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THE LINEAR METHOD FOR SOLVING THE SCATTERING PROBLEM IN AN INHOMOGENEOUS MEDIUM: THE CASE OF TE POLARIZED

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Abstract

In this article the electromagnetic waves scattered from an inhomogeneous medium are considered when the electromagnetic waves are polarized in the case of transverse electric. Using the Rellich lemma, the uniqueness of the solution of the direct scattering problem is proved. In order to show the existence of the solution of this problem, the operator equations are constructed and the Riesz theory which provides the existence of the inverse operator is used. Furthermore, for solution of the inverse scattering problems, an interior boundary value problem is considered. Finally, a linear integral equation is obtained whose the solution yield the support of the scattering object.

Keywords: Electromagnetic wave, Far-field pattern, Linear method, Scattering theory.

1. Introduction

The scattering problems of time-harmonic waves which are acoustic or electromagnetic waves are the basic problems in the scattering theory. These problems have been considered by many writers as direct and indirect scattering problems [2-19, 22-24].

Before the inverse scattering problems with regard to the direct scattering problems, the most important questions are the uniqueness and the existence of the solution of the direct scattering problem. Gerlach and Kress [17] , Colton, Kress and Monk [8] have proved the uniqueness of the solution by using Green's theorems and the unique continuation property of solution. Furthermore, they have showed the existence of solution by using the jump relations of the single-layer and double-layer in the potential theory and the integral equations. For the transmission boundary value problem, this results have been proved by Colton and Piana [9] .

In the inverse scattering theory, the most importance thing is scattered far-field model. In the 1980's, the inverse scattering problem of determining the unknown scattering obstacle from information about the far- field data was considered by Angel, Colton and Kirsch [2] , Tobocman [23] and many more mathematicians. Integral equations or Green's formulas were used to reformulate the inverse obstacle problem by these researchers.

For the solution of the inverse scattering problem, a method is the linear method which was suggested, firstly, by Colton and Kirsch [10]. Then the method is used by Colton, Kress and Monk [8], Colton and Piana [9], Colton, Piana and Potthast [11], Colton, Giebermann and Monk [12], Colton, Coyle and Monk [13], Cakoni, Colton and Monk [3], Cakoni, Colton and Haddar [4], Cakoni and Colton [5], Colton [14] and Colton and Kress [15]. This method is mathematically established by placing a network on the unknown domain by solving a linear integral equation for each point on this network and then determining the shape of the domain from the information about the solutions for this set of integral equations. To apply this method, first, the far field operatör $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) = \int_{\Omega} u_\infty(\hat{x},\mathrm{d}) g(\mathrm{d}) ds(\mathrm{d}), \qquad \hat{x},\mathrm{d} \in \Omega$$

where $\Omega = \{x \in IR^2 : |x| = 1\}$. Then the Regulation method [16] is used to solving of the linear integral equation $(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, y)$, where $\Phi_\infty(\hat{x}, y) = \dfrac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}.y}$ is the far-field model of the function $\Phi(x, y) = \dfrac{i}{4} H_0^{(1)}(k|x-y|)$ for $x \neq y$ [1]. According to this method, for $\forall \varepsilon > 0$, there exists a function $g = g(.,y) \in L^2(\Omega)$ such that $\|Fg - \Phi_\infty\| < \varepsilon$ and both $\|g(.,y)\|$ and $\|v_g(.,y)\|$ become unbounded as y approaches the boundary of the scatterer, where $v_g(x) = \int_{\Omega} e^{ikx.\mathrm{d}} g(\mathrm{d}) ds(\mathrm{d})$ is the Herlogtz wave function with kernel $g(.,y)$ [16]. The Herlogtz kernel $g(.,y)$ is determined from $(Fg)(\hat{x}) = \int_{\Omega} u_\infty(\hat{x},\mathrm{d}) g(\mathrm{d}) ds(\mathrm{d})$ for y on a grid containing the scatterer. Thus, the boundary of the unknown domain can be found as the locus of points y, where $\|g(.,y)\|_{L^2(\Omega)}$ begins to increase sharply.

Now, we consider the following problem:

We investigated an electromagnetic scattering problem in an inhomogenous medium when the incident wave is polarized parallel to the axis of infinite cylinder representing the scatterer and the magnetic field has only one component in the direction of the axis to the cylinder. This is the referred to as the transverse electric mode (briefly, TE mode) in scattering theory [9,22,24]. The electromagnetic waves can be obtained by using the Maxwell equations [16]. We assume that D is a simply-connected bounded domain in IR^2 with $C^2(\partial D)$ and which the domain is the cross section of the cylinder. For the time-harmonic electromagnetic waves, the scattering is defined by the Maxwell equations

$$curlE_0 - ikH_0 = 0 \qquad curlH_0 + ikE_0 = 0, \qquad \text{in } IR^2 \setminus \bar{D} \tag{1}$$

$$curlE - ikH = 0 \qquad curlH + ikn(x)E = 0, \quad \text{in} \ \ D \tag{2}$$

and the boundary conditions

$$\nu \times H_0 = \nu \times H, \qquad on \ \ \partial D \tag{3}$$

$$\nu \times curlE_0 + \frac{k}{\lambda}(\nu \times E_0)\times \nu = \frac{1}{n_0}(\nu \times curlE) + \frac{k}{\lambda}(\nu \times E)\times \nu, \ \text{on} \ \partial D \tag{4}$$

where k is positive wave number and ν is outward unit normal vector on ∂D. Let (E_0, H_0) and (E, H) be electromagnetic fields outside and inside the cylinder, respectively. Let be $\lambda \in C^{0,\alpha}(\partial D)$ and $\operatorname{Im}\lambda \geq 0$. $n(x)$ is the index of refraction defined by

$$n(x) = \frac{1}{\varepsilon_0}\left(\varepsilon(x) + \frac{i\sigma(x)}{\omega}\right)$$

where ε_0 is the constant permittivity in $IR^2 \setminus \bar{D}$, $\varepsilon(x)$ and $\sigma(x)$ are the permittivity and the conductivity of the cylinder, respectively, and ω is the frequency of the electromagnetic waves. We assume that $n(x)$ satisfies the following conditions:

(i) $n(x) \in C^2(IR^2)$ and $n(x) = n_0 > 0$ for $x \in IR^2 \setminus D$, where $n_0 (\neq 1) \in IR$

(ii) $\operatorname{Im} n(x) \geq 0$ and $D_0 = \{x \in D : \operatorname{Im} n(x) > 0\} \neq \varnothing$. (5)

If the electromagnetic wave is polarized in the transverse electric mode, the scalar fields u_0 and u can be defined as $H_0 = (H_{0_1}, H_{0_2}, H_{0_3}) = (0,0,u_0)$ and $H = (H_1, H_2, H_3) == (0,0,u)$. Thus, the Maxwell equations 1-2 and the boundary conditions 3-4 are equivalent to the Helmholtz equations and the boundary conditions in the following

$$\Delta u_0 + k^2 u_0 = 0, \qquad \text{in} \ \ IR^2 \setminus \bar{D} \tag{6}$$

$$\nabla . \left(\frac{1}{n}\nabla u\right) + k^2 u = 0, \qquad \text{in} \ \ D \tag{7}$$

$$u_0 - u = 0, \qquad \text{on} \ \ \partial D \tag{8}$$

$$\frac{\partial u_0}{\partial \nu} - \frac{1}{n_0}\frac{\partial u}{\partial \nu} + \lambda k\left(u_0 - \frac{1}{n_0}u\right) = 0, \qquad \text{on} \ \ \partial D. \tag{9}$$

The exterior field u_0 can be written in the form

$$u_0(x) = u^i(x) + u^s(x),$$

where $\mathrm{d} \in \Omega = \{x \in IR^2 : |x| = 1\}$ and $u^i(x) = e^{ikx.\mathrm{d}}$ is the incident plane wave with incident direction *d*. The scattered wave u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{10}$$

uniformly in all directions $\hat{x} = \dfrac{x}{|x|}$ with $r = |x|$. This condition guarantees that the scattered wave has the asymptotic behaviour

$$u^s(x) = u_\infty(\hat{x}, \mathrm{d})\frac{e^{ikr}}{\sqrt{r}} + O\left(r^{-3/2}\right)$$

as $r = |x| \to \infty$, where u_∞ is known as the far-field pattern of the scattered wave and is defined in the form

$$u_\infty(\hat{x}, \mathrm{d}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left[u(y)\frac{\partial e^{-ik\hat{x}.y}}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) e^{-ik\hat{x}.y} \right] ds(y) \;, \hat{x} \in \Omega \quad [16].$$

2. The Direct Scattering Problem

The scattering of time-harmonic plane waves by a simply connected bounded domain $D \subset IR^2$ is formed with the following direct scattering problem. For given $f \in C^{1,\alpha}(\partial D)$ and $\lambda, g \in C^{0,\alpha}(\partial D)$ from Hölder spaces with exponent $0 < \alpha < 1$, this problem is to find a pair of functions $u_0 \in C^2\left(IR^2 \setminus \overline{D}\right) \cap C^1\left(IR^2 \setminus D\right)$ and $u \in C^2(D) \cap C^1\left(\overline{D}\right)$ such that

$$\Delta u_0 + k^2 u_0 = 0, \qquad \text{in } IR^2 \setminus \overline{D} \tag{11}$$

$$\nabla.\left(\frac{1}{n}\nabla u\right) + k^2 u = 0, \qquad \text{in } D \tag{12}$$

$$u_0 - u = f \;, \qquad \text{on } \partial D \tag{13}$$

$$\frac{\partial u_0}{\partial \nu} - \frac{1}{n_0}\frac{\partial u}{\partial \nu} + \lambda k\left(u_0 - \frac{1}{n_0}u\right) = g, \qquad \text{on } \partial D, \tag{14}$$

where k is positive wave number and ν is the unit outward to ∂D. n_0 and n are defined in the conditions (i)-(ii) of (5). u_0 satisfies the Sommerfeld radiation condition (10), i.e.

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u_0}{\partial r}-iku_0\right)=0\,, \tag{15}$$

where $r=|x|$. For simplicity, we will always suppose that $\operatorname{Im}\lambda\ge 0$ on ∂D.

Theorem 2.1. The solution of the boundary value problem 11-15 is unique.

Proof. We suppose that the solution of the problem 11-15 is not unique. Let $u_0=u_{0_1}-u_{0_2}$ and $u=u_1-u_2$. Thus u_0 and u satisfy the homogeneous boundary value problem 6-9.
We first show that

$$\lim_{r\to\infty}\int_{\Omega_r}|u_0|^2\,ds=0\,, \tag{16}$$

where Ω_r denotes the circle with the radius r and centered in the origin and ds is the arc lenght. To achieve this, from the Sommerfeld radiation condition 15, we have

$$\lim_{r\to\infty}\int_{\Omega_r}\left[\left|\frac{\partial u_0}{\partial\nu}\right|^2+k^2|u_0|^2+2k\operatorname{Im}\left(u_0\frac{\partial\overline{u_0}}{\partial\nu}\right)\right]ds=\lim_{r\to\infty}\int_{\Omega_r}\left|\frac{\partial u_0}{\partial\nu}-iku_0\right|^2ds=0\,. \tag{17}$$

We take r large enough such that $D\subset\Omega_r$. Applying Green's theorem [16] in the domain $D_r=\left\{y\in IR^2\setminus\overline{D}\,:|y|<r\right\}$, we have

$$\int_{D_r}-k^2|u_0|^2\,dy+\int_{D_r}|gradu_0|^2\,dy=-\int_{\Omega_r}u_0\frac{\partial\overline{u_0}}{\partial\nu}ds(y)+\int_{\partial D}u_0\frac{\partial\overline{u_0}}{\partial\nu}ds(y)\,.$$

Taking imaginary parts of this equation, from the equation 17, we obtain

$$\lim_{r\to\infty}\int_{\Omega_r}\left[\left|\frac{\partial u_0}{\partial\nu}\right|^2+k^2|u_0|^2\right]ds=-2k\operatorname{Im}\int_{\partial D}u_0\frac{\partial\overline{u_0}}{\partial\nu}ds\,. \tag{18}$$

Applying Divergence theorem [18] to the function $u\left(\frac{1}{\overline{n}}\nabla\overline{u}\right)$, from the condition (i) of (5) and boundary conditions 8-9, we get

$$\int_D \left[-k^2|u|^2+\frac{1}{\overline{n}}|\nabla u|^2\right]dy=\int_{\partial D}\left[u_0\frac{\partial\overline{u_0}}{\partial\nu}+\overline{\lambda}k|u_0|^2-\frac{\overline{\lambda}k}{n_0}|u_0|^2\right]ds(y).$$

Taking imaginary parts of this equation, we have

$$\operatorname{Im}\int_D \frac{1}{\overline{n}}|\nabla u|^2\,dy=\operatorname{Im}\int_{\partial D}u_0\frac{\partial\overline{u_0}}{\partial\nu}ds(y)+\operatorname{Im}\int_{\partial D}\left(1-\frac{1}{n_0}\right)\overline{\lambda}k|u_0|^2\,ds(y). \tag{19}$$

From the condition (ii) of 5, the left-hand of equation 19 is positive or zero. Since $\operatorname{Im}\overline{\lambda}\le 0$ on ∂D, again from the condition (i) of 5, the last integral in the right-hand of equation 19 is negative or zero. Thus we obtain

$$\operatorname{Im}\int_{\partial D}u_0\frac{\partial\overline{u_0}}{\partial\nu}ds(y)\ge 0.$$

The equation 18 becomes

$$\lim_{r\to\infty}\int_{\Omega_r}\left[\left|\frac{\partial u_0}{\partial\nu}\right|^2+k^2|u_0|^2\right]ds\le 0. \tag{20}$$

Since the left-hand of equation 20 is positive or zero, we get the equation 16. From Rellich's lemma [16], $u_0=0$ in $IR^2\setminus\overline{D}$ and so $u_0=\dfrac{\partial u_0}{\partial\nu}=0$ in $IR^2\setminus D$ from the Theorem 3.12 in [7]. From the conditions 8-9, we obtain $u=\dfrac{\partial u}{\partial\nu}=0$ on ∂D. From the unique continuation principle (see : Theorem 8.6 in [16]), we obtain $u=0$ in D.

We will now apply the Riesz's theory (the inverse operator's existence) for compact operators [7,18] to demonstrate the existence of solution to the boundary value problem 11-15. With the change of variables $u(x)=\sqrt{n(x)}w(x)$, the boundary value problem 11-15 takes form

$$\Delta u_0+k^2u_0=0,\ \text{in}\ IR^2\setminus\overline{D} \tag{21}$$

$$\Delta w+\left(k^2n+p\right)w=0,\quad \text{in}\ D \tag{22}$$

$$u_0-\sqrt{n_0}\,w=f,\quad \text{on}\ \partial D \tag{23}$$

$$\frac{\partial u_0}{\partial\nu}-\frac{1}{\sqrt{n_0}}\frac{\partial w}{\partial\nu}+\lambda k\left(u_0-\frac{1}{\sqrt{n_0}}w\right)=g,\quad \text{on}\ \partial D \tag{24}$$

where

$$p(x) = -\sqrt{n(x)}\Delta\frac{1}{\sqrt{n(x)}} \ . \tag{25}$$

Then for $\psi,\phi \in C(\partial D)$ and $\psi_1 \in C(D)$, let's define the following functions

$$u_0(x) = \int_{\partial D}\left[\frac{\partial\Phi_0(x,y)}{\partial\nu(y)}\psi(y) + \Phi_0(x,y)\phi(y)\right]ds(y), \qquad x \in IR^2 \setminus \partial D \tag{26}$$

$$w(x) = \int_{\partial D}\left[\sqrt{n_0}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi(y) + \Phi(x,y)\phi(y)\right]ds(y) + \int_D \Phi(x,y)\rho(y)\psi_1(y)dy, \ x \in IR^2 \setminus \partial D \tag{27}$$

where $\rho(x) = k^2 n_0 - [k^2 n(x) + p(x)]$ and the functions $\Phi_0(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|)$ and $\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k\sqrt{n_0}|x-y|)$, $x \neq y$ in IR^2 are the fundemantal solutions of the Helmholtz equations which are $\Delta u + k^2 u = 0$ and $\Delta u + k^2 n_0 u = 0$, respectively, where $H_0^{(1)}$ is the Hankel function of the first kind and the zero order. The functions u_0 defined by equation 26 and w defined by equation 27 satisfies the problem 21-24 and the Sommerfeld radiation condition 15.

We introduce the following integral operators:

The operators K,S,T and K' are defined from $C(\partial D)$ to $C(\partial D)$, such that

$$(K\psi)(x) = 2\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi(y)ds(y), \qquad x \in \partial D \tag{28}$$

$$(S\phi)(x) = 2\int_{\partial D}\Phi(x,y)\phi(y)ds(y), \qquad x \in \partial D \tag{29}$$

$$(T\psi)(x) = 2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi(y)ds(y), \quad x \in \partial D \tag{30}$$

$$(K'\phi)(x) = 2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\Phi(x,y)\phi(y)ds(y), \quad x \in \partial D \tag{31}$$

The operators $K^{\wedge}$ and $S^{\wedge}$ are defined from $C(\partial D)$ to $C(D)$, such that

$$(K^{\wedge}\psi)(x) = 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) ds(y), \quad x \in D \tag{32}$$

$$(S^{\wedge}\phi)(x) = 2\int_{\partial D} \Phi(x,y)\phi(y) ds(y), \quad x \in D \tag{33}$$

The operators S_ρ and K_ρ^ι are defined from $C(D)$ to $C(\partial D)$, such that

$$(S_\rho \psi_1)(x) = 2\int_D \Phi(x,y)\rho(y)\psi_1(y) dy, \quad x \in \partial D \tag{34}$$

$$(K_p^\iota \psi_1)(x) = 2\frac{\partial}{\partial \nu(x)} \int_D \Phi(x,y)\rho(y)\psi_1(y) dy, \quad x \in \partial D \tag{35}$$

Finally, the operatör $S_\rho^\wedge$ be defined from $C(D)$ to $C(D)$, such that

$$(S_p^\wedge \psi_1)(x) = 2\int_D \Phi(x,y)\rho(y)\psi_1(y) dy, \quad x \in D. \tag{36}$$

Let K_0, S_0, T_0 and K_0^ι show the operators corresponding to K, S, T and K^ι, respectively, with Φ replaced by Φ_0.

Theorem 2.2. The functions u_0 and w defined by equations 26-27 are restricted to $IR^2 \setminus \overline{D}$ and D, respectively. Then the functions $\psi, \phi \in C(\partial D)$ and $\psi_1 \in C(D)$ satisfy the following integral equations

$$(K_0 - n_0 K)\psi + (1+n_0)\psi + (S_0 - \sqrt{n_0} S)\phi - \sqrt{n_0} S_\rho \psi_1 = 2f, \text{ on } \partial D \tag{37}$$

$$(T_0 - T)\psi + \left(K_0^\iota - \frac{1}{\sqrt{n_0}} K^\iota\right)\phi - \left(1 + \frac{1}{\sqrt{n_0}}\right)\phi - \frac{1}{\sqrt{n_0}} K_\rho^\iota \psi_1$$
$$+\lambda k\left[(K_0 - K)\psi + \left(S_0 - \frac{1}{\sqrt{n_0}} S\right)\phi + 2\psi - \frac{1}{\sqrt{n_0}} S_\rho \psi_1\right] = 2g, \text{ on } \partial D \tag{38}$$

$$\sqrt{n_0} K^{\wedge}\psi + S^{\wedge}\phi + S_\rho^\wedge \psi_1 - 2\psi_1 = 0, \quad \text{in } D. \tag{39}$$

Proof. Firstly, we will obtain the integral equation 37. When $x \in IR^2 \setminus \overline{D} \to x \in \partial D$, the limit value of u_0 in equation 26 is

$$u_0^+(x) = \int_{\partial D} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} \psi(y) ds(y) + \frac{1}{2}\psi(x) + \int_{\partial D} \Phi_0(x,y)\phi(y) ds(y).$$

When $x \in D \to x \in \partial D$, the limit value of w in equation 27 is

$$w^-(x) = \sqrt{n_0}\left[\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) ds(y) - \frac{1}{2}\psi(x)\right] + \int_{\partial D} \Phi(x,y)\phi(y) ds(y)$$
$$+\int_D \Phi(x,y)\rho(y)\psi_1(y) dy.$$

From the condition 23 and the operators 28, 29,34, we obtain

$$2f(x) = (K_0\psi)(x) + \psi(x) + (S_0\phi)(x) - n_0(K\psi)(x) + n_0\psi(x) - \sqrt{n_0}(S\phi)(x) - \sqrt{n_0}(S_\rho\psi_1)(x).$$

Thus, for $\forall x \in \partial D$, the equation 37 is obtained.

Now, we will obtain the integral equation 38. We take the derivative of the function u_0 in the direction ν. When $x \in IR^2 \setminus \overline{D} \to x \in \partial D$, the limit value of $\frac{\partial u_0}{\partial \nu}$ is

$$\frac{\partial u_0^+}{\partial \nu}(x) = \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} \psi(y) ds(y) + \frac{\partial}{\partial \nu(x)} \int_{\partial D} \Phi_0(x,y)\phi(y) ds(y) - \frac{1}{2}\phi(x).$$

We take the derivative of the function w in the direction ν. When $x \in D \to x \in \partial D$, the limit value of $\frac{\partial w}{\partial \nu}$ is

$$\frac{\partial w^-}{\partial \nu}(x) = \sqrt{n_0}\frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) ds(y) + \frac{\partial}{\partial \nu(x)} \int_{\partial D} \Phi(x,y)\phi(y) ds(y)$$
$$+\frac{1}{2}\phi(x) + \frac{\partial}{\partial \nu(x)} \int_D \Phi(x,y)\rho(y)\psi_1(y) dy.$$

From the condition 24 and the operators 28, 31, 34, 35 we have

$$2g(x) = (T_0\psi)(x) + (K_0'\phi)(x) - \phi(x) - (T\psi)(x) - \frac{(K'\phi)(x)}{\sqrt{n_0}} - \frac{\phi(x)}{\sqrt{n_0}} - \frac{(K_\rho'\psi_1)(x)}{\sqrt{n_0}}$$

$$+\lambda(x)k\left[(K_0\psi)(x)+\psi(x)+(S_0\phi)(x)-(K\psi)(x)+\psi(x)-\frac{(S\phi)(x)}{\sqrt{n_0}}-\frac{(S_\rho\psi_1)(x)}{\sqrt{n_0}}\right].$$

Thus, for $\forall x\in\partial D$, the equation 38 is obtained.

Finally, for the integral equation 39, if we write the operators 32, 33 and 36 in the function w defined by equation 27, then we obtain

$$2w(x)=\sqrt{n_0}\left(K^\wedge\psi\right)(x)+\left(S^\wedge\phi\right)(x)+\left(S_\rho^\wedge\psi_1\right)(x).$$

Since $w(x)\in C^2(D)\cap C^1(\overline{D})$ and $\psi_1(x)\in C(D)$, we can write $w(x)=\psi_1(x)$. Thus, we satisfy the equation 39 for $\forall x\in D$.

Equations 37 - 39 can be written in operator notation as

$$[A+B]\begin{bmatrix}\psi\\ \phi\\ \psi_1\end{bmatrix}=\begin{bmatrix}2f\\ 2g\\ 0\end{bmatrix}, \tag{40}$$

where the matrixes A and B are described in the following forms

$$A=\begin{bmatrix}(1+n_0)I & 0 & 0\\ 2\lambda kI & -\left(1+\frac{1}{\sqrt{n_0}}\right)I & 0\\ 0 & 0 & -2I\end{bmatrix}$$

and

$$B=\begin{bmatrix}K_0-n_0K & S_0-\sqrt{n_0}S & -\sqrt{n_0}S_\rho\\ (T-T_0)+\lambda k(K_0-K) & \left(K_0'-\frac{1}{\sqrt{n_0}}K'\right)+\lambda k\left(S_0-\frac{1}{\sqrt{n_0}}S\right) & -\frac{1}{\sqrt{n_0}}\left(K_\rho'+\lambda kS_\rho\right)\\ \sqrt{n_0}K^\wedge & S^\wedge & S_\rho^\wedge\end{bmatrix}.$$

The operator A clearly has a bounded inverse [2,4]. The operators in the matrix B are weakly singular operators . Thus, the operator B is compact in the space $C(\partial D)\times C(\partial D)\times C(D)$ [7].

In the following theorem, we will denote that the $A+B$ operator is injective.

Theorem 2.3. The boundary value problem 11-15 has a unique solution.

Proof. Let us consider the problem 21-24. From the uniqueness theorem 2.1, if $f = g = 0$ then $u_0 = 0$ in $IR^2 \setminus \overline{D}$ and $u = 0$ in D. Since $u = \sqrt{n}w$, then $w = 0$ in D, where the functions u_0 and w are defined by equations 26 and 27, respectively. From the equation 39 , we have $w - \psi_1 = 0$ for $x \in D$ and so $\psi_1 = 0$. Thus, the equations 37 and 38 reduce to

$$(K_0 - n_0 K)\psi + (1 + n_0)\psi + (S_0 - \sqrt{n_0} S)\phi = 0,$$

$$(T - T_0)\psi + \left(K_0' - \frac{1}{\sqrt{n_0}} K' \right)\phi - \left(1 + \frac{1}{\sqrt{n_0}} \right)\phi + \lambda k \left[\left(K_0 - K \right)\psi + \left(S_0 - \frac{1}{\sqrt{n_0}} S \right)\phi + 2\psi \right] = 0.$$

Using the jump relations of potential theory [16], we obtain

$$u_0^+ - u_0^- = \psi \qquad \frac{\partial u_0^+}{\partial \nu} - \frac{\partial u_0^-}{\partial \nu} = -\phi, \qquad \text{on } \partial D$$

$$w^+ - w^- = \sqrt{n_0}\psi \qquad \frac{\partial w^+}{\partial \nu} - \frac{\partial w^-}{\partial \nu} = -\phi, \qquad \text{on } \partial D.$$

Since $u_0 = 0$ in $IR^2 \setminus \overline{D}$ and $w = 0$ in D, then $u_0^+ = \frac{\partial u_0^+}{\partial \nu} = w^- = \frac{\partial w^-}{\partial \nu} = 0$. Thus, we have

$$u_0^- + \frac{1}{\sqrt{n_0}} w^+ = 0 \qquad \frac{\partial u_0^-}{\partial \nu} + \frac{\partial w^+}{\partial \nu} = 0, \qquad \text{on } \partial D \tag{41}$$

Since n_0 is real, from the Divergence theorem and equation 41, we have

$$\operatorname{Im} \int_{\partial D} w^+ \overline{\frac{\partial w^+}{\partial \nu}} ds = \operatorname{Im} \sqrt{n_0} \int_{\partial D} u_0^- \overline{\frac{\partial u_0^-}{\partial \nu}} ds = 0.$$

Since the function w is radiating solution of the Helmholtz equation for $x \in IR^2 \setminus \overline{D}$, from the Rellich's lemma, $w = 0$ in $IR^2 \setminus \overline{\mathrm{D}}$ and so $\frac{\partial w}{\partial \nu} = 0$. Since $w^+ = \frac{\partial w^+}{\partial \nu} = 0$ on ∂D and from equation 41, $u_0^- = \frac{\partial u_0^-}{\partial \nu} = 0$. Then we obtain $\psi = \phi = 0$ .Since $\psi = \phi = \psi_1 = 0$, the $A + B$ operatör is injective [18]. Since B is compact and $A + B$ is injective, the inhomogeneous system 40 has a unique solution ψ, ϕ, ψ_1 from the fundamental results of the Riesz's theory for compact operators (see: Theorem 1.16, Corollary 1.17 and Corollary 1.20 in [7]). Finally the boundary-value problem 21-24 has a unique solution.

To formulate the linear method, firstly, we consider the interior boundary-value problem.

3. The Interior Boundary Value Problem

The interior boundary value problem is to find the functions $u_0, u \in C^2(D) \cap C^1(\overline{D})$ to the differential equations

$$\Delta u_0 + k^2 u_0 = 0, \quad \text{in } D \tag{42}$$

$$\nabla . \left(\frac{1}{n} \nabla u \right) + k^2 u = 0, \quad \text{in } D \tag{43}$$

and the boundary conditions

$$u_0 - u = f, \quad \text{on } \partial D \tag{44}$$

$$\frac{\partial u_0}{\partial \nu} - \frac{1}{n_0} \frac{\partial u}{\partial \nu} + \lambda k \left(u_0 - \frac{1}{n_0} u \right) = g, \quad \text{on } \partial D . \tag{45}$$

Theorem 3.1 Let $D_0 = \{x \in D : \operatorname{Im} n(x) > 0\}$ be different from empty set. The solution of the interior boundary value problem 42–45 is unique.

Proof. Let $u_0, u \in C^2(D) \cap C^1(\overline{D})$ be the solution of the homogeneous interior boundary value problem, that is, assume $f = g = 0$. Then, applying of the Divergence theorem to the function $\overline{u} \left(\frac{1}{n} \nabla u \right)$ and using the condition (i) of 5 and homogeneous boundary conditions, we obtain

$$\int_D \left[\frac{1}{n} |\nabla u|^2 - k^2 |u|^2 \right] dy = \int_{\partial D} \overline{u} \frac{1}{n} \frac{\partial u}{\partial \nu} ds(y) = \int_{\partial D} \overline{u}_0 \frac{\partial u_0}{\partial \nu} ds(y) + \int_{\partial D} \lambda k \left(1 - \frac{1}{n_0} \right) |u_0|^2 ds(y)$$
$$= \int_D \left[\frac{1}{\overline{n}} |\nabla u|^2 - k^2 |u|^2 \right] dy + \int_{\partial D} \left(1 - \frac{1}{n_0} \right) \left(\lambda - \overline{\lambda} \right) k |u_0|^2 ds(y) .$$

Taking imaginary parts of this equation, we have

$$\operatorname{Im} \int_D \frac{1}{n} |\nabla u|^2 dy = \operatorname{Im} \int_D \frac{1}{\overline{n}} |\nabla u|^2 dy + \operatorname{Im} \int_{\partial D} \left(1 - \frac{1}{n_0} \right) \left(\lambda - \overline{\lambda} \right) k |u_0|^2 ds(y) . \tag{46}$$

Since $\operatorname{Im}\frac{1}{n} \le 0$ in D, the left-hand of equation 46 is negative or zero. Since $\operatorname{Im}\frac{1}{\overline{n}} \ge 0$ in D and $\operatorname{Im}\left(\lambda - \overline{\lambda}\right) \ge 0$ on ∂D, due to the condition (i) of 5, the right side of equation 46 is pozitive or zero. Thus, we get

$$\operatorname{Im}\int_D \frac{1}{\overline{n}}|\nabla u|^2 \, dy = 0 .$$

Since $\operatorname{Im}\frac{1}{\overline{n}} > 0$ in $D_0 \subset D$, then $\nabla u = 0$. Since u satisfies the equation 43, then $u = 0$ in D. From the unique continuation principle, we have $u = \frac{\partial u}{\partial \nu} = 0$ on ∂D. Also since u_0 satisfies the equation 42, from the homogeneous boundary conditions and the Helmholtz representation, $u_0 = 0$ in D.

We will show the existence of the solution of the interior boundary value problem 42-45. Again, using the change of variables $u(x) = \sqrt{n(x)}w(x)$, the interior boundary value problem 42-45 takes form

$$\Delta u_0 + k^2 u_0 = 0, \qquad \text{in } D \tag{47}$$

$$\Delta w + (k^2 n + p)w = 0, \quad \text{in } D \tag{48}$$

$$u_0 - \sqrt{n_0}\,w = f, \quad \text{on } \partial D \tag{49}$$

$$\frac{\partial u_0}{\partial \nu} - \frac{1}{\sqrt{n_0}}\frac{\partial w}{\partial \nu} + \lambda k\left(u_0 - \frac{1}{\sqrt{n_0}}w\right) = g, \quad \text{on } \partial D \tag{50}$$

where the function p is defined by equation 25. Now, for $\psi, \phi \in C(\partial D)$ ve $\psi_1 \in C(D)$, we use the function $u_0(x)$ for $x \in IR^2 \setminus \partial D$ defined by equation 26 and let's define the following function,

$$w(x) = \int_{\partial D}\left[\sqrt{n_0}\,\frac{\partial \Phi(x,y)}{\partial \nu(y)}\psi(y) - \Phi(x,y)\phi(y)\right]ds(y) + \int_D \Phi(x,y)\rho(y)\;\psi_1(y)dy, \quad x \in IR^2 \setminus \partial D \tag{51}$$

where the functions Φ_0, Φ ve ρ are defined as Section 2.

Theorem 3.2. Let the functions u_0 and w defined by equations 26 and 51 , respectively, are restricted to D.

Then the functions $\psi, \phi \in C(\partial D)$ and $\psi_1 \in C(D)$ satisfy the integral equations

$$(K_0 - n_0 K)\psi - (1-n_0)\psi + (S_0 + \sqrt{n_0} S)\varphi - \sqrt{n_0} S_\rho \psi_1 = 2f \ , \quad \text{on} \ \partial D \tag{52}$$

$$(T_0 - T)\psi + \left(K_0' + \frac{1}{\sqrt{n_0}} K' \right)\phi + \left(1 + \frac{1}{\sqrt{n_0}} \right)\phi - \frac{1}{\sqrt{n_0}} K_\rho' \psi_1$$
$$+\lambda k \left[\left(K_0 - K \right)\psi + \left(S_0 + \frac{1}{\sqrt{n_0}} S \right)\phi - \frac{1}{\sqrt{n_0}} S_\rho \psi_1 \right] = 2g \ , \ \text{on} \ \partial D \tag{53}$$

$$\sqrt{n_0} K^\wedge \psi - S^\wedge \phi + S_\rho^\wedge \psi_1 - 2\psi_1 = 0 \ , \quad in \quad D. \tag{54}$$

To prove, the similar way as Theorem 2.2 can be done .

Theorem 3.3 The interior boundary value problem 42-45 has a unique solution.

Proof. For the proof, we will examine the interior boundary value problem 47-50. From the uniqueness theorem 3.1, if $f = g = 0$ then $u_0 = w = 0$ in D. Since $\sqrt{n_0} K^\wedge \psi - S^\wedge \phi + S_\rho^\wedge \psi_1 = 2w$, from the equation 54, $\psi_1 = 0$. Thus, the equations 52 and 53 reduce to

$$(K_0 - n_0 K)\psi - (1-n_0)\psi + (S_0 + \sqrt{n_0} S)\phi = 0$$

and

$$(T_0 - T)\psi + \left(K_0' + \frac{1}{\sqrt{n_0}} K' \right)\phi + \left(1 + \frac{1}{\sqrt{n_0}} \right)\phi + \lambda k \left[\left(K_0 - K \right)\psi + \left(S_0 + \frac{1}{\sqrt{n_0}} S\phi \right) \right] = 0 \ .$$

From the jump relations, we obtain

$$u_0^+ - u_0^- = \psi \qquad \frac{\partial u_0^+}{\partial \nu} - \frac{\partial u_0^-}{\partial \nu} = -\phi \ , \qquad \text{on} \quad \partial D$$

$$w^+ - w^- = \sqrt{n_0}\psi \qquad \frac{\partial w^+}{\partial \nu} - \frac{\partial w^-}{\partial \nu} = \phi \ , \qquad \text{on} \quad \partial D.$$

Since $u_0 = w = 0$ in D, then $u_0^- = \dfrac{\partial u_0^-}{\partial \nu} = w^- = \dfrac{\partial w^-}{\partial \nu} = 0$. Thus, we have

$$u_0^+ - \frac{1}{\sqrt{n_0}} w^+ = 0 \qquad \frac{\partial u_0^+}{\partial \nu} + \frac{\partial w^+}{\partial \nu} = 0 \ , \qquad \text{on} \quad \partial D.$$

Thanks to n_0 positive real constant in condition (i) of 5, we obtain

$$\operatorname{Im}\int_{\partial D} u_0^+ \frac{\partial \overline{u_0^+}}{\partial \nu} ds = -\frac{1}{\sqrt{n_0}} \operatorname{Im}\int_{\partial D} w^+ \frac{\partial \overline{w^+}}{\partial \nu} ds . \tag{55}$$

The two integrals in equation 55 is pozitive or zero. Since u_0 and w are radiating solution of the Helmholtz equation for $x \in IR^2 \setminus \overline{D}$, from the Rellich's lemma, we have either $u_0 = 0$ or $w = 0$. Thus we have either $u_0^+ = \frac{\partial u_0^+}{\partial \nu} = 0$ or $w^+ = \frac{\partial w^+}{\partial \nu} = 0$ on ∂D. Then, we obtain $\psi = \phi = 0$. Thus, the existence of the solution of the interior boundary value problem 47-50 is obtained from the fundamental results of the Riesz's theory.

4. The Linear Method for The Inverse Scattering Problem

We will formulate the linear method for the solution of the inverse scattering problem defined by the boundary value problem 6-10. This problem is associated with the determine the support $\overline{D}$ of $n(x) - n_0$ from the information about the far-field pattern $u_\infty(\hat{x}, d)$ in the section 1. For $\forall \varepsilon > 0$, there exists a solution $g_y \in L^2(\Omega)$ such that

$$\left\| \int_\Omega u_\infty(\hat{x}, \mathrm{d}) g_y(\mathrm{d}) ds(\mathrm{d}) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}.y} \right\|_{L^2(\Omega)} < \varepsilon \quad \text{for } y \in D.$$

When $y \to \partial D$, both $\|g_y\|_{L^2(\Omega)}$ and $\|v_g\|_{L^2(D)}$ become unbounded [10,11].

First of all, we shall form the integral equation for the linear method. We will come up with a basic solution that provides equation 48. Let be $\Omega_\varepsilon = \{ y : |x-y| \le \varepsilon \} \subset D$. We take the integral

$$I(x,z) = \int_D \Phi(x,y) m(y) \Gamma(y,z) dy, \qquad z \in IR^2 \tag{56}$$

where $\Gamma \in C^2(D) \cap C^1(\overline{D})$. Let $I(x,z)$ be the solution of equation 48 and

$$m(y) = -n_0 + n(y) + \frac{p(y)}{k^2} \tag{57}$$

Let be

$$I(x,z)=\int_{\Omega_\varepsilon}\Phi(x,y)m(y)\Gamma(y,z)dy+\int_{D\backslash\Omega_\varepsilon}\Phi(x,y)m(y)\Gamma(y,z)dy \quad =I_1(x,z)+I_2(x,z).$$

Since $\Delta\Phi(x,y)+k^2n_0\Phi(x,y)=0$ for $x\neq y$, then we get $\left(\Delta+k^2n_0\right)I_2(x,z)=0$. Hence

$$\left(\Delta+k^2n_0\right)I(x,z)=\left(\Delta+k^2n_0\right)I_1(x,z)$$
$$=\Delta\int_{\Omega_\varepsilon}\Phi(x,y)m(y)\Gamma(y,z)dy+k^2n_0\int_{\Omega_\varepsilon}\Phi(x,y)m(y)\Gamma(y,z)dy\,. \quad (58)$$

Applying the divergence theorem to the first integral on the right-hand of equation 58, we get

$$\int_{\Omega_\varepsilon}\Delta\Phi(x,y)m(y)\Gamma(y,z)dy=-\int_{\partial\Omega_\varepsilon}\nabla_x\Phi(x,y)\nu(y)m(y)\Gamma(y,z)ds(y), \qquad \nabla_x=-\nabla_y$$

$$=\int_{\partial\Omega_\varepsilon}\nabla_y\left[\frac{i}{4}H_0^{(1)}\left(k\sqrt{n_0}\,|x-y|\right)\right]\nu(y)m(y)\Gamma(y,z)ds(y)$$

$$=\int_0^{2\pi}-\frac{\partial}{\partial\varepsilon}\frac{i}{4}H_0^{(1)}\left(k\sqrt{n_0}\varepsilon\right)m(x+\varepsilon\theta)\Gamma(x+\varepsilon\theta,z)\varepsilon d\theta$$

$$=\int_0^{2\pi}\frac{ik\sqrt{n_0}}{4}H_1^{(1)}\left(k\sqrt{n_0}\varepsilon\right)m(x+\varepsilon\theta)\Gamma(x+\varepsilon\theta,z)\varepsilon d\theta\,.$$

Thus, from $\lim_{\varepsilon\to0}\sqrt{n_0}\varepsilon H_1^{(1)}\left(k\sqrt{n_0}\varepsilon\right)=-\frac{i}{\pi}\left(\frac{2}{k}\right)$ given in [20] , we have

$$\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon}\Delta\Phi(x,y)m(y)\Gamma(y,z)dy=m(x)\Gamma(x,z)\,. \quad (59)$$

Applying the mean value theorem [1] to the second integral on the right-hand of equation 58, we get

$$\int_{\Omega_\varepsilon}\Phi(x,y)m(y)\Gamma(y,z)dy=\Phi(x,a)m(a)\Gamma(a,z)\left|\int_{\Omega_\varepsilon}dy\right|, \qquad 0<x-a<\varepsilon$$

$$=\frac{i\pi\varepsilon^2}{4}H_0^{(1)}\left(k\sqrt{n_0}\,|x-a|\right)m(a)\Gamma(a,z).$$

Therefore

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\Phi(x,y)m(y)\Gamma(y,z)\,dy=0 \tag{60}$$

From equations 59, 60 and function 57, the equation 58 takes the form

$$\left(\Delta+k^2\left[n(x)+\frac{p(x)}{k^2}-m(x)\right]\right)I(x,z)=m(x)\Gamma(x,z) \ .$$

Since $I(x,z)$ satisfies the equation 48 and $\Phi(x,y)$ is a solution of the Helmholtz equation,

$$\Gamma(x,z)=\Phi(x,z)-k^2I(x,z)$$

satisfies the equation 48. If we write the integral 56 in the last equation, then we obtain the Lippmann Schwinger integral equation [16]

$$\Gamma(x,z)=\Phi(x,z)+\int_D\Phi(x,y)\left[k^2n_0-\left(k^2n(y)+p(y)\right)\right]\Gamma(y,z)\,dy \ .$$

Thus $\Gamma(x,z)$ is a basic solution for the equation 48. From the Theorem 8.3 given in [16], the solution of $\Gamma(x,z)$ is a solution of the following problem

$$\Delta w+\left(k^2n+p\right)w=0\,, \qquad x\in IR^2 \tag{61}$$

$$w(x)=\Phi(x,z)+w^s(x) \tag{62}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial w^s}{\partial r}-ikw^s\right)=0\,. \tag{63}$$

With the change of variables $u(x)=\sqrt{n(x)}w(x)$, the problem 61-63 is takes form

$$\nabla.\left(\frac{1}{n}\nabla u\right)+k^2u=0\,, \qquad x\in IR^2 \tag{64}$$

$$u(x)=\sqrt{n(x)}\Phi(x,z)+u^s(x)\,, \tag{65}$$

where $u^s(x)$ satisfies the Sommerfeld radiation condition 10.

From the Theorem 8.7 given in [16] and the condition (ii) of 5, the problem 64-65 has at most one solution. Thus, the orijinal problem 61-63 has at most one solution and the Fredholm alternative [18] guarantee the existence of a fundamental solution for the equation 48.

Secondly, we will give the following lemma.

Lemma 4.1. Let D be a bounded domain with $C^2\left(\partial D\right)$, $x^* \in \partial D$ and $B_R = \left\{x \in IR^2 : \left|x - x^*\right| \leq R\right\}$ If the function $u \in C^2\left(D\right) \cap C^1\left(\overline{D}\right)$ is the solution of the following equation

$$\nabla.\left(\frac{1}{n}\nabla u\right) + k^2 u = 0 \qquad \text{in } D, \tag{66}$$

then there exists a constant $C > 0$ such that

$$\left\|u\right\|_{C(\partial D)} \leq C\left(\left\|\frac{\partial u}{\partial \nu}\right\|_{C(\partial D)} + \left\|u\right\|_{C(\partial D \setminus B_R)}\right). \tag{67}$$

Proof. The proof can be done in the similar way to proof of Lemma 4.4 given in [19]. Let $\lambda \in C^{0,\alpha}\left(\partial D\right)$ be positive function with support $\partial D \setminus B_R$. Now, we will show that any solution of equation (66) satisfying the boundary condition

$$\frac{\partial u}{\partial \nu} - \lambda k u = g \tag{68}$$

must vanish identically in D. We suppose that the solution of the problem 66 and 68 is not unique i.e. let $u = u_1 - u_2$. Thus the function u satisfy the homogeneous boundary condition

$$\frac{\partial u}{\partial \nu} - \lambda k u = 0. \tag{69}$$

We take the homogeneous problem 66 and 69. By applying the divergence theorem to the function $\bar{u}\left(\frac{1}{n}\nabla u\right)$ and then taking imaginary parts, we have

$$\operatorname{Im}\int_D \left[-k^2\left|u\right|^2 + \frac{1}{n}\left|\nabla u\right|^2\right]dy = \operatorname{Im}\frac{1}{n_0}\int_{\partial D}\lambda k\left|u\right|^2 ds\left(y\right). \tag{70}$$

From $\operatorname{Im}\frac{1}{n} \le 0$, the left-hand of equation 70 is negatif or zero. From $\operatorname{Im}\lambda \ge 0$ and $n_0 \in IR$ on ∂D, the right-hand of equation 70 is positive or zero. Moreover, since $\lambda \ne 0$, $u = 0$ on $\partial D \setminus B_R$, the boundary condition 69 implies that $\frac{\partial u}{\partial \nu} = 0$. From the unique continuation principle, we obtain that $u = 0$ in D. Thus, the problem 66 and 68 has at most one solution.

To show existence of the solution of the boundary value problem 66 and 68, we use the inverse operator's existence theorem [18]. Firstly, we define the function

$$\Phi_1(x,y) = \sqrt{\frac{n(x)}{n_0}}\Gamma(x,y)$$

and let this function be the fundamental solution to equation 66. With the function Φ in the operators S and K' which were defined in the operators 29 and 31 replaced by Φ_1. Therefore, for $\phi \in C(\partial D)$, we define the function

$$u(x) = \int_{\partial D} \Phi_1(x,y)\phi(y)\,ds(y), \qquad x \in IR^2 \setminus \partial D.$$

The function u restricted to D solves the problem 66 and 68. The function ϕ satisfies the integral equation

$$K'\phi + \phi - \lambda kS\phi = 2g \quad \text{on } \partial D. \tag{71}$$

This integral equation is obtained from the jump relations and the boundary condition 68. If $g = 0$, since $u = 0$ in D, from the unique continuation principle, then $u^- = \frac{\partial u^-}{\partial \nu} = 0$. From the continuity of the single-layer potential and the uniqueness of the solution of the exterior Dirichlet problem given in [16], we have that $u^- = u^+ = \frac{\partial u^+}{\partial \nu} = 0$. From the jump relations, we obtain that $\frac{\partial u^+}{\partial \nu} - \frac{\partial u^-}{\partial \nu} = -\phi$. Thus, $\phi = 0$. This ensures the existence of the solution. That is, since the homogeneous equation $(I + K' - \lambda kS)\phi = 0$ has to the solution $\phi = 0$, the operatör $I + K' - \lambda kS$ is injective. Thus, from the inverse operator's existence theorem, the inhomogeneous equation 71 for all $g \in C(\partial D)$ has a unique solution and the solution depends continuously on the function g. Since the inverse operator $(I + K' - \lambda kS)^{-1} : C(\partial D) \to C(\partial D)$ exists and bounded, then the constant $C_1 > 0$ exists such that

$$\|u\|_{C(\partial D)} \le C_1 \|g\|_{C(\partial D)}. \tag{72}$$

From the boundary condition 68 and since the function λ is support $\partial D \setminus B_R$, then

$$\|g\|_{C(\partial D)} = \left\| \frac{\partial u}{\partial \nu} - \lambda k u \right\|_{C(\partial D)} \le \left\| \frac{\partial u}{\partial \nu} \right\|_{C(\partial D)} + c\|u\|_{C(\partial D \setminus B_R)}, \qquad c > 0 .$$

Writing the last inequality in the inequality 72, we get the inequality 67.

Teorem 4.2. If the sequences $u_{0,j}$ and u_j are solutions of the interior boundary value problem

$$\Delta u_{0,j} + k^2 u_{0,j} = 0, \qquad \text{in } D \tag{73}$$

$$\nabla.\left(\frac{1}{n}\nabla u_j\right) + k^2 u_j = 0, \qquad \text{in } D \tag{74}$$

$$u_{0,j} - u_j = -\Phi_0\left(.,y_j\right), \qquad \text{on } \partial D \tag{75}$$

$$\frac{\partial u_{0,j}}{\partial \nu} - \frac{1}{n_0}\frac{\partial u_j}{\partial \nu} + \lambda k\left(u_{0,j} - \frac{1}{n_0}u_j\right) = -\frac{\partial \Phi_0\left(.,y_j\right)}{\partial \nu} - \lambda k \Phi_0\left(.,y_j\right), \qquad \text{on } \partial D \tag{76}$$

Then

$$\lim_{j\to\infty}\left\|u_{0,j}\right\|_{C^1(\partial D)} = \infty, \tag{77}$$

where the sequences y_j are defined by

$$y_j = y^* - \frac{R}{j}\nu\left(y^*\right) \tag{78}$$

for $R > 0$ is sufficently small and y^* is a point on ∂D.

Proof. We assume that there exists a positive constant c_1 such that

$$\left\|u_{0,j}\right\|_{C^1(\partial D)} \le c_1, \qquad j \to \infty \tag{79}$$

For $R > 0$ sufficently small and $y^* \in \partial D$, we take the set of points in $IR^2 \setminus \overline{D}$ defined with the sequences

$$z_j = y^* + \frac{R}{j}\nu\left(y^*\right) .$$

Let's define the sequence

$$u_j = u_j + \sqrt{n}\Gamma\left(.,z_j\right) \quad \text{in } D. \tag{80}$$

From the boundary condition 75 and the sequence 80, we obtain

$$u_{0,j} - u_j = -\left[\Phi_0\left(.,y_j\right) + \sqrt{n_0}\Gamma\left(.,z_j\right)\right] \quad \text{on } \partial D. \tag{81}$$

Again from the boundary condition 76 and the derivative of the sequence 80 in the direction ν, we obtain

$$\frac{\partial u_{0,j}}{\partial \nu} - \frac{1}{n_0}\frac{\partial u_j}{\partial \nu} + \lambda k\left(u_{0,j} - \frac{1}{n_0}u_j\right) = -\left[\frac{\partial \Phi_0\left(.,y_j\right)}{\partial \nu} + \frac{1}{\sqrt{n_0}}\frac{\partial \Gamma\left(.,z_j\right)}{\partial \nu}\right]$$
$$-\lambda k\left[\Phi_0\left(.,y_j\right) + \frac{1}{\sqrt{n_0}}\Gamma\left(.,z_j\right)\right] \quad \text{on } \partial D. \tag{82}$$

The right-hand of equations 81 and 82 are defined, respectively, by the sequences

$$f_j = \Phi_0\left(.,y_j\right) + \sqrt{n_0}\Gamma\left(.,z_j\right) \quad \text{on } \partial D,$$

$$g_j = \frac{\partial \Phi_0\left(.,y_j\right)}{\partial \nu} + \frac{1}{\sqrt{n_0}}\frac{\partial \Gamma\left(.,z_j\right)}{\partial \nu} + \lambda k\left[\Phi_0\left(.,y_j\right) + \frac{1}{\sqrt{n_0}}\Gamma\left(.,z_j\right)\right] \quad \text{on } \partial D.$$

Let the disk B_r and λ be as defined as the Lemma 4.1. Then there exists a constant $c_2 > 0$ such that

$$\left\| f_j \right\|_{C(\partial D \backslash B_R)} \le \sup_{. \in \partial D \backslash B_R} \left|\Phi_0\left(.,y_j\right)\right| + \sup_{. \in \partial D \backslash B_R} \left|\sqrt{n_0}\Gamma\left(.,z_j\right)\right| \le c_2. \tag{83}$$

The norm of sequence g_j is given by the following inequality

$$\left\|g_j\right\|_{C(\partial D)} \le \left\|\frac{\partial \Phi_0\left(.,y_j\right)}{\partial \nu} + \frac{1}{\sqrt{n_0}}\frac{\partial \Gamma\left(.,z_j\right)}{\partial \nu}\right\|_{C(\partial D)} + \left\|\lambda k\left[\Phi_0\left(.,y_j\right) + \frac{1}{\sqrt{n_0}}\Gamma\left(.,z_j\right)\right]\right\|_{C(\partial D)}.$$

Taking the first norm on the right-hand of the above inequation and using as in the proof of Lemma 4.2 [8], there exists a constant $c_3 > 0$ such that

$$\left\|\frac{\partial \Phi_0\left(.,y_j\right)}{\partial \nu} + \frac{1}{\sqrt{n_0}}\frac{\partial \Gamma\left(.,z_j\right)}{\partial \nu}\right\|_{C(\partial D)} \le c_3 .$$

Thus, the λ with support $\partial D \setminus B_R$, there exists a constant $c_4 > 0$ such that

$$\left\|g_j\right\|_{C(\partial D)} \le c_3 + \sup_{.\in \partial D\setminus B_R} \left| \lambda k \left[\Phi_0\left(.,y_j\right) + \frac{1}{\sqrt{n_0}}\Gamma\left(.,z_j\right)\right] \right| \le c_4 . \tag{84}$$

From the Lemma 4.1, there exists a constant $c_5 > 0$ such that

$$\left\|u_j\right\|_{C(\partial D)} \le c_5 \left(\left\|u_j\right\|_{C(\partial D\setminus B_R)} + \left\|\frac{\partial u_j}{\partial \nu}\right\|_{C(\partial D)} \right) . \tag{85}$$

From the boundary condition 75, the assumption 79 and the inequality 83, we obtain

$$\left\|u_j\right\|_{C(\partial D\setminus B_R)} \le \left\|u_{0,j}\right\|_{C(\partial D\setminus B_R)} + \left\|f_j\right\|_{C(\partial D\setminus B_R)} \le c_1 + c_2 . \tag{86}$$

From the boundary conditions 75, 76, the equation 81, the assumption 79, the inequalities 84 and 86, there exists a constant $c_6 > 0$ such that

$$\begin{aligned} \left\|\frac{\partial u_j}{\partial \nu}\right\|_{C(\partial D)} &\le \left|n_0\right| \left\|\frac{\partial u_{0,j}}{\partial \nu} + \lambda k u_{0,j}\right\|_{C(\partial D)} + \left\|-\lambda k u_j\right\|_{C(\partial D)} + \left|n_0\right| \left\|g_j\right\|_{C(\partial D)} \\ &\le \left|n_0\right| \left[\left\|\frac{\partial u_{0,j}}{\partial \nu}\right\|_{C(\partial D)} + \left\|\lambda k u_{0,j}\right\|_{C(\partial D\setminus B_R)} \right] + \left\|-\lambda k u_j\right\|_{C(\partial D\setminus B_R)} + \left|n_0\right| c_4 \le c_6 . \end{aligned} \tag{87}$$

When we write the inequalities 86 and 87 in the inequality 85, we obtain the following inequality

$$\left\|u_j\right\|_{C(\partial D)} \le c_7 \quad , \quad c_7 > 0 . \tag{88}$$

For the sequence f_j , we have

$$\left\|f_j\right\|_{C(\partial D)} = \left\|\Phi_0\left(.,y_j\right) + \sqrt{n_0}\Gamma\left(.,z_j\right)\right\|_{C(\partial D)} = \left\|u_j - u_{0,j}\right\|_{C(\partial D)} \le \left\|u_j\right\|_{C(\partial D)} + \left|-1\right| \left\|u_{0,j}\right\|_{C(\partial D)} .$$

From the assumption 79 and the inequality 88, $\left\|f_j\right\|_{C(\partial D)}$ is bounded which is a contradiction. Because f_j is nondefined in $\partial D \cap B_R$ and bounded according to the norm on $C\left(\partial D \setminus B_R\right)$. Therefore, $\left\|u_{0,j}\right\|_{C^1(\partial D)}$ is unbounded as $j \to \infty$.

To formulate the linear method for the solution of the inverse scattering problem, we will benefit from the information about the far-field model $u_\infty(\hat{x},d)$, where $\hat{x}=\frac{x}{|x|}$ and d are unit vector on the unit circle Ω. Recall that for this end, the Herglotz wave function in the form

$$v_g(x)=\int_\Omega e^{ikx.\mathrm{d}}g(\mathrm{d})ds(\mathrm{d}) \tag{89}$$

is a solution of the Helmholtz equation , where $g\in L^2(\Omega)$ is the kernel of v_g. Our aim is to show that there exists a function $g=g(.,y_j)\in L_2(\Omega)$ such that

$$\left\|\int_\Omega u_\infty(\hat{x},\mathrm{d})g(\mathrm{d})ds(\mathrm{d})-\frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}.y_j}\right\|_{L^2(\Omega)}<\varepsilon \qquad \text{for } \forall\varepsilon>0,$$

where $y_j\in D$ is defined by sequence 78. We will also show that it is $\lim_{j\to\infty}\left\|g(.,y_j)\right\|_{L^2(\Omega)}=\infty$. Thus, the boundary of D is characterized by points where the norm $\left\|g(.,y_j)\right\|_{L^2(\Omega)}$ is unlimited.

Theorem 4.3 There exists $g=g(.,y_j)\in L^2(\Omega)$ such that

$$\left\|\int_\Omega u_\infty(\hat{x},\mathrm{d})g(\mathrm{d})ds(\mathrm{d})-\frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}.y_j}\right\|_{L^2(\Omega)}<\varepsilon, \quad \text{for } \forall\varepsilon>0 \tag{90}$$

and $\lim_{j\to\infty}\left\|g(.,y_j)\right\|_{L^2(\Omega)}=\infty$. Moreover, if v_g is the Herglotz wave function defined by function 89, then $\lim_{j\to\infty}\left\|v_g(.,y_j)\right\|_{L^2(D)}=\infty$.

Proof. From Theorem 3.3, the interior boundary value problem 73-76 has a solution which is not generally a Herglotz wave function. However, a Herglotz wave function $U_{0,j}$ with kernel g is shown to exist and this function approaches $u_{0,j}$ in $C^1(\overline{D})$ given in [8,11]. Let u_0 be show the total field, solving the original exterior boundary value problem 6-10, and the functions u_0^* and u^* be defined $u_0^*(y)=u_0(y,-\hat{x})$ and $u^*(y)=u(y,-\hat{x})$, respectively. From the reciprocity relation [16] and the far-field pattern $u_\infty(\hat{x},\mathrm{d})$, we obtain

$$\int_{\Omega} u_\infty(\hat{x},\mathrm{d}) g(\mathrm{d}) ds(\mathrm{d}) = \int_{\Omega} u_\infty(-\mathrm{d},-\hat{x}) g(\mathrm{d}) ds(\mathrm{d})$$

$$= \int_{\Omega} \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \left\{ \int_{\partial D} \left[u_0^*(y) \frac{\partial e^{-ik\mathrm{d}.y}}{\partial \nu} - e^{-ik\mathrm{d}.y} \frac{\partial u_0^*}{\partial \nu}(y) \right] ds(y) \right\} g(\mathrm{d}) ds(\mathrm{d})$$

$$= \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left[u_0^*(y) \frac{\partial U_{0,j}}{\partial \nu}(y) - \frac{\partial u_0^*}{\partial \nu}(y) U_{0,j}(y) \right] ds(y). \quad (91)$$

Since $U_{0,j} \approx u_{0,j}$ in $C^1(\overline{D})$, the integral on the right-hand of equation 91 become

$$\int_{\partial D} \left[u_0^*(y) \frac{\partial U_{0,j}}{\partial \nu}(y) - \frac{\partial u_0^*}{\partial \nu}(y) U_{0,j}(y) \right] ds(y) \approx \int_{\partial D} \left[u_0^*(y) \frac{\partial u_{0,j}}{\partial \nu}(y) - \frac{\partial u_0^*}{\partial \nu}(y) u_{0,j}(y) \right] ds(y).$$

Applying the conditions 75-76 and then the conditions 8-9 , respectively, the last equation is in the form below

$$\int_{\partial D} \left(u_0^*(y) \frac{\partial U_{0,j}}{\partial \nu}(y) - \frac{\partial u_0^*}{\partial \nu}(y) U_{0,j}(y) \right) ds(y) \approx \frac{1}{n_0} \int_{\partial D} \left[u^*(y) \frac{\partial u_j}{\partial \nu}(y) - \frac{\partial u^*}{\partial \nu}(y) u_j(y) \right] ds(y)$$
$$- \int_{\partial D} \left[u_0^*(y) \frac{\partial \Phi_0}{\partial \nu}(y, y_j) - \frac{\partial u_0^*}{\partial \nu}(y) \Phi_0(y, y_j) \right] ds(y). \quad (92)$$

Let's apply the Divergens theorem to the first integral on the right hand of equation 92. We get

$$\frac{1}{n_0} \int_{\partial D} \left[u^*(y) \frac{\partial u_j}{\partial \nu}(y) - \frac{\partial u^*}{\partial \nu}(y) u_j(y) \right] ds(y)$$
$$= \int_{\partial D} \left[u(y,-\hat{x}) \frac{1}{n_0} \nabla u_j(y) \nu(y) - u_j(y) \frac{1}{n_0} \nabla u(y,-\hat{x}) \nu(y) \right] ds(y)$$
$$= \int_{D} \left[div\left[u(y,-\hat{x}) \frac{1}{n(y)} \nabla u_j(y) \right] - div\left[u_j(y) \frac{1}{n(y)} \nabla u(y,-\hat{x}) \right] \right] dy$$
$$= \int_{D} \left[u(y,-\hat{x}) \left(-k^2 u_j(y) \right) - u_j(y) \left(-k^2 u(y,-\hat{x}) \right) \right] dy = 0. \quad (93)$$

From the Helmholtz representation and the Green's formula, the last integral in the right-hand of equation 92 is

$$\int_{\partial D} \left[u_0(y,-\hat{x}) \frac{\partial \Phi_0}{\partial \nu}(y, y_j) - \frac{\partial u_0}{\partial \nu}(y,-\hat{x}) \Phi_0(y, y_j) \right] ds(y) = -u_0(y_j, -\hat{x}) = e^{-ik\hat{x}.y_j}. \quad (94)$$

From the equations 93 and 94, the equation 92 is in the form below

$$\int_{\partial D}\left[u_0^*(y)\frac{\partial U_{0,j}}{\partial \nu}(y)-\frac{\partial u_0^*}{\partial \nu}(y)U_{0,j}(y)\right]ds(y)\approx e^{-ik\hat{x}.y_j} \tag{95}$$

When we write the equation 95 in the equation 91, we get

$$\int_{\Omega}u_\infty(\hat{x},\mathrm{d})g(\mathrm{d})ds(\mathrm{d})\approx\frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}.y_j}\ .$$

Hence there is a function $g\in L^2(\Omega)$ that satisfies the equation 90. We assume that $\left\|g(.,y_j)\right\|_{L^2(\Omega)}$ is bounded as $j\to\infty$. Hence $\left\|U_{0,j}\right\|_{C^1(D)}$ is bounded. This implies that $\left\|u_{0,j}\right\|_{C^1(D)}$ is bounded as $j\to\infty$. This result is contradict with the Theorem 4.2. Thus, the theorem is proved.

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References

[1] Adams, R.A. (1998) A Complete Course Calculus 4th ed., Addison-Wesley, Canada,834-835.
[2] Angell, T.S., Colton, D., Kirsch, A. (1982) The three dimensional inverse scattering problem for acoustic waves, J.Diff. Equations, 46: 46-58.
[3] Cakoni, F., Colton, D., Monk, P. (2001) The direct and inverse scattering problems for partially coated obstacles, Inverse Problems, 17: 1997-2015.
[4] Cakoni, F., Colton, D., Haddar, H. (2002) The linear sampling method for anisopratic media", J. Comput. Appl. Math., 146: 285-299.
[5] Cakoni, F., Colton, D. (2003) The linear sampling method for craks, Inverse Problems, 19: 279-295.
[6] Cakoni, F., Colton, D. (2007) Inequalities in inverse scattering theory, J.Inv. III-Posed Problems, 15: 483-491 (2007).
[7] Colton, D., Kress, R. (1983) Integral Equations Methods in Scattering Theory. John Wiley, New York, 2-106.
[8] Colton, D., Kress, R., Monk, P. (1997) Inverse scattering from an orthotropic medium, J. Comput. Apply Math., 81: 269-298.
[9] Colton, D., Piana, M. (1998) The simple method for solving the electromagnetic inverse scattering problem : the case of TE polarized waves, Inverse Problems, 14: 597-614.
[10] Colton, D., Kirsch, A. (1996) A simple method for solving inverse scattering problems in the resonance region, Inverse Problems, 12: 383-393.
[11] Colton, D., Piana, M., Potthast, R. (1997) A simple method using Morozov's discrepancy principle for solving inverse scattering problems, Inverse Problems, 13: 1477-1493.
[12] Colton, D., Giebermann, K., Monk, P. (2000) A regularized sampling method for solving three – dimensional inverse scattering problems, SIAM J.Sci. Comput., 21 (6): 2316 – 2330.

[13] Colton, D., Coyle, J., Monk, P. (2000) Recent developments in inverse acoustic scattering theory, SIAM Review, 42 (3): 369-414.
[14] Colton, D. (2003) Inverse acoustic and electromagnetic scattering theory, Inverse Problems, 47: 67-110.
[15] Colton, D., Kress, R. (2006) Using fundamental solutions in inverse scattering theory, Inverse Problems, 22(3): 49-66.
[16] Colton, D., Kress, R. (1992) Inverse Acoustic and Electromagnetic Scattering Theory, Springer – Verlag, Berlin, 2-86,147-148,207-240.
[17] Gerlach, T., Kress, R. (1996) Uniqueness in inverse obstacle scattering with conductive boundary condition, Inverse Problems 12: 619-625.
[18] Kress, R. (1989) Linear Integral Equations, Springer-Verlag, Berlin, 14-45.
[19] Kirsch, A., Kress, R. (1993) Uniqueness in inverse obstacle scattering, Inverse Problems, 9: 285-299.
[20] Lebedev, N.N. (1972) Special Functions and Their Applications, Silverman, R.A., Dover Publications, Newyork, (107,134-135).
[21] Qin, H.H., Colton, D. (2011) The inverse scattering problem for cavities with impedance boundary condition, Adv. Comp. Math., 36:157-174 .
[22] Seydou, F. (2001) Profile inversion in scattering theory: the TE case, J. Comput. Appl. Math., 137: 49-60.
[23] Tobocman, W. (1989) Inverse acoustic wave scattering in two dimensions from impenetrable targets, Inverse Problems, 5: 1131-1144.
[24] Torun, G., Anar, İ.E. (2005) The electromagnetic scattering problem: the case of TE polarized waves, ELSEVIER Appl. Math. And Comput.,169: 339-354.

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ON CO-FILTERS IN CO-QUASIORDERED RESIDUATED SYSTEMS

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Abstract

Residuated relational systems have been the focus of many researchers in the past decade. In this article, as a continuation of [9], we focused on residuated relational systems $\langle A, \cdot, \rightarrow, 1, \not\preccurlyeq \rangle$ ordered under co-quasiorder relation $'\not\preccurlyeq'$ within the Bishop's constructive framework. In this report we give some new results on co-filters in such relational systems by more depth and deeper analyzing of the connection between the internal operation $'\cdot'$ and $'\rightarrow'$ with the co-quasiorder relation.

Keywords: Bishop's constructive mathematics; Set with apartness; Co-quasiordered residuated system; Co-filter.

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1 Introduction

Although in the last decade the concept of residual relational systems is in the focus of many researchers (for example, [3, 4]), there are still not many research reports on such algebraic structures.

Definition 1.1. ([4], Definition 2.1) A residuated relational system is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R \rangle$, where $\langle A, \cdot, \rightarrow, 1 \rangle$ is an algebra of type $\langle 2, 2, 0 \rangle$ and R is a binary relation on A and satisfying the following properties:

(1) $\langle A, \cdot, 1 \rangle$ is a commutative monoid;

(2) $(\forall x \in A)((x, 1) \in R)$;

(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \iff (x, y \rightarrow z) \in R)$.

They referred to the operation $'\cdot'$ as multiplication, to $'\rightarrow'$ as its residuum and to condition (3) as residuation.

The concept of residual relational system ordered under a quasi-order relation can be found in Bonzio's dissertation [3] from 2015 and in one of his articles [4] from 2018 (done together with I. Chajda). In the forthcoming articles [11, 12] this author introduced and analyzed concepts of ideals and filters in such systems. In the aforementioned texts, authors observed the relational system $\langle A, \cdot, \rightarrow, 1, R \rangle$ where R was a quasi-order relation.

In our article [9], we are developed this concept within the Bishop's constructive framework [1, 2, 5, 6, 13]. Observed and analyzed is residuated relational system with a set with apartness as the carrier of the algebraic construction, and additionally R was a co-quasiorder relation on the set A. With this article, as a continuation of our article [9, 10], we complements our researches on algebraic structures within Bishop's principled-philosophical orientation (see, for example [7, 8]).

The Constructive algebra abounds in specific behavior of algebraic structures determined on sets with apartness. Additionally, the ordered algebraic structures constructed on sets with apartness are also very interesting. Particularly, there is a possibility that an algebraic structure is ordered under a co-order (under a co-quasiorder) relation instead an order (or a quasi-order) relation.

In this article we continue our analysis of co-quasiordered residuated systems launched in [9] and [10]. Second, we continue to analyze the concept of co-filters in such systems and proved some new properties of this concept.

2 Preliminaries

2.1 The research framework

The setting of this research is the Bishop's constructive mathematics [**Bish**] in the seance of the following books [1], [2], [5], [6] and [13] - a mathematics based on the Intuitionistic logic [**IL**] (See [13]) and principled-philosophical orientation on Bishop's constructive mathematics.

Let $(S, =, \neq)$ be a constructive set in the sense of Bishop [1], Mines et all. [6], Troelstra and van Dalen [13]. On set $S = (S, =, \neq)$ in this mathematics we look as on a relational system with an one binary relation extensive with respect to the equality in the following sense

$$= \circ \neq \subseteq \neq \text{ and } \neq \circ = \subseteq \neq,$$

where $'\circ'$ is the standard operation between relations. The relation $\neq$ is a binary relation on S with the following properties:

$$\neg(x \neq x),\; x \neq y \Longrightarrow y \neq x,\; x \neq z \Longrightarrow x \neq y \vee y \neq z,$$
$$x \neq y \wedge y = z \Longrightarrow x \neq z.$$

It is called *apartness*. Let S and T be two sets with apartness, then the relation $\neq$ on $S \times T$ is defined by

$$(x, y) \neq (u, v) \Longleftrightarrow (x \neq u \vee y \neq v)$$

for any $x, u \in S$ and any $y, v \in T$.

Let Y be a subset of S and $x \in S$. We put it the following notation $\lhd$ as a relation between an element x and subset Y with (For more details on this relation, the readers can see the following texts [7, 8])

$$x \lhd Y \Longleftrightarrow (\forall y \in Y)(\, x \neq y).$$

Following the orientation in books [1], [2], [5] we define a subset

$$Y^{\lhd} = \{x \in S : x \lhd Y\}$$

of S called the *complement of* Y *in* S.

For subset Y of S we say that it is a *strongly extensional subset* if

$$(\forall x, y \in S)(y \in Y \Longrightarrow x \neq y \;\vee\; x \in Y).$$

For a relation R on S it is called a strongly extensional if

$$(\forall x, y, z, u \in S)((x, y) \in R \Longrightarrow ((x, y) \neq (z, u) \vee (z, u) \in R))$$

holds. For example, for a mapping $f : S \longrightarrow T$ it is called a strongly extensional (shortly: *se-mapping*) if holds

$$(\forall x, y \in S)(f(x) \neq f(y) \Longrightarrow x \neq y).$$

2.2 Co-quasiorder relation

The constructive notion of a co-quasiorder relation is the dual notion to the classical notion of a quasi-order relation. Let $(S, =, \neq)$ be a set with apartness. A consistent and co-transitive relation $\not\leq$ defined on S is called a *co-quasiorder* ([7, 8]):

$(\forall c, y \in S)(x \not\leq y \implies x \neq y)$ (consistency)

$(\forall x, y, z \in S)(x \not\leq z \implies (x \not\leq y \vee y \not\leq z))$ (co-transitivity).

We accept that the empty set $\emptyset$ is also a co-quasiorder relation on set S. The strong complement $\not\leq^{\lhd}$ of a co-quasiorder $\not\leq$ has the well known property.

Lemma 2.1. ([7], Lemma 2.2) If $\not\leq$ is a co-quasiorder on S, then the relation $\not\leq^{\lhd} = \{(x, y) \in S \times S : (x, y) \lhd \sigma\}$ is a quasi-order on S.

2.3 Co-quasiordered residuated systems

In our papers [9, 10], following the ideas of Bonzio [3] and Bonzio and Chajda [4], we introduced and analyzed the notion of residuated relational systems ordered under a co-quasiorder - a residuated relational systems $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ where R is a co-quasiorder relation on set $(A, =, \neq)$. In the article [9] we introduced and analyzed the concept of co-filters in such systems, and in the text [10] we introduced and analyzed the concept of co-ideals.

If R is a co-quasiorder relation on set $(A, =, \neq)$, then the axiom (2) in Definition 1.1 gives $(1, 1) \in R \subseteq \neq$ which is a contradiction. That is why we transformed this axiom into the next formula

(2') $(\forall x \in A)(x \neq 1 \Longrightarrow (x, 1) \in R)$.

Let $(A, =, \neq)$ be a set with apartness. A co-quasiordered residuated system is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where the axiom (2') is replaced by (2) and where R is a co-quasiorder on A.

Definition 2.1. ([9], Definition 2.1) A co-quasiordered residuated relational system is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \not\leq\rangle$, where $A = (A, =, \neq)$ is a set with apartness and where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and $\not\leq$ is a co-quasiorder relation on A and satisfying the following properties:

(1) $\langle A, \cdot, 1\rangle$ is a commutative monoid;

(2') $(\forall x \in A)(x \neq 1 \Longrightarrow x \not\leq 1)$;

(3) $(\forall x, y, z \in A)(x \cdot y \not\leq z \iff x \not\leq y \rightarrow z)$.

We will refer to the operation $' \cdot '$ as multiplication, to $' \rightarrow '$ as its residuum and to condition (3) as residuation.

Apart from the difference in the carrier of this constructed algebraic structure, the difference between the residuated relational system in our definition and the definition in texts [3, 4] is in the strong extensionality of the internal binary operations in A. Let us note that the internal operations $' \cdot '$ and $' \rightarrow '$ are total strongly extensional function from $A \times A$ into A:

$(\forall a, b, a', b' \in A)(a \cdot b \neq a' \cdot b' \Longrightarrow (a, b) \neq (a', b'))$,

$(\forall a, b, a', b' \in A)(a \rightarrow b \neq a' \rightarrow b' \Longrightarrow (a, b) \neq (a', b'))$.

Proposition 2.1. ([9], Proposition 2.3) Let $\mathfrak{A}$ be a co-quasiordered residuated relational system. Then

$$(\forall x, y \in A)(x \not\leq y \iff 1 \neq x \rightarrow y).$$

In the following theorem we shown that the co-quasiorder $' \not\leq '$ is compatible with the internal operation $' \cdot '$.

Theorem 2.1. ([9], Theorem 2.1) Let $\mathfrak{A}$ be a co-quasiordered residuated system. Then

$$(\forall x, y, a, b \in A)((a \cdot x \not\leq a \cdot y \vee x \cdot b \not\leq y \cdot b) \implies x \not\leq y).$$

In the following theorem we shown that the co-quasiorder $' \not\leq '$ is left compatible and right anti-compatible with the internal operation $' \rightarrow '$.

Theorem 2.2. ([9], Theorem 2.2) Let $\mathfrak{A}$ be a co-quasiordered residuated system. Then

(a) $(\forall x, y, a \in A)(a \to x \not\leqslant a \to y \implies x \not\leqslant y)$.

(b) $(\forall x, y, b \in A)(y \to b \not\leqslant x \to b \implies x \not\leqslant y)$.

Speaking by the language of classical algebra, when we speak of the compatibility of the internal binary operations $' \cdot '$ and $' \to '$ with the relation $' \not\leqslant '$, we mean on the cancellativity of these operations with respect to $' \not\leqslant '$.

The algebraic system ordered under co-quasiorder relation thus determined was in the focus of our forthcoming work [10], also.

3 Further developing the idea of co-filters

The following is valid

Lemma 3.1. Let $\langle A, \cdot, \to, 1, \not\leqslant \rangle$ be a co-quasiordere resuated system. The relation $\not\leqslant^{\lhd}$ is a quasi-order on the monoid $(A, \cdot)$ compatible with the internal operation in A.

Proof. As is known (see, for example [7], Lemma 2.1), $\not\leqslant^{\lhd}$ is a quasi-order relation on the set A. Let $x, y, a, u, v \in A$ be arbitrary elements such that $x \not\leqslant^{\lhd} y$ and $u \not\leqslant v$. Then

$$u \not\leqslant a \cdot x \lor a \cdot x \not\leqslant a \cdot y \lor a \cdot y \not\leqslant u$$

by co-transitivity of $\not\leqslant$. Thus $u \neq a \cdot x \lor a \cdot y \neq v$ because the option $a \cdot x \not\leqslant a \cdot y$ implies $x \not\leqslant y$ by Theorem 2.1 and according to consistency of $\not\leqslant$. So, we have $(a \cdot x, a \cdot y) \neq (u, v) \in \not\leqslant$. This means $a \cdot x \not\leqslant^{\lhd} a \cdot y$. Therefore, the relation $\not\leqslant^{\lhd}$ is left compatible with the internal operation in A.
The implication of $x \not\leqslant^{\lhd} y \implies x \cdot a \not\leqslant^{\lhd} y \cdot a$ can be prove by analogy with the previous evidence. ■

Corollary 3.1. If $\not\leqslant \cap \not\leqslant^{-1} = \emptyset$, then

(4) $(\forall x \in A)(1 \not\leqslant^{\lhd} x)$ and

(5) $(\forall x, y \in A)(x \not\leqslant^{\lhd} x \cdot y$ and $y \not\leqslant^{\lhd} x \cdot y)$.

Proof. Let $x, u, v \in A$ be arbitrary elements such that $u \not\leqslant v$ and $x \neq 1$. Then $x \not\leqslant 1$ by (2') and

$$u \not\leqslant v \implies (u \not\leqslant 1 \lor 1 \not\leqslant x \lor x \not\leqslant v).$$

Since the second option is impossible because $x \not\leqslant 1$ and $\not\leqslant \cap \not\leqslant^{-1} = \emptyset$, we have $(1, x) \neq (u, v) \in \not\leqslant$. So, it means $1 \not\leqslant^{\lhd} x$.
Since $1 \not\leqslant^{\lhd} x$ by the first evidence of this proof, it follows $1 \cdot y \not\leqslant^{\lhd} x \cdot y$ by Lemma 3.1. So, $y \not\leqslant^{\lhd} x \cdot y$ holds. The claim $y \not\leqslant^{\lhd} x \cdot y$ can prove by analogy to the previous claim. ■

It should be noted here that the condition $\not\leqslant \cap \not\leqslant^{-1} = \emptyset$ is not always satisfied. In what follows, we will always assume that this condition is fulfilled.

It is shown in [9], Proposition 2.1, that condition (3) implies condition

(6) $(\forall x, yz \in A)(x \cdot y \not\leqslant^{\lhd} z \iff x \not\leqslant^{\lhd} y \to z)$.

Naturally, the reverse implication does not valid in general case.

In our forthcoming article [10], Proposition 5, is proven.

Proposition 3.1. Classes $L_{\not\leqslant}(a) = \{y \in A : a \not\leqslant y\}$ $(a \in A)$ are strongly extensional subsets of A such that $a \lhd L_{\not\leqslant}(a)$, $1 \in L_{\not\leqslant}(a)$ and following formula is valid

(L) $(\forall u, v \in A)(v \in L_{\not\leqslant}(a) \implies (u \not\leqslant v \lor u \in L_{\not\leqslant}(a)))$.

In addition, these left classes of the relation $\not\leqslant$ have the following properties:

Proposition 3.2. Let $\langle A, \cdot, \to, 1, \not\leqslant \rangle$ be a co-quasiordere resuated system with $\not\leqslant \cap \not\leqslant^{-1} = \emptyset$ and $a, b \in A$. Then

(7) $(\forall x, y \in A)(x \cdot y \in L_{\not\leqslant}(a) \implies (x \in L_{\not\leqslant}(a) \land y \in L_{\not\leqslant}(a)))$;

(8) $(\forall x, y \in A)(x \not\leqslant y \implies x \to y \in L_{\not\leqslant}(a))$;

(9) $L_{\not\leqslant}(a) \cup L_{\not\leqslant}(b) \subseteq L_{\not\leqslant}(a \cdot b)$.

Proof. (7) Let $x, y \in A$ be arbitrary elements such that $x \cdot y \in L_{\not\preceq}(a)$. Then $a \not\preceq x \cdot y$. Thus $a \not\preceq x \vee x \not\preceq x \cdot y$ and $a \not\preceq y \vee y \not\preceq x \cdot y$ by co-transitivity of $\not\preceq$. Since the second option is impossible by (5), we have $x \in L_{\not\preceq}(a)$ and $y \in L_{\not\preceq}(a)$.
(8) Let $x, y \in A$ arbitrary elements such that $x \not\preceq y$. Then $x \not\preceq a \cdot x \vee a \cdot x \not\preceq y$ by co-transitivity of $\not\preceq$. Thus $a \cdot x \not\preceq y$ because the first option is impossible by (5). So, $a \not\preceq x \to y$ by (3). Therefore, $x \to y \in L_{\not\preceq}(a)$.
(9) If $t \in L_{\not\preceq}(a)$, then $a \not\preceq t$. Thus $a \not\preceq a \cdot b \vee a \cdot b \not\preceq y$. So, we have $t \in L_{\not\preceq}(a \cdot b)$ by (5). From this follows $L_{\not\preceq}(a) \cup L_{\not\preceq}(b) \subseteq L_{\not\preceq}(a \cdot b)$ immediately. ■

Corollary 3.2. Let $\langle A, \cdot, \to, 1, \not\preceq\rangle$ be a co-quasiordere resuated system with $\not\preceq \cap \not\preceq^{-1} = \emptyset$ and $a \in A$. Then

(10) $(\forall x, y \in A)(y \in L_{\not\preceq}(a) \implies (x \to y \in L_{\not\preceq}(a) \vee x \in L_{\not\preceq}(a)))$.

Proof. Let $x, y \in A$ be arbitrary elements such that $y \in L_{\not\preceq}(a)$. Then $x \not\preceq y \vee x \in L_{\not\preceq}(a)$ by (L). Thus $x \to y \in L_{\not\preceq}(a) \vee x \in L_{\not\preceq}(a)$ by (8). ■

In the article [9], we have developed the idea of co-filters in these algebraic systems. In addition, we have shown some of the significant features of these substructures in a residuated relational system ordered under a co-quasiorder.

Definition 3.1. ([9], Definition 2.2) A subset G of A is a co-filter of a residuated system $\mathfrak{A}$ ordered under a co-quasiorder $\not\preceq$ if the following conditions hold

(G1) $(\forall x, y \in A)(x \cdot y \in G \Longrightarrow x \in G \ \vee \ y \in G)$;

(G2) $(\forall x, y \in A)(y \in G \Longrightarrow (x \not\preceq y \vee x \in G)$.

Condition (G1) speaks that a co-filter G is a co-subgroupoid in $(A, \cdot)$.

Lemma 3.2. ([9]) Any co-filter G of a co-quasiordered residuated system $\mathfrak{A}$ is a strongly extensional subset in A.

Our first theorem correlate condition (G2) to condition (G1).

Theorem 3.1. Let $\mathfrak{A}$ be a co-quasiordered residuated system and G be a co-filter in $\mathfrak{A}$. Then $(G2) \implies (G1)$.

Proof. Let $x, y \in A$ be arbitrary elements such that $x \cdot y \in G$. Then $x \not\preceq x \cdot y \vee x \in G$ by (G2). Since the first option is impossible by (5), we have $x \in G$. The second part $x \cdot y \in G \implies y \in G$ of the proof of this theorem can be obtained analogously to the first part. ■

Corollary 3.3. Any co-filter G of a co-quasiordered residuated system $\mathfrak{A} = \langle A, \cdot, 1, \to, \not\preceq\rangle$ is a consistent subset in A.

Corollary 3.4. If G is a non empty co-filter in a co-quasiordered residuated system $\mathfrak{A}$, then $1 \in G$.

Theorem 3.2. Let A be a co-quasiordered residuated system and G be a subset of A. Then the condition (G2) is equivalent to the condition

(G3) $(\forall x, y, z \in A)(z \in G \implies (x \not\preceq y \to z \vee x \cdot y \in G))$.

Proof. $(G2) \implies (G3)$: Suppose (G2) holds and let $x, y, z \in A$ be arbitrary element such that $z \in G$. Then $z \in G \implies (x \cdot y \not\preceq z \vee x \cdot y \in G)$. Thus $x \not\preceq y \to z \vee x \cdot y \in G$ by (3). So, the condition (G3) is proven.
$(G3) \implies (G2)$. Opposite, let the condition (G3) be a valid formula in $\mathfrak{A}$ and let $x, y \in A$ be arbitrary elements such that $y \in G$. Then $y \in G \implies (x \not\preceq 1 \to y \vee x \cdot 1 \in G$ by (G3) where we put $z = 1$. Thus $y \in G \implies (x \not\preceq y \vee x \in G)$ by (1) and (3). So, the condition (G2) is a valid formula in $\mathfrak{A}$. ■

Subsets $L_{\not\preceq}(a)$ $(a \in A)$ are co-filters in a residuated relational system $\mathfrak{A}$ ordered under a co-quasiorder $\not\preceq$ according to (L) and (7), Therefore, the family $\mathfrak{G}(A)$ of all co-filters in $\mathfrak{A}$ is not empty.

Theorem 3.3. The family $\mathfrak{G}(A)$ of all co-filters of a co-quasiordered residuated system $\mathfrak{A}$ forms a complete lattice.

Proof. (i) Let $x, y \in A$ be arbitrary elements. Thus

$$\begin{aligned} y \in \bigcup \mathfrak{G} &\Longleftrightarrow (\exists G \in \mathfrak{G})(y \in G) \\ &\Longrightarrow (\exists G \in \mathfrak{G})(x \not\leqslant y \ \vee \ x \in G) \\ &\Longrightarrow x \not\leqslant y \ \vee \ x \in \bigcup \mathfrak{G}. \end{aligned}$$

(ii) Let $\mathfrak{B}$ be the families of all co-ideals contained in $\bigcap \mathfrak{G}$. Then $\bigcup \mathfrak{B}$ is the maximal co-filter contained in $\bigcap \mathfrak{G}$, according to the first part of this evidence.
(iii) If we put $\sqcup \mathfrak{G} = \bigcup \mathfrak{G}$ and $\sqcap \mathfrak{G} = \bigcup \mathfrak{B}$, then $(\mathfrak{G}(A), \sqcup, \sqcap)$ is a complete lattice. ■

Corollary 3.5. For each subset B of A, there is the maximal co-filter of $\mathfrak{A}$ contained in B.

Corollary 3.6. For elements $a_1, ..., a_n \in A$, there is the maximal co-filter K of $\mathfrak{A}$ such that $a_1 \lhd K$, ..., $a_n \lhd K$.

If T is a subset of A, then $\bigcup_{t \in T} L_{\not\leqslant}(t)$ is a co-filter in $\mathfrak{A}$, by Theorem 3.3. We call such a co-filter a *normal* co-filter. We will write $T^U = \bigcup_{t \in T} L_{\not\leqslant}(t)$ in this case.

Proposition 3.3. Let $\mathfrak{A}$ be a co-quasiordered residuated system. Then the union of any family of normal co-filters in $\mathfrak{A}$ is a normal co-filter in $\mathfrak{A}$.

Proof. The assertion of this proposition is a direct consequence of the following equality $(\bigcup_{i \in I} T_i)^U = \bigcup_{i \in I} T_i^U$. ■

Corollary 3.7. The family of all normal co-filters in $\mathfrak{A}$ forms join semi-lattice.
However, the intersection of two normal co-filters is not a co-filter in the general case.

In the following proposition we give one upper measure for a non-empty co-filter.

Proposition 3.4. For any non empty co-filter G in a co-quasiordered residuated system $\mathfrak{A}$ the following $G \subseteq \bigcup_{a \lhd G} L_{\not\leqslant}(a)$ holds.

Proof. Let $a \in A$ be an arbitrary element such that $a \lhd G$. Then from $t \in G$ follows $a \not\leqslant t \vee a \in G$ by (G2). Since the second option is impossible by hypothesis, we have $t \in L_{\not\leqslant}(a)$. Thus $G \subseteq \bigcup_{a \lhd G} L_{\not\leqslant}(a)$.

■

In order to offer one lower measure of a co-filter in a co-quasiprdered residuated system $\mathfrak{A}$, we need the notion of right class $R_{\not\leqslant}(b)$ of relation $\not\leqslant$ generated by the element $b \in A$: $R(b) = \{x \in A : x \not\leqslant b\}$.

Proposition 3.5. For any non empty co-filter G in a co-quasiordered residuated system $\mathfrak{A}$ the following $\bigcup_{b \in G} R(b)^{\lhd} \subseteq G$ holds.

Proof. Let $t \in A$ be an arbitrary element such that $t \in \bigcup_{b \in G} R(b)^{\lhd}$. Then there exists an element $b \in G$ such that $t \lhd R(b)$. Thus from (G2): $b \in G \Longrightarrow (t \not\leqslant b \vee y \in G)$ follows $t \in G$ because $\neg(t \not\leqslant b)$ by the hypothesis. Therefore, we have $\bigcup_{b \in G} R(b)^{\lhd} \subseteq G$. ■

4 Final reflection

Bishop's constructive mathematics includes the following two aspects:
(1) The Intuitionistic logic and
(2) The principled-philosophical orientations of constructivism.
Intuitionistic logic does not accept the TND principle as an axiom. In addition, Intuitionistic logic does not accept the validity of the 'double negation' principle. This makes it possible to have a difference relation in sets which is not a negation of the equality relation. Therefore, we accept that in Bishop's constructive mathematics we consider set S as one relational system $(S, =, \neq)$. In Bishop's constructive algebra we always encounter the following two problems:

(a) How to choose a predicate (or more predicates) between several classically equivalent ones by which an algebraic concept is determined.

(b) Since every predicate has at least one of its duals, how to construct a dual of the algebraic concept defined with a given predicate(s).

In this case, we are faced with the problem of describing a residuated relational system based on a set with apartness as the carrier for constructing an algebraic structure. By our orientation that in this construction, groupoid $(A, \cdot)$ is ordered under a a co-quasiorder relation instead of a quasi-order relation, a significantly different logical-sets framework is formed. In addition to the above, in this report we have described some of the important features of a class of substructures (in this case - the class of co-filers) in residuated relational systems constructed on sets with apartness in which both internal binary operations are strongly extensional functions.

The problem encountered by authors working within Bishop's constructive framework is that when developing concepts of new ideas and defining their interrelationships with respect to the permissible rules of conclusion in [**IL**], they must always strive for the results obtained to be correlated with the corresponding results that exist or can be obtained in the classical case.

References

[1] Bishop, E. (1967) Foundations of Constructive Analysis, New York: McGraw-Hill.

[2] Bishop, E. and Bridges, D. S. (1985) Constructive Analysis. Grundlehren der mathematischen Wissenschaften 279, Berlin: Springer.

[3] Bonzio, S. (2015) Algebraic structures from quantum and fuzzy logics. Ph.D Thesis. Cagliari: Universit'a degli studi di Cagliari.

[4] Bonzio, S/ and Chajda, I. (2018) Residuated relational systems. Asian-European J. Math., 11(2)(2018): 1850024 (14 pages)

[5] Bridges, D. S. and Richman, F. (1987) Varieties of Constructive Mathematics. Cambridge: London Mathematical Society Lecture Notes, No. 97, Cambridge University Press.

[6] Mines, R. Richman, F. and Ruitenburg, W. (1988) A Course of constructive algebra. New York: Springer.

[7] Romano, D. A. (2018) Co-ideals and co-filters in ordered set under co-quasiorder. Bull. Int. Math. Virtual Inst., 8(1): 177–188.

[8] Romano, D. A. (2019) Some algebraic structures with apartness, A review. J. Int. Math. Virtual Inst., 9(2): 361–395.

[9] Romano, D. A. (2019) Co-quasiordered residuated systems, An introduction. Asian-European J. Math., 12(5): ID 1950073

[10] Romano, D. A. (to appear) Co-ideals in a residuated relational system ordered under a co-quasiorder. Available at https://www.researchgate.net/profile/Daniel Romano

[11] Romano, D. A. (to appear) Filters in residuated relational system ordered under quasi-order. Available at https://www.researchgate.net/profile/Daniel Romano

[12] Romano, D. A. (to appear) Ideals in quasi-ordered residuated system. Available at https://www.researchgate.net/profile/Daniel Romano

[13] Troelstra, A. S. and van Dalen, D. (1988) Constructivism in Mathematics: An Introduction. Amsterdam: North-Holland.

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A STUDY OF ORDERED BI-GAMMA-HYPERIDEALS IN ORDERED GAMMA-SEMIHYPERGROUPS

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Abstract

The main purpose of this paper is to investigate ordered Γ-semihypergroups in the general terms of ordered Γ-hyperideals. We introduce ordered (generalized) (m,n)-Γ-hyperideals in ordered Γ-semihypergroups. Then, we characterize ordered Γ-semihypergroup by ordered (generalized) $(0,2)$-Γ-hyperideals, ordered (generalized) $(1,2)$-Γ–hyperideals and ordered (generalized) 0-minimal $(0,2)$-Γ-hyperideals. Furthermore, we investigate the notion of ordered (generalized) $(0,2)$-bi-Γ-hyperideals, ordered 0-$(0,2)$ bisimple ordered Γ-semihypergroups and ordered 0-minimal (generalized) $(0,2)$-bi-Γ-hyperideals in ordered Γ-semihyperoups. It is proved that an ordered Γ-semihypergroup S with a zero 0 is 0-$(0,2)$-bisimple if and only if it is left 0-simple.

Keywords: Algebraic hyperstructure; Γ-subsemihypergroup; bisimple; ordered Γ-semihypergroup; ordered bi-Γ-hyperideal; ordered (m,n)-Γ–hyperideal; ordered $(0,2)$-Γ-hyperideal.

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1 Introduction

The theory of (m,n)-ideal in semigroups was given by Lajos [44] as a generalization of left(resp. right) ideals in semigroups. Thereafter, the notion of generalized bi-ideal [(or generalized (1,1)-ideal] was introduced in semigroups also by Lajos [43] as a generalization of bi-ideals in semigroups. Then, various authors investigated these concepts [1], [2], [19], [27], [28], [29], [30], [31]. Akram, Yaqoob and Khan studied (m,n)-hyperideals in LA-semihypergroups [25]. Hila et al. [23], [47] investigated quasi-hyperideals and bi-hyperideals in semihypergroups.

The concept of hyperstructure was given by Marty [20], at the 8th Congress of Scandinavian Mathematics. He formulated hypergroups and began to derive its properties and results. Now, the notion of algebraic hyperstructures has become a highly fruitful branch in algebraic theory and it has wide applications in various branches of mathematics and applied science. For detailed review of the notion of hyperstructures, readers are referred to [8], [13], [18], [23], [33], [35], [37], [38], [39], [40], [42].

Recently, Basar et al. studied different aspects of ideal theoretic results in ordered semihypergroups [3], [4], [5], [6], [7] [41].

Later on, many algebraists have developed semihypergroups as the simplest algebraic hyperstructures with closure and associative properties. Semihypergroups (hypergroups) have been found useful for dealing with problems in different domains of algebraic hyperstructures. Many mathematicians studied various aspects of semihypergroups (hypergroups), for instance, Kondo and Lekkoksung [26], Bonansinga and Corsini [35], Leoreanu [49], Davvaz [8], Pibaljommee and Davvaz [9], Davvaz [10], [11], Freni [14], and Salvo [32]. The applications of semihypergroups (hypergroups) to areas such as graph theory, optimization theory, theory of discrete event dynamical systems, automata theory, generalized fuzzy computation, formal language theory, coding theory and analysis of computer programs have been extensively studied in the literature [12].

Then connection between hyperstructures and ordered sets has been investigated by many researchers. Heidari and Davvaz [15], [18] studied ordered hyperstructures. One main aspect of this theory, known as El-hyperstructures, was studied by Chvalina and Novak [21], [34]. Conard studied ordered semigroups [36]. The concept of ordered semihypergroups was studied in [9], [22], [47], [48]. Heideri et al. [16], [17], [45], [46] studied Γ-semihypergroups. We assume that the reader is familiar with some terminology in theory of semihypergroup and other related notions. What follows now are some definitions and preliminaries in the theory of Γ-semihypergroups that we need for formulation and proof of our main results.

Let H be a nonempty set, then the mapping $\circ : H \times H \to H$ is called hyperoperation or join operation on H, where $P^{\star}(H) = P(H) \setminus \{0\}$ is the set of all nonempty subsets of H. Let A and B be two nonempty sets. Then a hypergroupoid $(S, \circ)$ is called a Γ-semihypergroups if for every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$,

$$x \circ \alpha \circ (y \circ \beta \circ z) = (x \circ \alpha \circ y) \circ \beta \circ z,$$

i.e.,

$$\bigcup_{u \in y \circ \alpha \circ z} x \circ \alpha \circ u = \bigcup_{v \in x \circ \alpha \circ y} v \circ \beta \circ z.$$

A Γ-semihypergroup $(S, \circ)$ together with a partial order " $\leq$ " on S that is compatible with Γ-semihypergroup operation such that for all $x, y, z \in S$, we have

$$x \leq y \Rightarrow z \circ \alpha \circ x \leq z \circ \beta \circ y \text{ and } \mathrm{x} \circ \alpha \circ \mathrm{z} \leq \mathrm{y} \circ \beta \circ \mathrm{z},$$

ia called an ordered Γ-semihypergroup. For subsets A, B of an ordered Γ-semihypergroup S, the product set $A \circ \Gamma \circ B$ of the pair (A, B) relative to S is defined as below:

$$A \circ \Gamma \circ B = \{a \circ \gamma \circ b : a \in A, b \in B, \gamma \in \Gamma\},$$

and for $A \subseteq S$, the product set $A \circ \Gamma \circ A$ relative to S is defined as $A^2 = A \circ \Gamma \circ A$.

For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m \text{ for some } m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$.

Let $A \subseteq S$. Then, for a non-negative integer m, the power of A is defined by $A^m = A \circ \Gamma \circ A \circ \Gamma \circ A \circ \Gamma \circ A \cdots$, where A occurs m times. Note that the power vanishes if $m = 0$. So, $A^0 \circ \Gamma \circ S = S = S \circ \Gamma \circ A^0$.

In what follows we denote ordered Γ-semihypergroup $(S, \circ, \Gamma, \leq)$ by S unless otherwise specified.

Suppose S is an ordered Γ-semihypergroup and I is a nonempty subset of S. Then, I is called an ordered right (resp. left) Γ-hyperideal of S if

(i) $I \circ \Gamma \circ S \subseteq I (resp.\ S \circ \Gamma \circ I \subseteq I)$,

(ii) $a \in I, b \leq a$ for $b \in S \Rightarrow b \in I$.

Equivalent Definition:

(i) $I \circ \Gamma \circ S \subseteq I$ (resp. $S \circ \Gamma \circ I \subseteq I$).

(ii) $(I] = I$.

An ordered Γ-hyperideal I of S is both a right and a left ordered Γ-hyperideal of an ordered Γ-semihypergroup S. A right, left or (two-sided) ordered Γ-hyperideal I of S is called proper if $I \neq S$.

Definition 1.1: Let S be a Γ-semihypergroup and A be a nonempty subset of S, then A is called a generalized (m, n)-Γ-hyperideal of S if $A^m \Gamma S \Gamma A^n \subseteq A$, where m, n are arbitrary non-negative integers. Notice that if A is a sub-Γ-semihypergroup of S, then A is called an (m, n)-Γ-hyperideal of S.

Definition 1.2. Suppose A is a sub-Γ-semihypergroup (resp. nonempty subset) of an ordered Γ-semihypergroup S. Then, A is called an (resp. generalized) (m, n)-Γ-hyperideal of S if (i) $A^m \circ \Gamma \circ S \circ \Gamma \circ A^n \subseteq A$, and (ii) for $b \in A$, $s \in S$, $s \leq b \Rightarrow s \in A$.

Observe that in the above Definition 1.2., if we put $m = n = 1$, then A is called an ordered (generalized) bi-Γ-hyperideal of S. Furthermore, if $m = 0$ and $n = 2$, then we find an ordered (generalized) $(0, 2)$-Γ-hyperideal of S. In a similar manner, we can derive an ordered (generalized) $(1, 2)$-Γ-hyperideal and an ordered (generalized) $(2, 1)$-Γ-hyperideal of S.

Let $(S, \circ, \Gamma, \leq)$ be an ordered Γ-semihypergroup and A, B be nonempty subsets of S, then we easily have the following:

(i) $A \subseteq (A]$;

(ii) If $A \subseteq B$, then $(A] \subseteq (B]$;

(iii) $(A] \circ \Gamma \circ (B] \subseteq (A \circ \Gamma \circ B]$;

(iv) $(A] = ((A]]$;

(v) $((A] \circ \Gamma \circ (B]] = (A \circ \Gamma \circ B]$;

(vi) For every left (resp. right) ordered Γ-hyperideal T of S, $(T] = T$.

If A is a nonempty subset of S, $(A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]$ is an ordered (generalized) bi-Γ-hyperideal of S, we depict the proof of it as follows:

$$((A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]] = (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]$$

$$\text{and } (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \circ \Gamma \circ S \circ \Gamma \circ (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]$$
$$= (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \circ \Gamma \circ (S] \circ \Gamma \circ (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]$$
$$\subseteq (A^2 \circ \Gamma \circ S \circ \Gamma \circ A^2 \cup A^2 \circ \Gamma \circ S \circ \Gamma \circ A \circ \Gamma \circ S \circ \Gamma \circ A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2 \circ \Gamma \circ S \circ \Gamma \circ A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2 \circ \Gamma \circ S \circ \Gamma \circ A \circ \Gamma \circ S \circ \Gamma \circ A^2]$$
$$\subseteq (A \circ \Gamma \circ S \circ \Gamma \circ A^2]$$
$$\subseteq (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2].$$

2 Main Results

In the current section, we now study ideal theory in ordered Γ-semihypergroups. We obtain many equivalent conditions based on ordered Γ-hyperideal, ordered (0, 2)-Γ-hyperideal, ordered bi-Γ-hyperideal. We begin with the following:

Lemma 2.1: The following assertions are equivalent for a subset A of an ordered Γ-semihypergroup S:

(i) A is an ordered (generalized) $(0,2)$-Γ-hyperideal of S;

(ii) A is an ordered left Γ-hyperideal of some ordered left Γ-hyperideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose A is an ordered (generalized) $(0,2)$-Γ-hyperideal of an ordered Γ-semihypergroup S. Then, we obtain the following:

$$\begin{aligned}(A \cup S \circ \Gamma \circ A] \circ \Gamma \circ A &= (A^2 \cup S \circ \Gamma \circ A^2] \\ &\subseteq (A] \\ &= A,\end{aligned}$$

and

$$((A] = (A],$$

therefore, A is an ordered left Γ-hyperideal of ordered left Γ-hyperideal $(A \cup S \circ \Gamma \circ A]$ of S.
$(ii) \Rightarrow (i)$. Suppose L is an ordered left Γ-hyperideal of S and B is an ordered left Γ-hyperideal of L. Then, we have

$$\begin{aligned}S \circ \Gamma \circ A^2 &\subseteq S \circ \Gamma \circ L \circ \Gamma \circ \Gamma \circ A \\ &\subseteq L \circ \Gamma \circ A \\ &\subseteq A.\end{aligned}$$

Suppose $b \in A$ and $s \in S$ are such that $s \leq b$. As $b \in L$, we get $s \in L$ and so $s \in A$. Hence, A is an ordered (generalized) $(0,2)$-Γ-hyperideal of S.

Theorem 2.2. Let A be a subset of an ordered Γ-semihypergroup S. Then the following results are equivalent:

(i) A is an ordered (generalized) $(1,2)$-Γ-hyperideal of S;

(ii) A is an ordered left Γ-hyperideal of some ordered (generalized) bi-Γ-hyperideal of S;

(iii) A is an ordered (generalized) bi-Γ-hyperideal of some left ordered Γ-hyperideal of S;

(iv) A is an ordered (generalized) $(0,2)$-Γ-hyperideal of some ordered right Γ-hyperideal of S;

(v) A is an ordered right-Γ-hyperideal of some ordered (generalized) $(0,2)$-Γ-hyperideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose A is an ordered (generalized) $(1,2)$-Γ-hyperideal of S. This means A is a sub-Γ-semihypergroup (nonempty subset) of S and $A \circ \Gamma \circ S \circ \Gamma \circ A^2 \subseteq A$. Therefore,

$$\begin{aligned}(A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \circ \Gamma \circ A &= (B^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \circ \Gamma \circ (A] \\ &\subseteq (A^3 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^3] \\ &\subseteq (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A] = A.\end{aligned}$$

Clearly, if $b \in A$, $s \in (S^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]$ so that $s \leq b$ then, $s \in A$. Hence, A is an ordered left Γ-hyperideal of ordered (generalized) bi-Γ-hyperideal $(A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]$ of S.

$(ii) \Rightarrow (iii)$. Suppose A is an ordered left Γ-hyperideal of some ordered (generalized) bi-Γ-hyperideal B of S. Recall that $(A \cup S \circ \Gamma \circ A]$ is an ordered left Γ-hyperideal of S. According to our hypothesis,

$$\begin{aligned} A \circ (A \cup S \circ \Gamma \circ A] \circ B &\subseteq (A] \circ \Gamma \circ (A \cup S \circ \Gamma \circ A] \circ \Gamma \circ (A] \\ &\subseteq (A^3 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A \cup A \circ \Gamma \circ S \circ \Gamma \circ A \circ \Gamma \circ A] \\ &\subseteq (A \cup A \circ \Gamma \circ A] \\ &\subseteq (A] \\ &= A. \end{aligned}$$

Suppose $b \in A$, $s \in (A \cup S \circ \Gamma \circ A]$ such that $s \leq b$. As, $b \in A$, $b \in B$. So, $s \in B$ and therefore, $s \in A$. Hence, A is an ordered (generalized) bi-Γ-hyperideal of left ordered hyperideal $(A \cup S \circ \Gamma \circ A]$ of S.

$(iii) \Rightarrow (iv)$. Suppose A is an ordered (generalized) bi-Γ-hyperideal of some left ordered Γ-hyperideal L of S. This implies that $B \subseteq L$, $A \circ \Gamma \circ L^1 \circ \Gamma \circ B \subseteq A$ and $S \circ \Gamma \circ L \subseteq L$. Therefore,

$$\begin{aligned} (A \cup A \circ \Gamma \circ S] \circ \Gamma \circ A^2 &\subseteq (A \cup A \circ \Gamma \circ S] \circ (A^2] \\ &\subseteq (A^3 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A \cup A \circ \Gamma \circ S \circ L \circ \Gamma \circ A] \\ &\subseteq (A \cup A \circ \Gamma \circ L \circ \Gamma \circ A] \\ &\subseteq (A] = A. \end{aligned}$$

Furthermore, suppose that $b \in A$, $s \in (A \cup A \circ \Gamma \circ S]$ such that $s \leq b$, so $b \in L$. Then, $s \in L$, therefore, $s \in A$. Hence, A is an ordered (generalized) $(0,2)$-Γ-hyperideal of the ordered right Γ-hyperideal $(A \cup A \circ \Gamma \circ S]$ of S.

$(iv) \Rightarrow (v)$. Suppose A is an ordered (generalized) $(0,2)$-Γ-hyperideal of some ordered right Γ-hyperideal R of S. This implies that $A \subseteq R$, $R \circ \Gamma \circ A^2 \subseteq A$ and $R \circ \Gamma \circ S \subseteq R$. Then,

$$\begin{aligned} A \circ \Gamma \circ (A \cup S \circ \Gamma \circ A^2] &\subseteq (A] \circ \Gamma \circ (A \cup S \circ \Gamma \circ A^2] \\ &\subseteq (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A \cup R \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A \cup R \circ \Gamma \circ A^2] \\ &\subseteq (A] = A. \end{aligned}$$

Let $b \in A$, $s \in (A \cup S \circ \Gamma \circ A^2]$ such that $s \leq b$. Then, $b \in R$, so $s \in R$, thus $s \in B$. Hence, B is an ordered right Γ-hyperideal of the (generalized) $(0,2)$-Γ-hyperideal $(B \cup S \circ \Gamma \circ B^2]$ of S.

$(v) \Rightarrow (i)$. Suppose A is an ordered right Γ-hyperideal of an ordered (generalized) $(0,2)$-Γ-hyperideal R of S. This further shows that $A \subseteq R$, $A \circ \Gamma \circ R \subseteq A$ and $S \circ \Gamma \circ R^2 \subseteq R$. Then, we have the following:

$$\begin{aligned} A \circ S \circ \Gamma \circ A^2 &\subseteq A \circ \Gamma \circ S \circ \Gamma \circ R^2 \\ &\subseteq A \circ R \\ &\subseteq A. \end{aligned}$$

Suppose $b \in A$, $s \in S$ such that $s \leq b$. Since $b \in R$, so $s \in B$. Therefore, A is an ordered (generalized) $(1,2)$-Γ-hyperideal of S. Hence, A is an ordered (generalized) bi-Γ-hyperideal of S.

Lemma 2.3. A sub-Γ-semihypergroup (nonempty subset) A of an ordered Γ-semihypergroup S such that $A = (A]$ is an ordered (generalized) $(1,2)$-Γ-hyperideal of S if and only if there exists an ordered (generalized) $(0,2)$-Γ-hyperideal L of S and an ordered right Γ-hyperideal R of S so that $R \circ \Gamma \circ L^2 \subseteq A \subseteq R \cap L$.

Proof. Suppose A is an ordered (generalized)$(1,2)$-Γ-hyperideal of S. We know that $(A\cup S\circ\Gamma\circ A^2]$ and $(A\cup A\circ\Gamma\circ S]$ are an ordered (generalized) $(0,2)$-Γ-hyperideal and an ordered right Γ-hyperideal of S, respectively. Furthermore, assume $L=(A\cup S\circ\Gamma\circ A^2]$ and $R=(A\cup A\circ\Gamma\circ S]$. Then, we have the following:

$$\begin{aligned} R\circ\Gamma\circ L^2 &\subseteq (A^3\cup A^2\circ\Gamma\circ S\circ\Gamma\circ A^2\cup A\circ\Gamma\circ S\circ\Gamma\circ A^2\cup A\circ\Gamma\circ S\circ\Gamma\circ A\circ\Gamma\circ S\circ\Gamma\circ A^2] \\ &\subseteq (A^3\cup A\circ\Gamma\circ S\circ\Gamma\circ A^2] \\ &\subseteq (A]=A. \end{aligned}$$

Hence, $R\subseteq R\cap L$.

Conversely, suppose R is an ordered right Γ-hyperideal of S and L is an ordered (generalized) $(0,2)$-Γ-hyperideal of S so that $R\circ\Gamma\circ L^2\subseteq A\subseteq R\cap L$. Then, we have the following:

$$\begin{aligned} A\circ\Gamma\circ S\circ\Gamma\circ A^2 &\subseteq (R\cap L)\circ\Gamma\circ\Gamma\circ S\circ\Gamma\circ(R\cap L)\circ\Gamma\circ\Gamma\circ(R\cap L) \\ &\subseteq R\circ\Gamma\circ S\circ\Gamma\circ L^2 \\ &\subseteq R\circ\Gamma\circ L^2 \\ &\subseteq A. \end{aligned}$$

Hence, A is an ordered (generalized) $(1,2)$-Γ-hyperideal of S.

Definition 2.4. An ordered (generalized) $(0,2)$-bi-Γ-hyperideal B of S is called 0-minimal if $B\neq\{0\}$, $\{0\}$ is the only ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S properly contained in B.

Lemma 2.5. Suppose L is an ordered 0-minimal left Γ-hyperideal of an ordered Γ-semihypergroup S with 0 and I is a sub-Γ-semihypergroup (nonempty subset) of L such that $I=(I]$. Then, I is an ordered (generalized) $(0,2)$-Γ-hyperideal of S contained in L if and only if $(I\circ\Gamma\circ I]=\{0\}$ or $I=L$.

Proof. Suppose I is an ordered (generalized) $(0,2)$-Γ-hyperideal of S contained in L. As $(S\circ\Gamma\circ I^2]$ is an ordered left Γ-hyperideal of S and $(S\circ\Gamma\circ I^2]\subseteq I\subseteq L$, we obtain the following:
$(S\circ\Gamma\circ I^2]=\{0\}$ or $(S\circ\Gamma\circ I^2]=\{L\}$.
If $(S\circ\Gamma\circ I^2]=L$, then $L=(S\circ\Gamma\circ I^2]\subseteq(I]$. So, $I=L$. Suppose $(S\circ I^2]=\{0\}$. As $S\circ(I^2]\subseteq(S\circ\Gamma\circ I^2]=\{0\}\subseteq(I^2]$, then $(I^2]$ is an ordered left Γ-hyperideal of S contained in L. By the minimality of L, we obtain $(I^2]=\{0\}$ or $(I^2]=L$. If $(I^2]=L$, then $I=L$. Therefore, $I^2=\{0\}$ or $I=L$.
The converse part is straightforward.

Lemma 2.6. Suppose M is an ordered 0-minimal (generalized) $(0,2)$-Γ-hyperideal of an ordered Γ-semihypergroup S with a zero 0. Then $(M^2]=\{0\}$ or M is an ordered 0-minimal left Γ-hyperideal of S.

Proof. Since $M^2\subseteq M$ and

$$\begin{aligned} S\circ\Gamma\circ(M^2]^2 &= S\circ\Gamma\circ(M^2]\circ\Gamma\circ(M^2] \\ &\subseteq (S\circ\Gamma\circ M^2]\circ\Gamma\circ(M^2] \\ &\subseteq (M]\circ\Gamma\circ(M^2] \\ &\subseteq (M^2]. \end{aligned}$$

Then, we obtain $(M^2]$ is an ordered (generalized) $(0,2)$-Γ-hyperideal of S contained in M. Therefore, $(M^2]=\{0\}$ or $(M^2]=M$. Suppose $(M^2]=M$. Since

$$\begin{aligned} S\circ M &= S\circ\Gamma\circ(M^2] \\ &\subseteq (S\circ\Gamma\circ M^2] \\ &\subseteq (M]=M. \end{aligned}$$

It follows that M is an ordered left Γ-hyperideal of S. Suppose B is an ordered left Γ-hyperideal of S contained in M. Therefore,

$$\begin{aligned} S \circ \Gamma \circ B^2 &\subseteq B^2 \\ &\subseteq B \\ &\subseteq M. \end{aligned}$$

Hence, B is an ordered (generalized) $(0,2)$-Γ-hyperideal of S contained in M and so, $B = \{0\}$ or $B = M$.

Corollary 2.7 Suppose S is an ordered Γ-semihypergroup without a zero 0. Then, M is an ordered minimal (generalized) $(0,2)$-Γ-hyperideal of S if and only if M is an ordered minimal left Γ-hyperideal of S.

Proof. It follows by Lemma 2.5 and Lemma 2.6.

Lemma 2.8. Suppose S is an ordered Γ-semihypergroup without a zero 0. Further, suppose that M is a nonempty subset of S. Then, the following results are equivalent:

(i) M is an ordered (generalized) minimal $(2,1)$-Γ-hyperideal of S;

(ii) M is an ordered (generalized) minimal bi-Γ-hyperideal of S.

Proof. Suppose S is an ordered Γ-semihypergroup without zero and M is an ordered minimal (generalized) $(2,1)$-Γ-hyperideal of S. Then, $(M^2 \circ \Gamma \circ S \circ \Gamma \circ M] \subseteq M$ and so $(M^2 \circ \Gamma \circ S \circ \Gamma \circ M]$ is an ordered (generalized) $(2,1)$-Γ-hyperideal of S. Therefore, we obtain $(M^2 \circ \Gamma \circ S \circ \Gamma \circ M] = M$. As

$$\begin{aligned} M \circ \Gamma \circ S \circ \Gamma \circ M &= (M^2 \circ \Gamma \circ S \circ \Gamma \circ M] \circ \Gamma \circ S \circ \Gamma \circ M \\ &\subseteq (M^2 \circ \Gamma \circ S \circ \Gamma \circ M \circ \Gamma \circ S \circ \Gamma \circ M] \\ &\subseteq (M^2 \circ \Gamma \circ S \circ \Gamma \circ M] = M, \end{aligned}$$

we have that M is an ordered (generalized) bi-Γ-hyperideal of S. Let there exist an ordered (generalized) bi-Γ-hyperideal A of S contained in M. Then, $A^2 \circ S \circ A \subseteq A^2 \subseteq A \subseteq M$, therefore, A is an ordered (generalized) $(2,1)$-Γ-hyperideal of S contained in M. Using the minimality of M, we obtain $A = M$.

Conversely, suppose M is an ordered minimal (generalized) bi-Γ-hyperideal of S. Then, M is an ordered (generalized) $(2,1)$-Γ-hyperideal of S. Suppose T is an ordered (generalized) $(2,1)$-hyperideal of S contained in M. As

$$\begin{aligned} (T^2 \circ \Gamma \circ S \circ \Gamma \circ T] \circ \Gamma \circ S \circ \Gamma \circ (T^2 \circ S \circ T] &\subseteq (T^2 \circ (S \circ T \circ \Gamma \circ S \circ \Gamma \circ T^2 \circ \Gamma \circ S) \circ \Gamma \circ T] \\ &\subseteq (T^2 \circ \Gamma \circ S \circ \Gamma \circ T], \end{aligned}$$

we obtain $(T^2 \circ \Gamma \circ S \circ \Gamma \circ T]$ is an ordered (generalized) bi-Γ-hyperideal of S. This shows that $(T^2 \circ \Gamma \circ S \circ \Gamma \circ T] = M$. As $M = (T^2 \circ \Gamma \circ S \circ \Gamma \circ T] \subseteq (T] = T$, $M = T$. Hence, M is an ordered minimal (generalized) $(2,1)$-Γ-hyperideal of S.

Definition 2.9. A sub-Γ-semihypergroup (nonempty subset) A of an ordered Γ-semihypergroup S is called an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S if A is an ordered (generalized) bi-Γ-hyperideal of S and also an ordered (generalized) $(0,2)$-Γ-hyperideal of S.

Lemma 2.10. Suppose A is a subset of an ordered Γ-semihypergroup S. Then, the following conditions are equivalent:

(i) B is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S;

(ii) B is an ordered Γ-hyperideal of some ordered left Γ-hyperideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose A is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S. This shows that $A \circ \Gamma \circ S \circ \Gamma \circ B \subseteq B$ and $S \circ \Gamma \circ A^2 \subseteq A$. Then, we have

$$\begin{aligned} S \circ (A^2 \cup S \circ \Gamma \circ A^2] &\subseteq (S \circ \Gamma \circ A^2 \cup S^2 \circ \Gamma \circ A^2] \\ &\subseteq (S \circ A^2] \\ &\subseteq (A^2 \cup S \circ \Gamma \circ A^2] \end{aligned}$$

Therefore, $(A^2 \cup S \circ \Gamma \circ A^2]$ is an ordered left Γ-hyperideal of S. As

$$\begin{aligned} A \circ (A^2 \cup S \circ \Gamma \circ A^2] &\subseteq (A^3 \cup A \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A] \\ &= A, \end{aligned}$$

$(A^2 \cup S \circ \Gamma \circ A^2] \circ \Gamma \circ B \subseteq (A^3 \cup S \circ \Gamma \circ A^3] \subseteq (A] = A$. Hence, A is an ordered Γ-hyperideal of left ordered hyperideal $(A^2 \cup S \circ \Gamma \circ A^2]$ of S.
$(ii) \Rightarrow (i)$. Suppose A is an ordered Γ-hyperideal of some ordered left Γ-hyperideal L of S. By Lemma 2.1, A is an ordered (generalized) $(0,2)$-Γ-hyperideal of S, and hence, A is an ordered (generalized) bi-Γ-hyperideal of S.

Theorem 2.11. Suppose A is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-hyperideal of an ordered Γ-semihypergroup S with a zero 0. Then, exactly one of the followings cases arises:

(i) $A = \{0, b\}$, $(b \circ \Gamma \circ S \circ \Gamma \circ b] = \{0\}$;

(ii) $A = (\{0, b\}]$, $b^2 = 0$, $(b \circ S \circ \Gamma \circ b] = A$;

(iii) $(S \circ \Gamma \circ b^2] = A$ for all $b \in A \setminus \{0\}$.

Proof. Suppose A is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-hyperideal of an ordered Γ-semihypergroup S. Furthermore, suppose $b \in A \setminus \{0\}$. Then, $(S \circ \Gamma \circ b^2] \subseteq A$ and $(S \circ \Gamma \circ b \circ \Gamma \circ b]$ is an ordered left Γ-hyperideal of S, therefore, $(S \circ b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S. Hence, $(S \circ \Gamma \circ b^2] = \{0\}$ or $(S \circ b^2] = A$.
Let $(S \circ \Gamma \circ b^2] = \{0\}$. As $b^2 \in A$, we obtain either $b^2 = b$ or $b^2 = 0$ or $b^2 \in A \setminus \{0, b\}$. If $b^2 = b$, then $b = 0$. This is a contradiction. Let $b^2 \in A \setminus \{0, b\}$. Then,

$$\begin{aligned} S \circ \Gamma \circ (\{0, b^2\}]^2 &\subseteq (\{0, S \circ \Gamma \circ b^2\}] = (\{0\}] \cup (S \circ \Gamma \circ b^2] \\ &= \{0\} \\ &\subseteq (\{0\} \cup b^2], \end{aligned}$$

$$\begin{aligned} (\{0\} \cup b^2] \circ \Gamma \circ S \circ \Gamma \circ (\{0\} \cup b^2] &\subseteq (b^2 \circ \Gamma \circ S \circ \Gamma \circ b^2] \\ &\subseteq (S \circ \Gamma \circ b^2] = \{0\} \\ &\subseteq \{0, b^2\}. \end{aligned}$$

So, $(\{0\} \cup b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S contained in A, and we obtain that $(\{0\} \cup b^2] \neq \{0\}$, $(\{0\} \cup b^2] \neq A$. This is also not possible as A is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-hyperideal of S. Therefore, $b^2 = \{0\}$ and hence by Lemma 2.10, $A = (\{0, b\}]$. Now, since we have $(b \circ \Gamma \circ S \circ \Gamma \circ b]$ is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S contained in A, we get $(b \circ \Gamma \circ S \circ \Gamma \circ b] = \{0\}$ or $(b \circ \Gamma \circ S \circ \Gamma \circ b] = A$. So, $(S \circ \Gamma \circ b^2] = \{0\}$ and it implies that either $A = \{0, b\}$

and $(b \circ \Gamma \circ S \circ \Gamma \circ b] = \{0\}$ or $A = \{0, b\}$, $b^2 = \{0\}$ and $(b \circ \Gamma \circ S \circ \Gamma \circ b] = A$. If $(S \circ \Gamma \circ b^2] \neq \{0\}$, then $(S \circ \Gamma \circ b^2] = A$.

Corollary 2.12. Suppose B is an ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-hyperideal of an ordered Γ-semihypergroup S with a zero 0 so that $(B^2] \neq \{0\}$. Then, $B = (S \circ \Gamma \circ b^2]$ for every $b \in B \setminus \{0\}$.

Definition 2.13. An ordered Γ-semihypergroup S with a zero 0 is called 0-$(0, 2)$- bisimple if (i) $(S^2] \neq \{0\}$, and $\{0\}$ is the only ordered proper (generalized) $(0, 2)$-bi-Γ-hyperideal of S.

Corollary 2.14. An ordered Γ-semihypergroup S with a zero 0 is 0-$(0, 2)$-bisimple if and only if $(S \circ \Gamma \circ s^2] = S$ for every $s \in S \setminus \{0\}$.

Proof. If S is 0-$(0, 2)$-bisimple, then $(S \circ S] \neq \{0\}$ and S is an ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-hyperideal. By Corollary 2.12., we have $S = (S \circ \Gamma \circ s^2]$ for every $s \in S \setminus \{0\}$.

Conversely, suppose $S = (S \circ \Gamma \circ s^2]$ for every element $s \in S \setminus \{0\}$ and further suppose that A is an ordered (generalized) $(0, 2)$-bi-Γ-hyperideal of S such that $A \neq \{0\}$. Suppose $b \in A \setminus \{0\}$. Then, $S = (S \circ \Gamma \circ b^2] \subseteq (S \circ \Gamma \circ A^2] \subseteq (A] = A$, therefore $S = A$. Since, $S = (S \circ \Gamma \circ b^2] \subseteq (S \circ \Gamma \circ S] = (S^2]$, we obtain $\{0\} \neq S = (S \circ \Gamma \circ S] = (S^2]$. Hence, S is 0-$(0, 2)$-bi-simple. The proof is complete.

Theorem 2.15. An ordered Γ-semihypergroup S with a zero 0 is 0-$(0, 2)$-bisimple if and only if S is left 0-simple.

Proof. We recall that every ordered left Γ-hyperideal A of an ordered Γ-semihypergroup S is an ordered 0-$(0, 2)$-bi-Γ-hyperideal of S. So, $A = \{0\}$ or $A = S$. Therefore, if S is 0-$(0, 2)$-bisimple then S is left 0-simple.

Conversely, if S is left 0-simple then, $(S \circ s] = S$ for every $s \in S \setminus \{0\}$ from which it follows that

$$\begin{aligned} S &= (S \circ \Gamma \circ s] \\ &= ((S \circ \Gamma \circ s] \circ \Gamma \circ s] \\ &\subseteq ((S \circ \Gamma \circ s^2]] \\ &= (S \circ \Gamma \circ s^2]. \end{aligned}$$

Therefore, using Corollary 2.14, S is 0-$(0, 2)$-bisimple. The proof is complete.

Theorem 2.16. Suppose A is an ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-hyperideal of an ordered Γ-semihypergroup S. Then, either $(A \circ \Gamma \circ A] = \{0\}$ or A is left 0-simple.

Proof. Suppose $(A \circ A] \neq \{0\}$. Then, by Corollary 2.12, we obtain $(S \circ \Gamma \circ b^2] = A$ for every $b \in A \setminus \{0\}$. As $b^2 \in A \setminus \{0\}$ for every $b \in A \setminus \{0\}$, we obtain $b^4 = (b^2)^2 \in A \setminus \{0\}$. Suppose $b \in A \setminus \{0\}$. As, $(A \circ \Gamma \circ b^2] \circ \Gamma \circ S \circ \Gamma \circ (A \circ \Gamma \circ b^2] \subseteq (A \circ \Gamma \circ A \circ \Gamma \circ b^2] \subseteq (A \circ \Gamma \circ b^2]$ and

$$\begin{aligned} S \circ (A \circ \Gamma \circ b^2]^2 &\subseteq (S \circ \Gamma \circ A \circ \Gamma \circ b^2 \circ \Gamma \circ A \circ \Gamma \circ b^2] \\ &\subseteq (S \circ \Gamma \circ A^2 \circ \Gamma \circ b^2] \\ &\subseteq (A \circ \Gamma \circ b^2], \end{aligned}$$

we get that $(A \circ b^2]$ is an ordered (generalized) $(0, 2)$-bi-Γ-hyperideal of S contained in A. Therefore, $(A \circ \Gamma \circ b^2] = \{0\}$ or $(A \circ \Gamma \circ b^2] = A$. As, $b^4 \in A \circ b^2 \subseteq (A \circ b^2]$, and $b^4 \in A \setminus \{0\}$, we obtain $(A \circ \Gamma \circ b^2] = A$. By Corollary 2.14 and Theorem 2.15, it follows that A is left 0-simple. The proof is complete.

3 Conclusion

In the current paper, we enriched ideal theory in ordered Γ-semihypergroups. We derived various equivalent conditions related to ordered Γ-hyperideals, ordered (0, 2)-Γ-hyperideals, ordered bi-Γ-hyperideals. We introduced ordered (generalized) (m,n)-Γ-hyperideals in ordered Γ-semihypergroups. Then, we characterized ordered Γ-semihypergroup in terms of ordered (generalized) $(0,2)$-Γ-hyperideals, ordered (generalized) $(1,2)$-Γ-hyperideals and ordered (generalized) 0-minimal $(0,2)$-Γ-hyperideals. Furthermore, we studied the notion of ordered (generalized) $(0,2)$-bi-Γ-hyperideals, ordered 0-$(0,2)$ bisimple ordered Γ-semihypergroups and ordered 0-minimal (generalized) $(0,2)$-bi-Γ-hyperideals in ordered Γ-semihyperoups. It is shown that an ordered Γ-semihypergroup S with a zero 0 is 0-$(0,2)$-bisimple if and only if it is left 0-simple.

References

[1] Abul Basar and M. Y. Abbasi, Some Properties of Q-Fuzzy Ideals in po-Γ-Semigroups, Palestine Journal of Mathematics, 7(2), 2018, 505–511.

[2] Abul Basar and M. Y. Abbasi, on generalized bi-Γ-ideals in Γ-semigroups, Quasigroups and related systems, 23, 2015, 181–186.

[3] Abul Basar, M. Y. Abbasi and Sabahat Ali Khan, An introduction of theory of involutions in ordered semihypergroups and their weakly prime hyperideals, The Journal of the Indian Mathematical Society, 86,2019, 230-240.

[4] Abul Basar, A note on (m, n)-Γ-ideals of ordered LA-Γâ“ semigroups, Konuralp Journal of Mathematics, 7(1), 2019, 107-111.

[5] Abul Basar, Application of (m, n)-Γâ“ Hyperideals in Characterization of LA-Γ- Semihypergroups, Discussion Mathematicae General Algebra and Applications, 39(1), 2019, 135-147.

[6] Abul Basar, M. Y. Abbasi and Bhavanari Satyanarayana, On generalized Γ-hyperideals in ordered Γ-semihypergroups, Fundamental Journal of Mathematics and Applications, 2(1), 2019, 18-23.

[7] Abul Basar, Shahnawaz Ali, Mohammad Yahya Abbasi, Bhavanari Satyanarayana and Poonam Kumar Sharma, On some hyperideals in ordered semihypergroups, Journal of New Theory, 29, 2019.

[8] B. Davvaz, Fuzzy hyperideals in semihypergroups, Ital. J. Pure Appl. Math. N, 8, 2000, 67–74.

[9] B. Pibaljommee and B. Davvaz, Characterizations of (fuzzy) bi-hyperideals in ordered semihypergroups, J. Intell. Fuzzy Syst. 28, 2015, 2141–2148.

[10] B. Davvaz, Some results on congruences in semihypergroups, Bull. Malyas. Math. Sci. So., 23, 2000, 53–58.

[11] B. Davvaz, Characterization of subsemihypergroups by various triangular norms, Czech. Math. J., 55(4), 2005, 923–932.

[12] B. Davvaz, semihypergroup Theory, Academic Press, 2016.

[13] C. Gutan, Simplifiable semihypergroups, Algebraic hyperstructures and applications (Xanthi, 1990), 103–111, World Sci. Publ., Teaneck, NJ, 1991.

[14] D. Freni, Minimal order semihypergroups of type U on the right, II, J. Algebra, 340, 2011, 77–80.

[15] D. Heidari and B. Davvaz, On ordered hyperstructures, U. P. B. Sci. Bull. Series A, 73, 2011, 85–96.

[16] D. Heidari, S.O. Dehkordi and B. Davvaz, Γ-semihypergroups and their properties, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 72 (1), 2010.

[17] D. Heidari and B. Davvaz, Γ-hypergroups and Γ-semihypergroups associated to binary relations, Iranian Journal of Science and Technology, 35 (2), 2011.

[18] D. Heidari and B. Davvaz. On ordered hyperstructures. Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 73(2), 2011.

[19] D. N. Krgovic, On 0-Minimal $(0, 2)$-bi-ideals of semigroups, Publications De L' Institut Mathematique, Nouvelle serie, tome, 31(45), 1982, 103-107.

[20] F. Marty, Sur uni generalization de la notion de groupe, 8th Congress Math. Scandinaves, Stockholm, 1934, 45–49.

[21] J. Chvalina, Commutative hypergroups in the sense of Marty and ordered sets, in: Proc. Summer School, Gen. Algebra ordered Sets, Olomouc (Czech Republic), 1994, 19-30.

[22] J. Tang., A. Khan and Y. F. Luo, Characterization of semisimple ordered semihypergroups in terms of fuzzy hyperideals, J. Intell. Fuzzy Syst., 30, 2016, 1735-1753.

[23] K. Hilla, B. Davvaz and K. Naka, On quasi-hyperideals in semihypergroups, Commun. Algebra, 39, 2011, 41–83.

[24] K. Hila and J. Dine, On hyperideals on left almost semihypergroups, ISRN Algebra, 2011.

[25] M. Akram, N. Yaqoob and M. Khan, On (m, n)-hyperideals in LA-semihypergroups, Applied mathematical Sciences, 7, 2013, 2187 – 2191.

[26] M. Kondo and N. Lekkoksung, On intra-regular ordered Γ-semihypergroups, Int. J. Math. Anal., 7(28), 2013, 1379–1386.

[27] M. Y. Abbasi and Abul Basar, A note on ordered bi-Γ-ideals in intra-regular ordered Γ-semigroups, Afrika Mathematica, 27, 2016, 1403–1407.

[28] M. Y. Abbasi and Abul Basar, On Generalizations of ideals in LA-Γ-semigroups, Southeast Asian Bulletin of Mathematics, (39), 2015, 1-12.

[29] M. Y. Abbasi and A. Basar, Some properties of ordered 0-minimal (0, 2)-bi-Γ-ideals in po-Γ-semigroups, Hacettepe Journal of Mathematics and Statistics, 2(44), 2015, 247–254.

[30] M. Y. Abbasi and Abul Basar, Weakly prime ideals in involution po-Γ-semigroups, Kyungpook Mathematical Journal, 54, 2014, 629-638.

[31] M. Y. Abbasi and Abul Basar, On Ordered Quasi-Gamma-Ideals of Regular Ordered Gamma-Semigroups, Algebra, 2013.

[32] M. D. Salvo, D. Freni and G. Lo Faro, Fully simple semihypergroups, J. Algebra, 399, 2014, 358–377.

[33] M. Bakhshi and R.A. Borzooei, Ordered polygroups, Ratio Math, 24, 2013, 31-40.

[34] M. Novak, EL-hyperstructures, Ratio Math., 23, 2012, 65–80.

[35] P. Bonansinga and P. Corsini, On semihypergroup and hypergroup homomorphisms, Boll. Un. Mat. Ital. B., 1(2), 1982, 717–727.

[36] P. Conrad, Ordered semigroups, Nagoya Math. J. 16, 1960.

[37] P. Corsini, Sur les semi-hypergroupes, Atti Soc. Pelorit. Sci. Fis. Math. Nat. 26(4), 1980.

[38] P. Corsini, Prolegomena of Hypergroup Theory, second ed., Aviani Editore, Tricesimo, 1993.

[39] P. Corsini, Prolegomena of hypergroup theory, Second edition, Aviani editor, 1993.

[40] P. Corsini and V. Leoreanu, Applications of hyperstructure theory, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[41] P. Corsini and V. Leoreanu-Fotea, Applications of hyperstructure theory, Advances in Mathematics, Kluwer Academic Publisher, 2003.

[42] S. Hoskova, Upper order hypergroups as a reflective subcategory of subquasiorder hypergroups, Ital. J. Pure Appl. Math., 20, 2006, 2015–222.

[43] S. Lajos, Notes on generalized bi-ideals in semigroups, Soochow J. Math.,10, 1984, 55–59.

[44] S. Lajos, Generalized ideals in semigroups, Acta Sci. Math.,2, 1961, 217–222.

[45] S. Mirvakili, S.M. Anvariyeh, B. Davvaz, Γ-semihypergroups and regular relations, J. Math. 2013.

[46] S. M. Anvariyeh, S. Mirvakili and B. Davvaz, On Γ-hyperideals in Γ-semihypergroups, Carpathian J. Math. 26, 201.

[47] T. Changphas and B. Davvaz, Bi-hyperideals and Quasi-hyperideals in ordered semihypergroups, Ital. J. Pure Appl. Math.-N, 35, 2015, 493–508.

[48] T. Changphas and B. Davvaz, Properties of hyperideals in ordered semihypergroups, Ital. J. Pure Appl. Math.-N, 33, 2014, 425–432.

[49] V. Leoreanu, About the simplifiable cyclic semihypergroups, Italian J. Pure Appl. Math., 7, 2000, 69–76.

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A FORMULAS FOR THE OPERATOR NORM AND THE EXTENSION OF A LINEAR FUNCTIONAL ON A LINEAR SUBSPACE OF A HILBERT SPACE

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Abstract

In this paper, formulas are given both for the operator norm and for the extension of a linear functional defined on a linear subspace of a Hilbert space and the results are illustrated with examples.

Keywords: Linear functional; Cauchy-Schwarz inequality; norm; operator norm; Hilbert spaces; Hahn-Banach theorem; Riesz representation theorem; orthogonal vectors; orthogonal decomposition.

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1 Introduction

Linear functionals occupy quite important place in mathematics in terms of both theory and application. The weak and weak-star topologies, which are fundamental and substantial subject in functional analysis, are generated by families of linear functionals. They are important in the theory of differential equations, potential theory, convexity and control theory [6]. Linear functionals play fundamental role in characterizing the topological closure of sets and therefore they are important for approximation theory. They play a very important role in defining vector valued analytic functions, generalizing Cauchy integral theorem and Liouville theorem. Therefore the need arises naturally to construct linear functionals with certain properties. The construction is usually achieved by defining the linear functional on a subspace of a normed linear space where it is easy to verify the desired properties and then extending it to the whole space with retaining the properties. This is not always easy in the case of general normed linear spaces. We specialize to linear functionals defined on the subspaces of a Hilbert space and provide formulas (Theorem 2) both for the operator norms and norm preserving linear extensions of linear functionals.

We start with basic definitions and results and fixed notations that will be used in the sequel. We denote the field of the real numbers $\mathbb{R}$ or the field of the complex numbers $\mathbb{C}$ by $\mathbb{F}$. We denote the absolute value function by $|.|$ defined on the field $\mathbb{F}$. So for $x \in \mathbb{R}$, if $x < 0$ then $|x| = -x$ and if $x \geq 0$ then $|x| = x$. For $z = x + iy \in \mathbb{C}$ we have $|z| = \sqrt{x^2 + y^2}$. The complex number $\overline{z} = x - iy$ is the complex conjugate of the number $z = x + iy$.

Definition 1.1. Let X be a linear space over the field $\mathbb{F}$ and $\|.\| : X \mapsto \mathbb{R}$ be a function. If the function $\|.\|$ satisfies the following properties

1. $\|0\| = 0$ and $\|x\| > 0$ for every $x \in X \setminus \{0\}$ (positivity definiteness);

2. $\|\alpha x\| = |\alpha| \, \|x\|$ for every $\alpha \in \mathbb{F}$ and $x \in X$ (homogeneity); and
3. $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$ (triangular inequality);

then the function $\|.\|$ is called a norm on the space X and the pair $(X, \|.\|)$ is called a normed linear space.

Example 1.2. Let $p \in [1, \infty)$ and $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \ldots, n\}$. $\mathbb{R}^n$ is a linear (vector) space over the field $\mathbb{R}$ with componentwise addition and scaler multiplication. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ define $\|x\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$ and $\|x\|_\infty = \max\{|x_j| : j = 1, \ldots, n\}$. Then for $1 \le p \le \infty$ the functions $\|.\|_p$ are norms on the space $\mathbb{R}^n$. Hence $(\mathbb{R}^n, \|.\|_p)$ is a normed linear spaces for each $p \in [1, \infty]$.

Inner products spaces are very important sources of normed linear spaces.

Definition 1.3. Let X be a linear space over the field $\mathbb{F}$. If the function $(.,.) : X \times X \mapsto \mathbb{F}$ satisfies the following properties

1. $(x, x) \ge 0$ for all $x \in X$ and $(x, x) = 0$ if and only if $x = 0$;
2. $(x, y) = \overline{(y, x)}$ for every $x, y \in X$;
3. $(x + y, z) = (x, z) + (y, z)$ for every $x, y, z \in X$; and
4. $(\alpha x, y) = \alpha(x, y)$ for every $\alpha \in \mathbb{F}$ and for every $x, y \in X$;

then the function $(.,.)$ is called an inner product on X and the pair $(X, (.,.))$ is called a inner product space over the field $\mathbb{F}$. The number $\|x\| = \sqrt{(x, x)}$ is called the norm of the vector $x \in X$. If $x, y \in X$ and $(x, y) = 0$ then the vectors x and y are called orthogonal vectors.

The inner product generates the most important inequality in mathematics, namely the Cauchy-Schwarz inequality.

Theorem 1.4. [Cauchy-Schwarz inequality][2, 4] Let $(X, (.,.))$ be an inner product space over the field $\mathbb{F}$. Then for every $x, y \in X$, $|(x, y)| \le \sqrt{(x, x)}\sqrt{(y, y)} = \|x\| \, \|y\|$. The equality occurs if and only if the vectors x and y are linearly dependent.

From the Cauchy-Schwarz inequality it follows that the function $\|x\| = \sqrt{(x, x)}$ is a norm on the space X. This norm is called the norm generated by the inner product function $(.,.)$. If the normed linear space $(X, \|.\|))$ is a Banach space, that is, if every Cauchy sequence in X converges to a point in X, the inner product space $(X, (.,.))$ is called a Hilbert space.

Example 1.5. For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ the function $(.,.) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ defined by $(x, y) = \sum_{j=1}^n x_j y_j$ is an inner product on the space $\mathbb{R}^n$. The norm generated by this inner product is the Euclidean norm $\|x\|_2 = \sqrt{(x, x)} = \left(\sum_{j=1}^n x_j^2\right)^{1/2}$. The inner product space $(\mathbb{R}^n, \|.\|)$ is a Hilbert space.

On normed linear spaces the primary objects of study are the linear operators and linear functionals which play central role in functional analysis.

Definition 1.6. Let $(X, \|.\|)$ and $(Y, \|.\|')$ be normed linear spaces over the same field $\mathbb{F}$ and $T : X \mapsto Y$ be a mapping. If $T(\alpha x + \beta y) = \alpha T(x) + \beta T(Y)$ for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$ then T is called a linear operator. A linear operator T is called bounded if there is a real constant $M > 0$ such that $\|T(x)\|' \le M \|x\|$ for all $x \in X$. If T is a bounded linear operator the number $\|T\|_{op} = \|T\| =$

$\inf\left\{M : \|T(x)\|' \leq M\|x\| \text{ for all } x \in X\right\}$ is called a operator norm of T. The equivalent definition of operator norm is given by the formulas

$$\begin{aligned}\|T\|_{op} = \|T\| &= \sup\left\{\frac{\|T(x)\|'}{\|x\|} : \|x\| \neq 0\right\} = \sup\left\{\|T(x)\|' : \|x\| \leq 1\right\}\\ &= \sup\left\{\|T(x)\|' : \|x\| = 1\right\}\end{aligned}$$

The following is a very useful result for the computation of operator norms of linear operators.

Lemma 1.7. [Computation of operator norm] Let $(X, \|.\|)$ and $(Y, \|.\|')$ be normed linear spaces over the same field $\mathbb{F}$, $T : X \mapsto Y$ be a bounded linear operator and $M \geq 0$ be a real constant. If for every $x \in X$, $\|T(x)\|' \leq M\|x\|$ and $\|T(x_0)\|' = M\|x_0\|$ for a vector $x_0 \in X \setminus \{0\}$ then the operator norm of T is $\|T\|_{op} = M$.

Proof. If for every $x \in X$, $\|T(x)\|' \leq M\|x\|$ then by the definition of operator norm $\|T\|_{op} \leq M$. On the other hand if for a vector $x_0 \in X \setminus \{0\}$, $\|T(x_0)\|' = M\|x_0\|$ then by the definition of operator norm we have $M\|x_0\| = \|T(x_0)\|' \leq \|T\|_{op}\|x_0\|$ so that $M \leq \|T\|_{op}$. Therefore $\|T\|_{op} = M$.

Remark 1.8. We note that in the finite dimensional case the operator norm can be computed by the method of Lagrange multipliers with constraints.

It is now a classical result that a linear operator is bounded if and only if it is continuous. The set $\mathcal{B}(X, Y)$ of bounded linear operators is a linear space over $\mathbb{F}$ with pointwise addition and scaler multiplication and $\|.\|_{op}$ is a norm on $\mathcal{B}(X, Y)$. If $(Y, \|.\|')$ is a Banach space then the space $(\mathcal{B}(X, Y), \|.\|_{op})$ is a Banach space.

A bounded linear operator $f : (X, \|.\|) \mapsto (\mathbb{F}, |.|)$ is called a **bounded linear functional**. The Banach space $(X^*, \|.\|_{op})$ of bounded linear functional is called dual or conjugate space of the normed linear space $(X, \|.\|)$.

Example 1.9. Let $(X, (.,.))$ be an inner product space over the field $\mathbb{F}$ and $a \in X$ be a fixed vector. Then $f : (X, \|.\|) \mapsto (\mathbb{F}, |.|), f(x) = (x, a)$ is a bounded linear functional and $\|f\|_{op} = \|a\|$.

Linear functionals are important in terms of generating and characterizing linear subspaces. If $(X, \|.\|)$ is a normed linear space over the field $\mathbb{F}$ and $\ell : X \mapsto \mathbb{F}$ is a linear functional then the the kernel or the null space $\ker(\ell) = \{x \in X : \ell(x) = 0\}$ of the linear functional ℓ is a linear subspace of X. It is well-known that a linear functional is continuous if and only if its null space is closed. On the other hand we have the following simple result which shows the relations between linear subspaces and linear functionals.

We recall that a linear subspace W of a linear space X is called a codimension one linear subspace if the dimension of the quotient space $X \setminus W$ is $\dim(X \setminus W) = 1$.

Lemma 1.10. Let $(X, \|.\|)$ be normed linear space over the field $\mathbb{F}$. Then W is a codimension one linear subspace of the space X if and only if there is a linear functional $\ell : X \mapsto \mathbb{F}$ such that $W = \ker(\ell)$.

Proof. Since the null space of a linear operator is a linear subspace if $W = \ker \ell$ for a linear functional $\ell : X \mapsto \mathbb{F}$, then W is a linear subspace of the space X. Conversely we assume that W is a codimension one linear subspace of the space X. Let $x_0 \in X \setminus W$ be arbitrary and $M = \{\alpha x_0 : \alpha \in \mathbb{F}\}$ be the linear subspace of X generated by the vector x_0. Then $X = W \oplus M$. Since each $x \in X$ has a unique representation of the form $x = w_x + \alpha_x x_0$ where $w_x \in W$ and $\alpha_x x_0 \in M$ the function $\ell : X \mapsto \mathbb{F}, \ell(x) = \ell(w_x + \alpha_x x_0) = \alpha_x$ is a linear functional with $\ker(\ell) = W$.

There are two fundamental results about bounded linear functionals, namely the Hahn-Banach theorem and the Riesz representation theorem. The Hahn-Banach theorem, one of the indispensable tools of modern analysis, play the central role in the investigation of geometric and analytic properties of bounded linear functionals. The Riesz representation theorem completely characterizes the bounded linear functionals on certain normed linear spaces. We state a version of each of these theorems that we need in what follows.

Theorem 1.11. [Hahn-Banach][2, 1, 7, 3] Let $(X, \|.\|)$ be normed linear space over the field $\mathbb{F}$ and W be a linear subspace of X. If $f : (W, \|.\|) \mapsto (\mathbb{F}, |.|)$ is a bounded linear functional then there is a bounded linear functional $F : (X, \|.\|) \mapsto (\mathbb{F}, |.|)$ such that $F|_W = f$, that is for all $x \in W, F(x) = f(x)$ and $\|F\|_{op} = \|f\|_{op}$.

Remark 1.12. The linear functional F is called a norm preserving linear functional extension of the linear functional f. The important and the difficult part of the theorem is to get the norm preserving linear extension. Otherwise it is well-known that there are many linear extensions of f easy to construct.

Theorem 1.13. [Riesz representation theorem] [2, 5, 3, 7] Let $(X, (.,.))$ be a Hilbert space over the field $\mathbb{F}$ and $\|.\|$ be the norm generated by the inner product. Then a function $f : (X, \|.\|) \mapsto (\mathbb{F}, |.|)$ is a bounded linear functional if and only if there is a unique vector $a \in X$ such that $f(x) = (x, a)$ for all $x \in X$. Furthermore, the operator norm of the linear functional f is $\|f\|_{op} = \|a\|$.

Remark 1.14. By Lemma 1.10 and the Riesz representation theorem in a Hilbert space $(X, (.,.))$ a codimension one linear subspace W of the space X is of the form $W = \{x \in X : \ell_a(x) = (x, a) = 0\}$ where $a \in X$ is a fixed vector.

On finite dimensional normed linear spaces, the Riesz representation theorem provides more concrete information about the structure of linear functionals. In this context, we state a version of the Riesz representation theorem for the finite dimensional spaces and give its proof for the sake of completeness.

Theorem 1.15. [Riesz representation theorem] Let $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then a function $f : (\mathbb{R}^n, \|.\|_p) \mapsto (\mathbb{R}, |.|)$ is a bounded linear functional if and only if there is constant vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $f(x) = (x, a) = a_1x_1 + \cdots + a_nx_n$ for every $x \in \mathbb{R}^n$. Furthermore, the operator norm of f is $\|f\|_{op} = \|a\|_q$.

Proof. We first assume that for a constant vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ $f(x) = (x, a) = a_1x_1 + \cdots + a_nx_n$. Since the inner product is a linear functional with respect to the first variable it follows that f is a linear functional. On the other hand we assume that $f : (\mathbb{R}^n, \|.\|_p) \mapsto (\mathbb{R}, |.|)$ is a linear functional. If $f \equiv 0$ then for the vector $a = 0$, f is the required form $f(x) = a_1x_1 + \cdots + a_nx_n = (x, a)$. Therefore we may assume that $f \neq 0$. For $j = 1, 2, \ldots, n$ let $e_j = (0, \ldots, 0, \underset{j}{1}, 0 \ldots, 0)$. The set $\mathcal{B} = \{e_1, \ldots, e_n\}$ is a standard (Hamel) basis of the space $\mathbb{R}^n$. Hence every vector $x \in \mathbb{R}^n$ has a unique representation of the form $x = \sum_{j=1}^n x_je_j$. For $j = 1, 2, \ldots, n$ let $a_j = f(e_j)$. $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Since f is a linear functional we have $f(x) = \sum_{j=1}^n x_jf(e_j) = \sum_{j=1}^n x_ja_j = a_1x_1 + \cdots + a_nx_n = (x, a)$. So f is the required form.

For $1 \le p < \infty$ and for each $x \in \mathbb{R}^n$ by the Cauchy-Schwarz inequality we have $|f(x)| \le \|a\|_q \|x\|_p$. If $p = \infty$ then $q = 1$ and $|f(x)| \le \|a\|_1 \|x\|_\infty = \|a\|_q \|x\|_p$. Hence by the definition of operator norm $\|f\|_{op} \le \|a\|_q$.

For $1 \le p < \infty$, if $a_j = 0$ we define $x_j(0) = 0$, and if $a_j \neq 0$ we define $x_j(0) = \frac{|a_j|^q}{a_j}$ and let $x(0) = (x_1(0), \ldots, x_n(0))$. Since $\|x(0)\|_p = \|a\|_q^{q/p}$ and $|f(x(0))| = \|a\|_q^q = \|a\|_q^{q(\frac{1}{p}+\frac{1}{q})} = \|a\|_q \|a\|_q^{\frac{q}{p}} = \|a\|_q \|x(0)\|_p$ from the Lemma1.7 it follows that $\|f\|_{op} = \|a\|_q$.

For $p = \infty$, if $a_j \ge 0$ we define $x_j(0) = 1$ and if $a_j < 0$ we define $x_j(0) = -1$ and let $x(0) = (x_1(0), \ldots, x_n(0))$. Since $\|x(0)\|_\infty = 1$ and $|f(x(0))| = \|a\|_1 = \|a\|_1 \|x(0)\|_\infty$ from the Lemma 1.7 it follows that $\|f\|_{op} = \|a\|_1$. Therefore we have $\|f\|_{op} = \|a\|_q$ for all $1 \le p \le \infty$.

2 Operator Norms and Extension of Linear Functionals

The Hahn-Banach theorem states that a bounded linear functional on a linear subspace of a normed linear space can be extended to the whole space without changing its operator norm. On the other hand, the Riesz representation theorem provides formulas both for the linear functional and its operator norm on a Hilbert space. But, as far as I know there is no such a formula for the operator norm of a linear functional defined on a linear subspace of a normed linear space.

By analyzing the orthogonal decomposition theorem and the Riesz representation theorem [5], [7](4.11 Theorem, 4.12 Theorem) we get two methods of the unique norm preserving linear extension of a linear functional defined on a closed linear subspace a Hilbert space. We note and state these methods without proofs.

Lemma 2.1. Let $(X,(.,.))$ be a Hilbert space over the field $\mathbb{F}$, $\|.\|$ be the norm generated by the inner product, W be a closed linear subspace of the space X and $f:(W,\|.\|)\mapsto(\mathbb{F},|.|)$ be a nontrivial bounded linear functional. Let $M=\ker(f)$ be the null space of f and $M^{\perp}=\{x\in W:(x,y)=0 \text{ for all } y\in M\}$ be the orthogonal complement of the space M in W. Choose any vector $x_0\in M^{\perp}\setminus\{0\}$ and let $a=\frac{\overline{f(x_0)}}{\|x_0\|^2}x_0$. Then $f(x)=(x,a)$ for all $x\in W$, $\|f\|_{op}=\|a\|$ and the norm preserving linear extension of the functional f is the linear functional $F:(X,\|.\|)\mapsto(\mathbb{F},|.|), F(x)=(x,a)$.

Lemma 2.2. Let $(X,(.,.))$ be a Hilbert space over the field $\mathbb{F}$, $\|.\|$ be the norm generated by the inner product, W be a closed linear subspace of the space X and $f:(W,\|.\|)\mapsto(\mathbb{F},|.|)$ be a bounded linear functional. Let $p:X\mapsto W$ be the orthogonal projection of the space X onto the space W. Then $F:(X,\|.\|)\mapsto(\mathbb{F},|.|), F(x)=f\circ p(x)=f(p(x))$ is the norm preserving linear functional extension of the functional f.

The applications of these methods, without doubt, requires certain amount of work. In the case of a bounded linear functional defined on a codimension one subspace of a Hilbert space we provide simple formula both for the operator norm and for the norm preserving linear functional extension.

Theorem 2.3. [Formula for the operator norm and linear extension] Let $(X,(.,.))$ be a Hilbert space over the field $\mathbb{F}$, $\|.\|$ be the norm generated by the inner product, $a,b\in X\setminus\{0\}$ be fixed vectors, $W=\{x\in X:\ell_b(x)=(x,b)=0\}$ be a linear subspace of the space X and $f_a:(W,\|.\|)\mapsto(\mathbb{F},|.|)$, $f_a(x)=(x,a)$ be a linear functional. Then the operator norm of the linear functional f_a is $\|f_a\|_{op}=\left\|a-\frac{(a,b)}{\|b\|^2}b\right\|=\frac{1}{\|b\|}\sqrt{\|a\|^2\|b\|^2-|(a,b)|^2}$ and the norm preserving extension of the linear functional f_a is the linear functional $F:(X,\|.\|)\mapsto(\mathbb{F},|.|), F(x)=f_a(x)-\frac{(a,b)}{\|b\|^2}\ell_b(x)=\left(x,a-\frac{(a,b)}{\|b\|^2}b\right)$.

Proof. Since for each $x\in W$, $\ell_b(x)=0$ we have $F(x)=f_a(x)$. So the function F is an extension of the function f_a. By the Riesz representation theorems F is a linear functional on the space X and its operator norm is $\|F\|_{op}=\left\|a-\frac{(a,b)}{\|b\|^2}b\right\|$. Since by the properties of the inner product

$$\|F\|_{op}=\left\|a-\frac{(a,b)}{\|b\|^2}b\right\|=\sqrt{\left(a-\frac{(a,b)}{\|b\|^2}b,a-\frac{(a,b)}{\|b\|^2}b\right)}$$

$$=\sqrt{\|a\|^2-\frac{2|(a,b)|^2}{\|b\|^2}+\frac{|(a,b)|^2\|b\|^2}{\|b\|^4}}$$

$$=\sqrt{\|a\|^2-\frac{|(a,b)|^2}{\|b\|^2}}=\frac{1}{\|b\|}\sqrt{\|a\|^2\|b\|^2-|(a,b)|^2}$$

it suffices to show that $\|f_a\|_{op} = \left\| a - \frac{(a,b)}{\|b\|^2} b \right\|$. Since for $x \in W$, $\ell_b(x) = 0$ by the Cauchy-Schwarz inequality we have

$$|f_a(x)| = \left| f_a(x) - \frac{(a,b)}{\|b\|^2} \ell_b(x) \right| = \left| (x,a) - \frac{(a,b)}{\|b\|^2}(x,b) \right|$$
$$= \left| \left(x, a - \frac{(a,b)}{\|b\|^2} b \right) \right| \leq \left\| a - \frac{(a,b)}{\|b\|^2} b \right\| \|x\| .$$

By the definition of operator norm $\|f_a\|_{op} \leq \left\| a - \frac{(a,b)}{\|b\|^2} b \right\|$. Since the equality holds in Cauchy-Schwarz inequality when $x = a - \frac{(a,b)}{\|b\|^2} b \in W$ and $\left| f_a \left(a - \frac{(a,b)}{\|b\|^2} b \right) \right| = \left\| a - \frac{(a,b)}{\|b\|^2} b \right\|^2 = \left\| a - \frac{(a,b)}{\|b\|^2} b \right\| \|x\|$ it follows from the Lemma 1.7 that

$$\|f_a\|_{op} = \left\| a - \frac{(a,b)}{\|b\|^2} b \right\| = \frac{1}{\|b\|} \sqrt{\|a\|^2 \|b\|^2 - |(a,b)|^2}.$$

Remark 2.4. Since $a = \frac{(a,b)}{\|b\|^2} b + a - \frac{(a,b)}{\|b\|^2} b$ and $\left(a - \frac{(a,b)}{\|b\|^2} b, b \right) = 0$, the vector $a - \frac{(a,b)}{\|b\|^2} b$ is the component of the vector a orthogonal to the vector b. This observation gives the following results.

Corollary 2.5. In Theorem 2.3, if the vectors a and b are orthogonal, that is $(a,b) = 0$ then the operator norm of the linear functional f_a is $\|f_a\|_{op} = \|a\|$ and its norm preserving extension is the linear functional $F : (X, \|.\|) \mapsto (\mathbb{F}, |.|), F(x) = f_a(x)$.

Corollary 2.6. In Theorem 2.3., if the vectors a and b are collinear, that is $b = ta$ for a scaler $t \in \mathbb{F}$ then $f_a \equiv 0$, its operator norm $\|f_a\|_{op} = 0$ and its norm preserving extension is the linear functional $F : (X, \|.\|) \mapsto (\mathbb{F}, |.|), F(x) = 0$.

In the following result we assume that $\dim(\mathbb{R}^n) = n \geq 2$.

Corollary 2.7. Let $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \setminus \{0\}$ be fixed vectors, $W = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \ell_b(x) = (x,b) = b_1x_1 + b_2x_2 + \cdots + b_nx_n = 0\}$ be a linear subspace and $f_a : (W, \|.\|_2) \mapsto (\mathbb{R}, |.|)$, $f_a(x) = (x,a) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ be a linear functional. Then the operator norm of the linear functional f_a is $\|f_a\|_{op} = \sqrt{\|a\|_2^2 - \frac{(a,b)^2}{\|b\|_2^2}} = \frac{1}{\|b\|_2} \sqrt{\|a\|_2^2 \|b\|_2^2 - (a,b)^2}$ and its norm preserving extension is the linear functional $F : (\mathbb{R}^n, \|.\|_2) \mapsto (\mathbb{R}, |.|), F(x) = f_a(x) - \frac{(a,b)}{\|b\|_2^2} \ell_b(x) = \left(x, a - \frac{(a,b)}{\|b\|_2^2} b \right)$.

Remark 2.8. Since the computation of an operator norm is an extremum value problem we note that Corollary 2.7 may be used to solve certain type of extremum value problems.

Example 2.9. Let $W = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$ be a linear subspace of the space $\mathbb{R}^3$. Find the operator norm of the linear functional $f : (W, \|.\|_2) \mapsto (\mathbb{R}, |.|), f(x) = 2x_1 + 3x_3$ and its norm preserving linear functional extension $F : (\mathbb{R}^3, \|.\|_2) \mapsto (\mathbb{R}, |.|)$.

Solution. For the vectors $b = (b_1, b_2, b_3) = (1,1,1)$, $a = (a_1, a_2, a_3) = (2,0,3) \in \mathbb{R}^3$ we have $W = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ell_b(x) = (x,b) = x_1 + x_2 + x_3 = 0\}$ and $f(x) = f_a(x) = (x,a) = 2x_1 + 3x_3$. Since $(a,b) = 5$, $\|a\|_2 = \sqrt{13}$ and $\|b\|_2 = \sqrt{3}$ by the Corollary 2.7 the operator norm of the linear functional f is $\|f\|_{op} = \frac{1}{\sqrt{3}} \sqrt{39 - 25} = \sqrt{\frac{14}{3}} = \frac{\sqrt{42}}{3}$ and its norm preserving linear functional extension is $F : (\mathbb{R}^3, \|.\|_2) \mapsto (\mathbb{R}, |.|)$ $F(x) = f_a(x) - \frac{(a,b)}{\|b\|_2^2} \ell_b(x) = 2x_1 + 3x_3 - \frac{5}{3}(x_1 + x_2 + x_3) = \frac{1}{3}(x_1 - 5x_2 + 4x_3)$.

We use Lemma 2.1 and Lemma 2.2 to give alternative solutions of this example.

Alternative solution. $(W, \|.\|_2)$ is a Hilbert space. The kernel or the null space of the linear functional f is the linear subspace $M = \ker(f) = \left\{\alpha\left(1, -\frac{1}{3}, -\frac{2}{3}\right) : \alpha \in \mathbb{R}\right\}$ and its orthogonal complement in W is the linear subspace $M^{\perp} = \{\alpha(1, -5, 4) : \alpha \in \mathbb{R}\}$. Choose $x_0 = (1, -5, 4)$. and let $a = \frac{\overline{f(x_0)}}{\|x_0\|_2^2} x_0 = \frac{1}{3}(1, -5, 4)$. Then by Lemma 2.1 we have $f(x) = (x, a) = \frac{1}{3}(x_1 - 5x_2 + 4x_3)$ for all $x \in W$. By the Riesz representation theorem $\|f\|_{op} = \|a\|_2 = \frac{\sqrt{42}}{3}$ and by the uniqueness of extension the norm preserving linear extension of the linear functional f is the linear functional $F : (\mathbb{R}^3, \|.\|_2) \mapsto (\mathbb{R}, |.|)$, $F(x) = (x, a) = \frac{1}{3}(x_1 - 5x_2 + 4x_3)$.

Alternative solution 2. Since the space W is the kernel of the linear functional $\ell : \mathbb{R}^3 \mapsto \mathbb{R}, \ell(x) = x_1 + x_2 + x_3$ it is a closed codimension one linear subspace of $\mathbb{R}^3$ and hence $\dim W = 2$. The orthogonal projection of the space $\mathbb{R}^3$ onto the space W is the bounded linear operator $p : \mathbb{R}^3 \mapsto W$, $p(x) = \left(\frac{2x_1 - x_2 - x_3}{3}, \frac{2x_2 - x_1 - x_3}{3}, \frac{2x_3 - x_2 - x_1}{3}\right)$. So by Lemma 2.2 the norm preserving linear extension of the linear functional f is the linear functional $F : (\mathbb{R}^3, \|.\|_2) \mapsto (\mathbb{R}, |.|)$,

$$
\begin{aligned}
F(x) =& f(p(x)) = f\left(\frac{2x_1 - x_2 - x_3}{3}, \frac{2x_2 - x_1 - x_3}{3}, \frac{2x_3 - x_2 - x_1}{3}\right) \\
=& 2\left(\frac{2x_1 - x_2 - x_3}{3}\right) + 3\left(\frac{2x_3 - x_2 - x_1}{3}\right) \\
=& \frac{1}{3}(x_1 - 5x_2 + 4x_3) = ((x_1, x_2, x_3), \frac{1}{3}(1, -5, 4)).
\end{aligned}
$$

By the Riesz representation theorem $\|f\|_{op} = \|F\|_{op} = \left\|\frac{1}{3}(1, -5, 4)\right\|_2 = \frac{\sqrt{42}}{3}$.

We give a different solution of this example which is also important in terms of the method used.

Alternative solution 3. By the definition of operator norm combined probably with the method of Lagrange multipliers we have

$$
\begin{aligned}
\|f\|_{op} =& \sup\{|f(x)| : x = (x_1, x_2, x_3) \in W, \|x\|_2 = 1\} \\
=& \sup\left\{|2x_1 + 3x_3| : x = (x_1, x_2, x_3) \in \mathbb{R}^3, x_1 + x_2 + x_3 = 0, \|x\|_2 = 1\right\} \\
=& \sup\{2x_1 + 3x_3 : x_1, x_3 \geq 0, x_1 + x_2 + x_3 = 0, \|x\|_2 = 1\} = \frac{\sqrt{42}}{3}.
\end{aligned}
$$

By the Hahn-Banach theorem there is at least one norm preserving linear functional extension F of f to the space $(\mathbb{R}^3, \|.\|_2)$. By the Riesz representation theorem this extension is of the form $F(x) = (x, a) = a_1x_1 + a_2x_2 + a_3x_3$ where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ is a constant vector and $\|F\|_{op} = \|a\|_2$. For $x \in W$ by solving the linear extension equality $2x_1 + 3x_3 = f(x) = F(x) = a_1x_1 + a_2x_2 + a_3x_3$ and the operator norm equality $\frac{\sqrt{42}}{3} = \|f\|_{op} = \|F\|_{op} = \|a\|_2 = \sqrt{a_1^2 + a_2^2 + a_3^2}$ simultaneously we get $a_1 = \frac{1}{3}, a_2 = \frac{-5}{3}$ and $a_3 = \frac{4}{3}$. Therefore the unique norm preserving linear extension of the linear functional f is the linear functional $F(x) = \frac{1}{3}(x_1 - 5x_2 + 4x_3)$.

Example 2.10. Let $X = \mathcal{P}_3(\mathbb{R})$ be the linear space of all real polynomial functions of degree at most 3. The function $(.,.) : X \times X \mapsto \mathbb{R}$ defined by $(p, q) = \int_{-1}^{1} p(x)q(x)dx$ is an inner product on X and it generates the norm $\|p\| = \sqrt{(p, p)} = \left(\int_{-1}^{1}(p(x))^2dx\right)^{1/2}$. The inner product space $(X, (.,.))$ is a Hilbert space. Let $W = \{p \in X : \int_{-1}^{1}(p(x) + xp(x))dx = 0\}$ and $\ell : (W, \|.\|) \mapsto (\mathbb{R}, |.|), \ell(p) = \int_{-1}^{1}(p(x) + x^2p(x))dx$. Show that W is a linear subspace of X and ℓ is bounded linear functional. Find the operator norm of the functional ℓ and its norm preserving linear extension to the space X.

Solution. For the polynomial functions $a : \mathbb{R} \mapsto \mathbb{R}, a(x) = 1 + x^2$ and $b : \mathbb{R} \mapsto \mathbb{R}, b(x) = 1 + x$ we have $W = \{p \in X : \int_{-1}^{1}(p(x) + xp(x))dx = 0\} = \{p \in X : (p, b) = 0\}$ and $\ell(p) = \int_{-1}^{1}(p(x) + x^2p(x))dx = \int_{-1}^{1}(1 + x^2)p(x)dx = (p, a)$. Since the inner product is a bounded linear functional with respect to the first variable it follows that W is a closed codimension one linear subspace of the space X and ℓ is a bounded linear functional. Since

$$\begin{aligned}\|b\| &= \left(\int_{-1}^{1}(b(x))^2dx\right)^{1/2} = \left(\int_{-1}^{1}(1+x)^2dx\right)^{1/2}\\ &= \left(\int_{0}^{2}t^2dt\right)^{1/2} = \sqrt{\frac{t^3}{3}\,|_0^2} = \frac{2\sqrt{6}}{3} \quad \text{and}\\ (a,b) &= \int_{-1}^{1}a(x)b(x)dx = \int_{-1}^{1}(1+x+x^2+x^3)dx\\ &= \left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}\right)|_{-1}^{1} = 2+\frac{2}{3} = \frac{8}{3},\end{aligned}$$

by the Theorem 2.3 the operator norm of the linear functional ℓ is

$$\begin{aligned}\|\ell\|_{op} &= \left\|a-\frac{(a,b)}{\|b\|^2}b\right\| = \|a-b\| = \left(\int_{-1}^{1}(a(x)-b(x))^2dx\right)^{1/2}\\ &= \left(\int_{-1}^{1}(x^2-x)^2dx\right)^{1/2} = \left(\int_{-1}^{1}(x^4-2x^3+x^2)dx\right)^{1/2}\\ &= \sqrt{\left(\frac{x^5}{5}-\frac{2x^4}{4}+\frac{x^3}{3}\right)|_{-1}^{1}} = \sqrt{\frac{2}{5}+\frac{2}{3}} = \frac{4\sqrt{15}}{15}\end{aligned}$$

and its norm preserving linear functional extension is the linear functional $L : (X, \|.\|) \mapsto (\mathbb{R}, |.|)$ defined by

$$\begin{aligned}L(p) &= \left(p, a-\frac{(a,b)}{\|b\|^2}b\right) = (p, a-b)\\ &= \int_{-1}^{1}(a(x)-b(x))p(x)dx = \int_{-1}^{1}(x^2-x)p(x)dx.\end{aligned}$$

The following example shows that our formula works not just for finite dimensional Hilbert spaces but also works for infinite dimensional Hilbert spaces.

Example 2.11. Let $X = L_2([0,1]) = \{f : f : [0,1] \mapsto \mathbb{R},$ Lebesque measurable and $\|f\|_2 = \left(\int_0^1(f(x))^2dx\right)^{1/2}\}$ be the linear space of Lebesque square integrable functions. The function $(.,.) : X \times X \mapsto \mathbb{R}$ defined by $(f,g) = \int_0^1 f(x)g(x)dx$ is an inner product on X and it generates the norm $\|f\| = \|f\|_2$. Let $W = \{f \in X : \int_0^1 f(x)dx = 0\}$ and $\ell : (W, \|.\|) \mapsto (\mathbb{R}, |.|), \ell(f) = \int_0^1 x^2 f(x)dx$. Show that W is a linear subspace of X and ℓ is bounded linear functional on W. Find the operator norm of the functional ℓ and its norm preserving linear extension to the space X.

Solution. For the functions $a : [0,1] \mapsto \mathbb{R}, a(x) = x^2$ and $b : [0,1] \mapsto \mathbb{R}, b(x) = 1$ we have $W = \{f \in X : \int_0^1 f(x)dx = 0\} = \{f \in X : (f,b) = 0\}$ and $\ell(f) = \int_0^1 x^2 f(x)dx = (f,a)$. Since the inner product is a bounded linear functional with respect to the first variable it follows that W is a codimension one linear subspace of the space X and ℓ is a bounded linear functional. Since $\|b\| = \left(\int_0^1(b(x))^2dx\right)^{1/2} = \left(\int_0^1 1^2dx\right)^{1/2} = \sqrt{x\,|_0^1} = 1$ and $(a,b) = \int_0^1 a(x)b(x)dx = \int_0^1 x^2dx = \frac{x^3}{3}\,|_0^1 = \frac{1}{3}$ by the Theorem 2.3 the operator norm of the linear functional ℓ is

$$\begin{aligned}
\|\ell\|_{op} &= \left\|a - \frac{(a,b)}{\|b\|^2}b\right\| = \left\|a - \frac{b}{3}\right\| = \left(\int_0^1 \left(a(x) - \frac{b(x)}{3}\right)^2 dx\right)^{1/2} \\
&= \left(\int_0^1 \left(x^2 - \frac{1}{3}\right)^2 dx\right)^{1/2} = \left(\int_0^1 \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx\right)^{1/2} \\
&= \sqrt{\left(\frac{x^5}{5} - \frac{2}{9}x^3 + \frac{1}{9}x\right) |_0^1} = \sqrt{\left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right) - 0} \\
&= \sqrt{\frac{9-5}{9 \cdot 5}} = \sqrt{\frac{4}{9 \cdot 5}} = \frac{2}{3\sqrt{5}} = \frac{2\sqrt{5}}{15}
\end{aligned}$$

and its norm preserving linear functional extension is the linear functional $L : (X, \|.\|) \mapsto (\mathbb{R}, |.|)$, $L(p) = \left(f, a - \frac{(a,b)}{\|b\|^2}b\right) = \left(p, a - \frac{b}{3}\right) = \int_0^1 (a(x) - b(x))f(x)dx = \int_0^1 \left(x^2 - \frac{1}{3}\right) f(x)dx$.

We end the paper with the following question.

Question. Can we remove the codimension one hypothesis in Theorem 2.3? Is it possible to generalize these results to normed linear spaces under some smoothness conditions.

References

[1] B. Beauzamı, *Introduction to Banach Spaces and Their Geometry*, North Holland Publishing Com. Inc., Amsterdam, 1982.

[2] Béla Bollobás, *Linear Anaysis*, Cambridge Univ. Press, Cambridge, 1990.

[3] J. B. Conway, *A Course in Functional Analysis*, second ed., Springer-Verlag, New York, 1990.

[4] A. N. Kolmogorov & S. V. Fomin, *Introductory Real Analysis*, Dover Publications, New York, 1970.

[5] R. Larsen, *Functional anaysis*, M. Dekker, New York, 1973.

[6] L. Narici, E. Beckenstein, *The Hahn-Banach theorem: the lifes and times*, Topology and its Applications 77(1997), 193-211

[7] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Company, New York, 1987.

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MEASURE OF NONCOMPACTNESS FOR NONLINEAR HILFER FRACTIONAL DIFFERENTIAL EQUATION IN BANACH SPACES

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Abstract

This paper deals with nonlinear fractional differential equation with boundary value problem conditions. We investigate the existence of solutions in Banach spaces with Hilfer derivative. To obtain such result we apply Mönch's fixed point theorem and the technique of measures of noncompactness. At the end an example is given.

Keywords: Fractional differential equation; Hilfer fractional derivative; Kuratowski measures of noncompactness; Mönch fixed point theorems; Banach space.

MSC 2010: 26A33; 34B25; 34B15.

1 Introduction

In recent years, several papers have been devoted to the study of the existence of solutions for fractional differential equations, among others we refer the readers to the following references: Agarwal et al. [5, 4], Abbas et al. [3, 2], Sandeep et al. [32], Furati et al.[20] , Benchohra et al. [17, 18], Gu et al. [21]. Moreover, it has been proved that differential models involving derivatives of fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in many fields, for instance, about physics, control theory, rheology, chemistry, and so on (see the monograph of Kilbas and al. [25], Hilfer and al. [22, 23], and Samko and al. [30]).

In this paper we focus on the existence of solutions of the following boundary value problem for a nonlinear fractional differential equation,

$$D^{\alpha,\beta}_{a^+}y(t) = f(t, y(t)), t \in J := [0, T]. \tag{1.1}$$

with the fractional boundary conditions

$$\begin{aligned} &I^{1-\gamma}y(0) = y_0, \ \ I^{3-\gamma-2\beta}y'(0) = y_1, \\ &I^{1-\gamma}y(\eta) = \lambda(I^{1-\gamma}y(T)), \gamma = \alpha + \beta - \alpha\beta. \end{aligned} \tag{1.2}$$

where $D^{\alpha,\beta}_{0^+}$ is the Hilfer fractional derivative, $0 < \alpha < 1$,$0 \le \beta \le 1$, $0 < \lambda < 1$, $0 < \eta < T$ and let E be a Banach space space with norm $\|.\|$, $f : J \times E \times E \times E \times E \to E$ is given continuous function and satisfying some assumptions that will be specified later. We will use the technique of measures of noncompactness. which is often used in several branches of nonlinear analysis. Especially , that technique turns out to be a very useful tool in existence for several types of integral equations; details are found in Akhmerov et al. [7], Alvàrez [8], Banaŝ et al. [10, 11, 12, 13, 14, 15, 16], Benchohra et al. [17, 18], Mönch [27], Szufla [31].

The main idea used here is that on the Banach space E, we can not use Ascoli-Arzela theorem to prove the compactness of the operator, so we use the technique of measure of nocompactness to conclude.

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative [2, 3, 19], and other problems with Hilfer-Hadamard fractional derivative; see [1, 2, 33, 34]. Many existence results were established by the use of technics of nonlinear analysis such as Banach fixed point theorem, Schaefer's fixed point theorem, Lerayâ-Schauder nonlinear alternative, etc ..., and the technique of measures of noncompactness, see [4, 5, 6, 18, 15, 16].

In 2008, Benchohra et al. [17], considered the existence of solutions of an initial value problem for a nonlinear fractional differential equation

$$\begin{cases} D^r y(t) = f(t, y), & \text{for each } t \in J = [0, T], 1 < r < 2 \\ y(0) = y_0, y'(0) = y_1, & . \end{cases} \tag{1.3}$$

where D^r is the Caputo fractional derivative, $f : J \times E \to E$ is a given function, and E is a Banach space. They obtained results for solutions by using Mönch's fixed point theorem and the technique of measures of noncompactness.

In 2018, S. Abbas et al. [2], studied the existence of solutions for the following coupled system of Hilfer fractional differential equations

$$\begin{cases} D_0^{\alpha_1,\beta_1} u(t) = f_1(t, u(t), v(t)), & t \in J = [0, T] \\ D_0^{\alpha_2,\beta_2} v(t) = f_2(t, u(t), v(t)), \end{cases} \tag{1.4}$$

with the following initial conditions

$$\begin{cases} I_0^{1-\gamma_1} u(0) = \phi_1 \\ I_0^{1-\gamma_2} v(0) = \phi_2, \end{cases} \tag{1.5}$$

where $T > 0$, $\alpha_i \in (0, 1)$, $\beta_i \in [0, 1]$, $\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, $\phi_i \in E$, $f_i : I \times E \times E \to E$; $i = 1, 2$, are given functions, E is a real (or complex) Banach space with a norm $\|.\|$, $I_0^{1\gamma_i}$ is the left- sided mixed Riemann-Liouville integral of order $1 - \gamma_i$, and $D_0^{\alpha_i,\beta_i}$ is the generalized Riemann-Liouville derivative (Hilfer) operator of order α_i and type β_i: $i = 1, 2$. They obtained results for solutions by using the technique of measure of noncompactness and the fixed point theory.

In 2018, D.Vivek et al. [34], studied the existence, uniqueness and stability analysis of Hilfer-Hadamard type fractional neutral pantograph equations with boundary conditions of the form

$$\begin{cases} D_{1^+}^{\alpha,\beta} x(t) = f(t, x(t), x(\lambda t), D_{1^+}^{\alpha,\beta} x(\lambda t)), & t \in J = [0, T]. \\ I_{1^+}^{1-\gamma} x(1) = a, I_{1^+}^{1-\gamma} x(T) = b, & \gamma = \alpha + \beta - \alpha\beta. \end{cases} \tag{1.6}$$

where $D_{1^+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative, $0 < \alpha < 1$, $0 \le \beta \le 1$, $0 < \lambda < 1$. Let E be a Banach space, $f : J \times E \times E \times E \to E$ is a given continuous function. They obtained results for solutions by using Schaefer's fixed point theorem.

The principal goal of this paper is to prove the existence of solutions for the problem (1.1)-(1.2) using Mönch's fixed point theorem and its related Kuratowski measure of noncompactness.

2 Preliminaires

In what follows we introduce definitions, notations, and preliminary facts which are used in the sequel.

For more details, we refer to [4, 5, 7, 9, 11, 19, 20, 21, 22, 23, 24, 25, 26, 31, 32].
Denote by $C(J,E)$ the Banach space of continuous functions $y : J \to E$, with the usual supremum norm

$$\|y\|_\infty = \sup\{\|y(t)\|, t \in J\}.$$

Let $L^1(J,E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_J y(t)dt.$$

$AC^1(J,E)$ denotes the space of functions $y : J \to E$, whose first derivative is absolutely continuous.

Definition 2.1. [20] Let $J = [0,T]$ be a finite interval and γ as a real such that $0 \le \gamma < 1$. We introduce the weighted space $C_{1-\gamma}(J,E)$ of continuous functions f on $(0,T]$ as

$$C_{1-\gamma}(J,E) = \{f : (0,T] \to E : (t-a)^{1-\gamma} f(t) \in C(J,E)\}.$$

In the space $C_{1-\gamma}(J,E)$, we define the norm

$$\|f\|_{C_{1-\gamma}} = \|(t-a)^{1-\gamma} f(t)\|_C, C_0(J,E) = C(J,E).$$

Definition 2.2. [20] Let $0 < \alpha < 1, 0 \le \beta \le 1$, the weighted space $C^{\alpha,\beta}_{1-\gamma}(J,E)$ is defined by

$$C^{\alpha,\beta}_{1-\gamma}(J,E) = \{f : (0,T] \to \mathbb{R} : D^{\alpha,\beta}_{0^+} f \in C_{1-\gamma}(J,E)\}, \gamma = \alpha + \beta - \alpha\beta$$

and

$$C^1_{1-\gamma}(J,E) = \{f : (0,T] \to \mathbb{R} : f' \in C_{1-\gamma}(J,E)\}, \gamma = \alpha + \beta - \alpha\beta$$

with the norm

$$\|f\|_{C^1_{1-\gamma}} = \|f\|_C + \|f'\|_{C_{1-\gamma}}. \tag{2.1}$$

One have, see [20], $D^{\alpha,\beta}_{0^+} f = I^{\beta(1-\alpha)}_{0^+} D^{\gamma}_{0^+} f$ and $C^{\gamma}_{1-\gamma}(J,E) \subset C^{\alpha,\beta}_{1-\gamma}(J,E), \gamma = \alpha + \beta - \alpha\beta, 0 < \alpha < 1, 0 \le \beta \le 1$. Moreover, $C_{1-\gamma}(J,E)$ is complete metric space of all continuous functions mapping J into E with the metric d defined by

$$d(y_1, y_2) = \|y_1 - y_2\|_{C_{1-\gamma}(J,E)} := \max_{t \in J} |(t-a)^{1-\gamma}[y_1(t) - y_2(t)]|$$

for details see [20].

Notation 2.3. For a given set V of functions $v : J \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, t \in J,$$

and

$$V(J) = \{v(t) : v \in V, t \in J\}.$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definion 2.4. ([7, 11]). Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0,\infty]$ defined by

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } diam(B_i) \le \epsilon\}; \text{ here } B \in \Omega_E.$$

This measure of noncompactness satisfies some important properties [7, 11]:
(a) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
(b) $\mu(B) = \mu(\overline{B})$.
(c) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.
(d) $\mu(A + B) \leq \mu(A) + \mu(B)$
(e) $\mu(cB) = |c|\mu(B); c \in \mathbb{R}$.
(f) $\mu(convB) = \mu(B)$.

Now, we give some results and properties of fractional calculus. Definition 2.5. [26] Let $(0, T]$ and $f : (0, \infty) \to \mathbb{R}$ is a real valued continuous function. The Riemann-Liouville fractional integral of a function f of order $\alpha \in \mathbb{R}^+$ is denoted as $I^{\alpha}_{0+} f$ and defined by

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, t > 0. \tag{2.2}$$

where $\Gamma(\alpha)$ is the Euler's Gamma function.

Definion 2.6. [25] Let $(0, T]$ and $f : (0, \infty) \to \mathbb{R}$ is a real valued continuous function. The Riemann-Liouville fractional derivative of a function f of order $\alpha \in \mathbb{R}^+_0 = [0, +\infty)$ is denoted as $D^{\alpha}_{0+} f$ and defined by

$$D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)ds. \tag{2.3}$$

where $n = [\alpha] + 1$, and $[\alpha]$ means the integral part of α, provided the right hand side is pointwise defined on $(0, \infty)$.

Definion 2.7. [25] The Caputo fractional derivative of function f with order $\alpha > 0, n - 1 < \alpha < n, n \in \mathbb{N}$ is defined by

$$^{C}D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, t > 0. \tag{2.4}$$

In [22], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and Caputo derivatives as specific cases (see also [23, 24]).

Definion 2.8. [22] The Hilfer fractional derivative $D^{\alpha,\beta}_{0+}$ of order α $(n - 1 < \alpha < n)$ and type β $(0 \leq \beta \leq 1)$ is defined by

$$D^{\alpha,\beta}_{0+} = I^{\beta(n-\alpha)}_{0+} D^n I^{(1-\beta)(n-\alpha)}_{0+} f(t) \tag{2.5}$$

where I^{α}_{0+} and D^{α}_{0+} are Riemann-Liouville fractional integral and derivative defined by 2.2 and 2.3, respectively.

Remark 2.9. (See [19]) Hilfer fractional derivative interpolates between the R-L (2.3, if $\beta = 0$) and Caputo (2.4, if $\beta = 1$) fractional derivatives since

$$D^{\alpha,\beta}_{0+} = \begin{cases} DI^{1-\alpha} = D^{\alpha}_{0+}, \beta = 0, \quad I^{1-\alpha}D =^{C} D^{\alpha}_{0+}, \beta = 1, \\ I^{1-\alpha}D =^{C} D^{\alpha}_{0+}, \beta = 1, \end{cases}$$

Lemma 2.10. Let $0 < \alpha < 1, 0 \leq \beta \leq 1, \gamma = \alpha + \beta - \alpha\beta$, and $f \in L^1(J, E)$.
The operator $D^{\alpha,\beta}_{0+}$ can be written as

$$\begin{aligned} D^{\alpha,\beta}_{0+} f(t) &= \left(I^{\beta(1-\alpha)}_{0+} \frac{d}{dt} I^{(1-\gamma)}_{0+} f \right)(t) \\ &= I^{\beta(1-\alpha)}_{0+} D^{\gamma} f(t), \quad t \in J. \end{aligned}$$

Moreover, the parameter γ satisfies

$$0 < \gamma \le 1, \gamma \ge \alpha, \gamma > \beta, 1 - \gamma < 1 - \beta(1-\alpha).$$

Lemma 2.11. Let $0 < \alpha < 1, 0 \le \beta \le 1, \gamma = \alpha + \beta - \alpha\beta$, If $D_{0^+}^{\beta(1-\alpha)} f$ exists and in $L^1(J, E)$, then

$$D_{0^+}^{\alpha,\beta} I_{0^+}^{\alpha} f(t) = I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\beta(1-\alpha)} f(t), \text{ for a.e. } t \in J.$$

Furthermore, if $f \in C_{1-\gamma}(J, E)$ and $I_{0^+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}^1(J, E)$, then

$$D_{0^+}^{\alpha,\beta} I_{0^+}^{\alpha} f(t) = f(t), \text{ for a.e. } t \in J.$$

Lemma 2.12. Let $0 < \alpha < 1, 0 \le \beta \le 1, \gamma = \alpha + \beta - \alpha\beta$, and $f \in L^1(J, E)$. If $D_{0^+}^{\gamma} f$ exists and in $L^1(J, E)$, then

$$\begin{aligned} I_{0^+}^{\alpha} D_{0^+}^{\alpha,\beta} f(t) &= I_{0^+}^{\gamma} D_{0^+}^{\gamma} f(t) \\ &= f(t) - \frac{I_{0^+}^{1-\gamma} f(0^+)}{\Gamma(\gamma)} t^{\gamma-1}, \quad t \in J. \end{aligned}$$

Lemma 2.13. [25] For $t > a$, we have

$$\begin{aligned} I_{0^+}^{\alpha} (t-a)^{\beta-1}(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta+\alpha-1} \\ D_{0^+}^{\alpha} (t-a)^{\beta-1}(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}, \end{aligned} \tag{2.6}$$

Lemma 2.14. Let $\alpha > 0, 0 \le \beta \le 1$, so the homogeneous differential equation with Hilfer fractional order

$$D_{0^+}^{\alpha,\beta} h(t) = 0 \tag{2.7}$$

has a solution

$$h(t) = c_0 t^{\gamma-1} + c_1 t^{\gamma+2\beta-2} + c_2 t^{\gamma+2(2\beta)-3} + ... + c_n t^{\gamma+n(2\beta)-(n+1)}.$$

Definion 2.15. A map $f : J \times E \to E$ is said to be Caratheodory if
(i) $t \mapsto f(t, u)$ is measurable for each $u \in E$;
(ii) $u \mapsto F(t, u)$ is continuous for almost all $t \in J$.

The following theorems will play a major role in our analysis.

Theorem 2.16. ([5, 32]). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication
$V = \overline{conv} N(V)$ or $V = N(V) \cup 0 \Rightarrow \mu(V) = 0$
holds for every subset V of D, then N has a fixed point.

Lemma 2.17. ([32]). Let D be a bounded, closed and convex subset of the Banach space $C(J, E)$, G a continuous function on $J \times J$ and f a function from $J \times E \longrightarrow E$ which satisfies the Caratheodory conditions, and suppose there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $t \in J$ and each bounded set $B \subset E$, we have

$$\lim_{h \to 0^+} \mu(f(J_{t,h} \times B)) \le p(t)\mu(B); \text{ here } J_{t,h} = [t-h, t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_J G(s,t) f(s, y(s)) ds : y \in V\right\}\right) \le \int_J \|G(t,s)\| p(s) \mu(V(s)) ds.$$

3 Main results

First of all, we define what we mean by a solution of the BVP (1.1)-(1.2).

Definition 3.1. A function $y \in C_{1-\gamma}(J,E)$ is said to be a solution of the problem (1.1)- (1.2) if y satisfies the equation $D^{\alpha,\beta}_{a^+}y(t) = f(t, y(t))$ on J , and the conditions $I^{1-\gamma}y(0) = y_0, I^{3-\gamma-2\beta}y'(0) = y_1$, and $I^{1-\gamma}y(\eta) = \lambda(I^{1-\gamma}y(T))$.

Lemma 3.2. Let $f : J \times E \times E \times E \times E \to E$ be a function such that $f \in C_{1-\gamma}(J,E)$ for any $y \in C_{1-\gamma}(J,E)$. A function $y \in C^{\gamma}_{1-\gamma}(J,E)$ is a solution of the integral equation

$$y(t) = I^{\alpha}f(t,y(t)) + \frac{y_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{y_1}{\Gamma(\gamma+2\beta-1)}t^{\gamma+2\beta-2} + \zeta(\beta,\gamma,\eta,\lambda)$$
$$\left[y_0(\lambda-1) + \frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1 + \lambda I^{\alpha-\gamma+1}f(T,y(T)) - I^{\alpha-\gamma+1}f(\eta,y(\eta))\right]t^{\gamma+2(2\beta)-3} \tag{3.1}$$

if and only if y is a solution of the Hilfer fractional BVP

$$D^{\alpha,\beta}_{a^+}y(t) = f(t,y(t)), t \in J := [0,T], \tag{3.2}$$

with the fractional boundary conditions

$$\begin{aligned} &I^{1-\gamma}y(0) = y_0, \ \ I^{3-\gamma-2\beta}y'(0) = y_1, \\ &I^{1-\gamma}y(\eta) = \lambda(I^{1-\gamma}y(T)), \gamma = \alpha+\beta-\alpha\beta. \end{aligned} \tag{3.3}$$

Proof. Assume y satisfies (3.1). Then Lemma 2.18 implies that

$$y(t) = c_0t^{\gamma-1} + c_1t^{\gamma+2\beta-2} + c_2t^{\gamma+2(2\beta)-3} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds.$$

for some constants $c_0, c_1, c_2 \in \mathbb{R}$.
From (3.3), by Lemma 2.16 (2.6) , we have

- $I^{1-\gamma}y(0) = y_0$ implies that $c_0 = \frac{y_0}{\Gamma(\gamma)}$
- $I^{3-\gamma-2\beta}y'(0) = y_1$ implies that $c_1 = \frac{y_1}{\Gamma(\gamma+2\beta-1)}$
- $I^{1-\gamma}y(1) = \lambda(I^{1-\gamma}y(T))$ implies that

$$\begin{aligned} \left(I^{1-\gamma}y\right)(\eta) &= (I^{1-\gamma}\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1})(\eta)+(I^{1-\gamma}\frac{y_1}{\Gamma(\gamma)}t^{\gamma+2\beta-2})(\eta) + c_2\left(I^{1-\gamma}t^{\gamma+2(2\beta)-3}\right)(\eta) + I^{\alpha-\gamma+1}f(\eta,y(\eta)) \\ &= y_0 + \frac{y_1}{\Gamma(2\beta)}\eta^{2\beta-1} + c_2\frac{\Gamma(\gamma+2(2\beta)-2)}{\Gamma(4\beta-1)}\eta^{4\beta-2} + I^{\alpha-\gamma+1}f(\eta,y(\eta)) \\ \left(I^{1-\gamma}y\right)(T) &= (I^{1-\gamma}\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1})(T) + (I^{1-\gamma}\frac{y_1}{\Gamma(\gamma+2\beta-1)}t^{\gamma+2\beta-2})(T) + c_2\left(I^{1-\gamma}t^{\gamma+2(2\beta)-3}\right)(T) \\ &+ I^{\alpha-\gamma+1}f(T,y(T)) \\ &= y_0 + \frac{y_1}{\Gamma(2\beta)}T^{2\beta-1} + c_2\frac{\Gamma(\gamma+2(2\beta)-2)}{\Gamma(4\beta-1)}T^{4\beta-2} + I^{\alpha-\gamma+1}f(T,y(T)) \\ \lambda\left(I^{1-\gamma}y\right)(T) &= \lambda y_0 + \frac{\lambda y_1}{\Gamma(2\beta)}T^{2\beta-1} + c_2\frac{\lambda\Gamma(\gamma+2(2\beta)-2)}{\Gamma(4\beta-1)}T^{4\beta-2} + \lambda I^{\alpha-\gamma+1}f(T,y(T)) \end{aligned}$$

that is,

$$c_2 = \zeta(\beta,\gamma,\eta,\lambda)\left[y_0(\lambda-1) + \frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1 + \lambda I^{\alpha-\gamma+1}f(T,y(T)) - I^{\alpha-\gamma+1}f(\eta,y(\eta))\right]$$

With

$$\zeta(\beta,\gamma,\eta,\lambda)=\frac{\Gamma(4\beta-1)}{\Gamma(\gamma+4\beta-2)(\eta^{4\beta-2}-\lambda T^{4\beta-2})}$$

The following hypotheses will be used in the sequel.
(H1) $f: J\times E\to E$ satisfies the Caratheodory conditions;
(H2) There exists $p\in L^1(J,\mathbb{R}^+)\cap C(J,\mathbb{R}^+)$, such that,

$$\|f(t,y)\|\leq p(t)\|y\|,\ \text{for } t\in J \text{ and each } y\in E;$$

(H3) For each $t\in J$ and each bounded set $B\subset E$, we have

$$\lim_{h\to 0^+}\mu(f(J_{t,h}\times B))\leq t^{1-\gamma}p(t)\mu(B);\ \text{here } J_{t,h}=[t-h,t]\cap J.$$

Theorem 3.3. Assume that conditions (H1)-(H3) hold. Let

$$p^*=\sup_{t\in J}p(t).$$

If

$$p^*\left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left[|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right]T^{2(2\beta)-2}\right]<1 \tag{3.4}$$

then the BVP (1.1)-(1.2) has at least one solution.

Proof. We transform the problem (1.1)-(1.2) into a fixed point problem, then we consider the operator $N: C_{1-\gamma}(J,E)\to C_{1-\gamma}(J,E)$ defined by

$$N(y)(t)=I^{\alpha}f(t,y(t))+\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1}+\frac{y_1}{\Gamma(\gamma+2\beta-1)}t^{\gamma+2\beta-2}+\zeta(\beta,\gamma,\eta,\lambda)$$
$$\left[y_0(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1+\lambda I^{\alpha-\gamma+1}f(T,y(T))-I^{\alpha-\gamma+1}f(\eta,y(\eta))\right]t^{\gamma+2(2\beta)-3}$$

Clearly, the fixed points of the operator N are solutions of the problem (1.1)-(1.2). Let

$$R\geq\frac{\frac{y_0}{\Gamma(\gamma)}+\frac{y_1T^{2\beta-1}}{\Gamma(\gamma+2\beta-1)}+|\zeta(\beta,\gamma,\eta,\lambda)|\left(\|y_0\||\lambda-1|+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right)}{1-p^*\left(\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}-\frac{|\zeta(\beta,\gamma,\eta,\lambda)|T^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1})\right)} \tag{3.5}$$

and consider

$$D=\{y\in C_{1-\gamma}(J,E):\|y\|\leq R\}.$$

The subset D is closed, bounded and convex. We shall show that the assumptions of Theorem 2.4 are satisfied. The proof will be given in three steps.

1-First we show that N is continuous:

Let y_n be a sequence such that $y_n\to y$ in $C_{1-\gamma}(J,E)$. Then for each $t\in J$,

$$\|t^{1-\gamma}(N(y_n)(t)-N(y)(t))\|\leq\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s,y_n(s))-f(s,y(s))\|ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|t^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}$$
$$\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}\|f(s,y_n(s))-f(s,y(s))\|ds+\int_0^{\eta}(\eta-s)^{\alpha-\gamma}\|f(s,y_n(s))-f(s,y(s))\|ds\right]$$
$$\leq\left(\frac{t^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|t^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1})\right)\|f(s,y_n(s))-f(s,y(s))\|$$
$$\leq\left(\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|T^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1})\right)\|f(s,y_n(s))-f(s,y(s))\|$$

Since f is of Caratheodory type, then by the Lebesgue dominated convergence theorem we have

$$\|N(y_n) - N(y)\|_\infty \to 0 \text{ as } n \to \infty.$$

2-Second we show that N maps D into itself:

Take $y \in D$, by (H2), we have, for each $t \in J$ and assume that $Ny(t) \neq 0$.

$$\begin{aligned}
\|t^{1-\gamma}N(y)(t)\| &\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s))\|ds + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}t^{2\beta-1} \\
&+ |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]t^{4\beta-2} \\
&+ \frac{|\zeta(\beta,\gamma,\eta,\lambda)|t^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}\left[|\lambda|\int_0^T (T-s)^{\alpha-\gamma}\|f(s,y(s))\|ds + \int_0^\eta (\eta-s)^{\alpha-\gamma}\|f(s,y(s))\|ds\right] \\
&\leq \frac{T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\|y\|ds + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}T^{2\beta-1} \\
&+ |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]T^{4\beta-2} \\
&+ \frac{T^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}|\zeta(\beta,\gamma,\eta,\lambda)|\left[|\lambda|\int_0^T (T-s)^{\alpha-\gamma}p(s)\|y\|ds + \int_0^1 (1-s)^{\alpha-\gamma}p(s)\|y\|ds\right] \\
&\leq \frac{RT^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)ds + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}T^{2\beta-1} \\
&+ |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]T^{4\beta-2} \\
&+ \frac{RT^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}|\zeta(\beta,\gamma,\eta,\lambda)|\left[|\lambda|\int_0^T (T-s)^{\alpha-\gamma}p(s)ds + \int_0^\eta (\eta-s)^{\alpha-\gamma}p(s)ds\right] \\
&\leq \frac{Rp^*T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}T^{2\beta-1} \\
&+ |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]T^{4\beta-2} \\
&+ \frac{Rp^*T^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}|\zeta(\beta,\gamma,\eta,\lambda)|\left[|\lambda|\int_0^T (T-s)^{\alpha-\gamma}ds + \int_0^\eta (\eta-s)^{\alpha-\gamma}ds\right] \\
&\leq \frac{Rp^*T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}T^{2\beta-1} \\
&+ |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]T^{4\beta-2} \\
&+ |\zeta(\beta,\gamma,\eta,\lambda)|\left[\frac{|\lambda|Rp^*T^{\alpha-\gamma+4\beta-1}}{\Gamma(\alpha-\gamma+2)} + \frac{Rp^*\eta^{\alpha-\gamma+1}T^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}\right] \\
&\leq R.
\end{aligned}$$

3-Finally we show that $N(D)$ is bounded and equicontinuous:

By Step 2, it is obvious that $N(D) \subset C_{1-\gamma}(J,E)$ is bounded. For the equicontinuity of $N(D)$, let $t_1, t_2 \in J$, $t_1 < t_2$ and $y \in D$, so $t_2^{1-\gamma}Ny(t_2) - t_1^{1-\gamma}Ny(t_1) \neq 0$. Then

$$
\begin{aligned}
\|t_2^{1-\gamma}Ny(t_2)-t_1^{1-\gamma}Ny(t_1)\| &\leq \frac{1}{\Gamma(\gamma+2\beta-1)}\|y_1t_2^{2\beta-1}-y_1t_1^{2\beta-1}\|+|\zeta(\beta,\gamma,\eta,\lambda)|\\
&\left\|\left[y_0|\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1+|\lambda|I^{\alpha-\gamma+1}f(T,y(T))-I^{\alpha-\gamma+1}f(\eta,y(\eta))\right]\right\|\\
&(t_2^{2(2\beta)-2)}-t_1^{2(2\beta)-2)})\\
&+\left\|\frac{t_2^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds-\frac{t_1^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))ds\right\|\\
&\leq \frac{1}{\Gamma(\gamma+2\beta-1)}\|y_1\|(t_2^{2\beta-1}-t_1^{2\beta-1})+|\zeta(\beta,\gamma,\eta,\lambda)|\\
&\left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right](t_2^{2(2\beta)-2)}-t_1^{2(2\beta)-2)})\\
&+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}\|f(s,y(s))\|ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}\|f(s,y(s))\|ds\right]\\
&(t_2^{2(2\beta)-2)}-t_1^{2(2\beta)-2)})\\
&+\frac{1}{\Gamma(\alpha)}\left[t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}\|f(s,y(s))\|ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}\|f(s,y(s))\|ds\right.\\
&\left.+t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|f(s,y(s))\|ds\right]\\
&\leq \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}(t_2^{2\beta-1}-t_1^{2\beta-1})+|\zeta(\beta,\gamma,\eta,\lambda)|\\
&\left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|+\frac{1}{\Gamma(\alpha-\gamma+1)}\right.\\
&\left.\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)\|y\|ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)\|y\|ds\right]\right](t_2^{2(2\beta)-2)}-t_1^{2(2\beta)-2)})\\
&+\frac{1}{\Gamma(\alpha)}\left[t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}p(s)\|y\|ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}p(s)\|y\|ds\right.\\
&\left.+t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}p(s)\|y\|ds\right]\\
&\leq \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}(t_2^{2\beta-1}-t_1^{2\beta-1})+|\zeta(\beta,\gamma,\eta,\lambda)|\\
&\left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|+\frac{R}{\Gamma(\alpha-\gamma+1)}\right.\\
&\left.\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)ds\right]\right](t_2^{2(2\beta)-2)}-t_1^{2(2\beta)-2)})\\
&+\frac{R}{\Gamma(\alpha)}\left[t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}p(s)ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}p(s)ds\right.\\
&\left.+t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}p(s)ds\right]\\
&\leq \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}(t_2^{2\beta-1}-t_1^{2\beta-1})\\
&+|\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right.\\
&\left.+\frac{Rp^*}{\Gamma(\alpha-\gamma+1)}\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}ds\right]\right](t_2^{2(2\beta)-2)}-t_1^{2(2\beta)-2)})\\
&+\frac{Rp^*}{\Gamma(\alpha)}\left[t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}ds+t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right]
\end{aligned}
$$

$$\leq \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}(t_2^{2\beta-1}-t_1^{2\beta-1})$$
$$+|\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|+\frac{Rp^*(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1})}{\Gamma(\alpha-\gamma+2)}\right]$$
$$(t_2^{2(2\beta)-2)}-t_1^{2(2\beta)-2)})+\frac{Rp^*}{\Gamma(\alpha+1)}(t_2^{\alpha-\gamma+1}-t_1^{\alpha-\gamma+1}).$$

As $t_1\to t_2$, the right hand side of the above inequality tends to zero.
Hence $N(D)\subset D$.

Now we show that the implication holds:

Let $V\subset D$ such that $V=\overline{conv}(N(V)\cup\{0\})$.

We have $V(t)\subset\overline{conv}(N(V)\cup\{0\})$ for all $t\in J$. $NV(t)\subset ND(t)$, $t\in J$ is bounded and equicontinuous in E, the function $t\to v(t)=\mu(V(t))$ is continuous on J .
By assumption (H2), and the properties of the measure μ we have for each $t\in J$.

$$t^{1-\gamma}v(t)\leq\mu(t^{1-\gamma}N(V)(t)\cup\{0\}))\leq\mu(t^{1-\gamma}(NV)(t))$$
$$\leq\mu\left[t^{1-\gamma}\left[I^\alpha f(t,y(t))+\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1}+\frac{y_1}{\Gamma(\gamma)}t^{\gamma+2\beta-2}+\zeta(\beta,\gamma,\eta,\lambda)\right.\right.$$
$$\left.\left.\left(y_0(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1+\lambda I^{\alpha-\gamma+1}f(T,y(T))-I^{\alpha-\gamma+1}f(\eta,y(\eta))\right)t^{\gamma+2(2\beta)-3}\right]\right]$$
$$\leq\mu\left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}f(s,y(s))ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}f(s,y(s))ds\right)t^{2(2\beta)-2}\right]$$
$$\leq\left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mu(f(s,y(s)))ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}\mu(f(s,y(s)))ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}\mu(f(s,y(s)))ds\right)t^{2(2\beta)-2}\right]$$
$$\leq\left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)\mu(V(s))ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)\mu(V(s))ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)\mu(V(s))ds\right)t^{2(2\beta)-2}\right]$$
$$\leq\left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)v(s)ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)v(s)ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)v(s)ds\right)t^{2(2\beta)-2}\right]$$
$$\leq\|v\|\left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)ds\right)t^{2(2\beta)-2}\right]$$
$$\leq p^*\|v\|\left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}ds\right)t^{2(2\beta)-2}\right]$$
$$\leq p^*\|v\|\left[\frac{t^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right)t^{2(2\beta)-2}\right]$$

This means that

$$\|v\|\leq p^*\|v\|\left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right)T^{2(2\beta)-2}\right]$$

By $p^*\left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left[|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right]T^{2(2\beta)-2}\right]<1$ it follows that $\|v\|=0$, that is $v(t)=0$ for each $t\in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D. Applying now Theorem 2.16, we conclude that N has a fixed point which is a solution of the problem (1.1)-(1.2).

4 Example

We consider the problem for Hilfer fractional differential equations of the form:

$$\begin{cases} D^{\alpha,\beta}y(t)=f(t,y(t)),(t,y)\in([0,1],\mathbb{R}), \\ I^{1-\gamma}y(0)=y_0, I^{3-\gamma-2\beta}y'(0)=y_1, I^{1-\gamma}y(\eta)=\lambda(I^{1-\gamma}y(T)) \end{cases} \tag{4.1}$$

Here

$$\alpha=\frac{1}{2},\qquad \beta=\frac{1}{2},\qquad \gamma=\frac{3}{4},$$
$$\lambda=\frac{1}{2},\qquad \eta=\frac{1}{4},\qquad T=1.$$

With

$$f(t,yt))=\frac{1}{4}+\frac{ct^2}{e^{t+4}}|y(t)|,\quad t\in[0,1]$$

and

$$c=\frac{e^3}{10}\sqrt{\pi}$$

Clearly, the function f is continuous. For each $y\in E$ and $t\in[0,1]$, we have

$$\|f(t,y(t))\|\leq\frac{ct^2}{e^{t+4}}\|y\|$$

Hence, the hypothesis (H2) is satisfied with $p^*=ce^{-3}$. We shall show that condition 3.4 holds with $T=1$. Indeed,

$$p^*\left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left[|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right]T^{2(2\beta)-2}\right]<1$$

Simple computations show that all conditions of Theorem 3.1 are satisfied. It follows that the problem 4.1 has a solution defined on [0,T].

References

[1] S. Abbas, M. Benchohra, M. Bohner, Weak Solutions for Implicit Differential Equations with Hilfer-Hadamard Fractional Derivative, Advances in Dynamical Systems and Applications, Volume 12, Number 1, pp. 1-16 (2017)

[2] S. Abbas, M. Benchohra, J.E. Lazreg, J. J. Nieto, On a coupled system of Hilfer and Hilfer-Hadamard fractional defferential equation in Banach spaces, J. Nonlinear Funct. Anal. 2018 (2018), Article ID 12.

[3] S. Abbas, M. Benchohra, J.E. Lazreg, Y. Zhou, A Survey on Hadamard and Hilfer fractional differential equations:analysis and stability, Chaos, Solitons Fractals 102 (2017), 47-71.

[4] R.P. Agarwal, M. Benchohra, D. Seba, On the application of measure of noncompactness to the existence of solutions for fractional differential equations, Results Math. 55:3-4 (2009) 221-230.

[5] R. P. Agarwal, M. Meehan and D. ORegan, Fixed Point Theory and Applications, Cambridge Tracts in Mathematics, 141, Cambridge University Press, Cambridge, 2001.

[6] A. Aghajani, A. M. Tehrani, D. O'Regan, Some New Fixed Point Results via the Concept of Measure of Noncompactness, Filomat 29:6 (2015), 1209-1216.

[7] R. R. Akhmerov, M. I. Kamenskii, A. S. Patapov, A. E. Rodkina and B. N. Sadovskii, Measures of Noncompactness and Condensing Operators, trans. from the Russian by A. Iacob, Birkhauser Verlag, Basel, 1992.

[8] J. C. Alvàrez, Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces, Rev. Real. Acad. Cienc. Exact. Fis. Natur. Madrid 79 (1985), 53-66.

[9] J. M. Ayerbee Toledano, T. Dominguez Benavides, G. Lopez Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Operator Theory, Advances and Applications, vol 99, Birkhäuser, Basel, Boston, Berlin, 1997.

[10] J. Banas, Applications of measures of weak noncompactness and some classes of operators in the theory of functional equations in the Lebesgue space, Nonlinear Anal. 30 (1997), 3283-3293.

[11] J. Banas and K. Goebel, Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, Volume 60, Marcel Dekker, New York, 1980.

[12] J. Banas, M. Jleli, M. Mursaleen, B. Samet,C. Vetro, Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness, Springer Nature Singapore Pte Ltd. 2017.

[13] J. Banas, R. Nalepa, On a measure of noncompactness in the space of functions with tempered increments, J. Math. Anal. Appl. 435 (2016) 1634-1651.

[14] J. Banas and L. Olszowy, Measures of noncompactness related to monotonicity, Comment. Math. 41 (2001), 13-23.

[15] J. Banas and B. Rzepka, An application of a measure of noncompactness in the study of asymptotique stability, Appl. Math. Lett. 16 (2003), 1-6.

[16] J. Banas and K. Sadarangani, On some measures of noncompactness in the space of continuous functions, Nonlinear Anal. 68 (2008), 377-383.

[17] M. Benchohra, J. Henderson, D. Seba, Measure of noncompactness and fractional differential equations in Banach spaces, Commun. Appl. Anal. 12 (2008) 419-428.

[18] M. Benchohra, G.M. N'Guérékata, D. Seba, Measure of noncompactness and nondensely defined semilinear functional differential equations with fractional order, CUBO A Math. J. 12:3 (2010) 33-46.

[19] K. M. Furati, M. D. Kassim and N.e.Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, Computers Math. Appl,(2012), 1616-1626.

[20] K. M. Furati, M. D. Kassim, N. Tatar, Non-existence of global solutions for a differential equation involving Hilfer fractional derivative, Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 235, pp. 1-10.

[21] H. Gu, J.J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, Appl. Math. Comput. 257 (2015), 344-354.

[22] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, (2000).

[23] R.Hilfer, Y.Luchko, Z. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann-Lioville fractional derivative, Fractional calculus and Applications Analysis, 12 (2009), 289-318.

[24] [15] R. Kamocki, C. Obcznnski, On fractional Cauchy-type problems containing Hilfer derivative, Electronic Journal of Qualitative of Differential Equations, 50 (2016), 1-12.

[25] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies 204, Editor: Jan Van Mill, Elsevier, Amsterdam, The Netherlands, (2006).

[26] C. Kou, J. Liu, and Y. Ye, Existence and uniqueness of solutions for the Cachy-type problems of fractional differential equaitions, Discrete Dyn. Nat. Soc., Article ID 142175, (2010), 1-15.

[27] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, Nonlinear Anal. 4 (1980), 985-999.

[28] H. K. Nashine,R. Arab, R. P Agarwal,M De la Sen, Positive solutions of fractional integral equations by the technique of measure of noncompactness, Journal of Inequalities and Applications(2017) 2017:225 .

[29] H.Rebai,D.Seba, Weak Solutions for Nonlinear Fractional Differential Equation with Fractional Separated Boundary Conditions in Banach Spaces, Filomat 32:3 (2018), 1117-1125.

[30] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon 1993.

[31] Sandeep P. Bhairat, Existence and continuation of solutions of Hilfer fractional differential equations, Journal of Mathematical Modeling, Vol. 7, No. 1, 2019, pp. 1-20.

[32] S. Szufla, On the application of measure of noncompactness to existence theorems, Rend. Sem. Mat. Univ. Padova 75 (1986), 1-14.

[33] D. Vivek, K. Kanagarajan, E. M. Elsayed,Nonlocal initial value problems for implicit differential equations with Hilfer-Hadamard fractional derivative, Nonlinear Analysis: Modelling and Control, Vol. 23, No. 3, 341-360.

[34] D. Vivek, K. Kanagarajan,S. Sivasundaram, On the behavior of solutions of Hilfer-Hadamard type fractional neutral pantograph equations with boundary conditions, Communications in Applied Analysis, 22, No. 2 (2018), 211-232.

[35] J.-R. Wang, Y. Zhang, Nonlocal initial value problems for differential equations with Hilfer fractional derivative, Appl. Math. Comput. 266 (2015), 850-859.

[36] M. Yang, Q.-R. Wang, Approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions, Math. Methods Appl. Sci. 40 (2017), 1126-1138.