\upsilon}(f)+T_{\upsilon+1}(f)+\cdots+T_{n+\upsilon-1}(f)}{n}-f\right\|=0\quad (\textit{uniformly in } \upsilon)$$

  i.e., $T_n(f)$ is almost convergent to f,
- *Assume that $\mathcal{A}=\{C_1\}=\{[c_{nw}]\}$ where $c_{nw}=1/n$ if $1\leq w\leq n$; $c_{nw}=0$ if otherwise. Assume further that $(l_1)-(l_3)$ hold for $\mathcal{A}=\{C_1\}$ (Cesàro matrix). Then, for all $f\in BUC(\mathbb{R})$,*

$$\lim_{n\to\infty}\left\|\frac{T_1(f)+T_2(f)+\cdots+T_n(f)}{n}-f\right\|=0$$

  i.e., $T_n(f)$ is arithmetic mean convergent to f,
- *Suppose that $\mathcal{A}=\{I\}$ and $(l_1)-(l_3)$ hold. Then, for all $f\in BUC(\mathbb{R})$,*

$$\lim_{n\to\infty}\|T_n(f)-f\|=0$$

  i.e., $T_n(f)$ is uniformly convergent to f, where $T_n(f)$ is given in (4.2).

Similar corollaries also hold for generalized sampling series

$$S_w(f;x)=\sum_{k\in\mathbb{Z}}f\left(\tfrac{k}{w}\right)\chi(wx-k).$$

Now, we will give a specific kernel of $l_{k,w}$, which satisfies $(l_1)-(l_3)$ respectively. Take $\mathcal{A}=\{C_1\}$, and then define $l_{k,w}$ as follows:

$$l_{k,w}=\frac{(-1)^{w}+1}{2^{w(|k|)}}\left(\frac{2^{w}-1}{2^{w}+1}\right).$$

It is easy to see that (l_1) and (l_2) are satisfied from the following calculations:

$$\sup_{n\in\mathbb{N}}\sum_{w=1}^{n}\frac{1}{n}\sum_{k\in\mathbb{Z}}|l_{k,w}|\leq\sup_{n\in\mathbb{N}}\sum_{w=1}^{n}\frac{1}{n}\sum_{k\in\mathbb{Z}}\frac{2}{2^{w|k|}}$$

$$= \sup_{n\in\mathbb{N}} \sum_{w=1}^{n} \frac{2}{n}\left(\frac{2^w+1}{2^w-1}\right)$$
$$\leq \sup_{n\in\mathbb{N}} \sum_{w=1}^{n} \frac{6}{n}$$
$$= 6$$

and since $l_{k,w} > 0$, from the previous statement

$$\lim_{n\to\infty}\left|\sum_{w=1}^{n}\frac{1}{n}\sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| \leq \lim_{n\to\infty}\left|\sum_{w=1}^{n}\frac{(-1)^w}{n}\right| = 0.$$

On the other hand, for (l_3), for any integer $r \geq 1$, we get

$$\sum_{w=1}^{n}\frac{1}{n}\sum_{|k|\geq r}|l_{k,w}| = \sum_{w=1}^{n}\frac{1}{n}\left(\frac{(-1)^w+1}{2^w+1}\right)\frac{2^{w+1}}{2^{wr}}$$

where

$$\lim_{w\to\infty}\left(\frac{(-1)^w+1}{2^w+1}\right)\frac{2^{w+1}}{2^{wr}} = 0. \tag{4.3}$$

Then, since (4.3) is convergent to 0, its arithmetic mean is too, namely,

$$\lim_{n\to\infty}\sum_{w=1}^{n}\frac{1}{n}\left(\frac{(-1)^w+1}{2^w+1}\right)\frac{2^{w+1}}{2^{wr}} = 0,$$

which implies (l_3). For the behaviour of $l_{k,w}$, see Figure 1 ($k = 0, \cdots, 5$ and $w = 1, \cdots, 6$) which is symmetric for k. But in the classical sense, $l_{k,w}$ does not

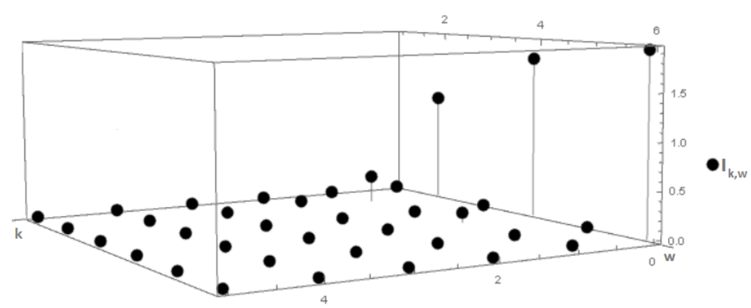

FIGURE 1. The kernel function $l_{k,w}$

satisfy the condition of (**A1**) since

$$\left|\sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| = (-1)^w + 1$$

is divergent. Therefore, our approximation is not trivial.

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ON THE STABILITY ANALYSIS OF THE TIME-FRACTIONAL VARIABLE ORDER KLEIN-GORDON EQUATION AND SOME NUMERICAL SIMULATIONS

SİNAN DENİZ

Abstract. In this paper, the Klein - Gordon equation is generalized using the concept of the variational order derivative. We try to construct the Crank-Nicholson scheme for numerical solutions of the modified Klein- Gordon equation. Stability analysis of the Crank-Nicholson scheme is examined and analyzed to prove the proposed method is stable for solving the time-fractional variable order Klein- Gordon equation. Numerical examples are also given for illustration.

1. Introduction

In recent years, fractional calculus and especially fractional differential equations (FDEs) have been extensively used for many different fields of mathematical physics such as relaxation processes,control theory of dynamical systems, viscoelasticity, diffusion and so on [1–5]. The main reason why they are so important is that a realistic modeling of many physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional derivatives. Besides, quite a number of different methods have been enhanced to analyze many different types of fractional differential equations for showing the importance of the fractional calculus [6–11]. On the other hand, stability analysis of fractional differential equations has attracted much attention over the past decade. Atangana has analyzed the stability of numerical solutions for many different types of FDEs such as groundwater flow equation [12], Schrödinger equation [13] and telegraph equation [14]. In [15], Zhang et. al. have examined the stability of FDEs, including linear FDEs, nonlinear FDEs and the FDEs with time-delay.

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As it is well known, partial differential equations are encountered frequently in many fields of applied physics [16–23]. One of them is Klein - Gordon equation which models many problems in quantum mechanics, condensed matter physics, etc. A Josephson junction, the motion of rigid pendula attached to a stretched wire can be described by sine Klein-Gordon equation and a non-local version of them are properly modeled by the fractional version of them [24]. In [25], Sweilam et al. has constructed a new and effective numerical scheme, namely weighted average nonstandard finite difference method, for analyzing the time variable-order fractional of nonlinear Klein-Gordon equation and so on.

In this paper, we investigate the stability of the linear time-fractional variable order Klein-Gordon equation:

$$D_{tt}^{\alpha(x,t)} y(x,t) - y_{xx}(x,t) + \mu y(x,t) = 0,\ 1 < \alpha(x,t) \leq 2, \mu > 0, \tag{1}$$

with the conditions

$$y(x,0) = \delta(x), y_t(x,0) = 0;\ 0 \leq t \leq T, 0 \leq x \leq L \tag{2}$$

where $\delta(x)$ is a real-valued continuous function.

2. Some basic information for the variable order fractional derivative

In this section, we give some basic definitions that we need for our analysis. For much more details about fractional analysis we refer to the books and papers in [26–28].

Definition 2.1. Let $0 < \alpha(x,t) < 1$ for all $(x,t) \in [a,b]$ and $f \in L_1[a,b]$. Then

$$_aI_t^{\alpha(.,.)}(f(t)) = \int_a^t \frac{1}{\Gamma[\alpha(t,x)]}(t-x)^{\alpha(t,x)-1} f(x)dx\ (t > a) \tag{3}$$

and

$$_bI_t^{\alpha(.,.)}(f(t)) = \int_t^b \frac{1}{\Gamma[\alpha(t,x)]}(x-t)^{\alpha(x,t)-1} f(x)dx\ (t > b) \tag{4}$$

are called the left and right Riemann-Liouville integral of variable fractional order $\alpha(.,.)$ respectively.

Definition 2.2. Let $aI_t^{1-\alpha(.,.)}f \in C[a,b]$ and $0 < \alpha(x,t) < 1$ for all $(x,t) \in [a,b]$. Then

$$_aD_t^{\alpha(.,.)}(f(t)) = \frac{d}{dt}\int_a^t \frac{1}{\Gamma[1-\alpha(t,x)]}(t-x)^{-\alpha(t,x)} f(x)dx\ (t > a) \tag{5}$$

and

$$ {}_bD_t^{\alpha(.,.)}\left(f(t)\right)=\frac{d}{dt}\int\limits_t^b\frac{1}{\Gamma\left[1-\alpha(x,t)\right]}(x-t)^{-\alpha(t,x)}f(x)dx\ (t<b) \tag{6} $$

are called the left and right Riemann-Liouville derivative of variable fractional order $\alpha(.,.)$ respectively.

Definition 2.3. Let f be a real valued differentiable function and $\alpha(x)\in C(0,1]$. Then the Caputo variable order differential operator is given by

$$ D_0^{\alpha(x)}\left(f(x)\right)=\frac{1}{\Gamma\left[1-\alpha(x)\right]}\int\limits_0^x\frac{df(t)}{dt}(x-t)^{-\alpha(t)}dt. \tag{7} $$

3. Crank-Nicholson Scheme for numerical solutions

The numbers of the works for numerical solutions of different types of fractional differential equations have begun to increase considerably in recent years. A few of the most important ones of them can be found in [13, 14, 29–31].

In this section, we construct the Crank-Nicholson scheme for the fractional Klein-Gordon equation by taking $x_l=lh, t_j=j\tau, Mh=L, N\tau=T, 0\le l\le M, 0\le j\le N$ where M,N are grid points, h,τ are step size and time respectively. Under these assumptions, Crank-Nicholson scheme can be presented by giving the following discretizations:

$$ y=\frac{1}{2}\left(y(x_l,t_{j+1})+y(x_l,t_j)\right) \tag{8} $$

$$ \begin{aligned} y_{xx}=\frac{\partial^2 y}{\partial x^2}=\frac{1}{2}&\left(\frac{y(x_{l+1},t_{j+1})-2y(x_l,t_{j+1})+y(x_{l-1},t_{j+1})}{h^2}\right)+\\ &\frac{1}{2}\left(\frac{y(x_{l+1},t_j)-2y(x_l,t_j)+y(x_{l-1},t_j)}{h^2}\right)+O(h^2) \end{aligned} \tag{9} $$

$$ \begin{aligned} &D_{tt}^{\alpha(x,t)}y=\frac{\partial^{{\alpha_l}^{j+1}}y(x_l,t_{j+1})}{\partial t^{{\alpha_l}^{j+1}}}=\frac{\tau^{-\alpha^{j+1}}}{\Gamma(2-\alpha_l^{j+1})}\times\\ &\left[\begin{array}{l} y(x_l,t_{j+1})-y(x_l,t_j)+\\ \sum\limits_{n=1}^{j}\left(y(x_l,t_{j-n+1})-y(x_l,t_{j-n})\right)\left((n+1)^{(1-\alpha_l^{j+1})}-n^{(1-\alpha_l^{j+1})}\right) \end{array}\right] \end{aligned} \tag{10} $$

Substituting (8), (9),(10) into the fractional Klein-Gordon equation (1) yields

$$
\begin{gathered}
\frac{\tau^{-\alpha^{j+1}}}{\Gamma(2-\alpha_l^{j+1})}\left[\begin{array}{l} y(x_l,t_{j+1})-y(x_l,t_j)+ \\ \displaystyle\sum_{n=1}^{j}\left(y(x_l,t_{j-n+1})-y(x_l,t_{j-n})\right)\left((n+1)^{(1-\alpha_l^{j+1})}-n^{(1-\alpha_l^{j+1})}\right) \end{array}\right] \\
-\left(\begin{array}{l} \dfrac{1}{2}\left(\dfrac{y(x_{l+1},t_{j+1})-2y(x_l,t_{j+1})+y(x_{l-1},t_{j+1})}{h^2}\right)+ \\ \dfrac{1}{2}\left(\dfrac{y(x_{l+1},t_j)-2y(x_l,t_j)+y(x_{l-1},t_j)}{h^2}\right) \end{array}\right) \\
+\mu\left(\frac{1}{2}\left(y(x_l,t_{j+1})+y(x_l,t_j)\right)\right)=0
\end{gathered}
\tag{11}
$$

Multiplying both sides of (11) with

$$
\frac{\Gamma(2-\alpha_l^{j+1})}{\tau^{-\alpha^{j+1}}}=\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})
$$

we get

$$
\begin{gathered}
y(x_l,t_{j+1})-y(x_l,t_j)+ \\
\sum_{n=1}^{j}\left(y(x_l,t_{j-n+1})-y(x_l,t_{j-n})\right)\left((n+1)^{(1-\alpha_l^{j+1})}-n^{(1-\alpha_l^{j+1})}\right) \\
-\frac{\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})}{2h^2}\left(\begin{array}{l} y(x_{l+1},t_{j+1})-2y(x_l,t_{j+1})+y(x_{l-1},t_{j+1})+ \\ y(x_{l+1},t_j)-2y(x_l,t_j)+y(x_{l-1},t_j) \end{array}\right) \\
+\frac{\mu\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})}{2}\left(y(x_l,t_{j+1})+y(x_l,t_j)\right)=0
\end{gathered}
\tag{12}
$$

and by making the following change of variables

$$
\begin{gathered}
y(x_l,t_j)=y_l^j,\quad R_l^{j+1}=\frac{\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})}{2h^2},\quad S_l^{j+1}=\frac{\mu\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})}{2} \\
c_n^{l,j+1}=(n+1)^{(1-\alpha_l^{j+1})}-n^{(1-\alpha_l^{j+1})},\quad d_n^{l,j+1}=c_{n-1}^{l,j+1}-c_n^{l,j+1}
\end{gathered}
\tag{13}
$$

Eq. (11) becomes

$$
\begin{gathered}
R_l^{j+1}\left(\, y_{l+1}^{j+1}-2y_l^{j+1}+y_{l-1}^{j+1}+y_{l+1}^{j}-2y_l^{j}+y_{l-1}^{j}\,\right)- \\
\sum_{n=1}^{j}\left[y_l^{j-n+1}-y_l^{j-n}\right]c_n^{l,j+1}+S_l^{j+1}\left(y_l^{j+1}+y_l^{j}\right)+y_l^{j+1}-y_l^{j}=0.
\end{gathered}
\tag{14}
$$

4. Stability analysis for Crank-Nicholson scheme

Stability analysis is a very important concept in solving many types of linear or nonlinear differential equations [32–34]. In order to examine the stability analysis of the Crank-Nicholson scheme defined above, we now take that $\varepsilon_l^j = y_l^j - Y_l^j$ where Y_l^j is the approximate numerical solution at the point (x_l, t_j) and

$$\varepsilon^j = \left[\varepsilon_1^j, \varepsilon_2^j, ..., \varepsilon_{M-1}^j\right]^T \tag{15}$$

with

$$\varepsilon^j(x) = \begin{cases} \varepsilon_l^j & \text{if} \quad x_l - h/2 < x \leq x_l + h/2, l = 1, 2, ..., M-1 \\ 0 & \text{if} \quad L - h/2 < x \leq L \end{cases} \tag{16}$$

for $l = 1, 2, ..., M-1$, $j = 1, 2, ..., N$. Thereby, one can use the Fourier series to state the function $\varepsilon^j(x)$ as:

$$\varepsilon^j(x) = \sum_{m=-\infty}^{m=\infty} \delta_m(m) exp\left[2i\pi mj/L\right] \tag{17}$$

where

$$\delta_j(x) = \frac{1}{L}\int_0^L \rho^j exp\left[2i\pi mx/L\right] dx. \tag{18}$$

Before going through a detailed analysis, we give the following remarks which will be necessary for stability conditions.

Remarks 4.1. One can set up the following properties for all $l = 1, 2, .., M-1$.

$$\begin{aligned} &i.\ R_l^{j+1}, S_l^{j+1} > 0 \\ &ii.\ 0 \leq d_n^{l,j} \leq d_{n-1}^{l,j} \\ &iii.\ 0 \leq c_n^{l,j} \leq 1, \sum_{n=0}^{j-1} c_{n+1}^{l,j+1} = 1 - d_n^{l,j+1}. \end{aligned} \tag{19}$$

Using the previous notations, one can present the error done while applying the Crank-Nicholson scheme to solve the given fractional Klein-Gordon equation (1) as:

$$\begin{aligned} &R_l^{j+1}\left(\ \varepsilon_{l+1}^{j+1} - 2\varepsilon_l^{j+1} + \varepsilon_{l-1}^{j+1} + \varepsilon_{l+1}^j - 2\varepsilon_l^j + \varepsilon_{l-1}^j\ \right) - \\ &\sum_{n=1}^{j}\left[\varepsilon_l^{j-n+1} - \varepsilon_l^{j-n}\right] c_n^{l,j+1} + S_l^{j+1}\left(\varepsilon_l^{j+1} + \varepsilon_l^j\right) + \varepsilon_l^{j+1} - \varepsilon_l^j. \end{aligned} \tag{20}$$

In order to show the equation (20) more briefly, the term ε_l^j can be represented in the delta-exponential form as:

$$\varepsilon_l^j = \delta_j exp\left[i\theta lj\right]. \tag{21}$$

where θ represents a real spatial wave number. Using (21) for $j = 0$, we get

$$\begin{aligned} &R_l^1 \left(\varepsilon_{l+1}^1 - 2\varepsilon_l^1 + \varepsilon_{l-1}^1 + \varepsilon_{l+1}^0 - 2\varepsilon_l^0 + \varepsilon_{l-1}^0 \right) + \\ &\sum_{n=1}^{0} \left[\varepsilon_l^{1-n} - \varepsilon_l^{-n}\right] c_n^{l,1} + S_l^1 \left(\varepsilon_l^1 + \varepsilon_l\right) + \varepsilon_l^1 - \varepsilon_l^0 = 0. \end{aligned} \tag{22}$$

Eq. (22) can be arranged as:

$$\delta_1 = \delta_0 \frac{1 + 4R_l^1 \sin^2\left(\frac{h\theta}{2}\right) - 2S_l^1 \sin^2\left(\frac{h\theta}{2}\right)}{1 + 4R_l^1 \sin^2\left(\frac{h\theta}{2}\right) + 2S_l^1 \sin^2\left(\frac{h\theta}{2}\right)} \tag{23}$$

and one can similarly obtain

$$\delta_{j+1} = \frac{\delta_j \left(1 + 4R_l^{k+1} \sin^2\left(\frac{h\theta}{2}\right) - 2S_l^{k+1} sin^2\left(\frac{h\theta}{2}\right)\right) - \sum_{n=0}^{j-1} d_{n+1}^{1,j+1} \delta_{j-n} + d_j^{1,j+1} \delta_0}{1 + 4R_l^{k+1} \sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1} \sin^2\left(\frac{h\theta}{2}\right)} \tag{24}$$

for $j = 0, 1, 2, ...$ We must now prove that the inequality $|\delta_j| \leq |\delta_0|$ holds for all $j = 1, 2, ...$ to accomplish the proof of the stability of numerical solutions.It is easy to see that the inequality is true for $j = 1$, because

$$\begin{aligned} |\delta_1| = |\delta_0| &\left| \frac{1 + 4R_l^1 \sin^2\left(\frac{h\theta}{2}\right) - 2S_l^1 \sin^2\left(\frac{h\theta}{2}\right)}{1 + 4R_l^1 \sin^2\left(\frac{h\theta}{2}\right) + 2S_l^1 \sin^2\left(\frac{h\theta}{2}\right)} \right| \leq \\ |\delta_0| &\left| \frac{1 + 4R_l^1 \sin^2\left(\frac{h\theta}{2}\right) + 2S_l^1 \sin^2\left(\frac{h\theta}{2}\right)}{1 + 4R_l^1 \sin^2\left(\frac{h\theta}{2}\right) + 2S_l^1 \sin^2\left(\frac{h\theta}{2}\right)} \right| = |\delta_0|. \end{aligned} \tag{25}$$

On the basis of induction, we now suppose that

$$|\delta_{j+1}| = \left| \frac{\delta_j \left(1 + 4R_l^{k+1} \sin^2\left(\frac{h\theta}{2}\right) - 2S_l^{k+1} sin^2\left(\frac{h\theta}{2}\right)\right) - \sum_{n=0}^{j-1} d_{n+1}^{1,j+1} \delta_{j-n} + d_j^{1,j+1} \delta_0}{1 + 4R_l^{k+1} \sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1} \sin^2\left(\frac{h\theta}{2}\right)} \right| \tag{26}$$

for $m = 2, 3, ...j$. Implementing the triangle inequality, the equality (26) turns into

$$|\delta_{j+1}| \leq \frac{|\delta_j| \left(\left|1 + 4R_l^{k+1} \sin^2\left(\frac{h\theta}{2}\right) - 2S_l^{k+1} sin^2\left(\frac{h\theta}{2}\right)\right|\right) + \sum_{n=0}^{j-1} |d_{n+1}^{1,j+1}||\delta_{j-n}| + |d_j^{1,j+1} \delta_0|}{\left|1 + 4R_l^{k+1} \sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1} \sin^2\left(\frac{h\theta}{2}\right)\right|}. \tag{27}$$

Using the induction hypothesis, we get

$$|\delta_{j+1}| \leq |\delta_0| \left[\frac{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) - 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right| + \sum_{n=0}^{j-1} |d_{n+1}^{1,j+1}| + |d_j^{1,j+1}|}{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|} \right]. \tag{28}$$

By taking advantage of Remark 1, we finally obtain the inequality

$$\begin{aligned} |\delta_{j+1}| &\leq |\delta_0| \left[\frac{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) - 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|}{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|} \right] \\ &\leq |\delta_0| \left[\frac{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|}{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|} \right] = |\delta_0| \end{aligned} \tag{29}$$

thus,

$$|\delta_{j+1}| \leq |\delta_0|,$$

and the proof is completed.

5. Numerical examples

In this section, we give some numerical simulations for the approximate solution of the time-fractional variable order Klein-Gordon equation.

Example 1. Consider the problem (1) with $\mu = 0.9$, $\alpha(x,t) = 0.04tanh(x^3+t) - sin^2(5x^4t - 9x^2)$ and $\delta(x) = 0.08\cos(x^3)$. The error surface figures of approximate solutions are depicted for different N's and for $h = 0.0002$. As can be seen from the figures 1 and 2, the larger the N, the smaller the error.

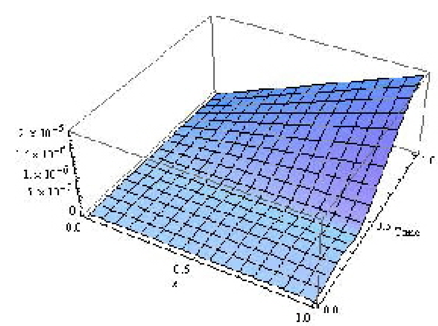

Figure 1. The error surface figures for $N = 40$

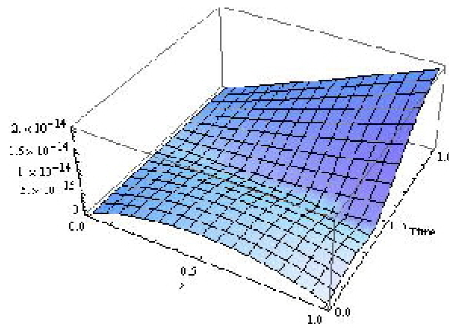

Figure 2. The error surface figures for $N = 80$

Example 2. As a second example, let us consider the problem (1) with $\mu = 0.8$, $\alpha(x,t) = 2 - sin^2(x^5 t + t^7)$ and $\delta(x) = x + \sec(x^{0.7})$. The error surface figures of approximate solutions are displayed for different N's and for $h = 0.00012$. Again, it is clear from the figures 3 and 4, we have smaller errors for the larger the N.

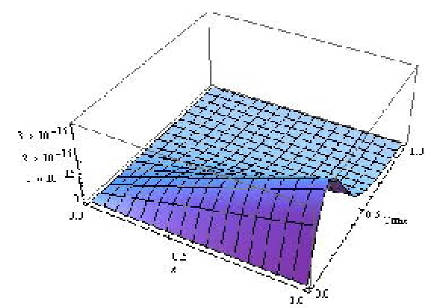

Figure 3. The error surface figures for $N = 80$

Example 3. As a final example, let us now consider the problem (1) with $\mu = 0.5$, $\alpha(x,t) = 1 - cos^2(x + t^3)$ and $\delta(x) = \sin(x)$. Figures of the approximate solutions are sketched for different N's and for $h = 0.0005$. A slight difference between these solutions can be seen from the simulations from Fig. 5 to 8 for $N = 10$ to $N = 70$. In addition to that, the error surface figure of approximate solution for $N = 80$ is demonstrated in Figure 9.

6. Results and discussion

We have modified the time-fractional variable order Klein-Gordon equation to analyze the concept of the variable order derivative. We apply the Crank-Nicholson method to solve the new modified equation numerically. Stability of this method is

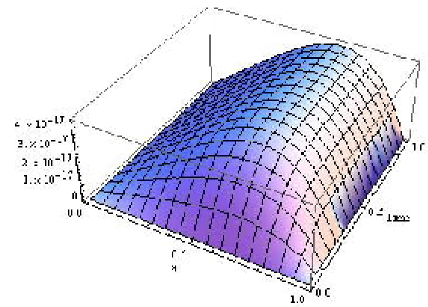

FIGURE 4. The error surface figures for $N = 85$

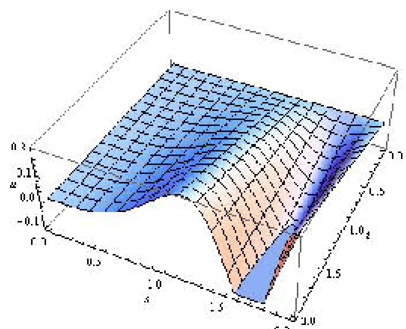

FIGURE 5. Numerical solution to problem (1) for $N = 10$

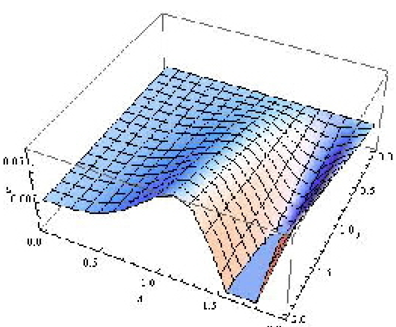

FIGURE 6. Numerical solution to problem (1) for $N = 30$

studied and reached by proving some inequalities. Some numerical examples have been also given for illustration. It can be concluded that Crank-Nicholson method

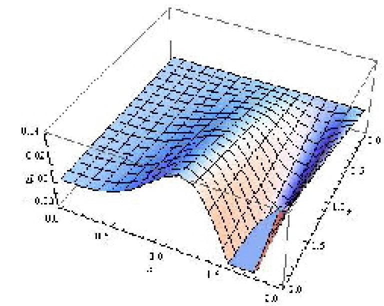

FIGURE 7. Numerical solution to problem (1) for $N = 50$

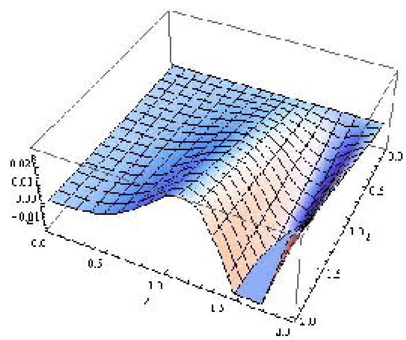

FIGURE 8. Numerical solution to problem (1) for $N = 70$

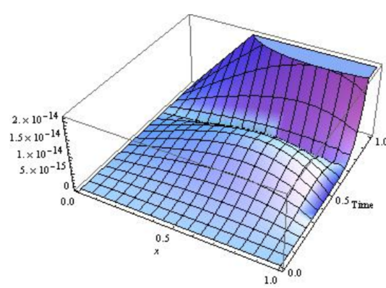

FIGURE 9. The error surface figures for $N = 80$

can be safely implemented to solve the time-fractional variable order Klein-Gordon equation.

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HALIT ORHAN AND MURAT ÇAĞLAR

We draw attention to some corrections in the section of "**Applications of Jack's Lemma"** which appear in the above-mentioned paper. Our results have changed due to the lack of the $\frac{1}{\tau(s)}$ factor on the left side of equation (in page 2166, line 24). So, we correct them in the following:

- In page 2166, line 19: $\cdots < 2\mu - \cdots$ should be $\cdots < 2\mu\tau(s) - \sqrt{2(1-\cos\theta)}$.
- In page 2166, line 22: $\cdots < \mu + \cdots$ should be $\cdots < \mu\tau(s) + \sqrt{2(1-\cos\theta)}$.
- In page 2166, line 24: $\dfrac{f_\tau(z)}{z} - \cdots = \cdots$ should be $\frac{1}{\tau(s)}\left(\dfrac{f_\tau(z)}{z} - e^{i\theta}\dfrac{g_\tau(z)}{z} - (1-e^{i\theta})\right) = \mu w(z)$.
- In page 2167, line 2: $\left|f'_\tau(z) - e^{i\theta}g'_\tau(z)\right| = \cdots$ should be $\frac{1}{\tau(s)}\left|f'_\tau(z) - e^{i\theta}g'_\tau(z)\right| = \cdots$.
- In page 2167, line 4: $\cdots < 2\mu - \cdots$ should be $\cdots < 2\mu\tau(s) - \sqrt{2(1-\cos\theta)}$.
- In page 2167, line 9: $\cdots = \left|\frac{1}{\tau(s)}(1-e^{i\theta}) + \cdots\right|$ should be $\cdots = \left|(1-e^{i\theta}) + \mu\tau(s)e^{i\theta}(1+k)\right|$.
- In page 2167, line 10: $\geq \mu(1+k) - \cdots$ should be $\geq \mu\tau(s)(1+k) - \left|1-e^{i\theta}\right|$.
- In page 2167, line 11: $\geq 2\mu - \cdots$ should be $\geq 2\mu\tau(s) - \sqrt{2(1-\cos\theta)}$.
- In page 2167, line 15: $\cdots = \left|\frac{1}{\tau(s)}(1-e^{i\theta}) + \cdots\right|$ should be $\cdots = \left|(1-e^{i\theta}) + \mu\tau(s)w(z)\right|$.
- In page 2167, line 16: $\leq \frac{1}{\tau(s)}\left|1-e^{i\theta}\right| + \cdots$ should be $\leq \left|1-e^{i\theta}\right| + \mu\tau(s)\left|w(z)\right|$.
- In page 2167, line 17: $< \mu + \cdots$ should be $< \mu\tau(s) + \sqrt{2(1-\cos\theta)}$.
- In page 2167, line 20: $\cdots < 2\mu - \frac{\sqrt{2}}{\tau(s)}$ should be $\cdots < 2\mu\tau(s) - \sqrt{2}$.

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- In page 2167, line 22: $\cdots < \mu + \frac{\sqrt{2}}{\tau(s)}$ should be $\cdots < \mu\tau(s) + \sqrt{2}$.
- In page 2168, line 2: $\cdots > \frac{1}{\tau(s)}(1-\cos\theta) - \dfrac{3\mu}{4}$ should be $\cdots > (1-\cos\theta) - \dfrac{3\mu}{4}\tau(s)$.
- In page 2168, line 4: $\cdots > \frac{1}{\tau(s)}(1-\cos\theta) - \frac{\mu}{2}$ should be $\cdots > (1-\cos\theta) - \frac{\mu}{2}\tau(s)$.
- In page 2168, line 6: $\dfrac{f_\tau(z)}{z} - \cdots = \cdots$ should be
- $\frac{1}{\tau(s)}\left(\dfrac{f_\tau(z)}{z} - e^{i\theta}\dfrac{g_\tau(z)}{z} - (1-e^{i\theta})\right) = \mu\frac{w(z)}{1-w(z)}$.
- In page 2168, line 8: $\cdots = \frac{1}{\tau(s)}(1-e^{i\theta}) + \cdots$ should be
$$\cdots = (1-e^{i\theta}) + \mu\tau(s)\frac{w(z)}{1-w(z)} + \mu\tau(s)\frac{zw'(z)}{(1-w(z))^2}.$$
- In page 2168, line 14: $\cdots = \mathrm{Re}\left(\frac{1}{\tau(s)}(1-e^{i\theta}) + \cdots\right)$ should be
$$\cdots = \mathrm{Re}\left((1-e^{i\theta}) + \mu\tau(s)\frac{e^{i\theta}}{1-e^{i\theta}} + \mu\tau(s)\frac{ke^{i\theta}}{(1-e^{i\theta})^2}\right).$$
- In page 2168, line 15: $= \frac{1}{\tau(s)}(1-\cos\theta) - \cdots$ should be
$= (1-\cos\theta) - \dfrac{\mu}{2}\tau(s) - k\mu\tau(s)\dfrac{1}{2(1-\cos\theta)}$.
- In page 2168, line 16: $\leq \frac{1}{\tau(s)}(1-\cos\theta) - \cdots$ should be $\leq (1-\cos\theta) - \dfrac{\mu}{2}\tau(s) - \dfrac{\mu}{4}\tau(s)$.
- In page 2168, line 17: $= \frac{1}{\tau(s)}(1-\cos\theta) - \dfrac{3\mu}{4}$ should be $= (1-\cos\theta) - \dfrac{3\mu}{4}\tau(s)$.
- In page 2168, line 23: $\cdots > \frac{1}{\tau(s)}(1-\cos\theta) - \frac{\mu}{2}$ should be $\cdots > (1-\cos\theta) - \frac{\mu}{2}\tau(s)$
- In page 2169, line 1: $\cdots > \frac{1}{\tau(s)} - \dfrac{3\mu}{4}$ should be $\cdots > 1 - \dfrac{3\mu}{4}\tau(s)$.
- In page 2169, line 2: $\cdots > \frac{1}{\tau(s)} - \dfrac{3\mu}{4}$ should be $\cdots > 1 - \dfrac{\mu}{2}\tau(s)$.
- In page 2169, line 3: $\cdots > \frac{1}{\tau(s)} - \dfrac{3(1-\beta)}{2}$ should be $\cdots > 1 - \dfrac{3(1-\beta)}{2}\tau(s)$.
- In page 2169, line 4: $\cdots > \frac{1}{\tau(s)} + \beta - 1$ should be $\cdots > 1 + (\beta-1)\,\tau(s)$.

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