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# Smarandache Curves According to Sabban Frame of the anti-Salkowski Indicatrix Curve

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**Abstract**

The aim of this paper is to define Smarandache curves according to the Sabban frame belonging to the spherical indicatrix curve of the anti-Salkowski curve. We also illustrate these curves with the Maple program and calculate the geodesic curvatures of these curves.

## 1. Introduction

Erich Salkowski (1881-1943), a German mathematician. In 1909, he defined curve families with non-constant $\tau$ and constant curvature $\kappa$ [1]. Later J. Monterde constructed a method for closed curves and the properties of anti-Salkowski curve used in [2]. For authors worked on the anti-Salkowski curve also can be seen in [3]-[7]. When the Frenet vectors of any curve are taken as the position vector, then the regular curves generated by these vectors are called Smarandache curves [8]. Smarandache curves in Euclidean 3-space are defined and some features of these curves are given in [9]. For some authors worked on the Smarandache curve also may be seen in [10, 11]. In 1990, the geodesic curve of a spherical curve is calculated by J. Koenderink with the Sabban frame of the spherical indicatrix curves in [12]. Then the Smarandache curves obtained from Sabban frame are defined and geodesic curvatures of these curves are given in [13].
In this study, Smarandache curves are defined according to the Sabban frames belonging to the spherical indicatrix curves of each of the $T,N,B$ Frenet vectors of the anti-Salkowski curve. The geodesic curvatures of these curves are then calculated.

## 2. Preliminaries

In the Euclidean 3-space $E^3$, the Frenet frame of any curve $\alpha$ is given by $\{T,N,B\}$. For an arbitrary curve $\alpha \in E^3$, with the first and second curvatures, $\kappa$ and $\tau$ respectively, the Frenet apparatus are given by [14]

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N.$$

Accordingly, the spherical indicatrix curves of Frenet vectors are $(T)$, $(N)$ and $(B)$ respectively.These equations of curves are given by [14]

$$\alpha_T(s) = T(s), \quad \alpha_N(s) = N(s), \quad \alpha_B(s) = B(s).$$

Let $\gamma : I \to S^2$ be a unit speed spherical curve. We denote $s$ as the arc-length parameter of $\gamma$. Let us denote by [14]

$$\gamma(s) = \gamma(s), \quad t(s) = \gamma'(s), \quad d(s) = \gamma(s) \wedge t(s).$$

We call $t(s)$ a unit tangent vector of $\gamma$. $\{\gamma, t, d\}$ frame is called the Sabban frame of $\gamma$ on $S^2$. Then we have the following spherical Frenet formulae of $\gamma$ :

$$\gamma' = t, \quad t' = -\gamma + \kappa_g d, \quad d' = -\kappa_g t \tag{2.1}$$

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where is called the geodesic curvature of $\kappa_g$ on $S^2$ and

$$\kappa_g = \langle t', d\rangle, \tag{2.2}$$

[12, 13].

**Definition 2.1.** *(anti-Salkowski curve) [2]. For any $m \in \mathbb{R}$ with $m \neq \mp\frac{1}{\sqrt{3}}$ , 0, let us define the space curve*

$$\begin{aligned} \beta_m(s) &= \Bigg(\frac{n}{2(4n^2-1)m}\Big(n(1-4n^2+3\cos(2ns))\cos(s)+(2n^2+1)\sin(s)\sin(2ns)\Big), \\ &\quad \frac{n}{2(4n^2-1)m}\Big(n(1-4n^2+3\cos(2ns))\sin(s)-(2n^2+1)\cos(s)\sin(2ns), \frac{n^2-1}{4n}(2ns+\sin(2ns))\Big)\Bigg) \end{aligned}$$

where $n = \frac{m}{\sqrt{1+m^2}}$. *The Frenet apparatus are*

$$\begin{cases} \kappa = \tan(ns), \quad \tau = 1, \quad \|\gamma_m(s)\| = \frac{cos(ns)}{\sqrt{1+m^2}} \\ T(s) = -\Big(\cos(s)\sin(ns)-n\sin(s)\cos(ns), \sin(s)\sin(ns)+n\cos(s)\cos(ns), \frac{n}{m}\cos(ns)\Big), \\ N(s) = n\left(\frac{\sin(s)}{m}, -\frac{\cos(s)}{m}, 1\right), \\ B(s) = \Big(-\cos(s)\cos(ns)-n\sin(s)\sin(ns), -\sin(s)\cos(ns)+n\cos(s)\sin(ns), \frac{n}{m}\sin(ns)\Big). \end{cases}$$

*The shape of this curve is given in Figure (2.1)*

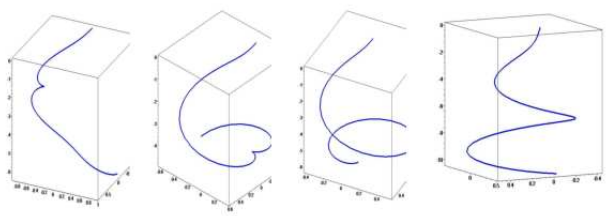

**Figure 2.1:** anti-Salkowski Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5,5]$

Let $(\alpha), (\delta)$ and $(\zeta)$ be spherical indicatrix curves of tangent, principal normal and binormal vectors belonging to anti-Salkowski curve, respectively. Using the equations (2.1) and (2.2), Sabban apparatus belonging to these curves is given by

$$\begin{aligned} T &= T\,, \quad T_T = N\,, \quad T\wedge T_T = B, \\ T' &= T_T, \quad T_T' = -T + \frac{1}{\tan(ns)}(T\wedge T_T), \quad (T\wedge T_T)' = -\frac{1}{\tan(ns)}T_T, \\ K_g^T &= \frac{1}{\kappa} = \frac{1}{\tan(ns)}. \end{aligned} \tag{2.3}$$

$$T(s) = (\cos(s)\sin(ns)-n\sin(s)\cos(ns), \sin(s)\sin(ns)+n\cos(s)\cos(ns), \frac{n}{m}\cos(ns)), \tag{2.4}$$

$$T_T(s) = n\left(\frac{\sin(s)}{m}, -\frac{\cos(s)}{m}, 1\right),$$

$$(T\wedge T_T)(s) = -(\cos(s)\cos(ns)+n\sin(s)\sin(ns), \sin(s)\cos(ns)-n\cos(s)\sin(ns), \frac{n}{m}\sin(ns)).$$

$$\begin{aligned} N &= N\,, \quad T_N = \frac{-\tan(ns)T+B}{\sqrt{\tan^2(ns)+1}}\,, \quad N\wedge T_N = \frac{T+\tan(ns)B}{\sqrt{\tan^2(ns)+1}}, \\ N' &= T_N\,, \quad T_N' = \frac{\tan(ns)'}{\sqrt{\tan^2(ns)+1}}N + N\wedge T_N\,, \quad (T\wedge T_T)' = \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}T_N, \\ K_g^N &= \frac{-\kappa'}{\sqrt{\kappa^2+1}} = \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}. \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
N(s) &= \Big(\frac{n\sin(s)}{m},-\frac{n\cos(s)}{m},n\Big),\\
T_N(s) &= \frac{1}{\sqrt{\tan^2(ns)+1}}\Big(-\cos(s)\cos(ns)-n\sin(s)\sin(ns)-\tan(ns)(-\cos(s)\sin(ns)+n\sin(s)\cos(ns)),\\
&\quad -\tan(ns)(-\sin(s)\sin(ns)-n\cos(s)\cos(ns))-\sin(s)\cos(ns)+n\cos(s)\sin(ns),\frac{2n}{m}\sin(ns)\Big), \qquad (2.6)\\
(N\wedge T_N)(s) &= \frac{1}{\sqrt{\tan^2(ns)+1}}\Big(\tan(ns)(-\cos(s)\cos(ns)-n\sin(s)\sin(ns))-\cos(s)\sin(ns)+n\sin(s)\cos(ns),-\sin(s)\sin(ns)\\
&\quad +\tan(ns)(-\sin(s)\cos(ns)+n\cos(s)\sin(ns))-n\cos(s)\cos(ns),\frac{n}{m}\tan(ns)\sin(ns)-\frac{n}{m}\cos(ns)\Big).
\end{aligned}
$$

$$
\begin{aligned}
B &= B, \quad T_B=-N, \quad B\wedge T_B=T,\\
B' &= T_B\,, \quad B_T'=-B+\tan(ns)(B\wedge T_B)\,,\\
(B\wedge T_B)' &= \tan(ns)T_B, \quad K_g^B=\kappa=\tan(ns). \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
B(s) &= -\Big(\cos(s)\cos(ns)+n\sin(s)\sin(ns),\sin(s)\cos(ns)-n\cos(s)\sin(ns),\frac{n}{m}\sin(ns)\Big),\\
T_B(s) &= -\Big(\frac{n\sin(s)}{m},-\frac{n\cos(s)}{m},n\Big), \qquad (2.8)\\
(B\wedge T_B)(s) &= -\Big(\cos(s)\sin(ns)-n\sin(s)\cos(ns),\sin(s)\sin(ns)+n\cos(s)\cos(ns),-\frac{n}{m}\cos(ns)\Big).
\end{aligned}
$$

## 3. Smarandache curves according to the Sabban frame belonging to spherical indicatrix curve of the anti-Salkowski curve

**Definition 3.1.** *Let $\alpha=\alpha(s)$ be a curve and $\{T,T_T,T\wedge T_T\}$ be Sabban frame of this curve. Then $TT_T$-Smarandache curve is given by*

$$\alpha_1(s) = \frac{1}{\sqrt{2}}(T+T_T). \qquad (3.1)$$

*According to equation (2.4) we can parameterize the $\alpha_1(s)$-Smarandache curve as in the following form*

$$\alpha_1(s) = \frac{1}{\sqrt{2}}\Big(-\cos(s)\sin(ns)+n\sin(s)\cos(ns)+\frac{n}{m}\sin(s),-\sin(s)\sin(ns)-n\cos(s)\cos(ns)-\frac{n}{m}\cos(s),-\frac{n}{m}\cos(ns)+n\Big).$$

*The shape of this curve is given in Figure (3.1)*

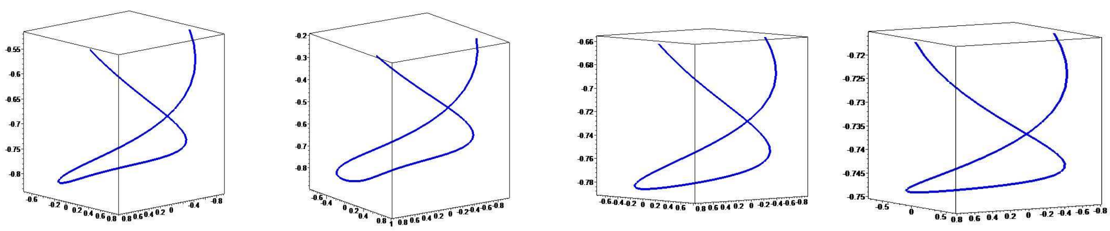

**Figure 3.1:** $TT_T$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

**Theorem 3.2.** *The geodesic curvature $K_g^{\alpha_1}$ according to $\alpha_1(s)$-Smarandache curve is*

$$K_g^{\alpha_1} = \frac{\tan^4(ns)}{(2\tan(ns)+1)^{\frac{5}{2}}}(\chi_1-\chi_2+2\tan(ns)\chi_3)\,,$$

*where the coefficients $\chi_1,\chi_2$ and $\chi_3$ are*

$$
\begin{aligned}
\chi_1 &= -2-\frac{1}{\tan^2(ns)}+\frac{1}{\tan(ns)}\Big(\frac{1}{\tan(ns)}\Big)',\\
\chi_2 &= -2-\frac{1}{\tan(ns)}\Big(\frac{1}{\tan(ns)}\Big)'-\frac{3}{\tan^2(ns)}-\frac{1}{\tan^4(ns)},\\
\chi_3 &= \frac{2}{\tan(ns)}+\Big(\frac{2}{\tan(ns)}\Big)'+\frac{1}{\tan^3(ns)}.
\end{aligned}
$$

*Proof.* If we take the derivative of (3.1) and from the equation (2.3) we get

$$(T_T)_{\alpha_1}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-T+T_T+\frac{1}{\tan(ns)}(T\wedge T_T)), \tag{3.2}$$

if we take the norm of (3.2) we have

$$\frac{ds^*}{ds}=\frac{\sqrt{2\tan^2(ns)+1}}{\tan(ns)\sqrt{2}}.$$

We obtain the tangent of $\alpha_1(s)$-Smarandahce curve as in

$$(T_T)_{\alpha_1} = \frac{1}{\sqrt{2\tan^2(ns)+1}}(-\tan(ns)T+\tan(ns)T_T+(T\wedge T_T)). \tag{3.3}$$

The derivative of (3.2) is

$$(T_T)'_{\alpha_1} = \frac{1}{\sqrt{2\tan^2(ns)+1}}(\chi_1 T+\chi_2 T_T+\chi_3(T\wedge T_T)).$$

From equations (3.1) and (3.3) we have

$$(T\wedge T_T)_{\alpha_1} = \frac{1}{\sqrt{2\tan^2(ns)+1}}(T-T_T+2\tan(ns)(T\wedge T_T)).$$

So the geodesic curvature from the equation (2.3) is

$$K_g^{\alpha_1} = \frac{\tan^4(ns)}{(2\tan(ns)+1)^{\frac{5}{2}}}\left(\chi_1-\chi_2+2\tan(ns)\chi_3\right).$$

□

**Definition 3.3.** *Let $\alpha=\alpha(s)$ be a curve and $\{T,T_T,T\wedge T_T\}$ be Sabban frame of this curve. Then $T(T\wedge T_T)$-Smarandache curve is given by*

$$\alpha_2(s) = \frac{1}{\sqrt{2}}(T+(T\wedge T_T)). \tag{3.4}$$

*According to equation (2.4) we can parameterize the $\alpha_2(s)$-Smarandache curve as in the following form*

$$\alpha_2(s) = \frac{1}{\sqrt{2}}\Big(-\cos(s)(\cos(ns)-\sin(ns))+n\sin(s)(\cos(ns)+\sin(ns)),$$
$$\sin(s)(\cos(ns)-\sin(ns))-n\cos(s)(\cos(ns)+\sin(ns)),-\frac{n}{m}(\cos(ns)+\sin(ns))\Big).$$

*The shape of this curve is given in Figure (3.2)*

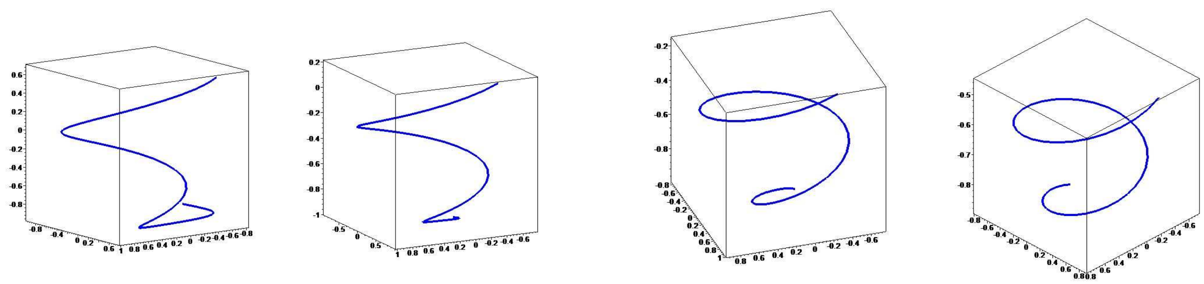

**Figure 3.2:** $T(T\wedge T_T)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

**Theorem 3.4.** *The geodesic curvature $K_g^{\alpha_2}$ according to $\alpha_2(s)$-Smarandache curve is given by*

$$K_g^{\alpha_2} = \frac{\tan(ns)+1}{\tan(ns)}.$$

*Proof.* If we take the derivative of (3.4) and from the equation (2.3) we get,

$$(T_T)_{\alpha_2}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(T_T-\frac{1}{\tan(ns)}T_T), \tag{3.5}$$

if we take the norm of (3.5), $\frac{ds^*}{ds}=\frac{\tan(ns)-1}{\tan(ns)\sqrt{2}}$ we have, We obtain the tangent of $\alpha_2(s)$-Smarandahce curve as in

$$(T_T)_{\alpha_2}=T_T. \tag{3.6}$$

The derivative in the (3.6) is

$$(T_T)'_{\alpha_2} \cdot \frac{ds^*}{ds} = \frac{\sqrt{2}}{\tan(ns)-1}(-\tan(ns)T + (T \wedge T_T)).$$

From equations (3.4) and (3.6) we have

$$(T \wedge T_T)_{\alpha_2} = \frac{1}{\sqrt{2}}(-T + (T \wedge T_T)).$$

So the geodesic curvature from the equation (2.3) is

$$K_g^{\alpha_2} = \frac{\tan(ns)+1}{\tan(ns)}.$$

□

**Definition 3.5.** *Let $\alpha = \alpha(s)$ be a curve and $\{T, T_T, T \wedge T_T\}$ be Sabban frame of this curve. Then $T_T(T \wedge T_T)$-Smarandache curve is given by*

$$\alpha_3(s) = \frac{1}{\sqrt{2}}(T_T + (T \wedge T_T)). \tag{3.7}$$

*According to equation (2.4) we can parameterize the $\alpha_4(s)$-Smarandache curve as in the following form*

$$\alpha_3(s) = \frac{1}{\sqrt{2}}\Big(\cos(s)\cos(ns) + n\sin(s)\sin(ns) + \frac{n}{m}\sin(s), \sin(s)\cos(ns) - n\cos(s)\sin(ns) - \frac{n}{m}\cos(s), -\frac{n}{m}\sin(ns) + n\Big).$$

*The shape of this curve is given in Figure (3.3)*

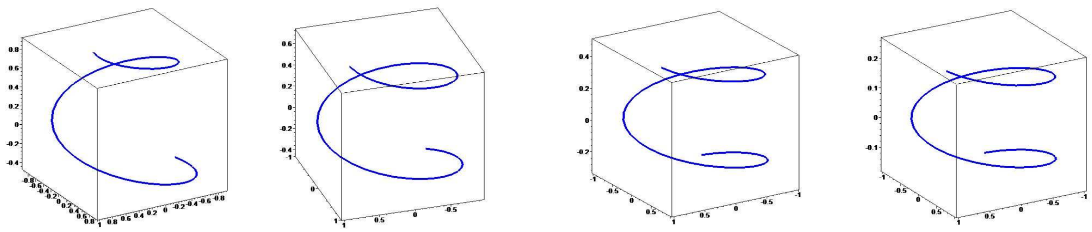

**Figure 3.3:** $T_T(T \wedge T_T)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5..5]$

**Theorem 3.6.** *The geodesic curvature $K_g^{\alpha_3}$ according to $\alpha_3(s)$-Smarandache curve is given by*

$$K_g^{\alpha_3} = \frac{\tan^4(ns)}{(1+2\tan^2(ns))^{\frac{5}{2}}}(2\chi_4 - \tan(ns)\chi_5 + \tan(ns)\chi_6),$$

*where the coefficients $\chi_4, \chi_5$ and $\chi_6$ are*

$$\chi_4 = \frac{2}{\tan(ns)}\Big(\frac{1}{\tan(ns)}\Big)' + \frac{1}{\tan(ns)} + \frac{2}{\tan^3(ns)},$$

$$\chi_5 = -1 - \Big(\frac{1}{\tan(ns)}\Big)' - \frac{3}{\tan^2(ns)} - \frac{2}{\tan^4(ns)},$$

$$\chi_6 = -\frac{1}{\tan^2(ns)} + \Big(\frac{1}{\tan(ns)}\Big)' - \frac{2}{\tan^4(ns)}.$$

*Proof.* If we take the derivative of (3.7) and from the equation (2.3) we get

$$(T_T)_{\alpha_3} \cdot \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\Big(-T - \frac{1}{\tan(ns)}T_T + \frac{1}{\tan(ns)}(T \wedge T_T)\Big), \tag{3.8}$$

if take the norm of (3.8) we have, $\frac{ds^*}{ds} = \frac{\sqrt{\tan^2(ns)+2}}{\tan(ns)\sqrt{2}}$. We obtain the tangent of $\alpha_3(s)$-Smarandahce curve as in

$$(T_T)_{\alpha_3} = \frac{1}{\sqrt{\tan^2(ns)+2}}(-\tan(ns)T - T_T + (T \wedge T_T)). \tag{3.9}$$

The derivative of (3.9) is

$$(T_T)'_{\alpha_3} = \frac{\tan^4(ns)\sqrt{2}}{(\tan^2(ns)+2)^2}(\chi_4 T + \chi_5 T_T + \chi_6(T \wedge T_T)).$$

From equations (3.7) and (3.9) we have

$$(T\wedge T_T)_{\alpha_3} = \frac{1}{\sqrt{2(\tan^2(ns)+2)}}(2T-\tan(ns)T_T+\tan(ns)(T\wedge T_T)).$$

So the geodesic curvature from the equation (2.3) is

$$K_g^{\alpha_3} = \frac{\tan^4(ns)}{(1+2\tan^2(ns))^{\frac{5}{2}}}\left(2\chi_4-\tan(ns)\chi_5+\tan(ns)\chi_6\right).$$

□

**Definition 3.7.** *Let $\alpha=\alpha(s)$ be a curve and $\{T,T_T,T\wedge T_T\}$ be Sabban frame of this curve. Then $TT_T(T\wedge T_T)$-Smarandache curve is given by*

$$\alpha_4(s) = \frac{1}{\sqrt{3}}(T+T_T+(T\wedge T_T)). \tag{3.10}$$

*According to equation (2.4) we can parameterize the $\alpha_1(s)$-Smarandache curve as in the following form*

$$\begin{aligned}\alpha_4(s) = \frac{1}{\sqrt{3}}\Big(&\cos(s)(\cos(ns)-\sin(ns))+n\sin(s)(\cos(ns)+\sin(ns))+\frac{n}{m}\sin(s),\\ &\sin(s)(\cos(ns)-\sin(ns))-n\cos(s)(\cos(ns)+\sin(ns))-\frac{n}{m}\cos(s),-\frac{n}{m}(\cos(ns)+\sin(ns))+n\Big).\end{aligned}$$

*The shape of this curve is given in Figure (3.4)*

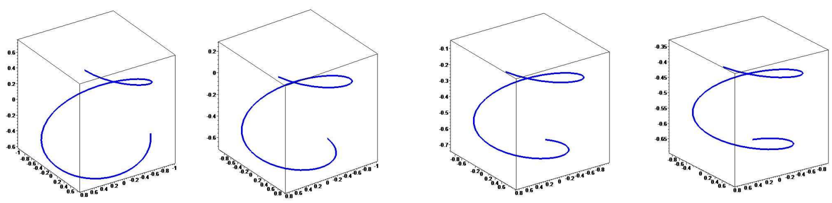

**Figure 3.4:** $TT_T(T\wedge T_T)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

**Theorem 3.8.** *The geodesic curvature $K_g^{\alpha_4}$ according to $\alpha_4(s)$-Smarandache curve is given as*

$$K_g^{\alpha_4} = \frac{\tan^4(ns)\left((2-\tan(ns))\chi_7-(1+\tan(ns))\chi_8+(2\tan(ns)-1)\chi_9\right)}{(4\sqrt{2}(\tan^2(ns)-\tan(ns)+1)^2)^{\frac{5}{2}}},$$

*where the coefficients $\chi_6,\chi_7$ and $\chi_8$ are*

$$\begin{aligned}\chi_7 &= -(\frac{1}{\tan(ns)})'+\frac{2}{\tan(ns)}(\frac{1}{\tan(ns)})'-2+\frac{4}{\tan(ns)}-\frac{4}{\tan^2(ns)}+\frac{2}{\tan^3(ns)},\\ \chi_8 &= -(\frac{1}{\tan(ns)})'-\frac{1}{\tan(ns)}(\frac{1}{\tan(ns)})'-2-\frac{4}{\tan^2(ns)}+\frac{2}{\tan(ns)}+\frac{2}{\tan^3(ns)}-\frac{2}{\tan^4(ns)},\\ \chi_9 &= \frac{1}{\tan(ns)}(\frac{1}{\tan(ns)})'+\frac{2}{\tan(ns)}-\frac{4}{\tan^2(ns)}+(\frac{2}{\tan(ns)})'+\frac{4}{\tan^3(ns)}-\frac{2}{\tan^4(ns)}.\end{aligned}$$

*Proof.* If we take the derivative of (3.10) and from the equation (2.3) we get,

$$(T_T)_{\alpha_4}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}(-T+(1-\frac{1}{\tan(ns)})T_T+\frac{1}{\tan(ns)}(T\wedge T_T)), \tag{3.11}$$

if we take the norm of (3.11) we have,

$$\frac{ds^*}{ds}=\frac{\sqrt{2(\tan^2(ns)-\tan(ns)+1)}}{\tan(ns)\sqrt{3}}.$$

We obtain the tangent of $\alpha_4(s)$-Smarandahce curve as in

$$(T_T)_{\alpha_4} = \frac{(-\tan(ns)T+(\tan(ns)-1)T_T+(T\wedge T_T))}{\sqrt{2(\tan^2(ns)-\tan(ns)+1)}}. \tag{3.12}$$

The derivative of (3.12) is

$$(T_T)'_{\alpha_4} = \frac{\tan^2(ns)\sqrt{3}(\chi_7T+\chi_8T_T+\chi_9(T\wedge T_T))}{4(\tan^2(ns)-\tan(ns)+1)^2}.$$

From equations (3.10) and (3.12) we have

$$(T\wedge T_T)_{\alpha_4} = \frac{(-\tan(ns)+2)T-(\tan(ns)+1)T_T+(2\tan(ns)-1)(T\wedge T_T)}{\sqrt{6(\tan^2(ns)-\tan(ns)+1)}}.$$

So the geodesic curvature from the equation (2.3) is

$$K_g^{\alpha_4} = \frac{\tan^4(ns)\left((2-\tan(ns))\chi_7-(1+\tan(ns))\chi_8+(2\tan(ns)-1)\chi_9\right)}{(4\sqrt{2}(\tan^2(ns)-\tan(ns)+1)^2)^{\frac{5}{2}}}.$$

□

**Definition 3.9.** *Let $\delta=\delta(s)$ be a curve and $\{N,T_N,N\wedge T_N\}$ be Sabban frame of this curve. Then $NT_N$-Smarandache curve is given by*

$$\delta_1(s) = \frac{1}{\sqrt{2}}(N+T_N).$$

*According to equation (2.6) we can parameterize the $\delta_1(s)$-Smarandache curve as in the following form*

$$\begin{aligned}\delta_1(s) = \frac{1}{\sqrt{2}}\Big(&-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\sin(ns)+n\sin(s)\cos(ns))+\frac{n\sin(s)}{m}-\cos(s)\cos(ns)-n\sin(s)\sin(ns),\\ &-\sin(s)\cos(ns)+n\cos(s)\sin(ns)-\frac{n\cos(s)}{m}-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\sin(ns)-n\cos(s)\cos(ns)),\\ &\frac{n\tan(ns)}{m\sqrt{\tan^2(ns)+1}}\cos(ns)+n\Big).\end{aligned}$$

*The shape of this curve is given in Figure (3.5)*

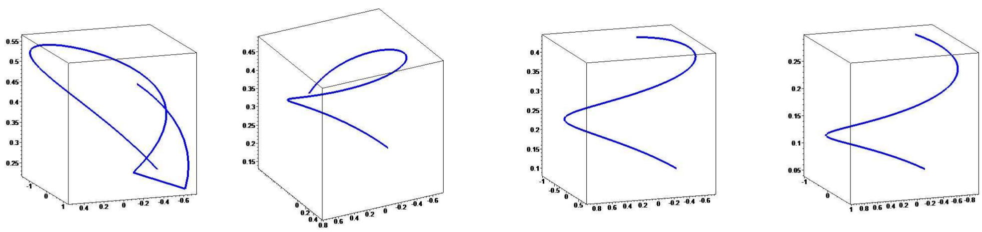

**Figure 3.5:** $NT_N$-Smarandache Curves , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

**Theorem 3.10.** *The geodesic curvature $K_g^{\delta_1}$ according to $\delta_1(s)$-Smarandache curve is given by*

$$K_g^{\delta_1} = \frac{(1+\tan^2(ns))\left(-\tan(ns)'\chi_{10}+\tan(ns)'\chi_{11}+2\sqrt{\tan^2(ns)+1}\chi_{12}\right)}{(2\sqrt{1+\tan^2(ns)}-(\tan(ns)')^2)^{\frac{5}{2}}},$$

*where the coefficients $\chi_{10}$, $\chi_{11}$ and $\chi_{12}$ are*

$$\begin{aligned}\chi_{10} &= -2-\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^2+\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)',\\ \chi_{11} &= -2-\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)'-3\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^2-\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^4,\\ \chi_{12} &= \frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}}+2\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)'+\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^3.\end{aligned}$$

*Proof.* If we take the derivative of equation (3.13) and from the equation (2.5) we have

$$(T_N)_{\delta_1}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\Big(-N+T_N+\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(N\wedge T_N)\Big), \tag{3.13}$$

if we take the norm of equation (3.13) we get

$$\frac{ds^*}{ds} = \frac{\sqrt{2(1+\tan^2(ns))-(\tan(ns)')^2}}{\sqrt{2}\sqrt{1+\tan^2(ns)}}.$$

We obtain the tangent of $\delta_1(s)$-Smarandahce curve as in

$$(T_N)_{\delta_1} = \frac{-\sqrt{\tan^2(ns)+1}\,N+\sqrt{\tan^2(ns)+1}\,T_N-\tan(ns)'(N\wedge T_N)}{\sqrt{2(1+\tan^2(ns))-(\tan(ns)')^2}}. \tag{3.14}$$

The derivative of (3.13) is

$$(T_N)'_{\delta_1} = \frac{(\tan^2(ns)+1)\sqrt{2}(\chi_{10}N+\chi_{11}T_N+\chi_{12}(N\wedge T_N))}{(2(\tan^2(ns)+1)-(\tan(ns)')^2)^2}.$$

From equations (3.13) and (3.14) we have

$$(N\wedge T_N)_{\delta_1} = \frac{(1+\tan^2(ns))^4(-\tan(ns)'(N-T_N)+2\sqrt{1+\tan^2(ns)}(N\wedge T_N))}{\sqrt{2(2(1+\tan^2(ns))-(\tan(ns)')^2)}}.$$

So the geodesic curvature from the equation (2.5) is

$$K_g^{\delta_1} = \frac{(1+\tan^2(ns))\left(-\tan(ns)'\chi_{10}+\tan(ns)'\chi_{11}+2\sqrt{\tan^2(ns)+1}\chi_{12}\right)}{(2\sqrt{1+\tan^2(ns)}-(\tan(ns)')^2)^{\frac{5}{2}}}.$$

□

**Definition 3.11.** *Let $\delta=\delta(s)$ be a curve and $\{N,T_N,N\wedge T_N\}$ be Sabban frame of this curve. Then $N(N\wedge T_N)$-Smarandache curve is given by*

$$\delta_2(s) = \frac{1}{\sqrt{2}}(N+(N\wedge T_N)). \tag{3.15}$$

*According to equation (2.6) we can parameterize the $\delta_2(s)$-Smarandache curve as in the following form*

$$\begin{aligned}\delta_2(s) = \frac{1}{\sqrt{2}}\Big(&\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\cos(ns)-n\sin(s)\sin(ns))-\cos(s)\sin(ns)+n\sin(s)\cos(ns)+\frac{n\sin(s)}{m},\\ &+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\cos(ns)+n\cos(s)\sin(ns))-\sin(s)\sin(ns)-n\cos(s)\cos(ns)-\frac{n\cos(s)}{m},\\ &\frac{n\tan(ns)}{m\sqrt{\tan^2(ns)+1}}\sin(ns)-\frac{n}{m}\cos(ns)+n\Big).\end{aligned}$$

*The shape of this curve is given in Figure (3.6)*

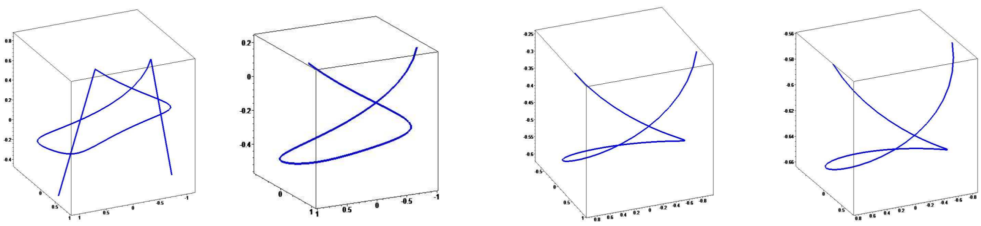

**Figure 3.6:** $N(N\wedge T_N)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

**Theorem 3.12.** *The geodesic curvature $K_g^{\delta_2}$ according to $\delta_2(s)$-Smarandache curve is given by*

$$K_g^{\delta_2} = \frac{\sqrt{\tan(ns)^2+1}-\tan(ns)'}{\sqrt{\tan(ns)^2+1}}.$$

*Proof.* If we take the derivative of equation (3.15) and from the equation (2.5) we get

$$(T_N)_{\delta_2}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(T_N-\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}T_N), \tag{3.16}$$

if we take the norm of equation (3.16) we have

$$\frac{ds^*}{ds}=\frac{\sqrt{\tan(ns)^2+1}+\tan(ns)'}{\sqrt{2}\sqrt{\tan(ns)^2+1}}.$$

We obtain the tangent of $\delta_2(s)$-Smarandahce curve as in

$$(T_N)_{\delta_2} = T_N. \tag{3.17}$$

The derivative of (3.17) is

$$(T_N)'_{\delta_2} = \frac{\sqrt{2}(-\sqrt{\tan^2(ns)+1}N - \tan(ns)'(N\wedge T_N))}{\sqrt{\tan(ns)^2+1}+\tan(ns)'}.$$

From equations (3.15) and (3.17) we have

$$(N\wedge T_N)_{\delta_2} = \frac{1}{\sqrt{2}}(-N+(N\wedge T_N)).$$

So the geodesic curvature from the equation (2.5) is

$$K_g^{\delta_2} = \frac{\sqrt{\tan(ns)^2+1}-\tan(ns)'}{\sqrt{\tan(ns)^2+1}}.$$

□

**Definition 3.13.** *Let $\delta=\delta(s)$ be a curve and $\{N,T_N,N\wedge T_N\}$ be Sabban frame of this curve. Then $T_N(N\wedge T_N)$-Smarandache curve ($\delta_3(s)$-Smarandache curve) is given by*

$$\delta_3(s) = \frac{1}{\sqrt{2}}(T_N+(N\wedge T_N)). \tag{3.18}$$

*According to equation (2.6) we can parameterize the $\delta_3(s)$-Smarandache curve as in the following form*

$$\begin{aligned}\delta_3(s) = \frac{1}{\sqrt{2}}\Big(&-\cos(s)\cos(ns)-n\sin(s)\sin(ns)-\cos(s)\sin(ns)+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\cos(ns)-n\sin(s)\sin(ns))\\ &-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\sin(ns)+n\sin(s)\cos(ns))+n\sin(s)\cos(ns),-\sin(s)\sin(ns)-n\cos(s)\cos(ns)\\ &-\sin(s)\cos(ns)+n\cos(s)\sin(ns)+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\cos(ns)+n\cos(s)\sin(ns))\\ &-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\sin(ns)-n\cos(s)\cos(ns)),\frac{n\tan(ns)}{m\sqrt{\tan^2(ns)+1}}(\cos(ns)+\sin(ns))-\frac{n}{m}\cos(ns)\Big)\end{aligned}$$

.

*The shape of this curve is given in Figure (3.7)*

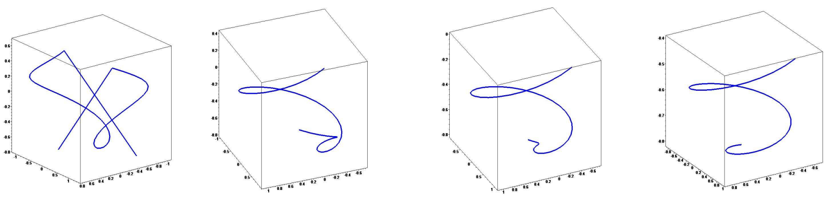

**Figure 3.7:** $T_N(N\wedge TN)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

**Theorem 3.14.** *The geodesic curvature $K_g^{\delta_3}$ according to $\delta_3(s)$-Smarandache curve is given by*

$$K_g^{\delta_3} = \frac{(\tan^2(ns)+1)^4((-2\tan(ns)')\chi_{13}-\sqrt{\tan(ns)^2+1}(\chi_{14}-\chi_{15}))}{(1+\tan^2(ns)+(-\tan(ns)')^2)^{\frac{5}{2}}},$$

*where the coefficients $\chi_{13}$, $\chi_{14}$ and $\chi_{15}$ are*

$$\begin{aligned}\chi_{13} &= 2\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})'+\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}+2(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^3,\\ \chi_{14} &= -1-(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})'-(\frac{-3\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^2-(\frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^4,\\ \chi_{15} &= -(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^2+(\frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}})'-(\frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^4.\end{aligned}$$

*Proof.* If we take the derivative of equation (3.18) and from the equation (2.5) we get

$$(T_N)_{\delta_3}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\Big(-N-\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}T_N+\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(N\wedge T_N)\Big), \tag{3.19}$$

if we take the norm of equation (3.19) we have

$$\frac{ds^*}{ds}=\frac{\sqrt{1+\tan^2(ns)+2(-\tan(ns)')^2}}{\sqrt{2}\sqrt{\tan^2(ns)+1}}.$$

We obtain the tangent of $\delta_3(s)$-Smarandahce curve as in

$$(T_N)_{\delta_3} = \frac{-\sqrt{\tan^2(ns)+1}N+\tan(ns)'T_N-\tan(ns)'(N\wedge T_N)}{\sqrt{1+\tan^2(ns)+2(-\tan(ns)')^2}}. \tag{3.20}$$

The derivative of (3.20) is

$$(T_N)'_{\delta_3} = \frac{\sqrt{2}(\tan^2(ns)+1)^2}{(1+\tan^2(ns)+2(-\tan(ns)')^2)^2}(\chi_{13}N+\chi_{14}T_N+\chi_{15}(N\wedge T_N)).$$

From equations (3.18) and (3.20) we have

$$(N\wedge T_N)_{\delta_3} = \frac{(-2\tan(ns)'N-\sqrt{1+\tan^2(ns)}T_N+\sqrt{1+\tan^2(ns)}(N\wedge T_N))}{\sqrt{2(1+\tan^2(ns)+2(-\tan(ns)')^2)}}.$$

So the geodesic curvature from the equation (2.5) is

$$K_g^{\delta_3} = \frac{(\tan^2(ns)+1)^4((-2\tan(ns)')\chi_{13}-\sqrt{\tan(ns)^2+1}(\chi_{14}-\chi_{15}))}{(1+\tan^2(ns)+(-\tan(ns)')^2)^{\frac{5}{2}}}.$$

□

**Definition 3.15.** *Let $\delta=\delta(s)$ be a curve and $\{N,T_N,N\wedge T_N\}$ be Sabban frame of this curve. Then $NT_N(N\wedge T_N)$-Smarandache curve ($\delta_4(s)$-Smarandache curve) is given by*

$$\delta_4(s) = \frac{1}{\sqrt{3}}(N+T_N+(N\wedge T_N)). \tag{3.21}$$

*According to equation (2.6) we can parameterize the $\delta_4(s)$-Smarandache curve as in the following form*

$$\begin{aligned}\delta_4(s) = \frac{1}{\sqrt{3}}\Big(&-\cos(s)\cos(ns)-n\sin(s)\sin(ns)-\cos(s)\sin(ns)+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\cos(ns)-n\sin(s)\sin(ns))+\frac{n\sin(s)}{m}\\ &-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\sin(ns)+n\sin(s)\cos(ns))+n\sin(s)\cos(ns),-\sin(s)\sin(ns)\\ &+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\cos(ns)+n\cos(s)\sin(ns))-n\cos(s)\cos(ns)-\sin(s)\cos(ns)+n\cos(s)\sin(ns)\\ &-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\sin(ns)-n\cos(s)\cos(ns))-\frac{n\cos(s)}{m},\frac{n\tan(ns)}{m\sqrt{\tan^2(ns)+1}}(\cos(ns)+\sin(ns))-\frac{n}{m}\cos(ns)+n\Big).\end{aligned}$$

*The shape of this curve is given in Figure (3.8)*

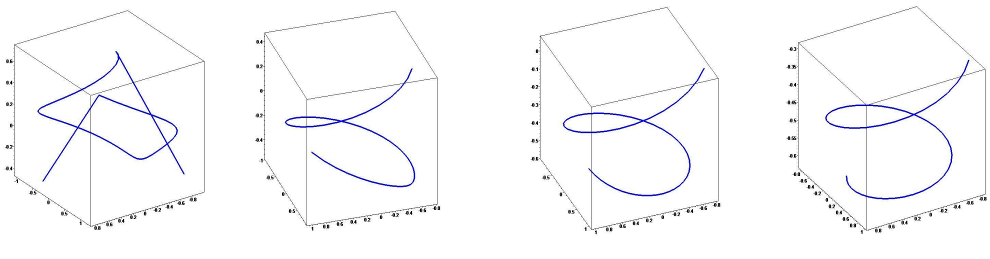

**Figure 3.8:** $TT_N(T\wedge T_N)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

**Theorem 3.16.** *The geodesic curvature $K_g^{\delta_4}$ according to $\delta_4(s)$-Smarandache curve is given by*

$$K_g^{\delta_4} = \frac{\left((-2\tan(ns)' - \sqrt{\tan(ns)^2+1})\chi_{16} - \chi_{17}(\sqrt{\tan(ns)^2+1} - \tan(ns)')\right)}{(4\sqrt{2}(1+\tan^2(ns)+\sqrt{1+\tan^2(ns)}\tan(ns)'+(-\tan(ns)')^2)^2)^{\frac{5}{2}}}$$

$$+\frac{(2\sqrt{\tan(ns)^2+1}+\tan(ns)')\chi_{18}}{(4\sqrt{2}(1+\tan^2(ns)+\sqrt{1+\tan^2(ns)}\tan(ns)'+(-\tan(ns)')^2)^2)^{\frac{5}{2}}},$$

*where the coefficients $\chi_{16}$, $\chi_{17}$ and $\chi_{18}$ are*

$$\begin{aligned}
\chi_{16} &= (\frac{\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' + \frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' - 2 + \frac{-4\tan(ns)'}{\sqrt{\tan^2(ns)+1}} + (\frac{4\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^2 + (\frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^3,\\
\chi_{17} &= -(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' - \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' - 2 + (\frac{4\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^2 - \frac{2\tan(ns)'}{\sqrt{\tan^2(ns)+1}} - (\frac{2\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^3\\
&\quad + (\frac{2\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^4,\\
\chi_{18} &= \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' + \frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}} - 4(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^2 + 2(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' + 4(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^3\\
&\quad - 2(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^4.
\end{aligned}$$

*Proof.* If we take the derivative of equation (3.21) and from the equation (2.5) we have

$$(T_N)_{\delta_4}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}(-N + (1 - \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})T_N + \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(N\wedge T_N)), \tag{3.22}$$

if we take the norm of equation (3.22) we get

$$\frac{ds^*}{ds} = \frac{\sqrt{2(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)}}{\sqrt{3}\sqrt{\tan^2(ns)+1}}.$$

We obtain the tangent of $\delta_4(s)$-Smarandahce curve as in

$$(T_N)_{\delta_4} = \frac{-\sqrt{\tan^2(ns)+1}N + (\sqrt{\tan^2(ns)+1}+\tan(ns)')T_N - \tan(ns)'(N\wedge T_N)}{\sqrt{2(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)}}. \tag{3.23}$$

The derivative of (3.23) is

$$(T_N)'_{\delta_4} = \frac{\sqrt{3}(\chi_{16}N + \chi_{17}T_N + \chi_{18}(N\wedge T_N))}{4(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)^2}.$$

From equations (3.21) and (3.23) we have

$$(N\wedge T_N)_{\delta_4} = \frac{(-(\sqrt{\tan^2(ns)+1}+2\tan(ns)')N - (\sqrt{\tan^2(ns)+1}-\tan(ns)')T_N)}{\sqrt{6(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)}}$$

$$+\frac{(2\sqrt{\tan^2(ns)+1}+\tan(ns)')(N\wedge T_N)}{\sqrt{6(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)}}.$$

So the geodesic curvature from the equation (2.5) is

$$K_g^{\delta_4} = \frac{\left((-2\tan(ns)' - \sqrt{\tan(ns)^2+1})\chi_{16} - \chi_{17}(\sqrt{\tan(ns)^2+1} - \tan(ns)')\right)}{(4\sqrt{2}(1+\tan^2(ns)+\sqrt{1+\tan^2(ns)}\tan(ns)'+(-\tan(ns)')^2)^2)^{\frac{5}{2}}}$$

$$+\frac{(2\sqrt{\tan(ns)^2+1}+\tan(ns)')\chi_{18}}{(4\sqrt{2}(1+\tan^2(ns)+\sqrt{1+\tan^2(ns)}\tan(ns)'+(-\tan(ns)')^2)^2)^{\frac{5}{2}}}.$$

$\square$

**Definition 3.17.** *Let $\zeta = \zeta(s)$ be a curve and $\{B, T_B, B \wedge T_B\}$ be Sabban frame of this curve. Then $BT_B$-Smarandache curve ($\zeta_1(s)$-Smarandache curve) is given by*

$$\zeta_1(s) = \frac{1}{\sqrt{2}}(B + T_B). \tag{3.24}$$

*According to equation (2.8) we can parameterize the $\zeta_1(s)$-Smarandache curve as in the following form*

$$\zeta_1(s) = \frac{1}{\sqrt{2}}\Big(-\cos(s)\cos(ns) - n\sin(s)\sin(ns) + \frac{n}{m}\sin(s), -\sin(s)\cos(ns) - n\cos(s)\sin(ns) - \frac{n}{m}\cos(s), \frac{n}{m}\sin(ns) + n\Big).$$

*The shape of this curve is given in Figure (3.9)*

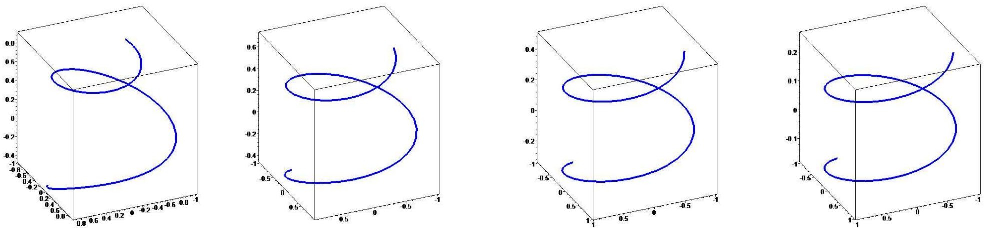

**Figure 3.9:** $BT_B(B \wedge T_B)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5, 5]$

**Theorem 3.18.** *The geodesic curvature $K_g^{\zeta_1}$ according to $\zeta_1(s)$-Smarandache curve is*

$$K_g^{\zeta_1} = \frac{1}{(2 + (\tan(ns))^2)^{\frac{5}{2}}}(\chi_{19}\tan(ns) - \chi_{20}\tan(ns) + 2\chi_{21}),$$

*where the coefficients $\chi_{19}$, $\chi_{20}$ and $\chi_{21}$ are*

$$\begin{aligned}
\chi_{19} &= -2 - \tan^2(ns) + \tan(ns)\tan(ns)', \\
\chi_{20} &= -2 - \tan(ns)\tan(ns)' - 3\tan^2(ns) - \tan^4(ns), \\
\chi_{21} &= 2\tan(ns) + 2\tan(ns)' + \tan^3(ns).
\end{aligned}$$

*Proof.* If we take the derivative of equation (3.24) and from the equation (2.7) we get

$$(T_B)_{\zeta_1} \cdot \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-B + T_B + \tan(ns)(B \wedge T_B)), \tag{3.25}$$

if we take the norm of equation (3.25) we have

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{2 + \tan^2(ns)}.$$

We obtain the tangent of $\zeta_1(s)$-Smarandahce curve as in

$$(T_B)_{\zeta_1} = \frac{1}{\sqrt{2 + \tan^2(ns)}}(-B + T_N + \tan(ns)(B \wedge T_B)). \tag{3.26}$$

The derivative of (3.26) is

$$(T_B)'_{\zeta_1} \cdot \frac{ds^*}{ds} = \frac{\sqrt{2}}{(2 + \tan^2(ns))^2}(\chi_{19}B + \chi_{20}T_B + \chi_{21}(B \wedge T_B)).$$

From equations (3.24) and (3.26) we have

$$(B \wedge T_B)_{\zeta_1} = \frac{1}{\sqrt{4 + 2\tan^2(ns)}}(\tan(ns)N - \tan(ns)T_B + 2(B \wedge T_B)).$$

So the geodesic curvature from the equation (2.7) is

$$K_g^{\zeta_1} = \frac{1}{(2 + (\tan(ns))^2)^{\frac{5}{2}}}(\chi_{19}\tan(ns) - \chi_{20}\tan(ns) + 2\chi_{21}).$$

□

**Definition 3.19.** *Let $\zeta = \zeta(s)$ be a curve and $\{B, T_B, B \wedge T_B\}$ be Sabban frame of this curve. Then $B(B \wedge T_B)$-Smarandache curve ($\zeta_2(s)$-Smarandache curve) is given by*

$$\zeta_2(s) = \frac{1}{\sqrt{2}}(B + (B \wedge T_B)). \tag{3.27}$$

*According to equation (2.8) we can parameterize the $\zeta_2(s)$-Smarandache curve as in the following form*

$$\begin{aligned}\zeta_2(s) = \frac{1}{\sqrt{2}}\Big(&-\cos(s)(\cos(ns)-\sin(ns)) - n\sin(s)(\cos(ns)+\sin(ns)), -\sin(s)(\cos(ns)-\sin(ns)) + n\cos(s)(\cos(ns)+\sin(ns)),\\ &\frac{n}{m}(\cos(ns)+\sin(ns))\Big).\end{aligned}$$

*The shape of this curve is given in Figure (3.10)*

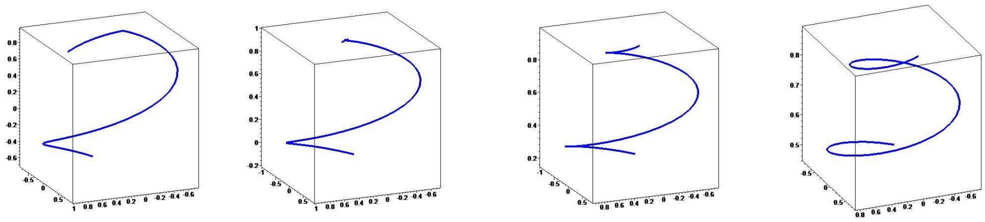

**Figure 3.10:** $B(B \wedge T_B)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5, 5]$

**Theorem 3.20.** *The geodesic curvature $K_g^{\zeta_2}$ according to $\zeta_2(s)$-Smarandache curve is*

$$K_g^{\zeta_2} = 1 + \tan(ns).$$

*Proof.* If we take the derivative of equation (3.27) and from the equation (2.7) we get

$$(T_B)_{\zeta_2} \cdot \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(T_B - \tan(ns)T_B), \tag{3.28}$$

if we take the norm in equation (3.28), $\frac{ds^*}{ds} = \frac{1-\tan(ns)}{\sqrt{2}}$. We obtain the tangent of $\zeta_2(s)$-Smarandahce curve as in

$$(T_B)_{\zeta_2} = T_B. \tag{3.29}$$

The derivative of (3.29) is

$$(T_B)'_{\zeta_2} \cdot \frac{ds^*}{ds} = -B + \tan(ns)(B \wedge T_B).$$

From eqnarrays (3.27) and (3.29) we have

$$(B \wedge T_B)_{\zeta_2} = \frac{1}{\sqrt{2}}(-B + (B \wedge T_B)).$$

So the geodesic curvature from the equation (2.7) is

$$K_g^{\zeta_2} = 1 + \tan(ns).$$

□

**Definition 3.21.** *Let $\zeta = \zeta(s)$ be a curve and $\{B, T_B, B\wedge T_B\}$ be Sabban frame of this curve. Then $T_B(B\wedge T_B)$-Smarandache curve ($\zeta_3(s)$-Smarandache curve) is given by*

$$\zeta_3(s) = \frac{1}{\sqrt{2}}(T_B + (B\wedge T_B)). \tag{3.30}$$

*According to equation (2.8) we can parameterize the $\zeta_3(s)$-Smarandache curve as in the following form*

$$\zeta_3(s) = \frac{1}{\sqrt{2}}\Big(\cos(s)\sin(ns) - n\sin(s)\cos(ns) + \frac{n}{m}\sin(s), \sin(s)\sin(ns) + n\cos(s)\cos(ns) - \frac{n}{m}\cos(s), \frac{n}{m}\cos(ns) + n\Big).$$

*The shape of this curve is given in Figure (3.11)*

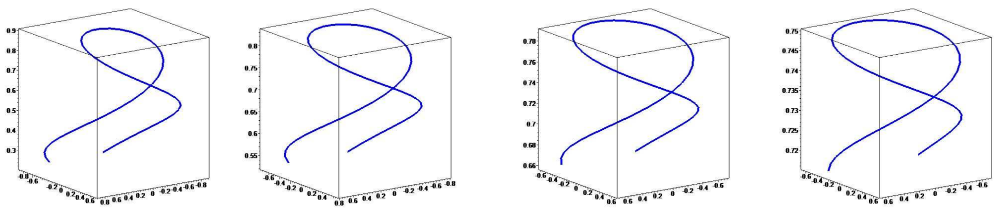

**Figure 3.11:** $T_B(B\wedge T_B)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5, 5]$

**Theorem 3.22.** *The geodesic curvature $K_g^{\zeta_3}$ according to $\zeta_3(s)$-Smarandache curve is*

$$K_g^{\zeta_3} = \frac{1}{(1+2(\tan(ns))^2)^{\frac{5}{2}}}\left(2\tan(ns)\chi_{22} - \chi_{23} + \chi_{24}\right),$$

*where the coefficients $\chi_{22}$, $\chi_{23}$, $\chi_{24}$ are*

$$\begin{aligned}
\chi_{22} &= 2\tan(ns)\tan(ns)' + \tan(ns) + 2\tan^3(ns),\\
\chi_{23} &= -1 - \tan(ns)' - 3\tan^2(ns) - 2\tan^4(ns),\\
\chi_{24} &= -\tan^2(ns) + 2\tan(ns)' - 2\tan^4(ns).
\end{aligned}$$

*Proof.* If we take the derivative of equation (3.30) and from the equation (2.7) we get

$$(T_B)_{\zeta_3}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-B - \tan(ns)T_B + \tan(ns)(B\wedge T_B)), \tag{3.31}$$

if we take the norm of eqnarray (3.31) we have

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{1+2\tan^2(ns)}.$$

We obtain the tangent of $\zeta_3(s)$-Smarandahce curve as in

$$(T_B)_{\zeta_3} = \frac{1}{\sqrt{1+2\tan^2(ns)}}(-B - \tan(ns)T_N + \tan(ns)(B\wedge T_B)). \tag{3.32}$$

The derivative of (3.32) is

$$(T_B)'_{\zeta_3}\cdot\frac{ds^*}{ds} = \frac{\sqrt{2}}{(1+2\tan^2(ns))^2}(\chi_{22}B + \chi_{23}T_B + \chi_{24}(B\wedge T_B)).$$

From equations (3.30) and (3.32) we have

$$(B\wedge T_B)_{\zeta_3} = \frac{1}{\sqrt{2+4\tan^2(ns)}}(2\tan(ns)B - T_B + (B\wedge T_B)).$$

So the geodesic curvature from the equation (2.7) is

$$K_g^{\zeta_3} = \frac{1}{(1+2(\tan(ns))^2)^{\frac{5}{2}}}\left(2\tan(ns)\chi_{22} - \chi_{23} + \chi_{24}\right).$$

□

**Definition 3.23.** *Let $\zeta = \zeta(s)$ be a curve and $\{B, T_B, B\wedge T_B\}$ be Sabban frame of this curve. Then $BT_B(B\wedge T_B)$-Smarandache curve ($\zeta_4(s)$-Smarandache curve) is given by*

$$\zeta_4(s) \quad = \quad \frac{1}{\sqrt{3}}(B + T_B + (B\wedge T_B)). \tag{3.33}$$

*According to equation (2.8) we can parameterize the $\zeta_4(s)$-Smarandache curve as in the following form*

$$\zeta_4(s) \quad = \quad \frac{1}{\sqrt{3}}\Big(-\cos(s)(\cos(ns)-\sin(ns)) - n\sin(s)(\cos(ns)+\sin(ns)) + \frac{n}{m}\sin(s), -\sin(s)(\cos(ns)-\sin(ns)) + n\cos(s)(\cos(ns) + \sin(ns)) - \frac{n}{m}\cos(s), \frac{n}{m}(\cos(ns)+\sin(ns)) + n\Big).$$

*The shape of this curve is given in Figure (3.12)*

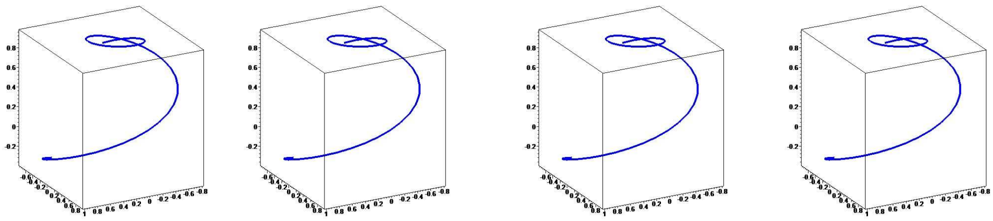

**Figure 3.12:** $BT_B(B\wedge T_B)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5, 5]$

**Theorem 3.24.** *The geodesic curvature $K_g^{\zeta_4}$ according to $\zeta_4(s)$-Smarandache curve is*

$$K_g^{\zeta_4} \quad = \quad \frac{(\chi_{25}(2\tan(ns)-1) + \chi_{26}(-1-\tan(ns)) + \chi_{27}(2-\tan(ns)))}{(4\sqrt{2}(1-\tan(ns)+\tan^2(ns))^2)^{\frac{5}{2}}},$$

*where the coefficients $\chi_{25}$, $\chi_{26}$, $\chi_{27}$ are*

$$\begin{aligned}
\chi_{25} &= -\tan(ns)' + 2\tan(ns)\tan(ns)' - 2 + 4\tan(ns) - 4\tan^2(ns) \\
&\quad + 2\tan^3(ns), \\
\chi_{26} &= -\tan(ns)' - \tan(ns)\tan(ns)' - 2 - 4\tan^2(ns) + 2\tan(ns) \\
&\quad + 2\tan^3(ns) - 2\tan^4(ns), \\
\chi_{27} &= \tan(ns)\tan(ns)' + 2\tan(ns) - 4\tan^2(ns) + 2\tan(ns)' + 4\tan^3(ns) \\
&\quad - 2\tan^4(ns).
\end{aligned}$$

*Proof.* If we take the derivative of equation (3.33) and from the equation (2.7) we have

$$(T_B)_{\zeta_4}\cdot\frac{ds^*}{ds} \quad = \quad \frac{1}{\sqrt{3}}(-B + (1-\tan(ns))T_B + \tan(ns)(B\wedge T_B)), \tag{3.34}$$

if we take the norm of equation (3.34)

$$\frac{ds^*}{ds} \quad = \quad \frac{1}{\sqrt{3}}\sqrt{2(1-\tan(ns)+\tan^2(ns))}.$$

We obtain the tangent of $\zeta_4(s)$-Smarandahce curve as in

$$(T_B)_{\zeta_4} \quad = \quad \frac{1}{\sqrt{2(1-\tan(ns)+\tan^2(ns))}}(-B + (1-\tan(ns))T_B + \tan(ns)(B\wedge T_B)). \tag{3.35}$$

The derivative of (3.35) is

$$(T_B)'_{\zeta_3}\cdot\frac{ds^*}{ds} \quad = \quad \frac{\sqrt{3}}{4(1-\tan(ns)+\tan^2(ns))^2}(\chi_{25}B + \chi_{26}T_B + \chi_{27}(B\wedge T_B)).$$

From equations (3.33) and (3.35) we have

$$(B\wedge T_B)_{\zeta_4} \quad = \quad \frac{((-1+2\tan(ns))B + (-1-\tan(ns))T_B + (2-\tan(ns))(B\wedge T_B))}{\sqrt{6(1-\tan(ns)+\tan^2(ns))}}.$$

So the geodesic curvature from the equation (2.7) is

$$K_g^{\zeta_4} \quad = \quad \frac{(\chi_{25}(2\tan(ns)-1) + \chi_{26}(-1-\tan(ns)) + \chi_{27}(2-\tan(ns)))}{(4\sqrt{2}(1-\tan(ns)+\tan(ns)^2)^2)^{\frac{5}{2}}}.$$

□

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# Two-Grid Iterative Method for a Class of Fredholm Functional Integral Equations based on the Radial Basis Function Interpolation

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**Abstract**

In this paper, we discuss a two-grid iterative method for solving a class of Fredholm functional integral equations based on the radial basis function interpolation. Firstly, the existence and uniqueness of the solution are proved by Banach fixed point theorem. Secondly, the algorithm and convergence analysis of the radial basis function approximation method is given on the coarse grid. Thirdly, the fine grid iterative solution and convergence results are obtained. Finally, the validity and reliability of the theoretical analysis are verified by two numerical experiments.

## 1. Introduction

Integral equations are widely used in quantum physics, engineering design, astronomy, geography, biomedicine and other fields, so it is of great application value to explore the solution of integral equations.

For a long time, the algorithms of integral equations have been widely concerned and studied. Many different methods have been used to approximate the solutions of some integral equations [1, 2]. In recent years, F.Muller and W.Varnhorn [3] have studied approximation and numerical solution of Fredholm integral equations of second kind using quasi-interpolation. Some convergence analysis for 2-dimensional Fredholm integral equation with complex factors by Meshless method were introduced in [4]. Application of Legendre wavelets for solving a class of functional integral equations were discussed in [5]. Chelyshkov collocation approach was developed in [6] for solving linear weakly singular Volterra integral equations. In addition, we know that the computational complexity of numerical integral discretization depends on the diameter $h$ of mesh generation. The calculated workload is usually $O(n^3)$, where $n = 1/h$. Therefore, the construction of a suitable two-grid algorithm can solve the difficulty of computational complexity. Two-grid method is a discretization technique based on two meshes of different sizes, which has been concerned by many researchers [7]-[10] for a long time. Therefore, it is very necessary to enrich the efficient algorithms of different types of integral equations.

In this paper, we consider the following as a class of Fredholm functional integral equations

$$u(x) - A(x)u(\alpha(x)) = f(x) + \lambda \int_\Omega k(x,s)u(s)ds, \quad x \in \Omega, \tag{1.1}$$

where $\Omega$ is the bounded closed area in $R^d, d = 1,2,3$, $x = (x_1, x_2, \cdots, x_d)$, $s = (s_1, s_2, \cdots, s_d)$, $\alpha(x) = (\alpha_1(x), \alpha_2(x), \cdots, \alpha_d(x))$, and $A(x), \alpha(x), f(x), k(x,s)$ are properly smooth known functions, $u(x) \in R$ is the unknown function.

The contents of this article are as follows. Section 2 contains the proofs of the existence and uniqueness of the exact solution for (1.1). In section 3, we discuss the radial basis function approximation method and convergence results on the coarse grid for (1.1). Section 4, is devoted to the two-grid iterative method and convergence results on the fine grid for (1.1). In the last section, the correctness of the theory is proved by two numerical examples.

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## 2. The existence and uniqueness of solution

In this section, the conditions which provides the existence and uniqueness of exact solution of (1.1) are given by using Banach fixed point theorem.

**Theorem 2.1.** *Assume that $\Omega$ is a suitable Banach space and $A(x),\alpha(x),f(x)\in C(\Omega),k(x,s)\in C(\Omega\times\Omega)$. If the following conditions are satisfied*

$$\begin{cases} (i)\alpha(x)\in\Omega, \quad for\ x\in\Omega, \\ (ii)\|A(x)\|_\infty+|\lambda|\cdot\|\int_\Omega|k(x,s)|ds\|_\infty=\gamma<1, \end{cases}$$

*where $\|\cdot\|_\infty=\max|\cdot|$. Then the (1.1) has a unique solution.*

*Proof.* Let $T$ be a mapping from $C(\Omega)$ to $C(\Omega)$ with

$$Tu(x)=A(x)u(\alpha(x))+f(x)+\lambda\int_\Omega k(x,s)u(s)ds,$$

for $u\in C(\Omega)$. Let $u_1,u_2$ be two solutions of (1.1), then we have

$$\begin{aligned}\|Tu_1-Tu_2\|_\infty &=\|A(x)[u_1(\alpha(x))-u_2(\alpha(x))]+\lambda\int_\Omega k(x,s)[u_1(s)-u_2(s)]ds\|_\infty \\ &\le(\|A(x)\|_\infty+|\lambda|\cdot\|\int_\Omega|k(x,s)|ds\|_\infty)\cdot\|u_1-u_2\|_\infty \\ &\le\gamma\cdot\|u_1-u_2\|_\infty.\end{aligned}$$

Note that $0<\gamma<1$, by the Banach fixed point theorem, then $T$ is a contractive mapping on $(C(\Omega),\|\cdot\|_\infty)$. So there exists the unique solution $u^*\in C(\Omega)$ such that $Tu^*=u^*$, and (1.1) has a unique solution. □

## 3. The radial basis function interpolation and convergence on the coarse grid

In this section, we give the algorithm of the radial basis function interpolation for (1.1), and obtain the convergence theorem in the infinite norm sense.
First of all, Assume $\{x^i\}_{i=1}^m,x^i=(x_1^i,x_2^i,\cdots,x_d^i)\in\Omega$ is a series of irregular observation points of $u(x)$ on $\Omega$, and let the basis of the radial basis function is $\varphi_1(x),\varphi_2(x),\cdots,\varphi_m(x)$, where $\varphi_i(x)=\varphi(\|x-x^i\|_2),i=1,2,\cdots,m$. Note that $r=\|x-x^i\|_2$. There are three common radial basis functions (see [11]):
(1)Gaussian distribution function $\varphi(r)=e^{-a^2r^2}$;
(2)MQ function $\varphi(r)=(c^2+r^2)^b$ and IMQ function $\varphi(r)=(c^2+r^2)^{-b}(b>0)$;
(3)Thin plate splines function $\varphi(r)=r^{2k-d}logr$.
Now, we construct vector space $V_m(\Omega)=span\{\varphi_1(x),\varphi_2(x),\cdots,\varphi_m(x)\}$. And let the radial basis function interpolation $u_m^I(x)=\sum\limits_{i=1}^m c_i\varphi_i(x)\in V_m$, satisfying $u_m^I(x^i)=u(x^i),i=1,2,\cdots,m$. From [11], we can get the following error estimation between the radial basis function and the exact solution:

$$\|u(x)-u_m^I(x)\|_\infty\le ch^{\frac{\varepsilon}{2}},$$

where $h=\sup\limits_{x\in\Omega}\min\limits_{1\le i\le m}\|x-x^i\|$.
Therefore, let $u(x)=u_m^I(x)+\varepsilon_m(x)$ where $\varepsilon_m(x)$ is the interpolation remainder of $u(x)$ on $V_m(\Omega)$ . Then we can obtain

$$u_m^I(x)+\varepsilon_m(x)-A(x)[u_m^I(\alpha(x))+\varepsilon_m(\alpha(x))]=f(x)+\lambda\int_\Omega k(x,s)[u_m^I(s)+\varepsilon_m(s)]ds. \tag{3.1}$$

Ignore the error terms $\varepsilon_m(x)$ and $\varepsilon_m(\alpha(x))$ and substitute $u_m^I(x)=\sum\limits_{i=1}^m c_i\varphi_i(x)$ into (3.1), we get the approximate equation of (1.1):

$$\sum_{i=1}^m c_i\varphi_i(x)-A(x)\sum_{i=1}^m c_i\varphi_i(\alpha(x))=f(x)+\sum_{i=1}^m c_i\lambda\int_\Omega k(x,s)\varphi_i(s)ds.$$

Remove the items then we have

$$\sum_{i=1}^m c_i[\varphi_i(x)-A(x)\varphi_i(\alpha(x))-\lambda\int_\Omega k(x,s)\varphi_i(s)ds]=f(x). \tag{3.2}$$

Let $\psi_i(x)=\varphi_i(x)-A(x)\varphi_i(\alpha(x))-\lambda\int_\Omega k(x,s)\varphi_i(s)ds,\ \ i=1,2,\cdots,m$, then (3.2) can be written as

$$\sum_{i=1}^m c_i\psi_i(x)=f(x). \tag{3.3}$$

Since $\varphi_1(x),\varphi_2(x),\cdots,\varphi_m(x)$ are linear independent functions on $V_m(\Omega)$ and satisfies the condition of Theorem 2.1, which implies $\psi_1(x),\psi_2(x),\cdots,\psi_m(x)$ are also linear independent functions on $V_m(\Omega)$. Here's the proof.
Because

$$\sum_{i=1}^m c_i\psi_i(x)=\sum_{i=1}^m c_i\varphi_i(x)-A(x)\sum_{i=1}^m c_i\varphi_i(\alpha(x))-\lambda\int_\Omega k(x,s)\sum_{i=1}^m c_i\varphi_i(s)ds,$$

and

$$\|\sum_{i=1}^{m} c_i\varphi_i(\alpha(x))\|_\infty \le \|\sum_{i=1}^{m} c_i\varphi_i(x)\|_\infty, \quad \alpha(x)\in\Omega.$$

From Theorem 2.1 and trigonometric inequality we can get

$$(1-\gamma)\cdot\|\sum_{i=1}^{m} c_i\varphi_i(x)\|_\infty \le \|\sum_{i=1}^{m} c_i\psi_i(x)\|_\infty \le (1+\gamma)\cdot\|\sum_{i=1}^{m} c_i\varphi_i(x)\|_\infty \tag{3.4}$$

For $x\in\Omega$, assume that $\sum\limits_{i=1}^{m} c_i\psi_i(x)=0$, then we obtain $\|\sum\limits_{i=1}^{m} c_i\psi_i(x)\|_\infty=0$. From (3.4) we have $\|\sum\limits_{i=1}^{m} c_i\varphi_i(x)\|_\infty=0$. Therefore,

$$\sum_{i=1}^{m} c_i\varphi_i(x)=0.$$

And because $\varphi_1(x),\varphi_2(x),\cdots,\varphi_m(x)$ are linear independent functions on $V_m(\Omega)$, so

$$c_1=c_2=\cdots=c_n=0.$$

The proof is complete.

(3.3) take the collocation points $x=x^j=(x_1^j,x_2^j,\cdots,x_d^j), j=1,2,\cdots,m$, then we get $\sum\limits_{i=1}^{m} c_i\psi_i(x^j)=f(x^j)$. Written in matrix form is

$$G_m\cdot C_m=F_m, \tag{3.5}$$

where $C_m=(c_1,c_2,\cdots,c_m)^T$, $F_m=(f(x^1),f(x^2),\cdots,f(x^m))^T$, $G_m=(g_{ij})_{m\times m}$, $g_{ij}=\psi_i(x^j),\ i,j=1,2,\cdots,m.$
Let $G_m$ is a nonsingular matrix, then $G_m^{-1}$ exists. So there is

$$C_m^*=G_m^{-1}\cdot F_m=(c_1^*,c_2^*,\cdots,c_m^*)^T.$$

Therefore, $u_m^*(x)=\sum\limits_{i=1}^{m} c_i^*\varphi_i(x)$ is called the approximate solution of radial basis function of (1.1).

**Theorem 3.1.** *Assume that $u(x)$ is the exact solution and $u_m^*(x)$ is the approximate solution of radial basis function, then*

$$\|u-u_m^*\|_\infty \le ch^{\frac{e}{2}}+\sum_{i=1}^{m}\|\varphi_i(x)\|_\infty\cdot\|G_m^{-1}\|_\infty\cdot\|R_m(\varepsilon)\|_\infty,$$

*where $R_m(\varepsilon)=-\varepsilon_m(x)+A(x)\varepsilon_m(\alpha(x))+\lambda\int_\Omega k(x,s)\varepsilon(s)ds$, and for $\forall x\in\Omega$, we have*

$$\lim_{h\to 0} u_m^*(x)=u(x).$$

*Proof.* For (3.1) and $u_m^I(x)=\sum\limits_{i=1}^{m} c_i\varphi_i(x)$, we can get

$$\sum_{i=1}^{m} c_i\varphi_i(x)+\varepsilon_m(x)-A(x)[\sum_{i=1}^{m} c_i\varphi_i(\alpha(x))+\varepsilon_m(\alpha(x))]=f(x)+\lambda\int_\Omega k(x,s)[\sum_{i=1}^{m} c_i\varphi_i(s)+\varepsilon_m(s)]ds,$$

abbreviated in the following form

$$\sum_{i=1}^{m} c_i\psi_i(x)=f(x)+[-\varepsilon_m(x)+A(x)\varepsilon_m(\alpha(x))+\lambda\int_\Omega k(x,s)\varepsilon_m(s)ds].$$

Now we write down the above equation as

$$\sum_{i=1}^{m} c_i\psi_i(x)=f(x)+R_m(\varepsilon),$$

where $R_m(\varepsilon)=-\varepsilon_m(x)+A(x)\varepsilon_m(\alpha(x))+\lambda\int_\Omega k(x,s)\varepsilon_m(s)ds$. The corresponding matrix form is

$$G_m\cdot C_m=F_m+R_m(\varepsilon). \tag{3.6}$$

Subtracting (3.5) from (3.6)to get $G_m(C_m-C_m^*)=R_m(\varepsilon)$, so

$$\|C_m-C_m^*\|_\infty \le \|G_m^{-1}\|_\infty\cdot\|R_m(\varepsilon)\|_\infty.$$

Then for

$$\begin{aligned}\|u-u_m^*\|_\infty &= \|u-u_m^I+u_m^I-u_m^*\|_\infty\\ &\le \|u-u_m^I\|_\infty+\|u_m^I-u_m^*\|_\infty\\ &\le ch^{\frac{e}{2}}+\|\textstyle\sum_{i=1}^{m}(c_i-c_i^*)\varphi_i(x)\|_\infty\\ &\le ch^{\frac{e}{2}}+\textstyle\sum_{i=1}^{m}|c_i-c_i^*|\|\varphi_i(x)\|_\infty\\ &\le ch^{\frac{e}{2}}+\textstyle\sum_{i=1}^{m}\|\varphi_i(x)\|_\infty\cdot\|G_m^{-1}\|_\infty\cdot\|R_m(\varepsilon)\|_\infty.\end{aligned}$$

Since $\|R_m(\varepsilon)\|_\infty\to 0$, $\|u-u_m^*\|_\infty\to 0$.

□

## 4. Two-grid iterative method and convergence analysis

As we all know, for the discretization of numerical integration of (1.1), the computational complexity depends on the diameter $h$ of mesh subdivision, and the workload of numerical calculation is usually $O(n^3)$, where $n = 1/h$. Therefore, computational complexity can be well solved by constructing an appropriate two-layer grid algorithm.

In this section, a new fixed point iterative approximation method is introduced in three steps to obtain two-grid iterative solution on the fine grid. The iterative algorithm is as follows

**Algorithm 1.**

**Step 1.** Select a series of irregular observation points of $u(x)$ on the fine grid as $(y^1, y^2, \cdots, y^n), y^i = (y_1^i, y_2^i, \cdots, y_d^i), i = 1,2,\cdots,n, d = 1,2$ *or* $3$.

**Step 2.** Approximate solution of Radial basis function on the fine grid is

$$u_n^{(0)}(x) = \sum_{i=1}^{n} l_i \Phi_i(x),$$

where $\Phi_i(x) = \varphi(\|x - y^i\|), i = 1,2,\cdots,n$.

**Step 3.** Take the initial value $u_n^{(0)}(x) = u_m^*(x)$ and construct iterative scheme

$$u_n^{(k+1)}(x) = A(x)u_n^{(k)}(\alpha(x)) + f(x) + \lambda \int_\Omega k(x,s)u_n^{(k)}(s)ds, \quad k = 0,1,2,\cdots. \tag{4.1}$$

The error estimation and convergence results between the fine grid approximate solution of radial basis function $u_n^I(x)$ and the two-grid iterative solution $u_n^{(k+1)}(x)$ are given below.

**Theorem 4.1.** *Let $u_n^I(x)$ is the fine grid approximate solution of radial basis function and $u_n^{(k+1)}(x)$ is the $(k+1)$th iterative solution determined by (4.1), then*

$$\|u_n^I(x) - u_n^{(k+1)}(x)\|_\infty \le \gamma^{k+1} \cdot \|u_n(x) - u_n^{(0)}(x)\|_\infty,$$

*where $\gamma < 1$ as in Theorem 2.1.*
*For $\forall x \in \Omega$, there holds*

$$\lim_{k\to\infty} u_n^{(k)}(x) = u_n^I(x).$$

*Proof.* Replace $u_n^I(x) = \sum_{i=1}^{n} c_i \varphi_i(x)$ in (1.1) and get

$$u_n^I(x) = A(x)u_n^I(\alpha(x)) + f(x) + \lambda \int_\Omega k(x,s)u_n^I(s)ds. \tag{4.2}$$

Subtracting (4.1) from (4.2), we have

$$u_n^I(x) - u_n^{(k+1)}(x) = A(x)[u_n^I(\alpha(x)) - u_n^{(k)}(\alpha(x))] + \lambda \int_\Omega k(x,s)[u_n^I(s) - u_n^{(k)}(s)]ds,$$

then

$$\begin{aligned}\|u_n^I(x) - u_n^{(k+1)}(x)\|_\infty &= \|A(x)[u_n^I(\alpha(x)) - u_n^{(k)}(\alpha(x))] + \lambda \textstyle\int_\Omega k(x,s)[u_n^I(s) - u_n^{(k)}(s)]ds\|_\infty \\ &\le (\|A(x)\|_\infty + |\lambda| \cdot \|\textstyle\int_\Omega |k(x,s)|ds\|_\infty) \cdot \|u_n^I - u_n^{(k)}\|_\infty \\ &\le \gamma \cdot \|u_n^I - u_n^{(k)}\|_\infty.\end{aligned}$$

In this way,

$$\|u_n^I(x) - u_n^{(k+1)}(x)\|_\infty \le \gamma^{k+1} \cdot \|u_n^I - u_n^{(0)}\|_\infty$$

can be obtained by progressive recursion.
From Theorem 2.1, we get $0 < \gamma < 1$, such that

$$\lim_{k\to\infty} \gamma^{k+1} = 0.$$

For $x \in \Omega$, there exists

$$\lim_{k\to\infty} u_n^{(k)}(x) = u_n^I(x).$$

□

## 5. Numerical examples

In this section, two numerical examples ($d = 2$) are given to illustrate the feasibility and validity of the above algorithm and its convergence analysis. The exact solution is compared with the two-grid iterative solution and its error estimation in the infinite norm sense is provided by using Matlab 2015a.

Selecting $\{(m,n)\} = \{(8,32),(8,64),(16,64)\}$, we calculate the maximum error $\|u_n(x,y) - u_n^{(k)}(x,y)\|_\infty = \max\limits_{(x_i,y_i)\in\Omega} |u_n(x_i,y_i) - u_n^{(k)}(x_i,y_i)|$ between the fine grid radial basis interpolation solution $u_n(x,y)$ and two-grid iterative solution $u_n^{(k)}(x,y)$ with $\{k\} = \{3,4,5,6\}$.

**Example 5.1.** *Consider the following Fredholm functional integral equation*

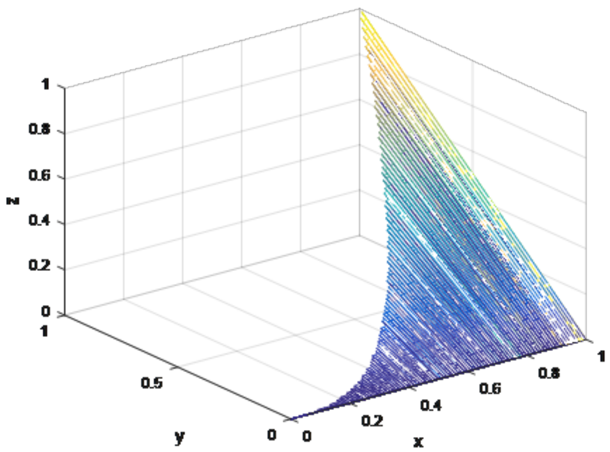
$$u(x,y) - \frac{x+y}{16} u(\alpha_1(x), \alpha_2(y)) = f(x,y) + \frac{1}{20}\int_\Omega (x+y)u(s_1,s_2)ds_1ds_2, \tag{5.1}$$

*where* $\Omega = \{(x,y)|0 \le x \le 1, 0 \le y \le x^2\}$, $\alpha_1(x) = \frac{1}{4}x$, $\alpha_2(x) = \frac{1}{16}x^2$ *and* $f(x,y) = xy - \frac{1}{240}x - \frac{1}{240}y - \frac{1}{16}xy^2( fracx64 + \frac{y}{64})$. (5.1) *has an exact solution* $u(x,y) = xy$.

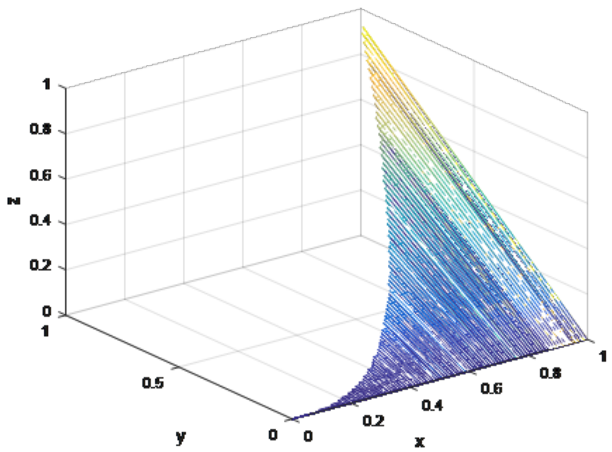
We solve (5.1) by two-grid method based on radial basis interpolation, and our experimental results can be seen from Table 1. The results were obtained by using Gaussian distribution function ($a = 1$) and IMQ function ($c = \sqrt{2}, b = 1$), respectively. Next, the exact solution $u(x,y)$ and the two-grid iterative solution $u_n^{(k)}(x,y)$ which are given and can be seen from Figure 5.1(a) and Figure 5.1(b) while $(m,n,k) = (8,64,6)$.

| | | | $\|u_n - u_n^{(k)}\|_\infty$ | | | | | $\|u_n - u_n^{(k)}\|_\infty$ | | | | | $\|u_n - u_n^{(k)}\|_\infty$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| m | n | k | Gaussian | IMQ | m | n | k | Gaussian | IMQ | m | n | k | Gaussian | IMQ |
| 8 | 32 | 3 | 9.1698e-04 | 1.3462e-04 | 8 | 64 | 3 | 1.5243e-03 | 9.3253e-04 | 16 | 64 | 3 | 1.2412e-05 | 7.5523e-06 |
| | | 4 | 2.2979e-05 | 2.1310e-05 | | | 4 | 2.4130e-04 | 1.4762e-04 | | | 4 | 1.9648e-06 | 1.1955e-06 |
| | | 5 | 1.9648e-06 | 3.3734e-06 | | | 5 | 3.8198e-05 | 2.3368e-05 | | | 5 | 3.1103e-07 | 1.8925e-07 |
| | | 6 | 5.7583e-07 | 5.3401e-07 | | | 6 | 6.0467e-06 | 3.6992e-06 | | | 6 | 4.9236e-08 | 2.9958e-08 |

**Table 1:** The maximum error $\|u_n(x) - u_n^{(k)}(x)\|_\infty$ for Example 5.1.




(a) The exact solution $u(x,y)$ of example 5.1




(b) The two-grid iterative solution $u_n^{(k)}(x,y)$ of example 5.1

**Example 5.2.** *Consider the following Fredholm functional integral equation*

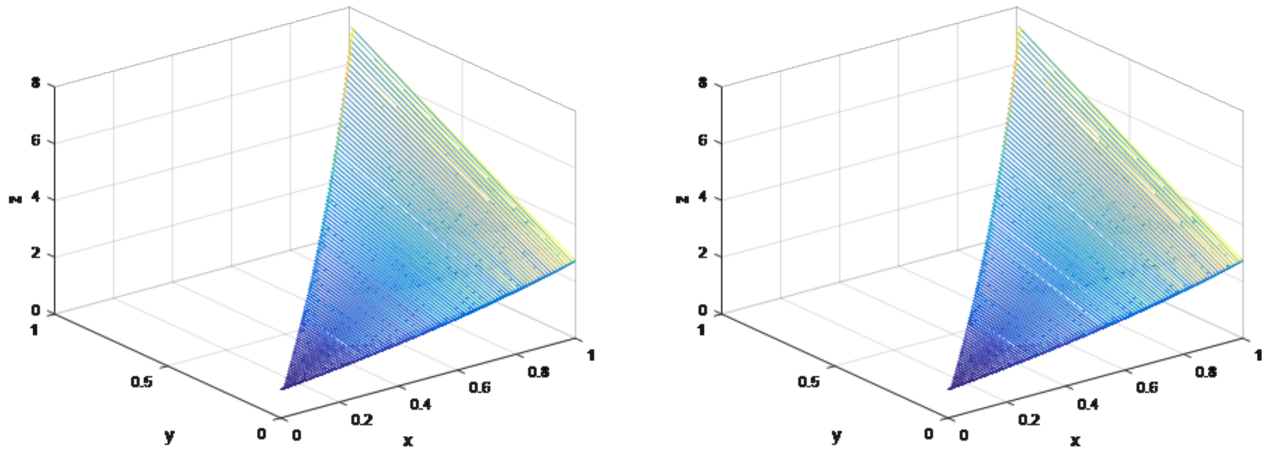
$$u(x,y) - \frac{1}{10}e^{xy}u(\alpha_1(x), \alpha_2(y)) = f(x,y) + \frac{1}{10}\int_\Omega xys_1s_2u(s_1,s_2)ds_1ds_2, \tag{5.2}$$

*where* $\Omega = \{(x,y)|0 \le x \le 1, 0 \le y \le x\}$, $\alpha_1(x) = x$, $\alpha_2(x) = \frac{1}{2}x$ *and* $f(x,y) = e^{x+y} - \frac{1}{10}e^{x+\frac{y}{2}}e^{xy} - \frac{xy}{20}$. (5.2) *has an exact solution* $u(x,y) = e^{x+y}$.

We also solve (5.2) by two-grid method based on radial basis interpolation, and the experimental results can be seen from Table 2. The results were obtained by using Gaussian distribution function ($a = 1$) and MQ function ($c = 4, b = \frac{1}{2}$), respectively. Next, the exact solution $u(x,y)$ and the two-grid iterative solution $u_n^{(k)}(x,y)$ which are given and can be seen from Figure 5.1(c) and Figure 5.1(d) while $(m,n,k) = (8,64,6)$.

| | | | $\|u_n - u_n^{(k)}\|_\infty$ | | | | | $\|u_n - u_n^{(k)}\|_\infty$ | | | | | $\|u_n - u_n^{(k)}\|_\infty$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| m | n | k | Gaussian | IMQ | m | n | k | Gaussian | IMQ | m | n | k | Gaussian | IMQ |
| 8 | 32 | 3 | 5.3422e-03 | 1.5322e-03 | 8 | 64 | 3 | 8.7424e-03 | 1.6395e-03 | 16 | 64 | 3 | 4.4892e-04 | 9.7401e-05 |
| | | 4 | 1.6000e-03 | 4.5476e-04 | | | 4 | 2.6000e-03 | 4.8660e-04 | | | 4 | 1.3324e-04 | 2.8909e-05 |
| | | 5 | 4.7488e-04 | 1.3497e-04 | | | 5 | 7.7168e-04 | 1.4442e-04 | | | 5 | 3.9546e-05 | 8.5802e-06 |
| | | 6 | 1.4094e-04 | 4.0059e-05 | | | 6 | 2.2903e-04 | 4.2864e-05 | | | 6 | 1.1737e-05 | 2.5466e-06 |

**Table 2:** The maximum error $\|u_n(x) - u_n^{(k)}(x)\|_\infty$ for Example 5.2.




(c) The exact solution $u(x,y)$ of example 5.2

(d) The two-grid iterative solution $u_n^{(k)}(x,y)$ of example 5.2

## 6. Conclusion

In this paper, a new two-grid method based on the radial basis function interpolation for solving a class of Fredholm functional integral equations, which has practical value is presented. The algorithm and convergence analysis of two-grid iterative solution are given.
At the same time, the method can greatly reduce the computational workload. Our numerical results can successfully prove the correctness of the proposed error estimation. Extending this method to other integral equations is our further research.

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# Three Equivalent $n$-Norms on the Space of $p$-Summable Sequences

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**Abstract**

Given a normed space, one can define a new $n$-norm using a semi-inner product $g$ on the space, different from the $n$-norm defined by Gähler. In this paper, we are interested in the new $n$-norm which is defined using such a functional $g$ on the space $\ell^p$ of $p$-summable sequences, where $1 \leq p < \infty$. We prove particularly that the new $n$-norm is equivalent with the one defined previously by Gunawan on $\ell^p$.

## 1. Introduction

On a normed space $(X, \|\cdot\|)$, let $g : X^2 \to \mathbb{R}$ be the functional defined by the formula

$$g(x,y) := \frac{1}{2}\|x\|\left[\tau_+(x,y) + \tau_-(x,y)\right],$$

with

$$\tau_\pm(x,y) := \lim_{t\to 0^\pm} \frac{\|x+ty\| - \|x\|}{t}.$$

Then, one may check that $g$ satisfies the following properties:

(1) $g(x,x) = \|x\|^2$ for every $x \in X$;
(2) $g(\alpha x, \beta y) = \alpha\beta\, g(x,y)$ for every $x,y \in X$ and $\alpha, \beta \in \mathbb{R}$;
(3) $g(x, x+y) = \|x\|^2 + g(x,y)$ for every $x,y \in X$;
(4) $|g(x,y)| \leq \|x\|\,\|y\|$ for every $x,y \in X$.

Assuming that the $g$-functional is linear in the second argument then $[y,x] = g(x,y)$ is a *semi-inner product* on $X$. Note that all vector spaces in text are assumed to be over $\mathbb{R}$. For example, one may observe that the functional

$$g(x,y) := \|x\|_p^{2-p}\sum_k |x_k|^{p-1}\operatorname{sgn}(x_k)\, y_k, \quad x := (x_k), y := (y_k) \in \ell^p$$

is a semi-inner product on $\ell^p$, $1 \leq p < \infty$ [1].

**Remark 1.1.** *Note that not all vector spaces have the property that the $g$-functional is linear in the second argument. If the normed space is smooth, then the $g$-functional is linear in the second argument. A normed spaces with the property that the $g$-functional is linear in the second argument is referred to as normed spaces of $(G)$-type [2].*

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By using a semi-inner product $g$, Miličić [3] introduced the following orthogonality relation on $X$: $x$ is said to be $g$-orthogonal to $y$, denoted by $x \perp_g y$, provided that $g(x,y) = 0$. For more recent works, see in [4, 5].
Recently, Nur and Gunawan in [6] defined a 2-norm on $X$ by

$$\|x_1,x_2\|_g := \sup_{\|y_j\|\le 1,\, j=1,2} \begin{vmatrix} g(y_1,x_1) & g(y_2,x_1) \\ g(y_1,x_2) & g(y_2,x_2) \end{vmatrix}.$$

Similarly, we can define an $n$-norm (with $n \ge 2$) using the semi-inner product $g$ on $X$. An *n-norm* on $X$ is a mapping $\|\cdot,\dots,\cdot\| : X\times\cdots\times X \longrightarrow \mathbb{R}$ which satisfies the following four properties:

(1) $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent;
(2) $\|x_1,\dots,x_n\|$ is invariant under permutation;
(3) $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$ for every $x_1,\dots,x_n \in X$ and for every $\alpha \in \mathbb{R}$;
(4) $\|x_1,\dots,x_{n-1},y+z\| \le \|x_1,\dots,x_{n-1},y\| + \|x_1,\dots,x_{n-1},z\|$ for every $x,y,z \in X$.

The pair $(X,\|\cdot,\dots,\cdot\|)$ is called an *n-normed space*.
The theory of 2-normed spaces was initially introduced by Gähler [7] in the 1960's. Meanwhile, the theory of $n$-normed spaces for $n \ge 2$ was developed in [8]-[10]. See [11]-[15] for recent results on this subject.
On the space $\ell^p$ of $p$-summable sequences, where $1 \le p < \infty$, the following $n$-norm

$$\|x_1,\dots,x_n\|_p := \left[\frac{1}{n!}\sum_{k_1}\cdots\sum_{k_n}\left(\operatorname{abs}\begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n}\end{vmatrix}\right)^p\right]^{\frac{1}{p}} \tag{1.1}$$

is defined by Gunawan in [16]. As shown in [17, 18], this $n$-norm is equivalent with the one formulated by Gähler in [8]-[10], namely

$$\|x_1,\dots,x_n\|'_p := \sup_{\|y_j\|_{p'}\le 1,\, j=1,\dots,n} \begin{vmatrix} \sum_k x_{1k}y_{1k} & \cdots & \sum_k x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk}y_{1k} & \cdots & \sum_k x_{nk}y_{nk}\end{vmatrix}, \tag{1.2}$$

where $p'$ denotes the dual exponent of $p$. Precisely, we have the following theorem.

**Theorem 1.2.** *[19] For every $x_1,\dots,x_n \in \ell^p$ $(1 \le p < \infty)$, we have*

$$(n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p \le \|x_1,\dots,x_n\|'_p \le (n!)^{\frac{1}{p}}\|x_1,\dots,x_n\|_p.$$

In this article, we shall first prove that, on $\ell^p$ $(1 \le p < \infty)$, the new 2-norm $\|\cdot,\cdot\|_g$ is equivalent with the 2-norm $\|\cdot,\cdot\|_p$ which is defined in (1.1). Using this result, we can prove that the 2-normed space $(\ell^p,\|\cdot,\cdot\|_g)$ is complete. We then extend the result for all $n \ge 2$.

## 2. Main results

Before we discuss the equivalence between the two 2-norms on $\ell^p$ $(1 \le p < \infty)$, we need some definitions. Let $(X,\|\cdot\|)$ be a normed space. We define the $g$-orthogonal projection of a vector $y$ on a subspace $S$ of $X$ as follows.

**Definition 2.1.** *[20] Let $y \in X$ and $S = span\{x_1,\dots,x_m\}$ be a subspace of $X$ with $\Gamma(x_1,\dots,x_m) = \det[g(x_i,x_j)] \ne 0$. The $g$-orthogonal projection of $y$ on $S$, which we denote by $y_S$, is defined by*

$$y_S := -\frac{1}{\Gamma(x_1,\dots,x_m)}\begin{vmatrix} 0 & x_1 & \cdots & x_m \\ g(x_1,y) & g(x_1,x_1) & \cdots & g(x_1,x_m) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_m,y) & g(x_m,x_1) & \cdots & g(x_m,x_m)\end{vmatrix},$$

*and its $g$-orthogonal complement $y - y_S$ is given by*

$$y - y_S = \frac{1}{\Gamma(x_1,\dots,x_m)}\begin{vmatrix} y & x_1 & \cdots & x_m \\ g(x_1,y) & g(x_1,x_1) & \cdots & g(x_1,x_m) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_m,y) & g(x_m,x_1) & \cdots & g(x_m,x_m)\end{vmatrix}.$$

Observe here that $x_i \perp_g y - y_S$ for every $i = 1,\dots,m$. Note that, if $S = \operatorname{span}\{x\}$, then

$$y_S = \frac{g(x,y)}{\|x\|^2}x,$$

and $y - y_S$ is the $g$-orthogonal complement $y$ on $S$. It is clear here that $x \perp_g y - y_S$.

Next, let $x_1,\dots,x_n \in X$ be a set of $n$ linearly independent vectors. We may construct a *left $g$-orthogonal sequence* $x_1^*,\dots,x_n^*$ with $x_1^* := x_1$, and

$$x_i^* := x_i - (x_i)_{S_{i-1}}, \tag{2.1}$$

where $S_{i-1} = \text{span}\left\{x_1^*,\dots,x_{i-1}^*\right\}$ for $i = 2,\dots,n$. Observe here that $x_i^* \perp_g x_j^*$ for $i < j$ (see [15, 20]).
For $X = \ell^p$ $(1 \le p < \infty)$, we have relation for the $n$-norm $\|x_1,\dots,x_n\|_p$ and the 'volume' of the $n$-dimensional parallelepiped spanned by $\{x_1,\dots,x_n\}$ in $\ell^p$, namely $V(x_1,\dots,x_n) = \prod_{i=1}^{n} \|x_i^*\|_p$, as follows.

**Theorem 2.2.** *[19] Let $\{x_1,\dots,x_n\}$ be a set of linearly independent vectors in $\ell^p$ $(1 \le p < \infty)$. Then we have*

$$(n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p \le V(x_{i_1},\dots,x_{i_n}) \le (n!)^{\frac{1}{p}}\|x_1,\dots,x_n\|_p$$

*for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$.*

Note that the value of $V(x_1,\dots,x_n)$ may not be invariant under permutation of $(x_1,\dots,x_n)$ because $g(\cdot,\cdot)$ may not be symmetry. The above theorem states that all possible values of $V(x_{i_1},\dots,x_{i_n})$ lie between two multiples of $\|x_1,\dots,x_n\|_p$, independent of the permutation.

### 2.1. The equivalence between two 2-norms

Let us consider Gunawan's definition and Gähler's definition of 2-norm on $\ell^p$ $(1 \le p < \infty)$, namely:

$$\|x_1,x_2\|_p = \left[\sum_{k_1}\sum_{k_2}\left(\text{abs}\begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2}\end{vmatrix}\right)^p\right]^{\frac{1}{p}}$$

and

$$\|x_1,x_2\|_p' := \sup_{\|y_j\|_{p'}\le 1,\, j=1,2}\begin{vmatrix} \sum\limits_k x_{1k}y_{1k} & \sum\limits_k x_{1k}y_{2k} \\ \sum\limits_k x_{2k}y_{1k} & \sum\limits_k x_{2k}y_{2k}\end{vmatrix}.$$

Meanwhile, Nur and Gunawan's 2-norm is given by

$$\|x_1,x_2\|_{g,p} = \sup_{\|y_j\|_p\le 1,\, j=1,2}\begin{vmatrix} \|y_1\|_p^{2-p}\sum\limits_k |y_{1k}|^{p-1}\text{sgn}(y_{1k})x_{1k} & \|y_2\|_p^{2-p}\sum\limits_k |y_{2k}|^{p-1}\text{sgn}(y_{2k})x_{1k} \\ \|y_1\|_p^{2-p}\sum\limits_k |y_{1k}|^{p-1}\text{sgn}(y_{1k})x_{2k} & \|y_2\|_p^{2-p}\sum\limits_k |y_{2k}|^{p-1}\text{sgn}(y_{2k})x_{2k}\end{vmatrix}.$$

**Remark 2.3.** *Using properties of determinants, the above 2-norm may be rewritten as*

$$\|x_1,x_2\|_{g,p} = \sup_{\|y_j\|_p\le 1,\, j=1,2}\frac{1}{2}\prod_{j=1}^{2}\|y_j\|_p^{2-p}\sum_{k_1}\sum_{k_2}\begin{vmatrix} |y_{1k_1}|^{p-1}sgn\left(y_{1k_1}\right) & |y_{1k_2}|^{p-1}sgn\left(y_{1k_2}\right) \\ |y_{2k_1}|^{p-1}sgn\left(y_{2k_1}\right) & |y_{2k_2}|^{p-1}sgn\left(y_{2k_2}\right)\end{vmatrix}\begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2}\end{vmatrix}.$$

*For $p = 2$, we observe that*

$$\|x_1,x_2\|_{g,2} = \sup_{\|y_j\|_2\le 1,\ j=1,2}\frac{1}{2}\sum_{k_1}\sum_{k_2}\begin{vmatrix} y_{1k_1} & y_{1k_2} \\ y_{2k_1} & y_{2k_2}\end{vmatrix}\begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2}\end{vmatrix}.$$

*One may then verify that the three 2-norms $\|\cdot,\cdot\|_2$, $\|\cdot,\cdot\|_2'$ and $\|\cdot,\cdot\|_{g,2}$ are identical (see [6, 12]).*

For other values of $p$, we have the following theorem.

**Theorem 2.4.** *For every $x_1,x_2 \in \ell^p$ $(1 \le p < \infty)$, we have*

$$2^{\frac{1}{p}-1}\|x_1,x_2\|_p \le \|x_1,x_2\|_{g,p} \le \|x_1,x_2\|_p' \le 2^{\frac{1}{p}}\|x_1,x_2\|_p.$$

*Proof.* For $j = 1,2$, let $y_j \in \ell^p$ with $\|y_j\|_p \le 1$. Take $u_j = (u_{jk})$ with $u_{jk} = \|y_j\|_p^{2-p}|y_{jk}|^{p-1}\text{sgn}(y_{jk})$. We observe that $u_j \in \ell^{p'}$ with $\|u_j\|_{p'} = \|y_j\|_p$. As a consequence, we have $\|x_1,x_2\|_{g,p} \le \|x_1,x_2\|_p'$. By using Theorem 1.2, we obtain

$$\|x_1,x_2\|_{g,p} \le \|x_1,x_2\|_p' \le 2^{\frac{1}{p}}\|x_1,x_2\|_p.$$

Next, assume that $\{x_1,x_2\}$ is linearly independent. Using the process in (2.1), we obtain the left $g$-orthogonal set $\left\{x_1^*,x_2^*\right\}$. Then, by Theorem 2.2, we have

$$2^{\frac{1}{p}-1}\|x_1,x_2\|_p \le V(x_1,x_2) = \|x_1^*\|_p\|x_2^*\|_p.$$

For $j=1,2$, let $y_j=\frac{x_j^*}{\|x_j^*\|_p}$, so that $\|y_j\|_p=1$. It follows from the properties of semi-inner product $g$ and matrix determinants that

$$\begin{aligned}\begin{vmatrix} g(y_1,x_1) & g(y_2,x_1) \\ g(y_1,x_2) & g(y_2,x_2) \end{vmatrix} &= \begin{vmatrix} \frac{1}{\|x_1^*\|_p} g(x_1^*,x_1^*) & \frac{1}{\|x_2^*\|_p} g(x_2^*,x_1^*) \\ \frac{1}{\|x_1^*\|_p} g(x_1^*,x_2^*) & \frac{1}{\|x_2^*\|_p} g(x_2^*,x_2^*) \end{vmatrix} \\ &= \|x_1^*\|_p \|x_2^*\|_p = V(x_1,x_2) \\ &\geq 2^{\frac{1}{p}-1} \|x_1,x_2\|_p .\end{aligned}$$

By the definition of $\|\cdot,\cdot\|_{g,p}$, we conclude that $\|x_1,x_2\|_{g,p}\geq 2^{\frac{1}{p}-1}\|x_1,x_2\|_p$. Combining with the previous inequalities, we have

$$2^{\frac{1}{p}-1}\|x_1,x_2\|_p \leq \|x_1,x_2\|_{g,p} \leq \|x_1,x_2\|'_p \leq 2^{\frac{1}{p}} \|x_1,x_2\|_p.$$

Note that if $\{x_1,x_2\}$ is a linearly dependent set, then all the 2-norms are equal 0, and so we have the equalities. □

**Corollary 2.5.** *For $1\leq p<\infty$, the three 2-norms $\|\cdot,\cdot\|_{g,p}$, $\|\cdot,\cdot\|'_p$, and $\|\cdot,\cdot\|_p$ are pairwise equivalent.*

Since $(\ell^p,\|\cdot,\cdot\|_p)$ is a 2-Banach space [1], we obtain the following corollary.

**Corollary 2.6.** *For $1\leq p<\infty$, the 2-normed space $(\ell^p,\|\cdot,\cdot\|_{g,p})$ is a 2-Banach space.*

### 2.2. The equivalence between two $n$-norms

All results in above subsection can be extended to $n$-normed spaces for any $n\geq 2$. Suppose that $g$ is a semi-inner product on $(X,\|\cdot\|)$. Consider the following mapping $\|\cdot,\dots,\cdot\|_g$ on $X\times\cdots\times X$:

$$\|x_1,\dots,x_n\|_g = \sup_{\|y_j\|\leq 1,\, j=1,\dots,n} \begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n) \end{vmatrix} = \sup_{\|y_j\|\leq 1,\, j=1,\dots,n} \det[g(y_j,x_i)]. \tag{2.2}$$

If $\|y_j\|\leq 1$ for $j=1,\dots,n$, then $\det[g(y_j,x_i)]\leq n!\prod_{i=1}^{n}\|x_i\|$. Note that the factor $n!$ comes from the number of terms in the expansion of $\det[g(y_j,x_i)]$. The following fact tells us that $\|\cdot,\dots,\cdot\|_g$ is a finite number.

**Fact 2.7.** *The inequality*

$$\|x_1,\dots,x_n\|_g \leq n!\prod_{i=1}^{n}\|x_i\|$$

*holds whenever $x_1,\dots,x_n\in X$.*

Moreover, we have the following result.

**Proposition 2.8.** *The mapping (2.2) defines an $n$-norm on $X$.*

*Proof.* It is obvious that, if $\{x_1,\dots,x_n\}$ is linearly dependent, then we have $\|x_1,\dots,x_n\|_g=0$. Conversely, if $\|x_1,\dots,x_n\|_g=0$, then the rows of the matrix $[g(y_j,x_i)]$ are linearly dependent for every $y_1,\dots,y_n\in X$ with $\|y_j\|\leq 1,\ j=1,\dots,n$. This happens only if $x_1,\dots,x_n$ are linearly dependent.
Next, by using the properties of supremum and matrix determinants, we obtain the invariance of $\|x_1,\dots,x_2\|_g$ under permutation. Furthermore, we have $\|\alpha x_1,\dots x_n\|_g=|\alpha|\,\|x_1,\dots,x_n\|_g$ for $\alpha\in\mathbb{R}$.
Finally, for arbitrary $x_0,x_1,\dots x_n\in X$, we obtain

$$\begin{aligned}\|x_0+x_1,\dots,x_n\|_g &= \sup_{\|y_j\|\leq 1,\ j=1,\dots,n} \begin{vmatrix} g(y_1,x_0+x_1) & \cdots & g(y_n,x_0+x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n) \end{vmatrix} \\ &\leq \sup_{\|y_j\|\leq 1, j=1,\dots,n} \begin{vmatrix} g(y_1,x_0) & \cdots & g(y_n,x_0) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n) \end{vmatrix} + \sup_{\|y_j\|\leq 1, j=1,\dots,n} \begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n) \end{vmatrix} \\ &= \|x_0,\dots,x_n\|_g + \|x_1,\dots,x_n\|_g.\end{aligned}$$

This completes the proof. □

The following theorem holds for an inner product space $(X,\langle\cdot,\cdot\rangle)$.

**Theorem 2.9.** *If $(X,\langle\cdot,\cdot\rangle)$ is a real inner product space, then the two n-norms $\|\cdot,\dots,\cdot\|_g$ in (2.2) and $\|\cdot,\dots,\cdot\|_s$ given by*

$$\|x_1,\dots,x_n\|_s := \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_n,x_1\rangle \\ \vdots & \ddots & \vdots \\ \langle x_1,x_n\rangle & \cdots & \langle x_n,x_n\rangle \end{vmatrix}^{\frac{1}{2}}$$

*are identical.*

*Proof.* On the inner product space $X$, the functional $g(\cdot,\cdot)$ is identical with the inner product $\langle\cdot,\cdot\rangle$. Therefore,

$$\|x_1,\dots,x_n\|_g = \sup_{\|y_j\|\le 1,\, j=1,\dots,n} \begin{vmatrix} \langle y_1,x_1\rangle & \cdots & \langle y_n,x_1\rangle \\ \vdots & \ddots & \vdots \\ \langle y_1,x_n\rangle & \cdots & \langle y_n,x_n\rangle \end{vmatrix}.$$

Now, applying the generalized Cauchy-Schwarz inequality [21] and Hadamard's inequality [22], we get

$$\|x_1,\dots,x_n\|_g \le \sup_{\|y_j\|\le 1,\, j=1,\dots,n} \|x_1,\dots,x_n\|_s\|y_1,\dots,y_n\|_s \le \|x_1,\dots,x_n\|_s.$$

Conversely, suppose that $\{x_1,\dots,x_n\}$ is linearly independent. Using the Gram-Schmidt process, we get the orthogonal set $\{x_1',\dots,x_n'\}$. Because the determinant of the Gram matrix of a linearly independent set being equal to the Gram matrix of the associated orthogonal set (obtained using Gram-Schmidt process), we have $\|x_1,\dots,x_n\|_s = \|x_1',\dots,x_n'\|_s = \|x_1'\|\cdots\|x_n'\|$. For $j=1,\dots,n$, let $y_j = \frac{x_j'}{\|x_j'\|}$, so that $\|y_j\| = 1$. Then, by the properties of the inner product and matrix determinants, we obtain

$$\begin{aligned} \begin{vmatrix} \langle y_1,x_1\rangle & \cdots & \langle y_n,x_1\rangle \\ \vdots & \ddots & \vdots \\ \langle y_1,x_n\rangle & \cdots & \langle y_n,x_n\rangle \end{vmatrix} &= \begin{vmatrix} \langle y_1,x_1'\rangle & \cdots & \langle y_n,x_1'\rangle \\ \vdots & \ddots & \vdots \\ \langle y_1,x_n'\rangle & \cdots & \langle y_n,x_n'\rangle \end{vmatrix} = \frac{1}{\|x_1'\|\cdots\|x_n'\|} \begin{vmatrix} \langle x_1',x_1'\rangle & \cdots & \langle x_n',x_1'\rangle \\ \vdots & \ddots & \vdots \\ \langle x_1',x_n'\rangle & \cdots & \langle x_n',x_n'\rangle \end{vmatrix} \\ &= \|x_1'\|\cdots\|x_n'\| = \|x_1,\dots,x_n\|_s. \end{aligned}$$

Thus, $\|x_1,\dots,x_n\|_g \ge \|x_1,\dots,x_n\|_s$. Hence we conclude that $\|x_1,\dots,x_n\|_g = \|x_1,\dots,x_n\|_s$ whenever $\{x_1,\dots,x_n\}$ is linearly independent. If $\{x_1,\dots,x_n\}$ is linearly dependent, then $\|x_1,\dots,x_n\|_g = \|x_1,\dots,x_n\|_s = 0$. □

**Remark 2.10.** *Note that, in an inner product space, we have the well-known Hadamard's inequality [22]*

$$\|x_1,\dots,x_n\|_g = \|x_1,\dots,x_n\|_s \le \|x_1\|\cdots\|x_n\|,$$

*which is better than that in Fact* **??**.

For $X=\ell^p$ $(1\le p<\infty)$, we rewrite the formula in (2.2) as

$$\|x_1,\dots,x_n\|_{g,p} = \sup_{\|y_j\|_p\le 1,\, j=1,\dots,n} \begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n) \end{vmatrix}.$$

Substituting $g(y_j,x_i) = \|y_j\|_p^{2-p}\sum_k |y_{jk}|^{p-1}\mathrm{sgn}(y_{jk})x_{ik}$ and using the properties of determinants, we have

$$\begin{aligned} \|x_1,\dots,x_n\|_{g,p} &= \sup_{\|y_j\|_p\le 1,\, j=1,\dots,n} \begin{vmatrix} \|y_1\|_p^{2-p}\sum\limits_k |y_{1k}|^{p-1}\mathrm{sgn}(y_{1k})x_{1k} & \cdots & \|y_n\|_p^{2-p}\sum\limits_k |y_{nk}|^{p-1}\mathrm{sgn}(y_{nk})x_{1k} \\ \vdots & \ddots & \vdots \\ \|y_1\|_p^{2-p}\sum\limits_k |y_{1k}|^{p-1}\mathrm{sgn}(y_{1k})x_{nk} & \cdots & \|y_n\|_p^{2-p}\sum\limits_k |y_{nk}|^{p-1}\mathrm{sgn}(y_{nk})x_{nk} \end{vmatrix} \\ &= \sup_{\|y_j\|_p\le 1,\, j=1,\dots,n} \prod_{j=1}^n \|y_j\|_p^{2-p} \sum_{k_1}\cdots\sum_{k_n}\prod_{j=1}^n |y_{jk_j}|^{p-1}\mathrm{sgn}\left(y_{jk_j}\right) \begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{vmatrix}. \end{aligned} \tag{2.3}$$

**Corollary 2.11.** *For $p=2$, the three n-norms $\|\cdot,\dots,\cdot\|_2$ in (1.1), $\|\cdot,\dots,\cdot\|_2'$ in (1.2) and $\|\cdot,\dots,\cdot\|_{g,2}$ in (2.3) are identical.*

For $p\ne 2$, we have the following generalization of Theorem 2.4.

**Theorem 2.12.** *For every $x_1,\dots,x_n\in\ell^p$ $(1\le p<\infty)$, we have*

$$(n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p \le \|x_1,\dots,x_n\|_{g,p} \le \|x_1,\dots,x_n\|_p' \le (n!)^{\frac{1}{p}}\|x_1,\dots,x_n\|_p.$$

*Proof.* For each $j=1,\dots,n$, let $y_j \in \ell^p$ with $\|y_j\|_p \le 1$. Then take $u_j=(u_{jk})$ with $u_{jk}=\|y_j\|_p^{2-p}|y_{jk}|^{p-1}\operatorname{sgn}(y_{jk})$. We observe that $u_j \in \ell^{p'}$ with $\|u_j\|_{p'}=\|y_j\|_p \le 1$. As a consequence, we have

$$\|x_1,\dots,x_n\|_{g,p} \le \|x_1,\dots,x_n\|_p'.$$

By using Theorem 1.2, we obtain

$$\|x_1,\dots,x_n\|_{g,p} \le \|x_1,\dots,x_n\|_p' \le (n!)^{\frac{1}{p}}\|x_1,\dots,x_n\|_p.$$

Conversely, suppose that $\{x_1,\dots,x_n\}$ is a linearly independent set. Using $x_1^*=x_1$ and so forth as in (2.1), we obtain the left $g$ -orthogonal set $\{x_1^*,\dots,x_n^*\}$. Then, it follows from Theorem 2.2 that

$$(n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p \le V(x_1,\dots,x_n)=\|x_1^*\|_p\cdots\|x_n^*\|_p.$$

For $j=1,\dots,n$, let $y_j=\frac{x_j^*}{\|x_j^*\|_p}$, so that $\|y_j\|_p=1$. Next, using the properties of matrix determinants and the semi-inner product $g$, we have

$$\begin{aligned}
\begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n) \end{vmatrix} &= \begin{vmatrix} \frac{1}{\|x_1^*\|_p} g(x_1^*,x_1^*) & \cdots & \frac{1}{\|x_n^*\|_p} g(x_n^*,x_1^*) \\ \vdots & \ddots & \vdots \\ \frac{1}{\|x_1^*\|_p} g(x_1^*,x_n^*) & \cdots & \frac{1}{\|x_n^*\|_p} g(x_n^*,x_n^*) \end{vmatrix} \\
&= \|x_1^*\|_p\cdots\|x_n^*\|_p = V(x_1,\dots,x_n) \\
&\ge (n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p,
\end{aligned}$$

whence $\|x_1,\dots,x_n\|_{g,p} \ge (n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p$. Combining with the previous inequalities, we obtain

$$(n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p \le \|x_1,\dots,x_n\|_{g,p} \le \|x_1,\dots,x_n\|_p' \le (n!)^{\frac{1}{p}}\|x_1,\dots,x_n\|_p.$$

If $\{x_1,\dots,x_n\}$ is linearly dependent, then all the $n$-norms vanish and so we have the equalities. □

**Corollary 2.13.** *For $1 \le p < \infty$, the three n-norms $\|\cdot,\dots,\cdot\|_{g,p}$, $\|\cdot,\dots,\cdot\|_p'$ and $\|\cdot,\dots,\cdot\|_p$ are equivalent.*

Knowing that the space $(\ell^p,\|\cdot,\dots,\cdot\|_p)$ is an $n$-Banach space in [16], we have a generalization of Corollary 2.6 as follows.

**Corollary 2.14.** *For $1 \le p < \infty$, the space $(\ell^p,\|\cdot,\dots,\cdot\|_{g,p})$ is an n-Banach space.*

## 3. Concluding remarks

In this paper, a new $n$-norm is defined using a semi-inner product $g$ on $\ell^p$ for $1 \le p < \infty$. Accordingly, on the space $\ell^p$ $(1 \le p < \infty)$, we have three different $n$-norms, namely Gähler's $n$-norm $\|\cdot,\dots,\cdot\|_p'$ defined in [8]-[10], Gunawan's $n$-norm $\|\cdot,\dots,\cdot\|_p$ defined in [16], and $\|\cdot,\dots,\cdot\|_{g,p}$ defined here in (2.3). In Corollary 2.13, we have just seen that the three $n$-norms on $\ell^p$ are equivalent. As expected, the case where $p=2$ is special. Here, the three $n$-norms on $\ell^2$ are identical.
In addition to the above three $n$-norms, we also have a formula for another $n$-norm using the semi-inner product $g$ on $\ell^p$ $(1 \le p < \infty)$, namely

$$\|x_1,\dots,x_n\|_{g,p}^{\circ} = \sup_{\|y_1,\dots,y_n\|_p \le 1} \begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n) \end{vmatrix}.$$

Since $g(y_j,x_i)=\|y_j\|_p^{2-p}\sum\limits_k |y_{jk}|^{p-1}\operatorname{sgn}(y_{jk})x_{ik}$, we obtain

$$\begin{aligned}
\|x_1,\dots,x_n\|_{g,p}^{\circ} &= \Bigg[\sup_{\|y_1,\dots,y_n\|_p \le 1} \frac{1}{n!}\prod_{j=1}^{n}\|y_j\|_p^{2-p} \times \\
&\quad \times \sum_{k_1}\cdots\sum_{k_n} \begin{vmatrix} |y_{1k_1}|^{p-1}\operatorname{sgn}(y_{1k_1}) & \cdots & |y_{1k_n}|^{p-1}\operatorname{sgn}(y_{1k_n}) \\ \vdots & \ddots & \vdots \\ |y_{nk_1}|^{p-1}\operatorname{sgn}(y_{nk_1}) & \cdots & |y_{nk_n}|^{p-1}\operatorname{sgn}(y_{nk_n}) \end{vmatrix} \begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{vmatrix}\Bigg].
\end{aligned}$$

Note that, for $p=2$, we have $\|x_1,\dots,x_n\|_{g,2}=\|x_1,\dots,x_n\|_{g,2}^{\circ}$. For other values of $p$, we can show that

$$\|x_1,\dots,x_n\|_{g,p} \le (n!)^{2-\frac{1}{p}}\|x_1,\dots,x_n\|_{g,p}^{\circ}.$$

Indeed, assuming that $x_1,\dots,x_n$ are linearly independent, let $x_1^*,\dots,x_n^*$ be the vectors obtained from $x_1,\dots,x_n$ through the process in (2.1). By taking $y_j = \frac{x_j^*}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}}$ $(j=1,\dots,n)$, we obtain $\|y_1,\dots,y_n\|_p = 1$. Next, using the properties of matrix determinants and the semi-inner product $g$, we have

$$\begin{aligned}\begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n)\end{vmatrix} &= \begin{vmatrix} \frac{1}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}} g(x_1^*,x_1^*) & \cdots & \frac{1}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}} g(x_n^*,x_1^*) \\ \vdots & \ddots & \vdots \\ \frac{1}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}} g(x_1^*,x_n^*) & \cdots & \frac{1}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}} g(x_n^*,x_n^*)\end{vmatrix} \\ &= \frac{\|x_1^*\|_p^2 \cdots \|x_n^*\|_p^2}{\|x_1^*,\dots,x_n^*\|_p}.\end{aligned}$$

Since $\|x_1,\dots,x_n\|_p \le (n!)^{1-\frac{1}{p}} \|x_1^*\|_p \cdots \|x_n^*\|_p$ by Theorem 2.2 and $\|x_1^*,\dots,x_n^*\|_p = \|x_1,\dots,x_n\|_p$, we obtain

$$\|x_1,\dots,x_n\|_{g,p}^{\circ} \ge (n!)^{\frac{2}{p}-2} \|x_1,\dots,x_n\|_p .$$

Moreover, using Theorem 2.12, we have

$$\|x_1,\dots,x_n\|_{g,p} \le (n!)^{2-\frac{1}{p}} \|x_1,\dots,x_n\|_{g,p}^{\circ} .$$

It follows from this inequality that the convergence of a sequence in $\|\cdot,\dots,\cdot\|_{g,p}^{\circ}$ implies the convergence in $\|\cdot,\dots,\cdot\|_{g,p}$, and hence also in $\|\cdot,\dots,\cdot\|_p$. Unfortunately, up to now, we do not know if the converse is true.

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# Lyapunov Exponents of One Dimensional Chaotic Dynamical Systems via a General Piecewise Spline Maximum Entropy Method

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**Abstract**

In this paper, we study the computation of Lyapunov exponents for deterministic dynamical systems via a general piecewise spline maximum entropy method. We present a comparison of computations of Lyapunov exponents between a piecewise linear, a piecewise quadratic and a piecewise cubic maximum entropy methods. In order to compute Lyapunov exponents for deterministic maps, we also compute density functions of their invariant measures via piecewise spline maximum entropy method.

## 1. Introduction

In a chaotic dynamical system, inaccuracies in specifying the initial state of the system are rapidly amplified in time and therefore, it is impossible to predict the long term system state. If nearby trajectories of a dynamical system diverges exponentially then the dynamical system possesses chaotic behaviour [1]-[3]. The rate of increase of perturbations of initial conditions is described by the Lyapunov exponent [4]. The Lyapunov exponent of a dynamical system classifies the dynamics of the system. A dynamical system with positive Lyapunov exponent exhibits chaotic nature of the system. Let $\tau : [0,1] \to [0,1]$ be a measure preserving deterministic dynamical system (map) and $\mu$ be a $\tau$-invariant measure on the probability space $[0,1]$. For any $x \in [0,1]$, the Lyapunov exponent $h(x)$ of the trajectory of $x$ is defined as (see, for example, [5])

$$\begin{aligned} h(x) &= \lim_{n\to\infty} \ln(|((\tau^n(x))'|)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1}{n}\Big(\ln|\tau'(x)| + \ln|\tau'(\tau(x)| + \cdots + \ln|\tau'(\tau^{n-1}(x)|\Big) \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \ln|\tau'(\tau^i(x))| \end{aligned} \tag{1.1}$$

provided the limit exists. If the function $\ln(\tau'(x))$ is integrable and $\tau$ is ergodic, then the Birkhoff Ergodic Theorem [6] gaurantees that for almost all $x \in [0,1]$ the above limit exists, it is a constant (say, $l$) and if $f$ is the density of the invariant measure $d\mu(x) = f(x)d\lambda(x)$, then the Lyapunov exponent

$$l = \int_0^1 \ln|\tau'(x)|d\mu(x) = \int_0^1 \ln|\tau'(x)|f(x)d\lambda(x), \tag{1.2}$$

where $\lambda$ is the underlying Lebesgue measure on $[0,1]$. Therefore, if the invariant measure $\mu$ or the density function $f$ is known, then one can calculate the Lyapunov exponent of $\tau$ in (1.1). Unfortunately, except for some cases, the analytical formula of the invariant measure

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$\mu$ or the density function $f$ for most of the deterministic maps is not known. Therefore, the computation of Lyapunov exponent havily rely on numerical approximation of invariant measures or density functions of the corresponding deterministic maps. Computations of Lyapunov exponents for deterministic dynamical systems are well studied by many researchers. A spatial average estimation method for Lyapunov exponents of deterministic maps is presented in [7] by G. Froyland. A numerical scheme is described in [8] for the $n$ Lyapunov exponents of an $n$-dimensional unknown dynamical system. In [4], A. Boyarsky presented a matrix method for the approximation of Lyapunov exponents and invariant measures. Lyapunov exponents from observed time series is studied in [9]. For Jacobian-based estimates of Lyapunov exponents from data, convergence rates and data requirements are studied in [10].

In the context of information theory, the principle of maximum entropy [11] was introduced by E. T. Jaynes [12] in 1957. Since then, mathematicians, physicists and engineers have widely used the maximum entropy method in many different directions for solving problems in mathematics, mathematical physics and other related branches of Science and Engineering. A maximum entropy method was described by Lawrence R. Mead and N. Papanicolaou in [11] for solving moment problems. For finite approximation of the Frobenius-Perron operator of deterministic dynamical systems the maximum entropy methods were described by many authors [13]-[18] and [19]. In [20], C. Bose and R. Murray presented dynamical conditions for convergence of a maximum entropy method for Frobenius-Perron operator equations. For approximation of invariant measures for position dependent random maps we have described maximum entropy methods in [21, 22]. An iterative maximum entropy method is presented in [23] for Lyapunov exponents and invariant densities for deterministic chaotic maps. In this paper, we compute Lyapunov exponents via a general piecewise spline maximum entropy method. Moreover, we compare our results between piecewise linear, piecewise quadratic and piecewise cubic maximum entropy methods for the computation of Lyapunov exponents.

In Section 2, we present a general piecewise spline maximum entropy method for approximation of invariant measures of deterministic dynamical systems. Moreover, we present convergence analysis of maximum entropy method. In Section 3, we present calculations of Lyapunov exponents of deterministic dynamical systems using general piecewise spline maximum entropy method. We present two numerical examples with a comparison between piecewise linear, piecewise quadratic and piecewise cubic maximum entropy method.

## 2. A general piecewise spline maximum entropy optimization method for invariant measures of deterministic chaotic dynamical systems

Let $(I=[0,1],\mathscr{B},\lambda)$ be a measure space, where $\mathscr{B}$ is a $\sigma$-algebra on $I=[0,1]$ and $\lambda$ is the Lebesgue measure on $\mathscr{B}$. Let $\tau: I\to I$ be a deterministic map such that $\tau$ has a unique absolutely continuous invariant measure $\mu^*$ with density $f^*$. Using (1.2) one can find the actual Lyapunov exponent $L$ for $\tau$. In this section, first we revisit a general piecewise spline maximum entropy optimization method for the approximation $f_n$ of the density function $f^*$. In the next section, using the approximate density $f_n$ we compute an approximate Lyapunov exponent $l_n$. We also present the convergence analysis of the general piecewise maximum entropy method. We closely follow [6] and [13].

Let

$$\cdots < x_{-2} < x_{-1} < x_0 = 0 < x_1 < x_2 < \dots$$

be an infinite set of nodes on the real line. The B-splines of degree $k$ are defined recursively as follows:

$$B_i^0(x)=\begin{cases} 1, & x\in[x_i,x_{i+1}),\\ 0, & x\notin[x_i,x_{i+1}), \end{cases} \quad ,i\in\{\dots,-2,-1,0,1,2,\dots\}$$

and

$$B_i^k(x)=\frac{x-x_i}{x_{i+k}-x_i}B_i^{k-1}(x)+\frac{x_{i+k+1}-x}{x_{i+k+1}-x_{i+1}}B_{i+1}^{k-1}(x), i\in\{\dots,-2,-1,0,1,2,\dots\}.$$

Each $B_i^k(x)$ is a piecewise polynomial of degree $k$ and $B_i^k(x)$ are called $B$-splines of degree $k$. The family $\{B_i^k\}$ of $B$-splines satisfies the following properties (see Proposition 1 in [13]):

**Properties of $B$-splines:**

1. If $x\notin[x_i,x_{i+k+1})$, then $B_i^k(x)=0$;
2. If $x\in(x_i,x_{i+k+1})$, then $B_i^k(x)>0$;
3. $\sum_i B_i^k(x)=1$ for all $x$;
4. For fixed $k$, the set $\{B_i^k|_{[x_o,x_n]}\}: -k\le i\le n-1\}$ of functions constitute a basis for the space $\Delta_n^k$ consisting of all functions in $C^{k-1}[x_0,x_n]$ which are piecewise polynomials of degree $\le k$ on the $n$ subintervals $[x_0,x_1],[x_1,x_2],\dots,[x_{n-1},x_n]$.

Let $\tau: I\to I$ be a deterministic map such that $\tau$ has a unique absolutely continuous invariant measure $\mu^*$ with density $f^*$. A particular choice of $\tau$ is a Lasota – Yorke map described in [24]. Note that the invariant density $f^*$ of the unique acim $\mu^*$ is the fixed point of the Frobenius-Perron operator $P_T$. In the following, we describe a general spline maximum entropy approximation scheme for $f^*$.

Let $\mathscr{D}$ be the set of all densities, that is,

$$\mathscr{D}=\{f\in L^1(0,1) \text{ such that } \mathrm{f}\geq 0 \text{ and } \parallel \mathrm{f}\parallel_1=\int_0^1 \mathrm{f(x)d\lambda(x)}=1\}.$$

The Boltzmann entropy [6] of $f\geq 0$ is defined by

$$H(f)=-\int_I f(x)\log f(x)d\lambda(x). \tag{2.1}$$

For properties of H see [6]. Using the Gibbs inequality

$$u-u\log u\leq v-u\log v, u,v\geq 0,$$

it can be shown that

$$\int_I f(x)\log f(x)d\lambda(x)\geq \int_I f(x)\log g(x)d\lambda(x)\ \forall f,g\in\mathscr{D}. \tag{2.2}$$

The above inequality in (2.2) leads to the following optimization problem [14]:

$$\max H(f) \text{ such that } \mathrm{f}\in\mathscr{D} \text{ and } \int_{\mathrm{I}}\mathrm{f(x)g_n(x)d\lambda(x)=m_n,\ 1\leq n\leq N}, \tag{2.3}$$

where $m_1,m_2,\cdots,m_N$ are given moments of the unknown density with respect to the moment function $\{g_1,g_2,...,g_N\}\subset L^\infty(I)$, respectively.

**Proposition 2.1.** *[6] Suppose that $a_1,a_2,\cdots,a_N$ are real numbers such that the function*

$$f_N(x)=\frac{e^{\sum_{n=1}^N a_n g_n(x)}}{\int_0^1 e^{\sum_{n=1}^N a_n g_n(x)}d\lambda(x)}$$

*satisfies the constraints in (2.3), that is,*

$$\frac{\int_0^1 g_i(x)e^{\sum_{n=1}^N a_n g_n(x)}d\lambda(x)}{\int_0^1 e^{\sum_{n=1}^N a_n g_n(x)}d\lambda(x)}=m_i,\ i=1,2,\ldots,N.$$

*Then $f_N$ is a unique solution of the maximum entropy problem (2.3).*

**Proof:** See [6].

Let $\mathscr{P}^{(n)}=\{I_1,I_2,\ldots,I_n\}$ partition of $[0,1]$ into $n$ equal subintervals, where $I_i=[b_{i-1},b_i]$, $b_i=ih, i=1,2,\ldots,n, b_0=0, b_n=1, h=\frac{1}{n}$. Without loss of generality and for convenience, we consider $n=2^s=l\cdot q$, where $l$ is a positive even integer and $q$ is the number of the sub-intervals of the partition $\mathscr{J}$ on which the maps $\tau$ is piecewise one-to-one and monotonic. Consider $2k$ additional nodes: $b_{-j}=-jh, b_{n+j}=(n+j)h, j=1,2,\ldots,k$. These nodes are needed to express all the involved $B$-splines for the state space $I=[0,1]$. Moreover, for fixed $k$, the set

$$\{B_i^k|_{[0,1]}\}: i=-k,-k+1,\ldots,0,1,2,\ldots,n-1\}$$

of functions constitute a basis for the space $\Delta_n^k$ consisting of all functions in $C^{k-1}[0,1]$ which are piecewise polynomials of degree $\leq k$ on the $n$ subintervals $[b_0,b_1],[b_1,b_2],\ldots,[b_{n-1},b_n]$ of $I=[0,1]$. $\Delta_n^k$ has dimension $n+k$ and $B_k^k, B_{-k+1}^k,\ldots,B_0^k,B_1^k,\ldots,B_{n-1}^k$ are elements of the basis for $\Delta_n^k$.

Let $f^*$ be an unique density function of the acim $\mu^*$ for the map $\tau$. Then the moments of $f^*$ with respect to $B$-spline $B_i^k, i=-k,-k+1,\ldots,0,1,2,\ldots,n-1$ is

$$m_i=\int_0^1 f^*(x)B_i^k(x)d\lambda(x). \tag{2.4}$$

**Proposition 2.2.**

$$\sum_{i=-k}^{n-1} m_i=1.$$

**Proof:**

$$\begin{aligned}\sum_{i=-k}^{n-1} m_i &= \sum_{i=-k}^{n-1}\int_0^1 f^*(x)B_i^k(x)d\lambda(x)=\int_0^1 f^*(x)\sum_{i=-k}^{n-1}B_i^k(x)d\lambda(x)\\ &= \int_0^1 f^*(x)d\lambda(x)=1,\end{aligned}$$

since $\sum_{i=-k}^{n-1}B_i^k(x)=1$, by Property (iii).

**Proposition 2.3.** *Suppose that $m_{-k}, m_{-k+1}, \dots, m_0, m_1, \dots, m_{n-1}$ are moments defined in (2.4) of the probability density function $f^*$ of the map $\tau$. If $a_{-k}, a_{-k+1}, \dots, a_0, a_1, \dots a_{n-1}$ are constants which satisfy*

$$\int_0^1 B_i^k(x) e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} d\lambda(x) = m_i, \; i = -k, -k+1, \dots, 0, 1, \dots, n-1,$$

*then*

$$f_n^k(x) = e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} \tag{2.5}$$

*is a density.*

*Proof.*

$$\begin{aligned}
\int_0^1 f_n^k(x) d\lambda(x) &= \int_0^1 e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} d\lambda(x) \\
&= \int_0^1 e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} \sum_{i=-k}^{n-1} B_i^k(x) d\lambda(x) \\
&= \sum_{i=-k}^{n-1} \int_0^1 B_i^k(x) e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} d\lambda(x) \\
&= \sum_{i=-k}^{n-1} m_i = 1
\end{aligned}$$

If we solve the maximum entropy problem

$$\max H(f) \text{ such that } f \in \mathscr{D} \text{ and } \int_I f(x) B_i^k(x) d\lambda(x) = m_i, \; -k \le i \le n-1,$$

then $f_n^k(x) = e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)}$ is a solution of (2.6) such that there exists constants $a_{-k}, a_{-k+1}, \dots, a_0, a_1, a_2, \dots, a_{n-1}$ satisfying

$$\int_0^1 B_i^k(x) e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} d\lambda(x) = m_i, \; i = -k, -k+1, \dots, 0, 1, \dots, n-1. \tag{2.6}$$

□

Now, using the Birkhoff's Ergodic Theorem (see below), we describe a method for estimating the moments $m_{-k}, m_{-k+1}, \dots, m_0, m_1, \dots, m_{n-1}$ in (2.6) for the unknown invariant density $f^*$ of a map $\tau$:

**Theorem 2.4.** *If $\mu^*$ is $\tau$-invariant, $\mu^*$ is absolutely continuous and unique among absolutely continuous invariant measures, $P_\tau$ satisfies*

$$\| P_\tau^n f \|_{BV} \le A \| f \|_{BV} + B \| f \|_1,$$

*where $A > 0$ and $B > 0$ are constants defined in [24]. Then for $\mu^*$ almost every $x$ with probability 1,*

$$\frac{1}{M} \sum_{i=0}^{M-1} f(\tau^i(x)) \to \mu^*(f),$$

*for any $f \in L^1([0,1])$. Moreover, if $[0,1$ is a probability space (that is, $\mu^*$ is a probability measure on $[0,1]$) and $\tau$ is ergodic, then there exists a function $\hat{f} \in L^1([0,1])$ such that*

$$\lim_{M \to \infty} \frac{1}{M} \sum_{i=0}^{M-1} f(\tau^i(x)) = \hat{f}(x), \forall x \in [0,1] \mu^* - a.e.$$

*and $\hat{f}$ is constant and $\hat{f} = \int_0^1 f(x) d\mu^*(x)$.*

Note that $f^*$ is the density of the map $\tau$ with respect to the acim $\mu^*$. Thus, $d\mu^*(x) = f^*(x) d\lambda(x)$. If we replace $f$ by $B_i^k, i = -k, -k+1, \dots, 0, 1, 2, \dots, n-1$ and $\hat{f}$ by $m_i$ then from Equation (2.7) we obtain

$$m_i = \lim_{M \to \infty} \frac{1}{M} \sum_{j=0}^{M-1} B_i^k(\tau^j(x)), \forall x \in [0,1] \; a.e., i = k, -k+1, \dots, 0, 1, \dots, n-1. \tag{2.7}$$

For large $M$, define

$$\hat{m}_i = \frac{1}{M} \sum_{j=0}^{M-1} B_i(\tau^j(x)), \forall x \in [0,1] \; a.e., i = k, -k+1, \dots, 0, 1, \dots, n-1.$$

Note that the choice of $x$ almost surely doesn't matter asymptotically. Now, we consider the following normalized approximation for moments (for convenience, we denote them by $m_i$.):

$$m_i \approx \frac{\hat{m}_i}{\sum_{i=-k}^{n-1} \hat{m}_i}, i = k, -k+1, \dots, 0, 1, \dots, n-1.$$

Note that if $a_{-k}, a_{-k+1}, \ldots, a_0, a_2, \ldots, a_{n-1}$ satisfy (2.5) then $a_{-k}, a_{-k+1}, \ldots, a_0, a_2, \ldots, a_{n-1}$ also satisfy

$$\frac{\int_0^1 B_i^k(x) e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} d\lambda(x)}{\int_0^1 e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} d\lambda(x)} = m_i, \; i = -k, -k+1, \ldots, 0, 1, 2, \ldots, n-1.$$

The nonlinear equations in (2.5) form the following system of $n+k$ nonlinear equations:

$$G(a) = 0, \tag{2.8}$$

where $G : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ defined by

$$\begin{aligned} G_i(a_{-k}, a_{-k+1}, \ldots, a_0, a_1, \ldots, a_{n-1}) \quad &= \quad \int_0^1 B_i^k(x) e^{\sum_{i=-k}^{n-1} a_i \phi_i^k(x)} d\lambda(x) - m_i, \\ & \quad\quad i = -k, -k+1, \ldots, 0, 1, \ldots, n-1. \end{aligned}$$

The Jacobian matrix $G' = \left(g'_{i,j}\right)_{-k \le i,j \le n-1}$ of $G$ is defined by

$$g'_{i,j} = \frac{\partial G_i}{\partial a_j} = \int_0^1 B_i^k(x) e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} B_j^k(x) d\lambda(x), \; i, j = -k, -k+1, \ldots, 0, 1, \ldots, n-1.$$

**Proposition 2.5.** *The system (2.5) has a unique solution $a = (a_{-k}, a_{-k+1}, \ldots, a_0, a_1, \ldots, a_{n-1})$.*

*Proof.* Since support of $B_i^k$ and $B_j^k$ are disjoint for $|i-j| > k$, it is clear that Jacobian matrix $G'$ of $G$ is symmetric. Let

$$\beta = (\beta_{-k}, \beta_{-k+1}, \ldots, \beta_0, \beta_1, \ldots, \beta_{n-1}) \in \mathbb{R}^{n+k}.$$

Then,

$$\begin{aligned} \beta G' \beta^T \quad &= \quad \sum_{i=-k}^{n-1} \sum_{j=-k}^{n-1} \beta_i \beta_j \int_0^1 B_i^k(x) e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} B_j^k(x) d\lambda(x) \\ &= \quad \int_0^1 \sum_{i=-k}^{n-1} \sum_{j=-k}^{n-1} \beta_i B_i^k(x) e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} B_j^k(x) \beta_j d\lambda(x) \\ &= \quad \int_0^1 \left( \sum_{i=-k}^{n-1} \beta_i B_i^k(x) \right) e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} \left( \sum_{j=-k}^{n-1} B_j^k(x) \beta_j \right) d\lambda(x) > 0. \end{aligned}$$

Thus, $G'$ is positive definite. Let

$$g(a_{-k}, a_{-k+1}, \ldots, a_0, a_1, \ldots, a_{n-1}) = \int_0^1 e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)} d\lambda(x) - \sum_{i=-k}^{n-1} a_i m_i.$$

Now, consider the following global minimization problem:

$$\min_{(a_{-k}, a_{-k+1}, \ldots, a_0, a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n+k}} g(a_{-k}, a_{-k+1}, \ldots, a_0, a_1, \ldots, a_{n-1}) \tag{2.9}$$

It can be shown that the system (2.5) of equations is the gradient equation of the global minimization problem (2.9). Since $G'$ is positive definite, the objective function $g$ is strictly convex on $\mathbb{R}^{n+k}$. Moreover, $g(a_{-k}, a_{-k+1}, \ldots, a_0, a_1, \ldots, a_{n-1})$ is coercive on its domain. Thus, the above global unconditioned convex programming problem (2.9) has a unique solution and therefore, the system (2.5) has a unique solution. □

**Algorithm**: Choose $n$ and $k$. Calculate the moments $m_i, i = -k, -k+1, \ldots, 0, 1 \ldots, n-1$ using (2.8) and use these moments $m_i, i = -k, -k+1, \ldots, 0, 1 \ldots, n-1$ to find $a = (a_{-k}, a_{-k+1}, \ldots, a_0, a_1, \ldots, a_{n-1})$. Then the solution of the the maximum moment problem (2.6) is

$$f_n^k(x) = e^{\sum_{i=-k}^{n-1} a_i B_i^k(x)}$$

which is an approximation of the density function $f^*$ of the acim $\mu^*$ for the map $\tau$.

For convergence analysis, we also assume that the unique invariant measure $f^*$ of $\tau$ is used in Equation (2.4) for the calculations of moments $m_i, i = -k, -k+1, \ldots, 0, 1, \ldots, n-1$. Our convergence analysis is based on the following general convergence theory for moment problem developed by Borwein and Lewis in [25].

Let $X$ be a locally convex topological vector space with nested sequence of compact subsets $\{G_n\}$. Let $W : X \to [-\infty, \infty)$ be a functional with compact level sets. Let $g_n$ be an optimal solution of $\max\{W(h) : h \in G_n\}$ and $g_\infty$ be the unique optimal solution of the limiting problem $\max\{W(h) : h \in \cap_{n=1}^{\infty} G_n\}$ with $W(g_\infty) > -\infty$. It was proved in [25] that $\lim_{n\to\infty} g_n = g_\infty$ under the topology of $X$ and $\lim_{n\to\infty} W(g_n) = W(g_\infty)$. For our B-spline maximum entropy method of degree $k$, we partition the interval $[0,1]$ in the following special way so that the feasible sets are monotonically decreasing. First we divide $[0,1]$ into $k+1$ equal subintervals. Then we divide one subinterval of the current partition into $k+1$ equal parts at each step in succession. Thus the corresponding $n+k$-dimensional spaces $\Delta_n^k$ of spline functions of degree $k$ are monotonically increasing, which guarantees that the feasible sets of the maximum entropy method are monotonically decreasing. Furthermore, by property of the Boltzmann entropy in the previous section, these feasible sets are weakly compact in $L^1(0,1)$. Since $L^1(0,1)$ is a locally convex topological vector space in the weak topology, the above weak convergence implies the weak convergence of our method. Thus, we have the following theorem:

**Theorem 2.6.** *Let $\tau : I \to I$ be a deterministic dynamical system with the unique fixed point $f^*$ of the Frobenius - Perron operator $P_\tau$ satisfying $H(f^*) > -\infty$ and $f_n$ be the sequence of solutions in (2.10). Then $\lim_{n\to\infty} f_n = f^*$ weakly and $\lim_{n\to\infty} H(f_n) = H(f^*)$.*

Note that a functional $W$ on a normed space $X$ into $[-\infty, \infty)$ is called a **Kadec** if $x_n \to x$ weakly and $W(x_n) \to W(x)$ imply $x_n \to x$ strongly. It can be shown that Boltzmann entropy functional $H$ in (2.1) is Kadec (see [25]). Thus, we have the following strong convergence result.

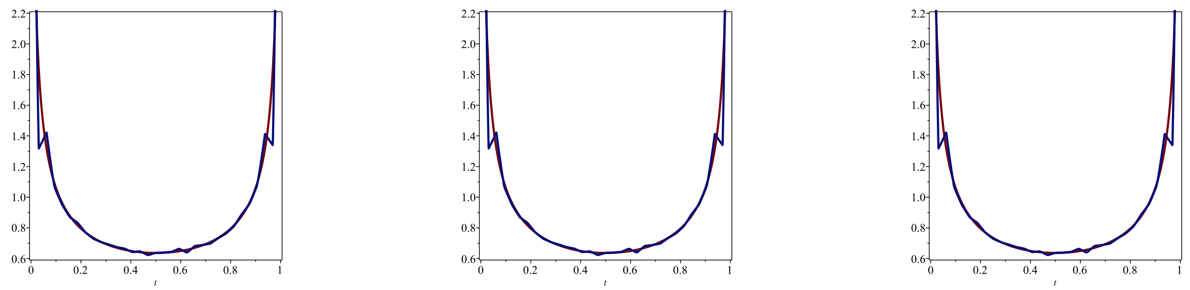
**Theorem 2.7.** *Let $\tau : I \to I$ be a deterministic dynamical system with the unique fixed point $f^*$ of the Frobenius - Perron operator $P_\tau$ satisfying $H(f^*) > -\infty$ and $f_n$ be the sequence of solutions in (2.10). Then $\lim_{n\to\infty} \parallel f_n - f^* \parallel_1 = 0$.*

## 3. Numerical examples for Lyapunov exponents and invariant measures of one dimensional chaotic maps via piecewise spline maximum entropy method

In this section, we give two the results from two numerical examples to illustrate our method. Our first example uses the well-known logistic map for which the density function $f*$ of the acim is known and hence hence the analytical Lyapunov exponent $l$ is also known. The second example uses a piecewise polynomial mapping whose corresponding density function is not known. For both of these examples we apply our piecewise spline maximum entropy method using piecewise linear, piecewise quadratic, and piecewise cubic splines.

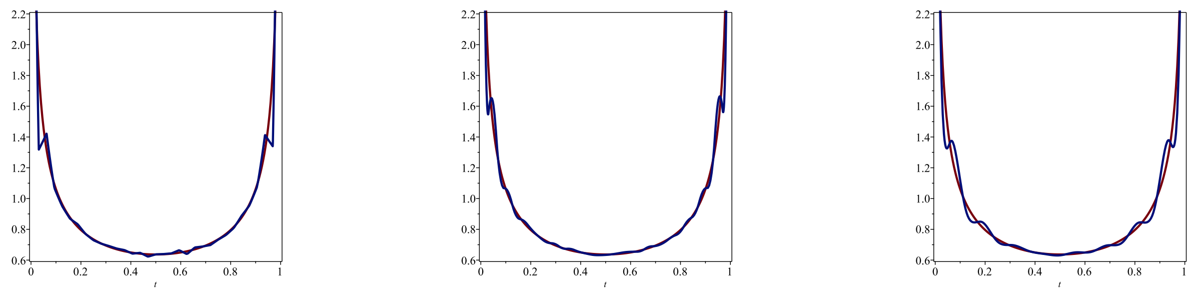
**Example 3.1.** *We consider the well known logistic map $\tau : [0,1] \to [0,1]$ defined by $\tau(x) = 4x(1-x)$. The actual density $f^*$ of $\tau$ is given by $f^*(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$. The logistic map $\tau$ is topologically conjugate to the tent map and it is known that the Lyapunov exponent for the logistic map $\tau$ is $l = \ln 2 = 0.693147$. Now, we apply (a) piecewise linear maximum entropy method; (b) piecewise quadratic maximum entropy method; (c) piecewise cubic maximum entropy method and we compute approximate Lyapunov exponent $l_n$, error $E_n = |l_n - l|$, the approximate density function $f_n$ and the $L^1-$norm $\parallel f_n - f^* \parallel_1$ .*

*In Figure 3.1, using (a) piecewise linear maximum entropy method; (b) piecewise quadratic maximum entropy method and (c) piecewise cubic maximum entropy method, we present the graph of the actual density function $f^*(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ (red) and the graph of the approximate density function $f_n$ (blue). Gauss quadrature method is used for integrations. $500{,}000$ iterations are used for the approximation of moments.*



**(a)** A subfigure **(b)** A subfigure **(c)** A subfigure

**Figure 3.1:** This is lot of figures arranged side by side in matrix form with captions for each and a main caption



**(a)** **(b)** **(c)**

**Figure 3.2:** Approximate density $f_n$ of the density function $f^*$ of invariant measure $\mu^*$ for the map $\tau$ via piecewise spline maximum entropy method: Figure 3.1 (a) approximate density $f_{32}$ (blue) via piecewise linear maximum entropy method and the actual density $f^*$ (red); Figure 3.1 (b) approximate density $f_{32}$ (blue) via piecewise quadratic maximum entropy method and the actual density $f^*$ (red); Figure 3.1 (c) approximate density $f_{16}$ (blue) via piecewise cubic maximum entropy method and the actual density $f^*$ (red).

| $n$ | $l_n$ *(piecewise linear)* | $\lvert l_n - l\rvert$ *(piecewise linear)* | $\parallel f_n - f^* \parallel_1$ *(piecewise linear)* |
|---|---|---|---|
| 4 | 0.68775 | 0.00539 | 0.22098 |
| 8 | 0.69363 | 0.00049 | 0.15821 |
| 16 | 0.69355 | 0.00040 | 0.10494 |
| 32 | 0.69361 | 0.00046 | 0.07745 |

| $n$ | $l_n$ *(piecewise quad)* | $\lvert l_n - l\rvert$ *(piecewise quad)* | $\parallel f_n - f^* \parallel_1$ *(piecewise quad)* |
|---|---|---|---|
| 4 | 0.67348 | 0.01965 | 0.17346 |
| 8 | 0.69092 | 0.00222 | 0.12631 |
| 16 | 0.693443 | 0.0002 | 0.08778 |
| 32 | 0.69352 | 0.00037 | 0.05108 |

| $n$ | $l_n$ *(piecewise cubic)* | *(piecewise cubic)* | $\parallel f_n - f^* \parallel_1$ *(piecewise cubic)* |
|---|---|---|---|
| 4 | 0.70605 | 0.01290 | 0.15001 |
| 8 | 0.69570 | 0.00255 | 0.10621 |
| 16 | 0.69391 | 0.00255 | 0.06343 |
| 32 | 0.69352 | 0.00037 | 0.05108 |

*The density function* $f^*(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$ *of the invariant measure* $\mu^*$ *is known for the logistic map* $\tau(x) = 4x(1-x)$*. Therefore, instead of (2.7) one can use (2.4) for the calculations of moments. In the following tables we present some approximate Lyapunov exponents* $l_n$ *and error* $|l_n - l|$ *using piecewise linear, piecewise quadratic and piecewise cubic maximum entropy method, where we have used (2.4) for moments.*

| *piecewise spline method* | $n$ | $l_n$ | $\lvert l_n - l\rvert$ |
|---|---|---|---|
| *piecewise linear* | 4 | 0.68812 | 0.00503 |
| *piecewise linear* | 32 | 0.69363 | 0.00048 |
| *piecewise quad* | 32 | 0.69349 | 0.00034 |
| *piecewise cubic* | 32 | 0.69352 | 0.00037 |

**Example 3.2.** *We consider the map* $\tau : [0,1] \to [0,1]$ *defined by*

$$\tau(x) = \begin{cases} \frac{3}{2}x^2 + \frac{5}{4}x, & 0 \le x < \frac{1}{2}, \\ \frac{3}{2}(1-x)^2 + \frac{5}{4}(1-x), & \frac{1}{2} \le x \le 1, \end{cases}$$

*The map* $\tau$ *is is a piecewise expanding map. However, analytical density function of* $\tau$ *is not known. We apply (a) piecewise linear maximum entropy method; (b) piecewise quadratic maximum entropy method; (c) piecewise cubic maximum entropy method and we compute approximate Lyapunov exponent* $l_n$.

| $n$ | $l_n$ *(piecewise linear)* | $l_n$ *(piecewise quadratic)* | $l_n$ *(piecewise cubic)* |
|---|---|---|---|
| 4 | 0.63118 | 0.63104 | 0.63132 |
| 8 | 0.63105 | 0.63104 | 0.63114 |
| 16 | 0.63105 | 0.63104 | 0.63105 |

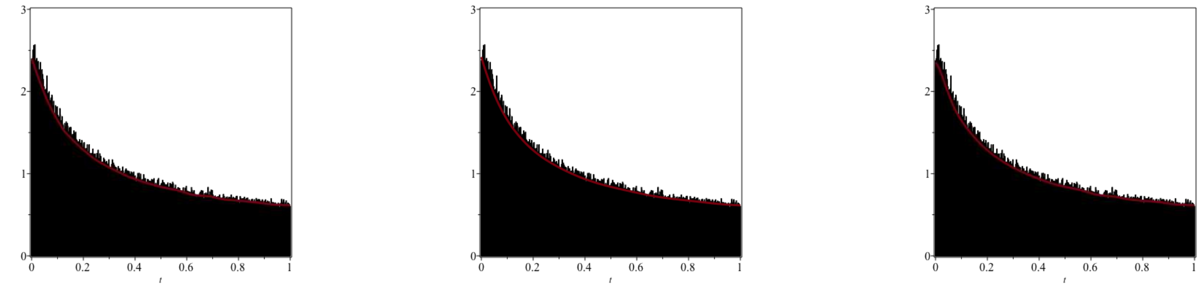
*In Figure 3.2, using (a) piecewise linear maximum entropy method; (b) piecewise quadratic maximum entropy method and (c) piecewise cubic maximum entropy method, we present the graph of the histogram (with* 500,000 *iterations) of an approximate density function (black) and the graph of the approximate density function* $f_n$ *(red). Gauss quadrature method is used for integrations.*

**Example 3.3.** *We consider the map* $\tau : [0,1] \to [0,1]$ *defined by*

$$\tau(x) = \begin{cases} \frac{2x}{1-x}, & 0 \le x < \frac{1}{3}, \\ \frac{1-x}{2x}, & \frac{1}{3} \le x \le 1, \end{cases}$$

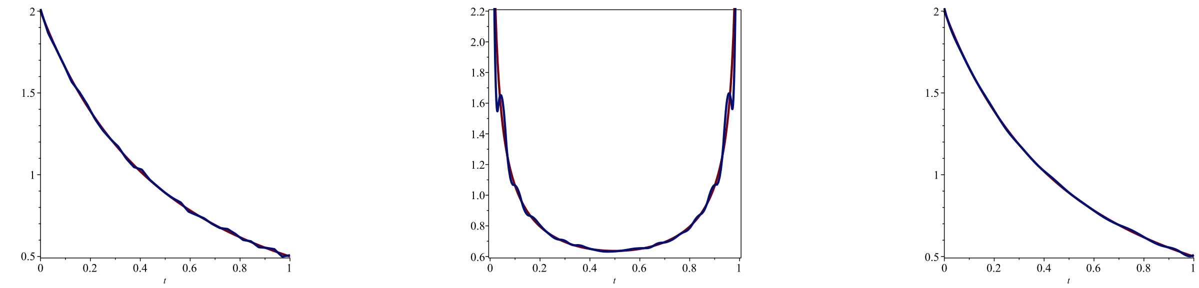
*The actual density* $f^*$ *of* $\tau$ *is given by* $f^*(x) = \dfrac{2}{(1-x)^2}$*. The Lyapunov exponent for the map* $\tau$ *is* $l \approx= 0.693147$*. Now, we apply (a) piecewise linear maximum entropy method; (b) piecewise quadratic maximum entropy method; (c) piecewise cubic maximum entropy method and we compute approximate Lyapunov exponent* $l_n$*, error* $E_n = |l_n - l|$.

*In Figure 3.3, using (a) piecewise linear maximum entropy method; (b) piecewise quadratic maximum entropy method and (c) piecewise cubic maximum entropy method, we present the graph of the actual density function* $f^*(x) = \dfrac{2}{(1-x)^2}$*. (red) and the graph of the approximate density function* $f_n$ *(blue). Gauss quadrature method is used for integrations.* 500,000 *iterations are used for the approximation of moments.*



**(a)** **(b)** **(c)**

**Figure 3.3:** Histogram and approximate density via piecewise spline maximum entropy method: Figure 3.2 (a) the histogram of the density function of the map $\tau$ with 500,000 points on the trajectory of the map $\tau$ with 1000 subintervals for $[0,1]$. and the graph of the approximate density function $f_{16}$ via piecewise linear maximum entropy method; Figure 3.2 (b) the same histogram and the graph of the approximate density function $f_8$ via piecewise quadratic maximum entropy method; Figure 3.2 (c) the same histogram and the graph of the approximate density function $f_{16}$ via piecewise cubic maximum entropy method;



**(a)** **(b)** **(c)**

**Figure 3.4:** Approximate density $f_n$ of the density function $f^*$ of invariant measure $\mu^*$ for the map $\tau$ via piecewise spline maximum entropy method: Figure 3.3 (a) approximate density $f_{32}$ (blue) via piecewise linear maximum entropy method and the actual density $f^*$ (red); Figure 3.2 (b) approximate density $f_{32}$ (blue) via piecewise quadratic maximum entropy method and the actual density $f^*$ (red); Figure 3.3 (c) approximate density $f_{16}$ (blue) via piecewise cubic maximum entropy method and the actual density $f^*$ (red).

| $n$ | $l_n$ *(piecewise linear)* | $\lvert l_n - l\rvert$ *(piecewise linear)* |
|---|---|---|
| 4 | 0.69323 | 0.00009 |
| 8 | 0.69316 | 0.00001 |
| 16 | 0.69316 | 0.00001 |
| 32 | 0.69314 | 0.0000008 |

.

| $n$ | $l_n$ *(piecewise quad)* | $\lvert l_n - l\rvert$ *(piecewise quad)* |
|---|---|---|
| 4 | 0.69323 | 0.00008 |
| 8 | 0.69314 | 0.00005 |
| 16 | 0.69313 | 0.00001 |
| 32 | 0.69314 | 0.000001 |

.

| $n$ | $l_n$ *(piecewise cubic)* | *(piecewise cubic)* |
|---|---|---|
| 4 | 0.69300 | 0.00014 |
| 8 | 0.69325 | 0.00010 |
| 16 | 0.69314 | 0.00003 |

.

*The density function* $f^*(x) = \dfrac{2}{(1-x)^2}$ *of the invariant measure* $\mu^*$ *is known for the map* $\tau$*. Therefore, instead of (2.7) one can use (2.4) for the calculations of moments. In the following tables we present some approximate Lyapunov exponents* $l_n$ *and error* $|l_n - l|$ *using piecewise linear and piecewise quadratic maximum entropy method, where we have used (2.4) for moments.*

| *piecewise spline method* | $n$ | $l_n$ | $\|l_n - l\|$ |
|---|---|---|---|
| *piecewise linear* | 4 | 0.69324 | 0.00008 |
| *piecewise linear* | 16 | 0.69314 | 0.000005 |
| *piecewise quad* | 16 | 0.69314 | 0.000005 |

## 4. Conclusion

In this paper, we study numerical computations of Lyapunov exponents for deterministic chaotic dynamical systems in one dimension. First, we discuss the fact that the Lyapunov exponent is one of the key tools for determining whether a deterministic dynamical system is chaotic or not. Moreover, we show that the computation of Lyapunov exponents of dynamical systems depends on the computation of invariant measures of dynamical systems. Then, we study a general piecewise spline maximum entropy method for the computation of Lyapunov exponents and invariant measures for deterministic dynamical systems. We present a proof of convergence of the general piecewise spline maximum entropy method. The general piecewise spline maximum entropy method includes piecewise linear, piecewise quadratic, piecewise cubic and higher order maximum entropy methods. Finally, we present two examples where we compute Lyapunov exponents of dynamical systems via piecewise linear, piecewise quadratic and piecewise cubic maximum entropy methods. Moreover, we compute invariant measures of the dynamical systems. In the first example we present errors between the numerical results and analytical results (both for Lyapunov exponent and invariant measures). The numerical examples show that the piecewise spline maximum entropy method is a useful method for the computation of Lyapunov exponents and invariant measures for deterministic dynamical systems. In future we plan to study the speed of convergence of the piecewise spline maximum entropy method for the computation of Lyapunov exponents and invariant measures.

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# Symmetry Analysis and Conservation Laws of the Boundary Value Problems for Time-Fractional Generalized Burgers' Differential Equation

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**Abstract**

Many physical phenomena in nature can be described or modeled via a differential equation or a system of differential equations. In this work, we restrict our attention to research a solution of fractional nonlinear generalized Burgers' differential equations. Thereby we find some exact solutions for the nonlinear generalized Burgers' differential equation with a fractional derivative, which has domain as $\mathbb{R}^2 \times \mathbb{R}^+$. Here we use the Lie groups method. After applying the Lie groups to the boundary value problem we get the partial differential equations on the domain $\mathbb{R}^2$ with reduced boundary and initial conditions. Also, we find conservation laws for the nonlinear generalized Burgers' differential equation.

## 1. Introduction

The research of exact solutions plays an important role in the study of nonlinear systems. Many methods as the inverse scattering method [1], Hirota bilinear method [2], Lie symmetry analysis [3, 4], CK (Clarkson-Kruskal) method [5, 6], etc. have been developed to find these exact physically significant solutions of the partial differential equation, although this is rather difficult. Our work in this area is to use Lie transformation methods and its analysis to search exact solutions to fractional nonlinear partial differential equations. It is known that the Lie group method is a powerful and direct approach to the construction of exact solutions of nonlinear differential equations. Essentially, the symmetry analysis is aimed at using the symmetry of the equation. The process thus obtained reduces the complexity of the given equation. Even though physical phenomena are mostly based on searching the solution of the underlying nonlinear model equations, it is too difficult to find a general solution of the fractional nonlinear partial differential equation. There is no existing general theory for nonlinear partial differential equations. While there is no existing general theory for nonlinear partial differential equations, many special cases have yielded to appropriate changes of variable [7–11]. In fact, transformations are perhaps the most powerful tool currently available in this area [12–14]. Ivanova, Sophocleous and Tracin in [15] investigated the Lie symmetry analysis of (2+1) - dimensional variable coefficient Burgers differential equation of the form

$$u_t = A(t)u_{xx} + B(t)u_{yy} + uu_x.$$

They obtained the symmetries, according them conservation laws and some analytical solutions for above equation. Later Abd-el-Malek and Amin in [16] studied the symmetry analysis of the generalized (1+1)-dimensional Burgers differential equation in the form

$$u_t + \alpha(u^n)_x = \beta g(t)(u^n)_{xx},$$

with boundary and initial conditions $u(0,x) \longrightarrow \infty$, for $x > 0$, $u(t,0) = \gamma r(t)$, for $t > 0, \gamma \neq 0$, and $\lim\limits_{x\to\infty} u(t,x) \longrightarrow \infty$, for $t > 0$.
Some recent studies of Burgers differential equation the reader can see in [17, 18].
In this research, we show the applying of Lie group analysis to study (2+1)-dimensional time-fractional generalized Burgers' differential equation with boundary and initially conditions:

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$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}}+P\nabla(u^{n})=Rg(t)\triangle(u^{n}), \tag{1.1}$$

$$\begin{array}{ll} u(0,x,y)\longrightarrow\infty, & (x,y)\in\mathbb{R}^{2} \\ u(t,0,0)=\Phi(t), & t\in[0,\infty), \\ \lim\limits_{(x,y)\to(\infty,\infty)} u(t,x,y)\longrightarrow\infty, & t\in[0,\infty). \end{array} \tag{1.2}$$

Here$(x,y)\in\mathbb{R}^{2}$, $t\in\mathbb{R}^{+}$, $0<\alpha\leq 1$, $n>1$, $P,R\neq 0$ and $\frac{\partial^{\alpha}u}{\partial t^{\alpha}}$ is a fractional derivative which is considered in the Riemann–Liouville terms as [19]

$$\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}}=\begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{0}^{t}\frac{f(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, & if \quad \alpha\notin\mathbb{N}, \quad n-1<\alpha<n, \quad n\in\mathbb{N}, \\ \frac{d^{n}}{dt^{n}}f(t), & if \quad \alpha,n\in\mathbb{N}. \end{cases}$$

Moreover we investigate the conservation laws for above equation by using Ibragimov's theorem for fractional derivative equations [7,20]. The Lie group or Lie symmetry analysis allows us to see that the underlying symmetry algebra of the equation reduce the dimension, it is since each of the time-fractional equations is invariant under time translation symmetry. So, by using the Lie symmetry, we show that the fractional partial differential equation with the domain $\mathbb{R}^{2}\times\mathbb{R}^{+}$ can be transformed into a nonlinear fractional partial differential equation with the domain $\mathbb{R}^{2}$.

## 2. Symmetry analysis for time fractional partial differential equation

Consider a time-fractional partial differential equations with three independent variables $x>0$, $y>0$, and $t>0$ as following:

$$F(x,y,t,u,\partial_{t}^{\alpha}u,u_{x},u_{y},u_{xx},u_{yy})=0, \quad 0<\alpha\leq 1, \tag{2.1}$$

where $\partial_{t}^{\alpha}u$ is Riemann–Liouville fractional derivative of $u$.
A one parameter Lie symmetry transformations acting on a space of three independent variables $(t,x,y)$ and depended variable $u$ are determined as

$$\begin{aligned} \bar{t}&=t+\varepsilon\tau(t,x,y,u)+O(\varepsilon^{2}), \\ \bar{x}&=x+\varepsilon\xi_{1}(t,x,y,u)+O(\varepsilon^{2}), \\ \bar{y}&=y+\varepsilon\xi_{2}(t,x,y,u)+O(\varepsilon^{2}), \\ \bar{u}&=u+\varepsilon\eta(t,x,y,u)+O(\varepsilon^{2}), \end{aligned} \tag{2.2}$$

where $\varepsilon>0$ is an infinitesimal group parameter. The infinitesimal generator associated with the above transformations can be written as:

$$X=\xi_{1}(t,x,y,u)\frac{\partial}{\partial x}+\xi_{2}(t,x,y,u)\frac{\partial}{\partial y}+\tau(t,x,y,u)\frac{\partial}{\partial t}+\eta(t,x,y,u)\frac{\partial}{\partial u}$$

with $\xi_{1}=\frac{d\bar{x}}{d\varepsilon}\mid_{\varepsilon=0}$, $\xi_{2}=\frac{d\bar{y}}{d\varepsilon}\mid_{\varepsilon=0}$, $\tau=\frac{d\bar{t}}{d\varepsilon}\mid_{\varepsilon=0}$ and $\eta=\frac{d\bar{u}}{d\varepsilon}\mid_{\varepsilon=0}$. According to the infinitesimal invariant criterion (2.2), prolongation $pr^{(\alpha,2)}X$ to equation (2.1) has the form

$$pr^{(\alpha,2)}X(E)\mid_{E=0}=0, \qquad E=F(t,x,y,u,\partial_{t}^{\alpha}u,u_{x},u_{xx})=0,$$

here the operator $pr^{(\alpha,2)}X$ takes the following form

$$pr^{(\alpha,2)}X=X+\eta_{\alpha}^{t}\partial_{\partial_{t}^{\alpha}u}+\eta_{1}^{x}\partial_{u_{x}}+\eta_{2}^{x}\partial_{u_{xx}}+\eta_{1}^{y}\partial_{u_{y}}+\eta_{2}^{y}\partial_{u_{yy}},$$

where

$$\eta_{\alpha}^{t}=D_{t}^{\alpha}(\eta)+\xi_{1}D_{t}^{\alpha}(u_{x})-D_{t}^{\alpha}(\xi_{1}u_{x})+\xi_{2}D_{t}^{\alpha}(u_{y})-D_{t}^{\alpha}(\xi_{2}u_{y})+D_{t}^{\alpha}(uD_{t}(\tau))-D_{t}^{\alpha+1}(\tau u)+\tau D_{t}^{\alpha+1}(u),$$

$$\begin{aligned} \eta_{1}^{x}&=D_{x}\eta-u_{x}D_{x}\xi_{1}-u_{y}D_{x}\xi_{2}-u_{t}D_{x}\tau, \\ \eta_{2}^{x}&=D_{x}\eta_{1}^{x}-u_{xx}D_{x}\xi_{1}-u_{xy}D_{x}\xi_{2}-u_{xt}D_{x}\tau, \\ \eta_{1}^{y}&=D_{y}\eta-u_{x}D_{y}\xi_{1}-u_{y}D_{y}\xi_{2}-u_{t}D_{y}\tau, \\ \eta_{2}^{y}&=D_{y}\eta_{1}^{y}-u_{yx}D_{y}\xi_{1}-u_{yy}D_{y}\xi_{2}-u_{yt}D_{y}\tau, \end{aligned} \tag{2.3}$$

with $D_{i}$ is the total derivative

$$D_{i}=\partial_{i}+u_{i}\partial_{u}+u_{it}\partial_{u_{t}}+u_{jt}\partial_{u_{t}}+u_{ii}\partial_{u_{i}}+u_{jj}\partial_{u_{j}}+\ldots.$$

and $D_{t}^{\alpha}$ is a fractional derivative operator with respect to $t$.

The expression for $\eta_1^x$, $\eta_1^y$, $\eta_2^x$, and $\eta_2^y$ in (2.3) can be easily obtained [4,21], here we concentrate our attention on $\eta_\alpha^t$. Using the generalized Leibnitz rule, that was given in [22]

$$D_t^\alpha(f(t)g(t)) = \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}f(t)D_t^n g(t), \qquad \binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.$$

So we get

$$\xi_1 D_t^\alpha(u_x) - D_t^\alpha(\xi_1 u_x) = -\sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u_x)D_t^n(\xi_1),$$
$$\xi_2 D_t^\alpha(u_y) - D_t^\alpha(\xi_2 u_y) = -\sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u_y)D_t^n(\xi_2),$$

and

$$D_t^\alpha(uD_t(\tau)) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u) = -\alpha D_t(\tau)D_t^\alpha(u) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{\alpha-n}(u)D_t^{n+1}(\tau).$$

Thereby we get the expression

$$\eta_\alpha^t = D_t^\alpha(\eta) - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u_x)D_t^n(\xi_1) - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u_y)D_t^n(\xi_2) - \alpha D_t(\tau)D_t^\alpha(u) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{\alpha-n}(u)D_t^{n+1}(\tau).$$

According to the compound function of the chain rule [23] we get

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}(-g(t))^r \frac{d^m}{dt^m}\left(g(t)^{k-r}\right)\frac{d^k f(g)}{dg^k}.$$

Thus infinitesimal $\eta_\alpha^t$ takes a form

$$\eta_\alpha^t = \tfrac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha(\tau_t + u_t\tau_u))\tfrac{\partial^\alpha u}{\partial t^\alpha} - u\tfrac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu$$
$$+\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\tfrac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}\tau\right]D_t^{\alpha-n}u - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi_1)D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^{\alpha-n}(u_y)D_t^n(\xi_2),$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}(-u)^r}{\Gamma(n+1-\alpha)}\frac{\partial^m}{\partial t^m}\left(u^{k-r}\right)\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

## 3. Symmetry analysis for time-fractional nonlinear generalized Burgers' differential equation

After some easy mathematical transformations our equation (1.1) can be written in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) = nRg(t)((n-1)u^{n-2}(u_x^2+u_y^2) + u^{n-1}(u_{xx}+u_{yy})). \tag{3.1}$$

By substitution of transformations (2.2) and (2.3) into (3.1) and equating the multiplier of $\varepsilon$ to zero we get that, for the fractional nonlinear generalized Burgers' differential equation (3.1) the invariance criterion takes the form

$$\begin{aligned}\eta_\alpha^t + n(n-1)Pu^{n-2}\eta(u_x+u_y) + nPu^{n-1}(\eta_1^x+\eta_1^y) - n(n-1)(n-2)Rg(t)u^{n-3}\eta(u_x^2+u_y^2)\\ -2n(n-1)Rg(t)u^{n-2}(\eta_1^x+\eta_1^y) - n(n-1)Rg'(t)u^{n-2}(u_x^2+u_y^2)\tau\\ -n(n-1)Rg(t)u^{n-2}(u_{xx}+u_{yy})\eta - nRg'(t)u^{n-1}(u_{xx}+u_{yy})\tau - nRg(t)u^{n-1}(\eta_2^x+\eta_2^y) = 0.\end{aligned} \tag{3.2}$$

Substituting the extended infinitesimals (2.3) into the equation (3.2) we get following system of differential equations:

$$\partial_u\xi_1 = \partial_u\xi_2 = \partial_t\xi_1 = \partial_t\xi_2 = \partial_u\tau = \partial_x\tau = \partial_y\tau = \eta_{uu} = 0,$$
$$P\eta(n^2-n) + Rg(t)\partial_x\eta(2n-2n^2) + u(nP\alpha\partial_t\tau - nP(\partial_y\xi_1+\partial_x\xi_1) + nRg(t)(\partial_{yy}\xi_1+\partial_{xx}\xi_1) + 2nRg(t)\partial_{xu}\eta) = 0,$$
$$P\eta(n^2-n) + Rg(t)\partial_y\eta(2n-2n^2) + u(nP\alpha\partial_t\tau - nP(\partial_y\xi_2+\partial_x\xi_2) + nRg(t)(\partial_{yy}\xi_2+\partial_{xx}\xi_2) + 2nRg(t)\partial_{yu}\eta) = 0,$$
$$Rg(t)\eta(n-n^2) + u(-nRg'(t)\tau - nR\alpha g(t)\partial_t\tau + 2nRg(t)\partial_x\xi_1) = 0,$$
$$Rg(t)\eta(n-n^2) + u(-nRg'(t)\tau - nR\alpha g(t)\partial_t\tau + 2nRg(t)\partial_x\xi_2) = 0,$$
$$\partial_{ut}\eta - \tfrac{\alpha-1}{2}\partial_{tt}\tau = 0,$$
$$\tfrac{\partial^\alpha \eta}{\partial t^\alpha} - u\tfrac{\partial^\alpha \eta_u}{\partial t^\alpha} + nPu^{n-1}(\partial_x\eta+\partial_y\eta) - nRg(t)u^{n-1}(\partial_{xx}\eta+\partial_{yy}\eta) = 0.$$

In below we study some cases and obtain generating infinitesimal operators for classification of solutions of the equation.
*Case 1:* For arbitrary $g(t)$ and $0<\alpha\leq 1$ we get infinitesimals as

$$\xi_1 = c_1,\quad \xi_2 = c_2,\quad \tau = 0,\quad \eta = 0,$$

here $c_1$ and $c_2$ are arbitrary constants and there are two infinitesimal operators

$$X_1 = \frac{\partial}{\partial x} \qquad X_2 = \frac{\partial}{\partial y}.$$

*Case 2:* For $g(t)=1$ we get following infinitesimals

$$\xi_1 = c_1,\quad \xi_2 = c_2,\quad \tau = c_3 t + c_4,\quad \eta = -\tfrac{\alpha c_3}{n-1}u,$$

here $c_1, c_2, c_3$, and $c_4$ are arbitrary constants and thus we obtain two additional infinitesimal operators

$$X_3 = \frac{\partial}{\partial t}, \qquad X_4 = t\frac{\partial}{\partial t} + \frac{\alpha}{1-n}u\frac{\partial}{\partial u}.$$

*Case 3:* For $g(t)=t^b$ with $b\neq 0$ we have infinitesimals as

$$\xi_1 = c_5 x + c_6,\quad \xi_2 = c_5 x + c_7,\quad \tau = c_5\tfrac{t}{b},\quad \eta = \tfrac{b-\alpha}{b(n-1)}c_5 u,$$

here $c_5, c_6$, and $c_7$ are arbitrary constants and there is one additional infinitesimal operator

$$X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{t}{b}\frac{\partial}{\partial t} + \frac{b-\alpha}{b(n-1)}u\frac{\partial}{\partial u}.$$

*Case 4:* For $g(t)=e^t$ we obtain following infinitesimals in a form

$$\xi_1 = c_6 x + c_7,\quad \xi_2 = c_6 y + c_8,\quad \tau = c_6,\quad \eta = \tfrac{c_6}{n-1}u,$$

here $c_6, c_7$ and $c_8$ are arbitrary constants and we have one additional infinitesimal operator

$$X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + \frac{1}{n-1}u\frac{\partial}{\partial u}.$$

## 4. Symmetry reductions of the time fractional nonlinear generalized Burgers' differential equation

Now, we obtain similarity reductions and present the reduced nonlinear fractional ordinary differential equations. Also we classify the corresponding group invariant solutions of the fractional nonlinear generalized Burgers' equation.
*Case 2:* For $g(t)=1$ we have four infinitesimal operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial t}, \qquad X_4 = t\frac{\partial}{\partial t} + \frac{\alpha}{1-n}u\frac{\partial}{\partial u}.$$

The similarity variables for infinitesimal operator $X_1$ and $X_2$ can be found by solving the corresponding characteristic equation

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0} \qquad \frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}.$$

Thus we obtain the similarity reduction $u=\phi(t)$, by substituting which into (1.1) we get

$$D_t^{\alpha}\phi(t) = 0.$$

Thereby the exact solution of time fractional nonlinear generalized Burgers' differential equation (1.1) with $X_1$ and $X_2$ is

$$u(t,x,y) = ct^{\alpha-1},$$

where $c$ is arbitrary constant.
For infinitesimal operator $X_3$ the corresponding characteristic equation is in a form

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}.$$

This equation gives us a similarity reduction $u=\phi(x,y)$, by substituting which into (1.1) we have

$$u(t,x,y)=0.$$

And the similarity variables for infinitesimal operator $X_4$ can be found by solving the corresponding characteristic equation

$$\frac{dx}{0}=\frac{dy}{0}=\frac{dt}{t}=\frac{du}{\frac{\alpha u}{1-n}}.$$

Here we have $u=t^{\frac{\alpha}{1-n}}\phi(x,y)$ similarity reduction, by substituting it into (1.1) we get

$$\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}s^{\frac{\alpha}{1-n}}\phi(x,y)ds+nPt^{\frac{\alpha n}{1-n}}(\phi(x,y))^{n-1}(\partial_x\phi(x,y)+\partial_y\phi(x,y))$$

$$-Rt^{\frac{\alpha n}{1-n}}(n(\phi(x,y))^{n-1}(\partial_{xx}\phi(x,y)+\partial_{yy}\phi(x,y))+n(n-1)(\phi(x,y))^{n-2}((\partial_x\phi(x,y))^2+\partial_y\phi(x,y))^2))=0.$$

After some easy transformations we obtain following nonlinear ordinary differential equation

$$\frac{\Gamma\left(1+\frac{\alpha}{(1-n)}\right)}{\Gamma\left(1-\alpha+\frac{\alpha}{(1-n)}\right)}\phi(x,y)+nP(\phi(x,y))^{n-1}(\partial_x\phi(x,y)+\partial_y\phi(x,y))-Rn(\phi(x,y))^{n-1}(\partial_{xx}\phi(x,y)+\partial_{yy}\phi(x,y))$$

$$-Rn(n-1)(\phi(x,y))^{n-2}((\partial_x\phi(x,y))^2+\partial_y\phi(x,y))^2)=0.$$

*Case 3:* For $g(t)=t^b$ with $b\neq 0$ we have three infinitesimal operators

$$X_1=\frac{\partial}{\partial x},\qquad X_2=\frac{\partial}{\partial y},\qquad X_3=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{t}{b}\frac{\partial}{\partial t}+\frac{b-\alpha}{b(n-1)}u\frac{\partial}{\partial u}.$$

The third infinitesimal operator by solving the corresponding characteristic equations

$$\frac{dx}{x}=\frac{dy}{y}=\frac{bdt}{t}=\frac{bdu}{\frac{b-\alpha}{n-1}u},$$

gives us the similarity reduction

$$u(x,y,t)=t^{\frac{b-\alpha}{n-1}}\omega(p_1,p_2),$$

where $p_1=xt^{-b}$ and $p_2=yt^{-b}$.

*Case 4:* And lastly for $g(t)=e^t$ we have three infinitesimal operators

$$X_1=\frac{\partial}{\partial x},\qquad X_2=\frac{\partial}{\partial y},\qquad X_3=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{\partial}{\partial t}+\frac{1}{(n-1)}u\frac{\partial}{\partial u}.$$

The third infinitesimal operator gives us the corresponding characteristic equations

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{1}=\frac{du}{\frac{1}{n-1}u},$$

and a similarity reduction

$$u(x,y,t)=e^{\frac{1}{n-1}t}\nu(q_1,q_2),$$

here $q_1=xe^{-t}$, and $q_2=ye^{-t}$.

## 5. Conservation laws

In this section we will construct the conservation laws of time-fractional nonlinear generalized Burgers' differential equation (2.1) by using Ibragimov's theorem [24,25]. Ibragimov proved this theorem for differential equations with integer order. And it was applied to fractional differential equations [26,27].

We will search a vector field $C=(C^t,C^x,C^x)$, where $C^t=C^t(t,x,y,u,u_x,u_y,\ldots)$, $C^x=C^x(t,x,y,u,u_x,u_y,\ldots)$, and $C^y=C^y(t,x,y,u,u_x,u_y,\ldots)$ is conserved vector for (3.1) on all its solution if it satisfies the following conservation equation $D_t(C^t)+D_x(C^x)+D_y(C^y)=0$, where $C^t$, $C^x$, and $C^y$ are conservation laws for equation (2.1). A formal Lagrangian function for (2.1) is given by

$$L=v(t,x,y)E. \tag{5.1}$$

Here $v(t,x,y)$ is a new dependent variable and

$$E=\frac{\partial^\alpha u}{\partial t^\alpha}+nPu^{n-1}(u_x+u_y)-n(n-1)Rg(t)u^{n-2}(u_x^2+u_y^2)+nRg(t)u^{n-1}(u_{xx}+u_{yy}).$$

The Euler–Lagrange operator with respect to $u$ is defined by [27,28]

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+(D_t^\alpha)^*\frac{\partial}{\partial D_t^\alpha u}-D_x\frac{\partial}{\partial u_x}-D_y\frac{\partial}{\partial u_y}+D_x^2\frac{\partial}{\partial u_{xx}}+D_y^2\frac{\partial}{\partial u_{yy}}-\ldots,$$

where $(D_t^\alpha)^*$ is adjoint operator of $D_t^\alpha$ that has a form

$$(D_t^\alpha)^*=(-1)_t^nI_T^{n-\alpha}(D_t^n).$$

By using Euler–Lagrange operator we can define an adjoint equation of equation (3.1) as

$$\frac{\delta L}{\delta u}=0. \tag{5.2}$$

After calculations, the equation (5.2) takes a form

$$\frac{\delta L}{\delta u}=(D_t^{\alpha})^{*}v-nPu^{n-1}(v_x+v_y)-nRg(t)u^{n-1}(v_{xx}+v_{yy}).$$

So, we say that generalized Burgers' equation is nonlinearly self-adjoint if the adjoint equation (5.2) is satisfied for all solution $u$ of equation (3.1) upon a substitution $v=\varphi(t,x,y,u)$ and $\varphi(t,x,y,u)\neq 0$. This substitution allows us use formal Lagrangian as usual classical Lagrangian and construct the conservation laws.
Thus, $x$-component conservation laws for the equation (3.1) have the form [28]

$$C_i^x=\xi_1 L+W_i\left(\frac{\partial L}{\partial u_x}-D_x\frac{\partial L}{\partial u_{xx}}\right)+D_x(W_i)\left(\frac{\partial L}{\partial u_{xx}}\right),$$

here $W_i=\eta^i-\xi_1^i u_x-\xi_2^i u_y-\tau^i u_t$. $y$-component conservation laws for the equation (3.1) have the form [28]

$$C_i^y=\xi_2 L+W_i\left(\frac{\partial L}{\partial u_y}-D_y\frac{\partial L}{\partial u_{yy}}\right)+D_y(W_i)\left(\frac{\partial L}{\partial u_{yy}}\right).$$

And $t$-component conservation laws for the equation (3.1) have the form

$$C_i^t=\sum_{k=0}^{m-1}(-1)^k D_t^{\alpha-1-k}(W_i)D_t^k\left(\frac{\partial L}{\partial D_t^{\alpha}u}\right)-(-1)^m J\left(W_i,D_t^m\frac{\partial L}{\partial D_t^{\alpha}u}\right),$$

for $m-1<\alpha<m$ and $J$ is a integral

$$J(f,g)=\frac{1}{\Gamma(m-\alpha)}\int_0^t\int_t^T\frac{f(x,y,s)g(x,y,p)}{(p-s)^{\alpha+1-m}}dpds.$$

Thus, by using (5.1) and above formulas we can find $C^x$, $C^y$, and $C^t$ for our problem.
*Case 1:* For arbitrary $g(t)$ we have $W_1=u_x$ and $W_2=u_y$ that gives us

$$\begin{aligned}
C_1^t&=vD_t^{\alpha-1}(u_x)+J(u_x,v_t),\\
C_2^t&=vD_t^{\alpha-1}(u_y)+J(u_y,v_t),\\
C_1^x&=v\left(\tfrac{\partial^{\alpha}u}{\partial t^{\alpha}}+nPu^{n-1}(u_x+u_y)-n(n-1)Rg(t)u^{n-2}(u_x^2+u_y^2)+nRg(t)u^{n-1}(u_{xx}+u_{yy})\right)+\\
&\quad nRg(t)u^{n-1}u_{xx}v-3n(n-1)Rg(t)u^{n-2}v(u_x)^2+nPu^{n-1}u_xv-nRg(t)u^{n-1}u_x(u_xv_u+v_x),\\
C_2^x&=nRg(t)u^{n-1}u_{xy}v-3n(n-1)Rg(t)u^{n-2}vu_xu_y+nPu^{n-1}u_yv-nRg(t)u^{n-1}u_y(u_xv_u+v_x),\\
C_1^y&=nRg(t)u^{n-1}u_{xy}v-3n(n-1)Rg(t)u^{n-2}vu_xu_y+nPu^{n-1}u_xv-nRg(t)u^{n-1}u_x(u_yv_u+v_y),\\
C_2^y&=v\left(\tfrac{\partial^{\alpha}u}{\partial t^{\alpha}}+nPu^{n-1}(u_x+u_y)-n(n-1)Rg(t)u^{n-2}(u_x^2+u_y^2)+nRg(t)u^{n-1}(u_{xx}+u_{yy})\right)+\\
&\quad nRg(t)u^{n-1}u_{yy}v-3n(n-1)Rg(t)u^{n-2}v(u_y)^2+nPu^{n-1}u_yv-nRg(t)u^{n-1}u_y(u_yv_u+v_y).
\end{aligned}$$

*Case 2:* For $g(t)=t^b$ we get $W_1=u_x$, $W_2=u_y$, and $W_3=xu_x+yu_y+\frac{t}{b}u_t+\frac{\alpha-b}{b(1-n)}u$, thus the corresponding conservation laws are like:

$C_1^t$ and $C_2^t$ are the same.

$$\begin{aligned}
C_3^t&=tv\left(\tfrac{\partial^{\alpha}u}{\partial t^{\alpha}}+nPu^{n-1}(u_x+u_y)-n(n-1)Rg(t)u^{n-2}(u_x^2+u_y^2)+nRg(t)u^{n-1}(u_{xx}+u_{yy})\right)+\\
&\quad vD_t^{\alpha-1}\left(xu_x+yu_y+\tfrac{t}{b}u_t+\tfrac{\alpha-b}{b(1-n)}u\right)+J\left(xu_x+yu_y+\tfrac{t}{b}u_t+\tfrac{\alpha-b}{b(1-n)}u,v_t\right),\\
C_1^x&=v\left(\tfrac{\partial^{\alpha}u}{\partial t^{\alpha}}+nPu^{n-1}(u_x+u_y)-n(n-1)Rt^bu^{n-2}(u_x^2+u_y^2)+nRt^bu^{n-1}(u_{xx}+u_{yy})\right)+\\
&\quad nRt^bu^{n-1}u_{xx}v-3n(n-1)Rt^bu^{n-2}v(u_x)^2+nPu^{n-1}u_xv-nRt^bu^{n-1}u_x(u_xv_u+v_x),\\
C_2^x&=nRt^bu^{n-1}u_{xy}v-3n(n-1)Rt^bu^{n-2}vu_xu_y+nPu^{n-1}u_yv-nRt^bu^{n-1}u_y(u_xv_u+v_x),\\
C_3^x&=xv\left(\tfrac{\partial^{\alpha}u}{\partial t^{\alpha}}+nPu^{n-1}(u_x+u_y)-n(n-1)Rt^bu^{n-2}(u_x^2+u_y^2)+nRt^bu^{n-1}(u_{xx}+u_{yy})\right)+\\
&\quad nRt^bu^{n-1}v\left(u_x+\tfrac{\alpha-b}{b(1-n)}u_x+\tfrac{t}{b}u_{xt}+yu_{xy}+xu_{xx}\right)+\left(\tfrac{\alpha-b}{b(1-n)}u+\tfrac{t}{b}u_t+yu_y+xu_x\right)\\
&\quad\left(-3n(n-1)Rt^bu^{n-2}vu_x+nPu^{n-1}v-nRt^bu^{n-1}(u_xv_u+v_x)\right),\\
C_1^y&=nRt^bu^{n-1}u_{xy}v-3n(n-1)Rt^bu^{n-2}vu_xu_y+nPu^{n-1}u_xv-nRt^bu^{n-1}u_x(u_yv_u+v_y),\\
C_2^y&=v\left(\tfrac{\partial^{\alpha}u}{\partial t^{\alpha}}+nPu^{n-1}(u_x+u_y)-n(n-1)Rt^bu^{n-2}(u_x^2+u_y^2)+nRt^bu^{n-1}(u_{xx}+u_{yy})\right)+\\
&\quad nRt^bu^{n-1}u_{yy}v-3n(n-1)Rt^bu^{n-2}v(u_y)^2+nPu^{n-1}u_yv-nRt^bu^{n-1}u_y(u_yv_u+v_y),\\
C_3^y&=yv\left(\tfrac{\partial^{\alpha}u}{\partial t^{\alpha}}+nPu^{n-1}(u_x+u_y)-n(n-1)Rt^bu^{n-2}(u_x^2+u_y^2)+nRt^bu^{n-1}(u_{xx}+u_{yy})\right)+\\
&\quad nRt^bu^{n-1}v\left(u_y+\tfrac{\alpha-b}{b(1-n)}u_y+\tfrac{t}{b}u_{yt}+yu_{yy}+xu_{xy}\right)+\left(\tfrac{\alpha-b}{b(1-n)}u+\tfrac{t}{b}u_t+yu_y+xu_x\right)\\
&\quad\left(-3n(n-1)Rt^bu^{n-2}vu_y+nPu^{n-1}v-nRt^bu^{n-1}(u_yv_u+v_y)\right).
\end{aligned}$$

*Case 3:* For $g(t)=1$ we obtain $W_1=u_x$, $W_2=u_y$, $W_3=u_t$, and $W_4=tu_t+\frac{\alpha}{1-n}u$, thus the corresponding conservation laws are in the following form:

$C_1^t$ and $C_2^t$ is the same.

$$C_3^t = v(\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) - n(n-1)Rg(t)u^{n-2}(u_x^2+u_y^2) + nRg(t)u^{n-1}(u_{xx}+u_{yy})) + vD_t^{\alpha-1}(u_t) + J(u_t, v_t),$$

$$C_4^t = tv(\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) - n(n-1)Rg(t)u^{n-2}(u_x^2+u_y^2) + nRg(t)u^{n-1}(u_{xx}+u_{yy})) + vD_t^{\alpha-1}(tu_t + \frac{\alpha}{1-n}u) + J(tu_t + \frac{\alpha}{1-n}u, v_t),$$

$$C_1^x = v(\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) - n(n-1)Rg(t)u^{n-2}(u_x^2+u_y^2) + nRg(t)u^{n-1}(u_{xx}+u_{yy})) + nRu^{n-1}u_{xx}v - 3n(n-1)Ru^{n-2}(u_x)^2v + nPu^{n-1}u_xv - nRu^{n-1}u_x(u_xv_u+v_x),$$

$$C_2^x = nRu^{n-1}vu_{xy} - 3n(n-1)Ru^{n-2}u_xu_yv + nPu^{n-1}u_yv - nRu^{n-1}u_y(u_xv_u+v_x)),$$

$$C_3^x = nRu^{n-1}vu_{xt} - 3n(n-1)Ru^{n-2}u_xu_tv + nPu^{n-1}u_tv - nRu^{n-1}u_t(u_xv_u+v_x)),$$

$$C_4^x = nRu^{n-1}v(tu_{xt} + \frac{\alpha}{n-1}u_x) + (tu_t + \frac{\alpha}{n-1}u)(-3n(n-1)Ru^{n-2}u_xv + nPu^{n-1}v - nRu^{n-1}(u_xv_u+v_x)),$$

$$C_1^y = nRu^{n-1}vu_{xy} - 3n(n-1)Ru^{n-2}u_xu_yv + nPu^{n-1}u_xv - nRu^{n-1}u_x(u_yv_u+v_y)),$$

$$C_2^y = v(\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) - n(n-1)Rg(t)u^{n-2}(u_x^2+u_y^2) + nRg(t)u^{n-1}(u_{xx}+u_{yy})) + nRu^{n-1}u_{yy}v - 3n(n-1)Ru^{n-2}(u_y)^2v + nPu^{n-1}u_yv - nRu^{n-1}u_y(u_yv_u+v_y)),$$

$$C_3^y = nRu^{n-1}vu_{yt} - 3n(n-1)Ru^{n-2}u_yu_tv + nPu^{n-1}u_tv - nRu^{n-1}u_t(u_yv_u+v_y))),$$

$$C_4^y = nRu^{n-1}v(tu_{yt} + \frac{\alpha}{n-1}u_y) + (tu_t + \frac{\alpha}{n-1}u(-3n(n-1)Ru^{n-2}u_yv + nPu^{n-1}v - nRu^{n-1}(u_yv_u+v_y))).$$

*Case 4:* For $g(t) = e^t$ we have $W_1 = u_x$, $W_2 = u_y$, and $W_2 = xu_x + yu_y + u_t + \frac{1}{1-n}u$, and thus the corresponding conservation laws are:

$C_1^t$ and $C_2^t$ is the same.

$$C_3^t = v(\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) - n(n-1)Re^tu^{n-2}(u_x^2+u_y^2) + nRe^tu^{n-1}(u_{xx}+u_{yy})) + vD_t^{\alpha-1}(xu_x + yu_y + u_t + \frac{1}{1-n}u) + J(xu_x + yu_y + u_t + \frac{1}{1-n}u, v_t),$$

$$C_1^x = v(\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) - n(n-1)Re^tu^{n-2}(u_x^2+u_y^2) + nRe^tu^{n-1}(u_{xx}+u_{yy})) + nRe^tu^{n-1}u_{xx}v - 3n(n-1)Re^tu^{n-2}(u_x)^2v + nPu^{n-1}u_xv - nRe^tu^{n-1}u_x(u_xv_u+v_x),$$

$$C_2^x = nRe^tu^{n-1}u_{xy}v - 3n(n-1)Re^tu^{n-2}u_xu_yv + nPu^{n-1}u_yv - nRe^tu^{n-1}u_y(u_xv_u+v_x),$$

$$C_3^x = vx(\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) - n(n-1)Re^tu^{n-2}(u_x^2+u_y^2) + nRe^tu^{n-1}(u_{xx}+u_{yy})) + nRe^tu^{n-1}(u_x + xu_{xx} + yu_{xy} + u_{tx} + \frac{1}{1-n}u_x) + \left(xu_x + +yu_y + u_t + \frac{1}{1-n}u\right)(-3n(n-1)Re^tu^{n-2}u_xv + nPu^{n-1}v - nRe^tu^{n-1}(u_xv_u+v_x)),$$

$$C_1^y = nRe^tu^{n-1}u_{yx}v - 3n(n-1)Re^tu^{n-2}u_yu_xv + nPu^{n-1}u_xv - nRe^tu^{n-1}u_x(u_yv_u+v_y),$$

$$C_2^y = v(\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) - n(n-1)Re^tu^{n-2}(u_x^2+u_y^2) + nRe^tu^{n-1}(u_{xx}+u_{yy})) + nRe^tu^{n-1}u_{yy}v - 3n(n-1)Re^tu^{n-2}(u_y)^2v + nPu^{n-1}u_yv - nRe^tu^{n-1}u_y(u_yv_u+v_y),$$

$$C_3^y = vy(\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}(u_x+u_y) - n(n-1)Re^tu^{n-2}(u_x^2+u_y^2) + nRe^tu^{n-1}(u_{xx}+u_{yy})) + nRe^tu^{n-1}(xu_{xy} + yu_y + u_{yy} + u_{ty} + \frac{1}{1-n}u_y) + \left(xu_x + yu_y + u_t + \frac{1}{1-n}u\right)(-3n(n-1)Re^tu^{n-2}u_yv + nPu^{n-1}v - nRe^tu^{n-1}(u_yv_u+v_y)).$$

## 6. Symmetry analysis for boundary value problem

In this section, we will discuss the symmetry analysis for the boundary value problem. Lie symmetry analysis is one of the most widely-applicable methods of finding exact solutions of differential equations, but it was not widely used for solving boundary value problems. And the reason is the initial and boundary conditions are usually are not invariant under any obtained Lie symmetry method transformations [3, 29–31]. So, for partial differential equations, an invariant solution found by applying symmetry transformation solves a given boundary value problem, when the symmetry transformation leaves invariant all boundary conditions and the domain of the boundary value problem.

Now, let the $Q$-condition symmetry

$$Q = \xi_1(x,y,t,u)\frac{\partial}{\partial x} + \xi_2(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u}, \tag{6.1}$$

with

$$Q_k(u_t - F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial y^k}))|_M = 0,$$

and the manifold $M = \{u_t - F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial y^k}) = 0, Q(u) = 0\}$ is admitted by the boundary value problem defined on a domain $\Omega$:

$$u_t = F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial y^k}\right), \qquad (x,y,t) \in \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+, \tag{6.2}$$

$$d_a(x,y,t) = 0 : B_a\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^{k-1} u}{\partial x^{k-1}},\frac{\partial^{k-1} u}{\partial y^{k-1}}\right) = 0, \qquad a = 1,...,p. \tag{6.3}$$

$$l_c(x,y,t)=\infty : L_c(x,y,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^{k-1}u}{\partial x^{k-1}},\frac{\partial^{k-1}u}{\partial y^{k-1}})=0, \qquad c=1,...,p_\infty. \tag{6.4}$$

Here $B_a(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^{k-1}u}{\partial x^{k-1}},\frac{\partial^{k-1}u}{\partial y^{k-1}})$ boundary condition on $d_a(x,y,t)$. Suppose that the above boundary value problem has a unique solution.

So, for the manifold $M=\{l_c(x,y,t)=\infty, L_c(x,y,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial},...,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial y^k})=0\}$ there exist a smooth bijective transformation that maps the manifold $M$ into

$$M^*=\{l_c^*(x^*,y^*,t^*)=\infty, L_c^*(x^*,y^*,t^*,u,\frac{\partial u}{\partial x^*},\frac{\partial u}{\partial},...,\frac{\partial^{k^*}u}{\partial (x^*)^{k^*}},\frac{\partial^{k^*}u}{\partial (y^*)^{k^*}})=0\}.$$

**Definition 6.1.** *The symmetry $Q$ which has the form (6.1) is allowed by the boundary value problem (6.2)-(6.4) if:*

- $Q_{(k)}(u_t-F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial y^k}))=0$ *for* $u_t=F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial y^k})$;
- $Qd_a(x,y,t)=0$ *for* $d_a(x,y,t)=0,\ a=1,...,p$;
- $Q_{(k)}B_a(x,y,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial y^k})=0$ *for* $B_a(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial y^k})=0$ *on* $d_a(x,y,t)=0,\ a=1,..,p$;
- *there exist a smooth bijective transformation that maps the manifold $M$ into $M^*$ of the same dimensionality;*
- $Q^*l_c^*(x^*,y^*,t^*)=0$ *for* $l_c^*(x^*,y^*,t^*)=0,\ c=1,...,p_\infty$;
- $Q^*_{(k^*)}L_c(x,y,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},...,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial y^k})=0$ *for* $l_c(x,y,t)=\infty,\ c=1,...,r.$

Let us consider our fractional partial differential equation (1.1) with $\alpha=1$, which defined on the domain $0\le t<\infty$, $x>0$, and $y>0$ with initial and boundary conditions

$$u_t+Pnu^{n-1}(u_x+u_y)=nRg(t)(u^{n-1}(u_{xx}+u_{yy}+(n-1)u^{n-2}((u_x)^2+(u_y)^2)),\quad n>1,\quad P,R\neq 0 \tag{6.5}$$

$$\begin{aligned} &u(x,y,0)\longrightarrow\infty, && (x,y)\in\mathbb{R}^+\times\mathbb{R}^+,\\ &u(0,0,t)=\Phi(t), && t\in[0,\infty],\\ &u(x,y,t)_{(x,y)\to(0,0)}\longrightarrow\infty, && t\in[0,\infty].\end{aligned}$$

The problem (6.5) for $g(t)=e^t$ with boundary and initially conditions is not invariant. But it is invariant for $g(t)=1$ and $g(t)=t^b$, $b>0$. As we found before the equation (1.1) with $g(t)=t^b$ has an infinitesimal generator

$$X=(c_1+c_2x)\frac{\partial}{\partial x}+(c_3+c_2y)\frac{\partial}{\partial y}+c_2\frac{t}{b}\frac{\partial}{\partial t}+c_2\frac{b-\alpha}{b(n-1)}u\frac{\partial}{\partial u}.$$

So, after applying $X$ to boundary condition as $\xi_1(0)=0$ for $x=0$ and $\xi_2(0)=0$ for $y=0$ we get $c_1=c_3=0$ and $c_2\left(\frac{b-1}{b(n-1)}\Phi(t)-t\frac{d\Phi}{dt}\right)=0$, where $\Phi(t)=Kt^{\frac{b-1}{n-1}}$, $K$ is arbitrary constant.

According above definition let assume $t^*=t$, $x^*=1/x$, $y^*=1/y$, and $u^*=u$ bijective transformation which maps $M=\{x\to\infty,y\to\infty,u\to\infty\}$ to $M^*=\{x^*\to 0,y^*\to 0,u^*\to 0\}$. This transformation maps the infinitesimal operator $X$ to $X^*$. Thus, $X=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{t}{b}\frac{\partial}{\partial t}+\frac{b-1}{b(n-1)}\frac{\partial}{\partial u}$ infinitesimal operator with $X^*=-x^*\frac{\partial}{\partial x^*}-y^*\frac{\partial}{\partial y^*}+\frac{t^*}{b}\frac{\partial}{\partial t^*}-\frac{b-1}{b(n-1)}u^*\frac{\partial}{\partial u^*}$ leaves invariant the equation (6.5) with boundary and initially conditions:

$$\begin{aligned} &u(t,x,y)_{t\to 0}\longrightarrow\infty, && (x,y)\in\mathbb{R}^+\times\mathbb{R}^+,\\ &u(t,0,0)=Kt^{\frac{b-1}{n-1}}, && t\in[0,\infty],\\ &u(t,x,y)_{(x,y)\to(0,0)}\longrightarrow\infty, && t\in[0,\infty].\end{aligned}$$

Which give us $u=t^{\frac{b-1}{n-1}}f(r_1,r_2)$, where $r_1=\frac{x}{t^b}$, $r_2=\frac{y}{t^b}$ transformation, after applying that we get

$$\begin{cases} \frac{b-1}{n-1}f-b(r_1f_{r_1}+r_2f_{r_2})+nPf^{n-1}(f_{r_1}+f_{r_2})-R(nf^{n-1}(f_{r_1r_1}+f_{r_2r_2})+n(n-1)f^{n-2}(f_{r_1}^2+f_{r_2}^2))=0,\\ f(r_1,r_2)_{(r_1,r_2)\to(0,0)}\longrightarrow 0,\\ \lim\limits_{(r_1,r_2)\longrightarrow(\infty,\infty)}f(r_1,r_2)=K,\end{cases}$$

boundary value problem of partial differential equation with two independent variable.

And the equation (1.1) with $g(t)=1$ have an infinitesimal generator

$$X=c_1\frac{\partial}{\partial x}+c_2\frac{\partial}{\partial y}+(c_3t+c_4)\frac{t}{b}\frac{\partial}{\partial t}+c_4\frac{1}{1-n}\frac{\partial}{\partial u}.$$

Thus, as $\xi_1(0)=0$ for $x=0$, $\xi_2(0)=0$ for $y=0$, and $\tau(0)=0$ for $t=0$ we get

$$c_1=0,\quad c_2=0,\quad c_4=0,\quad and\quad c_3\left(\frac{1}{1-n}\Phi(t)-t\frac{d\Phi}{dt}\right)=0.$$

So, $\Phi(t)=Ct^{\frac{1}{1-n}}$, $C$ is arbitrary constant. According above, the $X=t\frac{\partial}{\partial t}+\frac{1}{1-n}\frac{\partial}{\partial u}$ with $X^*=t^*\frac{\partial}{\partial t^*}-\frac{1}{1-n}u^*\frac{\partial}{\partial u^*}$ infinitesimal operator leaves invariant the equation (6.5) with boundary and initially conditions:

$$\begin{aligned} &u(t,x,y)_{t\to 0}\longrightarrow\infty, && (x,y)\in\mathbb{R}^+\times\mathbb{R}^+,\\ &u(t,0,0)=Ct^{\frac{1}{1-n}}, && t\in[0,\infty],\\ &u(t,x,y)_{(x,y)\to(0,0)}\longrightarrow\infty, && t\in[0,\infty].\end{aligned}$$

Which give us $u=t^{\frac{1}{1-n}}h(x,y)$ transformation, after applying that we get

$$\begin{cases} \frac{1}{1-n}h+nPh^{n-1}(h_x+h_y)-R(nh^{n-1}(h_{xx}+h_{yy})+n(n-1)h^{n-2}(h_x^2+h_y^2))=0,\\ h(x,y)_{(x,y)\to(0,0)}\longrightarrow 0,\\ \lim\limits_{(x,y)\longrightarrow(\infty,\infty)}h(x,y)=C,\end{cases}$$

boundary value problem of partial differential equation with two independent variable.

## 7. Conclusion

In this work, we presented the application of Lie group analysis to study time-fractional nonlinear generalized Burgers' differential equations. So, we found some exact solutions of nonlinear generalized Burgers' differential equation with fractional derivative here we used the method of Lie groups method. Also, we obtained the conservation laws for corresponding cases of the function $g(t)$. After applying the Lie groups we got boundary value problems with reduced dimension for special cases of $g(t)$. Moreover we defined conditions which leave invariant the boundary value problem (1.1)-(1.2) for $g(t)=t^{\lambda}$ and $g(t)=1$ with $\alpha=1$. The symmetry method on fractional boundary value problem is our future research.


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# Spinor Representations of Involute Evolute Curves in $\mathbb{E}^3$

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**Abstract**

In this paper, we have obtained spinor with two complex components representations of Involute Evolute curves in $\mathbb{E}^3$. Firstly, we have given the spinor equations of Frenet vectors of two curves which are parameterized by arc-length and have an arbitrary parameter. Moreover, we have chosen that these curves are Involute Evolute curves and have matched these curves with different spinors. Then, we have investigated the answer of question "How are the relationships between the spinors corresponding to the Involute Evolute curves in $\mathbb{E}^3$?". Finally, we have given an example which crosscheck to theorems throughout this study.

## 1. Introduction

The theory of spinors, especially used in applications to electron spin and theory of relativity in quantum mechanics, was expressed by B. L. van der Waerden in 1929. The introduction of spinors is one of the most difficult topics in quantum mechanics. Even if the spin-1/2 is considered, some fundamental sections of spinors, such as the effects of rotation on spinors, turn out to be difficult to explain. Spinors appear to be closely related to the theory of the electromagnetic theory. According to physicists spinors are multilinear transformations. Thanks to this feature, spinors are mathematical entities somewhat like tensors and allow a more general treatment of the notion of invariance under rotation and Lorentz boosts. Spinors can also be used without reference to relativity, but they arise naturally in discussions of the Lorentz group. Moreover, for a spinor it can be say that it is the most basic sort of mathematical object that can be Lorentz-transformed, [1]-[3]. On the other hand, the basic knowledge of spinor theory is based on earlier years, indeed, if we consider the relationship between spinors and Euler's parameters it is date back to 1776. Spinors are vectorial objects and there is no their multilinear features for mathematicians. Also, spinors have one-index. In discussing vectors and tensors there are two ways in which we can proceed; the geometrical and analytical. To use the geometrical approach, we describe each kind of quantity in terms of its magnitudes and directions; in the analytical treatment, we use components.

Spinors were first studied by Elie Cartan in a geometrical sense. Cartan was one of the founders of Lie group theory which is one of the most important topics of mathematics and which has many physical applications. So, Cartan's study is a very impressive reference in terms of the geometry of the spinors since this gives the spinor representation of the basic geometric definitions [1]. In geometrical meaning, another study was made by Vivarelli. In that study, Vivarelli established a one-to-one linear relationship between the quaternions and spinors. In addition, using the relationship between the rotations in quaternions and three-dimensional Euclidean space, Vivarelli actually obtained the spinor representation of the rotations in Euclidean space, [4]. Castillo, on the other hand, examined the spinor formulation of the curve theory, one of the important subjects of differential geometry. In that study, Castillo gave a spinor corresponding to a mutually orthogonal vector triads in three-dimensional Euclidean space and thus obtained a spinor representation of the Frenet frame and curvatures of a curve, [5]. Based on that study, Kişi and Tosun obtained the spinor formulation of the Darboux frame on a directed surface in three-dimensional Euclidean space, [6]. Similarly, in [7], the spinor Bishop equations of the curves in $\mathbb{E}^3$ have been expressed.

Ketenci et. al investigated the answer of question "How does a spinor correspond to a mutually orthogonal vector triad in three dimensional Minkowski space $\mathbb{E}^3_1$?". Thus, they introduced hyperbolic spinors. Based on this, they matched the hyperbolic spinors which have hyperbolic components up with Frenet frame of a curve in Minkowski space $\mathbb{E}^3_1$, [8]. Then, Erişir et. al obtained the spinor representation of the Bishop frame, an alternative frame, of a curve in the three-dimensional Minkowski space, and the spinor formulation of the relationship between

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Frenet frame and Bishop frame, [9]. Also, the Darboux frame on the oriented surface in $\mathbb{E}_1^3$ was obtained by the aid of hyperbolic spinors, [10]. Moreover, in [11] Tarakçıoğlu et. al considered the Vivarelli's study and they gave a different approach to the relationship between the split quaternions and rotations in Minkowski space $\mathbb{R}_1^3$. In addition, they obtained an automorphism of the split quaternion algebra H' corresponding to a rotation in $\mathbb{R}_1^3$. Then, they gave the relationship between the hyperbolic spinors and rotations in $\mathbb{R}_1^3$.

In this paper, we have studied on spinors with two complex components and we have given spinor representations of Involute Evolute curves in $\mathbb{E}^3$. Firstly, we have introduced spinor representation of Frenet vectors of any unit-speed curve in three dimensional Euclidean space $\mathbb{E}^3$. Then, we have obtained spinor equations of the curve which is not parameterized by arc-length and considered the Involute Evolute curves corresponding the different spinors. Thus, we have investigated the answer of question "How are the relations between the spinors corresponding to the Involute Evolute curves in $\mathbb{E}^3$?". Finally, we have given an example.

## 2. Preliminaries

### 2.1. Involute Evolute curves in $\mathbb{E}^3$

It is well known that if a curve is differentiable at the each point of an open interval then a set of mutually orthogonal unit vectors can be constructed. These vectors are called tangent $\boldsymbol{T}$, normal $\boldsymbol{N}$ and binormal $\boldsymbol{B}$ unit vectors or the Serret-Frenet frame, collectively. So, let us consider that the regular curve $(\alpha)$ which is the differentiable function so that $\alpha : I \to \mathbb{E}^3$, $(I \subseteq \mathbb{R})$ has the arbitrary parameter $t$. Moreover, for $\forall t \in I$ the Frenet vectors on the point $\alpha(t)$ of the curve $(\alpha)$ are given by $\{\boldsymbol{T}(t), \boldsymbol{N}(t), \boldsymbol{B}(t)\}$. So, these Frenet vectors are obtained by the following equations

$$\begin{aligned} \boldsymbol{T} &= \frac{1}{\|\alpha'\|}\alpha', \\ \boldsymbol{N} &= \boldsymbol{B} \times \boldsymbol{T}, \\ \boldsymbol{B} &= \frac{1}{\|\alpha' \times \alpha''\|}(\alpha' \times \alpha'') \end{aligned}$$

where " $'$ " is the derivative with respect to arbitrary parameter $t$, $\kappa$ and $\tau$ are the curvature and torsion of this curve $(\alpha)$, [12].

Moreover, the Frenet derivative formulas of this curve $(\alpha)$ are given by

$$\begin{aligned} \boldsymbol{T}' &= \|\alpha'\| \, \kappa \boldsymbol{N}, \\ \boldsymbol{N}' &= \|\alpha'\| \left(-\kappa \boldsymbol{T} + \tau \boldsymbol{B}\right), \\ \boldsymbol{B}' &= -\|\alpha'\| \, \tau \boldsymbol{N}, \end{aligned} \tag{2.1}$$

[12].

Now, we know that if there is equation $\|\alpha'(s)\| = 1$ for $\forall s \in I$ on the point $\alpha(s)$ of the curve $\alpha : I \to \mathbb{E}^3$, $(I \subseteq \mathbb{R})$, the curve $(\alpha)$ is called as the curve parameterized by arc-length parameter $s$. So, the Frenet vectors of the curve $(\alpha)$ parameterized by arc-length parameter can be obtained by

$$\begin{aligned} \boldsymbol{T} &= \alpha', \\ \boldsymbol{N} &= \frac{1}{\|\alpha''\|}\alpha'', \\ \boldsymbol{B} &= \boldsymbol{T} \times \boldsymbol{N} \end{aligned} \tag{2.2}$$

where " $'$ " is the derivative with respect to arc-length parameter $s$. Moreover, the Frenet formulas of this curve are as

$$\begin{aligned} \boldsymbol{T}' &= \kappa \boldsymbol{N}, \\ \boldsymbol{N}' &= -\kappa \boldsymbol{T} + \tau \boldsymbol{B}, \\ \boldsymbol{B}' &= -\tau \boldsymbol{N}, \end{aligned} \tag{2.3}$$

[12].

The Involute Evolute curves in $\mathbb{E}^3$ are well known and one of the most studied curve pairs in elementary differential geometry. So, for the Involute Evolute curves, the following definition and theorems can be given.

**Definition 2.1.** *Let the curve $\alpha : I \to \mathbb{E}^3$ be parameterized by arc-length parameter and the curve $\beta : I \to \mathbb{E}^3$ be any curve which has an arbitrary parameter. Moreover, the Frenet frames of these curves $(\alpha)$ and $(\beta)$ are considered that $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ and $\{\boldsymbol{T}^*, \boldsymbol{N}^*, \boldsymbol{B}^*\}$, respectively. So, if there is equation*

$$\langle \boldsymbol{T}, \, \boldsymbol{T}^* \rangle = 0,$$

*then the curve $(\beta)$ is called the involute of the curve $(\alpha)$ and the curve $(\alpha)$ is called the evolute of the curve $(\beta)$, [12].*

**Theorem 2.2.** *Let the curve $\beta, \alpha : I \to \mathbb{E}^3$ be consider Involute Evolute curves, respectively. Then, the distance between mutual points of these curves is*

$$d(\alpha(s), \, \beta(s)) = \; |c - s| \, .$$

*So, it can be written*

$$\beta(s) = \alpha(s) + (c - s)\boldsymbol{T}(s), \tag{2.4}$$

*[12].*

**Theorem 2.3.** *Let the Frenet frames of the Involute Evolute curves $(\beta,\alpha)$ be $\{\boldsymbol{T}^*,\boldsymbol{N}^*,\boldsymbol{B}^*\}$ and $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$, respectively. So, the relationship between these Frenet frames is*

$$\begin{aligned} \boldsymbol{T}^* &= \boldsymbol{N}, \\ \boldsymbol{N}^* &= \tfrac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\kappa\boldsymbol{T}+\tau\boldsymbol{B},\right) \\ \boldsymbol{B}^* &= \tfrac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau\boldsymbol{T}+\kappa\boldsymbol{B},\right) \end{aligned} \tag{2.5}$$

*[12].*

**Theorem 2.4.** *Let the Frenet curvatures of the Involute Evolute curves $(\beta,\alpha)$ be $\kappa^*,\tau^*$ and $\kappa,\tau$, respectively. So, the relationship between these Frenet curvatures is*

$$\begin{aligned} \kappa^* &= \tfrac{\sqrt{\kappa^2+\tau^2}}{|(c-s)\kappa|} \\ \tau^* &= \tfrac{\kappa\tau'-\kappa'\tau}{|(c-s)\kappa|(\kappa^2+\tau^2)} \end{aligned}$$

*[12].*

## 2.2. Spinors

In this section, spinors introduced by Cartan [1], which is a basic study in geometric sense, are given. Afterwards, the spinors in the study given by Del Castillo and Barrales are mentioned, [5].

Consider that $\boldsymbol{x}=(x_1,\,x_2,\,x_3)\in\mathbb{C}^3$ is the isotropic vector and $\mathbb{C}^3$ is the three-dimensional complex vector space. So, we obtain ${x_1}^2+{x_2}^2+{x_3}^2=0$. The set of isotropic vectors in the vector space $\mathbb{C}^3$ forms a two-dimensional surface in the space $\mathbb{C}^2$. If this two-dimensional surface is parameterized by $\xi_1$ and $\xi_2$ coordinates, then $x_1={\xi_1}^2-{\xi_2}^2$, $x_2=i({\xi_1}^2+{\xi_2}^2)$, $x_3=-2\xi_1\xi_2$ is obtained. It is seen from the solution of this equation that $\xi_1=\pm\sqrt{\frac{x_1-ix_2}{2}}$, $\xi_2=\pm\sqrt{\frac{-x_1-ix_2}{2}}$. It is seen that; in the complex vector space $\mathbb{C}^3$, each isotropic vector corresponds to two vectors, $(\xi_1,\,\xi_2)$ and $(-\xi_1,\,-\xi_2)$ in the space $\mathbb{C}^2$. Conversely; both vectors so given in space $\mathbb{C}^2$ correspond to the same isotropic vector $\boldsymbol{x}$. Cartan expressed that the two-dimensional complex vectors $\xi=(\xi_1,\,\xi_2)$ described in this way are called as spinors. In addition, Cartan emphasized that spinors are not only two-dimensional complex vectors, but also represent a three-dimensional complex isotropic vectors, [1].

Consider the $SO(3)$, the group of rotations around the origin in the three-dimensional real vector space $\mathbb{R}^3$, and the $SU(2)$, the group of $2x2$ dimensional unitary matrices. As is known, the $SO(3)$ group is homomorphic to the $SU(2)$ group, [5, 13, 14]. By means of this homomorphism and spinors introduced by Cartan, the isotropic vector $\boldsymbol{a}+i\boldsymbol{b}$ is matched with spinor

$$\xi=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

where $\boldsymbol{a},\,\boldsymbol{b}\in\mathbb{R}^3$. So, we have $x_1={\xi_1}^2-{\xi_2}^2$, $x_2=i({\xi_1}^2+{\xi_2}^2)$ and $x_3=-2\xi_1\xi_2$ where $\boldsymbol{a}+i\boldsymbol{b}=(x_1,\,x_2,\,x_3)\in\mathbb{C}^3$, [1, 5]. As it is known, Pauli matrices form a basis for $2x2$-dimensional Hermitian and unitary matrices. Using the Pauli matrices and the matrix $C=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, the matrices $\sigma$ are generated as $\sigma_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\sigma_2=\begin{pmatrix}i&0\\0&i\end{pmatrix}$ and $\sigma_3=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$, [5, 15]. On the other hand, the mate $\hat{\xi}$ of the spinor $\xi$ is obtained as

$$\hat{\xi}=-\begin{pmatrix}0&1\\-1&0\end{pmatrix}\overline{\xi}=-\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\overline{\xi_1}\\ \overline{\xi_2}\end{pmatrix}=\begin{pmatrix}-\overline{\xi_2}\\ \overline{\xi_1}\end{pmatrix}.$$

Throughout this information, we obtain that

$$\begin{aligned} \boldsymbol{a}+i\boldsymbol{b} &= \xi^t\sigma\xi, \\ \boldsymbol{c} &= -\hat{\xi}^t\sigma\xi \end{aligned}$$

where $\boldsymbol{a}+i\boldsymbol{b}$ is the isotropic vector in the space $\mathbb{C}^3$ and $\boldsymbol{c}\in\mathbb{R}^3$, [5].

When necessary operations are considered, it is seen that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are equal in length and these vectors are mutually orthogonal, [5]. Moreover, the following proposition can be given.

**Proposition 2.5.** *Let two arbitrary spinors be $\xi$ and $\phi$. Then, the following statements are hold;*

$$\begin{array}{ll} i) & \overline{\phi^t\sigma\xi}=-\hat{\phi}^t\sigma\hat{\xi} \\ ii) & \widehat{\lambda\phi+\mu\xi}=\overline{\lambda}\hat{\phi}+\overline{\mu}\hat{\xi} \\ iii) & \hat{\hat{\xi}}=-\xi \\ iv) & \phi^t\sigma\xi=\xi^t\sigma\phi \end{array}$$

*where $\lambda,\,\mu\in\mathbb{C}$, [5].*

Now, let a curve parameterized by arc-length be $\alpha : I \to \mathbb{E}^3$, $(I \subseteq \mathbb{R})$. So, $\|\alpha'(s)\| = 1$ where $s$ is the arc-length parameter of the curve $(\alpha)$. Moreover, we consider that the Frenet vectors of this curve are $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ and the spinor $\xi$ corresponds to the Frenet vectors $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$. Thus, the following equations

$$\begin{aligned} \boldsymbol{N} + i\boldsymbol{B} &= \xi^t \sigma \xi = ({\xi_1}^2 - {\xi_2}^2,\ i({\xi_1}^2 + {\xi_2}^2),\ -2\xi_1\xi_2), \\ \boldsymbol{T} &= -\widehat{\xi}^t \sigma \xi = (\xi_1 \overline{\xi_2} + \overline{\xi_1}\xi_2,\ i(\xi_1\overline{\xi_2} - \overline{\xi_1}\xi_2),\ |\xi_1|^2 - |\xi_2|^2) \end{aligned} \tag{2.6}$$

can be written where $\overline{\xi}^t \xi = 1$ since these vectors are mutually orthogonal, [5]. Moreover, the following theorem can be given.

**Theorem 2.6.** *If the spinor $\xi$ with two complex components represents the triad $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of a curve parameterized by its arc-length $s$ the Frenet equations are equivalent to the single spinor equation*

$$\frac{d\xi}{ds} = \frac{1}{2}(-i\tau\xi + \kappa\hat{\xi})$$

*where $\kappa$ and $\tau$ denote the curvature and torsion of the curve, respectively, [5].*

## 3. Main theorems and proofs

In this section, first of all, we have expressed that the spinor representations of each Frenet vectors $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of a unit-speed curve $(\alpha)$, separately. In addition that, we have considered that the curve $(\beta)$ which has not arc-length parameter and a different spinor is corresponded to the Frenet vectors $\{\boldsymbol{N}^*, \boldsymbol{B}^*, \boldsymbol{T}^*\}$ of this curve. Moreover, we have given the spinor equations of this curve. Then, we have regarded that the curves $(\beta, \alpha)$ are Involute Evolute curves and obtained the relationship between spinors corresponding to these curves with theorems. Finally, we have given an example.

Let $\alpha : \mathrm{I} \to \mathbb{E}^3$ be arbitrary unit-speed curve and the Frenet vectors of this curve be $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$. So, the following theorem can be given.

**Theorem 3.1.** *Let the Frenet vectors of the unit-speed curve $\alpha : I \to \mathbb{E}^3$ be $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$. We assume that the spinor $\xi$ is corresponded to this curve $(\alpha)$, So, the spinor equations of these Frenet vectors are*

$$\begin{aligned} &\boldsymbol{T} = -\hat{\xi}^t \sigma \xi, \\ &\boldsymbol{N} = \tfrac{1}{2}\left(\xi^t \sigma \xi - \hat{\xi}^t \sigma \hat{\xi}\right), \\ &\boldsymbol{B} = -\tfrac{i}{2}\left(\xi^t \sigma \xi + \hat{\xi}^t \sigma \hat{\xi}\right). \end{aligned}$$

*Proof.* Let the spinor $\xi$ be correspond to the Frenet curve $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of unit-speed curve $(\alpha)$. Then, considering the equations (2.2) and (2.6) for the tangent vector on the point $\alpha(s)$ of $(\alpha)$ the following equation

$$\boldsymbol{T} = \alpha' = -\hat{\xi}^t \sigma \xi \tag{3.1}$$

can be written. If we calculate the derivative of the equation (3.1) and make necessary arrangement, we obtain that

$$\alpha'' = \frac{\kappa}{2}\left(\xi^t \sigma \xi - \hat{\xi}^t \sigma \hat{\xi}\right).$$

On the other hand, let us consider the equation (2.2), So, we obtain that the spinor equation of the normal vector $\boldsymbol{N}$ of the curve $(\alpha)$ is obtain that

$$\boldsymbol{N} = \frac{1}{2}\left(\xi^t \sigma \xi - \hat{\xi}^t \sigma \hat{\xi}\right). \tag{3.2}$$

Similarly, using the equations (2.3), (3.1) and (3.2), we have

$$\boldsymbol{N}' = \frac{1}{2}\left[-i\tau\left(\xi^t \sigma \xi + \hat{\xi}^t \sigma \hat{\xi}\right) + 2\kappa \hat{\xi}^t \sigma \xi\right]$$

and

$$\frac{1}{2}\left[-i\tau\left(\xi^t \sigma \xi + \hat{\xi}^t \sigma \hat{\xi}\right) + 2\kappa \hat{\xi}^t \sigma \xi\right] = -\kappa\left(-\hat{\xi}^t \sigma \xi\right) + \tau \boldsymbol{B}.$$

And finally, the spinor equation of the binormal tangent of the curve $(\alpha)$ is

$$\boldsymbol{B} = -\frac{i}{2}\left(\xi^t \sigma \xi + \hat{\xi}^t \sigma \hat{\xi}\right). \tag{3.3}$$

So, the proof ends. $\square$

Indeed, if we consider the first equality in the equation (2.6), we see that $\boldsymbol{N} = Re(\xi^t \sigma \xi)$ and $\boldsymbol{B} = Im(\xi^t \sigma \xi)$. So, considering complex numbers we obtain that

$$\begin{aligned} &\boldsymbol{N} = \tfrac{1}{2}\left(\xi^t \sigma \xi + \overline{\xi^t \sigma \xi}\right), \\ &\boldsymbol{B} = -\tfrac{i}{2}\left(\xi^t \sigma \xi - \overline{\xi^t \sigma \xi}\right). \end{aligned}$$

Finally, using the Proposition 2.5, we reach the equations (3.2) and (3.3).

Moreover, the spinor equations of these vectors can be written in terms of components as follows since they will be used operations after that

$$\begin{aligned}
\boldsymbol{T} &= \left(\xi_1\overline{\xi_2}+\overline{\xi_1}\xi_2, i(\xi_1\overline{\xi_2}-\overline{\xi_1}\xi_2), |\xi_1|^2-|\xi_2|^2\right),\\
\boldsymbol{N} &= \tfrac{1}{2}\left(\xi_1{}^2-\xi_2{}^2-\overline{\xi_2}^2+\overline{\xi_1}^2, i\left(\xi_1{}^2+\xi_2{}^2-\overline{\xi_1}^2-\overline{\xi_2}^2\right), -2\xi_1\xi_2-2\overline{\xi_1\xi_2}\right),\\
\boldsymbol{B} &= -\tfrac{i}{2}\left(\xi_1{}^2-\xi_2{}^2+\overline{\xi_2}^2-\overline{\xi_1}^2, i\left(\xi_1{}^2+\xi_2{}^2+\overline{\xi_1}^2+\overline{\xi_2}^2\right), -2\xi_1\xi_2+2\overline{\xi_1\xi_2}\right).
\end{aligned} \tag{3.4}$$

On the other hand, let us take any curve $(\beta)$ which is not parameterized by arc length. Moreover, let the Frenet vectors of this curve be $\{\boldsymbol{N}^*, \boldsymbol{B}^*, \boldsymbol{T}^*\}$ and the different spinor corresponding the curve $(\beta)$ be $\phi$. So, similar to the equation (2.6), we can write

$$\begin{aligned}
\boldsymbol{N}^*+i\boldsymbol{B}^* &= \phi^t\sigma\phi,\\
\boldsymbol{T}^* &= -\widehat{\phi}^t\sigma\phi.
\end{aligned}$$

So, the spinor equations of this curve $(\beta)$ can be written by components as

$$\begin{aligned}
\boldsymbol{T}^* &= (\phi_1\overline{\phi_2}+\overline{\phi_1}\phi_2, i(\phi_1\overline{\phi_2}-\overline{\phi_1}\phi_2), |\phi_1|^2-|\phi_2|^2),\\
\boldsymbol{N}^* &= \tfrac{1}{2}\left(\phi_1{}^2-\phi_2{}^2+\overline{\phi_1}^2-\overline{\phi_2}^2, i\left(\phi_1{}^2+\phi_2{}^2-\overline{\phi_1}^2-\overline{\phi_2}^2\right), -2\left(\phi_1\phi_2+\overline{\phi_1\phi_2}\right)\right),\\
\boldsymbol{B}^* &= -\tfrac{i}{2}\left(\phi_1{}^2-\phi_2{}^2+\overline{\phi_2}^2-\overline{\phi_1}^2, i\left(\phi_1{}^2+\phi_2{}^2+\overline{\phi_1}^2+\overline{\phi_2}^2\right), -2\left(\phi_1\phi_2-\overline{\phi_1\phi_2}\right)\right)
\end{aligned} \tag{3.5}$$

for the curve $(\beta)$. In addition that, the equation (3.5) provides the equation (2.1). So, we can give the following theorem.

**Theorem 3.2.** *Let the Frenet vectors of the curve $(\beta)$ which is not parameterized by arc length be $\{\boldsymbol{N}^*, \boldsymbol{B}^*, \boldsymbol{T}^*\}$ and the spinor corresponding to this curve be $\phi$. So, the Frenet equation of this curve in terms of a single spinor equation is written by*

$$\frac{d\phi}{ds} = \frac{\|\beta'\|}{2}(-i\tau^*\phi+\kappa^*\hat{\phi}).$$

*Proof.* Let the Frenet vectors $\{\boldsymbol{N}^*, \boldsymbol{B}^*, \boldsymbol{T}^*\}$ of the curve $(\beta)$ be correspond to the spinor $\phi$. We know that $\{\phi, \hat{\phi}\}$ is the basis for the spinor with two complex components. So, it can be written

$$\frac{d\phi}{ds} = f\phi+g\hat{\phi} \tag{3.6}$$

where the functions $f$ and $g$ are arbitrary, complex-valued functions. On the other hand, using the equations (2.1), (3.5) and (3.6) we obtain that

$$\|\beta'\|\left(-\kappa^*\boldsymbol{T}^*-i\tau^*\left(\boldsymbol{N}^*+i\boldsymbol{B}^*\right)\right) = 2f\left(\boldsymbol{N}^*+i\boldsymbol{B}^*\right)-2g\boldsymbol{T}^*.$$

So, we have

$$f = \frac{-i\tau^*\|\beta'\|}{2}, \quad g = \frac{\kappa^*\|\beta'\|}{2}. \tag{3.7}$$

Finally, if we consider the equations (3.6) and (3.7), we obtain that the Frenet vectors of the curve $(\beta)$ in terms of a single spinor equation as

$$\frac{d\phi}{ds} = \frac{\|\beta'\|}{2}(-i\tau^*\phi+\kappa^*\hat{\phi})$$

where $\kappa^*$ and $\tau^*$ are curvature and torsion of curve $(\beta)$. □

Now, we express the spinor representation of Involute Evolute curves. Let us consider the curves $\alpha, \beta : I \to \mathbb{E}^3$ and the Frenet vectors $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ and $\{\boldsymbol{T}^*, \boldsymbol{N}^*, \boldsymbol{B}^*\}$ of the curves $(\alpha)$ and $(\beta)$, respectively. Moreover, the curves $(\beta, \alpha)$ are considered that Involute Evolute curves and the spinors $\phi$ and $\xi$ are corresponded to the Involute Evolute curves $(\beta, \alpha)$, respectively. So, we can give the following theorem.

**Theorem 3.3.** *Let the curves $\beta, \alpha : I \to \mathbb{E}^3$ be Involute Evolute curves which have the Frenet vectors $\{\boldsymbol{N}^*, \boldsymbol{B}^*, \boldsymbol{T}^*\}$ and $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$, respectively. Moreover, the spinors corresponding to the Frenet vectors of these curves $(\alpha)$ and $(\beta)$ are considered as $\xi$ and $\phi$, respectively. So, the relationship between the spinor equations of Involute Evolute curves is*

$$\overline{\phi}\overline{\xi}^t\phi = C\phi\overline{\phi}^t\xi$$

*where* $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

*Proof.* We consider that the curves $\alpha, \beta : I \to \mathbb{E}^3$ are Involute Evolute curves which have the Frenet vectors $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ and $\{\boldsymbol{N}^*, \boldsymbol{B}^*, \boldsymbol{T}^*\}$, respectively. So, we know that relationship between the tangent vectors of these curves is $\langle \boldsymbol{T}, \boldsymbol{T}^*\rangle = 0$. Thus, using this relation and the first equations in the equations (3.4), (3.5), we obtain

$$\left(\xi_1\overline{\xi_2}+\overline{\xi_1}\xi_2\right)\left(\phi_1\overline{\phi_2}+\overline{\phi_1}\phi_2\right)-(\xi_1\overline{\xi_2}-\overline{\xi_1}\xi_2)(\phi_1\overline{\phi_2}-\overline{\phi_1}\phi_2)+\left(|\xi_1|^2-|\xi_2|^2\right)\left(|\phi_1|^2-|\phi_2|^2\right) = 0$$

and

$$2\xi_1\overline{\xi_2}\overline{\phi_1}\phi_2+2\overline{\xi_1}\xi_2\phi_1\overline{\phi_2}+|\xi_1|^2|\phi_1|^2-|\xi_1|^2|\phi_2|^2-|\xi_2|^2|\phi_1|^2+|\xi_2|^2|\phi_2|^2=0. \tag{3.8}$$

Also, if the equation (3.8) is written as matrix product, the following equation can be written

$$\begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}\begin{pmatrix}\overline{\phi_1}\\ \overline{\phi_2}\end{pmatrix}\begin{pmatrix}\overline{\xi_1} & \overline{\xi_2}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}+\begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}\begin{pmatrix}\phi_2\\ -\phi_1\end{pmatrix}\begin{pmatrix}\overline{\phi_1} & \overline{\phi_2}\end{pmatrix}\begin{pmatrix}\overline{\xi_2}\\ -\overline{\xi_1}\end{pmatrix}=0$$

where the spinors $\xi$ and $\phi$ are written as column matrix like these $\xi=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ and $\phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$. Finally, we have

$$\overline{\phi}\overline{\xi}^t\phi=C\phi\overline{\phi}^t\hat{\xi}$$

where $C=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. □

So, we obtain the spinor equations of Involute Evolute curves. After that, we will call the spinors corresponding to Involute Evolute curves as Involute Evolute spinors. The other relationship between Involute Evolute spinors can be given following theorem.

**Theorem 3.4.** *Let the curves $\beta,\alpha: I\to\mathbb{E}^3$ be Involute Evolute curves and the spinors corresponding to the Frenet vectors of these curves are considered as $\phi$ and $\xi$, respectively. So, the relationship between Involute Evolute spinors is*

$$\xi^t\overline{\phi}\overline{\xi}^t\phi=\frac{1}{2}.$$

*Proof.* We know that for the spinors $\xi$ and $\phi$ there is relationship $|\xi_1|^2+|\xi_2|^2=1$ and $|\phi_1|^2+|\phi_2|^2=1$ since the Frenet vectors corresponding to these spinors are unit vectors. So, we can write

$$\left(|\xi_1|^2+|\xi_2|^2\right)\left(|\phi_1|^2+|\phi_2|^2\right)=1.$$

So, we have

$$|\xi_1|^2|\phi_1|^2+|\xi_1|^2|\phi_2|^2+|\xi_2|^2|\phi_1|^2+|\xi_2|^2|\phi_2|^2=1. \tag{3.9}$$

If we use the equations (3.8) and (3.9), we obtain

$$\left(\xi_1\overline{\phi_1}+\xi_2\overline{\phi_2}\right)\left(\overline{\xi_2}\phi_2+\overline{\xi_1}\phi_1\right)=\frac{1}{2}.$$

Moreover, if the last equation is written by matrix product by the help of $\xi=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$, the equation can be obtain

$$\xi^t\overline{\phi}\overline{\xi}^t\phi=\frac{1}{2}.$$

□

Now, the expression of the Involute Evolute spinors in terms of each other can be written as follows.

**Theorem 3.5.** *Let the spinors $\phi$ and $\xi$ be Involute Evolute spinors. So, the expression of spinor $\phi$ in terms of the spinor $\xi$ is*

$$\begin{array}{l}{\phi_1}^2=\frac{\kappa-i\tau}{2\sqrt{\kappa^2+\tau^2}}\left(\xi_1-\bar{\xi}_2\right)^2\\ {\phi_2}^2=\frac{\kappa-i\tau}{2\sqrt{\kappa^2+\tau^2}}\left(\bar{\xi}_1+\xi_2\right)^2\end{array}.$$

*Proof.* We consider that the equations (3.4) and (3.5) are written in the second equation of (2.5). So, we find the equations

$$\frac{1}{2}\left({\phi_1}^2-{\phi_2}^2-\bar{\phi}_2^2+\bar{\phi}_1^2\right)=\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\kappa\left(\xi_1\overline{\xi_2}+\overline{\xi_1}\xi_2\right)-i\frac{\tau}{2}\left({\xi_1}^2-{\xi_2}^2+\overline{\xi_2}^2-\overline{\xi_1}^2\right)\right), \tag{3.10}$$

$$\frac{1}{2}\left({\phi_1}^2+{\phi_2}^2-\bar{\phi}_1^2-\bar{\phi}_2^2\right)=\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\kappa\left(\xi_1\overline{\xi_2}-\overline{\xi_1}\xi_2\right)-i\frac{\tau}{2}\left({\xi_1}^2+{\xi_2}^2+\overline{\xi_1}^2+\overline{\xi_2}^2\right)\right) \tag{3.11}$$

and

$$-\phi_1\phi_2-\bar{\phi}_1\bar{\phi}_2=\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\kappa\left(\xi_1\bar{\xi}_1-\xi_2\overline{\xi_2}\right)+i\tau\left(\xi_1\xi_2-\overline{\xi_1\xi_2}\right)\right). \tag{3.12}$$

Similarly, if the equations (3.4) and (3.5) are written in the third equation of (2.5), we find that

$$-\frac{i}{2}\left({\phi_1}^2-{\phi_2}^2+\bar{\phi}_2^2-\bar{\phi}_1^2\right)=\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau\left(\xi_1\overline{\xi_2}+\overline{\xi_1}\xi_2\right)-i\frac{\kappa}{2}\left({\xi_1}^2-{\xi_2}^2+\overline{\xi_2}^2-\overline{\xi_1}^2\right)\right), \tag{3.13}$$

$$-\frac{i}{2}\left({\phi_1}^2+{\phi_2}^2+\bar{\phi}_1^2+\bar{\phi}_2^2\right)=\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau\left(\xi_1\overline{\xi_2}-\overline{\xi_1}\xi_2\right)-i\frac{\kappa}{2}\left({\xi_1}^2+{\xi_2}^2+\overline{\xi_1}^2+\overline{\xi_2}^2\right)\right) \tag{3.14}$$

and

$$i\left(\phi_1\phi_2-\bar{\phi}_1\bar{\phi}_2\right)=\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau\left(\xi_1\bar{\xi}_1-\xi_2\overline{\xi_2}\right)-i\kappa\left(-\xi_1\xi_2+\overline{\xi_1\xi_2}\right)\right). \tag{3.15}$$

If we consider the equations (3.10) and (3.13), we obtain

$${\phi_1}^2-{\phi_2}^2=\frac{\kappa-i\tau}{2\sqrt{\kappa^2+\tau^2}}\left(\left(\xi_1-\overline{\xi_2}\right)^2-\left(\overline{\xi_1}+\xi_2\right)^2\right).$$

Similarly, from the equations (3.11) and (3.14), we have

$${\phi_1}^2+{\phi_2}^2=\frac{\kappa-i\tau}{2\sqrt{\kappa^2+\tau^2}}\left(\left(\xi_1-\overline{\xi_2}\right)^2+\left(\overline{\xi_1}+\xi_2\right)^2\right).$$

Finally, from the equations (3.12) and (3.15), we get

$$\phi_1\phi_2=\frac{\kappa-i\tau}{2\sqrt{\kappa^2+\tau^2}}\left(\xi_1-\overline{\xi_2}\right)\left(\overline{\xi_1}+\xi_2\right).$$

So, we can write

$$\begin{aligned}{\phi_1}^2&=\tfrac{\kappa-i\tau}{2\sqrt{\kappa^2+\tau^2}}\left(\xi_1-\bar{\xi}_2\right)^2,\\ {\phi_2}^2&=\tfrac{\kappa-i\tau}{2\sqrt{\kappa^2+\tau^2}}\left(\bar{\xi}_1+\xi_2\right)^2.\end{aligned}$$

□

Now, we give an example.

**Example 3.6.** *Let the unit-speed curve $\alpha: I\to\mathbb{E}^3$ be $\alpha(s)=\left(\frac{1}{\sqrt{2}}\cos s,\frac{1}{\sqrt{2}}\sin s,\frac{1}{\sqrt{2}}s\right)$. So, if we use the equation (2.2), for the Frenet vectors $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ of the curve $(\alpha)$, we calculate as*

$$\begin{aligned}\boldsymbol{T}(s)&=\left(-\tfrac{1}{\sqrt{2}}\sin s,\tfrac{1}{\sqrt{2}}\cos s,\tfrac{1}{\sqrt{2}}\right),\\ \boldsymbol{N}(s)&=\left(-\cos s,-\sin s,\ 0\right),\\ \boldsymbol{B}(s)&=\left(\tfrac{1}{\sqrt{2}}\sin s,\ -\tfrac{1}{\sqrt{2}}\cos s,\ \tfrac{1}{\sqrt{2}}\right).\end{aligned} \tag{3.16}$$

*Moreover, from the equation (2.3) we obtain the curvature and torsion of this curve as*

$$\kappa=\frac{1}{\sqrt{2}},\qquad \tau=\frac{1}{\sqrt{2}}.$$

*Now, we consider that the Frenet vectors $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ are corresponded to the spinor $\xi$. So, from the equations (3.4) and (3.16), we get*

$$\begin{aligned}\xi_1&=\tfrac{1}{2}\sqrt{\tfrac{2+\sqrt{2}}{2}}\left(\sqrt{1-\cos s}+i\sqrt{1+\cos s}\right),\\ \xi_2&=-\tfrac{1}{2}\sqrt{\tfrac{2+\sqrt{2}}{2}}\left(\sqrt{1+\cos s}+i\sqrt{1-\cos s}\right),\end{aligned}$$

*In addition that, from Theorem 2.6, we have $\frac{d\xi}{ds}=\frac{\sqrt{2}}{4}\left(-i\xi+\hat{\xi}\right)$.*
*Now, we regard that the involute curve of unit-speed curve $(\alpha)$ is $(\beta)$ which has not arc-length parameter. So, if we look the equation (2.4), then the curve $(\beta)$ is written by*

$$\beta(s)=\frac{1}{\sqrt{2}}\left(\cos s-(c-s)\sin s,\sin s+(c-s)\ \cos s,c\right).$$

*Then, we obtain the Frenet vectors and curvature, torsion of this curve*

$$\begin{aligned}\boldsymbol{T}^*(s)&=\left(-\cos s,-\sin s,0\right),\\ \boldsymbol{N}^*(s)&=\left(\sin s,-\cos s,0\right),\\ \boldsymbol{B}^*(s)&=(0,0,1)\end{aligned}$$

*and*

$$\kappa^*=\frac{\sqrt{2}}{(c-s)},\qquad \tau^*=0.$$

*Finally, we get the spinor corresponded to involute curves*

$$\begin{aligned}\phi_1&=\tfrac{1}{2}\left(\sqrt{1+\sin s}+i\sqrt{1-\sin s}\right),\\ \phi_2&=-\tfrac{1}{2}\left(\sqrt{1-\sin s}+i\sqrt{1+\sin s}\right).\end{aligned}$$

*So, the Frenet equation of this curve in terms of a single spinor equation is written by $\frac{d\phi}{ds}=\frac{1}{2}\hat{\phi}$.*

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# On Directed Baire Spaces

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**Abstract**

We study directed Baire spaces and their relevant topological properties. A characterization of directed Baire spaces is given using point finite family of $G_\delta-$sets. Further, we prove that the product of directed Baire space with a metric hereditarily directed Baire space is a downward-directed Baire space. Finally, it is established that the product of a Baire space with a hereditarily metric Volterra space is again a Volterra space.

## 1. Introduction

A topological space $X$ is a Baire space (resp. second category) if intersection of any sequence of dense open subsets of $X$ is dense (resp. non-empty). It follows from the definition that the intersection of countably many dense $G_\delta-$sets of Baire space (resp. second category) $X$ must be dense (resp. non-empty) in $X$ [1]. The properties of Baire spaces and characterizations are studied in [2]. A family $\mathscr{B}$ of non-empty open subsets of a topological space is said to be *pseudo base* [3] ($\pi-$*base*) if every non-empty open set contains at least one member of $\mathscr{B}$. A space $X$ is called a $P-$space [4] if every countable intersection of open subsets of $X$ is open. A directed set [5](or a directed preorder or a filtered set) is a non-empty set $\Delta$ together with a reflexive and transitive binary relation $\leq$ (that is, a preorder), with the additional property that every pair of elements has an upper bound. In other words, for any $a$ and $b$ in $\Delta$, there must exists a $c \in \Delta$ with $a \leq c$ and $b \leq c$. In this article, we consider only the directed set in which every two elements of it are comparable. A space $X$ is a directed Baire space if intersection of family of dense $G_\delta-$subsets $\{D_\alpha \mid \alpha \in \Delta\}$ of $X$ is dense and weakly directed Baire space if intersection of family of dense $G_\delta-$subsets $\{D_\alpha \mid \alpha \in \Delta\}$ of $X$ is non-empty where $\Delta$ is a directed set. A space $X$ is called downward-directed Baire space if intersection of the family of decreasing dense $G_\delta-$subsets $\{D_\alpha \mid \alpha \in \Delta\}$ of $X$ is dense, where $\Delta$ is a directed set. The following Example 1.1 shows the existence of directed Baire spaces.

**Example 1.1.** *Consider $X = [0,\infty)$ with the topology having $\mathscr{B} = \{[0,a) \mid a \neq 0 \in X\}$ as its basis. In this space, the intersection of any family of dense $G_\delta-$subsets of $X$ is dense.*

By definition itself it is clear that every directed Baire space is Baire, but there are Baire spaces which are not directed Baire, refer Example 1.2.

**Example 1.2.** *Consider $\mathbb{R}$ with usual metric. Since $\mathbb{R}$ is a complete metric space, it is a Baire space and hence second category. Since $\mathbb{Q} \cup \{\alpha\}$ is countable and each singleton sets of $\mathbb{R}$ is closed, $\mathbb{Q} \cup \{\alpha\}$ is an $F_\sigma-$set and its complement is a dense $G_\delta-$subset of $\mathbb{R}$ for every irrational $\alpha \in \mathbb{R}$. Hence there exists a family of dense $G_\delta-$sets such that their intersection is not dense (in particular empty set) namely $(\mathbb{Q} \cup \{\alpha\})^c$ where $\alpha$ runs over irrationals. Hence $\mathbb{R}$ with usual metric is neither directed Baire space nor weakly directed Baire.*

Also, by definition itself it is clear that every directed Baire space is weakly directed Baire, but the converse does not hold as shown by Example 1.3.

**Example 1.3.** *Consider $\mathbb{R}$ with the topology obtained from the basis $\mathscr{B} = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{0\}$. Since every dense set contains $\{0\}$, the intersection of every family of dense $G_\delta-$sets is non-empty. For every irrational $\alpha \in \mathbb{R}$ the set $(\mathbb{Q} - \{0\}) \cup \{\alpha\}$ is countable, and each*

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*singleton sets of $\mathbb{R}$ is closed, the above defined set is an $F_\sigma-$set and its complement is a dense $G_\delta-$subset. Hence there is a family of dense $G_\delta-$sets such that their intersection is not dense (which equals $\{0\}$) namely, $(\mathbb{Q}-\{0\}\cup\{\alpha\})^c$ where $\alpha$ runs over irrationals. Hence this topological space is weakly directed Baire, Baire and second category but not directed Baire.*

**Example 1.4.** *There is a space which is weakly directed Baire and hence second category but not Baire and directed Baire. Let $X=\mathbb{Q}\cup(1,2)$ where $\mathbb{Q}$ denotes the set of the rational numbers in $(0,1)$ of the real line. Topologize $X$ by the subbasis $\{\{(a,b)\mid a,b\in(1,2)\}\cup\mathbb{Q}\}$. Then $X$ is not Baire because the open set $\mathbb{Q}$ is of first category. But $X$ is weakly directed Baire in itself as the open subset $(1,2)$ is.*

**Example 1.5.** *Consider $(X,\tau)$ where $X=[0,\infty)$ and the topology $\tau$ has $\{[a,\infty)-F\mid a\in X$ and $F$ is a finite subset of $X\}$ as its basis. By its construction, it is of first category, so it is none of the Baire, second category, directed Baire and weakly directed Baire.*

**Example 1.6.** *There is a space which is second category but not weakly directed Baire, Baire and directed Baire. Topologize $X=\mathbb{Q}\cup(0,2)$ by the subbasis $\{\{(a,b)\mid a,b\in(0,2)\}\cup\mathbb{Q}\}$. Since $\mathbb{Q}$ is a first category set, $X$ fails to be a Baire space and so $X$ is not a directed Baire space. But $X$ is of second category in itself as the open subset $(1,2)$ is. Since $\mathbb{Q}$ is countable, $\{r_1,r_2,r_3\ldots\}$ be the sequential arrangements of $\mathbb{Q}$ and $I$ is the collection of all irrationals in $(0,2)$. Define $H_i^\alpha=\{r_i\}\cup\{I-\{\alpha\}\}$ where $\alpha$ is irrational in $(0,2)$. $\{H_i^\alpha\mid i\in\mathbb{N}$ and $\alpha\in I\}$ is the collection of dense $G_\delta-$sets in $X$ whose intersection is empty.*

**Theorem 1.7.** *Every compact $p-$space $X$ is a directed Baire space.*

*Proof.* Suppose $\{U_\alpha\mid\alpha\in\Delta\}$ is a family of dense $G_\delta-$subsets of $X$, and take $U$ as an open subset of $X$. It is enough to show that $U\cap(\bigcap_\alpha U_\alpha)\neq\emptyset$. Since in a $p-$space, every $G_\delta-$sets are open, it is possible to construct a non-empty open subset $V_\alpha$ of $X$ such that $V_\alpha\subset U\cap U_\alpha$. Defining recursively we have non-empty open subsets $\{V_\alpha\}$ of $X$ such that $V_{\alpha+1}\subset V_\alpha\cap U_{\alpha+1}$, for each $\alpha$ and $V_{\alpha+1}$ is the successor of $V_\alpha$. Suppose $\bigcap_\alpha V_\alpha=\emptyset$. Then define $W_\alpha=X-cl(V_\alpha)$, so that $\{W_\alpha\}$ is an open cover of $X$. As $X$ is compact, we find a finite sub-cover $\{W_{\alpha_1},W_{\alpha_2},\ldots W_{\alpha_n}\}$ such that $\{W_{\alpha_k}\}$ covers $X$. Since $W_\alpha=X-cl(V_\alpha)$ we have $cl(V_\alpha)\subset int(X-W_\alpha)=X-cl(W_\alpha)$. Hence $V_\alpha\subset X-cl(W_\alpha)$ for every $\alpha$. But $\emptyset=\bigcap_{k=1}^{n}(X-cl(W_{\alpha_k}))\supset\bigcap_{n=1}^{n}V_{\alpha_k}=V_{\alpha_n}$, which is a contradiction. Thus, $\emptyset\neq\bigcap_\alpha V_\alpha\subset U\cap(\bigcap_\alpha U_\alpha)$. □

**Corollary 1.8.** *Every submaximal compact space is a directed Baire space.*

*Proof.* Since in submaximal spaces, every dense set is open, the proof follows. □

Since every compact $p-$spaces are also Baire spaces, by Theorem 1.7, there are spaces which are Baire, directed Baire, weakly directed Baire and second category. Example 1.2 shows that there are Baire spaces which are not weakly directed Baire space, Example 1.4 shows that there are weakly directed Baire space which are neither Baire nor directed Baire spaces. The following arrow diagram shows the relation between the four spaces namely, Baire, second category, directed Baire and weakly directed Baire.

$$\begin{tikzcd}
\textit{Baire} \arrow[r] & \textit{second category} \\
\textit{directed Baire} \arrow[u] \arrow[r] & \textit{weakly directed Baire} \arrow[u]
\end{tikzcd}$$

Hence Baire, second category, directed Baire and weakly directed Baire are independent concepts.

## 2. Characterization for directed Baire space

The hereditary property of (resp. weakly) directed Baire spaces need not be true for arbitrary spaces. Here, we prove that for some classes of subspaces, these properties are hereditary. This gives us a new characterization for directed Baireness of spaces.

**Theorem 2.1.** *In a directed Baire space $X$, if $H\subset X$ and $A\subset H$ where $A$ is a $G_\delta$ set implies that $int(\overline{A})$ is dense in $H$, then $H$ is a directed Baire space.*

*Proof.* Since $int(\overline{A})$ is dense in $H$, we have $\overline{H}\subset\overline{(H\cap int\overline{A})}$. For, $x\in\overline{H}$ implies $U_x\cap H\neq\emptyset$ for every neighborhood $U_x$ of $x$. Therefore $y\in U_x\cap H$ is a neighborhood of $y$ in $H$, Since $int(\overline{A})$ is dense in $H$, $(U_x\cap H)\cap int(\overline{A})\neq\emptyset$ so that $U_x\cap(H\cap int(\overline{A}))\neq\emptyset$. Hence $\overline{H}\subset\overline{(H\cap int\overline{A})}$. Since $A\subset H$, $\overline{A}\subset\overline{H}\subset\overline{H\cap int(\overline{A})}$. But $\overline{A}\subset\overline{H}\subset\overline{H\cap int(\overline{A})}\subset\overline{A}$. Therefore, $\overline{A}=\overline{H}=\overline{H\cap int(\overline{A})}$.
Let $\{D_\alpha:\alpha\in\Delta\}$ be a family of dense $G_\delta-$subsets in $H$. Then $\overline{H\cap D_\alpha}=\overline{H}$ for every $\alpha$. Define $A^+=A\cup(X-\overline{H})$, and $D_\alpha^+=D_\alpha\cup(X-\overline{H})$ for every $\alpha$. Then $A^+$ and $D_\alpha^+$ are dense $G_\delta-$sets in $X$ for every $\alpha$ and $X$ is directed Baire $A^+\cap(\bigcap_\alpha D_\alpha)$ is dense in $X$. Hence $(A\cap(\bigcap_\alpha D_\alpha))\cup(X-\overline{H})$ is dense in $X$.
Now $\overline{A}=\overline{H}$ implies $int(\overline{A})\subset\overline{(A\cap(\bigcap_\alpha D_\alpha))}$. Suppose not, there is an element $a\in int(\overline{A})$ with $a\notin\overline{(A\cap(\bigcap_\alpha D_\alpha))}$. That is, there exists $U_a$ such that $U_a\cap(A\cap(\bigcap_\alpha D_\alpha))=\emptyset$. For every $a\in int(\overline{A})$, there exists $V_a$ such that $V_a\cap(X-\overline{H})=\emptyset$. Since $a\in int(\overline{A})$, there exists an open set $V_a$, $V_a\subset\overline{A}$ so that $V_a\subset\overline{H}$. Take $W_a=U_a\cap V_a$. Then $W_a\cap((A\cap(\bigcap_\alpha D_\alpha))\cup(X-\overline{H}))=\emptyset$, which is a contradiction. $\overline{H}\subset\overline{(H\cap int(\overline{A}))}\subset\overline{(int(\overline{A}))}\subset\overline{A\cap(\bigcap_\alpha D_\alpha)}\subset\overline{H\cap(\bigcap_\alpha D_\alpha)}\subset\overline{H}$. Therefore, $\bigcap_\alpha D_\alpha$ is dense in $H$. □

**Remark 2.2.** *Observe that $H$ satisfies the hypothesis of Theorem 2.1 if $H$ is open or a regular closed or a dense $G_\delta$ subset of $X$.*

**Corollary 2.3.** *$X$ is a directed Baire space if and only if each non-empty open subspace is a weakly directed Baire space.*

*Proof.* If $X$ is directed Baire, then every non-empty open subspace is also directed Baire and hence weakly directed Baire. Conversely, suppose that each non-empty open subspace is weakly directed Baire. Let $\{D_\alpha \mid \alpha \in \Delta\}$ be a family of dense $G_\delta-$subsets of $X$. If $O$ is a non-empty open subset of $X$, then $O \cap D_\alpha$ are $G_\delta-$subsets of $O$ which are dense in $O$. Then $\bigcap_\alpha (O \cap D_\alpha) \neq \emptyset$ so that $O \cap (\bigcap_\alpha D_\alpha) \neq \emptyset$. Therefore, $\bigcap_\alpha D_\alpha$ is dense in $X$. □

**Remark 2.4.** *$X$ is directed Baire if and only if each non-empty open subspaces $U$ of $X$ cannot be written as the union of any family of nowhere dense $F_\sigma-$sets in $U$.*

**Theorem 2.5.** *If $\mathscr{O}$ is a family of open subsets of $X$ whose union is dense in $X$, then the following hold.*
*(a) If there is some non-empty $U_1 \in \mathscr{O}$ such that $U_1$ is weakly directed Baire, then $X$ is weakly directed Baire.*
*(b) If each member of $\mathscr{O}$ is directed Baire, then $X$ is directed Baire.*

*Proof.* (a) Let $U_1$ be a weakly directed Baire set in $\mathscr{O}$ and $\{O_\alpha \mid \alpha \in \Delta\}$ be a family of dense $G_\delta-$subsets of $X$. Then $U_1 \cap O_\alpha$ are dense $G_\delta-$sets in $U_1$. Since $U_1$ is weakly directed Baire, $\bigcap_\alpha (U_1 \cap O_\alpha) \neq \emptyset$ in $U_1$ which implies that $U_1 \cap (\bigcap_\alpha O_\alpha) \neq \emptyset$ so that $\bigcap_\alpha O_\alpha \neq \emptyset$. Hence $X$ is a weakly directed Baire space.
(b) Let $V_1$ be a non-empty open subset of $X$ and $\{O_\alpha \mid \alpha \in \Delta\}$ be a family of dense $G_\delta-$subsets of $X$. Since $\cup\{U_1 \mid U_1 \in \mathscr{O}\}$ is dense in $X$, $V_1 \cap U_1 \neq \emptyset$ for some $U_1 \in \mathscr{O}$. By hypothesis, $V_1 \cap U_1$ is a weakly directed Baire subspace of $U_1$. Now $O_\alpha \cap (U_1 \cap V_1)$ are dense $G_\delta-$sets of $U_1 \cap V_1$, $\bigcap_\alpha O_\alpha \cap (U_1 \cap V_1) \neq \emptyset$. Therefore, $(\bigcap_\alpha O_\alpha) \cap V_1 \neq \emptyset$. Hence $X$ is directed Baire. □

Now we characterize directed Baire spaces in terms of point finite $G_\delta-$cover of $X$. A family $\mathscr{U} = \{U_\alpha \mid \alpha \in \mathscr{I}\}$ is said to be point finite in a topological space $X$ if every point of $X$ lies in only finite members of $\mathscr{U}$, and it is locally finite at $x \in X$ if every neighborhood of $x$ intersects only finite members of $\mathscr{U}$.

**Theorem 2.6.** *A space $X$ is directed Baire if and only if every point finite $G_\delta-$cover of $X$ is locally finite at a dense set of points.*

*Proof.* Let $\mathscr{W} = \{U_\alpha \mid \alpha \in \Delta\}$ be a point finite $G_\delta-$cover of $X$ and $U$ be a non-empty open subset of $X$. Assume that $\mathscr{W}$ is not locally finite at any point of $U$. If $\mathscr{V} = \{V_\alpha\}$, $V_\alpha = U_\alpha \cap U$, then each open set in $\mathscr{W}$ intersects many members of $\mathscr{V}$. Put $\mathscr{F} = \{F_\alpha \mid F_\alpha \subset \Delta$ and $\Delta - F_\alpha$ is finite$\}$. Let $\mathscr{J}$ be the index set of the family $\mathscr{F}$. Now for each $J \in \mathscr{J}$, define $X_J = Bd(\cup\{V_\beta \mid \beta \in F_J\})$. Each $X_J$ is closed and $int(X_J) = \emptyset$, so that each $X_J$ is nowhere dense. Let $x \in U$. Since $\mathscr{W} = \{U_\alpha\}$ is point finite, there exists a $J' \in \mathscr{J}$ such that $x$ belongs to the members of $\{V_\alpha \mid \alpha \in \Delta - F_{J'}\}$, but no other members of $\mathscr{V}$. So $x \notin \cup\{V_\beta \mid \beta \in F_{J'}\}$. If $V$ is an open set containing $x$, then $V$ intersects some members of $\{V_\beta \mid \beta \in F_{J'}\}$, since $\mathscr{V} = \{V_\alpha\}$ is not locally finite at any point of $U$. Since $x \notin \cup\{V_\beta \mid \beta \in F_{J'}\}$, $x \in X_{J'}$. Hence $U = \cup(U \cap X_J)$, which is a contradiction, by Remark 2.4. Conversely, let $U$ be a non-empty open subset of $X$. Suppose $X$ is not directed Baire, $U = \cup X_\alpha$, where $int(\overline{X_\alpha}) = \emptyset$ for each $\alpha$ in the index set $\Delta$, by Remark 2.4. Let $U_0 = X$ and for each $\alpha \in \Delta$ define $U_\alpha = U - \bigcup_{\beta \leq \alpha} \overline{X_\beta}$. Let $\mathfrak{U} = \{U_\alpha\}$ where $\alpha \in \Delta$, which is a point finite $G_\delta-$cover of $X$. Then $\mathfrak{U}$ is locally finite at some $x$ in $U$. Let $O$ be an open set of $x$ such that $x \in O \subset U$. Since $int(X_\alpha) = \emptyset$, $O \nsubseteq \bigcup_{\beta \leq \alpha} \overline{X_\beta}$ for each $\alpha$. Thus, $O$ must intersect every member of $\mathfrak{U}$, which is a contradiction to locally finiteness of the point finite $G_\delta-$cover $\mathfrak{U}$. □

Blumberg [6] showed that for every real valued function $f$ defined on the real line $\mathbb{R}$, there exists a dense subset $D$ of $\mathbb{R}$ such that $f|_D$ is continuous. We will say that space $X$ has Blumberg's property with, respect to $Y$ if for every function $f : X \to Y$, there exists a dense subset $D$ of $X$ such that $f|_D$ is continuous. It is known [7] that for a metric space $X$, $X$ is a Baire space if and only if $X$ has Blumberg's property with respect to the reals. In Theorem 2.7, the similar result is proved for directed Baire space.

**Theorem 2.7.** *Let $Y$ contain an infinite discrete subset $D = \{y_\alpha \mid \alpha \in \Delta\}$. If $X$ satisfies Blumberg's property with respect to $Y$, then $X$ is a directed Baire space.*

*Proof.* Let $\mathbb{D} = \{y_\alpha \mid \alpha \in \Delta\}$ be a infinite discrete subset of $Y$. If $X$ is not a directed Baire space, then there is an open set $U$ in $X$ such that $U = (\bigcup_\alpha U_\alpha)$. Define a function $f : X \to Y$ as follows: let $f(x) = y_{\alpha_0}$ for each $x \in X - U$, where $y_{\alpha_0} \in D$ and let $f(x) = y_\beta$ for each $x \in U$, where $\beta = \min\{\alpha \mid x \in U_\alpha\}$. From the construction of the function $f$, $f|_D$ is not continuous for every dense subset $D$ of $X$. □

## 3. Product of directed Baire spaces

A directed Baire space in which every closed subspace is also directed Baire space is called a hereditarily directed Baire space. We discuss the product of directed Baire spaces. The following Lemma 3.1 is useful in the sequel.

**Lemma 3.1.** *Let $Y$ be a topological space, $(A,d)$ be a metric space and $C$ be a dense $G_\delta-$subset of $Y \times A$. Then given any finite subset $F$ of $A$, $\varepsilon > 0$ and non-empty open set $O$ of $Y$, there exists a finite subset $A'$ of $A$ and a dense $G_\delta-$subset $C_Y$ of $O$ such that*
*(i) for each $z \in F$, there exists $a \in A'$ with $d(z,a) < \varepsilon$*
*(ii) $C_Y \times A' \subseteq C$.*

*Proof.* For the given finite subset $F$ of $A$, define an open subset $V = \cup B(z,\varepsilon)$ of $A$ where union runs over the points of $F$. Since $C$ is dense in $Y \times A$, $(O \times V) \cap C \neq \emptyset$. Then $C_Y = P_X((O \times V) \cap C)$ and $A' \subset P_Y((O \times V) \cap C)$ are the requirements. □

**Theorem 3.2.** *If $X$ is a directed Baire space and $Y$ is a metrizable hereditarily directed Baire space, then $X \times Y$ is a downward-directed Baire space.*

*Proof.* Let $\{D_\alpha \mid \alpha \in \Delta\}$ be a family of decreasing dense $G_\delta-$sets in $X \times Y$. We prove that $\bigcap_\alpha D_\alpha$ is dense in $X \times Y$. Let $G$ and $H$ be any non-empty open sets in $X$ and $Y$, respectively. To prove $[\bigcap_\alpha D_\alpha] \cap (G \times H) \neq \emptyset$. Let $\{s_\alpha \mid \alpha \in \Delta\}$ be a net in $[0,\infty)$ with usual metric, which converges to 0.
Let $\alpha_1$ be the least member of $\Delta$. Since $D_{\alpha_1}$ is dense $(G \times H) \cap D_{\alpha_1} \neq \emptyset$. Define a dense $G_\delta-$set of $G$, $X_{\alpha_1} = P_X((G \times H) \cap D_{\alpha_1})$ and $Z_{\alpha_1} = \{y\}$, where $y \in P_Y((G \times H) \cap D_{\alpha_1})$.
By Lemma 3.1, for any finite subset $Z_{\alpha_1}$ of $Y$, non-empty open set $G$ in $X$, dense $G_\delta-$set $D_\beta$ of $X \times Y$ and $s_\beta > 0$, we can find a finite subset $Y_\beta$ of $Y$ and a dense $G_\delta$ subset $X_\beta$ of $G$ such that
(i) for each $z \in Z_{\alpha_1}$, one can find $y \in Y_\beta$ with $d(z,y) < s_\beta$
(ii) $X_\beta \times Y_\beta \subseteq D_\beta$. Then we define
(iii) $Z_\beta = Z_{\alpha_1} \cup Y_\beta$, where $\beta$ is the successor of $\alpha_1$.
Continuing in this way, we reached a family of dense $G_\delta-$subsets $\{X_\alpha \mid \alpha \in \Delta\}$ of $G$. Since $X$ is a directed Baire space, $\bigcap_\alpha X_\alpha \neq \emptyset$. Choose $x \in \bigcap_\alpha X_\alpha$ and define, for each $\alpha \in \Delta$, the dense $G_\delta-$subsets $\{W_\alpha \mid \alpha \in \Delta\}$ of $Y$ so that $\{x\} \times W_\alpha = (\{x\} \times Y) \cap D_\alpha$.
Let $Z^+ = \bigcup_\alpha Z_\alpha \subset H$. Since $Y$ is hereditarily directed Baire, $\overline{Z^+}$ is directed Baire which implies $W_\alpha \cap Z^+$ is dense in $Z^+$ for each $\alpha \in \Delta$. For, let $z \in Z^+$, $\alpha \in \Delta$ and $\varepsilon > 0$ be given. Since the net $(s_\alpha)$ converges to 0, for the neighborhood $[0,\varepsilon)$ of 0, we can find $\delta \in \Delta$ such that $0 \leq s_\alpha < \varepsilon$ for every $\alpha > \delta$. Choose $\rho \in \Delta$ sufficiently large so that $\rho > \alpha$, $s_\rho < \varepsilon$ and $z \in Z_\rho$. There is an element $y \in Y_{\rho_1}$ with $d(y,z) < s_{\rho_1} < s_\delta < \varepsilon$ and $(x,y) \in D_{\rho_1} \cap (\{x\} \times Y)$, which implies that $y \in W_{\rho_1} \subset W_\alpha$, where $\rho_1$ is the successor of $\rho$. Thus, $y \in B(z,\varepsilon) \cap (W_\alpha \cap Z^+)$ so that $B(z,\varepsilon) \cap (W_\alpha \cap Z^+)$ is non-empty. Choosing $y \in (\bigcap_\alpha W_\alpha) \cap H \cap \overline{Z^+}$, we get that $(x,y) \in [\bigcap_\alpha D_\alpha] \cap (G \times H)$. □

**Theorem 3.3.** *Let X and Y be directed Baire spaces. If either of the space has a countable pseudo base, their product is directed Baire.*

*Proof.* Assume that $X \times Y$ is not directed Baire. We can find an open set $G \times H$ in the product space such that $(G \times H) \cap (\bigcap_\alpha D_\alpha) = \emptyset$ where $\{D_\alpha \mid \alpha \in \Delta\}$ is a family of dense $G_\delta-$ sets in $X \times Y$. Since $D_\alpha$ are $G_\delta-$sets, $D_\alpha = \bigcap_{n=1}^{\infty} D_\alpha^n$ where $D_\alpha^n$ are open in $X \times Y$. Since $D_\alpha$ is dense, each $D_\alpha^n$ is also dense.
Let $\{V_k\}$ be a countable pseudo base for $Y$. Now for each $n,k$ and $\alpha$, define $h_\alpha^{n,k} = D_\alpha^n \cap (U \times V_k)$. Also, define $H_\alpha^{n,k} = P_X(h_\alpha^{n,k})$ so that $H_\alpha^{n,k}$ are open. Also, $D_\alpha^n$ is dense in $G \times H$ implies $D_\alpha^n \cap (G \times V_k)$ is dense in $G \times V_k$ which implies $h_\alpha^{n,k}$ is dense in $G \times V_k$. For any open set $U_1$ in $G$, $U_1 \times V_k$ is an open set in $U \times V_k$. Therefore, $(U_1 \times V_k) \cap h_\alpha^{n,k} \neq \emptyset$ implies $U_1 \cap P_X(h_\alpha^{n,k}) \neq \emptyset$ which implies $U_1 \cap H_\alpha^{n,k} \neq \emptyset$. Therefore, each $H_\alpha^{n,k}$ is dense in $G$. Since $X$ is directed Baire, $G$ will become directed Baire, by Remark 2.4.
Since $G$ is directed Baire, $\bigcap_{n,k} [G \cap H_\alpha^{n,k}]$ are dense in $G$ and so $\bigcap_{n,k} [G \cap H_\alpha^{n,k}] \neq \emptyset$. Therefore, there exists some $a \in G$ with $a \in \bigcap_{n,k} [G \cap H_\alpha^{n,k}]$ which gives $a \in H_\alpha^{n,k}$ for every $n,k$.
Define $D_\alpha^n(a) = \{b \in H \mid (a,b) \in D_\alpha^n\}$. For each $V_k$, $(a,b) \in D_\alpha^n \cap (G \times V_k)$ for all $n,k$ implies $(a,b) \in \bigcap_n [D_\alpha^n \cap (G \times V_k)]$ which gives that $(a,b) \in D_\alpha \cap (G \times V_k)$. Therefore, there is some $b \in V_k$ with $(a,b) \in D$ so that $b \in V_k$ such that $b \in D_\alpha^n(a)$. Therefore, $D_\alpha^n(a) \cap V_k \neq \emptyset$. Therefore, $D_\alpha^n(a)$ is dense in $H$. Also, $D_\alpha^n(a)$ is an open set.
Since $Y$ is directed Baire, $H$ is also directed Baire and hence $\bigcap_{n,\alpha} D_\alpha^n(a) \neq \emptyset$. Therefore, we can find $z \in H$ with $z \in \bigcap_{n,\alpha} D_\alpha^n(a)$ and hence $(a,z) \in \bigcap_{n,k} D_\alpha^n = D_\alpha$ which is not possible. Thus, $\bigcap_\alpha D_\alpha \neq \emptyset$ and so $G \times H$ is a weakly directed Baire space. □

## 4. Product of Volterra spaces

In 1993, the class of Volterra spaces was introduced by Gauld and Piotrowski [8]. A topological space $(X,\tau)$ is said to be *Volterra* [8, 9] (resp. *weakly Volterra* [8]) if the intersection of any two dense $G_\delta-$ sets in $X$ is dense (resp. non-empty). By the definition itself, every Baire space is Volterra and every space of second category is weakly Volterra. Is there exists a Baire space $X$ whose square $X^2$ is not Baire ? The first space with such properties, constructed under the Continuum Hypothesis, is due to Oxtoby [3]. This example was improved to an absolute one by Cohen [10] relying on forcing. Finally, Fleissner and Kunen [11] constructed a metrizable Baire space $X$ whose square $X^2$ is not Baire in ZFC by direct combinatorial arguments. Gauld, Greenwood and piotrowski [12], using stationary sets in the result of Flessner proved that there exists a metric Baire space whose square is not even Weakly Volterra. Spadaro [13] proved that the product of a hereditarily volterra space and a hereditarily Baire space may fail to be weakly volterra. In [14], Moors proved that "The Product of a Baire space with a hereditarily Baire metric space is Baire". In that proof, he use Choquet game [15]-[17] played on $X$ to get a non-empty subset for any given sequence of dense open sets in $X$.

**Theorem 4.1.** *If X is Baire and Y is metrizable hereditarily Volterra, then $X \times Y$ is a Volterra space.*

*Proof.* Suppose that $C$ and $D$ are two dense $G_\delta-$sets in $X \times Y$. Let $G$ and $H$ be non-empty open sets in $X$ and $Y$, respectively. To prove $(C \cap D) \cap (G \times H) \neq \emptyset$. Since $C$ and $D$ are dense $G_\delta-$sets, $C = \bigcap_{n=1}^{\infty} C_n$ and $D = \bigcap_{n=1}^{\infty} D_n$, where $\{C_n\}$ and $\{D_n\}$ are decreasing sequence of open dense sets in $X \times Y$.
Denseness of $C$ gives that $(G \times H) \cap C \neq \emptyset$. Define a dense $G_\delta-$set of $G$, $C_1 = P_X((G \times H) \cap C)$ and $Z_1^C = \{b^C\}$, where $b^C \in P_Y((G \times H) \cap C)$.
Also, since $D$ is dense, $(G \times H) \cap D \neq \emptyset$. Define a dense $G_\delta-$set of $G$, $D_1 = P_X((G \times H) \cap D)$ and $Z_1^D = \{b^D\}$, where $b^D \in P_Y((G \times H) \cap C)$.
Also, define $Z_1 = Z_1^C \cup Z_1^D$.
By Lemma 3.1, for a finite set $Z_1$ of $Y$, non-empty open set $G$ in $X$, dense $G_\delta-$set $C$ of $X \times Y$ and $\frac{1}{2} > 0$, there is a finite subset $Z_2^C$ of $Y$ and a dense $G_\delta$ subset $C_2$ of $G$ such that
(i) for every $a \in Z_1$, there is some $b \in Z_2^C$ with $d(a,b) < \frac{1}{2}$
(ii) $C_2 \times Z_2^C \subseteq C$.

Also, by Lemma 3.1, for a finite set $Z_1$ of $Y$, non-empty open set $G$ in $X$, dense $G_\delta-$set $D$ of $X \times Y$, and $\frac{1}{2} > 0$ there is a finite subset $Z_2^D$ of $Y$ and a dense $G_\delta$ subset $D_2$ of $G$ such that
(i) for every $a \in Z_1$, there is some $b \in Z_2^D$ with $d(a,b) < \frac{1}{2}$
(ii) $D_2 \times Z_2^D \subseteq D$.
Define $Z_2 = Z_1 \cup Z_2^C \cup Z_2^D$.
Continuing in this way, for every $n \in \mathbb{D}$, by Lemma 3.1, given any finite subset $Z_{n-1}$ of $Y$, non-empty open set $G$ in $X$, dense $G_\delta-$set $C$ of $X \times Y$ and $\frac{1}{n} > 0$, there is a finite subset $Z_n^C$ of $Y$ and a dense $G_\delta$ subset $C_n$ of $G$ such that
(i) for every $a \in Z_{n-1}$, there is some $b \in Z_n^C$ with $d(a,b) < \frac{1}{n}$
(ii) $C_n \times Z_n^C \subseteq C$.
Also, given any finite subset $Z_{n-1}$ of $Y$, non-empty open set $G$ in $X$, dense $G_\delta-$set $D$ of $X \times Y$ and $\frac{1}{n} > 0$, there is a finite subset $Z_n^D$ of $Y$ and a dense $G_\delta$ subset $D_n$ of $G$ such that
(i) for every $a \in Z_{n-1}$, there is some $b \in Z_n^D$ with $d(a,b) < \frac{1}{n}$
(ii) $D_n \times Z_n^D \subseteq D$.
Define $Z_n = Z_{n-1} \cup Z_n^C \cup Z_n^D$.
The countable collection $\{C_n \mid n \in \mathbb{D}\} \cup \{D_n \mid n \in \mathbb{D}\}$ of dense $G_\delta-$subsets can be enumerated as a sequence of dense $G_\delta-$sets $\{H_i \mid i \in \mathbb{D}\}$ of $G$. Since every $H_i$ is a dense $G_\delta-$set, $H_i = \bigcap_{j=1}^{\infty} H_i^j$ where $H_i^j$ is a dense open set in $G$. Since a countable union of countable set the family $\{H_i^j \mid i,j \in \mathbb{D}\}$ also can be enumerated as a sequence of dense open sets $\{O_m \mid m \in \mathbb{D}\}$. Since $X$ is a Baire space, the open subset $G$ is also a Baire space. Therefore, $\bigcap_{m=1}^{\infty} O_m \neq \emptyset$.
Choose $s \in \bigcap_{m=1}^{\infty} O_m$ and define, $C(s) = \{t \in H \mid (s,t) \in C\}$ and $D(s) = \{t \in H \mid (s,t) \in D\}$. Now $C(s) = (\bigcap_m C_m)(s) = \bigcap_m [C_m(s)]$, because $t \in (\bigcap_m C_m)(s) \Leftrightarrow (s,t) \in \bigcap_m C_m \Leftrightarrow (s,t) \in C_m$ for all $m \Leftrightarrow t \in C_m(s)$ for all $m \Leftrightarrow t \in \bigcap_m [C_m(s)]$. Therefore, $C(s)$ is a $G_\delta-$set. Similarly, $D(s)$ is also a $G_\delta-$set.
Let $S = \bigcup_{n=1}^{\infty} Z_n \subset H$. Since $Y$ is hereditarily Volterra, $\overline{S}$ is Volterra and hence $C(s) \cap S$ and $D(s) \cap S$ are dense in $S$.
For, $a \in Z$, and $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ sufficiently large so that $1/N < \varepsilon$ and $a \in Z_{N-1}$. There is some $t \in Z_N^C$ such that $d(t,a) < 1/N < \varepsilon$ and $C_N \times Z_N^C \subseteq C$. Hence $(s,t) \in (C_N \times Y) \cap (\{s\} \times Y) \subset C$, which implies that $t \in C(s)$. Thus, $t \in B(a,\varepsilon) \cap C(s) \cap Z \neq \emptyset$. Similarly, $D(s) \cap Z$ is also dense in $Z$. Choosing $t \in C(s) \cap D(s) \cap H \cap \overline{Z}$, we get $(s,t) \in C \cap D \cap (G \times H)$. Hence $C \cap D$ is dense in the product space. □

In Theorem 4.1 above, the hereditary property cannot be dropped, since Fleissner and Kunen [11] constructed a metrizable Baire space $X$ whose square $X^2$ is not Baire. Since Spadaro [13], shows that the product of a hereditarily volterra space and a hereditarily Baire space may fail to be weakly volterra the metrizability of the Volterra space cannot be dropped in the above theorem.
Piotrowski raised a question that, "*Whether* $X \times [0,1]$ *is Volterra or not*? for any Volterra space $X$". As a partial answer to this question, in Corollary 4.2 below, we consider a subfamily of Volterra spaces consisting of metrizable hereditarily Volterra space, and proved that cartesian product of $X$ and $[0,1]$ is again a Volterra space.

**Corollary 4.2.** *If $(X,\tau)$ is a metrizable hereditarily Volterra space, then $X \times [0,1]$ is also a Volterra space.*

*Proof.* It is well known that, a subset $A$ of a complete metric space $(M,d)$ is complete if and only if $A$ is a closed subset of $M$ and consequently, $[0,1]$ is complete. Since $[0,1]$ is Baire, $X \times [0,1]$ is Volterra, by Baire Category Theorem and Theorem 4.1. □

## 5. Conclusion

In this paper, we have introduced the concepts of directed Baire and weakly directed Baire spaces. Since every compact p-spaces are also Baire spaces, we have proved that there are spaces which are Baire, directed Baire, weakly directed Baire and second category. Also, it is shown that there are Baire spaces which are not weakly directed Baire space and there are weakly directed Baire space which are neither Baire nor directed Baire spaces, by giving examples. Hence we have proved that the concepts namely, Baire, second category, directed Baire and weakly directed Baire are independent. We have proved that the product of directed Baire spaces is also Directed Baire if either of the space has a countable pseudo base. Also, we have provided partial answer for the question raised by Piotrowski regarding product of Volterra spaces. The results of this article can also be applied on generalized topological spaces and ideal topological spaces by some suitable modifications. We hope that this work will provide the basis for further study on directed Baire spaces.

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# On Dual-Hyperbolic Numbers with Generalized Fibonacci and Lucas Numbers Components

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**Abstract**

Dual-hyperbolic Fibonacci and Lucas numbers with Fibonacci and Lucas coefficients are introduced by Cihan et al. and some identities and theorems are given regarding modules and conjugates of these numbers. Later, generating function and Binet's formula with the help of this generating function have been derived. Also, Binet formula, Cassini's, Catalan's, d'Ocagne's, Honsberger and Tagiuri identities are found for dual-hyperbolic numbers with generalized Fibonacci and Lucas coefficients. While these operations are being done, we will benefit from the well-known Fibonacci and Lucas identitites. Moreover, it is seen that the results which are obtained for the values $p = 1$ and $q = 0$ corresponds to the theorems in the article by Cihan et al. [1].

## 1. Introduction

Italian mathematician Leonardo Fibonacci's Liber Abaci was one of the most important books on mathematics in the Middle Ages. Through this book mathematicians introduced Fibonacci number sequence concept. Several studies have been conducted with respect to Fibonacci numbers and Fibonacci quaternions [2]-[5].
Dual-hyperbolic numbers with Fibonacci and Lucas coefficients which is constitutes a new number system have been introduced by Cihan and her colleagues [1]. In this article, the dual-hyperbolic number system has been generalized based on the article [1].
Firstly, addition, multiplication, modules and conjugates of these numbers have been defined and the fundamental identities for these numbers regarding these operations have been proven. Then, we have defined generating function and this function helped us to find Binet's formula. Additionally, d'Ocagne's, Honsberger, Tagiuri, Catalan identities have been obtained and Cassini's identity has been given in case of $r = 1$ for the Catalan identity. Finally, we have discussed special cases and have given examples.

## 2. Preliminaries

The Fibonacci and Lucas numbers have many interesting properties and applications. Initial conditions for the Fibonacci and Lucas numbers are defined as follows respectively

$$F_0 = 0, \quad F_1 = 1, \ldots, \quad F_{n+1} = F_n + F_{n-1}, \quad n \geq 1$$

and

$$L_0 = 2, \quad L_1 = 1, \ldots, \quad L_{n+1} = L_n + L_{n-1}, \quad n \geq 1$$

where $F_n$ and $L_n$ denote the $n$-th Fibonacci and Lucas numbers, respectively.
Binet formula for the $n$-th Fibonacci and Lucas numbers are given by the following relation

$$F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n), \qquad L_n = \alpha^n + \beta^n, \quad n \geq 1$$

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(see [2]).
On the other hand, Horadam introduced generalized Fibonacci numbers with the initial conditions as follows

$$H_1 = p, \quad H_2 = p+q, \quad p, q \in Z$$

where the recurrence relation is defined by

$$H_n = H_{n-1} + H_{n-2}, \quad n \geq 3,$$

or

$$H_n = (p-q)F_n + qF_{n+1}.$$

In the above equation, if we take $p=1$ and $q=0$, then the generalized Fibonacci number becomes Fibonacci number. If we take $p=1$ and $q=2$, then the generalized Lucas number becomes Lucas number. Furthermore, Horadam investigated Binet formula for the $n$-th generalized Fibonacci number such that

$$H_n = \frac{1}{2\sqrt{5}}\left(\alpha^n - \mu\beta^n\right)$$

(see [2]).

The set of dual-hyperbolic numbers is defined by

$$DH = \left\{w = z_1 + z_2\varepsilon | z_1, z_2 \in H \, where, \; \varepsilon^2 = 0, \varepsilon \neq 0\right\}.$$

If we consider two hyperbolic numbers $z_1 = x_1 + x_2 j$ and $z_2 = y_1 + y_2 j$, then any dual-hyperbolic number can be written as

$$w = x_1 + x_2 j + y_1\varepsilon + y_2 j\varepsilon.$$

There exist five different conjugates and these conjugates are given as follows

$$\begin{array}{ll}
|\omega|^{\dagger_1} = \bar{z}_1 + \varepsilon\bar{z}_2, & \text{hyperbolic conjugation} \\
|\omega|^{\dagger_2} = z_1 - \varepsilon z_2, & \text{dual conjugation} \\
|\omega|^{\dagger_3} = \bar{z}_1 - \varepsilon\bar{z}_2, & \text{coupled conjugation} \\
|\omega|^{\dagger_4} = \bar{z}_1\left(1 - \varepsilon\frac{z_2}{z_1}\right) \quad (\omega \in DH - A), & \text{dual} - \text{hyperbolic conjugation} \\
|\omega|^{\dagger_5} = z_2 - \varepsilon z_1, & \text{anti} - \text{dual conjugation}
\end{array}$$

where "$-$ " denotes the standard complex conjugation and the zero divisors of $DH$ is defined by the set $A$ [6]. Namely, $DH - A$ is a multiplicative group. The dual hyperbolic numbers form a commutative ring with 0 characteristic. Unlike quaternions, the multiplication of dual-hyperbolic numbers with generalized Fibonacci and Lucas number has a commutative ring structure. However, multiplication of dual-hyperbolic numbers with generalized Fibonacci and Lucas number constitutes two-dimensional Complex Clifford and 4-dimensional Real Clifford algebra structure.

## 3. Properties of Dual-Hyperbolic numbers with generalized Fibonacci and Lucas coefficients

The dual-hyperbolic Fibonacci and Lucas numbers are defined as

$$DHF_n = F_n + F_{n+1}j + F_{n+2}\varepsilon + F_{n+3}j\varepsilon$$

and

$$DHL_n = L_n + L_{n+1}j + L_{n+2}\varepsilon + L_{n+3}j\varepsilon$$

respectively. Here, $F_n$ and $L_n$ are the $n$-th generalized Fibonacci number and Lucas numbers respectively and $\varepsilon$ denotes dual unit $\left(\varepsilon^2 = 0, \; \varepsilon \neq 0\right)$, $j$ denotes imaginary unit $\left(j^2 = 1\right)$, $j\varepsilon$ denotes imaginary-dual unit $\left(j\varepsilon^2 = 0\right)$. After these numbers have been defined in the article [1], some identities regarding the modules, conjugates have been obtained for dual-hyperbolic Fibonacci and Lucas numbers. Then, negadual-hyperbolic Fibonacci, negadual-hyperbolic Lucas, d'Ocagne's, Cassini, Catalan identities and the correspondence of Binet formula have been given for these numbers. Now, Let's define the dual-hyperbolic number system with generalized Fibonacci and Lucas coefficients by considering the study [1].

**Definition 3.1.** *$H_n$ is called as n-th Fibonacci number which have either $H_n = H_{n-1} + H_{n-2}$, $n \geq 3$ or $H_n = (p-q)F_n + qF_{n+1}$ the recurrence relations and depending on the initial values such that*

$$H_1 = p, \; H_2 = p+q, \; H_3 = 2q+3p, \ldots \qquad (p, q \in Z).$$

*Then, the sets of generalized Fibonacci and Lucas sequences are defined*

$$DHX = \left\{DHX_n = R_n + \varepsilon R_n^* = \left(H_n + jH_{n+1}\right) + \varepsilon\left(H_{n+2} + jH_{n+3}\right) \;|\, H_n \text{ Generalized Fibonacci Number}\right\}$$

*and*

$$DHY = \left\{DHY_n = P_n + \varepsilon P_n^* = \left(V_n + jV_{n+1}\right) + \varepsilon\left(V_{n+2} + jV_{n+3}\right) \;|\, V_n \text{ Generalized Lucas Number}\right\}$$

*where $\varepsilon$ $\left(\varepsilon^2 = 0, \; \varepsilon \neq 0\right)$, $j$ $\left(j^2 = 1\right)$ and $i\varepsilon$ $\left((j\varepsilon)^2 = 0\right)$, denote dual unit, imaginary unit and dual-imaginary unit, respectively. So the base elements of dual-hyperbolic numbers with generalized Fibonacci and Lucas coefficients are $(1, \; j, \; \varepsilon, \; j\varepsilon)$. Multiplication scheme of these base elements are given in Table 1.*

| $\times$ | $1$ | $j$ | $\varepsilon$ | $j\varepsilon$ |
|---|---|---|---|---|
| $1$ | $1$ | $j$ | $\varepsilon$ | $j\varepsilon$ |
| $j$ | $j$ | $1$ | $j\varepsilon$ | $\varepsilon$ |
| $\varepsilon$ | $\varepsilon$ | $j\varepsilon$ | $0$ | $0$ |
| $j\varepsilon$ | $j\varepsilon$ | $\varepsilon$ | $0$ | $0$ |

**Table 1:** Multiplication scheme of dual-hyperbolic units

If two dual-hyperbolic numbers with generalized Fibonacci coefficients are $DHX_n^1 = R_n + R_n^*\varepsilon = H_n + H_{n+1}j + H_{n+2}\varepsilon + H_{n+3}j\varepsilon$ and $DHX_n^2 = K_n + K_n^*\varepsilon = G_n + G_{n+1}j + G_{n+2}\varepsilon + G_{n+3}j\varepsilon$ then the addition, subtraction and multiplication operations of these numbers are defined as

$$\begin{aligned} DHX_n^1 \pm DHX_n^2 &= (R_n + R_n^*\varepsilon) \pm (K_n + K_n^*\varepsilon) \\ &= (H_n + H_{n+1}j + H_{n+2}\varepsilon + H_{n+3}j\varepsilon) \pm (G_n + G_{n+1}j + G_{n+2}\varepsilon + G_{n+3}j\varepsilon) \\ &= (H_n \pm G_n) + (H_{n+1} \pm G_{n+1})j + (H_{n+2} \pm G_{n+2})\varepsilon + (H_{n+3} \pm G_{n+3})j\varepsilon \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} DHX_n^1 \times DHX_n^2 &= (R_n + R_n^*\varepsilon) \times (K_n + K_n^*\varepsilon) \\ &= (H_n + H_{n+1}j + H_{n+2}\varepsilon + H_{n+3}j\varepsilon) \times (G_n + G_{n+1}j + G_{n+2}\varepsilon + G_{n+3}j\varepsilon) \\ &+ (H_nG_n + H_{n+1}G_{n+1}) + (H_nG_{n+1} + H_{n+1}G_n)j \\ &+ (H_nG_{n+2} + H_{n+1}G_{n+3} + H_{n+3}G_{n+1} + H_{n+2}G_n)\varepsilon \\ &+ (H_nG_{n+3} + H_{n+1}G_{n+2} + H_{n+2}G_{n+1} + H_{n+3}G_n)j\varepsilon \end{aligned} \tag{3.2}$$

respectively. Also, any dual-hyperbolic number with generalized Fibonacci coefficient can be expressed as follows

$$DHX_n = R_n + R_n^*\varepsilon = (H_n + H_{n+1}j) + (H_{n+2} + H_{n+3}j)\varepsilon. \tag{3.3}$$

This yields five different conjugates. Thus, these five different conjugates can be defined as follows

$$DHX_n^{\dagger_1} = (H_n - H_{n+1}\,j) + (H_{n+2} - H_{n+3}\,j)\ \varepsilon, \qquad \text{hyperbolic conjugation} \tag{3.4}$$

$$DHX_n^{\dagger_2} = (H_n + H_{n+1}\,j) - (H_{n+2} + H_{n+3}\,j)\,\varepsilon, \qquad \text{dual conjugation} \tag{3.5}$$

$$DHX_n^{\dagger_3} = (H_n - H_{n+1}\,j) - (H_{n+2} - H_{n+3}\,j)\,\varepsilon, \qquad \text{coupled conjugation} \tag{3.6}$$

$$DHX_n^{\dagger_4} = (H_n - H_{n+1}\,j) \times \left(1 - \frac{H_{n+2} + H_{n+3}\,j}{H_n + H_{n+1}\,j}\varepsilon\right), \qquad \text{dual} - \text{hyperbolic conjugation} \tag{3.7}$$

$$DHX_n^{\dagger_5} = (H_{n+2} + H_{n+3}\,j) - (H_n + H_{n+1}\,j)\,\varepsilon, \qquad \text{anti} - \text{dual conjugation.} \tag{3.8}$$

Five different norms can be given for dual-hyperbolic numbers with generalized Fibonacci coefficients thanks to the definition of conjugates.

**Definition 3.2.** *Let $DHX_n$ be a dual-hyperbolic number with generalized Fibonacci coefficient. In this case, j-th modulus of $DHX_n$ are denoted by $|DHX_n|^2_{\dagger_i}$, $(j = 1,2,3,4,5)$ and are given as follows*

$$\begin{aligned} |DHX_n|^2_{\dagger_1} &= DHH_n \times DHH_n^{\dagger_1} \\ |DHX_n|^2_{\dagger_2} &= DHH_n \times DHH_n^{\dagger_2} \\ |DHX_n|^2_{\dagger_3} &= DHH_n \times DHH_n^{\dagger_3} \\ |DHX_n|^2_{\dagger_4} &= DHH_n \times DHH_n^{\dagger_4} \\ |DHX_n|^2_{\dagger_5} &= DHH_n \times DHH_n^{\dagger_5} \end{aligned}$$

**Proposition 3.3.** *Let $DHX_n$ be a dual-hyperbolic number with generalized Fibonacci coefficient. Then, the following identities are satisfied:*

$$DHX_n + DHX_n^{\dagger_1} = 2\,(H_n + H_{n+2}\varepsilon) \tag{3.9}$$

$$DHX_n \times DHX_n^{\dagger_1} = (H_n^2 - H_{n+1}^2) + 2\varepsilon\,(H_nH_{n+2} - H_{n+1}H_{n+3}) \tag{3.10}$$

$$DHX_n + DHX_n^{\dagger_2} = 2\,(H_n + H_{n+1}\,j) \tag{3.11}$$

$$DHX_n \times DHX_n^{\dagger_2} = [(2p-q)\,H_{2n+1} - eF_{2n+1}] + 2H_nH_{n+1}\,j \tag{3.12}$$

$$DHX_n + DHX_n^{\dagger_3} = 2\left(H_n + H_{n+3}\,j\varepsilon\right) \tag{3.13}$$

$$DHX_n \times DHX_n^{\dagger_3} = (H_n^2 - H_{n+1}^2) + j\varepsilon\left[2e(-1)^{n+1}\right] \tag{3.14}$$

$$DHX_n + DHX_n^{\dagger_4} = 2H_n + \varepsilon\frac{2H_{n+1}}{H_n^2 - H_{n+1}^2}\left[\left(H_{n+3}H_n - H_{n+1}H_{n+2}\right) + j\left(H_{n+2}H_n - H_{n+1}H_{n+3}\right)\right] \tag{3.15}$$

$$DHX_n \times DHX_n^{\dagger_4} = H_n^2 - H_{n+1}^2 \tag{3.16}$$

$$DHX_n + DHX_n^{\dagger_5} = (H_n + H_{n+2}) + (H_{n+1} + H_{n+3})\,j + H_{n+1}\varepsilon + H_{n+2}\,j\varepsilon \tag{3.17}$$

$$\begin{aligned} DHX_n \times DHX_n^{\dagger_5} = &\left(H_nH_{n+2} + H_{n+1}H_{n+3}\right) + j\left(H_nH_{n+3} + H_{n+1}H_{n+2}\right) \\ &+\varepsilon\left(H_{n+2}^2 + H_{n+3}^2 - H_{n+1}^2 - H_n^2\right) + 2j\varepsilon\left(H_{n+3}H_{n+2} - H_{n+1}H_n\right) \end{aligned} \tag{3.18}$$

*Proof.* (3.9): Using equations (3.1), (3.3) and (3.4), we obtain

$$DHX_n + DHX_n^{\dagger_1} = 2\left(H_n + H_{n+2}\varepsilon\right).$$

Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that $DHX_n + DHX_n^{\dagger_1} = 2\left(F_n + F_{n+2}\varepsilon\right)$.

(3.10): Considering equations (3.2), (3.3) and (3.4), the result is found by

$$\begin{aligned} DHX_n \times DHX_n^{\dagger_1} &= \left(H_n^2 + H_{n+1}^2\right) + 2\left(H_nH_{n+2} + H_{n+1}H_{n+3}\right)\varepsilon \\ &= \left(H_n^2 - H_{n+1}^2\right) + 2\varepsilon\left(H_nH_{n+2} - H_{n+1}H_{n+3}\right). \end{aligned}$$

Here, If the values $p = 1,\ q = 0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that $DHX_n \times DHX_n^{\dagger_1} = F_{2n+1} + 2F_{2n+3}\varepsilon$.

(3.11): From the equations (3.1), (3.3) and (3.5), we can reach the following identity

$$DHX_n + DHX_n^{\dagger_2} = 2\left(H_n + H_{n+1}\,j\right).$$

Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number $H_n$, $DHX_n + DHX_n^{\dagger_1} = 2\left(F_n + F_{n+2}\,j\right)$.

(3.12): Using the equations (3.2), (3.3), (3.5), using the identity $H_{n-1}^2 + H_n^2 = (2p-q)\,H_{2n-1} - eF_{2n-1}$ (see.ref. [2]) and simplifying we have

$$DHX_n \times DHX_n^{\dagger_2} = \left[(2p-q)\,H_{2n+1} - eF_{2n+1}\right] + 2H_nH_{n+1}\,j.$$

Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that $DHX_n \times DHX_n^{\dagger_2} = F_{2n+1} + 2F_nF_{n+1}\,j$.

(3.13): We can write the following equation by using the equations (3.1), (3.3) and (3.6)

$$DHX_n + DHX_n^{\dagger_3} = 2\left(H_n + H_{n+3}\,j\varepsilon\right).$$

Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that $DHX_n + DHX_n^{\dagger_3} = 2\left(F_n + F_{n+3}\,j\varepsilon\right)$.

(3.14): From equations (3.2), (3.3) and (3.6), we have

$$DHX_n \times DHX_n^{\dagger_3} = (H_n^2 - H_{n+1}^2) + j\varepsilon\left[2e(-1)^{n+1}\right].$$

While we are obtaining the above equation, the identity $H_nH_{n+r+1} - H_{n-s}H_{n+r+s+1} = (-1)^{n+s}eF_sF_{r+s+1}$ has been used [2]. Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that $DHX_n \times DHX_n^{\dagger_3} = -F_{n-1}F_{n+2} - 2(-1)^n\,j\varepsilon$.

(3.15): If we take into account the equations (3.1), (3.3) and (3.7), then the following identity can be easily seen

$$DHX_n + DHX_n^{\dagger_4} = 2H_n + \varepsilon\frac{2H_{n+1}}{H_n^2 - H_{n+1}^2}\left[\left(H_{n+3}H_n - H_{n+1}H_{n+2}\right) + j\left(H_{n+2}H_n - H_{n+1}H_{n+3}\right)\right].$$

Here, If the values $p=1,q=0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that

$$DHX_n+DHX_n^{\dagger_4}=2F_n+\varepsilon\frac{2F_{n+1}}{F_n^2-F_{n+1}^2}\left[(F_{n+3}F_n-F_{n+1}F_{n+2})+j\left(F_{n+2}F_n-F_{n+1}F_{n+3}\right)\right].$$

(3.16): By making the necessary operations with the help of the equations (3.2), (3.3), (3.7) and rearranging the last equation, the following identity can be given

$$DHX_n\times DHX_n^{\dagger_4}=H_n^2-H_{n+1}^2.$$

Here, If the values $p=1,q=0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that $DHX_n\times DHX_n^{\dagger_4}=F_n^2-F_{n+1}^2$.

(3.17): Considering the equations (3.1), (3.3) and (3.8), we have

$$DHX_n+DHX_n^{\dagger_5}=(H_n+H_{n+2})+(H_{n+1}+H_{n+3})j+H_{n+1}\varepsilon+H_{n+2}j\varepsilon.$$

Here, If the values $p=1,q=0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that $DHX_n+DHX_n^{\dagger_5}=F_n+F_{n+2}+(F_{n+1}+F_{n+3})J+F_{n+1}\varepsilon+F_{n+2}j\varepsilon$.

(3.18): If we use equations (3.2), (3.3), (3.8) and make the necessary calculations, then the rearranged equation yields

$$\begin{aligned}DHX_n\times DHX_n^{\dagger_5}=&(H_nH_{n+2}+H_{n+1}H_{n+3})+j\left(H_nH_{n+3}+H_{n+1}H_{n+2}\right)\\&+\varepsilon\left(H_{n+2}^2+H_{n+3}^2-H_{n+1}^2-H_n^2\right)+2j\varepsilon\left(H_{n+3}H_{n+2}-H_{n+1}H_n\right).\end{aligned}$$

Here, If the values $p=1,q=0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that

$$\begin{aligned}DHX_n\times DHX_n^{\dagger_5}=&(F_nF_{n+2}+F_{n+1}F_{n+3})+j\left(F_nF_{n+3}+F_{n+1}F_{n+2}\right)\\&+\varepsilon\left(F_{n+2}^2+F_{n+3}^2-F_{n+1}^2-F_n^2\right)+2j\varepsilon\left(F_{n+3}F_{n+2}-F_{n+1}F_n\right).\end{aligned}$$

□

**Theorem 3.4.** *Let $DHX_n$ and $DHX_{n-1}$ be two dual-hyperbolic numbers with generalized coefficients. There exist the following identities for these numbers and their conjugates:*

***i)*** $\left(DHX_n\times DHX_n^{\dagger_1}\right)+\left(DHX_{n-1}\times DHX_{n-1}^{\dagger_1}\right)=-\left[(2p-q)H_{2n}-eF_{2n}\right]+2\varepsilon\left(-H_{n+1}^2\right)$

***ii)*** $DHX_n^2=2H_nDHX_n-\left(DHX_n\times DHX_n^{\dagger_1}\right)+2\varepsilon\left(H_{n+2}^2-H_{n+1}H_{n+3}\right)+2j\varepsilon\left(H_{n+1}H_{n+2}\right)$

***iii)*** $\begin{aligned}DHX_n^2+DHX_{n-1}^2=&\,2(2p-q)DHX_{2n-1}-DHX_n\times DHX_n^{\dagger_1}-DHX_{n-1}\times DHX_{n-1}^{\dagger_1}+(2p-q)\left(2H_{2n+3}\varepsilon+2H_{2n+2}j\varepsilon\right)\\&-e\left(2F_{2n-1}+2F_{2n}j+2\left(F_{2n+3}+F_{2n+1}\right)\varepsilon+4F_{2n+2}i\varepsilon\right)-2H_{n+1}^2\varepsilon\end{aligned}$

***iv)*** $DHY_n\times DHX_n^{\dagger_1}-DHY_n^{\dagger_1}\times DHX_n=(-1)^n\left[\left(4p^2-8pq+8q^2\right)+\left(-6p^2\right)j\varepsilon\right].$

*Proof.* ***i)*** By using identity $H_{n-1}^2+H_n^2=(2p-q)H_{2n-1}-eF_{2n-1}$ [2] and considering the equations (3.2), (3.3) and the above equations which have been defined by Horadam, the proof can be seen easily.

***ii)*** Considering the equation (3.1), the proof can be easily seen.

***iii)*** From the identity $H_nH_m+H_{n+1}H_{m+1}=(2p-q)H_{m+n+1}-eF_{m+n+1}$ [3] and equation (3.1), the proof is completed.

***iv)*** Using the equation (3.1) and the identity $L_nF_m=F_{m+n}+(-1)^mF_{m-n}$ [2], the desired result is obtained. Also, the equations given in Proposition 2.2. in the article [1] are specially obtained by giving values $p=1,q=0$ in the equations we have found above.

***i)*** $\left(DHX_n\times DHX_n^{\dagger_1}\right)+\left(DHX_{n-1}\times DHX_{n-1}^{\dagger_1}\right)=-F_{2n}+2\varepsilon\left(-F_{n+1}^2\right)$

***ii)*** $DHX_n^2=2H_nDHX_n-\left(DHX_n\times DHX_n^{\dagger_1}\right)+2\varepsilon\left(F_{n+2}^2-F_{n+1}F_{n+3}\right)+2j\varepsilon\left(F_{n+1}F_{n+2}\right)$

***iii)*** $DHX_n^2+DHX_{n-1}^2=4DHX_{2n-1}-DHX_n\times DHX_n^{\dagger_1}-DHX_{n-1}\times DHX_{n-1}^{\dagger_1}+\left[-2F_{2n-1}+2\left(F_{2n+3}-F_{2n+1}\right)\varepsilon-2F_{2n}j\right]-2F_{n+1}^2\varepsilon$

***iv)*** $DHY_n\times DHX_n^{\dagger_1}-DHY_n^{\dagger_1}\times DHX_n=(-1)^n\left[4+(-6)j\varepsilon\right]$ □

**Theorem 3.5.** *Let $DHX_n$ be a dual-hyperbolic number with generalized coefficient. Then, the following identities are valid:*

***1)*** $DHX_n+DHX_{n+1}=DHX_{n+2}$

***2)*** $(DHX_n)^2=2\left(H_nDHX_n\right)+2\left(H_{n+1}DHX_{n+1}\right)-\left[\left(H_n^2+H_{n+1}^2\right)+2\left(H_{n+1}H_{n+2}\right)j+2\left(H_{n+1}H_{n+3}\right)j\varepsilon\right]$

***3)*** $-DHX_n+DHX_{n+1}j+DHX_{n+2}\varepsilon-DHX_{n+3}j\varepsilon=H_{n+1}$

**4)** $$(DHX_n \times DHX_m)+(DHX_{n+1}\times DHX_{m+1}) = (2p-q)\left[\ (H_{m+n+1}+H_{m+n+3})+2H_{m+n+2}j+2(H_{m+n+3}+H_{m+n+5})\varepsilon+4H_{m+n+4}i\varepsilon\ \right] \\ -e\left[\ (F_{m+n+1}+F_{m+n+3})+2F_{m+n+2}j+2(F_{m+n+3}+F_{m+n+5})\varepsilon+4F_{m+n+4}j\varepsilon\ \right]$$

**5)** $$DHX_n{}^2+DHX_{n-1}^2 = [(2p-q)(H_{2n+1}+H_{2n-1})-e(F_{2n+1}+F_{2n-1})]+2j[(2p-q)H_{2n}-eF_{2n}] \\ +2\varepsilon[(2p-q)(H_{2n+3}+H_{2n+1})-e(F_{2n+3}+F_{2n+1})]+4j\varepsilon[(2p-q)H_{2n+2}-eF_{2n+2}]$$

*Proof.* ***1)*** Let $DHX_n$ and $DHX_{n+1}$ be two dual-hyperbolic numbers with generalized coefficients. In this case, taking into account that the equation (3.1), we get

$$DHX_n+DHX_{n+1}=H_{n+2}+H_{n+3}j+H_{n+4}\varepsilon+H_{n+5}j\varepsilon=DHX_{n+2}.$$

Here, if the values $p=1$ and $q=0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that $DHX_n+DHX_{n+1}=DHF_{n+2}$.

***2)*** Let $DHX_n$ be dual-hyperbolic numbers with generalized coefficients. If the equation (3.2) is used, then the following equality is obtained

$$\begin{aligned} DHX_n^2 &= [(H_n+H_{n+1}j)+(H_{n+2}+H_{n+3}j)\varepsilon]\times[(H_n+H_{n+1}j)+(H_{n+2}+H_{n+3}j)\varepsilon] \\ &= 2(H_nDHX_n)+2(H_{n+1}DHX_{n+1})-\left[\left(H_n^2+H_{n+1}^2\right)+2(H_{n+1}H_{n+2})j+2(H_{n+1}H_{n+3})j\varepsilon\right]. \end{aligned}$$

Here, if the values $p=1$ and $q=0$ are specially taken in the generalized Fibonacci number $H_n$, then $(DHX_n)^2=2(F_nDHF_n)+2(F_{n+1}DHF_{n+1})-\left[\left(F_n^2+F_{n+1}^2\right)+2(F_{n+1}F_{n+2})j+2(F_{n+1}F_{n+3})j\varepsilon\right]$ is found.

***3)*** By considering the equation (3.1) and doing some algebraic calculations, we obtain

$$\begin{aligned} -DHX_n+DHX_{n+1}j+DHX_{n+2}\varepsilon-DHX_{n+3}j\varepsilon &= -[(H_n+H_{n+1}j)+(H_{n+2}+H_{n+3}j)\varepsilon] \\ &\quad+[(H_{n+1}+H_{n+2}j)+(H_{n+3}+H_{n+4}j)\varepsilon]j \\ &\quad+[(H_{n+2}+H_{n+3}j)+(H_{n+4}+H_{n+5}j)\varepsilon]\varepsilon \\ &\quad-[(H_{n+3}+H_{n+4}j)+(H_{n+5}+H_{n+6}j)\varepsilon]i\varepsilon \\ &= H_{\mathrm{n}+1}. \end{aligned}$$

Here, if the values $p=1$ and $q=0$ are specially taken in the generalized Fibonacci number $H_n$,

$$-DHX_n+DHX_{n+1}j+DHX_{n+2}\varepsilon-DHX_{n+3}j\varepsilon=F_{\mathrm{n}+1}$$

is found.

***4)*** Follows from the identity $H_nH_m+H_{n+1}H_{m+1}=(2p-q)H_{m+n+1}-eF_{m+n+1}$ (see ref. [3]) and using the equation (3.2), we achieve that

$$\begin{aligned} (DHX_n\times DHX_m)+(DHX_{n+1}\times DHX_{m+1}) &= [(H_n+H_{n+1}j)+(H_{n+2}+H_{n+3}j)\varepsilon]\times[(H_m+H_{m+1}j)+(H_{m+2}+H_{m+3}j)\varepsilon] \\ &\quad+[(H_{n+1}+H_{n+2}j)+(H_{n+3}+H_{n+4}j)\varepsilon]\times[(H_{m+1}+H_{m+2}j)+(H_{m+3}+H_{m+4}j)\varepsilon] \\ &= (2p-q)\begin{bmatrix} (H_{m+n+1}+H_{m+n+3})+2H_{m+n+2}j+2(H_{m+n+3}+H_{m+n+5})\varepsilon \\ +4H_{m+n+4}i\varepsilon \end{bmatrix} \\ &\quad-e[(F_{m+n+1}+F_{m+n+3})+2F_{m+n+2}j+2(F_{m+n+3}+F_{m+n+5})\varepsilon+4F_{m+n+4}j\varepsilon] \end{aligned}$$

Here, If the values $p=1$ and $q=0$ are specially taken in the generalized Fibonacci number $H_n$

$$(DHX_n\times DHX_m)+(DHX_{n+1}\times DHX_{m+1})=(F_{m+n+1}+F_{m+n+3})+2F_{m+n+2}j+2(F_{m+n+3}+F_{m+n+5})\varepsilon+4F_{m+n+4}j\varepsilon$$

***5)*** Considering the identity $H_nH_m+H_{n+1}H_{m+1}=(2p-q)H_{m+n+1}-eF_{m+n+1}$ (see ref. [3]) and the equation (3.2), we reach the result. Here, If the values $p=1$ and $q=0$ are specially taken in the generalized Fibonacci number $DHX_n{}^2+DHX_{n-1}^2=(F_{2n+1}+F_{2n-1})+2F_{2n}j+2(F_{2n+3}+F_{2n+1})\varepsilon+4F_{2n+2}j\varepsilon$. □

**Theorem 3.6.** *Let $DHX_n$ and $DHL_n$ be dual-hyperbolic Fibonacci and dual-hyperbolic Lucas numbers with generalized Fibonacci and Lucas coefficients, respectively. For $n\geq 0$, there exist the following relationships between these numbers:*
***1)*** $DHX_{n+1}+DHX_{n-1}=pDHL_n+qDHL_n$

***2)*** $DHX_{n+2}-DHX_{n-2}=pDHL_n+qDHL_n$

*Proof.* Equations ***1)*** and ***2)*** are found by taking the identity $H_{n+1}+H_{n-1}=pL_n+qL_{n-1}$ (see ref. [4]) and using the recurrence relation $H_n=(p-q)F_n+qF_{n+1}$, respectively.

$$\begin{aligned} DHX_{n+1}+DHX_{n-1} &= (H_{n+1}+H_{n+2}j+H_{n+3}\varepsilon+H_{n+4}j\varepsilon)+(H_{n-1}+H_nj+H_{n+1}\varepsilon+H_{n+3}j\varepsilon) \\ &= (H_{n+1}+H_{n-1})+(H_{n+2}+H_n)j+(H_{n+3}+H_{n+1})\varepsilon+(H_{n+4}+H_{n+3})j\varepsilon \\ &= (pL_n+qL_{n-1})+(pL_{n-1}+qL_n)j+(pL_{n+2}+qL_{n+1})\varepsilon+(qL_{n+3}+qL_{n+2})j\varepsilon \\ &= pDHL_n+qDHL_n \end{aligned}$$

$$\begin{aligned} DHX_{n+2}-DHX_{n-2} &= (H_{n+2}+H_{n+3}j+H_{n+4}\varepsilon+H_{n+5}j\varepsilon)-(H_{n-2}+H_{n-1}j+H_{n}\varepsilon+H_{n+1}j\varepsilon) \\ &= (H_{n+2}-H_{n-2})+(H_{n+3}-H_{n-1})\,j+(H_{n+4}-H_{n})\,\varepsilon+(H_{n+5}-H_{n+1})\,j\varepsilon \\ &= (pL_n+qL_{n-1})+(pL_{n-1}+qL_n)\,j+(pL_{n+2}+qL_{n+1})\,\varepsilon+(qL_{n+3}+qL_{n+2})\,j\varepsilon \\ &= pDHL_n+qDHL_n \end{aligned}$$

Here, if the values $p=1, q=0$ are specially taken in the generalized Fibonacci number $H_n$, then the desired results are obtained. □

**Theorem 3.7.** *The sums of the dual-hyperbolic numbers with generalized Fibonacci coefficients satisfy the following relations:*

***1)*** $\sum\limits_{s=1}^{n} DHX_s = DHX_{n+2}-DHX_2$

***2)*** $\sum\limits_{s=0}^{p} DHX_{s+n}+DHX_{x+1} = DHX_{n+p+2}$

***3)*** $\sum\limits_{s=1}^{n} DHX_{2s-1} = DHX_{2n}-DHX_0$

***4)*** $\sum\limits_{s=1}^{n} DHX_{2s} = DHX_{2n+1}-DHX_1$

*Proof.* Using the identity $\sum\limits_{t=a}^{n} H_t = H_{n+2}-H_{a+1}$ (see ref. [4]), the proof can be seen easily as follows

***1)*** $\sum\limits_{s=1}^{n} DHX_s = \sum\limits_{s=1}^{n} H_s + j\sum\limits_{s=1}^{n} H_{s+1}+\varepsilon\sum\limits_{s=1}^{n} H_{s+2}+j\varepsilon\sum\limits_{s=1}^{n} H_{s+3} = DHX_{n+2}-DHX_2$

***2)*** $\sum\limits_{s=0}^{p} DHX_{n+s}+DHX_{n+1} = \sum\limits_{s=1}^{n} H_{n+s}+H_{n+1}+j\sum\limits_{s=1}^{n} H_{n+s+1}+H_{n+2}+\varepsilon\sum\limits_{s=1}^{n} H_{n+s+2}+H_{n+3}+j\varepsilon\sum\limits_{s=1}^{n} H_{n+s+3}+H_{n+4} = DHX_{n+p+2}$

***3)*** $\sum\limits_{s=1}^{n} DHX_{2s-1} = \sum\limits_{s=1}^{n} H_{2s-1}+j\sum\limits_{s=1}^{n} H_{2s}+\varepsilon\sum\limits_{s=1}^{n} H_{2s+1}+j\varepsilon\sum\limits_{s=1}^{n} H_{2s+2} = DHX_{2n}-DHX_0$

***4)*** $\sum\limits_{s=1}^{n} DHX_{2s} = \sum\limits_{s=1}^{n} H_{2s}+j\sum\limits_{s=1}^{n} H_{2s+1}+\varepsilon\sum\limits_{s=1}^{n} H_{2s+2}+j\varepsilon\sum\limits_{s=1}^{n} H_{2s+3} = DHX_{2n+1}-DHX_1$

Also, if we consider the values $p=1, q=0$ in the generalized Fibonacci number $H_n$, then the above equations becomes as follows:

***1)*** $\sum\limits_{s=1}^{n} DHX_s = DHF_{n+2}-DHF_2$

***2)*** $\sum\limits_{s=0}^{p} DHX_{s+n}+DHX_{x+1} = DHF_{n+p+2}$

***3)*** $\sum\limits_{s=1}^{n} DHX_{2s-1} = DHF_{2n}-DHF_0$

***4)*** $\sum\limits_{s=1}^{n} DHX_{2s} = DHF_{2n+1}-DHF_1$ □

Now, let's find correspondence of the Binet formula for the dual-hyperbolic Fibonacci numbers which helps to find golden ratio.

**Theorem 3.8.** *Let $DHX_n$ be dual-hyperbolic number with generalized Fibonacci coefficient. For $m,n \geq 1$, the Binet formula for this number is given by*

$$DHX_n = \frac{\bar{\alpha}\,\alpha^n - \bar{\beta}\,\beta^n}{\alpha-\beta}$$

*where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$ and the coefficients $\bar{\alpha}$, $\bar{\beta}$ are as follows*

$$\bar{\alpha} = (p-q\beta)+[p(1-\beta)+q]\,j+[p(2-\beta)+q(1-\beta)]\,\varepsilon+[p(3-2\beta)+q(2-\beta)]\,j\varepsilon$$

*and*

$$\bar{\beta} = (p-q\alpha)+[p(1-\alpha)+q]\,j+[p(2-\alpha)+q(1-\alpha)]\,\varepsilon+[p(3-2\alpha)+q(2-\alpha)]\,j\varepsilon.$$

*Proof.* If $t_1$ and $t_2$ denote the roots of characteristic equation $t^2-t-1=0$ associated to the recurrence relation $DHX_n+DHX_{n+1}=DHX_{n+2}$. Then, these roots can be found as $\alpha = t_1 = \frac{1+\sqrt{5}}{2}$ and $\beta = t_2 = \frac{1-\sqrt{5}}{2}$. Note that, $\alpha+\beta=1, \alpha.\beta=-1$ and $\alpha-\beta=\sqrt{5}$. Therefore, the general term of the dual-hyperbolic number sequence with generalized Fibonacci coefficients may be expressed in the form:

$$DHX_n = A\alpha^n + B\beta^n$$

for some coefficients $A$ and $B$. For $n=0$ and $n=1$, the following equalities can be written

$$DHX_0 = (q,\ p,\ p+q,\ 2p+q)$$

and

$$DHX_1 = (p,\ p+q,\ 2p+q,\ 3p+2q).$$

Also, if we give to $n$ the values $n=0$ and $n=1$, we get

$$DHX_0 = A + B$$

and

$$DHX_1 = \alpha A + \beta B.$$

Then, solving this system of linear equations, we have

$$A = \frac{DHX_1 - \beta DHX_0}{\alpha - \beta} \qquad \textit{and} \qquad B = \frac{\alpha DHX_0 - DHX_1}{\alpha - \beta}$$

where some coefficients $\bar{\alpha}$ and $\bar{\beta}$ are

$$\bar{\alpha} = (p - q\beta) + [p(1-\beta) + q]\, j + [p(2-\beta) + q(1-\beta)]\, \varepsilon + [p(3-2\beta) + q(2-\beta)]\, j\varepsilon$$

and

$$\bar{\beta} = (p - q\alpha) + [p(1-\alpha) + q]\, j + [p(2-\alpha) + q(1-\alpha)]\, \varepsilon + [p(3-2\alpha) + q(2-\alpha)]\, j\varepsilon.$$

□

**Theorem 3.9.** *The generating function for dual-hyperbolic number with generalized coefficients is*

$$g(x) = \frac{1}{1-x-x^2} \sum_{s=0}^{3} (DHX_s + DHX_{s-1}x) e_s.$$

*Proof.* Assuming that the generating function for dual-hyperbolic number with generalized coefficients becomes

$$g(x) = \sum_{n=0}^{\infty} P_n x^n.$$

such that

$$P_n = (DHX_n,\ DHX_{n+1},\ DHX_{n+2}\ ,DHX_{n+3}).$$

Multiplying the generating function by $x$ and $x^2$, the following equalities can be written

$$\begin{aligned} xg(x) &= P_0 x + P_1 x^2 + \ldots + P_{n-1} x^n + \ldots \\ x^2 g(x) &= P_0 x^2 + P_1 x^3 + \ldots + P_{n-2} x^n + \ldots \end{aligned}$$

After some algebraic calculations, we obtain

$$g(x) = \frac{1}{1-x-x^2} \sum_{s=0}^{3} (P_0 + (P_1 - P_0)x).$$

This completes the proof. □

Now, let's write the Binet formula in terms of the generating function which has been obtained in Theorem 3.9.

**Theorem 3.10.** *Binet formula for the dual-hyperbolic numbers with generalized Fibonacci coefficients is*

$$P_n = P_1 H_n + P_0 H_{n-1}.$$

*Proof.* Let's we take the relation

$$P_n = A\alpha^n + B\beta^n.$$

Putting $n=0$ and $n=1$ in the above equation, $A$ and $B$ are obtained by

$$A = \frac{P_1 - \beta P_0}{\alpha - \beta}, \quad B = \frac{\alpha P_0 - P_1}{\alpha - \beta}$$

In this case, $P_n$ can be rewritten as

$$P_n = \frac{1}{\alpha - \beta} [(P_1 - \beta P_0)\alpha^n + (\alpha P_0 - P_1)\beta^n].$$

When the equalities of $P_0$ and $P_1$ is written in Theorem 3.9 and is arranged, $P_n$ is found as

$$P_n = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) \sum_{s=0}^{3} DHX_{s+1} e_s + \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) \sum_{s=0}^{3} DHX_s e_s.$$

Finally

$$P_n = P_1 H_n + P_0 H_{n-1}$$

is obtained. □

Let us express the Catalan identity which is one of the most known identities of Fibonacci numbers.

**Theorem 3.11.** *(Catalan's Identity)*
*For $n \geq r$, the relation*

$$DHX_n^2 - DHX_{n+r} \times DHX_{n-r} = (-1)^{n-r} \mu F_r^2 \left[ j + 3j\varepsilon \right]$$

*is verified.*

*Proof.* Squaring $DHX_n$, multiplying $DHX_{n+r}$ and $DHX_{n-r}$ and noting that $H_{m+k}H_{n-k} - H_m H_n = (-1)^{n-k+1} \mu F_k F_{m+k-n}$ [7], the following equalities are obtained

$$DHX_n^2 = H_n^2 + H_{n+1}^2 + 2H_nH_{n+1}\,j + 2\left(H_nH_{n+2} + H_{n+1}H_{n+3}\right)\varepsilon + 2\left(H_{n+1}H_{n+2} + H_nH_{n+3}\right) j\varepsilon$$

and

$$\begin{aligned} DHX_{n+r} \times DHX_{n-r} = {} & H_{n+r}H_{n-r} + H_{n+r+1}H_{n-r+1} + \left(H_{n+r+1}H_{n-r} + H_{n+r}H_{n-r+1}\right) j \\ & + \left(H_{n+r}H_{n-r+2} + H_{n+r+1}H_{n-r+3} + H_{n-r+2}H_{n-r} + H_{n-r+3}H_{n-r+1}\right)\varepsilon \\ & + \left(H_{n+r+1}H_{n-r+2} + H_{n+r}H_{n-r+3} + H_{n+r+3}H_{n-r} + H_{n+r+2}H_{n-r+1}\right) j\varepsilon. \end{aligned}$$

Adding the above equations gives us the proof. Writting $p = 1$ and $q = 0$ in the Catalan identity for dual-hyperbolic numbers with generalized coefficients, Catalan identity for dual-hyperbolic numbers with Fibonacci coefficients is found. Namely

$$DHX_n^2 - DHX_{n+r} \times DHX_{n-r} = (-1)^{n-r} F_r^2 \left( j + 3j\varepsilon \right).$$

□

Let's give Cassini identity for generalized dual-hyperbolic numbers as a special case of Catalan identity.

**Theorem 3.12.** *(Cassini's Identity)*
*Let $DHX_n$ be the dual-hyperbolic number with generalized Fibonacci coefficients. For $n \geq 1$, we have*

$$DHX_n^2 - \left(DHX_{n+1} \times DHX_{n-1}\right) = (-1)^{n-1} \mu \left( j + 3j\varepsilon \right).$$

*Proof.* For $r = 1$, we see that the identity in Theorem 3.11 becomes the desired identity.Putting $p = 1$ and $q = 0$ in the above identity, we get

$$DHX_n^2 - \left(DHX_{n+1} \times DHX_{n-1}\right) = (-1)^{n-1} \left( j + 3j\varepsilon \right).$$

This identity is Cassini formula for dual-hyperbolic numbers. □

**Theorem 3.13.** *(Honsberger Identity)*
*For $n, m \geq 0$, the Honsberger identity for the dual-hyperbolic number with generalized coefficient $DHX_n$ is given by*

$$\begin{aligned} \left(DHX_{k-1} \times DHX_n\right) + \left(DHX_k \times DHX_{n+1}\right) = {} & \left[(2p-q)\left(H_{k+n} + H_{k+n+2}\right) - e\left(F_{k+n} + F_{k+n+2}\right)\right] \\ & + 2j\left[(2p-q)H_{k+n+1} - eF_{k+n+1}\right] \\ & + 2\varepsilon\left[(2p-q)\left(H_{k+n+2} + H_{k+n+4}\right) - e\left(F_{k+n+2} + F_{k+n+4}\right)\right] \\ & + 4j\varepsilon\left[(2p-q)H_{k+n+3} - eF_{k+n+3}\right]. \end{aligned}$$

*Proof.* If we take into consider the equations (3.1), (3.2) and use the identity $H_nH_m + H_{n+1}H_{m+1} = (2p-q)H_{m+n+1} - eF_{m+n+1}$ (see ref. [3]), we complete the proof. If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number $H_n$, then the following identity is found

$$\begin{aligned} \left(DHX_{k-1} \times DHX_n\right) + \left(DHX_k \times DHX_{n+1}\right) = {} & \left(F_{k+n} + F_{k+n+2}\right) + 2F_{k+n+1}\,j \\ & + 2\left(F_{k+n+2} + F_{k+n+4}\right)\varepsilon + 4F_{k+n+3}\,j\varepsilon. \end{aligned}$$

□

**Theorem 3.14.** *(Tagiuri Identity)*
*Let $DHX_n$ be the dual-hyperbolic number with generalized Fibonacci coefficients. For m $n, m \geq 1$, Tagiuri's identity is as follows:*

$$\left(DHX_{m+k} \times DHX_{n-k}\right) - \left(DHX_m \times DHX_n\right) = (-1)^{n-k-1} \mu F_k F_{m+k-n} \left( j + 3j\varepsilon \right).$$

*Proof.* The proof can be easily seen by using the identity $H_{m+k}H_{n-k} - H_mH_n = (-1)^{n-k+1} \mu F_k F_{m+k-n}$ [7] and equations (3.1) and (3.2). If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number $H_n$, then it is concluded that

$$\left(DHX_{m+k} \times DHX_{n-k}\right) - \left(DHX_m \times DHX_n\right) = (-1)^{n-k-1} F_k F_{m+k-n} \left( j + 3j\varepsilon \right).$$

□

**Theorem 3.15.** *(d'Ocagne's Identity)*
*Let $DHX_n$ be the dual-hyperbolic number with generalized Fibonacci coefficients. For $m > n$, $m \in N$ and $n \in Z$, we have*

$$\left(DHX_{m+k} \times DHX_{n-k}\right) - \left(DHX_m \times DHX_n\right) = \mu F_{m-n} (-1)^n \left( j + 3j\varepsilon \right).$$

*Proof.* Using identity $H_{m+k}H_{n-k} - H_mH_n = (-1)^{n-k+1} \mu F_k F_{m+k-n}$ [7] and the equations (3.1) and (3.2), the proof is completed. If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number $H_n$, then the following identity is found

$$\left(DHX_{m+k} \times DHX_{n-k}\right) - \left(DHX_m \times DHX_n\right) = F_{m-n} (-1)^n \left( j + 3j\varepsilon \right).$$

□

# 4. Conclusion

Our main aim in this study was to generalize the study which was done on dual-hyperbolic numbers. It was seen that, some theorems were obtained as a result of this generalization and they corresponded to the theorems in the article [1] for $p = 1$, $q = 0$. Also, the generating function was obtained and the Binet formula was given with the help of the generating function. Unlike the identities which was given in the article [1], Honsberger and Tagiuri identities were proved. At the same time, special cases of these identities were discussed. Because of the fact that generalized Fibonacci and Lucas coefficient dual-hyperbolic number system have commutative algebra structure, five different conjugates can be defined. As a result, in addition to the identities related to the conjugates which we obtained in Proposition 3.4, the following identities are given.

**i)**

$$
\begin{aligned}
\left(DHX_n \times DHX_n^{\dagger_2}\right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_2}\right) &= (2p-q)\left[H_{2n-1}+H_{2n+2}\right] - e\left[F_{2n-1}+F_{2n+2}\right] + 2jH_nH_{n-1} \\
\left(DHX_n \times DHX_n^{\dagger_3}\right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_3}\right) &= -(1+2j)\left[(2p-q)H_{2n} - eF_{2n}\right] \\
\left(DHX_n \times DHX_n^{\dagger_4}\right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_4}\right) &= -\left[(2p-q)H_{2n} - eF_{2n}\right] \\
\left(DHX_n \times DHX_n^{\dagger_5}\right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_5}\right) &= \left[(2p-q)\left(H_{2n+3}+H_{2n+1}\right) - e\left(F_{2n+3}+F_{2n+1}\right)\right] \\
&\quad +2j\left[(2p-q)H_{2n+2} - eF_{2n+2}\right] \\
&\quad +\varepsilon\left[(2p-q)\left(2H_{2n}+H_{2n+5}\right) - e\left(2F_{2n+3}+F_{2n+5}\right)\right] \\
&\quad +2j\varepsilon\left[(2p-q)\left(H_{2n+2}+H_{2n}\right) - e\left(F_{2n+2}+F_{2n}\right)\right]
\end{aligned}
$$

**ii)**

$$DHX_n^2 = 2H_nDHX_n - DHX_nDHX_n^{\dagger_2} + 2H_{n+1}\left(H_{n+1}+H_nj+H_{n+2}\varepsilon+H_{n+2}j\varepsilon\right)$$

$$DHX_n^2 = 2H_nDHX_n - DHX_nDHX_n^{\dagger_3} + 2\left(H_{n+2}H_{n+1}\varepsilon+H_nH_{n+3}j\varepsilon\right)$$

$$DHX_n^2 = 2H_nDHX_n - DHX_nDHX_n^{\dagger_4} + 2H_{n+2}\left(H_n\varepsilon+H_{n+1}j\varepsilon\right)$$

$$
\begin{aligned}
DHX_n^2 &= 2H_nDHX_n - DHX_nDHX_n^{\dagger_5} + \left(H_{n+2}H_{n-1}+H_{n+2}H_n+H_{n+1}H_{n+3}\right) \\
&\quad +\left(H_{n+3}H_n+H_{n+1}H_{n+2}\right)j + \left(2H_{n+2}H_n+H_{n+2}^2+H_{n+3}^2-H_{n+1}^2-H_n^2\right)\varepsilon \\
&\quad +2\left(H_{n+2}H_{n+1}+H_{n+2}H_{n+3}-H_{n+1}H_n\right)j\varepsilon
\end{aligned}
$$

**iii)**

$$
\begin{aligned}
DHX_n^2+DHX_{n-1}^2 &= 2(2p-q)DHX_{2n-1} - DHX_nDHX_n^{\dagger_2} - DHX_{n-1}DHX_{n-1}^{\dagger_2} \\
&\quad +(2p-q)\left(2H_{2n+1}+2H_{2n+3}\varepsilon+2H_{2n+2}j\varepsilon\right) \\
&\quad -e\left(2F_{2n-1}+2F_{2n+1}+F_{2n}j+2\left(F_{2n+3}+F_{2n+1}\right)\varepsilon+4F_{2n+2}j\varepsilon\right)+2\left(H_nH_{n-1}\right)j
\end{aligned}
$$

$$
\begin{aligned}
DHX_n^2+DHX_{n-1}^2 &= 2(2p-q)DHX_{2n-1} - DHX_nDHX_n^{\dagger_3} - DHX_{n-1}DHX_{n-1}^{\dagger_3} \\
&\quad +(2p-q)\left(-2H_{2n}j+2H_{2n+3}\varepsilon+2H_{2n+2}j\varepsilon\right) \\
&\quad -e\left(2F_{2n-1}+2\left(F_{2n+3}+F_{2n+1}\right)\varepsilon+4F_{2n+2}j\varepsilon\right)
\end{aligned}
$$

$$
\begin{aligned}
DHX_n^2+DHX_{n-1}^2 &= 2(2p-q)DHX_{2n-1} - DHX_nDHX_n^{\dagger_4} - DHX_{n-1}DHX_{n-1}^{\dagger_4} \\
&\quad +2(2p-q)\left(H_{2n+3}\varepsilon+H_{2n+2}j\varepsilon\right) \\
&\quad -2e\left(F_{2n-1}+F_{2n}j+\left(F_{2n+3}+F_{2n+1}\right)\varepsilon+2F_{2n+2}j\varepsilon\right)
\end{aligned}
$$

$$
\begin{aligned}
DHX_n^2+DHX_{n-1}^2 &= 2(2p-q)DHX_{2n-1} - DHX_nDHX_n^{\dagger_5} - DHX_{n-1}DHX_{n-1}^{\dagger_5} \\
&\quad +(2p-q)\left[H_{2n+4}+2H_{2n+2}j+\left(2H_{2n+3}+2H_{2n}+H_{2n+5}\right)\varepsilon+2\left(2H_{2n+2}+H_{2n}\right)j\varepsilon\right] \\
&\quad -e\left[F_{2n+3}+2F_{2n+1}+F_{2n-1}+2\left(F_n+F_{2n+2}\right)+\left(4F_{2n+3}+F_{2n+5}\right)\varepsilon+\left(6F_{2n+2}+F_{2n}\right)j\varepsilon\right]
\end{aligned}
$$

**iv)**

$$DHY_n \times DHX_n^{\dagger_2} - DHY_n^{\dagger_2} \times DHX_n = 4(-1)^n\left[2p^2\varepsilon+p^2j\varepsilon\right]$$

$$DHY_n \times DHX_n^{\dagger_3} - DHY_n^{\dagger_3} \times DHX_n = 4(-1)^n\left[\left(p^2-2pq+2q^2\right)j-2p^2\varepsilon\right]$$

$$
\begin{aligned}
DHY_n \times DHX_n^{\dagger_4} - DHY_n^{\dagger_4} \times DHX_n &= \left[4(-1)^{n-1}\left(p^2+2pq-2q^2\right)\right]j \\
&\quad + \frac{8(-1)^n p^2}{\left(V_n+V_{n+1}i\right)\left(H_n+H_{n+1}i\right)}\left[\begin{array}{l} -p^2F_{2n+1}+pq\left(-2F_{2n+2}-F_{2n}\right) \\ +q^2\left(F_{2n+1}+F_{2n}\right) \end{array}\right]j\varepsilon
\end{aligned}
$$

$$DHY_n \times DHX_n^{\dagger_5} - DHY_n^{\dagger_5} \times DHX_n = 2(-1)^{n-1}p^2j$$

The proofs of these identities are easily seen by following the similar ways in the proof of Theorem 3.4. Finally, the special values $p = 1$ and $q = 0$ provide the above equations.

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# The Exact Solutions of Conformable Fractional Partial Differential Equations Using New Sub Equation Method

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**Abstract**

In this article, authors employed the new sub equation method to attain new traveling wave solutions of conformable time fractional partial differential equations. Conformable fractional derivative is a well behaved, applicable and understandable definition of arbitrary order derivation. Also this derivative obeys the basic properties that Newtonian concept satisfies. In this study authors obtained the exact solution for KDV equation where the fractional derivative is in conformable sense. New solutions are obtained in terms of the generalized version of the trigonometric functions.

## 1. Introduction

Fractional differential equations (FDEs) are generalized form of the integer order differential equations. In the last decades, researchers have worked hard for obtaining analytic solutions of nonlinear FDEs. Nonlinear FDEs are often used to describe many problems arising in many fields such as physics, chemistry, engineering, heat transfer, applied mathematics, control theory et all. [1]-[4]. So, many authors presented very strong methods to solve FDEs. For instance Kurt et. al. [5] studied the solutions of time fractional Whitham-Broer-Kaup Equation by using homotopy analysis method where the fractional terms are described in Caputo sense. Tasbozan et. al. [6] employed the finite element method for attaining the approximate solutions of diffusion equation where the derivatives are in Riemann-Liouville sense. Celik et. al. [7] utilized Crank-Nicolson scheme to get the the numerical solutions of fractional diffusion equation. As it is seen from the given references, all the obtained results are numerical solutions for the considered nonlinear equations. Because, the analytical methods can not be applied to the nonlinear equations which involves Caputo, Riemann-Liouville and Riesz fractional derivative definitions. On the contrary, conformable fractional detivative definition gives us chance to get the exact solutions of nonlinear FDEs by using new wave transformation [8] and the chain rule [9]. For example Eslami and Rezazadeh [10] used the first integral method to obtain analytic solutions of time fractional Wu-Zhang system. Aminikhah et. al. [11] obtained analytic solutions of fractional regularized long-wave equations using sub-equation method. Osman et al. [12] employed the unified method to get the analytic solutions of conformable time fractional Schrödinger equation with perturbation terms. For further details please see the references [13]-[34]. In this paper, we handle the Korteweg-de Vries equation with a source that provides a sixth order differential equation.

$$D_x^6u+20D_xuD_x^4u+40D_x^2uD_x^3u+120D_xu^2D_x^2u+D_x^3D_t^{\mu}u+8D_xuD_xD_t^{\mu}u+4D_t^{\mu}uD_x^2u=0. \tag{1.1}$$

## 2. Conformable fractional calculus

R. Khalil et. al. [32] presented the definition of conformable fractional derivative as follows.

**Definition 2.1.** *$\mu^{th}$ order "conformable fractional derivative" of function g which is defined as $g:[0,\infty)\to\mathbb{R}$ can be dedicated as*

$$T_\mu(g)(t)=\lim_{\varepsilon\to 0}\frac{g(t+\varepsilon t^{1-\mu})-g(t)}{\varepsilon}$$

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*for all $t>0,\ \alpha\in(0,1)$. Assuming thatg is $\mu$- differentiable over some $(0,a)$ where $a>0$ and $\lim_{t\to 0^+} g^{(\mu)}(t)$ exists, then $g^{(\mu)}(0)=\lim_{t\to 0^+} g^{(\mu)}(t)$.*

The other fractional derivative definitions such as Caputo, Riemann-Liouville, Grünwald-Letnikov and etc. do not satisfy basic principles which are provided by Newtonian type derivative. For instance

1. Assume that $\lambda$ is a constant and $\alpha\in R$. Then $D_a^{\mu}(\lambda)\neq 0$ for Riemann-Liouville derivative.
2. The Riemann-Liouville and Caputo derivatives do not provide the derivative of the product of two functions.
3. $D_a^{\mu}(fg)\neq fD_a^{\mu}(g)+gD_a^{\mu}(f)$.
4. The Riemann-Liouville and Caputo derivatives do not do not provide the derivative of the quotient of two functions
5. $D_a^{\mu}\left(\frac{f}{g}\right)\neq\frac{gD_a^{\mu}(f)-fD_a^{\mu}(g)}{g^2}$.

This new definition satisfies the properties which are given in the following theorem.

**Theorem 2.2.** *Let $\mu\in(0,1)$ and $f,g$ be $\mu-$ differentiable at point $t>0$. Then*

1. *$T_{\mu}(af+bg)=aT_{\mu}(f)+bT_{\mu}(g)$, for all $a,b\in\mathbb{R}$*
2. *$T_{\mu}(t^p)=pt^{p-\mu}$ for all $p\in\mathbb{R}$.*
3. *$T_{\mu}(\lambda)=0$ for all constant function $f(t)=\lambda$.*
4. *$T_{\mu}(fg)=fT_{\mu}(g)+gT_{\mu}(f)$.*
5. *$T_{\mu}\left(\frac{f}{g}\right)=\frac{gT_{\mu}(g)-fT_{\mu}(f).}{g^2}$.*
6. *If $f$ is differentiable, then $T_{\mu}(f)(t)=t^{1-\mu}\frac{df}{dt}$.*

## 3. The new sub-equation method

Consider that the general form of nonlinear fractional partial differential equation can be expressed as

$$H\left(u,\frac{\partial^{\mu}u}{\partial t^{\mu}},\frac{\partial u}{\partial x},u\frac{\partial u}{\partial x},u^2\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\dots\right)=0. \tag{3.1}$$

Using the wave transform $\xi=kx+w\frac{t^{\mu}}{\mu}$ where $k$ and $w$ are constants and chain rule [9] in Eq. (3.1), the independent variables and can be changed into single variable. So Eq. (3.1) can be rewritten as

$$P(u,u'(\xi),u''(\xi),\dots). \tag{3.2}$$

Consider that $u(\xi)$ can be written as a polynomial in $Q(\xi)$

$$u(\xi)=\sum_{j=0}^{n}a_jQ^j(\xi), \tag{3.3}$$

where $a_j\,(0\leq j\leq n)$ are constant coefficients to be determined after and $Q(\xi)$ provides first order linear ODE of the form

$$Q'(\xi)=Ln(A)\left(\alpha+\beta Q(\xi)+\sigma Q^2(\xi)\right),\quad A\neq 0,1, \tag{3.4}$$

where $\alpha,\beta,\sigma$ are constants. Moreover , Eq. has the following traveling wave solutions.
**Family 1.**If $\beta^2-4\alpha\sigma<0$ and $\sigma\neq 0$, then we have

$$\begin{aligned}
Q_1(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\tan_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right),\\
Q_2(\xi) &= -\frac{\beta}{2\sigma}-\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\cot_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right),\\
Q_3(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\left(\tan_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\pm\sqrt{pq}\sec_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\right),\\
Q_4(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\left(-\cot_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\pm\sqrt{pq}\csc_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\right),\\
Q_5(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4\sigma}\left(\tan_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4}\xi\right)-\cot_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4}\xi\right)\right).
\end{aligned}$$

**Family 2.**Suppose that $\beta^2-4\alpha\sigma>0$ and $\sigma\neq 0$,

$$\begin{aligned}
Q_6(\xi) &= -\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\tanh_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right),\\
Q_7(\xi) &= -\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\coth_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right),\\
Q_8(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\left(-\tanh_A\left(\sqrt{\beta^2-4\alpha\sigma}\xi\right)\pm i\sqrt{pq}\operatorname{sech}_A\left(\sqrt{\beta^2-4\alpha\sigma}\xi\right)\right),\\
Q_9(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\left(-\coth_A\left(\sqrt{\beta^2-4\alpha\sigma}\xi\right)\pm \sqrt{pq}\operatorname{csch}_A\left(\sqrt{\beta^2-4\alpha\sigma}\xi\right)\right),\\
Q_{10}(\xi) &= -\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{4\sigma}\left(\tanh_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{4}\xi\right)+\coth_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{4}\xi\right)\right).
\end{aligned}$$

**Family 3.**Consider that $\alpha\sigma>0$ and $\beta=0$,

$$\begin{aligned}
Q_{11}(\xi) &= \sqrt{\frac{\alpha}{\sigma}}\tan_A\left(\sqrt{\alpha\sigma}\xi\right),\\
Q_{12}(\xi) &= -\sqrt{\frac{\alpha}{\sigma}}\cot_A\left(\sqrt{\alpha\sigma}\xi\right),\\
Q_{13}(\xi) &= \sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(2\sqrt{\alpha\sigma}\xi\right)\pm\sqrt{pq}\sec_A\left(2\sqrt{\alpha\sigma}\xi\right)\right),\\
Q_{14}(\xi) &= \sqrt{\frac{\alpha}{\sigma}}\left(-\cot_A\left(2\sqrt{\alpha\sigma}\xi\right)\pm\sqrt{pq}\csc_A\left(2\sqrt{\alpha\sigma}\xi\right)\right),\\
Q_{15}(\xi) &= \frac{1}{2}\sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)-\cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)\right).
\end{aligned}$$

**Family 4.**Regard that $\alpha\sigma<0$ and $\beta=0$,

$$\begin{aligned}
Q_{16}(\xi) &= -\sqrt{-\frac{\alpha}{\sigma}}\tanh_A\left(\sqrt{-\alpha\sigma}\xi\right),\\
Q_{17}(\xi) &= -\sqrt{-\frac{\alpha}{\sigma}}\coth_A\left(\sqrt{-\alpha\sigma}\xi\right),\\
Q_{18}(\xi) &= \sqrt{-\frac{\alpha}{\sigma}}\left(-\tanh_A\left(2\sqrt{-\alpha\sigma}\xi\right)\pm i\sqrt{pq}\operatorname{sech}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right),\\
Q_{19}(\xi) &= \sqrt{-\frac{\alpha}{\sigma}}\left(-\coth_A\left(2\sqrt{-\alpha\sigma}\xi\right)\pm \sqrt{pq}\operatorname{csch}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right),\\
Q_{20}(\xi) &= -\frac{1}{2}\sqrt{-\frac{\alpha}{\sigma}}\left(\tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)+\coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)\right).
\end{aligned}$$

**Family 5.** When $\beta=0$ and $\sigma=\alpha$,

$$\begin{aligned}
Q_{21}(\xi) &= \tan_A\left(\alpha\xi\right),\\
Q_{22}(\xi) &= -\cot_A\left(\alpha\xi\right),\\
Q_{23}(\xi) &= \tan_A\left(2\alpha\xi\right)\pm\sqrt{pq}\sec_A\left(2\alpha\xi\right),\\
Q_{24}(\xi) &= -\cot_A\left(2\alpha\xi\right)\pm\sqrt{pq}\csc_A\left(2\alpha\xi\right),\\
Q_{25}(\xi) &= \frac{1}{2}\left(\tan_A\left(\frac{\alpha}{2}\xi\right)-\cot_A\left(\frac{\alpha}{2}\xi\right)\right).
\end{aligned}$$

**Family 6.** If $\beta=0$ and $\sigma=-\alpha$, chosen

$$\begin{aligned}
Q_{26}(\xi) &= -\tanh_A\left(\alpha\xi\right),\\
Q_{27}(\xi) &= -\coth_A\left(\alpha\xi\right),\\
Q_{28}(\xi) &= -\tanh_A\left(2\alpha\xi\right)\pm i\sqrt{pq}\operatorname{sech}_A\left(2\alpha\xi\right),\\
Q_{29}(\xi) &= -\coth_A\left(2\alpha\xi\right)\pm\sqrt{pq}\operatorname{csch}_A\left(2\alpha\xi\right),\\
Q_{30}(\xi) &= -\frac{1}{2}\left(\tanh_A\left(\frac{\alpha}{2}\xi\right)+\coth_A\left(\frac{\alpha}{2}\xi\right)\right).
\end{aligned}$$

**Family 7.**While $\beta^2=4\alpha\sigma$,

$$Q_{31}(\xi) = \frac{-2\alpha(\beta\xi Ln(A)+2)}{\beta^2\xi Ln(A)}.$$

**Family 8.**When $beta = k$ , $\alpha = mk(m \neq 0)$ and $\sigma = 0$,

$$Q_{32}(\xi) = A^{k\xi} - m.$$

**Family 9.**When $\beta = \sigma = 0$,

$$Q_{33}(\xi) = \alpha\xi Ln(A).$$

**Family 10.**When $\beta = \alpha = 0$,

$$Q_{34}(\xi) = \frac{-1}{\sigma\xi Ln(A)}.$$

**Family 11.**When $\alpha = 0$ and $\beta \neq 0$,

$$\begin{aligned}
Q_{35}(\xi) &= -\frac{p\beta}{\sigma(\cosh_A(\beta\xi) - \sinh_A(\beta\xi) + p)}, \\
Q_{36}(\xi) &= -\frac{q\beta}{\sigma(\cosh_A(\beta\xi) - \sinh_A(\beta\xi) + q)}, \\
Q_{37}(\xi) &= -\frac{\beta(\sinh_A(\beta\xi) + \cosh_A(\beta\xi))}{\sigma(\sinh_A(\beta\xi) + \cosh_A(\beta\xi) + q)},
\end{aligned}$$

**Family 12.**When $\beta = k$ , $\sigma = mk(m \neq 0), p = q$ and $\alpha = 0$,

$$Q_{38}(\xi) = \frac{pA^{k\xi}}{p - mqA^{k\xi}}.$$

**Remark 3.1.** *The generalized version of the trigonometric functions and the generalized types of the hypergeometric functions are declared as [33]*

$$\begin{aligned}
\sinh_A(\xi) &= \frac{pA^{\xi} - qA^{-\xi}}{2}, & \cosh_A(\xi) &= \frac{pA^{\xi} + qA^{-\xi}}{2}, \\
\tanh_A(\xi) &= \frac{pA^{\xi} - qA^{-\xi}}{pA^{\xi} + qA^{-\xi}}, & \coth_A(\xi) &= \frac{pA^{\xi} + qA^{-\xi}}{pA^{\xi} - qA^{-\xi}}, \\
\operatorname{sech}_A(\xi) &= \frac{2}{pA^{\xi} + qA^{-\xi}}, & \operatorname{csch}_A(\xi) &= \frac{2}{pA^{\xi} - qA^{-\xi}}, \\
\sin_A(\xi) &= \frac{pA^{i\xi} - qA^{-i\xi}}{2i}, & \cos_A(\xi) &= \frac{pA^{i\xi} + qA^{-i\xi}}{2}, \\
\tan_A(\xi) &= -i\frac{pA^{i\xi} - qA^{-i\xi}}{pA^{i\xi} + qA^{-i\xi}}, & \cot_A(\xi) &= i\frac{pA^{i\xi} + qA^{-i\xi}}{pA^{i\xi} - qA^{-i\xi}}, \\
\sec_A(\xi) &= \frac{2}{pA^{i\xi} + qA^{-i\xi}}, & \csc_A(\xi) &= \frac{2i}{pA^{i\xi} - qA^{-i\xi}},
\end{aligned}$$

where $p, q > 0$ are constants and $\xi$ is an independent variable. In addition, by considering the balance between the highest order derivative linear term and nonlinear terms appearing in ODE (3.2), the positive integer $n$ can be defined. Replacing Eq. (3.3) into ODE (3.2), using Eq. (3.4), and equalizing the coefficients of all the powers of $Q(\xi)$ to zero, we will obtain an equation system in terms of $k, w$ and $a_j$ $(0 \le j \le n)$. From this obtained system the values for $k, w$ and $a_j$ can be found with the aid of a computer software. Replacing the obtained values of $k, w$ and $a_j$ into Eq.(3.3), we may acquire all possible solutions of Eq. (3.1).

## 4. Analytic results for time fractional KdV6 equation with conformable derivative

Using the wave transformation and applying chain rule [9]

$$u(x,t) = u(\xi), \qquad \xi = kx + w\frac{t^{\mu}}{\mu}. \tag{4.1}$$

Eq. (1.1) is transferred to

$$k^6u^{(vi)}(\xi) + k^3wu^{iv}(\xi) + 6k^2w\left(u'(\xi)\right)^2 + 20k^5u^{iv}(\xi)u'(\xi) + 40k^5u''(\xi)u'''(\xi) + 12k^2wu'(\xi)u''(\xi) = 0$$

where the prime symbolizes the known derivative of function $u(\xi)$ with respect to $\xi$. Integrating the above equation once and making some algebraic calculations led to

$$k^6u^{(v)}(\xi)+k^3wu'''(\xi)+3k^2w(u')^2+5k^5u'''u+20k^5(u'')^2+12k^2wuu'=0. \tag{4.2}$$

Considering the homogeneous balance between $u^2u'$ and $u^{(5)}$ in Eq. (4.2) we obtain $n+5=3(n+1)$; then $n=1$;so we can write Eq. (3.3) as

$$u(\xi)=a_0+a_1Q(\xi). \tag{4.3}$$

Subrogating Eq. (4.3) with (3.4) into Eq. (4.2) and gathering all the same power of $Q(\xi)$ together, the left hand side of Eq. (4.2) turns into a polynomial of $Q(\xi)$. Equalizing the each coefficient of the same power of $Q(\xi)$ to zero led to an equation system. Solving the obtained system due to unknowns variables $a_0$, $a_1$ and $w$, the solutions can be concluded as

$$w=\frac{{a_1}^3(\beta^2-4\alpha\sigma)}{\sigma^3Ln(A)}, \qquad k=-\frac{a_1}{\sigma Ln(A)}. \tag{4.4}$$

Putting the solution set (4.4) with (4.1) into (4.3) and solutions of Eq. (1.1), can be expressed as

**Case 1.**If $\beta^2-4\alpha\sigma<0$ and$\sigma\neq0$, then we have

$$\begin{aligned}
u_1(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\tan_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right)\right),\\
u_2(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}-\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\cot_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right)\right),\\
u_3(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\left(\tan_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\pm\sqrt{pq}\sec_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\right)\right),\\
u_4(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\left(-\cot_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\pm\sqrt{pq}\csc_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\right)\right),\\
u_5(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4\sigma}\left(\tan_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4}\xi\right)-\cot_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{{a_1}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.

**Case 2.**Suppose that $\beta^2-4\alpha\sigma>0$ and $\sigma\neq0$,

$$\begin{aligned}
u_6(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\tanh_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right)\right),\\
u_7(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\coth_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right)\right),\\
u_8(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\left(-\tanh_A\left(\sqrt{\beta^2-4\alpha\sigma}\xi\right)\pm i\sqrt{pq}\,\mathrm{sech}_A\left(\sqrt{\beta^2-4\alpha\sigma}\xi\right)\right)\right),\\
u_9(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\left(-\coth_A\left(\sqrt{\beta^2-4\alpha\sigma}\xi\right)\pm\sqrt{pq}\,\mathrm{csch}_A\left(\sqrt{\beta^2-4\alpha\sigma}\xi\right)\right)\right),\\
u_{10}(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{4\sigma}\left(\tanh_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{4}\xi\right)+\coth_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{4}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{{a_1}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.

**Case 3.**Consider that $\alpha\sigma>0$ and $\beta=0$,

$$\begin{aligned}
u_{11}(\xi) &= a_0+a_1\left(\sqrt{\frac{\alpha}{\sigma}}\tan_A\left(\sqrt{\alpha\sigma}\xi\right)\right),\\
u_{12}(\xi) &= a_0-a_1\left(\sqrt{\frac{\alpha}{\sigma}}\cot_A\left(\sqrt{\alpha\sigma}\xi\right)\right),\\
u_{13}(\xi) &= a_0+a_1\left(\sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(2\sqrt{\alpha\sigma}\xi\right)\pm\sqrt{pq}\sec_A\left(2\sqrt{\alpha\sigma}\xi\right)\right)\right),\\
u_{14}(\xi) &= a_0+a_1\left(\sqrt{\frac{\alpha}{\sigma}}\left(-\cot_A\left(2\sqrt{\alpha\sigma}\xi\right)\pm\sqrt{pq}\csc_A\left(2\sqrt{\alpha\sigma}\xi\right)\right)\right),\\
u_{15}(\xi) &= a_0+a_1\left(\frac{1}{2}\sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)-\cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.
**Case 4.**Regard that $\alpha\sigma<0$ and $\beta=0$,

$$\begin{aligned}
u_{16}(\xi) &= a_0-a_1\left(\sqrt{-\frac{\alpha}{\sigma}}\tanh_A\left(\sqrt{-\alpha\sigma}\xi\right)\right),\\
u_{17}(\xi) &= a_0-a_1\left(\sqrt{-\frac{\alpha}{\sigma}}\coth_A\left(\sqrt{-\alpha\sigma}\xi\right)\right),\\
u_{18}(\xi) &= a_0+a_1\left(\sqrt{-\frac{\alpha}{\sigma}}\left(-\tanh_A\left(2\sqrt{-\alpha\sigma}\xi\right)\pm i\sqrt{pq}\operatorname{sech}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right)\right),\\
u_{19}(\xi) &= a_0+a_1\left(\sqrt{-\frac{\alpha}{\sigma}}\left(-\coth_A\left(2\sqrt{-\alpha\sigma}\xi\right)\pm \sqrt{pq}\operatorname{csch}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right)\right),\\
u_{20}(\xi) &= a_0-a_1\left(\frac{1}{2}\sqrt{-\frac{\alpha}{\sigma}}\left(\tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)+\coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.
**Case 5.**When $\beta=0$ and $\sigma=\alpha$,

$$u_{21}(\xi)=a_0+a_1\tan_A(\alpha\xi),$$

$$\begin{aligned}
u_{22}(\xi) &= a_0-a_1\cot_A(\alpha\xi),\\
u_{23}(\xi) &= a_0+a_1\left(\tan_A(2\alpha\xi)\pm\sqrt{pq}\sec_A(2\alpha\xi)\right),\\
u_{24}(\xi) &= a_0+a_1\left(-\cot_A(2\alpha\xi)\pm\sqrt{pq}\csc_A(2\alpha\xi)\right),\\
u_{25}(\xi) &= a_0+a_1\left(\frac{1}{2}\left(\tan_A\left(\frac{\alpha}{2}\xi\right)-\cot_A\left(\frac{\alpha}{2}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.
**Case 6.**If $\beta=0$ and $\sigma=-\alpha$, chosen

$$\begin{aligned}
u_{26}(\xi) &= a_0-a_1\tanh_A(\alpha\xi),\\
u_{27}(\xi) &= a_0-a_1\coth_A(\alpha\xi),\\
u_{28}(\xi) &= a_0+a_1\left(-\tanh_A(2\alpha\xi)\pm i\sqrt{pq}\operatorname{sech}_A(2\alpha\xi)\right),\\
u_{29}(\xi) &= a_0+a_1\left(-\coth_A(2\alpha\xi)\pm\sqrt{pq}\operatorname{csch}_A(2\alpha\xi)\right),\\
u_{30}(\xi) &= a_0-\frac{a_1}{2}\left(\tanh_A\left(\frac{\alpha}{2}\xi\right)+\coth_A\left(\frac{\alpha}{2}\xi\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.
**Case 11.**When $\alpha=0$ and $\beta\neq0$,

$$\begin{aligned}
u_{31}(\xi) &= a_0-\frac{pa_1\beta}{\sigma\left(\cosh_A(\beta\xi)-\sinh_A(\beta\xi)+p\right)},\\
u_{32}(\xi) &= a_0-\frac{qa_1\beta}{\sigma\left(\cosh_A(\beta\xi)-\sinh_A(\beta\xi)+q\right)},\\
u_{33}(\xi) &= a_0-\frac{a_1\beta\left(\sinh_A(\beta\xi)+\cosh_A(\beta\xi)\right)}{\sigma\left(\sinh_A(\beta\xi)+\cosh_A(\beta\xi)+q\right)},
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.

**Case 12.**When $\beta=k$ , $\sigma=mk(m\neq0), p=q$ and$\alpha=0$,

$$u_{34}(\xi)=a_0+\frac{pa_1A^{k\xi}}{p-mqA^{k\xi}}.$$

## 5. Conclusion

In this manuscript the new sub-equation method successfully applied to time fractional KdV6 equation. Analytic solutions of the nonlinear KdV6 equation are successfully obtained. Also wave transform and chain rule are used, so the nonlinear conformable FDE changes into differential equation with integer order derivative. As it can be from the obtained results new sub-equation method is a reliable, efficient and applicable tool for obtaining the exact solutions of fractional partial differential equations in conformable sense.

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# Conservation Laws for a Model with both Cubic and Quadratic Nonlinearity

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**Abstract**

In this paper, the conservation laws for a model with both quadratic and cubic nonlinearity

$$m_t = bu_x + \frac{1}{2}a\left[\left(u^2 - u_x^2\right)m\right]_x + \frac{1}{2}c\left(2m\cdot u_x + m_x\cdot u\right); \quad m = u - u_{xx}$$

are considered for the six cases of coefficients. By using a variational derivative approach, conservation laws were constructed. The computations to derive multipliers and conservation law fluxes are conducted by using a Maple-based package which is called GeM.

## 1. Introduction

In this paper, we consider the conservation laws for the model

$$m_t = bu_x + \frac{1}{2}a\left[\left(u^2 - u_x^2\right)m\right]_x + \frac{1}{2}c\left(2m\cdot u_x + m_x\cdot u\right); \quad m = u - u_{xx}, \tag{1.1}$$

where $a$, $b$ and $c$ are arbitrary constants. Eq. (**??**) models the one-way propagation of a fluid that lies on a horizontal flat bottom.
Conservation laws, indicating that a certain measurable property (as mass, momentum or charge) of an isolated physical system does not change as the system evolves over time, are of fundamental importance in nonlinear science. The study of the conservation laws of the KdV equation was a milestone in the exploration of some techniques that include Miura transformation, Lax pair, inverse scattering transform, bi-Hamiltonian structures, for solving evolutionary equations [**?**]. Conservation laws have several applications in the field of differential equations. For example, Lax [**?**] proved global existence theorems by using conservation laws, DiPerna [**?**] used extra conservation laws for the decay of shock waves, and stability problems were considered by Benjamin. In [**?**], they were used for studying cracks and dislocations in elasticity (for more information see [**?**]). The existence of solitons is also closely related to the existence of an infinite number of conservation laws of partial differential equations and is a predictor of complete integrability.
There are many powerful methods used to find conservation laws such as Laplace's direct technique [**?**], Noether's theorem [**?**], the characteristic form (also known as multiplier or integrating factor) given by Steudel [**?**]. In this paper, we use the multiplier approach among these techniques to derive conservation laws and conserved quantities corresponding to the six different cases of coeffficients of Eq. (**??**). The multiplier approach will be explained in detail in the next section.
The emergence of symbolic computational packages provides great satisfaction in the performance of complex and tedious calculations. Over the past decades, researchers have focused on developing symbolic computational packages working with either *Maple* or *Mathematica* which are based on different approaches to conservation laws. Many computational packages have recently been developed, and we can classify these packages on the environment in which they work in two parts:

1. Packages which are based on *Mathematica* [**?**] environment: Goktas and Hereman developed **condens.m** [**?**], Adams and Hereman developed **TransPDEDensity.m** [**?**], and Poole and Hereman developed **ConservationLawsMD.m** [**?**].
2. Packages which are based on *Maple* environment: Cheviakov developed **GeM** [**?**, **?**], Anderson and Cheb-Terrab developed **Vessiot suite** [**?**] , Rocha Filho and Figueiredo developed **SADE** [**?**].

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**GeM** package [**?**] will be used in the present paper to find the conservation laws of Eq. (**??**) in the six different cases of coeffficients. **GeM** package is developed to find the conservation laws and symmetries of differential equations. There exists a determining system for obtaining multipliers (and hence conservation laws) for any partial differential equation. To obtain symmetries, this package, firstly obtain an overdetermined system of determining equations, afterwards this system is simplified by a Rif package routines, and then a *Maple* command gives all symmetry generators of differential equations. For the conservation laws, **GeM** package firstly obtain a determining system for multipliers, afterwards the obtained system is simplified by Rif package to get multipliers. Once the multipliers are obtained, the fluxes are constructed by the direct method, homotopy methods or scaling symmetry formula.

Eq. (**??**) is studied in [**?**] where they mainly interested in peakon, weak kink ank kink-peakon interactional solutions. To show that Eq. (**??**) is completely integrable, they present the Lax representation, bi-Hamiltonian structure and infinitely many conservation laws for Eq. (**??**). In [**?**] the conservation laws are obtained explicitly only for $b=0, a\neq 0, c\neq 0$ case.

According to the different cases of coefficients, Eq. (**??**) reduces to the following six cases:

1. Case $(b\neq 0, a\neq 0, c\neq 0)$:
   $m_t = bu_x + \frac{1}{2}a\left[\left(u^2-u_x^2\right)m\right]_x + \frac{1}{2}c\left(2m\cdot u_x + m_x\cdot u\right)$, which is a linear combination of CH and mCH or generalized CH equation, see (Qiao, Xia, and Li [e-print arXiv:1205.2028v3 (2012)]).
2. Case $(b\neq 0, a=0, c=-2)$:
   $m_t = bu_x - (2mu_x + m_x u)$,wich is a quadratic nonlinear equation.
3. Case $(b\neq 0, a=-2, c=0)$:
   $m_t = bu_x - \left[\left(u^2-u_x^2\right)m\right]_x$, which is a cubic nonlinear equation.
4. Case $(b=0, a\neq 0, c\neq 0)$:
   $m_t = \frac{1}{2}a\left[m\left(u^2-u_x^2\right)\right]_x + \frac{1}{2}c\left(2mu_x + m_x u\right)$, which known as FQXL model.
5. Case $(b=0, a=-2, c=0)$ :
   $m_t = -\left(2mu_x + m_x u\right)$, which is known as Camassa -Holm equation (CH).
6. Case $(b=0, a=0, c=-2)$:
   $m_t = -\left[\left(u^2-u_x^2\right)m\right]_x$, which known as modified Camassa-Holm equation (mCH).

In the present paper, the conservation laws of the above six cases of the coefficients are computed explicitly using GeM Maple routines which are based on multiplier method. The multipliers are used to make the system being studied get a divergence form, then by equating this divergence to zero one can obtain a conservation law. For the convenience, these are explained in detail in Section 2. The computations are performed in Section 3, and the results are summarized in the last section.

## 2. Basic concepts on the method proposed

To compute conserved densities and fluxes, we use a multiplier approach based on the fact that the Euler operator eliminates a total divergence. Let $u$ be dependent variable and $t, x$ be independent variables.

1. Consider an *nth*-order partial differential equation
   $$G(t,x,u,u_t,u_x,u_{tt},u_{tx},\ldots) = 0. \tag{2.1}$$
2. The standard Euler operator $E_u$ is defined as
   $$E_u = \frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \mathrm{D}_t\frac{\partial}{\partial u_t} - \mathrm{D}_x\frac{\partial}{\partial u_x} + \mathrm{D}_t^2\frac{\partial}{\partial u_{tt}} + \mathrm{D}_x^2\frac{\partial}{\partial u_{xx}} - \ldots.$$
   where $D_t$ and $D_x$ are the total differentiation operators which are given by:
   $$\begin{aligned} \mathrm{D}_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \ldots, \\ \mathrm{D}_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \ldots. \end{aligned}$$
   The Euler operator tests whether an expression is a total derivative without using any integration by parts [**?**].
3. A vector $T=(T^t, T^x)$ is defined as the conserved vector of (**??**) if $\mathrm{D}_t T^t + \mathrm{D}_x T^x = 0$ holds for all solutions of (**??**), where $T^t$ is *conserved density* and $T^x$ is associated *flux*. The divergence expression $\mathrm{D}_t T^t + \mathrm{D}_x T^x = 0$ is called the local conservation law for (**??**).
4. A multiplier $\lambda$ of (**??**) is a function on the solution space which satisfies
   $$\mathrm{D}_t T^t + \mathrm{D}_x T^x = \lambda G \tag{2.2}$$
   for any function $u(x,t)$ [**?**, **?**]. The multipliers may be chosen to depend on both the variables (independent and dependent) and derivatives up to a certain order.
5. The multipliers may be determined by taking the variational derivative of (**??**)
   $$\frac{\delta}{\delta u}(\lambda G) = 0, \tag{2.3}$$
   where $\delta/\delta s$ is the Euler operator defined as above. The conserved vectors can be derived using (**??**) after computing the multipliers from (**??**).

## 3. Multipliers and conservation laws for the six cases via GeM Maple Routines

In this section, we use a multiplier approach technique for deriving conservation laws and conserved quantities corresponding to the six cases of Eq. (**??**) via Maple-based GeM package.
We start with the sixth case $(b=0, a=0, c=-2)$. Consider the following model with both cubic and quadratic nonlinearity:

$$m_t = bu_x + \frac{1}{2}a\left[\left(u^2-u_x^2\right)m\right]_x + \frac{1}{2}c\left(2m\cdot u_x + m_x\cdot u\right); m = u - u_{xx}, \tag{3.1}$$

By using the following Maple command

```
> restart : with(PDEtools) : declare(m(t,x),u(t,x)) :
> pde := diff(m(t,x),t) - b*diff(u(t,x),x) - (1/2)*a*(diff((u(t,x)^2 - (diff(u(t,x),x))^2)*m(t,x),x)) - (1/2)*c*(2*m*(diff(u(t,x),x)) + u*(diff(m(t,x),x))) = 0 :
> m(t,x) = u(t,x) - (diff(u(t,x),x,x)) :
> CH := eval(pde, {b = 0, a = 0, c = -2, m(t,x) = u(t,x) - (diff(u(t,x),x,x))});
```

we can rewrite equation (**??**) as follows:

$$CH := u_t - u_{txx} + 2\left(u-u_{xx}\right)u_x + u\left(u_x - u_{xxx}\right) = 0 \tag{3.2}$$

where $u = u(t,x)$ . We will explain the case $(b=0, a=0, c=-2)$ in detail along with GeM Maple routines given in [**?**, **?**]. Dependent and independent variables and Eq. (**??**) can be defined in GeM by the following commands:

```
> read("H:/gem32_12.mpl") :
> with(linalg) : With(GeM);
> gem_decl_vars(indeps = [t,x], deps = [u(t,x)], freeconst = [b,a,c]);
> gem_decl_eqs([diff(u(t,x),t) - (diff(u(t,x),x,x,t)) + ((2*(u(t,x) - (diff(u(t,x),x,x))))*(diff(u(t,x),x)) + (diff(u(t,x),x) - (diff(u(t,x),x,x,x)))*u(t,x)) = 0], solve_for = [diff(u(t,x),t)]);
```

Let us take the multipliers as $\lambda = \lambda\left(t,x,u,u_t,u_x,u_{xx},u_{xxx}\right)$. The Maple routines to be used in GeM for which the multipliers will be obtained from are

```
> det_CH := gem_conslaw_det_eqs([t,x,u(t,x),diff(u(t,x),t),diff(u(t,x),x),diff(u(t,x),x,x),diff(u(t,x),x,x,x)]);
> CL_CH_mult := gem_conslaw_multipliers();
> simpli_CH := DEtools[rifsimp](det_CH, CL_CH_mult, mindim = 1);
```

For the simplified form of multipliers, the determining equations are

$$\lambda_{uu} = \frac{3\lambda_{u_{xx}}}{2u-2u_{xx}}, \lambda_{uu_{xx}} = -\frac{3\lambda_{u_{xx}}}{2u-2u_{xx}}, \lambda_{u_{xx}u_{xx}} = \frac{3\lambda_{u_{xx}}}{2u-2u_{xx}}, \lambda_t = 0, \lambda_x = 0, \lambda_{u_t} = 0, \lambda_{u_x} = 0, \lambda_{u_{xxx}} = 0 \tag{3.3}$$

The following Maple command is used to solve the system (**??**)

```
> multipliers_CH_sol := pdsolve(simpli_CH[Solved]);
```

which yields

$$\lambda\left(t,x,u,u_t,u_x,u_{xx},u_{xxx}\right) = \_C2u + \_C3 + \frac{\_C1}{\sqrt{-u+u_{xx}}}.$$

Here $\_C1, \_C2$ and $\_C3$ are arbitrary constants. There arise three linearly independent conservation laws from the following multipliers

$$\lambda^{(1)} = 1, \lambda^{(2)} = u, \lambda^{(3)} = \frac{1}{\sqrt{-u+u_{xx}}} \tag{3.4}$$

The next task is the construction of conservation laws from the multipliers given in (**??**). The *direct method* is used to compute the flux expressions in the Maple command

```
> gem_get_CL_fluxes(multipliers_CH_sol);
```

.

We have the conservation law fluxes which are presented in the following table for the multipliers (**??**) :

| Case | Multiplier | Fluxes |
|---|---|---|
| $b=0$ <br> $a=0$ <br> $c=-2$ | $\lambda^{(1)}=1$ | $T^t = u - u_{xx}$ <br> $T^x = \frac{3}{2}u^2 - uu_{xx} - \frac{1}{2}u_x^2$ |
| | $\lambda^{(2)}=u$ | $T^t = \frac{1}{2}u^2 - uu_{xx} - \frac{1}{2}u_x^2$ <br> $T^x = u^3 - u_{xx}u^2 + u_t u_x$ |
| | $\lambda^{(3)} = \frac{1}{\sqrt{-u+u_{xx}}}$ | $T^t = -2\sqrt{-u+u_{xx}}$ <br> $T^x = -2\sqrt{-u+u_{xx}}u$ |

The homotopy formulas will be employed for the fluxes since the multipliers do not contain arbitrary constants. The following Maple command is used for *first homotopy formula*

```
> gem_get_CL_fluxes(multipliers_CH_sol, method = "Homotopy1");
```

to get conservation law fluxes

| Case | Multiplier | Fluxes |
|---|---|---|
| $b=0$<br>$a=0$<br>$c=-2$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$<br>$T^x=\frac{3}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{u(u-u_{xx})}{2}$<br>$T^x=u^3-u_{xx}u^2-\frac{1}{2}u_{tx}u+\frac{1}{2}u_xu_t$ |
| | $\lambda^{(3)}=\frac{1}{\sqrt{-u+u_{xx}}}$ | $T^t=-2\sqrt{-2u+2u_{xx}}$<br>$T^x=-2u\sqrt{-2u+2u_{xx}}$ |

For *second homotopy* formula, the Maple command

$> gem\_get\_CL\_fluxes(multipliers\_CH\_sol, method = \text{"}Homotopy2\text{"});$

yields the expressions for conservation law fluxes which are presented in the following table :

| Case | Multiplier | Fluxes |
|---|---|---|
| $b=0$<br>$a=0$<br>$c=-2$ | $\lambda^{(1)}=1$ | $T^t=u-\frac{u_{xx}}{3}$<br>$T^x=\frac{3}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2-\frac{2}{3}u_{tx}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{1}{2}u^2-\frac{1}{3}uu_{xx}+\frac{1}{6}u_x^2$<br>$T^x=u^3-u_{xx}u^2-\frac{2}{3}uu_{tx}+\frac{1}{3}u_tu_x$ |
| | $\lambda^{(3)}=\frac{1}{\sqrt{-u+u_{xx}}}$ | $T^t=-\frac{2\sqrt{-u+u_{xx}}\left(u^3+\left(-\frac{13u_{xx}}{6}-\frac{u_{xxxx}}{6}\right)u^2+\left(\frac{3u_{xx}^2}{2}+\frac{u_{xx}u_{xxxx}}{6}-\frac{5\left(u_x-\frac{3u_{xxx}}{5}\right)(u_x-u_{xxx})}{12}\right)u+\frac{(-2u_{xx}^2+u_x(u_x-u_{xxx}))u_{xx}}{6}\right)}{(u-u_{xx})^3}$<br>$T^x=-\frac{1}{(u-u_{xx})^3}\left(2\sqrt{-u+u_{xx}}\left(u^4-3u^3u_{xx}+\left(3u_{xx}^2-\frac{5u_{tx}}{6}+\frac{u_{txxx}}{6}\right)u^2-\frac{u_{xx}(4u_{xx}u_{tx}+(u_x+u_{xx})u_t-2u_{txx}u_x)}{6}\right)\right)$ |

Now, by repeating the previous processes for the cases (1,2,3,4 and 5), we find multipliers and conserved vectors using the *direct method* and *first homotopy method*, which are given in Tables **??**, **??**, respectively.

| Case | Multiplier | Fluxes |
|---|---|---|
| $b\neq 0$<br>$a\neq 0$<br>$c\neq 0$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$<br>$T^x=-\frac{au^3}{2}+\frac{(2au_{xx}-3c)u^2}{4}+\frac{\left(2au_x^2+2cu_{xx}-4b\right)u}{4}-\frac{\left(au_{xx}-\frac{c}{2}\right)u_x^2}{2}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$<br>$T^x=-\frac{3au^4}{8}+\frac{(4au_{xx}-4c)u^3}{8}+\frac{\left(2au_x^2+4cu_{xx}-4b\right)u^2}{8}-\frac{u_x^2u_{xx}au}{2}+\frac{u_x\left(u_x^3a+8u_t\right)}{8}$ |
| | $\lambda^{(3)}=\frac{(-2u+2u_{xx})a-c}{\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}}$ | $T^t=-2\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}$<br>$T^x=\left(au^2-au_x^2+cu\right)\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}$ |
| $b\neq 0$<br>$a=0$<br>$c=-2.$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$<br>$T^x=\frac{3u^2}{2}+\frac{(-2b-2u_{xx})u}{2}-\frac{u_x^2}{2}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$<br>$T^x=u^3+\frac{(-b-2u_{xx})u^2}{2}+u_tu_x$ |
| | $\lambda^{(3)}=\frac{2}{\sqrt{2b-4u+4u_{xx}}}$ | $T^t=-\sqrt{2b-4u+4u_{xx}}$<br>$T^x=-\sqrt{2b-4u+4u_{xx}}u$ |
| $b\neq 0$<br>$a=-2$<br>$c=0$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$<br>$T^x=u^3-u_{xx}u^2+\left(-u_x^2-b\right)u+u_x^2u_{xx}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$<br>$T^x=\frac{3u^4}{4}-u_{xx}u^3+\frac{\left(-2u_x^2-2b\right)u^2}{4}+u_x^2u_{xx}u-\frac{u_x^4}{4}+u_tu_x$ |
| | $\lambda^{(3)}=\frac{-u+u_{xx}}{\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}$ | $T^t=-\frac{\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}{2}$<br>$T^x=-\frac{\left(u^2-u_x^2\right)\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}{2}$ |
| $b=0$<br>$a\neq 0.$<br>$c\neq 0$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$<br>$T^x=-\frac{au^3}{2}+\frac{(2au_{xx}-3c)u^2}{4}+\frac{\left(2au_x^2+2cu_{xx}\right)u}{4}-\frac{\left(au_{xx}-\frac{c}{2}\right)u_x^2}{2}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$<br>$T^x=-\frac{3au^4}{8}+\frac{(4au_{xx}-4c)u^3}{8}+\frac{\left(2au_x^2+4cu_{xx}\right)u^2}{8}-\frac{u_x^2u_{xx}au}{2}+\frac{u_x\left(2u_x^3a+8u_t\right)}{8}$ |
| | $\lambda^{(3)}=\frac{(-2u+2u_{xx})a-c}{\sqrt{(u-u_{xx})((u-u_{xx})a+c)}}$ | $T^t=-2\sqrt{(u-u_{xx})((u-u_{xx})a+c)}$<br>$T^x=\sqrt{(u-u_{xx})((u-u_{xx})a+c)}\left(au^2-au_x^2+cu\right)$ |
| $b=0$<br>$a=-2$<br>$c=0$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$<br>$T^x=(u-u_x)(u+u_x)(u-u_{xx})$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$<br>$T^x=-\frac{u_x^4}{4}-\frac{u(u-2u_{xx})u_x^2}{2}+u_tu_x+\frac{3u^4}{4}-u_{xx}u^3$ |
| | $\lambda^{(3)}=\frac{1}{(u-u_{xx})^2}$ | $T^t=-\frac{1}{u-u_{xx}}$<br>$T^x=\frac{3u^2-4uu_{xx}+u_x^2}{u-u_{xx}}$ |

**Table 1:** Multipliers and conserved vectors using direct method

| Case | Multiplier | Fluxes |
|---|---|---|
| $b\neq 0$ <br> $a\neq 0$ <br> $c\neq 0$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$ <br> $T^x=-\frac{au^3}{2}+\frac{(2au_{xx}-3c)u^2}{4}+\frac{\left(2au_x^2+2cu_{xx}-4b\right)u}{4}-\frac{\left(au_{xx}-\frac{c}{2}\right)u_x^2}{2}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{u(u-u_{xx})}{2}$ <br> $T^x=-\frac{3au^4}{8}+\frac{(4au_{xx}-4c)u^3}{8}+\frac{\left(2au_x^2+4cu_{xx}-4b\right)u^2}{8}+\frac{\left(-4u_x^2u_{xx}a-4u_{tx}\right)u}{8}+\frac{u_x\left(u_x^3a+4u_t\right)}{8}$ |
| | $\lambda^{(3)}=\frac{(-2u+2u_{xx})a-c}{\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}}$ | $T^t=2\sqrt{b}-2\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}$ <br> $T^x=\left(au^2-au_x^2+cu\right)\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}$ |
| $b\neq 0$ <br> $a=0$ <br> $c=-2.$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$ <br> $T^x=\frac{3u^2}{2}+\frac{(-2b-2u_{xx})u}{2}-\frac{u_x^2}{2}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{u(u-u_{xx})}{2}$ <br> $T^x=u^3+\frac{(-b-2u_{xx})u^2}{2}-\frac{uu_{tx}}{2}+\frac{u_tu_x}{2}$ |
| | $\lambda^{(3)}=\frac{2}{\sqrt{2b-4u+4u_{xx}}}$ | $T^t=\sqrt{2}\sqrt{b}-\sqrt{2b-4u+4u_{xx}}$ <br> $T^x=-\sqrt{2b-4u+4u_{xx}}u$ |
| $b\neq 0$ <br> $a=-2$ <br> $c=0$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$ <br> $T^x=u^3-u_{xx}u^2+\left(-u_x^2-b\right)u+u_x^2u_{xx}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{u(u-u_{xx})}{2}$ <br> $T^x=\frac{3u^4}{4}-u_{xx}u^3+\frac{\left(-2u_x^2-2b\right)u^2}{4}+\frac{\left(4u_x^2u_{xx}-2u_{tx}\right)u}{4}-\frac{u_x^4}{4}+\frac{u_tu_x}{2}$ |
| | $\lambda^{(3)}=\frac{-u+u_{xx}}{\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}$ | $T^t=\frac{\sqrt{-b}}{2}-\frac{\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}{2}$ <br> $T^x=\frac{(u-u_x)(u+u_x)\left(-2u^2+4uu_{xx}-2u_{xx}^2+b\right)}{2\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}$ |
| $b=0$ <br> $a\neq 0.$ <br> $c\neq 0$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$ <br> $T^x=-\frac{au^3}{2}+\frac{(2au_{xx}-3c)u^2}{4}+\frac{\left(2au_x^2+2cu_{xx}\right)u}{4}-\frac{\left(au_{xx}-\frac{c}{2}\right)u_x^2}{2}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{u(u-u_{xx})}{2}$ <br> $T^x=-\frac{3au^4}{8}+\frac{(4au_{xx}-4c)u^3}{8}+\frac{\left(2au_x^2+4cu_{xx}\right)u^2}{8}+\frac{\left(-4au_x^2u_{xx}-4u_{tx}\right)u}{8}+\frac{u_x\left(au_x^3+4u_t\right)}{8}$ |
| | $\lambda^{(3)}=\frac{(-2u+2u_{xx})a-c}{\sqrt{(u-u_{xx})^2a+cu-cu_{xx}}}$ | $T^t=-2\sqrt{(u-u_{xx})^2a+cu-cu_{xx}}$ <br> $T^x=\left(au^2-au_x^2+cu\right)\sqrt{(u-u_{xx})^2a+cu-cu_{xx}}$ |
| $b=0$ <br> $a=-2$ <br> $c=0$ | $\lambda^{(1)}=1$ | $T^t=u-u_{xx}$ <br> $T^x=u^3-u_{xx}u^2-u_x^2u+u_x^2u_{xx}$ |
| | $\lambda^{(2)}=u$ | $T^t=\frac{u(u-u_{xx})}{2}$ <br> $T^x=\frac{3u^4}{4}-u_{xx}u^3-\frac{u_x^2u^2}{2}+\frac{\left(8u_x^2u_{xx}-4u_{tx}\right)u}{8}+\frac{u_x\left(-2u_x^3+4u_t\right)}{8}$ |
| | $\lambda^{(3)}=\frac{4u-4u_{xx}}{\sqrt{-2(u-u_{xx})^2}}$ | $T^t=-2\sqrt{-2(u-u_{xx})^2}$ <br> $T^x=\left(-2u^2+2u_x^2\right)\sqrt{-2(u-u_{xx})^2}$ |

**Table 2:** Multipliers and conserved vectors using the first homotopy formula

# 4. Conclusion

The conservation laws for Eq. (**??**) with both quadratic and cubic nonlinearity for the six cases of coefficients ($(b\neq 0, a\neq 0, c\neq 0)$,$(b\neq 0, a=0, c=-2)$,$(b\neq 0, a=-2, c=0)$,$(b=0, a\neq 0, c\neq 0)$,$(b=0, a=-2, c=0)$ and $(b=0, a=0, c=-2)$) are constructed via a Maple package called GeM. The conservation laws $\rho_t=F_x$ of Eq. (**??**) were obtained in [**?**]. But, they were given explicitly only for $b=0, a\neq 0, c\neq 0$ case. In the present paper, the conservation laws of all the above six cases are computed explicitly. Three multipliers are obtained by defining the multipliers of the form $\lambda=\lambda\left(t,x,u,u_t,u_x,u_{xx},u_{xxx}\right)$ in GeM Maple routines. More multipliers may be computed in the case of including higher order derivatives in the multipliers. Direct method and homotopy formula are used to compute the fluxes for each cases. The fluxes obtained here can be used to construct the solutions of Eq. (**??**).

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# New Advances in Kotzig's Conjecture

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**Abstract**

In 1973 Kotzig conjectures that the complete graph $K_{2n+1}$ can be cyclically decomposed into $2n+1$ copies of any tree of size $n$. Rosa proved that this decomposition exists if and only if there exists a $\rho$-labeling of the tree. In this work we prove that if $T'$ is a graceful tree, then any tree $T$ obtained from $T'$ by attaching a total of $k \geq 1$ pendant vertices to any collection of $r$ vertices of $T'$, where $1 \leq r \leq k$, admits a $\rho$-labeling. As a consequence of this result, many new families of trees with this kind of labeling are produced, which indicates the strong potential of this result. Moreover, the technique used to prove this result, gives us an indication of how to determine whether a given tree of size $n$ decomposes the complete graph $K_{2n+1}$. We also prove the existence of a $\rho$-labeling for two subfamilies of lobsters and present a method to produce $\rho$-labeled trees attaching pendant vertices and pendant copies of the path $P_3$ to some of the vertices of any graceful tree.
In addition, for any given tree $T$, we use bipartite labelings to show that this tree is a spanning tree of a graph $G$ that admits an $\alpha$-labeling. This is not a new result; however, the construction presented here optimizes (reduces) the size of $G$ with respect to all the similar results that we found in the literature.

## 1. Introduction

A *decomposition* (or *edge-decomposition*) of the complete graph $K_n$ is a system $R$ of subgraphs such that any edge of $K_n$ belongs to exactly one of the subgraphs in $R$. Suppose that the vertices of $K_n$ are labeled $0,1,\dots,n-1$; let $ij \in E(K_n)$, a *turning of the edge ij* is the increase of both labels by one, i.e., the edge $(i+1)(j+1)$, where the addition is taken modulo $n$. A *turning of a subgraph* $G$ of $K_n$ is the simultaneous turning of all the edges of $G$. A decomposition $R$ of $K_n$ is called *cyclic* when for every $G$ in $R$, the turning $G'$ of $G$ is also in $R$.
In 1963, Ringel [1] presented the following conjecture: If $T$ is a tree of size $n$, then the complete graph $K_{2n+1}$ is edge-decomposable into $2n+1$ copies of $T$. Ten years later, Kotzig [2] stated the following variation of this conjecture: The complete graph $K_{2n+1}$ can be cyclically decomposed into $2n+1$ subgraphs isomorphic to a given tree with $n$ edges. In 1966, Rosa [3] introduced four valuations (or labelings) of the vertices of a graph that can be used to find a cyclic decomposition of $K_{2n+1}$. A *difference vertex labeling* of a graph $G$ of size $n$ is an injective mapping $f: V(G) \to S$, where $S$ is a set of nonnegative integers, such that every edge $uv$ of $G$ has assigned a *weight* defined by $|f(v)-f(u)|$. All labelings considered in this work are difference vertex labelings. Rosa's valuations can be described as follows.
Suppose that $f$ is a labeling of a graph $G$ of size $n$. Let $L_f$ be the set of labels assigned by $f$ to the vertices of $G$ and $W_f$ be the set of weights induced by $f$ on the edges of $G$. Consider the following conditions.

(a) $L_f \subseteq \{0,1,\dots,n\}$
(b) $L_f \subseteq \{0,1,\dots,2n\}$
(c) $W_f = \{1,2,\dots,n\}$
(d) $W_f = \{w_1,w_2,\dots,w_n\}$ where $w_i = i$ or $w_i = 2n+1-i$ for every $1 \leq i \leq n$
(e) it exists $\lambda$ in $\{0,1,\dots,n\}$, such that for any arbitrary edge $uv$ of $G$, either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$. (The number $\lambda$ is called the *boundary value* of $f$.)

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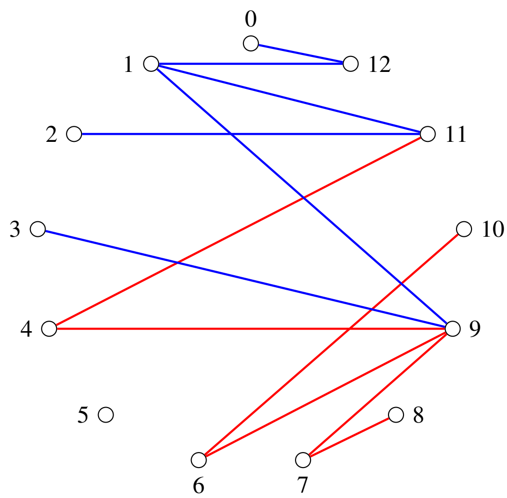


**Figure 1.1:** A $\rho$-labeling of $S(3,2)$ and its 8th turning

| +0 | +1 | +2 | +3 | +4 | +5 | +6 | +7 | +8 | +9 | +10 | +11 | +12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0-12 | 1-0 | 2-1 | 3-2 | 4-3 | 5-4 | 6-5 | 7-6 | 8-7 | 9-8 | 10-9 | 11-10 | 12-11 |
| 1-12 | 2-0 | 3-1 | 4-2 | 5-3 | 6-4 | 7-5 | 8-6 | 9-7 | 10-8 | 11-9 | 12-10 | 0-11 |
| 1-11 | 2-12 | 3-0 | 4-1 | 5-2 | 6-3 | 7-4 | 8-5 | 9-6 | 10-7 | 11-8 | 12-9 | 0-10 |
| 2-11 | 3-12 | 4-0 | 5-1 | 6-2 | 7-3 | 8-4 | 9-5 | 10-6 | 11-7 | 12-8 | 0-9 | 1-10 |
| 1-9 | 2-10 | 3-11 | 4-12 | 5-0 | 6-1 | 7-2 | 8-3 | 9-4 | 10-5 | 11-6 | 12-7 | 0-8 |
| 3-9 | 4-10 | 5-11 | 6-12 | 7-0 | 8-1 | 9-2 | 10-3 | 11-4 | 12-5 | 0-6 | 1-7 | 2-8 |

**Table 1:** $\rho$-labeling of $S(3,2)$ and all its turnings

When $f$ satisfies the conditions (a), (c), and (e), it is called an $\alpha$-*labeling* (and $G$ is designated an $\alpha$-*graph*). If $f$ satisfies the conditions (a) and (c), then it is called a $\beta$-*labeling* or *graceful labeling* (and $G$ is named a *graceful graph*). The function $f$ is a $\sigma$-*labeling* if (b) and (c) hold (in this case $G$ is a $\sigma$-*graph*). A $\rho$-*labeling* must satisfies the conditions (b) and (d); in this case, $G$ is named a $\rho$-*graph*. Thus, every $\alpha$-graph is a graceful graph, every graceful graph is a $\sigma$-graph, and every $\sigma$-graph is a $\rho$-graph. Using these labelings, Rosa [3] proved the following theorem.

**Theorem 1.1.** *A cyclic decomposition of the complete graph $K_{2n+1}$ into subgraphs isomorphic to a given graph $G$ of size $n$ exists if and only if there exists a $\rho$-labeling of $G$.*

Consequently, Kotzig's conjecture can be stated in terms of $\rho$-labelings as follows.

**Conjecture 1.** *Every tree of size $n$ is a $\rho$-tree.*

The tree $S(3,2)$, obtained by attaching a pendant vertex to every leaf of the star $S_3 \cong K_{1,3}$, is the smallest tree that is not an $\alpha$-tree. Suppose that the vertices of $K_{1,3}$ are labeled $0,1,\ldots,12$. Thus, every column in Table 1 shows the adjacencies of the vertices of $S(3,2)$ within $K_{1,3}$, being the first column a $\rho$-labeling of this tree and every column after that corresponds to a turning of the previous labeled graph. In Figure 1.1 we show, in blue, the $\rho$-labeling of $S(3,2)$ used to create the cyclic decomposition of $K_{1,3}$, together with its 8th turning, represented in red.

In Section 2 we show that a tree $T$ admists a $\rho$-labeling if it has a graceful subtree $T'$, such that $T'$ can be obtained by deleting a number of leaves of $T$. Given that several families of graceful trees are known, this result allows us to expand, considerably, the number of trees or families of trees that admit a $\rho$-labeling, therefore, decompose the complete graph $K_{2n+1}$. Also here, we show two subfamilies of 3-distance trees that admit $\rho$-labelings. In addition, we study the existence of $\rho$-labelings for trees obtained from smaller graceful trees by attaching copies of the path $P_3$ to some selected vertices of the base graceful trees; these selected vertices may be chosen almost randomly.
In [4], Barrientos and Krop represented a tree as an ordered rooted tree to calculate its excess $\varepsilon(T)$. This parameter was used in [5] to find a $\rho$-labeling for any tree $T$ that containing a branch that is a caterpillar of size at least $\varepsilon(T)$. In Section 3, we use the parameter $\varepsilon(T)$ to show the existence of an $\alpha$-graph of size $n+\varepsilon(T)$ that contains $T$ as a spanning tree.
All graphs used in this paper are finite, with no loops nor multiple edges. We follow the notation and terminology used in [6] and [7].

## 2. Constructing $\rho$-graphs from graceful graphs

Several families of $\rho$-trees are known. Gallian [7] mentions that in an unpublished work of Caro et al., [8], it was proven that all graphs with at most 11 edges have a $\rho$-labeling as well as lobsters. Késdy [9] defined a stunted tree as follows: a tree of size $n$ is *stunted* if its edges can be linearly ordered $e_1, e_2, \ldots, e_n$ so that $e_1$ and $e_2$ share a vertex and, for all $3 \le j \le n$, $e_j$ shares a vertex with at least one $e_k$ such that $2k \le j-1$. He proved that if $p = 2n+1$ is prime, then every stunted tree of size $n$ has a $\rho$-labeling.
A *spider* is a tree that has at most one vertex, called the *center*, of degree greater than 2. Bahls et al. [10], proved that spiders for which the lengths of every path from the center to a leaf differ by at most one, are graceful. A *comet* is a spider where all the paths used have equal

length. El-Zanati et al. [11], proved that trees of diameter at most 5, lobsters, and comets admit a more restrictive type of $\rho$-labeling, called $\rho^+$-labeling. Essentially, $\rho^+$-labelings are $\rho$-labelings with the extra condition of being bipartite (see Section 3 for a formal definition of bipartite labeling). This extra condition is what it makes their result a novelty.

### 2.1. The first expansion

Let $T$ be a tree of positive size $q$ with at least $k$ leaves and $T'$ be any tree of size $q-k$ obtained from $T$ by deleting $k$ of its leaves. We claim that $T$ admits a $\rho$-labeling when $T'$ is a graceful tree.

**Theorem 2.1.** *Let $T'$ be any tree obtained from $T$ by deleting any $k$ of its leaves. If $T'$ is graceful, then $T$ admits a $\rho$-labeling.*

*Proof.* Let $T$ be a tree of size $q$ with at least $k$ leaves and $T'$ be a graceful tree obtained by deleting $k$ leaves from $T$. Suppose that $f$ is a graceful labeling of $T'$. Thus, the labels assigned by $f$ to the vertices of $T'$ are $0,1,\dots,q-k$ and the weights induced by $f$ on the edges of $T'$ are $1,2,\dots,q-k$. Let $v_1,v_2,\dots,v_r$ be the vertices of $T'$ that when seen as vertices of $T$ corresponds to those incident to the edges of $T$ that were deleted to form $T'$. Without loss of generality we assume that $f(v_1)>f(v_2)>\cdots>f(v_r)$. Thus, $f(v_1)\le q-k$ and $f(v_r)\ge 0$.

In any $\rho$-labeling of a graph of size $q$, the labels are taken from $\{0,1,\dots,2q\}$. When the labeled $T'$ is seen inside $T$, its labeling can be extended to a $\rho$-labeling of $T$ as follows. There are $k$ leaves of $T$ that have not been labeled yet; none of the integers in $\{q-k+1,q-k+2,\dots,2q\}$ have been used as a label; none of the integers $q-k+1,q-k+2,\dots,q$ correspond to the weight of an edge of $T$. Since we are constructing a $\rho$-labeling of $T$, instead of these weights we use their complements with respect to $2q+1$, that is, $q+k,q+k-1,\dots,q+1$, respectively. For every $i\in\{1,2,\dots,r\}$, let $s_i$ be the number of edges incident to $v_i$ that were deleted from $T$ to form $T'$. Starting with $v_1$, these edges are going to have, the still not assigned, $s_i$ largest weights, in the list $q+k,q+k-1,\dots,q+1$, and so on. In order to see that this is possible, note that the label $q+k+f(v_1)\le q+k+q-k\le 2q$ can be assigned to a leaf adjacent to $v_r$. This means that given the graceful labeling of $T'$ and the set of integers that have not been used as labels, we can produce edges whose weights are $q+k,q+k-1,\dots,q+1$.
All the pendant unlabeled vertices attached to $v_i$, for every $1\le i\le r$, are labeled with consecutive integers, inducing consecutive weights. If $m$ is the smallest label assigned to a leaf adjacent to $v_i$, inducing the weight $w=q+t\in\{q+1,q+2,\dots,q+k\}$ for some $1\le t<k$, then the largest label assigned to a leaf adjacent to $v_{i+1}$ is at most $m-2$. In fact, given that $m-f(v_i)=q+t$, equivalently, $m=f(v_i)+q+t$, when $x$ is the largest label used on a leaf adjacent to $v_{i+1}$, then

$$\begin{aligned}
x-f(v_{i+1}) &= q+t-1\\
x &= f(v_{i+1})+q+t-1\\
x &< (f(v_i)+q+t)-1\\
x &< m-1\\
x &\le m-2.
\end{aligned}$$

Hence, there is always an integer in $\{q+1,q+2,\dots,2q\}$ that can be assigned as the label of a leaf, attached to $v_i$, to produce the required weight for that edge. Therefore, we have obtained a $\rho$-labeling of $T$.

□

In the rest of this section we show some results that can be obtained as a direct consequence of Theorem 2.1.

By definition, a *lobster* is a tree such that the deletion of all its pendant edges results in a caterpillar and caterpillars are graceful [3], therefore, using Theorem 2.1 we may prove the following corollary.

**Corollary 2.2.** *All lobsters admit $\rho$-labelings.*

In Figure 2.1 we show an example of this labeling for a lobster $L$ of size 38. The edges of $L$ represented by red lines are the ones that we deleted to obtain a caterpillar. As we mentioned at the beginning of this section, the fact that lobsters are $\rho$-trees has been proven in [8] and [11]; however, the earliest proof of it was given by Huang and Rosa in 1978 [12].

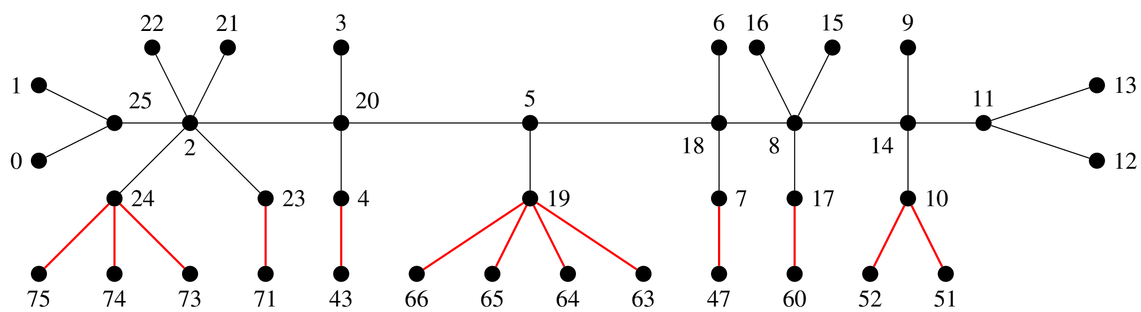


**Figure 2.1:** $\rho$-labeling of a lobster of size 38

In the same line, we have the following two results. Morgan [13] introduced the concept of $k$-distance tree as follows. Let $P$ be any of the longest paths in a tree $T$; $T$ is a *$k$-distance tree* if every vertex is at distance at most $k$ from $P$. We refer to $P$ as the *central path* of $T$. From this definition we have that paths, caterpillars, and lobsters are 0-, 1-, and 2-distance trees, respectively. Recall that a subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. In [14], Burzio and Ferrarese have shown that the tree obtained from any graceful tree, by subdividing every edge the same number of times, is also graceful. We use this result to prove the existence of a $\rho$-labeling for the members of two subfamilies of 3-distance trees.

Let $T$ be a 3-distance tree. The symbols $T^*$ and $P^*$ are used here to designate the lobster obtained from $T$ by deleting all its leaves and the central path of $T^*$, respectively.

**Theorem 2.3.** *Let $T$ be a 3-distance tree. If for every pair of vertices in $P^*$ of degree at least 3, the distance between them is even, then $T$ is a $\rho$-tree.*

*Proof.* Let assume that $T$ is a 3-distance tree such that for every pair of vertices $u, v$ in $P^*$, of degree at least 3, $\text{dist}(u, v)$ is even. Let $X$ be the subset of $V(T^*)$ that contains all the leaves at distance 1 from $P^*$. Then for every $x \in X$, there exists $y_x \in V(T^*)$ such that $xy_x \in E(T)$. Suppose that for every $x \in X$, the edge $xy_x$ has been added to $E(T^*)$; thus, each vertex $y_x$ is at distance 2 from $P^*$, which implies that the new tree, denoted by $L$, is a lobster where all the leaves not in $P^*$ are at distance 2 from $P^*$ and the vertices of degree at least 3 in $P^*$ are separated by an even number of edges. Then, the lobster $L$ can be obtained by subdividing once every edge of a caterpillar. Using the result in [14], we know that $L$ is a graceful tree. Since $T$ can be obtained from $L$ by attaching some pendant vertices, a $\rho$-labeling of $T$ can be constructed applying Theorem 2.1 to a gracefully labeled copy of $L$. $\square$

Similarly, but as a direct consequence of Theorem 2.1, we can prove that when $T$ is a 3-distance tree where every leaf at distance 2 from the central path is at distance 3 from a leaf and for every pair of vertices of degree at least 3 in $P^*$ the distance between them is a positive multiple of 3, then $T$ admits a $\rho$-labeling. We just need to observe that the underneath $T$ is created by subdividing twice every edge of a caterpillar. So, this graph is graceful and $T$ is a $\rho$-graph by Theorem 2.1

**Corollary 2.4.** *Let $T$ be a 3-distance tree. If for every pair of vertices in $P^*$ of degree at least 3, the distance between them is a positive multiple of 3, and every leaf at distance 2 from $P^*$ is at distance 3 from a leaf, then $T$ is a $\rho$-tree.*

Using the help of a computer, Fang [15] has shown that all trees of size up to 34 are graceful, therefore, they admit a $\rho$-labeling. This fact helps us to prove the following corollary of Theorem 2.1.

**Corollary 2.5.** *Let $T$ be a tree of size $q$ with at least $k$ leaves. Any tree $T'$ of size $q-k$, obtained by deleting $k$ leaves from $T$, admits a $\rho$-labeling if $q-k \leq 34$.*

In [16], Eshghi and Azimi presented an algorithm to find graceful labelings of larger graphs. They verified this method with all trees with 30, 35, and 40 vertices. Hence, the result in Corollary 2.3 is also valid when $q-k = 39$.

It is well-known that all trees with at most four leaves are graceful [17], [18], and [19]. Based on this result we have the following corollary of Theorem 2.1.

**Corollary 2.6.** *If $T$ is a tree such that the removal of all its leaves results in a tree with at most four leaves, then $T$ admists a $\rho$-labeling.*

It is also known that trees with diameter at most 5 are graceful [18], [20]. If $T$ is a tree of diameter 5, it can be represented as a rooted tree where the root is any of the central vertices; thus, only one of the branches coming out of the root has height 3. Suppose now that $A$ is a tree of diameter 6. When $A$ is seen as a rooted tree, where the root is its central vertex, more than one of the branches coming out of the root has height 3, if for all, except one of these branches, the leaves in level 3 are deleted, a tree $T$ of diameter 5 is obtained, which is graceful. Therefore, we can obtain a $\rho$-labeling of $T$ applying the procedure described in Theorem 2.1. Furthermore, the same idea can be used to construct a $\rho$-labeling of any tree $A^*$ of diameter 7, in this case, we delete all the leaves in levels 3 and 4 to obtain a tree of diameter 5; hence we can apply Theorem 2.1. Thus, we have proven the following corollary.

**Corollary 2.7.** *If $T$ is a tree of diameter at most 7, then $T$ admits a $\rho$-labeling.*

There are several families of graceful rooted trees, for example, Bermond and Sotteau [21] proved that every rooted tree, in which every level contains vertices of the same degree, is graceful. A tree satisfying this condition is called *symmetric*. This result allows us to prove the following corollary.

**Corollary 2.8.** *If $T$ is a rooted tree with the property that the removal of some of its leaves results in a symmetric tree, then $T$ admits a $\rho$-labeling.*

In [22], Balbuena et al., proved that all trees having an even or quasi even degree sequence are graceful. They obtained a graceful labeling for this type of graphs by representing them as rooted trees. They worked with rooted trees of diameter $D$, where the root is the central vertex and has odd degree, and all the interior vertices have even degree (even degree sequence). The case of a quasi even degree sequence is similar, except that the vertices in the penultimate level have odd degree.

**Corollary 2.9.** *If $T$ is a tree with the property that the removal of some of its leaves results in a tree with an even or quasi even degree sequence, then $T$ admist a $\rho$-labeling.*

Sethuraman and Jesintha [23] proved that all rooted trees, in which every level contains leaves and the degrees of the internal vertices in the same level are equal, are graceful.

**Corollary 2.10.** *If $T$ is a rooted tree with the property that the removal of some of its leaves results in a rooted tree where every level contains leaves and the internal vertices, in that level, have the same degree, then $T$ admits a $\rho$-labeling.*

## 2.2. The second expansion

We open this part exploring the existence of a $\rho$-labeling of a tree $T'$ of size $n+2k$ obtained, from a graceful tree $T$ of size $n$, identifying an end-vertex of a copy of the path $P_3$ to each of $k$ selected vertices of $T$.

**The Labeling Scheme**

Suppose that $f$ is a graceful labeling of a tree $T$ of size $n$. Thus, the labels assigned by $f$ to the vertices of $T$ are the integers $0,1,\dots,n$ and the induced weights are $1,2,\dots,n$. This labeling will be extended to a $\rho$-labeling of a tree of size $n+2k$.

Let $v_1,v_2,\dots,v_k$ be distinct vertices of $T$. We assume that $f(v_1)>f(v_2),\dots,>f(v_k)$. For every $1\le i\le k$, we say that the vertex set of the $i$th copy of $P_3$ is $\{v_i,u_i,w_i\}$ and the edge set is $\{v_iu_i,u_iw_i\}$. Then the labeling $f$ of $T$ is extended to include the vertices $u_i$ and $w_i$, in the following way:

$$f(u_i)=f(v_i)+n+4k+1-i$$

and

$$f(w_i)=f(v_i)+3k-2i+1.$$

We claim that $f$ is a $\rho$-labeling of $T'$ when $f(v_k)\ge n-k$.

**Theorem 2.11.** *Let $f$ be a graceful labeling of a tree $T$ of size $n$ and $v_1,v_2,\dots,v_k$ be distinct vertices of $T$. If $\min\{f(v_i):1\le i\le k\}\ge n-k$, then the tree $T'$, obtained attaching an end-vertex of $P_3$ to each $v_i$, is a $\rho$-tree.*

*Proof.* Suppose that the graceful labeling $f$ of $T$ has been extended to all the vertices of $T'$ using the labeling scheme presented above. First, we verify that the weights induced by $f$ satisfy the conditions required by a $\rho$-labeling.

The edge $v_iu_i$ has weight

$$f(u_i)-f(v_i)=(f(v_i)+n+4k+1-i)-f(v_i)=n+4k+1-i.$$

Since $1\le i\le k$, these weights form the set $\{n+4k,n+4k-1,\dots,n+3k+1\}$. But the complement of each of these numbers is $(2n+4k+1)-(n+4k+1-i)=n+i$, which implies these weights are equivalent to the integers in $\{n+1,n+2,\dots,n+k\}$.

The edge $u_iw_i$ has weight

$$f(u_i)-f(w_i)=(f(v_i)+n+4k+1-i)-(f(v_i)+3k-2i+1)=n+k+i.$$

Then, these weights form the set $\{n+k+1,n+k+2,\dots,n+2k\}$.

Since the weights induced on the edges of $T$ are $1,2,\dots,n$, the weights induced by $f$ on the edges of $T'$ satisfy the conditions required by a $\rho$-labeling.

Now we analyze the injectivity of the function $f$. When $f$ is restricted to the vertices in $T$ it is injective because it is a graceful labeling. Each extension of $f$ to the vertices $u_i$ and $w_i$ is defined as a linear function, which implies that we just need to check that all the labels used are different.

Note that $\min\{f(u_i):1\le i\le k\}=f(v_k)+n+3k+1$ and $\max\{f(u_i):1\le i\le k\}=f(v_1)+n+4k$. Since $f(v_k),f(v_1)\in\{0,1,\dots,n\}$, we get

$$\{f(u_i):1\le i\le k\}\subseteq\{n+3k+1,n+3k+2,\dots,2n+4k\}$$

and

$$\{0,1,\dots,n\}\cap\{f(u_i):1\le i\le k\}=\varnothing.$$

On the other side, $\min\{f(w_i):1\le i\le k\}=f(v_k)+k+1$ and $\max\{f(w_i):1\le i\le k\}=f(v_1)+3k-1$. Thus,

$$\{f(w_i):1\le i\le k\}\cap\{f(u_i):1\le i\le k\}=\varnothing$$

and

$$\{0,1,\dots,n\}\cap\{f(w_i):1\le i\le k\}=\varnothing$$

when $\min\{f(w_i):1\le i\le k\}>n$, that is, when $f(v_k)+k+1>n$; which is equivalent to say that $f(v_k)>n-k-1$ or $f(v_k)\ge n-k$. But this condition is one of the hypotheses. Therefore, the final labeling of $T'$ is in fact a $\rho$-labeling as we claimed. □

Suppose that $T'$ has been labeled using the scheme presented above. Then the edges $v_iu_i$ have weights $n+4k+1-i$ for every $1\le i\le k$. The edges $u_iw_i$ have weights $n+2k+1-i$; that is, the weights $n+k+1,n+k+2,\dots,n+2k$.

Recall that if $v_1,v_2,\dots,v_k$ are the selected vertices of $T$, then $f(v_1)>f(v_2)>\cdots>f(v_k)$. Thus, the edge $u_kv_k$ has weight $n+3k+1$, which implies that $f(u_k)=f(v_k)+n+3k+1$. The edge $u_kw_k$ has weight $n+k+1$ and $f(w_k)=f(u_k)-n-k-1$, then

$$f(w_k)=f(v_k)+n+3k+1-n-k-1=f(v_k)+2k.$$

But $f(w_k)$ is the smallest label assigned to a vertex $w_i$, then it must be larger than $n$, that is, $f(v_k)+2k\ge n+1$, in other terms, $f(v_k)\ge n+1-2k$.

This implies that if the smallest label of the selected vertices of $T$ is at least $n+1-2k$, the tree $T'$ is a $\rho$-tree. Since 0 is the smallest possible value for $f(v_k)$, we get that when $2k\ge n+1$, any possible tree $T'$, obtained from $T$ by attaching $k$ copies of $P_3$, is a $\rho$-tree. In this way, we have proven the next theorem.

**Theorem 2.12.** *If $T'$ is obtained from a graceful tree $T$ of size $n$ attaching a copy of $P_3$ to each of $k$ selected vertices of $T$ and $2k \geq n+1$, then $T'$ is a $\rho$-tree.*

Even when this result is not as strong as Theorem 2.1, it produces $\rho$-labelings for many trees. For instance, if $n = 10$, then $6 \leq k \leq 11$; this means that for any graceful tree of size 10, it is possible to construct $\sum_{k=6}^{11} C(10,k) = 2^{10}$ $\rho$-labeled trees. In general we have that there are $\sum_{\lceil \frac{n+1}{2} \rceil}^{n+1} C(n,k) = 2^n$ obtained from a gracefully labeled tree of size $n$. In Figure 2.2 we have an example for each $k \in \{6,7,8,9,10,11\}$, where $T$ is a gracefully labeled tree of size $n = 10$.

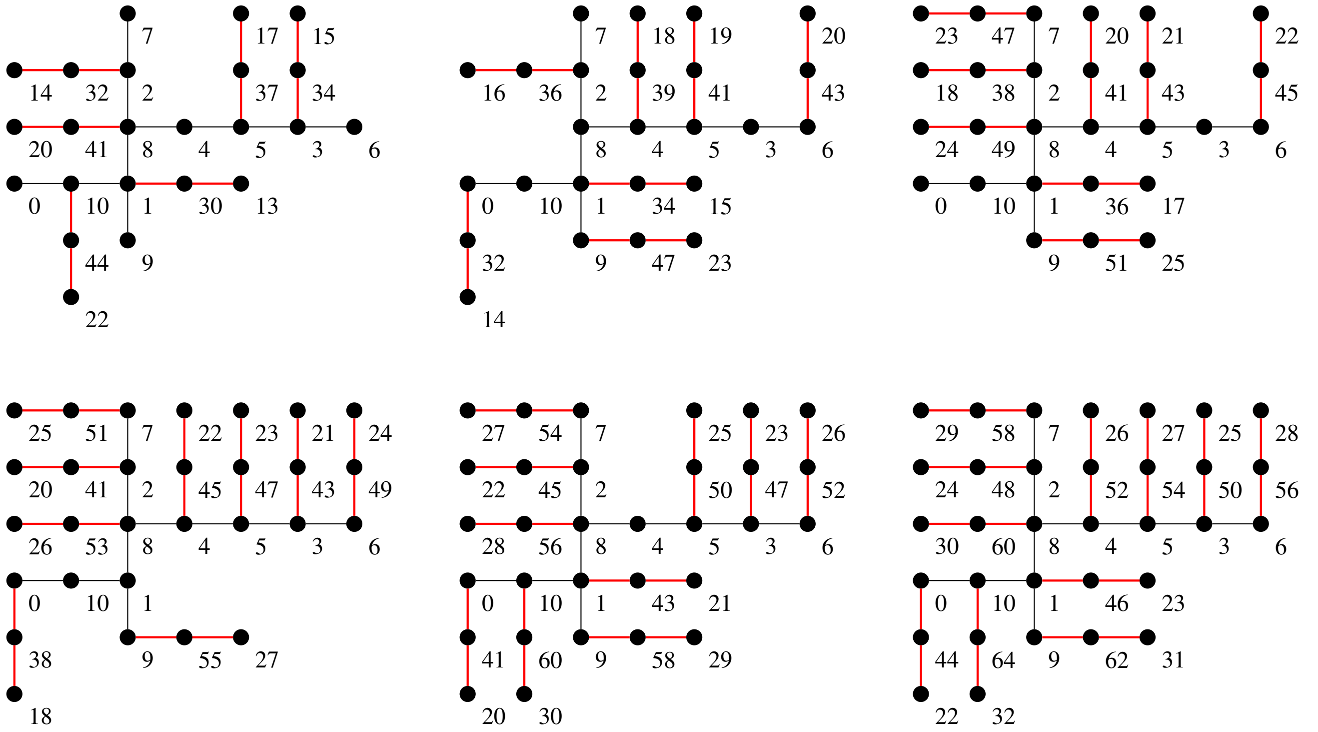

**Figure 2.2:** Some of the $2^{10}$ $\rho$-labeled trees obtained from a graceful tree of size 10

By $\mathscr{T}_n$ we denote the family of all $\rho$-labeled trees which $\rho$-labeling is obtained following the labeling scheme used in Theorem 2.1.

**Theorem 2.13.** *If $T''$ is obtained from $T'$ in $\mathscr{T}_n$ by attaching a pendant vertex to each of $k$ selected vertices of $T'$, where $2k \geq n+1$ and the labels of any pair of selected vertices are at least two units apart, then $T''$ is a $\rho$-tree.*

*Proof.* Assume that $T$ is a graceful tree of size $n$ and $T'$ is any tree obtained by attaching any number of pendant vertices to any number of vertices of $T$. Suppose that $t$ is the total number of pendant vertices attached to $T$ to form $T'$. Let $f$ be the $\rho$-labeling of $T'$ procured using Theorem 2.1 and $v_1, v_2, \ldots, v_k$ be distinct pendant vertices of $T'$ such that for each $1 \leq i \leq k$, $f(v_i) - f(v_{i+1}) \geq 2$. Note that the labels assigned to the pendant vertices introduced to form $T'$ are in the set $\{n+t+2k+1, n+t+2k+2, \ldots, n+2t+2k\}$. Thus, the integers in the set $L = \{n+1, n+2, \ldots, n+t+2k\}$ have not been used as labels of $T'$.

As we did in the previous theorem, the goal is that the edges generated connecting each $v_i$, for $1 \leq i \leq k$, to a pendant vertex, have weights $n+t+k, n+t+k-1, \ldots, n+t+1$, respectively. To achieve this goal, we label the pendant vertices with integers from $L$. Note that $f(v_k) \leq n+t+2k+1$ and the pendant edge associated with $v_k$ has weight $n+t+1$, so the pendant vertex has a label at least $x$, where

$$\begin{aligned}(n+2t+2k) - x &= n+t+k+1\\ x &= 2k,\end{aligned}$$

since $2k \geq n+1$, this integer has not been assigned as a label, hence it can be used now and no repetition of labels happens. Similarly, the pendant edge associated with $v_1$ has weight $n+t+k$, so the pendant vertex has a label at most $y$, where

$$\begin{aligned}(n+2t+2k) - y &= n+t+k+1\\ y &= t+k-1.\end{aligned}$$

Thus, we conclude that is possible to label all the pendant vertices with integers from $L$, producing the weights $n+t+1, n+t+2, \ldots, n+t+k$. Consequently, the labeling of $T''$ is in fact a $\rho$-labeling. □

In Figure 2.3 we present an example of this labeling for a graceful tree of size 6, with $t = 21$ and $k = 9$. The blue edges are those produced by the first expansion, and the red edges are the ones created by the second expansion of the graceful tree represented with black edges.a

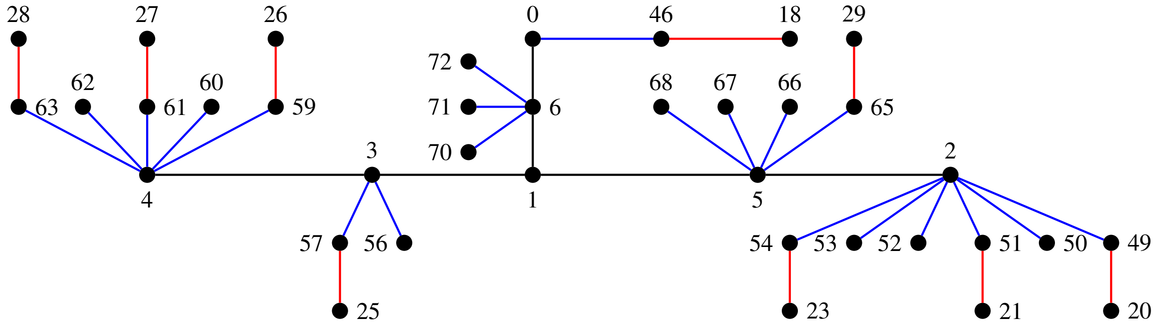


**Figure 2.3:** A $\rho$-labeled tree obtained from a subdivision of the star $S_3$

## 3. Every tree is a spanning tree of an $\alpha$-graph

In this section we show that any tree $T$ of size $n$ is a spanning tree of an $\alpha$-graph of size $n+\varepsilon(T)$, where $\varepsilon(T)$ is the excess of $T$, that is, a parameter associated with the left-layered representation of $T$ introduced in [4].
Let $T$ be a rooted tree, for every vertex $v$ of $T$, $\gamma(v)$ is the number of levels in $T$ where $v$ has at least one descendant. We say that $T$ is a *left-layered tree* if the vertices on each level of $T$ adhere to the following rules:

(i) If $u$ and $v$ are siblings of degree one, the order of $u$ and $v$ is arbitrary.
(ii) If $u$ and $v$ are siblings and $\gamma(u)<\gamma(v)$, then $u$ is placed to the left of $v$.
(iii) If $u$ and $v$ are siblings, $\gamma(u)=\gamma(v)$, and $\deg(u)\geq\deg(v)$, then $u$ is placed to the left of $v$.
(iv) If $u$ and $v$ are siblings such that $u$ is placed to the left of $v$, then the descendants of $u$ are placed to the left of the descendants of $v$.

In Figure 3.1 we show the left-layered representation of a tree of size 58. The edges of this tree are represented with black lines.

Let $T_r$ be the left-layered representation of a rooted tree $T$ with root $r$. Thus, $L_k=\{v_j^k : 1\leq j\leq n_k\}$ is the set of all vertices of $T$ at distance $k$ from $r$, that is, the vertices of $T_r$ placed on level $k$. We assume that $v_j^k$ is placed to the left of $v_{j+1}^k$, for all $1\leq j\leq n_k-1$. The *excess* of $L_k$, denoted $\Omega_k$, is defined to be

$$\Omega_k=\begin{cases} 0 & \text{if } k=0,h,\\ n_k-n_{k,0}-1 & \text{if } 1\leq k\leq h-1,\end{cases}$$

where $n_{k,0}$ is the number of vertices in $L_k$ with no children and $h$ is the height of $T_r$. The *excess of* $T_r$, denoted by $\mathrm{ex}(T_r)$, is given by $\mathrm{ex}(T_r)=\sum_{k=0}^{h}\Omega_k$. The *excess of* $T$, denoted by $\varepsilon(T)$, is defined to be

$$\varepsilon(T)=\min\{\mathrm{ex}(T_r) : \text{for all } r\in V(T)\}.$$

Thus, if $T$ is a caterpillar or a path, $\varepsilon(T)=0$, and it is obtained when $r$ is chosen to be any of the vertices of maximum eccentricity in $T$. The tree in Figure 3.1, represented with the black edges, has excess 18; the vertices used to calculate this parameter are in black.

A *bipartite labeling* of a tree $T$ of size $n$ is an injective mapping $f: V(G)\to\{0,1,\ldots,s\}$, with $s\geq n$, such that all induced weights are distinct and the labels assigned to the vertices in one of the stable sets of $T$ are smaller than the labels assigned to the vertices of the other stable set. When $s=n$, the bipartite labeling is indeed an $\alpha$-labeling of $T$. In [5] we introduced a bipartite labeling of $T$ where $s=n+\varepsilon(T)$. For the sake of completeness, we describe this labeling again. We will use it to prove that every tree $T$ of size $n$ and excess $\varepsilon(T)$ is a spanning tree of an $\alpha$-graph of order $n+1$ and size $n+\varepsilon(T)$.
Recall that on every level, the vertices are ordered from the right to the left. If $h$ is the height of $T$, then the labels assigned to the vertices on each level are consecutive integers. On the levels $L_h, L_{h-2},\ldots$ they are in increasing order; on the levels $L_{h-1}, L_{h-3},\ldots$ they are in decreasing order. The smaller labels are on the levels $L_h, L_{h-2},\ldots$.
The first label on $L_h$ is 0 and the first label on $L_{h-2i}$ is the addition of the largest label on $L_{h-2i+2}$ and $1+\Omega_{h-2i+1}$.
The first label on $L_{h-1}$ is $n+\varepsilon(T)$ and the first label on $L_{h-1-2i}$ is the subtraction of the smallest label on $L_{h+1-2i}$ and $1+\Omega_{h-2i}$.

In Figure 3.1 we exhibit an example of this bipartite labeling.

**Theorem 3.1.** *Any tree $T$ of size $n$ is a spanning tree of an $\alpha$-graph of size $n+\varepsilon(T)$.*

*Proof.* Suppose that $T$ has been labeled using the bipartite labeling $f$ described before. Thus, the labels assigned belong to the set $\{0,1,\ldots,n+\varepsilon(T)\}$ and the induced weights are in the set $\{1,2,\ldots,n+\varepsilon(T)\}$. Let $v_1$ and $v_2$ be two consecutive vertices on level $L_k$ such that they are not siblings, this implies the existence of two consecutive vertices, $u_1$ and $u_2$, on level $L_{k-1}$ such that $u_1v_1, u_2v_2\in E(T)$.
If $h$ and $k$ have the same parity, then $f(v_1)+1=f(v_2)$ and $f(u_1)-1=f(u_2)$. Thus, the weight of $u_1v_1$, is $f(u_1)-f(v_1)$, which is two units larger than the weight of $u_2v_2$, because

$$f(u_2)-f(v_2)=f(u_1)-1-(f(v_1)+1)=f(u_1)-f(v_1)-2.$$

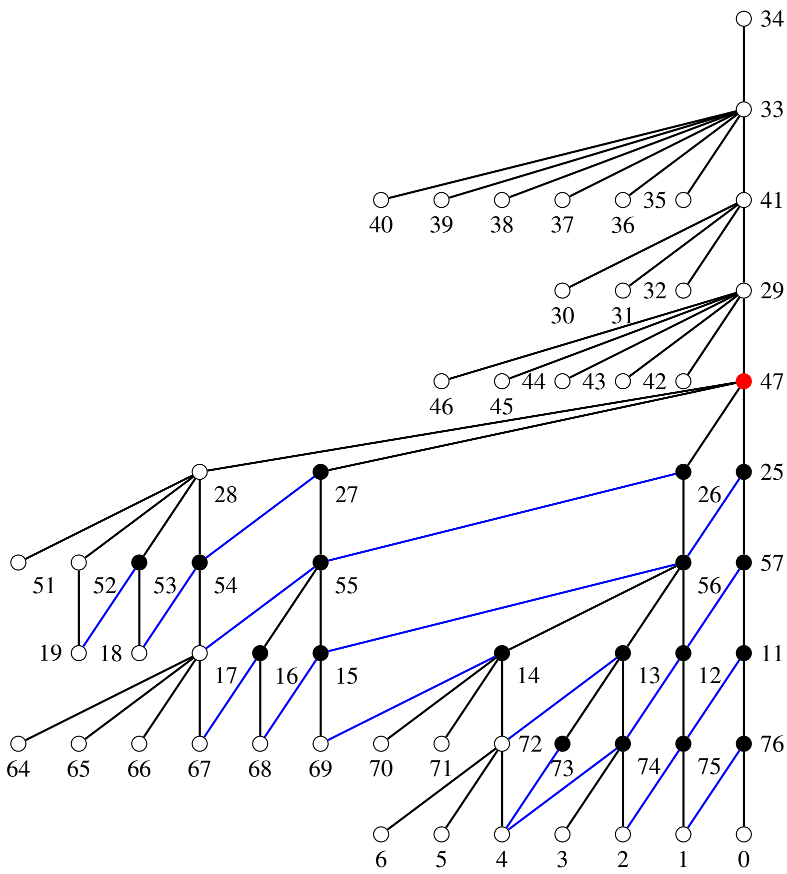


**Figure 3.1:** Embedding of a tree of size 58 and excess 18 in an $\alpha$-graph of size 76

Hence, the integer $f(u_1)-f(v_1)-1$ is not the weight of any edge of $T$. If the vertices $v_2$ and $u_1$ are connected (or the vertices $v_1$ and $u_2$), we create an edge of weight $f(u_1)-f(v_1)-1$.

If $h$ and $k$ have different parity, then $f(v_2)=f(v_1)-1$ and $f(u_1)+1=f(u_2)$. Thus, the weight of $u_1v_1$, is $f(u_1)-f(v_1)$, and again, it is two units larger than the weight of $u_2v_2$, because

$$f(v_2)-f(u_2)=f(v_1)-1-(f(u_1)+1)=f(v_1)-f(u_1)-2.$$

So, the integer $f(v_1)-f(u_1)-1$ is not the weight of any edge of $T$. If the vertices $v_2$ and $u_1$ are connected (or the vertices $v_1$ and $u_2$), we create an edge of weight $f(v_1)-f(u_1)-1$.

If this process of introducing a new edge is done every time that is needed, the resulting graph has $T$ as a spanning tree and its labeling induces the weights $1,2,\ldots,n+\varepsilon(T)$. Since the original labeling is bipartite, the labeling of the new graph is an $\alpha$-labeling.

□

In Figure 3.1 we show an example of this labeling. The black edges correspond to a tree $T$ of size 58, $\varepsilon(T)=18$, and $h=9$, $\Omega_9=\Omega_4=\Omega_3=\Omega_2=\Omega_1=\Omega_0=0$, $\Omega_8=4$, $\Omega_7=6$, $\Omega_6=5$, and $\Omega_5=3$. The missing weights are 27, 29, 31, 34, 36, 38, 41, 45, 51, 53, 59, 62, 69, 70, 73, and 75. If the blue edges are added to the labeled tree, they form an $\alpha$-labeled graph of size $58+18=76$ that contains $T$ as a spanning tree.

Note that for every missing weight in the original bipartite labeling of $T$, there are two possible new edges with that weight, therefore, there are at least $2^{\varepsilon(T)}$ hosts for $T$ that are $\alpha$-graphs. As far as we know, the $\alpha$-graph obtained in this way, is the smallest $\alpha$-graph that has $T$ as a spanning tree.

Other authors have studied similar problems; for example, Rao and Sahoo [24] proved that every connected graph of order $n$ can be embedded as an induced subgraph in a graceful Eulerian graph of size $3^n$. In [25], Sethuraman and Ragukumar introduced some algorithms that allow them to prove, among other results, that any tree of size $n$ is a spanning tree of an $\alpha$-graph of size $M$. They provided an example for a tree of size 20 embedded in an $\alpha$-graph of size 55. This tree has excess 5; thus, when we apply our labeling scheme to this tree, the associated $\alpha$-graph has size 25. We have not been able to determine the value of $M$ in [25] but we believe that for any given tree $T$ of size $n$, the inequality $n+\varepsilon(T)<M$ holds.

## 4. Conclusion

Since there are many known families of graceful trees and any graceful tree can be used to construct infinitely many larger $\rho$-trees, Theorem 2.1 is the most powerful result in the context of $\rho$-labelings. Then it was natural to ask whether is possible to weaken the condition of being graceful imposed to the starting tree. As a result of that analysis, we obtained the last theorems in Section 2. In particular, Theorem 2.13 tells us that for any $k$ large enough, there are $k$ end-vertices in $T'$ that can be extended one more time, to obtain a new $\rho$-tree. Is there any other type of tree that can be used, instead of $P_3$, to attach to all or some of the vertices of a tree $T$ to create new $\rho$-trees? The result in Section 3 gives us an upper bound for the size of the smaller $\alpha$-graph that contains $T$ as a spanning tree. Further studies of this type should include the examination of how good this bound is, which is better than any other bound known to date.

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# Applicable Multiplicative Calculus Using Multiplicative Modulus Function

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**Abstract**

The classical calculus is viewed as additive calculus based on addition in the real line. Another known multiplicative calculus corresponding to multiplication in the positive real axis has been precisely introduced. Abstract multiplicative integration through positive measures has been newly introduced. Results of multiplicative differentiation and integration have been obtained for completion, when some of them have been obtained through multiplicative modulus function. Results have been obtained also for abstract multiplicative measure integration.

## 1. Introduction

Arithmetic mean of real numbers is considered in elementary statistics and geometric mean of positive real numbers is also considered in elementary statistics. The first one is a mean for addition and the second one is a mean for multiplication, and both are considered as applicable. The usual absolute value function is a function for addition. A new absolute value function, which was just mentioned in [1], has been extremely used for multiplication in [2]. The classical calculus of Newton and Leibnitz is based on addition. Another calculus was also known for multiplication, and it became an important part of research since the publication of the book [3] and the article [1], which provides a good introduction for multiplicative calculus. It has been extended in many directions; fractional derivative, complex derivative, integral transformations, differential equations and applications for science and engineering in [1], [4]-[18]. It has been established in [19, 20] that multiplicative calculus would also be applicable. The author believes that some precision is required in the introduction of multiplicative calculus, because it is also treated as a course meant for undergraduate students in view of articles like [21]. This has been done in this article. Moreover, multiplicative modulus function used in [2] has been applied to derive some new, but elementary results. Multiplicative integration using positive measures has also been defined precisely and some fundamental results have been obtained. The author could not find any article in literature for multiplicative abstract measure integration, even though there is an advanced research article [22] for Lebesgue measure integration. It should be observed that some changes have been done in this article in conventional notations for multiplicative calculus. Let us begin with a definition of the classical absolute value of real numbers. Let us use the notations R and P for the set of real numbers and the set of positive real numbers, respectively.

**Definition 1.1.** *If $x \in R$, then additive absolute value of $x$ is denoted by $|x|$, and defined as the number* $\max\{x, -x\}$.

Let us now present the notation and definition of multiplicative absolute value given in [2].

**Definition 1.2.** *If $x \in P$, then multiplicative absolute value of $x$ is denoted by $|x|_\times$, and defined as the number* $\max\{x, x^{-1}\}$.

**Proposition 1.3.** *Let $x, y \in R$, and let $u, v \in P$. Then $|uv|_\times \leq |u|_\times |v|_\times$, $|\log u| = log|u|_\times$, and* $\exp|x| = |\exp x|_\times$.

*Proof.* Direct verification. □

The second section discusses fundamental results for multiplicative differentiation for completion, the third section discusses multiplicative Riemann integration with a precise definition, and the fourth section discusses newly introduced multiplicative abstract measure integration.

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## 2. Multiplicative differentiation

**Definition 2.1.** *Let $f:[a,b]\to R$ be a real valued function and let $x_0\in[a,b]$. Then the derivative of $f$ exists at $x_0$, if $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ exists. This limit is denoted by $f^{'}(x_0)$ or $\frac{df(x_0)}{dx}$ or $\frac{df}{dx}|_{x=x_0}$ or $\frac{df(x)}{dx}|_{x=x_0}$ , and it is called the derivative of $f$ at $x_0$.*

**Definition 2.2.** *Let $F:[a,b]\to(0,\infty)$ be a positive real valued function and let $x_0\in[a,b]$. Then the m-derivative of $F$ exists at $x_0$, if $\lim_{x\to x_0}\left(\frac{F(x)}{F(x_0)}\right)^{\frac{1}{x-x_0}}$ exists, and it is not equal to zero. This limit is denoted by $F^{|}(x_0)$ or $\frac{DF(x_0)}{Dx}$ or $\frac{DF}{Dx}|_{x=x_0}$ or $\frac{DF(x)}{Dx}|_{x=x_0}$, and it is called the m-derivative of $F$ at $x_0$.*

**Remark 2.3.** *These definitions can be extended to other intervals of types $[a,b),(a,b),(a,b]$, naturally. One sided derivatives can also be defined. The higher order derivatives can also be defined.*

**Lemma 2.4.** *Let $F,[a,b],x_0$ be as in Definition 2.2. Then $\frac{DF(x_0)}{Dx}$ exists if and only if $\frac{d\log F(x_0)}{dx}$ exists, and in this case, $\frac{DF(x_0)}{Dx}=\exp\left(\frac{d\log F(x_0)}{dx}\right)$.*

*Proof.* The proof follows from the relation $\exp\left(\frac{\log F(x)-\log F(x_0)}{x-x_0}\right)=\left(\frac{F(x)}{F(x_0)}\right)^{\frac{1}{x-x_0}}$. □

**Corollary 2.5.** *$\frac{DF(x_0)}{Dx}$ exists if and only if $\frac{dF(x_0)}{dx}$ exists, for $F$ and $x_0$ given in Lemma 2.4.*

*Proof.* Use the relation $F(x)=\exp\log F(x)$, and the chain rule. □

**Lemma 2.6.** *Let $f,[a,b],x_0$ be as in Definition 2.1. Then $\frac{df(x_0)}{dx}$ exists if and only if $\frac{D\exp(f(x_0))}{Dx}$ exists, and in this case, $\frac{df(x_0)}{dx}=\log\left(\frac{D\exp(f(x_0))}{Dx}\right)$.*

*Proof.* The proof follows from the relation $\log\left(\frac{\exp(f(x))}{\exp(f(x_0))}\right)^{\frac{1}{x-x_0}}=\frac{f(x)-f(x_0)}{x-x_0}$. □

**Remark 2.7.** *Formally, $\frac{D}{Dx}=\exp\frac{d}{dx}\log$, and $\frac{d}{dx}=\log\frac{D}{Dx}\exp$. For any integer $n\geq 2$, it can be verified formally that $\frac{D^n}{Dx^n}=\exp\frac{d^n}{dx^n}\log$, and $\frac{d^n}{dx^n}=\log\frac{D^n}{Dx^n}\exp$.*

Let us try to use these formal relations and let us convert some results of Chapter 5 in [23].

**Proposition 2.8.** *Let $F,[a,b],x_0$ be as in Definition 2.2. Suppose $\frac{DF(x_0)}{Dx}$ exists. Then $F$ is continuous at $x_0$.*

*Proof.* Let $M=\frac{DF(x_0)}{Dx}$. Then $0<M<\infty$, and $M^{x-x_0}\to 1$ as $x\to x_0$. Therefore, $\frac{F(x)}{F(x_0)}\to 1$ and hence $F(x)\to F(x_0)$ as $x\to x_0$.
*Another Proof:*
Since $\frac{d\log F(x_0)}{dx}$ exists, $\log F(x_0)$ is continuous at $x_0$, and hence $F(x)$ is continuous at $x_0$. □

**Proposition 2.9.** *Let $F,G$ be positive real valued functions on $[a,b]$ and m-differentiable at a point $x_0$ in $[a,b]$. Then $\frac{D(FG)(x_0)}{Dx}$ exists and it is $\frac{DF(x_0)}{Dx}\frac{DG(x_0)}{Dx}$.*

*Proof.* It follows from Definition 2.2. □

**Theorem 2.10.** *Suppose $f$ is a real valued function on $[a,b]$. Let $x_0\in[a,b]$. Suppose $f^{'}(x_0)$ exists. Let $G$ be a positive real valued function on an interval $I$ which contains the range of $f$. Suppose $G$ is m-differentiable at the point $f(x_0)$. Let $H(t)=G(f(t))$, for all $t\in[a,b]$. Then $H$ is m-differentiable at $x_0$, and $H^{|}(x_0)=(G^{|}(f(x_0))^{f^{'}(x_0)}$.*

*Proof.* By Lemma 2.4,

$$
\begin{aligned}
(G^{|}(f(x_0))^{f'(x_0)} &= \left(\exp\left(\frac{d\log G(f(x_0))}{dy}\right)\right)^{\frac{df(x_0)}{dx}} \text{ (with } y\in I)\\
&= \exp\left(\frac{df(x_0)}{dx}\frac{d(\log G)f(x_0)}{dy}\right)\\
&= \exp\left(\frac{d((\log G)\circ f)(x_0)}{dx}\right)\\
&= \frac{DH(x_0)}{Dx}.
\end{aligned}
$$

*Another Proof:*
Use Proposition 2.8 and the relation

$$
\left(\frac{H(x)}{H(x_0)}\right)^{\frac{1}{x-x_0}}=\left(\left(\frac{G(f(x))}{G(f(x_0))}\right)^{\frac{1}{f(x)-f(x_0)}}\right)^{\frac{f(x)-f(x_0)}{x-x_0}}
$$

for the case $f(x)\neq f(x_0)$. Separate the case $f(x)=f(x_0)$. □

**Lemma 2.11.** *Let $F$ be a positive real valued function defined on $[a,b]$. If $F$ has a local maximum at a point $x_0$ in $(a,b)$ and if $\frac{DF(x_0)}{Dx}$ exists, then $\frac{DF(x_0)}{Dx}=1$.*

*Proof.* Suppose $\frac{DF(x_0)}{Dx}$ exists. Then $\frac{d\log F(x_0)}{dx}$ exists. Suppose $F$ has a local maximum at $x_0$ in $(a,b)$. Then $\frac{d\log F(x_0)}{dx}=0$, and hence $\frac{DF(x_0)}{Dx}=\exp\left(\frac{d\log F(x_0)}{dx}\right)=1$.
*Direct Proof:*
$\left(\frac{F(x)}{F(x_0)}\right)^{\frac{1}{x-x_0}}\geq 1$, if $x<x_0$, and $\left(\frac{F(x)}{F(x_0)}\right)^{\frac{1}{x-x_0}}\leq 1$, if $x>x_0$, when $x$ is in a suitable neighborhood of $x_0$. □

**Theorem 2.12.** *Suppose $F$ and $G$ are continuous positive real valued functions on $[a,b]$ which are m-differentiable in $(a,b)$. Then there is a point $x_0$ in $(a,b)$ such that $\left(\frac{F(b)}{F(a)}\right)^{\frac{G'(x_0)}{G(x_0)}}=\left(\frac{G(b)}{G(a)}\right)^{\frac{F'(x_0)}{F(x_0)}}$*

*Proof.* Use Theorem 5.9 in [23], and Lemma 2.4 to find a point $x_0$ in $(a,b)$ such that $(\log F(b)-\log F(a))\frac{G'(x_0)}{G(x_0)}=(\log G(b)-\log G(a))\frac{F'(x_0)}{F(x_0)}$ □

**Proposition 2.13.** *Suppose $F$ is a continuous positive real valued function on $[a,b]$ which is m-differentiable in $(a,b)$. Then $\left(\frac{F(b)}{F(a)}\right)^{\frac{1}{b-a}}=\frac{DF(x_0)}{Dx}$, for some $x_0\in(a,b)$.*

*Proof.* By the mean value theorem, there is a point $x_0\in(a,b)$ such that $\log F(b)-\log F(a)=(b-a)\frac{d\log F(x_0)}{dx}$. □

**Theorem 2.14.** *Suppose $F$ is an m-differentiable positive real valued function on $(a,b)$.*
*(a) If $F^{|}(x)\geq 1$ for all $x\in(a,b)$, then $F$ is monotonically increasing in $(a,b)$.*
*(b) If $F^{|}(x)=1$ for all $x\in(a,b)$, then $F$ is a constant function in $(a,b)$.*
*(c) If $F^{|}(x)\leq 1$ for all $x\in(a,b)$, then $F$ is monotonically decreasing in $(a,b)$.*

*Proof.* Apply Theorem 5.11 in [23] to $\log F$ function. □

**Theorem 2.15.** *Suppose $F$ is an m-differentiable positive real valued function on $[a,b]$. Let $\lambda$ be a constant such that $F^{|}(a)<\lambda<F^{|}(b)$. Then there is a point $x_0$ in $(a,b)$ such that $F^{|}(x_0)=\lambda$.*

*Proof.* Observe that $\exp\left(\frac{d\log F(a)}{dx}\right)<\lambda<\exp\left(\frac{d\log F(b)}{dx}\right)$ and hence $\frac{d\log F(a)}{dx}<\log\lambda<\frac{d\log F(b)}{dx}$ . By Theorem 5.12 in [23], there is a $x_0\in(a,b)$ such that $\frac{d\log F(x_0)}{dx}=\log\lambda$ . In this case, $F^{|}(x_0)=\lambda$. □

**Theorem 2.16.** *Let $F$ be a positive real valued function on $[a,b]$, and $n$ be a positive integer such that $\frac{D^{n-1}F(t)}{Dx^{n-1}}$ is continuous on $[a,b]$ and $\frac{D^nF(t)}{Dx^n}$ exists for every $t\in(a,b)$. Let $\alpha,\beta$ be points in $[a,b]$. Then there is a point $x_0$ between $\alpha$ and $\beta$ such that*
$$\frac{F(\beta)}{F(\alpha)}=\sum_{k=1}^{n-1}\left(\frac{D^kF(\alpha)}{Dx^k}\right)^{\frac{(\beta-\alpha)^k}{k!}}+\left(\frac{D^nF(x_0)}{Dx^n}\right)^{\frac{(\beta-\alpha)^n}{n!}}$$

*Proof.* By Remark 2.7, $\frac{d^{n-1}\log F}{dx^{n-1}}$ is continuous on $[a,b]$ and $\frac{d^n\log F(t)}{dx^n}$ exists for every $t\in(a,b)$. By Theorem 5.15 in [23], there is a point $x_0$ between $\alpha$ and $\beta$ such that $f(\beta)-f(\alpha)=\sum_{k=1}^{n-1}\frac{d^kf(\alpha)}{dx^k}\frac{(\beta-\alpha)^k}{k!}+\frac{d^nf(x_0)}{dx^n}\frac{(\beta-\alpha)^n}{n!}$, where $f=\log F$. The result follows from this relation. □

## 3. Multiplicative Riemann integration

**Definition 3.1.** *Let $[a,b]$ be a given interval. A partition $P=x_0,x_1,\ldots,x_n$ of $[a,b]$ satisfies $a=x_0\leq x_1\leq\ldots\leq x_{n-1}\leq x_n=b$. Let $D$ be the collection of all partitions of $[a,b]$. This collection $D$ is a directed set, directed by a relation $\leq$ defined by: $P_1\leq P_2$ if and only if $P_1\subseteq P_2$. Let $F:[a,b]\to(0,\infty)$ be a function such that $m\leq F(x)\leq M$, for some $m>0$ and $M<\infty$, for all $x\in[a,b]$. To each partition $P=x_0,x_1,\ldots,x_n$ of $[a,b]$, fix $t_i\in[x_{i-1},x_i]$, for $i=1,2,\ldots,n$, define $F_P=\prod_{i=1}^nF(t_i)^{(x_i-x_{i-1})}$ . Suppose all nets $(F_P)_{P\in D}$ converge uniformly to a common number $p\in(0,\infty)$ in the following sense: For every $\varepsilon>0$, there is a partition $P_0$ of $[a,b]$ such that $|F_P-p|<\varepsilon$, for all partitions $P\geq P_0$ in $D$, and for all selections of $t_i$. The number $p$ is called the m-Riemann integral of $F$ of $[a,b]$, and it is denoted by $M_a^bF(x)Dx$, or simply, $M_a^bF$. In this case, let us say that $F$ is m-Riemann integrable over $[a,b]$.*

**Remark 3.2.** *Let $F$ be as in Definition 3.1. Then $F$ is m-Riemann integrable over $[a,b]$ if and only if $\log F$ is Riemann integrable over $[a,b]$. Moreover,*
$$M_a^bF(x)Dx=\exp\int_a^b\log F(x)dx,$$
*in this case. Let $f$ be a bounded real valued function on $[a,b]$. Then $f$ is Riemann integrable over $[a,b]$ if and only if $\exp f$ is m-Riemann integrable over $[a,b]$. Moreover,*
$$\int_a^bf(x)dx=\log M_a^b\exp f(x)Dx,$$

*in this case.*

**Lemma 3.3.** *Let $F:[a,b]\to(0,\infty)$, $G:[a,b]\to(0,\infty)$ be functions which are m-Riemann integrable over $[a,b]$. Then the pointwise multiplication function $FG:[a,b]\to(0,\infty)$ is also m-Riemann integrable over $[a,b]$, and $M_a^b(FG)(x)Dx=(M_a^bF(x)Dx)(M_a^bG(x)Dx)$. If $F(x)=c>0$, for every $x\in[a,b]$, then $M_a^bF(x)Dx=c^{b-a}$.*

*Proof.* It follows from Definition 3.1. □

Let us recall that for every number $x\in(0,\infty)$, $|x|_\times=\max\{x,x^{-1}\}$, and for every $x,y\in(0,\infty)$, $|xy|_\times\le|x|_\times|y|_\times$. Direct verification is applicable.

**Lemma 3.4.** *Let $F:[a,b]\to(0,\infty)$ be m-Riemann integrable. Then, $|M_a^bF(x)Dx|_\times\le M_a^b|F(x)|_\times Dx$.*

*Proof.* It follows from Definition 3.1. □

**Theorem 3.5.** *Suppose $f:[a,b]\to(0,\infty)$ be m-Riemann integrable. For $a\le x\le b$, define $F(x)=M_a^xf(t)Dt$. Then, $F$ is continuous on $[a,b]$. Moreover, if $f$ is continuous at a point $x_0$ of $[a,b]$, then $F$ is m-differentiable at $x_0$, and $F^{|}(x_0)=f(x_0)$.*

*Proof.* $\int_a^x\log f(t)dt$ is a continuous function of $x$ in $[a,b]$, by Theorem 6.20 in [23] and Remark 3.2. Thus, $F(x)=\exp\int_a^x\log f(t)dt$ is a continuous function of $x$ in $[a,b]$. Also, if $f$ is continuous at a point $x_0$ of $[a,b]$, then $\left.\frac{d\int_a^x\log f(t)dt}{dx}\right|_{x=x_0}=\log f(x_0)$, by Theorem 6.20 in [12]. Thus, $F^{|}(x_0)=f(x_0)$, when $f$ is continuous at $x_0$. □

**Theorem 3.6.** *Let $f$ be an m-Riemann integrable function on $[a,b]$. Suppose there is an m-differentiable function $F$ on $[a,b]$ such that $F^{|}=f$. Then, $M_a^bf(x)Dx=\frac{F(b)}{F(a)}$.*

*Proof.* The function $\log f$ is Riemann integrable over $[a,b]$. The relation $F^{|}=f$ implies that $\exp\frac{d\log F}{dx}=f$ or $\frac{d\log F}{dx}=\log f$. By Theorem 6.21 in [23], $\int_a^b\log f(x)dx=\log F(b)-\log F(a)$, and hence $M_a^bf(x)Dx=\frac{F(b)}{F(a)}$. □

## 4. Multiplicative abstract measure integration

**Definition 4.1.** *Let $(X,\mathfrak{M},\mu)$ be a positive measure space. See [24]. Let $s:X\to[1,\infty)$ be a simple measurable function and let $s=\sum_{i=1}^n\alpha_i\chi_{A_i}$, where $\alpha_i$ are distinct real numbers and $A_i$ are pairwise disjoint sets. For $A\in\mathfrak{M}$, define $M_AsD\mu=\Pi_{i=1}^n{\alpha_i}^{\mu(A\cap A_i)}$. It is $+\infty$ if and only if some $\alpha_i>1$ with $\mu(A\cap A_i)=+\infty$. Let $f:X\to[1,\infty]$ be a measurable function. For $A\in\mathfrak{M}$, define $M_AfD\mu=\sup_{1\le s\le f}M_AsD\mu$, where $s$ represents a simple measurable function. Let us say that $f:X\to(0,\infty)$ is m-absolutely m-integrable, if $M_A|f|_\times D\mu<\infty$. To each measurable $f:X\to(0,\infty)$, let us define $f_\times^+:X\to[1,\infty)$, and $f_\times^-:X\to[1,\infty)$ by $f_\times^+(x)=\max\{1,f(x)\}$, and $f_\times^-(x)=\max\{1,f(x)^{-1}\}$, for every $x\in X$. Then $|f|_\times=f_\times^+f_\times^-$, and $f=\frac{f_\times^+}{f_\times^-}$. If $f$ is absolutely m-integrable over $X$, then let us define $M_AfD\mu=\frac{M_Af_\times^+D\mu}{M_Af_\times^-D\mu}$, for every $A\in\mathfrak{M}$.*

**Remark 4.2.** *If $f\in L^1(\mu)$ and $f$ is real valued, then $\exp f$ is m-absolutely m-integrable on $X$, and $\log M_X\exp fD\mu=\int_Xfd\mu$. If $F$ is m-absolutely m-integrable on $X$, then $\log F\in L^1(\mu)$ and $M_XFD\mu=\exp\int_X\log Fd\mu$.*

**Theorem 4.3.** *Let $(f_n)_{n=1}^\infty$ be a sequence of measurable functions on $X$ and $f$ be a measurable function $X$ such that*
*(a) $1\le f_1(x)\le f_2(x)\le\ldots\le\infty$, for every $x\in X$, and*
*(b) $f_n(x)\to f(x)$ as $n\to\infty$, for every $x\in X$.*
*Then $M_Xf_nD\mu\to M_XfD\mu$, as $n\to\infty$.*

*Proof.* Apply the classical monotone convergence theorem to the functions $\log f_n$ and $\log f$ and use Remark 4.2. □

**Corollary 4.4.** *Let $f_n:X\to[1,\infty]$ be measurable for $n=1,2,\ldots$, and let $f(x)=\prod_{n=1}^\infty f_n(x)$, for every $x\in X$. Then $M_XfD\mu=\prod_{n=1}^\infty M_Xf_nD\mu$.*

*Proof.* Observe that $M_Xf_1f_2D\mu=(M_Xf_1D\mu)(M_Xf_2D\mu)$. Extend this relation to finite products $f_1f_2\ldots f_n$. Now, apply the previous Theorem 4.3. □

The following proposition can be verified directly.

**Proposition 4.5.** *Let $f,g$ be m-absolutely m-integrable positive real valued functions on $X$. Let $c>0$ be a scalar. Then $M_XfgD\mu=(M_XfD\mu)(M_XgD\mu)$ and $M_Xf^cD\mu=(M_XfD\mu)^c$.*

**Proposition 4.6.** *Let $f_n:X\to[1,\infty]$ be measurable, for each $n=1,2,\ldots$, Then*

$$M_X\lim_{n\to\infty}\inf f_nD\mu\le\lim_{n\to\infty}\inf M_Xf_nD\mu.$$

*Proof.* Apply the classical Fatou's lemma to the functions $\log f_n$ and use Remark 4.2. □

**Theorem 4.7.** *Suppose $(f_n)_{n=1}^\infty$ is a sequence of positive real valued measurable functions on $X$ such that $f(x)=\lim_{n\to\infty}f_n(x)$ for some $f(x)>0$, for every $x\in X$. Suppose there is an m-integrable function $g:X\to[1,\infty)$ such that $|f_n|_\times\le g(x)$, $\forall n=1,2,\ldots$, $\forall x\in X$. Then $f$ is an m-absolutely integrable function, $\lim_{n\to\infty}M_X|\frac{f_n}{f}|_\times D\mu=1$, and $\lim_{n\to\infty}M_Xf_nD\mu=M_XfD\mu$.*

*Proof.* Let $F_n = \log f_n$ and $F = \log f$. Let $G = \log g$. Then $\lim_{n\to\infty} F_n(x) = F(x)$, $\forall x \in X$, and $G \in L^1(\mu)$. Also, $|F_n(x)| = |\log f_n(x)| = \log|f_n|_\times \le \log g = G$, $\forall x \in X$. By the classical dominated convergence theorem, $\int_X F_n d\mu \to \int_X F d\mu$ and $\lim_{n\to\infty}\int_X |F_n - F| d\mu = 0$, as $n \to \infty$. Then $\exp\int_X \log f_n d\mu \to \exp\int_X \log f d\mu$, as $n \to \infty$, and $\lim_{n\to\infty}\exp\int_X |\log(\frac{f_n}{f})| d\mu = \lim_{n\to\infty}\exp\int_X \log|\frac{f_n}{f}|_\times d\mu = 1$. Thus, $\lim_{n\to\infty} M_X f_n D\mu = M_X f D\mu$, and
$\lim_{n\to\infty} M_X |\frac{f_n}{f}|_\times D\mu = 1$. □

**Proposition 4.8.** *Suppose $f : X \to (0,\infty)$ is an m-absolutely m-integrable function. Then $|M_X f D\mu|_\times \le M_X |f|_\times D\mu$.*

*Proof.* It follows from definitions. □

**Proposition 4.9.** *(a) Let $f : X \to [1,\infty]$ be a measurable function on $X$ such that $M_X f D\mu = 1$. Then $f = 1$ almost everywhere on $X$.*
*(b) Let $f : X \to (0,\infty)$ be m-absolutely m-integrable on $X$. Suppose $M_E f D\mu = 1$, $\forall\, E \in \mathfrak{M}$. Then $f = 1$ almost everywhere on $X$.*
*(c) Let $f : X \to (0,\infty)$ be m-absolutely m-integrable on $X$. Suppose $|M_X f D\mu|_\times = M_X |f|_\times D\mu$. Then there is a constant $\alpha$ such that $\alpha f = |f|_\times$ almost everywhere on $X$.*

*Proof.* Use Proposition 1.3 along with Theorem 1.39 in [24] for $\log f$. □

## 5. Conclusions

Theorem 11.33 in [23] states that a Lebesgue integrable bounded real valued function on $[a,b]$ is Riemann integrable over the interval if and only if the function is continuous almost everywhere in that interval with respect to classical Lebesgue measure. One can state a corresponding result for m-Riemann integrable functions. Such observations may give a hope to transform every known result in additive calculus and abstract measure integration theory through logarithmic exponential transformations. But, it is not true. Theorems 2.10 and 2.12 reveal difficulties in transforming chain rule and a generalized mean value theorem which is used for establishing L'Hospital's rules. It is yet to be tried for replacement of exponential transformation and its inverse logarithmic transformation by means of other transformations like Lorentz transformation and its inverse Lorentz transformation of the theory of Einstein's special relativity. It is yet to be tried to extend results which provide generalizations of research work in differentiation, integration and measure theory like the ones provided in [25]-[30].

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