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ON $I-$DEFERRED STATISTICAL CONVERGENCE IN TOPOLOGICAL GROUPS

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Abstract. In this paper, the concepts of $I-$deferred statistical convergence of order α and $I-$deferred statistical convergence of order (α,β) in topological groups were defined. Also some inclusion relations between $I-$statistical convergence of order α, $I-$deferred statistical convergence of order α, $I-$statistical convergence of order (α,β) and $I-$deferred statistical convergence of order (α,β) in topological groups are given.

1. INTRODUCTION

The idea of statistical convergence was given by Zygmund [38] in the first edition of his monograph puplished in Warsaw in 1935. The consept of statistical convergence was introduced by Steinhaus [30] and Fast [13] and later reintroduced by Schoenberg [28] independently. Later on it was further investigated from the sequence space point of view and linked with summability theory by Çakallı ([2],[3],[4],[5],[6]), Çınar et al. [7], Et et al. ([9],[10],[11],[12],[24]), Fridy [14], Fridy and Orhan [15], Işık and Akbaş [17], Salat [22], Savaş [23], Sengul et. al. ([31],[32],[33],[34]), Srivastava and Et [29], Yıldız [37] and many others.

Let X be a non-empty set. Then a family of sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I is additive *i.e.* $A, B \in I$ implies $A \cup B \in I$ and hereditary, *i.e.* $A \in I$, $B \subset A$ implies $B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a *filter* of X if and only if (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$ and (iii) $A \in F$, $A \subset B$ implies $B \in F$.

An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$.

A non-trivial ideal I is said to be *admissible* if $I \supset \{\{x\} : x \in X\}$.

If I is a non-trivial ideal in $X (X \neq \phi)$ then the family of sets $F(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter of X, called the *filter associated with I*.

Throughout the paper I will stand for a non-trivial admissible ideal of $\mathbb{N}$.

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The idea of $I-$convergence of real sequences was introduced by Kostyrko *et al.* [19] and also independently by Nuray and Ruckle [21] (who called it generalized statistical convergence) as a generalization of statistical convergence. Later on $I-$convergence was studied in ([20],[26],[27],[25],[35],[36]).

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan in [16] and after then statistical convergence of order α and strong $p-$Cesàro summability of order α studied by Çolak [8].

In 1932, R.P. Agnew [1] defined the deferred Cesaro mean $D_{p,q}$ of the sequence $x=(x_k)$ by

$$(D_{p,q}x)_n = \frac{1}{q(n)-p(n)} \sum_{p(n)+1}^{q(n)} x_k$$

where $(p(n))$ and $(q(n))$ are sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = +\infty. \tag{1.1}$$

Let K be a subset of $\mathbb{N}$, and denote the set $\{k : p(n) < k \le q(n), k \in K\}$ by $K_{p,q}(n)$. Deferred density of K is defined by

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{q(n)-p(n)} |K_{p,q}(n)| \tag{1.2}$$

whenever the limit exists (finite or infinite). The vertical bars in (1.2) indicate the cardinality of the set $K_{p,q}(n)$.

A real valued sequence $x=(x_k)$ is said to be deferred statistical convergent to l, if

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} |\{p(n) < k \le q(n) : |x_k - l| \ge \varepsilon\}| = 0$$

for every $\varepsilon > 0$. If $q(n) = n$, $p(n) = 0$ then deferred statistical convergence coincides statistical convergence [18].

2. $I-$DEFERRED STATISTICAL CONVERGENCE OF ORDER α IN TOPOLOGICAL GROUPS

In this section, some inclusion relations between $I-$statistical convergence, $I-$statistical convergence of order α and $I-$deferred statistical convergence of order α in topological groups are given.

Definition 2.1. *Let $(p(n))$ and $(q(n))$ be two sequences of non-negative integers satisfying the conditions (1.1), X be an abelian topological Hausdorf group, $(x(k))$ be a sequence of real numbers and α be a positive real number such that $0<\alpha\le 1$. The sequence $x=(x(k))$ is said to be $DS^{\alpha}_{p,q}(X,I)-$statistically convergent in topological groups to l (or $I-$deferred statistically convergent sequences of order α in topological groups to l) if there is a real number l for each neighbourhood U of 0 such that*

$$\left\{ n\in\mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} |\{p(n) < k \le q(n) : x(k) - l \notin U\}| \ge \delta \right\} \in I.$$

In this case we write $DS^{\alpha}_{p,q}(I) - \lim x(k) = l$ or $x(k) \to l \left(DS^{\alpha}_{p,q}(I)\right)$. The set of all $DS^{\alpha}_{p,q}(X,I)-$ statistically convergent sequences in topological groups will be denoted by $DS^{\alpha}_{p,q}(X,I)$. If $\alpha = 1$, then $I-$deferred statistical convergence *of order α coincides then $I-$deferred statistical convergence in topological groups ($DS_{p,q}(X,I)-$ convergence) and if $q(n)=n$, $p(n)=0$ then $I-$deferred statistical convergence of order α coincides $I-$statistical convergence of order α in topological groups ($S^{\alpha}(X,I)-$convergence). If $q(n)=n$, $p(n)=0$ and $\alpha=1$, then $I-$deferred statistical convergence of order α coincides $I-$statistical convergence in topological groups ($S(X,I)-$convergence).*

Theorem 2.1. *Let $(p(n))$ and $(q(n))$ be two sequences of non-negative integers satisfying the conditions (1.1) and α,β be positive real numbers such that $0<\alpha\leq\beta\leq 1$ then $DS^{\alpha}_{p,q}(X,I)\subseteq DS^{\beta}_{p,q}(X,I)$ and the inclusion is strict.*

Proof. Omitted. □

Theorem 2.1 yields the following corollary.

Corollary 2.2. *If a sequence is $DS^{\alpha}_{p,q}(X,I)-$statistically convergent of order α to l, then it is $DS_{p,q}(X,I)-$statistically convergent to l.*

Theorem 2.3. *Let $(p(n))$ and $(q(n))$ be two sequences of non-negative integers satisfying the conditions (1.1) and α be a positive real number such that $0<\alpha\leq 1$. If $\liminf_n \frac{q(n)}{p(n)}>1$, then $S^{\alpha}(X,I)\subset DS^{\alpha}_{p,q}(X,I)$.*

Proof. Suppose that $\liminf_n \frac{q(n)}{p(n)}>1$; then there exists an $a>0$ such that $\frac{q(n)}{p(n)}\geq 1+a$ for sufficiently large n, which implies that

$$\frac{q(n)-p(n)}{q(n)}\geq\frac{a}{1+a}\Longrightarrow\left(\frac{q(n)-p(n)}{q(n)}\right)^{\alpha}\geq\left(\frac{a}{1+a}\right)^{\alpha}\Longrightarrow\frac{1}{q(n)^{\alpha}}\geq\frac{a^{\alpha}}{(1+a)^{\alpha}}\frac{1}{(q(n)-p(n))^{\alpha}}.$$

If $S^{\alpha}(I)-\lim_{k\to\infty}x(k)=l$, then for each neighbourhood U of 0 and for sufficiently large n, we have

$$\begin{aligned}\frac{1}{q(n)^{\alpha}}\left|\{k\leq q(n):x(k)-l\notin U\}\right| &\geq \frac{1}{q(n)^{\alpha}}\left|\{p(n)<k\leq q(n):x(k)-l\notin U\}\right|\\ &\geq \frac{a^{\alpha}}{(1+a)^{\alpha}}\frac{1}{(q(n)-p(n))^{\alpha}}\left|\{p(n)<k\leq q(n):x(k)-l\notin U\}\right|.\end{aligned}$$

Therefore, we can write

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))^{\alpha}}\left|\{p(n)<k\leq q(n):x(k)-l\notin U\}\right|\geq\delta\right\}\\ \subseteq&\left\{n\in\mathbb{N}:\frac{1}{q(n)^{\alpha}}\left|\{k\leq q(n):x(k)-l\notin U\}\right|\geq\delta\frac{a^{\alpha}}{(1+a)^{\alpha}}\right\}\in I.\end{aligned}$$

This implies that $S^{\alpha}(X,I)\subset DS^{\alpha}_{p,q}(X,I)$. □

Theorem 2.4. *Let $(p(n))$ and $(q(n))$ be two sequences of non-negative integers satisfying the conditions (1.1) and α be a positive real number such that $0<\alpha\leq 1$. If $\liminf_n \frac{(q(n)-p(n))^{\alpha}}{n}>0$ and $q(n)<n$, then $S(X,I)\subset DS^{\alpha}_{p,q}(X,I)$.*

Proof. For each neighbourhood U of 0, we have

$$\{k\leq n:x(k)-l\notin U\}\supset\{p(n)<k\leq q(n):x(k)-l\notin U\}.$$

Therefore,

$$\begin{aligned}\frac{1}{n}\left|\{k\leq n: x(k)-l\notin U\}\right| &\geq \frac{1}{n}\left|\{p(n)<k\leq q(n): x(k)-l\notin U\}\right| \\ &= \frac{(q(n)-p(n))^{\alpha}}{n}\frac{1}{(q(n)-p(n))^{\alpha}}\left|\{p(n)<k\leq q(n): x(k)-l\notin U\}\right|.\end{aligned}$$

Hence, we can write

$$\begin{aligned}&\left\{n\in\mathbb{N}: \frac{1}{(q(n)-p(n))^{\alpha}}\left|\{p(n)<k\leq q(n): x(k)-l\notin U\}\right|\geq\delta\right\} \\ \subseteq\ &\left\{n\in\mathbb{N}: \frac{1}{n}\left|\{k\leq n: x(k)-l\notin U\}\right|\geq\delta\frac{(q(n)-p(n))^{\alpha}}{n}\right\}\in I.\end{aligned}$$

Consequently, $S(X,I)\subset DS_{p,q}^{\alpha}(X,I)$. □

Theorem 2.5. *Let $(p(n)),(q(n)),(p'(n)),(q'(n))$ be four sequences of non-negative integers such that $p(n)<q(n)$, $p'(n)<q'(n)$ and $q(n)-p(n)\leq q'(n)-p'(n)$ for all $n\in\mathbb{N}$, let U be any neighbourhood of 0 and let α and β be such that $0<\alpha\leq\beta\leq 1$.*

(i) If

$$\lim_{n\to\infty}\inf\frac{(q(n)-p(n))^{\alpha}}{(q'(n)-p'(n))^{\beta}}>0 \tag{2.1}$$

then $DS_{p',q'}^{\beta}(X,I)\subseteq DS_{p,q}^{\alpha}(X,I)$,

(ii) If

$$\lim_{n\to\infty}\frac{q'(n)-p'(n)}{(q(n)-p(n))^{\beta}}=1 \tag{2.2}$$

then $DS_{p,q}^{\alpha}(X,I)\subseteq DS_{p',q'}^{\beta}(X,I)$.

Proof. (i) Let (2.1) be satisfied. For given $\varepsilon>0$ and each neighbourhood U,W of 0 such that $W\subset U$, we have

$$\{p'(n)<k\leq q'(n): x(k)-l\notin W\}\supseteq\{p(n)<k\leq q(n): x(k)-l\notin U\},$$

and so

$$\begin{aligned}&\frac{1}{(q'(n)-p'(n))^{\beta}}\left|\{p'(n)<k\leq q'(n): x(k)-l\notin W\}\right| \\ \geq\ &\frac{(q(n)-p(n))^{\alpha}}{(q'(n)-p'(n))^{\beta}}\frac{1}{(q(n)-p(n))^{\alpha}}\left|\{p(n)<k\leq q(n): x(k)-l\notin U\}\right|\end{aligned}$$

for all $n\in\mathbb{N}$, where $p(n)<q(n)$, $p'(n)<q'(n)$ and $q(n)-p(n)\leq q'(n)-p'(n)$.

Then we can write

$$\begin{aligned}&\left\{n\in\mathbb{N}: \frac{1}{(q(n)-p(n))^{\alpha}}\left|\{p(n)<k\leq q(n): x(k)-l\notin U\}\right|\geq\delta\right\} \\ \subseteq\ &\left\{n\in\mathbb{N}: \frac{1}{(q'(n)-p'(n))^{\beta}}\left|\{p'(n)<k\leq q'(n): x(k)-l\notin W\}\right|\geq\delta\frac{(q(n)-p(n))^{\alpha}}{(q'(n)-p'(n))^{\beta}}\right\}\in I.\end{aligned}$$

This completes the proof.

(ii) Omitted. □

Corollary 2.6. *Let* $(p(n)), (q(n)), (p'(n)), (q'(n))$ *be four sequences of non-negative integers such that* $p(n) < q(n)$, $p'(n) < q'(n)$ *and* $q(n) - p(n) \leq q'(n) - p'(n)$ *for all* $n \in \mathbb{N}$ *and* $0 < \alpha \leq 1$.

If (2.1) *holds then,*

(i) $DS^{\alpha}_{p',q'}(X,I) \subseteq DS^{\alpha}_{p,q}(X,I)$,

(ii) $DS_{p',q'}(X,I) \subseteq DS^{\alpha}_{p,q}(X,I)$,

(iii) $DS_{p',q'}(X,I) \subseteq DS_{p,q}(X,I)$.

If (2.2) *holds then,*

(i) $DS^{\alpha}_{p,q}(X,I) \subseteq DS^{\alpha}_{p',q'}(X,I)$,

(ii) $DS^{\alpha}_{p,q}(X,I) \subseteq DS_{p',q'}(X,I)$,

(iii) $DS_{p,q}(X,I) \subseteq DS_{p',q'}(X,I)$.

3. $I-$DEFERRED STATISTICAL CONVERGENCE OF ORDER(α,β) IN TOPOLOGICAL GROUPS

In this section, the results which were given in the previous section are generalized. Some inclusion relations between $I-$statistical convergence of order (α,β) and $I-$deferred statistical convergence of order (α,β) in topological groups are given.

Definition 3.1. *Let* $(p(n))$ *and* $(q(n))$ *be two sequences of non-negative integers satisfying the conditions* (1.1)*,* X *be an abelian topological Hausdorf group,* $(x(k))$ *be a sequence of real numbers and* α, β *be positive real numbers such that* $0 < \alpha \leq \beta \leq 1$*. The sequence* $x = (x(k))$ *is said to be* $I-$*deferred statistical convergent of order* (α,β) *in topological groups to* l *(or* $DS^{\alpha,\beta}_{p,q}(X,I)-$*statistically convergent to* l *), if there is a real number* l*, for each neighbourhood* U *of* 0 *such that*

$$\left\{ n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}} \left| \{ p(n) < k \leq q(n) : x(k) - l \notin U \} \right|^{\beta} \geq \delta \right\} \in I.$$

In this case we write $DS^{\alpha,\beta}_{p,q}(I) - \lim x(k) = l$ *or* $x(k) \to l \left(DS^{\alpha,\beta}_{p,q}(I) \right)$*. The set of all* $DS^{\alpha,\beta}_{p,q}(X,I)-$*statistically convergent sequences in topological groups will be denoted by* $DS^{\alpha,\beta}_{p,q}(X,I)$*. If* $q(n) = n$, $p(n) = 0$ *and* $\alpha = \beta = 1$*, then* $I-$*deferred statistical convergence of order* (α,β) *coincides* $I-$*statistical convergence in topological groups (*$S(X,I)-$*convergence).*

Theorem 3.1. *Let* $(p(n))$ *and* $(q(n))$ *be two sequences of non-negative integers satisfying the conditions* (1.1) *and* $\alpha_1, \alpha_2, \beta_1$ *and* β_2 *be positive real numbers such that* $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$*, then* $DS^{\alpha_1,\beta_2}_{p,q}(X,I) \subseteq DS^{\alpha_2,\beta_1}_{p,q}(X,I)$ *and the inclusion is strict.*

Proof. Omitted. □

Theorem 3.2. *Let* $(p(n))$ *and* $(q(n))$ *be two sequences of non-negative integers satisfying the conditions* (1.1) *and* α, β *be two positive real numbers such that* $0 < \alpha \leq \beta \leq 1$*. If* $\liminf_n \frac{q(n)}{p(n)} > 1$*, then* $S^{\alpha,\beta}(X,I) \subset DS^{\alpha,\beta}_{p,q}(X,I)$.

Proof. The proof is similar to that of Theorem 2.3. □

Theorem 3.3. *Let $(p(n)), (q(n)), (p'(n))$ and $(q'(n))$ be four sequences of non-negative integers such that $p(n) < q(n), p'(n) < q'(n)$ and $q(n) - p(n) \leq q'(n) - p'(n)$ for all $n \in \mathbb{N}$, let U be any neighbourhood of 0 and let $\alpha_1, \alpha_2, \beta_1$ and β_2 be such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$.*

(i) If

$$\lim_{n\to\infty} \inf \frac{(q(n) - p(n))^{\alpha_1}}{(q'(n) - p'(n))^{\alpha_2}} > 0 \tag{3.1}$$

then $DS_{p',q'}^{\alpha_2,\beta_2}(X, I) \subseteq DS_{p,q}^{\alpha_1,\beta_1}(X, I)$,

(ii) If

$$\lim_{n\to\infty} \frac{q'(n) - p'(n)}{(q(n) - p(n))^{\alpha_2}} = 1 \tag{3.2}$$

then $DS_{p,q}^{\alpha_1,\beta_2}(X, I) \subseteq DS_{p',q'}^{\alpha_2,\beta_1}(X, I)$.

Proof. (*i*) Let $\lim_{n\to\infty} \inf \frac{(q(n)-p(n))^{\alpha_1}}{(q'(n)-p'(n))^{\alpha_2}} > 0$. For given $\varepsilon > 0$ and each neighbourhood U, W of 0 such that $W \subset U$, we have

$$\begin{aligned} &\frac{1}{(q'(n) - p'(n))^{\alpha_2}} \left|\{p'(n) < k \leq q'(n) : x(k) - l \notin W\}\right|^{\beta_2} \\ \geq\ & \frac{(q(n) - p(n))^{\alpha_1}}{(q'(n) - p'(n))^{\alpha_2}} \frac{1}{(q(n) - p(n))^{\alpha_1}} \left|\{p(n) < k \leq q(n) : x(k) - l \notin U\}\right|^{\beta_1} \end{aligned}$$

for all $n \in \mathbb{N}$.

Therefore, we can write

$$\begin{aligned} &\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha_1}} \left|\{p(n) < k \leq q(n) : x(k) - l \notin U\}\right|^{\beta_1} \geq \delta\right\} \\ \subseteq\ & \left\{n \in \mathbb{N} : \frac{1}{(q'(n) - p'(n))^{\alpha_2}} \left|\{p'(n) < k \leq q'(n) : x(k) - l \notin W\}\right|^{\beta_2} \geq \delta \frac{(q(n) - p(n))^{\alpha_1}}{(q'(n) - p'(n))^{\alpha_2}}\right\} \in I. \end{aligned}$$

This completes the proof.

(*ii*) Omitted. □

Corollary 3.4. *Let $(p(n)), (q(n)), (p'(n))$ and $(q'(n))$ be four sequences of non-negative integers such that $p(n) < q(n)$, $p'(n) < q'(n)$ and $q(n) - p(n) \leq q'(n) - p'(n)$ for all $n \in \mathbb{N}$ and $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$.*

If (3.1) holds then,

(i) $DS_{p',q'}^{\alpha_2}(X, I) \subseteq DS_{p,q}^{\alpha_1}(X, I)$ for $\beta_1 = \beta_2 = 1$,

(ii) $DS_{p',q'}(X, I) \subseteq DS_{p,q}^{\alpha_1}(X, I)$ for $\alpha_2 = \beta_1 = \beta_2 = 1$,

(iii) $DS_{p',q'}(X, I) \subseteq DS_{p,q}(X, I)$ for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$.

If (3.2) holds then,

(i) $DS_{p,q}^{\alpha_1}(X, I) \subseteq DS_{p',q'}^{\alpha_2}(X, I)$ for $\beta_1 = \beta_2 = 1$,

(ii) $DS_{p,q}^{\alpha_1}(X, I) \subseteq DS_{p',q'}(X, I)$ for $\alpha_2 = \beta_1 = \beta_2 = 1$,

(iii) $DS_{p,q}(X, I) \subseteq DS_{p',q'}(X, I)$ for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$.

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SPECTRAL DISJOINTNESS AND INVARIANT SUBSPACES

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IN MEMORY OF RISTEARD TIMONEY

Abstract. Spectral disjointness confers a certain mutual independence on pairs of Banach algebra elements. Necessary and sufficient for full spectral disjointness of diagonal elements is that the structural idempotent is a holomorphic function of a block diagonal matrix, while a partial left-right spectral disjointness is sufficient for membership of the double commutant. For bounded linear Banach space operators with an invariant subspace, spectral disjointness for the restriction and quotient operators implies both hyperinvariance and reducing.

1. BLOK STRUCTURE

Our "spectral disjointness" applies to pairs of operators defined on different spaces, and we need a somewhat elaborate algebraic framework for them: accordingly, we look at matrices with block structure.

If G is a ring, with identity I, then [7] an idempotent

$$Q = Q^2 \in G$$

imposes a *block structure* on G:

$$G \cong \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

where A and B are rings with identity in their own right, while M and N are bimodules over A and B; there are also bilinear mappings

$$(m,n) \mapsto m \cdot n \ (M \times N \to A) \ ; \ (m,n) \mapsto n \cdot m \ (M \times N \to B)$$

The structure is laid bare by formal multiplication of 2×2 matrices. We can take

$$A = QGQ \ ; \ M = QG(I-Q) \ ; \ N = (I-Q)GQ \ ; \ B = (I-Q)G(I-Q)$$

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The identity I, the structural idempotent Q and a generic element $T \in G$ are now given by block matrices:

1.5 $$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ; \ Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ ; \ T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} .$$

The commutant of the structural idempotent is the subring of *block diagonals*,

$$\mathrm{comm}(Q) = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \subseteq G$$

In the notation of (1.5),

1.7 $$QT = TQ \Longleftrightarrow T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ .$$

More generally [13] there are upper and lower *block triangles*:

$$QT = QTQ \Longleftrightarrow T = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \begin{pmatrix} A & M \\ O & B \end{pmatrix}$$

$$TQ = QTQ \Longleftrightarrow T = \begin{pmatrix} a & 0 \\ n & b \end{pmatrix} \in \begin{pmatrix} A & O \\ N & B \end{pmatrix}$$

2. INVERTIBILITY

An element $T \in G$ is said to be *invertible*, written $T \in G^{-1}$, if there is another element $T^{-1} \in G$, for which

$$T^{-1}T = I = TT^{-1}$$

More generally if

$$T'T = I$$

then we say that $T \in G^{-1}_{left}$ is *left invertible* and $T' \in G^{-1}_{right}$ is *right invertible*; we observe

$$G^{-1} = G^{-1}_{left} \cap G^{-1}_{right}$$

that the invertible group is the intersection of the left and right invertible semigroups. In general it is quite a complicated business to express the invertibility or otherwise of an element $T \in G$ in terms of the contributing elements $a \in A$, $m \in M$, $n \in N$ and $b \in B$ of (1.5); for the block diagonals of (1.7) it is however rather simple:

2.4 $$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G^{-1}_{left} \Longleftrightarrow a \in A^{-1}_{left} \ \& \ b \in B^{-1}_{left}$$

and

2.5 $$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G^{-1}_{right} \Longleftrightarrow a \in A^{-1}_{right} \ \& \ b \in B^{-1}_{right} \ ,$$

and hence

$$T \in G^{-1} \Longleftrightarrow a \in A^{-1} \ \& \ b \in B^{-1}$$

For upper block triangles [6] something more subtle obtains:

2.7 $$a \in A^{-1}_{left} \ \& \ b \in B^{-1}_{left} \Longrightarrow \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in G^{-1}_{left} \Longrightarrow a \in A^{-1}_{left}$$

2.8 $$a \in A^{-1}_{right} \ \& \ b \in B^{-1}_{right} \Longrightarrow \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in G^{-1}_{right} \Longrightarrow b \in B^{-1}_{right}$$

Also

2.9 $$T \in G^{-1}_{left} \ \& \ a \ \in A^{-1}_{right} \Longrightarrow b \in B^{-1}_{left}$$

and

2.10 $$T \in G^{-1}_{right} \ \& \ b \ \in B^{-1}_{left} \Longrightarrow a \in A^{-1}_{right} \ .$$

It follows, that of the three assertions

$$T \in G^{-1} \ ; \ a \in A^{-1} \ ; \ b \in B^{-1}$$

any two imply the third.

3. SPECTRUM

If the rings G, A and B are complex linear algebras, then invertibility breeds *spectrum*

3.1 $$\sigma^{left}_G(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin G^{-1}_{left}\} \ ,$$

and

3.2 $$\sigma^{right}_G(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin G^{-1}_{right}\} \ ,$$

and then

$$\sigma_G(T) = \sigma^{left}_G(T) \cup \sigma^{right}_G(T)$$

with corresponding notation for $\sigma_A(a)$ and $\sigma_B(b)$. Thus, for a block diagonal $T \in G$, we can rewrite (2.4) and (2.5) in the form

$$\sigma^{left}_G(T) = \sigma^{left}_A(a) \cup \sigma^{left}_B(b)$$

and

$$\sigma^{right}_G(T) = \sigma^{right}_A(a) \cup \sigma^{right}_B(b)$$

For upper block triangles $T \in G$, (2.7) and (2.8) take the form

$$\sigma^{left}_A(a) \subseteq \sigma^{left}_G(T) \subseteq \sigma^{left}_A(a) \cup \sigma^{left}_B(b)$$

and

$$\sigma^{right}_B(b) \subseteq \sigma^{right}_G(T) \subseteq \sigma^{right}_B(b) \cup \sigma^{right}_A(a)$$

Also (2.9) and (2.10) take the form

$$\sigma^{left}_B(b) \subseteq \sigma^{left}_G(T) \cup \sigma^{right}_A(a)$$

and

$$\sigma^{right}_A(a) \subseteq \sigma^{right}_G(T) \cup \sigma^{left}_B(b)$$

It follows that, of the three sets

$$\sigma_G(T) \ ; \ \sigma_A(a) \ ; \ \sigma_B(b)$$

each is a subset of the union of the other two:

$$\sigma_G(T) \subseteq \sigma_A(a) \cup \sigma_B(b) \cup \big(\sigma_A(a) \cap \sigma_B(b)\big)$$

We can improve on this: by (2.7)-(2.10) we have ([6] Theorem 3.1, Theorem 3.2)

$$\sigma_A(a) \cup \sigma_B(b) = \sigma_G(T) \cup \big(\sigma^{right}_A(a) \cap \sigma^{left}_B(b)\big)$$

4. SPECTRAL DISJOINTNESS

When the linear algebras G, A and B are complex Banach algebras, then the spectral theory begins to bite. When the structural idempotent $Q = Q^2 \in G$ is bounded, then it is necessary and sufficient, for spectral disjointness

4.1 $$\sigma_A(a) \cap \sigma_B(b) = \emptyset \ ,$$

that

4.2 $$Q \in \mathrm{Holo}(T) \ :$$

the structural idempotent is a holomorphic function of the generic $T \in G$ of (1.5). This of course means that there exists a holomorphic function $f : U \to \mathbf{C}$ defined on an open neighbourhood of the spectrum $\sigma_G(T) = \sigma_A(a) \cup \sigma_B(b)$ for which

$$Q = f(T) = \frac{1}{2\pi i} \oint_{\sigma(T)} f(z)(zI - T)^{-1} dz$$

is given by the Cauchy integral formula. Inspecting the contour integral, which winds $+1$ times round the spectrum of T, it is sufficient, and obviously necessary, that Q lies in the closed subalgebra generated by all rational functions of T: this is generated by the polynomials in T, together with all possible inverses $(\lambda I - T)^{-1}$. To see why the disjointness (4.1) gives (4.2), it is sufficient to take the *characteristic function*

$$f = \chi_K \ with \ K = \sigma_A(a)$$

Conversely if $Q = f(T)$ then $a = f(1)$ and $b = f(0)$ and hence, by the spectral mapping theorem,

$$\sigma_A(a) \cap \sigma_B(b) \subseteq f^{-1}(1) \cap \ f^{-1}(0) = \emptyset$$

Since the block diagonal T is in the commutant of the idempotent Q, it follows that generally the idempotent Q is also in the commutant of the block diagonal T. If however it turns out ([7] Theorem 1;[10]) that the idempotent Q is a holomorphic function of T, then it follows that the idempotent is in the *double commutant* of the block diagonal:

4.6 $$Q \in \mathrm{comm}^2(T) \ .$$

In finite dimensions, in particular for matrices, it turns out [14] that everything in the double commutant of T is a polynomial in T, and hence (4.6) and (4.2) are equivalent. In general Banach algebras, as we shall see, (4.2) is strictly stronger than (4.6). This whole argument extends [13] to upper and lower block triangles.

We might notice here another "spectral disjointness": if for example $f = p/q$ is a rational function, with "relatively prime" polynomials p and q,

$$f = \frac{p}{q} \in H = C(\Omega) \ with \ \Omega = D_f = \mathbf{C} \setminus q^{-1}(0)$$

then necessary and sufficent for $f(T)$ to exist is

$$\sigma_H(f) \cap \sigma_G(T) = \emptyset$$

none of the poles $q^{-1}(0)$ of f can be in the spectrum of T. For example

$$f = z^{-1} \Longrightarrow \sigma_H(f) = \{0\}$$

thus

$$0 \notin \sigma_G(T) \Longleftrightarrow T \in G^{-1}$$

5. PARTIAL SPECTRAL DISJOINTNESS

In Banach algebras we claim ([7] Theorem 2;[10]) that a weaker "left,right" spectral disjointness is sufficient for the double commutant property :

5.1 $$\sigma_A^{left}(a) \cap \sigma_B^{right}(b) = \emptyset$$

and

5.2 $$\sigma_A^{right}(a) \cap \sigma_B^{left}(b) = \emptyset$$

are together sufficient for (4.6). Specifically we claim that (5.1) implies

$$L_a - R_b \in B(M)_{left}^{-1}$$

the generalized inner derivation $L_a - R_b \in E = B(M)$ has a bounded left inverse. This is the spectral mapping theorem in two variables. With no need of tensor product theory

$$\sigma_E^{left}(L_a, R_b) \subseteq \sigma_E^{left}(L_a) \times \sigma_E^{left}(R_b) \subseteq \sigma_A^{left}(a) \times \sigma_B^{right}(b)$$

and then, since L_a and R_b commute, by the spectral mapping theorem

$$0 \in \sigma_E^{left}(L_a - R_b) \Longrightarrow 0 \in \sigma_A^{left}(a) - \sigma_B^{right}(b)$$

and the spectral disjointness (5.1) excludes 0 from the right hand side. If the inner derivation $L_a - R_b$ has a bounded left inverse then it is also "bounded below", and hence in particular one-to-one: if $m \in M$ there is implication

$$am = mb \Longrightarrow m = 0$$

This is one step on the way to the double commutivity (4.6). If instead (5.2) holds then instead the generalized derivation $L_b - R_a \in F = B(N)$ is left invertible and hence also one-one. Now for arbitrary $(c, u, v, d) \in A \times M \times N \times B$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c & u \\ v & d \end{pmatrix} - \begin{pmatrix} c & u \\ v & d \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ac - ca & au - ub \\ bv - va & bd - db \end{pmatrix}$$

It follows that if $S = \begin{pmatrix} c & u \\ v & d \end{pmatrix}$ commutes with $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ then c commutes with a and d commutes with b, while

$$(L_a - R_b)u = 0 \in M \textit{ and } (L_b - R_a)v = 0 \in N$$

The condition (5.1) therefore ensures that S is a lower block triangle, while (5.2) makes it an upper block triangle, and together they put it in the commutant of Q, givng the inclusion

$$\text{comm}(T) \subseteq \text{comm}(Q)$$

which is equivalent to (4.6).

The condition (5.2) also says that $L_a - R_b$ has a bounded right inverse in $E = B(M)$ and hence is onto:

$$M = aM + Mb$$

which confers a certain splitting "left,right exactness" [9] on the pair (a, b). Dually (5.1) says that also

$$N = bN + Na$$

Notice also [10] that left,right spectral disjointness makes block triangles "similar" to their block diagonals.

6. LINEAR OPERATORS

If the linear algebra $G = L(X)$ is all the linear operators on a linear vector space X, then an *invariant subspace* for $T \in G$ is a subspace $Y \subseteq X$ for which

6.1 $$T(Y) \subseteq Y \subseteq X \ .$$

In the purely linear environment, this will confer block structure on the algebra $L(X)$. For Banach algebra structure we need a Banach space, and to look at bounded operators $T \in B(X)$; evidently we will only be interested in invariant subspaces $Y \subseteq X$ which are norm closed. It is now not clear that this confers block structure on $G = B(X)$: it is necessary that the invariant subspace is also *complemented*. We can however still mount a similar discussion, courtesy of the *quotient*:

$$X/Y = \{[x]_Y \equiv x + Y : x \in X\}$$

the set of cosets $x + Y$, normed by the distance function:

$$\|[x]_Y\| = \mathrm{dist}(x, Y) = \inf\{\|x - y\| : y \in Y\}$$

Now if (6.1) holds then the operator $T \in G = L(X)$ has a *restriction*

$$T_Y \in L(Y)$$

and a *quotient*

$$T_{/Y} \in L(X/Y)$$

defined by setting, for each $y \in Y$ and each $x \in X$,

$$T_Y(y) = Ty \ ; \ T_{/Y}([x]_Y) = [Tx]_Y$$

When $T \in B(X)$ is bounded on a Banach space and $Y \subseteq X$ is closed, then both the restriction and the quotient are also bounded.

As in the block matrix situation the invertibility of $T \in G = L(X)$, $T_Y = a \in A = L(Y)$ and $T_{/Y} = b \in B = L(X/Y)$ are mutually constrained. In the purely linear environment, necessary and sufficient for two-sided invertibility is that an operator is both one-one and onto; for bounded operators on Banach space this continues to be the case, courtesy of the "Open Mapping Theorem". To see the mutual constraints observe [2] the implications

6.7 $$T_Y, T_{/Y} \text{ one-one} \Longrightarrow T \text{ one-one} \Longrightarrow T_Y \text{ one-one} \ ;$$

6.8 $$T_Y, T_{/Y} \text{ onto} \Longrightarrow T \text{ onto} \Longrightarrow T_{/Y} \text{ onto} \ ;$$

6.9 $$T \text{ one-one} \ , \ T_Y \text{ onto} \Longrightarrow T_{/Y} \text{ one-one} \ ;$$

6.10 $$T \text{ onto} \ , \ T_{/Y} \text{ one-one} \Longrightarrow T_Y \text{ onto} \ .$$

To verify these implications, express non singularity properties of T_Y and $T_{/Y}$ in terms of the whole space X:

$$T_Y \text{ one-one} \Longleftrightarrow T^{-1}(0) \cap Y \subseteq O \equiv \{0\}$$

$$T_Y \text{ onto} \Longleftrightarrow Y \subseteq T(Y)$$

$$T_{/Y} \text{ one-one} \Longleftrightarrow T^{-1}(Y) \subseteq Y$$

$$T_{/Y} \text{ onto} \Longleftrightarrow X \subseteq Y + T(X)$$

7. SPECTRAL THEORY

The spectrum of $T \in G$ is the same as always:

$$\sigma(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin G^{-1}\}$$

The *point spectrum* or *eigenvalues* of $T \in G$ is

$$\pi(T) = \{\lambda \in \mathbf{C} : (T - \lambda I)^{-1}(0) \neq \{0\}\} \subseteq \sigma^{left}(T)$$

The *defect spectrum* is in a sense dual to the point spectrum:

$$\pi'(T) = \{\lambda \in \mathbf{C} : (T - \lambda I)(X) \neq X\} \subseteq \sigma^{right}(T)$$

Evidently

$$\sigma(T) = \pi(T) \cup \pi'(T)$$

this is true both for $G = L(X)$ and for $G = B(X)$. From the implications (6.7)-(6.10) it follows that

7.5 $$\sigma(T) \subseteq \sigma(T_Y) \cup \sigma(T_{/Y}) \subseteq \sigma(T) \cup (\sigma(T_Y) \cap \sigma(T_{/Y})) \ .$$

It follows that disjointness

7.6 $$\sigma(T_Y) \cap \sigma(T_{/Y}) = \emptyset$$

implies equality

$$\sigma(T) = \sigma(T_Y) \cup \sigma(T_{/Y})$$

We see (7.6), in the Banach space situation, as a significant property of the invariant subspace $T(Y) \subseteq Y \subseteq X$: when it holds we shall describe the subspace Y as *spectrally invariant*.

Barnes ([1] Proposition 4) has an improvement (*cf* (3.11)) on the right hand side of (7.5): by (6.7)-(6.10)

$$\sigma(T_Y) \cup \sigma(T_{/Y}) = \sigma(T) \cup (\pi'(T_Y) \cap \pi(T_{/Y}))$$

8. PARTIALLY HYPERINVARIANT SUBSPACES

When $T \in G = B(X)$ is a bounded operator on a Banach space X then we describe a subspace $Y \subseteq X$ as an "invariant subspace" for T provided it is norm closed and satisfies the inclusion (6.1). We describe it as *hyperinvariant* provided

8.1 $$\text{comm}(T)Y \subseteq Y \ :$$

this means that there is implication, for $S \in G$,

$$ST = TS \Longrightarrow S(Y) \subseteq Y \subseteq X$$

More generally we shall describe a subspace $Y \subseteq X$ as *comm-square invariant* for $T \in G$ provided

8.3 $$\text{comm}^2(T)Y \subseteq Y \ .$$

More generally still we will say that Y is *holomorphically invariant* for T when

8.4 $$\text{Holo}(T)Y \subseteq Y \ .$$

Evidently this is the same as *inverse invariant*, in the sense that if $\lambda \in \mathbf{C}$ there is implication

$$T - \lambda I \in G^{-1} \Longrightarrow (T - \lambda I)^{-1}Y \subseteq Y$$

There is obvious implication

$$(8.1) \Longrightarrow (8.3) \Longrightarrow (8.4) \Longrightarrow (6.1)$$

It turns out [2] that none of these three implications is reversible; the counterexamples can all be taken to be 2×2 matrices of familiar operators such as the forward and backward shift. It also turns out that a spectrally invariant subspace $Y \subseteq X$, in the sense of (7.6), is hyperinvariant, in the sense (8.1), and also *reducing*: this means that it has an invariant *complement*, in the sense of a closed subspace $Z \subseteq X$ for which

$$Y + Z = X \ , \ Y \cap Z = O \equiv \{0\} \ , \ T(Z) \subseteq Z$$

In general ([2] Example 5) neither of hyperinvariant and reducing implies the other; also ([2] Example 4) hyperinvariant and reducing do not together imply spectral invariance (7.6).

9. BLOCK STRUCTURE FOR OPERATORS

Associated with an invariant subspace $T(Y) \subseteq Y \subseteq X$ for a linear operator $T \in L(X)$ we have a family of block triangular matrices of operators

$$T_U = \begin{pmatrix} T_Y & U \\ 0 & T_{/Y} \end{pmatrix} : \begin{pmatrix} Y \\ X/Y \end{pmatrix} \to \begin{pmatrix} Y \\ X/Y \end{pmatrix}$$

with

9.2 $$U \in L(X/Y, Y) \ ;$$

in the bottom left hand corner we have (*cf* [2] (0.3))

9.3 $$K_Y T J_Y = T_{/Y} K_Y J_Y = K_Y J_Y T_Y = 0 \in L(Y, X/Y) \ .$$

If $f \in \text{Holo}(\sigma(T_Y) \cup \sigma(T_{/Y}))$ then, with

9.4 $$T'_U = \begin{pmatrix} T_Y & T_Y U - U T_{/Y} \\ 0 & T_{/Y} \end{pmatrix}, Q_U = \begin{pmatrix} I_Y & U \\ 0 & 0_{/Y} \end{pmatrix},$$

we have 8

9.5 $$f(T'_U) = \begin{pmatrix} f(T_Y) & f(T_Y)U - Uf(T_{/Y}) \\ 0 & f(T_{/Y}) \end{pmatrix},$$

and also ([13] Theorem 1) necessary and sufficient for spectral invariance (7.6) is that

9.6 $$Q_U \in \text{Holo}(T'_U) \ .$$

As in the block diagonal case, the weaker left,right disjointness conditions (5.1) and (5.2) are ([13] Theorem 3) together sufficient for membership of the double commutant:

9.7 $$Q_U \in \text{comm}^2(T'_U) \ .$$

This turns out ([2] Theorem 7) to be helpful towards a sort of converse [3] to Lomonosov's theorem.

10. PRIMES AND EUCLID

We observe [10] a curious analogy between the spectral theory of operators and the prime factorization of integers: if we write

$$n = p_1^{\nu_1(n)} p_2^{\nu_2(n)} \dots p_k^{\nu_k(n)}$$

for the *prime factorization* of $n \in \mathbf{N} \subseteq \mathbf{Z}$, with

$$p = (p_1, p_2, p_3, \dots) = (2, 3, 5, 7, 11, 13, \dots)$$

for the usual sequence of prime numbers, then it is tempting to interpret

$$\{p_j : j \in \mathbf{N}, \nu_j(n) \neq 0\} = \varpi(n)$$

as some kind of "spectrum" of $n \in \mathbf{N}$. For example

$$n = 1 \Longleftrightarrow \varpi(n) = \emptyset$$

$n \in 1 + \mathbf{N}$ is a prime power if and only if $\varpi(n)$ is a singleton,

$$\#\varpi(n) = 1$$

and is square free if and only if every prime factor occurs with multiplicity one:

$$j \in \mathbf{N} \Longrightarrow \nu_j(n) \leq 1$$

If $\{m, n\} \subseteq 1 + \mathbf{N}$ then ([16] Corollary 4.1.3, Theorem 7.2.2)

10.7 $$\varpi(mn) = \varpi(m) \cup \varpi(n) \ ,$$

and, by the *Euclidean algorithm*, spectral disjointness gives rise to a sort of "exactness":

$$\varpi(m) \cap \varpi(n) = \emptyset \Longrightarrow 1 \in \mathbf{Z}m + n\mathbf{Z}$$

The background motivation, stimulated by Rosenthal-cubed [16], would be to try and apply linear algebra intuitions to elementary number theory. In another direction, Read [15], using essentially (10.7) as the definition, shows that all Banach algebra primes "have closed range".

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PPF DEPENDENT FIXED POINTS OF GENERALIZED WEAKLY CONTRACTION MAPS VIA C_G-SIMULATION FUNCTIONS

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Abstract. In this paper, we introduce the notion of generalized weakly $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$contraction maps with respect to the C_G-simulation function and prove the existence of PPF dependent fixed points of nonself maps in Banach spaces. For such maps, PPF dependent fixed points may not be unique. We provide an example to illustrate this phenomenon.

1. INTRODUCTION AND PRELIMINARIES

In fixed point theory, Banach contraction principle is one of the well known basic fundemental result and it gives an idea for the existence of fixed points with uniqueness in complete metric spaces. In 1997, Alber and Gurre-Delabriere [1] introduced weakly contractive maps which are extensions of contraction maps and obtained fixed point results in the setting of Hilbert spaces. Rhoades [9] extended this concept to metric spaces. Based on this idea, many authors generalized and extended the contraction maps and weakly contractive maps by introducing new functions like $\alpha-$admissible maps, $C-$class function, simulation function etc., for more details we refer [2, 10, 14, 18].

Throughout this paper, we denote the real line by $\mathbb{R}$, $\mathbb{R}^+ = [0,\infty)$, and $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Z}$ is the set of integers.

In 2011, Choudhury, Konar, Rhoades and Metiya [16] introduced the notion of generalized weakly contractive mapping as follows and proved the existence of fixed points of generalized weakly contractive mappings in complete metric spaces.

Definition 1.1. [16] *Let (X,d) be a metric space, T a self-mapping of X. We shall call T a generalized weakly contractive mapping if for any $x,y \in X$,*
$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max\{d(x,y),d(y,Ty)\}),$$

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where
(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous monotone increasing function with* $\psi(t) = 0 \iff t = 0$,
(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function with* $\phi(t) = 0 \iff t = 0$,
(iii) $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$.

Theorem 1.1. [16] *Let* (X, d) *be a complete metric space,* T *a generalized weakly contractive self-mapping of* X. *Then* T *has a unique fixed point.*

In 2012, Samet, Vetro and Vetro [30] introduced the concept of $\alpha-$admissible mappings as follows.

Definition 1.2. [30] *Let* (X, d) *be a metric space. Let* $T : X \to X$ *and* $\alpha : X \times X \to \mathbb{R}^+$ *be two functions. Then* T *is said to be an* $\alpha-$*admissible mapping if*

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1 \tag{1.1}$$

for all $x, y \in X$.

In 2013, Karapınar, Kumam and Salimi [23] introduced the notion of triangular $\alpha-$admissible mappings as follows.

Definition 1.3. [23] *Let* T *be a self-mapping of* X *and let* $\alpha : X \times X \to \mathbb{R}^+$ *be a function. Then* T *is said to be a triangular* $\alpha-$*admissible mapping if*

$$\begin{aligned} \alpha(x, y) \geq 1 &\implies \alpha(Tx, Ty) \geq 1 \text{ and} \\ \alpha(x, z) \geq 1,\ \alpha(z, y) \geq 1 &\implies \alpha(x, y) \geq 1 \end{aligned} \tag{1.2}$$

for all $x, y, z \in X$.

In 2014, Ansari [2] introduced the concept of $C-$class function as follows.

Definition 1.4. [2] *A mapping* $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is called a* $C-$*class function if it is continuous and for any* $s, t \in \mathbb{R}^+$, *the function* G *satisfies the following conditions:*
(i) $G(s, t) \leq s$ *and*
(ii) $G(s, t) = s$ *implies that either* $s = 0$ *or* $t = 0$.
The family of all $C-$*class functions is denoted by* Δ.

Example 1.1. [2] *The following functions belong to* Δ.
(i) $G(s, t) = s - t$ *for all* $s, t \in \mathbb{R}^+$.
(ii) $G(s, t) = ks$ *for all* $s, t \in \mathbb{R}^+$ *where* $0 < k < 1$.
(iii) $G(s, t) = \frac{s}{(1+t)^r}$ *for all* $s, t \in \mathbb{R}^+$ *where* $r \in \mathbb{R}^+$.
(iv) $G(s, t) = s\beta(s)$ *for all* $s, t \in \mathbb{R}^+$ *where* $\beta : \mathbb{R}^+ \to [0, 1)$ *is continuous.*

In 2015, Khojasteh, Shukla and Radenović [24] introduced the notion of simulation function and proved the existence of fixed points of Z_H-contractions in complete metric spaces.

Definition 1.5. [24] *A function* $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is said to be a simulation function if it satisfies the following conditions:*
(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(t, s) < s - t$ *for all* $t, s > 0$;
(ζ_3) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then* $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of all simulation functions in the sense of Definition 1.5 by Z_H.

Example 1.2. [24, 22] *Let $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function with $\phi_i(t) = 0$ if and only if $t = 0$ for $i = 1, 2, 3$. Then the following functions $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ belong to Z_H.*
(i) $\zeta(t,s) = \frac{s}{s+1} - t$ *for all* $t, s \in \mathbb{R}^+$.
(ii) $\zeta(t,s) = \lambda s - t$ *for all* $t, s \in \mathbb{R}^+$ *and* $0 < \lambda < 1$.
(iii) $\zeta(t,s) = \phi_1(s) - \phi_2(t)$ *for all* $t, s \in \mathbb{R}^+$, *where* $\phi_1(t) < t \le \phi_2(t)$ *for all* $t > 0$.

Definition 1.6. [24] *Let (X, d) be a metric space, $T : X \to X$ be a mapping and $\zeta \in Z_H$. Then T is called a Z_H-contraction with respect to ζ if*

$$\zeta(d(Tx, Ty), d(x, y)) \ge 0 \tag{1.3}$$

for all $x, y \in X$.

Theorem 1.2. [24] *Let (X, d) be a complete metric space and $T : X \to X$ be a Z_H-contraction with respect to ζ. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$ where $x_n = Tx_{n-1}$ for any $n \in \mathbb{N}$ converges to the fixed point of T.*

In 2015, Nastasi and Vetro [4] proved the existence of fixed points in complete metric spaces by using simulation functions and a lower semicontinuous function.

Theorem 1.3. [4] *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that there exist a simulation function ζ and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that*

$$\zeta(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), d(x, y) + \varphi(x) + \varphi(y)) \ge 0 \tag{1.4}$$

for any $x, y \in X$. Then T has a unique fixed point u such that $\varphi(u) = 0$.

In 2018, Cho [14] introduced the notion of generalized weakly contractive mappings in metric spaces and proved the existence of its fixed points in complete metric spaces.

Definition 1.7. [14] *Let (X, d) be a metric space, T a self-mapping of X. Then T is called a generalized weakly contractive mapping if*

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \le \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, T, \varphi)) \tag{1.5}$$

for all $x, y \in X$, where
(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and* $\psi(t) = 0 \iff t = 0$,
(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a lower semicontinuous function and* $\phi(t) = 0 \iff t = 0$,
(iii) $m(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty), \frac{1}{2}[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)]\}$,
(iv) $l(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), \ d(y, Ty) + \varphi(y) + \varphi(Ty)\}$ *and*
(v) $\varphi : X \to \mathbb{R}^+$ *is a lower semicontinuous function.*

Theorem 1.4. [14] *Let (X, d) be a complete metric space. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

In 2018, Liu, Ansari, Chandok and Radenović [25] generalized the simulation function introduced by Khojasteh, Shukla and Radenović [24] by using $C-$class functions with C_G property.

Definition 1.8. [25] *A mapping $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ has the property C_G if there exists an $C_G \geq 0$ such that*
(i) for any $s, t \in \mathbb{R}^+$, $G(s,t) > C_G$ implies $s > t$, and
(ii) $G(t,t) \leq C_G$ for all $t \in \mathbb{R}^+$.

Example 1.3. [25] *The following functions are elements of Δ that have property C_G for all $t, s \in \mathbb{R}^+$:*
(i) $G(s,t) = s - t, C_G = r, r \in \mathbb{R}^+$,
(ii) $G(s,t) = s - \frac{(2+t)t}{1+t}, C_G = 0$,
(iii) $G(s,t) = \frac{s}{1+kt}, k \geq 1, C_G = \frac{r}{1+k}, r \geq 2$.

Definition 1.9. [25] *A function $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a C_G-simulation function if it satisfies the following conditions:*
(ζ_4) $\zeta(0,0) = 0$;
(ζ_5) $\zeta(t,s) < G(s,t)$ *for all $t, s > 0$ where $G \in \Delta$ has property C_G;*
(ζ_6) *if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_G$.*

We denote the set of all C_G-simulation functions by Z_G.

Example 1.4. [25] *The following functions ζ belong to Z_G.*
(i) Let $k \in \mathbb{R}$ be such that $k < 1$ and $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kG(s,t) - t$, here $C_G = 0$.
(ii) Let $k \in \mathbb{R}$ be such that $k < 1$ and let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kG(s,t)$, here $C_G = 1$.
(iii) We define $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $\zeta(t,s) = \lambda s - t$, where $\lambda \in (0,1)$ and $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $G(s,t) = s - t$ for any $s, t \in \mathbb{R}^+$.
Clearly $\zeta(0,0) = 0$ and $G \in \Delta$ with $C_G = 0$.
Clearly $\zeta(t,s) = \lambda s - t < s - t = G(s,t)$ and hence ζ satisfies (ζ_5).
If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = k > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then
$\limsup_{n\to\infty} \zeta(t_n, s_n) = \limsup_{n\to\infty} (\lambda s_n - t_n) = \lambda k - k = (\lambda - 1)k < 0$.
Therefore ζ satisfies (ζ_6) and hence $\zeta \in Z_G$.

In 1977, Bernfeld, Lakshmikantham and Reddy [12] introduced the concept of fixed point for mappings that have different domains and ranges which is called PPF (Past, Present and Future) dependent fixed point, for more details we refer [6, 11, 17, 19, 21, 26].

Let $(E, ||.||_E)$ be a Banach space and we denote it simply by E. Let $I = [a,b] \subseteq \mathbb{R}$ and $E_0 = C(I,E)$, the set of all continuous functions on I equipped with the supremum norm $||.||_{E_0}$ and we define it by $||\phi||_{E_0} = \sup_{a \leq t \leq b} ||\phi(t)||_E$ for $\phi \in E_0$.

For a fixed $c \in I$, the Razumikhin class R_c of functions in E_0 is defined by $R_c = \{\phi \in E_0 / \, ||\phi||_{E_0} = ||\phi(c)||_E\}$. Clearly every constant function from I to E belongs to R_c so that R_c is a non-empty subset of E_0 .

Definition 1.10. [12] *Let R_c be the Razumikhin class of continuous functions in E_0. We say that* *(i) the class R_c is algebraically closed with respect to the difference if $\phi - \psi \in R_c$ whenever $\phi, \psi \in R_c$.*

(ii) the class R_c is topologically closed if it is closed with respect to the topology on E_0 by the norm $||.||_{E_0}$.

The Razumikhin class of functions R_c has the following properties.

Theorem 1.5. [5] *Let R_c be the Razumikhin class of functions in E_0. Then*

i) $E_0 = \bigcup_{c\in[a,b]} R_c$.

ii) *for any $\phi \in R_c$ and $\alpha \in \mathbb{R}$, we have $\alpha\phi \in R_c$.*

iii) *the Razumikhin class R_c is topologically closed with respect to the norm defined on E_0.*

iv) $\bigcap_{c\in[a,b]} R_c = \{\phi \in E_0 \mid \phi : I \to E \text{ is constant}\}$.

Definition 1.11. [12] *Let $T : E_0 \to E$ be a mapping. A function $\phi \in E_0$ is said to be a PPF dependent fixed point of T if $T\phi = \phi(c)$ for some $c \in I$.*

Definition 1.12. [12] *Let $T : E_0 \to E$ be a mapping. Then T is called a Banach type contraction if there exists $k \in [0, 1)$ such that $||T\phi - T\psi||_E \le k\,||\phi - \psi||_{E_0}$ for all $\phi, \psi \in E_0$.*

Theorem 1.6. [12] *Let $T : E_0 \to E$ be a Banach type contraction. Let R_c be algebraically closed with respect to the difference and topologically closed. Then T has a unique PPF dependent fixed point in R_c.*

Definition 1.13. [28] *Let $c \in I$. Let $T : E_0 \to E$ and $\alpha : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be an α_c-admissible mapping if*

$$\alpha(\phi(c), \psi(c)) \ge 1 \implies \alpha(T\phi, T\psi) \ge 1 \tag{1.6}$$

for all $\phi, \psi \in E_0$.

In 2013, Hussain, Khaleghizadeh, Salimi and Akbar [21] introduced the concept of α_c-admissible mapping with respect to μ_c and proved theorems for the existence of PPF dependent fixed points and PPF dependent coincidence points for contractive mappings in Banach spaces.

Definition 1.14. [21] *Let $c \in I$ and $T : E_0 \to E$. Let $\alpha, \mu : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be an α_c-admissible mapping with respect to μ_c if*

$$\alpha(\phi(c), \psi(c)) \ge \mu(\phi(c), \psi(c)) \implies \alpha(T\phi, T\psi) \ge \mu(T\phi, T\psi) \tag{1.7}$$

for all $\phi, \psi \in E_0$.

Note that, if we take $\mu(x, y) = 1$ for all $x, y \in E$ then α_c-admissible mapping with respect to μ_c is an α_c-admissible mapping. If we take $\alpha(x, y) = 1$ for all $x, y \in E$ in (1.7) then we say that T is a μ_c-*subadmissible mapping*.

In 2014, Ćirić, Alsulami, Salimi and Vetro [13] introduced the concept of triangular α_c-admissible mapping with respect to μ_c as follows.

Definition 1.15. [13] *Let $c \in I$ and $T : E_0 \to E$. Let $\alpha, \mu : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be a triangular α_c-admissible mapping with respect* *to* μ_c *if*

$$\begin{cases} \text{(i) } \alpha(\phi(c),\psi(c)) \geq \mu(\phi(c),\psi(c)) \implies \alpha(T\phi,T\psi) \geq \mu(T\phi,T\psi) \\ \qquad\qquad\qquad\qquad \text{and} \\ \text{(ii) } \alpha(\phi(c),\psi(c)) \geq \mu(\phi(c),\psi(c)),\ \alpha(\psi(c),\varphi(c)) \geq \mu(\psi(c),\varphi(c)) \\ \qquad \implies \alpha(\phi(c),\varphi(c)) \geq \mu(\phi(c),\varphi(c)). \end{cases} \tag{1.8}$$

for all $\phi,\psi,\varphi \in E_0$.

Lemma 1.7. [13] *Let* T *be a triangular* α_c-*admissible mapping with respect to* μ_c. *We define the sequence* $\{\phi_n\}$ *by* $T\phi_n = \phi_{n+1}(c)$ *for all* $n \in \mathbb{N}\cup\{0\}$, *where* $\phi_0 \in R_c$ *is such that* $\alpha(\phi_0(c),T\phi_0) \geq \mu(\phi_0(c),T\phi_0)$. *Then* $\alpha(\phi_m(c),\phi_n(c)) \geq \mu(\phi_m(c),\phi_n(c))$ *for all* $m,n \in \mathbb{N}$ *with* $m < n$.

Remark. *If* $\mu(x,y) = 1$ *for any* $x,y \in E$ *in Lemma 1.7, we get the following lemma.*

Lemma 1.8. *Let* T *be a triangular* α_c-*admissible mapping. We define the sequence* $\{\phi_n\}$ *by* $T\phi_n = \phi_{n+1}(c)$ *for all* $n \in \mathbb{N}\cup\{0\}$, *where* $\phi_0 \in R_c$ *is such that* $\alpha(\phi_0(c),T\phi_0) \geq 1$. *Then* $\alpha(\phi_m(c),\phi_n(c)) \geq 1$ *for all* $m,n \in \mathbb{N}$ *with* $m < n$.

Remark. *If* $\alpha(x,y) = 1$ *for any* $x,y \in E$ *in Lemma 1.7, we get the following lemma.*

Lemma 1.9. *Let* T *be a triangular* μ_c-*subadmissible mapping. We define the sequence* $\{\phi_n\}$ *by* $T\phi_n = \phi_{n+1}(c)$ *for all* $n \in \mathbb{N}\cup\{0\}$, *where* $\phi_0 \in R_c$ *is such that* $\mu(\phi_0(c),T\phi_0) \leq 1$. *Then* $\mu(\phi_m(c),\phi_n(c)) \leq 1$ *for all* $m,n \in \mathbb{N}$ *with* $m < n$.

Lemma 1.10. [7] *Let* $\{\phi_n\}$ *be a sequence in* E_0 *such that* $||\phi_n - \phi_{n+1}||_{E_0} \to 0$ *as* $n \to \infty$. *If* $\{\phi_n\}$ *is not a Cauchy sequence, then there exists an* $\epsilon > 0$ *and two subsequences* $\{\phi_{m_k}\}$ *and* $\{\phi_{n_k}\}$ *of* $\{\phi_n\}$ *with* $m_k > n_k > k$ *such that* $||\phi_{n_k} - \phi_{m_k}||_{E_0} \geq \epsilon$, $||\phi_{n_k} - \phi_{m_k-1}||_{E_0} < \epsilon$ *and*

i) $\lim_{k\to\infty} ||\phi_{n_k} - \phi_{m_k+1}||_{E_0} = \epsilon$, ii) $\lim_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k}||_{E_0} = \epsilon$,

iii) $\lim_{k\to\infty} ||\phi_{n_k} - \phi_{m_k}||_{E_0} = \epsilon$, iv) $\lim_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} = \epsilon$.

In Section 2, we introduce the notion of generalized weakly $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$contraction map with respect to a C_G-simulation function $\zeta \in Z_G$ and prove the existence of PPF dependent fixed points of these maps in Banach spaces(Theorem 2.1) which is the main result of this paper. For such maps, PPF dependent fixed points may not be unique. In Section 3, we draw some corollaries and an example is provided to illustrate our main result.

2. EXISTENCE OF PPF DEPENDENT FIXED POINTS

We denote

$\Psi = \{\xi \mid \xi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing and $\xi(t) = 0 \iff t = 0\}$

and

$\Phi = \{\eta \mid \eta : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\eta(t) = 0 \iff t = 0\}$.

Based on the results of [4, 14, 16] we introduce a notion of generalized weakly $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$contraction map with respect to $\zeta \in Z_G$ as follows.

Definition 2.1. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function and* $\zeta \in Z_G$. *If there exist* $\xi \in \Psi, \eta \in \Phi, \alpha : E \times E \to \mathbb{R}^+,\ \mu : E \times E \to (0,\infty)$, *and a lower semicontinuous* *function* $\varphi : E \to \mathbb{R}^+$ *such that*

$$\zeta(\alpha(\phi(c), \psi(c))\xi(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)), \\ \mu(\phi(c), \psi(c))(\xi(M(\phi, \psi)) - \eta(N(\phi, \psi)))) \geq C_G \tag{2.1}$$

for all $\phi, \psi \in E_0$*, where* $\xi(t) > \eta(t)$ *for any* $t > 0$*,*

$$M(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi), \\ ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi), \\ \tfrac{1}{2}[||\phi(c) - T\psi||_E + \varphi(\phi(c)) + \varphi(T\psi) + ||\psi(c) - T\phi||_E + \varphi(\psi(c)) + \varphi(T\phi)]\}$$

and

$$N(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$

then we say that T *is a generalized weakly* $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$*contraction map with respect to* ζ*.*

Remark. *(i) If* $\varphi(x) = 0$ *for any* $x \in E$ *in the inequality (2.1) then* T *is called a generalized weakly* $Z_{G,\alpha,\mu,\xi,\eta}-$*contraction map with respect to* ζ*.*

(ii) If $\varphi(x) = 0, \mu(x, y) = 1 = \alpha(x, y)$ *for any* $x, y \in E$ *in the inequality (2.1) then* T *is called a generalized weakly* $Z_{G,\xi,\eta}-$*contraction map with respect to* ζ*.*

(iii) If $\varphi(x) = 0, \mu(x, y) = 1 = \alpha(x, y)$ *for any* $x, y \in E$ *and* $\xi(t) = t$ *for any* $t \in \mathbb{R}^+$ *in the inequality (2.1) then* T *is called a generalized weakly* $Z_{G,\eta}-$*contraction map with respect to* ζ*.*

Theorem 2.1. *Let* $c \in I$*. Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *T is a generalized weakly $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$contraction map with respect to ζ,*
(ii) *T is a triangular α_c-admissible mapping and triangular μ_c-subadmissible mapping,*
(iii) *R_c is algebraically closed with respect to the difference,*
(iv) *if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty, \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$ then $\alpha(\phi_n(c), \phi(c)) \geq 1$ and $\mu(\phi_n(c), \phi(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$ and*
(v) *there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$ and $\mu(\phi_0(c), T\phi_0) \leq 1$.*

Then T *has a PPF dependent fixed point* $\phi^* \in R_c$ *such that* $\varphi(\phi^*(c)) = 0$*.*

Proof. From (v) we have $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$ and $\mu(\phi_0(c), T\phi_0) \leq 1$. Let $\{\phi_n\}$ be a sequence in R_c defined by

$$T\phi_n = \phi_{n+1}(c) \tag{2.2}$$

for any $n = 0, 1, 2, 3...$.
Since R_c is algebraically closed with respect to the difference, we have

$$||\phi_{n+1} - \phi_n||_{E_0} = ||\phi_{n+1}(c) - \phi_n(c)||_E \tag{2.3}$$

for any $n = 0, 1, 2, 3...$.
Since T is triangular α_c-admissible and triangular μ_c-subadmissible mappings, by Lemma 1.8 and Lemma 1.9 we have

$$\alpha(\phi_m(c), \phi_n(c)) \geq 1 \\ \text{and} \\ \mu(\phi_m(c), \phi_n(c)) \leq 1 \tag{2.4}$$

for any $m, n \in \mathbb{N}$ with $m < n$.
If there exists $n \in \mathbb{N} \cup \{0\}$ such that $\phi_n = \phi_{n+1}$ then $T\phi_n = \phi_{n+1}(c) = \phi_n(c)$ and hence $\phi_n \in R_c$ is a PPF dependent fixed point of T.

Suppose that $\phi_n \neq \phi_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$.
If either $M(\phi_n, \phi_{n+1}) = 0$ or $N(\phi_n, \phi_{n+1}) = 0$ then the result is trivial.
Suppose that $M(\phi_n, \phi_{n+1}) \neq 0$ and $N(\phi_n, \phi_{n+1}) \neq 0$.
We consider

$$\begin{aligned}
M(\phi_n, \phi_{n+1}) = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\
&||\phi_n(c) - T\phi_n||_E + \varphi(\phi_n(c)) + \varphi(T\phi_n),\\
&||\phi_{n+1}(c) - T\phi_{n+1}||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_{n+1}),\\
&\tfrac{1}{2}[||\phi_n(c) - T\phi_{n+1}||_E + \varphi(\phi_n(c)) + \varphi(T\phi_{n+1}) +\\
&\qquad ||\phi_{n+1}(c) - T\phi_n||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_n)]\}\\
= \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\
&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\
&||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)),\\
&\tfrac{1}{2}[||\phi_n - \phi_{n+2}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+2}(c)) +\\
&\qquad ||\phi_{n+1} - \phi_{n+1}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+1}(c))]\}\\
= \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\
&||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))\}
\end{aligned}$$

and

$$\begin{aligned}
N(\phi_n, \phi_{n+1}) = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\
&||\phi_{n+1}(c) - T\phi_{n+1}||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_{n+1})\}\\
= \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\
&||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))\}.
\end{aligned}$$

Suppose that

$$\begin{aligned}
\max\{||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))\}\\
= ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)).
\end{aligned}$$

Clearly $M(\phi_n, \phi_{n+1}) = N(\phi_n, \phi_{n+1}) = ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))$.
Since $\phi_{n+1} \neq \phi_{n+2}$, we have $||\phi_{n+1} - \phi_{n+2}||_{E_0} > 0$ and hence
$||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)) > 0$ and which implies that
$\xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) > 0$.
Therefore

$$\begin{aligned}
&\alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1}))\\
&\qquad = \alpha(\phi_n(c), \phi_{n+1}(c))\xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) > 0.
\end{aligned}$$

Since $\xi(t) > \eta(t)$ for any $t > 0$ we have $\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1})) > 0$ and hence $\mu(\phi_n(c), \phi_{n+1}(c))(\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1}))) > 0$.
From (2.1), we have

$$\begin{aligned}
C_G \leq \zeta(&\alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1})),\\
&\mu(\phi_n(c), \phi_{n+1}(c))(\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1}))))\\
< G(&\mu(\phi_n(c), \phi_{n+1}(c))(\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1}))),\\
&\alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1}))). \quad \text{(by } (\zeta_5))
\end{aligned}$$

Now by the property C_G , we get

$$\begin{aligned}
&\mu(\phi_n(c), \phi_{n+1}(c))(\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1})))\\
&\qquad > \alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1})).
\end{aligned}$$

Clearly

$$\begin{aligned}
\xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) &= \xi(M(\phi_n, \phi_{n+1}))\\
&> \xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1}))
\end{aligned}$$

$$\geq \mu(\phi_n(c),\phi_{n+1}(c))(\xi(M(\phi_n,\phi_{n+1}))-\eta(N(\phi_n,\phi_{n+1})))$$
$$>\alpha(\phi_n(c),\phi_{n+1}(c))\xi(||T\phi_n-T\phi_{n+1}||_E+\varphi(T\phi_n)+\varphi(T\phi_{n+1}))$$
$$\geq \xi(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))),$$

a contradiction.
Therefore
$||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c)) > ||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))$
and hence $M(\phi_n,\phi_{n+1}) = N(\phi_n,\phi_{n+1}) = ||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))$.
Let $d_n = ||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))$.
Then the sequence $\{d_n\}$ is a decreasing sequence and hence convergent.
Let $\lim_{n\to\infty} d_n = k$ (say). Suppose that $k>0$.
Since $\phi_n \neq \phi_{n+1}$ we have $d_n = ||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c)) > 0$
and which implies that $\xi(d_n) = \xi(||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))) > 0$.
Similarly $\eta(d_n) > 0$. Clearly $M(\phi_n,\phi_{n+1}) = N(\phi_n,\phi_{n+1}) = d_n$ and hence
$\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)) > 0$.
Similarly $d_{n+1} > 0$ and which implies that $\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1}) > 0$.
From (2.1), we have

$$\begin{aligned} C_G &\leq \zeta(\alpha(\phi_n(c),\phi_{n+1}(c))\xi(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))), \\ &\qquad \mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n))) \\ &= \zeta(\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1}),\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n))) \\ &< G(\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)),\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1})). \text{ (by}(\zeta_5)) \end{aligned} \tag{2.5}$$

Now by the property C_G , we get that
$\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)) > \alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1})$.
Clearly

$$\begin{aligned} \xi(d_n) &> \xi(d_n)-\eta(d_n) \\ &\geq \mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)) \\ &> \alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1}) \\ &\geq \xi(d_{n+1}). \end{aligned}$$

On applying limits as $n\to\infty$, we get that
$\lim_{n\to\infty}\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)) = \lim_{n\to\infty}\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1}) = \xi(k) > 0$.
On applying limit superior to (2.5), we get that

$$\begin{aligned} C_G &\leq \limsup_{n\to\infty}\zeta(\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1}),\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n))) \\ &< C_G, \text{ (by } (\zeta_6)) \end{aligned}$$

a contradiction.
Therefore $k=0$ and hence $\lim_{n\to\infty}[||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))] = 0$.
That is

$$\lim_{n\to\infty}||\phi_n-\phi_{n+1}||_{E_0} = 0 \text{ and } \lim_{n\to\infty}\varphi(\phi_n(c)) = 0. \tag{2.6}$$

We now show that the sequence $\{\phi_n\}$ is a Cauchy sequence in R_c.
Suppose that the sequence $\{\phi_n\}$ is not a Cauchy sequence.
Then there exists an $\epsilon > 0$ and two subsequences $\{\phi_{m_k}\}$ and $\{\phi_{n_k}\}$ of $\{\phi_n\}$ with $m_k > n_k > k$ such that $||\phi_{n_k}-\phi_{m_k}||_{E_0} \geq \epsilon$, $||\phi_{n_k}-\phi_{m_k-1}||_{E_0} < \epsilon$ and by Lemma 1.10 we have,

$$\begin{aligned} &\lim_{k\to\infty}||\phi_{n_k}-\phi_{m_k}||_{E_0} = \epsilon \text{ and} \\ &\lim_{k\to\infty}||\phi_{n_k}-\phi_{m_k+1}||_{E_0} = \epsilon = \lim_{k\to\infty}||\phi_{n_k+1}-\phi_{m_k}||_{E_0} \\ &\qquad = \lim_{k\to\infty}||\phi_{n_k+1}-\phi_{m_k+1}||_{E_0}. \end{aligned} \tag{2.7}$$

Let $d_{n_k m_k} = ||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c))$.
Then from (2.6) and (2.7) it follows that
$\lim_{k\to\infty} d_{n_k m_k} = \epsilon = \lim_{k\to\infty} d_{n_k+1 m_k+1}$.
Since ξ is continuous, we get that

$$\lim_{k\to\infty} \xi(d_{n_k+1 m_k+1}) = \xi(\epsilon) > 0. \tag{2.8}$$

We consider

$$\begin{aligned} M(\phi_{n_k}, \phi_{m_k}) = \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)),\\ &||\phi_{n_k}(c) - T\phi_{n_k}||_E + \varphi(\phi_{n_k}(c)) + \varphi(T\phi_{n_k}),\\ &||\phi_{m_k}(c) - T\phi_{m_k}||_E + \varphi(\phi_{m_k}(c)) + \varphi(T\phi_{m_k}),\\ &\tfrac{1}{2}[||\phi_{n_k}(c) - T\phi_{m_k}||_E + \varphi(\phi_{n_k}(c)) + \varphi(T\phi_{m_k}) +\\ &\quad ||\phi_{m_k}(c) - T\phi_{n_k}||_E + \varphi(\phi_{m_k}(c)) + \varphi(T\phi_{n_k})]\}\\ = \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)),\\ &||\phi_{n_k} - \phi_{n_k+1}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{n_k+1}(c)),\\ &||\phi_{m_k} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{m_k}(c)) + \varphi(\phi_{m_k+1}(c)),\\ &\tfrac{1}{2}[||\phi_{n_k} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k+1}(c)) +\\ &\quad ||\phi_{m_k} - \phi_{n_k+1}||_{E_0} + \varphi(\phi_{m_k}(c)) + \varphi(\phi_{n_k+1}(c))]\}\\ = \max\{&d_{n_k m_k}, d_{n_k n_k+1}, d_{m_k m_k+1}, \tfrac{1}{2}[d_{n_k m_k+1} + d_{m_k n_k+1}]\}. \end{aligned}$$

On applying limits as $k \to \infty$, we get that $\lim_{k\to\infty} M(\phi_{n_k}, \phi_{m_k}) = \epsilon$.

We consider

$$\begin{aligned} N(\phi_{n_k}, \phi_{m_k}) = \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)),\\ &||\phi_{m_k}(c) - T\phi_{m_k}||_E + \varphi(\phi_{m_k}(c)) + \varphi(T\phi_{m_k})\}\\ = \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)),\\ &||\phi_{m_k} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{m_k}(c)) + \varphi(\phi_{m_k+1}(c))\}\\ = \max\{&d_{n_k m_k}, d_{m_k m_k+1}\}. \end{aligned}$$

On applying limits as $k \to \infty$, we get that $\lim_{k\to\infty} N(\phi_{n_k}, \phi_{m_k}) = \epsilon$.

Since ξ, η are continuous, we have
$\lim_{k\to\infty} \xi(M(\phi_{n_k}, \phi_{m_k})) = \xi(\epsilon) > 0$ and $\lim_{k\to\infty} \eta(N(\phi_{n_k}, \phi_{m_k})) = \eta(\epsilon) > 0$.
Therefore

$$\lim_{k\to\infty} (\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) = \xi(\epsilon) - \eta(\epsilon) > 0. \tag{2.9}$$

(since $\xi(t) > \eta(t)$ for $t > 0$)
From (2.8) and (2.9), there exists $k_1 \in \mathbb{N}$ such that

$$\begin{aligned} &\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k})) > \tfrac{\xi(\epsilon)-\eta(\epsilon)}{2} > 0\\ &\text{and}\\ &\xi(d_{n_k+1 m_k+1}) > \tfrac{\eta(\epsilon)}{2} > 0 \end{aligned} \tag{2.10}$$

for any $k \geq k_1$.
From (2.4), we have

$$\begin{aligned} &\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1 m_k+1}) \geq \xi(d_{n_k+1 m_k+1}) > 0 \text{ and}\\ &\mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) > 0. \end{aligned} \tag{2.11}$$

for any $k \geq k_1$.
For any $k \geq k_1$, from (2.1) we have

$$\begin{aligned} C_G \leq \zeta(&\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(||T\phi_{n_k} - T\phi_{m_k}||_E + \varphi(T\phi_{n_k}) + \varphi(T\phi_{m_k})),\\ &\mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k})))) \end{aligned}$$

$$
\begin{aligned}
&= \zeta(\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{n_k+1}(c)) + \varphi(\phi_{m_k+1}(c))),\\
&\qquad \mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))))\\
&= \zeta(\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1 m_k+1}),\\
&\qquad \mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))))
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
&< G(\mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))),\\
&\qquad \alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1 m_k+1})). \quad \text{(by (2.11) and } (\zeta_5))
\end{aligned}
$$

Now by the property C_G, we have

$$
\mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) > \alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1 m_k+1}). \tag{2.13}
$$

Clearly

$$
\begin{aligned}
\xi(M(\phi_{n_k}, \phi_{m_k})) &> \xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))\\
&\geq \mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k})))\\
&> \alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1 m_k+1}) \ \text{(by(2.13))}\\
&\geq \xi(d_{n_k+1 m_k+1}).
\end{aligned}
$$

On applying limits as $k \to \infty$, we get that

$$
\begin{aligned}
&\lim_{k\to\infty} \mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k})))\\
&\qquad = \lim_{k\to\infty} \alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1 m_k+1}) = \xi(\epsilon) > 0.
\end{aligned} \tag{2.14}
$$

On applying limit superior as $k \to \infty$ to (2.12), by (2.13) ,(2.14) and (ζ_6) we get

$$
\begin{aligned}
C_G \leq \limsup_{k\to\infty} &\zeta(\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1 m_k+1}),\\
&\mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))))\\
< C_G,&
\end{aligned}
$$

a contradiction.
Therefore the sequence $\{\phi_n\}$ is a Cauchy sequence in R_c.
Since E_0 is complete, there exists $\phi^* \in E_0$ such that $\phi_n \to \phi^*$.
Since R_c is topologically closed, we have $\phi^* \in R_c$.
Clearly $||\phi^*||_{E_0} = ||\phi^*(c)||_E$. (since $\phi^* \in R_c$)
Since φ is lower semicontinuous function, we have
$\varphi(\phi^*(c)) \leq \liminf_{n\to\infty} \varphi(\phi_n(c)) = 0$ and hence $\varphi(\phi^*(c)) = 0$.
We now show that $T\phi^* = \phi^*(c)$. Suppose that $T\phi^* \neq \phi^*(c)$.
From (2.4) we have $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$.
From (iv) we get that $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ and $\mu(\phi_n(c), \phi^*(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$.
We consider

$$
\begin{aligned}
M(\phi_n, \phi^*) = \max\{&||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)),\\
&||\phi_n(c) - T\phi_n||_E + \varphi(\phi_n(c)) + \varphi(T\phi_n),\\
&||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*),\\
&\tfrac{1}{2}[||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) +\\
&\qquad ||\phi^*(c) - T\phi_n||_E + \varphi(\phi^*(c)) + \varphi(T\phi_n)]\}\\
= \max\{&||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)),\\
&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\
&||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*),\\
&\tfrac{1}{2}[||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) +
\end{aligned}
$$

$$||\phi^* - \phi_{n+1}||_{E_0} + \varphi(\phi^*(c)) + \varphi(\phi_{n+1}(c))]\}$$

and

$$N(\phi_n, \phi^*) = \max\{||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)), \\ ||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*)\}.$$

If either $M(\phi_n, \phi^*) = 0$ or $N(\phi_n, \phi^*) = 0$ then $T\phi^* = \phi^*(c)$, a contradiction.

Therefore $M(\phi_n, \phi^*) > 0$ and $N(\phi_n, \phi^*) > 0$.

Clearly $M(\phi_n, \phi^*) \geq N(\phi_n, \phi^*)$.

Since $\xi(t) > \eta(t)$ for $t > 0$ we have $\xi(M(\phi_n, \phi^*)) \geq \xi(N(\phi_n, \phi^*)) > \eta(N(\phi_n, \phi^*))$ and hence $\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)) > 0$.

Clearly

$$\mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) > 0. \tag{2.15}$$

If $||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*) = 0$ then $\phi_{n+1}(c) = T\phi_n = T\phi^*$.

On applying limits as $n \to \infty$, we get $\phi^*(c) = T\phi^*$, a contradiction.

Therefore $||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*) > 0$ and hence $\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) > 0$.

Clearly

$$\alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) > 0. \tag{2.16}$$

From (2.1) we have

$$\begin{aligned} C_G &\leq \zeta(\alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)), \\ &\qquad \mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)))) \\ &< G(\mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))), \\ &\qquad \alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*))). \end{aligned}$$

Now by the property C_G, we get that

$$\begin{aligned} &\mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) \\ &\quad > \alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)). \end{aligned} \tag{2.17}$$

On applying limits as $n \to \infty$ to $M(\phi_n, \phi^*)$ and $N(\phi_n, \phi^*)$, we get that $\lim_{n\to\infty} M(\phi_n, \phi^*) = ||\phi^*(c) - T\phi^*||_E + \varphi(T\phi^*) = \lim_{n\to\infty} N(\phi_n, \phi^*)$.

Since ξ is continuous, we get that

$\lim_{n\to\infty} \xi(M(\phi_n, \phi^*)) = \xi(||\phi^*(c) - T\phi^*||_E + \varphi(T\phi^*)) > 0$. (since $T\phi^* \neq \phi^*(c)$)

Clearly

$$\begin{aligned} \xi(M(\phi_n, \phi^*)) &> \xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)) \\ &\geq \mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) \\ &> \alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) \\ &\geq \xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) \\ &= \xi(||\phi_{n+1}(c) - T\phi^*||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi^*)). \end{aligned}$$

On applying limits as $n \to \infty$, we get

$$\begin{aligned} &\lim_{n\to\infty} \alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) \\ &= \lim_{n\to\infty} \mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) \\ &= \xi(||\phi^*(c) - T\phi^*||_E + \varphi(T\phi^*)) > 0. \end{aligned}$$

From (2.1) we have

$$\begin{aligned} C_G &\leq \zeta(\alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)), \\ &\qquad \mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)))). \end{aligned}$$

On applying limit superior as $n \to \infty$, by (ζ_6) we get that

$$C_G \leq \limsup_{n\to\infty} \zeta(\alpha(\phi_n(c),\phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)), \mu(\phi_n(c),\phi^*(c))(\xi(M(\phi_n,\phi^*)) - \eta(N(\phi_n,\phi^*)))) < C_G,$$

a contradiction.

Therefore $T\phi^* = \phi^*(c)$ and hence $\phi^* \in R_c$ is a PPF dependent fixed point of T such that $\varphi(\phi^*(c)) = 0$. □

3. COROLLARIES AND EXAMPLES

Corollary 3.1. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *T is a generalized weakly $Z_{G,\alpha,\mu,\xi,\eta}-$contraction map with respect to ζ,*

(ii) *T is a triangular α_c-admissible mapping and triangular μ_c-subadmissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty, \alpha(\phi_n(c),\phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c),\phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N}\cup\{0\}$ then $\alpha(\phi_n(c),\phi(c)) \geq 1$ and $\mu(\phi_n(c),\phi(c)) \leq 1$ for any $n \in \mathbb{N}\cup\{0\}$ and*

(v) *there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c),T\phi_0) \geq 1$ and $\mu(\phi_0(c),T\phi_0) \leq 1$.*

Then T has a PPF dependent fixed point in R_c.

Proof. By taking $\varphi(x) = 0$ for any $x \in E$ in Theorem 2.1 we obtain the desired result. □

By choosing $\alpha(x,y) = 1 = \mu(x,y)$ for any $x,y \in E$ in Corollary 3.1 we get the following corollary.

Corollary 3.2. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *T is a generalized weakly $Z_{G,\xi,\eta}-$contraction map with respect to ζ and*

(ii) *R_c is algebraically closed with respect to the difference.*

Then T has a PPF dependent fixed point in R_c.

By choosing $\xi(t) = t$ for any $t \in \mathbb{R}^+$ in Corollary 3.2 we get the following corollary.

Corollary 3.3. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *T is a generalized weakly $Z_{G,\eta}-$contraction map with respect to ζ and*

(ii) *R_c is algebraically closed with respect to the difference.*

Then T has a PPF dependent fixed point in R_c.

By choosing $\alpha(x,y) = 1 = \mu(x,y)$ for any $x,y \in E$, $\xi(t) = t$ for any $t \in \mathbb{R}^+$ and $C_G = 0$ in Theorem 2.1 we get the following corollary.

Corollary 3.4. *Let $c \in I$ and $\zeta \in Z_G$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *if there exist $\eta \in \Phi$ and a lower semicontinuous function $\varphi : E \to \mathbb{R}^+$ such that*

$$\zeta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi), M(\phi,\psi) - \eta(N(\phi,\psi))) \geq 0$$

for any $\phi,\psi \in E_0$, where $\eta(t) < t$ for any $t > 0$,

$M(\phi,\psi) = \max\{||\phi-\psi||_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)), ||\phi(c)-T\phi||_E+\varphi(\phi(c))+\varphi(T\phi),$

$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi),$$
$$\frac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\},$$
$$N(\phi,\psi) = \max\{||\phi-\psi||_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)), ||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi)\}$$

and

(ii) *R_c is algebraically closed with respect to the difference.*

Then T has a PPF dependent fixed point $\phi^* \in R_c$ such that $\varphi(\phi^*(c)) = 0$.

By choosing $\varphi(x) = 0$ for any $x \in E$ in Corollary 3.4 we get the following corollary.

Corollary 3.5. *Let $c \in I$ and $\zeta \in Z_G$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *if there exists $\eta \in \Phi$ such that*
$$\zeta(||T\phi - T\psi||_E, M(\phi,\psi) - \eta(N(\phi,\psi))) \geq 0$$
for any $\phi,\psi \in E_0$, where $\eta(t) < t$ for any $t > 0$,
$$M(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\phi(c) - T\phi||_E, ||\psi(c) - T\psi||_E,$$
$$\frac{1}{2}[||\phi(c) - T\psi||_E + ||\psi(c) - T\phi||_E]\},$$
$$N(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\psi(c) - T\psi||_E\}$$
and

(ii) *R_c is algebraically closed with respect to the difference.*

Then T has a PPF dependent fixed point in R_c.

By choosing $\zeta(t,s) = \lambda s - t, G(s,t) = s - t$ for any $s,t \in \mathbb{R}^+, C_G = 0$ and $\lambda \in (0,1)$ in Theorem 2.1 we get the following corollary.

Corollary 3.6. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *if there exist $\xi \in \Psi, \eta \in \Phi, \alpha : E \times E \to \mathbb{R}^+,\ \mu : E \times E \to (0,\infty), \lambda \in (0,1)$ and a lower semicontinuous function $\varphi : E \to \mathbb{R}^+$ such that*

$$\alpha(\phi(c),\psi(c))\xi(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq \lambda\mu(\phi(c),\psi(c))(\xi(M(\phi,\psi)) - \eta(N(\phi,\psi))) \tag{3.1}$$

for any $\phi,\psi \in E_0$, where $\xi(t) > \eta(t)$ for any $t > 0$,
$M(\phi,\psi) = \max\{||\phi-\psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi),$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi),$$
$$\frac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\},$$
$$N(\phi,\psi) = \max\{||\phi-\psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\},$$

(ii) *T is a triangular α_c-admissible mapping and triangular μ_c-subadmissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty, \alpha(\phi_n(c),\phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c),\phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N}\cup\{0\}$ then $\alpha(\phi_n(c),\phi(c)) \geq 1$ and $\mu(\phi_n(c),\phi(c)) \leq 1$ for any $n \in \mathbb{N}\cup\{0\}$ and*

(v) *there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$ and $\mu(\phi_0(c), T\phi_0) \leq 1$.*

Then T has a PPF dependent fixed point $\phi^* \in R_c$ such that $\varphi(\phi^*(c)) = 0$.

By choosing $\xi(t) = t,\ t \in \mathbb{R}^+$ in Corollary 3.6 we get the following corollary.

Corollary 3.7. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *if there exist* $\eta \in \Phi, \alpha : E \times E \to \mathbb{R}^+,\ \mu : E \times E \to (0, \infty), \lambda \in (0, 1)$ *and a lower semicontinuous function* $\varphi : E \to \mathbb{R}^+$ *such that*

$$\alpha(\phi(c), \psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq \lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi)) \tag{3.2}$$

for any $\phi, \psi \in E_0$*, where* $\eta(t) < t$ *for any* $t > 0$,

$M(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi),$
$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi),$
$\frac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\},$

$N(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)),$
$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\},$

(ii) T *is a triangular* α_c-*admissible mapping and triangular* μ_c-*subadmissible mapping,*
(iii) R_c *is algebraically closed with respect to the difference,*
(iv) *if* $\{\phi_n\}$ *is a sequence in* E_0 *such that* $\phi_n \to \phi$ *as* $n \to \infty, \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *then* $\alpha(\phi_n(c), \phi(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *and*
(v) *there exists* $\phi_0 \in R_c$ *such that* $\alpha(\phi_0(c), T\phi_0) \geq 1$ *and* $\mu(\phi_0(c), T\phi_0) \leq 1$.

Then T *has a PPF dependent fixed point* $\phi^* \in R_c$ *such that* $\varphi(\phi^*(c)) = 0$.

By choosing If $\varphi(x) = 0$ for any $x \in E$ in Corollay 3.7 we get the following corollary.

Corollary 3.8. *Let* $c \in I$*. Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exist* $\eta \in \Phi, \alpha : E \times E \to \mathbb{R}^+,\ \mu : E \times E \to (0, \infty)$ *and* $\lambda \in (0, 1)$ *such that*

$$\alpha(\phi(c), \psi(c))||T\phi - T\psi||_E \leq \lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \tag{3.3}$$

for any $\phi, \psi \in E_0$*, where* $\eta(t) < t$ *for any* $t > 0$,

$M(\phi, \psi) = \max\{||\phi - \psi||_{E_0}, ||\phi(c) - T\phi||_E, ||\psi(c) - T\psi||_E,$
$\frac{1}{2}[||\phi(c) - T\psi||_E + ||\psi(c) - T\phi||_E]\},$

$N(\phi, \psi) = \max\{||\phi - \psi||_{E_0}, ||\psi(c) - T\psi||_E\},$

(ii) T *is a triangular* α_c-*admissible mapping and triangular* μ_c-*subadmissible mapping,*
(iii) R_c *is algebraically closed with respect to the difference,*
(iv) *if* $\{\phi_n\}$ *is a sequence in* E_0 *such that* $\phi_n \to \phi$ *as* $n \to \infty, \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *then* $\alpha(\phi_n(c), \phi(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *and*
(v) *there exists* $\phi_0 \in R_c$ *such that* $\alpha(\phi_0(c), T\phi_0) \geq 1$ *and* $\mu(\phi_0(c), T\phi_0) \leq 1$.

Then T *has a PPF dependent fixed point in* R_c.

By choosing $\alpha(x, y) = 1 = \mu(x, y)$ for any $x, y \in E$ in Corollay 3.6 we get the following corollary.

Corollary 3.9. *Let* $c \in I$*. Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exist* $\xi \in \Psi, \eta \in \Phi, \lambda \in (0, 1)$ *and a lower semicontinuous function* $\varphi : E \to \mathbb{R}^+$ *such that*

$$\xi(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq \lambda(\xi(M(\phi, \psi)) - \eta(N(\phi, \psi))) \tag{3.4}$$

for any $\phi, \psi \in E_0$, *where* $\xi(t) > \eta(t)$ *for any* $t > 0$,

$M(\phi,\psi) = \max\{||\phi-\psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi),$
$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi),$
$\frac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\},$

$N(\phi,\psi) = \max\{||\phi-\psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)),$
$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\},$

(ii) R_c *is algebraically closed with respect to the difference.*
Then T *has a PPF dependent fixed point* $\phi^* \in R_c$ *such that* $\varphi(\phi^*(c)) = 0$.

By choosing $\varphi(x) = 0$ for any $x \in E$ in Corollay 3.9 we get the following corollary.

Corollary 3.10. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exist* $\xi \in \Psi, \eta \in \Phi$ *and* $\lambda \in (0,1)$ *such that*

$$\xi(||T\phi - T\psi||_E) \le \lambda(\xi(M(\phi,\psi)) - \eta(N(\phi,\psi))) \tag{3.5}$$

for any $\phi, \psi \in E_0$, *where* $\xi(t) > \eta(t)$ *for any* $t > 0$,

$M(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\phi(c) - T\phi||_E, ||\psi(c) - T\psi||_E,$
$\frac{1}{2}[||\phi(c) - T\psi||_E + ||\psi(c) - T\phi||_E]\},$

$N(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\psi(c) - T\psi||_E\},$

(ii) R_c *is algebraically closed with respect to the difference.*
Then T *has a PPF dependent fixed point in* R_c.

By choosing $\xi(t) = t$ for any $t \in \mathbb{R}^+$ in Corollary 3.10 we get the following corollary.

Corollary 3.11. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exist* $\eta \in \Phi$ *and* $\lambda \in (0,1)$ *such that*

$$||T\phi - T\psi||_E \le \lambda(M(\phi,\psi) - \eta(N(\phi,\psi)))$$

for any $\phi, \psi \in E_0$, *where* $\eta(t) < t$ *for any* $t > 0$,

$M(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\phi(c) - T\phi||_E, ||\psi(c) - T\psi||_E,$
$\frac{1}{2}[||\phi(c) - T\psi||_E + ||\psi(c) - T\phi||_E]\},$

$N(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\psi(c) - T\psi||_E\}$

and

(ii) R_c *is algebraically closed with respect to the difference.*
Then T *has a PPF dependent fixed point in* R_c.

We present the following example in support of Theorem 2.1, which suggests that under the hypotheses of Theorem 2.1, T may have more than one fixed point.

Example 3.1. *Let* $E = \mathbb{R}$, $c = 1 \in I = [\frac{1}{2}, 2] \subseteq \mathbb{R}$, $E_0 = C(I,E)$.
We define $T : E_0 \to E, \alpha : E \times E \to \mathbb{R}^+, \mu : E \times E \to (0,\infty)$ *by*

$$T\phi = \begin{cases} -2 & \text{if} \quad \phi(c) \le 0 \\ \frac{3\phi(c)-4}{2} & \textit{if} \quad 0 \le \phi(c) < \frac{1}{2} \\ -\frac{1}{2} & \text{if} \quad \phi(c) \ge \frac{1}{2}, \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1 & \textit{if} \quad x \ge y \\ 0 & \text{if} \quad x < y, \end{cases}$$

and

$$\mu(x,y) = \begin{cases} \frac{1}{\sqrt{2}} & \textit{if} \quad x \ge y \\ 2 & \text{if} \quad x < y. \end{cases}$$

We first prove that T is an α_c-admissible mapping.
For any $\phi,\psi\in E_0$, we suppose that $\alpha(\phi(c),\psi(c))\geq 1$.
From the definition of α, we get $\phi(c)\geq\psi(c)$.
Case (i): *Suppose that $0\leq\phi(c),\psi(c)<\frac{1}{2}$.*
Clearly $3\phi(c)-4\geq 3\psi(c)-4$ and which implies that $\frac{3\phi(c)-4}{2}\geq\frac{3\psi(c)-4}{2}$.
Therefore $T\phi\geq T\psi$ and hence $\alpha(T\phi,T\psi)\geq 1$.
Case (ii): *Suppose that $\phi(c),\psi(c)\geq\frac{1}{2}$.*
Clearly $T\phi=-\frac{1}{2}=T\psi$ and which implies that $\alpha(T\phi,T\psi)\geq 1$.
Case (iii): *Suppose that $\phi(c),\psi(c)\leq 0$.*
Clearly $T\phi=-2=T\psi$ and which implies that $\alpha(T\phi,T\psi)\geq 1$.
Case (iv): *Suppose that $0\leq\phi(c)<\frac{1}{2}$ and $\psi(c)\leq 0$.*
Since $\phi(c)\geq 0$ we have $T\phi=\frac{3\phi(c)-4}{2}\geq -2=T\psi$ and which implies that $\alpha(T\phi,T\psi)\geq 1$.
Case (v): *Suppose that $\phi(c)\geq\frac{1}{2}$ and $\psi(c)\leq 0$.*
Clearly $T\phi=-\frac{1}{2}>-2=T\psi$ and which implies that $\alpha(T\phi,T\psi)\geq 1$.
Case (vi): *Suppose that $\phi(c)\geq\frac{1}{2}$ and $0\leq\psi(c)<\frac{1}{2}$.*
Since $\psi(c)\leq 1$ we have $T\phi=-\frac{1}{2}\geq\frac{3\psi(c)-4}{2}=T\psi$ and which implies that $\alpha(T\phi,T\psi)\geq 1$.
From the above cases, we get that T is an α_c-admissible mapping.
For any $\phi,\psi,\gamma\in E_0$, we suppose that $\alpha(\phi(c),\psi(c))\geq 1$ and $\alpha(\psi(c),\gamma(c))\geq 1$.
From the definition of α, we get $\phi(c)\geq\psi(c)\geq\gamma(c)$.
Therefore $\phi(c)\geq\gamma(c)$ and hence $\alpha(\phi(c),\gamma(c))\geq 1$.
Therefore T is a traingular α_c-admissible mapping.
Similarly, we can prove that T is a triangular μ_c-subadmissible mapping.
Let $\lambda=\frac{1}{\sqrt{2}}$. Then $\lambda\in(0,1)$.
We define $\varphi:E\to\mathbb{R}^+$ by

$$\varphi(x)=\begin{cases}0 & \textit{if }\ x\leq 0\\ x & \text{if }\ 0\leq x<\frac{1}{2}\\ 0 & \text{if }\ x\geq\frac{1}{2}.\end{cases}$$

Clearly φ is a lower semicontinuous function.
We define $\eta:\mathbb{R}^+\to\mathbb{R}^+$ by $\eta(t)=\frac{t}{2}$ for any $t\in\mathbb{R}^+$. Clearly $\eta\in\Phi$.
Let $\phi,\psi\in E_0$.
If $\phi(c)<\psi(c)$ then from the definition of α, the inequality (3.2) trivially holds.
Without loss of generality, we assume that $\phi(c)\geq\psi(c)$.
From the definition of α, we get $T\phi\geq T\psi$.
We consider
$||T\phi-T\psi||_E+\varphi(T\phi)+\varphi(T\psi)\leq T\phi-T\psi+T\phi+T\psi=2\ T\phi$.
Therefore

$$\alpha(\phi(c),\psi(c))(||T\phi-T\psi||_E+\varphi(T\phi)+\varphi(T\psi))\leq 2\ T\phi. \tag{3.6}$$

Also we have

$$\begin{aligned}M(\phi,\psi)=\max\{&||\phi-\psi||_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)),||\phi(c)-T\phi||_E+\varphi(\phi(c))+\varphi(T\phi),\\ &||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi),\\ &\tfrac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\}\\ \geq\max\{&||\phi-\psi||_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)),||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi)\}\end{aligned}$$

which implies that

$$\begin{aligned} M(\phi,\psi)-\eta(N(\phi,\psi)) &\geq \tfrac{1}{2}\max\{\|\phi-\psi\|_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)),\\ &\qquad \|\psi(c)-T\psi\|_E+\varphi(\psi(c))+\varphi(T\psi)\}\\ &\geq \tfrac{1}{2}\max\{\|\phi(c)-\psi(c)\|_E+\varphi(\phi(c))+\varphi(\psi(c)),\\ &\qquad \|\psi(c)-T\psi\|_E+\varphi(\psi(c))+\varphi(T\psi)\}\\ &= \tfrac{1}{2}\max\{\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)),\\ &\qquad \|\psi(c)-T\psi\|_E+\varphi(\psi(c))+\varphi(T\psi)\}.\\ &\qquad\qquad \textit{(since } \phi(c)\geq\psi(c)\textit{)} \end{aligned}$$

Therefore

$$M(\phi,\psi)-\eta(N(\phi,\psi)) \geq \tfrac{1}{2}\max\{\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)), \|\psi(c)-T\psi\|_E+\varphi(\psi(c))+\varphi(T\psi)\}. \tag{3.7}$$

Case (i): Suppose that $T\psi=\psi(c)$.
If $\psi\in R_c$ then ψ is a PPF dependent fixed point of T and hence the result holds.
Let us suppose $\psi\notin R_c$.
We define $\psi_1 : I\to E$ by $\psi_1(x)=\psi(c)$, $x\in I$. Clearly $\psi_1\in R_c$.
From the definition of T, we have

$$T\psi_1=\begin{cases} -2 & \text{if } \psi_1(c)\leq 0\\ \frac{3\psi_1(c)-4}{2} & \textit{if } 0\leq\psi_1(c)<\frac{1}{2}\\ -\frac{1}{2} & \text{if } \psi_1(c)\geq\frac{1}{2}. \end{cases}$$

That is

$$T\psi_1=\begin{cases} -2 & \text{if } \psi(c)\leq 0\\ \frac{3\psi(c)-4}{2} & \textit{if } 0\leq\psi(c)<\frac{1}{2}\\ -\frac{1}{2} & \text{if } \psi(c)\geq\frac{1}{2}. \end{cases}$$

Therefore $T\psi_1=T\psi=\psi(c)=\psi_1(c)$.
Hence ψ_1 is a PPF dependent fixed point of T in R_c and the result follows.
Case (ii): Suppose that $\psi(c)<T\psi$.
From the definition of T we have $\psi(c)<-2$ and hence $T\psi=-2$.
Since $\phi(c)\geq\psi(c)$ we have $\phi(c)\leq 0$ or $0\leq\phi(c)<\frac{1}{2}$ or $\phi(c)\geq\frac{1}{2}$.
Suppose that $\phi(c)\leq 0$. Clearly $T\phi=-2$.
From (3.7) we have

$$\begin{aligned} M(\phi,\psi)-\eta(N(\phi,\psi)) &\geq \tfrac{1}{2}\max\{\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)),\\ &\qquad \|\psi(c)-T\psi\|_E+\varphi(\psi(c))+\varphi(T\psi)\}\\ &= \tfrac{1}{2}\max\{\phi(c)-\psi(c), T\psi-\psi(c)\}\\ &\qquad \textit{(since } \varphi(\phi(c))=\varphi(\psi(c))=\varphi(T\psi)=0\textit{)}\\ &\geq \tfrac{1}{2}\max\{0, T\psi-\psi(c)\}\geq\tfrac{1}{2}\max\{0,T\psi-\phi(c)\}.\\ &\qquad \textit{(since } \phi(c)\geq\psi(c)\implies -\psi(c)\geq-\phi(c)\textit{)} \end{aligned}$$

If $\phi(c)<T\psi$ then $T\psi-\phi(c)>0$ and hence
$M(\phi,\psi)-\eta(N(\phi,\psi))\geq\frac{1}{2}(T\psi-\phi(c))=-1-\frac{\phi(c)}{2}$.
Clearly
$\lambda\mu(\phi(c),\psi(c))(M(\phi,\psi)-\eta(N(\phi,\psi)))\geq-\frac{1}{2}-\frac{\phi(c)}{4}\geq 2\ T\phi.$

(since $-\frac{1}{2}-\frac{\phi(c)}{4}\geq-4 \iff \phi(c)\leq 14$)

If $\phi(c)>T\psi$ then $T\psi-\phi(c)<0$ and hence
$M(\phi,\psi)-\eta(N(\phi,\psi))\geq 0>-4=2\ (-2)=2\ T\phi.$
Suppose that $0\leq\phi(c)<\frac{1}{2}$. Clearly $T\phi=\frac{3\phi(c)-4}{2}$.
From (3.7) we have
$M(\phi,\psi)-\eta(N(\phi,\psi))\geq\frac{1}{2}\max\{\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)),$

$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \phi(c), T\psi - \psi(c)\}$$
(since $\varphi(\psi(c)) = \varphi(T\psi) = 0$*)*
$$= \tfrac{1}{2}\max\{2\phi(c) - \psi(c), T\psi - \psi(c)\}$$
$$\geq \tfrac{1}{2}\max\{2\psi(c) - \psi(c), T\psi - \psi(c)\}$$
$$= \tfrac{1}{2}\max\{\psi(c), T\psi - \psi(c)\}$$
$$= \tfrac{1}{2}(T\psi - \psi(c)) = -1 - \tfrac{\psi(c)}{2} \geq -1 - \tfrac{\phi(c)}{2}.$$
(since $\psi(c) < -2$ *and* $T\psi - \psi(c) > 0$*)*

Clearly
$\lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq -\frac{1}{2} - \frac{\phi(c)}{4} \geq 2\ T\phi.$
(since $-\frac{1}{2} - \frac{\phi(c)}{4} \geq 3\phi(c) - 4 \iff \phi(c) \leq \frac{14}{13}$*)*

Suppose that $\phi(c) \geq \frac{1}{2}$*. Clearly* $T\phi = -\frac{1}{2}$*.*
From (3.7) we have
$$M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \tfrac{1}{2}\max\{\phi(c) - \psi(c), T\psi - \psi(c)\}$$
(since $\varphi(\phi(c)) = \varphi(\psi(c)) = \varphi(T\psi) = 0$*)*
$$= \tfrac{1}{2}(\phi(c) - \psi(c))$$
(since $\phi(c) > T\psi$ *we have* $\phi(c) - \psi(c) > T\psi - \psi(c) > 0$*)*
$$> 0.$$

Clearly
$\lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) > 0 > -1 = 2(-\frac{1}{2}) = 2\ T\phi.$

Case (iii): Suppose that $\psi(c) > T\psi$*.*
From the definition of T *we have* $0 \leq \psi(c) < \frac{1}{2}$ *or* $-2 < \psi(c) \leq 0$ *or* $\psi(c) \geq \frac{1}{2}$*.*
Sub-case (i): Suppose that $0 \leq \psi(c) < \frac{1}{2}$*. Clearly* $T\psi = \frac{3\psi(c)-4}{2} < 0$*.*
Since $\phi(c) \geq \psi(c)$ *we have either* $0 \leq \phi(c) < \frac{1}{2}$ *or* $\phi(c) \geq \frac{1}{2}$*.*
Suppose that $0 \leq \phi(c) < \frac{1}{2}$*. Clearly* $T\phi = \frac{3\phi(c)-4}{2}$
From (3.7) we have
$$M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \phi(c) + \psi(c), \psi(c) - T\psi + \psi(c)\}$$
(since $T\psi < 0$ *we have* $\varphi(T\psi) = 0$*)*
$$= \tfrac{1}{2}\max\{2\phi(c), 2\psi(c) - T\psi\} \geq \tfrac{1}{2}max\{2\psi(c), 2\psi(c) - T\psi\}.$$
(since $\phi(c) \geq \psi(c)$*)*
$$= \psi(c) - \tfrac{T\psi}{2}.\ \textit{(since } T\psi < 0\textit{)}$$

Clearly
$\lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c)}{2} - \frac{T\psi}{4} = \frac{\psi(c)}{2} - \frac{3\psi(c)-4}{8} = \frac{\psi(c)+4}{8} \geq 2T\phi.$
(since $\phi(c) \geq \psi(c)$ *and* $\frac{\psi(c)+4}{8} \geq 3\phi(c) - 4 \iff \psi(c) \leq \frac{36}{23}$*)*

Suppose that $\phi(c) \geq \frac{1}{2}$*. Clearly* $T\phi = -\frac{1}{2}$*.*
From (3.7) we have
$$M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \psi(c), \psi(c) - T\psi + \psi(c)\}$$

(since $\phi(c) \geq \frac{1}{2}$ *and* $T\psi < 0$ *we have* $\varphi(\psi(c)) = \varphi(T\psi) = 0$*)*

$= \frac{1}{2}\max\{\phi(c), 2\psi(c) - T\psi\} \geq \frac{1}{2}max\{\psi(c), 2\psi(c) - T\psi\}.$ *(since* $\phi(c) \geq \psi(c)$*)*

$= \psi(c) - \frac{T\psi}{2}$. *(since* $T\psi < 0$*)*

Clearly

$$\lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c)}{2} - \frac{T\psi}{4} = \frac{\psi(c)}{2} - \frac{3\psi(c)-4}{8}$$

$$= \frac{\psi(c)+4}{8} \geq 2(-\frac{1}{2}) = 2\ T\phi.$$

(since $\frac{\psi(c)+4}{8} \geq -1 \iff \psi(c) \geq -12$)

<u>*Sub-case (ii)*</u>*: Suppose that* $-2 < \psi(c) \leq 0$. *Clearly* $T\psi = -2$.

Since $\phi(c) \geq \psi(c)$ *we have either* $-2 < \phi(c) \leq 0$ *or* $0 \leq \phi(c) < \frac{1}{2}$ *or* $\phi(c) \geq \frac{1}{2}$.

Suppose that $-2 < \phi(c) \leq 0$. *Clearly* $T\phi = -2$.

From (3.7) we have

$$M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$

$$= \frac{1}{2}\max\{\phi(c) - \psi(c), \psi(c) - T\psi\}$$

(since $\phi(c), \psi(c), T\psi < 0$ *we have* $\varphi(T\psi) = \varphi(\phi(c)) = \varphi(\psi(c)) = 0$*)*

$\geq \frac{1}{2}\max\{0, \psi(c) + 2\} = \frac{\psi(c)+2}{2}$. *(since* $\psi(c) + 2 > 0$*)*

Clearly

$$\lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c)+2}{4} \geq -4 = 2\ T\phi.$$

(since $\frac{\psi(c)+2}{4} \geq -4 \iff \psi(c) \geq -18$*)*

Suppose that $0 \leq \phi(c) < \frac{1}{2}$. *Clearly* $T\phi = \frac{3\phi(c)-4}{2}$.

From (3.7) we have

$$M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$

$$= \frac{1}{2}\max\{\phi(c) - \psi(c) + \phi(c), \psi(c) - T\psi\}$$

(since $\psi(c), T\psi \leq 0$ *we have* $\varphi(T\psi) = \varphi(\psi(c)) = 0$*)*

$\geq \frac{1}{2}\max\{\phi(c), \psi(c) + 2\} \geq \frac{1}{2}max\{\psi(c), \psi(c) + 2\}$ *(since* $\psi(c) + 2 > 0$*)*

$= \frac{\psi(c)+2}{2}$.

Clearly

$$\lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c)+2}{4} \geq 2\ T\phi.$$

(since $\phi(c) \geq \psi(c)$ *and* $\frac{\psi(c)+2}{4} \geq 3\phi(c) - 4 \iff \psi(c) \leq \frac{18}{11}$)

Suppose that $\phi(c) \geq \frac{1}{2}$. *Clearly* $T\phi = -\frac{1}{2}$.

From (3.7) we have

$$M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$

$$= \frac{1}{2}\max\{\phi(c) - \psi(c), \psi(c) - T\psi\}$$

(since $\psi(c), T\psi \leq 0$ *and* $\phi(c) \geq \frac{1}{2}$ *we have* $\varphi(T\psi) = \varphi(\phi(c)) = \varphi(\psi(c)) = 0$*)*

$\geq \frac{1}{2}\max\{0, \psi(c) + 2\} = \frac{\psi(c)+2}{2}$. *(since* $\psi(c) + 2 > 0$*)*

Clearly

$\lambda\mu(\phi(c),\psi(c))(M(\phi,\psi)-\eta(N(\phi,\psi)))\geq\frac{\psi(c)+2}{4}\geq 2\ T\phi.$
(since $\frac{\psi(c)+2}{4}\geq -1 \iff \psi(c)\geq -6$*)*

Sub-case (iii): Suppose that $\psi(c)\geq\frac{1}{2}$*. Clearly* $T\psi=-\frac{1}{2}$*.*
Since $\phi(c)\geq\psi(c)$ *we have* $\phi(c)\geq\frac{1}{2}$*. Clearly* $T\phi=-\frac{1}{2}$*.*
From (3.7) we have

$$M(\phi,\psi)-\eta(N(\phi,\psi))\geq\frac{1}{2}\max\{\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)),\ ||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi)\}$$
$$=\frac{1}{2}\max\{\phi(c)-\psi(c),\psi(c)-T\psi\}$$

(since $T\psi\leq 0$ *and* $\psi(c),\phi(c)\geq\frac{1}{2}$ *we have* $\varphi(T\psi)=\varphi(\phi(c))=\varphi(\psi(c))=0$*)*

$$\geq\frac{1}{2}\max\{0,\psi(c)+\tfrac{1}{2}\}=\frac{\psi(c)}{2}+\frac{1}{4}.$$ *(since* $\psi(c)+\frac{1}{2}>0$*)*

Clearly
$\lambda\mu(\phi(c),\psi(c))(M(\phi,\psi)-\eta(N(\phi,\psi)))\geq\frac{\psi(c)}{4}+\frac{1}{8}\geq 2\ T\phi.$
(since $\frac{\psi(c)}{4}+\frac{1}{8}\geq -1 \iff \psi(c)\geq -\frac{9}{2}$*)*

From all the above cases, we get

$$\lambda\mu(\phi(c),\psi(c))(M(\phi,\psi)-\eta(N(\phi,\psi)))\geq\alpha(\phi(c),\psi(c))(||T\phi-T\psi||_E+\varphi(T\phi)+\varphi(T\psi)).$$

Therefore the inequality (3.2) is holds.
Let $\{\phi_n\}$ *be a sequence in* E_0 *such that* $\alpha(\phi_n(c),\phi_{n+1}(c))\geq 1$ *and* $\mu(\phi_n(c),\phi_{n+1}(c))\leq 1$ *for any* $n\in\mathbb{N}\cup\{0\}$*.*
Then from the definition of α*, we have* $\phi_n(c)\geq\phi_{n+1}(c)$ *for any* $n\in\mathbb{N}\cup\{0\}$ *and hence convergent. Since* $\mathbb{R}$ *is complete, there exists* $r\in\mathbb{R}$ *such that* $\phi_n(c)\to r$ *as* $n\to\infty$*.*
We define $\gamma:I\to E$ *by* $\gamma(x)=r, x\in I$*. Then* $\gamma\in R_c$ *and* $\gamma(c)=r$*.*
Therefore $\phi_n(c)\to\gamma(c)$ *as* $n\to\infty$*. Clearly* $\phi_n(c)\geq\gamma(c)$ *for any* $n\in\mathbb{N}\cup\{0\}$*.*
From the definition of α *and* μ*, we get* $\alpha(\phi_n(c),\gamma(c))\geq 1$ *and* $\mu(\phi_n(c),\gamma(c))\leq 1$ *for any* $n\in\mathbb{N}\cup\{0\}$*. Therefore the condition (iv) is satisfied.*
For any $n\in\mathbb{R}$*, we define* $\phi_n:I\to E$ *by*

$$\phi_n(x)=\begin{cases} nx^2 & \textit{if}\ \ x\in[\frac{1}{2},1]\\ \frac{n}{x^2} & \text{if}\ \ x\in[1,2].\end{cases}$$

Clearly $\phi_n\in E_0, ||\phi_n||_{E_0}=||\phi_n(c)||_E$ *and hence* $\phi_n\in R_c$ *for any* $n\in\mathbb{R}$*.*
Let $\mathrm{F}_0=\{\phi_n\mid n\in\mathbb{R}\}$*. Then* $\mathrm{F}_0\subseteq R_c$ *and* F_0 *is algebraically closed with respect to the difference.*
Clearly $\phi_2(c)\geq T\phi_2$ *and hence* $\alpha(\phi_2(c),T\phi_2)\geq 1$ *and* $\mu(\phi_2(c),T\phi_2)\leq 1$*.*
Therefore the condition (v) is satisfied.

Therefore T *satisfies all the hypotheses of Corollary 3.7 which in turn* T *satisfies all the hypotheses of Theorem 2.1 with* $\zeta(t,s)=\lambda s-t, G(s,t)=s-t,\xi(t)=t$ *for any* $s,t\in\mathbb{R}^+, C_G=0$ *and* $\lambda=\frac{1}{\sqrt{2}}\in(0,1)$ *and hence* $\phi_{-2}\in R_c$ *is a PPF dependent fixed point of* T *such that* $\varphi(\phi_{-2}(c))=0$*.*
We define $\gamma_1:I\to E$ *by*

$$\gamma_1(x)=\begin{cases} -2x & \textit{if}\ \ x\in[\frac{1}{2},1]\\ 2x-4 & \text{if}\ \ x\in[1,2].\end{cases}$$

Clearly $||\gamma_1||_{E_0} = 2 = ||\gamma_1(c)||_E$ *and hence* $\gamma_1 \in R_c$.
We observe that $T\gamma_1 = \gamma_1(c)$. *(since* $\gamma_1(c) = -2 < 0$*, we have* $T\gamma_1 = -2 = \gamma_1(c)$*)*
Therefore $\gamma_1 \in R_c$ *is another PPF dependent fixed point of* T *such that* $\varphi(\gamma_1(c)) = 0$.

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ON CONVEX OPTIMIZATION IN HILBERT SPACES

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Abstract. In this paper, convex optimization techniques are employed for convex optimization problems in infinite dimensional Hilbert spaces. A first order optimality condition is given. Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $x \in \mathbb{R}^n$ be a local solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$. Then $f'(x, d) \geq 0$ for every direction $d \in \mathbb{R}^n$ for which $f'(x, d)$ exists. Moreover, Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $x^* \in \mathbb{R}^n$. If x^* is a local minimum of f, then $\nabla f(x^*) = 0$. A simple application involving the Dirichlet problem is also given. Lastly, we have given optimization conditions involving positive semi-definite matrices.

1. INTRODUCTION

Studies on convex optimization have been carried out by many mathematicians and it still remains interesting. Convex operators, convex vector-functions among others, that is, mappings defined on a convex subset of a vector space and with values in an ordered vector space, have been intensively studied in the last years, mainly in connection with optimization problems and mathematical programming in ordered vector spaces (see [1], [3], [5]). The normality of the cone is essential in the proofs of the continuity properties of convex vector-functions. Lipschitz properties of continuous convex vector functions defined on an open convex subset of a normed space and with values in a normed space ordered by a normal cone have also been considered [6]. Equicontinuity results for pointwise bounded families of continuous convex mappings have also been studied with many interesting results obtained. It has been shown that a pointwise bounded family of continuous convex mappings, defined on an open convex subset of a Banach space X and with values in a normed space Y ordered by a normal cone, is locally equi-Lipschitz on X. Equicontinuity and equi-Lipschitz results for families of continuous convex mappings defined on open convex subsets of Baire topological vector spaces or of barrelled locally convex spaces and taking values in a topological vector space respectively in a locally convex space, ordered by a normal cone have also been obtained [7]. We are concerned here with the classical results on optimization of convex functionals in infinite-dimensional real Hilbert spaces. When working with infinite-dimensional spaces, a basic difficulty is that, unlike the case in finite-dimension, being closed and bounded

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does not imply that a set is compact. In reflexive Banach spaces, this problem is mitigated by working in weak topologies and using the result that the closed unit ball is weakly compact. This in turn enables mimicking some of the same ideas in finite-dimensional spaces when working on unconstrained optimization problems. It is the goal of these note to provide a concise coverage of the problem of minimization of a convex function on a Hilbert space ([8]-[10]). The focus is on real Hilbert spaces, where there is further structure that makes some of the arguments simpler. Namely, proving that a closed and convex set is also weakly sequentially closed can be done with an elementary argument, whereas to get the same result in a general Banach space we need to invoke Mazur's Theorem. The ideas discussed in this brief note are of great utility in theory of Partial Differential Equations, where weak solutions of problems are sought in appropriate Sobolev's spaces [2]. After a brief review of the requisite preliminaries, we develop the main results. Though, the results in this note are classical, we provide proofs of key theorems for a self contained presentation. A simple application, regarding the Dirichlet problem, is provided for the purposes of illustration. Also, we recall an important point about notions of compactness and sequential compactness in weak topologies [4]. It is common knowledge that compactness and sequential compactness are equivalent in metric spaces. The situation is not obvious in the case of weak topology of an infinite-dimensional normed linear space [6]. Lastly, we give optimization conditions involving positive semi-definite matrices.

2. PRELIMINARIES

Definition 2.1. *A sequence x_n in a Banach space B is said to converge to $x \in B$ if $\lim_{n\to\infty} x_n = x$. Also a sequence x_n in a Hilbert space H converges weakly to x if, $\lim_{n\to\infty}\langle x_n, u\rangle = \langle x, u\rangle$, $\forall u \in H$. We use the notation $x_n \rightharpoonup x$ to mean that x_n converges weakly to x.*

Definition 2.2. *A set $D \subseteq \mathbb{R}^n$ is bounded if there exists a constant $M > 0$ such that $\|x\| < M$, for all $x \in D$. The set Dis said to be compact if it is closed and bounded.*

Example 2.1. *A closed interval $[a, b]$ is bounded in $\mathbb{R}$, and is therefore also compact. The circle and its interior $\{(x, y)|x^2 + y^2 \leq 1\}$ is a closed set in $\mathbb{R}^2$, and is also bounded, and therefore it is compact. The interval $[0,\infty)$ is closed in $\mathbb{R}$, as its complement $(-\infty, 0)$ is open, but it is not bounded, so it is not compact either.*

Definition 2.3. *A real valued function f on a Banach space B is lower semi-continuous (LSC) if $f(x) \leq \liminf_{n\to\infty} f(x_n)$ for all sequences x_n in B such that $x_n \to x$ (strongly) and weakly sequentially lower semi-continuous (weakly sequentially LSC) if $x_n \rightharpoonup x$.*

Definition 2.4. *A non-empty set W is said to be convex if for all $\beta \in [0, 1]$ and $\forall\, x, y \in W$ $\beta x + (1 - \beta)y \in W$. Let X be a metric space and $W \subseteq X$ a non-empty convex set. A function $f : W \to \mathbb{R}$ is convex if for all $\beta \in [0, 1]$ and $\forall\, x, y \in W$*

$$f(\beta x + (1 - \beta)y) \leq \beta f(x) + (1 - \beta)f(y).$$

Remark. *We note that the function f in the above definition is called strictly convex if the above inequality is strict for $x \neq y$ and $\beta \in (0, 1)$. A function f is convex if and only if its epigraph, epi(f), is convex whereby epi$(f) := f(x, r) \in$*

$dom(f) \times \mathbb{R} : f(x) \leq r$. An optimization problem is convex if both the objective function and feasible set are convex.

Definition 2.5. *Let $\mathbb{R}^n$ be an n-dimensional real space and $W \subseteq \mathbb{R}^n$. A point $x^* \in \mathbb{R}^n$ is called a* global minimizer *of the optimization problem* $\min_{x \in W} f(x)$, *if $x^* \in W$ and $f(x^*) \leq f(x)$, for all $x \in W$.*

Definition 2.6. *Let $\mathbb{R}^n$ be an n-dimensional real space and $W \subseteq \mathbb{R}^n$. A point $x^* \in \mathbb{R}^n$ is called a* local minimizer *of the optimization problem* $\min_{x \in W} f(x)$, *if there exists a neighbourhood N of x^* such that x^* is a global minimizer of the problem $\mathcal{P} = \min_{x \in W \cap N} f(x)$. That is there exists $\varepsilon > 0$ such that $f(x^*) \leq f(x)$, whenever $x^* \in W$ satisfies $\|x^* - x\| \leq \varepsilon$.*

Remark. *Any local minimizer of a convex optimization problem is a global minimizer.*

Theorem 2.1. *(Weierstrass Extreme Value Theorem) Every continuous function on a compact set attains its extreme values on that set.*

Proposition 2.2. *Let B be a Banach space and $f : B \to \mathbb{R}$. Then the following are equivalent. (i). f is (weakly sequentially) LSC.*
(ii). $epi(f)$, is (weakly sequentially) closed.

Remark. *$f : B \to \mathbb{R}$ is coercive if for all $x \in B$, $\lim_{\|x\| \to \infty} f(x) = \infty$. As an example, the function $f(x, y) = x^2 + y^2$ is coercive, as $\lim_{\|x\| \to \infty} f(x, y) = \lim_{\|x\| \to \infty} \|x\|^2 + \infty$. Also, A linear function is never coercive. For instance, a linear function on $\mathbb{R}^2$ has the form $f(x, y) = ax + by + c$, for constants a, b and c, and is equal to c along the line defined by the equation $ax + by = 0$. Since $\|x\| \to \infty$ along this line, but $f(x, y) = c$ along this line, $f(x, y)$ is not coercive. As these examples show, in order for a function to be coercive, it must approach $+\infty$ along any path within $\mathbb{R}^n$ on which $\|x\|$ becomes infinite.*

Proposition 2.3. *Let $f(x)$ be a continuous function defined on all of $\mathbb{R}^n$. If $f(x)$ is coercive, then $f(x)$ has a global minimizer. Furthermore, if the first partial derivatives of $f(x)$ exist on all of $\mathbb{R}^n$, then any global minimizers of $f(x)$ can be found among the critical points of $f(x)$.*

Lemma 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous on all of $\mathbb{R}^n$. The function f is coercive if and only if for every $\beta \in \mathbb{R}$ the set $\{x | f(x) \leq \beta\}$ is compact.*

Proof. First we need to show that the coercivity of f implies the compactness of the sets $\{x | f(x) \leq \beta\}$. We note that the continuity of f implies the closedness of the sets $\{x | f(x) \leq \beta\}$. Therefore, it suffices to show that any set of the form $\{x | f(x) \leq \beta\}$ is bounded. We prove this by contradiction. Suppose to the contrary that there is an $\beta \in \mathbb{R}$ such that the set $S = \{x | f(x) \leq \beta\}$ is unbounded. Then there must exist a sequence $\{x^r\} \subset S$ with $\|x^r\| \to \infty$. But then, by the coercivity of f, we must also have $f(x^r) \to \infty$. This contradicts the fact that $f(x^r) \leq \beta$ for all $r = 1, 2, ...$ Hence the set S must be bounded. Conversely, assume that that each of the sets $\{x | f(x) \leq \beta\}$ is bounded and let $\{x^r\} \subset \mathbb{R}$ be such that $\|x^r\| \to \infty$. Assume that there exists a subsequence of the integers $J \subset \mathbb{N}$ such that the set $\{f(x^r)\}_J$ is bounded above. Then there exists $\beta \in \mathbb{R}$ such that $\{f(x^r)\}_J \subset \{x | f(x) \leq \beta\}$. But this cannot be the case since each of the sets $\{x | f(x) \leq \beta\}$ is bounded while every subsequence of the sequence $\{x^r\}$ is unbounded by definition. Therefore, the set $\{f(x^r)\}_J$ cannot be bounded, and so the sequence $\{f(x^r)\}$ contains no bounded subsequence, that is $f(x^r) \to \infty$. □

Corollary 2.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous on all of $\mathbb{R}^n$. If f is coercive, then f has at least one global minimizer.*

Proof. Let $\beta \in \mathbb{R}$ be chosen so that the set $S = \{x|f(x) \leq \beta\}$ is non-empty. By coercivity, this set is compact. By Weierstrass's Theorem, the problem $\min\{f(x)|x \in S\}$ has at least one global solution. It is easy to see that the set of global solutions to the problem $\min\{f(x)|x \in S\}$ is a global solution to $\mathcal{P}$ and this completes the proof. □

Remark. *We note that coercivity hypothesis is stronger than as strictly required in order to establish the existence of a solution. Indeed, a global minimizer must exist if there exist one non-empty compact lower level set. We do not need all of them to be compact. However, in practice, coercivity is a sufficiency.*

Proposition 2.6. *Let H be an infinite dimensional real separable Hilbert space and let $W \subseteq H$ be a (strongly) closed and convex set. Then, W is weakly sequentially closed.*

Proof. Let the sequence $x_n \rightharpoonup x$ be in W. It only suffices to show that $x \in W$ by showing that $x = \phi_W(x)$, where $\phi_W(x)$ is the projection of x into the closed convex set W. Indeed, we know that the projection $\phi_W(x)$ satisfies the variational inequality, $\langle x - \phi_W(x), y - \phi_W(x)\rangle \leq 0$, for all $y \in W$.
So,

$$\langle x - \phi_W(x), x_n - \phi_W(x)\rangle \leq 0, \ \forall\, n. \tag{2.1}$$

But, $x_n \rightharpoonup x$ be in W so we have,

$$\begin{aligned} \|x - \phi_W(x)\|^2 &= \langle x - \phi_W(x), x - \phi_W(x)\rangle \\ &= \lim_{n\to\infty} \langle x - \phi_W(x), x_n - \phi_W(x)\rangle \end{aligned}$$

Hence, by Equation 2.1 we have $\|x - \phi_W(x)\| = 0$. That is, $x = \phi_W(x)$. □

Lemma 2.7. *Let $f : H \to \mathbb{R}$ be a LSC convex function. Then f is weakly LSC.*

Proof. We know that f is convex iff $epi(f)$ is convex. Moreover, $epi(f)$ is strongly closed because f is (strongly) LSC. By proposition 2.6 we have that $epi(f)$ is weakly sequentially closed implying that f is weakly sequentially LSC.

□

3. MAIN RESULT

Theorem 3.1. *Let H be an infinite dimensional real separable Hilbert space and $W \subseteq H$ be a weakly sequentially closed and bounded set. Let $f : W \to \mathbb{R}$ be weakly sequentially LSC. Then f is bounded from below and has a minimizer on W.*

Proof. The proof has two steps:
(i). f is bounded below.
(ii). There exists a minimizer in W.

Step(i): Suppose that f is not bounded from below. Then there exist a sequence $x_n \in W$ such that $f(x_n) < -n$ for all n. But W is bounded so x_n has a weakly convergent subsequence x_{n_i} Furthermore, W is weakly sequentially closed therefore $x \in W$. Then, since f is weakly sequentially LSC we have $f(x) \leq \liminf_{n\to\infty} f(x_{n_i}) = -\infty$ which is a contradiction. Hence, f is bounded from below.

Step(ii): Let $x_n \in W$ be a minimizing sequence for f that is $f(x_n) \to \inf_W f(x)$. Let $\lambda := \inf_W f(x)$. Since W is bounded and weakly sequentially closed, it follows that x_n has a weakly convergent subsequence has a weakly convergent subsequence $x_{n_i} \in W$. But f is weakly sequentially LSC so we have

$$\lambda \leq f(x^*) \leq \liminf f(x_{n_i}) = \lim f(x_{n_i}) = \lambda$$

So, $f(x^*) = \lambda$ □

Corollary 3.2. *Let H be an infinite dimensional real separable Hilbert space and $W \subseteq H$ be a weakly sequentially closed and bounded set. Let $f : W \to \mathbb{R}^n$ be non-empty and closed, and that $f : W \to \mathbb{R}^n$ is LSC and coercive. Then the optimization problem $\inf_{x\in W} f(x)$ admits at least one global minimizer.*

Proof. With an analogy to the proof of Theorem 3.1 the proof of coercivity is sufficient. □

Theorem 3.3. *A function that is strictly convex on W has a unique minimizer on W.*

Proof. Assume the contrary, that $f(x)$ is convex yet there are two points $x, y \in W$ such that $f(x)$ and $f(y)$ are local minima. Because of the convexity of W every point on the secant line $\beta x + (1-\beta)y$ is in W. Without loss of generality suppose $f(x) \geq f(y)$ if this is not the case, simply relabel the points. We then have $\beta f(x) + (1-\beta)f(y) < f(y), \forall\, \beta \in (0,1)$. But f is strictly convex, we also have $f(\beta x + (1-\beta)y) < f(x), \forall\, \beta \in (0,1)$. Taking β arbitrarily close to 0 along the secant line, $z = \beta x + (1-\beta)y$ remains in W(since W is convex) and $f(z)$ remains strictly below $f(x)$ (because f is strictly convex). Therefore, there is no open ball B containing x such that $f(x) < f(z), \forall\, z(B \cap W) \setminus x$. Therefore, x is not a local minimizer, which is a contradiction. □

In this last part we give an optimality conditions. We give the first order condition for optimality here. Consider the function $\psi : \mathbb{R} \to \mathbb{R}$ given by $\psi(t) = f(x + td)$ for some choice of x and d in $\mathbb{R}^n$. The key variational object in this context is the directional derivative of f at a point x in the direction d given by

$$f'(x, d) = \lim_{t\downarrow 0} \frac{f(x + td) - f(x)}{t}.$$

When f is differentiable at the point $x \in \mathbb{R}^n$, then $f'(x, d) = \nabla f(x)^T d = \psi'(0)$. The next two results give us an optimality condition.

Proposition 3.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $x \in \mathbb{R}^n$ be a local solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$. Then $f'(x, d) \geq 0$ for every direction $d \in \mathbb{R}^n$ for which $f'(x, d)$ exists.*

Theorem 3.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $x^* \in \mathbb{R}^n$. If x^* is a local minimum of f, then $\nabla f(x^*) = 0$.*

Proof. We know that every differentiable function is continuous so by Proposition 3.4 we have we have

$$0 \leq f'(x^*, d) = \nabla f(x^*)^T d,$$

for all $d \in \mathbb{R}^n$. Taking $d = -\nabla f(x^*)$ we obtain $0 \leq -\nabla f(x^*)^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 \leq 0$. Therefore, $\nabla f(x^*) = 0$. □

Example 3.1. *Consider the Dirichlet problem: $-\triangle u = f$, in W and $u = 0$, on ∂W, where $W \subset \mathbb{R}^n$ is a bounded domain, and $f \in L^2(W)$. It is well known that this problem has a weak solution which is convex and continuous, and coercive. Thus, the existence of a unique minimizer is ensured by application of Theorem 3.5.*

In the next results we consider positive definite matrices. We use concepts from linear algebra to obtain simpler, more intuitive criteria for determining whether a symmetric matrix, such as the Hessian of a function at a point, is positive or negative definite or semi-definite. Let T be an $n \times n$ symmetric matrix. A nonzero vector $x \in \mathbb{R}^n$ is an eigenvector of T if there exists a scalar λ such that $Tx = \lambda x$. The scalar λ is called an eigenvalue of T corresponding to x. From the equation $Tx - \lambda x = (T - \lambda I)x = 0$, and the fact that $x \neq 0$ it follows that the matrix $T - \lambda I$ is not invertible. Therefore, any eigenvalue λ of T satisfies $det(T - \lambda I) = 0$. This determinant is a polynomial of degree n in λ, which is called the characteristic polynomial. Therefore, the eigenvalues can be found by computing the characteristic polynomial, and then computing its roots. For a general matrix T, the eigenvalues may be real or complex, because a polynomial with real coefficients can have complex roots, but the eigenvalues of a symmetric matrix T are real. Furthermore, if T is symmetric, there exists an orthogonal matrix P, meaning that $P^t P = I$, such that $T = PDP^t$, where D is a diagonal matrix whose diagonal entries are the eigenvalues of T. The columns of P are orthonormal vectors, meaning that they are orthogonal and are of magnitude 1. They are also the eigenvectors of T. The following result follows immediately.

Theorem 3.6. *Let T be a symmetric matrix on a real Hilbert space. Then the following conditions hold:*
(i). T is positive definite if and only if all of its eigenvalues are positive;
(ii). T is negative definite if and only if all of its eigenvalues are negative;
(iii). T is positive semi-definite if and only if all of its eigenvalues are nonnegative;
(iv). T is negative semi-definite if and only if all of its eigenvalues are non-positive;
(v). $T$is indefinite if and only if at least one of its eigenvalues is positive and at least one of its eigenvalues is negative.

Proof. The proof is trivial. □

Next we demonstrate the use of these conditions for optimization in the next example.

Example 3.2. *Let $f(x, y, z) = x^2 + y^2 + z^2 - 4xy$. Then we have $\nabla f(x, y, z) = (2x - 4y, 2y - 4x, 2z)$, which yields the critical point $(0, 0, 0)$, and*
$Hf(x, y, z) = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. *This matrix has the characteristic polynomial $detHf(x, y, z) - \lambda I = (2 - \lambda)(\lambda + 2)(\lambda - 6)$. Therefore, the eigenvalues are $2, -2$ and 6, which means that the Hessian is indefinite. We conclude that $(0, 0, 0)$ is a saddle point, and there are no global maximizers or minimizers.*

4. CONCLUSION

This work is geared to its extension to portfolio optimization, whereby applications to stochastic optimization with regarding Cox-Ross-Rubinstein model and Hamilton-Jacobi-Bellman Equation will be considered.

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TRIPLE POSITIVE SOLUTIONS FOR A NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we investgate the existence of three positive solutions of a nonlinear fractional differential equations with multi-point and multi-strip boundary conditions. The existence result is obtained by using the Leggett-Williams fixed point theorem. An example is also given to illustrate our main results.

1. INTRODUCTION

Differential equations with fractional derivative have been used to model problems in many fields of science and technology as the mathematical modeling of systems, processes in the fields of physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, finance, etc. (see[3, 11, 15, 16, 17, 21, 25, 26, 28, 31, 36] and the references therein).

Several definitions of fractional derivative have been presented to the literature, amongst are; Riemann-Liouville, Caputo and Grunwald-Letnikov definitions, Atangana-Baleanu operator [4], Liouville-Caputo [22], Caputo-Fabrizio [9], the conformable derivative [18].

Many authors have studied the existence and the multiplicity of solutions of fractional boundary value problems by different approaches. We refer the reader to ([2, 5, 6, 10, 12]). Furthermore, the research in numerical approximations and analytical techniques for the solution of different boundary value problems for time-fractional equation has attracted by ([28, 34, 35, 37]).

Fractional-order multipoint or integral boundary value problems constitute a very interesting and important class of problems. They have been research topics from several authors ([1, 7, 13, 23, 29, 30, 32, 33]). It is worth mentioning that, in 2012, Cabada and Wang [8] investigate the existence of positive solutions of the following nonlinear fractional differential equations with integral boundary value

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conditions:

$$\begin{cases} {}^{C}D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)\, ds, \end{cases} \tag{1.1}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative and $f : [0,1] \times [0,\infty) \to [0,\infty)$ by using the Guo–Krasnoselskii fixed point theorem.

In 2014, Zhou and Jiang [38] studied the existence of positive solutions of the following problem:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u'(0) - \beta u(\xi) = 0, \ u'(1) + \sum_{i=1}^{m-3} \gamma_i u(\eta_i) = 0, \end{cases} \tag{1.2}$$

where α is a real number with $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 \le \gamma_i \le 1$, $i = 1, 2, ..., m-3$, $0 \le \xi < \eta_1 < \eta_2 < ... < \eta_{m-3} \le 1$, and D_{0+}^{α} denotes the Caputo's derivative. They used the fixed point index theory and Krein-Rutman theorem.

In 2016, Guo et al.[14] investigate the existence of at least three positive solutions to the problem

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \ u'(1) = \sum_{j=1}^{\infty} \eta_j u\xi_j, \end{cases}$$

where $2 < \alpha \le 3$, $\eta_j \ge 0$, $0 < \xi_1 < \xi_2 ... < \xi_{j-1} < \xi_j < ... < 1$ $(j = 1, 2, ...)$ and ${}^{C}D_{0+}^{\alpha}$ is the standard Caputo derivative. They applying the Avery–Peterson's fixed point theorem to obtain the existence of multiple positive solutions .

Motivated and inspired by the works mentioned above, we are concerned with the existence of multiple positive solutions of the following nonlinear fractional differential equations with multi-stip conditions

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) + h(t) f(t, u(t)) = 0, \quad t \in (0,1), \\ u^{(i)}(0) = 0, \ i = 2, ..., n-1, \\ u'(0) = \sum_{i=1}^{m-2} b_i u'(\eta_i), \ u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\, ds, \end{cases} \tag{1.3}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivatives, $n-1 < \alpha \le n$, $n \ge 3$ is an integer. Using the Leggett-Williams fixed point theorem, we provide sufficient conditions for the existence of multiple (at least three) positive solutions for the above boundary value problems.

In the remainder, we assume the following conditions:

(H_1) $0 = \eta_0 < \eta_1 < \eta_2 ... < \eta_{m-2} < 1$, $a_i \ge 0$, $b_i \ge 0$, $(i = 1, ..., m-2)$, $0 \le \sum_{i=1}^{m-2} b_i < 1$ and $0 \le \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1}) < 1$, where $m > 2$ is an integer;

(H_2) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous;

(H_3) $h : (0,1) \to [0,+\infty)$ is continuous, and $h(t)$ does not identically vanish on any subinterval of $(0,1)$. Furthermore h satisfies $0 < \int_0^1 h(t)\, dt < +\infty$.

2. PRELIMINARIES

For the reader's convenience, we present some necessary definitions and relations for fractional-order derivatives and integrals, which can be found in [22, 28].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : (0,+\infty) \to \mathbb{R}$ is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the right side is pointwise defined on $(0,+\infty)$ *where* $\Gamma(\cdot)$ *is the Gamma function.*

Definition 2.2. *For a function* $f:[0,+\infty)\to\mathbb{R}$*, the Caputo derivative of fractional order* α *is defined as*

$$^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)\,ds,\quad n=[\alpha]+1,$$

where $[\alpha]$ *denotes the integer part of the real number* α*, provided the right side is pointwise defined on* $(0,+\infty)$.

Lemma 2.1. *Let* $\alpha>0$ *and* $u\in AC^N[0,1]$*. Then the fractional differential equation*

$$^{C}D^{\alpha}u(t)=0,$$

has a unique solution

$$u(t)=c_0+c_1t+c_2t^2+...+c_{N-1}t^{N-1},\quad c_i\in\mathbb{R},\ i=1,2,...,N,$$

where N *is the smallest integer greater than or equal to* α.

Remark 1. *The following property (Dirichlet's formula) of the fractional calculus is well known ([26] p.57)*

$$I^{\nu}I^{\mu}y(t)=I^{\nu+\mu}y(t),\quad t\in[0,1],\ y\in L(0,1),\ \nu+\mu\geq 1,$$

which has the form

$$\int_0^t(t-s)^{\nu-1}\left(\int_0^s(s-\tau)^{\mu-1}y(\tau)\,d\tau\right)ds=\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}\int_0^t(t-s)^{\nu+\mu-1}y(s)\,ds$$

Definition 2.3. *Let* E *be a real Banach space. A nonempty convex closed set* $K\subset E$ *is said to be a cone provided that*

(i) $au\in K$ *for all* $u\in K$ *and all* $a\geq 0$*, and*

(ii) $u,-u\in K$ *implies* $u=0$.

Definition 2.4. *The map* α *is defined as a nonnegative continuous concave functional on a cone* K *of a real Banach space* E *provided that* $\alpha:K\to[0,+\infty)$ *is continuous and*

$$\alpha(tx+(1-t)y)\geq t\alpha(x)+(1-t)\alpha(y)$$

for all $x,y\in K$ *and* $0\leq t\leq 1$.

Let $0<a<b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha,a,b)$ by

$$P_r=\{x\in K\mid \|x\|<r\}$$

and

$$P(\alpha,a,b)=\{x\in K\mid a\leq\alpha(x),\|x\|\leq b\}.$$

Theorem 2.2. [19] *Let* $A:\overline{P_c}\to\overline{P_c}$ *be a completely continuous operator and let* α *be a nonnegative continuous concave functional on* K *such that* $\alpha(x)\leq\|x\|$ *for all* $x\in\overline{P_c}$*. Suppose there exist* $0<a<b<d<c$ *such that*

(C_1) $\{x\in P(\alpha,b,d)\mid\alpha(x)>b\}\neq\emptyset$ *and* $\alpha(Ax)>a$ *for* $x\in P(\alpha,b,d)$*,*

(C_2) $\|Ax\|<a$ *for* $\|x\|\leq a$*, and*

(C_3) $\alpha(Ax)>b$ *for* $x\in P(\alpha,b,c)$ *with* $\|Ax\|>d$.

Then A *has at least three fixed points* x_1*,* x_2 *and* x_3 *in* $\overline{P_c}$ *such that*

$\|x_1\| < a$, $b < \alpha(x_2)$, *and* $\|x_3\| > a$ *with* $\alpha(x_3) < b$.

Lemma 2.3. *For $y \in C[0,1]$, the following boundary value problem*

$$\begin{cases} {}^{C}D_{0+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1), \\ u^{(i)}(0) = 0, \ i = 2, ..., n-1, \\ u'(0) = \sum_{i=1}^{m-2} b_i u'(\eta_i), \ u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\,ds \end{cases} \tag{2.1}$$

has the unique solution

$$u(t) = c_0 + c_1 t - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \tag{2.2}$$

where

$$\begin{aligned} c_0 = {} & \frac{\int_0^1 (1-s)^{\alpha-1} y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right)\Gamma(\alpha)} - \frac{\sum_{i=1}^{m-2} a_i \left[\int_0^{\eta_i} (\eta_i - s)^{\alpha} - \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^{\alpha}\right] y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right)\Gamma(\alpha+1)} \\ & + \frac{\left(1 - \sum_{i=1}^{m-2} a_i \frac{\eta_i^2 - \eta_{i-1}^2}{2}\right) \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right)\left(1 - \sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}, \\ c_1 = {} & -\frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}. \end{aligned} \tag{2.3}$$

Proof. In view of Definition 2.1 and Lemma 2.1, it is clear that equation 2.1 is equivalent to the integral form

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + c_0 + c_1 t + ... + c_{n-1} t^{n-1},$$

where $c_0, c_1, ..., c_{n-1} \in \mathbb{R}$ are arbitrary constants.

Next, using the initial conditions: $u^{(i)}(0) = 0$, $i = 2, ..., n-1$, we get

$$c_2 = c_3 = ... = c_{n-1} = 0,$$

that is,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + c_0 + c_1 t. \tag{2.4}$$

So we get

$$u'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s)\,ds + c_1. \tag{2.5}$$

By $u'(0) = \sum_{i=1}^{m-2} b_i u'(\eta_i)$, we obtain

$$c_1 = -\frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}. \tag{2.6}$$

Integrating the equation 2.4 from η_{i-1} to η_i for $0 \leq \eta_{i-1} \leq \eta_i \leq 1$, $i = 1, ..., m-2$, and using Remark 1, we get

$$\begin{aligned}
\int_{\eta_{i-1}}^{\eta_i} u(t)\,dt &= -\frac{1}{\Gamma(\alpha)} \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\,d\tau \right) ds + c_0 \int_{\eta_{i-1}}^{\eta_i} ds + c_1 \int_{\eta_{i-1}}^{\eta_i} sds \\
&= -\frac{1}{\Gamma(\alpha)} \left[\int_0^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\,d\tau \right) ds + \int_{\eta_{i-1}}^{0} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\,d\tau \right) ds \right] \\
&\quad + c_0 \int_{\eta_{i-1}}^{\eta_i} ds + c_1 \int_{\eta_{i-1}}^{\eta_i} sds \\
&= -\frac{1}{\Gamma(\alpha+1)} \int_0^{\eta_i} (\eta_i - s)^{\alpha} y(s)\,ds + \frac{1}{\Gamma(\alpha+1)} \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^{\alpha} y(s)\,ds \\
&\quad + c_0 (\eta_i - \eta_{i-1}) + c_1 \frac{\eta_i^2 - \eta_{i-1}^2}{2}.
\end{aligned}$$

Then, by the condition $u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\,ds$, we get

$$\begin{aligned}
-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds + c_0 + c_1 &= -\frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{m-2} a_i \int_0^{\eta_i} (\eta_i - s)^{\alpha} y(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{m-2} a_i \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^{\alpha} y(s)\,ds \\
&\quad + c_0 \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1}) + c_1 \sum_{i=1}^{m-2} a_i \frac{\eta_i^2 - \eta_{i-1}^2}{2}.
\end{aligned}$$

Which implies

$$\begin{aligned}
c_0 &= \frac{\int_0^1 (1-s)^{\alpha-1} y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right) \Gamma(\alpha)} - \frac{\sum_{i=1}^{m-2} a_i \left[\int_0^{\eta_i} (\eta_i - s)^{\alpha} - \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^{\alpha} \right] y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right) \Gamma(\alpha+1)} \\
&\quad + \frac{\left(1 - \sum_{i=1}^{m-2} a_i \frac{\eta_i^2 - \eta_{i-1}^2}{2}\right) \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right) \left(1 - \sum_{i=1}^{m-2} b_i\right) \Gamma(\alpha-1)}.
\end{aligned}$$

□

Remark 2. *i) Assume that* (H_1) *hold. Then, for* $y \in C([0,1])$ *and* $y(t) \geq 0$ *by (2.5) and (2.6), we obtain* $u'(t) < 0$ *and*

$$u''(t) = -\frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} y(s)\,ds < 0. \tag{2.7}$$

ii) If we assume that (H_1) *hold, we have*

$$0 \leq \sum_{i=1}^{m-2} a_i \left(\eta_i^2 - \eta_{i-1}^2\right) \leq \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1}) < 1.$$

Lemma 2.4. *Let* (H_1) *satisfied. If* $y(t) \in C[0,1]$ *satisfying* $y(t) \geq 0$*, then the function* u *of (2.2) satisfies* $u(t) \geq 0$*.*

Proof. From Remark 2, $u(t)$ is concave and non-increasing on $[0,1]$. Then

$$\max_{0\leq t\leq 1} u(t) = u(0), \quad \min_{0\leq t\leq 1} u(t) = u(1). \tag{2.8}$$

From the concavity of u , we have

$$\frac{u(\eta_1)}{\eta_1} \geq \frac{u(\eta_2)}{\eta_2} \geq \ldots \geq \frac{u(\eta_{i-1})}{\eta_{i-1}} \geq \frac{u(\eta_i)}{\eta_i} \geq \ldots \geq \frac{u(1)}{1} \tag{2.9}$$

and

$$\int_{\eta_{i-1}}^{\eta_i} u(s)\,ds \geq \frac{1}{2}(\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1})), \tag{2.10}$$

where $\frac{1}{2}(\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1}))$ is the area of the trapezoid under the curve $u(t)$ from $t = \eta_{i-1}$ to $t = \eta_i$ for $i = 1, 2, ..., m-2$. Multiplying both sides of the inequality (2.10) with a_i and combining conditions (2.9), (2.10) and $u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\,ds$, we get

$$\begin{aligned} u(1) &\geq \frac{1}{2}\sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1})) \\ &\geq \frac{1}{2}\sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})(\eta_i u(1) + \eta_{i-1} u(1)) \\ &= \frac{1}{2}\sum_{i=1}^{m-2} a_i \left(\eta_i^2 - \eta_{i-1}^2\right) u(1). \end{aligned}$$

If $u(1) < 0$, we get

$$2 \leq \sum_{i=1}^{m-2} a_i \left(\eta_i^2 - \eta_{i-1}^2\right).$$

This contradicts the fact that $\sum_{i=1}^{m-1} a_i \left(\eta_i^2 - \eta_{i-1}^2\right) < 1$. Then $u(1) \geq 0$. Therefore, we get $u(t) \geq 0$ for $t \in [0,1]$. The proof is complete. □

Lemma 2.5. *Let (H_1) hold. If $y \in C([0,1])$ and $y \geq 0$, then the unique solution u of the problem (2.1) satisfies*

$$\min_{t\in[0,1]} u(t) \geq \gamma \|u\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})(2 - \eta_i - \eta_{i-1})}{2 - \sum_{i=1}^{m-2} a_i \left(\eta_i^2 - \eta_{i-1}^2\right)}. \tag{2.11}$$

Proof. From Remark 2, u is concave and nonincreasing on $[0,1]$. This implies that

$$\|u\| = u(0), \quad \min_{t\in[0,1]} u(t) = u(1)$$

and

$$u(0) \leq u(1) + \frac{u(1) - u(t)}{1-t}(0-1)$$

or

$$u(0)(1-t) \leq u(1)(1-t) + u(t) - u(1). \tag{2.12}$$

By integrating the both sides of the inequality (2.12) from $t = \eta_{i-1}$ to $t = \eta_i$, we have

$$u(0)\int_{\eta_{i-1}}^{\eta_i}(1-t)\,dt \le u(1)\int_{\eta_{i-1}}^{\eta_i}(1-t)\,dt + \int_{\eta_{i-1}}^{\eta_i}u(t)\,dt - u(1)\int_{\eta_{i-1}}^{\eta_i}dt$$

and by the condition $u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\,ds$, we get

$$\begin{aligned} u(0) &\le u(1)\left[1 + \frac{1-\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})}{\sum_{i=1}^{m-2} a_i\left(\eta_i - \eta_{i-1} - \frac{1}{2}\left(\eta_i^2 - \eta_{i-1}^2\right)\right)}\right] \\ &\le u(1)\left[\frac{2-\sum_{i=1}^{m-2} a_i\left(\eta_i^2 - \eta_{i-1}^2\right)}{\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})(2-(\eta_i + \eta_{i-1}))}\right]. \end{aligned}$$

Thus

$$\min_{t\in[0,1]} u(t) \ge \frac{\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})(2-\eta_i - \eta_{i-1})}{2-\sum_{i=1}^{m-2} a_i\left(\eta_i^2 - \eta_{i-1}^2\right)}u(0).$$

□

Let $E = C([0,1])$ be a Banach space of all continuous real functions on $[0,1]$ equipped with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$ for $u \in E$, and define

$$K = \{u \in E \mid u \textit{ is nonnegative concave and nonincreasing on } [0,1]\}.$$

It is obvious that K is a cone.

Define the operator $A : E \to E$ as follows:

$$\begin{aligned} Au(t) = {} & \frac{\int_0^1 (1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})\right)\Gamma(\alpha)} \\ & - \frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i}(\eta_i - s)^{\alpha} - \int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right] h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})\right)\Gamma(\alpha+1)} \\ & + \frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2 - \eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i \int_0^{\eta_i}(\eta_i - s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\ & - \frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i}(\eta_i - s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}t \\ & - \frac{\int_0^t (t-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\Gamma(\alpha)}. \end{aligned} \tag{2.13}$$

Then u is a solution of the boundary value problem (1.3) if and only if it is a fixed point of the operator A.

Lemma 2.6. *Assume that $(H_1)-(H_3)$ hold. Then the operator $A : E \to E$ is completely continuous.*

Proof. Let $u \in K$, then $Au(t) \ge 0$, $(Au)'(t) \le 0$ and $(Au)''(t) \le 0$, $0 \le t \le 1$, consequently, $A : K \to K$. In view of continuity of $h(t)$ and $f(t,u)$, we get A is continuous.

Take $N \subset K$ be bounded, that is, there exists a positive constant l for any $u \in N$, such that $\|u\| \leq l$. Let $L = \max_{t\in[0,1],u\in[0,l]} f(t,u) + 1$, then, for any $u \in N$, we have

$$\begin{aligned}
Au(t) \leq& \frac{\int_0^1 (1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} \\
&+ \frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
\leq& L\left[\frac{\int_0^1 h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}\right. \\
&\left.+\frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\eta_i^{\alpha-2} \int_0^{\eta_i} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}\right].
\end{aligned}$$

Hence, $A(N)$ is uniformly bounded. Now, we will prove that $A(N)$ is equicontinuous. For each $u \in N$, $0 \leq \tau_1 < \tau_2 \leq 1$, we have

$$\begin{aligned}
&|(Au)(\tau_2)-(Au)(\tau_1)| \\
=&\left|\frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}\tau_2 + \frac{\int_0^{\tau_2} (\tau_2-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\Gamma(\alpha)}\right. \\
&\left.-\frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}\tau_1 - \frac{\int_0^{\tau_1} (\tau_1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\Gamma(\alpha)}\right| \\
\leq& \frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}(\tau_2-\tau_1) \\
&+\left|\frac{\int_0^{\tau_2} (\tau_2-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\Gamma(\alpha)} - \frac{\int_0^{\tau_1} (\tau_1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\Gamma(\alpha)}\right| \\
\leq& \frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}(\tau_2-\tau_1)
\end{aligned}$$

$$+\left|\frac{\int_0^{\tau_1}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]h(s)f(s,u(s))ds}{\Gamma(\alpha)}\right|$$
$$+\left|\frac{\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}h(s)f(s,u(s))ds}{\Gamma(\alpha)}\right|$$
$$\leq\frac{L\sum_{i=1}^{m-2}b_i\eta_i^{\alpha-2}\int_0^{\eta_i}h(s)ds}{\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)}(\tau_2-\tau_1)$$
$$+\frac{L\int_0^{\tau_1}h(s)ds}{\Gamma(\alpha)}\left(\tau_2^{\alpha-1}-\tau_1^{\alpha-1}\right)+\frac{L\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds}{\Gamma(\alpha)}(\tau_2-\tau_1)^{\alpha-1}.$$
$$\leq L\left(\frac{\sum_{i=1}^{m-2}b_i\eta_i^{\alpha-2}\int_0^{\eta_i}h(s)ds}{\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)}(\tau_2-\tau_1)+\frac{\int_0^{\tau_1}h(s)ds}{\Gamma(\alpha)}\left(\tau_2^{\alpha-1}-\tau_1^{\alpha-1}\right).\right.$$
$$\left.+\frac{\int_{\tau_1}^{\tau_2}h(s)ds}{\Gamma(\alpha)}(\tau_2-\tau_1)^{\alpha-1}\right).$$

Therefore, $A(N)$ is equicontinuous. Applying the Arzela -Ascoli theorem, we conclude that A is a completely continuous operator. The proof is completed. □

3. MAIN RESULTS

In this section, we discuss the existence of triple positive solutions of the Problem (1.3). We define the nonnegative continuous concave functional on K by

$$\alpha(u)=\min_{0\leq t\leq 1}u(t).$$

It is obvious that, for each $u\in K$, $\alpha(u)\leq\|u\|$. For convenience, we use the following notation. Let

$$M=\frac{\int_0^1 h(s)ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}+\frac{\sum_{i=1}^{m-2}a_i\eta_{i-1}^{\alpha}\int_0^{\eta_{i-1}}h(s)ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$
$$+\frac{\left(1-\sum_{i=1}^{m-2}a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2}b_i\eta_i^{\alpha-2}\int_0^{\eta_i}h(s)ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)},$$
$$m=\frac{\int_0^1(1-s)^{\alpha-1}h(s)ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}-\frac{\sum_{i=1}^{m-2}a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right]h(s)ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$
$$+\frac{\left(1-\sum_{i=1}^{m-2}a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2}b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2}h(s)ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)}$$
$$-\frac{\sum_{i=1}^{m-2}b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2}h(s)ds}{\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)}-\frac{\int_0^1(1-s)^{\alpha-1}h(s)ds}{\Gamma(\alpha)}.$$

Theorem 3.1. *Suppose that the conditions $(H_1)-(H_3)$ hold. In addition, assume there exist non-negative numbers a, b and c such that $0<a<b<\gamma c$, and $f(t,u)$ satisfies the following growth conditions:*

(H_4) $f(t,u) \le \frac{c}{M}$, *for all* $(t,u) \in [0,1] \times [0,c]$,
(H_5) $f(t,u) \le \frac{a}{M}$, *for all* $(t,u) \in [0,1] \times [0,a]$,
(H_6) $f(t,u) > \frac{b}{m}$, *for all* $(t,u) \in [0,1] \times \left[b, \frac{b}{\gamma}\right]$.
Then the boundary value problems (1.3) have at least three positive solutions u_1, u_2 *and* u_3 *such that*
$\|u_1\| < a,\ b < \alpha(u_2),\ \|u_3\| > a,$ *with* $\alpha(u_3) < b$.

Proof. From Lemma 2.6, the operator $A : K \to K$ is completely continuous. Now, we prove that $A : \overline{P_c} \to \overline{P_c}$. For $u \in \overline{P_c}$, we have $\|Au\| = Au(0)$. Then

$$\begin{aligned}
Au(0) &= \frac{\int_0^1 (1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha} - \int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right] h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad + \frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i \int_0^{\eta_i}(\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
&\le \frac{\int_0^1 h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2} a_i\eta_{-1}^{\alpha}\int_0^{\eta_{i-1}} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad + \frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\eta_i^{\alpha-2}\int_0^{\eta_i} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
&\le \frac{c}{M}\left(\frac{\int_0^1 h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}\right. \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i\eta_{-1}^{\alpha}\int_0^{\eta_{i-1}} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad \left.+\frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\eta_i^{\alpha-2}\int_0^{\eta_i} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}\right) \\
&\le c.
\end{aligned}$$

Thus, $\|Au\| \le c$. Consequently, $A : \overline{P_c} \to \overline{P_c}$.

In a completely analogous manner, the condition (H_5) implies that the condition (C_2) of Theorem 2.2 is satisfied for A.

Now, we show that condition (C_1) of Theorem 2.2 is satisfied. Since $\alpha\left(\frac{b}{\gamma}\right) = \frac{b}{\gamma} > b$, then $\left\{u \in P\left(\alpha, b, \frac{b}{\gamma}\right) \mid \alpha(u) > b\right\} \ne \emptyset$. If $u \in P\left(\alpha, b, \frac{b}{\gamma}\right)$, then $b \le u(s) \le \frac{b}{\gamma}$, $s \in [0,1]$.

By condition (H_6), we get

$$\begin{aligned}
\alpha\left(\left(Au\right)(t)\right) = \min_{0\leq t\leq 1}\left(\left(Au\right)(t)\right) = \left(Au\right)(1) \\
&= \frac{\int_0^1 (1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right] h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad + \frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
&\quad - \frac{\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} - \frac{\int_0^1 (1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\Gamma(\alpha)} \\
&\geq \frac{b}{m}\left(\frac{\int_0^1 (1-s)^{\alpha-1} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}\right. \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right] h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad + \frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
&\quad \left. - \frac{\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} - \frac{\int_0^1 (1-s)^{\alpha-1} h(s)\,ds}{\Gamma(\alpha)}\right) \\
&\geq b.
\end{aligned}$$

Therefore, condition (C_1) of Theorem 2.2 is satisfied.

For the condition (C_3) of the Theorem 2.2, we can verify it easily under our assumptions using Lemma 2.5. Here

$$\alpha(Au) = \min_{0\leq t\leq 1}(Au)(t) \geq \gamma\frac{b}{\gamma} = b$$

as long as if $u \in P(\alpha,b,c)$, with $\|Au\| > \frac{b}{\gamma}$.

Therefore, the condition (C_3) of Theorem 2.2 is satisfied. By Theorem 2.2, there exist three positive solutions u_1, u_2 and u_3 such that $\|u_1\| < a$, $b < \alpha(u_2(t))$ and $\|u_3\| > a$, with $\alpha(u_3(t)) < b$. □

4. EXAMPLE

Consider the boundary value problem

$$\begin{cases} D_{0+}^{2,5}u(t) + (1-t)f(t,u(t)) = 0, \quad t\in(0,1), \\ u''(0) = 0, \\ u'(0) = 0,1u'(0,4) + 0,02u'(0,6) + 0,05u'(0,8), \\ u(1) = 0,01\int_0^{0,4} u(s)\,ds + 0,02\int_{0,4}^{0,6} u(s)\,ds + 0,4\int_{0,6}^{0,8} u(s)\,ds \end{cases} \tag{4.1}$$

where

$$f(t,u)=\begin{cases} e^{-\frac{t}{8}}\left(\frac{u^3}{144}+3+\ln(4u+3)\right),\ 0\le t\le 1,\ 0\le u\le 3,\\ e^{-\frac{t}{8}}\left(\frac{51}{16}+\ln 15+25\sqrt{u-3}\right),\ 0\le t\le 1,\ 3<u\le 150,\\ e^{-\frac{t}{8}}\left(\frac{51}{16}+\ln 15+25\sqrt{147}+\sqrt{u-150}\right),\ 0\le t\le 1,\ 3<u\le 150.\end{cases}$$

To show the problem (4.1) has at least three positive solutions, we apply Theorem 3.1 with $\alpha=2.5$, $m=5$, $b_1=0.1$, $b_2=0.02$, $b_3=0.05$, $a_1=0.01$, $a_2=0.02$, $a_3=0.4$, $\eta_1=0.4$, $\eta_2=0.6$, $\eta_3=0.8$.

Then, by direct calculations, we can obtain that

$$1-\sum_{i=1}^{3}b_i=0.83,\quad 1-\sum_{i=1}^{3}a_i(\eta_i-\eta_{i-1})=0.912,\quad 1-\sum_{i=1}^{3}a_i\left(\eta_i^2-\eta_{i-1}^2\right)=0.9412,$$

$$\gamma=0.0310242\quad ,M=0.495731\quad ,m=0.16194.$$

If we choose $a=3$, $b=4$ and $c=160$, we obtain

$$f(t,u)\le 312.166719\le\frac{c}{M}\approx 322.7557,\ 0\le t\le 1,\ 0\le u\le 160,$$

$$f(t,u)\le 5.896\le\frac{a}{M}\approx 6.052,\ 0\le t\le 1,\ 0\le u\le 3,$$

$$f(t,u)\ge 27.2652274\ge\frac{b}{m}\approx 24.7005\ 0\le t\le 1,\ 4\le u\le 128.931608.$$

Thus by Theorem 3.1 the problem (1.3) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\|u_1\|<3,\quad 4<\alpha(u_2(t)),\ and\ \|u_3\|>3,\ with\ \alpha(u_3(t))<4.$$

5. CONCLUSION

In this paper, some results on the existence and multiplicity of solutions for a nonlinear higher order fractional differential equation involving the left Caputo fractional derivative with both multi-point and multi-strip boundary conditions are obtained. Under sufficient conditions, we have applied the Leggett-Williams fixed point theorem to obtain the existence of at least three positive solutions. An example is given to show the applicability of our results.

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