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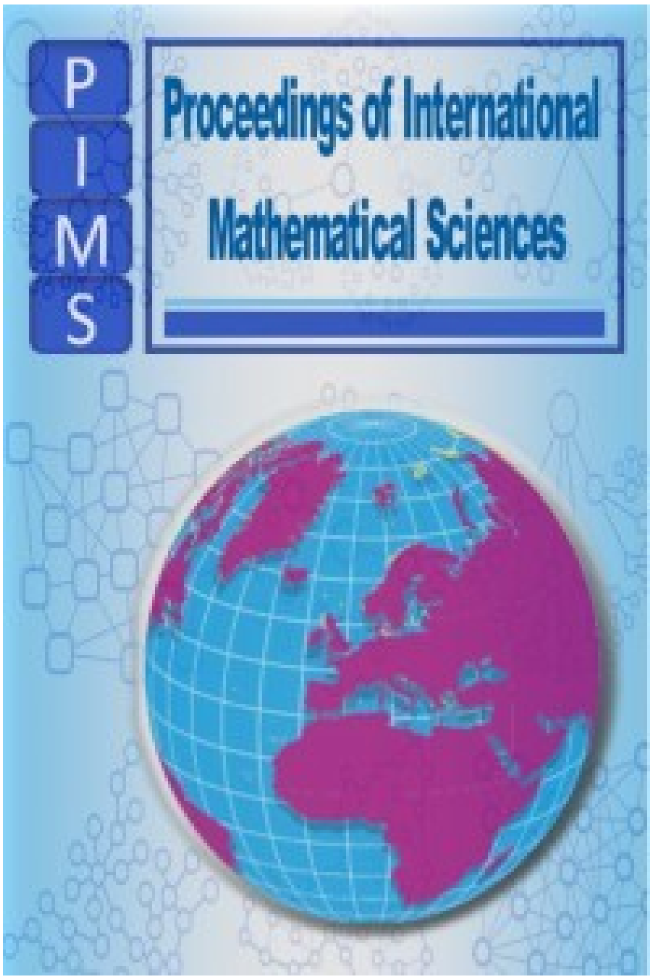

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ON THE λ_h^α−STATISTICAL CONVERGENCE OF THE FUNCTIONS DEFINED ON THE TIME SCALE

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Abstract. In this paper, we have introduced the concepts λ_h^α-density of a subset of the time scale $\mathbb{T}$ and λ_h^α-statistical convergence of order α $(0 < \alpha \leq 1)$ of $\Delta-$ measurable function f defined on the time scale $\mathbb{T}$ with the help of modulus function h and $\lambda = (\lambda_n)$ sequences. Later, we have discussed the connection between classical convergence, λ-statistical convergence and λ_h^α-statistical convergence. In addition, we have seen that f is strongly λ_h^α-Cesaro summable on T then f is λ_h^α-statistical convergent of order α.

1. Introduction

The concept of statistical convergence which is a generalization of classical convergence was first given by Zygmund [21] and later were introduced independently by Steinhaus [18] and Fast [4]. This concept is discussed under different names in spaces such as topological space, cone metric space, Banach space, time scale (see [10],[11],[12],[13],[15],[16],[17],[18],[19],[20],[26],[24],[25],[34],[41],[43]). Mursaleen [27] introduced the notion of λ-statistical convergence by using the sequence $\lambda = (\lambda_n)$ and then λ-statistical convergence on the time scales was introduced by Yılmaz et al[33] . The order of statistical convergence of a sequence of positive linear operators was introduced by Gadjiev and Orhan [36]. Later, Çolak [37] introduced and investigated the statistical convergence of order α $(0 < \alpha \leq 1)$ and strong p-Cesaro summability of order α of number sequences.

The time scale calculus was first introduced by Hilger in his Ph.D. thesis in 1988 (see [8],[9],[22]). In later years, the integral theory on time scales was given by Guseinov [7], and further studies were developed by Cabada-Vivero [3] and Rzezuchowski [16]. Recently, Seyyidoğlu and Tan [17] defined the density of the subset of the time scale. By using this definition, they gave $\Delta-$convergence and $\Delta-$Cauchy concepts for a real valued function defined on time scale. On the other side, the modulus function was first introduced by Nakano [14]. Aizpuru et al.[1] defined a new density concept with the help of a modulus function and obtained

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a new convergence concept between ordinary convergence and statistical convergence. Gürdal and Özgür [6] introduced ideal h-statistical convergence and ideal h-statistical Cauchy concepts in normed space using the modulus function h and ideals.

In this paper, we have aimed to define λ_h^α-statistical convergence of $\Delta-$ measurable functions of order α $(0<\alpha\leq 1)$ defined on the time scale by using modulus function h and $\lambda=(\lambda_n)$ sequences in light of works of Seyyidoğlu and Tan [17] and others [7], [2].

2. Prelimineries

The statistical convergence concept is based on the asymptotic (natural) density of a subset B in $\mathbb{N}$ (the set of positive integers) which is defined as

$$\delta(B)=\lim_{n\to\infty}\frac{|\{k\leq n:\ k\in B\}|}{n}, \tag{2.1}$$

where $|B|$ denotes the number of elements in B (see [29],[4],[5]). It has been generalized to α-density of a subset $B\subset\mathbb{N}$ and given the definition of $\alpha-$statistically convergence ($\alpha\in(0,1]$) by Colak [37]. The notion of λ-statistical convergence was introduced by Mursaleen [27] using the sequence $\lambda=(\lambda_n)$ which is a non-decreasing sequence of positive numbers tending to ∞ as $n\to\infty$ such that $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$, and $I_n=[n-\lambda_n+1,n]$. Lets denote by Λ the set of $\lambda=(\lambda_n)$ sequences. The λ- density of $B\subset\mathbb{N}$ is defined by

$$\delta_\lambda(B)=\lim_{n\to\infty}\frac{|\{k\in I_n:k\in B\}|}{\lambda_n} \tag{2.2}$$

and $\delta_\lambda(B)$ reduces to the natural density $\delta(B)$ in case of $\lambda_n=n$ for all $n\in\mathbb{N}$ (see [33]). A sequence $x=(x_n)$ is said to be λ- statistically convergent to L of order α ($\alpha\in(0,1]$) if for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{|\{k\in I_n:|x_k-L|\geq\epsilon\}|}{(\lambda_n)^\alpha}=0. \tag{2.3}$$

In this case, we write $s_{\lambda^\alpha}-limx=L$ (see [33],[27],[38],[28],[45],[46],[44]) and we denote by S_{λ^α} the set of λ^α- statistically convergent sequences of order α . If $\lambda_n=n$, S_{λ^α} reduces to S^α the set of statistically convergent number sequences of order α.

On the other hand, we recall that $h:[0,\infty)\to[0,\infty)$ is called modulus function, or simply modulus, if it is satisfies:

(1) $h(s)=0$ *if and only if* $s=0$,
(2) $h(s+p)\leq h(s)+h(p)$ *for every* $s,p\in[0,\infty)$,
(3) h *is increasing*,
(4) h *is continuous from the right at* 0.

A modulus may be bounded or unbounded . For instance, $h(x)=x^p$, where $0<p\leq 1$, is unbounded, but $h(x)=\frac{x}{1+x}$ is bounded (see [39], [23]).

Let h be an unbounded modulus function. The $\lambda_h^\alpha-$density of order α $(0<\alpha\leq 1)$ of a set $B\subseteq\mathbb{N}$ is defined by

$$\delta^{\lambda_h^\alpha}(B)=\lim_{n\to\infty}\frac{h(|\{n-\lambda_n+1\leq k\leq n:k\in B\}|)}{h((\lambda_n)^\alpha)} \tag{2.4}$$

whenever this limit exists.

In this study, we shall give a notion of $\lambda_h^\alpha-$statistical convergence on any time scales and its properties. Throughout this paper, we consider the time scales which are unbounded from above and have a minimum point. Lets remember some concepts.

A nonempty closed subset of $\mathbb{R}$ is called a time scale and is denoted by $\mathbb{T}$. We suppose that a time scale has the topology inherited from $\mathbb{R}$ with the standart topology. For $t \in \mathbb{T}$, we consider the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. In this definition, we take $\inf \emptyset = sup\mathbb{T}$. For $t \in \mathbb{T}$ with $a \leq b$, it is defined the interval $[a, b]$ in $\mathbb{T}$ by $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Let $\mathbb{T}$ be a time scale. Denote by $\mathcal{F}$ the family of all left-closed and right-open intervals of $\mathbb{T}$ of the form $[a, b) = \{t \in \mathbb{T} : a \leq t < b\}$ with $a, b \in \mathbb{T}$ and $a \leq b$. It is clear that the interval $[a, a)$ is an empty set, $\mathcal{F}$ is semiring of subsets of $\mathbb{T}$. Let $m : \mathcal{F} \to [0, \infty)$ be the set function on $\mathcal{F}$ that assings to each interval $[a, b)$ its lenght $b - a, m([a, b)) = b - a$. Then m is a countably additive measure on $\mathcal{F}$. We denote by μ_Δ the Caratheodory extension of the set function m associated with family $\mathcal{F}$ (for the Caratheodory extension see [17]) and is denoted by μ_Δ, the Lebesgue Δ-measure on $\mathbb{T}$, and that is a countably additive measure . In this case, it is known that if $a \in \mathbb{T} - \{max\mathbb{T}\}$, then the single point set $\{a\}$ is Δ-measurable and $\mu_\Delta(a) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leq b$ then $\mu_\Delta(a, b)_\mathbb{T} = b - \sigma(a)$. If $a, b \in \mathbb{T} - \{max\mathbb{T}\}$, $a \leq b$; $\mu_\Delta(a, b]_\mathbb{T} = \sigma(b) - \sigma(a)$ and $\mu_\Delta[a, b]_\mathbb{T} = \sigma(b) - a$. It can be easily seen that the measure of a subset of $\mathbb{N}$ is equal to its cardinality (see [17],[32]).

Turan and Duman [30] introduced the concept of statistical convergence of Δ-measurable real-valued functions defined on time scales as follows. Suppose that Ω be a Δ-measurable subset of $\mathbb{T}$. Then, the set $\Omega(t)$ is defined by $\Omega(t) =: \{s \in [t_0, t]_\mathbb{T} : s \in \Omega\}$ for $t \in \mathbb{T}$. In this case, the density of Ω on $\mathbb{T}$ can be defined as

$$\delta_\mathbb{T}(\Omega) = \lim_{t\to\infty} \frac{\mu_\Delta(\Omega(t))}{\mu_\Delta([t_0, t]_\mathbb{T})} \tag{2.5}$$

provided that the limit exists. In case of $\mathbb{T} = \mathbb{N}$, this reduces to the classical concept of asymptotic density. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta-$ measurable function. Then, f is statistically convergent to a real number L on $\mathbb{T}$ if for every $\epsilon > 0$, $\delta_\mathbb{T}(\{t \in \mathbb{T} : \ |f(t) - L| \geq \epsilon\}) = 0$. In this case, it can be written $s_\mathbb{T} - \lim_{t\to\infty} f(t) = L$.

Later, the λ-statistical convergence on time scale was introduced by Yılmaz et al [33], [31]. It is said that f is $\lambda-$statistically convergent on $\mathbb{T}$ to a real number L if

$$\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}(\{s \in [t - \lambda_t + t_0, t]_\mathbb{T} : \ |f(s) - L| \geq \epsilon\})}{\mu_{\Delta_\lambda}([t - \lambda_t + t_0, t]_\mathbb{T})} = 0 \tag{2.6}$$

for every $\epsilon > 0$. In this case, we can writes $s_\mathbb{T}^\lambda - \lim_{t\to\infty} f(t) = L$. The set of all $\lambda-$statistically convergence functions on $\mathbb{T}$ will be denoted by $S_\mathbb{T}^\lambda$. Here and afterwards Δ_λ shows that Δ depends on λ.

3. Main Results

Definition 3.1. *Let Ω be a Δ_λ-measurable subset of $\mathbb{T}$, h be an unbounded modulus function and α be any real number $(0 < \alpha \leq 1)$. Then, one defines the set $\Omega(t, \lambda)$*

by $\Omega(t,\lambda) =: \{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : s \in \Omega\}$ *for* $t \in \mathbb{T}$. *In this case, the* $\lambda_h^\alpha-$*density of* Ω *on* $\mathbb{T}$ *of order* α *can be defined as*

$$\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = \lim_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)} \tag{3.1}$$

provided that the limit exists.

We can easily get $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = \delta_{\mathbb{T}}^{h^\alpha}(\Omega)$ if $\lambda_t = t$ and $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = \delta_{\mathbb{T}}^{\lambda^\alpha}(\Omega)$ if we take $h(x) = x$ on $\mathbb{T}$.

Definition 3.2. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a* Δ_λ*- measurable function. Then, one says that* f *is* $\lambda_h^\alpha-$ *statistically convergent to a real number* L *of order* α $(0 < \alpha \leq 1)$ *on* $\mathbb{T}$ *if*

$$\lim_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)-L| \geq \epsilon\}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)} = 0 \tag{3.2}$$

for every $\epsilon > 0$.

In this case, one writes $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty} f(t) = L$. The set of all $\lambda_h^\alpha-$ statistically convergence functions on $\mathbb{T}$ will be denoted by $S_{\mathbb{T}}^{\lambda_h^\alpha}$.

If we take $\lambda_t = t$ in (8), we get classical statistically convergent on $\mathbb{T}$ to a real number L, for the function f which is defined by [17],[30] in (7). This shows that our results are generalizations of classical conclusions.

As will be noted that, when $\alpha = 1$, $\lambda_h^\alpha-$density of Ω on $\mathbb{T}$ of order α returns to λ_h-density. In case $h(x) = x$, $\lambda_h^\alpha-$density becomes $\lambda^\alpha-$density. If $\alpha = 1$ and $h(x) = x$, then $\lambda_h^\alpha-$density reduces to $\lambda-$density of Ω on $\mathbb{T}$.

The equality $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) + \delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\setminus\Omega) = 1$ does not hold for α $(0 < \alpha \leq 1)$ and an unbounded modulus h, in general. For instance, if we take $h(x) = x^p$, $0 < p \leq 1$, $0 < \alpha < 1$ and $\Omega = \{2n : n \in \mathbb{N}\}$, then $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = \delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\setminus\Omega) = \infty$. Also, finite sets have zero $\lambda_h^\alpha-$density for any unbounded modulus h and α $(0 < \alpha \leq 1)$ (see [30], [38]).

Lemma 3.1. *Let* α $(0 < \alpha \leq 1)$ *be any real number,* Ω *be a* Δ_λ*-measurable subset of* $\mathbb{T}$ *and* h *be an unbounded modulus function. If* $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = 0$*, then* $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\setminus\Omega) \neq 0$.

Proof. Let α $(0 < \alpha \leq 1)$ be any given real number and the equality $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = 0$ be valid for any unbounded modulus h. Suppose that $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\setminus\Omega) = 0$. Let us say $\Omega(t,\lambda)_{\mathbb{T}} =: \{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : s \in \Omega(t)\}$ for $t \in \mathbb{T}$ and $\mathbb{T}\setminus\Omega(t,\lambda)_{\mathbb{T}} =: \{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : s \in \mathbb{T}\setminus(\Omega)(t)\}$ for $t \in \mathbb{T}$. Since $\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}) = \mu_{\Delta_\lambda}(\Omega(t,\lambda)_{\mathbb{T}}) + \mu_{\Delta_\lambda}(\mathbb{T}\setminus\Omega(t,\lambda)_{\mathbb{T}})$ for $t \in \mathbb{T}$ and h is subadditive, we have

$$h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})) \leq h(\ \mu_{\Delta_\lambda}\Omega(t,\lambda)_{\mathbb{T}}) + h(\ \mu_{\Delta_\lambda}(\mathbb{T}\setminus\Omega(t,\lambda)_{\mathbb{T}})) \tag{3.3}$$

Hence we may write

$$\begin{aligned} &\lim_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)} \\ \leq\ &\lim_{t\to\infty} \frac{h(\ \mu_{\Delta_\lambda}\Omega(t,\lambda)_{\mathbb{T}})}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)} + \lim_{t\to\infty} \frac{h(\ \mu_{\Delta_\lambda}(\mathbb{T}\setminus\Omega(t,\lambda)_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)}. \end{aligned} \tag{3.4}$$

Since $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = 0$ and $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\diagdown\Omega) = 0$, the right side of the inequality is zero and thus

$$\lim_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)} = 0.$$

This is a contradiction. Because $\frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)} \geq 1$ for α $(0<\alpha\leq 1)$ and therefore

$$\lim_{t\to\infty} \frac{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)} \geq 1. \tag{3.5}$$

□

For any unbounded modulus h and $0<\alpha\leq 1$, if $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = 0$ then $\delta_{\mathbb{T}}^{\lambda^\alpha}(\Omega) = 0$, but the inverse of this does not need to be true ([40]). Namely, a set having zero α-density for some α $(0<\alpha\leq 1)$ might have non-zero λ_h^α−density for some unbounded modulus h, with the same α. Similarly a set having zero λ− density might have non-zero λ_h^α−density for some unbounded modulus h and $0<\alpha\leq 1$. For example, let $h(x) = log(x+1)$ and $\Omega = \{1,4,9,...\}$. Then $\delta^{\lambda}(\Omega) = 0$ and $\delta_{\mathbb{T}}^{\lambda^\alpha}(\Omega) = 0$ for $1/2<\alpha\leq 1$, but $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) \geq \delta_{\mathbb{T}}^{\lambda_h}(\Omega) = 1/2$ and therefore $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) \neq 0$.

If $\Phi\subseteq\mathbb{T}$ has zero λ_h^α−density for some unbounded modulus h and for some α $(0<\alpha\leq 1)$, then it has zero λ^α−density and hence zero λ−density (see [3]).

Lemma 3.2. [40] *Let h be an unbounded modulus and $\Phi\subseteq\mathbb{T}$. If $0<\alpha\leq\beta\leq 1$, then $\delta_{\mathbb{T}}^{\lambda_h^\beta}(\Phi) \leq \delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Phi)$.*

Thus, for any unbounded modulus h and $0<\alpha\leq\beta\leq 1$, if Φ has zero λ_h^α−density in that case, it has zero λ_h^β−density. Specially, a set having zero λ_h^α−density for some α $(0<\alpha\leq 1)$ has zero λ_h−density. But, the inverse is not correct. For instance, let $h(x) = x^p$ for $0<p\leq 1$ and $\Phi = \{1,4,9,...\}$. Then

$$\begin{aligned}
\delta_{\mathbb{T}}^{\lambda_h}(\Phi) &= \lim_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}\Phi(t,\lambda)_{\mathbb{T}})}{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))} && (3.6)\\
&\leq \lim_{t\to\infty} \frac{h(\lceil\sqrt{\Phi(t,\lambda)}\rceil)}{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))} && (3.7)\\
&= \lim_{t\to\infty} \frac{(\lceil\sqrt{\Phi(t,\lambda)}\rceil)^p}{(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^p} = 0
\end{aligned}$$

but, if we get $0<\alpha\leq 1/2$,

$$\begin{aligned}
\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Phi) &= \lim_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}\Phi(t,\lambda)_{\mathbb{T}})}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)} && (3.8)\\
&= \lim_{t\to\infty} \frac{(\lceil\sqrt{\Phi(t,\lambda)}\rceil)^p}{((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)^p} = \infty
\end{aligned}$$

where $\lceil r\rceil$ denotes the integer part of number r.

Proposition 3.3. *Let $f,g:\mathbb{T}\to\mathbb{R}$ be a Δ_λ- measurable functions such that $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty} f(t) = L_1$ and $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty} g(t) = L_2$. Then the following statements hold:*

i) $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty}(f(t)+g(t)) = L_1 + L_2$,

ii) $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty}(cf(t)) = cL_1$.

Proof. It is easy to prove and we omit it. □

Theorem 3.4. $S_{\alpha\mathbb{T}}^{h} \subseteq S_{\mathbb{T}}^{\lambda_h^\alpha}$ *if and only if*

$$\liminf_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t_0,t]_{\mathbb{T}})^\alpha)} > 0 \tag{3.9}$$

Proof. For given $\epsilon > 0$, we have

$$h(\mu_\Delta(\{s\in[t_0,t]_{\mathbb{T}} : \ |f(s)-L| \geq \epsilon\})) \supset h(\mu_\Delta(\{s\in[t-\lambda_t+t_0,t]_{\mathbb{T}} : \ |f(s)-L| \geq \epsilon\})).$$

Then

$$\begin{aligned} &\frac{h(\mu_{\Delta_\lambda}(\{s\in[t_0,t]_{\mathbb{T}} : \ |f(s)-L| \geq \epsilon\}))}{h((\mu_{\Delta_\lambda}([t_0,t]_{\mathbb{T}})^\alpha)} \\ \geq \ &\frac{h(\mu_{\Delta_\lambda}(\{s\in[t-\lambda_t+t_0,t]_{\mathbb{T}} : \ |f(s)-L| \geq \epsilon\}))}{h((\mu_{\Delta_\lambda}([t_0,t]_{\mathbb{T}})^\alpha)} \\ = \ &\frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t_0,t]_{\mathbb{T}})^\alpha)} \frac{1}{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))} \\ &h(\mu_{\Delta_\lambda}(\{s\in[t-\lambda_t+t_0,t]_{\mathbb{T}} : \ |f(s)-L| \geq \epsilon\})) \end{aligned}$$

Hence by using (3.9) and taking the limit as $t\to\infty$, we get $s_{\mathbb{T}}^{\frac{\alpha}{h}} - \lim_{t\to\infty} f(s) \to L$ implies $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty} f(s) = L$. □

The definition of $p-Cesaro$ summability on time scales was given by Turan and Duman [30] as follows.

Definition 3.3. [30] *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function and $0<p<\infty$. Then, f is strongly $p-Cesaro$ summable on $\mathbb{T}$ if there exists some $L\in\mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{1}{(\mu_\Delta([t_0,t]_{\mathbb{T}}))} \int_{[t_0,t]_{\mathbb{T}}} |f(s)-L|^p \Delta s = 0. \tag{3.10}$$

The set of all $p-Cesaro$ summable functions on $\mathbb{T}$ is denoted by $[W_p]_{\mathbb{T}}$.

We need to emphasize that measure theory on time scales was first constructed by Guseinov [7] and *Lebesque* $\Delta - integral$ on time scales introduced by Cabada and Vivero [35].

Definition 3.4. *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ_λ- measurable function, $\lambda\in\Lambda$ and $0<p<\infty$. We say that f is strongly $\lambda_h^\alpha - Cesaro$ summable on $\mathbb{T}$ if there exists some $L\in\mathbb{R}$ such that*

$$\lim_{t\to\infty}\frac{1}{(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \Delta s=0. \tag{3.11}$$

In this case we write $[W,\lambda_h^\alpha]_{\mathbb{T}}-lim\ f(s)=L$. The set of all strongly $\lambda_h^\alpha-$Cesaro summable functions on $\mathbb{T}$ will be denoted by $[W,\lambda_h^\alpha]_{\mathbb{T}}$. If we take $h(x)=x^p$ and $\alpha=1$ then we get $[W,\lambda_p]_{\mathbb{T}}$ the set of all strongly $\lambda_p-Cesaro$ summable functions on $\mathbb{T}$ (see [33]).

Lemma 3.5. *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ_λ- measurable function and $\Omega(t,\lambda)=\{\ s\in[t-\lambda_t+t_0,t]_{\mathbb{T}}: h(|f(s)-L|)\geq\epsilon\ \}$ for $\epsilon>0$. In this case, we have*

$$\begin{aligned} h(\mu_{\Delta_\lambda}(\Omega(t,\lambda))) &\leq \frac{1}{\epsilon}\int\limits_{\Omega(t,\lambda)} h(|f(s)-L|)\ \Delta s & (3.12)\\ &\leq \frac{1}{\epsilon}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \ \Delta s & (3.13)\end{aligned}$$

Proof. It can be proved by using similar method with [30]. □

Theorem 3.6. *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ_λ- measurable function, $\lambda\in\Lambda$, $L\in\mathbb{R}$ and $0<p<\infty$. Then we get:*

i) $[W,\lambda_h^\alpha]_{\mathbb{T}}.\subset s_{\mathbb{T}}^{\lambda_h^\alpha}$.

ii) If f is strongly $\lambda_h^\alpha-Cesaro$ summable to L, then $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim\limits_{t\to\infty}f(t)=L$.

iii) If $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim\limits_{t\to\infty}f(t)=L$ and f is a bounded function, then f is strongly $\lambda_h^\alpha-Cesaro$ summable to L.

Proof. i) Let $\epsilon>0$ and $[W,\lambda_h^\alpha]_{\mathbb{T}}-lim\ f(s)=L$. We can write

$$\begin{aligned}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \Delta s &\geq \int\limits_{\Omega(t,\lambda)} h(|f(s)-L|)\ \ \Delta s & (3.14)\\ &\geq \epsilon\ \ h(\mu_{\Delta_\lambda}(\Omega(t,\lambda))). & (3.15)\end{aligned}$$

Therefore, $[W,\lambda_h^\alpha]_{\mathbb{T}}-lim\ f(s)=L$ implies $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim\limits_{t\to\infty}f(s)=L$.

ii) Let f is strongly $\lambda_h^\alpha-Cesaro$ summable to L. For given $\epsilon>0$, let $\Omega(t,\lambda)$ $=\{\ s\in[t-\lambda_t+t_0,t]_{\mathbb{T}}: h(|f(s)-L|)\geq\epsilon\ \}$ on time scale $\mathbb{T}$. Then, it follows from lemma 9

$$\epsilon\ h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))\leq\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \Delta s.$$

Dividing both sides of the last equality by $h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))$ and taking limit as $t\to\infty$, we obtain

$$\begin{aligned} &\lim_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)} & (3.16)\\ \leq\ &\frac{1}{\varepsilon}\lim_{t\to\infty}\frac{1}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \Delta s=0 \end{aligned}$$

□

which yields that $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim_{t\to\infty} f(t)=L$.

iii) Let f be bounded and $\lambda_h^\alpha-$statistically convergent to L on $\mathbb{T}$. Then, there exists a positive number M such that $|f(s)|\leq M$ for all $s\in\mathbb{T}$, and also

$$\lim_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}=0$$

where $\Omega(t,\lambda)=\{\ s\in[t-\lambda_t+t_0,t]_{\mathbb{T}}:h(|f(s)-L|)\geq\epsilon\ \}$ as stated before. Since

$$\begin{aligned}
&\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \Delta s\\
=&\int\limits_{\Omega(t,\lambda)} h(|f(s)-L|)\ \ \Delta s+\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}/\Omega(t,\lambda)} h(|f(s)-L|)\ \ \Delta s \qquad (3.17)\\
\leq&\ (h(M)+h(|L|))\int\limits_{\Omega(t,\lambda)}\Delta s+\epsilon\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}/\Omega(t,\lambda)}\Delta s\\
=&\ (h(M)+h(|L|))\ h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))+\epsilon\ \ h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \ \Delta s \qquad (3.18)\\
\leq&\ (h(M)+h(|L|))\lim_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}+\epsilon
\end{aligned}$$

Since $\epsilon>0$ is arbitrary, the proof follows from (3.16) and (3.18).

Theorem 3.7. *Let f be a Δ_λ-measurable function. Then, $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim_{t\to\infty} f(t)=L$ if and only if there exists a Δ_λ-measurable set $\Omega\subseteq\mathbb{T}$ such that $\delta^{\lambda_h^\alpha}(\Omega)=1$ and $\lim_{t\to\infty}$ $h(|f(t)-L|)=0,\ \ (t\in\Omega(t,\lambda))$.*

Proof. It can be easily proved by using similar way in Theorem 3.9 of Turan and Duman (see, [30]). □

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WEAK CONE-COMPLETENESS OF DIRECT SUMS IN LOCALLY CONVEX CONES

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Abstract. We consider the weak cone-completeness in locally convex cones and prove that the direct sum of a family of weakly cone-complete separated locally convex cones is weakly cone-complete. We conclude that a direct sum cone topology is barreled whenever its components are weakly cone-complete and separated with the countable bases.

1. Introduction

The notions of barreledness and weak cone completeness in locally convex cones have been defined and investigated by W. Roth in [8]. Various topics of locally convex cones have been studied from the direct sum point of view in [2-7]. In this paper, we discuss the direct sum topology of weakly cone complete locally convex cones and show that it is barreled if its components are separated and carry the countable bases.

An *ordered cone* is a set $\mathcal{P}$ endowed with an addition $(a, b) \longmapsto a+b$ and a scalar multiplication $(\alpha, a) \longmapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$, and for the scalar multiplication the usual associative and distributive properties hold, that is, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$, $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. In addition, the cone $\mathcal{P}$ carries a (partial) order, i.e., a reflexive transitive relation $\leq$ that is compatible with the algebraic operations, that is $a \leq b$ implies $a+c \leq b+c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. For example, the extended scalar field $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers is a preordered cone. We consider the usual order and algebraic operations in $\overline{\mathbb{R}}$; in particular, $\alpha + \infty = +\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. In any cone $\mathcal{P}$, equality is obviously such an order, hence all results about ordered cones apply to cones without order structures as well.

A *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, i.e., a subset of positive elements that is directed downward, closed for addition and multiplication by (strictly) positive scalars. The elements v of $\mathcal{V}$ define *upper* (*lower*) *neighborhoods* for the elements of $\mathcal{P}$ by $v(a) = \{b \in \mathcal{P} :$

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$b \le a+v\}$ (respectively, $(a)v = \{b \in \mathcal{P} : a \le b+v\}$), creating the *upper*, respectively *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology.* We assume all elements of $\mathcal{P}$ to be *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \le a+\rho v$ for some $\rho > 0$. Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$. For a locally convex cone $(\mathcal{P}, \mathcal{V})$ the collection of all sets $\widetilde{v} \subseteq \mathcal{P}^2$, where $\widetilde{v} = \{(a, b) : a \le b + v\}$ for all $v \in \mathcal{V}$, defines a *convex quasi-uniform structure* on $\mathcal{P}$. On the other hand, every convex quasi-uniform structure leads to a full locally convex cone, including $\mathcal{P}$ as a subcone and induces the same convex quasi-uniform structure. For details see [1, Ch I, 5.2]. For cones $\mathcal{P}$ and $\mathcal{Q}$, a map $t : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator*, if $t(a + b) = t(a) + t(b)$ and $t(\alpha a) = \alpha t(a)$ for all $a, b \in \mathcal{P}$ and $\alpha \ge 0$. If $\mathcal{V}$ and $\mathcal{W}$ are abstract neighborhood systems on $\mathcal{P}$ and $\mathcal{Q}$, a linear operator $t : \mathcal{P} \to \mathcal{Q}$ is called *uniformly continuous* if for every $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $t(a) \le t(b)+w$ whenever $a \le b+v$. Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$. Endowed with the neighborhood system $\varepsilon = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. The set of all uniformly continuous linear functionals $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ is a cone called the *dual cone* of $\mathcal{P}$ and denoted by $\mathcal{P}^*$.

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *weakly cone-complete* if for all $b \in \mathcal{P}$ and $v \in \mathcal{V}$, every sequence $(a_i)_{n\in\mathbb{N}}$ in $v(b) \cap (b)v$ that converges to b in the symmetric topology of $\mathcal{P}$ and $\eta_i > 0$ such that $\sum_{i=1}^{\infty} \eta_i = 1$ there is $a \in v(b) \cap (b)v$ such that $\mu(a) = \sum_{i=1}^{\infty} \eta_i \mu(a_i)$ for all $\mu \in \mathcal{P}^*$ with $\mu(b) < +\infty$. A convex subset U of $\mathcal{P}^2$ is called *barrel*, if it satisfies the following properties:

(U_1) For every $b \in \mathcal{P}$ there is a neighborhood $v \in \mathcal{V}$ such that for every $a \in v(b) \cap (b)v$ there is a $\lambda > 0$ such that $(a, b) \in \lambda\mathrm{U}$.

(U_2) If $(a, b) \notin \mathrm{U}$, then there is a $\mu \in \mathcal{P}^*$ such that $\mu(c) \le \mu(d) + 1$ for all $(c, d) \in \mathrm{U}$ and $\mu(a) > \mu(b) + 1$.

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *barreled* if for every barrel U and every element $b \in \mathcal{P}$ there is a neighborhood $v \in \mathcal{V}$ and a $\lambda > 0$ such that $(a, b) \in \lambda\mathrm{U}$ for all $a \in v(b) \cap (b)v$. A subset $\mathcal{V}_0$ of $\mathcal{V}$ is a *neighborhood base*, if for every $v \in \mathcal{V}$ there is $v_0 \in \mathcal{V}_0$ such that $v_0 \le v$. Every weakly cone complete locally convex cone with a countable neighborhood base is barreled [8, Theorem 2.3].

2. Weak cone-completeness and direct sums

Let $\mathcal{P}_\gamma$, $\gamma \in \Gamma$ be cones and put $\mathcal{P} = \times_{\gamma\in\Gamma}\mathcal{P}_\gamma$. For elements $a, b \in \mathcal{P}$, $a = \times_{\gamma\in\Gamma}a_\gamma, b = \times_{\gamma\in\Gamma}b_\gamma$ and $\alpha \ge 0$ we set $a+b = \times_{\gamma\in\Gamma} (a_\gamma+b_\gamma)$ and $\alpha a = \times_{\gamma\in\Gamma} (\alpha a_\gamma)$. With these operations $\mathcal{P}$ is a cone which is called the *product cone* of $\mathcal{P}_\gamma$. The subcone of the product cone $\mathcal{P}$ spanned by $\cup\mathcal{P}_\gamma$ (more precisely, by $\cup j_\gamma(\mathcal{P}_\gamma)$, where $j_\gamma : \mathcal{P}_\gamma \to \mathcal{P}$ is the injection mapping) is said to be the *direct sum cone* of $\mathcal{P}_\gamma$ and denoted by $\mathcal{Q} = \sum_{\gamma\in\Gamma} \mathcal{P}_\gamma$. If $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$, $\gamma \in \Gamma$ be a family of locally convex cones, then $\mathcal{W} = \times_{\gamma\in\Gamma}\mathcal{V}_\gamma$ leads to the *finest locally convex cone topology* on $\mathcal{Q}$ such that the all injection mappings j_γ are uniformly continuous:

Definition 2.1. For elements $a, b \in \mathcal{Q}$, $a = \sum_{\gamma\in\Delta} a_\gamma$, $b = \sum_{\gamma\in\Theta} b_\gamma$ and $w \in \mathcal{W}$, $w = \times_{\gamma\in\Gamma}v_\gamma$, we set

$$a \le_\Gamma b + w$$

if $a_\gamma \le_\gamma b_\gamma + \alpha v_\gamma$ *for all* $\gamma \in \Delta \cup \Theta$, *where* $\sum_{\gamma\in\Delta\cup\Theta} \alpha_\gamma \le 1$.

The subsets $\{(a,b) \in \mathcal{Q}^2 : a \leq_\Gamma b + w\}$ for all $w \in \mathcal{W}$ describe the finest convex quasi-uniform structure on $\mathcal{Q}$ which makes every injection mapping uniformly continuous. According to [1, Ch I, 5.4], there exists a full cone $\mathcal{Q} \oplus \mathcal{W}_0$ with abstract neighborhood system $\mathrm{W} = \{0\} \oplus \mathcal{W}$, whose neighborhoods yield the same quasi-uniform structure on $\mathcal{Q}$. The elements $w \in \mathcal{W}, w = \times_{\gamma\in\Gamma} v_\gamma$ form a basis for W in the following sense: For every $\mathrm{w} \in \mathrm{W}$ there is $w \in \mathcal{W}$ such that $a \leq_\Gamma b + w$ for $a, b \in \mathcal{Q}$ implies that $a \leq_\Gamma b \oplus \mathrm{w}$. The locally convex cone topology on $\mathcal{Q}$ induced by $\mathcal{W}$ is called the *locally convex direct sum cone of* $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ and denoted by $(\mathcal{Q}, \mathcal{W})$. For details see [3].

Proposition 2.1. *If $\mathcal{Q} = \sum_{\gamma\in\Gamma} \mathcal{P}_\gamma$ is the locally convex direct sum cone, then*

(a) *if $b \in \mathcal{Q}$ and $(a_i)_{i\in\mathbb{N}} \subset \mathcal{Q}$ converges to b in the symmetric topology of $\mathcal{Q}$, then for each $\gamma \in \Gamma$, $(\varphi_\gamma(a_i))_{i\in\mathbb{N}}$ converges to $\varphi_\gamma(b)$ in the symmetric topology of $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$,*

(b) *$\mathcal{Q}^* = \times_{\gamma\in\Gamma} \mathcal{P}^*_\gamma$, where $\mathcal{Q}^*$ is the dual cone of $(\mathcal{Q}, \mathcal{W})$.*

Proof. (a) Fix $\gamma \in \Gamma$ and let $v_\gamma \in \mathcal{V}_\gamma$. If we set $w = \times_{\xi\in\Gamma} v_\xi$, where $v_\xi = v_\gamma$ for $\xi = \gamma$ and $v_\xi \in \mathcal{V}_\gamma$ otherwise, then $a \leq_\Gamma b + w$ for $a, b \in \mathcal{Q}$ implies that $\varphi_\gamma(a) \leq \varphi_\gamma(b) + v_\gamma$, i.e., φ_γ is uniformly continuous. For (b), see Theorem 3.10 in [7]. □

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. A subset A of $\mathcal{P}$ is *bounded* in the weak topology $\sigma(\mathcal{P}, \mathcal{P}^*)$ if, $-\infty < \inf_{\mu\in F, x\in A} \mu(x) \leq \sup_{\mu\in F, x\in A} \mu(x) < +\infty$ for all finite sets $F \subset \mathcal{P}^*$ [5, 6]. The cone $\mathcal{P}$ is *separated* if $\bar{a} = \bar{b}$ for $a, b \in \mathcal{P}$ implies $a = b$, where $\overline{a}$ is the closure of a with respect to the lower topology of $\mathcal{P}$; for example $\overline{\mathbb{R}}$ with the neighborhoods system $\varepsilon = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$ is separated [1, Ch I, 3.12]. A locally convex cone direct sum cone is separated if and only if its components are separated [7, Corollary 3,3].

Lemma 2.2. *Suppose $(\mathcal{Q}, \mathcal{W})$ is the locally convex direct sum cone of $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$, $b \in \mathcal{Q}$ and let $\mathcal{Q}^*_b$ be the subcone of all $\mu \in \mathcal{Q}^*$ with $\mu(b) < +\infty$. If $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is separated for all $\gamma \in \Gamma$ and $(a_i)_{i\in\mathbb{N}} \subset \mathcal{Q}$ converges to b in the symmetric topology of $\mathcal{Q}$, then there is a finite subset Δ of Γ such that for each $i \in \mathbb{N}$, $\varphi_\gamma(a_i) = 0$ for all $\gamma \in \Gamma \setminus \Delta$.*

Proof. Let $\mu_1, ..., \mu_n \in \mathcal{Q}^*_b$ and let $w' \in \mathcal{W}$ such that $w' \leq w$ and $\mu_i \in w'^\circ$ for all $i = 1, 2, ..., n$. By the assumption, there is $i_0 \in \mathbb{N}$ such that $a_i \in w'(b) \cap (b)w'$ for all $i \geq i_0$ which yields

$$-\infty < \inf_{1\leq j\leq n, i\geq i_0} \mu_j(a_i) \leq \sup_{1\leq j\leq n, i\geq i_0} \mu_j(a_i) < +\infty,$$

i.e., $\{a_i : i \geq i_0\}$ is $\sigma(\mathcal{Q}, \mathcal{Q}^*_b)$-bounded so, by [6, Theorem 2.6], there is a finite set $\Delta \subset \Gamma$ such that $\{a_i : i \geq i_0\} \subseteq \sum_{\gamma\in\Delta} \varphi_\gamma\{a_i : i \geq i_0\}$. □

Theorem 2.3. *The direct sum cone topology $(\mathcal{Q}, \mathcal{W}) = (\sum_{\gamma\in\Gamma} \mathcal{P}_\gamma, \times_{\gamma\in\Gamma} \mathcal{V}_\gamma)$ is weakly cone-complete, whenever for each $\gamma \in \Gamma$, $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is separated and weakly cone-complete.*

Proof. Suppose $b \in \mathcal{Q}$, $b = \sum_{\gamma\in\Delta} b_\gamma$, $w \in \mathcal{W}$, $w = \times_{\gamma\in\Gamma} v_\gamma$ and let $(a_i)_{i\in\mathbb{N}} \subset w(b) \cap (b)w$ converges to b in the symmetric topology of $\mathcal{Q}$ and $\eta_i > 0$ such that $\sum_{i=1}^{\infty} \eta_i = 1$. Using the Lemma 2.2, we may assume that $a_i = \sum_{\gamma\in\Delta} \varphi_\gamma(a_i)$ for all $i \in \mathbb{N}$. By Proposition 2.1 (a), the sequence $(\varphi_\gamma(a_i))_{i\in\mathbb{N}} \subset (\alpha_\gamma v_\gamma)(b_\gamma) \cap (b_\gamma)(\alpha_\gamma v_\gamma)$ converges to b_γ in the symmetric topology of $\mathcal{P}_\gamma$ for all $\gamma \in \Delta$, where $\sum_{\gamma\in\Delta} \alpha_\gamma \leq 1$;

so from the weak cone-completeness of $\mathcal{P}_\gamma$ there exists $a_\gamma \in (\alpha_\gamma v_\gamma)(b_\gamma) \cap (b_\gamma)(\alpha_\gamma v_\gamma)$ such that

$$\mu_\gamma(a_\gamma) = \sum_{i=1}^{\infty} \eta_i \mu_\gamma(\varphi_\gamma(a_i))$$

for all $\mu_\gamma \in \mathcal{P}_\gamma^*$ with $\mu_\gamma(b_\gamma) < \infty$. Then $a := \sum_{\gamma\in\Delta} a_\gamma \in w(b) \cap (b)w$ and for every $\mu \in \mathcal{Q}^*$ with $\mu(b) < +\infty$, we have $\mu = \times_{\gamma\in\Gamma}\mu_\gamma$ by Proposition 2.1 (b), where $\mu_\gamma \in \mathcal{P}_\gamma^*$ such that $\mu_\gamma(b_\gamma) < +\infty$ for all $\gamma \in \Delta$. Thus

$$\begin{aligned} \mu(a) &= \sum_{\gamma\in\Delta}\sum_{i=1}^{\infty} \eta_i \mu_\gamma(\varphi_\gamma(a_i)) = \sum_{i=1}^{\infty} \eta_i \sum_{\gamma\in\Delta} \mu_\gamma(\varphi_\gamma(a_i)) \\ &= \sum_{i=1}^{\infty} \eta_i \mu(\sum_{\gamma\in\Delta} \varphi_\gamma(a_i)) = \sum_{i=1}^{\infty} \eta_i \mu(a_i), \end{aligned}$$

i.e., $(\mathcal{Q}, \mathcal{W})$ is weakly cone-complete. □

By combining Theorem 2.3 and [8, Theorem 2.3], we have:

Corollary 2.4. *A direct sum cone topology is barreled, whenever its components are separated and weakly cone-complete with the countable bases.*

Example 2.1. (i) If we consider $(\overline{\mathbb{R}}, \varepsilon)$, $\varepsilon = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, then $\overline{\mathbb{R}}^* = \mathbb{R}_+ \cup \{\overline{0}\}$; where $\overline{0}(x) = 0$ for all $x \in \mathbb{R}$ and $\overline{0}(+\infty) = +\infty$ [9, Example 2.2]. Let $b \in \overline{\mathbb{R}}$, $\epsilon > 0$, $(a_i)_{i\in\mathbb{N}} \subset \epsilon(b) \cap \epsilon(b)$ converges to b in the symmetric topology of $(\overline{\mathbb{R}}, \varepsilon)$ and let $\eta_i > 0$ such that $\sum_{i=1}^{\infty} \eta_i = 1$. If $b = +\infty$, then for $a = +\infty$ the assertion holds. If $b \in \mathbb{R}$ then $a := \sum_{i=1}^{\infty} \eta_i a_i \in \epsilon(b) \cap (b)\epsilon$ and for every $\mu \in \overline{\mathbb{R}}^*$, $\mu = \lambda$ for some $\lambda > 0$, hence $\mu(a) = \lambda(\sum_{i=1}^{\infty} \alpha_i a_i) = \sum_{i=1}^{\infty} \alpha_i \mu(a_i)$, i.e., $(\overline{\mathbb{R}}, \varepsilon)$ is weakly cone-complete.

(ii) We consider $\mathcal{Q} = \sum_{n\in\mathbb{N}} \overline{\mathbb{R}}$ with the countable neighborhood system $\mathcal{W} = \times_{n\in\mathbb{N}}\varepsilon$. For elements $a, b \in \mathcal{Q}$, $a = \sum_{n\in\Delta} a_n$, $b = \sum_{n\in\Theta} b_n$, the direct sum neighborhood $w \in \mathcal{W}$, $w = \times_{n\in\mathbb{N}}\epsilon_n$ on $\mathcal{Q}$ is defined by

$$a \leq_{\mathbb{N}} b + w \quad \text{if} \quad a_n \leq b_n + \alpha_n\varepsilon_n \quad (\text{for all} \quad n \in \Delta \cup \Theta)$$

where $\sum_{n\in\Delta\cup\Theta} \alpha_n \leq 1$. Suppose $b \in \mathcal{Q}$, $b = \sum_{n\in\Delta_{\mathbb{R}}} b_n + \sum_{n\in\Delta\setminus\Delta_{\mathbb{R}}}(+\infty)$, where $\Delta = \{n \in \mathbb{N} : b_n \neq 0\}$, $\Delta_{\mathbb{R}} = \{n \in \Delta : b_n \in \mathbb{R}\}$ and let $w \in \mathcal{W}$, $w = \times_{n\in\mathbb{N}}\epsilon_n$. Let $(a_i)_{i\in\mathbb{N}} \subset w(b) \cap (b)w$, $a_i = \sum_{n\in\Delta^i_{\mathbb{R}}} a^i_n + \sum_{n\in\Delta_i\setminus\Delta^i_{\mathbb{R}}}(+\infty)$ for all $i \in \mathbb{N}$ such that $(a_i)_{i\in\mathbb{N}}$ converges to b in the symmetric topology of $\mathcal{Q}$ and for $\eta_i > 0$, let $\sum_{i=1}^{\infty} \eta_i = 1$. Without loss of generality we may assume that $\Delta_i = \Delta$ and $\Delta^i_{\mathbb{R}} = \Delta_{\mathbb{R}}$ for all $i \in \mathbb{N}$. Then

$$a := \sum_{n\in\Delta_{\mathbb{R}}}\sum_{i=1}^{\infty} \eta_i a^i_n + \sum_{n\in\Delta\setminus\Delta_{\mathbb{R}}}\sum_{i=1}^{\infty} \eta_i(+\infty) \in w(b) \cap (b)w$$

and for every $\mu \in \mathcal{Q}^*$ with $\mu(b) < +\infty$, we have $\mu = \times_{n\in\mathbb{N}}\mu_n$ by Proposition 2.1 (b), where

$$\mu_n = \begin{cases} \lambda_n \text{ (some } \lambda_n > 0) & \text{if} \quad b_n \in \mathbb{R}, \\ 0_{\overline{\mathbb{R}}^*} & \text{if} \quad b = +\infty. \end{cases}$$

Thus

$$\begin{aligned}
\mu(a) &= \sum_{n\in\Delta_{\mathbb{R}}}\lambda_n(\sum_{i=1}^{\infty}\eta_i a_n^i)+\sum_{n\in\Delta\setminus\Delta_{\mathbb{R}}}0_{\overline{\mathbb{R}}^*}(\sum_{i=1}^{\infty}\eta_i(+\infty))\\
&= \sum_{i=1}^{\infty}\eta_i\lambda_n(\sum_{n\in\Delta_{\mathbb{R}}}a_n^i)+\sum_{i=1}^{\infty}\eta_i 0_{\overline{\mathbb{R}}^*}(\sum_{n\in\Delta\setminus\Delta_{\mathbb{R}}}(+\infty))\\
&= \sum_{i=1}^{\infty}\eta_i\mu(a_i),
\end{aligned}$$

i.e., $(\sum_{n\in\mathbb{N}}\overline{\mathbb{R}}, \times_{n\in\mathbb{N}}\varepsilon)$ is weakly cone-complete.

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AN INVERSE DIFFUSION-WAVE PROBLEM DEFINED IN HETEROGENEOUS MEDIUM WITH ADDITIONAL BOUNDARY MEASUREMENT

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Abstract. This paper deals with an inverse problem to determine a space-dependent coefficient in a one-dimensional time fractional diffusion-wave equation defined in heterogeneous medium with additional boundary measurement. Then, we construct the explicit finite difference scheme for the direct problem based on the equivalent partial integro-differential equation and Simpson's rule. Using the matrix analysis and mathematical induction, we prove that our scheme is stable and convergent . The least squares method with homotopy regularization is introduced to determine the space-dependent coefficient, and an inversion algorithm is performed by one numerical example. This inversion algorithm is effective at least for this inverse problem.

1. Introduction

In this paper, we consider the following equation:

$$^{c}\mathcal{D}_t^{\alpha} u(x,t) = \frac{\partial}{\partial x}\left(D(x)\frac{\partial u(x,t)}{\partial x}\right) + f(x,t), \quad 0 < x < L,\ 0 < t \leq T, \tag{1.1}$$

with the initial conditions

$$u(x,0) = \psi(x), \quad u_t(x,0) = \varphi(x), \quad 0 \leq x \leq L, \tag{1.2}$$

and the Neumann boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0, \quad 0 \leq t \leq T, \tag{1.3}$$

where $u(x,t)$ denotes state variable at space point x and time t, and $1 < \alpha < 2$ is called fractional order of the derivative in time, $D(x)$ is the space-dependent

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coefficient, $f(x,t)$ is a source term, and ${}^{c}\mathcal{D}_t^{\alpha}u(x,t)$ means the Caputo derivative defined by:

$$ {}^{C}\mathcal{D}_t^{\alpha}u(x,t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{\partial^2 u(x,s)}{\partial s^2}ds. \tag{1.4}$$

In this study, we are concerned with the inverse problem of approximating the unknown space-dependent coefficient $D(x)$, while the initial functions $\psi(x)$ and $\varphi(x)$ and the source term $f(x,t)$ are considered as known functions. To determine the set of functions (u,D) in the inverse problem (1.1)-(1.3), we need an over-specified condition:

$$u(x,T)=\eta(x), \quad 0<x<L, \tag{1.5}$$

is used.

2. The direct problem

The direct problem is composed by Eq. (1.1) with the initial and boundary value conditions (1.2) and (1.3).

2.1. The explicit finite difference scheme.

Firstly, we have the following lemma:

Lemma 2.1. *([2, 3]) Let $\alpha\in\,]1,2[$ and $y\in\mathcal{C}^2([0,T])$ with $T>0$. Then, we have*

(1) ${}_0^C\mathcal{D}_t^{\alpha-1}\left(\mathcal{I}_t^{\alpha-1}y(t)\right)=y(t)$,
(2) $\mathcal{I}_t^{\alpha-1}\left({}_0^C\mathcal{D}_t^{\alpha}y(t)\right)=\mathcal{I}_t^{\alpha-1}\left[{}_0^C\mathcal{D}_t^{\alpha-1}(y'(t))\right]=y'(t)-y'(0)$,

where ${}_0^C\mathcal{D}_t^{\alpha}$ is the Caputo fractional derivative operator defined in (1.4) and $\mathcal{I}_t^{\alpha-1}$ is the Riemann-Liouville integral operator defined as

$$\mathcal{I}_t^{\alpha-1}g(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}g(s)\,ds. \tag{2.1}$$

Based on this lemma, we have the following theorem.

Theorem 2.2. *([2, 5]). The equation (1.1) is equivalent to the following partial integro-differential equation*

$$u_t(x,t)=\varphi(x)+\mathcal{I}_t^{\alpha-1}\left[D'(x)u_x(x,t)+D(x)u_{xx}(x,t)\right]+\mathcal{I}_t^{\alpha-1}f(x,t), \tag{2.2}$$

where $F(x,t)=\mathcal{I}_t^{\alpha-1}f(x,t)$.

We consider $\Omega_\tau=\{t_n : t_n=n\tau, 0\le n\le N\}$ a uniform mesh of the interval $[0,T]$ with $\tau=T/N$ and using Simpson's rule [3], we obtain the following lemma.

Lemma 2.3. *If $g\in\mathcal{C}^4([0,T])$ and $\alpha\in\,]1,2[$, then*

$$\mathcal{I}_t^{\alpha-1}g(t_n)=\frac{\tau^{\alpha-1}}{3\Gamma(\alpha-1)}\sum_{k=1}^{n}\omega_k g(t_{n-k})+O\left(\tau^5\right),$$

where $\omega_1=5,\ \omega_k=6k^{\alpha-2}, k=2,\dots,n-2,\ \omega_{n-1}=5(n-1)^{\alpha-2},\ \omega_n=n^{\alpha-2}$.

Let $\Omega_h = \{x_i / x_i = ih, 0 \leq i \leq M\}$ is a uniform mesh of the interval $[0, L]$ with $h = L/M$ and $M \in \mathbb{N}^*$. Suppose $u = \{u_i^n / 0 \leq i \leq M, 0 \leq n \leq N\}$ is a grid function on $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Considering the Eq.(2.2) at the point (x_i, t_n) and with Lemma 2.3, we obtain an explicit scheme for (2.2) in the following matrix form:

$$\begin{cases} U^0 = \psi, \\ U^1 = (I + 5A)\, U^0 + \tau\varphi + \frac{\tau^\alpha}{3\Gamma(\alpha-1)} f^0, \\ U^n = (I + 5A)\, U^{n-1} + \tau\varphi + \frac{\tau^\alpha}{3\Gamma(\alpha-1)} \sum_{k=1}^{n} \omega_k f^{n-k} + \displaystyle\sum_{k=2}^{n} \omega_k A U^{n-k}. \end{cases} \tag{2.3}$$

Here I is the $M-1$ order identity matrix. Where $U^n = \left(u_1^n, u_2^n, \ldots, u_{M-1}^n\right)^T, \varphi = \left(\varphi_1, \varphi_2, \ldots, \varphi_{M-1}\right)^T, \psi = \left(\psi\left(x_1\right), \ldots, \psi\left(x_{M-1}\right)\right)^T$, $f^n = \left(f_1^n, f_2^n, \ldots, f_{M-1}^n\right)^T$ and $A = (a_{ij}), i, j = 1, 2, \ldots, M-1$ is defined by

$$A = \begin{pmatrix} -p_1 & p_1 & 0 & \ldots & 0 & 0 \\ p_2 - q_2 & q_2 - 2p_2 & p_2 & \ldots & 0 & 0 \\ 0 & p_3 - q_3 & q_3 - 2p_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & q_{M-2} - 2p_{M-2} & p_{M-2} \\ 0 & 0 & 0 & \ldots & p_{M-1} - q_{M-1} & q_{M-1} - p_{M-1} \end{pmatrix} \tag{2.4}$$

2.2. **Stability and convergence.** Firstly, we have the following lemma.

Lemma 2.4. *Suppose that $D : [0, L] \to \mathbb{R}_+$ is a continuously differentiable function on $]\,0, L\,[$. Then, the matrix A given by (2.4) is negative definite, and we have*

$$a_{ii} = -\sum_{j=1, j\neq i}^{M-1} |a_{ij}|, \quad \|A\| \leq \frac{4\tau^\alpha}{3h^2\Gamma(\alpha-1)} \max_{0\leq x\leq L} D(x). \tag{2.5}$$

By utilizing linear difference scheme (2.3), we can easily get

$$\begin{cases} E^0 = \tilde{\psi} - \psi \\ E^1 = (I + 5A)\, E^0, \\ E^n = (I + 5A)\, E^{n-1} + \displaystyle\sum_{k=2}^{n} \omega_k A E^{n-k}, \end{cases} \tag{2.6}$$

where $\tilde{\psi}$ denotes the initial function with noises, $E^n = \tilde{U}^n - U^n$ denotes the solutions error for the n-th step iteration, and $n = 1, \ldots, N$.

Theorem 2.5. *The explicit difference scheme defined by (2.3) is unconditionally stable.*

We denote $e_i^n = u(x_i, t_n) - u_i^n$, $i = 1, \ldots, M-1$, $n = 1, \ldots, N$, where $u(x_i, t_n)$ is the exact solution of the direct problem (1.1)-(1.3) at mesh point (x_i, t_n) and u_i^n is the solution of the difference scheme (2.3) also at (x_n, t_n), and $e^n = \left(e_1^n, e_2^n, \ldots, e_{M-1}^n\right)^T$. Note that $e_i^0 = u(x_i, 0) - \psi(x_i) = 0$. We have

$$\begin{cases} e^1 = R^1, \\ e^n = (I + 5A)\, e^{n-1} + \displaystyle\sum_{k=2}^{n} \omega_k A e^{n-k} + R^n, \end{cases} \tag{2.7}$$

where $R^n = \left(R_1^n, R_2^n, \ldots, R_{M-1}^n\right)^T$ denotes the truncated term.

Theorem 2.6. *The solution of the explicit difference scheme (2.3) is convergent to the exact solution of the direct problem (1.1)-(1.3) as $h, \tau \to 0$ for finite time domain.*

3. The inverse problem

The inverse problem is formulated by: the fractional diffusion-wave equation (1.1), the initial conditions (1.2), the boundary conditions (1.3) and the additional condition (1.4). For the solution of the inverse problem, suppose that the function $D \in \mathcal{C}(0, L)$. Let V be a subspace of $\mathcal{C}(0, L)$ of finite dimension s and $(\eta_i(x))$, $i = 1, \ldots, s$ une base de V. We can write the diffusion-wave coefficient $D(x)$ by:

$$D(x) = \sum_{i=1}^{s} p_i \eta_i(x). \tag{3.1}$$

For $D(x)$ given, the direct problem (1.1)-(1.3) admits a unique solution noted by $u(x, t, D)$. To find $D(x)$ just find the vector $P = (p_1, p_2, \ldots, p_s)^T \in \mathbb{R}^s$. Let $\beta > 0$, we notice $S_\beta = \{P \in \mathbb{R}^s : \|P\| \leq \beta\}$ the admissible set of unknowns P.

3.1. Nonlinear least squares problem.

To solve the inverse problem we solve a nonlinear least squares problem:

$$\begin{cases} \min \Phi(P) = \|u(L, t; P) - \psi(t)\|_2^2, \quad 0 < t \leq T. \\ P \in S_\beta \end{cases} \tag{3.2}$$

The objective function Φ continuous and convex on the set S_β closed and bounded. Therefore, according to Weierstrass theorem, the problem (3.2) admits at least one solution. On the other hand the problem (3.2) is ill-posed so that the problem admits several solutions. For uniqueness, using Homotopy regularization [1], we consider the following regularized problem:

$$\begin{cases} \min \Phi_\lambda(P) = (1-\lambda)\|u(L, t; P) - \psi(t)\|_2^2 + \lambda\|P\|_2^2, \\ P \in S_\beta, \end{cases} \tag{3.3}$$

where $0 < \lambda < 1$ is the regularization parameter. To get P^j, we assume that $P^{j+1} = P^j + \delta P^j, \quad j = 0, 1, \ldots$. We need to determine a regularized vector $\delta P^j = \left(\delta p_1^j, \delta p_2^j, \ldots, \delta p_s^j\right)^T$. Using Taylor's approximation to order one, we find:

$$u(L, t; P + \delta P) \approx u(L, t; P) + \nabla_P^T u(L, t; P) \cdot \delta P. \tag{3.4}$$

From (3.3) and (3.4) the objective function of the regularized problem becomes:

$$F_\lambda(\delta P) = (1-\lambda)\left\|\nabla_P^T u(L, t; P).\delta P - (\psi(t) - u(L, t; P))\right\|_2^2 + \lambda\|\delta P\|_2^2. \tag{3.5}$$

By the finite difference method, we obtain:

$$\nabla_P^T u(L, t_n; P).\delta P \approx \sum_{i=0}^{s} \frac{u(L, t_n; (p_0, \ldots, p_i + \tau, \ldots, p_s)) - u(L, t_n; p)}{\tau}.\delta p_i. \tag{3.6}$$

We define the matrix $H = (h_{ni})_{N \times (s+1)}$ by:

$$h_{ni} = \frac{u(L, t_n; (p_0, \ldots, p_i + \tau, \ldots, p_s)) - u(L, t_n; p)}{\tau}. \tag{3.7}$$

Let $U = (u(L,t_1;p), u(L,t_2;p), \dots, u(L,t_N;p))^T$, $\Psi = (\psi(t_1), \psi(t_2), \dots, \psi(t_N))^T$. Using (3.6) and (3.7), we can write (3.5) in the form:

$$F_\lambda(\delta P) = (1-\lambda)\left\| H\delta P - (\Psi - U)\right\|_2^2 + \lambda \left\|\delta P\right\|_2^2. \tag{3.8}$$

We have the following equivalence result:

Proposition 3.1. *([4, 5]).*

✓ *δP a minimum point of F_λ if only if δP solution of the normal equation:*

$$(1-\lambda)H^T H\delta P + \lambda\delta P = H^T(\Psi - U). \tag{3.9}$$

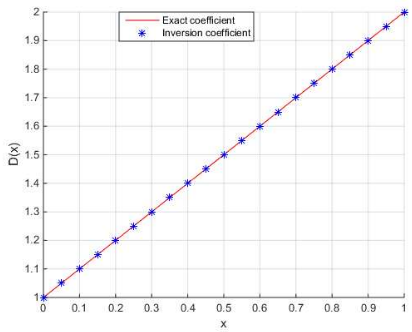
✓ *For all $0 < \lambda < 1$, the normal equation (3.9) has a unique solution.*

Algorithm 1 (Inversion algorithm, [4, 5])

1: Give an initial value P, the step τ, α, λ and ε,
2: Solve the scheme (2.3) to get $u(\ell, t_n; P)$ and $u(\ell, t_n; (p_0, \dots, p_i + \tau, \dots, p_s))$, for all $n = 1, 2, \dots, N$ et $i = 0, 1, \dots, s$
3: Calculate the matrix H and the vectors U, Ψ,
4: Calculate a regularization vector δP by: $\delta P = \left[(1-\lambda)\, H^T H + \lambda I\right]^{-1} H^T(\Psi - U)$.
5: If $\|\delta P\|_2 \le \varepsilon$ stop, and $P + \delta P$ is considered a solution. Otherwise, go to step 2 by replacing P with $P + \delta P$.

3.2. **Numerical test.** ([4, 5])

- $T = 1$, $L = 1$, $\varphi(x) = x^2(1-x)^2$, $\psi(x) = 0$, $\lambda = 0.01$,
- $f(x,t) = \dfrac{2x^2(1-x)^2 t^{2-\alpha}}{\Gamma(3-\alpha)} - (1+t^2)(16x^3 - 6x^2 - 8x + 2)$,
- $D(x) = 1 + x$, $P^0 = (1,1)$. $\tau = 0.4$, $\varepsilon = 10^{-6}$, $\alpha = 1.8$,
- $M = 20$, $N = 1000$.



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COLOMBEAU TYPE ALGEBRA OF PSEUDO ALMOST PERIODIC GENERALIZED FUNCTIONS

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Abstract. The aim of this work is to introduce and to study an algebra of pseudo almost periodic generalized functions containing the classical pseudo almost periodic functions as well as pseudo almost periodic distributions.

1. Introduction and preliminaries

The concept of pseudo almost periodicity is a generalization of Bohr almost periodicity, it has been introduced by C. Zhang, see [8]. The algebra $\mathcal{G}$ of generalized functions of Colombeau give an answer to the problem of multiplication of distributions. For a detailed study of these generalized functions see the book [6]. An algebra of almost periodic generalized functions of Colombeau type containing classical Bohr almost periodic functions and almost periodic Schwartz distributions has been introduced and studied in [3]. As mentioned in the abstract, the first aim of this work is to introduce and to study an algebra of pseudo almost periodic generalized functions of Colombeau type containing Zhang pseudo almost periodic functions as well as pseudo almost periodic Schwartz distributions. In section 1, we recall the basic definitions and results that we shall use in this work. The main results of paper are given in the next section. First, we construct the space of smooth pseudo almost periodic functions and we recall the algebra $\mathcal{G}_{L^\infty}$ of bounded generalized functions in which we study the pseudo almost periodicity. Next, we define the space $\mathcal{M}_{pap}$ of pseudo almost periodic moderate elements and the space $\mathcal{N}_{pap}$ of pseudo almost periodic negligible elements. The main properties of $\mathcal{M}_{pap}$ and $\mathcal{N}_{pap}$ are summarized in Proposition 2. The new algebra $\mathcal{G}_{pap}$ of pseudo almost periodic generalized functions of Colombeau type is given in Definition 5. A characterization of elements of $\mathcal{G}_{pap}$ similar to the classical result for pseudo almost periodic Schwartz distributions is given by Proposition 4. By means of convolution with a mollifier $\rho \in \Sigma$, we show that the space of pseudo almost periodic Schwartz distributions $\mathcal{B}'_{pap}$. can be embedded by the map i_{pap} into the algebra $\mathcal{B}_{pap}$. By defining the canonical embedding σ_{pap} between $\mathcal{B}_{pap}$ and $\mathcal{G}_{pap}$, Proposition 6, shows

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that we have two ways to embed the space $\mathcal{B}_{pap}$ into $\mathcal{G}_{pap}$ by i_{pap} and by σ_{pap} . Finally, another result is Proposition 7, in which we give an extension of the classical Bohl-Bohr's Theorem. We refer the reader to [3], [4] and [5] from which the results of this paper were inspired. In this paper we consider functions and distributions defined on $\mathbb{R}$. Recall $(\mathcal{C}_b, \| \quad \|_{L^\infty})$ the Banach algebra of bounded and continuous complex valued functions on $\mathbb{R}$ endowed with the norm $\| \quad \|_{L^\infty}$ of uniform convergence on $\mathbb{R}$. The space $\mathcal{C}_{ap}$ of almost periodic functions on $\mathbb{R}$, which was introduced by H. Bohr, is the closed subalgebra of $(\mathcal{C}_b, \| \quad \|_{L^\infty})$ that contains all the functions f, satisfying: for any $\varepsilon > 0$, the set

$$\left\{ \tau \in \mathbb{R} : \sup_{x \in \mathbb{R}} |f(x+\tau) - f(x)| < \varepsilon \right\}, \tag{1.1}$$

is relatively dense in $\mathbb{R}$. In [8], C. Zhang introduced an extension of the almost periodic functions. Set

$$\mathcal{C}_0 = \left\{ f \in \mathcal{C}_b : \lim_{t \longrightarrow +\infty} \frac{1}{2t} \int_{-t}^{t} |f(x)|\, dx = 0 \right\}. \tag{1.2}$$

Definition 1.1. *A function $f \in \mathcal{C}_b$ is called pseudo almost periodic if it can be written as $f = g + h$, where $g \in \mathcal{C}_{ap}$ and $h \in \mathcal{C}_0$.*

The above decomposition is unique, so the functions g and h are called respectively the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f. Denote by $\mathcal{C}_{pap}$ the set of all such functions. Then we have $\mathcal{C}_{ap} \subset \mathcal{C}_{pap} \subset \mathcal{C}_b$.

Now, we recall Schwartz almost periodic distributions, see [7]. Let $p \in [1, +\infty]$, the space

$$\mathcal{D}_{L^p} := \left\{ \varphi \in \mathcal{C}^\infty : \varphi^{(j)} \in L^p, \forall j \in \mathbb{Z}_+ \right\}, \tag{1.3}$$

endowed with the topology defined by the countable family of norms

$$|\varphi|_{k,p} := \sum_{j \le k} \left\| \varphi^{(j)} \right\|_{L^p}, k \in \mathbb{Z}_+, \tag{1.4}$$

is a differential Frechet subalgebra of $\mathcal{C}^\infty$. The topological dual of $\mathcal{D}_{L^1}$, denoted by $\mathcal{D}'_{L^\infty}$, is called the space of bounded distributions.

Definition 1.2. *A distribution $T \in \mathcal{D}'_{L^\infty}$ is called almost periodic if the set $\{\tau_h T, h \in \mathbb{R}\}$ of translated of T is relatively compact in $\mathcal{D}'_{L^\infty}$. The space of Schwartz almost periodic distributions is denoted by $\mathcal{B}'_{ap}$.*

Define

$$\mathcal{B}'_0 := \left\{ T \in \mathcal{D}'_{L^\infty} : \lim_{t \longrightarrow +\infty} \frac{1}{2t} \int_{-t}^{t} |(T * \varphi)(x)|\, dx = 0, \forall \varphi \in \mathcal{D} \right\}. \tag{1.5}$$

Definition 1.3. *A distribution $T \in \mathcal{D}'_{L^\infty}$ is called pseudo almost periodic if it can be written as $T = R + S$, where $R \in \mathcal{B}'_{ap}$ and $S \in \mathcal{B}'_0$. The space of all such distributions is denoted by $\mathcal{B}'_{pap}$.*

The above decomposition is unique and we have $\mathcal{B}'_{ap} \subset \mathcal{B}'_{pap} \subset \mathcal{D}'_{L^\infty}$.

Theorem 1.1. *Let $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent :*
(i) $T \in \mathcal{B}'_{pap}$.
(ii) $T * \varphi \in \mathcal{C}_{pap}, \forall\ \varphi \in \mathcal{D}$.
(iii) $\exists k \in \mathbb{Z}_+, \exists \left(f_j\right)_{j \leq k} \subset \mathcal{C}_{pap} : T = \sum_{j \leq k} f_j^{(j)}$.

2. Results

In this section, we introduce the algebra of pseudo almost periodic generalized functions of Colombeau type and we give their main properties.

Definition 2.1. *The space of smooth pseudo almost periodic functions on $\mathbb{R}$, is denoted and defined by*

$$\mathcal{B}_{pap} := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in \mathcal{C}_{pap}, \forall j \in \mathbb{Z}_+ \right\}. \tag{2.1}$$

We give some, easy to prove, properties of the space $\mathcal{B}_{pap}$.

Proposition 2.1. *(i) $\mathcal{B}_{pap}$ is a closed subalgebra of $\mathcal{D}_{L^\infty}$ stable by derivation.*
(ii) If $T \in \mathcal{B}'_{pap}$ and $\varphi \in \mathcal{B}_{pap}$, then $\varphi T \in \mathcal{B}'_{pap}$.
(iii) $\mathcal{B}_{pap} * \mathcal{D}'_{L^1} \subset \mathcal{B}_{pap}$.
(iv) $\mathcal{B}_{pap} = \mathcal{D}_{L^\infty} \cap \mathcal{C}_{pap}$.

Let $I =]0, 1]$, $\varepsilon \in I$, and

$$\mathcal{M}_{L^\infty} = \left\{ \left(u_\varepsilon\right)_\varepsilon \in \left(\mathcal{D}_{L^\infty}\right)^I : \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \left|u_\varepsilon\right|_{k,\infty} = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0 \right\}, \tag{2.2}$$

$$\mathcal{N}_{L^\infty} = \left\{ \left(u_\varepsilon\right)_\varepsilon \in \left(\mathcal{D}_{L^\infty}\right)^I : \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+,\ \left|u_\varepsilon\right|_{k,\infty} = O\left(\varepsilon^{m}\right),\ \varepsilon \longrightarrow 0 \right\}. \tag{2.3}$$

The algebra of bounded generalized functions on $\mathbb{R}$, is denoted and defined by the quotient algebra

$$\mathcal{G}_{L^\infty} := \frac{\mathcal{M}_{L^\infty}}{\mathcal{N}_{L^\infty}}, \tag{2.4}$$

An element u of $\mathcal{G}_{L^\infty}$ is an equivalence class, that is, $u = [(u_\varepsilon)_\varepsilon] = \left(u_\varepsilon\right)_\varepsilon + \mathcal{N}_{L^\infty}$. Following the construction of the algebra $\mathcal{G}_{ap}$ of almost periodic generalized functions, see [3], we define the space of pseudo almost periodic moderate elements

$$\mathcal{M}_{pap} = \left\{ \left(u_\varepsilon\right)_\varepsilon \in \left(\mathcal{B}_{pap}\right)^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \left|u_\varepsilon\right|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0 \right\}, \tag{2.5}$$

and the space of pseudo almost periodic negligible elements

$$\mathcal{N}_{pap} = \left\{ \left(u_\varepsilon\right)_\varepsilon \in \left(\mathcal{B}_{pap}\right)^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, \left|u_\varepsilon\right|_{k,\infty} = O\left(\varepsilon^{m}\right), \varepsilon \longrightarrow 0 \right\}. \tag{2.6}$$

The main properties of $\mathcal{M}_{pap}$ and $\mathcal{N}_{pap}$ are summarized in the following proposition.

Proposition 2.2. *(i) The space $\mathcal{M}_{pap}$ is a subalgebra of $\left(\mathcal{B}_{pap}\right)^I$.*
(ii) The space $\mathcal{N}_{pap}$ is an ideal of $\mathcal{M}_{pap}$.

Proof. (i) It follows from the fact that $\mathcal{B}_{pap}$ is a differential algebra.

(ii) Let $\left(u_\varepsilon\right)_\varepsilon \in \mathcal{N}_{pap}$ and $\left(v_\varepsilon\right)_\varepsilon \in \mathcal{M}_{pap}$, then $\forall k \in \mathbb{Z}_+, \exists m' \in \mathbb{Z}_+, \exists c_1 > 0, \exists \varepsilon_0 \in I, \forall \varepsilon < \varepsilon_0, \left|v_\varepsilon\right|_{k,\infty} < c_1 \varepsilon^{-m'}$. Take $m \in \mathbb{Z}_+$, then for $m'' = m + m', \exists c_2 > 0$ such that $\left|u_\varepsilon\right|_{k,\infty} < c_2 \varepsilon^{m''}$. Since the family of the norms $\left|\ \ \right|_{k,\infty}$ is compatible with the algebraic structure of $\mathcal{D}_{L^\infty}$, then $\forall k \in \mathbb{Z}_+, \exists c_k > 0$ such that $|u_\varepsilon v_\varepsilon|_{k,\infty} \leq c_k |u_\varepsilon|_{k,\infty} |v_\varepsilon|_{k,\infty}$, consequently $|u_\varepsilon v_\varepsilon|_{k,\infty} < c_k c_2 \varepsilon^{m''} c_1 \varepsilon^{-m'} \leq c\varepsilon^m$, where $c = c_1 c_2 c_k$. Hence $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{N}_{pap}$. □

Definition 2.2. *The algebra of pseudo almost periodic generalized functions is defined as the quotient algebra*

$$\mathcal{G}_{pap} := \frac{\mathcal{M}_{pap}}{\mathcal{N}_{pap}}. \tag{2.7}$$

We have the following results.

Proposition 2.3. $\mathcal{G}_{ap} \hookrightarrow \mathcal{G}_{pap} \hookrightarrow \mathcal{G}_{L^\infty}$.

A characterization of elements of $\mathcal{G}_{pap}$ is given by the following result.

Proposition 2.4. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}$, the following assertions are equivalent :*
(i) u is pseudo almost periodic.
(ii) $u_\varepsilon * \varphi \in \mathcal{B}_{pap}$, $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$.

Proof. $(i) \Longrightarrow (ii)$: If $u \in \mathcal{G}_{pap}$, then for every $\varepsilon \in I$ we have $u_\varepsilon \in \mathcal{B}_{pap}$, the result (iii) of Proposition (2.1) gives $u_\varepsilon * \varphi \in \mathcal{B}_{pap}, \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$.

$(ii) \Longrightarrow (i)$: Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{L^\infty}$ and $u_\varepsilon * \varphi \in \mathcal{B}_{pap}$, $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$, then from Theorem (1.1) $-$ (ii) it follows that $u_\varepsilon \in \mathcal{B}_{pap}$, it suffices to show that

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0, \tag{2.8}$$

which follows from the fact that $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{L^\infty}$. If $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{L^\infty}$ and $u_\varepsilon * \varphi \in \mathcal{B}_{pap}$, $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$, we obtain the same result, because $\mathcal{N}_{L^\infty} \subset \mathcal{M}_{L^\infty}$. □

Remark. *The characterization (ii) does not depend on representatives.*

The space $\mathcal{B}_{pap}$ is canonically embedded into $\mathcal{G}_{pap}$, i.e.

$$\begin{array}{rccc} \sigma_{pap}: & \mathcal{B}_{pap} & \longrightarrow & \mathcal{G}_{pap} \\ & f & \longrightarrow & [(f)_\varepsilon] = (f)_\varepsilon + \mathcal{N}_{pap} \end{array} \tag{2.9}$$

Set $\Sigma = \left\{ \rho \in \mathcal{S} : \int\limits_{\mathbb{R}} \rho(x)\, dx = 1 \text{ and } \int\limits_{\mathbb{R}} x^\alpha \rho(x)\, dx = 0, \forall \alpha \geq 1 \right\}$ and $\rho_\varepsilon(.) = \frac{1}{\varepsilon}\rho\left(\frac{\cdot}{\varepsilon}\right), \varepsilon > 0$.

Proposition 2.5. *For $\rho \in \Sigma$, the map*

$$\begin{array}{rccc} i_{pap}: & \mathcal{B}'_{pap} & \longrightarrow & \mathcal{G}_{pap} \\ & T & \longrightarrow & (T * \rho_\varepsilon)_\varepsilon + \mathcal{N}_{pap}, \end{array} \tag{2.10}$$

is a linear embedding which commutes with derivatives.

Proof. Let $T \in \mathcal{B}'_{pap}$, from Theorem (1.1) $-$ (iii), $\exists (f_\beta)_\beta \subset \mathcal{C}_{pap}$ such that $T = \sum\limits_{\beta \leq m} f_\beta^{(\beta)}$, so $\forall \alpha \in \mathbb{Z}_+$,

$$\left|\left(T^{(\alpha)} * \rho_\varepsilon\right)(x)\right| \leq \sum_{\beta \leq m} \frac{1}{\varepsilon^{\alpha+\beta}} \|f_\beta\|_{L^\infty} \int\limits_{\mathbb{R}} \left|\rho^{(\alpha+\beta)}(y)\right| dy,$$

consequently, $\exists c>0$ such that $\sup\limits_{x\in\mathbb{R}}\left|\left(T^{(\alpha)}*\rho_\varepsilon\right)(x)\right|\leq\frac{c}{\varepsilon^{\alpha+m}}$, hence, $\exists c'>0$ such that

$$|T*\rho_\varepsilon|_{m',\infty}=\sum_{\alpha\leq m'}\sup_{x\in\mathbb{R}}\left|\left(T^{(\alpha)}*\rho_\varepsilon\right)(x)\right|\leq\frac{c'}{\varepsilon^{m+m'}},\text{ where } c'=\sum_{\alpha\leq m'}\frac{c}{\varepsilon^{\alpha}},\qquad(2.11)$$

which shows that $(T*\rho_\varepsilon)_\varepsilon\in\mathcal{M}_{pap}$. Let $(T*\rho_\varepsilon)_\varepsilon\in\mathcal{N}_{pap}$, then $\lim\limits_{\varepsilon\longrightarrow0}T*\rho_\varepsilon=0$ in $\mathcal{D}'_{L^\infty}$, but we have also $\lim\limits_{\varepsilon\longrightarrow0}T*\rho_\varepsilon=T$ in $\mathcal{D}'_{L^\infty}$, this mean that i_{pap} is an embedding. The linearity of i_{pap} it results from the fact that the convolution is linear and that $i_{pap}\left(T^{(j)}\right)=\left(T^{(j)}*\rho_\varepsilon\right)_\varepsilon=(T*\rho_\varepsilon)_\varepsilon^{(j)}=\left(i_{pap}(T)\right)^{(j)}$. □

The following result shows that there are tow ways to embed the space $\mathcal{B}_{pap}$ into $\mathcal{G}_{pap}$.

Proposition 2.6. *The following diagram*

$$\begin{array}{ccc}\mathcal{B}_{pap} & \longrightarrow & \mathcal{B}'_{pap}\\ & \sigma_{pap}\searrow & \downarrow i_{pap}\\ & & \mathcal{G}_{pap}\end{array}\qquad(2.12)$$

is commutative.

Proof. Let $f\in\mathcal{B}_{pap}$, we must show that $(f*\rho_\varepsilon-f)_\varepsilon\in\mathcal{N}_{pap}$. Indeed, by Taylor's formula and the fact that $\rho\in\Sigma$, we have

$$\|f*\rho_\varepsilon-f\|_{L^\infty}\leq\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\left|\frac{(-y)^m}{m!}f^{(m)}(x-\theta\varepsilon y)\rho(y)\,dy\right|\varepsilon^m.$$

Then $\exists C_m>0$ such that $\|f*\rho_\varepsilon-f\|_{L^\infty}\leq C_m\left\|f^{(m)}\right\|_{L^\infty}\|y^m\rho\|_{L^1}\varepsilon^m$. The same result can be obtained for all the derivatives of f. Hence $(f*\rho_\varepsilon-f)_\varepsilon\in\mathcal{N}_{pap}$. □

We have the following generalized version of the classical Bohl-Bohr's Theorem.

Proposition 2.7. *A primitive of a pseudo almost periodic generalized function is pseudo almost periodic if and only if it is bounded generalized function.*

Proof. Let $u=[(u_\varepsilon)_\varepsilon]\in\mathcal{G}_{pap}$ and U its primitive, i.e. $U=[(U_\varepsilon)_\varepsilon]$. where $U_\varepsilon(x)=\int_{x_0}^{x}u_\varepsilon(t)\,dt$ and $x_0\in\mathbb{R}$. If $U\in\mathcal{G}_{pap}$, then by Proposition (2.3), $U\in\mathcal{G}_{L^\infty}$. Conversely, let $x_0\in\mathbb{R}$ and assume that $U=[(U_\varepsilon)_\varepsilon]\in\mathcal{G}_{L^\infty}$, then by definition $\forall\varepsilon\in I,\forall x\in\mathbb{R},U_\varepsilon(x)=\int_{x_0}^{x}u_\varepsilon(t)\,dt\in\mathcal{D}_{L^\infty}$, which show that U_ε is a bounded primitive of $u_\varepsilon\in\mathcal{C}_{pap}$. From the classical case, we deduce that $U_\varepsilon\in\mathcal{C}_{pap}$, i.e. $\forall\varepsilon\in I,U_\varepsilon\in\mathcal{C}_{pap}\cap\mathcal{D}_{L^\infty}=\mathcal{B}_{pap}$, Proposition (2.1)$-(iv)$. Moreover, $(U_\varepsilon)_\varepsilon\in\mathcal{M}_{L^\infty}$, i.e. $\forall k\in\mathbb{Z}_+,\exists m\in\mathbb{Z}_+,|U_\varepsilon|_{k,\infty}=O\left(\varepsilon^{-m}\right),\varepsilon\longrightarrow0$. Thus, $(U_\varepsilon)_\varepsilon\in\mathcal{M}_{pap}$ and $U\in\mathcal{G}_{pap}$. The result is independent on representatives. □

3. Conclusion

This work has allowed us to lift the concept of pseudo almost periodicity to the level of generalized functions.The results obtained are the first steps to go on studying other problems. Some of them, the uniqueness of the decomposition of a pseudo almost periodic generalized function, the composition of tempered generalized function with pseudo almost periodic generalized function, the convolution and some results of existence for the linear differential equations in the framework of pseudo almost periodic generalized functions.

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ON THE STABILIZATION FOR A CLASS OF DISTRIBUTED BILINEAR SYSTEMS USING BOUNDED CONTROLS

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Abstract. This paper considers the question of the output stabilization for a class of infinite dimensional bilinear systems evolving on a spatial domain Ω. Then, we give sufficient conditions for exponential, strong and weak stabilization of the output of such systems. Examples and simulations illustrate the efficiency of such controls.

1. Introduction

In this paper, we consider the following bilinear system

$$\begin{cases} \dot{z}(t) = Az(t) + v(t)Bz(t), \quad t \geq 0, \\ z(0) = z_0, \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ generates a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a Hilbert space H, endowed with norm and inner product denoted, respectively, by $\|.\|$ and $\langle ., . \rangle$, $v(.) \in V_{ad}$ (the admissible controls set) is a scalar valued control and $B : H \to H$ is a linear bounded operator. The problem of feedback stabilization of distributed system (1.1) was studied in many works that lead to various results. In [1], it was shown that the control

$$v(t) = -\langle z(t), Bz(t) \rangle, \tag{1.2}$$

weakly stabilizes system (1.1) provided that B be a weakly sequentially continuous operator such that, for all $\psi \in H$, we have

$$\langle BS(t)\psi, S(t)\psi \rangle = 0, \quad \forall\ t \geq 0 \Longrightarrow \psi = 0, \tag{1.3}$$

and if (1.3) is replaced by the following assumption

$$\int_0^T |\langle BS(s)\psi, S(s)\psi \rangle| ds \geq \gamma \|\psi\|^2, \quad \forall\ \psi \in H,\ (\text{for some } \gamma, T > 0), \tag{1.4}$$

then control (1.2) strongly stabilizes system (1.1) (see [2]). In [3], the authors show that when the resolvent of A is compact, B self-adjoint and monotone, then strong

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stabilization of system (1.1) is proved using bounded controls. Let the output state space Y be a Hilbert space with inner product $\langle .,. \rangle_Y$ and the corresponding norm $\|.\|_Y$, and let $C \in \mathcal{L}(H, Y)$ be an output operator. The system (1.1) is augmented with the output

$$w(t) := Cz(t). \tag{1.5}$$

The output stabilization means that $w(t) \to 0$ as $t \to +\infty$ using suitable controls. In the case when $Y = H$ and $C = I$, one obtains the classical stabilization of the state. When $C \neq I$, the output stabilization for distributed systems was studied in many works: in [14], authors considered the output exponential stabilization for one-dimensional wave equations with boundary control. In [4], authors considered output stabilization for Kirchhoff-type equation with boundary control. They studied the existence and uniqueness of solution of system and the strong stabilization of such equation was proved. In [6], authors established the output stabilization for a class of nonlinear systems with boundary control. They investigated the existence of solution and the exponential stabilization of such systems. In [7], author studied weak and strong output stabilization for semilinear systems using controls that do not take into account the output operator. In [11], authors considered exponential, strong and weak output stabilization of semilinear systems. If $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be the system evolution domain and $\omega \subset \Omega$, when $C = \chi_\omega$, the restriction operator to a subregion ω of Ω, one is concerned with the behaviour of the state only in a subregion of the system evolution domain. This is what we call regional stabilization. The notion of regional stabilization is useful in systems theory since there exist systems which are not stabilizable on the whole domain but stabilizable on some subregion ω. Moreover stabilizing a systems on a subregion is cheaper than stabilizing it on the whole domain [12]. In [13], regional stabilization for bilinear systems was studied using decomposition of system (1.1) into regional stable and regional unstable subsystems, therefore regional stabilization of system (1.1) turns out to stabilizing its unstable part. In [10], authors proved regional strong and weak stabilization of bilinear systems with unbounded control operator. In [9], authors considered regional weak, strong and exponential stabilization of bilinear systems with control operator B assumed to be bounded with respect to the graph norm of the operator A. In this paper, we study the exponential, strong and weak stabilization of the output (1.5) using bounded controls. Then, we develop sufficient conditions that allow exponential, strong and weak stabilization of the output of such system. Illustrations by examples and simulations are given. The approach is based on the decay of an adapted function, the exact and weak observability conditions, and semigroup properties. The paper is organized as follows. The second section discusses sufficient conditions to achieve exponential, strong and weak stabilization of the output (1.5). In the third section, we give illustrating examples. The fourth section is devoted to simulations.

2. Output stabilization

In this section, we develop sufficient conditions that allow exponential, strong and weak stabilization of the output (1.5).

Definition 2.1. *The output (1.5) is said to be:*
1. weakly stabilizable, if there exists a control $v(.) \in V_{ad}$ such that for any initial condition $z_0 \in H$, the corresponding solution $z(t)$ of system (1.1) is global and

satisfies

$$\langle Cz(t), \psi \rangle_Y \to 0, \quad \forall \psi \in Y, \quad \text{as } t \to \infty,$$

2. strongly stabilizable, if there exists a control $v(.) \in V_{ad}$ such that for any initial condition $z_0 \in H$, the corresponding solution $z(t)$ of system (1.1) is global and verifies

$$\|Cz(t)\|_Y \to 0, \quad \text{as } t \to \infty,$$

3. exponentially stabilizable, if there exists a control $v(.) \in V_{ad}$ such that for any initial condition $z_0 \in H$, the corresponding solution $z(t)$ of system (1.1) is global and there exist $\alpha, \beta > 0$ such that

$$\|Cz(t)\|_Y \leq \alpha e^{-\beta t} \|z_0\|, \quad \forall t > 0.$$

Remark. *It is clear that exponential stability of (1.5) implies strong stability of (1.5) implies weak stability of (1.5).*

2.1. **Exponential stabilization.** In this subsection, we develop sufficient conditions for exponential stabilization of the output (1.5).
The following result concerns the exponential stabilization of (1.5).

Theorem 2.1. *Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H and B is a bounded control operator. If the conditions:*
1. $\mathcal{R}e(\langle C^*CAy, y\rangle) \leq 0$, $\forall y \in D(A)$,
2. $\mathcal{R}e(\langle C^*CBy, y\rangle\langle By, y\rangle) \geq 0$, $\forall y \in H$,
3. there exist $T, \gamma > 0$, such that

$$\int_0^T |\langle C^*CBS(t)y, S(t)y\rangle| dt \geq \gamma \|Cy\|_Y^2, \quad \forall\, y \in H, \tag{2.1}$$

hold, then the control

$$v(t) = \begin{cases} -\frac{\langle C^*CBz(t), z(t)\rangle}{\|z(t)\|^2} & \text{if } z(t) \neq 0 \\ 0 & \text{if } z(t) = 0, \end{cases} \tag{2.2}$$

exponentially stabilizes the output (1.5).

Proof. System (1.1) has a unique weak solution $z(t)$ (see [8]) defined on a maximal interval $[0, t_{\max}]$ by

$$z(t) = S(t)z_0 + \int_0^t g(z(s))S(t-s)Bz(s)ds, \tag{2.3}$$

where

$$g(z(t)) = \begin{cases} -\frac{\langle C^*CBz(t), z(t)\rangle}{\|z(t)\|^2} & \text{if } z(t) \neq 0 \\ 0 & \text{if } z(t) = 0. \end{cases}$$

Since $(S(t))_{t\geq 0}$ is a semigroup of contractions, we deduce

$$\frac{d}{dt}\|z(t)\|^2 \leq 2g(z(t))\langle Bz(t), z(t)\rangle.$$

Integrating this inequality over the interval $[0, t]$, we have

$$\|z(t)\|^2 - \|z(0)\|^2 \leq 2\int_0^t g(z(s))\langle Bz(s), z(s)\rangle ds.$$

Using hypothesis 2 of Theorem 2.1, it follows that

$$\|z(t)\| \leq \|z_0\|. \tag{2.4}$$

For all $z_0 \in H$ and $t \geq 0$, we have

$$\begin{aligned}\langle C^*CBS(t)z_0, S(t)z_0\rangle &= \langle C^*CBz(t), z(t)\rangle - \langle C^*CBz(t), z(t) - S(t)z_0\rangle \\ &+ \langle C^*CBS(t)z_0 - C^*CBz(t), S(t)z_0\rangle.\end{aligned}$$

Since B is bounded, then

$$|\langle C^*CBS(t)z_0, S(t)z_0\rangle| \leq |\langle C^*CBz(t), z(t)\rangle| + 2\alpha\|B\|\|z(t) - S(t)z_0\|\|z_0\|, \quad (2.5)$$

where α is a positive constant.
Using (2.4), we deduce

$$|\langle C^*CBz(t), z(t)\rangle| \leq |g(z(t))|\|z(t)\|\|z_0\|, \ \forall t \in [0, T]. \quad (2.6)$$

While from (2.3) and using Schwartz's inequality, we obtain

$$\|z(t) - S(t)z_0\| \leq \|B\| \left(T \int_0^T |g(z(t))|^2 \|z(t)\|^2 dt \right)^{\frac{1}{2}}. \quad (2.7)$$

Integrating (2.5) over the interval $[0, T]$ and taking into account (2.6) and (2.7), we have

$$\begin{aligned}\int_0^T |\langle C^*CBS(t)z_0, S(t)z_0\rangle| dt &\leq 2\alpha T^{\frac{3}{2}} \|B\|^2 \|z_0\| \left(\int_0^T |g(z(t))|^2 \|z(t)\|^2 dt \right)^{\frac{1}{2}} \\ &+ T^{\frac{1}{2}} \|z_0\| \left(\int_0^T |g(z(t))|^2 \|z(t)\|^2 dt \right)^{\frac{1}{2}}. \quad (2.8)\end{aligned}$$

Let us consider the nonlinear semigroup $U(t)z_0 := z(t)$ (see [1]). Replacing z_0 by $U(t)z_0$ in (2.8), and using the superposition properties of the semigroup $(U(t))_{t\geq 0}$, we deduce that

$$\begin{aligned}\int_0^T |\langle C^*CBS(s)U(t)z_0, S(s)U(t)z_0\rangle| ds &\leq 2\alpha T^{\frac{3}{2}} \|B\|^2 \|U(t)z_0\| \quad (2.9) \\ &\times \left(\int_t^{t+T} |g(U(s)z_0)|^2 \|U(s)z_0\|^2 ds \right)^{\frac{1}{2}} \\ &+ T^{\frac{1}{2}} \|U(t)z_0\| \left(\int_t^{t+T} |g(U(s)z_0)|^2 \|U(s)z_0\|^2 ds \right)^{\frac{1}{2}}.\end{aligned}$$

Thus, by using (2.1) and (2.9), it follows that

$$\gamma \|CU(t)z_0\|_Y \leq M \left(\int_t^{t+T} |g(U(s)z_0)|^2 \|U(s)z_0\|^2 ds \right)^{\frac{1}{2}}, \quad (2.10)$$

where $M = (2\alpha T\|B\|^2 + 1)T^{\frac{1}{2}}$ is a positive constant depending on $\|z_0\|$ and T.
From hypothesis 1 of Theorem 2.1, we have

$$\frac{d}{dt} \|CU(t)z_0\|_Y^2 \leq -2|g(U(t)z_0)|^2 \|U(t)z_0\|^2. \quad (2.11)$$

Integrating (2.11) from nT and $(n+1)T$, $(n \in \mathbb{N})$, we obtain

$$\|CU(nT)z_0\|_Y^2 - \|CU((n+1)T)z_0\|_Y^2 \geq 2 \int_{nT}^{(n+1)T} |g(U(s)z_0)|^2 \|U(s)z_0\|^2 ds.$$

Using (2.10), (2.11) and the fact that $\|CU(t)z_0\|_Y$ decreases, it follows

$$\left(1+2\left(\frac{\gamma}{M}\right)^2\right)\|CU((n+1)T)z_0\|_Y^2 \leq \|CU(nT)z_0\|_Y^2.$$

Then

$$\|CU((n+1)T)z_0\|_Y \leq \beta\|CU(nT)z_0\|_Y,$$

where $\beta = \frac{1}{\left(1+2\left(\frac{\gamma}{M}\right)^2\right)^{\frac{1}{2}}}$. By recurrence, we show that $\|CU(nT)z_0\|_Y \leq \beta^n\|Cz_0\|_Y$.
Taking $n = E(\frac{t}{T})$ the integer part of $\frac{t}{T}$, we deduce that

$$\|CU(t)z_0\|_Y \leq Re^{-\sigma t}\|z_0\|,$$

where $R = \alpha\left(1+2\left(\frac{\gamma}{M}\right)^2\right)^{\frac{1}{2}}$, with $\alpha > 0$ and $\sigma = \frac{\ln\left(1+2\left(\frac{\gamma}{M}\right)^2\right)}{2T} > 0$, which achieves the proof. ☐

2.2. **Strong stabilization.** The following result will be used to prove strong stabilization of the output (1.5).

Theorem 2.2. *Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H and $B : H \to H$ is a bounded linear operator. If the conditions:*
1. $\mathcal{R}e\big(\langle C^*CA\psi,\psi\rangle\big) \leq 0,\ \forall\psi\in D(A)$,
2. $\mathcal{R}e\big(\langle C^*CB\psi,\psi\rangle\langle B\psi,\psi\rangle\big) \geq 0,\ \forall\psi\in H$,
hold, then control

$$v(t) = -\frac{\langle C^*CBz(t),z(t)\rangle}{1+|\langle C^*CBz(t),z(t)\rangle|}, \tag{2.12}$$

allows the estimate

$$\left(\int_0^T |\langle C^*CBS(s)z(t),S(s)z(t)\rangle|ds\right)^2 = \Theta\left(\int_t^{t+T}\frac{|\langle C^*CBz(s),z(s)\rangle|^2}{1+|\langle C^*CBz(s),z(s)\rangle|}ds\right), \quad \text{as } t\to+\infty. \tag{2.13}$$

Proof. We have

$$\frac{1}{2}\frac{d}{dt}\langle Cz(t),Cz(t)\rangle_Y = \mathcal{R}e\big(\langle CAz(t),Cz(t)\rangle_Y\big) + \mathcal{R}e\big(v(t)\langle CBz(t),Cz(t)\rangle_Y\big).$$

Then

$$\frac{1}{2}\frac{d}{dt}\langle Cz(t),Cz(t)\rangle_Y = \frac{1}{2}\frac{d}{dt}\|Cz(t)\|_Y^2 = \mathcal{R}e\big(\langle C^*CAz(t),z(t)\rangle\big)+\mathcal{R}e\big(v(t)\langle C^*CBz(t),z(t)\rangle\big).$$

From hypothesis 1 of Theorem 2.2, we have

$$\frac{1}{2}\frac{d}{dt}\|Cz(t)\|_Y^2 \leq \mathcal{R}e\big(v(t)\langle C^*CBz(t),z(t)\rangle\big).$$

In order to make the function $\frac{1}{2}\|Cz(t)\|_Y^2$ nonincreasing, we consider the control

$$v(t) = -\frac{\langle C^*CBz(t),z(t)\rangle}{1+|\langle C^*CBz(t),z(t)\rangle|},$$

so that the resulting closed-loop system is

$$\dot{z}(t) = Az(t) + f(z(t)),\ z(0) = z_0, \tag{2.14}$$

where $f(z) = -\frac{\langle C^*CBz,z\rangle Bz}{1+|\langle C^*CBz,z\rangle|}$, $\forall\, z \in H$.
Since f is locally Lipschitz, then system (2.14) has a unique mild solution $z(t)$ (see Theorem 1.4, pp 185 in [8]) defined on a maximal interval $[0, t_{max}]$ by

$$z(t) = S(t)z_0 + \int_0^t S(t-s)f(z(s))ds. \tag{2.15}$$

Because of the contractions of the semigroup (i.e $\mathcal{R}e\big(\langle A\psi,\psi\rangle\big) \leq 0,\ \forall\, \psi \in D(A)$), we have

$$\frac{d}{dt}\|z(t)\|^2 \leq -2\frac{\langle C^*CBz(t),z(t)\rangle\langle Bz(t),z(t)\rangle}{1+|\langle C^*CBz(t),z(t)\rangle|}.$$

Integrating this inequality over the interval $[0,t]$, we deduce

$$\|z(t)\|^2 - \|z(0)\|^2 \leq -2\int_0^t \frac{\langle C^*CBz(s),z(s)\rangle\langle Bz(s),z(s)\rangle}{1+|\langle C^*CBz(s),z(s)\rangle|}ds.$$

Using condition 2 of Theorem 2.2, it follows that

$$\|z(t)\| \leq \|z_0\|. \tag{2.16}$$

From hypothesis 1 of Theorem 2.2, we have

$$\frac{d}{dt}\|Cz(t)\|_Y^2 \leq -2\frac{|\langle C^*CBz(t),z(t)\rangle|^2}{1+|\langle C^*CBz(t),z(t)\rangle|}.$$

Integrating this inequality, we deduce

$$\|Cz(t)\|_Y^2 - \|Cz(0)\|_Y^2 \leq -2\int_0^t \frac{|\langle C^*CBz(s),z(s)\rangle|^2}{1+|\langle C^*CBz(s),z(s)\rangle|}ds. \tag{2.17}$$

While from (2.15) and using Schwartz inequality, we obtain

$$\|z(t)-S(t)z_0\| \leq \|B\|\|z_0\|\left(T\int_0^t \frac{|\langle C^*CBz(s),z(s)\rangle|^2}{1+|\langle C^*CBz(s),z(s)\rangle|}ds\right)^{\frac{1}{2}},\ \forall t \in [0,T]. \tag{2.18}$$

Since B is bounded and C continuous, we have

$$|\langle C^*CBS(s)z_0, S(s)z_0\rangle| \leq 2K\|B\|\|z(s)-S(s)z_0\|\|z_0\|+|\langle C^*CBz(s),z(s)\rangle|, \tag{2.19}$$

where K is a positive constant. Replacing z_0 by $z(t)$ in (2.18) and (2.19), we deduce

$$\begin{aligned} |\langle C^*CBS(s)z(t),S(s)z(t)\rangle| &\leq 2K\|B\|^2\|z_0\|^2\left(T\int_t^{t+T}\frac{|\langle C^*CBz(s),z(s)\rangle|^2}{1+|\langle C^*CBz(s),z(s)\rangle|}ds\right)^{\frac{1}{2}} \\ &+ |\langle C^*CBz(t+s),z(t+s)\rangle|, \quad \forall t \geq s \geq 0. \end{aligned}$$

Integrating this relation over $[0,T]$ and using Cauchy-Schwartz, we obtain

$$\begin{aligned} \int_0^T |\langle C^*CBS(s)z(t),S(s)z(t)\rangle| ds &\leq \left(2K\|B\|^2T^{\frac{3}{2}} + T\left(1+K\|B\|\|z_0\|^2\right)\right) \\ &\times \left(\int_t^{t+T}\frac{|\langle C^*CBz(s),z(s)\rangle|^2}{1+|\langle C^*CBz(s),z(s)\rangle|}ds\right)^{\frac{1}{2}}, \end{aligned}$$

which achieves the proof. □

The following result gives sufficient conditions for strong stabilization of the output (1.5).

Theorem 2.3. *Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H and B is a bounded linear operator. If the assumptions 1, 2 of Theorem 2.2 and*

$$\int_0^T |\langle C^*CBS(t)\psi, S(t)\psi\rangle| dt \geq \gamma \|C\psi\|_Y^2, \ \ \forall \ \psi \in H, \ (for\ some\ T, \gamma > 0), \quad (2.20)$$

hold, then control (2.12) strongly stabilizes the output (1.5) with decay estimate

$$\|Cz(t)\|_Y = \Theta\left(\frac{1}{\sqrt{t}}\right), \ \ as \ \ t \longrightarrow +\infty. \quad (2.21)$$

Proof. Using (2.17), we deduce

$$\|Cz(kT)\|_Y^2 - \|Cz((k+1)T)\|_Y^2 \geq 2\int_{kT}^{k(T+1)} \frac{|\langle C^*CBz(t), z(t)\rangle|^2}{1+|\langle C^*CBz(t), z(t)\rangle|} dt, \ \ k \geq 0.$$

From (2.13) and (2.20), we have

$$\|Cz(kT)\|_Y^2 - \|Cz((k+1)T)\|_Y^2 \geq \beta \|Cz(kT)\|_Y^4, \quad (2.22)$$

where $\beta = \frac{\gamma^2}{2\left(2K\|B\|^2T^{\frac{3}{2}} + T\left(1+K\|B\|\|z_0\|^2\right)\right)^2}$. Taking $s_k = \|Cz(kT)\|_Y^2$, the inequality (2.22) can be written as

$$\beta s_k^2 + s_{k+1} \leq s_k, \ \ \forall k \geq 0.$$

Since $s_{k+1} \leq s_k$, we obtain

$$\beta s_{k+1}^2 + s_{k+1} \leq s_k, \ \ \forall k \geq 0.$$

Taking $p(s) = \beta s^2$ and $q(s) = s - (I+p)^{-1}(s)$ in Lemma 3.3, page 531 in [5], we deduce

$$s_k \leq x(k), \ \ k \geq 0,$$

where $x(t)$ is the solution of equation $x'(t) + q(x(t)) = 0, \ \ x(0) = s_0$.
Since $x(k) \geq s_k$ and $x(t)$ decreases give $x(t) \geq 0$, $\forall t \geq 0$. Furthermore, it is easy to see that $q(s)$ is an increasing function such that

$$0 \leq q(s) \leq p(s), \forall s \geq 0.$$

We obtain $-\beta x(t)^2 \leq x'(t) \leq 0$, which implies that

$$x(t) = \Theta(t^{-1}), \ \text{as}\ t \to +\infty.$$

Finally the inequality $s_k \leq x(k)$, together with the fact that $\|Cz(t)\|_Y$ decreases, we deduce the estimate

$$\|Cz(t)\|_Y = \Theta\left(\frac{1}{\sqrt{t}}\right), \ \ \text{as} \ \ t \longrightarrow +\infty.$$

□

2.3. **Weak stabilization.** The following result provides sufficient conditions for weak stabilization of the output (1.5).

Theorem 2.4. *Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H and B is a compact operator. If the conditions:*
1. $\mathcal{R}e\left(\langle C^*CA\psi, \psi\rangle\right) \leq 0$, $\forall \psi \in D(A)$,
2. $\mathcal{R}e\left(\langle C^*CB\psi, \psi\rangle \langle B\psi, \psi\rangle\right) \geq 0$, $\forall \psi \in H$,
3. $\langle C^*CBS(t)\psi, S(t)\psi\rangle = 0, \ \ \forall t \geq 0 \Longrightarrow C\psi = 0$,
hold, then control (2.12) weakly stabilizes the output (1.5).

Proof. Let us consider the nonlinear semigroup $\Gamma(t)z_0 := z(t)$ and let (t_n) be a sequence of real numbers such that $t_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$.
From (2.16), $\Gamma(t_n)z_0$ is bounded in H, then there exists a subsequence $(t_{\phi(n)})$ of (t_n) such that

$$\Gamma(t_{\phi(n)})z_0 \rightharpoonup \psi, \text{ as } n \to \infty.$$

Since B is compact and C continuous, we have

$$\lim_{n\to+\infty} \langle C^*CBS(t)\Gamma(t_{\phi(n)})z_0, S(t)\Gamma(t_{\phi(n)})z_0\rangle = \langle C^*CBS(t)\psi, S(t)\psi\rangle.$$

For all $n \geq$, we set

$$\Lambda_n(t) := \int_{\phi(n)}^{\phi(n)+t} \frac{|\langle C^*CB\Gamma(s)z_0, \Gamma(s)z_0\rangle|^2}{1+|\langle C^*CB\Gamma(s)z_0, \Gamma(s)z_0\rangle|} ds.$$

It follows that $\forall t \geq 0$, $\Lambda_n(t) \to 0$ as $n \to +\infty$.
Using (2.13), we deduce

$$\lim_{n\to+\infty} \int_0^t |\langle C^*CBS(s)\Gamma(t_{\phi(n)})z_0, S(s)\Gamma(t_{\phi(n)})z_0\rangle| ds = 0.$$

Hence, by the dominated convergence Theorem, we have

$$\int_0^t |\langle C^*CBS(s)\psi, S(s)\psi\rangle| ds = 0.$$

We conclude that

$$\langle C^*CBS(s)\psi, S(s)\psi\rangle = 0, \quad \forall s \in [0, t].$$

Using condition 3 of Theorem 2.4, we deduce that

$$C\Gamma(t_{\phi(n)})z_0 \rightharpoonup 0, \quad \text{as } n \longrightarrow +\infty. \tag{2.23}$$

On the other hand, it is clear that (2.23) holds for each subsequence $(t_{\phi(n)})$ of (t_n) such that $C\Gamma(t_{\phi(n)})z_0$ weakly converges in Y. This implies that $\forall\varphi \in Y$, we have $\langle C\Gamma(t_n)z_0, \varphi\rangle \to 0$ as $n \longrightarrow +\infty$ and hence

$$C\Gamma(t)z_0 \rightharpoonup 0, \quad \text{as } t \longrightarrow +\infty.$$

□

3. Examples

Example 3.1. *Let Ω denote a bounded open subset of $\mathbb{R}^n$, and consider the following wave equation*

$$\begin{cases} \dfrac{\partial^2 z(x,t)}{\partial t^2} - \Delta z(x,t) = v(t)\dfrac{\partial z(x,t)}{\partial t} & \Omega\times]0,+\infty[\\ z(x,t) = 0 & \partial\Omega\times]0,+\infty[\\ z(x,0) = z_0(x), \ \dfrac{\partial z(x,0)}{\partial t} = z_1(x) & \Omega. \end{cases} \tag{3.1}$$

This system has the form of equation (1.1) if we set $H = H_0^1(\Omega) \times L^2(\Omega)$ with $\langle (y_1, z_1), (y_2, z_2)\rangle = \langle y_1, y_2\rangle_{H^1(\Omega)} + \langle z_1, z_2\rangle_{L^2(\Omega)}$, $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$.

We consider the output operator $C = I$, we have A is skew-adjoint on H and the assumption (2.20) holds (see [2]). Then the control

$$v(t) = -\frac{\|\frac{\partial z(.,t)}{\partial t}\|^2_{L^2(0,1)}}{1+\|\frac{\partial z(.,t)}{\partial t}\|^2_{L^2(0,1)}}, \tag{3.2}$$

strongly stabilises system (3.1) with the decay estimate

$$\|(z(.,t), \frac{\partial z(.,t)}{\partial t})\|_H = \Theta(\frac{1}{\sqrt{t}}), \quad as \ t \longrightarrow +\infty.$$

Example 3.2. *Let us consider a system defined on $\Omega =]0,1[$ by*

$$\begin{cases} \frac{\partial z(x,t)}{\partial t} = Az(x,t) + v(t)a(x)z(x,t) & \Omega\times]0,+\infty[\\ z(x,0) = z_0(x) & \Omega, \end{cases} \tag{3.3}$$

where $H = L^2(\Omega)$, $Az = -z$, and $a \in L^\infty(]0,1[)$ such that $a(x) \geq 0$ a.e on $]0,1[$ and $a(x) \geq c > 0$ on subregion ω of Ω and $v(.) \in L^\infty(0,+\infty)$ the control function. System (3.3) is augmented with the output

$$w(t) = \chi_\omega z(t), \tag{3.4}$$

where $\chi_\omega : L^2(\Omega) \longrightarrow L^2(\omega)$, the restriction operator to ω and χ^*_ω is the adjoint operator of χ_ω. The operator A generates a semigroup of contractions on $L^2(\Omega)$ given by $S(t)z_0 = e^{-t}z_0$. For all $z_0 \in L^2(\Omega)$ and $T = 2$, we obtain

$$\begin{aligned} \int_0^2 \langle \chi^*_\omega \chi_\omega BS(t)z_0, S(t)z_0\rangle dt &= \int_0^2 e^{-2t}dt \int_\omega a(x)|z_0|^2dx \\ &\geq \beta\|\chi_\omega z_0\|^2_{L^2(\omega)}, \end{aligned}$$

with $\beta = c\displaystyle\int_0^2 e^{-2t}dt > 0$.
Then the control

$$v(t) = -\frac{\displaystyle\int_\omega a(x)|z(x,t)|^2dx}{1+\displaystyle\int_\omega a(x)|z(x,t)|^2dx},$$

strongly stabilizes the output (3.4) with decay estimate

$$\|\chi_\omega z(t)\|_{L^2(\omega)} = \Theta\left(\frac{1}{\sqrt{t}}\right), \quad as \ t \longrightarrow +\infty.$$

Example 3.3. *Consider a system defined in $\Omega =]0,+\infty[$, and described by*

$$\begin{cases} \frac{\partial z(x,t)}{\partial t} = -\frac{\partial z(x,t)}{\partial x} + v(t)Bz(x,t) & \Omega\times]0,+\infty[\\ z(x,0) = z_0(x) & \Omega, \end{cases} \tag{3.5}$$

where $Az = -\dfrac{\partial z}{\partial x}$ *with domain* $D(A) = \{z \in H^1(\Omega) \mid z(0) = 0,\ z(x) \to 0$ *as* $x \to +\infty\}$ *and* $Bz = \displaystyle\int_0^1 z(x)dx$. *The operator* A *generates a semigroup of contractions*

$$(S(t)z_0)(x) = \begin{cases} z_0(x-t) & if\, x \geq t \\ 0 & if\, x < t. \end{cases}$$

Let $\omega =]0,1[$ *be a subregion of* Ω *and system (3.5) is augmented with the output*

$$w(t) = \chi_\omega z(t). \tag{3.6}$$

We have

$$\begin{aligned} \mathcal{R}e(\langle \chi_\omega^* \chi_\omega Az, z \rangle) &= -\mathcal{R}e(\int_0^1 z'(x)z(x)dx) \\ &= -\frac{z^2(1)}{2} \leq 0, \end{aligned}$$

so, the assumption 1 of Theorem 2.4 holds. The operator B *is compact and verifies*

$$\langle \chi_\omega^* \chi_\omega BS(t)z_0, S(t)z_0 \rangle = \left(\int_0^{1-t} z_0(x)dx \right)^2, \quad 0 \leq t \leq 1.$$

Thus

$$\langle \chi_\omega^* \chi_\omega BS(t)z_0, S(t)z_0 \rangle = 0, \quad \forall t \geq 0 \implies z_0(x) = 0,\ a.e\ \ on\, \omega.$$

Then, the control

$$v(t) = -\frac{\left(\displaystyle\int_0^1 z(x,t)dx \right)^2}{1 + \left(\displaystyle\int_0^1 z(x,t)dx \right)^2}, \tag{3.7}$$

weakly stabilizes the output (3.6).

4. Simulations

Consider system (3.5) with $z(x,0) = \sin(\pi x)$, and augmented with the output (3.6).

• For $\omega =]0,2[$, figure 1 shows that the state is stabilized on ω with error equals 3.4×10^{-4}, and the evolution of control function is given by figure 2.

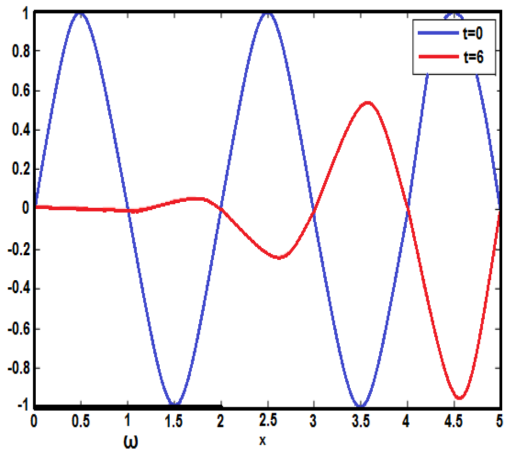

FIGURE 1. The stabilization of the state on $\omega =]0,2[$.

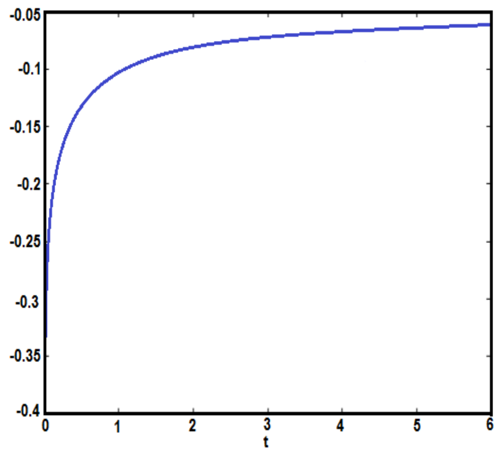

FIGURE 2. Evolution of control function.

• For $\omega =]0,3[$, figure 3 shows that the state is stabilized on ω with error equals 7.8×10^{-4} and the evolution of control is given by figure 4.

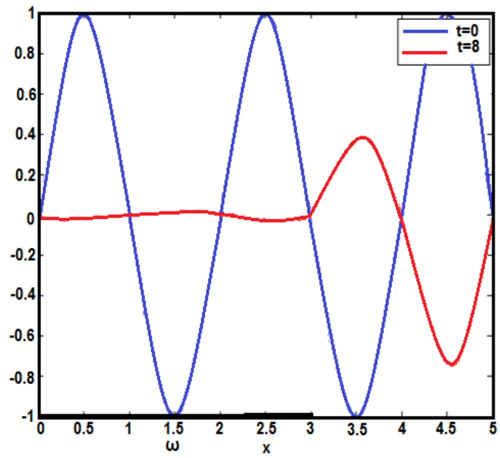

FIGURE 3. The stabilization of the state on $\omega =]0,3[$.

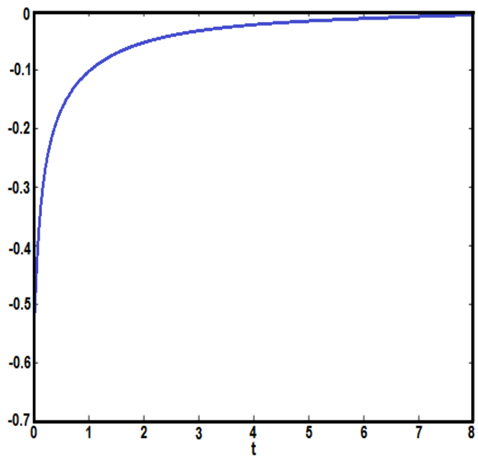

FIGURE 4. Evolution of control function.

5. CONCLUSION

The output stabilization of bilinear systems is discussed. Under sufficient conditions, we give bounded controls depending on the output operator that exponentially, strongly and weakly stabilizes the output of such systems. Numerical simulations illustrate the efficiency theoretical results. This work gives an opening to others questions, this is the case of output stabilization of semilinear systems with bounded controls.

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