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The Topological Connectivity of the Independence Complex of Circular-Arc Graphs

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Abstract

Let us denoted the topological connectivity of a simplicial complex C plus 2 by $\eta(C)$.
Let ψ be a function from class of graphs to the set of positive integers together with ∞.
Suppose ψ satisfies the following properties:

1. $\psi(K_0)$=0.
2. For every graph G there exists an edge $e=(x,y)$ of G such that

$$\psi(G-e)\geq\psi(G)$$

(where $G-e$ is obtained from G by the removal of the edge e), and

$$\psi(G-N(\{x,y\}))\geq\psi(G)-1$$

then

$$\eta(\mathscr{I}(G))\geq\psi(G)$$

(where $(G-N(\{x,y\}))$ is obtained from G by the removal of all neighbors of x and y (including, of course, x and y themselves).
Let us denoted the maximal function satisfying the conditions above by ψ_0.
Berger [3] prove the following conjecture:

$$\eta(\mathscr{I}(G))=\psi_0(G)$$

for trees and completements of chordal graphs.

Kawamura [2] proved conjecture, for chordal graphs. Berger [3] proved Conjecture for trees and completements of chordal graphs. In this article I proved the following theorem: Let G be a circular-arc graph G if $\psi_0(G)\leq 2$ then $\eta(\mathscr{I}(G))\leq 2$. Prior the attempt to verify the previously mentioned cases, we need a few preparations which will be discussed in the introduction.

1. Introduction

A non-empty collection C of sets is called a *simplicial complex* if it is hereditary, namely if $\sigma\in C$ and $\tau\subseteq\sigma$ imply $\tau\in C$. It is well known that every simplicial complex has a unique (up to homeomorphism) geometric realisation, namely an embedding in some space $\mathbb{R}^n$, in which every simplex $\sigma\in C$ is realized as a homeomorph of a simplex in $\mathbb{R}^n$.

The simplicial join of X and Y. Denoted by $X*Y$ is the (geometric realization of) a simplicial complex defined by

$$X*Y=\{\sigma\cup\tau\mid\sigma\in X\tau\in Y\}$$

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Although we use the same symbol $*$ to mean the join of graphs and the join of simplicial complex, this causes no confusion in the seqel. the join of simplicial complex X with a singleton v is the cone of X with apex v, denoted C_vX. the suspension of X the join of S^0 and X, is denoted by susp X.

We shall identify a complex with its geometric realisation. *The topological connectivity* of a simplicial complex C is the largest number c such that for every number $k \leq c$, every embedding of the k-dimensional sphere S^k is extendable to an embedding of the $k+1$-dimensional ball B^{k+1} in C. The connectivity of a complex may be infinite. The connectivity of C, plus 2,is denoted by $\eta(C)$ .the reason for this definition is that the addition of 2 makes the formulation of some results more elegant.

For a graph G with the vertex set V, a subset A of V is said to be independent if no two distinct vertices of A are adjacent. A subset S of V is said to be dominating in G if every vertex $v \in V$ is adjacent to a vertex of S. The domination number $\gamma(G)$ is the minimum of the cardinality of dominating sets of $G : \gamma(G) = min\{|S||S \textit{ is dominating in } G\}$.
For a graph H, we denote by $\mathscr{I}(H)$ the simplicial complex consisting of all independent sets of vertices in H. As before, let $V_i, i \leq m$ be a partition of the vertex set of a graph G. Given a vertex $v \in V(G)$, we write $i(v)$. for the index i for which $v \in V_i$. For a set Z of vertices we write $I(Z) = \{i(z) : z \in Z\}$. As usual, we denote the set $\{1,...,m\}$ by $[m]$. Given a subset I of $[m]$, we write V_I for $\bigcup_{i\in I} V_i$.
Consider a graph G with a partition $V_1,V_2,...,V_m$ of its vertex set. A choice of one vertex from each set V_i is called an *independent system of representative* (ISR) if the selected vertices are non-adjacent in G.
Existence of ISR :

Theorem 1.1. *If for all* $I \subseteq [m]$

$$\eta(\mathscr{I}(G[V_I])) \geq |I|$$

then the partition $(V_i)\,(i \leq m)$ *of* $V(G)$ *has an ISR.*

To exemplify these notions and the above theorem, consider a bipartite graph G, whose sides are the two parts V_1 and V_2 of the given partition. In this case there exists an ISR if and only if the graph is not complete bipartite. But not being complete bipartite means the existence if a connection in $\mathscr{I}(G)$ between the two simplices V_1 and V_2 of $\mathscr{I}(G)$. Thus, not being complete bipartite is tantamount to $\mathscr{I}(G)$ being connected, which, in the above terminology, means being 0-connected, which means that $\eta(\mathscr{I}(G)) \geq 2$. Thus is this example, the condition of the theorem is not only sufficient, but also necessary (Enough to inseart one arc between tow simplices to be connected).

Theorem 1.2. *Let* ψ *be a function from class of graphs to the set of positive integers together with* ∞. *Suppose* ψ *satisfies the following properties:*

1. $\psi(K_0)$*=0.*
2. *For every graph G there exists an edge e=(x,y) of G such that*

$$\psi(G-e) \geq \psi(G)$$

(where $G-e$ *is obtained from G by the removal of the edge e), and*

$$\psi(G-N(\{x,y\})) \geq \psi(G)-1$$

then

$$\eta(\mathscr{I}(G)) \geq \psi(G)$$

(where $(G-N(\{x,y\}))$ is obtained from G by the removal of all neighbors of x and y (including, of course, x and y themselves).
In fact, there is a maximal function ψ_0 satisfying the conditions of the theorem. It is best described in terms of a game between two players, (I) and (II). Player (I) want to maximize the function ψ in the theorem(and hence prove that ψ is large), while player(II) wants to minimize ψ. Player (I) selects an edge $e = (x,y)$ in the graph given at the present stage of the game. Player (II) chooses between two possibilities: he or she either (1) deletes e from the graph, or else (2) deletes all neighbors of x and y (including, of course, x and y themselves). the game ends when either there remains an isolated vertex, in which case ψ is defined as ∞, or there are no remaining vertices, in which case ψ is defined as the number of moves of player (II) of type (2).
We define $\psi_0(G)$ as the maximal value of $\psi(G)$ player (I) can achieve in the game. Theorem 1.2 then states that $\eta(\mathscr{I}(G)) \geq \psi(G)$. Aharoni, Berger and Ziv [1] suggested the following *conjecture*:

Conjecture 1.3.

$$\eta(\mathscr{I}(G)) = \psi_0(G)$$

Kawamura [2] proved the conjecture for chordal graphs. Now we define a chordal graph: A graph is *chordal* if each of its cycles of four or more nodes has a chord , which is an edge joining two nodes that are not adjacent in the cycle. Chordal graphs are know to be perfect graphs. They are sometimes also called triangulated graphs.
The chordal graphs are the intersection graphs of subtrees of a tree A classical theorem of Dirac [4] states that For each chordal graph G, there exists a vertex , called a *simplicial vertex*, such that $N(v)$ is a complete graph.

In this article I proved the obove conjecture for Circular-arc graphs. The *circular-arc graphs*, defined as the intersection graphs of a set of arcs on the circle. Such a graph has one vertex for each arc in the set, and an edge between every pair of vertices corresponding to arcs that intersect.

Formally, let

$$I_1, I_2 \dots, I_n \subseteq S^1$$

be a set of arcs. Then the corresponding circular-arc graph is $G = (V, E)$, where

$$V = \{I_1, I_2 \dots, I_n\}$$

and

$$\{I_\alpha, I_\beta\} \in E \Leftrightarrow I_\alpha \bigcap I_\beta \neq \emptyset$$

A family of arcs that corresponds to G is called an *arc model* of G. In [3] the following was proved:

Theorem 1.4. *Let G be a minimal counterexample for conjecture 1.3. Then G has no vertex of degree 1.*

Corollary 1.5. *Conjecture 1.3 holds for every tree.*

The following was also proved in [3]

Theorem 1.6. *If G is a connected chordal graph then $\psi_0(\overline{G}) = \infty$.*

Theorems 1.6 and Corollary 1.5 of Berger prove Conjecture 1.3 for trees and completements of chordal graphs. Kawamura [2] proved it for chordal graphs.

Theorem 1.7. *Let G be a chordal graph. then $\mathscr{I}(G)$ is either contractible or is homotopy equivalent to the wedge of finitely many spheres $\bigvee S^{k_t}$. where $k_t \geq \gamma(G) - 1$ for each k_t.*

Conversely, all finite wedges of phares appear as homotopy types of independence complex of chordal graphs.
thus

Observation 1.8. *(1) Let L be a simplicial complex and let $k_1, \dots, k_r$ be subcomplexes of L (repetitions allowed). Take mutually distinct points $u_1, \dots, u_r$and v with $\{u_1, \dots, u_r, v\} \cup L = \phi$. then the union*

$$X = C_v L \cup \bigcup_{i=1}^{r} C_{u_i} K_i$$

Subject to the conditions

$$\begin{aligned} C_{u_i} K_i \cap C_v L &= K_i and \\ C_{u_i} K_i \cap C_{u_j} L &= K_i \cap K_j (i, j = 1, \dots r, i \neq j). \end{aligned}$$

is homotopy equivalent to $\bigvee_{i=1}^{r} suspK_i$.
(2)The suspension susp X is contractible for each contractible complex X.
(3)$susp(\vee S^{k_r}) \backsimeq \vee S^{k_r+1}$.

The following lemma provides us with a clear view on the structure of independence complexes of chordal graphs. For a vertex v of a graph G, let $I_v(G)$be the subcomplex generated by the independent sets containing v:

$$I_v(G) = \{A | \text{ there exists a simplex } B \in I(G) \text{ such that } v \in B \text{ and } A \subset B\}.$$

Lemma 1.9. *Let G be a graph and let v be a simplicial vertex of G. Enumerate $N(v)$as $N(v) = \{w_1, \dots, w_r\}$. then we have the following.*

$$I(G) \simeq \bigvee suspI(G - N[w_i]).$$

For each chordal garph G, we define $\eta(G) \in \{0, 1, \dots, \infty\}$ as follows: if $I(G)$ is contractible, then let $\eta(G) = \infty$ and, if $I(G)$ is homotopy equivalent to the wedge $\bigvee S^{i_r}$, then let $\eta(G) = mini_r$, the minimum dimension of the associated spheres. under this notation theorem1.7 and lemmaa1.9 yield the following corollary.

Corollary 1.10. *Let G be a chordal graph.*
(1) We have an inequality $\eta(G) \geq \gamma(G) - 1$. (2) Let v be a simplicial vertex of G and enumerate $N(v)$ as $N(v) = \{w_1, \dots, w_r\}$. then we have $\eta(G) = min\{\eta(g - n[w_i]) | i = 1, \dots, r\} + 1$. here we make a convention that $min\{\infty, \dots, \infty\} + 1 = \infty$.
For the path P_n with n edges, $I(P_n)$ is contractible if $n \equiv 0(mod3)$ and is homotopy equivalent to the sphere of dimension $\lfloor \frac{n}{3} \rfloor$. hence we have

$$\eta(P_n) = \begin{cases} \infty & if \quad n \equiv 0(mod3) \\ \lfloor \frac{n}{3} \rfloor & otherwise \quad . \end{cases}$$

It shoud be noted that*Kazuhiro Kawwamura* [2] was proved above for chordal graphes *(Theorem 1.7, Observation 1.8, Lemma 1.9 and Corollary 1.10).*

2. New Result

In this article I proved the following theorem, about circular-arc graphes.

Theorem 2.1. *Let G be a circular-arc graph, if exist that $\psi_0(G) \leq 2$ then $\eta(\mathscr{I}(G)) \leq 2$.*

3. Some simple graphs

3.1. Path Grap, P_n

Definition 3.1. *In graph theory, Graph (V,E) with $n+1$ vertices, which is a simple path is called path graph and is denoted P_n. Formally, a set V of vertices $V=\{1,2,...,n+1\}$, and a set E of edges $E=\{\{i,i+1\}|i=1,2,...,n\}$.*

(see Fig. 3.1).

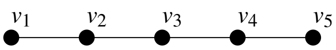


Fig: 3.1

Before we begin we want to define the concept of *retraction*:

Definition 3.2. *Let X be a topological space, let $A \subseteq X$, and let $i : A \hookrightarrow X$ be the inclusion map. A continuous map $f : X \to A$ is called a **retraction** if*

$$f|_A = id_A;$$

Correspondingly, A is called a retract, a deformation retraction of X.

In the first stage we will explain the previously mention corollary 1.10: , but will begin by adding a chart which summarizes the main points regarding the corollary 1.10

$$\eta(P_n)=\psi_0(P_n)=\begin{cases}\infty & if \quad n\equiv 0(mod 3)\\ \lfloor \frac{n}{3}\rfloor & otherwise \quad .\end{cases}$$

	Figure (P_n)	$\mathscr{I}(P_n)$	Figure $\mathscr{I}(P_n)$	$\eta(P_n)$
P_1	v_1 — v_2	The independent complex disconnected	v_1 v_2	$\eta(P_1)=1$
P_2	v_1 — v_2 — v_3	The independent complex disconnected	v_2; v_1 — v_3	$\eta(P_2)=1$
P_3	v_1 — v_2 — v_3 — v_4	The independent complex is contractible.	v_2 — v_4 — v_1 — v_3	$\eta(P_3)=\infty$
P_4	v_1 — v_2 — v_3 — v_4 — v_5	Now we can contractible v_3 into v_4	(see Fig 1)	$\eta(P_4)=2$

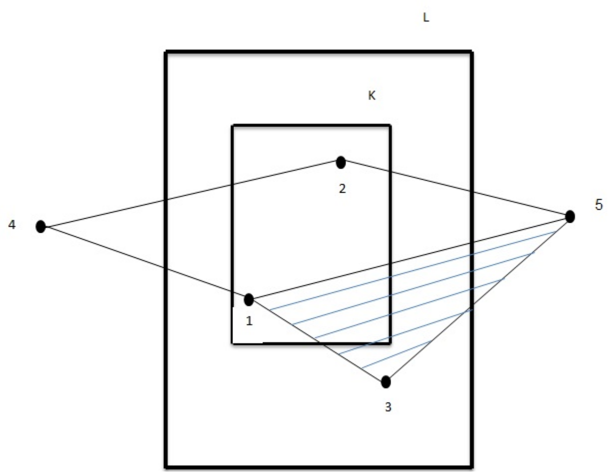


Figure 1: The independent complex of P_5.

Let's start from P_1 and P_2, the independent complex is not connected, then:

$$\eta(P_1)=\eta(P_2)=1.$$

In the case P_3 the $\mathscr{I}(P_3)$ is contractible, then $\eta(P_3) = \infty$.
In the case P_4, let us denote the vertecies $v_1, v_2 ... v_5$. In the $\mathscr{I}(P_4)$ (See Fig 1), let L be a set that contains $\{v_1, v_2, v_3\}$, denote by K the set of vertecies which are connected to v_4, and v_5 connected to the elements in L. Now we can contractible v_3 into v_4,(and this does not change homotopy).

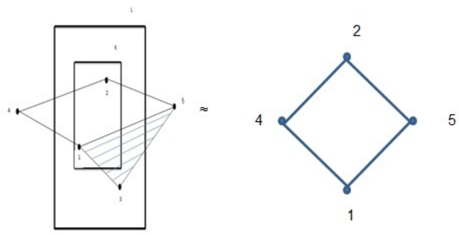


Figure 2: contractibtion v_3 into v_4.

then

$$\eta(susp(\mathscr{I}(P_4))) = 2.$$

(See Fig. 2).
In the case of P_7, we denote the vertecies $\{v_1, v_2, ..., v_7\}$. We will construct the independent complex as shown in the picture below (See Fig.3), we notice that $N(v_5) \subseteq N(v_7)$, then we can contract v_5 into v_7.

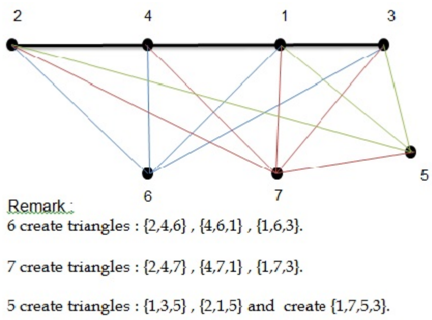


Figure 3: The independent complex of P_7.

After contraction, the result is Figure 4.

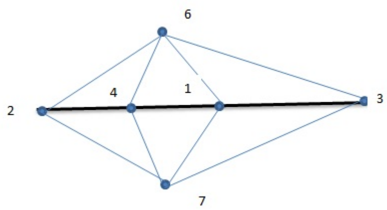


Figure 4: Contraction v_5 into v_7.

In the case of P_{10}, we denote the vertecies $v_1, v_2, ..., v_{10}$. We construct the independent complex, we notice that $N(v_8) \subseteq N(v_{10})$, then we can contract v_8 into v_{10}.
Now can prove the above case using induction, if we have two arcs $U \subseteq V$, (Fig 5), (If all the neighbors of U contained in the neighbors of V) we can contract all independent sets in U into V and they remain independent.

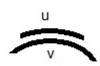


Figure 5: $U \subseteq V$

3.2. Circle Grap, C_n

Now we want to know the topology connectivity of a *Circle Graph*. At first I want to deine the *Circle Graph*.

Definition 3.3. *A graph with n vertexes which resembles a circle, is called a* ***Circle Graph*** *and is denoted by C_n. Formally, the graph set of vertexes is $V = \{0,1,...,n-1\}$ and the arc's set is $E = \{\{i,(i+1) mod\ n\ \}|i = 0,1,...,n-1\}$.*

We will start case C_5, and we will finde its topological connectivity:

3.2.1. C_5

In the first stage we built the independent complex of C_5 (Figure 6), and the $\mathscr{I}(C_5)$ is connected, but not simply connected then

$$\eta(C_5) = 2.$$

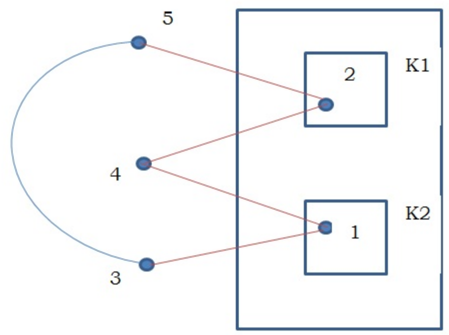


Figure 6: $\mathscr{I}(C_5)$

To get ψ_0 we want to remove an arc, Then the new graph is P_4. Then

$$\eta(C_5) = \psi_0(C_5) = 2.$$

3.2.2. C_6

At this stage we will show that

$$\eta(C_5) = \psi_0(C_5) = 2.$$

C_6 contains the vertexes $\{v_1, v_2, ..., v_6\}$. At the independent complex of C_6 (Figure 7), the $N(v_4)$-(v_1 and v_2) denote by $K1$, the $N(v_6)$-(v_2 and v_3) denote by $K2$, the $N(v_5)$-(v_1, v_2 and v_3) denote by L.

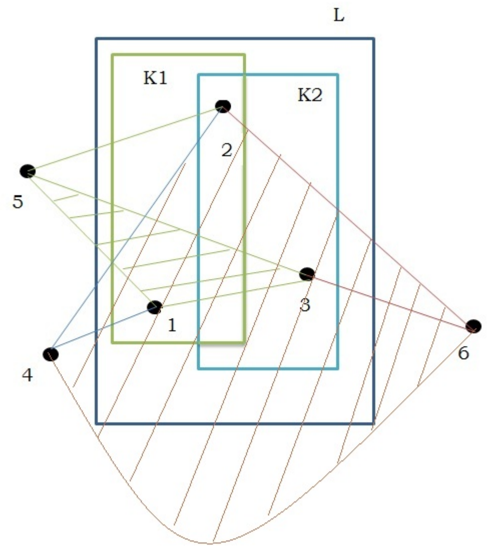
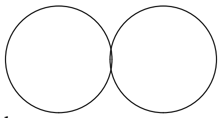


Figure 7: $\mathscr{I}(C_6)$

Then

$$\mathscr{I}(c_6) \cong susp \begin{pmatrix} 2 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \bigvee susp \begin{pmatrix} 2 \\ \cdot \\ \cdot \\ 3 \end{pmatrix} =$$



For the same reason of the case C_5: $\psi_0 = 2$, then

$$\psi_0(C_6) = \eta(C_6) = 2.$$

3.2.3. C_8

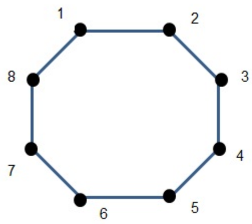


Figure 8: (C_8)

Constructing the independent complex of C_8 is not simple.
We denote the vertexes of C_8 by $\{v_1, v_2, ..., v_8\}$, see figure 8.
We will construct the independent complex as follows:
We denote the neighbors of the vertex v_6 by $K1$, and denote the neighbors of the vertex v_8 by $K2$. The *suspension* of the form *F*igure 9

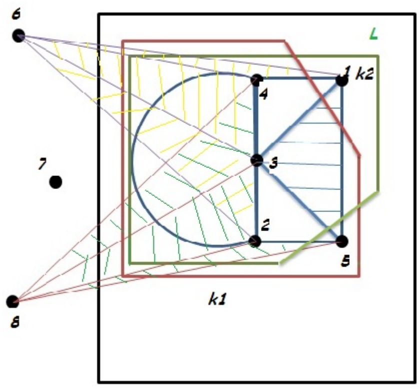


Figure 9: The first stage of constructing $\mathscr{I}(C_8)$

is homotope to:

$$\mathscr{I}(fig9) \cong susp \begin{pmatrix} 2 \\ 4 \\ 1 \\ 3 \end{pmatrix} \bigvee susp \begin{pmatrix} 4 \\ 2 \\ 5 \\ 3 \end{pmatrix}$$

and we have the arc $\{v_6, v_8\}$. Ohterwise the independent complex is contactible and the topological connectivity will be ∞.
Will complete construction of independent complex of C_8, and we will obtain the form seen in *Fig.*10.

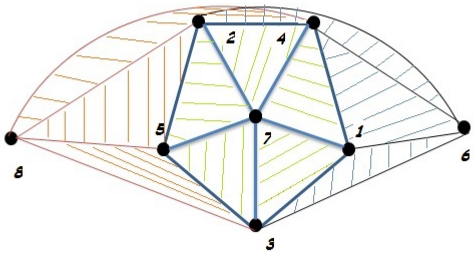
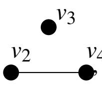
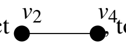


Figure 10: $\mathscr{I}(C_8)$

We have put two contractible independent complexes with anther contractible, that is the neighbors of v_7 (*S*ee *F*ig. 11).

Figure 11: Contraction

We will contract the form Fig 11, to three vertexes v_2, v_4 and v_3  and one arc $\{v_2, v_4\}$. We will contract  to one vertex denoted by v_{24}.
The independent complex of C_8 after the contraction, looks like Fig. 12, and that homotopy to S^2.
Then

$$\eta(C_8) = 3.$$

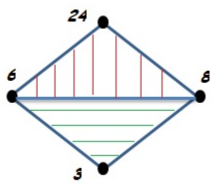


Figure 12: Contractible $\mathscr{I}(C_8)$

3.2.4. General Case C_n

Getting back to the last case (C_8), and denoting v_8 by n, v_7 by $n-1$ and v_6 by $n-2$.
We have seen something else: if we ignore the relationship between n and $n-2$, then we have obtained:

$$\mathscr{I}(C_n) \cong Susp(P_{n-5}) \bigvee Susp(P_{n-5}) \bigvee Susp(Susp(P_{n-6})).$$

and the value of $\eta(C_n)$:

$$\eta(C_n) = min(\eta(P_{n-5}) + 1, \eta(P_{n-6}) + 2).$$

We have three cases:

1. If $n \equiv 0(mod3)$

   then $\eta(P_{n-6}) = \infty$, then the minimum is $\eta(P_{n-5}) + 1 = \lceil \frac{n-5}{3} \rceil + 1$, and the result is:

   $$\lceil \frac{n-5}{3} \rceil + 1 = \frac{n}{3}.$$

2. If $n \equiv 1(mod3)$

   then we get the minimum in $\eta(P_{n-5}) + 1 = \lceil \frac{n-5}{3} \rceil + 1$ and the result is:

   $$\lceil \frac{n-5}{3} \rceil + 1 = \lfloor \frac{n}{3} \rfloor.$$

3. If $n \equiv 2(mod3)$

   then we get the minimum in $\eta(P_{n-6})+2 = \lceil \frac{n-6}{3} \rceil + 2$ and the result is:

   $$\lceil \frac{n-6}{3} \rceil + 2 = \lceil \frac{n}{3} \rceil.$$

In the end, we have obtained the same result in the three cases:

$$\frac{n}{3}.$$

$$\eta(C_n) = min(\eta(P_{n-5})+1, \eta(P_{n-6})+2) = \frac{n}{3}.$$

4. Proof of theorem 2.1

Now we want to prove the theorem 2.1.
Let us denote the largest independent set in the cycle arc graph G by α. We need to prove that:
if $\psi_0 \leq 2$ then $\eta \leq 2$.

1. If $\alpha = 1$ then $\psi_0 = \eta = 1$, because G is a complete graph.
2. If $\alpha = 2$, the independent complex of G is $\bar{G}$, ($\mathscr{I}(G) = \bar{G}$).
   (a) If $\bar{G}$ not connected graph, then:

   $$\psi_0 = \eta = 1$$

   because G contains a bipartite graph.

   (b) If $\bar{G}$ is a tree [3] then:

   $$\psi_0 = \eta = \infty.$$

   (c) If $\bar{G}$ is a connected graph but not a tree, then:

   $$\psi_0 = \eta = 2.$$

   In order to see that $\eta = 2$, note that there is a circle in $\bar{G} = \mathscr{I}(G)$ and we can not fill it. In order to see that $\psi_0 = 2$ it is enough to show that $\psi_0 \geq 2$ and this can be achieved by using the strategy of always choosing an edge between two vertices of distance 2 in $\bar{G}$.
3. If $\alpha = 3$

   Recall that the collection of all simplices of $\mathscr{I}(G)$ of dimension less than or equal to d is called the $d-skeleton$ of $\mathscr{I}(G)$, and is denoted by $\mathscr{I}(G)^{(d)}$. In particular, $\bar{G}$ is the $1-skeleton$ of $\mathscr{I}(G)$.
   We now define a continuous function $f: \bar{G} \to S^1$ (where $\bar{G}$ is viewed as a topological space). The function maps each vertex to the middle of the appropriate segment, and maps each edge to an arc between the mid-segments according to the following rule. If there exists a third segment disjoint from both segments, then the arc goes through this third segment (see fig 13).

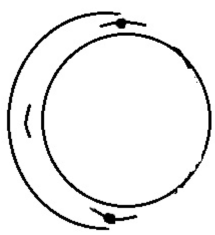
   

   Figure 13: mapping vertics to mid-segment, and edges between mid-segment through the third segment.

   The image of the triangle in $\bar{G}$ is arounded S^1 exactly twice.

   Now the image of every triangle in $\bar{G}$ goes around S^1 an even number of times, and therefore it is impossible to fill a cycle in $\bar{G}$ that goes around S^1 an odd number of times.
4. $\alpha \geq 4$
   Let v_1,v_2,v_3,v_4 be independent vertices.
   Given sphare, and a segment corresponding to v_1, v_1 that does not contain any other segmet, but can stand out from side. We will review cutoff v_1 from all outstandingIt can stick out from the left side and the right side, (see fig.14).

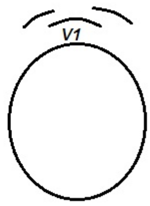


Figure 14: We will review cutoff v_1 from all aspects.

In the case of containment we remove the arc between the two segments (see fig.15).

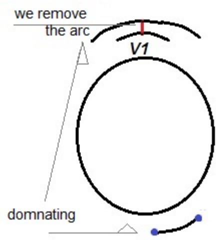


Figure 15: we remove the arc between the two segments.

For example, the first player chooses an arc, the second player removes all its neighbors, then: remains $\psi_0 = 1$, and the new graph is chordal.
In the case of a sticking out from the left side or the right side:
We begin from the segment in the least standing out to the longest, which we cannot remove, this vertex we denote by x.(See fig.16).

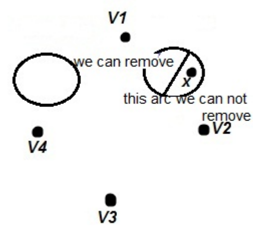


Figure 16: The longest arc, we can not remove.

If we remove $\{v_1,x\}$ and all its neighbors, then a dominating vertex will exist, which we denote by y.
We claim that the vertices x and y is dominating set.
If x is connected to v_1, v_2 and v_3, then we can remove $\{x,v_2\}$. We can not disconnect y from v_3, because, the result would be bipartite graph, (we have a dominating vertex), (see fig.17).

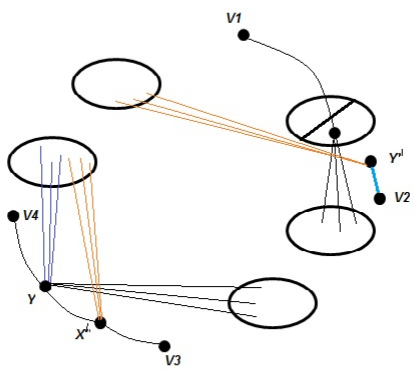


Figure 17: The structure of the graph.

y sticks out more than x' and y' sticks out more than x. Therefore, y and y' is a dominating set. then: (fig.18)

Figure 18: y and y' is a dominating set.

Therefore, there is a square we cannot fill.

Then we proved that: If

$$\psi_0(G) \leq 2$$

then

$$\eta(\mathscr{I}(G)) \leq 2.$$

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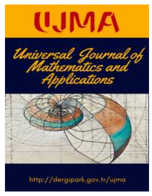

Dynamics of a Host-Parasitoid Model Related to Pennycuick Growth Form

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Abstract

In this study, the dynamical results of the model by obtaining the steady states existing in the host-parasitoid model were given. Also, some results relating to steady states of the model by depending the parameter made from biological assumptions were obtained.

1. Introduction

Stability analysis which examines the dynamics of the populations plays an important role in population dynamics. For local asymptotic stability, solutions must approach a steady state under initial conditions close to the steady state. In global asymptotic stability, solutions must approach a steady state under all initial conditions. Since a globally attractive equilibrium point is locally attractive, a globally asymptotically stable steady state is locally asymptotically stable.
It is well known that the Allee effect plays an important role in the stability analysis of the steady states of a population dynamic model (see, for instance,[4, 8, 9]). The Allee effect, first introduced by Allee [8], represents a negative density dependence when the population growth rate is reduced at low population size. It may be due to a number of sources including difficulties in finding mates, inbreeding depression, food exploitation, predator avoidance of defense, and social dysfunction at small population sizes. In recent years, the studies on stability of population model with different forms derived from biological facts have attracted many mathematicians [3, 4, 5, 6, 7, 9].
Many ecological models consisting interspecific interactions are generated by differential and difference equations. Especially, the discrete-time ecological form with non-overlapping populations are better formulated than continuous-time form. The host-parasitoid models are one of such forms which are studied intensively in the last few decades. One of the earliest applications of discrete-time models including host-parasitoid interaction was obtained by Nicholson and Bailey who applied it to the parasitoid *Encarsia formosa* and the host *Trialeurodes vaporariorum* in 1935 [1, 2]. Parasitoids are parasites which lay their eggs to host larvae and pupae. Hosts escaping parasitism increase their generation. The searching efficiency of parasitoid increases the number of the parasitized host. The successful parasitized hosts die, but the eggs laid by the parasites can survive for future generations. The general host-parasitoid model proposed by Nicholson-Bailey is presented in the following form

$$\begin{aligned} N_{t+1} &= rN_t e^{-cP_t} \\ P_{t+1} &= eN_t(1-e^{-cP_t}). \end{aligned}$$

where r and e are positive parameters. This model assumes as follows:
N_t is the density of host species in generation t;

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P_t is the density of parasitoid species in generation t;
r is the number of eggs laid by a host that survive through the larvae, pupae and adult stages;
e is the number of eggs laid by parasitoid on a single host that survive through larvae, pupae, adult stages;
c is the searching efficiency.
Also, e^{-cP_t} is a fraction of hosts that are not parasitized according to number of encounters under the law of mass action and Poisson distribution.
In this paper, we will investigate the dynamics of a host-parasitoid interaction connected to Pennycuick growth form [10] with different modifications as follows:

$$\begin{aligned} H_{t+1} &= \frac{(1+ae^{b})H_t}{1+ae^{bH_t}}e^{-cP_t} \ ; \quad a,b,c>0 \\ P_{t+1} &= H_t(1-e^{-cP_t}). \end{aligned} \tag{1}$$

Here, H_t is the host population at time t; P_t is the parasitoid population at time t. The growth rate of the host population in the absence of the parasitoid, $\frac{(1+ae^{b})}{1+ae^{bH_t}}$, is associated with the Pennycuick function comes from Pennycuick et al [10].
The aim of this study is to find steady states of the model (1) with and without Allee effect and immigration parameter; and is to investigate the locally asymptotically stability of these steady states.
This paper is regulated as: In Section II, we investigated the steady states of the model (1), and analyzed the locally asymptotically stability of the model (1). In Section III, the steady states of host parasitoid model (1) was examined with immigration parameter. Also, the locally asymptotically stability of this points was investigated. Section IV gives the locally asymptotically stability of the steady states of the host-parasitoid model (1) with Allee effect. Finally, the conclusion is presented.

2. Steady states of the model (1)

In this section, we will obtain the steady states of model (1) by using $H_t = H_{t+1} = H^*$ and $P_t = P_{t+1} = P^*$ as follows:

$$\begin{aligned} H^* &= \frac{(1+ae^{b})H^*}{1+ae^{bH^*}}e^{-cP^*} \ ; \quad a,b,c>0 \\ P^* &= H^*(1-e^{-cP^*}). \end{aligned} \tag{2}$$

Then, we have the following theorem.

Theorem 2.1. *The model (1) has the steady states* $(0,0)$, $(1,0)$ *and* (H^*,P^*).

Proof. It is clearly seen that $(0,0)$ is a steady state for model (1). Let's take $H^* \neq 0$ and $P^* = 0$. Then we have

$$H^* = \frac{(1+ae^{b})H^*}{1+ae^{bH^*}}. \tag{3}$$

by from (2). So, we can see that Eq.(3) is provided for $H^* = 1$. Then $(1,0)$ is steady state of the model (1). Now, we must show that the model (1) has the steady state (H^*,P^*) such that $H^* \neq 0$ and $P^* \neq 0$. If the first equality in (2) is considered, we can write

$$H^* = \frac{(1+ae^{b})H^*}{1+ae^{bH^*}}e^{-cP^*}$$

$$\Rightarrow e^{-cP^*} = \frac{1+ae^{bH^*}}{1+ae^{b}} \tag{4}$$

$$\Rightarrow P^* = \frac{-1}{c}\ln\frac{1+ae^{bH^*}}{1+ae^{b}}. \tag{5}$$

If the following inequality is provided

$$0 < \frac{1+ae^{bH^*}}{1+ae^{b}} < 1 \tag{6}$$

then $P^* > 0$ in Eq.(5). We obtain

$$0 < H^* < 1 \tag{7}$$

by from inequality (6). If we combine Eq.(4) with the second equation of (2), then we get

$$P^* = H^*\left(1-\frac{1+ae^{bH^*}}{1+ae^{b}}\right)$$

If P^* is written in the first equation in (2), then we obtain

$$1+ae^{b} = (1+ae^{bH^*})e^{cH^*\left(1-\frac{1+ae^{bH^*}}{1+ae^{b}}\right)}. \tag{8}$$

Let's write the following function by using the right side of (8) such that $H^* = x$

$$f(x) = (1+ae^{bx})e^{cx(1-\frac{1+ae^{bx}}{1+ae^{b}})}.$$

Since (1,0) is a steady state of the model (1), we can easily see that $x = 1$ is a solution of the Eq.(8). By considering (7), let's investigate some other points providing Eq.(8) apart from $x = 1$. In this way, if the derivation of the function $f(x)$ is calculated, we get

$$f'(x) = e^{cx(1-\frac{1+ae^{bx}}{1+ae^{b}})}\left[abe^{bx} + (1+ae^{bx})\left(\frac{ace^{b}}{1+ae^{b}} - \frac{ace^{bx}+abce^{bx}x}{1+ae^{b}}\right)\right].$$

From $f'(x) = 0$,

$$1 = \frac{(1+abe^{bx})}{abe^{bx}}\left(\frac{ace^{bx}+abce^{bx}x}{1+ae^{b}} - \frac{ace^{b}}{1+ae^{b}}\right). \tag{9}$$

is obtained. Let the function in the right side in Eq.(9) be $F(x)$. Since $F(x)$ is increasing ($F(x) = \infty$ as $x \to \infty$), Eq.(9) has an interaction point. Also, since $f(0) = 1+a$, $f'(0) = ab + \frac{(1+a)ace^{b}}{1+ae^{b}} > 0$, this critical point is a local maximum for $f(x)$. From this and by considering inequality (7), $f'(1) < 0$ must be provided. If this inequality is solved, we get the condition $c > 1$.

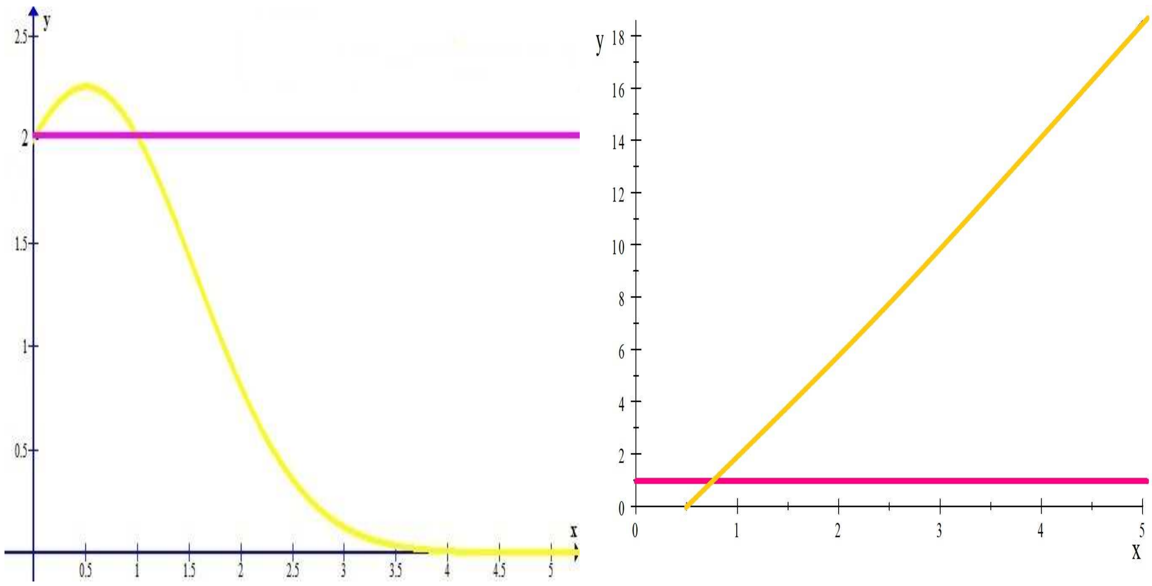

Figure 1: a & b

Figure 2.1- (a): Graphs of $(1+ae^{b})$ *and function* $f(x)$ *where* $a = 1$, $b = 0.02$ *and* $c \approx 45.161$.
Figure 2.1- (b): Graphs showing the intersection point $x = 0.58208$ *where* $a = 1$, $b = 0.02$ *and* $c \approx 45.161$.
In Figure 2.1-(a), it is easily seen that the function $f(x)$ has a critical point. If the graphics of the functions Eq. (2.8) are drawn on the same coordinate plane, we observe this interaction point in Fig 2.1-(b). □

Corollary 2.2. *For the model (1), the following statements hold true:*
(a)-If $c \le 1$, *then the model (1) has the steady states* $(0,0)$ *and* $(1,0)$.
(b)-If $c > 1$, *then the model (1) has the steady states* $(0,0)$, $(1,0)$ *and* (H^*, P^*).

2.1. Stability analysis of model (1)

In this section, we will investigate the locally asymptotically stability conditions of the steady states of (1).

Theorem 2.3. *For the steady states of the model (1), the following statements hold true.*
(a)-The steady state $(0,0)$ *is not locally asymptotically stable.*
(b)-If $2+2ae^{b}-abe^{b} > 0$ *and* $c < 1$, *then the steady state* $(1,0)$ *is locally asymptotically stable.*
(c)-If $c > 1$ *and under additional conditions, then the steady state* (H^*, P^*) *is locally asymptotically stable.*

Proof. (a)-If the model (1) is considered, then we can write

$$F(H_t, P_t) = \frac{(1+ae^{b})H_t}{1+ae^{bH_t}}e^{-cP_t} \;;\; a,b,c > 0$$
$$G(H_t, P_t) = H_t(1-e^{-cP_t}).$$

Firstly, let' s consider $c \le 1$. If the Jacobian matrix of model (1) is created in the neighborhood of $(0,0)$, then we have

$$J_{(0,0)} = \begin{bmatrix} 1+ae^{b} & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of $J_{(0,0)}$ are $\sigma_1 = 1+ae^b$ and $\sigma_2 = 0$. So, $(0,0)$ is locally asymptotically stable if

$$|\sigma_1| < 1 \text{ and } |\sigma_2| < 1 \tag{10}$$

hold. Since $ae^b > 0$, one of the inequalities in (10) is not provided. Namely, the steady state $(0,0)$ is not locally asymptotically stable.
(b)-Similarly, if the Jacobian matrix of model (1) is created in the neighborhood of $(1,0)$, then we have

$$J_{(1,0)} = \begin{bmatrix} \frac{1+ae^b-abe^b}{(1+ae^b)} & -c \\ 0 & c \end{bmatrix}.$$

From this $(1,0)$, is locally asymptotically stable if

$$|c| < 1 \text{ and } \left|\frac{1+ae^b-abe^b}{(1+ae^b)}\right| < 1. \tag{11}$$

hold. We know that $c > 0$. If the (11) is solved, we obtain

$$c < 1,\ abe^b > 0 \text{ and } 2+2ae^b-abe^b > 0\ .$$

Since $abe^b > 0$ is always true, $(1,0)$ is locally asymptotically stable under the condition

$$c < 1 \text{ and } 2+2ae^b-abe^b > 0\ . \tag{12}$$

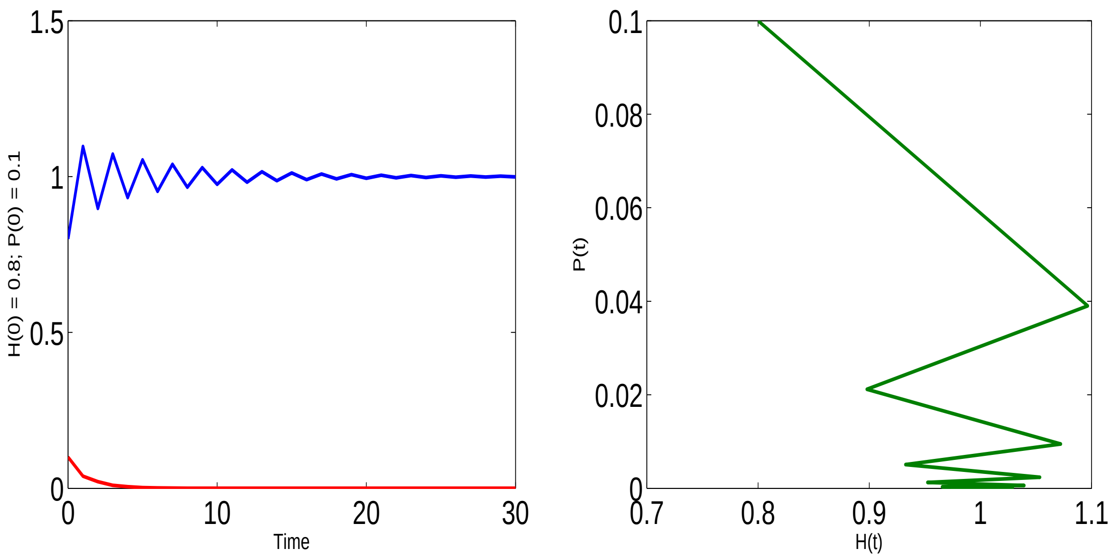

Figure 2: a & b

Figure 2.2-(a): Time series diagram of the model (1) where $a = 1.75$, $b = 2$ and $c = 0.5$. The initial conditions are $H_0 = 0.8$ and $P_0 = 0.1$.
Figure 2.2-(b): Phase diagram of the model (1) where $a = 1.75$, $b = 2$ and $c = 0.5$. The initial conditions are $H_0 = 0.8$ and $P_0 = 0.1$.
(c)-Finally, let' s consider that $c > 1$. The entries of the Jacobian matrix which is evaluated in the neighborhood of (H^*,P^*) can be written as follows:

$$J_{11} = e^{-cP^*}(1+ae^b)\frac{\left[(1+ae^{bH^*}-abe^{bH^*}H^*\right]}{(1+ae^{bH^*})^2}$$
$$J_{12} = -ce^{-cP^*}\frac{(1+ae^b)H^*}{(1+ae^{bH^*})}$$
$$J_{21} = (1-e^{-cP^*})$$
$$J_{22} = cH^*e^{-cP^*}.$$

From the definition of the determinant and the trace of the matrix $J_{(H^*,P^*)}$, we can write

$$trJ_{(H^*,P^*)} = e^{-cP^*}\left(\frac{(1+ae^b)\left[1+ae^{bH^*}-abe^{bH^*}H^*\right]}{(1+ae^{bH^*})^2}+cH^*\right)$$
$$\det J_{(H^*,P^*)} = e^{-2cP^*}\frac{(1+ae^b)\left[1+ae^{bH^*}-abe^{bH^*}H^*\right]}{(1+ae^{bH^*})^2}cH^* + ce^{-cP^*}\frac{(1+ae^b)H^*}{(1+ae^{bH^*})}(1-e^{-cP^*}).$$

respectively. If the following inequality (see [2])

$$|trJ| < 1+\det J < 2. \tag{13}$$

is provided, then (H^*,P^*) is locally asymptotically stable.
By using the inequality (13), we get that (H^*,P^*) is locally asymptotically stable if

$$e^{-cP^*}\left(\frac{(1+ae^b)\left[1+ae^{bH^*}-abe^{bH^*}H^*\right]}{(1+ae^{bH^*})^2}\right)\left(1-e^{-cP^*}cH^*\right)+cH^*e^{-cP^*}-ce^{-cP^*}\frac{(1+ae^b)H^*}{(1+ae^{bH^*})}(1-e-cP^*)<1$$

$$e^{-2cP^*}\frac{(1+ae^b)\left[1+ae^{bH^*}-abe^{bH^*}H^*\right]}{(1+ae^{bH^*})^2}cH^*+ce^{-cP^*}\frac{(1+ae^b)H^*}{(1+ae^{bH^*})}(1-e^{-cP^*})<1 \quad (14)$$

$$e^{-cP^*}\left(\frac{(1+ae^b)\left[1+ae^{bH^*}-abe^{bH^*}H^*\right]}{(1+ae^{bH^*})^2}\right)\left(1+e^{-cP^*}cH^*\right)+cH^*e^{-cP^*}+ce^{-cP^*}\frac{(1+ae^b)H^*}{(1+ae^{bH^*})}(1-e^{-cP^*})>-1 \quad (15)$$

such that $c>1$.

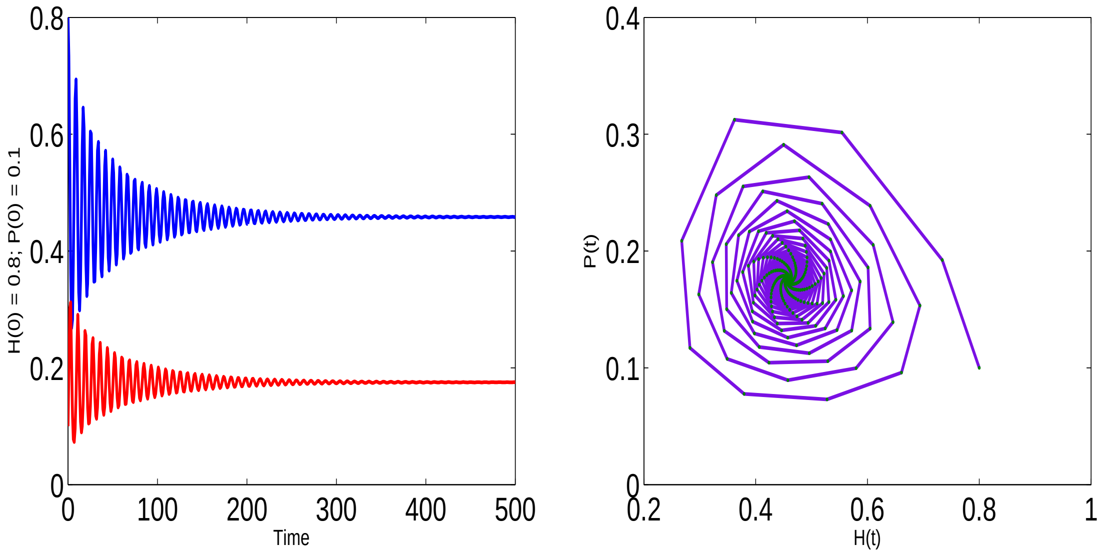

Figure 3: a & b

Figure 2.3-(a): Time series diagram of the model (1) where $a=1$, $b=1.25$ and $c=2.75$. The initial conditions are $H_0=0.8$ and $P_0=0.1$.
Figure 2.3-(b): Phase Diagram of the model (1) where $a=1$, $b=1.25$ and $c=2.75$ The initial conditions are $H_0=0.8$ and $P_0=0.1$. □

Corollary 2.4. *If the inequality $c\leq 1$ is provided, then the model (1) has the steady states $(0,0)$ and $(1,0)$. The steady state $(0,0)$ is always unstable. The steady state $(1,0)$ is only unique locally asymptotically stable point under condition (12).*

Corollary 2.5. *If the inequality $c>1$ is provided, then the model (1) has the steady states $(0,0)$, $(1,0)$ and (H^*,P^*). The steady state (H^*,P^*) is only unique locally asymptotically stable point under conditions (14).*

3. Steady states of the model (1) with immigration parameter

We will investigate the steady states of the model subject to the parameter β into host population in the model (1). Then, the general discrete-time host-population model is

$$\begin{aligned}H_{t+1}&=\frac{(1+ae^b)H_t}{1+ae^{bH_t}}e^{-cP_t}+\beta\ ;\ a,b,c>0\\ P_{t+1}&=H_t(1-e^{-cP_t}).\end{aligned} \quad (16)$$

Here, $\beta\in(1,\infty)$ is a diffusive force which called as immigration ([3, 4]). Now, let's examine the steady states of the model (16). From $H_t=H_{t+1}=H_1^*$ and $P_t=P_{t+1}=P_1^*$, we can write

$$\begin{aligned}H_1^*&=\frac{(1+ae^b)H_1^*}{1+ae^{bH_1^*}}e^{-cP_1^*}+\beta\ ;\ a,b,c>0\\ P_1^*&=H_1^*(1-e^{-cP_1^*}).\end{aligned} \quad (17)$$

Then, we have the following theorem.

Theorem 3.1. *If $0<\frac{(1+ae^{bH_1^*})(H_1^*-\beta)}{(1+ae^b)H_1^*}<1$ is provided, then the model (16) has the steady state $(H_1^*,0)$ and (H_1^*,P_1^*) such that $H_1^*>\beta$. Otherwise, $(H_1^*,0)$ is unique steady state.*

Proof. It is clearly seen that there is not the steady state $(0,0)$. Since $\beta \in (1,\infty)$; it must be $H_1^* \neq 0$. Let's take $H_1^* \neq 0$ and $P_1^* = 0$. Then we can write

$$H_1^* = \frac{(1+ae^b)H_1^*}{1+ae^{bH_1^*}} + \beta$$

$$\Rightarrow (1+ae^b) = \frac{(1+ae^{bH_1^*})(H_1^*-\beta)}{H_1^*} \quad , \quad H_1^* > \beta \tag{18}$$

from Eq.(17). Let's define the following the function such that $H_1^* = x$,

$$g(x) = \frac{(1+ae^{bx})(x-\beta)}{x} \quad , \quad x \neq 0 \tag{19}$$

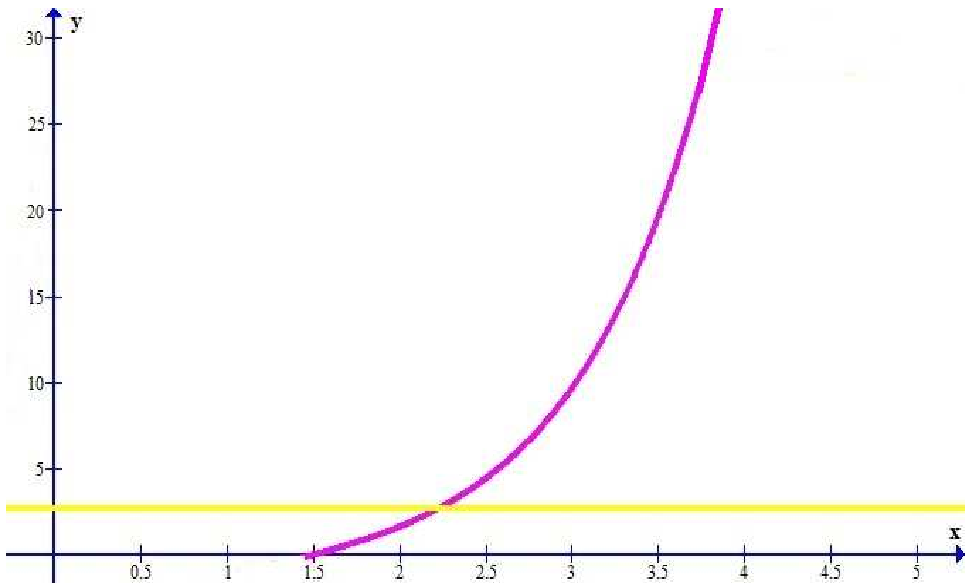

Figure 4: Graphs showing the intersection point

Figure 3.1 Graphs showing the intersection point H_1^* in (18) where $a = 0.5$, $b = 1.2$ and $\beta = 1.5$.
From the $g'(x) = 0$, we have

$$abe^{bx}(1-\frac{\beta}{x}) = -\frac{\beta}{x^2}(1+ae^{bx}). \tag{20}$$

Eq.(20) has not an interaction point. Also, we can seen that $g(x) = \infty$ as $x \to \infty$. Since the function $g(x)$ is increasing, Eq.(18) has an interaction point H_1^*.
Now, let's investigate other points (H_1^*, P_1^*) of the model (16) such that $H_1^* \neq 0$ and $P_1^* \neq 0$. If the first equality in (16) is considered, we can write

$$H_1^* = \frac{(1+ae^b)H_1^*}{1+ae^{bH_1^*}} e^{-cP_1^*} + \beta$$
$$\Rightarrow (1+ae^{bH_1^*})(H_1^*-\beta) = (1+ae^b)H_1^* e^{-cP_1^*}$$

$$e^{-cP_1^*} = \frac{(1+ae^{bH_1^*})(H_1^*-\beta)}{(1+ae^b)H_1^*} \tag{21}$$

$$\Rightarrow P_1^* = -\frac{1}{c}\ln\frac{(1+ae^{bH_1^*})(H_1^*-\beta)}{(1+ae^b)H_1^*} \tag{22}$$

If the following inequality is provided

$$0 < \frac{(1+ae^{bH_1^*})(H_1^*-\beta)}{(1+ae^b)H_1^*} < 1 \tag{23}$$

then $P_1^* > 0$ in (22). Also, we have

$$\beta < H_1^* \text{ and } (1+ae^{bH_1^*})(1-\frac{\beta}{H_1^*}) < 1+ae^b$$

from inequality (23). If we combine (21) with the second equation of (16), then we obtain

$$P_1^* = H_1^*(1 - \frac{(1+ae^{bH_1^*})(H_1^* - \beta)}{(1+ae^b)H_1^*}).$$

If P_1^* is written in the first equation in the model (16), then we obtain

$$H_1^* = \frac{(1+ae^b)H_1^*}{1+ae^{bH_1^*}} e^{-cH_1^*(1-\frac{(1+ae^{bH_1^*})(H_1^*-\beta)}{(1+ae^b)H_1^*})} + \beta$$

$$\Rightarrow 1 + ae^b = (1+ae^{bH_1^*})(1 - \frac{\beta}{H_1^*}) e^{cH_1^*(1-\frac{(1+ae^{bH_1^*})(H_1^*-\beta)}{(1+ae^b)H_1^*})} \tag{24}$$

If the previous similar operations are done, it is seen that Eq.(24) has an interaction point. Let's write the following function by using the right side (24)

$$h(x) = (1+ae^{bx})(1 - \frac{\beta}{x}) e^{cx(1-\frac{(1+ae^{bx})(x-\beta)}{(1+ae^b)x})}$$

for $H_1^* = x$. Here, $h(x) \to 0$ as $x \to \infty$.

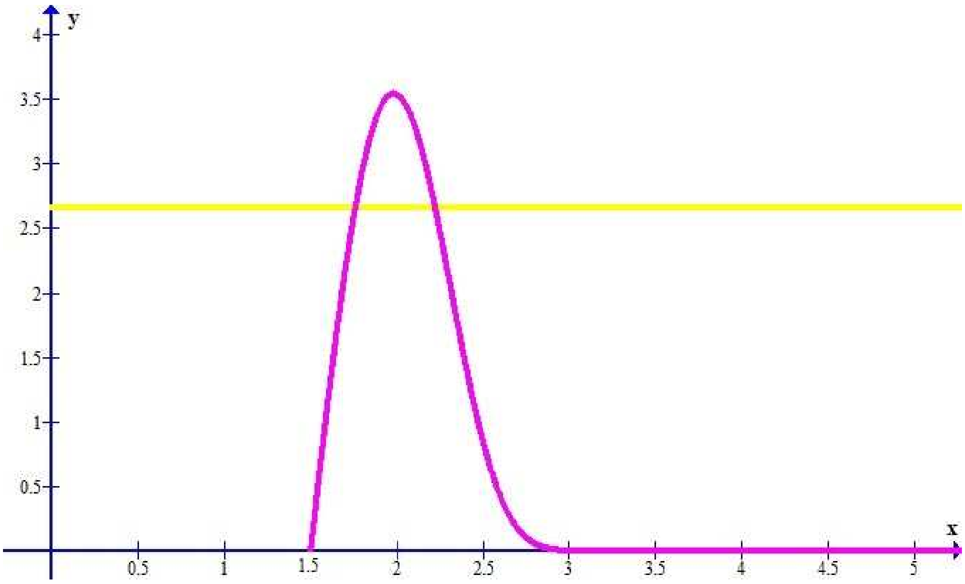

Figure 5: Graphs of $(1+ae^b)$ and function $h(x)$

Figure 3.2 Graphs of $(1+ae^b)$ *and function* $h(x)$ *where* $a = 0.5$, $b = 1.2$, $c = 1$ *and* $\beta = 1.5$. □

3.1. Stability analysis of model (1) with immigration parameter

In this section, we will investigate the locally asymptotically stability conditions of steady states of (16). If the model (16) is considered, we can write

$$F(H_t, P_t) = \frac{(1+ae^b)H_t}{1+ae^{bH_t}} e^{-cP_t} + \beta \ ; \ a,b,c > 0$$
$$G(H_t, P_t) = H_t(1 - e^{-cP_t}).$$

Then, we have the following theorem.

Theorem 3.2. *For the steady states of the model (16), the following statements hold true:*
(a)-Assume that the inequality (23) is not provided. The steady state $(H_1^*, 0)$ *of the model (16) has the locally asymptotically stable if the conditions (14) are provided for* $(H_1^*, 0)$.
(b)-Assume that the inequality (23) is provided and $(H_1^*, 0)$ *unstable. The steady state* (H_1^*, P_1^*) *of the model (16) has the locally asymptotically stable if the conditions (14) are provided for* (H_1^*, P_1^*).

Proof. (a)-On the assumption, $(H_1^*, 0)$ is unique steady state of the model (16). If the Jacobian matrix, evaluated in the neighborhood of $(H_1^*, 0)$, is written, we get

$$J_{(H_1^*,0)} = \begin{bmatrix} \frac{(1+ae^b)[(1+ae^{bH_1^*})-abe^{bH_1^*}H_1^*]}{(1+ae^{bH_1^*})^2} & -c\frac{(1+ae^b)H_1^*}{1+ae^{H_1^*}} \\ 0 & cH_1^* \end{bmatrix}.$$

The eigenvalues of $J_{(H_1^*,0)}$ are $\lambda_1 = \frac{(1+ae^b)[(1+ae^{bH_1^*})-abe^{bH_1^*}H_1^*]}{(1+ae^{bH_1^*})^2}$ and $\lambda_2 = cH_1^*$. Consequently, $(H_1^*,0)$ is locally asymptotically stable if

$$\left|\frac{(1+ae^b)[(1+ae^{bH_1^*})-ae^{bH_1^*}H_1^*]}{(1+ae^{bH_1^*})^2}\right| < 1 \text{ and } |cH_1^*| < 1.$$

We know that $H_1^* > 1$ from Theorem 5 and $c > 0$. If the last inequalities are clearly written, we have the following inequalities

$$(1+ae^{bH_1^*})^2 - (1+ae^b)[(1+ae^{bH_1^*})-ae^{bH_1^*}H_1^*] > 0 \tag{25}$$

$$(1+ae^{bH_1^*})^2 + (1+ae^b)[(1+ae^{bH_1^*})-ae^{bH_1^*}H_1^*] > 0$$

$$cH_1^* < 1.$$

(b)-On the assumption, we must consider that the conditions (25) are not provided. Then, we can investigate locally asymptotic stability conditions for (H_1^*,P_1^*). The locally asymptotic stability conditions founded for the steady state (H^*,P^*) of the model (1) are also applied to stability of the steady state (H_1^*,P_1^*). So, if the conditions (14) are re-written for (H_1^*,P_1^*), then we get as follows:

$$e^{-cP_1^*}\left(\frac{(1+ae^b)\left[1+ae^{bH_1^*}-abe^{bH_1^*}H_1^*\right]}{(1+ae^{bH_1^*})^2}\right)\left(1-e^{-cP_1^*}cH_1^*\right)+cH_1^*e^{-cP_1^*}-ce^{-cP_1^*}\frac{(1+ae^b)H_1^*}{(1+ae^{bH_1^*})}(1-e^{-cP_1^*}) < 1$$

$$e^{-2cP_1^*}\frac{(1+ae^b)\left[1+ae^{bH_1^*}-abe^{bH_1^*}H_1^*\right]}{(1+ae^{bH_1^*})^2}cH_1^*+ce^{-cP_1^*}\frac{(1+ae^b)H_1^*}{(1+ae^{bH_1^*})}(1\text{-}e^{-cP_1^*}) < 1 \tag{26}$$

$$e^{-cP_1^*}\left(\frac{(1+ae^b)\left[1+ae^{bH_1^*}-abe^{bH_1^*}H_1^*\right]}{(1+ae^{bH_1^*})^2}\right)\left(1+e^{-cP_1^*}cH_1^*\right)+cH_1^*e^{-cP_1^*}+ce^{-cP_1^*}\frac{(1+ae^b)H_1^*}{(1+ae^{bH_1^*})}(1-e^{-cP_1^*}) > -1 \tag{27}$$

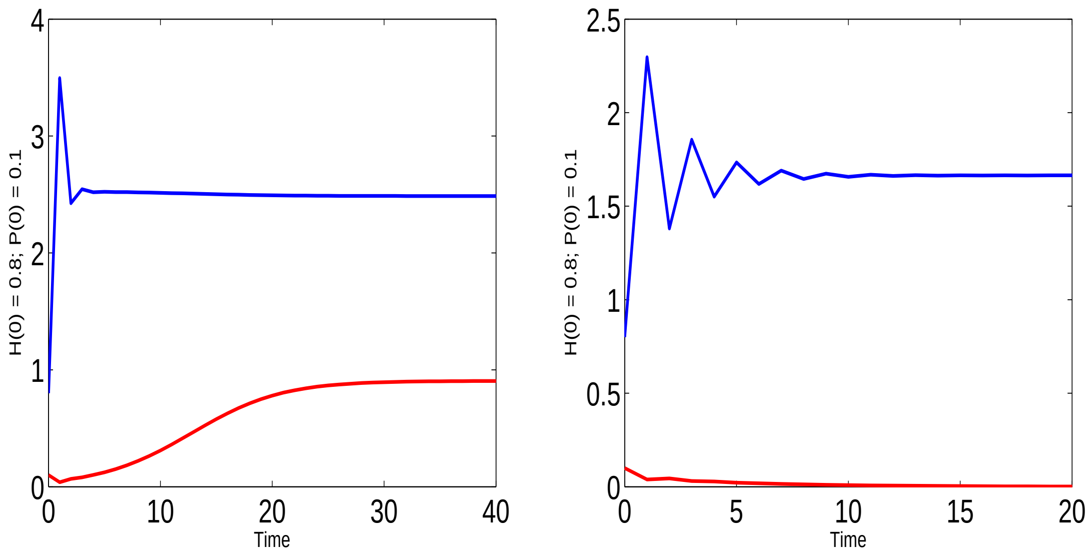

Figure 6: Time series diagram of the model (1)

Figure 3.3. (a): Time series diagram of the model (1) where $a = 1$, $b = 2$, $c = 0.5$ and $\beta = 2.4$.*The initial conditions are* $H_0 = 0.8$ *and* $P_0 = 0.1$.
Figure 3.3. (b): Time series diagram of the model (1) where $a = 1$, $b = 2$, $c = 0.5$ and $\beta = 1.2$.*The initial conditions are* $H_0 = 0.8$ *and* $P_0 = 0.1$. □

Corollary 3.3. *The steady states* $(0,0)$, $(1,0)$ *and* (H^*,P^*) *of the model (1) under immigration parameter disappear. The model (1) which subject to immigration parameter appears the steady states* $(H_1^*,0)$ *and* (H_1^*,P_1^*). *When inequality (23) is not provided,* $(H_1^*,0)$ *is unique steady state of the model (16), and it is locally asymptotically stable under (25). Otherwise, the model (16) has steady states* $(H_1^*,0)$ *and* (H_1^*,P_1^*). *If* $(H_1^*,0)$ *unstable, then the steady state* (H_1^*,P_1^*) *is locally asymptotically stable under (26).*

4. The stability analysis of model (1) with Allee effect

We will investigate the steady states of the model by including the Allee effect $\alpha(H_t)$ into host population in the model (1). Then, the general discrete-time host-population model is as follows:

$$\begin{aligned} H_{t+1,\alpha} &= \frac{\alpha^* H_t}{1+ae^{bH_t}} e^{-cP_t} \alpha(H_t)\ ;\ \ a,b,c>0 \\ P_{t+1} &= H_t(1-e^{-cP_t}), \end{aligned} \tag{28}$$

where $\alpha^* = (1+ae^b)/\alpha$ such that $\alpha : \alpha(H_t) > 0$. Therefore, it is clear that the model (1) and the model (28) have the same steady states. The following assumptions on the Allee function α are derived from biological facts:
(i) If there are no partners, there is no reproduction. Mathematically speaking, the Allee function is zero when the population density is zero.
(ii) Allee effect increases as density increases. Mathematically speaking, the derivatives of the Allee function are always positive for all positive values.
(iii) Allee effect disappear at high densities. Namely, limit of the Allee function approaches to 1 as the population size increases.

Theorem 4.1. $(0,0)$ *is unique locally asymptotically stable steady state of the model (28).*

Proof. The entries of the Jacobian matrix associated with the model (28) are given as follows:

$$\begin{aligned} J_{11,\alpha} &= \frac{\alpha^* e^{-cP_t}}{(1+ae^{bH_t})^2}([\alpha(H_t)+\alpha'(H_t)H_t](1+ae^{bH_t}) \\ &-abe^{bH_t}\alpha(H_t)H_t) \\ J_{12,\alpha} &= -\alpha^* ce^{-cP_t}\frac{H_t\alpha(H_t)}{(1+ae^{bH_t})} \\ J_{21,\alpha} &= (1-e^{-cP_t}) \\ J_{22,\alpha} &= cH_te^{-cP_t}. \end{aligned} \tag{29}$$

The Jacobian matrix of the model (28) about the steady state $(1,0)$ is

$$J_{\alpha(1,0)} = \begin{bmatrix} \frac{\alpha^*}{(1+ae^b)^2}([\alpha(1)+\alpha'(1)](1+ae^b)-abe^b\alpha(1)) & \frac{-\alpha^*\alpha(1)}{(1+ae^b)} \\ 0 & c \end{bmatrix}.$$

Consequently, since $\lambda_{1,\alpha} = \frac{\alpha^*}{(1+ae^b)^2}([\alpha(1)+\alpha'(1)](1+ae^b)-abe^b\alpha(1))$ and $\lambda_{2,\alpha}=c$, the steady state $(1,0)$ is locally asymptotically stable if

$$\left|\frac{\alpha^*}{(1+ae^b)^2}([\alpha(1)+\alpha'(1)](1+ae^b)-\alpha^* abe^b\alpha(1))\right| < 1 \text{ and } c<1$$

Also, by using (29), the entries of the Jacobian matrix of the model (28) of about (H^*,P^*) are given as follows:

$$\begin{aligned} J_{\alpha,11} &= \frac{\alpha^* e^{-cP^*}}{(1+ae^{bH^*})^2}([\alpha(H^*)+\alpha'(H^*)H^*](1+ae^{bH^*})-abe^{bH^*}\alpha(H^*)H^*) \\ J_{\alpha,12} &= -\alpha^* ce^{-cP_t}\frac{H^*\alpha(H^*)}{(1+ae^{bH^*})} \\ J_{\alpha,21} &= (1-e^{-cP^*}) \\ J_{\alpha,22} &= cH^*e^{-cP^*}. \end{aligned}$$

From the definition of the determinant and the trace of the matrix $J_{(H^*,P^*)}$, we have

$$\begin{aligned} trJ_{\alpha(H^*,P^*)} &= \frac{\alpha^* e^{-cP^*}}{(1+ae^{bH^*})^2}([\alpha(H^*)+\alpha'(H^*)H^*](1+ae^{bH^*}) \\ &-abe^{bH^*}\alpha(H^*)H^*)+cH^*e^{-cP^*} \\ \det J_{\alpha(H^*,P^*)} &= \frac{\alpha^* e^{-2cP^*}cH^*}{(1+ae^{bH^*})^2}([\alpha(H^*)+\alpha'(H^*)H^*](1+ae^{bH^*})-abe^{bH^*}\alpha(H^*)H^*) \\ &+\alpha^* ce^{-cP_t}\frac{H^*\alpha(H^*)}{(1+ae^{bH^*})}(1-e^{-cP^*}). \end{aligned}$$

The (13) yields the following inequality

$$\left|\begin{array}{c} \frac{\alpha^* e^{-cP^*}}{(1+ae^{bH^*})^2}([\alpha(H^*)+\alpha'(H^*)H^*](1+ae^{bH^*}) \\ -abe^{bH^*}\alpha(H^*)H^*+cH^*e^{-cP^*}]) \end{array}\right| < 1+\frac{\alpha^* e^{-2cP^*}cH^*}{(1+ae^{bH^*})^2}([\alpha(H^*)+\alpha'(H^*)H^*](1+ae^{bH^*})-abe^{bH^*}\alpha(H^*)H)$$

$$\alpha^* e^{-cP^*}\frac{\alpha(H^*)H^*}{(1+ae^{bH^*})}(1-e^{-cP^*}) < 2$$

such that $c > 1$. Similarly, the Jacobian matrix of the model (28) about the steady state $(0,0)$ is

$$J_{\alpha(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $(0,0)$ is locally asymptotically stable steady state in every situation. □

Corollary 4.2. *The model (1) with and without Allee effect have the same steady states. As the steady state $(1,0)$ and (H^*,P^*) become unstable with Allee effect, the steady state $(0,0)$ becomes locally asymptotically stable. So, $(0,0)$ is unique locally asymptotically stable steady state in the model (1) which subject to Allee effect.*

5. Conclusion

In this paper, we investigated the steady states of the model (1) with and without immigration parameter and Allee effect. Also, we examined the locally asymptotically stability of steady states of this models. So, we have reached some dynamical consequences which give conditions on stability of the steady states.

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Compact Finite Differences Method for FitzHugh-Nagumo Equation

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Abstract

In this paper, we developed the compact finite differences method to find approximate solutions for the FitzHugh-Nagumo (F-N) equations. To the best of our knowledge, until now there is no compact finite difference solutions have been reported for the FitzHugh-Nagumo equation arising in gene propagation and model. We have given numerical example to demonstrate the validity and applicability.

1. Introduction

Nonlinear systems play a pivotal role in the mathematical modeling of scientific and engineering problems. The FitzHugh-Nagumo equation is a nonlinear reaction-diffusion equation that models an active pulse transmission line simulating a nerve axon [9] and it is used area of population genetics [4], circuit theory, and other fields [1], [8]. It is expressed as

$$u_t - u_{xx} + u(1-u)(\rho - u) = 0 \tag{1.1}$$

where $0 < \rho < 1$ and $u(x,t)$ is of the unknown function depending on the temporal variable t and the spatial variable x. If we take $\rho = -1$, then Eq. (1.1) converts into the Newell-Whitehead equation.

$$u_t - u_{xx} + u^3 - u = 0$$

F-N equation (1.1) combines diffusion, and nonlinearity which is controlled by the term $u(1-u)(\rho - u)$. Many physicists and mathematicians have paid much attention to the Fitzhugh-Nagumo equation in recent years due to its importance in mathematical physics. Shih et al.[10] studied this equation and showed its applications in the field of population and circuit theory. In a study, the authors of a paper [6] examined the F-N equation and derived a novel series of exact solutions with the aid of the first integral technique. In another investigation, Abbasbandy [15] applied the homotopy anlaysis approach to obtain the soliton solution of the F-N equation. In an attempt, the authors of an article [13] investigated the variational method for solving both the Nagumo telegraph and the Nagumo reaction-diffusion partial differential equations. Jacobi elliptic function has been presented by Nucci and Clarkson [11] to obtain the solution of the F-N equation. In another study, Jackson [3] examined the semi-discrete estimates for the F-N equations. Moreover, the semi-analytical techniques have been successfully applied by Dehghan et al.[9], to present the approximate solution of the standard F-N equation.

In this work, we aim to investigate a generalized Fitzhugh-Nagumo equation with time-dependent coefficients and linear dispersion term given by

$$u_t + \alpha(t)u_x - \beta(t)u_{xx} + \gamma(t)u(1-u)(\rho - u) = 0, \quad (x,t) \in [A,B] \times [0,T] \tag{1.2}$$

boundary condition

$$u(A,t) = g_1(t), \quad u(B,t) = g_2(t), \quad t \in [0,T]$$

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initial condition

$$u(x,0)=f(t), \quad x\in[A,B]$$

where $\alpha(t),\beta(t)$ and $\gamma(t)$ are arbitrary functions of t. $\alpha(t),\beta(t)$ and $\gamma(t)$ are all real-valued functions. For $\alpha(t)=0$ and $\beta(t)=\gamma(t)=1$, Eq. (1.2) will be reduced to the standard Fitzhugh-Nagumo equation (1.1).
The time-dependent Fitzhugh-Nagumo equation cannot be integrated by the classical integration methods. Triki and Wazwaz [7] examined a generalized F-N equation exhibiting time varying coefficients and linear dispersion term. Jacobi-Gauss-Lobatto collocation method has been applied for generalized F-N equation by Bhrawy [2].
Compact finite difference methods are techniques used in applied mathematics and scientific computing to numerically solve linear and nonlinear differential equations. Mohanty et al. [14] used new two-level implicit compact operator method for the solution of Burgers-Huxley equation. In a study, the authors of a paper [5] derived solution of the parabolic problems with delay using compact finite difference methods. Wang et al. [17] applied compact finite difference scheme to study the coupled Gross-Pitaevskii equations. In another investigation, Wu and Xu [18] derived the solutions of 2D Helmholtz equation with the compact sixth-order finite difference scheme.
Although there are many methods to construct the compact schemes, Pade Appoximation Method and Taylor Series Method which are the two basic approximations came into prominence. Many researchers are using higher order compact finite difference schemes to solve differential equations. This is because significant improvements to the accuracy of numerical solutions have been obtained by using fourth or sixth order compact finite difference schemes. Another advantage is that the high accuracy is obtained on coarser grids which ensures greater computational efficiency [12]. In this study, compact finite differences schemes for the first and second derivative approximations are constructed both for the inner points and the boundary points by using the Taylor approximation. Along the spatial coordinate , first and second derivatives are replaced with the fifth order compact schemes for the inner points and the sixth order compact schemes for the boundary points. The paper is arranged as follows: In Section 2, compact finite difference method for F-N equation is presented. In Section 3, numerical results for different problems are presented in tables and conclusion is given in Section 4.

2. Compact Finite Differences Method

Compact finite difference method is a special finite difference method which uses the values of the function and its derivatives only at three consecutive points. The independent variable nodes are given as $x_i=h(i-1)$ where $h=(b-a)/(N-1)$, in the interval of $[a,b]$ for $1\le i\le N$. If the function values at the nodes are given as $f_i=f(x_i)$ the finite differences approximation to the first derivative $f_i^{'}$ at the node indexed by i depends on the function values at the neighbor nodes [16].
The approximation for the first derivative of the function is expressed as in the following

$$f_{i-2}^{'}+\eta f_{i-1}^{'}+f_i^{'}+\eta f_{i+1}^{'}+\gamma f_{i+2}^{'}=c\frac{f_{i+3}-f_{i-3}}{6h}+b\frac{f_{i+2}-f_{i-2}}{4h}+a\frac{f_{i+1}-f_{i-1}}{2h} \tag{2.1}$$

where $f_i=f(x_i)$ and coefficients can be determined by the Taylor expansion. To get the fourth order tridiagonal schemes, the coefficients γ and c are set to be 0. The order of accuracy for the approximations for the first and second derivatives are calculated as $O(h^5)$ for inner points and $O(h^6)$ for the boundary points. The approximation for the second derivative of the function is expressed as in the following

$$\gamma f_{i-2}^{''}+\eta f_{i-1}^{''}+f_i^{''}+\eta f_{i+1}^{''}+\gamma f_{i+2}^{''}=c\frac{f_{i+3}-2f_i+f_{i-3}}{9h^2}+b\frac{f_{i+2}-2f_i+f_{i-2}}{4h^2}+a\frac{f_{i+1}-2f_i+f_{i-1}}{h^2} \tag{2.2}$$

For boundary points, the approximation is written in the following form

$$f_1^{'}+\eta f_2^{'}=\frac{1}{h}(af_1+bf_2+cf_3+df_4)$$
$$f_1^{''}+\eta f_2^{''}=\frac{1}{h^2}(af_1+bf_2+cf_3+df_4+ef_5)$$

The equations (2.1) and (2.2) can easily be adapted for the boundary point $i=N$. The compact schemes for the first and second derivative approximations for the interior and boundary points with their calculated coefficients can be seen below.

For the first derivative

$$\frac{h}{3}(-17u_1^{'}-14u_2^{'}+u_3^{'})=u_0+8u_1-9u_2 \tag{2.3}$$
$$\frac{h}{3}(u_{i-1}^{'}+4u_i^{'}+u_{i+1}^{'})=-u_{i-1}+u_{i+1} \tag{2.4}$$
$$\frac{h}{3}\left(\frac{1}{8}u_{N-3}^{'}-\frac{5}{8}u_{N-2}^{'}+\frac{19}{8}u_{N-1}^{'}+\frac{9}{8}u_N^{'}\right)=-u_{N-1}+u_N \tag{2.5}$$

For the second derivative

$$\frac{h^2}{12}(14u_1^{''}-5u_2^{''}+4u_3^{''}-5u_4^{''})=u_0-2u_i+u_{i+1} \tag{2.6}$$
$$\frac{h^2}{12}(u_{i-1}^{''}+10u_i^{''}+u_{i+1}^{''})=u_{i-1}-2u_i+u_{i+1} \tag{2.7}$$
$$\frac{h^2}{12}(-u_{N-4}^{''}+4u_{N-3}^{''}-5u_{N-2}^{''}+14u_{N-1}^{''})=u_{N-2}-2u_{N-1}+u_N \tag{2.8}$$

By taking 7 nodes, the matrices obtained from (2.3), (2.4) and (2.5) are as in the following.

$$A_1 = \begin{pmatrix} -\frac{17h}{3} & -\frac{14h}{3} & \frac{h}{3} & 0 & 0 \\ \frac{h}{3} & \frac{4h}{3} & \frac{h}{3} & 0 & 0 \\ 0 & \frac{h}{3} & \frac{4h}{3} & \frac{h}{3} & 0 \\ 0 & 0 & \frac{h}{3} & \frac{4h}{3} & \frac{h}{3} \\ 0 & \frac{h}{24} & \frac{-5h}{24} & \frac{19h}{24} & \frac{9h}{24} \end{pmatrix}, \quad U' = \begin{pmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \\ u_5' \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 8 & -9 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix U' having the approximation values of the first derivative at each node is calculated via the equation (2.9). The following calculations are carried out to obtain the matrix U' after applying the LU decomposition technique to the known matrix A_1

$$A_1 U' = K_1 U + H_1 \tag{2.9}$$

LU decomposition technique is applied to the known matrix A_1 and other calculations are as in the following.

$$\begin{aligned} (L_0 U_0) U' &= K_1 U + H_1 \\ L_0^{-1} L_0 U_0 U' &= L_0^{-1} K_1 U + L_0^{-1} H_1 \\ T_1 &= L_0^{-1} K_1 \\ G_1 &= L_0^{-1} H_1 \\ U_0 U' &= T_1 U + G_1 \\ U_0^{-1} U_0 U' &= U_0^{-1} T_1 U + U_0^{-1} G_1 \\ S_1 &= U_0^{-1} G_1 \\ C_1 &= U_0^{-1} T_1 \end{aligned}$$

$$U' = C_1 U + S_1 \tag{2.10}$$

Using the compact schemes in (2.6), (2.7) and (2.8), the matrices below are obtained to get the matrix U'' having the approximation values of the second derivative at each node and similar calculations are carried out for that. U' and U'' are inserted as a first and second derivatives of the function while constructing the discretization scheme.

$$A_2 = \begin{pmatrix} \frac{14h^2}{12} & -\frac{5h^2}{12} & \frac{4h^2}{12} & -\frac{h^2}{12} & 0 \\ \frac{h^2}{12} & \frac{10h^2}{12} & \frac{h^2}{12} & 0 & 0 \\ 0 & \frac{h^2}{12} & \frac{10h^2}{12} & \frac{h^2}{12} & 0 \\ 0 & 0 & \frac{h^2}{12} & \frac{10h^2}{12} & \frac{h^2}{12} \\ 0 & -\frac{h^2}{12} & \frac{4h^2}{12} & -\frac{5h^2}{12} & \frac{14h^2}{12} \end{pmatrix}, \quad U'' = \begin{pmatrix} u_1'' \\ u_2'' \\ u_3'' \\ u_4'' \\ u_5'' \end{pmatrix}$$

,

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}, \quad K_2 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ 0 \\ u_6 \end{pmatrix}$$

$$A_2 U'' = K_2 U + H_2$$

$$U'' = C_2 U + S_2 \tag{2.11}$$

2.1. Compact Finite Difference method for Fitzhugh-Nagumo equation

Equations (2.10) and (2.11) approximating to the first and second derivatives of functions u is substituted to the equation (1.2). Compact finite differences for spatial dimension and finite differences along the time axis are applied, consequently they are rearranged via explicit approximation and the following discretization scheme is obtained.

$$\frac{U_n^{j+1} - U_n^j}{\Delta t} + \alpha(t_j)(C_1 U_n^j + S_1) - \beta(t_j)(C_2 U_n^j + S_2) + \gamma(t_j) U_n^j (1 - U_n^j)(\rho - U_n^j)$$

While the approximations to the functions u via compact scheme is being constructed, the known boundary values are placed in the vectors H_1 and H_2. And the vectors S_1 and S_2 are calculated in each time step. Because they are dependent on the approximation values of u after the first step.

$$H_1 = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ 0 \\ u_N \end{pmatrix}$$

3. Numerical Results

In this section, solution for equation in (1.2) is obtained via compact finite differences method. To illustrate the efficiency of the compact finite differences method for the problem handled in this study, the maximum error which is defined by the equation below

$$L_\infty = \max_{1\leq j\leq N} |u(x_j,t) - U(x_j,t)|$$

where $u(x_j,t)$ and $U(x_j,t)$ refer to the exact solution and solution via compact finite differences method, respectively.

Example 1. Consider equation (1.2) with $\alpha(t) = 0, \beta(t) = 1, \gamma(t) = 1$:

$$u_t - u_{xx} + u(1-u)(\rho - u) = 0; \quad (x,t) \in [-10,10] \times [0,T] \tag{3.1}$$

subject to the boundary conditions

$$\begin{aligned} u(-10,t) &= \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2\sqrt{2}}(-10 - \frac{2\rho-1}{\sqrt{2}}t)) \\ u(10,t) &= \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2\sqrt{2}}(10 - \frac{2\rho-1}{\sqrt{2}}t)) \end{aligned}$$

and the initial condition

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{x}{2\sqrt{2}}), x \in [-10,10]$$

The exact solution of Eq. (3.1) is

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2\sqrt{2}}(x - \frac{2\rho-1}{\sqrt{2}}t))$$

The results for example 1 are displayed in Tables 4.1 and Figure 3.2. Numerical solutions at different T are presented for computational domain $[-10,10]$ using compact finite difference method in Table 1. And in Table 2, we exhibit the maximum absolute errors between exact and approximate solutions for $N = 12, 48, 64$. The Fig. 3.1 display the numerical results with exact one for $\rho = 0.75, \Delta t = 0.001, N = 24$ in 3D form up to time $T = 1$. The Fig. 3.2 compares the numerical solutions with exact one for different ρ values. As it can be seen from the Table 1 and Table 2 the compact finite differences method is very accurate.

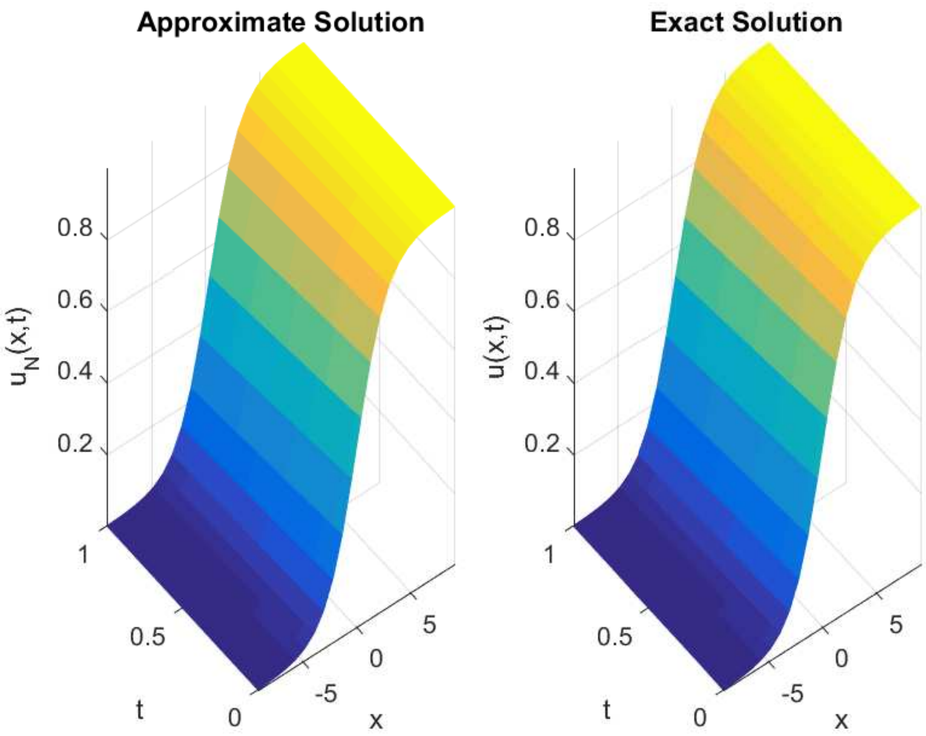

Figure 3.1: For $N = 24, \Delta t = 0.001, T = 1, \rho = 0.75$ solution of Fitzhugh-Nagumo equation for Example 1

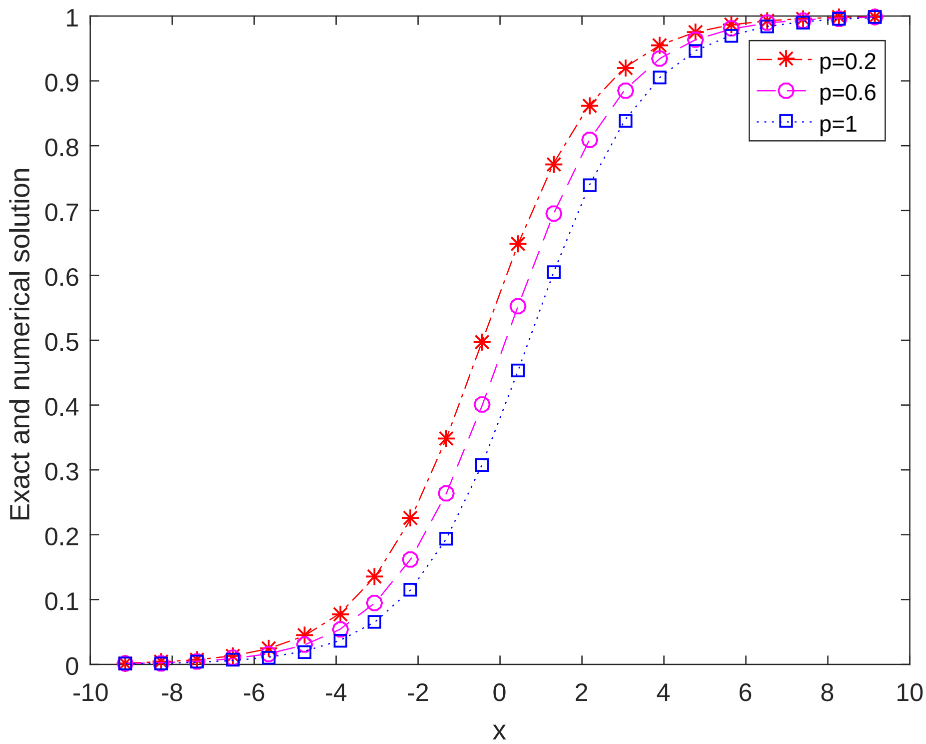

Figure 3.2: For $N = 24, \Delta t = 0.001, T = 1$ solution of Fitzhugh-Nagumo equation for Example 1

Table 2:

Example 2. In this example we examine the nonlinear time-dependent generalized F-H equation with time coefficients.

$$u_t + \cos(t)u_x - \cos(t)u_{xx} - 2\cos(t)(u(1-u)(\rho - u)) = 0; (x,t) \in [-10, 10] \times [0, T] \tag{3.2}$$

dt	T=0.2	T=1	T=2	T=3	T=4
dt=0.1	7.2090e-05	2.8991e-04	5.2054e-04	6.8524e-04	8.8408e-04
dt=0.01	2.3475e-05	8.0706e-05	1.1019e-04	1.5188e-04	1.7331e-04
dt=0.001	1.8739e-05	6.2242e-05	7.7129e-05	9.8812e-05	1.1203e-04
dt=0.0001	1.8267e-05	6.0398e-05	7.4378e-05	9.3508e-05	1.0590e-04

Table 1: Maximum error with $\rho = 0.75$ and $N = 24$ for Example 1

N	dt=0.01	dt=0.001	dt=0.0001
12	3.9857e-04	3.9300e-04	3.9244e-04
48	8.3749e-06	4.0905e-06	3.7343e-06
64	5.9363e-06	8.1794e-07	3.9098e-07

Table 2: Maximum error with $T = 0.2$ and $\rho = 0.75$ for Example 1

subject to the boundary conditions

$$u(-10,t) = \frac{\rho}{2} + \frac{\rho}{2}\tanh(\frac{\rho}{2}(-10-(3-\rho)\sin(t))),$$
$$u(10,t) = \frac{\rho}{2} + \frac{\rho}{2}\tanh(\frac{\rho}{2}(10-(3-\rho)\sin(t)))$$

and initial condition

$$u(x,0) = \frac{\rho}{2} + \frac{\rho}{2}\tanh(\frac{\rho x}{2}), x \in [-10,10]$$

The analytical solution of Eq.(3.2) is

$$u(x,t) = \frac{\rho}{2} + \frac{\rho}{2}\tanh(\frac{\rho}{2}(x-(3-\rho)\sin(t)))$$

The observed maximum absolute errors for different values of ρ are given in Table 3. The numerical results are illustrated in Fig. 3.3

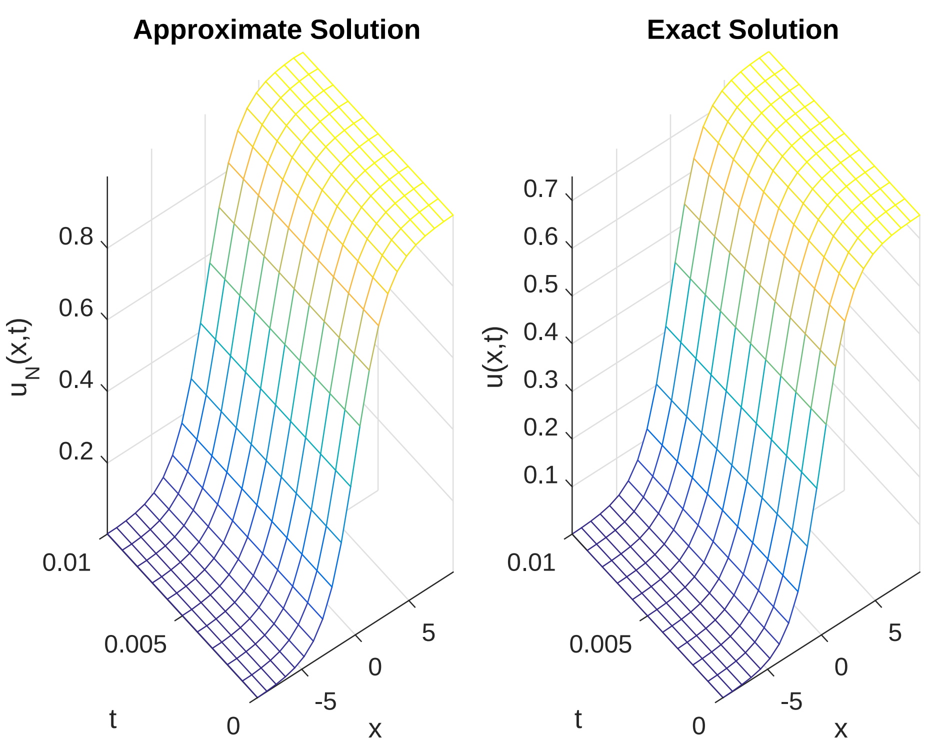

Figure 3.3: For $N = 24, \Delta t = 0.001, T = 0.01, \rho = 0.75$, solution of Fitzhugh-Nagumo equation for Example 2

ρ	dt=0.1	dt=0.01	dt=0.001
ρ=0.25	0.0449	0.0451	0.0451
ρ=0.5	0.1481	0.1489	0.1490
ρ=0.75	0.2682	0.2700	0.2701

Table 3: Maximum error with $T = 1$ and $N = 24$ for Example 2

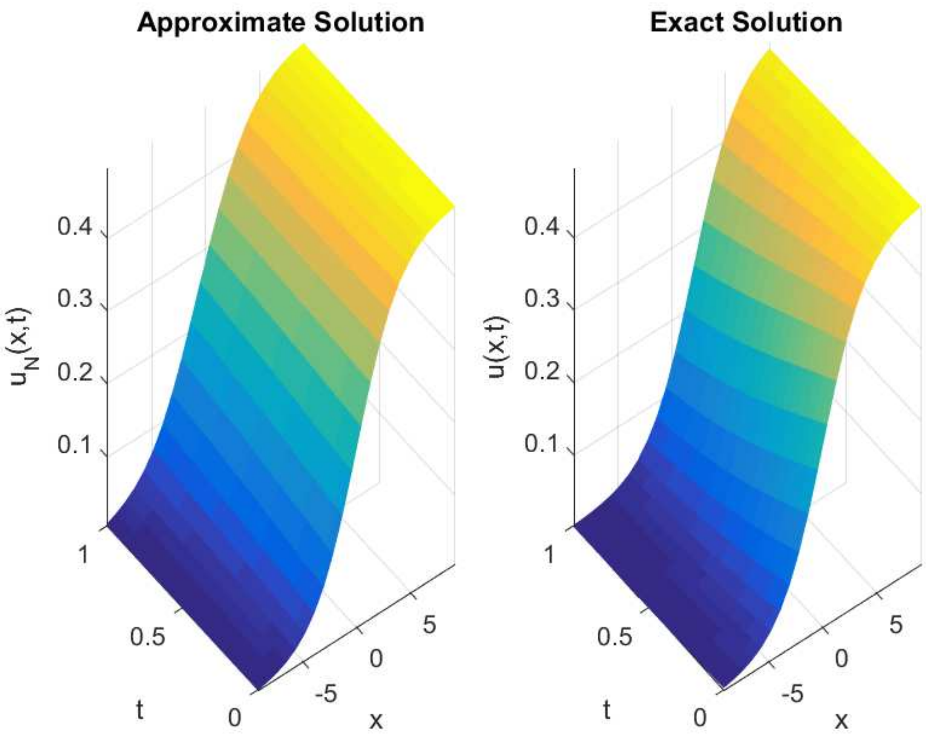

Figure 3.4: For $N = 24, \Delta t = 0.001, T = 0.01, \rho = 0.05$, solution of Fitzhugh-Nagumo equation for Example 2

4. Conclusion

In this study, compact finite differences method is used to solve Fitzhugh-Nagumo equation. To check the accuracy of the method two test problems available in the literature are solved. The maximum absolute errors are presented in Tables with different time step. From examples, we have observed that the choice of particular ρ values can affect accurate of the numerical solutions. It is suggested that the compact finite different method produced accurate solution.

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Some Relations Between the Riemann Zeta Function and the Generalized Bernoulli Polynomials of Level m

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Abstract

The main purpose of this paper is to show some relations between the Riemann zeta function and the generalized Bernoulli polynomials of level m. Our approach is based on the use of Fourier expansions for the periodic generalized Bernoulli functions of level m, as well as quadrature formulae of Euler-Maclaurin type. Some illustrative examples involving such relations are also given.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

It is a classical result due to Riemann that $\zeta(s)$ can be analytically continued to a meromorphic function on the whole complex plane with the only pole at $s = 1$, which is a simple pole with residue 1. Also, if we consider the classical Bernoulli polynomials given by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi,$$

and the classical Bernoulli numbers, $B_n = B_n(0)$, for all $n \geq 0$, then it is well known the following relation between $\zeta(s)$ and the Bernoulli polynomials:

$$\zeta(2k) = \frac{(-1)^{k-1}\pi^{2k}2^{2k-1}}{(2k)!}B_{2k}, \quad k \geq 1. \tag{1.1}$$

Euler's relation (1.1) provides an elegant formula for the explicit evaluation of $\zeta(2k)$, which shows the arithmetical nature of $\zeta(2k)$ (cf. eg., [3, 4, 5, 6]). However, for the zeta values $\zeta(2k+1)$ there is very little known information. For instance, in his paper of 1981 R. Apéry showed that $\zeta(3)$ is irrational, but for $k \geq 2$ the arithmetical nature of $\zeta(2k+1)$ remains open (cf. [1, 3, 4, 5, 6, 7] and the references thereof). In this contribution we are interested in exploring similar relations to (1.1) in the setting of the generalized Bernoulli polynomials of level m [15, 18]. In order to do that, we show some constraints of the use of Fourier expansions for the periodic generalized Bernoulli functions of level m, as well as, our approach which is mainly based on quadrature formulae of Euler-Maclaurin type.

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The outline of the paper as follows. Section 2 provides a short background about some relevant properties of the generalized Bernoulli polynomials of level m. Section 3 is devoted to show some constraints of the use of Fourier expansions for the periodic generalized Bernoulli functions of level m (see Theorems 3.2 and 3.3). Finally, Section 4 contains the basic ideas in order to obtain quadrature formulae of Euler-Maclaurin type based on generalized Bernoulli polynomials of level m (see Theorem 4.2). Also, in this section is proved a result that reveals an interesting property about the applications of the quadrature formulae of Euler-Maclaurin type based on these polynomials (see Theorem 4.3). As usual, throughout this paper the convention $0^0 = 1$ will be adopted and an empty sum will be interpreted to be zero.

2. Generalized Bernoulli polynomials of level m: some properties

For a fixed $m \in \mathbb{N}$, the generalized Bernoulli polynomials of level m are defined by means of the following generating function [14, 15, 18, 20, 21, 22]

$$\frac{z^m e^{xz}}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x)\frac{z^n}{n!}, \quad |z| < 2\pi \tag{2.1}$$

and, the generalized Bernoulli numbers of level m are defined by $B_n^{[m-1]} := B_n^{[m-1]}(0)$, for all $n \geq 0$. The generalized Bernoulli polynomials of level m also have been called hypergeometric Bernoulli polynomials [12]. It is clear that if $m = 1$ in (2.1), then we obtain the definition of the classical Bernoulli polynomials $B_n(x)$, and classical Bernoulli numbers, respectively, i.e., $B_n(x) = B_n^{[0]}(x)$, and $B_n = B_n^{[0]}$, respectively, for all $n \geq 0$.
It is not difficult to check that the first four generalized Bernoulli polynomials of level m are:

$$B_0^{[m-1]}(x) = m!,$$
$$B_1^{[m-1]}(x) = m!\left(x - \frac{1}{m+1}\right),$$
$$B_2^{[m-1]}(x) = m!\left(x^2 - \frac{2}{m+1}x + \frac{2}{(m+1)^2(m+2)}\right),$$
$$B_3^{[m-1]}(x) = m!\left(x^3 - \frac{3}{m+1}x^2 + \frac{6}{(m+1)^2(m+2)}x + \frac{6(m-1)}{(m+1)^3(m+2)(m+3)}\right).$$

The following results summarize some properties of the generalized Bernoulli polynomials of level m (cf. [14, 15, 13, 18]).

Proposition 2.1. *[18, Proposition 1] For a fixed $m \in \mathbb{N}$, let $\left\{B_n^{[m-1]}(x)\right\}_{n\geq 0}$ be the sequence of generalized Bernoulli polynomials of level m. Then the following statements hold:*

a) Summation formula. For every $n \geq 0$,

$$B_n^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]} x^{n-k}. \tag{2.2}$$

b) Differential relations (Appell polynomial sequences). For $n, j \geq 0$ with $0 \leq j \leq n$, we have

$$[B_n^{[m-1]}(x)]^{(j)} = \frac{n!}{(n-j)!} B_{n-j}^{[m-1]}(x). \tag{2.3}$$

c) Inversion formula. [15, Equation (2.6)] For every $n \geq 0$,

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(m+k)!} B_{n-k}^{[m-1]}(x). \tag{2.4}$$

d) Recurrence relation. [15, Lemma 3.2] For every $n \geq 1$,

$$B_n^{[m-1]}(x) = \left(x - \frac{1}{m+1}\right) B_{n-1}^{[m-1]}(x) - \frac{1}{n(m-1)!} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x).$$

e) Integral formulas.

$$\int_{x_0}^{x_1} B_n^{[m-1]}(x)dx = \frac{1}{n+1}\left[B_{n+1}^{[m-1]}(x_1) - B_{n+1}^{[m-1]}(x_0)\right] = \sum_{k=0}^{n} \frac{1}{n-k+1} \binom{n}{k} B_k^{[m-1]}((x_1)^{n-k+1} - (x_0)^{n-k+1}). \tag{2.5}$$

$$B_n^{[m-1]}(x) = n\int_0^x B_{n-1}^{[m-1]}(t)dt + B_n^{[m-1]}. \tag{2.6}$$

f) [15, Theorem 3.1] Differential equation. For every $n \geq 1$, the polynomial $B_n^{[m-1]}(x)$ satisfies the following differential equation

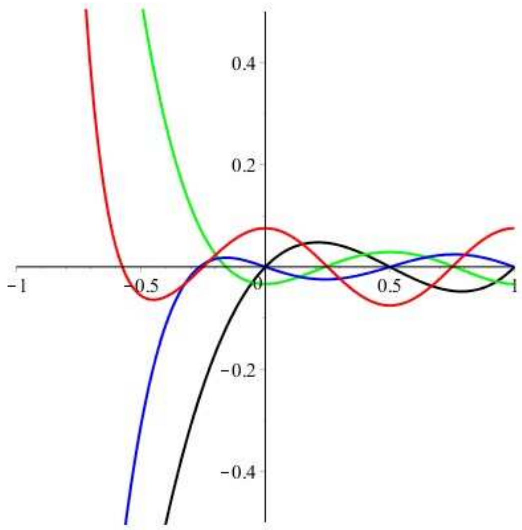
$$\frac{B_n^{[m-1]}}{n!}y^{(n)} + \frac{B_{n-1}^{[m-1]}}{(n-1)!}y^{(n-1)} + \cdots + \frac{B_2^{[m-1]}}{2!}y'' + (m-1)!\left(\frac{1}{m+1} - x\right)y' + n(m-1)!y = 0.$$

If we denote by $\mathbb{P}_n$ the linear space of polynomials with real coefficients and degree less than or equal to n, then (2.4) implies that

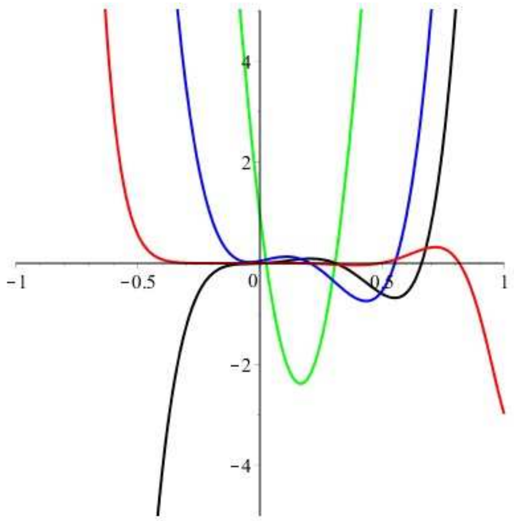
Proposition 2.2. *[18, Proposition 2] For a fixed $m \in \mathbb{N}$ and each $n \geq 0$, the set $\left\{B_0^{[m-1]}(x), B_1^{[m-1]}(x), \ldots, B_n^{[m-1]}(x)\right\}$ is a basis for $\mathbb{P}_n$, i.e.,*

$$\mathbb{P}_n = span B_0^{[m-1]}(x), B_1^{[m-1]}(x), \ldots, B_n^{[m-1]}(x).$$

We conclude this section showing in Figure 2.1 the plots of some generalized Bernoulli polynomials of level m.



(a) Level: $m = 1$. Degrees: $n = 3$ (black), $n = 4$ (green), $n = 5$ (blue), $n = 10$ (red).



(b) Level: $m = 5$. Degrees: $n = 2$ (green), $n = 5$ (blue), $n = 6$ (black), $n = 10$ (red).

Figure 2.1: Graphs of some generalized Bernoulli polynomials for the levels $m = 1$ (classical Bernoulli polynomials) and $m = 5$, respectively.

3. Fourier expansions and generalized Bernoulli polynomials of level m

For a fixed $m \in \mathbb{N}$, the periodic generalized Bernoulli functions of level m are defined as follows.

$$\begin{aligned} p_n^{[m-1]}(x) &= \frac{B_n^{[m-1]}(x)}{n!}, \quad 0 \leq x < 1, \\ p_n^{[m-1]}(x+1) &= p_n^{[m-1]}(x), \quad x \in \mathbb{R}. \end{aligned} \tag{3.1}$$

The functions $p_n^{[m-1]}(x)$ are continuous on $\mathbb{R}$ with continuous derivatives up to order $n-1$ only if $m = 1$ and $n > 2$.
In what follows, the symbol "$\sim$" is used to refer to the formal Fourier expansion for a given function on an interval, and it is not associated to some notion of convergence in particular, since as we know there are several kinds of convergence involved with the notion of Fourier expansion associated to a given function.
For $m = 1$ the Fourier expansions for the periodic generalized Bernoulli functions of level m coincide with the Fourier expansions for the periodic Bernoulli functions, i.e.,

$$p_1^{[0]}(x) = p_1(x) \sim -\sum_{k=1}^{\infty} \frac{2\sin(2\pi kx)}{2\pi k}, \tag{3.2}$$

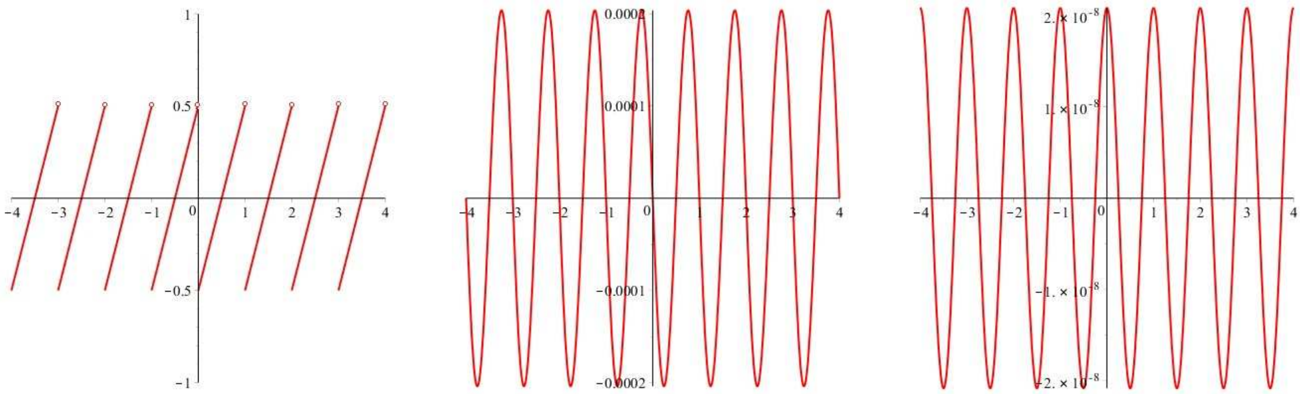
$$p_{2r}^{[0]}(x) = p_{2r}(x) = (-1)^{r-1} \sum_{k=1}^{\infty} \frac{2\cos(2\pi kx)}{(2\pi k)^{2r}}, \tag{3.3}$$

$$p_{2r+1}^{[0]}(x) = p_{2r+1}(x) = (-1)^{r-1} \sum_{k=1}^{\infty} \frac{2\sin(2\pi kx)}{(2\pi k)^{2r+1}}, \tag{3.4}$$

with $r \geq 1$.
Notice that by a well known result on the uniform convergence of Fourier expansions (see, for instance, [10, 17, 23]), the Fourier series (3.3) and (3.4) are uniformly convergent, while this does not hold for the Fourier expansion (3.2), since

$$p_1(0) = p_1(1) = -\frac{1}{2}, \quad \text{and} \quad \lim_{\varepsilon \to 0^+} p_1(1-\varepsilon) = \frac{1}{2},$$

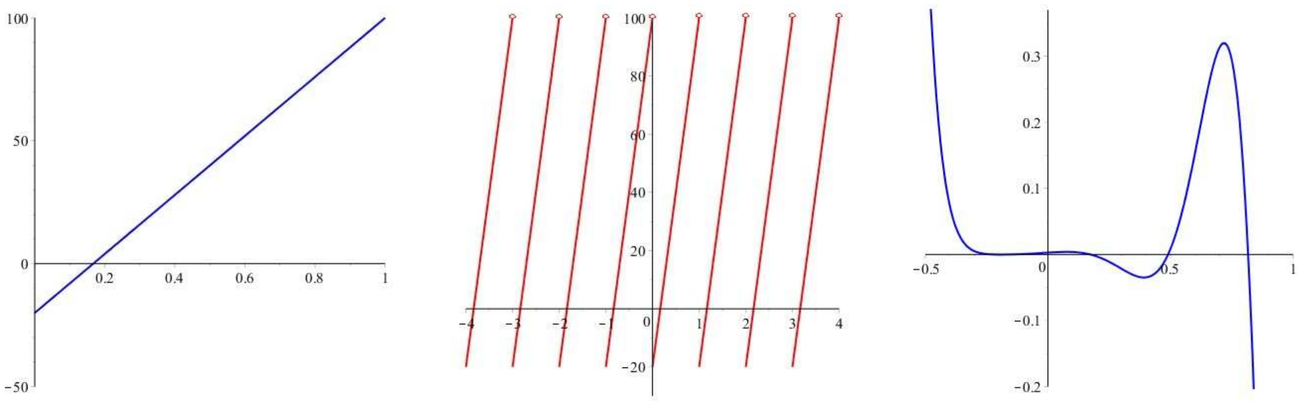
whereas the Fourier expansion (3.2) assumes the value 0 at both $x = 0$ and $x = 1$. Figure 3.1 shows the plots for some periodic Bernoulli functions.



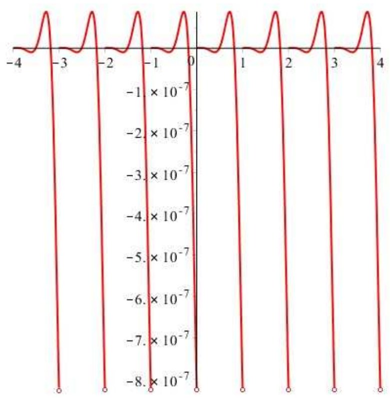
(a) Graph of $p_1(x)$. **(b)** Graph of $p_5(x)$. **(c)** Graph of $p_{10}(x)$.

Figure 3.1: Periodic Bernoulli functions for $n = 1, 5, 10$.

It is important to note that the sequence of functions $\{p_n(x)\}_{n\geq 2} \subset C^{n-2}(-\infty,\infty)$ (when $n = 2$, we are using the notation $C^0(-\infty,\infty) = C(-\infty,\infty)$), because the Bernoulli numbers satisfy the equality $B_n = (-1)^n B_n(1)$, for any $n \geq 0$ (see e.g., [3, Proposition 4.9]), $B_n = 0$, if $n \geq 3$ is odd, and by the condition of periodicity (3.1) with $m = 1$. In Figure 3.2 the plots for several generalized Bernoulli polynomials of level $m = 5$ and their corresponding periodic generalized Bernoulli functions are shown.



(a) Graph of $B_1^{[4]}(x)$. **(b)** Graph of $p_1^{[4]}(x)$. **(c)** Graph of $B_{10}^{[4]}(x)$.



(d) Graph of $p_{10}^{[4]}(x)$.

Figure 3.2: Generalized Bernoulli polynomials of level $m = 5$ and their corresponding periodic generalized Bernoulli functions for $n = 1, 10$.

It is worthy to mention that for $m, n > 1$ the functions $p_n^{[m-1]}(x)$ are only differentiable on $\mathbb{R} \setminus \mathbb{Z}$ -unlike what happens when $m = 1$ and $n > 2$ are considered (cf. [17, Chap. 3, Sec. 3.2])-. Thus, from (3.1) and (2.3) we deduce that $\left[p_{n+1}^{[m-1]}(x)\right]' = p_n^{[m-1]}(x)$ for each $x \in (k, k+1)$, $k \in \mathbb{Z}$. Hence,

$$\left[p_{n+1}^{[m-1]}(x)\right]' = p_n^{[m-1]}(x), \quad \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Z}. \tag{3.5}$$

Also, the periodic generalized Bernoulli functions of level m are integrable function on $[0, 1]$. Therefore, they satisfy Dirichlet conditions for the existence of their Fourier expansions [10, 23].

For a fixed $m \in \mathbb{N}$ we note that $p_1^{[m-1]}(x)$ has the following Fourier coefficients:

$$a_{0,1}^{[m-1]} = 2\int_0^1 p_1^{[m-1]}(x)\,dx = \frac{m!}{2}\left(\frac{m-1}{m+1}\right),$$
$$a_{k,1}^{[m-1]} = 2\int_0^1 p_1^{[m-1]}(x)\cos(2\pi kx)\,dx = 0,$$
$$b_{k,1}^{[m-1]} = 2\int_0^1 p_1^{[m-1]}(x)\sin(2\pi kx)\,dx = -\frac{2m!}{2\pi k},.$$

with $k \geq 1$. Thus, $p_1^{[m-1]}(x)$ has the Fourier expansion

$$p_1^{[m-1]}(x) \sim \frac{m!}{2}\left(\frac{m-1}{m+1}\right) - \sum_{k=1}^{\infty} \frac{2m!\sin(2\pi kx)}{2\pi k}. \tag{3.6}$$

For $x \in (0,1)$, let us integrate the series (3.6) formally, term by term:

$$\begin{aligned}\int_0^x p_1^{[m-1]}(t)\,dt &= \frac{m!}{2}\left(\frac{m-1}{m+1}\right)x - \sum_{k=1}^{\infty} \frac{2m!}{2\pi k}\int_0^x \sin(2\pi kt)\,dt \\ &= \frac{m!}{2}\left(\frac{m-1}{m+1}\right)x - \sum_{k=1}^{\infty} \frac{2m!}{(2\pi k)^2}(1-\cos(2\pi kx)) \\ &= \frac{m!}{2}\left(\frac{m-1}{m+1}\right)x - \frac{m!}{2\pi^2}\zeta(2) + \sum_{k=1}^{\infty} \frac{2m!\cos(2\pi kx)}{(2\pi k)^2}. \end{aligned} \tag{3.7}$$

From (2.5) we have

$$\int_0^x p_1^{[m-1]}(t)\,dt = p_2^{[m-1]}(x) - \frac{B_2^{[m-1]}}{2}. \tag{3.8}$$

Hence, the substitution of (3.8) into (3.7) yields the following expansion for $p_2^{[m-1]}(x)$

$$p_2^{[m-1]}(x) = \frac{B_2^{[m-1]}}{2} + \frac{m!}{2}\left(\frac{m-1}{m+1}\right)x - \frac{m!}{2\pi^2}\zeta(2) + \sum_{k=1}^{\infty} \frac{2m!\cos(2\pi kx)}{(2\pi k)^2}. \tag{3.9}$$

Since, $p_2^{[m-1]}(x)$ has the following Fourier coefficients:

$$a_{0,2}^{[m-1]} = \frac{m!}{3}\left(\frac{m-1}{(m+1)^2}\right)\left(\frac{m^2+2m-2}{m+2}\right),$$
$$a_{k,2}^{[m-1]} = 2\int_0^1 p_2^{[m-1]}(x)\cos(2\pi kx)\,dx = \frac{2m!}{(2\pi k)^2},$$
$$b_{k,2}^{[m-1]} = 2\int_0^1 p_1^{[m-1]}(x)\sin(2\pi kx)\,dx = -\frac{m!}{2\pi k}\left(\frac{m-1}{m+1}\right),$$

with $k \geq 1$, then $p_2^{[m-1]}(x)$ has the Fourier expansion

$$p_2^{[m-1]}(x) = \frac{m!}{6}\left(\frac{m-1}{(m+1)^2}\right)\left(\frac{m^2+2m-2}{m+2}\right) + \sum_{k=1}^{\infty} \frac{2m!\cos(2\pi kx)}{(2\pi k)^2} - \sum_{k=1}^{\infty} \frac{m!(m-1)\sin(2\pi kx)}{2\pi k(m+1)}. \tag{3.10}$$

On comparing (3.9) and (3.10), for $x \in (0,1)$ we see that

$$\frac{m!}{2\pi^2}\zeta(2) - \frac{B_2^{[m-1]}}{2} + \frac{m!}{6}\left(\frac{m-1}{(m+1)^2}\right)\left(\frac{m^2+2m-2}{m+2}\right) = \frac{m!}{2}\left(\frac{m-1}{m+1}\right)x + \sum_{k=1}^{\infty} \frac{m!(m-1)\sin(2\pi kx)}{2\pi k(m+1)}. \tag{3.11}$$

If we put $x = \frac{1}{2}$ in (3.11), then we obtain

$$\zeta(2) = \frac{2\pi^2}{m!}\left[\frac{B_2^{[m-1]}}{2} + \frac{m!}{4}\left(\frac{m-1}{m+1}\right) - \frac{m!}{6}\left(\frac{m-1}{(m+1)^2}\right)\left(\frac{m^2+2m-2}{m+2}\right)\right]. \tag{3.12}$$

The relation (3.12) connects the zeta number $\zeta(2)$ with the generalized Bernoulli polynomial $B_2^{[m-1]}(x)$ for any $m > 1$. Notice that if $m = 1$ then (3.12) coincides with Euler's relation (1.1) for $k = 1$.
For example, if we take $m = 2$ then (3.12) becomes

$$\zeta(2) = \pi^2\left(\frac{B_2^{[1]}}{2} + \frac{1}{6} - \frac{1}{18}\right) = \frac{\pi^2}{6}.$$

Since on $[0,1]$, the polynomial $p_n(x)$ is symmetric about the midpoint $x=\frac{1}{2}$, when n is even, and it is antisymmetric about $x=\frac{1}{2}$, when n is odd; that is,

$$p_n(1-x)=(-1)^n p_n(x), \quad 0\le x\le 1, \quad n\ge 2. \tag{3.13}$$

It follows that when $m=1$, taking $x=0$ in (3.3) and evaluating $p_{2r}(0)$ from (3.1) and using (3.13), we obtain (cf. [17, Eq. (3.54)]):

$$\zeta(2r)=\sum_{n=1}^{\infty}\frac{1}{n^{2r}}=(-1)^{r-1}\pi^{2r}2^{2r-1}\frac{B_{2r}}{(2r)!}, \quad r\ge 1,$$

this last equation is precisely (1.1).
Next, we will use the notation $p_n^{[m-1]}(x^-)$ and $p_n^{[m-1]}(x^+)$ for representing the one-sided limits $\lim_{y\to x^-} p_n^{[m-1]}(y)$ and $\lim_{y\to x^+} p_n^{[m-1]}(y)$, respectively. The following Proposition provides the Fourier expansion for $p_n^{[m-1]}(x)$ when $m>1$.

Proposition 3.1. *For a fixed $m\in\mathbb{N}$ and any $n\in\mathbb{N}$, let $p_n^{[m-1]}(x)$ be the periodic generalized Bernoulli functions of level m. Then Fourier expansion for $p_n^{[m-1]}(x)$ on $[0,1]$ is given by*

$$p_n^{[m-1]}(x)\sim\frac{a_{0,n}^{[m-1]}}{2}+\sum_{k=1}^{\infty}a_{k,n}^{[m-1]}\cos(2\pi kx)+\sum_{k=1}^{\infty}b_{k,n}^{[m-1]}\sin(2\pi kx), \tag{3.14}$$

where

$$\frac{a_{0,n}^{[m-1]}}{2}=p_{n+1}^{[m-1]}(1^-)-p_{n+1}^{[m-1]}(0)=\frac{1}{(n+1)!}\left(B_{n+1}^{[m-1]}(1)-B_{n+1}^{[m-1]}\right). \tag{3.15}$$

And for $k\ge 1$:

$$a_{k,n}^{[m-1]}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor-1}(-1)^j\frac{2}{(2\pi k)^{2j+2}}\left(p_{n-2j-1}^{[m-1]}(1^-)-p_{n-2j-1}^{[m-1]}(0)\right) \tag{3.16}$$

$$=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor-1}(-1)^j\frac{2}{(2\pi k)^{2j+2}}\frac{\left(B_{n-2j-1}^{[m-1]}(1)-B_{n-2j-1}^{[m-1]}\right)}{(n-2j-1)!}, \tag{3.17}$$

$$b_{k,n}^{[m-1]}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^{j+1}\frac{2}{(2\pi k)^{2j+1}}\left(p_{n-2j}^{[m-1]}(1^-)-p_{n-2j}^{[m-1]}(0)\right) \tag{3.18}$$

$$=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^{j+1}\frac{2}{(2\pi k)^{2j+1}}\frac{\left(B_{n-2j}^{[m-1]}(1)-B_{n-2j}^{[m-1]}\right)}{(n-2j)!}. \tag{3.19}$$

Proof. For each $p_n^{[m-1]}(x)$ it is well known that its Fourier coefficients are given by

$$a_{0,n}^{[m-1]}=2\int_0^1 p_n^{[m-1]}(x)\,dx, \tag{3.20}$$

$$a_{k,n}^{[m-1]}=2\int_0^1 p_n^{[m-1]}(x)\cos(2\pi kx)\,dx, \tag{3.21}$$

$$b_{k,n}^{[m-1]}=2\int_0^1 p_n^{[m-1]}(x)\sin(2\pi kx)\,dx, \tag{3.22}$$

with $k\ge 1$. Then, (3.15) is a straightforward consequence of (3.20) and (3.5). For obtaining the relations (3.16) and (3.18) it suffices use integration by parts on the right-hand side of (3.21) and (3.22), respectively. So, we get

$$a_{k,n}^{[m-1]}=-\frac{1}{2\pi k}b_{k,n-1}^{[m-1]}, \tag{3.23}$$

$$b_{k,n}^{[m-1]}=-\frac{2}{2\pi k}\left(p_n^{[m-1]}(1^-)-p_n^{[m-1]}(0)\right)+\frac{1}{2\pi k}a_{k,n-1}^{[m-1]}. \tag{3.24}$$

Then replacing n by $n-1$ in (3.24) and substituting the result obtained into (3.23), we get the following recurrence relation

$$a_{k,n}^{[m-1]}+\frac{1}{(2\pi k)^2}a_{k,n-2}^{[m-1]}=\frac{2}{(2\pi k)^2}\left(p_{n-1}^{[m-1]}(1^-)-p_{n-1}^{[m-1]}(0)\right). \tag{3.25}$$

Analogously, we can obtain

$$b_{k,n}^{[m-1]}+\frac{1}{(2\pi k)^2}b_{k,n-2}^{[m-1]}=-\frac{2}{2\pi k}\left(p_n^{[m-1]}(1^-)-p_n^{[m-1]}(0)\right). \tag{3.26}$$

Finally, it follows from (3.25) and (3.26) that

$$a_{k,n}^{[m-1]} = \frac{2}{(2\pi k)^2}\left(p_{n-1}^{[m-1]}(1^-)-p_{n-1}^{[m-1]}(0)\right) - \frac{2}{(2\pi k)^4}\left(p_{n-3}^{[m-1]}(1^-)-p_{n-3}^{[m-1]}(0)\right) + \frac{2}{(2\pi k)^6}\left(p_{n-5}^{[m-1]}(1^-)-p_{n-5}^{[m-1]}(0)\right)$$
$$- \frac{2}{(2\pi k)^8}\left(p_{n-7}^{[m-1]}(1^-)-p_{n-7}^{[m-1]}(0)\right) + \cdots + (-1)^{\lfloor \frac{n}{2}\rfloor -1}\frac{2}{(2\pi k)^{2\lfloor \frac{n}{2}\rfloor}}\left(p_{n-(\lfloor \frac{n}{2}\rfloor -1)}^{[m-1]}(1^-)-p_{n-(\lfloor \frac{n}{2}\rfloor -1)}^{[m-1]}(0)\right),$$

and

$$b_{k,n}^{[m-1]} = -\frac{2}{2\pi k}\left(p_{n}^{[m-1]}(1^-)-p_{n}^{[m-1]}(0)\right) + \frac{2}{(2\pi k)^3}\left(p_{n-2}^{[m-1]}(1^-)-p_{n-2}^{[m-1]}(0)\right) - \frac{2}{(2\pi k)^5}\left(p_{n-4}^{[m-1]}(1^-)-p_{n-4}^{[m-1]}(0)\right)$$
$$+ \frac{2}{(2\pi k)^7}\left(p_{n-6}^{[m-1]}(1^-)-p_{n-6}^{[m-1]}(0)\right) + \cdots + (-1)^{\lfloor \frac{n}{2}\rfloor +1}\frac{2}{(2\pi k)^{2\lfloor \frac{n}{2}\rfloor+1}}\left(p_{n-2\lfloor \frac{n}{2}\rfloor}^{[m-1]}(1^-)-p_{n-2\lfloor \frac{n}{2}\rfloor}^{[m-1]}(0)\right).$$

From these last relations we obtain (3.17) and (3.19), respectively.

□

Theorem 3.2. *For a fixed $m \in \mathbb{N}$ and $n \in \mathbb{N}$, let $p_n^{[m-1]}(x)$ be the periodic generalized Bernoulli functions of level m. If $x \in (0,1)$, then the following identity holds.*

$$p_n^{[m-1]}(x) = \frac{m!(m-1)}{2(m+1)}\frac{x^n}{n!} + m!p_n(x) + \sum_{k=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2k}\left[(2k)!p_{2k}^{[m-1]}(0) + \frac{2(-1)^k m!(2k)!\zeta(2k)}{(2\pi)^{2k}}\right]\frac{x^{n-2k}}{n!}$$
$$+ \sum_{k=1}^{\lfloor \frac{n-1}{2}\rfloor} p_{2k-1}^{[m-1]}(0)\frac{x^{n-2k-1}}{(n-2k-1)!}. \tag{3.27}$$

Proof. Using Proposition 3.1 we obtain the following expression for $p_1^{[m-1]}(x)$:

$$p_1^{[m-1]}(x) = \frac{m!(m-1)}{2(m+1)} - m!\sum_{k=1}^{\infty}\frac{2\sin 2\pi kx}{2\pi k}, \quad \text{whenever } x \in (0,1). \tag{3.28}$$

Then in view of (3.1) and (2.6), we see that

$$p_n^{[m-1]}(x) = p_n^{[m-1]}(0) + \int_0^x p_{n-1}^{[m-1]}(t)\,dt, \quad \text{if } x \in [0,1). \tag{3.29}$$

Taking $n = 2$ and substituting (3.28) into (3.29), we get

$$p_2^{[m-1]}(x) = p_2^{[m-1]}(0) + \int_0^x p_1^{[m-1]}(t)\,dt = \frac{m!(m-1)}{2(m+1)}x + \left(2!p_2^{[m-1]}(0) - \frac{2m!2!}{(2\pi)^2}\zeta(2)\right)\frac{1}{2!} + m!p_2(x).$$

Similarly, for $n = 3$ we can deduce

$$p_3^{[m-1]}(x) = p_3^{[m-1]}(0) + \int_0^x p_2^{[m-1]}(t)\,dt = \frac{m!(m-1)}{2(m+1)}\frac{x^2}{2!} + \left(2!p_2^{[m-1]}(0) - \frac{2m!2!}{(2\pi)^2}\zeta(2)\right)\frac{1}{2!}x + p_3^{[m-1]}(0) + m!p_3(x).$$

Iterating this procedure (3.27) follows.

□

Recall that the Dirichlet convergence theorem [10, 17, 23] guarantees that the Fourier series (3.14) converges pointwise at $x \in \mathbb{Z}$ to the average of $p_n^{[m-1]}(x^+)$ and $p_n^{[m-1]}(x^-)$. Indeed, based on this fact we prove the next result.

Theorem 3.3. *For a fixed $m \in \mathbb{N}$ and any $r \in \mathbb{N}$, the following identity holds.*

$$\zeta(2r) = \frac{(-1)^{r-1}2^{2r-1}\pi^{2r}B_{2r}^{[m-1]}}{m!(2r)!} + \Delta_r^{[m-1]}, \tag{3.30}$$

where

$$\Delta_r^{[m-1]} = \frac{(-1)^{r-1}2^{2r-1}\pi^{2r}}{m!}\left[\frac{B_{2r}^{[m-1]}(1)-B_{2r}^{[m-1]}}{2(2r)!} - \frac{B_{2r+1}^{[m-1]}(1)-B_{2r+1}^{[m-1]}}{(2r+1)!} - \sum_{j=1}^{r-1}\frac{\left(B_{2r-2j+1}^{[m-1]}(1)-B_{2r-2j+1}^{[m-1]}\right)}{(2r-2j+1)!}\frac{B_{2j}}{(2j)!}\right] \tag{3.31}$$

$$= \frac{(-1)^{r-1}2^{2r-1}\pi^{2r}}{m!}\left[\frac{1}{2(2r)!}\sum_{k=0}^{2r-1}\binom{2r}{k}B_k^{[m-1]} - \frac{1}{(2r+1)!}\sum_{k=0}^{2r}\binom{2r+1}{k}B_k^{[m-1]} - \sum_{j=1}^{r-1}\sum_{k=0}^{2j}\binom{2j+1}{k}\frac{B_k^{[m-1]}B_{2j}}{(2j+1)!(2j)!}\right]. \tag{3.32}$$

Proof. Let us consider $n = 2r$ and $x = 0$ in (3.14). Since $x = 0$ is a point of discontinuity of $p_{2r}^{[m-1]}(x)$, by the Dirichlet convergence theorem [10, 17, 23] we have

$$\frac{p_{2r}^{[m-1]}(0^+) + p_{2r}^{[m-1]}(0^-)}{2} = \frac{a_{0,2r}^{[m-1]}}{2} + \sum_{k=1}^{\infty} a_{k,2r}^{[m-1]}. \tag{3.33}$$

Since

$$\frac{p_{2r}^{[m-1]}(0^+) + p_{2r}^{[m-1]}(0^-)}{2} = \frac{B_{2r}^{[m-1]} + B_{2r}^{[m-1]}(1)}{2(2r)!},$$

using (3.15) and (3.17), we can rewrite (3.33) as follows

$$\frac{B_{2r}^{[m-1]} + B_{2r}^{[m-1]}(1)}{2(2r)!} = \frac{1}{(2r+1)!}\left(B_{2r+1}^{[m-1]}(1) - B_{2r+1}^{[m-1]}\right) + \sum_{k=1}^{\infty}\sum_{j=0}^{r-1}(-1)^j \frac{2}{(2\pi k)^{2j+2}} \frac{\left(B_{2r-2j-1}^{[m-1]}(1) - B_{2r-2j-1}^{[m-1]}\right)}{(2r-2j-1)!}. \tag{3.34}$$

Taking into account that

$$p_{2j+2}(0) = (-1)^{j+1}\sum_{n=1}^{\infty}\frac{2}{(2\pi n)^{2j+2}},$$

the relation (3.34) can be expressed as

$$\frac{B_{2r}^{[m-1]} + B_{2r}^{[m-1]}(1)}{2(2r)!} = \frac{1}{(2r+1)!}\left(B_{2r+1}^{[m-1]}(1) - B_{2r+1}^{[m-1]}\right) + m!p_{2r}(0) + \sum_{j=0}^{r-2}\frac{\left(B_{2r-2j-1}^{[m-1]}(1) - B_{2r-2j-1}^{[m-1]}\right)}{(2r-2j-1)!}\frac{B_{2j+2}}{(2j+2)!}.$$

Or equivalently,

$$\frac{B_{2r}^{[m-1]} + B_{2r}^{[m-1]}(1)}{2(2r)!} = \frac{1}{(2r+1)!}\left(B_{2r+1}^{[m-1]}(1) - B_{2r+1}^{[m-1]}\right) + m!p_{2r}(0) + \sum_{j=1}^{r-1}\frac{\left(B_{2r-2j+1}^{[m-1]}(1) - B_{2r-2j+1}^{[m-1]}\right)}{(2r-2j+1)!}\frac{B_{2j}}{(2j)!}. \tag{3.35}$$

Now, from (3.35) we deduce that

$$\frac{2(-1)^{r-1}\zeta(2r)}{(2\pi)^{2r}} = \frac{B_{2r}^{[m-1]}(1) + B_{2r}^{[m-1]}}{2m!(2r)!} - \frac{B_{2r+1}^{[m-1]}(1) - B_{2r+1}^{[m-1]}}{m!(2r+1)!} - \frac{1}{m!}\sum_{j=1}^{r-1}\frac{\left(B_{2r-2j+1}^{[m-1]}(1) - B_{2r-2j+1}^{[m-1]}\right)}{(2r-2j+1)!}\frac{B_{2j}}{(2j)!}. \tag{3.36}$$

Hence, (3.36) takes the form:

$$\zeta(2r) = \frac{(-1)^{r-1}2^{2r-1}\pi^{2r}B_{2r}^{[m-1]}}{m!(2r)!} + \Delta_r^{[m-1]}, \tag{3.37}$$

where

$$\Delta_r^{[m-1]} = \frac{(-1)^{r-1}2^{2r-1}\pi^{2r}}{m!}\left[\frac{B_{2r}^{[m-1]}(1) + B_{2r}^{[m-1]}}{2(2r)!} - \frac{B_{2r+1}^{[m-1]}(1) - B_{2r+1}^{[m-1]}}{(2r+1)!} - \sum_{j=1}^{r-1}\frac{\left(B_{2r-2j+1}^{[m-1]}(1) - B_{2r-2j+1}^{[m-1]}\right)}{(2r-2j+1)!}\frac{B_{2j}}{(2j)!}\right].$$

Hence, $\Delta_r^{[m-1]}$ satisfies (3.31).
Finally, the substitution of (2.2) into the above expression for $\Delta_r^{[m-1]}$, and some suitable computations yield the identity (3.32). □

Notice that if $m = 1$ in (3.30) then we recover (1.1). It is not difficult to see that for $r = 1$ the identity (3.30) yields the same result than the identity (3.12).

4. Riemann zeta function and quadrature formulae of Euler-Maclaurin type

It is well known that using the Euler-Maclaurin summation formula (cf. [2, 8, 11], and [16, Chap. 2, Sec. 3, p. 30]) it is possible to deduce the following formula for the integral of the product of two classical Bernoulli polynomials

$$\int_0^1 B_s(t)B_r(t)dt = (-1)^{s+1}\frac{s!r!}{(s+r)!}B_{s+r}, \quad \text{where } r,s \geq 1. \tag{4.1}$$

Using integration by parts a similar formula to (4.1) has been deduced in [18]. More precisely, for an integer $r \geq 0$ and a closed interval $[a,b]$, let $C^r[a,b]$ denote the set of all r-times continuously differentiable functions defined on $[a,b]$. Then following result holds.

Lemma 4.1. *[18, Lemma 1] Let $r \geq 1$ and $f \in C^r[0,1]$. For a fixed $m \in \mathbb{N}$, we have*

$$\int_0^1 f(t)dt = \frac{1}{m!}\left[\sum_{k=1}^{r} A_k^{[m-1]}(f) + \frac{(-1)^r}{r!}\int_0^1 f^{(r)}(t)B_r^{[m-1]}(t)dt\right], \tag{4.2}$$

where

$$A_k^{[m-1]}(f) = \frac{(-1)^k}{k!}\left(f^{(k-1)}(0)B_k^{[m-1]} - f^{(k-1)}(1)B_k^{[m-1]}(1)\right), \qquad k = 1,\ldots,r.$$

Applying the substitution $f(t) = B_{r+n}^{[m-1]}(t)$ into (4.2) and taking into account (2.3), (2.6) we have

$$\int_0^1 B_r^{[m-1]}(t)B_n^{[m-1]}(t)\,dt = \frac{(-1)^{r+1}r!n!m!}{(r+n)!}\left[\frac{B_{r+n+1}^{[m-1]} - B_{r+n+1}^{[m-1]}(1)}{r+n+1} + \frac{1}{m!}\sum_{k=1}^{r} A_k^{[m-1]}\right], \tag{4.3}$$

where $r,n \geq 1$ and

$$A_k^{[m-1]} = \frac{(-1)^k}{k}\binom{r+n}{k-1}\left(B_{r+n-k+1}^{[m-1]}B_k^{[m-1]} - B_{r+n-k+1}^{[m-1]}(1)B_k^{[m-1]}(1)\right), \qquad k = 1,\ldots,r.$$

The expression (4.3) is the analogue of (4.1) in the setting of the generalized Bernoulli polynomials of level m. We strongly recommend to the interested reader see [18] for the corresponding proofs of the results mentioned above.
Let $L^2[0,1]$ be the space of the square-integrable functions on $[0,1]$, endowed with the norm

$$\|f\|_{L^2[0,1]} := \left(\int_0^1 |f(t)|^2 dt\right)^{1/2} = \langle f,f\rangle^{1/2},$$

where

$$\langle f,g\rangle := \int_0^1 f(t)g(t)dt, \text{ for every } f,g \in L^2[0,1].$$

It is not difficult to see that we can determine the norm $\|B_n^{[m-1]}\|_{L^2[0,1]}$ using (4.3), as

$$\begin{aligned}\|B_n^{[m-1]}\|^2_{L^2[0,1]} &= \frac{(n!)^2 m!(-1)^n}{(2n+1)!}(B_{2n+1}^{[m-1]}(1) - B_{2n+1}^{[m-1]}) \\ &\quad + (n!)^2(-1)^{n+1}\sum_{k=1}^{n}\frac{(-1)^k}{(2n+1-k)!k!}(B_{2n+1-k}^{[m-1]}B_k^{[m-1]} - B_{2n+1-k}^{[m-1]}(1)B_k^{[m-1]}(1)).\end{aligned} \tag{4.4}$$

From the trigonometric form of Fourier expansion for $f \in L^2[0,1]$ it is possible to deduce the following form of Parseval's identity:

$$\|f\|^2_{L^2[0,1]} = \frac{|a_0(f)|^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}|a_k(f)|^2 + |b_k(f)|^2, \tag{4.5}$$

where

$$a_k(f) = 2\int_0^1 f(x)\cos(2\pi kx)\,dx, \quad k \geq 0,$$
$$b_k(f) = 2\int_0^1 f(x)\sin(2\pi kx)\,dx, \quad k \geq 1.$$

Hence, using (4.4) we show how linear combinations of the values of $\zeta(2k)$ can be obtained by applying Parseval's identity (4.5) with the Fourier coefficients (3.15), (3.17) and (3.19) of the periodic generalized Bernoulli functions of level m.
Applying Parseval's identity (4.5) to $p_n^{[m-1]}(x)$ and using (3.15)-(3.19), we can deduce that

$$\begin{aligned}\|B_n^{[m-1]}\|^2_{L^2[0,1]} &= (n!)^2\left[\frac{\left(a_{0,n}^{[m-1]}\right)^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_{k,n}^{[m-1]}\right)^2 + \left(b_{k,n}^{[m-1]}\right)^2\right] \\ &= \frac{(B_{n+1}^{[m-1]}(1) - B_{n+1}^{[m-1]})^2}{(n+1)^2} + 2(n!)^2\sum_{k=1}^{\infty}\left[\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor-1}\frac{(-1)^j}{(2\pi k)^{2j+2}}\left(\frac{B_{n-2j-1}^{[m-1]}(1) - B_{n-2j-1}^{[m-1]}}{(n-2j-1)!}\right)\right]^2 \\ &\quad + 2(n!)^2\sum_{k=1}^{\infty}\left[\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^{j+1}}{(2\pi k)^{2j+1}}\left(\frac{B_{n-2j}^{[m-1]}(1) - B_{n-2j}^{[m-1]}}{(n-2j)!}\right)\right]^2.\end{aligned} \tag{4.6}$$

Comparing (4.4) with (4.6) we obtain the next equality:

$$\sum_{k=1}^{\infty} A_{k,n}^2 + B_{k,n}^2 = \frac{m!(-1)^n}{2(2n+1)!}(B_{2n+1}^{[m-1]}(1) - B_{2n+1}^{[m-1]}) - \frac{(B_{n+1}^{[m-1]}(1) - B_{n+1}^{[m-1]})^2}{2(n+1)^2}$$
$$+ \frac{(-1)^{n+1}}{2} \sum_{k=1}^{n} \frac{(-1)^k}{(2n+1-k)!k!}(B_{2n+1-k}^{[m-1]} B_k^{[m-1]} - B_{2n+1-k}^{[m-1]}(1) B_k^{[m-1]}(1)), \tag{4.7}$$

where

$$A_{k,n} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{(-1)^j}{(2\pi k)^{2j+2}} \left(\frac{B_{n-2j-1}^{[m-1]}(1) - B_{n-2j-1}^{[m-1]}}{(n-2j-1)!} \right),$$
$$B_{k,n} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{j+1}}{(2\pi k)^{2j+1}} \left(\frac{B_{n-2j}^{[m-1]}(1) - B_{n-2j}^{[m-1]}}{(n-2j)!} \right).$$

Furthermore, if $m = 1$ in (4.7) we recover (1.1). Following the ideas of [19] we can obtain a quadrature formulae of Euler-Maclaurin type based on generalized Bernoulli polynomials of level $m \in \mathbb{N} \setminus \{1\}$.

Theorem 4.2. *Let $r \geq 1$, $f \in C^r[a,b]$ and $m \in \mathbb{N}$. For a fixed $n \in \mathbb{N}$ let $x_j = a + jh$, $j = 0,1,\ldots,n$, where $h = \frac{b-a}{n}$, and $f_j^{(k-1)} = f^{(k-1)}(x_j)$, $k = 1,2,\ldots,r$. Then, the following composite trapezoidal rules hold.*

$$\int_a^b f(t)\,dt = \sum_{j=0}^{n-1} \sum_{k=1}^{r} \tilde{A}_{k,j}^{[m-1]}(f) + R_r^{[m-1]}(f), \tag{4.8}$$

where

$$\tilde{A}_{k,j}^{[m-1]}(f) = \frac{(-1)^{k+1}}{m!k!} h^k \left(f_{j+1}^{(k-1)} B_k^{[m-1]}(1) - f_j^{(k-1)} B_k^{[m-1]} \right), \qquad 1 \leq k \leq r,$$

and

$$R_r^{[m-1]}(f) = \frac{(-h)^r}{m!r!} \int_a^b f^{(r)}(t) B_r^{[m-1]} \left(\frac{t-a}{h} - \left\lfloor \frac{t-a}{h} \right\rfloor \right) dt.$$

Proof. Let $g \in C^r[0,1]$. By (4.2) we get

$$\int_0^1 g(t)\,dt = \frac{1}{m!} \sum_{k=1}^{r} \frac{(-1)^{k+1}}{k!} \left(g^{(k-1)}(1) B_k^{[m-1]}(1) - g^{(k-1)}(0) B_k^{[m-1]} \right) + \frac{(-1)^r}{m!r!} \int_0^1 g^{(r)}(t) B_r^{[m-1]}(t)\,dt. \tag{4.9}$$

Taking $g(t) = f(x_j + ht)$ it is easy to check that $g^{(k)}(t) = h^k f^{(k)}(x_j + ht)$ for $k = 1,2,\ldots,r$. Substituting $g^{(k-1)}(1)$, $g^{(k-1)}(0)$, $g^{(r)}(t)$ into (4.9), and making a suitable change of variable, we obtain that

$$\int_{x_j}^{x_{j+1}} f(t)\,dt = \frac{1}{m!} \sum_{k=1}^{r} \frac{(-1)^{k+1}}{k!} h^k \left(f^{(k-1)}(x_{j+1}) B_k^{[m-1]}(1) - f^{(k-1)}(x_j) B_k^{[m-1]} \right)$$
$$+ \frac{(-h)^r}{m!r!} \int_{x_j}^{x_{j+1}} f^{(r)}(t) B_r^{[m-1]} \left(\frac{t - x_j}{h} \right) dt, \tag{4.10}$$

whenever $j = 0,1,\ldots,n-1$. Next, adding all these terms for $j = 0,\ldots,n-1$ to both sides of (4.10), and nothing that if $x_j \leq t \leq x_{j+1}$ then $j \leq \frac{t-a}{h} \leq j+1$, we have

$$\int_a^b f(t)\,dt = \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} f(t)\,dt = \frac{1}{m!} \sum_{j=0}^{n-1} \sum_{k=1}^{r} \frac{(-1)^{k+1}}{k!} h^k \left(f^{(k-1)}(x_{j+1}) B_k^{[m-1]}(1) - f^{(k-1)}(x_j) B_k^{[m-1]} \right) + R_r^{[m-1]}(f).$$

From this last equation (4.8) follows. □

We conclude this section with a result that reveals an interesting property about the applications of the quadrature formulae of Euler-Maclaurin type (4.8). Using the approach given in [11, pp. 117-120], it is possible to provide a theorem comparing simultaneously the convergence of a series $\sum_{k=1}^{\infty} f(k)$ and an integral $\int_1^{\infty} f(x)dx$ in the setting of generalized Bernoulli polynomials of level m. In particular, with such a theorem we can estimate the values $\zeta(2k+1)$, for $k \geq 1$.
Let $r \geq 1$, $f \in C^r[1,\infty)$. For a fixed $m \in \mathbb{N}$, we will denote by

$$S(l) := \sum_{j=1}^{l} f(j), \tag{4.11}$$

$$\tilde{\sigma}_r^{[m-1]}(q_1) := f(q_1) + \frac{1}{m!} \sum_{k=1}^{r} \frac{(-1)^{k+1}}{k!} f^{(k-1)}(q_1) B_k^{[m-1]}, \tag{4.12}$$

$$\sigma_r^{[m-1]}(q_2) := \frac{1}{m!}\sum_{k=1}^{r}\frac{(-1)^{k+1}}{k!}f^{(k-1)}(q_2)B_k^{[m-1]}(1), \tag{4.13}$$

$$\rho_r^{[m-1]}(q_1,q_2) := \frac{1}{m!}\sum_{j=q_1+1}^{q_2-1}\sum_{k=2}^{r}\frac{(-1)^{k+1}}{k!}(B_k^{[m-1]}(1)-B_k^{[m-1]})f^{(k-1)}(j), \tag{4.14}$$

$$R_r^{[m-1]}(q_1,q_2) := \frac{(-1)^r}{r!}\int_{q_1}^{q_2} g_r^{[m-1]}(t)\,dt, \tag{4.15}$$

where $l,q_1,q_2\in\mathbb{N}$ and $g_r^{[m-1]}(t)=f^{(r)}(t)B_r^{[m-1]}(t-\lfloor t\rfloor)$. As well as, we will consider the following limits:

$$\begin{aligned}
S(\infty) &:= \lim_{l\to\infty} S(l),\\
\sigma_r^{[m-1]}(\infty) &:= \lim_{q_2\to\infty}\sigma_r^{[m-1]}(q_2),\\
\rho_r^{[m-1]}(q_1,\infty) &:= \lim_{q_2\to\infty}\rho_r^{[m-1]}(q_1,q_2),\\
R_r^{[m-1]}(q_1,\infty) &:= \lim_{q_2\to\infty}R_r^{[m-1]}(q_1,q_2),\\
e_r^{[m-1]}(q_1) &:= \rho_r^{[m-1]}(q_1,\infty),\\
\delta_r^{[m-1]}(q_1) &:= R_r^{[m-1]}(q_1,\infty).
\end{aligned}$$

For the reader's convenience, we recall the definition of Euler's constant for a function f (cf. [11, p. 118]). For $f\in C^r[1,\infty)$ and any $n\in\mathbb{N}$ let us consider the sequence

$$\gamma_n(f) := \sum_{i=1}^{n} f(i) - \int_1^n f(t)\,dt. \tag{4.16}$$

Euler's constant for function f is defined as the limit

$$\gamma(f) := \lim_{n\to\infty}\gamma_n(f), \tag{4.17}$$

whenever such limit exists and be finite.
The quadrature formulae of Euler-Maclaurin type (4.8) is also of theoretical interest. More precisely, the definitions (4.16), (4.17) and the formulae (4.8) imply the following result:

Theorem 4.3. *For a fixed $m\in\mathbb{N}$, every $r,p,n\in\mathbb{N}$ and $f\in C^r[1,\infty)$. Assume that $\rho_r^{[m-1]}(1,\infty)$, $\int_1^\infty |f^{(r)}(t)|\,dt$ converge, and the finite limit $\lambda_0 := \lim_{n\to\infty} f(n)$ exists, then*

(a) The integral $\int_1^\infty f(t)\,dt$ converges if and only if the series $\sum_{j=1}^\infty f(j)$ converges.
(b) If the integral $\int_1^\infty f(t)\,dt$ converges, then

$$\int_1^\infty f(t)\,dt = \int_1^p f(t)\,dt + S(\infty) - S(p-1) + \sigma_r^{[m-1]}(\infty) - \widetilde{\sigma}_r^{[m-1]}(p) + e_r^{[m-1]}(p) + \delta_r^{[m-1]}(p).$$

Notice that if $\rho_r^{[m-1]}(1,\infty)$ converges, then $\lim_{n\to\infty} f^{(k-1)}(n)=0$ for every $k=2,\ldots,r$.

Proof. Without loss of generality we can assume that $p\le n$. The substitution $a=p$, $b=n$ and $h=1$ into (4.8) and the use of (4.11)-(4.15) yield the identity

$$\int_p^n f(t)\,dt = S(n-1)-S(p-1)-f(n)+\sigma_r^{[m-1]}(n)-\widetilde{\sigma}_r^{[m-1]}(p)+\rho_r^{[m-1]}(p,n)+R_r^{[m-1]}(p,n), \tag{4.18}$$

where $S(0)=0$ by definition. The remainder $R_r^{[m-1]}(p,n)$ can be estimated by

$$\left|R_r^{[m-1]}(p,n)\right| \le \frac{\mu_r^{[m-1]}}{r!}\int_p^n |f^{(r)}(t)|\,dt, \tag{4.19}$$

where $\mu_r^{[m-1]}=\max\{|B_r^{[m-1]}(x)| : 0\le x\le 1\}$.
By (4.16) and the formula (4.18) we obtain

$$\gamma_n(f) = f(n)+\widetilde{\sigma}_r^{[m-1]}(1)-\sigma_r^{[m-1]}(n)-\rho_r^{[m-1]}(1,n)-R_r^{[m-1]}(1,n). \tag{4.20}$$

Our assumptions imply, according to (4.20), that the Euler's constant for the function f, $\gamma(f)$, exists and the next equality is satisfied:

$$\gamma(f) = \lambda_0+\widetilde{\sigma}_r^{[m-1]}(1)-\sigma_r^{[m-1]}(\infty)-\rho_r^{[m-1]}(1,\infty)-R_r^{[m-1]}(1,\infty). \tag{4.21}$$

Now, from (4.20) and (4.21) we have

$$\gamma(f)=\gamma_n(f)+\lambda_0-f(n)+\sigma_r^{[m-1]}(n)-\sigma_r^{[m-1]}(\infty)-e_r^{[m-1]}(n-1)-\delta_r^{[m-1]}(n), \tag{4.22}$$

where

$$\left|\delta_r^{[m-1]}(n)\right|\leq\frac{\mu_r^{[m-1]}}{r!}\int_n^\infty\left|f^{(r)}(t)\right|dt.$$

Thus, substituting (4.16) into (4.22) and using (4.11) we obtain

$$\int_1^n f(t)\,dt=S(n)-\gamma(f)+\lambda_0-f(n)+\sigma_r^{[m-1]}(n)-\sigma_r^{[m-1]}(\infty)-e_r^{[m-1]}(n-1)-\delta_r^{[m-1]}(n). \tag{4.23}$$

Finally, part (a) of Theorem 4.3 can be deduced from (4.23). In order to obtain part (b) of Theorem 4.3 it suffices to consider (4.18) and the equality $\int_p^n f(t)\,dt=\int_1^n f(t)\,dt-\int_1^p f(t)\,dt$.

□

The interested reader may consult the analogous result for $m=1$ in [11, Theorem 2].

Example 4.4. *To compute $\zeta(3)=\sum_{k=1}^\infty\frac{1}{k^3}$, we can put $m=5$, $r=2$, $p=100$, $f(x)=\frac{1}{x^3}$, $x\in[1,\infty)$ and apply part (b) of Theorem 4.3. Then, we obtain*

$$\begin{aligned}
S(99)&=1.2020064006596776104,\\
\sigma_2^{[4]}(p)&=\frac{5}{6p^3}+\frac{85}{84p^4},\\
\sigma_2^{[4]}(100)&=8.4345238095238095238\times10^{-7},\\
\sigma_2^{[4]}(\infty)&=0,\\
\widetilde{\sigma}_2^{[4]}(p)&=\frac{5}{6p^3}+\frac{1}{84p^4},\\
\widetilde{\sigma}_2^{[4]}(100)&=8.3345238095238095238\times10^{-7},\\
e_2^{[4]}(100)&=3.2836666500022217224\times10^{-7}.
\end{aligned}$$

Next, part (b) of Theorem 4.3 gives

$$\begin{aligned}
\zeta(3)&=\int_{100}^\infty\frac{dt}{t^3}+S(99)-\sigma_2^{[4]}(\infty)+\widetilde{\sigma}_2^{[4]}(100)-e_2^{[4]}(100)-\delta_2^{[4]}(100)\\
&=0.00005+1.2020064006596776104+(8.3345238095238095238)\times10^{-7}\\
&\quad-(3.2836666500022217224)\times10^{-7}-\delta_2^{[4]}(100)\\
&=1.2020560622930126102-\delta_2^{[4]}(100).
\end{aligned}$$

Since

$$\delta_2^{[4]}(100)\approx3.10296\times10^{-7},$$

we obtain the following estimates for $\zeta(3)$:

$$\zeta(3)\approx1.2020557519970993510. \tag{4.24}$$

In this case, our approximation is accurate up to five decimal places of $\zeta(3)=1.2020569031595942854\ldots$.
Since, for $p\geq1$,

$$\left|\delta_2^{[4]}(p)\right|\leq\frac{\mu_2^{[4]}}{2}\int_p^\infty\frac{12}{t^5}\,dt=\frac{850}{7p^4}.$$

Then,

$$\left|\delta_2^{[4]}(100)\right|\leq0.000001214285714,$$

and the estimate (4.24) could be refined in order to get an accurate up to six decimal places.

Example 4.5. *Now, we will estimate* $\zeta(3)=\sum_{k=1}^{\infty}\frac{1}{k^3}$, *taking* $m=2$, $r=2$, $p=20$, $f(x)=\frac{1}{x^3}$, $x\in[1,\infty)$ *and apply part (b) of Theorem 4.3 again. In this case, we have*

$$\begin{aligned}
S(19) &= 1.2020064006596776104,\\
\sigma_2^{[1]}(p) &= \frac{2}{3p^3}+\frac{7}{12p^4},\\
\sigma_2^{[1]}(20) &= 0.000086979166666666666667,\\
\sigma_2^{[1]}(\infty) &= 0,\\
\widetilde{\sigma}_2^{[1]}(p) &= \frac{2}{3p^3}+\frac{1}{12p^4},\\
\widetilde{\sigma}_2^{[1]}(20) &= 0.000083854166666666666667,\\
e_2^{[1]}(20) &= 0.000022447851778303 27.
\end{aligned}$$

From part (b) of Theorem 4.3 we get

$$\begin{aligned}
\zeta(3) &= \int_{20}^{\infty}\frac{dt}{t^3}+S(19)-\sigma_2^{[1]}(\infty)+\widetilde{\sigma}_2^{[1]}(20)-e_2^{[1]}(20)-\delta_2^{[1]}(20)\\
&= 0.00125+1.2007428419584369581+0.000083854166666666666667-0.000022447851778303 27-\delta_2^{[1]}(20)\\
&= 1.2020560522930126102-\delta_2^{[1]}(20).
\end{aligned}$$

Since

$$\delta_2^{[1]}(20)\approx 9.40\times 10^{-7},$$

we obtain the following numerical approximation of $\zeta(3)$

$$\zeta(3)\approx 1.2019663288791965826, \tag{4.25}$$

which only is accurate up to two decimal places of $\zeta(3)=1.2020569031595942854\ldots$

Example 4.6. *To estimate* $\zeta(5)=\sum_{k=1}^{\infty}\frac{1}{k^5}$, *we put* $m=2$, $r=6$, $p=30$, $f(x)=\frac{1}{x^5}$, $x\in[1,\infty)$ *and apply part (b) of Theorem 4.3. In this case, we have*

$$\begin{aligned}
S(29) &= 1.0369274253541474188,\\
\sigma_6^{[1]}(p) &= \frac{2}{3p^5}+\frac{35}{36p^6}+\frac{8}{9p^7}+\frac{77}{216p^8}\\
&\quad -\frac{26}{81p^9}-\frac{151}{270p^{10}},\\
\sigma_6^{[1]}(30) &= 2.8809650704405569010\times 10^{-8},\\
\sigma_6^{[1]}(\infty) &= 0,\\
\widetilde{\sigma}_6^{[1]}(p) &= \frac{2}{3p^5}+\frac{5}{36p^6}+\frac{1}{18p^7}-\frac{7}{216p^8}\\
&\quad -\frac{5}{81p^9}-\frac{1}{270p^{10}},\\
\widetilde{\sigma}_6^{[1]}(30) &= 2.7627849714267435143\times 10^{-8},\\
e_6^{[1]}(30) &= 6.48060252152\times 10^{-9}.
\end{aligned}$$

From part (b) of Theorem 4.3 we get

$$\begin{aligned}
\zeta(5) &= \int_{30}^{\infty}\frac{dt}{t^5}+S(29)-\sigma_3^{[1]}(\infty)+\widetilde{\sigma}_3^{[1]}(30)-e_3^{[1]}(30)-\delta_3^{[1]}(30)\\
&= 3.0864197530864197531\times 10^{-7}+1.0369274253541474188+2.7627902250673169740\times 10^{-8}\\
&\quad -6.47839130112\times 10^{-9}-\delta_6^{[1]}(30)\\
&= 1.0369277263337192158-\delta_6^{[1]}(30).
\end{aligned}$$

According to

$$\delta_6^{[1]}(30)\approx -3.9236379933251\times 10^{-51},$$

we obtain the following numerical approximation of $\zeta(5)$

$$\zeta(5)\approx 1.0369277263337192158. \tag{4.26}$$

So, our approximation is accurate up to seven decimal places of $\zeta(5)=1.0369277551433699263\ldots$

In [11, Example 5] the examples 4.4 and 4.5 are considered for the level $m=1$. Indeed, putting $r=2$ and $p=20$ the estimate (4.24) is also obtained. So, from a numerical viewpoint the level $m=1$ seems to provide a low computational cost.
Finally, the numerical evidence corresponding to the examples 4.4-4.6 suggests that when $m>1$ for obtaining higher precision for our approximations to the series $\sum_{j=1}^{\infty} f(j)$ we need only use higher values of r in part (b) of Theorem 4.3.

Example 4.7. *Using part (a) of Theorem 4.3 we can deduce that the series*

$$sum_{k=1}^{\infty} \frac{\cos(\sqrt{k})}{k}$$

converges, since

$$\int_1^{\infty} \frac{\cos(\sqrt{t})}{t}\, dt \approx -0.67480784580193626932...$$

The above approximation was performed using MAPLE 15. However, it is not difficult to show that the integral $\int_1^{\infty} \frac{\cos(\sqrt{t})}{t}\, dt$ *converges. Notice that*

$$2\int_1^b \frac{d(\sin(\sqrt{t}))}{\sqrt{t}} = \int_1^b \frac{\cos(\sqrt{t})}{t}\, dt,$$

and by the formula for integration by parts of Riemann-Stieltjes, we have:

$$2\left[\int_1^b \frac{d(\sin(\sqrt{t}))}{\sqrt{t}} + \int_1^b \sin(\sqrt{t})d\left(\frac{1}{\sqrt{t}}\right)\right] = 2\left(\frac{\sin(\sqrt{b})}{\sqrt{b}} - \sin(1)\right).$$

Consequently,

$$\int_1^b \frac{\cos(\sqrt{t})}{t}\, dt = 2\left(\frac{\sin(\sqrt{b})}{\sqrt{b}} - \sin(1)\right) + \int_1^b \frac{\sin(\sqrt{t})}{t^{3/2}}\, dt,$$

since $\lim_{b\to\infty} \frac{\sin(\sqrt{b})}{\sqrt{b}} = 0$ *and the integral* $\int_1^{\infty} \frac{\sin(\sqrt{t})}{t^{3/2}}\, dt$ *converges, then*

$$\int_1^{\infty} \frac{\cos(\sqrt{t})}{t}\, dt \quad \text{converges.}$$

We can provide another solution by using Dirichlet's test for improper integrals (see for instance, [11, Example 4] where a similar series is considered.)

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Solutions of a System of Two Higher-Order Difference Equations in Terms of Lucas Sequence

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Abstract

In this paper we give some theoretical explanations related to the representation for the general solution of the system of the higher-order rational difference equations

$$x_{n+1}=\frac{5y_{n-k}-5}{y_{n-k}},\qquad y_{n+1}=\frac{5x_{n-k}-5}{x_{n-k}},\qquad n,k\in\mathbb{N}_0,$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and the initial conditions x_{-k}, $x_{-k+1},\ldots$, x_0, y_{-k}, $y_{-k+1},\ldots$, y_0 are non zero real numbers such that their solutions are associated to Lucas numbers.
We also study the stability character and asymptotic behavior of this system.

1. Introduction

Giving theoretical explanations related to the exact solutions of most systems of the higher-order rational difference equations is sophisticated sometimes. Therefore, some of the recent papers give formulas for solutions to systems of difference equations and prove them by using only the method of induction.
The prime purpose of this work is to give some theoretical explanations related to the general solution of the system of the higher-order rational difference equations

$$x_{n+1}=\frac{5y_{n-k}-5}{y_{n-k}},\qquad y_{n+1}=\frac{5x_{n-k}-5}{x_{n-k}},\qquad n,k\in\mathbb{N}_0, \tag{1.1}$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and the initial conditions x_{-k}, $x_{-k+1},\ldots$, x_0, y_{-k}, $y_{-k+1},\ldots$, y_0 are non zero real numbers. The solutions of (1.1) are expressed using the famous Fibonacci and Lucas numbers.
The idea is establish the solution form of system (1.1) using appropriate transformation reducing the system into a system of linear type difference equations.
In [30], the authors give formulas for solutions of the equation

$$y_{n+1}=\frac{1+y_{n-1}}{y_ny_{n-1}},\qquad n\in\mathbb{N}_0,$$

and prove them by using only the method of induction. However, the formulas are not justified by some theoretical explanations.
Halim et al. [8] gave the solutions of the systems of difference equations

$$x_{n+1}=\frac{1}{\pm1\pm y_{n-k}},\quad y_{n+1}=\frac{1}{\pm1\pm x_{n-k}},\qquad n,k\in\mathbb{N}_0,$$

such that their solutions are associated to Fibonacci numbers.

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Also, in [7] Halim et al. establish the solution form of equation

$$y_{n+1}=\frac{\alpha+\beta y_{n-1}}{\delta y_n y_{n-1}}, \qquad n\in\mathbb{N}_0,$$

using appropriate transformation reducing the equation into a linear type difference equation, such that their solutions are associated to generalized Padovan numbers.
In [19], Stevic gave a theoretical explanation for the formula of solutions of the following difference equation

$$y_{n+1}=\frac{\alpha y_n+\beta}{\gamma y_n+\delta}, \qquad n\in\mathbb{N}_0,$$

where parameters $\alpha,\beta,\gamma,\delta$ and initial value y_0 are real numbers, such that their solutions are associated to generalized Fibonacci numbers. More details on this aspect can be simply found in refs. [1]-[3],[9]-[13], [19], [22]-[28], [30],[31].

2. Preliminaries

2.1. Linearized stability of the higher-order systems

Let f and g be two continuously differentiable functions:

$$f:\ I^{k+1}\times J^{k+1}\longrightarrow I, \quad g:\ I^{k+1}\times J^{k+1}\longrightarrow J,$$

where I, J are some interval of real numbers. For $n\in\mathbb{N}_0$, consider the system of difference equations

$$\begin{cases} x_{n+1}= & f\left(x_n,x_{n-1},\dots,x_{n-k},y_n,y_{n-1},\dots,y_{n-k}\right)\\ y_{n+1}= & g\left(x_n,x_{n-1},\dots,x_{n-k},y_n,y_{n-1},\dots,y_{n-k}\right)\end{cases} \tag{2.1}$$

where $n,k\in\mathbb{N}_0$, $(x_{-k},x_{-k+1},\dots,x_0)\in I^{k+1}$ and $(y_{-k},y_{-k+1},\dots,y_0)\in J^{k+1}$.
Define the map $H:\ I^{k+1}\times J^{k+1}\longrightarrow I^{k+1}\times J^{k+1}$ by

$$H(W)=(f_0(W),f_1(W),\dots,f_k(W),g_0(W),g_1(W),\dots,g_k(W))$$

where

$$W=(u_0,u_1,\dots,u_k,v_0,v_1,\dots,v_k)^T,$$

$$f_0(W)=f(W),f_1(W)=u_0,\dots,f_k(W)=u_{k-1},$$

$$g_0(W)=g(W),g_1(W)=v_0,\dots,g_k(W)=v_{k-1}.$$

Let

$$W_n=[x_n,x_{n-1},\dots,x_{n-k},y_n,y_{n-1},\dots,y_{n-k}]^T.$$

Then, we can easily see that system (2.1) is equivalent to the following system written in vector form

$$W_{n+1}=H(W_n), \qquad n\in\mathbb{N}_0, \tag{2.2}$$

that is

$$\begin{cases} x_{n+1}= & f\left(x_n,x_{n-1},\dots,x_{n-k},y_n,y_{n-1},\dots,y_{n-k}\right),\\ x_n= & x_n,\\ \vdots & \\ x_{n-k+1}= & x_{n-k+1},\\ y_{n+1}= & g\left(x_n,x_{n-1},\dots,x_{n-k},y_n,y_{n-1}\dots,,y_{n-k}\right),\\ y_n= & y_n,\\ \vdots & \\ y_{n-k+1}= & y_{n-k+1}.\end{cases}$$

Definition 2.1 (Equilibrium point)**.** *An equilibrium point $(\bar{x},\bar{y})\in I\times J$ of system (2.1) is a solution of the system*

$$\begin{cases} x=f\left(x,x,\dots,x,y,y,\dots,y\right),\\ y=g\left(x,x,\dots,x,y,y,\dots,y\right).\end{cases}$$

Furthermore, an equilibrium point $\overline{W}\in I^{k+1}\times J^{k+1}$ of system (2.2) is a solution of the system

$$W=H(W).$$

Definition 2.2 (Stability)**.** *Let $\overline{W}$ be an equilibrium point of system (2.2) and $\|\ .\ \|$ be any norm (e.g. the Euclidean norm).*

1. *The equilibrium point $\overline{W}$ is called stable (or locally stable) if for every $\varepsilon > 0$ there exist δ such that $\|W_0 - \overline{W}\| < \delta$ implies $\|W_n - \overline{W}\| < \varepsilon$ for $n \geq 0$.*
2. *The equilibrium point $\overline{W}$ is called asymptotically stable (or locally asymptotically stable) if it is stable and there exist $\gamma > 0$ such that $\|W_0 - \overline{W}\| < \gamma$ implies*
$$\lim_{n\to+\infty} W_n = \overline{W}.$$
3. *The equilibrium point $\overline{W}$ is said to be global attractor (respectively global attractor with basin of attraction a set $G \subseteq I^{k+1} \times J^{k+1}$, if for every W_0 (respectively for every $W_0 \in G$)*
$$\lim_{n\to+\infty} W_n = \overline{W}.$$
4. *The equilibrium point $\overline{W}$ is called globally asymptotically stable (respectively globally asymptotically stable relative to G) if it is asymptotically stable, and if for every W_0 (respectively for every $W_0 \in G$),*
$$\lim_{n\to+\infty} W_n = \overline{W}.$$
5. *The equilibrium point $\overline{W}$ is called unstable if it is not stable.*

Remark 2.3. *Clearly, $(\bar{x}, \bar{y}) \in I \times J$ is an equilibrium point for system (2.1) if and only if $\overline{W} = (\bar{x}, \bar{x}, \cdots, \bar{x}, \bar{y}, \bar{y}, \cdots, \bar{y}) \in I^{k+1} \times J^{k+1}$ is an equilibrium point of system (2.2).*

From here on, by the stability of the equilibrium points of system (2.1), we mean the stability of the corresponding equilibrium points of the equivalent system (2.2). The linearized system, associated to system (2.2), about the equilibrium point

$$\overline{W} = (\bar{x}, \bar{x}, \cdots, \bar{x}, \bar{y}, \bar{y}, \cdots, \bar{y}),$$

is given by

$$W_{n+1} = AW_n, \qquad n \in \mathbb{N}_0,$$

where A is the Jacobian matrix of the map H at the equilibrium point $\overline{W}$ given by

$$A = \begin{pmatrix}
\frac{\partial f_0}{\partial u_0}(\overline{W}) & \frac{\partial f_0}{\partial u_1}(\overline{W}) & \dots & \frac{\partial f_0}{\partial u_k}(\overline{W}) & \frac{\partial f_0}{\partial v_0}(\overline{W}) & \frac{\partial f_0}{\partial v_1}(\overline{W}) & \dots & \frac{\partial f_0}{\partial v_k}(\overline{W}) \\
\frac{\partial f_1}{\partial u_0}(\overline{W}) & \frac{\partial f_1}{\partial u_1}(\overline{W}) & \dots & \frac{\partial f_1}{\partial u_k}(\overline{W}) & \frac{\partial f_1}{\partial v_0}(\overline{W}) & \frac{\partial f_1}{\partial v_1}(\overline{W}) & \dots & \frac{\partial f_1}{\partial v_k}(\overline{W}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_k}{\partial u_0}(\overline{W}) & \frac{\partial f_k}{\partial u_1}(\overline{W}) & \dots & \frac{\partial f_k}{\partial u_k}(\overline{W}) & \frac{\partial f_k}{\partial v_0}(\overline{W}) & \frac{\partial f_k}{\partial v_1}(\overline{W}) & \dots & \frac{\partial f_k}{\partial v_k}(\overline{W}) \\
\frac{\partial g_0}{\partial u_0}(\overline{W}) & \frac{\partial g_0}{\partial u_1}(\overline{W}) & \dots & \frac{\partial g_0}{\partial u_k}(\overline{W}) & \frac{\partial g_0}{\partial v_0}(\overline{W}) & \frac{\partial g}{\partial v_1}(\overline{W}) & \dots & \frac{\partial g_0}{\partial v_k}(\overline{W}) \\
\frac{\partial g_1}{\partial u_0}(\overline{W}) & \frac{\partial g_1}{\partial u_1}(\overline{W}) & \dots & \frac{\partial g_1}{\partial u_k}(\overline{W}) & \frac{\partial g_1}{\partial v_0}(\overline{W}) & \frac{\partial g_1}{\partial v_1}(\overline{W}) & \dots & \frac{\partial g_1}{\partial v_k}(\overline{W}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial g_k}{\partial u_0}(\overline{W}) & \frac{\partial g_k}{\partial u_1}(\overline{W}) & \dots & \frac{\partial g_k}{\partial u_k}(\overline{W}) & \frac{\partial g_k}{\partial v_0}(\overline{W}) & \frac{\partial g_k}{\partial v_1}(\overline{W}) & \dots & \frac{\partial g_k}{\partial v_k}(\overline{W})
\end{pmatrix}.$$

Theorem 2.4. *(Linearized stability)*

1. *If all the eigenvalues of the Jacobian matrix A lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{W}$ of system (2.2) is asymptotically stable.*
2. *If at least one eigenvalue of the Jacobian matrix A have absolute value greater than one, then the equilibrium point $\overline{W}$ of system (2.2) is unstable.*

2.2. Lucas sequence

The integer sequence defined by the recurrence relation

$$L_{n+1} = L_n + L_{n-1}, \quad n \in \mathbb{N},$$

with the initial conditions $L_0 = 2$ and $L_1 = 1$, is known as the Lucas numbers and was named after François Edouard Anatole Lucas (1842-91). This is the same recurrence relation as for the Fibonacci sequence, but with different initial conditions ($F_0 = 0, F_1 = 1$). The first few terms of the recurrence sequence are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$. The Binet's formula for this recurrence sequence can easily be obtained and is given by

$$L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} (the\ so-called\ golden\ number), \quad \beta = \frac{1-\sqrt{5}}{2}.$$

That is,

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha.$$

For more informations associated with Lucas sequence, see [15] and [29].

3. Closed-Form solutions and stability of system (1.1)

For the rest of our discussion we assume L_n, the n-th Lucas number, to satisfy the recurrence equation

$$L_{n+1} = L_n + L_{n-1}, \quad n \in \mathbb{N}_0,$$

with initial conditions $L_0 = 2$ and $L_1 = 1$.

3.1. Linear second order differences equations with constants coefficients.

As is well-known, the equation

$$z_{n+1} + 5z_n + 5z_{n-1} = 0, \quad n \in \mathbb{N}_0, \tag{3.1}$$

(the homogeneous linear second order difference equation with constant coefficients), where z_0 and $z_{-1} \in \mathbb{R}$, is usually solved by using the characteristic roots λ_1 and λ_2 of the characteristic polynomial $\lambda^2 + 5\lambda + 5 = 0$, so

$$\lambda_1 = \sqrt{5}\beta, \qquad \lambda_2 = -\sqrt{5}\alpha,$$

and the formulas of general solution is

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n.$$

Using the initial conditions z_0 and z_{-1} with some calculations we get

$$c_1 = -\sqrt{5}\left(z_{-1} - \frac{z_0}{5}\lambda_1\right),$$

$$c_2 = -\sqrt{5}\left(\frac{z_0}{5}\lambda_2 - z_{-1}\right).$$

So,

$$z_n = \sqrt{5}\left(z_{-1}\left[\lambda_2^n - \lambda_1^n\right] - \frac{z_0}{5}\left[\lambda_2^{n+1} - \lambda_1^{n+1}\right]\right),$$

$$= \sqrt{5}\left(z_{-1}(\sqrt{5})^n\left[(-1)^n\alpha^n - \beta^n\right] - \frac{z_0(\sqrt{5})^{n+1}}{(\sqrt{5})^2}\left[(-1)^{n+1}\alpha^{n+1} - \beta^{n+1}\right]\right),$$

by put

$$N_n = \left((-1)^n\alpha^n - \beta^n\right),$$

is obtained that the general solution of equation (3.1) is

$$z_n = (\sqrt{5})^n\left[z_{-1}\sqrt{5}N_n - z_0N_{n+1}\right]. \tag{3.2}$$

Similarly, let

$$z_{n+1} - 5z_n + 5z_{n-1} = 0, \quad n \in \mathbb{N}_0, \tag{3.3}$$

so, by put

$$M_n = \left(\alpha^n - (-1)^n\beta^n\right),$$

is obtained that the general solution of equation (3.3) is

$$z_n = -(\sqrt{5})^n\left[\sqrt{5}z_{-1}M_n - z_0M_{n+1}\right]. \tag{3.4}$$

3.2. Linear system of second order difference equations with constant coefficients.

Let the linear system of second order difference equations

$$u_{n+1} = 5v_n - 5u_{n-1}, \quad v_{n+1} = 5u_n - 5v_{n-1}, \quad n \in \mathbb{N}_0. \tag{3.5}$$

From (3.5) we get

$$v_n = \frac{1}{5}\left(u_{n+1} + 5u_{n-1}\right). \tag{3.6}$$

We replace (3.6) in the second equation of the system (3.5), we get

$$\frac{1}{5}u_{n+2} - 3u_n + 5u_{n-2} = 0,$$

which can be written both as

$$\underbrace{(u_{n+2} - 5u_{n+1} + 5u_n)}_{s_{n+1}} + 5\underbrace{(u_{n+1} - 5u_n + 5u_{n-1})}_{s_n} + 5\underbrace{(u_n - 5u_{n-1} + 5u_{n-2})}_{s_{n-1}} = 0, \quad n \in \mathbb{N},$$

which is in the form of equation (3.1) and as

$$\underbrace{(u_{n+2}+5u_{n+1}+5u_n)}_{k_{n+1}}-5\underbrace{(u_{n+1}+5u_n+5u_{n-1})}_{k_n}+5\underbrace{(u_n+5u_{n-1}+5u_{n-2})}_{k_{n-1}}=0, \quad n\in\mathbb{N}, \tag{3.7}$$

which is in the form of equation (3.3). Form (3.4) and (3.2) we can write

$$\begin{cases} s_{2n}= & (\sqrt{5})^{2n}\left[5s_{-1}F_{2n}+s_0L_{2n+1}\right], \\ s_{2n+1}= & (\sqrt{5})^{2n+2}\left[s_{-1}L_{2n+1}+s_0F_{2n+2}\right]. \end{cases}$$

Hence

$$u_{2n+1}-5u_{2n}+5u_{2n-1}=(\sqrt{5})^{2n}[5(u_0-5u_{-1}+5u_{-2})F_{2n}+(u_1-5u_0+5u_{-1})L_{2n+1}], \tag{3.8}$$

and

$$u_{2n+2}-5u_{2n+1}+5u_{2n}=(\sqrt{5})^{2n+2}[(u_0-5u_{-1}+5u_{-2})L_{2n+1}+(u_1-5u_0+5u_{-1})F_{2n+2}].$$

Similarly, form (3.3) and (3.7) we can write

$$\begin{cases} k_{2n}= & -(\sqrt{5})^{2n}\left[5k_{-1}F_{2n}-k_0L_{2n+1}\right], \\ k_{2n+1}= & -(\sqrt{5})^{2n+2}\left[k_{-1}L_{2n+1}-k_0F_{2n+2}\right]. \end{cases}$$

Hence

$$u_{2n+1}+5u_{2n}+5u_{2n-1}=-(\sqrt{5})^{2n}\left[5(u_0+5u_{-1}+5u_{-2})F_{2n}-(u_1+5u_0+5u_{-1})L_{2n+1}\right], \tag{3.9}$$

and

$$u_{2n+2}-5u_{2n+1}+5u_{2n}=-(\sqrt{5})^{2n+2}\left[(u_0+5u_{-1}+5u_{-2})L_{2n+1}-(u_1+5u_0+5u_{-1})F_{2n+2}\right].$$

Now, by subtracting equation (3.9) from equation (3.8), we obtain

$$u_{2n}=-(\sqrt{5})^{2n}[5v_{-1}F_{2n}-u_0L_{2n+1}]. \tag{3.10}$$

Also, by equation (3.9) and equation (3.8), we obtain

$$v_{2n}=-(\sqrt{5})^{2n}\left[5u_{-1}F_{2n}-v_0L_{2n+1}\right]. \tag{3.11}$$

By a similar calculation, we obtain

$$u_{2n+1}=-(\sqrt{5})^{2n+2}[u_{-1}L_{2n+1}-v_0F_{2n+2}], \tag{3.12}$$

and

$$v_{2n+1}=-(\sqrt{5})^{2n}[v_{-1}L_{2n+1}-u_0F_{2n+2}]. \tag{3.13}$$

Now we consider the system of two first-order difference equations

$$z_{n+1}=\frac{5w_n-5}{w_n}, \qquad w_{n+1}=\frac{5z_n-5}{z_n}, \qquad n\in\mathbb{N}_0. \tag{3.14}$$

where the initial conditions z_0 and w_0 are non zero real numbers.
Through an analytical approach. We put

$$z_n=\frac{u_n}{v_{n-1}}, \quad w_n=\frac{v_n}{u_{n-1}}.$$

Hence we have the system

$$u_{n+1}=5v_n-5u_{n-1}, \quad v_{n+1}=5u_n-5v_{n-1}. \tag{3.15}$$

by formulas, (3.5), (3.2), (3.4), (3.10), (3.11), (3.12) and (3.13) is obtained that the general solution of system (3.15) is

$$\begin{cases} u_{2n} &= -(\sqrt{5})^{2n}\left[5v_{-1}F_{2n}-u_0L_{2n+1}\right], \\ u_{2n+1} &= -(\sqrt{5})^{2n+2}\left[u_{-1}L_{2n+1}-v_0F_{2n+2}\right], \\ v_{2n} &= -(\sqrt{5})^{2n}\left[5u_{-1}F_{2n}-v_0L_{2n+1}\right], \\ v_{2n+1} &= -(\sqrt{5})^{2n+2}\left[v_{-1}L_{2n+1}-u_0F_{2n+2}\right]. \end{cases}$$

From all above mentioned we see that the following theorem holds.

Theorem 3.1. *Let $\{z_n, w_n\}_{n\geq -1}$ be a solution of (3.14). Then, for $n = 2,3,\ldots,$*

$$\begin{cases} z_{2n} &= \dfrac{5F_{2n} - z_0 L_{2n+1}}{L_{2n-1} - z_0 F_{2n}}, \\ z_{2n+1} &= \dfrac{5L_{2n+1} - 5w_0 F_{2n+2}}{5F_{2n} - w_0 L_{2n+1}}, \\ w_{2n} &= \dfrac{5F_{2n} - w_0 L_{2n+1}}{L_{2n-1} - w_0 F_{2n}}, \\ w_{2n+1} &= \dfrac{5L_{2n+1} - 5z_0 F_{2n+2}}{5F_{2n} - z_0 L_{2n+1}}. \end{cases}$$

where $\{L_n\}_n$ is the Lucas sequence, $\{F_n\}_n$ is the Fibonacci sequence and the initial conditions z_0 and $w_0 \in \mathbb{R} - G_1$, with G_1 is the Forbidden Set of system (3.14) given by

$$G_1 = \bigcup_{n=-1}^{\infty} \{(z_0, w_0) : L_{2n-1} - z_0 F_{2n} = 0, \quad 5F_{2n} - w_0 L_{2n+1} = 0\}.$$

Let

$$\begin{cases} x_n^{(j)} = x_{(k+1)n-j}, \\ y_n^{(j)} = y_{(k+1)n-j}. \end{cases} \tag{3.16}$$

where $j \in \{0, 1, \ldots k\}$.
Using notation (3.16), we can write (1.1) as

$$\begin{cases} x_{n+1}^{(j)} = \dfrac{5y_n^{(j)} - 5}{y_n^{(j)}}, \\ y_{n+1}^{(j)} = \dfrac{5x_n^{(j)} - 5}{x_n^{(j)}}. \quad n \in \mathbb{N}, \end{cases}$$

for each $j \in \{0, 1, \ldots, k\}$.
So, from Theorem (3.1) we get for $n = 2,3,\ldots,$

$$\begin{cases} x_{2n}^{(j)} &= \dfrac{5F_{2n} - x_0^{(j)} L_{2n+1}}{L_{2n-1} - x_0^{(j)} F_{2n}}, \\ x_{2n+1}^{(j)} &= \dfrac{5L_{2n+1} - 5y_0^{(j)} F_{2n+2}}{5F_{2n} - y_0^{(j)} L_{2n+1}}, \\ y_{2n}^{(j)} &= \dfrac{5F_{2n} - y_0^{(j)} L_{2n+1}}{L_{2n-1} - y_0^{(j)} F_{2n}}, \\ y_{2n+1}^{(j)} &= \dfrac{5L_{2n+1} - 5x_0^{(j)} F_{2n+2}}{5F_{2n} - x_0^{(j)} L_{2n+1}}. \end{cases}$$

From all above mentioned we see that the following theorem holds.

Theorem 3.2. *Let $\{x_n, y_n\}_{n\geq -1}$ be a solution of (1.1). Then, for $n = 2,3,\ldots,$*

$$\begin{cases} x_{(k+1)2n-j} &= \dfrac{5F_{2n} - x_{-j} L_{2n+1}}{L_{2n-1} - x_{-j} F_{2n}}, \\ x_{(k+1)(2n+1)-j} &= \dfrac{5L_{2n+1} - 5y_{-j} F_{2n+2}}{5F_{2n} - y_{-j} L_{2n+1}}, \\ y_{(k+1)2n-j} &= \dfrac{5F_{2n} - y_{-j} L_{2n+1}}{L_{2n-1} - y_{-j} F_{2n}}, \\ x_{(k+1)(2n+1)-j} &= \dfrac{5L_{2n+1} - 5x_{-j} F_{2n+2}}{5F_{2n} - x_{-j} L_{2n+1}}. \end{cases}$$

where $j \in \{0, 1, \ldots, k\}$, $\{L_n\}_n$ the Lucas sequence, $\{F_n\}_n$ the Fibonacci sequence and the initial conditions x_{-k}, $x_{-k+1}, \ldots x_0$, y_{-k}, $y_{-k+1}, \ldots, y_1$ and $y_0 \in \mathbb{R} - G_j$, with G_j is the Forbidden Set of system (1.1) given by

$$G_j = \bigcup_{n=-1}^{\infty} \{(x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0,) : L_{2n-1} - x_{-j} F_{2n} = 0, \quad 5F_{2n} - y_{-j} L_{2n+1} = 0, \quad j = 0, 1, \ldots, k\}.$$

3.3. Global stability of positive solutions

In this section we study the global stability character of the solutions of system (1.1). It is easy to show that (1.1) has a unique real positive equilibrium point given by

$$E=(\bar{x},\bar{y})=\left(\sqrt{5}\alpha,\sqrt{5}\alpha\right),$$

where α is the golden number.
Let $I=J=(0,+\infty)$ and consider the functions

$$f:\ I^{k+1}\times J^{k+1}\longrightarrow I,\ g:\ I^{k+1}\times J^{k+1}\longrightarrow J$$

defined by

$$f(u_0,u_1,\ldots,u_k,v_0,v_1,\ldots,v_k)=\frac{5v_k-5}{v_k},$$
$$g(u_0,u_1,\ldots,u_k,v_0,v_1,\ldots,v_k)=\frac{5u_k-5}{u_k}.$$

Theorem 3.3. *The equilibrium point E is locally asymptotically stable.*

Proof. The the linearized system about the equilibrium point

$$\overline{W}=\left(\sqrt{5}\alpha,\ldots,\sqrt{5}\alpha,\sqrt{5}\alpha,\ldots,\sqrt{5}\alpha\right)\in I^{k+1}\times J^{k+1}$$

is given by

$$X_{n+1}=AX_n,\quad X_n=(x_n,x_{n-1},\ldots,x_{n-k},y_n,y_{n-1},\ldots,y_{n-k})^T,$$

and

$$A=\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \frac{1}{\alpha^2}\\ 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0\\ 0 & 0 & \ldots & \frac{1}{\alpha^2} & 0 & 0 & \ldots & 0\\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

So, after some elementary calculations, we get

$$P(\lambda)=\det(A-\lambda I_{2k+2})=\lambda^{2k+2}-\frac{1}{\alpha^4}.$$

Now, consider the two functions defined by

$$a(\lambda)=\lambda^{2k+2},\quad b(\lambda)=\frac{1}{\alpha^4}.$$

We have

$$|b(\lambda)|<|a(\lambda)|,\forall\lambda:|\lambda|=1.$$

Thus, by Rouche's Theorem, all zeros of $P(\lambda)=a(\lambda)-b(\lambda)=0$ lie in $|\lambda|<1$. So, by Theorem (2.4), we get that E is locally asymptotically stable. □

Theorem 3.4. *The equilibrium point E is globally asymptotically stable.*

Proof. Let $\{x_n,y_n\}_{n\geq -k}$ be a solution of (1.1). By Theorem (3.3) we need only to prove that E is global attractor, that is

$$\lim_{n\to\infty}(x_n,y_n)=E.$$

To do this, we prove that for $j=0,1,\ldots,k$ we have

$$\lim_{n\to+\infty}x_{(k+1)2n-j}=\lim_{n\to+\infty}x_{(k+1)(2n+1)-j}=\lim_{n\to+\infty}y_{(k+1)2n-j}=\lim_{n\to+\infty}y_{(k+1)(2n+1)-j}=\sqrt{5}\alpha.$$

For $j=0,1,\ldots,k$, it follows from Theorem (3.2) that

$$\lim_{n\to+\infty}x_{(k+1)2n-j}=\lim_{n\to+\infty}\frac{5F_{2n}-x_{-j}L_{2n+1}}{L_{2n-1}-x_{-j}F_{2n}}$$
$$=\lim_{n\to+\infty}\frac{5-x_{-j}\frac{L_{2n+1}}{F_{2n}}}{\frac{L_{2n-1}}{F_{2n}}-x_{-j}}.$$

Using

$$\lim_{n\to+\infty}\frac{F_{n+1}}{L_n}=\frac{\alpha}{\sqrt{5}},\quad \lim_{n\to+\infty}\frac{L_{n+1}}{F_n}=\sqrt{5}\alpha$$

we get

$$\lim_{n\to+\infty}x_{(k+1)2n-j}=\alpha\sqrt{5}.$$

Similarly we get

$$\lim_{n\to+\infty}x_{(k+1)(2n+1)-j}=\lim_{n\to+\infty}y_{(k+1)2n-j}=\lim_{n\to+\infty}y_{(k+1)(2n+1)-j}=\sqrt{5}\alpha.$$

□

3.4. Numerical confirmation

This subsection is included to verify and confirm the results we obtained in this work. As an illustration of our results, we consider the following numerical examples.

Example 3.5. *Let $k=0$ in system (1.1), then we obtain the system*

$$x_{n+1}=\frac{5y_n-5}{y_n},\qquad y_{n+1}=\frac{5x_n-5}{x_n},\qquad n\in\mathbb{N}_0. \tag{3.17}$$

Assume $x_0=0.7$ and $y_0=1.5$. (See Fig (3.1))

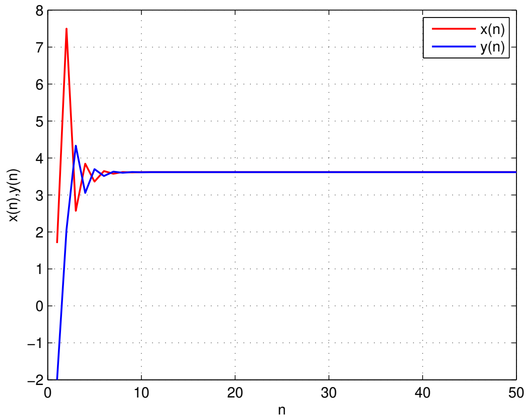

Figure 3.1: This figure shows that the solution of the system (3.17) is globally asymptotically stable

Example 3.6. *Let $k=2$ in system (1.1), then we obtain the system*

$$x_{n+1}=\frac{5y_{n-2}-5}{y_{n-2}},\qquad y_{n+1}=\frac{5x_{n-2}-5}{x_{n-2}},\qquad n\in\mathbb{N}_0. \tag{3.18}$$

Assume $x_{-2}=0.5,\ x_{-1}=0.7,\ x_0=1.6,\ y_{-2}=0.6,\ y_{-1}=-50$ and $y_0=1.7$. (See Fig(3.2))

Example 3.7. *Let $k=3$ in system (1.1), then we obtain the system*

$$x_{n+1}=\frac{5y_{n-3}-5}{y_{n-3}},\qquad y_{n+1}=\frac{5x_{n-3}-5}{x_{n-3}},\qquad n\in\mathbb{N}_0. \tag{3.19}$$

Assume $x_{-3}=0.8,\ x_{-2}=0.7,\ x_{-1}=0.6,\ x_0=0.9,\ y_{-3}=1.1,\ y_{-2}=1.8,\ y_{-1}=1.3$ and $y_0=1.6$. (See Fig(3.3))

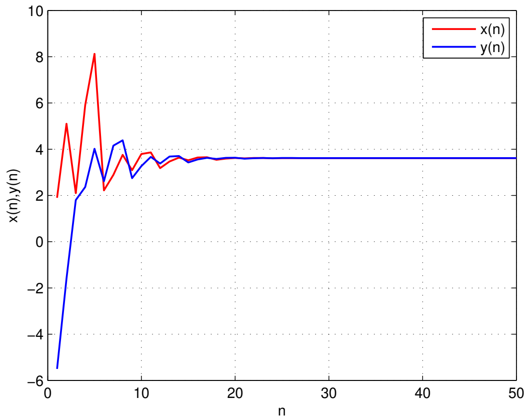

Figure 3.2: This figure shows that the solution of the system (3.18) is globally asymptotically stable

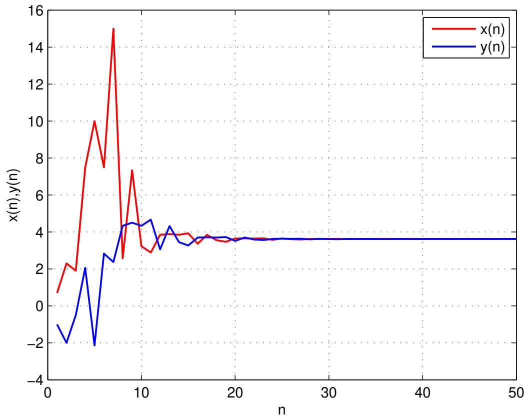

Figure 3.3: This figure shows that the solution of the system (3.19) is globally asymptotically stable

4. Conclusion

In this work, we have successfully established a theoretical explanation related to the closed-form solution of the system of two higher-order difference equations

$$x_{n+1} = \frac{5y_{n-k} - 5}{y_{n-k}}, \qquad y_{n+1} = \frac{5x_{n-k} - 5}{x_{n-k}}, \qquad n,k \in \mathbb{N}_0.$$

Also, we obtained stability results for the positive solutions of this system. Particularly, we have shown that the positive solutions of this system tends to a computable finite number, and is in fact expressible in terms of the well-known golden number.
This work we leave to the interested readers.

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Global Behavior of Two Rational Third Order Difference Equations

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Abstract

In this paper, we solve and study the global behavior of all admissible solutions of the two difference equations

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}}, \quad n = 0, 1, \ldots,$$

and

$$x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots,$$

where the initial values x_{-2}, x_{-1}, x_0 are real numbers.
We show that every admissible solution for the first equation converges to zero. For the other equation, we show that every admissible solution is periodic with prime period six. Finally we give some illustrative examples.

1. Introduction

In [11], the author determined the forbidden sets and discussed the global behaviors of solutions of the two difference equations

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - x_{n-2}}, \quad n = 0, 1, \ldots,$$

and

$$x_{n+1} = \frac{x_n x_{n-1}}{-x_n + x_{n-2}}, \quad n = 0, 1, \ldots,$$

where the initial values x_{-2}, x_{-1}, x_0 are real numbers.
In [2], the author determined the forbidden sets and discussed the global behaviors of solutions of the two difference equations

$$x_{n+1} = \frac{a x_n x_{n-1}}{\pm b x_{n-1} + c x_{n-2}}, \quad n = 0, 1, \ldots,$$

where a, b, c are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.
Elsayed in [19] studied the behavior of solutions of the nonlinear difference equation

$$x_{n+1} = a x_{n-1} + \frac{b x_n x_{n-1}}{c x_n + d x_{n-2}}, \quad n = 0, 1, \ldots,$$

where a, b, c, d are positive real constants and the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers. For more on difference equations (See [1, 3–10, 12–18, 20–28]) and the references therein.
In this paper, we study the two difference equations

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

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and

$$x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1} + x_{n-2}}, \quad n = 0, 1, ..., \tag{1.2}$$

where the initial values x_{-2}, x_{-1}, x_0 are real numbers.

2. The difference equation $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}}$

During this section, we suppose that

$$\lambda_- = \frac{1}{2} - \frac{\sqrt{5}}{2} \text{ and } \lambda_+ = \frac{1}{2} + \frac{\sqrt{5}}{2}.$$

2.1. Solution of Equation (1.1)

The transformation

$$y_n = \frac{x_{n-1}}{x_n}, \text{ with } y_{-1} = \frac{x_{-2}}{x_{-1}},\ y_0 = \frac{x_{-1}}{x_0} \tag{2.1}$$

reduces Equation (1.1) into the difference equation

$$y_{n+1} = \frac{1}{y_{n-1}} - 1,\ n = 0, 1, ... \tag{2.2}$$

By solving Equation (2.2) and after some calculations, the solution of Equation (1.1) can be obtained.

Theorem 2.1. *Let $\{x_n\}_{n=-2}^{\infty}$ be an admissible solution of Equation (1.1). Then*

$$x_n = \begin{cases} -\frac{v}{(x_0 f_{\frac{n-1}{2}} - x_{-1} f_{\frac{n+1}{2}})(x_{-1} f_{\frac{n+1}{2}} - x_{-2} f_{\frac{n+3}{2}})}, & n = 1, 3, ..., \\ \frac{v}{(x_0 f_{\frac{n}{2}} - x_{-1} f_{\frac{n}{2}+1})(x_{-1} f_{\frac{n}{2}} - x_{-2} f_{\frac{n}{2}+1})}, & n = 2, 4, ..., \end{cases} \tag{2.3}$$

where $v = x_0 x_{-1} x_{-2}$ and f_n is the solution of the difference equation

$$f_{n+2} = f_n + f_{n+1},\ f_0 = 0,\ f_1 = 1,\ n = 0, 1, ...$$

Proof. We can write the solution formula (2.3) as

$$x_{2m+1} = -\frac{v}{(x_0 f_m - x_{-1} f_{m+1})(x_{-1} f_{m+1} - x_{-2} f_{m+2})}$$

and (2.4)

$$x_{2m+2} = \frac{v}{(x_0 f_{m+1} - x_{-1} f_{m+2})(x_{-1} f_{m+1} - x_{-2} f_{m+2})}.$$

When $m = 0$,

$$\begin{aligned} x_1 &= -\frac{v}{(x_0 f_0 - x_{-1} f_1)(x_{-1} f_1 - x_{-2} f_2)} \\ &= \frac{v}{x_{-1}(x_{-1} - x_{-2})} = \frac{x_0 x_{-2}}{x_{-1} - x_{-2}}. \end{aligned}$$

Similarly

$$\begin{aligned} x_2 &= \frac{v}{(x_0 f_1 - x_{-1} f_2)(x_{-1} f_1 - x_{-2} f_2)} \\ &= \frac{v}{(x_0 - x_{-1})(x_{-1} - x_{-2})} = \frac{x_1 x_{-1}}{x_0 - x_{-1}}. \end{aligned}$$

Suppose that the solution formula (2.4) is true for $m > 0$. Then

$$\begin{aligned} \frac{x_{2m+1} x_{2m-1}}{x_{2m} - x_{2m-1}} &= \frac{\left(\frac{v}{(x_0 f_m - x_{-1} f_{m+1})(x_{-1} f_{m+1} - x_{-2} f_{m+2})}\right)\left(\frac{v}{(x_0 f_{m-1} - x_{-1} f_m)(x_{-1} f_m - x_{-2} f_{m+1})}\right)}{\frac{v}{(x_0 f_m - x_{-1} f_{m+1})(x_{-1} f_m - x_{-2} f_{m+1})} + \frac{v}{(x_0 f_{m-1} - x_{-1} f_m)(x_{-1} f_m - x_{-2} f_{m+1})}} \\ &= \frac{v}{(x_0 f_{m-1} - x_{-1} f_m)(x_{-1} f_{m+1} - x_{-2} f_{m+2}) + (x_0 f_m - x_{-1} f_{m+1})(x_{-1} f_{m+1} - x_{-2} f_{m+2})} \\ &= \frac{v}{(x_{-1} f_{m+1} - x_{-2} f_{m+2})(x_0(f_{m-1} + f_m) - x_{-1}(f_m + f_{m+1}))} \\ &= \frac{v}{(x_{-1} f_{m+1} - x_{-2} f_{m+2})(x_0 f_{m+1} - x_{-1} f_{m+2})} \\ &= x_{2m+2}. \end{aligned}$$

Similarly we can show that

$$\frac{x_{2m+2} x_{2m}}{a x_{2m+1} + b x_{2m}} = x_{2m+3}.$$

This completes the proof. □

It is clear for Equation (1.1) that if we start with the point $(x_0, x_{-1}, x_{-2}) \in \mathbb{R}^3$, we have the following:
If $x_0 = 0$ and $x_{-1}x_{-2} \neq 0$, then x_3 is undefined.
If $x_{-1} = 0$ and $x_0x_{-2} \neq 0$, then x_5 is undefined.
If $x_{-2} = 0$ and $x_0x_{-1} \neq 0$, then x_4 is undefined.
Therefore, any point $(x_0, x_{-1}, x_{-2}) \in \mathbb{R}^3$ with $x_0x_{-1}x_{-2} = 0$ belongs to the forbidden set of Equation (1.1).
The following result provides the forbidden set of Equation (1.1).

Theorem 2.2. *The forbidden set of equation* (1.1) *is*

$$F = \bigcup_{i=0}^{2}\{(u_0,u_{-1},u_{-2}) \in \mathbb{R}^3 : u_{-i} = 0\} \cup \bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2}) \in \mathbb{R}^3 : u_0 = u_{-1}\frac{f_{m+1}}{f_m}\} \cup \bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2}) \in \mathbb{R}^3 : u_{-1} = u_{-2}\frac{f_{m+1}}{f_m}\}.$$

Proof. The proof is clear using the arguments after Theorem (2.1) and formula (2.3). □

2.2. Global behavior of equation (1.1)

In this section, we shall give two invariant sets for Equation (1.1) and a result concerns the global behavior of the solutions of Equation (1.1). Consider the set

$$D_1 = \{(x,y,z) \in \mathbb{R}^3 : \frac{x}{1/\lambda_-^2} = -\frac{y}{1/\lambda_-} = z\}$$

and

$$D_2 = \{(x,y,z) \in \mathbb{R}^3 : \frac{x}{1/\lambda_+^2} = -\frac{y}{1/\lambda_+} = z\}.$$

Theorem 2.3. *The two sets D_1 and D_2 are invariant sets for Equation* (1.1).

Proof. Let $(x_0, x_{-1}, x_{-2}) \in D_1$. We show that $(x_n, x_{n-1}, x_{n-2}) \in D_1$ for each $n \in \mathbb{N}$. The proof is by induction on n. The point $(x_0, x_{-1}, x_{-2}) \in D_1$ implies

$$\frac{x_0}{1/\lambda_-^2} = -\frac{x_{-1}}{1/\lambda_-} = x_{-2}.$$

Now for $n = 1$, we have

$$x_1 = \frac{x_0x_{-2}}{x_{-1} - x_{-2}} = \frac{(1/\lambda_-)x_{-1}\lambda_-x_{-1}}{x_{-1} + \lambda_-x_{-1}} = \frac{x_{-1}}{\lambda_-^2}.$$

Then we have

$$\frac{x_1}{1/\lambda_-^2} = -\frac{x_0}{1/\lambda_-} = x_{-1}.$$

This implies that $(x_1, x_0, x_{-1}) \in D_1$.
Suppose now that $(x_n, x_{n-1}, x_{n-2}) \in D_1$. This means that

$$\frac{x_n}{1/\lambda_-^2} = -\frac{x_{n-1}}{1/\lambda_-} = x_{n-2}.$$

Then

$$x_{n+1} = \frac{x_nx_{n-2}}{x_{n-1} - x_{n-2}} = \frac{(1/\lambda_-)x_{n-1}\lambda_-x_{n-1}}{x_{n-1} + \lambda_-x_{n-1}} = \frac{x_{n-1}}{\lambda_-^2}.$$

This implies that $(x_{n+1}, x_n, x_{n-1}) \in D_1$. Therefore, D_1 is an invariant set for Equation (1.1).
By similar way, we can show that D_2 is an invariant set for Equation (1.1).
This completes the proof. □

Theorem 2.4. *Every admissible solution of Equation* (1.1) *converges to zero.*

Proof. Suppose that $\{x_n\}_{n=-2}^{\infty}$ is an admissible solution of Equation (1.1).
Using Formula (2.4), we can write

$$\begin{aligned} x_{2m+1} &= -\frac{\nu}{(x_0f_m - x_{-1}f_{m+1})(x_{-1}f_{m+1} - x_{-2}f_{m+2})} \\ &= -\frac{\nu}{f_mf_{m+1}(x_0 - x_{-1}\frac{f_{m+1}}{f_m})(x_{-1} - x_{-2}\frac{f_{m+2}}{f_{m+1}})}. \end{aligned} \tag{2.5}$$

But

$$\frac{f_{m+1}}{f_m} \to \lambda_+ \text{ and } f_m \to \infty \text{ as } m \to \infty.$$

This implies that

$$x_{2m+1} \to 0 \text{ as } m \to \infty.$$

Similarly, we can show that $x_{2m+2} \to 0,$ as $m \to \infty$.
Therefore, $x_n \to 0$ as $n \to \infty$. This completes the proof. □

Example (1)
Figure (2.1) shows that a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) with $x_{-2} = 2, x_{-1} = -0.2$ and $x_0 = 1$ converges to zero.

Example (2)
Figure (2.2) shows that a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) with $x_{-2} = -1, x_{-1} = -0.2$ and $x_0 = -1.8$ converges to zero.

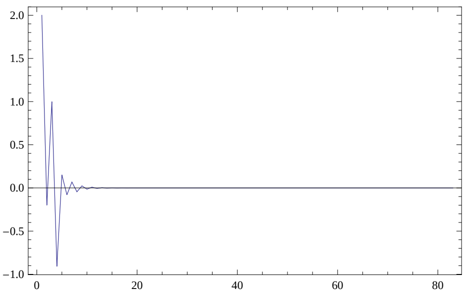

Figure 2.1: $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}}$

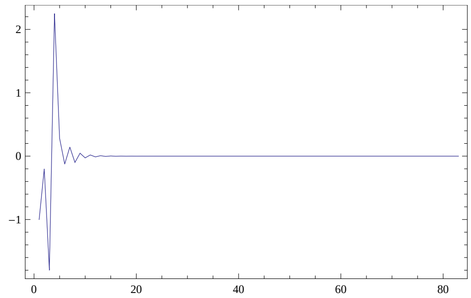

Figure 2.2: $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}}$

3. The difference equation $x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1} + x_{n-2}}$

In this section, we study the difference equation (1.2).

3.1. Solution of Equation (1.2)

The transformation (2.1) reduces Equation (1.2) into the difference equation

$$y_{n+1} = -\frac{1}{y_{n-1}} + 1, \ n = 0, 1, \ldots. \tag{3.1}$$

By solving Equation (3.1) and after some calculations, the solution of Equation (1.2) can be obtained.

Theorem 3.1. *Let $\{x_n\}_{n=-2}^{\infty}$ be an admissible solution of Equation (1.2). Then*

$$x_n = \begin{cases} \frac{\mu}{(\alpha_0 \cos\frac{(n-3)\pi}{6} - \beta_0 \sin\frac{(n-3)\pi}{6})(\alpha_{-1} \cos\frac{(n-1)\pi}{6} - \beta_{-1} \sin\frac{(n-1)\pi}{6})}, & n = 1, 3, \ldots, \\ \frac{\mu}{(\alpha_0 \cos\frac{(n-2)\pi}{6} - \beta_0 \sin\frac{(n-2)\pi}{6})(\alpha_{-1} \cos\frac{(n-2)\pi}{6} - \beta_{-1} \sin\frac{(n-2)\pi}{6})}, & n = 2, 4, \ldots, \end{cases} \tag{3.2}$$

where $\mu = x_0 x_{-1} x_{-2}$, $\alpha_0 = -x_0 + x_{-1}$, $\beta_0 = \frac{1}{\sqrt{3}}(x_0 + x_{-1})$, $\alpha_{-1} = -x_{-1} + x_{-2}$ *and* $\beta_{-1} = \frac{1}{\sqrt{3}}(x_{-1} + x_{-2})$.

Proof. We can write the given solution (3.2) as

$$x_{2m+1} = \frac{\mu}{\gamma_0(m-1)\gamma_{-1}(m)}$$

and (3.3)

$$x_{2m+2} = \frac{\mu}{\gamma_0(m)\gamma_{-1}(m)},$$

where

$$\gamma_0(m) = \alpha_0 \cos\frac{m\pi}{3} - \beta_0 \sin\frac{m\pi}{3}$$

and

$$\gamma_{-1}(m) = \alpha_{-1} \cos\frac{m\pi}{3} - \beta_{-1} \sin\frac{m\pi}{3}.$$

When $m = 0$,

$$\begin{aligned} x_1 &= \frac{\mu}{\gamma_0(-1)\gamma_{-1}(0)} = \frac{\mu}{(\alpha_0 \cos\frac{-\pi}{3} - \beta_0 \sin\frac{-\pi}{3})(\alpha_{-1})} \\ &= \frac{\mu}{\frac{1}{2}(\alpha_0 + \sqrt{3}\beta_0)(\alpha_{-1})} = \frac{\mu}{x_{-1}(-x_{-1} + x_{-2})} \\ &= \frac{x_0 x_{-2}}{-x_{-1} + x_{-2}}. \end{aligned}$$

Similarly

$$\begin{aligned} x_2 &= \frac{\mu}{\gamma_0(0)\gamma_{-1}(0)} = \frac{\mu}{\alpha_0\alpha_{-1}} \\ &= \frac{x_0x_{-1}x_{-2}}{(-x_0+x_{-1})(-x_{-1}+x_{-2})} \\ &= \frac{x_1x_{-1}}{-x_0+x_{-1}}. \end{aligned}$$

Suppose that the solution (3.3) is true for $m>0$.
Then

$$\begin{aligned} \frac{x_{2m+1}x_{2m-1}}{-x_{2m}+x_{2m-1}} &= \frac{\left(\frac{\mu}{\gamma_0(m-1)\gamma_{-1}(m)}\right)\left(\frac{\mu}{\gamma_0(m-2)\gamma_{-1}(m-1)}\right)}{-\frac{\mu}{\gamma_0(m-1)\gamma_{-1}(m-1)}+\frac{\mu}{\gamma_0(m-2)\gamma_{-1}(m-1)}} \\ &= \frac{\mu}{\gamma_{-1}(m)(-\gamma_0(m-2)+\gamma_0(m-1))}. \end{aligned}$$

But we can show that

$$\gamma_0(m-1)-\gamma_0(m-2)=\gamma_0(m),\ m=0,1,....$$

This implies that

$$\begin{aligned} \frac{x_{2m+1}x_{2m-1}}{-x_{2m}+x_{2m-1}} &= \frac{\mu}{\gamma_0(m)\gamma_{-1}(m)} \\ &= x_{2m+2}. \end{aligned}$$

Similarly we can show that

$$\frac{x_{2m+2}x_{2m}}{ax_{2m+1}+bx_{2m}} = x_{2m+3}.$$

This completes the proof. □

It is clear for Equation (1.2) that if we start with the point $(x_0,x_{-1},x_{-2})\in\mathbb{R}^3$, we have the following:
If $x_0=0$ and $x_{-1}x_{-2}\neq 0$, then x_3 is undefined.
If $x_{-1}=0$ and $x_0x_{-2}\neq 0$, then x_5 is undefined.
If $x_{-2}=0$ and $x_0x_{-1}\neq 0$, then x_4 is undefined.
Therefore, any point $(x_0,x_{-1},x_{-2})\in\mathbb{R}^3$ with $x_0x_{-1}x_{-2}=0$ belongs to the forbidden set of Equation (1.2).
The following result provides the forbidden set of Equation (1.2).

Theorem 3.2. *The forbidden set of equation* (1.2) *is*

$$\begin{aligned} F = &\bigcup_{i=0}^{2}\{(u_0,u_{-1},u_{-2})\in\mathbb{R}^3 : u_{-i}=0\}\cup\{(u_0,u_{-1},u_{-2})\in\mathbb{R}^3 : u_0=u_{-1}\}\cup \\ &\{(u_0,u_{-1},u_{-2})\in\mathbb{R}^3 : u_{-1}=u_{-2}\}. \end{aligned}$$

3.2. Global Behavior of Equation (1.2)

Theorem 3.3. *Every admissible solution for Equation* (1.2) *is periodic with prime period six.*

Proof. Suppose that $\{x_n\}_{n=-2}^{\infty}$ is an admissible solution for Equation (1.2).
It is clear that both the functions $\gamma_{-1}(m)$ and $\gamma_0(m)$ satisfy

$$\gamma_{-1}(m+3)=-\gamma_{-1}(m) \text{ and } \gamma_0(m+3)=-\gamma_0(m).$$

Then

$$\begin{aligned} x_{2(m+3)+1} &= \frac{\mu}{\gamma_0(m+2)\gamma_{-1}(m+3)} \\ &= \frac{\mu}{\gamma_0(m-1)\gamma_{-1}(m)} \\ &= x_{2m+1},\ m=-1,0,.... \end{aligned}$$

Similarly

$$\begin{aligned} x_{2(m+3)+2} &= \frac{\mu}{\gamma_0(m+3)\gamma_{-1}(m+3)} \\ &= \frac{\mu}{\gamma_0(m)\gamma_{-1}(m)} \\ &= x_{2m+2},\ m=-2,-1,..... \end{aligned}$$

Therefore, the solution $\{x_n\}_{n=-2}^{\infty}$ is periodic with prime period six. This completes the proof. □

Example (3)
Figure (3.1) shows that a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.2) with $x_{-2} = -3.2$, $x_{-1} = 2.8$ and $x_0 = 0.9$ is periodic with prime period six.

Example (4)
Figure (3.2) shows that a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.2) with $x_{-2} = 1.2$, $x_{-1} = 1.7$ and $x_0 = -0.2$ is periodic with prime period six.

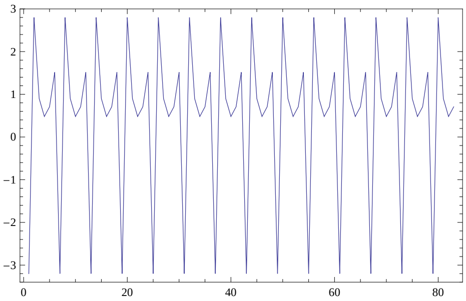

Figure 3.1: $x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1}+x_{n-2}}$

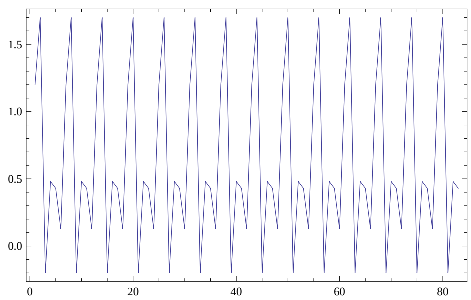

Figure 3.2: $x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1}+x_{n-2}}$

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Some Results on Nearly Cosymplectic Manifolds

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Abstract

The object of this paper is to study Ricci solitons under some curvature conditions in nearly cosymplectic manifolds.

1. Introduction

Cosympletic manifold is an odd dimensional counterpart of a Kähler manifold which is defined by Lipperman and Blair 1967 [9]. In parallel with Olzak's work [1], [2], Endo investigated the geometry of nearly cosymplectic manifolds [3].
Ricci soliton is a special solution to the Ricci flow introduced by Hamilton [10] in the year 1982. In [12], Sharma initiated the study of Ricci solitons in contact Riemannian geometry. Later, Tripathi [13], Nagaraja et al. [11] and others extensively studied Ricci solitons in contact metric manifolds. Ricci soliton in Riemanian manifold (M,g) is a natural generalization of an Einstein metric and is defined as a triple (g,V,λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$(\mathfrak{L}_V g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0 \tag{1.1}$$

where S is the Ricci tensor of M and $\mathfrak{L}_V$ denoted the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively.
In [16], [19], authors studied the properties of generalized recurrent manifolds where as the properties of generalized φ-recurrent manifolds have studied in [8], [16], [17] and [18].
In this paper we study some curvature conditions such that φ-recurrent, pseudo-projective φ-recurrent, concircular φ-recurrent and Ricci recurrent which characterize Ricci solitons in nearly cosymplectic manifolds.

2. Preliminaries

2.1. Nearly Cosymplectic Manifolds

Let (M,φ,ξ,η,g) be an $(2n+1)-$dimensional almost contact Riemannian manifold, where φ is a type of $(1,1)-$tensor field, ξ is the structure vector field, η is a $1-$form and g is the Riemannian metric. It is well known that the $(\varphi,\xi,\eta,g)-$structure satisfies the conditions [7] for any vector fields X and Y on M,

$$\varphi^2X=-X+\eta(X)\xi, \quad \eta(\xi)=1, \quad g(X,\xi)=\eta(X)$$

$$\eta(\varphi X)=0, \quad \varphi\xi=0, \tag{2.1}$$

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$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y). \tag{2.2}$$

A nearly cosymplectic manifold is an almost contact metric manifold (M,φ,ξ,η,g) such that

$$(\nabla_X\varphi)Y+(\nabla_Y\varphi)X=0, \tag{2.3}$$

for all vector fields X,Y. Clearly, this condition is equivalent to $(\nabla_X\varphi)X=0$. It is known that in a nearly cosymplectic manifold the Reeb vector field ξ is Killing and satisfies $\nabla_\xi\xi=0$ and η is a contact form $\nabla_\xi\eta=0$. The tensor field h of type $(1,1)$ defined by

$$\nabla_X\xi=hX, \tag{2.4}$$

is skew symmetric and anticommutes with φ. It satisfies

$$h\xi=0,\quad \eta\circ\varphi=0, \tag{2.5}$$

and the following formulas hold [3], [4]

$$g((\nabla_X\varphi)Y,hZ)=\eta(Y)g(h^2X,\varphi Z)-\eta(X)g(h^2Y,\varphi Z),$$

$$tr(h^2)=constant,$$

$$R(Y,Z)\xi=\eta(Y)h^2Z-\eta(Z)h^2Y, \tag{2.6}$$

$$S(Z,\xi)=-tr(h^2)\eta(Z), \tag{2.7}$$

where R, S, Q and η are the Riemannian curvature tensor type of $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci operator defined by $g(QX,Y)=S(X,Y)$.
Let (g,V,λ) be a Ricci soliton in a nearly cosymplectic manifold M. Taking $V=\xi$ then from (2.4) and (1.1), we have

$$S(X,Y)=-\lambda g(X,Y). \tag{2.8}$$

The above equation yields

$$QX=-\lambda X, \tag{2.9}$$

$$S(X,\xi)=\lambda\eta(X), \tag{2.10}$$

$$r=-\lambda n. \tag{2.11}$$

Also by definition of covariant derivative, we have

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi). \tag{2.12}$$

3. φ-Recurrent Nearly Cosymplectic Manifolds

Definition 3.1. *A nearly cosymplectic manifold is said to be φ-recurrent manifold [14] if there exist a non-zero $1-$form A such that*

$$\varphi^2((\nabla_W R)(X,Y)Z=A(W)R(X,Y)Z \tag{3.1}$$

for arbitrary vector fields X, Y, Z, W.
Let us consider a φ-recurrent nearly cosymplectic manifold. By virtue of (2.1) and (3.1), we have

$$-(\nabla_W R)(X,Y)Z+\eta((\nabla_W R)(X,Y)Z)\xi=A(W)R(X,Y)Z. \tag{3.2}$$

Theorem 3.2. *Let given Ricci soliton on nearly cosymplectic manifolds. Then there is not exist $\varphi-$recurrent nearly cosymplectic manifold.*

Proof. Contracting (3.2) with U, we obtain

$$-g((\nabla_W R)(X,Y)Z,U)+\eta((\nabla_W R)(X,Y)Z)\eta(U)=A(W)g(R(X,Y)Z,U). \tag{3.3}$$

Let e_i $(i=1,2,...,2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Taking $X=U=e_i$ in (3.3) and taking summation over i, $1\leq i\leq 2n+1$, we get

$$-(\nabla_W S)(Y,Z)=A(W)S(Y,Z). \tag{3.4}$$

Replacing Z by ξ in (3.4) and using (2.7), we have

$$-(\nabla_W S)(Y,\xi)=-tr(h^2)A(W)\eta(Y). \tag{3.5}$$

Using (2.7) and (2.4) in (2.12), we obtain

$$(\nabla_W S)(Y,\xi)=-[S(Y,hW)+tr(h^2)g(Y,hW)]. \tag{3.6}$$

In view of (3.5) and (3.6), we have

$$S(Y,hW)=-tr(h^2)[g(Y,hW)+A(W)\eta(Y)]. \tag{3.7}$$

Taking $Y=\xi$ in (3.7), we get

$$S(\xi,hW)=-tr(h^2)[g(Y,hW)+A(W)\eta(\xi). \tag{3.8}$$

Using (2.1), (2.5) and (2.8) in (3.8), we find

$$-\lambda g(hW,\xi)=tr(h^2)A(W),$$

$$tr(h^2)A(W)=0,$$

$$A(W)=0.$$

This is a contradiction. □

4. Generalized $\varphi-$Recurrent Nearly Cosymplectic Manifolds

Definition 4.1. *A nearly cosymplectic manifold is said to be generalized φ-recurrent manifold if its curvature tensor R satisfies the relation*

$$\varphi^2((\nabla_W R)(X,Y)Z)=A(W)R(X,Y)Z+B(W)\{g(Y,Z)X-g(X,Z)Y\}, \tag{4.1}$$

where A and B are $1-$forms and non-zero and these are defined by

$$A(W)=g(W,\rho_1),\quad B(W)=g(W,\rho_2),$$

and ρ_1,ρ_2 are unit vector fields associated with $1-$forms A, B respectively.

Theorem 4.2. *In a generalized φ-recurrent strictly nearly cosymplectic manifold (M_n,g), the associated vector fields ρ_1 and ρ_2 of the $1-$forms A and B respectively are co-directional.*

Proof. In consequence of (2.1), equation (4.1) becomes

$$-(\nabla_W R)(X,Y)Z+\eta((\nabla_W R)(X,Y)Z)\xi=A(W)R(X,Y)Z+B(W)\{g(Y,Z)X-g(X,Z)Y\}\ ,$$

from which it follows by taking inner product with U that

$$-g((\nabla_W R)(X,Y)Z,U)+\eta((\nabla_W R)(X,Y)Z)\eta(U)=A(W)g(R(X,Y)Z,U)+B(W)\{g(Y,Z)g(X,U)-g(X,Z)g(Y,U)\}. \tag{4.2}$$

Let $\{e_i\}$, $i=1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (4.2) and taking summation over $i, 1\leq i\leq 2n+1$, we get

$$-(\nabla_W S)(Y,Z)+\sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i)=$$
$$A(W)S(Y,Z)+2nB(W)g(Y,Z). \tag{4.3}$$

Again replacing Z by ξ in (4.3) and using (2.1) and (2.7), we get

$$-(\nabla_W S)(Y,\xi)+\sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i)=\{-trh^2A(W)+2nB(W)\}\eta(Y). \tag{4.4}$$

The second term of left hand side in (4.4) with (2.1)takes the form

$$\sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i)=\eta((\nabla_W R)(\xi,Y)\xi)\eta(\xi)=g((\nabla_W R)(\xi,Y)\xi,\xi). \tag{4.5}$$

Using (2.4), (2.5)and (2.6)in (4.5), we obtain

$$g((\nabla_W R)(\xi,Y)\xi,\xi)=0. \tag{4.6}$$

In view of (4.6), (4.4) becomes

$$(\nabla_W S)(Y,\xi)=\{tr(h^2)A(W)-2nB(W)\}\eta(Y). \tag{4.7}$$

The equation (2.12) with (2.4) and (2.7) takes the form

$$(\nabla_W S)(Y,\xi)=-tr(h^2)g(Y,hW)-S(Y,hW). \tag{4.8}$$

From equations (4.7) and (4.8), we find

$$-tr(h^2)g(Y,hW)-S(Y,hW)=(tr(h^2)A(W)-2nB(W))\eta(Y). \tag{4.9}$$

Replacing Y by ξ then using (2.5) in (4.9) we have

$$A(W)=(\frac{2n}{tr(h^2)})B(W).$$

This means that the vector fields ρ_1 and ρ_2 of the $1-$forms are co-directional. □

5. Ricci-Recurrent Nearly Cosymplectic Manifold

Theorem 5.1. *Let given Ricci soliton on nearly cosymplectic manifolds. Then there is not exist Ricci recurrent nearly cosymplectic manifold.*

Proof. A nearly cosymplectic manifold is said to be Ricci-recurrent manifold if there exist a non-zero $1-$form A such that

$$(\nabla_W S)(Y,Z)=A(W)S(Y,Z). \tag{5.1}$$

Replacing Z by ξ in (5.1) and using (2.7), we have

$$(\nabla_W S)(Y,\xi)=-tr(h^2)A(W)\eta(Y). \tag{5.2}$$

Using (2.4) and (2.7) in (2.12), we obtain

$$(\nabla_W S)(Y,\xi)=-[S(Y,hW)+tr(h^2)g(y,hW)]. \tag{5.3}$$

In view of (5.2) and (5.3), we have

$$S(Y,hW)=-tr(h^2)g(Y,hW)+tr(h^2)A(W)\eta(Y). \tag{5.4}$$

Taking $Y=\xi$ in (5.4), we get

$$A(W)=0.$$

It contradicts that $A\neq 0$. Thus, the proof is completed. □

6. Pseudo-projective $\varphi-$recurrent Nearly Cosymplectic Manifold

In a nearly cosymplectic manifold M, the pseudo-projective curvature tensor $\widetilde{P}$ is given by [20]

$$\widetilde{P}(X,Y)Z=aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y]-\frac{r}{2n+1}(\frac{a}{2n}+b)[g(Y,Z)X-g(X,Z)Y] \tag{6.1}$$

where a and b are constants such that $a,b\neq 0$.

Theorem 6.1. *Ricci soliton in a pseudo-projective φ-recurent nearly cosymplectic manifold (M,g) with 1-form non-zero A depends on the sign of $tr(h^2)$.*

Proof. A nearly cosymplectic manifold is said to be pseudo-projective φ-recurrent manifold if there exists a non-zero $1-$form A such that

$$\varphi^2((\nabla_W\widetilde{P})(X,Y)Z)=A(W)\widetilde{P}(X,Y)Z, \tag{6.2}$$

for arbitrary vector fields X, Y, Z, W. Let us consider a pseudo-projective φ-recurrent nearly cosymplectic manifold. By virtue of (2.1) and (6.2), we have

$$-(\nabla_W\widetilde{P})(X,Y)Z)+\eta((\nabla_W\widetilde{P})(X,Y)Z)\xi=A(W)\widetilde{P}(X,Y)Z). \tag{6.3}$$

Contracting (6.3) with U, we obtain

$$-g((\nabla_W\widetilde{P})(X,Y)Z,U)+\eta((\nabla_W\widetilde{P})(X,Y)Z)\eta(U)=A(W)g(\widetilde{P}(X,Y)Z,U). \tag{6.4}$$

Let e_i $(i=1,2,...,2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (6.4) and taking summation over i, $1\leq i\leq 2n+1$, we get

$$(\nabla_W S)(Y,Z)=A(W)\{S(Y,Z)-\frac{r}{2n+1}g(Y,Z)\}. \tag{6.5}$$

Replacing Z by ξ in (6.5) and using (2.1) and (2.7), we have

$$(\nabla_W S)(Y,\xi)=-A(W)\{tr(h^2)-\frac{r}{2n+1}\}\eta(Y). \tag{6.6}$$

Using (2.7) and (2.4) in (2.12), we obtain

$$(\nabla_W S)(Y,\xi)=-[S(Y,hX)+tr(h^2)g(Y,hX). \tag{6.7}$$

In view of(6.6) and (6.7), we have

$$S(Y,hX)=A(W)\{tr(h^2)+\frac{r}{2n+1}\}\eta(Y)-tr(h^2)g(Y,hX).$$

Taking $Y=\xi$ and using (2.5), (2.8), (2.11) we get

$$A(W)\{tr(h^2)-\frac{\lambda n}{2n+1}\}=0.$$

for non-zero $A(W)$ we find

$$\lambda=\frac{tr(h^2)(2n+1)}{n}.$$

Hence, the proof is completed. □

7. Concircular $\varphi-$Recurrent Nearly Cosymplectic Manifold

The Concircular curvature tensor of (M,g) is given by [21]

$$\widetilde{C}(X,Y)Z=R(X,Y)Z-\frac{r}{2n(2n+1)}[g(Y,Z)X-g(X,Z)Y]. \tag{7.1}$$

Definition 7.1. *A nearly cosymplectic manifold is said to be concircular $\varphi-$recurrent manifold if there exist a non-zero $1-$form A such that*

$$\varphi^2((\nabla_W\widetilde{C})(X,Y)Z)=A(W)\widetilde{C}(X,Y)Z. \tag{7.2}$$

for arbitrary vector fields X, Y, Z, W.

Theorem 7.2. *Ricci soliton in a concircular $\varphi-$recurrent nearly cosymplectic manifold M with $1-$form non-zero A depends on the sign of $tr(h^2)$.*

Proof. Let us consider a concircular $\varphi-$recurrent nearly cosymplectic manifold. By virtue of (2.1) and (7.2), we have

$$-(\nabla_W\widetilde{C})(X,Y)Z+\eta((\nabla_W\widetilde{C})(X,Y)Z)\xi=A(W)\widetilde{C}(X,Y)Z. \tag{7.3}$$

Contracting (7.3) with U, we obtain

$$-g((\nabla_W\widetilde{C})(X,Y)Z,U)+\eta((\nabla_W\widetilde{C})(X,Y)Z)\eta(U)=A(W)g(\widetilde{C}(X,Y)Z,U). \tag{7.4}$$

Let e_i $(i=1,2,...,2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (7.4) and taking summation over i, $1\leq i\leq 2n+1$, we get

$$(\nabla_W S)(Y,Z)=-A(W)\{S(Y,Z)-\frac{r}{2n+1}g(Y,Z)\}. \tag{7.5}$$

Replacing Z by ξ in (7.5) and using (2.1) and (2.7), for a constant r, we have

$$(\nabla_W S)(Y,\xi)=A(W)\eta(Y)\{tr(h^2)+\frac{r}{2n+1}\}. \tag{7.6}$$

Using (2.7) and (2.4) in (2.12), we obtain

$$(\nabla_W S)(Y,\xi)=-[S(Y,hW)+tr(h^2)g(Y,hW)]. \tag{7.7}$$

In view of (7.6) and (7.7), we have

$$S(Y,hW)=-\{tr(h^2)+\frac{r}{2n+1}\}A(W)\eta(Y)-tr(h^2)g(Y,hW). \tag{7.8}$$

Taking $Y=\xi$, and using (2.5) and (2.8) a characteristic vector field in (7.8), we get

$$A(W)\{tr(h^2)+\frac{r}{2n+1}\}=0. \tag{7.9}$$

Using (2.11) in (7.9), for non-vanishing A, we have

$$\lambda=\frac{tr(h^2)(2n+1)}{n}.$$

So, we have desired result. □

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On a Competitive System of Rational Difference Equations

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Abstract

This paper aims to investigate the global stability and the rate of convergence of positive solutions that converge to the equilibrium point of the system of difference equations in the modeling competitive populations in the form

$$x_{n+1}^{(1)}=\frac{\alpha x_{n-2}^{(1)}}{\beta+\gamma\prod\limits_{i=0}^{2}x_{n-i}^{(2)}},\ x_{n+1}^{(2)}=\frac{\alpha_1 x_{n-2}^{(2)}}{\beta_1+\gamma_1\prod\limits_{i=0}^{2}x_{n-i}^{(1)}},\ n=0,1,\ldots$$

where the parameters $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1$ are positive numbers and the initial conditions $x_{-i}^{(1)},x_{-i}^{(2)}$ are arbitrary non-negative numbers for $i\in\{0,1,2\}$.

1. Introduction

Difference equation or discrete dynamical system is a diverse field which impacts almost every branch of pure and applied mathematics. Every difference equation determines a dynamical system and vice versa. Recently, there has been a big interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology, see [10, 11]. Therefore, the asymptotic behavior of solutions of the system for rational difference equations has received huge interest, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 13].
This paper deals with the following two-dimensional system

$$x_{n+1}^{(1)}=\frac{\alpha x_{n-2}^{(1)}}{\beta+\gamma\prod\limits_{i=0}^{2}x_{n-i}^{(2)}},\ x_{n+1}^{(2)}=\frac{\alpha_1 x_{n-2}^{(2)}}{\beta_1+\gamma_1\prod\limits_{i=0}^{2}x_{n-i}^{(1)}},\ n=0,1,\ldots \tag{1.1}$$

where the parameters $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1$ are positive numbers and the initial conditions $x_{-i}^{(1)},x_{-i}^{(2)}$ are arbitrary non-negative numbers for $i\in\{0,1,2\}$. Actually, in [15] some dynamical behaviors of the system (1.1) has been studied. But, we notice that the authors have not examined various properties of system (1.1), namely, the global stability, the rate of convergence and the asymptotic behavior. Our aim in this paper is to give a complete picture as regards the global behavior of positive solutions of system (1.1). That is, we here study the global asymptotic stability of zero equilibrium and the rate of convergence of solutions.
The following the boundedness and the local stability results have obtained in [15].

Lemma 1.1. $(\bar{x}_1,\bar{x}_2)=(0,0)$ *is always an equilibrium point of system (1.1).*

Theorem 1.2. *If both* $\frac{\alpha}{\beta}<1$ *and* $\frac{\alpha_1}{\beta_1}<1$*, then every positive solution of system (1.1) is bounded.*

Theorem 1.3. *If both* $\frac{\alpha}{\beta}<1$ *and* $\frac{\alpha_1}{\beta_1}<1$*, then the zero equilibrium point of system (1.1) is locally asymptotically stable.*

In the present paper, we will provide some results about the global behavior and the rate of convergence of positive solutions that converge to the zero equilibrium point of the system (1.1), in the regions of parameters described in Theorem 1.3. In addition to this, we will present the

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use of *Poincaré's Theorem* and a devolopment of *Perron's Theorem* to conclude the precise asymptotics of positive solutions that converge to the equilibrium.
Consider the following one-dimensional system of difference equations

$$\left.\begin{array}{l} x_{n+1}=f_1(x_n,y_n),\ n=0,1,\ldots \\ y_{n+1}=f_2(x_n,y_n),\ n=0,1,\ldots \end{array}\right\} \tag{1.2}$$

where f_1, f_2 are continuous functions that maps some set I into I. The set I is an interval of real numbers. System (1.2) is competitive if $f_1(x,y)$ is non-decreasing in x and non-increasing in y and $f_2(x,y)$ is non-increasing in x and non-decreasing in y. System (1.2) is called anti-competitive system, if the functions f_1 and f_2 have monotonic character opposite to the monotonic character in competitive system. It is well know that the dynamical properties of competitive populations has received great interest from both theoretical and mathematical biologists [14] due to its universal commonness. Competitive and anti-competitive systems were studied by many authors (see [1, 4, 5, 8]). Especially, studying the rate of convergence of solutions of some systems of difference equations is a topic of big interest [2, 3, 9].
We state that the following theorems give precise information about the asymptotics of linear non-autonomous difference equations. Consider the scalar mth-order linear difference equation

$$y_{n+m}+p_1(n)y_{n+m-1}+\ldots+p_m(n)y_n=0 \tag{1.3}$$

where m is a positive integer and $p_i:\mathbb{Z}^+\to\mathbb{C}$ for $i\in\{1,\ldots,m\}$. Suppose that

$$q_i=\lim_{n\to\infty}p_i(n),\ \text{for } i=1,2,\ldots,m, \tag{1.4}$$

exist in $\mathbb{C}$. For the following limitting equation of (1.3)

$$y_{n+m}+q_1y_{n+m-1}+\ldots+q_my_n=0, \tag{1.5}$$

the asymptotics of solutions of (1.3) are given the following results. See [10, 13].

Theorem 1.4. *(Poincaré's Theorem) Consider (1.3) based on the condition (1.4). Let λ_i for $i=1,\ldots,m$ be the roots of the characteristic equation*

$$\lambda^m+q_1\lambda^{m-1}+\ldots+q_m=0 \tag{1.6}$$

of the limiting equation (1.5) under the condition that $|\lambda_i|\neq|\lambda_j|$ for $i\neq j$. If x_n is a positive solution of (1.3), then either $x_n=0$ for all large n or there exists an index $j\in\{1,\ldots,m\}$ such that

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lambda_j.$$

The releated results were obtained by Perron, and one of Perron's results was improved by Pituk, see [13].

Theorem 1.5. *Assume that (1.4) holds. If x_n is a positive solution of (1.3), then either eventually $x_n=0$ or*

$$\lim_{n\to\infty}\sup(|x_{nj}|)^{1/n}=|\lambda_j|,$$

where $\lambda_1,\ldots,\lambda_m$ are the roots (not necessarily distinct) of the characteristic equation (1.6).

Consider

$$Y_{n+1}=[A+B(n)]Y_n \tag{1.7}$$

where Y_n is an m-dimensional vector, $A\in C^{m\times m}$ is a constant matrix and

$$B:\mathbb{Z}^+\to C^{m\times m}$$

is a matrix function satisfying

$$\|B(n)\|\to 0,\ \text{when } n\to\infty, \tag{1.8}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm $\|.\|$. See [12].

Theorem 1.6. *(Pituk) Suppose that condition (1.8) holds for system (1.7). If Y_n is a solution of (1.7), then either*

$$Y_n=0$$

for all large n or

$$\theta=\lim_{n\to\infty}\|Y_n\|^{1/n}$$

exists and θ is equal to the modulus one of the eigenvalues of the matrix A.

Theorem 1.7. *(Pituk) Suppose that condition (1.8) holds for system (1.7). If Y_n is a solution of (1.7), then either*

$$Y_n=0$$

for all large n or

$$\theta=\lim_{n\to\infty}\frac{\|Y_{n+1}\|}{\|Y_n\|}$$

exists and θ is equal to the modulus one of the eigenvalues of the matrix A.

2. Global Behavior of the system (1.1)

In this section, we investigate the global behavior of the system (1.1).

Theorem 2.1. *If both $\frac{\alpha}{\beta} < 1$ and $\frac{\alpha_1}{\beta_1} < 1$, then the zero equilibrium point of system (1.1) is globally asymptotically stable.*

Proof. We know by Theorem 1.3 that the zero equilibrium point $(\bar{x}_1, \bar{x}_2) = (0,0)$ of the system (1.1) is locally asymptotically stable. So, it suffices to prove for any solution $\{(x_n^{(1)}, x_n^{(2)})\}_{n=-2}^{\infty}$ of system (1.1) that

$$\lim_{n\to\infty}(x_n^{(1)}, x_n^{(2)}) = (0,0).$$

From the boundedness result system (1.1) it is clear that it is sufficient to prove that $\{(x_n^{(1)}, x_n^{(2)})\}_{n=-2}^{\infty}$ is decreasing. We have that

$$0 \le x_{n+1}^{(1)} = \frac{\alpha x_{n-2}^{(1)}}{\beta + \gamma \prod_{i=0}^{2} x_{n-i}^{(2)}} < \frac{\alpha}{\beta} x_{n-2}^{(1)}$$

and

$$0 \le x_{n+1}^{(2)} = \frac{\alpha_1 x_{n-2}^{(2)}}{\beta_1 + \gamma_1 \prod_{i=0}^{2} x_{n-i}^{(1)}} < \frac{\alpha_1}{\beta_1} x_{n-2}^{(2)}.$$

By induction on n, one has

$$0 \le x_{3n+i}^{(1)} < (\frac{\alpha}{\beta})^{n+1} x_{-3+i}^{(1)},\ i = 1,2,3,$$

and

$$0 \le x_{3n+i}^{(2)} < (\frac{\alpha}{\beta})^{n+1} x_{-3+i}^{(2)},\ i = 1,2,3.$$

Thus, for $\frac{\alpha}{\beta} < 1$ and $\frac{\alpha_1}{\beta_1} < 1$, we can have

$$\lim_{n\to\infty}(x_n^{(1)}, x_n^{(2)}) = (0,0).$$

This completes the proof. □

3. Rate of Convergence

In this section, we will characterize the rate of convergence of a solution that converges to the equilibrium point

$$M = (\bar{x}_1, \bar{x}_2) = (0,0)$$

of the system (1.1).
Using Theorem 1.6 and 1.7, we obtain the following rate of convergence result.

Theorem 3.1. *Suppose that $\frac{\alpha}{\beta} < 1$ and $\frac{\alpha_1}{\beta_1} < 1$. Let $\{(x_n^{(1)}, x_n^{(2)})\}_{n=-2}^{\infty}$ be any positive solution of the system (1.1) such that*

$$\lim_{n\to\infty} x_n^{(1)} = \bar{x}_1,$$
$$\lim_{n\to\infty} x_n^{(2)} = \bar{x}_2$$

where $M = (\bar{x}_1, \bar{x}_2)$ and M is globally asymptotically stable. Then, the error vector

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix}_{6\times 1} = \begin{pmatrix} x_n^{(1)} - \bar{x}_1 \\ x_{n-1}^{(1)} - \bar{x}_1 \\ x_{n-2}^{(1)} - \bar{x}_1 \\ x_n^{(2)} - \bar{x}_2 \\ x_{n-1}^{(2)} - \bar{x}_2 \\ x_{n-2}^{(2)} - \bar{x}_2 \end{pmatrix}_{6\times 1}$$

of every positive solution of the system (1.1) satisfies both of the following asymptotic relations:

$$\lim_{n\to\infty} \|E_n\|^{1/n} = |\lambda_i J_F(M)|,\ \textit{for some } i = 1,2,\ldots,6$$
$$\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda_i J_F(M)|,\ \textit{for some } i = 1,2,\ldots,6$$

where

$$|\lambda_i J_F(M)|$$

is equal to the modulus one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium point M.

Proof. Let $\{(x_n^{(1)},x_n^{(2)})\}_{n=-2}^{\infty}$ be any positive solution of the system (1.1) such that

$$\lim_{n\to\infty} x_n^{(1)} = \bar{x}_1$$

and

$$\lim_{n\to\infty} x_n^{(2)} = \bar{x}_2.$$

To find the error terms, we have

$$x_{n+1}^{(1)} - \bar{x}_1 = \sum_{i=0}^{2} A_i(x_{n-i}^{(1)} - \bar{x}_1) + \sum_{i=0}^{2} B_i(x_{n-i}^{(2)} - \bar{x}_2)$$
$$x_{n+1}^{(2)} - \bar{x}_2 = \sum_{i=0}^{2} C_i(x_{n-i}^{(1)} - \bar{x}_1) + \sum_{i=0}^{2} D_i(x_{n-i}^{(2)} - \bar{x}_2).$$

Set

$$e_n^1 = x_n^{(1)} - \bar{x}_1,$$
$$e_n^2 = x_n^{(2)} - \bar{x}_2;$$

it follows that

$$e_{n+1}^1 = \sum_{i=0}^{2} A_i e_{n-i}^1 + \sum_{i=0}^{2} B_i e_{n-i}^2$$
$$e_{n+1}^2 = \sum_{i=0}^{2} C_i e_{n-i}^1 + \sum_{i=0}^{2} D_i e_{n-i}^2$$

where

$$A_0 = 0, A_1 = 0, A_2 = \frac{\alpha}{\beta + \gamma \prod_{i=0}^{2} x_{n-i}^{(2)}},$$

$$B_0 = -\frac{\alpha\gamma x_{n-2}^{(1)} x_{n-1}^{(2)} x_{n-2}^{(2)}}{(\beta + \gamma \prod_{i=0}^{2} x_{n-i}^{(2)})^2}, B_1 = -\frac{\alpha\gamma x_{n-2}^{(1)} x_n^{(2)} x_{n-2}^{(2)}}{(\beta + \gamma \prod_{i=0}^{2} x_{n-i}^{(2)})^2}, B_2 = -\frac{\alpha\gamma x_{n-2}^{(1)} x_n^{(2)} x_{n-1}^{(2)}}{(\beta + \gamma \prod_{i=0}^{2} x_{n-i}^{(2)})^2},$$

$$C_0 = -\frac{\alpha_1\gamma_1 x_{n-2}^{(2)} x_{n-1}^{(1)} x_{n-2}^{(1)}}{(\beta_1 + \gamma_1 \prod_{i=0}^{2} x_{n-i}^{(1)})^2}, C_1 = -\frac{\alpha_1\gamma_1 x_{n-2}^{(2)} x_n^{(1)} x_{n-2}^{(1)}}{(\beta_1 + \gamma_1 \prod_{i=0}^{2} x_{n-i}^{(1)})^2}, C_2 = -\frac{\alpha_1\gamma_1 x_{n-2}^{(2)} x_n^{(1)} x_{n-1}^{(1)}}{(\beta_1 + \gamma_1 \prod_{i=0}^{2} x_{n-i}^{(1)})^2},$$

$$D_0 = 0,\ D_1 = 0, D_2 = \frac{\alpha_1}{\beta_1 + \gamma_1 \prod_{i=0}^{2} x_{n-i}^{(1)}}.$$

Taking the limits, it is clear that

$$\lim_{n\to\infty} A_0 = 0,\ \lim_{n\to\infty} A_1 = 0 \text{ and } \lim_{n\to\infty} A_2 = \frac{\alpha}{\beta + \gamma\bar{x}_2^3}$$

$$\lim_{n\to\infty} B_0 = -\frac{\alpha\gamma\bar{x}_1\bar{x}_2^2}{(\beta + \gamma\bar{x}_2^3)^2},\ \lim_{n\to\infty} B_1 = -\frac{\alpha\gamma\bar{x}_1\bar{x}_2^2}{(\beta + \gamma\bar{x}_2^3)^2} \text{ and } \lim_{n\to\infty} B_2 = -\frac{\alpha\gamma\bar{x}_1\bar{x}_2^2}{(\beta + \gamma\bar{x}_2^3)^2},$$

$$\lim_{n\to\infty} C_0 = -\frac{\alpha_1\gamma_1\bar{x}_1^2\bar{x}_2}{(\beta_1 + \gamma_1\bar{x}_1^3)^2},\ \lim_{n\to\infty} C_1 = -\frac{\alpha_1\gamma_1\bar{x}_1^2\bar{x}_2}{(\beta_1 + \gamma_1\bar{x}_1^3)^2} \text{ and } \lim_{n\to\infty} C_2 = -\frac{\alpha_1\gamma_1\bar{x}_1^2\bar{x}_2}{(\beta_1 + \gamma_1\bar{x}_1^3)^2},$$

$$\lim_{n\to\infty} D_0 = 0,\ \lim_{n\to\infty} D_1 = 0 \text{ and } \lim_{n\to\infty} D_2 = \frac{\alpha_1}{\beta_1 + \gamma_1\bar{x}_1^3}.$$

That is

$$A_2 = \tfrac{\alpha}{\beta+\gamma\bar{x}_2^3} + \varsigma_n, \quad B_0 = -\tfrac{\alpha\gamma\bar{x}_1\bar{x}_2^2}{(\beta+\gamma\bar{x}_2^3)^2} + \tau_n, \quad B_1 = -\tfrac{\alpha\gamma\bar{x}_1\bar{x}_2^2}{(\beta+\gamma\bar{x}_2^3)^2} + \upsilon_n, \quad B_2 = -\tfrac{\alpha\gamma\bar{x}_1\bar{x}_2^2}{(\beta+\gamma\bar{x}_2^3)^2} + \delta_n$$
$$C_0 = -\tfrac{\alpha_1\gamma_1\bar{x}_1^2\bar{x}_2}{(\beta_1+\gamma_1\bar{x}_1^3)^2} + \lambda_n, \quad C_1 = -\tfrac{\alpha_1\gamma_1\bar{x}_1^2\bar{x}_2}{(\beta_1+\gamma_1\bar{x}_1^3)^2} + \varepsilon_n, \quad C_2 = -\tfrac{\alpha_1\gamma_1\bar{x}_1^2\bar{x}_2}{(\beta_1+\gamma_1\bar{x}_1^3)^2} + \eta_n, \quad D_2 = \tfrac{\alpha_1}{\beta_1+\gamma_1\bar{x}_1^3} + \varphi_n$$

where $\varsigma_n \to 0$, $\tau_n \to 0$, $\upsilon_n \to 0$, $\delta_n \to 0$, $\lambda_n \to 0$, $\varepsilon_n \to 0$, $\eta_n \to 0$, $\varphi_n \to 0$ for $n \to \infty$.
Thus, the limitting system of error terms about the equilibrium M can be written as follows:

$$E_{n+1} = (C + D(n))E_n,$$

where $E_n = (e_n^1, e_{n-1}^1, e_{n-2}^1, e_n^2, e_{n-1}^2, e_{n-2}^2)^T$,

$$C = \begin{pmatrix} 0 & 0 & \frac{\alpha}{\beta} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{6\times 6}, D_n = \begin{pmatrix} 0 & 0 & \varsigma_n & \tau_n & \upsilon_n & \delta_n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_n & \varepsilon_n & \eta_n & 0 & 0 & \varphi_n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 6}$$

and $\|D(n)\| \to 0$, when $n \to \infty$. This completes the proof. □

4. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this manuscript.

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