OVER 2,000 HEALTHY BREAKFAST RECIPES

CONFERENCE PROCEEDINGS OF SCIENCE AND TECHNOLOGY

HTTP://DERGIPARK.GOV.TR/CPOST

VOLUME II
ISSUE III
IECMSA 2019

Conference Proceedings of Science and Technology

ISSN 2651-544X

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019) Baku/Azerbaijan

CONFERENCE PROCEEDINGS OF SCIENCE AND TECHNOLOGY

Conference Proceedings of Science and Technology

ISSN: 2651-544X

Preface

I welcome you to the 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019) on August 27-30, 2019 in Baku, Azerbaijan. It is an honor for me to inform you that this conference is dedicated to the 100th Anniversary of the first university of Azerbaijan-Baku State University which is a leader of educational institutions, has a rich history and today it is known as one of the most famous scientific and educational centers of Azerbaijan Republic.

IECMSA-2019 is supported by Sakarya University, Baku State University, International Balkan University, Firat University, Tekirdag Namik Kemal University, Kocaeli University, Amasya University, Gazi University, and Turkic World Mathematical Society.

The series of IECMSA provides a highly productive forum for reporting the latest developments in the researches and applications of Mathematics. The previous seven conferences held annually since 2012 such that IECMSA-2012, Prishtine, Kosovo, IECMSA-2013, Sarajevo, Bosnia and Herzegovina, IECMSA-2014, Vienna, Austria, IECMSA-2015, Athens, Greece, IECMSA-2016, Belgrade, Serbia, IECMSA-2017, Budapest, Hungary, and IECMSA-2018, Kyiv, Ukraine.

The scientific committee members of IECMSA-2019 and the external reviewers invested significant time in analyzing and assessing multiple papers, consequently, they hold and maintain a high standard of quality for this conference. The scientific program of the conference features invited talks, followed by contributed oral and poster presentations in seven parallel sessions.

The conference program represents the efforts of many people. I would like to express my gratitude to all members of the scientific committee, external reviewers, sponsors and, honorary committee for their continued support to the IECMSA. I also thank the invited speakers for presenting their talks on current researches. Also, the success of IECMSA depends on the effort and talent of researchers in mathematics and its applications that have written and submitted papers on a variety of topics. So, I would like to sincerely thank all participants of IECMSA-2019 for contributing to this great meeting in many different ways. I believe and hope that each of you will get the maximum benefit from the conference.

Prof. Dr. Murat TOSUN
Chairman
On behalf of the Organizing Committee

Contents

Conference Proceedings of Science and Technology, 2(3), 2019, 164–168

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

On a*-*I*-open Sets and a Decomposition of Continuity

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Aynur Keskin Kaymakci [1*]

[1] *Department of Mathematics, Faculty of Science, Selcuk University, Campus, Konya, Turkey, ORCID:0000-0001-5909-8477*
* *Corresponding Author E-mail: akeskin@selcuk.edu.tr*

Abstract: In this paper, we introduce a new set namely a*-*I*-open set in ideal topological spaces. Besides, we give some properties and characterizations of it. We obtain that it is stronger than pre*-I-open set with b-open set and weaker than $\delta\beta_I$-open set. Finally, we give a decomposition of continuity by using a*-*I*-open set as stated the following:" $f:(X,\tau,I)\longrightarrow(Y,\varphi)$ is continuous if and only if it is a*-I-continuous and strongly A$_I$-continuous."

Keywords: a*-*I*-open set, Decomposition of continuity, Ideal.

1 Introduction and preliminaries

Topic of ideals in topological spaces has been studied since beginning of 20th century. It has won reputain and importance in citevai.Throughout this paper, we will denote topological spaces by (X,τ) and (Y,φ). For a subset A of a space (X,τ), the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. It is well known that a subset A of a space (X,τ) is said to be regular open citevel if $A=Int(Cl(A))$. A subset A of a space (X,τ) is said to be δ-$open$ citevel if for each $x\in A$ there exists a regular open set U such that $x\in U\subseteq A$. A is δ-$closed$ citevel if $(X$-$A)$ is δ-$open$. The set $\{x\in X\mid x\in U\subseteq A$ for some regular open set U of $X\}$ is called the δ-interior of A and is denoted by $Int_\delta(A)$ citevel. A point $x\in X$ is called a δ-cluster point of A if $A\cap Int(Cl(V))\neq\varnothing$ for each open set V containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\delta Cl(A)$ citevel. Of course, δ-$open$ sets form a topology τ^δ and then $\tau^\delta\subset\tau$ holds citevel.

An ideal I on X is defined as a nonempty collection of subsets of X satisfying the following two conditions:

(1) If $A\in I$ and $B\subset A$, then $B\in I$;

(2) If $A\in I$ and $B\in I$, then $A\cup B\in I$.

Let (X,τ) be a topological space and I an ideal on X . An ideal topological space is a topological space (X,τ) with an ideal I on X and is denoted by (X,τ,I). For a subset $A\subset X$, $A^*(I,\tau)=\{x\in X\mid U\cap A\notin I$ for each neighbourhood U of $x\}$ is called the local function of A with respect to I and τ (citekur). Throught this paper, we use A^* instead of $A^*(I,\tau)$. Besides, in citejan, authors introduced a new Kuratowski closure operator $Cl^*(.)$ defined by $Cl^*(A)=A\cup A^*(I,\tau)$ and obtained a new topology on X which is called an $*$-$topo\log y$. This topology is denoted by $\tau^*\ (I)$ which is finer than τ.

A point x in an ideal topological space is called δ_I-$cluster\ point$ of A if $Int(Cl^*(U)\cap A\neq\varnothing$ for each neighborhood U of x. The set of all δ_I-$cluster\ points$ of A is called the δ_I-$closure\ of\ A$ and will be denoted by $\delta Cl_I(A)$ citey/"uk. A is said to be δ_I-$closed$ citey/"uk if $A=\delta Cl_I(A)$. Of course, the complement of δ_I-open set is said δ_I-$closed$ citey/"uk. The family of all δ_I-$open$ sets in any ideal topological space (X,τ,I) form a topology $\tau^{\delta I}$ and then $\tau^{\delta I}\subset\tau$ holds citey/"uk.

Definition 1. *some label A subset A of an ideal topological space (X,τ,I) is said to be α-open citenja (resp. semi-open citelev, pre-open citemas1, b-open citeand (or γ open citeel-a), β-open citeabd) if $A\subset Int(Cl(Int(A)))$ (resp. $A\subset Cl(Int(A))$, $A\subset Int(Cl(A))$, $A\subset Int(Cl(A))\cup Cl(Int(A))$, $A\subset Cl(Int(Cl(A)))$).*

Definition 2. *some label A subset A of an ideal topological space (X,τ,I) is said to be pre-I-$open$ citedon (resp. $semi$-I-$open$ citehat1, α-I-$open$ citehat1, b-I-$open$ citeg/"ul, β-I-$open$ citehat 1) if $A\subset Int(Cl^*(A))$ (resp. $A\subset Cl^*(Int(A))$, $A\subset Int(Cl^*(Int(A)))$, $A\subset Cl^*(Int(A))\cup Int(Cl^*(A))$, $A\subset Cl(Int(Cl^*(A)))$).*

Definition 3. *A subset A of an ideal topological space (X,τ,I) is said to be δ-α-I-$open$ citehat 4, pre*-I-open citeeki (resp. semi*-I-open , $\delta\beta$-I-open citehat 4) if $A\subset Int(Cl(\delta Int_I(A)))$ (resp. $A\subset Int(\delta Cl_I(A))$, $A\subset Cl(\delta Int_I(A))$, $A\subset Cl(Int(\delta Cl_I(A)))$).*

Related to above definitions, one can find the following diagram in citehat 4. None of these implications are reversible in generally as shown in the related papers.

$$
\begin{array}{ccccccc}
\text{open} & \rightarrow & \alpha\text{-}I\text{-open} & \rightarrow & semi\text{-}I\text{-open} & & \\
\uparrow & & \downarrow \quad \searrow & & \downarrow \quad \searrow & & \\
\uparrow & & \downarrow pre\text{-}I\text{-open} & \rightarrow & \downarrow \ b\text{-}I\text{-open} & \rightarrow & \beta\text{-}I\text{-open} \\
\uparrow & & \downarrow \quad \downarrow & & \downarrow \quad \downarrow & & \downarrow \\
\uparrow & & \alpha\text{-open} \downarrow & \rightarrow & semi\text{-open} \downarrow & & \downarrow \\
\uparrow & & \uparrow \ \searrow \ \downarrow & & \searrow \quad \downarrow & & \downarrow \\
\uparrow & & \uparrow pre\text{-open} & \rightarrow & b\text{-open} & \rightarrow & \beta\text{-open} \\
\uparrow & & \uparrow \quad \searrow & & & \searrow & \downarrow \\
\uparrow & & \uparrow & pre^{*}\text{-}I\text{-open} & \rightarrow & a^{*}\text{-}I\text{-open} & \downarrow \\
\uparrow & & \uparrow \quad \nearrow & & & \searrow & \downarrow \\
\delta\text{-}I\text{-open} & \rightarrow & \delta\text{-}\alpha\text{-}I\text{-open} & \rightarrow & semi^{*}\text{-}I\text{-open} & \rightarrow & \delta\beta_I\text{-open}
\end{array}
$$

$Diagram\ II$

Lemma 1. *For a subset A of an ideal topological space (X, τ, I) , the following properties are hold:*
(1) If U is an open set, then $U \cap Cl^*(A) \subseteq Cl^*(U \cap A)$ citehat2,
(2) If U is an open set, then $\delta Cl_I(U)) = Cl(U)$ citehat3.

2 a*-*I*-open sets

In this section, to give a decomposition of open set we introduce a new set which name is a*-*I*-open set and obtain some properties and characterizations of it.

Definition 4. *A subset A of an ideal topological space (X, τ, I) is said to be an a*-I-open if $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A))$. The complement of an a*-I-open set said to be an a*-I-closed. It is obvious that A is an a*-I-closed if and only if $Cl(\delta Int_I(A)) \cap Int(Cl(A)) \subset A$.*

Corollary 1. *It is obtained from Definition 4, $\varnothing$ and X are both a*-I-open sets and a*-I-closed sets.*

Proposition 1. *Let (X, τ, I) be an ideal topological space. Then, the following properties are hold:*
(1) If A is pre*-I-open, then it is a*-I-open,
(2) If A is b-open, then it is a*-I-open,
(3) If A is a*-I-open, then it is $\delta\beta_I$-open.

Proof: The proof of (1) is clear from Definitions 1, 3 and 4. The others are obtained by using related set definitions.The following diagram is obtained by using Proposition 3 and several sets defined above. ☐

Remark 1. *The converses of each statements in Proposition 3 are not true in generally as shown in the next examples.*

Example 1. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\varnothing\}$.(1) Set $A = \{a, d\}$. Then, A is an a*-I-open but it is not pre*-I-open (2) Set $A = \{a, b\}$. Then, A is an a*-I-open but it is not b-open.*

Example 2. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\varnothing\}$. For $A = \{b, d\}$ is $\delta\beta_I$-open, but it isn't a*-I-open.*

We have the following diagram.

$$
\begin{array}{ccccccccc}
open & \rightarrow & \alpha\text{-}I\text{-}open & \rightarrow & semi\text{-}I\text{-}open & & \\
\uparrow & & \downarrow \quad \searrow & & \downarrow \quad \searrow & & \\
\uparrow & & \downarrow pre\text{-}I\text{-}open & \rightarrow & \downarrow \; b\text{-}I\text{-}open & \rightarrow & \beta\text{-}I\text{-}open \\
\uparrow & & \downarrow \qquad \downarrow & & \downarrow \quad \downarrow & & \downarrow \\
\uparrow & & \alpha\text{-}open \downarrow & \rightarrow & semi\text{-}open \downarrow & & \downarrow \\
\uparrow & & \uparrow \;\; \searrow \;\; \downarrow & & \searrow \quad \downarrow & & \downarrow \\
\uparrow & & \uparrow pre\text{-}open & \rightarrow & b\text{-}open & \rightarrow & \beta\text{-}open \\
\uparrow & & \uparrow \qquad \searrow & & & & \swarrow \\
\uparrow & & \uparrow \quad pre^{*}\text{-}I\text{-}open & & \rightarrow & \delta\beta_{I}\text{-}open & \\
\uparrow & & \uparrow \qquad \nearrow & & \nearrow & & \\
\delta\text{-}I\text{-}open & \rightarrow & \delta\text{-}\alpha\text{-}I\text{-}open & \rightarrow & semi^{*}\text{-}I\text{-}open & &
\end{array}
$$

Diagram I

Proposition 2. *For an ideal topological space (X, τ, I) and a subset A of X, the following property is hold: "If $I = \wp(X)$, then A is an a^*-I-open if and only if A is an b-open."*

Proof: Since sufficiency is stated in Proposition 3(2), we prove only necessity. Let $I = \wp(X)$. Then, $A^* = \varnothing$ and $Cl^*(A) = A \cup A^* = A$ for every subset A of X. So, we have $\delta Cl_I(A) = Cl(A)$. If A is an a^*-I-$open$ set, then we obtain that $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A)) \subset Int(Cl(A)) \cup Cl(Int(A))$ and hence every a^*-I-$open$ set is a b-$open$. □

Remark 2. *The notions of a^*-I-open set and β-open set are independent each other. Indeed in Example 2, set $A = \{b, d\}$ is β-open, but it isn't a^*-I-open. Besides in Example 1(2), set $A = \{a, b\}$ is an a^*-I-open but it is not β-open.*

Proposition 3. *Let (X, τ, I) be an ideal topological space with an arbitrary index set Δ. If $\{A_\alpha : \alpha \in \Delta\} \subset a^*IO(X, \tau)$, then $\cup\{A_\alpha : \alpha \in \Delta\} \in a^*IO(X, \tau)$.*

Proof: Since $\{A_\alpha : \alpha \in \Delta\} \subset a^*IO(X, \tau)$, $A_\alpha \subset Int(\delta Cl_I(A_\alpha)) \cup Cl(Int(A_\alpha))$ for every $\alpha \in \Delta$. Since δCl_I is a Kuratowski closure operator, we have

$$
\begin{aligned}
(\underset{\alpha\in\triangle}{\cup} A_\alpha) &\subset \left(\underset{\alpha\in\triangle}{\cup} Int(\delta Cl_I(A_\alpha)) \cup Cl(Int(A_\alpha))\right) \\
&= \left(\underset{\alpha\in\triangle}{\cup} Int(\delta Cl_I(A_\alpha))\right) \cup \left(\underset{\alpha\in\triangle}{\cup} Cl(Int(A_\alpha))\right) \\
&\subset Int\left(\underset{\alpha\in\triangle}{\cup} \delta Cl_I(A_\alpha)\right) \cup Cl\left(\underset{\alpha\in\triangle}{\cup} Int(A_\alpha)\right) \\
&\subset Int(\delta Cl_I\left(\underset{\alpha\in\triangle}{\cup} A_\alpha\right) \cup Cl(Int\left(\underset{\alpha\in\triangle}{\cup} A_\alpha\right).
\end{aligned}
$$
□

Proposition 4. *Let (X, τ, I) be an ideal topological space and A, U are subsets of X. If A is an a^*-I-open set and U is δ-I-open set. Then $(A \cap U)$ is an a^*-I-open set.*

Proof: Since A is an a^*-I-$open$ set and U is δ-I-$open$ set, we have $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A))$ and $U \subset \delta Int\,(U)$. By using some properties of closure, interior and δ-I-closure operations, we have

$$
\begin{aligned}
(A \cap U) &\subset ((Int(\delta Cl_I(A))) \cup Cl(Int(A))) \cap \delta Int_I(U) \\
&= (Int(\delta Cl_I(A)) \cap \delta Int_I(U)) \cup (Cl(Int(A)) \cap \delta Int_I(U)) \\
&\subseteq (Int(\delta Cl_I(A)) \cap Int(U)) \cup (Cl(Int(A)) \cap Int(U)) \\
&\subseteq Int\,[\delta Cl_I(A) \cap Int(U)] \cup Cl\,[Int(A) \cap Int(U)] \\
&\subseteq Int\,(\delta Cl_I\,(A \cap Int(U))) \cup Cl(Int(A \cap U)) \\
&\subseteq Int\,(\delta Cl_I\,(A \cap U)) \cup Cl(Int(A \cap U)).
\end{aligned}
$$

This shows that $(A \cap U)$ is an a^*-I-$open$ set. □

Definition 5. *A subset A of an ideal topological space (X, τ, I) is called*
(1) strongly t-I-set citeeki if $Int(\delta Cl_I\,(A) = Int\,(A)$,
(2) strongly A_I-set if $A = U \cap V$, where $U \in \tau$ and V is strongly t-I-set and $Int(\delta Cl_I\,(V) = Cl\,(Int\,(V))$.

Theorem 1. *The following properties hold for a subset A of an ideal topological space (X, τ, I):*
(1) If A is strongly t-I-set and $Int(\delta Cl_I\,(A) = Cl\,(Int\,(A))$, then it is strongly A_I-set,
(2) If A is open set, then it is strongly A_I-set.

Proof:
(1) : Since A is strongly t-I-set with $Int(\delta Cl_I\,(A) = Cl\,(Int\,(A))$ and $X \in \tau$, the proof of 1) is obvious.
(2) : Since X is strongly t-I-set with $Int(\delta Cl_I\,(X) = Cl\,(Int\,(X))$ and $A \in \tau$, the proof of 2) is obtained. ☐

Theorem 2. *For a subset A of (X, τ, I), the following properties are equivalent:*
(1) A is open,
(2) A is an a^*-I-open and strongly A_I-set.

Proof: $(1) \Longrightarrow (2)$: By Diagram II, every open set is a^*-I-$open$. Besides, we have every open set is strongly A$_I$-set according to Theorem 7(2).

$(2) \Longrightarrow (1)$: Let A is an a^*-I-$open$ and strongly A$_I$-set. Then, we have $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A))$ and strongly A$_I$-set if $A = U \cap V$, where $U \in \tau$ and V is strongly t-I-set and $Int(\delta Cl_I\,(V) = Cl\,(Int\,(V))$, respectively. Therefore, we have $A \subset Int(\delta Cl_I(U \cap V)) \cup Cl(Int(U \cap V)) \subseteq [Int(\delta Cl_I(U)) \cap Int(\delta Cl_I(V))] \cup [Cl(Int(U) \cap Cl(Int(V)] = [Int(\delta Cl_I(U)) \cap Cl(Int(V)] \cup [Cl(Int(U) \cap Cl(Int(V)]$. According to Lemma 1(2), since $U \in \tau$, it is obvious that $\delta Cl_I(U) = Cl(U)$ and $Int(\delta Cl_I(U)) = Int(Cl(U))$. So, we have

$A \subset [Int(Cl(U)) \cap Cl(Int(V)] \cup [Cl(Int(U) \cap Cl(Int(V)] = [Int(Cl(U)) \cup Cl(Int(U)] \cap Cl(Int(V)$. Consequently, since $A \subset U$, we obtain $A \subset U \cap \{[Int(Cl(U)) \cup Cl(Int(U)] \cap Cl(Int(V))\} = \{U \cap [Int(Cl(U)) \cup Cl(Int(U)]\} \cap Cl(Int(V)) = [(U \cap Int(Cl(U))) \cup (U \cap Cl(Int(U)))] \cap Cl(Int(V)) = U \cap Int(V) = Int(U \cap V) = Int(A)$. Hence A is an open. ☐

The notions of a^*-I-$open$ set and strongly A$_I$-set are independent each other as shown in the following examples.

Example 3. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $I = \{\varnothing, \{d\}\}$. For $A = \{a\}$, then it is a^*-I-open but it isn't strongly A_I-set.*

Example 4. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\varnothing, \{a\}\}$. For $A = \{b, d\}$, then it is strongly A_I-set but it isn't a^*-I-open.*

3 Decomposition of continuity

In this section, we introduce the notions of a*-I-continuity, strongly A$_I$-continuity and obtain a decomposition of continuity.

Definition 6. *A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be b-continuous citeel-a if $f^{-1}\,(V)$ is a b-open set in (X, τ) for every open set V in (Y, φ).*

Definition 7. *A function $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ is said to be pre*-I-continuous citeeki (resp. δ beta$_I$-continuous citehat4, a*-I-continuous strongly A_I-continuous) if $f^{-1}\,(V)$ is a pre*-I-open (resp. $\delta\beta_I$-open, a*-I-open set, strongly A_I-set).*
(resp. $\delta\beta_I$-open, a*-I-open set, strongly A_I

Proposition 5. *For a function f:$(X, \tau, I) \longrightarrow (Y, \varphi)$, the following properties are hold: (1) If f is pre*-I-continuous, then f is a*-I(2) If f is b-continuous, then f is a*-I-continuous, (3) If f is a*-I-continuous, then f is $\delta\beta_I$*

Proof: The proofs are omitted from Proposition 3 as consequences by using Definitions 6 and 7. ☐

Remark 3. *The converses of each statements in Proposition 9 are not true in generally as shown in the next examples.*

Example 5. *Let (X, τ, I) be an ideal topological space as same as in Example 1 and $Y = \{a, b\}$, $\varphi = \{Y, \varnothing, \{a\}\}$. (1) Let $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ be a function defined as $f(a) = f(d) = a$, $f(b) = f(c) = b$. Then f is a*-I-continuous, but it isn't pre*-I-continuous.*
(2) Let $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ be a function defined as $f(b) = f(d) = a$, $f(a) = f(c) = b$. Then f is a*-I-continuous, but it isn't b-continuous.

Example 6. *Let (X, τ, I) be an ideal topological space as same as in Example 2 and $Y = \{a, b\}$, $\varphi = \{Y, \varnothing, \{a\}\}$. Let $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ be a function defined as $f(a) = f(d) = a$, $f(b) = f(c) = b$. Then f is $\delta\beta_I$-continuous, but it isn't a*-I-continuous.*

It is known that a function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is continuous if $f^{-1}\,(V)$ is an open set in (X, τ) for every open set V in (Y, φ).

Theorem 3. *For a function $f : (X, \tau, I) \longrightarrow (Y, \varphi)$, the following statements are equivalent: (1) f is continuous, (2) f is a*-I-continuous and strongly A_I-continuous.*

Proof: This follows from Theorem 8. ☐

Acknowledgement

This work is supported by Scientific Research Projects Coordination Office(BAP) of Selcuk University with 19701234 number project.

4 References

[1] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, *β-open sets and β-continuous mapping*, Bull. Fac. Sci. Assiut Univ. **A12**(1) (1983), 77-90.
[2] D. Andrijević, *On b-open sets*, Mat. Vesnik, **48** (1996), 59-64.

[3] A. Caksu Güler, G. Aslim, *b-I-open sets and decomposition of continuity via idealization*, Proceedings of Institute of Mathematics and Mechanics, Natural Academy of Sciences of Azerbaijan, **22** (2005), 27-32.
[4] J. Dontchev, *On pre-I-open sets and decompositon of I-continuity*, Banyan Math. J., **2** (1996).
[5] E. Ekici, T. Noiri, *On subsets and decompositions of continuity in ideal topological spaces*, Arab. J. Sci. Eng. Sect. A Sci.,**34** (2009), 165-167.
[6] A.A. El-Atik, A study on some types of mappings on topological spaces, MSc. Thesis, Tanta University, Egypt, 1997.
[7] E. Hatir, T. Noiri, *On decompositions of continuity via idealization*, Acta Math. Hungar., **96**(4) (2002), 341-349.
[8] E. Hatir, A. Keskin, T. Noiri, *A note on strong β-I-open sets and strongly β-I-continuous functions*, Acta Math. Hungar., **108**(1-2) (2005), 87-94.
[9] E. Hatir, *A note on $\delta\alpha$-I-open sets and $semi^*$-I-open sets*, Math. Commun., **16** (2011), 433-445.
[10] E. Hatir, *On decompositions of continuity and complete continuity in ideal topological spaces*, Europan Journal of Pure and Applied Math., **6**(3) (2013), 352-362.
[11] D. Jankovič, T.R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, **97** (1990), 295-310.
[12] K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.
[13] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36-41.
[14] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proceedings of the Mathematical and Physical Society of Egypt, **53** (1982), 47-53.
[15] O. Njastad, *On Some Classes of Nearly Open Sets*, Pasific J. Math., **15** (1965), 961-970.
[16] R. Vaidyanathaswamy, *The localisation theory in set topology*, Proc. Indian Acad. Sci. Math. Sci., **20** (1945), 51-61.
[17] N. V. Velićko, *H-closed topological spaces*, Amer. Math. Soc. Transl., **78** (1968), 103-118.
[18] S. Yüksel, A. Acikgöz, T. Noiri, *On δ-I-continuous functions*, Turkish Journal of Mathematics, **29** (2005), 39-51.

Conference Proceedings of Science and Technology, 2(3), 2019, 169–172

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

The Space bv_k^θ and Matrix Transformations

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

G. Canan Hazar Güleç[1] M. Ali Sarigöl[2]*

[1] Department of Mathematics, Faculty of Science and Arts, Pamukkale University, Denizli, Turkey, ORCID:0000-0002-8825-5555
[2] Department of Mathematics, Faculty of Science and Arts, Pamukkale University, Denizli, Turkey, ORCID:0000-0002-9820-1024
** Corresponding Author E-mail: gchazar@pau.edu.tr*

Abstract: In this study, we introduce the space bv_k^θ, give its some algebraic and topological properties, and also characterize some matrix operators defined on that space. Also we extend some well known results.

Keywords: BK spaces, Matrix transformations, Sequence spaces.

1 Introduction

Let ω be the set of all complex sequences, ℓ_k and c be the sets of k-absolutely convergent series and convergent sequences, respectively. By bv we denote the space of all sequences of bounded variation, i.e.,

$$bv = \{x \in w : \Delta x \in \ell_k\}.$$

Let U and V be subspaces of w and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$, i.e.,

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

provided that the series is convergent for $n \geq 0$. Then, we say that A defines a matrix transformation from U into V, and denote it by $A \in (U, V)$ if the sequence $A(x) = (A_n(x)) \in V$ for every sequence $x \in U$, also the sets $U^\beta = \{\varepsilon = (\varepsilon_v) : \Sigma \varepsilon_v x_v \text{ converges for all } x \in U\}$ and

$$U_A = \{x \in \omega : A(x) \in U\} \tag{1}$$

are called the β dual of U and the domain of a matrix A in U. Further, $U \subset w$ is said to be a BK-space if it is a Banach space with continuous coordinates $p_n : U \to \mathbb{C}$ defined by $p_n(x) = x_n$ for $n \geq 0$. The sequence (e_v) is called a Schauder base (or briefly base) for a normed sequence space U if for each $x \in U$ there exist unique scalar coefficients (x_v) such that

$$\lim_{m\to\infty} \left\| x - \sum_{v=0}^{m} x_v e_v \right\| = 0,$$

and we write

$$x = \sum_{v=0}^{\infty} x_v e_v.$$

An infinite matrix $A = (a_{nv})$ is called a triangle if $a_{nn} \neq 0$ and $a_{nv} = 0$ for all $v > n$ for all n, v [1].
We define the notations Γ_c, Γ_∞ and Γ_s for $v = 1, 2, ...$, as follows:

$$\Gamma_c = \left\{ \varepsilon = (\varepsilon_v) : \lim_m \sum_{v=r}^{m} \varepsilon_v \text{ exists for } r = 1, 2, ... \right\},$$

$$\Gamma_\infty = \left\{ \varepsilon = (\varepsilon_v) : \sup_{m,r} \left| \sum_{v=r}^{m} \varepsilon_v \right| < \infty,\ r = 1, 2, ... \right\},$$

and

$$\Gamma_s = \left\{ \varepsilon = (\varepsilon_v) : \sup_m \sum_{r=1}^{m} \left| \theta_r^{-1/k^*} \sum_{v=r}^{m} \varepsilon_v \right|^{k^*} < \infty \right\},$$

where k^* is the conjugate of k, that is, $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k = 1$.

More recently some new sequence spaces by means of the matrix domain of a particular limitation method or absolute summability methods have been defined and studied by several authors in many research papers (see, for instance [2–8]). In this study, we introduce the space bv_k^{θ}, give its some algebraic and topological properties and characterize some matrix operators defined on that space. Also we extend some well known results.

The following lemmas are needed in proving our theorems.

Lemma 1. Let $1 \leq k < \infty$. Then, $A \in (\ell, \ell_k)$ if and only if

$$\sup_v \sum_{n=0}^{\infty} |a_{nv}|^k < \infty,$$

[9].

Lemma 2.
a-)

$$A \in (\ell, c) \Leftrightarrow (i) \quad \lim_n a_{nv} \text{ exists for each } v \text{, and } \quad (ii) \quad \sup_{n,v} |a_{nv}| < \infty.$$

b-) Let $1 < k < \infty$.Then $A \in (\ell_k, c) \Leftrightarrow (i)$ holds and

$$\sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty$$

[10].

2 The space bv_k^{θ} and matrix operators

In this section we introduce the space bv_k^{θ} as

$$bv_k^{\theta} = \left\{ x = (x_k) \in w : \left(\theta_n^{1/k^*} \triangle x_n \right) \in \ell_k \right\},$$

where (θ_n) is a sequence of nonnegative terms, $1 \leq k < \infty$ and $\triangle x_n = x_n - x_{n-1}$ for all n. Note that it includes some known spaces. For example, it is reduced to bv^k for $\theta_n = 1$ for all n and $bv_1^{\theta} = bv$, which have been studied by Malkowsky et al [11] and Jarrah and Malkowsky [6]. Moreover, recently, Başar et al [3] have defined the sequence space $bv\,(u, p)$ and proved that this space is linearly isomorphic to the space $\ell(p)$ of Maddox [12] as generalized to paranormed space.

It is redefined as $bv_k^{\theta} = (\ell_k)_A$ with the notation (1) , where the matrix A is defined by

$$a_{nv} = \begin{cases} -\theta_n^{1/k^*}, & v = n-1, \\ \theta_n^{1/k^*}, & v = n, \\ 0, & v \neq n, n-1. \end{cases}$$

Further, $\left| N_p^{\theta} \right|_k = \left(bv_k^{\theta} \right)_A$ and $|C_\alpha|_k = \left(bv_k^{\theta} \right)_B$ where A and B are Cesàro and Nörlund means of series Σx_n (see [8],[5, 13]).

Now we begin with topological properties of bv_k^{θ}, which also can be deduced from [3] .

Lemma 3. Let $1 \leq k < \infty$ and (θ_n) be a sequence of nonnegative numbers. Then,
a-) The space bv_k^{θ} is a BK-space and norm isomorphic to the space ℓ_k, $i.e.$, $bv_k^{\theta} \approx \ell_k$.
b-) $\left(bv_k^{\theta} \right)^{\beta} = \Gamma_c \cap \Gamma_s$ for $1 < k < \infty$ and $(bv)^{\beta} = \Gamma_c \cap \Gamma_\infty$ for $k = 1$.
c-) Define the sequence $b^{(j)} = \left(b_n^{(j)} \right)$ such that, for $j, n \geq 0$,

$$b_n^{(j)} = \begin{cases} \theta_j^{-1/k^*}, & n \geq j, \\ 0, & n < j. \end{cases}$$

Then, the sequence $b^{(j)} = \left(b_n^{(j)} \right)$ is the base of bv_k^{θ}.

Proof: a-) Since ℓ_k is a BK-space with respect to its usual norm and A is a triangle matrix, Theorem 4.3.2 of Wilansky [1, p. 61] gives the fact that bv_k^{θ} is a BK-space for $1 \leq k < \infty$. Now, consider $T : bv_k^{\theta} \to \ell_k$ defined by $y = T(x) = \left(\theta_n^{1/k^*} \Delta x_n \right)$ for all $x \in bv_k^{\theta}$. Then, it is clear that T is a linear operator, and surjective since, if $y = (y_n) \in \ell_k$, then $x = (x_n) = \left(\Sigma_{j=0}^n \theta_j^{-1/k^*} y_j \right) \in bv_k^{\theta}$, and also one to one. Further, it preserves the norm, since

$$\|T(x)\|_{\ell_k} = \left(\sum_{n=0}^{\infty} \theta_n^{k-1} |\triangle x_n|^k \right)^{1/k} = \|x\|_{bv_k^{\theta}},$$

which completes the proof.

b-) This part can be proved together with Lemma 2.

c-) Since the sequence $e^{(j)}$ is a base of ℓ_k, where $e^{(j)} = \left(e_n^{(j)}\right)_{n=0}^{\infty}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$, it is clear that the sequence $b^{(j)}$ is the base of bv_k^{θ}. In fact, we first note that $T^{-1}(e^{(j)}) = b^{(j)}$. Now, if $x \in bv_k^{\theta}$, then there exists $y \in \ell_k$ such that $y = T(x)$, and so it follows from (a) that

$$\left\| x - \sum_{j=0}^{m} x_j b^{(j)} \right\|_{bv_k^{\theta}} = \left\| y - \sum_{j=0}^{m} y_j e^{(j)} \right\|_{\ell_k} \to 0 \text{ as } m \to \infty,$$

and it is easy to see that the representation $x = \sum_{j=0}^{\infty} x_j b^{(j)}$ is unique. □

Theorem 1. *Let $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \geq 0$, (θ_n) be a sequence of nonnegative numbers and $1 \leq k < \infty$. Then, $A \in \left(bv, bv_k^{\theta}\right)$ if and only if*

$$\lim_{n\to\infty} \sum_{j=\nu}^{\infty} a_{nj} \text{ exists for each } v, \tag{2}$$

$$\sup_{n,v} \left| \sum_{j=v}^{\infty} a_{nj} \right| < \infty \tag{3}$$

and

$$\sup_{\nu} \sum_{n=0}^{\infty} \left| \theta_n^{1/k^*} \sum_{j=\nu}^{\infty} \left(a_{nj} - a_{n-1,j}\right) \right|^k < \infty. \tag{4}$$

Proof: $A \in \left(bv, bv_k^{\theta}\right)$ iff $\left(a_{nj}\right)_{j=0}^{\infty} \in bv^{\beta}$ and $A\left(x\right) \in bv_k^{\theta}$ for every $x \in bv$, and also, by Lemma 3, $\left(a_{nj}\right)_{j=0}^{\infty} \in bv^{\beta}$ iff (2) and (3) hold. Now, to prove necessity and sufficiency of the condition (4), consider the operators $B : bv \to \ell$ and $B' : bv_k^{\theta} \to \ell_k$ defined by

$$B_n\left(x\right) = \Delta x_n,\ B_n'(x) = \theta_n^{1/k^*} \Delta x_n,$$

respectively. As in Lemma 3, these operators are bijection and the matrices corresponding to these operators are triangles. Further, let $x \in bv$ be given. Then, $B\left(x\right) = y \in \ell$ iff $x = S\left(y\right)$, where S is the inverse of B and it is given by

$$s_{n\nu} = \begin{cases} 1,\ 0 \leq \nu \leq n, \\ 0, \quad\ \ \nu > n. \end{cases}$$

On the other hand, if any matrix $R = (r_{nv}) \in (\ell, c)$, then, the series $R_n(x) = \Sigma r_{nv} x_v$ is convergent uniformly in n, since, by Lemma 2, the remaining term tends to zero uniformly in n, that is,

$$\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \left(\sup_{n,v} |r_{nv}| \right) \sum_{v=m}^{\infty} |x_v| \to 0 \text{ as } m \to \infty,$$

and so

$$\lim_{n} R_n(x) = \sum_{v=0}^{\infty} \lim_{n} r_{nv} x_v. \tag{5}$$

Now, it is easily seen from (2) and (3) that $H = \left(h_{mr}^{(n)}\right) \in (\ell, c)$, which gives us, by (5) , that

$$A_n(x) = \lim_{m} \sum_{r=0}^{m} h_{mr}^{(n)} y_r = \sum_{r=0}^{\infty} \left(\sum_{v=r}^{\infty} a_{nv} \right) y_r,$$

converges for all $n \geq 0$,where, for $r, m = 0, 1, ...,$

$$h_{mr}^{(n)} = \begin{cases} \sum_{v=r}^{m} a_{nv} s_{vr}, 0 \leq r \leq m, \\ \qquad 0, r > m. \end{cases}$$

This shows that the mapping sequence $A(x) = (A_n(x))$ exists. On the other hand, since S is the infinite triangle matrix, it is clear that $A(x) = A\left(S(y)\right) \in bv_k^{\theta}$ for every $x \in bv$ iff $B'\left(A\left(S(y)\right)\right) \in \ell_k$, i.e., $\left(B'oAoS\right)(y) \in \ell_k$, which implies that $D = B'oAoS : \ell \to \ell_k$.

Therefore, it can be written that $A : bv \to bv_k^{\theta}$ iff $D : \ell \to \ell_k$, and also $D = B' o \widehat{A}$, where $\widehat{A} = AoS$. Now, a few calculations reveal that

$$\widehat{a}_{nv} = \sum_{j=v}^{\infty} a_{nj} s_{jv} = \sum_{j=v}^{\infty} a_{nj}$$

and so

$$d_{nv} = \sum_{j=0}^{n} b'_{nj} \widehat{a}_{jv} = \theta_n^{1/k^*} \sum_{j=\nu}^{\infty} \left(a_{nj} - a_{n-1,j}\right)$$

Now, let us apply Lemma 1 with the matrix D. Then, it can be easily obtained from the definition of the matrix D that $D : \ell \to \ell_k$ iff condition (4) holds. This completes the proof.

□

If A is an infinite triangle matrix in Theorem 1, then (2) and (3) hold, and so it reduces to the following result.

Corollary 1. If A is an infinite triangle matrix of complex numbers for all $n, v \geq 0$ and $1 \leq k < \infty$, then, $A \in \left(bv, bv_k^{\theta}\right)$ if and only if

$$\sup_{\nu} \sum_{n=0}^{\infty} \left| \theta_n^{1/k^*} \sum_{j=\nu}^{n} \left(a_{nj} - a_{n-1,j}\right) \right|^k < \infty.$$

Acknowledgement

This study is supported by Pamukkale University Scientific Research Projects Coordinatorship (Grant No. 2019KRM004-029).

3 References

[1] A. Wilansky, *Summability Through Functional Analysis*, North-Holland Mathematical Studies, **85**, Elsevier Science Publisher, 1984.
[2] A. M. Akhmedov, F. Başar, *The fine spectra of the difference operator Δ over the sequence space bv_p*, $(1 \leq p < \infty)$, Acta Math. Sin. (Engl. Ser.), **23**(10) (2007), 1757-1768.
[3] F. Başar, B. Altay, M. Mursaleen, *Some generalizations of the space bvp of p-bounded variation sequences*, Nonlinear Analysis **68**(2) (2008), 273–287.
[4] G.C.H. Güleç, *Compact Matrix Operators on Absolute Cesàro Spaces*, Numer. Funct. Anal. Optim., 2019. DOI: 10.1080/01630563.2019.1633665
[5] G. C. Hazar, M.A. Sarıgöl, *On absolute Nörlund spaces and matrix operators*, Acta Math. Sin. (Engl. Ser.), **34**(5) (2018), 812-826.
[6] A. M. Jarrah, E. Malkowsky, *BK spaces, bases and linear operators*, Rend. Circ. Mat. Palermo II, **52** (1998), 177-191.
[7] E. E. Kara, M. İlkhan, *Some properties of generalized Fibonacci sequence spaces*, Linear and Multilinear Algebra **64** (2016), 2208–2223.
[8] M. A. Sarıgöl, *Spaces of Series Summable by Absolute Cesàro and Matrix Operators*, Comm. Math Appl. **7**(1) (2016), 11-22.
[9] I. J. Maddox, *Elements of functinal analysis,* Cambridge University Press, London, New York, (1970).
[10] M. Stieglitz, H. Tietz, *Matrixtransformationen von Folgenraumen Eine Ergebnisübersicht*, Math Z. **154** (1977), 1-16.
[11] E. Malkowsky, V. Rakočević, S. Živković, *Matrix transformations between the sequence space bv^k and certain BK spaces*, Bull. Cl. Sci. Math. Nat. Sci. Math. **123**(27) (2002), 33–46.
[12] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. Oxford **18**(2) (1967), 345-355.
[13] M. F. Mears, *Absolute Regularity and the Nörlund Mean*, Annals of Math., **38**(3) (1937), 594-601.

Conference Proceedings of Science and Technology, 2(3), 2019, 173–179

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

On The Directional Associated Curves of Timelike Space Curve

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Gül Uğur Kaymanlı[1,] Cumali Ekici[2] Mustafa Dede[3]*

[1] *Department of Mathematics, Çankırı Karatekin University, Çankırı, Turkey, ORCID:0000-0003-4932-894X*
[2] *Department of Mathematics-Computer, Eskişehir Osmangazi University, Eskişehir, Turkey, ORCID:0000-0002-3247-5727*
[3] *Department of Mathematics, Kilis 7 Aralık University, Kilis, Turkey, ORCID:0000-0003-2652-637X*
** Corresponding Author E-mail: gulugurk@karatekin.edu.tr*

Abstract: In this work, the directional associated curves of timelike space curve in Minkowski 3-space by using q-frame are studied. We investigate quasi normal-binormal direction and donor curves of the timelike curve with q-frame. Finally, some new associated curves are constructed and plotted.

Keywords: Associated curves, Minkowski space, q-frame.

1 Introduction

The theory of curves is the one of the most important subject in differential geometry. The curves are represented in parametrized form and then their geometric properties and various quantities associated with them, such as curvature and arc length expressed via derivatives and integrals using the idea of vector calculus. There are special curves which are classical differential geometric objects. These curves are obtained by assuming a special property on the original regular curve. Some of them are Smarandache curves, curves of constant breadth, Bertrand curves, and Mannheim curves, associated curves, etc. Studying curves can be differed according to frame used for curve [1], [2], [3]. There are many studies on these special curves; for example, Choi and Kim in 2012 introduced the notion of the principal (binormal)-direction curve and principal (binormal)-donor curve of a Frenet curve and gave the relationship of curvature and torsion of its mates in both Euclidean and Minkowski spaces [4]-[5]. Also Macit and Duldul in 2014 worked on the new associated curves in $\mathbf{E^3}$ and $\mathbf{E^4}$ [6]. New associated curves by using the Bishop frame are obtained by some researches in [7], [8], [9] and [10]. In this paper, we give another approach to directional associated curves of timelike space curve with q-frame used in [11], [12], [13] and [14].

The aim of this study in this paper is to define n_q, b_q-direction curves and n_q, b_q-donor curves of timelike curve γ via the q-frame in $\mathbb{E}_1^3$ and give the relationship between q-curvatures and curvature and torsion of its mates in Minkowski space.

2 Preliminaries

Let $\alpha(t)$ be a space curve with a non-vanishing second derivative. The Frenet frame is defined as follows,

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{b} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \mathbf{n} = \mathbf{b} \wedge \mathbf{t}. \tag{1}$$

The curvature κ and the torsion τ are given by

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}. \tag{2}$$

The well-known Frenet formulas are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}, \tag{3}$$

where

$$v = \|\alpha'(t)\|. \tag{4}$$

In order to construct the 3D curve offset, Coquillart in [15] introduced the quasi-normal vector of a space curve. The quasi-normal vector is defined for each point of the curve, and lies in the plane perpendicular to the tangent of the curve at this point.

As an alternative to the Frenet frame, a new adapted frame called q-frame in both Euclidean and Minkowski space is defined by Ekici et all in [11] and [13]. Given a space curve $\alpha(t)$ the q-frame consists of three orthonormal vectors, the unit tangent vector $\mathbf{t}$, the quasi-normal vector $\mathbf{n}_q$ and the quasi-binormal vector $\mathbf{b}_q$. The q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{5}$$

where $\mathbf{k}$ is the projection vector, which can be chosen $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$ or $\mathbf{k} = (0, 0, 1)$.

A q-frame along a space curve is shown in Figure 1.

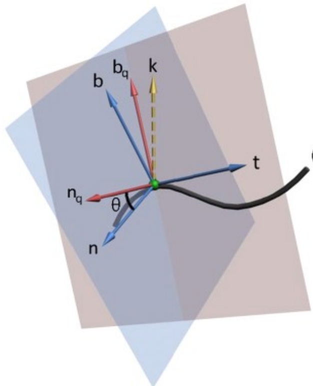

Fig. 1: The q-frame and Frenet frame

Since the derivation formula for the q-frame for the timelike curve in Minkowski space does not depend on projection vector being timelike or spacelike, we work on spacelike projection vector without loss of generality.

In [12], the variation equations of the directional q-frame for the timelike space curve when tangent vector (timelike), projection vector $\mathbf{k} = (0, 1, 0)$ (spacelike), quasi-normal vector (spacelike) and quasi-binormal vector (spacelike) are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{6}$$

where the q-curvatures are

$$k_1 = \langle \mathbf{t}', \mathbf{n}_q \rangle, \quad k_2 = \langle \mathbf{t}', \mathbf{b}_q \rangle, \quad k_3 = \langle \mathbf{n}_q', \mathbf{b}_q \rangle,$$

In the three dimensional Minkowski space $\mathbb{R}^3_1$, the inner product and the cross product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3_1$ are defined as

$$< \mathbf{u}, \mathbf{v} >= u_1 v_1 + u_2 v_2 - u_3 v_3 \tag{7}$$

and

$$\mathbf{u} \wedge \mathbf{v} = (u_3 v_2 - u_2 v_3, u_1 v_3 - u_3 v_1, u_1 v_2 - u_2 v_1) \tag{8}$$

where $e_1 \wedge e_2 = e_3$, $e_2 \wedge e_3 = -e_1$, $e_3 \wedge e_1 = -e_2$, respectively [16].

The norm of the vector $\mathbf{u}$ is given by

$$\|\mathbf{u}\| = \sqrt{|\langle u, u \rangle|}. \tag{9}$$

We say that a Lorentzian vector $\mathbf{u}$ is spacelike, lightlike or timelike if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ and $\mathbf{u} \neq 0$, $\langle \mathbf{u}, \mathbf{u} \rangle < 0$, respectively. In particular, the vector $\mathbf{u} = 0$ is spacelike.

An arbitrary curve $\alpha(s)$ in $\mathbb{R}^3_1$ can locally be spacelike, timelike or null(lightlike), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null.

A null curve α is parameterized by pseudo-arc s if $\langle \alpha''(s), \alpha''(s) \rangle = 1$. On the other hand, a non-null curve α is parameterized by arc-lenght parameter s if $\langle \alpha'(s), \alpha'(s) \rangle = \pm 1$ [17] and [18].

Then Frenet formulas of timelike curve may be written as

$$\frac{d}{dt} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{10}$$

where $v = \left\| \alpha'(t) \right\|$. The Minkowski curvature and torsion of timelike curve $\alpha(t)$ are obtained by

$$\kappa =< \mathbf{t}', \mathbf{n} >, \quad \tau =< \mathbf{n}', \mathbf{b} >,$$

respectively [16] and [19].

Let x and y be future painting (or post painting) timelike vectors in E_1^3 , then there is an unique real number $\theta \geq 0$ such that

$$\langle x, y\rangle = \|x\| \, \|y\| \cosh\theta.$$

This number is called the hyperbolic angle between the vectors x and y [19]. Let x and y be spacelike vectors in E_1^3 that span spacelike vector subspace. Then, there is an unique real number $\theta \geq 0$ such that

$$\langle x, y\rangle = \|x\| \, \|y\| \cos\theta.$$

This number is called the spacelike angle between the vectors x and y.

Let x be a spacelike and y be a timelike vectors in E_1^3 , then there is an unique real number $\theta \geq 0$ such that

$$\langle x, y\rangle = \|x\| \, \|y\| \sinh\theta.$$

This number is called the timelike angle between the vectors x and y [19]. The relation between Frenet ($\mathbf{n}$ is timelike) and q-frame ($\mathbf{t}$ is timelike) is given as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh\theta & \cosh\theta \\ 0 & \cosh\theta & \sinh\theta \end{bmatrix} . \begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix}, \tag{11}$$

where the angle is between $\mathbf{n}$ and $\mathbf{n}_q$.

Also the relation between q-curvatures and curvature and torsion are

$$k_1 = \kappa\sinh\theta, \;\; k_2 = \kappa\cosh\theta, \;\; k_3 = -d\theta + \tau. \tag{12}$$

The relation between Frenet ($\mathbf{b}$ is timelike) and q-frame ($\mathbf{t}$ is timelike) is given as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & -\sinh\theta & -\cosh\theta \end{bmatrix} . \begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix}, \tag{13}$$

where the angle is between $\mathbf{b}$ and $\mathbf{n}_q$.

Also the relation between q-curvatures and curvature and torsion are

$$k_1 = \kappa\cosh\theta, \;\; k_2 = \kappa\sinh\theta, \;\; k_3 = -d\theta - \tau. \tag{14}$$

3 Directional Associated Curves of Timelike Space Curve

In this section, we inverstigate $\mathbf{n}_q$ and $\mathbf{b}_q-$ direction and donor curves of the timelike curve with q-frame in $\mathbb{E}_1^3$. For a Frenet frame $\gamma : I \to \mathbb{E}_1^3$, consider a vector field V with q frame as follows:

$$V(s) = u(s)t(s) + v(s)n_q(s) + w(s)b_q(s), \tag{15}$$

where u, v, and w are functions on I satisfying

$$u^2(s) + v^2(s) - w^2(s) = 1. \tag{16}$$

Then, an integral curve $\overline{\gamma}(s)$, that is $V(\overline{\gamma}(s)) = \overline{\gamma}'(s)$, of V defined on I is a unit speed curve in $\mathbb{E}_1^3$.

Let γ be a timelike curve in $\mathbb{E}_1^3$. An integral curve of n_q is called n_q-direction curve of the timelike curve γ via q-frame.

Remark 1. *A n_q-direction curve is an integral curve of the equation (15) with* $u(s) = w(s) = 0, v(s) = 1.$

Let γ be a timelike curve in $\mathbb{E}_1^3$. An integral curve of b_q is called b_q-direction curve of the timelike curve γ via q-frame.

Remark 2. *A b_q-direction curve is an integral curve of the equation (15) with* $u(s) = v(s) = 0, w(s) = 1.$

3.1 $\mathbf{n}_q-$ *direction and donor curves of the timelike curve with q-frame*

Theorem 1. *Let γ be a timelike space curve in $\mathbb{E}_1^3$ with the q-curvatures k_1, k_2, k_3 and $\overline{\gamma}$ be the n_q-direction curve of γ with the q-curvature $\overline{k}_1, \overline{k}_2, \overline{k}_3$. Then we have*

$$\begin{gathered} \overline{t} = n_q, \;\; \overline{n}_q = -t, \;\; \overline{b}_q = b_q \\ \overline{k}_1 = |k_1| \;\; or \;\; \overline{k}_1 = \sqrt{|2k_3^2 - k_1^2|}, \;\; \overline{k}_2 = k_3, \;\; \overline{k}_3 = k_2. \end{gathered} \tag{17}$$

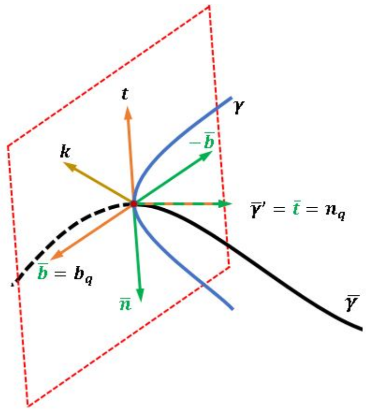

Fig. 2: n_q direction curve

Proof. By definition of n_q-direction curve of γ, we can write

$$\overline{\gamma}' = \bar{t} = n_q. \tag{18}$$

Geometrically, since $\overline{n}_q$ and t lie on the same plane, we can take $\overline{n}_q = -t$. The vectorial product of $\bar{t}$ and $\overline{n}_q$ is as follows:

$$\overline{b}_q = \overline{n}_q \times \bar{t} \tag{19}$$

therefore, $\overline{b}_q = b_q$. Differentiating the expression (18) and then taking its norm, we find

$$\overline{k}_1 = |k_1| \;\text{ or }\; \overline{k}_1 = \sqrt{|2k_3^2 - k_1^2|}. \tag{20}$$

Using definition of $q-$ curvatures and derivation formula of $q-$ frame, one can get $\overline{k}_2 = k_3$, and $\overline{k}_3 = k_2$.

Theorem 2. *Let γ be a timelike space curve in $\mathbb{E}_1^3$ with the q−curvatures k_1, k_2, k_3 and $\overline{\gamma}$ be the n_q-direction curve of the timelike curve γ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then we have*

$$\begin{aligned} &\bar{t} = n_q, \quad \overline{n} = -t, \quad \bar{b} = b_q \\ &\overline{\kappa} = \sqrt{|-k_1^2 + k_3^2|}, \quad \overline{\tau} = -k_2. \end{aligned} \tag{21}$$

Proof. By definition of n_q-direction curve of γ, we can write

$$\overline{\gamma}' = \bar{t} = n_q. \tag{22}$$

Differentiating the expression (22) and then taking its norm, we find

$$\overline{\kappa} = \sqrt{|-k_1^2 + k_3^2|} \tag{23}$$

Differentiation of the expressions (22) gives us

$$\overline{n} = -t. \tag{24}$$

The vectorial product of $\bar{t}$ and $\overline{n}$ is as follows:

$$\bar{b} = \overline{n} \times \bar{t}. \tag{25}$$

Using the expressions (22), (24) in (25) we find that

$$\bar{b} = b_q. \tag{26}$$

Finally, differentiating (26) and using (24) in it, we have

$$\overline{\tau} = -k_2. \tag{27}$$

Corollary 1. *Let γ be a timelike curve in $\mathbb{E}_1^3$ and $\overline{\gamma}$ be the n_q-direction curve of γ. The Frenet frame of $\overline{\gamma}$ is given in terms of the $q-$ frame as follows:*

$$\begin{aligned} &\bar{t}(s) = \overline{n}_q(s), \\ &\overline{n}(s) = -\sinh(\textstyle\int k_2(s)ds)\overline{n}_q(s) + \cosh(\int k_2(s)ds)\bar{b}_q(s), \\ &\bar{b}(s) = \cosh(\textstyle\int k_2(s)ds)\overline{n}_q(s) - \sinh(\int k_2(s)ds)\bar{b}_q(s). \end{aligned} \tag{28}$$

Proof. It is straightforwardly seen by substituting (23) and (27) into (11).

Corollary 2. *If the curve γ is a n_q-donor curve of the curve $\overline{\gamma}$ with the curvatures k_1, k_2, k_3, then the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of the timelike curve γ are given by*

$$\overline{\tau} = \sqrt{|-k_1^2 + k_3^2|}, \quad \overline{\kappa} = \pm k_2 + (\frac{k_3^2}{-k_1^2 + k_3^2})(\frac{k_1}{k_3})' \tag{29}$$

Proof. Taking the squares of (23) and (27)**,** then subtracting them side by side by using (12) gives us the equation (29).

Corollary 3. *Let γ be a timelike curve with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ in $\mathbb{E}_1^3$ and $\overline{\gamma}$ be the n_q-direction curve of γ with the curvatures k_1, k_2, k_3. Then it satisfies*

$$\frac{k_2}{k_1} = \coth\theta, \quad \frac{\overline{\tau}}{\overline{\kappa}} = \pm\frac{k_2}{\sqrt{-k_1^2 + k_3^2}} + \frac{k_3^2}{(-k_1^2 + k_3^2)^{\frac{3}{2}}}(\frac{k_1}{k_3})' \tag{30}$$

Proof. It is straightforwardly seen by substituting the expressions (23), (27) and (29) into (12).

3.2 $\mathbf{b}_q-$ *direction and donor curves of the timelike curve with q-frame*

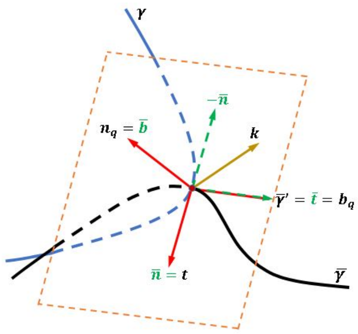

Fig. 3: b_q direction curve

Theorem 3. *Let γ be a timelike space curve in $\mathbb{E}_1^3$ with the q-curvatures k_1, k_2, k_3 and $\overline{\gamma}$ be the n_q-direction curve of γ with the q-curvature $\overline{k}_1, \overline{k}_2, \overline{k}_3$. Then we have*

$$\begin{gathered} \bar{t} = n_q, \ \ \overline{n}_q = -t, \ \ \bar{b}_q = b_q \\ \overline{k}_1 = |k_1| \ \ or \ \ \overline{k}_1 = \sqrt{|2k_3^2 - k_1^2|}, \ \ \overline{k}_2 = k_3, \ \ \overline{k}_3 = k_2. \end{gathered} \tag{31}$$

Proof. By definition of n_q-direction curve of γ, we can write

$$\overline{\gamma}' = \bar{t} = n_q. \tag{32}$$

Geometrically, since $\overline{n}_q$ and t lie on the same plane, we can take $\overline{n}_q = -t$. The vectorial product of $\bar{t}$ and $\overline{n}_q$ is as follows:

$$\bar{b}_q = \overline{n}_q \times \bar{t} \tag{33}$$

therefore, $\bar{b}_q = b_q$. Differentiating the expression (32) and then taking its norm, we find

$$\overline{k}_1 = |k_1| \ \text{ or } \ \overline{k}_1 = \sqrt{|2k_3^2 - k_1^2|}. \tag{34}$$

Using definition of $q-$ curvatures and derivation formula of $q-$ frame, one can get

$$\overline{k}_2 = k_3 \ \text{ and } \ \overline{k}_3 = k_2. \tag{35}$$

Theorem 4. *Let γ be a timelike space curve in $\mathbb{E}_1^3$ with the $q-$curvatures k_1, k_2, k_3 and $\overline{\gamma}$ be the b_q-direction curve of the timelike curve γ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then we have*

$$\begin{gathered} \bar{t} = b_q, \ \ \overline{n} = t, \ \ \bar{b} = n_q \\ \overline{\kappa} = \sqrt{|-k_2^2 + k_3^2|}, \ \ \overline{\tau} = -k_1. \end{gathered} \tag{36}$$

Proof. By definition of b_q-direction curve of γ, we can write

$$\overline{\gamma}' = \overline{t} = b_q. \tag{37}$$

Differentiating the expression (37) and then taking its norm, we find

$$\overline{\kappa} = \sqrt{|-k_2^2 + k_3^2|} \tag{38}$$

Differentiation of the expressions (37) with using of (38) gives us

$$\overline{n} = t. \tag{39}$$

The vectorial product of $\overline{t}$ and $\overline{n}$ is as follows:

$$\overline{b} = \overline{n} \times \overline{t}. \tag{40}$$

Using the expressions (37), (39) in (40) we find that

$$\overline{b} = n_q. \tag{41}$$

Finally, differentiating (41) and using definition of curvature, we have

$$\overline{\tau} = k_1 \tag{42}$$

which proves theorem.

Corollary 4. *Let γ be a timelike curve in $\mathbb{E}_1^3$ and $\overline{\gamma}$ be the b_q-direction curve of γ. The Frenet frame of $\overline{\gamma}$ is given in terms of the $q-$ frame as follows:*

$$\begin{aligned}
&\overline{t}(s) = \overline{b}_q(s),\\
&\overline{n}(s) = \cosh(\textstyle\int k_1(s)ds)\overline{n}_q(s) + \sinh(\int k_1(s)ds)\overline{b}_q(s),\\
&\overline{b}(s) = -\sinh(\textstyle\int k_1(s)ds)\overline{n}_q(s) - \cosh(\int k_1(s)ds)\overline{b}_q(s).
\end{aligned} \tag{43}$$

Proof. It is straightforwardly seen by substituting (38) and (42) into (13).

Corollary 5. *If the curve γ is a n_q-donor curve of the curve $\overline{\gamma}$ with the curvatures k_1, k_2, k_3, then the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of the timelike curve γ are given by*

$$\overline{\tau} = \sqrt{|-k_2^2 + k_3^2|}, \quad \overline{\kappa} = \pm k_1 + (\frac{k_3^2}{-k_2^2 + k_3^2})(\frac{k_2}{k_3})' \tag{44}$$

Proof. Taking the squares of (38) and (42)**,** then subtracting them side by side by using (14) gives us the equation (44).

Corollary 6. *Let γ be a timelike curve with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ in $\mathbb{E}_1^3$ and $\overline{\gamma}$ be the n_q-direction curve of γ with the curvatures k_1, k_2, k_3. Then it satisfies*

$$\frac{k_2}{k_1} = \tanh\theta, \quad \frac{\overline{\tau}}{\overline{\kappa}} = \pm\frac{k_1}{\sqrt{-k_2^2 + k_3^2}} + \frac{k_3^2}{(-k_2^2 + k_3^2)^{\frac{3}{2}}}(\frac{k_2}{k_3})' \tag{45}$$

Proof. It is straightforwardly seen by substituting the expressions (38), (42) and (44) into (14).

4 Examples

In this section, an example of directional associated curves of timelike space curve with q-frame are constructed and plotted.

Example 1. Consider a timelike curve

$$\gamma(t) = (-\frac{5}{9}\cosh(3t), \frac{4}{3}t, -\frac{5}{9}\sinh(3t)).$$

The Frenet frame vectors and curvatures are calculated by

$$\begin{aligned}
&\mathbf{t} = \big(-\tfrac{5}{3}\sinh(3t), \tfrac{4}{3}, -\tfrac{5}{3}\cosh(3t)\big),\\
&\mathbf{n} = \big(-\cosh(3t), 0, -\sinh(3t)\big),\\
&\mathbf{b} = \big(\tfrac{4}{3}\sinh(3t), -\tfrac{5}{3}, \tfrac{4}{3}\cosh(3t)\big),\\
&\kappa = 5, \quad \tau = 4.
\end{aligned}$$

The q-frame vectors and curvatures are obtained by

$$
\begin{aligned}
\mathbf{t} &= \big(-\tfrac{5}{3}\sinh(3t), \tfrac{4}{3}, -\tfrac{5}{3}\cosh(3t) \big), \\
\mathbf{n_q} &= \big(-\cosh(3t), 0, -\sinh(3t) \big), \\
\mathbf{b_q} &= \big(-\tfrac{4}{3}\sinh(3t), \tfrac{5}{3}, -\tfrac{4}{3}\cosh(3t) \big), \\
k_1 &= 5, \quad k_2 = 0, \quad k_3 = -4.
\end{aligned}
$$

n_q and b_q- direction curves of γ shown in Figure 4 are written as

$$
\begin{aligned}
\overline{\gamma} &= \big(-\tfrac{1}{3}\sinh(3t) + c_1, c_2, -\tfrac{1}{3}\cosh(3t) + c_3 \big), \\
\overline{\overline{\gamma}} &= \big(-\tfrac{4}{9}\cosh(3t) + c_4, \tfrac{5}{3}t + c_5, -\tfrac{4}{9}\sinh(3t) + c_6 \big),
\end{aligned}
$$

respectively.

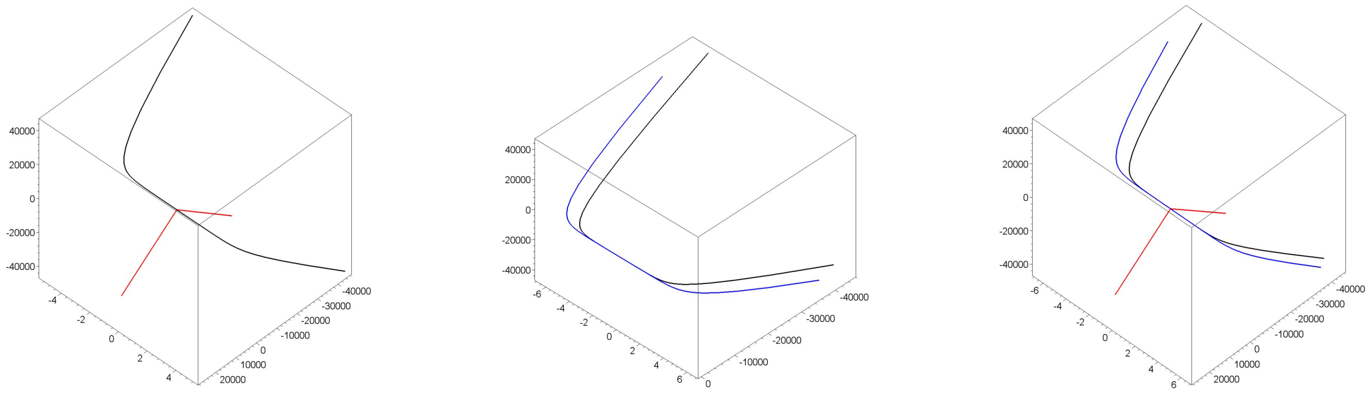

Fig. 4: Timelike curve (black), n_q direction curve (red) and b_q direction curve (blue) for $c_i = 0$.

All the figures in this study were created by using maple programme.

5 References

[1] R. L. Bishop, *There is more than one way to frame a curve*, Am. Math. Mon., **82**(3) (1975), 246-251.
[2] J. Bloomenthal, *Calculation Of Reference Frames Along A Space Curve*, Graphics gems, Academic Press Professional, Inc., San Diago, CA, 1990.
[3] H. Guggenheimer, *Computing frames along a trajectory*, Comput. Aided Geom. Des., **6** (1989), 77-78.
[4] J. H. Choi, Y. H. Kim, *Associated curves of a Frenet curve and their applications*, Appl. Math. Comput., **218**(18) (2012), 9116-9124.
[5] J. H. Choi, Y. H. Kim, A. T. Ali, *Some associated curves of Frenet non-lightlike curves in* $\mathbb{E}_1^3$. J Math Anal Appl., **394** (2012), 712-723.
[6] N. Macit, M. Düldül, *Some New Associated curves of a Frenet Curve in* $\mathbf{E^3}$ *and* $\mathbf{E^4}$, Turkish J. Math., **38** (2014), 1023-1037.
[7] T. Körpınar, M. T. Sarıaydın, E. Turhan, *Associated Curves According to Bishop Frame in Euclidean 3-space*, AMO. **15** (2015), 713-717.
[8] Y. Ünlütürk, S. Yılmaz, M. Çimdiker, S. Şimşek, *Associated curves of non-lightlike curves due to the Bishop frame of type-1 in Minkowski 3-space*, Adv. Model. Optim., **20**(1) (2018), 313-327.
[9] Y. Ünlütürk, S. Yılmaz, *Associated Curves of the Spacelike Curve via the Bishop Frame of type-2 in* $\mathbf{E_1^3}$, Journal of Mahani Mathematical Research Center, **8**(1-2) (2019), 1-12.
[10] S. Yılmaz, Characterizations of Some Associated and Special Curves to Type-2 Bishop Frame in $\mathbf{E^3}$, Kirklareli University Journal of Engineering and Science, **1** (2015), 66-77.
[11] M. Dede, C. Ekici, A. Görgülü, *Directional q-frame along a space curve*, IJARCSSE, **5** (2015) 775-780.
[12] M. Dede, G. Tarım, C. Ekici, *Timelike Directional Bertrand Curves in Minkowski Space*, 15th International Geometry Symposium, Amasya, Turkey 2017.
[13] C. Ekici, M. Dede, H. Tozak, *Timelike directional tubular surfaces*, Int. J. Mathematical Anal., 8(5) (2017), 1-11.
[14] G. U. Kaymanlı, C. Ekici, M. Dede, *Directional canal surfaces in E3*, Current Academic Studies in Natural Sciences and Mathematics Sciences, (2018) 63-80.
[15] S. Coquillart, *Computing offsets of B-spline curves*, Computer-Aided Design, **19**(6) (1987) 305-09.
[16] K. Akutagawa, S. Nishikawa, *The Gauss map and spacelike surfaces with prescribed mean curvature in Minkowski* 3*-space*, Tohoku Math. J. **42**(2) (1990), 67-82.
[17] W. B. Bonnor, *Null curves in a Minkowski space-time*, Tensor, N. S., 20(1969), 229-242.
[18] R. Lopez, *Differential geometry of curves and surfaces in Lorentz-Minkowski space*, Int Elect Journ Geom, **3**(2) (2010), 67-101.
[19] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

Conference Proceedings of Science and Technology, 2(3), 2019, 180–184

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

De-Moivre and Euler Formulae for Dual-Hyperbolic Numbers

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Mehmet Ali Güngör[1,*] *Elma Kahramani*[2]

[1] *Department of Mathematics, Faculty of Science and Arts, Sakarya University, Sakarya, Turkey, ORCID:0000-0003-1863-3183*
[2] *Department of Mathematics, Faculty of Science and Arts, Sakarya University, Sakarya, Turkey, ORCID:0000-0002-4017-0931*
** Corresponding Author E-mail: agungor@sakarya.edu.tr*

Abstract: In this study, we generalize the well-known formulae of de-Moivre and Euler of hyperbolic numbers to dual-hyperbolic numbers. Furthermore, we investigate the roots and powers of a dual-hyperbolic number by using these formulae. Consequently, we give some examples to illustrate the main results in this paper.

Keywords: Dual number, Hyperbolic number.

1 Introduction

The number systems of two- dimensional numbers have taken place in literature with a multi-perspective approach. The hyperbolic numbers were first introduced by J. Cockle [1] and elaborated by I.M. Yaglom [2]. At the end of the 20th century, O. Bodnar, A. Stakhov and I.S. Tkachenko revealed a hyperbolic function class with gold ratio [3]. In recent years, there have been a great number of studies referring to hyperbolic numbers [4]-[9]. One of the most important recent studies has been given by A. Harkin and J. Harkin and generalized trigonometry including complex, hyperbolic and dual numbers were studied [10]. Any hyperbolic number (or split complex number, perplex number, double number) $z = x + j\,y$ is a pair of real numbers (x, y), which consists of the real unit $+1$ and hyperbolic (unipotent) imaginary unit j satisfying $j^2 = 1, j \neq \pm 1$. Therefore, hyperbolic numbers are elements of two-dimensional real algebra

$$H = \left\{ z = x + jy \,|x, y \in R \text{ and } j^2 = 1\,(j \neq \pm 1) \right\}$$

which is generated by 1 and j. The module of a hyperbolic number z is defined by

$$|z| = \left\{ \begin{array}{lll} \mp\sqrt{x^2 - y^2} & ; & |x| \geq |y| \\ \mp\sqrt{y^2 - x^2} & ; & |x| \leq |y| \end{array} \right.$$

and its argument is $\varphi = \operatorname{arctanh}\left(\frac{y}{x}\right)$ and represented by $\arg(z)$. Any hyperbolic number z can be given by one of the following forms;

$$\begin{array}{l} \text{a}-)\ z = r\,(\cosh\varphi + j\sinh\varphi) \\ \text{b}-)\ z = r\,(\sinh\varphi + j\cosh\varphi)\,. \end{array}$$

The hyperbolic number given in (a) and (b) is called the first and second type hyperbolic number, respectively, see figure 1.

On the other hand, the developments in the number theory present us new number systems including the dual numbers which are expressed by the real and dual parts similar to hyperbolic numbers. This idea was first introduced by W. K. Clifford to solve some algebraic problems [11]. Afterwards, E. Study presented different theorems with his studies on kinematics and line geometry [12].

A dual number is a pair of real numbers which consists of the real unit $+1$ and dual unit ε satisfying $\varepsilon^2 = 0$ for $\varepsilon \neq 0$. Therefore, the dual numbers are elements of two-dimensional real algebra

$$D = \left\{ z = x + \varepsilon y \mid x, y \in R, \varepsilon^2 = 0, \varepsilon \neq 0 \right\}$$

which is generated by $+1$ and ε.

Similar to the hyperbolic numbers, the module of a dual number z is defined by $|z| = |x + \varepsilon y| = |x| = r$ and its argument is $\theta = \frac{y}{x}$ and represented by $\arg(z)$. The set of all points which satisfy the equation $|z| = |x| = r > 0$ and which are on the dual plane are the lines $x = \pm r$ [2]. This circle is called the Galilean circle on a dual plane. Let S be a circle centered with O and M be a point on S. If d is the line OM, and α is the angle δ_{Od}, a Galilean circle can be seen in the following figure 2.

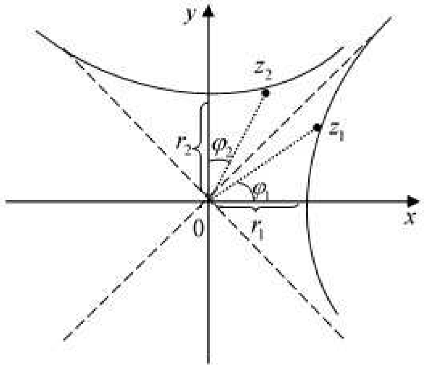

Fig. 1: Representation of hyperbolic numbers at a coordinate plane

So, one can easily see that

$$\text{cosg}\alpha = \frac{|OP|}{|OM|} = 1 \quad , \quad \text{sing}\alpha = \frac{|MP|}{|OM|} = \frac{\delta_{Od}}{1} = \alpha.$$

Moreover, the exponential representation of a dual number $z = x + \varepsilon y$ is in the form of $z = xe^{\varepsilon\alpha}$ where $\frac{y}{x}$ is dual angle and it is shown as $\arg(z) = \frac{y}{x} = \alpha$ [3]. In addition, from the definitions of Galilean cosine and sine, we realize

$$\text{cosg}(\alpha) = 1 \text{ and } \text{sing}(\alpha) = \frac{y}{x} = \alpha.$$

By considering the exponential rules, we write

$$\begin{aligned} \text{cosg}(x+y) &= \text{cosg}(x)\,\text{cosg}(y) - \varepsilon^2 \text{sing}(x)\,\text{sing}(y), \\ \text{sing}(x+y) &= \text{sing}(x)\,\text{cosg}(y) + \text{cosg}(x)\,\text{sing}(y), \\ \text{cosg}^2(x) + \varepsilon^2 \text{sing}^2(x) &= 1 \end{aligned}$$

[10].

E. Cho proved that de-Moivre formula for the hyperbolic numbers is admissible for quaternions [13]. Also, Yaylı and Kabadayı gave the de-Moivre formula for dual quaternions [14]. This formula was also investigated for the case of hyperbolic quaternions in [15]. In this study, we first introduce dual-hyperbolic numbers and algebraic expressions on dual hyperbolic numbers. We also generalize de-Moivre and Euler formulae given for hyperbolic and dual numbers to dual-hyperbolic numbers. Then we have found the roots and forces of the dual-hyperbolic numbers. Finally, the obtained results are supported by examples.

2 Dual-Hyperbolic numbers

A dual-hyperbolic number ω can be written in the form of hyperbolic pair (z_1, z_2) such that $+1$ is the real unit and ε is the dual unit. Thus, we denote dual-hyperbolic numbers set by

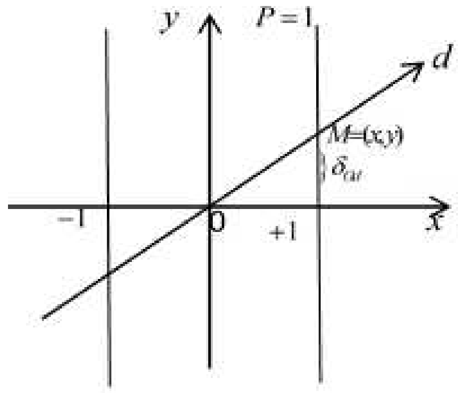

Fig. 2: Galilean unit circle

$$DH = \left\{ \omega = z_1 + \varepsilon z_2 \mid z_1, z_2 \in H \text{ and } \varepsilon^2 = 0,\ \varepsilon \neq 0 \right\}.$$

If we consider hyperbolic numbers $z_1 = x_1 + jx_2$ and $z_2 = x_3 + jx_4$, we represent a dual-hyperbolic number

$$\omega = x_1 + x_2 j + x_3 \varepsilon + x_4 \varepsilon j.$$

Here j, ε and εj are unit vectors in three-dimensional vectors space such that j is a hyperbolic unit, ε is a dual unit, and εj is a dual-hyperbolic unit [16]. So, the multiplication table of dual-hyperbolic numbers' base elements is given below.

$\times$	1	j	ε	$j\varepsilon$
1	1	j	ε	$j\varepsilon$
j	j	1	$j\varepsilon$	ε
ε	ε	$j\varepsilon$	0	0
$j\varepsilon$	$j\varepsilon$	ε	0	0

Table 1 Multiplication Table of Dual-Hyperbolic Numbers

We define addition and multiplication on dual-hyperbolic numbers as follows

$$\begin{array}{l} \omega_1 + \omega_2 = (z_1 \pm \varepsilon z_2) + (z_3 \pm \varepsilon z_4) = (z_1 \pm z_3) + \varepsilon (z_2 \pm z_4), \\ \omega_1 \times \omega_2 = (z_1 + \varepsilon z_2) \times (z_3 + \varepsilon z_4) = z_1 z_3 + \varepsilon (z_1 z_4 + z_2 z_3) \end{array}$$

where ω_1 and ω_2 are dual-hyperbolic numbers and $z_1, z_2, z_3, z_4 \in H$. On the other hand, the division of two dual-hyperbolic numbers is

$$\frac{\omega_1}{\omega_2} = \frac{z_1 + \varepsilon z_2}{z_3 + \varepsilon z_4} = \frac{z_1}{z_3} + \varepsilon \frac{z_2 z_3 - z_1 z_4}{{z_3}^2},$$

where $\mathrm{Re}\,(\omega_2) \neq 0$.

Thus, dual-hyperbolic numbers yield a commutative ring whose characteristic is 0. If we consider both algebraic and geometric properties of dual-hyperbolic numbers, we define five possible conjugations of dual-hyperbolic numbers. These are

$$\begin{array}{ll} \omega^{\dagger_1} = \bar{z}_1 + \varepsilon \bar{z}_2, & (\text{hyperbolic conjugation}), \\ \omega^{\dagger_2} = z_1 - \varepsilon z_2, & (\text{dual conjugation}), \\ \omega^{\dagger_3} = \bar{z}_1 - \varepsilon \bar{z}_2, & (\text{coupled conjugation}), \\ \omega^{\dagger_4} = \overline{z_1}\left(1 - \varepsilon \frac{z_2}{z_1}\right) & (\omega \in DH - A), \quad (\text{dual} - \text{hyperbolic conjugation}), \\ \omega^{\dagger_5} = z_2 - \varepsilon z_1, & (\text{anti} - \text{dual conjugation}), \end{array}$$

where "-" denotes the standard hyperbolic conjugation and the zero divisors of DH is defined by the set A [17].

In regards to these definitions, we give the following proposition for modules of dual-hyperbolic numbers.

Proposition 1. *Let $\omega = z_1 + \varepsilon z_2$ be a dual-hyperbolic number. Then we write*

$$\begin{array}{l} |\omega|^2_{\dagger_1} = \omega \times \omega^{\dagger_1} = |z_1|^2 + 2\varepsilon \mathrm{Re}\,(z_1 \bar{z}_2) \in D \\ |\omega|^2_{\dagger_2} = \omega \times \omega^{\dagger_2} = z_1^2 \in H \\ |\omega|^2_{\dagger_3} = \omega \times \omega^{\dagger_3} = |z_1|^2 - 2j\varepsilon \mathrm{Im}\,(z_1 \bar{z}_2) \in DH \\ |\omega|^2_{\dagger_4} = \omega \times \omega^{\dagger_4} = |z_1|^2 \in R \ (\omega \in DH - A) \\ |\omega|^2_{\dagger_5} = \omega \times \omega^{\dagger_5} = z_1 z_2 + \varepsilon \left(z_2^2 - z_1^2\right) \in DH \end{array}$$

[17].

3 De-Moivre and Euler formulae for Dual-Hyperbolic number

The exponential representation of a dual-hyperbolic number is $\omega = z_1\, e^{\frac{z_2}{z_1}\varepsilon}$, where $\omega = z_1 + \varepsilon z_2 \in DH$ is a dual-hyperbolic number and $(z_1 \neq 0)$. The dual-hyperbolic angle $\frac{z_2}{z_1}$ is called the argument of dual-hyperbolic number and it is denoted by $\arg \omega = \frac{z_2}{z_1} = \varphi$ [17].

Theorem 1. *Let $\omega = z_1 + \varepsilon z_2 \in DH - A$ be a dual-hyperbolic number and φ be the principal argument of ω. Every dual-hyperbolic number can be written in the form of*

$$\begin{aligned} w &= z_1 e^{\varepsilon \varphi} \\ &= z_1 \left(\mathrm{cosg}(\varphi) + \varepsilon \mathrm{sing}(\varphi)\right) = \left\{ \begin{array}{l} r\left(\cosh\varphi + j\sinh\varphi\right)\left(\mathrm{cosg}(\varphi) + \varepsilon \mathrm{sing}(\varphi)\right),\ |x_1| > |y_1| \\ r\left(\sinh\varphi + j\cosh\varphi\right)\left(\mathrm{cosg}(\varphi) + \varepsilon \mathrm{sing}(\varphi)\right),\ |y_1| > |x_1| \end{array} \right. \end{aligned}$$

such that $\mathrm{cosg}(\varphi) = 1$ *and* $\mathrm{sing}(\varphi) = \varphi$.

Proof: The exponential representation of a dual-hyperbolic number $\omega = z_1 + \varepsilon z_2 \in DH - A$ is $\omega = z_1\, e^{\frac{z_2}{z_1}\varepsilon}$, where dual-hyperbolic number $\frac{z_2}{z_1}$ is the principal argument φ. Thus, if we write ω in the form of

$$\omega = z_1 e^{\varepsilon\varphi} = z_1 \left(1 + \varepsilon\varphi + \frac{(\varepsilon\varphi)^2}{2!} + \frac{(\varepsilon\varphi)^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots\right)$$

from properties of the dual unit, we see that

$$\omega = z_1 e^{\varepsilon\varphi} = z_1 \left(1 + \varepsilon\varphi\right) = z_1 \left(\mathrm{cosg}(\varphi) + \varepsilon \mathrm{sing}(\varphi)\right).$$

Eventually, by considering each case of $|x_1| > |y_1|$ or $|y_1| > |x_1|$ if we substitute the hyperbolic number $z_1 = x_1 + j\, y_1 \in H$ into the last equation we get

$$\omega = \begin{cases} r\left(\cosh\varphi + j\sinh\varphi\right)\left(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right), & |x_1| > |y_1|, \\ r\left(\sinh\varphi + j\cosh\varphi\right)\left(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right), & |y_1| > |x_1|. \end{cases}$$

□

Theorem 2. *Let $\omega = z_1 + \varepsilon z_2 \in DH - A$ be a dual-hyperbolic number and $\arg\omega = \frac{z_2}{z_1} = \varphi$. Then $\frac{1}{e^{\varepsilon\varphi}} = e^{\varepsilon(-\varphi)}$.*

Proof: If we use the Euler formula for $\frac{1}{e^{\varepsilon\varphi}}$, we have

$$\begin{aligned} \tfrac{1}{e^{\varepsilon\varphi}} &= \frac{1}{\left(1+\varepsilon\varphi+\frac{(\varepsilon\varphi)^2}{2!}+\frac{(\varepsilon\varphi)^3}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots\right)} \\ &= \tfrac{1}{\cos g(\varphi)+\varepsilon\sin g(\varphi)}. \end{aligned}$$

If we multiply both the numerator and the denominator of the last fraction by $\mathrm{cosg}(\varphi) - \varepsilon\mathrm{sing}(\varphi)$, we get

$$\begin{aligned} \tfrac{1}{e^{\varepsilon\varphi}} &= \tfrac{1}{\mathrm{cosg}(\varphi)+\varepsilon\mathrm{sing}(\varphi)}\tfrac{(\mathrm{cosg}(\varphi)-\varepsilon\mathrm{sing}(\varphi))}{(\mathrm{cosg}(\varphi)-\varepsilon\mathrm{sing}(\varphi))} \\ &= \tfrac{\mathrm{cosg}(\varphi)-\varepsilon\mathrm{sing}(\varphi)}{\mathrm{cosg}^2(\varphi)}. \end{aligned}$$

If we consider equality $\mathrm{cosg}^2(\varphi) = 1$, we have

$$\frac{1}{e^{\varepsilon\varphi}} = \mathrm{cosg}(\varphi) - \varepsilon\mathrm{sing}(\varphi).$$

This gives us the relation

$$\frac{1}{e^{\varepsilon\varphi}} = \mathrm{cosg}(\varphi) - \varepsilon\mathrm{sing}(\varphi) = \mathrm{cosg}(-\varphi) + \varepsilon\mathrm{sing}(-\varphi).$$

As a consequence, we get $\frac{1}{e^{\varepsilon\varphi}} = e^{\varepsilon(-\varphi)}$. □

Theorem 3. *Let $\omega = z_1 + \varepsilon z_2 \in DH - A$ be a dual-hyperbolic number and $\omega = z_1 e^{\varepsilon\varphi} = z_1\left(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right)$ be its polar representation. Then, the equation*

$$\omega^n = \left(z_1 e^{\varepsilon\varphi}\right)^n = \left(z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right)^n = {z_1}^n \left(\mathrm{cosg}(n\varphi) + \varepsilon\mathrm{sing}(n\varphi)\right)$$

yields for all non-negative integers.

Proof: First, let's prove that de-Moivre formula is correct for $n \in N$. For this, under consideration the Galilean trigonometric identities, for $n = 2$ the dual-hyperbolic number $\omega = z_1 e^{\varepsilon\varphi} \in DH - A$ becomes

$$\begin{aligned} \left(z_1 e^{\varepsilon\varphi}\right)^2 &= z_1\left(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right) z_1\left(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right) \\ &= {z_1}^2\left(\mathrm{cosg}^2(\varphi) + \varepsilon\left(\mathrm{cosg}(\varphi)\mathrm{sing}(\varphi) + \mathrm{sing}(\varphi)\mathrm{cosg}(\varphi)\right)\right) \\ &= {z_1}^2\left(\mathrm{cosg}(2\varphi) + \varepsilon\mathrm{sing}(2\varphi)\right). \end{aligned}$$

Suppose that the equality is true for $n = k$, that is,

$$\left(z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right)^k = {z_1}^k\left(\mathrm{cosg}(k\varphi) + \varepsilon\mathrm{sing}(k\varphi)\right).$$

Then for the case $n = k + 1$, we find

$$\begin{aligned} \left(z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right)^{k+1} &= z_1\left(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right)^k\left(z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right) \\ &= {z_1}^k\left(\mathrm{cosg}(k\varphi) + \varepsilon\mathrm{sing}(k\varphi)\right) z_1\left(\mathrm{cosg}(k\varphi) + \varepsilon\mathrm{sing}(k\varphi)\right) \\ &= {z_1}^k\left(\mathrm{cosg}(k\varphi)\mathrm{cosg}(\varphi) + \varepsilon\left(\mathrm{cosg}(k\varphi)\mathrm{sing}(\varphi) + \mathrm{sing}(k\varphi)\mathrm{cosg}(\varphi)\right)\right) \\ &= {z_1}^{k+1}\left(\mathrm{cosg}((k+1)\varphi) + \varepsilon\mathrm{sing}((k+1)\varphi)\right). \end{aligned}$$

Here $z_1^k = r^k (\cosh(k\varphi) + j\sinh(k\varphi))$ for $|x_1| > |y_1|$ and $r = |z_1| = \mp\sqrt{{x_1}^2 - {y_1}^2}$. Moreover, $z_1^k = r^k (\sinh(k\varphi) + j\cosh(k\varphi))$ for $|y_1| > |x_1|$ and $r = |z_1| = \mp\sqrt{{y_1}^2 - {x_1}^2}$. On the other hand, for $\omega = z_1 e^{\varepsilon\varphi} \in DH - A$ and $n \in N$ we can write

$$\begin{aligned} w^{-n} &= {z_1}^{-n} (\text{cosg}(n\varphi) - \varepsilon\text{sing}(n\varphi)) \\ &= {z_1}^{-n} (\text{cosg}(-n\varphi) + \varepsilon\text{sing}(-n\varphi)) \,. \end{aligned}$$

Thus, for all $n \in Z$ we obtain

$$\omega^n = \left(z_1 e^{\varepsilon\varphi}\right)^n = (z_1(\text{cosg}(\varphi) + \varepsilon\text{sing}(\varphi))^n = {z_1}^n (\text{cosg}(n\varphi) + \varepsilon\text{sing}(n\varphi)) \,.$$

□

Theorem 4. *The n-th degree root of ω is*

$$\sqrt[n]{\omega} = \sqrt[n]{z} \left(\text{cosg}\left(\frac{\varphi}{n}\right) + \varepsilon\text{sing}\left(\frac{\varphi}{n}\right)\right)$$

where $\omega = z_1 + \varepsilon z_2 \in DH - A$ is a dual-hyperbolic number.

Proof: Polar representation of $\omega = z_1 + \varepsilon z_2 \in DH - A$ is $\omega = z_1 (\text{cosg}(\varphi) + \varepsilon\text{sing}(\varphi))$. From Theorem 3, we know that

$$\omega^n = \left(z_1 e^{\varepsilon\varphi}\right)^n = (z_1(\text{cosg}(\varphi) + \varepsilon\text{sing}(\varphi))^n = {z_1}^n (\text{cosg}(n\varphi) + \varepsilon\text{sing}(n\varphi)) \,.$$

So, we get

$$\begin{aligned} \sqrt[n]{\omega} = \omega^{\frac{1}{n}} &= z_1^{\frac{1}{n}} \left(\text{cosg}\left(\tfrac{1}{n}\varphi\right) + \varepsilon\text{sing}\left(\tfrac{1}{n}\varphi\right)\right) \\ &= \sqrt[n]{z_1} \left(\text{cosg}\left(\tfrac{\varphi}{n}\right) + \varepsilon\text{sing}\left(\tfrac{\varphi}{n}\right)\right) . \end{aligned}$$

This completes the proof. □

4 References

[1] J. Cockle *On a new imaginary in algebra*, London-Dublin-Edinburgh Philosophical Magazine **3**(34) (1849), 37-47.
[2] I. M. Yaglom, *A simple non-Euclidean geometry and its physical basis*, Springer-Verlag New York, 1979.
[3] S. Yüce, Z. Ercan, *On properties of the dual quaternions*, European Journal of Pure and Applied Mathematics **4**(2) (2011), 142-146.
[4] G. Sobczyk *The hyperbolic number plane*, The College Math. J., **26**(4) (1995), 268-280.
[5] F. Catoni, R. Cannata, V. Catoni, P. Zampetti, *Hyperbolic trigonometry in two-dimensional space-time geometry*, Nuovo Cimento della Societa Italiana di Fisica B **118** (2003), 475-491.
[6] S. Yüce, N. Kuruoğlu , *One-parameter plane hyperbolic motions*, Adv. Appl. Clifford Alg. **18**(2) (2018), 279-285.
[7] M. Akar, S. Yüce, S. Sahin, *On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers*, Journal of Computer Science Computational Mathematics, **8**(1) (2018), 279-285.
[8] S. Ersoy, M. Akyiğit, *One-parameter homothetic motion in the hyperbolic plane and Euler-Savary formula*, Adv. Appl. Clifford Alg. **21**(2) (2011), 297-317.
[9] D. P. Mandic, V. S. L. Goh, *Hyperbolic valued nonlinear adaptive filters: noncircularity, widely linear and neural models*, John Wiley-Sons., 2009.
[10] G. Helzer, *Special relativity with acceleration*, Amer. Math. Monthy **107**(3) (2000), 219-237.
[11] W. K. Clifford, *Preliminary sketch of bi-quaternions*, Proc. London Math. Soc. **4** (1873), 381-395.
[12] E. Study, *Geometrie der dynamen*, Leipzig, Germany, 1903.
[13] E. Cho, *De-MoivreâĂŹs formula for quaternions*, Appl. Math. Lett. **11**(6) (1998), 33-35.
[14] H. Kabadayı, Y. Yaylı, *De-Moivre's formula for dual quaternions*, Kuwait J. Sci. Technol. **38**(1) (2011), 15-23.
[15] I. A. Kösal, *A note on hyperbolic quaternions*, Universal Journal Of Mathematics and Applications **1**(3) (2018), 155-159.
[16] V. Majernik, *Multicomponent number systems*, Acta Physics Polonica A **3**(90) (1996), 491-498.
[17] F. Messelmi, *Dual-hyperbolic numbers and their holomorphic functions*, (2015), https://hal.archives-ouvertes.fr/hal-01114178.

Conference Proceedings of Science and Technology, 2(3), 2019, 185–188

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

Compact Operators in the Class $\left(bv_k^\theta, bv\right)$

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

M. Ali Sarıgöl[1,*]

[1] *Department of Mathematics Pamukkale University TR-20007 Denizli TURKEY ORCID:0000-0002-4107-4669*

** Corresponding Author E-mail: msarigol@pau.edu.tr*

Abstract: The space bv of bounded variation sequence plays an important role in the summability. More recently this space has been generalized to the space bv_k^θ and the class $\left(bv_k^\theta, bv\right)$ of infinite matrices has been characterized by Hazar and Sarıgöl $[2]$. In the present paper, for $1 < k < \infty$, we give necessary and sufficient conditions for a matrix in the same class to be compact, where θ is a sequence of positive numbers.

Keywords: Matrix transformations, Sequence spaces, bv_k^θ spaces.

1 Introduction

Let ω be the set of all complex sequences, ℓ_k and c be the set of k-absolutely convergent series and convergent sequences. In $[2]$, the space bv_k^θ has been defined by

$$bv_k^\theta = \left\{ x = (x_k) \in w : \sum_{n=0}^{\infty} \theta_n^{k-1} \left|\triangle x_n\right|^k < \infty,\ x_{-1} = 0 \right\},$$

which is a BK space for $1 \leq k < \infty$, where (θ_n) is a sequence of nonnegative terms and $\triangle x_n = x_n - x_{n-1}$ for all n.

Also, in the special case $\theta_n = 1$ for all n, it is reduced to bv^k, studied by Malkowsky, Rakočević and Živković $[1]$, and $bv_1^\theta = bv$.

Let U and V be subspaces of w and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$, i.e.,

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

provided that the series are convergent for $v, n \geq 0$. Then, A defines a matrix transformation from U into V, denoted by $A \in (U, V)$, if the sequence $Ax = (A_n(x)) \in V$ for all sequence $x \in U$.

Lemma 1.1 $([6])$. Let $1 < k < \infty$ and $1/k + 1/k^* = 1$. Then, $A \in (\ell_k, \ell)$ if and only if

$$\|A\|'_{(\ell_k,\ell)} = \left\{ \sum_{\nu=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} \right\}^{1/k^*} < \infty$$

and there exists $1 \leq \xi \leq 4$ such that $\|A\|'_{(\ell_k,\ell)} = \xi \|A\|_{(\ell_k,\ell)}$

If S and H are subsets of a metric space (X, d) and $\varepsilon > 0$, then S is called an ε-net of H , if, for every $h \in H$, there exists an $s \in S$ such that $d(h, s) < \varepsilon$; if S is finite, then the ε-net S of H is called a finite ε-net of H. By M_X, we denote the collection of all bounded subsets of X. If $Q \in M_X$, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\right\}.$$

The function $\chi : M_X \to [0, \infty)$ is called the Hausdorff measure of noncompactness $[5]$.

If X and Y are normed spaces, $\mathcal{B}(X, Y)$ states the set of all bounded linear operators from X to Y and is also a normed space according to the norm $\|L\| = \sup_{x \in S_X} \|L(x)\|$, where S_X is a unit sphere in X, $i.e.$, $S_X = \{x \in X : \|x\| = 1\}$. Further, a lineer operator $L : X \to Y$ is said to be compact if the sequence $(L(x_n))$ has convergent subsequence in Y for every bounded sequence $x = (x_n) \in X$. By $\mathcal{C}(X, Y)$ we denote the set of such operators.

The following results are need to compute Hausdorff measure of noncompactness.

Lemma 1.2 ([4]). Let X and Y be Banach spaces, $L \in \mathcal{B}(X,Y)$. Then, Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, is defined by

$$\|L\|_\chi = \chi\left(L\left(S_X\right)\right),$$

and

$$L \in \mathcal{C}(X,Y) \text{ iff } \|L\|_\chi = 0.$$

Lemma 1.3 ([5]). Let Q be a bounded subset of the normed space X where $X = \ell_k$ for $1 \leq k < \infty$.If $P_r : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, ..., x_r, 0, ...)$ for all $x \in X$, then

$$\chi(Q) = \lim_{r\to\infty} \sup_{x\in Q} \left\|\left(I - P_r\right)(x)\right\|,$$

where I is the identity operator on X.

Lemma 1.4 ([4]). Let X be normed sequence space, χ_T and χ denote Hausdorff measures of noncompactness on M_{X_T} and M_X, the collections of all bounded sets in X_T and X, respectively. Then,

$$\chi_T(Q) = \chi(T(Q)) \text{ for all } Q \in M_{X_T},$$

where T is an infinite triangle matrix.

2 Compact operators on the space bv_k^θ

More recently the class $\left(bv_k^\theta, bv\right)$, $1 < k < \infty$, has been characterized by Hazar and Sarıgöl [2] in the following form. In the present paper, by computing Hausdorff measure of noncompactness, we characterize compact operators in the same class.

Theorem 2.1. Let $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \geq 0$ and $1 < k < \infty$. Then, $A \in \left(bv_k^\theta, bv\right)$ if and only if

$$\lim_{n\to\infty} \sum_{j=\nu}^{\infty} a_{nj} \text{ exists for each } v \tag{2.1}$$

$$\sup_m \sum_{\nu=0}^{m} \left| \theta_\nu^{-1/k^*} \sum_{j=\nu}^{m} a_{nj} \right|^{k^*} < \infty \text{ for each } n \tag{2.2}$$

$$\sum_{\nu=0}^{\infty} \left(\sum_{n=0}^{\infty} \left| \theta_\nu^{1/k^*} \sum_{j=\nu}^{\infty} \left(a_{nj} - a_{n-1,j}\right) \right| \right)^{k^*} < \infty. \tag{2.3}$$

Also, for special case $\theta_v = 1$, it is reduced to the following result of [1].

Corollary 2.2. Let $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \geq 0$ and $1 < k < \infty$. Then, $A \in \left(bv^k, bv\right)$ if and only if (2.1) holds,

$$\sup_m \sum_{\nu=0}^{m} \left| \sum_{j=\nu}^{m} a_{nj} \right|^{k^*} < \infty \text{ for each } n,$$

$$\sum_{\nu=0}^{\infty} \left(\sum_{n=0}^{\infty} \left| \sum_{j=\nu}^{\infty} \left(a_{nj} - a_{n-1,j}\right) \right| \right)^{k^*} < \infty.$$

Now we give the following theorem.

Theorem 2.3. Let $1 < k < \infty$ and $\theta = (\theta_n)$ be a sequence of positive numbers. If $A \in \left(bv_k^\theta, bv\right)$, then there exists $1 \leq \xi \leq 4$ such that

$$\|A\|_\chi = \frac{1}{\xi}\lim_{r\to\infty}\left\{\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*}\right\}^{1/k^*}, \tag{2.4}$$

and $A \in \mathcal{C}\left(bv_k^\theta, bv\right)$ if and only if

$$\lim_{r\to\infty}\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*} = 0 \tag{2.5}$$

where

$$d_{nj} = \theta_j^{-1/k^*}\sum_{v=j}^{\infty}\left(a_{nv} - a_{n-1,v}\right)$$

Proof. Define $T_1 : bv_k^\theta \to \ell_k$ and $T_2 : bv \to \ell$ by $T_1(x) = \theta_v^{1/k^*}(x_v - x_{v-1})$ and $T_2(x) = x_v - x_{v-1},\ x_{-1} = 0$. Then, it clear that T_1 and T_2 are isomorhism preseving norms, $i.e.$, $\|x\|_{bv_k^\theta} = \|T_1(x)\|_{\ell_k}$ and $\|x\|_{bv} = \|T_2(x)\|_\ell$. So, bv_k^θ and bv are isometrically isomorhic to ℓ_k and ℓ, respectively, $i.e.$, $bv_k^\theta \simeq \ell_k$ and $bv \simeq \ell$. Now let $T_1(x) = y$ for $x \in bv_k^\theta$. Then, $x = T_1^{-1}(y) \in S_{bv_k^\theta}$ if and only if $y \in S_{\ell_k}$, where $S_X = \left\{x \in X : \|x\|_X = 1\right\}$. Also, it is seen easily (see [3]) that $T_2AT_1^{-1} = D$ and $A \in \left(bv_k^\theta, bv\right)$ iff $D \in (\ell_k, \ell)$. Further, by Lemma 1.1, there exists $1 \leq \xi \leq 4$ such that

$$\begin{aligned}
\|A\|_{\left(bv_k^\theta, bv\right)} &= \sup_{x\neq\theta}\frac{\|A(x)\|_{bv}}{\|x\|_{bv_k^\theta}} = \sup_{x\neq\theta}\frac{\left\|T_2^{-1}DT_1(x)\right\|_{bv}}{\|x\|_{bv_k^\theta}} \\
&= \sup_{x\neq\theta}\frac{\|D(y)\|_\ell}{\|y\|_{\ell_k}} = \|D\|_{(\ell_k,\ell)} \\
&= \frac{1}{\xi}\|D\|'_{(\ell_k,\ell)}
\end{aligned}$$

and so, by Lemmas 1.2, 1.3 and 1.4, we have

$$\begin{aligned}
\|A\|_\chi &= \chi\left(AS_{bv_k^\theta}\right) = \chi(T_2AS_{bv_k^\theta}) \\
&= \chi(DT_1S_{bv_k^\theta}) = \lim_{r\to\infty}\sup_{y\in S_{\ell_k}}\|(I - P_r)\,D(y)\|_\ell \\
&= \lim_{r\to\infty}\sup_{y\in S_{\ell_k}}\left\|D^{(r)}(y)\right\| = \lim_{r\to\infty}\left\|D^{(r)}\right\|_{(\ell_k,\ell)} \\
&= \frac{1}{\xi}\lim_{r\to\infty}\left\{\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*}\right\}^{1/k^*}
\end{aligned}$$

where $P_r : \ell \to \ell$ is defined by $P_r(y) = (y_0, y_1, ..., y_r, 0, ...)$, and

$$d_{nv}^{(r)} = \begin{cases} 0, & 0 \leq n \leq r \\ d_{nv}, & n > r \end{cases}$$

So the proof is completed by Lemma 1.2.

In the special case $\theta_n = 1$, the following result is immediate.

Corollary 2.4. Let $1 < k < \infty$. If $A \in \left(bv^k, bv\right)$, then there exists $1 \leq \xi \leq 4$ such that

$$\|A\|_\chi = \frac{1}{\xi}\lim_{r\to\infty}\left\{\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*}\right\}^{1/k^*}$$

and

$$A \in \mathcal{C}\left(bv_k, bv\right) \text{ iff } \lim_{r\to\infty}\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*} = 0$$

where

$$d_{nj} = \sum_{v=j}^{\infty}\left(a_{nv} - a_{n-1,v}\right)$$

Acknowledgement

The present paper was supported by the scientific and research center of Pamukkale University, Project No. 2019KKP067 (2019KRM004).

3 References

[1] E. Malkowsky, V. Rakočević, S. Živković, *Matrix transformations between the sequence space bv^k and certain BK spaces,* Bull. Cl. Sci. Math. Nat. Sci. Math., **123**(27) (2002), 33–46.
[2] G. C. Hazar, M. A. Sarıgöl, *The space bv_k^θ and matrix transformations*, 8th International Eurasian Converence on Mathematical Sciences and Applications (IECMSA 2019), 2019 (in press).
[3] G. C. Hazar, M. A. Sarıgöl, *On absolute Nörlund spaces and matrix operators*, Acta Math. Sin. (Engl. Ser.) **34**(5) (2018), 812-826.
[4] E. Malkowsky, V. Rakočević, *An introduction into the theory of sequence space and measures of noncompactness,* Zb. Rad. (Beogr) **9**(17) (2000), 143-234.
[5] V. Rakočević, *Measures of noncompactness and some applications*, Filomat, **12** (1998), 87-120.
[6] M. A. Sarıgöl, *Extension of Mazhar's theorem on summability factors*, Kuwait Jour. Sci., **42**(2) (2015), 28-35.
[7] M. Stieglitz, H. Tietz, *Matrixtransformationen von Folgenraumen Eine Ergebnisüberischt,* Math Z., **154** (1977), 1-16.

Conference Proceedings of Science and Technology, 2(3), 2019, 189–193

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

Deferred Statistical Convergence in Metric Spaces

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Mikail Et [1] *Muhammed Çınar* [2] *Hacer Şengül* [3]

[1] *Faculty of Science, Department of Mathematics, Firat University, Elazig, Turkey, ORCID:0000-0001-8292-7819*
[2] *Faculty of Education, Department of Mathematics Education, University of Mus Alparslan, Mus, Turkey, ORCID:0000-0002-0958- 0705*
[3] *Faculty of Education, Harran University, Sanliurfa, Turkey, ORCID:0000-0003-4453-0786*
** Corresponding Author E-mail: mikailet68@gmail.com*

Abstract: In this paper, the concept of deferred statistical convergence is generalized to general metric spaces, and some inclusion relations between deferred strong Cesàro summability and deferred statistical convergence are given in general metric spaces.

Keywords: Metric space, Statistical convergence, Deferred statistical convergence.

1 Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [2] and Fast [3] and then reintroduced independently by Schoenberg [4]. Over the years and under different names, statistical convergence has been discussed in the Theory of Fourier Analysis, Ergodic Theory, Number Theory, Measure Theory, Trigonometric Series, Turnpike Theory and Banach Spaces. Later on it was further investigated from the sequence spaces point of view and linked with summability theory by Gupta and Bhardwaj [5], Braha et al. [6], Çınar et al. [7], Connor [8], Et et al. ([9],[10],[11],[12],[13]), Fridy [14], Işık et al. ([15],[16],[17]), Mohiuddine et al. [18], Mursaleen et al. [19], Nuray [20], Nuray and Aydın [21], Salat [22], Şengül et al. ([23],[24],[25],[26]), Srivastava et al. ([27],[28]) and many others.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k),$$

provided that the limit exists, where $\chi_{\mathbb{E}}$ is the characteristic function of the set $\mathbb{E}$. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and that

$$\delta\left(\mathbb{E}^c\right) = 1 - \delta\left(\mathbb{E}\right).$$

A sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to L if, for every $\varepsilon > 0$, we have

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0.$$

In this case, we write

$$x_k \xrightarrow{\text{stat}} L \quad \text{as} \quad k \to \infty \qquad \text{or} \qquad S - \lim_{k\to\infty} x_k = L.$$

In 1932, Agnew [29] introduced the concept of deferred Cesàro mean of real (or complex) valued sequences $x = (x_k)$ defined by

$$(D_{p,q}x)_n = \frac{1}{(q\,(n) - p\,(n))} \sum_{k=p(n)+1}^{q(n)} x_k,\; n = 1, 2, 3, \ldots,$$

where $p = \{p\,(n)\}$ and $q = \{q\,(n)\}$ are the sequences of non-negative integers satisfying

$$p\,(n) < q\,(n) \text{ and } \lim_{n\to\infty} q\,(n) = \infty.$$

Let K be a subset of $\mathbb{N}$ and denote the set $\{k : p\,(n) < k \leq q\,(n)\,, k \in K\}$ by $K_{p,q}\,(n)$.

Deferred density of K is defined by

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |K_{p,q}(n)|, \text{ provided the limit exists,}$$

where, vertical bars indicate the cardinality of the enclosed set $K_{p,q}(n)$. If $q(n) = n$, $p(n) = 0$, then the deferred density coincides with natural density of K.

A real valued sequence $x = (x_k)$ is said to be deferred statistically convergent to L, if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_{p,q} - \lim x_k = L$. If $q(n) = n$, $p(n) = 0$, for all $n \in \mathbb{N}$, then deferred statistical convergence coincides with usual statistical convergence [30].

2 Main Results

In this section, we give some inclusion relations between statistical convergence, deferred strong Cesàro summability and deferred statistical convergence in general metric spaces.

Definition 1 Let (X, d) be a metric space and $\{p(n)\}$ and $\{q(n)\}$ be two sequences as above. A metric valued sequence $x = (x_k)$ is said to be $DS^d_{p,q}-$convergent (or deferred $d-$statistically convergent) to a if there is a real number $a \in X$ such that

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}| = 0.$$

In this case we write $DS^d_{p,q} - \lim x_k = a$ or $x_k \to a\left(DS^d_{p,q}\right)$. The set of all $DS^d_{p,q}-$statistically convergent sequences will be denoted by $DS^d_{p,q}$. If $q(n) = n$ and $p(n) = 0$, then deferred $d-$statistical convergence coincides $d-$statistical convergence.

Definition 2 Let (X, d) be a metric space and $\{p(n)\}$ and $\{q(n)\}$ be two sequences as above. A metric valued sequence $x = (x_k)$ is said to be strongly $Dw^d_{p,q}-$summable (or deferred strongly $d-$Cesàro summable) to a if there is a real number $a \in X$ such that

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} \sum_{p(n)+1}^{q(n)} d(x_k, a) = 0.$$

In this case we write $Dw^d_{p,q} - \lim x_k = a$ or $x_k \to a\left(Dw^d_{p,q}\right)$. The set of all strongly $Dw^d_{p,q}-$summable sequences will be denoted by $Dw^d_{p,q}$. If $q(n) = n$ and $p(n) = 0$, for all $n \in \mathbb{N}$, then deferred strong $d-$Cesàro summability coincides strong $d-$Cesàro summability.

Theorem 3 Let (X, d) be a linear metric space and $x = (x_k)$, $y = (y_k)$ be metric valued sequences, then

(i) If $DS^d_{p,q} - \lim x_k = x_0$ and $DS^d_{p,q} - \lim y_k = y_0$, then $DS^d_{p,q} - \lim(x_k + y_k) = x_0 + y_0$,

(ii)If $DS^d_{p,q} - \lim x_k = x_0$ and $c \in \mathbb{C}$, then $DS^d_{p,q} - \lim(cx_k) = cx_0$,

(iii) If $DS^d_{p,q} - \lim x_k = x_0$, $DS^d_{p,q} - \lim y_k = y_0$ and $x, y \in \ell_\infty$, then $DS^d_{p,q} - \lim(x_k y_k) = x_0 y_0$.

Theorem 4 $Dw^d_{p,q} \subseteq DS^d_{p,q}$ and the inclusion is strict.

Proof. First part of proof is easy, so omitted. To show the strictness of the inclusion, choose $q(n) = n, p(n) = 0$, for all $n \in \mathbb{N}$ and $a = 0$ and define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} \frac{\sqrt{n}}{2}, & k = n^2 \\ 0, & k \neq n^2 \end{cases}.$$

Then for every $\varepsilon > 0$, we have

$$\frac{1}{(q(n) - p(n))} |\{p(n) < k \leq q(n) : d(x_k, 0) \geq \varepsilon\}| \leq \frac{[\sqrt{n}]}{n} \to 0, \text{ as } n \to \infty,$$

where $d(x, y) = |x - y|$, that is $x_k \to 0\left(DS^d_{p,q}\right)$. At the same time, we get

$$\frac{1}{(q(n) - p(n))} \sum_{p(n)+1}^{q(n)} d(x_k, 0) \leq \frac{[\sqrt{n}]\,[\sqrt{n}]}{n} \to 1,$$

i.e. $x_k \nrightarrow 0\left(Dw^d_{p,q}\right)$. Therefore, $Dw^d_{p,q} \subseteq DS^d_{p,q}$ is strict.

Theorem 5 If $\liminf_n \frac{q(n)}{p(n)} > 1$, then $S^d \subset DS^d_{p,q}$.

Proof. Suppose that $\liminf_n \frac{q(n)}{p(n)} > 1$; then there exists a $\nu > 0$ such that $\frac{q(n)}{p(n)} \geq 1+\nu$ for sufficiently large n, which implies that

$$\frac{q(n)-p(n)}{q(n)} \geq \frac{\nu}{1+\nu} \Longrightarrow \frac{1}{q(n)} \geq \frac{\nu}{(1+\nu)} \frac{1}{(q(n)-p(n))}.$$

If $x_k \to a\left(S^d\right)$, then for every $\varepsilon > 0$ and for sufficiently large n, we have

$$\begin{aligned}\frac{1}{q(n)} \left|\{k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right| &\geq \frac{1}{q(n)} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right| \\ &\geq \frac{\nu}{(1+\nu)} \frac{1}{(q(n)-p(n))} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right|.\end{aligned}$$

This proves the proof.
"In the following theorem, by changing the conditions on the sequences (p_n) and (q_n) we give the same relation with Theorem 5."

Theorem 6 If $\lim_{n\to\infty} \inf \dfrac{(q(n)-p(n))}{n} > 0$ and $q(n) < n$, then $S^d \subseteq DS^d_{p,q}$.

Proof. Let $\lim_{n\to\infty} \inf \dfrac{(q(n)-p(n))}{n} > 0$ and $q(n) < n$, then for each $\varepsilon > 0$ the inclusion

$$\{k \leq n : d(x_k, a) \geq \varepsilon\} \supset \{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}$$

is satisfied and so we have the following inequality

$$\begin{aligned}\frac{1}{n} \left|\{k \leq n : d(x_k, a) \geq \varepsilon\}\right| &\geq \frac{1}{n} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right| \\ &= \frac{(q(n)-p(n))}{n} \frac{1}{(q(n)-p(n))} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right|.\end{aligned}$$

Therefore $S^d \subseteq DS^d_{p,q}$.

Theorem 7 Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be four sequences of non-negative integers such that

$$p'(n) < p(n) < q(n) < q'(n) \text{ for all } n \in \mathbb{N}, \tag{1}$$

then

(i) If

$$\lim_{n\to\infty} \frac{q(n)-p(n)}{q'(n)-p'(n)} = m > 0 \tag{2}$$

then $DS^d_{p',q'} \subseteq DS^d_{p,q}$,

(ii) If

$$\lim_{n\to\infty} \frac{q'(n)-p'(n)}{q(n)-p(n)} = 1 \tag{3}$$

then $DS^d_{p,q} \subseteq DS^d_{p',q'}$.

Proof. (i) Let (2) be satisfied. For given $\varepsilon > 0$ we have

$$\{p'(n) < k \leq q'(n) : d(x_k, a) \geq \varepsilon\} \supseteq \{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\},$$

and so

$$\begin{aligned}&\frac{1}{(q'(n)-p'(n))} \left|\{p'(n) < k \leq q'(n) : d(x_k, a) \geq \varepsilon\}\right| \\ &\geq \frac{(q(n)-p(n))}{(q'(n)-p'(n))} \frac{1}{(q(n)-p(n))} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right|.\end{aligned}$$

Therefore $DS^d_{p',q'} \subseteq DS^d_{p,q}$.

(ii) Omitted.

Theorem 8 Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$be four sequences of non-negative integers defined as in (1).

(i) If (2) holds then $Dw^d_{p',q'} \subset Dw^d_{p,q}$,

(ii) If (3) holds and $x = (x_k)$ be a bounded sequence, then $Dw^d_{p,q} \subset Dw^d_{p',q'}$.

Proof. Omitted.

Theorem 9 Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be four sequences of non-negative integers defined as in (1). Then

(i) Let (2) holds, if a sequence is strongly $Dw^d_{p',q'}-$summable to a, then it is $DS^d_{p,q}-$convergent to a,

(ii) Let (3) holds and $x = (x_k)$ be a bounded sequence, if a sequence is $DS^d_{p,q}-$convergent to a then it is strongly $Dw^d_{p',q'}-$summable to a.

Proof. (i) Omitted.

(ii) Suppose that $DS^d_{p,q} - \lim x_k = a$ and $(x_k) \in \ell_\infty$. Then there exists some $M > 0$ such that $d(x_k, a) < M$ for all k, then for every $\varepsilon > 0$ we may write

$$
\begin{aligned}
&\frac{1}{(q'(n) - p'(n))} \sum_{p'(n)+1}^{q'(n)} d(x_k, a) \\
&= \frac{1}{(q'(n) - p'(n))} \sum_{q(n)-p(n)+1}^{q'(n)-p'(n)} d(x_k, a) + \frac{1}{(q'(n) - p'(n))} \sum_{p(n)+1}^{q(n)} d(x_k, a) \\
&\leq \frac{\left(q'(n) - p'(n)\right) - (q(n) - p(n))}{(q'(n) - p'(n))} M + \frac{1}{(q'(n) - p'(n))} \sum_{p(n)+1}^{q(n)} d(x_k, a) \\
&\leq \left(\frac{q'(n) - p'(n)}{q(n) - p(n)} - 1\right) M + \frac{1}{(q(n) - p(n))} \sum_{\substack{p(n)+1 \\ d(x_k,a) \geq \varepsilon}}^{q(n)} d(x_k, a) \\
&+ \frac{1}{(q(n) - p(n))} \sum_{\substack{p(n)+1 \\ d(x_k,a) < \varepsilon}}^{q(n)} d(x_k, a) \\
&\leq \left(\frac{q'(n) - p'(n)}{q(n) - p(n)} - 1\right) M + \frac{M}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right| \\
&+ \frac{q'(n) - p'(n)}{q(n) - p(n)} \varepsilon.
\end{aligned}
$$

This completes the proof.

3 References

[1] A. Zygmund, *Trigonometric series*, Cambridge University Press, Cambridge, London and New York, 1979.

[2] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), 73–74.

[3] H. Fast, *Sur la convergence statistique*, Colloq. Math.,**2** (1951), 241–244.

[4] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.

[5] S. Gupta, V. K. Bhardwaj, *On deferred $f-$statistical convergence*, Kyungpook Math. J. **58**(1) (2018), 91–103.

[6] N. L. Braha, H. M. Srivastava, S. A. Mohiuddine, *A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean*, Appl. Math. Comput., **228** (2014), 162–169.

[7] M. Çınar, M. Karakaş, M. Et, *On pointwise and uniform statistical convergence of order α for sequences of functions*, Fixed Point Theory Appl. **33**(2013), 11.

[8] J. S. Connor, *The Statistical and strong p-Cesàro convergence of sequences*, Analysis, **8** (1988), 47–63.

[9] M. Et, A. Alotaibi, S. A. Mohiuddine, *On $(\Delta^m, I)-$statistical convergence of order α*, The Scientific World Journal, 2014, 535419 DOI: 10.1155/2014/535419.

[10] M. Et, S. A. Mohiuddine, A. Alotaibi, *On $\lambda-$statisti*cal convergence and strongly $\lambda-$summable functions of order α, J. Inequal. Appl. **469** (2013), 8.

[11] M. Et, B. C. Tripathy, A. J. Dutta, *On pointwise statistical convergence of order α of sequences of fuzzy mappings*, Kuwait J. Sci. **41**(3) (2014), 17–30.

[12] M. Et, R. Colak, Y. Altın, *Strongly almost summable sequences of order α*, Kuwait J. Sci. **41**(2), (2014), 35–47.

[13] E. Savaş, M. Et, *On $(\Delta^m_\lambda, I)-$statistical convergence of order α*, Period. Math. Hungar. **71**(2) (2015), 135–145.

[14] J. A. Fridy, *On statistical convergence*, Analysis, **5** (1985), 301–313.

[15] M. Işık, K. E. Akbaş, *On $\lambda-$statistical convergence of order α in probability*, J. Inequal. Spec. Funct. **8**(4) (2017), 57–64.

[16] M. Işık, K. E. Et, *On lacunary statistical convergence of order α in probability*, AIP Conference Proceedings 1676, 020045 (2015); doi: http://dx.doi.org/10.1063/1.4930471.

[17] M. Işık, K. E. Akbaş, *On Asymptotically Lacunary Statistical Equivalent Sequences of Order α in Probability*, ITM Web of Conferences **13**, 01024 (2017). DOI: 10.1051/itmconf/20171301024.

[18] S. A. Mohiuddine, A. Alotaibi, M. Mursaleen, *Statistical convergence of double sequences in locally solid Riesz spaces*, Abstr. Appl. Anal., **2002** (2012), Article ID 719729, 9 pp.

[19] M. Mursaleen, A. Khan, H. M. Srivastava, K. S. Nisar, *Operators constructed by means of q-Lagrange polynomials and A-statistical approximation*, Appl. Math. Comput., **219** (2013), 6911–6918.

[20] F. Nuray, *λ-strongly summable and λ-statistically convergent functions*, Iran. J. Sci. Technol. Trans. A Sci., **34** (2010), 335–338.

[21] F. Nuray, B. Aydin, *Strongly summable and statistically convergent functions*, Inform. Technol. Valdymas **1**(30) (2004), 74–76.

[22] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.

[23] H. Şengül, M. Et, *On $I-$lacunary statistical convergence of order α of sequences of sets*, Filomat **31**(8) (2017), 2403–2412.
[24] H. Şengül, *On Wijsman $I-$lacunary statistical equivalence of order (η, μ)*, J. Inequal. Spec. Funct. **9**(2) (2018), 92–101.
[25] H. Şengül, *On $S_{\alpha}^{\beta}(\theta)-$convergence and strong $N_{\alpha}^{\beta}(\theta, p)-$summability*, J. Nonlinear Sci. Appl. **10**(9) (2017), 5108–5115.
[26] H. Şengül, M. Et, *Lacunary statistical convergence of order (α, β) in topological groups*, Creat. Math. Inform. **2683** (2017), 339–344.
[27] H. M. Srivastava, M. Mursaleen, A. Khan, *Generalized equi-statistical convergence of positive linear operators and associated approximation theorems,* Math. Comput. Modelling **55** (2012), 2040–2051.
[28] H. M. Srivastava, M. Et, *Lacunary statistical convergence and strongly lacunary summable functions of order α*, Filomat **31**(6) (2017), 1573–1582.
[29] R. P. Agnew, *On deferred Cesàro mean*, Ann. Math.,**33** (1932), 413-421.
[30] M. Küçükaslan, M. Yılmaztürk *On deferred statistical convergence of sequences*, Kyungpook Math. J. **56** (2016), 357-366.

Conference Proceedings of Science and Technology, 2(3), 2019, 194–197

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

A New Type Generalized Difference Sequence Space $m(\phi, p)(\Delta_m^n)$

Mikail Et [1] *Rifat Colak* [2]

[1] *Faculty of Science, Department of Mathematics, Firat University, Elazig, Turkey, ORCID:0000-0001-8292-7819*
[2] *Faculty of Science, Department of Mathematics, Firat University, Elazig, Turkey, ORCID:0000-0001-8161-5186*
** Corresponding Author E-mail: mikailet68@gmail.com*

Abstract: Let (ϕ_n) be a non-decreasing sequence of positive numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$. The class of all sequences (ϕ_n) is denoted by Φ. The sequence space $m(\phi)$ was introduced by Sargent [1] and he studied some of its properties and obtained some relations with the space ℓ_p. Later on it was investigated by Tripathy and Sen [2] and Tripathy and Mahanta [3]. In this work, using the generalized difference operator Δ_m^n, we generalize the sequence space $m(\phi)$ to sequence space $m(\phi, p)(\Delta_m^n)$, give some topological properties about this space and show that the space $m(\phi, p)(\Delta_m^n)$ is a $BK-$space by a suitable norm. The results obtained are generalizes some known results.

Keywords: Difference sequence, BK-space, Symmetric space, Normal space.

1 Introduction

By w, we denote the space of all complex (or real) sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$. We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Also by ℓ_1 and ℓ_p; we denote the spaces of all absolutely summable and $p-$absolutely summable sequences, respectively.

Let $x \in w$ and $S(x)$ denotes the set of all permutation of the elements x_n, i.e. $S(x) = \{(x_{\pi(n)}) : \pi(n)$ is a permutation on $\mathbb{N}\}$. A sequence space E is said to be symmetric if $S(x) \subset E$ for all $x \in E$.

A sequence space E is said to be solid (normal) if $(y_n) \in E$, whenever $(x_n) \in E$ and $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$.

A sequence space E is said to be sequence algebra if $x.y \in E$, whenever $x, y \in E$.
A sequence space E is said to be perfect if $E = E^{\alpha\alpha}$.
It is well known that if E is perfect then E is normal.

A sequence space E with a linear topology is called a $K-$space provided each of the maps $p_i : E \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for each $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A $K-$space E is called an $FK-$space provided E is a complete linear metric space. An $FK-$space whose topology is normable is called a $BK-$space.

The notion of difference sequence spaces was introduced by Kızmaz [4] and it was generalized by Et and Çolak [5] for $X = \ell_\infty, c, c_0$ as follows:

Let n be a non-negative integer, then

$$\Delta^n(X) = \{x = (x_k) : (\Delta^n x_k) \in X\},$$

where $\Delta^n x_k = \Delta^{n-1}x_k - \Delta^{n-1}x_{k+1}$ for all $k \in \mathbb{N}$ and so $\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}$. Et and Çolak [5] showed that the sequence spaces $\Delta^n(c_0)$, $\Delta^n(c)$ and $\Delta^n(\ell_\infty)$ are $BK-$spaces with the norm

$$\|x\|_{\Delta 1} = \sum_{i=1}^{n} |x_i| + \|\Delta^n x\|_\infty .$$

After then, using a new difference operator Δ_m^n, Tripathy et al. ([6], [7], [8]) have defined a new type difference sequence space $\Delta_m^n(X)$ such as

$$\Delta_m^n(X) = \{x = (x_k) : (\Delta_m^n x_k) \in X\},$$

where $m, n \in \mathbb{N}$, $\Delta_m^0 x = x$, $\Delta_m^1 x = (x_k - x_{k+m})$, $\Delta_m^n x = (\Delta_m^n x_k) = \left(\Delta_m^{n-1}x_k - \Delta_m^{n-1}x_{k+m}\right)$ and so $\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^n \binom{n}{v} x_{k+mv}$, and give some topological properties about this space and show that the spaces $\Delta_m^n(X)$ are $BK-$spaces by the norm

$$\|x\|_{\Delta 2} = \sum_{i=1}^{mn} |x_i| + \|\Delta_m^n x\|_\infty$$

for $X = \ell_\infty, c$ and c_0. Recently, difference sequences have been studied in ([9],[10],[11],[12],[13],[14],[15],[16],[17],[18]) and many others.

2 Main results

In this section, we introduce a new class $m(\phi, p)(\Delta_m^n)$ of sequences , establish some inclusion relations and some topological properties. The obtained results are more general than those of Çolak and Et [19], Sargent [1] and Tripathy and Sen [2] .

The notation φ_s denotes the class of all subsets of $\mathbb{N}$, those do not contain more than s elements. Let (ϕ_n) be a non-decreasing sequence of positive numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$. The class of all sequences (ϕ_n) is denoted by Φ.

The sequence spaces $m(\phi)$ and $m(\phi, p)$ were introduced by Sargent [1], Tripathy and Sen [2] as follows, respectively

$$m(\phi) = \left\{ x = (x_k) \in w : \|x\|_{m(\phi)} = \sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty \right\},$$
$$m(\phi, p) = \left\{ x = (x_k) \in w : \|x\|_{m(\phi,p)} = \sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\phi_s} \left(\sum_{k \in \sigma} |x_k|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

Let $m, n \in \mathbb{N}$ and $1 \leq p < \infty$. Now we define the sequence space $m(\phi, p)(\Delta_m^n)$ as

$$m(\phi, p)(\Delta_m^n) = \left\{ x = (x_k) \in w : \sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \left| \Delta_m^n x_k \right|^p < \infty \right\}.$$

From this definition it is clear that $m(\phi, p)\left(\Delta_m^0\right) = m(\phi, p)$ and $m(\phi, 1)\left(\Delta_m^0\right) = m(\phi)$. In case of $m = 1$, we shall write $m(\phi, p)(\Delta^n)$ instead of $m(\phi, p)(\Delta_m^n)$ and in case of $p = 1$, we shall write $m(\phi)(\Delta_m^n)$ instead of $m(\phi, p)(\Delta_m^n)$. The sequence space $m(\phi, p)(\Delta_m^n)$ contains some unbounded sequences for $m, n \geq 1$. For example, the sequence $(x_k) = (k^n)$ is an element of $m(\phi, p)(\Delta_m^n)$ for $m = 1$, but is not an element of ℓ_∞.

Theorem 1. The space $m(\phi, p)(\Delta_m^n)$ is a Banach space with the norm

$$\|x\|_{\Delta_m^n} = \sum_{i=1}^{r} |x_i| + \sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\phi_s} \left(\sum_{k \in \sigma} \left| \Delta_m^n x_k \right|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \tag{1}$$

where $r = mn$ for $m \geq 1, n \geq 1$.

Proof. It is a routine verification that $m(\phi, p)(\Delta_m^n)$ is a normed linear space normed by (1) for $1 \leq p < \infty$. Let $\left(x^l\right)$ be a Cauchy sequence in $m(\phi, p)(\Delta_m^n)$, where $x^l = (x_k^l)_{k=1}^\infty = \left(x_1^l, x_2^l, ...\right) \in m(\phi, p)(\Delta_m^n)$, for each $l \in \mathbb{N}$. Then given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\left\|x^l - x^t\right\|_{\Delta_m^n} = \sum_{i=1}^{r} \left|x_i^l - x_i^t\right| + \sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\phi_s} \left(\sum_{k \in \sigma} \left| \Delta_m^n \left(x_k^l - x_k^t\right) \right|^p \right)^{\frac{1}{p}} < \varepsilon \tag{2}$$

for all $l, t > n_0$. Hence we obtain

$$\left|x_k^l - x_k^t\right| \to 0 \text{ as } l, t \to \infty, \text{ for each } k \in \mathbb{N}.$$

Therefore $(x_k^l)_{l=1}^\infty = \left(x_k^1, x_k^2, ...\right)$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, it is convergent, that is,

$$\lim_l x_k^l = x_k$$

for each $k \in \mathbb{N}$. Using these infinite limits $x_1, x_2, x_3, ...$ let us define the sequence $x = (x_k)$. We should show that $x \in m(\phi, p)(\Delta_m^n)$ and $\left(x^l\right) \to x$. Taking limit as $t \to \infty$ in (2) , we get

$$\left\|x^l - x\right\|_{\Delta_m^n} = \sum_{i=1}^{r} \left|x_i^l - x_i\right| + \sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\phi_s} \left(\sum_{k \in \sigma} \left| \Delta_m^n \left(x_k^l - x_k\right) \right|^p \right)^{\frac{1}{p}} < \varepsilon \tag{3}$$

for all $l \geq n_0$. This shows that $\left(x^l\right) \to x$ as $l \to \infty$. From (3) we also have

$$\sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l - x_k\right)\right|^p\right)^{\frac{1}{p}} < \varepsilon$$

for all $l \geq n_0$. Hence $x^l - x = \left(x_k^l - x_k\right)_k \in m(\phi,p)(\Delta_m^n)$. Since $x^l - x, x^l \in m(\phi,p)(\Delta_m^n)$ and $m(\phi,p)(\Delta_m^n)$ is a linear space, we have $x = x^l - \left(x^l - x\right) \in m(\phi,p)(\Delta_m^n)$. Therefore $m(\phi,p)(\Delta_m^n)$ is complete.

Theorem 2. The space $m(\phi,p)(\Delta_m^n)$ is a $BK-$space.

Proof. Omitted.

Theorem 3. [2] i) The space $m(\phi,p)$ is a symmetric space,
ii) The space $m(\phi,p)$ is a normal space.

Theorem 4. The sequence space $m(\phi,p)(\Delta_m^n)$ is not sequence algebra, is not solid and is not symmetric, for $m,n,p \geq 1$.

Proof. For the proof of the Theorem, consider the following examples:

Example 1. It is obvious that, if $x = \left(k^{n-2}\right)$, $y = \left(k^{n-2}\right)$ and $m = 1$, then $x, y \in m(\phi,p)(\Delta_m^n)$, but $x.y \notin m(\phi,p)(\Delta_m^n)$. Hence $m(\phi,p)(\Delta_m^n)$ is not a sequence algebra.

Example 2. It is obvious that, if $x = \left(k^{n-1}\right)$ and $m = 1$, then $x \in m(\phi,p)(\Delta_m^n)$, but $(\alpha_k x_k) \notin m(\phi,p)(\Delta_m^n)$ for $(\alpha_k) = \left((-1)^k\right)$. Hence $m(\phi,p)(\Delta_m^n)$ is not solid.

Example 3. Let us consider the sequence $x = \left(k^{n-1}\right)$. Then $x \in m(\phi,p)(\Delta_m^n)$ for $m = 1$. Let (y_k) be a rearrangement of (x_k) which is defined as follows:

$$y_k = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...\}.$$

Then $y \notin m(\phi,p)(\Delta_m^n)$. Hence $m(\phi,p)(\Delta_m^n)$ is not symmetric.

The following result is a consequence of Theorem 4.

Corollary 1. The sequence space $m(\phi,p)(\Delta_m^n)$ is not perfect, for $m,n,p \geq 1$.

Theorem 5. $m(\phi)(\Delta_m^n) \subset m(\phi,p)(\Delta_m^n)$ for each $m,n,p \geq 1$.

Proof. Omitted.

Theorem 6. $m(\phi,p)(\Delta_m^n) \subset m(\psi,p)(\Delta_m^n)$ if and only if $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$.

Proof. Suppose that $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$. Then $\phi_s \leq K\psi_s$ for every s and for some positive number K. If $x \in m(\phi,p)(\Delta_m^n)$, then,

$$\sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}} < \infty.$$

Now, we have

$$\sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\psi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}\%} < \sup_{s\geq 1}(K) \sup_{s\geq 1\quad k\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}} < \infty.$$

Hence $x \in m(\psi,p)(\Delta_m^n)$.

Conversely let $m(\phi,p)(\Delta_m^n) \subset m(\psi,p)(\Delta_m^n)$ and suppose that $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = \infty$. Then, there exists a sequence (s_i) of natural numbers such that $\lim\limits_i \left(\frac{\phi_{s_i}}{\psi_{s_i}}\right) = \infty$. Then, for $x \in m(\phi,p)(\Delta_m^n)$ we have

$$\sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\psi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}} \geq \sup_{i\geq 1}\left(\frac{\phi_{s_i}}{\psi_{s_i}}\right) \sup_{i\geq 1,\ \sigma\in\varphi_{s_i}} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}} = \infty.$$

Therefore $x \notin m(\psi,p)(\Delta_m^n)$. This contradict to $m(\phi,p)(\Delta_m^n) \subset m(\psi,p)(\Delta_m^n)$. Hence $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$.

From Theorem 6, we get the following result.

Corollary 2. $m(\phi, p)(\Delta_m^n) = m(\psi, p)(\Delta_m^n)$ if and only if $0 < \inf_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) \leq \sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$.

Theorem 7. $m(\phi, p)\left(\Delta_m^{n-1}\right) \subset m(\phi, p)(\Delta_m^n)$ and the inclusion is strict.

*Proof.*Let $x \in m(\phi, p)\left(\Delta_m^{n-1}\right)$. It is well known that, for $1 \leq p < \infty$, $|a+b|^p \leq 2^p\left(|a|^p + |b|^p\right)$. Hence, for $1 \leq p < \infty$, we have

$$\frac{1}{\phi_s}\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p \leq 2^p\left(\frac{1}{\phi_s}\sum_{k\in\sigma}\left|\Delta_m^{n-1}x_k\right|^p + \frac{1}{\phi_s}\sum_{k\in\sigma}\left|\Delta_m^{n-1}x_{k+1}\right|^p\right)$$

Hence $x \in m(\phi, p)(\Delta_m^n)$.

To show the inclusion is strict consider the following example.

Example 4. Let $\phi_n = 1$, for all $n \in \mathbb{N}$, $m = 1$ and $x = \left(k^{n-1}\right)$, then $x \in \ell_p(\Delta_m^n) \setminus \ell_p\left(\Delta_m^{n-1}\right)$.

Theorem 8. We have $\ell_p(\Delta_m^n) \subset m(\phi, p)(\Delta_m^n) \subset \ell_\infty(\Delta_m^n)$.

Proof. Since $m(\phi, p)(\Delta_m^n) = \ell_p(\Delta_m^n)$ for $\phi_n = 1$, for all $n \in \mathbb{N}$, then $\ell_p(\Delta_m^n) \subset m(\phi, p)(\Delta_m^n)$. Now assume that $x \in m(\phi, p)(\Delta_m^n)$. Then we have

$$\sup_{s\geq 1,\ \sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}} < \infty \text{ and so } \left|\Delta_m^n x_k\right| < K\phi_1,$$

for all $k \in \mathbb{N}$ and for some positive number K. Thus, $x \in \ell_\infty(\Delta_m^n)$.

Theorem 9. If $0 < p < q$, then $m(\phi, p)(\Delta_m^n) \subset m(\phi, q)(\Delta_m^n)$.

Proof. Proof follows from the following inequality

$$\left(\sum_{k=1}^n |x_k|^q\right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}}, \quad (0 < p < q).$$

3 References

[1] W. L. C. Sargent, *Some sequence spacess related to ℓ_p spaces*, J. London Math. Soc. **35** (1960), 161-171.
[2] B. C. Tripathy, M. Sen, *On a new class of sequences related to the space ℓ_p*, Tamkang J. Math. **33**(2) (2002), 167-171.
[3] B. C. Tripathy, S. Mahanta, *On a class of sequences related to the ℓ_p* space defined by Orlicz functions, Soochow J. Math. **29**(4) (2003), 379–391.
[4] H. Kızmaz, *On certain Sequence spaces*, Canad. Math. Bull. **24**(2) (1981), 169-176.
[5] M. Et, R. Çolak, *On generalized difference sequence spaces*, Soochow J. Math. **21**(4) (1995), 377-386.
[6] A. Esi, B. C. Tripathy, B. Sarma, *On some new type generalized difference sequence spaces*, Math. Slovaca **57**(5) (2007), 475–482.
[7] A. Esi, B. C. Tripathy, *A New Type Of Difference Sequence Spaces*, International Journal of Science & Technology **1**(1) (2006), 11-14.
[8] B . C. Tripathy, A. Esi, B. K. Tripathy, *On a new type of generalized difference Cesaro Sequence spaces*, Soochow J. Math. **31**(3) (2005), 333-340.
[9] Y. Altin, *Properties of some sets of sequences defined by a modulus function*, Acta Math. Sci. Ser. B Engl. Ed. **29**(2) (2009), 427–434.
[10] H. Altinok, M. Et, R. Çolak, *Some remarks on generalized sequence space of bounded variation of sequences of fuzzy numbers*, Iran. J. Fuzzy Syst. **11**(5) (2014), 39–46, 109.
[11] S. Demiriz, C. Çakan, *Some topological and geometrical properties of a new difference sequence space*. Abstr. Appl. Anal. 2011, Art. ID 213878, 14 pp.
[12] S. Erdem, S. Demiriz, *On the new generalized block difference sequence spaces*, Appl. Appl. Math., Special Issue No. **5** (2019), 68–83.
[13] M. Et, A. Alotaibi, S. A. Mohiuddine, *On (Δ^m, I)-statistical convergence of order α*, The Scientific World Journal, 2014, 535419 DOI: 10.1155/2014/535419.
[14] M. Et, M. Mursaleen, M. Işık, *On a class of fuzzy sets defined by Orlicz functions*, Filomat **27**(5) (2013), 789–796.
[15] M. Et, V. Karakaya, *A new difference sequence set of order α and its geometrical properties*, Abstr. Appl. Anal. 2014, Art. ID 278907, 4 pp.
[16] M. Karakaş, M. Et, V. Karakaya, *Some geometric properties of a new difference sequence space involving lacunary sequences*, Acta Math. Sci. Ser. B (Engl. Ed.) **33**(6) (2013), 1711–1720.
[17] M. A. Sarıgöl, *On difference sequence spaces*, J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math.-Phys. **10** (1987), 63–71.
[18] E. Savaş, M. Et, *On $(\Delta_\lambda^m, I)-$statistical convergence of order α*, Period. Math. Hungar. **71**(2) (2015), 135–145.
[19] R. Çolak, M. Et, *On some difference sequence sets and their topological properties*, Bull. Malays. Math. Sci. Soc. **28**(2) (2005), 125–130.

Conference Proceedings of Science and Technology, 2(3), 2019, 198–200

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

On Some Generalized Deferred Cesàro Means-II

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Mikail Et

[1] *Faculty of Science, Department of Mathematics, Firat University, Elazig, Turkey, ORCID: 0000-0001-8292-7819*

* *Corresponding Author E-mail: mikailet68@gmail.com*

Abstract: In this study, using the genealized difference operator Δ^m, we introduce some new sequence spaces and investigate some topological properties of these sequence spaces
Keywords: Difference sequence, Deferred Cesaro mean.

1 Introduction

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\|_\infty = \sup |x_k|$, where $k \in \mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers. Also by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely summable and $p-$absolutely summable sequences, respectively.

A sequence space X with a linear topology is called a $K-$space provided each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for each $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A $K-$space X is called an $FK-$space provided X is a complete linear metric space. An $FK-$space whose topology is normable is called a $BK-$space. We say that an $FK-$space X has AK (or has the AK property), if (e_k) (the sequence of unit vectors) is a Schauder bases for X.

The notion of difference sequence spaces was introduced by Kızmaz [**?**] and the notion was generalized by Et and Çolak [**?**]. Later on Et and Nuray [**?**] generalized these sequence spaces to the following sequence spaces:

Let X be any sequence space and let m be a non-negative integer. Then,

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

$\Delta^0 x = (x_k)$, $\Delta^m x = \left(\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}\right)$ and so $\Delta^m x_k = \sum\limits_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} x_{k+i}$. is a Banach space normed by

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x_k\|_\infty .$$

If $x \in X(\Delta^m)$ then there exists one and only one $y = (y_k) \in X$ such that

$$x_k = \sum_{i=1}^{k-m} (-1)^m \binom{k-i-1}{m-1} y_i = \sum_{i=1}^{k} (-1)^m \binom{k+m-i-1}{m-1} y_{i-m}, \quad y_{1-m} = y_{2-m} = \cdots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. Recently, a large amount of work has been carried out by many mathematicians regarding various generalizations of sequence spaces. For a detailed account of sequence spaces one may refer to ([2-13]).

In 1932, Agnew [**?**] introduced the concept of deferred Cesaro mean of real (or complex) valued sequences $x = (x_k)$ defined by

$$(D_{p,q}x)_n = \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} x_k, \; n = 1, 2, 3, \ldots,$$

where $p = \{p(n)\}$ and $q = \{q(n)\}$ are the sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty. \tag{1}$$

2 Topological Properties of $X(\Delta^m)$

In this section we prove some results involving the sequence spaces $C_0^d(\Delta^m), C_1^d(\Delta^m)$ and $C_\infty^d(\Delta^m)$.

Definition 1. *Let m be a fixed non-negative integer and let $\{p(n)\}$ and $\{q(n)\}$ be two sequences of non-negative integers satisfying the condition (1). We define the following sequence spaces:*

$$C_0^d(\Delta^m) = \left\{ x = (x_k) : \lim_n \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k = 0 \right\},$$
$$C_1^d(\Delta^m) = \left\{ x = (x_k) : \lim_n \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \left(\Delta^m x_k - L\right) = 0 \right\},$$
$$C_\infty^d(\Delta^m) = \left\{ x = (x_k) : \sup_n \left(\frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k \right) < \infty \right\}.$$

The above sequence spaces contain some unbounded sequences for $m \geq 1$, for example let $x = (k^m)$, then $x \in C_\infty^d(\Delta^m)$, but $x \notin \ell_\infty$.

Theorem 1. *The sequence spaces $C_0^d(\Delta^m), C_1^d(\Delta^m)$ and $C_\infty^d(\Delta^m)$ are Banach spaces normed by*

$$\|x\|_\Delta = \sum_{i=1}^m |x_i| + \sup_n \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k \right|.$$

Proof: Proof follows from Theorem **??** of Et and Nuray [**?**]. □

Theorem 2. *$X(\Delta^{m-1}) \subset X(\Delta^m)$ and the inclusion is strict for $X = C_0^d, C_1^d$ and C_∞^d.*

Proof: The inclusions part of the proof are esay. To see that the inclusions are strict, let $m = 2$ and $q(n) = n, p(n) = 0$ and consider a sequence defined by $x = \left(k^2\right)$, then $x \in C_1^d(\Delta^2)$, but $x \notin C_1^d(\Delta)$ (If $x = \left(k^2\right)$, then $\left(\Delta^2 x_k\right) = (2, 2, 2, ...)$. □

Theorem 3. *The inclusions $C_0^d(\Delta^m) \subset C_1^d(\Delta^m) \subset C_\infty^d(\Delta^m)$ are strict.*

Proof: First inclusion is esay. Second inclusion follows from the following inequality

$$\begin{aligned} \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k \right| &\leq \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k - L \right| + \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} L \right| \\ &\leq \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k - L \right| + L \end{aligned}$$

For strict the inclusion, observe that $x = (1, 0, 1, 0, ...) \in C_\infty^d(\Delta^m)$, but $x \notin C_1^d(\Delta^m)$, (If $x = (1, 0, 1, 0, ...)$, then $(\Delta^m x_k) = \left((-1)^{m+1} 2^{m+1}\right)$). □

Theorem 4. *$C_1^d(\Delta^m)$ is a closed subspace of $C_\infty^d(\Delta^m)$.*

Proof: Proof follows from Theorem **??** of Et and Nuray [**?**]. □

Theorem 5. $C_1^d(\Delta^m)$ *is a nowhere dense subset of* $C_\infty^d(\Delta^m)$.

Proof: Proof follows from the fact that $C_1^d(\Delta^m)$ is a proper and complete subspace of $C_\infty^d(\Delta^m)$. □

Theorem 6. $C_\infty^d(\Delta^m)$ *is not separable, in general.*

Proof: Suppose that $C_\infty^d(\Delta^m)$ is separable for some $m \geq 1$, for example let $m = 2$ and $q(n) = n, p(n) = 0$. In this case $C_\infty(\Delta^2)$ is separable. In Theorem **??**, Bhardwaj et al. [**?**] show that $C_\infty(\Delta^2)$ is not separable. So $C_\infty^d(\Delta^m)$ is not separable, in general. □

Theorem 7. $C_\infty^d(\Delta^m)$ *does not have Schauder basis. separable, in general.*

Proof: Proof follows from the fact that if a normed space has a Schauder basis, then it is separable. □

Theorem 8. $C_1^d(\Delta^m)$ *is separable*.

Proof: Proof follows from Theorem **??** of Et and Nuray [**?**]. □

3 Acknowledgement

This research was supported by Management Union of the Scientific Research Projects of Firat University under the Project Number: FUBAB FF.19.15. We would like to thank Firat University Scientific Research Projects Unit for their support.

4 References

[1] H. Kızmaz, *On certain sequence spaces*, Canad. Math. Bull. **24**(2) (1981), 169-176.
[2] M. Et, R. Colak, *On generalized difference sequence spaces*, Soochow J. Math. **21**(4) (1995), 377-386.
[3] M. Et, F. Nuray, Δ^m*−statistical convergence*, Indian J. Pure Appl. Math. **32**(6) (2001), 961–969.
[4] R. P. Agnew, *On deferred Cesàro means*, Ann. of Math. (2) **33**(3) (1932), 413–421.
[5] V. K. Bhardwaj, S. Gupta, R. Karan, *Köthe-Toeplitz duals and matrix transformations of Cesàro difference sequence spaces of second order*, J. Math. Anal. **5**(2) (2014), 1–11.
[6] B. Altay, F. Basar, *On the fine spectrum of the difference operator* Δ *on* c_0 *and* c, Inform. Sci. **168**(1-4) (2004), 217–224.
[7] Y. Altin, *Properties of some sets of sequences defined by a modulus function*, Acta Math. Sci. Ser. B Engl. Ed. **29**(2) (2009), 427–434.
[8] V. K. Bhardwaj, S. Gupta, *Cesàro summable difference sequence space*, J. Inequal. Appl., **2013**(315) (2013), 9.
[9] M.Candan, *Vector-valued FK-space defined by a modulus function and an infinite matrix*: Thai J. of Math **12**(1) (2014), 155-165.
[10] M. Et, *On some generalized Cesàro difference sequence spaces*, İstanbul Üniv. Fen Fak. Mat. Derg. **55/56** (1996/97), 221–229.
[11] M. Et, M. Mursaleen and M. Işık, *On a class of fuzzy sets defined by Orlicz functions*, Filomat **27**(5) (2013), 789–796.
[12] G.Kılınc, M. Candan, *Some Generalized Fibonacci Difference Spaces defined by a Sequence of Modulus Functions*, Facta Universitatis, Series: Mathematics and Informatics, **32**(1) (2017), 095-116.
[13] M. A. Sarıgöl, *On difference sequence spaces*, J. Karadeniz Tech. Univ., Fac. Arts Sci., Ser. Math.-Phys **10**, 63-71.

Conference Proceedings of Science and Technology, 2(3), 2019, 201–204

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

Solutions of Singular Differential Equations by means of Discrete Fractional Analysis

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Resat Yilmazer[1,*] *Gonul Oztas* [1]

[1] *Department of Mathematics, University of Firat, Elazig, Turkey, ORCID:0000-0002-5059-3882*

* *Corresponding Author E-mail: ryilmazer@firat.edu.tr*

Abstract: Recently, many researchers demonstrated the usefulness of fractional calculus in the derivation of particular solutions of linear ordinary and partial differential equation of the second order. In this study, we acquire new discrete fractional solutions of singular differential equations (homogeneous and nonhomogeneous) by using discrete fractional nabla operator $\nabla^{\upsilon}(0<\upsilon<1)$.

Keywords: Discrete fractional analysis, Nabla operator, Singular differential equations.

1 Introduction

The remarkably widely investigated subject of fractional and discrete fractional calculus has gained importance and popularity during the past three decades or so, due chiefly to its demonstrated applications in numerous seemingly diverse fields of science and engineering [1]-[4]. The analogous theory for discrete fractional analysis was initiated and properties of the theory of fractional differences and sums were established. Recently, many articles related to discrete fractional analysis have been published [5]-[9]. The fractional nabla operator have been applied to various singular ordinary and partial differential equations such as the second-order linear ordinary differential equation of hypergeometric type [10], the Bessel equation [11], the Hermite equation [12], the non- fuchsian differential equation [13], the hydrogen atom equation [14].

The aim of this article is to obtain new dfs of the singular differential equation by means of fractional calculus operator.

2 Preliminary and properties

Here we only give a very short introduction to the basic definitions in discrete fractional calculus. For more on the subject we refer the reader to [5, 13].

Let $\zeta\in\mathbb{R}^{+}$, $n\in\mathbb{Z}$, such that $n-1\leq\zeta<n$. The $\zeta^{th}-$ order fractional sum of F is defined as

$$\nabla_c^{-\zeta}F(t)=\frac{1}{\Gamma(\zeta)}\sum_{\tau=c}^{t}(t-\rho(\tau))^{\overline{\zeta-1}}F(\tau), \tag{1}$$

where $t\in\mathbb{N}_\alpha=\{\alpha,\alpha+1,\alpha+2,...\}$, $\alpha\in\mathbb{R}$, $\rho(t)=t-1$ is the backward jump operator.

The rising factorial power and rising function is given by

$$t^{\overline{n}}=t(t+1)(t+2)...(t+n-1),\ n\in\mathbb{N},\ t^{\overline{0}}=1,$$

$$t^{\overline{\zeta}}=\frac{\Gamma(t+\zeta)}{\Gamma(t)},\ \zeta\in\mathbb{R},\ t\in\mathbb{R}\backslash\{...,-2,-1,0\},\ 0^{\overline{\zeta}}=0. \tag{2}$$

Note that

$$\nabla\left(t^{\overline{\zeta}}\right)=\zeta t^{\overline{\zeta-1}}, \tag{3}$$

where $\nabla\phi(t)=\phi(t)-\phi(\sigma(t))=\phi(t)-\phi(t-1)$.

The $\zeta^{th}-$ order fractional difference of F is defined by

$$\nabla_c^{\zeta}F(t)=\nabla^{n}\left[\nabla_c^{\zeta-(n-\zeta)}F(t)\right]$$

$$=\nabla^{n}\left[\frac{1}{\Gamma(n-\zeta)}\sum_{\tau=c}^{t}(t-\sigma(\tau))^{\overline{n-\zeta-1}}F(\tau)\right], \tag{4}$$

where F is defined on $\mathbb{N}_\alpha$.
Lemma 1. (Linearity). Let F and G be analytic and single-valued functions. Then

$$[c_1F(t)+c_2G(t)]_\zeta = c_1F_\zeta(t)+c_2G_\zeta(t), \tag{5}$$

where c_1 and c_2 are constants, $\zeta\in\mathbb{R};\ t\in\mathbb{C}$.
Lemma 2. (Index law). Let ϕ be an analytic and single-valued function. The following equality holds

$$\left(F_\zeta(t)\right)_\eta = F_{\zeta+\eta}(t) = \left(F_\eta(t)\right)_\zeta \ \left(F_\zeta(t)\neq 0;\ F_\eta(t)\neq 0;\ \zeta,\eta\in\mathbb{R};\ t\in\mathbb{C}\right). \tag{6}$$

Lemma 3. (Leibniz Rule). Suppose that F and G are analytic and single-valued functions. Then

$$\nabla_0^\zeta(FG)(t) = \sum_{n=0}^{t}\binom{\zeta}{n}\left[\nabla_0^{\zeta-n}F(t-n)\right]\left[\nabla^n G(t)\right],\ \zeta\in\mathbb{R};\ t\in\mathbb{C}, \tag{7}$$

where $\nabla^n G(t)=G_n(t)$ is the ordinary derivative of G of order $n\in\mathbb{N}_0$.
Definition 4. μ shift operator is given by

$$\mu^n F(t) = F(t-n) \tag{8}$$

where $n\in\mathbb{N}$.

3 Main results

Theorem 1. Let $F\in\{F: 0\neq|F_\upsilon|<\infty;\ \upsilon\in\mathrm{R}\}$. Then the following homogeneous ordinary differential equation:

$$s(1-s)\ F_2+[(\alpha-2\gamma)\ s+\gamma+\sigma]\ F_1+\gamma(\alpha-\gamma+1)\ F=0,\quad (s\in\mathrm{C}\backslash\{0,1\}), \tag{9}$$

has particular solutions of the forms:

$$F = k\left\{s^{-(\upsilon\tau+\gamma+\sigma)}(1-s)^{-(\upsilon\tau+\gamma-\alpha-\sigma)}\right\}_{-(1+\upsilon)}, \tag{10}$$

and

$$F = ks^{1-(\gamma+\sigma)}\left\{s^{-(\upsilon\tau-\gamma-\sigma+2)}(1-s)^{-(\upsilon\tau+\gamma-\alpha-\sigma)}\right\}_{-(1+\upsilon)} \tag{11}$$

where $F_n=d^nF/ds^n\ (n=0,1,2)$, $F_0=F=F(s)$, $\alpha\neq 0$, γ, σ are given constants, k is an arbitrary constant and τ is a shift operator [15].
Proof. ***(i)*** When we operate ∇^υ to the both sides of (9), we readily obtain;

$$\nabla^\upsilon[F_2 s(1-s)]+\nabla^\upsilon\{F_1[(\alpha-2\gamma)\ s+\gamma+\sigma]\}+\nabla^\upsilon[F\gamma(\alpha-\gamma+1)]=0. \tag{12}$$

Using $(5)-(7)$ we have

$$\nabla^\upsilon[F_2 s(1-s)] = F_{2+\upsilon}s(1-s)+F_{1+\upsilon}\upsilon\tau(1-2s)-F_\upsilon\upsilon(\upsilon-1)\ \tau^2 \tag{13}$$

and

$$\nabla^\upsilon\{F_1[(\alpha-2\gamma)\ s+\gamma+\sigma]\} = F_{1+\upsilon}[(\alpha-2\gamma)\ s+\gamma+\sigma]+F_\upsilon\upsilon\tau(\alpha-2\gamma) \tag{14}$$

where τ is a shift operatÃűr. By substituting (13), (14) into the (12), we obtain

$$\begin{aligned}&F_{2+\upsilon}s(1-s)+F_{1+\upsilon}[\upsilon\tau(1-2s)+(\alpha-2\gamma)\ s+\gamma+\sigma]\\ &+F_\upsilon\left[\upsilon(1-\upsilon)\ \tau^2+\upsilon\tau(\alpha-2\gamma)+\gamma(\alpha-\gamma+1)\right]=0.\end{aligned} \tag{15}$$

Choose υ such that

$$\upsilon(1-\upsilon)\ \tau^2+\upsilon\tau(\alpha-2\gamma)+\gamma(\alpha-\gamma+1)=0,$$

$$\upsilon = \left[(\tau+\alpha-2\gamma)\pm\sqrt{(\tau+\alpha-2\gamma)^2+4\gamma(\alpha-\gamma+1)}\right]/2\tau. \tag{16}$$

From Eq. (16), one can easily see that

$$\left[(\tau+\alpha-2\gamma)^2\geq 4\gamma(-\alpha+\gamma-1)\right],$$

we have then

$$F_{2+\upsilon}s(1-s)+F_{1+\upsilon}[\upsilon\tau(1-2s)+(\alpha-2\gamma)\ s+\gamma+\sigma]=0, \tag{17}$$

from (15) and (16).
Next, writing:

$$F_{1+\upsilon} = f(s) \quad \left[F = f_{-(1+\upsilon)}\right], \tag{18}$$

we have

$$f_1 + f\left[\frac{\upsilon\tau(1-2s) + (\alpha - 2\gamma)\ s + \gamma + \sigma}{s(1-s)}\right] = 0, \tag{19}$$

from eqs. (17) and (18). A particular solution of linear ordinary differential equation (19):

$$f = ks^{-(\upsilon\tau+\gamma+\sigma)}(1-s)^{-(\upsilon\tau+\gamma-\alpha-\sigma)}. \tag{20}$$

Therefore, we obtain (10) from (18) and (20).
(ii) Set

$$F = s^{\eta}\Phi, \quad \Phi = \Phi(s). \tag{21}$$

The first and second derivatives of (21) are acquired as follows:

$$F_1 = \eta s^{\eta-1}\Phi + s^{\eta}\Phi_1 \tag{22}$$

and

$$F_2 = \eta(\eta-1)\ s^{\eta-2}\Phi + 2\eta s^{\eta-1}\Phi_1 + s^{\eta}\Phi_2. \tag{23}$$

Substitute $(21) - (23)$ into (9), we obtain

$$s(1-s)\ \Phi_2 + [(1-s)\ 2\eta + (\alpha-2\gamma)\ s + \gamma + \sigma]\ \Phi_1$$

$$+\left[\left(\left(\eta^2-\eta\right) + (\gamma+\sigma)\ \eta\right)\ s^{-1} - \left(\eta^2-\eta\right) + (\alpha-2\gamma)\ \eta + \gamma(\alpha-\gamma+1)\right]\ \Phi{=}0. \tag{24}$$

Choose η such that

$$\left(\eta^2-\eta\right) + (\gamma+\sigma)\ \eta = 0,$$

that is

$$\eta = 0, \quad \eta = 1 - (\gamma+\sigma).$$

In the case $\eta = 0$, we have the same results as i.

Let $\eta = 1-(\gamma+\sigma)$. From (21) and (24), we have

$$F = s^{1-(\gamma+\sigma)}\Phi \tag{25}$$

and

$$s(1-s)\ \Phi_2 + [(2\sigma+\alpha-2)\ s - (\gamma+\sigma-2)]\ \Phi_1 + [(1-\sigma)\ (\sigma+\alpha)]\ \Phi = 0 \tag{26}$$

respectively.
Applying the discrete operator ∇^{υ} to both sides of (26), we obtain

$$\Phi_{2+\upsilon}s(1-s) + \Phi_{1+\upsilon}\left[\upsilon\tau(1-2s) + (2\sigma+\alpha-2)\ s - (\gamma+\sigma-2)\right]$$

$$+\Phi_{\upsilon}\left[\upsilon(1-\upsilon)\ \tau^2 + \upsilon\tau(2\sigma+\alpha-2) + (1-\sigma)\ (\alpha+\sigma)\right] {=}0. \tag{27}$$

Choose υ such that

$$\upsilon(1-\upsilon)\ \tau^2 + \upsilon\tau(2\sigma+\alpha-2) + (1-\sigma)\ (\alpha+\sigma)\ {=}0,$$

$$\upsilon = \left[(\tau+2\sigma+\alpha-2) \pm \sqrt{(\tau+2\sigma+\alpha-2)^2 - 4(\sigma-1)\ (\alpha+\sigma)}\right]/2\tau. \tag{28}$$

From Eq. (28), one can get

$$\left[(\tau+2\sigma+\alpha-2)^2 \geq 4(\sigma-1)\ (\alpha+\sigma)\right],$$

then we have

$$\Phi_{2+\upsilon}s(1-s) + \Phi_{1+\upsilon}\left[\upsilon\tau(1-2s) + (2\sigma+\alpha-2)\ s - (\gamma+\sigma-2)\right] = 0, \tag{29}$$

from (27) and (28).

Next, by writing

$$\Phi_{1+\upsilon} = g(s), \quad \left[\Phi = g_{-(1+\upsilon)}\right], \tag{30}$$

we have

$$g_1 + g\left[\frac{\upsilon\tau(1-2s) + (2\sigma+\alpha-2)\ s - (\gamma+\sigma-2)}{s(1-s)}\right] = 0, \tag{31}$$

from (29) and (30) . A particular solution to this linear differential equation is given by

$$g = ks^{-(\upsilon\tau-\gamma-\sigma+2)}(1-s)^{-(\upsilon\tau+\gamma-\alpha-\sigma)}. \tag{32}$$

Thus we obtain the solution (11) from (25), (30) and (32).

4 Conclusion

In this article, we applied the nabla operator of discrete fractional analysis to the second order linear differential equations. We obtained the discrete fractional solutions of these equations via this new operator method.

Acknowledgement

This study was supported by Firat University Scientific Research Projects with unit FUBAP-FF.19.10. We would like to thank Firat University Scientific Research Projects Unit for their support.

5 References

[1] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, 1993.
[2] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.
[3] C. Goodrich, A. C. Peterson, *Discrete Fractional Calculus*, Berlin: Springer, 2015.
[4] R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[5] H. L., Gray, N., Zhang, *On a New Definition of the Fractional Difference, Mathematics of Computation*, **50** (182) (1988), 513-529.
[6] F. M. Atici, P.W. Eloe, *Discrete fractional calculus with the nabla operator*, Electronic Journal of Qualitative Theory of Differential Equations, Spec. Ed I, **3** (2009), 1-12.
[7] N. Acar, F. M. Atici, *Exponential functions of discrete fractional calculus*, Appl. Anal. Discrete Math. **7** (2013), 343-353.
[8] G. A. Anastassiou, *Right nabla discrete fractional calculus*, Int. J. Difference Equations, **6** (2011), 91-104.
[9] J. J. Mohan, *Analysis of nonlinear fractional nabla difference equations*, Int. J. Analysis Applications **7** (2015), 79-95.
[10] R. Yilmazer, et al., *Particular Solutions of the Confluent Hypergeometric Differential Equation by Using the Nabla Fractional Calculus Operator*, Entropy, **18** (49) (2016), 1-6.
[11] R. Yilmazer, O. Ozturk, *On Nabla Discrete Fractional Calculus Operator for a Modified Bessel Equation*, Therm. Sci., **22** (2018), S203-S209.
[12] R. Yilmazer, *Discrete fractional solution of a Hermite Equation*, Journal of Inequalities and Special Functions, **10** (1) (2019), 53-59.
[13] R. Yilmazer, *Discrete fractional solution of a non-homogeneous non-fuchsian differential equations*, Therm. Sci., **23** (2019), 121-127.
[14] R. Yilmazer, *$N-$ fractional calculus operator N^{μ} method to a modified hydrogen atom equation*, Math. Commun., **15** (2010), 489-501.
[15] W. G. Kelley, A. C. Peterson, *Difference Equations: An Introduction with Applications*,Academic Press, San Diego, 2001.

Conference Proceedings of Science and Technology, 2(3), 2019, 205–208

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Geometric Interpretation of Curvature Circles in Minkowski Plane

Kemal Eren[1,*] *Soley Ersoy*[2]

[1] *Fatsa Science High School, Ordu, Turkey, ORCID:0000-0001-5273-7897*
[2] *Department of Mathematics, Faculty of Sciences and Arts, Sakarya University, Sakarya, Turkey, ORCID:0000-0002-7183-7081*
** Corresponding Author E-mail: kemal.eren1@ogr.sakarya.edu.tr*

Abstract: In this study, we investigate the geometric interpretation of the curvature circles of motion at the initial position in Minkowski plane. We consider the equations of the circling-point and centering-point curves of one-parameter motion in Minkowski plane and then determine the positions of these curves relative to each other.

Keywords: Centering-point curve, Circling-point curve, Minkowski plane.

1 Introduction

The concept of instantaneous invariants was first given by Bottema to determine the geometric properties of a moving rigid body at a given moment. Therefore, the geometric and kinematic properties of planar motions in Euclidean space are investigated according to these invariants [1] and this method has also guided many studies in the field of kinematics [2–6]. Later, the instantaneous invariants were called B-invariants (Bottema-invariants) by Veldkamp [7]. Besides, Veldkamp found special geometrical ground curves such as the inflection curve, the circling-point curve and the centering-point curve with the help of B-invariants, as well as the intersection points of these curves, Ball and Burmester points [8, 9]. The special geometrical ground curves in Minkowski (Lorentz) plane and their intersection points were analyzed by recent studies [10, 11], however, the positions of these curves relative to each other have not been studied yet. Therefore, it is aimed to present the geometric interpretation of curvature circles relative to each other throughout one-parameter planar motion in Minkowski plane based on the above-mentioned studies.

2 Preliminaries

The Minkowski plane L is the plane R^2 endowed with the Lorentzian scalar product given by $\langle x,y\rangle = x_1y_1 - x_2y_2$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The norm of a vector is defined by $\|x\| = \sqrt{|\langle x,x\rangle|}$. An arbitrary vector $x \in L$ is called timelike if $\langle x,x\rangle < 0$, spacelike if $\langle x,x\rangle > 0$ or $x = 0$, lightlike if $\langle x,x\rangle = 0$ whereby $x \neq 0$. Two vectors x and y are said to be orthogonal if $\langle x,y\rangle = 0$. Let L_m be a Minkowski plane in continuous motion relative to a fixed Minkowski plane L_f. Then one-parameter planar motion L_m with respect to L_f is represented by

$$\begin{aligned} X &= x\cosh\theta + y\sinh\theta + a \\ Y &= x\sinh\theta + y\cosh\theta + b \end{aligned} \tag{1}$$

with respect to Cartesian frames of reference xoy and XOY in L_m and L_f, respectively. Here $a,\ b$ and θ are functions depending on time t. The position corresponding to $\varphi = 0$ of L_m is called initial position. The values for the initial position of the nth $(n = 0, 1, 2, \ldots)$ derivative of a function f of φ with respect to φ is denoted by f_n.

The Minkowski plane L_m is chosen to rotate with a constant angular velocity relative to the fixed Minkowski plane L_f, that is, $\theta = t$. The canonical relative system of motion is constructed by

$$a_0 = b_0 = a_1 = b_1 = a_2 = 0 \tag{2}$$

and the instantaneous invariants a_n and b_n characterize completely the infinitesimal properties of motion of Minkowski planes up the n-th order as

$$\begin{aligned} &X = x, \quad X' = y, \quad X'' = x, \quad X''' = y + a_3, \\ &Y = y, \quad Y' = x, \quad Y'' = y + b_2, \quad Y''' = x + b_3 \end{aligned} \tag{3}$$

for $t = 0$ [10, 11].

3 The curvature circles in Minkowski plane

In this section, let's first recall the definitions of curvature circles in Minkowski plane.

Definition 1. *The locus of the points of moving Minkowski plane L_m, whose curvature of the trajectory is constant at initial position, is called circling-point curve in Minkowski plane and denoted by cp.*

The equation of the circling-point curve cp in Minkowski plane is

$$\left(x^2 - y^2\right)(a_3x - b_3y) + 3x\left(x^2 - y^2 + y\right) = 0, \quad (x, y) \neq (0, 0) \tag{4}$$

where $(x, y) \neq (0, 0)$ or $x \neq \mp y$, [10, 11].

Definition 2. *The locus of the curvature centers of the points of moving Minkowski plane L_m is called centering-point curve in Minkowski plane and denoted by $\tilde{cp}$.*

The equation of the centering-point curve $\tilde{cp}$ in Minkowski plane is

$$\left(x^2 - y^2\right)(a_3x - b_3y) + 3xy = 0 \tag{5}$$

where $(x, y) \neq (0, 0)$ or $x \neq \mp y$, [10, 11].

Now, let us examine the positions of circling-point and centering-point curves relative to each other in Minkowski plane. The curve cp given by equation (4) and the curve $\tilde{cp}$ given by equation (5) can be arranged as

$$\left(x^2 - y^2\right)\left(\tfrac{(a_3+3)}{3}x - \tfrac{b_3}{3}y\right) + xy = 0$$

and

$$\left(x^2 - y^2\right)\left(\tfrac{a_3}{3}x - \tfrac{b_3}{3}y\right) + xy = 0,$$

respectively.

On the other hand, a third-order cubic curve γ in Minkowski plane can be given by

$$(\alpha x + \beta y)\left(x^2 - y^2\right) + xy = 0. \tag{6}$$

Let γ be an irreducible curve, this means that $\alpha\beta \neq 0$.

If $\alpha = \frac{a_3+3}{3}$ and $\beta = -\frac{b_3}{3}$ are satisfied, then the curve given by the equation (6) corresponds to the circling-point curve cp according to the canonical system in Minkowski plane.

Moreover, if there are the relations $\alpha = \frac{a_3}{3}$ and $\beta = -\frac{b_3}{3}$, then the curve given by the equation (6) corresponds to the centering-point curve $\tilde{cp}$ according to the canonical system in Minkowski plane.

Theorem 1. *The parametric equation of the curve γ is given by*

$$x = \frac{u}{(u^2 - 1)(\alpha + \beta u)}, \quad y = \frac{u^2}{(u^2 - 1)(\alpha + \beta u)} \tag{7}$$

where $u \neq \pm 1$.

Proof: If we substitute $y = ux$, such that $u \neq \pm 1$, in the equation (6), then we get $x^3(\alpha + \beta u)\left(1 - u^2\right) + ux^2 = 0$. Afterwards, some direct calculations completes the proof. □

Specifically, the parametric value $\frac{-\alpha}{\beta}$ corresponds to the infinity point of the curve γ. We can examine the reducible states of this curve in the following corollaries:

Corollary 1. *In Minkowski plane, the parametric equation of the curvature circle Γ_0, which is tangent to the curve γ along the axis y, is represented by*

$$x = \frac{1}{\beta(u^2 - 1)}, \quad y = \frac{u}{\beta(u^2 - 1)}. \tag{8}$$

Proof: If $\alpha = 0$ is taken in the equation (7) then the proof is obvious. □

Corollary 2. *In Minkowski plane, the parametric equation of the curvature circle Γ_1, which is tangent to the curve γ along the axis x, is given by*

$$x = \frac{u}{\alpha(u^2 - 1)}, \quad y = \frac{u^2}{\alpha(u^2 - 1)}. \tag{9}$$

Proof: Taking $\beta = 0$ in the equation (7) completes the proof. □

From the equation (8), the Cartesian equation of the curvature circle Γ_0 in Minkowski plane is represented as

$$\beta\left(x^2-y^2\right)+x=0. \tag{10}$$

Similarly, by taking the equation (9) the Cartesian equation of the curvature circle Γ_1 in Minkowski plane is given by

$$\alpha\left(x^2-y^2\right)+y=0. \tag{11}$$

Let the points A_i $(i=1,2,3)$ be on the curve γ. In that case, these points are given as

$$A_i=\left(\frac{u_i}{({u_i}^2-1)(\alpha+\beta u_i)},\quad \frac{{u_i}^2}{({u_i}^2-1)(\alpha+\beta u_i)}\right),\quad (i=1,2,3).$$

Theorem 2. *The points A_i $(i=1,2,3)$ with parametric value u_i $(i=1,2,3)$ are on the same line does not pass through the origin if and only if*

$$u_3u_2u_1=\frac{\alpha}{\beta}. \tag{12}$$

Proof: The points A_i are on the same line that does not pass through the origin if and only if the slopes of the lines A_1A_2 and A_2A_3 are equal the each other. Thus, there is the relationship

$$\frac{\frac{-{u_3}^2}{(1-{u_3}^2)(\alpha+\beta u_3)}+\frac{{u_2}^2}{(1-{u_2}^2)(\alpha+\beta u_2)}}{\frac{-u_3}{(1-{u_3}^2)(\alpha+\beta u_3)}+\frac{u_2}{(1-{u_2}^2)(\alpha+\beta u_2)}}=\frac{\frac{-{u_2}^2}{(1-{u_2}^2)(\alpha+\beta u_2)}+\frac{{u_1}^2}{(1-{u_1}^2)(\alpha+\beta u_1)}}{\frac{-u_2}{(1-{u_2}^2)(\alpha+\beta u_2)}+\frac{u_1}{(1-{u_1}^2)(\alpha+\beta u_1)}}.$$

In this manner, we get

$$\beta^2u_1{u_2}^2u_3+\beta\alpha\left(u_2\left(u_1u_3-1\right)\right)-\alpha^2.$$

If this equation is factored, we find

$$(\beta u_1u_2u_3-\alpha)=0 \text{ or } (\beta u_2+\alpha)=0.$$

So, we can write

$$u_1u_2u_3=\tfrac{\alpha}{\beta} \text{ or } u_2=\tfrac{-\alpha}{\beta}.$$

Here $u_2\neq\frac{-\alpha}{\beta}$ must be satisfied since the parametric value $\frac{-\alpha}{\beta}$ corresponds to the infinity point of the curve γ. □

If one of these three points is at the infinity, i.e., $u_3^*=\frac{-\alpha}{\beta}$, this means that this line is parallel to the asymptotes of the curve γ and cuts the curve at two points with the parameters u_1^* and u_2^*. Then the correlation between the parameters u_1^* and u_2^* is given by

$$u_1^*u_2^*=-1. \tag{13}$$

If the points A_1 and A_2 of the curve γ are represented with respect to the parameters u_1 and u_2, then the equation of the line A_1A_2 is found as

$$\left(\alpha\left(u_2+u_1\right)+\beta u_1u_2\left(u_1u_2+1\right)\right)x-\left(\alpha\left(u_1u_2+1\right)+\beta u_1u_2\left(u_2+u_1\right)\right)y+u_1u_2=0. \tag{14}$$

After the formation this equation we have

$$\alpha\left(\left(u_1+u_2\right)x-\left(u_1u_2+1\right)y\right)-\beta u_1u_2\left(-\left(u_1u_2+1\right)x+\left(u_2+u_1\right)y-\frac{1}{\beta}\right)=0. \tag{15}$$

If we denote the slopes of the lines d_1 and d_2 given by the equations

$$\left(u_1+u_2\right)x-\left(u_1u_2+1\right)y=0 \tag{16}$$

and

$$-\beta\left(u_1u_2+1\right)x+\beta\left(u_2+u_1\right)y-1=0 \tag{17}$$

by m_{d_1} and m_{d_2}, respectively, we see that these lines are perpendicular in Minkowski plane since there is the relationship $m_{d_1}m_{d_2}=1$. Hence, we can interpret that the line given by the equation (14) passes through the intersection of the lines d_1 and d_2 which are perpendicular to each other in the Minkowski plane.

Also, considering the equation of distance from a point to a line in the Minkowski plane we find the equation of the distance from origin to the line A_1A_2 as

$$d=\frac{|u_1u_2|}{\sqrt{\left|\left(-\alpha^2+\beta^2u_1^2u_2^2\right)\left(u_1^2-1\right)\left(u_2^2-1\right)\right|}} \tag{18}$$

where $u_i\neq\pm1$, $i=1,2$.

Let A_3 be a point with the parameter $-u_1$ on the curve γ. From the equation (18), the lines A_2A_1 and A_2A_3 have equal distance from origin, that is, the lines A_2A_1 and A_2A_3 are symmetrical according to the point A_2.

Now let's give the formation of the circles Γ_0 and Γ_1. Since the geometric location of the curvature centers of the curve cp is the centering-point curve $\tilde{cp}$, the curvature center of a point with the parameter u of the curve cp coincides with the same parameter point of the curve $\tilde{cp}$, [11]. Let A_1 and A_2 be two points on the curve cp. Also, let α_1 and α_2 be the centers of curvature of these points. If the points A_1 and A_2 are given by the parameters u_1 and u_2 , respectively, the equation of line A_1A_2 is found by writing $\alpha = \frac{a_3+3}{3}$, $\beta = -\frac{b_3}{3}$ in the equation (14) and the equation of line $\alpha_1\alpha_2$ is found by writing $\alpha = \frac{a_3}{3}$, $\beta = -\frac{b_3}{3}$ in the equation (14).

Thus, we get the equations of A_1A_2 and $\alpha_1\alpha_2$ lines as

$$((3 + a_3)(u_1 + u_2) - b_3u_1u_2(1 + u_1u_2))x - ((3 + a_3)(1 + u_1u_2) - b_3u_1u_2(u_1 + u_2))y - 3u_1u_2 = 0$$

and

$$(a_3(u_1 + u_2) - b_3u_1u_2(1 + u_1u_2))x - (a_3(1 + u_1u_2) - b_3u_1u_2(u_1 + u_2))y - 3u_1u_2 = 0,$$

respectively. Here, the lines A_1A_2 and $\alpha_1\alpha_2$ pass through the intersection of the lines given by the equations (16) and (17), which are perpendicular to each other in the Minkowski plane. Here, the equation (16) indicates a line and this line passes through the pole point P and the intersection point Q of the lines $\alpha_1\alpha_2$ and A_1A_2. The equation (17) refers to the equation of the line perpendicular to the line PQ passing through the point Q.

In case of $\alpha = 0$, by substituting the parameter equation (18) into the equation (17), for Γ_0 we get

$$u^2 - (u_2 + u_1)u + u_1u_2 = 0. \tag{19}$$

Corollary 3. *u_1 and u_2 (the roots of the equation (19)) give the parametric expression of the intersection points of circle Γ_0 with the line given by the equation (17).*

In addition, these points are on the PA_1 and PA_2 lines. Similarly, the above statements can be investigated for the curvature circle Γ_1 in Minkowski plane. For this, let's first examine the line passing through the pole point P perpendicular to the line PQ. This line is given by the following equation taking into consideration the equation (16) such that the product of the slopes of these lines is 1 and these lines pass from pole P:

$$(u_1 + u_2)y - (u_1u_2 + 1)x = 0.$$

If the above equation and (14) are considered together, the intersection point (is denoted by R) of this line with line A_1A_2 is on the line below

$$\alpha((u_1 + u_2)x - (u_1u_2 + 1)y) + u_1u_2 = 0. \tag{20}$$

So the line passing through the point R is parallel to the line PQ. By substituting the parameter equation of circle Γ_1 into the equation (20), we get

$$u^2 - (u_2 + u_1)u + u_1u_2 = 0. \tag{21}$$

The equation (21) is the previously obtained equation (19).

Corollary 4. *u_1 and u_2 (the roots of the equation (21)) give the parametric expression of the intersection point of the circle Γ_1 and the line given by equation (20).*

4 References

[1] O. Bottema, *On instantaneous invariants*, Proceedings of the International Conference for Teachers of Mechanisms, New Haven (CT): Yale University, 1961, 159–164.

[2] O. Bottema, *On the determination of Burmester points for five distinct positions of a moving plane; and other topics*, Advanced Science Seminar on Mechanisms, Yale University, July 6-August 3, 1963.

[3] O. Bottema, B. Roth, *Theoretical Kinematics*, New York (NY), Dover, 1990.

[4] B. Roth, *On the advantages of instantaneous invariants and geometric kinematics*, Mech. Mach. Theory, **89** (2015), 5–13.

[5] F. Freudenstein, *Higher path-curvature analysis in plane kinematics*, ASME J. Eng. Ind., **87** (1965), 184–190.

[6] F. Freudenstein, G. N. Sandor, *On the Burmester points of a plane*, ASME J. Appl. Mech., **28** (1961), 41–49.

[7] G. R. Veldkamp, *Curvature theory in plane kinematics* [Doctoral dissertation], Groningen: T.H. Delft, 1963.

[8] G. R. Veldkamp, *Some remarks on higher curvature theory*, J. Manuf. Sci. Eng., **89** (1967), 84–86.

[9] G. R. Veldkamp, *Canonical systems and instantaneous invariants in spatial kinematics*, J. Mech., **2** (1967) 329–388.

[10] K. Eren, S. Ersoy, *Circling-point curve in Minkowski plane*, Conference Proceedings of Science and Technology, **1**(1), (2018), 1–6.

[11] K. Eren, S. Ersoy, *A comparison of original and inverse motion in Minkowski plane*, Appl. Appl. Math., Special Issue No.**5** (2019), 56–67.

Conference Proceedings of Science and Technology, 2(3), 2019, 209–211

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

The Measurement of Success Distribution with Gini Coefficient

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Şüheda Güray[1,*]

[1] *Baskent University, Ankara, Turkey, ORCID: 0000-0002-9562-1461*
* *Corresponding Author E-mail: sguray@baskent.edu.tr*

Abstract: The aim of this study is to calculate and examine the distribution of the academic success of the students of the Faculty of Education academic years between 2014-2019 the courses of statistics and probability with lorenz curve and Gini coefficient. In this regard, Tomul, E [**?**] in Educational Inequality in Turkey: Gini to Evaluate According to the index, Erdem, E., Çoban, S. [**?**] 'provinces in Turkey in Measurement Based Education Inequality and Economic Development Relationship with Difference: Education Gini Explained with coefficients.

Keywords: Academic achievement, Gini coefficient, Lorenz diagram.

1 The importance of the study

The Gini coefficient, developed by the Italian Statistician Corrado Gini (1912), is also used to determine the inequality in economic literature [**?**] because it shows simplicity and distribution with a single coefficient [**?**] and the Gini Coefficient, which is a tool used to measure inequality, in different disciplines including health and education. [**?**].

Gini coefficient of 0 means absolute equality and a value of 1 means absolute inequality. Therefore, decreasing and increasing the coefficient over time indicates the decrease and increase of inequality. In this context; What is the success inequality of Gini Coefficient and how it is distributed according to the lessons and years?

The main questions that the study seeks to answer are: What does the Gini Coefficient Achievement Distribution of Academic Achievement of Elementary Mathematics Teacher Statistics Probability course mean between the academic years of 2014-2019?

The empirical data used in the study may vary in academic terms. The sample of the study was; the academic years between 2014-2019 consists of the number of students. The number of samples between 2014-2019 is the academic achievement data of 112 students. The sample distribution by year is 2014; 36 students, 2015; 16, students, 2016; 7 students, 2017; 6 students, 2018; 28 students, 2019; 19 students

Classes	2014	2015	2016	2017	2018	2019
Statistics and Probability	36	16	7	6	28	19

The Lorenz curve examines the relationship between a certain cumulative share of national income and the cumulative share of those who obtain it. The Lorenz curve is conceptually similar to the percentage slicing method; it relates the cumulative share of income to the cumulative share of individuals, rather than simply determining their share of income. The Lorenz curve is a graphical form that shows how much the percentage income groups receive from the income distribution [**?**]. However; the usefulness of the Lorenz curve helps us to present the inequality in income distribution by a single number, without needing to tell us how much the percentage of individual groups receive.

Gini Coefficient is a non-negative number less than 1. By calculating the area between the Lorenz curve and the 45-degree line giving full equality, a numerical value ranging from 0 to 1, namely the "Gini Coefficient", is found. Where the income distribution is most fair, A = 0. The closer the Gini Coefficient is to 0, the more fair the income distribution is. Family structure of the society, population structure, educational level, tax situation, the structure of the financial sector or industry and development indicators are some factors that may affect the income distribution in a country. In general, the Gini Coefficient, i.e. the income distribution, is interpreted as sufficient after 0.40 and worse after 0.50 [**?**].

In this study, Lorenz curve and the Gini coefficient previously used in an unused area, in the area of measurement and the evaluation of the final stage of evaluation. Between the academic years of 2014 and 2019, the Faculty of Education Mathematics Education in Primary Education teacher candidates Statistics Probability courses of academic achievement was evaluated as data notes.

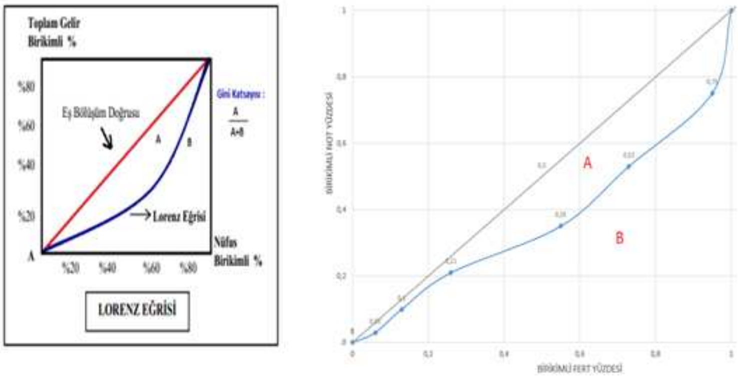

Fig. 1: Lorentz calculation chart of income and academic achievement

The Gini Coefficient Calculation of the Student Success of the year 2014 in Excel

Success Points	fi(student Frequency)	Number of Cumulative Students	percentage of cumulative students	Si(average grade)	cumulative average grade	cumulative grade average percentage	A
0	0	0	0	0	0	0	0
4≤ <17	2	2	0,06	10,5	10,5	0,03	0,000841751
17≤ <30	2	4	0,11	23,5	34	0,10	0,00356742
30≤ <43	4	8	0,22	36,5	70,5	0,203463203	0,01675485
43≤ <56	11	19	0,53	49,5	120	0,346320346	0,083994709
56≤ <69	7	26	0,72	62,5	182,5	0,526695527	0,084876543
69≤ <82	7	33	0,92	75,5	258	0,744588745	0,123597082
82≤ <95	3	36	1,00	88,5	346,5	1	0,072691198
							0,386323553

B	0,386323553	
A	0,113676447	GİNİ=A/(A+B)=**0,22**7352894
A+B	0,5	

In the results of the study, the academic achievement obtained with the Gini coefficient approach of the Elementary Mathematics Teacher Statistics Probability courses were distributed in the most fair year by year 2014 academic year, and in 2015 it moved away from the fair distribution (gini coefficient; 0.45). , 19 and 0.15 academic achievement (Gini Coefficient in general, i.e. the income distribution, up to 0.40 sufficient, 0.50 are interpreted as bad after we see).

The Geogbra Calculation of 2014

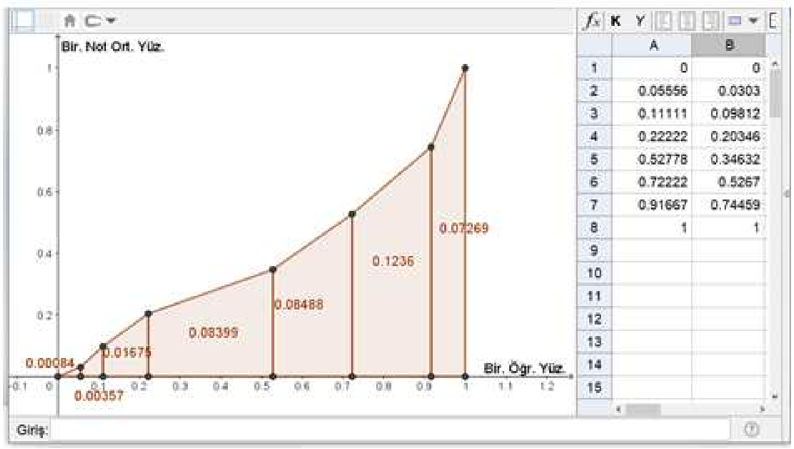

Fig. 2: Trapezoidal areas below the curve(A), A area (0,5- B);B=0,5-0,1136447=0,3863553 Gini=A/A+B = 0,1136447/(0,1136447+0,38635 53)=0,227352894

The Gini coefficient of 0.22 is that the elementary mathematics teachers' academic achievement is distributed to students fairly or 36 students share the achievement fairly. If the Gini coefficient is 0.45, it is suggested that the prospective mathematics teacher candidates did not distribute their academic achievement fairly in the courses of Statistics and Probability or 16 students could not share the achievement fairly, but they fit the expected situation in the other years, and further evaluations can be made by following the success of other courses in those years.

2 References

[1] E. Tomul, *Educational inequality in Turkey: an evaluation by Gini index, Education and Science*, **36**(160) (2011), 133-143.
[2] E. Erdem, S. Çoban, *Türkiyede İller bazında eğitim eşitsizliğinin ölçülmesi ve ekonomik gelişmişlik farklılıklarıyla İlişkisi; eğitimin Gini katsayıları*, 14. İstatistik Araştırma Sempozyumu, 5-6 Mayıs 2005, Ankara, 188-2004, 2006.
[3] F. Şenses, *İktisada (farklı bir) giriş, giriş iktisadı öğrencileri ve iktisada ilgi duyanlar için yardımcı kitap*, İletişim Yayınları, İstanbul, 2017.
[4] M. C. Brawn, *Using Ginu style indices to evalute the spatial patterns of heslt practitioners; theoretical considerations and application on the Slberta data*, Soc. Sci. Med., **38**(9) (1994), 1243-1256.
[5] Ş. Yazgan, *Kamu yatırımları dağılımının Gini katsayısı ile ölcülmesi:Türkiye üzerine bir uygulama,* IJEPHSS **1**(1) (2018), 1999-2017.

Conference Proceedings of Science and Technology, 2(3), 2019, 212–214

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Fractional Solutions of a k-hypergeometric Differential Equation

Resat Yilmazer [1,*] *Karmina K. Ali*[1,2]

[1] *Faculty of Science, Department of Mathematics, Firat University, Elazig, Turkey, ORCID:0000 0002 50593882*
[2] *Faculty of Science, Department of Mathematics, University of Zakho, Iraq, ORCID:0000-0002-3815-4457*
** Corresponding Author E-mail: rstyilmazer@gmail.com*

Abstract: In the present work, we study the second order homogeneous k-hypergeometric differential equation by utilizing the discrete fractional Nabla calculus operator. As a result, we obtained a novel exact fractional solution to the given equation.

Keywords: Discrete fractional, the k-hypergeometric differential equation, Nabla operator.

1 Introduction

Fractional calculus deal with derivatives and integrals of arbitrary orders, their applications seem in different areas of science such as physics, applied mathematics, chemistry, engineering [1–4]. Mathematical models have significant applications in physical and technical processing phenomena [5–9]. The solutions of the differential equations relevant to many interesting special functions in mathematics, physics, and engineering, such as the hypergeometric series [10], the zeta function [11], the continued fraction [12], the power series [13], the Fourier analysis [14]. The discrete fractional Nabla calculus operator have been applied to various singular ordinary equations such as the second-order linear ordinary differential equation of hypergeometric type [15], the modified Bessel differential equation [16], the radial equation of the fractional Schrödinger equation [17, 18], the Gauss equation [19], the non-Fuchsian differential equation [20], the Chebyshev's equation [21]. The aim of this study is to apply the Nabla calculus operator to a well-known ordinary differential equation k-hypergeometric equation [22], which is expressed by

$$kr\left(1-kr\right)\frac{d^2w}{dr^2}+\left[\alpha-\left(k+\rho+\sigma\right)kr\right]\frac{dw}{dr}-\rho\sigma w=v\left(r\right), \tag{1}$$

where $k\in\mathbb{R}^+, \alpha,\ \rho,\ \sigma\in\mathbb{R}^+$and $v\left(r\right)$ is holomorphic in an interval $D\subseteq\mathbb{C}$. If $k=1$ and the function $v\left(r\right)$ be vanishes identically, then Eq. (1) reduce to a linear homogenous hypergeometric ordinary differential equation (ODE) as follows

$$r\left(1-r\right)\frac{d^2w}{dr^2}+\left[\alpha-\left(1+\rho+\sigma\right)r\right]\frac{dw}{dr}-\rho\sigma w=0. \tag{2}$$

Many researchers have been studied the hypergeometric differential equation by different schemes, such as Kummer, presented the concurrent of hypergeometric equation in physical models [23]. Campos, finalize that this kind of equation contains complex calculations, and also the singularities of the differential equation are orderly. [24].

2 Preliminaries

Here, we have some imperative knowledge about the discrete fractional calculus theory and also some necessary notes, $\mathbb{N}$ is the set of natural numbers including zero, and $\mathbb{Z}$ is the set of integers. The $\mathbb{N}_b=\{b,b+1,b+2,...\}$ for $b\in\mathbb{Z}$. Let $f(t)$ and $g(t)$ are the real valued functions defined on $\mathbb{N}_0^+$. For more details see [15–21].
Definition 1. The rising factorial power is defined by

$$z^{\bar{n}}=t\left(z+1\right)\left(z+2\right)...\left(z+n-1\right),\ n\in\mathbb{N},\ z^{\bar{0}}=1.$$

Given α be a real number, then $z^{\bar{\alpha}}$ is expressed by

$$t^{\bar{\alpha}}=\frac{\Gamma\left(t+\alpha\right)}{\Gamma\left(t\right)}, \tag{3}$$

where $z\in\mathbb{R}\backslash\{...,-2,-1,0\}$, and $0^{\bar{\alpha}}=0$.
Let us symbolize that

$$\nabla\left(z^{\overline{\alpha}}\right)=\alpha z^{\overline{\alpha-1}}, \tag{4}$$

here $\nabla u(z) = u(z) - u(z-1)$. For $n = 2, 3, \ldots$ describe ∇^n by $\nabla^n = \nabla\nabla^{n-1}$.
Definition 2. The α^{th} order fractional sum of f is defined by

$$\nabla_b^{-\alpha} f(z) = \sum_{s=b}^{z} \frac{[s-\delta(z)]^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(s), \tag{5}$$

where $z \in \mathbb{N}_b,\ \delta(z) = z-1$ is backward jump operator.
Theorem 1. Let $f(z)$ and $g(z) :\ \mathbb{N}_0^+ \to \mathbb{R},\ \alpha,\ \beta > 0$, and $h,\ v$ are constants, then

$$\nabla^{-\alpha}\nabla^{-\beta} f(z) = \nabla^{-(\alpha+\beta)} f(z) = \nabla^{-\beta}\nabla^{-\alpha} f(z) \tag{6}$$

$$\nabla^{\alpha}[hf(z) + vg(z)] = h\nabla^{\alpha} f(z) + v\nabla^{\alpha} g(z) \tag{7}$$

$$\nabla\nabla^{-\alpha} f(z) = \nabla^{-(\alpha-1)} f(z) \tag{8}$$

$$\nabla^{-\alpha}\nabla f(z) = \nabla^{(1-\alpha)} f(z) - \begin{pmatrix} z+\alpha-2 \\ z-1 \end{pmatrix} f(0) \tag{9}$$

Lemma 1. For all $\alpha > 0,\ \alpha^{th}$ order fractional difference of the product fg is expressed by

$$\nabla_0^{\alpha}(fg)(z) = \sum_{n=0}^{z} \begin{pmatrix} \alpha \\ n \end{pmatrix} \left[\nabla_0^{\alpha-n} f(z-n)\right]\left[\nabla^n g(z)\right]. \tag{10}$$

Lemma 2. If the function $f(t)$ is single valued and analytic, then

$$[f_\alpha(z)]_\beta = f_{\alpha+\beta}(z) = \left[f_\beta(z)\right]_\alpha,\ \left[f_\alpha(z) \neq 0,\ f_\beta(z) \neq 0,\ \alpha,\ \beta \in \mathbb{R},\ z \in \mathbb{N}\right]. \tag{11}$$

3 Main results

Theorem 2. Let $w \in \{w : 0 \neq |w_\vartheta| < \infty,\ \vartheta \in \mathbb{R}\}$, and then the homogeneous k-hypergeometric equation is given by

$$w_2 kr(1-kr) + w_1[\alpha - (k+\rho+\sigma)kr] - w\rho\sigma = 0, \tag{12}$$

has a particular solution of the form

$$w = h\left\{(r)^{-\left(\frac{1}{k}(\vartheta\theta k+\alpha)\right)}(1-kr)^{-\left(\frac{1}{k}(\vartheta\theta k+\rho+\sigma-\alpha+k)\right)}\right\}_{-(\vartheta+1)},\ r \neq \left\{0, \frac{1}{k}\right\}. \tag{13}$$

where $w_m(r) = \frac{d^m w}{dr^m},\ (m = 0, 1, 2),\ w_0 = w(r)$, and $\alpha,\ \rho,\ \sigma$ are given constants as well as h is a constant of integration.
Proof. When we applied the discrete fractional calculus operator to both sides of Eq. (12), we have

$$\nabla^{\vartheta} w_2 kr(1-kr) + \nabla^{\vartheta} w_1[\alpha - (k+\rho+\sigma)kr] - \nabla^{\vartheta}(w\rho\sigma) = 0, \tag{14}$$

using Eq. (8), and Eq. (9) together with Eq. (14), one may obtain

$$\begin{aligned} &w_{\vartheta+2} kr(1-kr) + w_{\vartheta+1}[\vartheta\theta k(1-2kr) + \alpha - (k+\rho+\sigma)kr] \\ &+ w_\vartheta\left[-\vartheta(\vartheta-1)\theta^2 k^2 + \vartheta\theta(-(k+\rho+\sigma)k) - \rho\sigma\right] = 0, \end{aligned} \tag{15}$$

where θ is a shift operator.
We choose ϑ such that

$$\vartheta(\vartheta-1)\theta^2 k^2 + \vartheta\theta\left(k^2 + k\rho + k\sigma\right) + \rho\sigma = 0,$$

$$\vartheta = \frac{\left[\theta k - (k+\rho+\sigma) \pm \sqrt{\left((k+\rho+\sigma)-\theta k\right)^2 - 4\rho\sigma}\right]}{2\theta k}, \tag{16}$$

and let $(k+\rho+\sigma-\theta k)^2 \geq 4\rho\sigma$, then we have

$$w_{\vartheta+2} kr\,(1-kr) + w_{\vartheta+1}\left[\vartheta\theta k\,(1-2kr) + \alpha - (k+\rho+\sigma)\,kr\right] = 0, \tag{17}$$

and set

$$w_{\vartheta+1} = W = W\,(r)\,,\ \left(w = W_{-(\vartheta+1)}\right). \tag{18}$$

Therefore

$$W_1 + W\left[\frac{\vartheta\theta k\,(1-2kr) + \alpha - (k+\rho+\sigma)\,kr}{kr\,(1-kr)}\right] = 0, \tag{19}$$

by using Eq. (17), and Eq. (18), then the solution of the ODE Eq. (19) has the form

$$W = h\,(r)^{-\left(\frac{1}{k}(\vartheta\theta k+\alpha)\right)}(1-kr)^{-\left(\frac{1}{k}(\vartheta\theta k+\rho+\sigma-\alpha+k)\right)}. \tag{20}$$

4 Conclusion

In the present study, we applied the discrete fractional Nabla calculus operator to the homogeneous k-hypergeometric differential equation. As a result, we obtained a new exact discrete fractional solution.

5 References

[1] K. S. Miller, and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, 1993.
[2] K. Oldham, and J. Spanier, *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, Elsevier, 1974.
[3] I. Podlubny, *Matrix approach to discrete fractional calculus. Fractional calculus and applied analysis*, **3**(4) (2000), 359-386.
[4] H. T. Michael, *The Laplace transform in discrete fractional calculus*, Computers and Mathematics with Applications **62**(3) (2011) 1591-1601.
[5] M. N. Özişik, H. R. B. Orlande, M. J. Colac, R. M. Cotta, *Finite difference methods in heat transfer*, CRC press, 2017.
[6] P. T. Kuchment, *Floquet theory for partial differential equations*, Birkhäuser, 2012.
[7] A. H. Khater, M. H. M. Moussa, and S. F. Abdul-Aziz, *Invariant variational principles and conservation laws for some nonlinear partial differential equations with variable coefficients part II*, Chaos, Solitons and Fractals **15**(1) (2013), 1-13.
[8] P. Verdonck, *The role of computational fluid dynamics for artificial organ design*, Artificial organs **26**(7) (2002), 569-570.
[9] A. Mandelis, *Diffusion-wave fields: mathematical methods and Green functions*, Springer Science and Business Media, 2013.
[10] G. M. Viswanathan, *The hypergeometric series for the partition function of the 2D Ising model*, Journal of Statistical Mechanics: Theory and Experiment **2015**(7) (2015), 07004.
[11] C. M. Bender, C. B. Dorje, and P.M. Markus, *Hamiltonian for the zeros of the Riemann zeta function*, Physical Review Letters **118**(13) (2017), 130201.
[12] P. Flajolet, *Combinatorial aspects of continued fractions*, Discrete mathematics **306**(10-11) (2006), 992-1021.
[13] G. Plonka, D. Potts, G. Steidi, M. Tasche, *Fourier series, Numerical Fourier Analysis*, Birkhäuser, Cham, 1-59, 2018.
[14] J. W. Cooley, J. W. Tukey, *An algorithm for the machine calculation of complex fourier series*, Mathematics of computation, **19**(90), 297-301, 1965.
[15] R. Yilmazer, M. Inc, F. Tchier, D. Baleanu, *Particular solutions of the confluent hypergeometric differential equation by using the nabla fractional calculus operator,* Entropy **18**(2) (2016), 49.
[16] R. Yilmazer, and O. Ozturk, *On Nabla discrete fractional calculus operator for a modified Bessel equation*, Therm. Sci. **22** (2018) 203-209.
[17] R. Yilmazer, *N-fractional calculus operator N^{μ} method to a modified hydrogen atom equation*, Mathematical Communications **15**(2) (2010), 489-501.
[18] R. Yilmazer, *Discrete fractional solutions of a Hermite equation*, Journal of Inequalities and Special Functions, **10**(1) (2019), 53-59.
[19] R. Yilmazer, *Discrete fractional solution of a non-Homogeneous non-Fuchsian differential equations*, Thermal Science, **23**(1) (2019), 121-127.
[20] R. Yilmazer, M. Inc, and M. Bayram, *On discrete fractional solutions of Non-Fuchsian differential equations*, Mathematics **6**(12) (2018), 308.
[21] M. Inc and R. Yilmazer, *On some particular solutions of the Chebyshev's equation by means of Ña discrete fractional calculus operator*, Prog. Fract. Differ. Appl. **2**(2) (2016), 123-129.
[22] L. Shengfeng, and Y. Dong, *k-Hypergeometric series solutions to one type of non-homogeneous k-Hypergeometric Equations*, Symmetry **11**(2) (2019), 262.
[23] E. E. Kummer, *De integralibus quibusdam definitis et seriebus infinitis*, Journal für die reine und angewandte Mathematik **17** (1837), 228-242.
[24] L. Campos, *On some solutions of the extended confluent hypergeometric differential equation*, Journal of computational and applied mathematics **137**(1) (2001), 177-200.

