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# Contents

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# Differential Relations for the Solutions to the NLS Equation and Their Different Representations

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**Abstract**
Solutions to the focusing nonlinear Schrödinger equation (NLS) of order $N$ depending on $2N-2$ real parameters in terms of wronskians and Fredholm determinants are given. These solutions give families of quasi-rational solutions to the NLS equation denoted by $v_N$ and have been explicitly constructed until order $N=13$. These solutions appear as deformations of the Peregrine breather $P_N$ as they can be obtained when all parameters are equal to 0. These quasi rational solutions can be expressed as a quotient of two polynomials of degree $N(N+1)$ in the variables $x$ and $t$ and the maximum of the modulus of the Peregrine breather of order $N$ is equal to $2N+1$. Here we give some relations between solutions to this equation. In particular, we present a connection between the modulus of these solutions and the denominator part of their rational expressions. Some relations between numerator and denominator of the Peregrine breather are presented.

**Keywords:** Fredholm determinants, NLS equation, Peregrine breathers, Rogue waves, Wronskians
**2010 AMS:** 35B05, 35C99, 35Q55, 35L05, 76M99, 78M99

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## 1. Introduction

We consider the one dimensional focusing nonlinear Schrödinger equation (NLS) which can be written in the form

$$iv_t + v_{xx} + 2|v|^2 v = 0, \tag{1.1}$$

The first results concerning the NLS equation date from the works of Zakharov and Shabat in 1972 who solved it using the inverse scattering method [1, 2]. Its and Kotlyarov first constructed periodic and almost periodic algebro-geometric solutions to the focusing NLS equation in 1976 [3, 4]. Ma found in 1979 the first breather type solution of the NLS equation [5]. In 1983, the first quasi rational solutions of NLS equation were constructed by Peregrine [6]. In 1986, Eleonski, Akhmediev and Kulagin obtained the two-phase almost periodic solution to the NLS equation and got the first higher order analogue of the Peregrine breather[7, 8, 9]. Other analogues of the Peregrine breathers of order 3 and 4 were constructed using Darboux transformations, in a series of articles by Akhmediev et al. [10, 11, 12, 13].
Recently, many works about NLS equation have been published using different methods. We can quote the works of Matveev et al. [14, 15] in 2010 for the representation of the solutions in terms of wronskians; those of Gaillard [16, 17, 18] for the solutions given in terms of wronskians and Fredholm determinants, and their quasi-rational solutions limit of order $N$ depending on $2N-2$ parameters. Akhmediev gave quasi rational solutions using Darboux transformation in several papers [19, 20, 21]. Guo, Ling and Liu in 2012 gave an other representation of the solutions as a ratio of two determinants [22] using generalized Darboux transformation. A new approach has been done by Ohta and Yang in [23] using Hirota bilinear method. Smirnov [24]

gave solutions with an algebro-geometric approach. Other types of solutions were given by Zhao et al. in [25].

We give some relations between the modulus of these solutions and the denominator part of their rational expression. Some relations between numerator and denominator of the rational solutions are given.

## 2. Different representations of solutions to the NLS equation

### 2.1 Solutions of the NLS equation in terms of of Fredholm determinant

We have to define the following notations.
The terms $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ are functions of the parameters $\lambda_\nu$, $1 \leq \nu \leq 2N$; they are defined by the formulas :

$$\begin{aligned} &\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}};\\ &x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad r = 1,3. \end{aligned} \tag{2.1}$$

The parameters $-1 < \lambda_\nu < 1$, $\nu = 1,\ldots,2N$, are real numbers such that

$$\begin{aligned} &-1 < \lambda_{N+1} < \lambda_{N+2} < \ldots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \ldots < \lambda_1 < 1\\ &\lambda_{N+j} = -\lambda_j, \quad j = 1,\ldots,N. \end{aligned} \tag{2.2}$$

The condition (2.2) implies that

$$\kappa_{j+N} = \kappa_j, \quad \delta_{j+N} = -\delta_{j+N}, \quad \gamma_{j+N} = \gamma_j^{-1}, \quad x_{r,j+N} = x_{r,j}, \quad j = 1,\ldots,N.$$

Complex numbers $e_\nu$ $1 \leq \nu \leq 2N$ are defined in the following way :

$$\begin{aligned} &e_j = i\sum_{l=1}^{N-1} a_l (j\varepsilon)^{2l+1} - \sum_{l=1}^{N-1} b_l (j\varepsilon)^{2l+1},\\ &e_{j+N} = i\sum_{l=1}^{N-1} a_l (j\varepsilon)^{2l+1} + \sum_{l=1}^{N-1} b_l (j\varepsilon)^{2l+1},\\ &1 \leq j \leq N-1. \end{aligned} \tag{2.3}$$

$\varepsilon$, $a_\nu$, $b_\nu$, $\nu = 1\ldots 2N$ are arbitrary real numbers.
Let $I$ be the unit matrix, and

$$\varepsilon_j = j \quad 1 \leq j \leq N, \quad \varepsilon_j = N + j, \quad N+1 \leq j \leq 2N.$$

Let's consider the matrix $D_r = (d_{jk}^{(r)})_{1\leq j,k\leq 2N}$ defined by :

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \tag{2.4}$$

With these notations, the solution to the NLS equation takes the form [16, 17, 18] :

**Theorem 2.1.** *The function $v$ defined by*

$$v(x,t) = \frac{\det(I + D_3(x,t))}{\det(I + D_1(x,t))} e^{2it - i\varphi}.$$

*is a solution to the focusing NLS equation depending on $2N-1$ real parameters $a_j$, $b_j$, $\varepsilon$, $1 \leq j \leq N-1$ with the matrix $D_r = (d_{jk}^{(r)})_{1\leq j,k\leq 2N}$ defined by*

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu).$$

*where $\kappa_\nu$, $\delta_\nu$, $x_{r,\nu}$, $\gamma_\nu$, $e_\nu$ being defined in(2.1), (2.2) and (2.3).*

### 2.2 Wronskian representation

For this, we need to define the following notations :

$$\phi_{r,\nu} = \sin\Theta_{r,\nu}, \quad 1 \le \nu \le N, \quad \phi_{r,\nu} = \cos\Theta_{r,\nu}, \quad N+1 \le \nu \le 2N, \quad r = 1,3,$$

with the arguments

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2, \quad 1 \le \nu \le 2N.$$

The functions $\phi_{r,\nu}$ are defined by

$$\phi_{r,\nu} = \sin\Theta_{r,\nu}, \quad 1 \le \nu \le N, \quad \phi_{r,\nu} = \cos\Theta_{r,\nu}, \quad N+1 \le \nu \le 2N, \quad r = 1,3, \tag{2.5}$$

We denote $W_r(y)$ the wronskian of the functions $\phi_{r,1}, \dots, \phi_{r,2N}$ defined by

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_{r,\nu})_{\nu,\mu\in[1,\dots,2N]}].$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\mu\in[1,\dots,2N]}$ defined in (2.4). Then we have the following statement [17] :

**Theorem 2.2.**

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \dots, \phi_{r,2N})(0),$$

*where*

$$k_r(y) = \frac{2^{2N}\exp(i\sum_{\nu=1}^{2N}\Theta_{r,\nu})}{\prod_{\nu=2}^{2N}\prod_{\mu=1}^{\nu-1}(\gamma_\nu - \gamma_\mu)}.$$

With these notations, we have the following result [17] :

**Theorem 2.3.** *The function $v$ defined by*

$$v(x,t) = \frac{W_3(\phi_{3,1}, \dots, \phi_{3,2N})(0)}{W_1(\phi_{1,1}, \dots, \phi_{1,2N})(0)} e^{2it - i\varphi}.$$

*is a solution to the focusing NLS equation depending on $2N-1$ real parameters $a_j$, $b_j$, $\varepsilon$, $1 \le j \le N-1$ with $\phi_\nu^r$ defined in (2.5)*

$$\phi_{r,\nu} = \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), \quad 1 \le \nu \le N,$$
$$\phi_{r,\nu} = \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), \quad N+1 \le \nu \le 2N, \quad r = 1,3,$$

*$\kappa_\nu$, $\delta_\nu$, $x_{r,\nu}$, $\gamma_\nu$, $e_\nu$ being defined in(2.1), (2.2) and (2.3).*

We can give another representation of the solutions to the NLS equation depending only on terms $\gamma_\nu$, $1 \le \nu \le 2N$. From the relations (2.1), we can express the terms $\kappa_\nu$, $\delta_\nu$ and $x_{r,\nu}$ in function of $\gamma_\nu$, for $1 \le \nu \le 2N$ and we obtain :

$$\kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \le j \le N,$$
$$\kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = -\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j + i}{\gamma_j - i}, \quad N+1 \le j \le 2N.$$

We have the following new representation [17, 26] :

**Theorem 2.4.** *The function $v$ defined by*

$$v(x,t) = \frac{\det[(\partial_y^{\mu-1}\tilde{\phi}_{3,\nu}(0))_{\nu,\mu\in[1,\dots,2N]}]}{\det[(\partial_y^{\mu-1}\tilde{\phi}_{1,\nu}(0))_{\nu,\mu\in[1,\dots,2N]}]} e^{2it - i\varphi} \tag{2.6}$$

*is a solution to the NLS equation (1.1) depending on* $2N-1$ *real parameters* $a_j$, $b_j$, $\varepsilon$, $1 \leq j \leq N-1$. *The functions* $\tilde{\phi}_{r,\nu}$ *are defined by*

$$\begin{array}{l}
\tilde{\phi}_{r,j}(y) = \sin\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - i\frac{(r-1)}{2}\ln\frac{\gamma_j - i}{\gamma_j + i} + \gamma_j y - ie_j\right), \\
\tilde{\phi}_{r,N+j}(y) = \cos\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t + i\frac{(r-1)}{2}\ln\frac{\gamma_j - i}{\gamma_j + i} + \frac{1}{\gamma_j}y - ie_{N+j}\right), \\
\text{where } \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}},\ 1 \leq j \leq N. \\
\lambda_j \text{ is an arbitrary real parameter such that } 0 < \lambda_j < 1,\ \lambda_{N+j} = -\lambda_j,\ 1 \leq j \leq N. \\
\text{The terms } e_\nu \text{ are defined by (2.3),} \\
\text{where } a_j \text{ and } b_j \text{ are arbitrary real numbers, } 1 \leq j \leq N-1.
\end{array} \tag{2.7}$$

**Remark 2.1.** *In the formula (2.6), the determinants* $\det[(\partial_y^{\mu-1} f_\nu(0))_{\nu,\mu\in[1,\dots,2N]}]$ *are the wronskians of the functions* $f_1,\dots,f_{2N}$ *evaluated in* $y=0$. *In particular* $\partial_y^0 f_\nu$ *means* $f_\nu$.

### 2.3 Families of quasi-rational solutions of NLS equation in terms of a quotient of two determinants

The following notations are used :

$$X_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2,$$

$$Y_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2,$$

for $1 \leq \nu \leq 2N$, with $\kappa_\nu$, $\delta_\nu$, $x_{r,\nu}$ defined in (2.1).
Parameters $e_\nu$ are defined by (2.3).
Below the following functions are used :

$$\begin{array}{ll}
\varphi_{4j+1,k} = \gamma_k^{4j-1}\sin X_k, & \varphi_{4j+2,k} = \gamma_k^{4j}\cos X_k, \\
\varphi_{4j+3,k} = -\gamma_k^{4j+1}\sin X_k, & \varphi_{4j+4,k} = -\gamma_k^{4j+2}\cos X_k,
\end{array} \tag{2.8}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll}
\varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos X_{N+k}, & \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin X_{N+k}, \\
\varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos X_{N+k}, & \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin X_{N+k},
\end{array} \tag{2.9}$$

for $1 \leq k \leq N$.
We define the functions $\psi_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, the term $X_k$ is only replaced by $Y_k$.

$$\begin{array}{ll}
\psi_{4j+1,k} = \gamma_k^{4j-1}\sin Y_k, & \psi_{4j+2,k} = \gamma_k^{4j}\cos Y_k, \\
\psi_{4j+3,k} = -\gamma_k^{4j+1}\sin Y_k, & \psi_{4j+4,k} = -\gamma_k^{4j+2}\cos Y_k,
\end{array} \tag{2.10}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll}
\psi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos Y_{N+k}, & \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin Y_{N+k}, \\
\psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos Y_{N+k}, & \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin Y_{N+k},
\end{array} \tag{2.11}$$

for $1 \leq k \leq N$.
Then we get the following result [26] :

**Theorem 2.5.** *The function v defined by*

$$v(x,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})}e^{2it-i\varphi}$$

*is a quasi-rational solution of the NLS equation (1.1) depending on* $2N-2$ *real parameters* $a_j$, $b_j$, $1 \leq j \leq N-1$, *where*

$$\begin{aligned}
&n_{j1} = \varphi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial \varepsilon^{2k-2}}(x,t,0),\\
&n_{jN+1} = \varphi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial \varepsilon^{2k-2}}(x,t,0),\\
&d_{j1} = \psi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial \varepsilon^{2k-2}}(x,t,0),\\
&d_{jN+1} = \psi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial \varepsilon^{2k-2}}(x,t,0),\\
&2 \leq k \leq N,\ 1 \leq j \leq 2N
\end{aligned}$$

*The functions* $\varphi$ *and* $\psi$ *are defined in (2.8),(2.9), (2.10), (2.11).*

## 3. Structure of the multi-parametric solutions to the NLS equation of order $N$ depending on $2N-2$ parameters

### 3.1 The quotient of two polynomials of degree $(N(N+1)$ in $x$ and $t$ by an exponential depending on $t$

Here we present a result which states the structure of the quasi-rational solutions of the NLS equation. It was only conjectured in preceding works [16, 18]. Moreover we obtain here families of deformations of the $N$th Peregrine breather depending on $2N-2$ parameters.
In this section we use the notations defined in the previous sections. The functions $\varphi$ and $\psi$ are defined in (2.8), (2.9), (2.10), (2.11).
The structure of the quasi rational solutions to the NLS equation is given by [27] :

**Theorem 3.1.** *The function* $v$ *defined by*

$$v(x,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} e^{2it-i\varphi} \tag{3.1}$$

*is a quasi-rational solution of the NLS equation (1.1) quotient of two polynomials* $R(x,t)$ *and* $S(x,t)$ *depending on* $2N-2$ *real parameters* $a_j$ *and* $b_j$, $1 \leq j \leq N-1$.
$R(x,t)$ *and* $S(x,t)$ *are polynomials of degrees* $N(N+1)$ *in* $x$ *and* $t$.

**Remark 3.1.** *The polynomials* $R(x,t)$ *and* $S(x,t)$ *have the same coefficients of degrees* $N(N+1)$ *in* $2x$ *and* $4t$ *equal to* 1.
*The polynomial* $B(x,t)$ *does not have any real root.*

### 3.2 The structure of the Peregrine breather of order $n$

There is any freedom to choose $\gamma_j$ in such a way that the conditions on $\lambda_j$ are checked. We know from previous works [16, 18] that the (analogue) Peregrine breathers are obtained when all the parameters $a_j$ and $b_j$ are equal to 0. In order to get the more simple expressions in the determinants, we choose particular solutions in the previous families.

Here we choose $\gamma_j = j\varepsilon$ as simple as possible in order to have the conditions on $\lambda_j$ checked, and we have [26, 27] :

**Theorem 3.2.** *The function* $v_0$ *defined by*

$$v_{n,0}(x,t) = \left(\frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} e^{2it-i\varphi}\right)_{(a_j=b_j=0,\, 1\leq j\leq N-1)} \tag{3.2}$$

*is the Peregrine breather of order* $N$ *solution of the NLS equation (1.1) whose highest amplitude in modulus is equal to* $2N+1$.

**Remark 3.2.** *The previous result is given in the frame where the limit of the modulus of the solution when* $x$ *or* $t$ *tend to infinity is equal to* 1. *We know that if* $v(x,t)$ *is is a solution to the NLS equation then* $u(x,t) = av(ax, a^2t)$ *is also a solution to the NLS equation, for any arbitrary real* $a$.

**Remark 3.3.** *In (3.2), the matrices* $(n_{jk})_{j,k\in[1,2N]}$ *and* $(d_{jk})_{j,k\in[1,2N]}$ *are defined in (3.1).*

We have seen in previous section that solutions of NLS equation given by (2.7) can be written in function uniquely of terms $\gamma$. We recall that the terms $\gamma_j$ are given by $\gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}$, $1 \leq j \leq N$; $\lambda_j$ is an arbitrary real parameter such that $0 < \lambda_j < 1$, $\lambda_{N+j} = -\lambda_j$, $1 \leq j \leq N$.
We can rewrite the result given in (2.7) in a simplest formulation as follows [26, 27] :

**Theorem 3.3.** *The function $v$ defined by*

$$v(x,t) = \frac{\det((f_{jk}^{(3)})_{j,k\in[1,2N]})}{\det((f_{jk}^{(1)})_{j,k\in[1,2N]})} e^{2it-i\varphi}$$

*is a quasi-rational solution of the NLS equation (1.1) depending on $2N-2$ real parameters $a_j$, $b_j$, $1 \leq j \leq N-1$ where*

$$f_{jk}^{(r)} = \frac{\partial^{2(k-1)}}{\partial \varepsilon^{2(k-1)}} \left(\gamma^{4j-1} \sin\left[\frac{2\gamma}{1+\gamma^2}x + 4i\frac{\gamma(1-\gamma^2)}{(1+\gamma^2)^2}t - i\frac{r-1}{2}\ln\frac{\gamma-i}{\gamma+i} + \sum_{l=1}^{N-1}(a_l+ib_l)\varepsilon^{2l+1} + (j-1)\frac{\pi}{2}\right]\right)_{(\varepsilon=0)},$$
$$f_{jN+k}^{(r)} = \frac{\partial^{2(k-1)}}{\partial \varepsilon^{2(k-1)}} \left(\gamma^{2N-4j-1} \cos\left[\frac{2\gamma}{1+\gamma^2}x - 4i\frac{\gamma(1-\gamma^2)}{(1+\gamma^2)^2}t + i\frac{r-1}{2}\ln\frac{\gamma-i}{\gamma+i} + \sum_{l=1}^{N-1}(a_l-ib_l)\varepsilon^{2l+1} + (j-1)\frac{\pi}{2}\right]\right)_{(\varepsilon=0)},$$
$$1 \leq k \leq N, \quad 1 \leq j \leq 2N, \quad r \in \{1;3\}, \quad \varepsilon \in ]0;1[, \quad \gamma = \varepsilon(1-\varepsilon^2)^{1/2}.$$

**Remark 3.4.** *In the previous theorem, the expression $\frac{\partial^0}{\partial \varepsilon^0} f(x)$ means $f(x)$.*

The solution to the NLS equation can be written in the form

$$v_N(x,t) = \frac{R_N(x,t)}{S_N(x,t)} e^{2it} = \left(1 + \frac{A_N(x,t)}{B_N(x,t)}\right) e^{2it} \tag{3.3}$$

and the Peregrine breather in the form

$$v_{N,0}(x,t) = \frac{T_N(x,t)}{U_N(x,t)} e^{2it} = \left(1 + \frac{P_N(x,t)}{Q_N(x,t)}\right) e^{2it} \tag{3.4}$$

where the index 0 means that all the parameters are equal to 0.

## 4. Differential relation for the NLS equation

In previous works [26, 27], we have proven that the solutions $v_N$ to the NLS equation can be written in the form

$$v_N(x,t) = \left(1 + \frac{A_N(x,t)}{B_N(x,t)}\right) e^{2it}.$$

We have a very simple relation between the square of the modulus of $v_N$ and the denominator part $B_N$. This relation appears in a paper of Ling and Zhao [25] where the solutions to the NLS equation are given in the frame of the generalized Darboux transfomation. Here this result and its proof are given in a general frame by the following theorem :

**Theorem 4.1.** *The solutions $v_N(x,t) = \left(1 + \frac{A_N(x,t)}{B_N(x,t)}\right) e^{2it}$ to the NLS equation verify the following relation*

$$|v_N(x,t)|^2 = 1 + (\ln B_N(x,t))_{xx}, \tag{4.1}$$

*where the subscript $_{xx}$ means the double derivation with respect to $x$.*

*Proof.* For simplicity with omit the references to $N$ and $(x,t)$ to the solution $v$ and the polynomials $A$ and $B$. If we substitute $v$ by $\left(1 + \frac{A}{B}e^{2it}\right)$ in the expression $X = iv_t + v_{xx} + 2|v|^2 v$, we get

$$\frac{2}{B^3}\left(|A+B|^2(A+B) + AB_x^2 - AB_{xx}B\right)$$

$$+\frac{1}{B^2}\left(i(BA_t - AB_t) - 2B^2 - 2AB - 2A_xB_x + AB_{xx} + BA_{xx}\right) = 0$$

This can gives the two following relations

$$|A+B|^2(A+B)+(A+B)B_x^2-(A+B)B_{xx}B-(A+B)B^2=0$$

and

$$i(A_tB-AB_t)+(A_{xx}B-2A_xB_x+AB_{xx})+(B_{xx}B-B_x^2)=0$$

The first relation can be rewritten as

$$|A+B|^2+B_x^2-B_{xx}B-B^2=0$$

Then the square of the modulus of $v_N$ can be written as

$$|v_N|^2=\frac{|A+B|^2}{B^2}=1+(\ln B)_{xx},$$

which proves relation (4.1). □

## 5. Relations between rational part of the solutions to the NLS equation

With the preceding notations, we get the following statement

**Theorem 5.1.** *The polynomials of the solutions $v_N$ to the NLS equation defined by (3.3) $v_N(x,t)=\frac{R_N(x,t)}{S_N(x,t)}e^{2it}$ verify the following relations*

$$\begin{aligned}&(i(R_N)_t+(R_N)_{xx}-2R_N)S_N^2-((S_N)_{xx}+i(S_N)_t)R_NS_N\\&-2(R_N)_x(S_N)_xS_N+2((S_N)_x^2+R_N\overline{R_N})R_N=0.\end{aligned}$$

*Proof.* It is sufficient to replace in the equation (1.1) $v_N(x,t)$ by $\frac{R_N(x,t)}{S_N(x,t)}\ e^{2it}$. □

**Proposition 5.1.** *The coordinates of extrema $(x_0,t_0)$ of solutions $v_N$ to the NLS equation defined by (3.3) $v_N(x,t)=\frac{R_N(x,t)}{S_N(x,t)}e^{2it}$ verify the the following relations*

$$\begin{aligned}&(R_N)_x(x_0,t_0)\overline{R_N}(x_0,t_0)S_N(x_0,t_0)+(\overline{R_N})_x(x_0,t_0)R_N(x_0,t_0)S_N(x_0,t_0)\\&-2(S_N)_x(x_0,t_0)R_N(x_0,t_0)\overline{R_N}(x_0,t_0)=0,\end{aligned}$$

$$\begin{aligned}&(R_N)_t(x_0,t_0)\overline{R_N}(x_0,t_0)S_N(x_0,t_0)+(\overline{R_N})_t(x_0,t_0)R_N(x_0,t_0)S_N(x_0,t_0)\\&-2(S_N)_t(x_0,t_0)R_N(x_0,t_0)\overline{R_N}(x_0,t_0)=0.\end{aligned}$$

$$(R_N)_x(x_0,t_0)S_N(x_0,t_0)-(S_N)_x(x_0,t_0)R_N(x_0,t_0)=0.$$

$$(R_N)_t(x_0,t_0)S_N(x_0,t_0)-(S_N)_t(x_0,t_0)R_N(x_0,t_0)+2iS_N(x_0,t_0)R_N(x_0,t_0)=0.$$

*where $\overline{a}$ means the complex conjugate of $a$.*

*Proof.* It is sufficient to compute the partial derivatives of (1.1) $v_N(x,t)$ defined by $\frac{R_N(x,t)}{S_N(x,t)}\ e^{2it}$. □

**Remark 5.1.** *As a consequence of the result on the highest modulus of the $P_N$ breather defined by (3.4) $v_{N,0}(x,t)=\frac{T_N(x,t)}{U_N(x,t)}e^{2it}$, we get*

$$T_N(0,0)=(2N+1)U_N(0,0).$$

## 6. Conclusion

Different representations of the solutions to the NLS equation have been summarized in this paper, as well as the structure of the quasi rational solutions. Some differential relations have been given in this text for the NLS equation.
From different studies realized by the author, [26]-[32], it seems that the maximums of the modulus of the solutions to the NLS equation are in connection with the zeros of the Yablonski-Vorob'ev polynomials [33, 34].
It would be relevant to study this conjecture.
It would be also relevant to search other types of equations verified by the polynomials $(P_N, Q_N)$, $(R_N, S_N)$, $(A_N, B_N)$ or $(T_N, U_N)$.

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# Coincidence Point Theorems on $b$-Metric Spaces via $C_F$-Simulation Functions

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**Abstract**
In this paper, we investigate the existence and uniqueness of the coincidence points with the $C_F$-simulation function for two nonlinear operators on the $b$-metric space. Our results improve and generalize some of the results available in the literature.

**Keywords:** $b$-metric space, Fixed point, Simulation function
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## 1. Introduction

From a theoretical standpoint, there are many different ways to solve the problems encountered in mathematics and related sciences. In the recent years, the most remarkable theory is the fixed point theory which is used in many areas. The most known theory is the Banach contraction principle [1] and this theory has numerous applications in important areas (see [2], [3]).

Recently, Khojasteh et al. [4] introduced the concept of simulation function. Then, they introduced the non-linear $Z$-contraction of the simulation class of functions. The well known Banach contraction principle ensures the existence and uniqueness of fixed point of a contraction on a complete metric space. After this interesting principle, several authors generalized this principle by introducing the various contractions on metric spaces (see [4],[5]).

Now, we give some concepts and results from the literature used throughout the study.

**Definition 1.1.** *[6] Let $X$ be a non-empty set and let $d : X \times X \longrightarrow [0,\infty)$ be a function satisfying the following conditions:*

*(i)* $d(x,y) = 0 \Longleftrightarrow x = y$, *for all* $x,y \in X$ ,

*(ii)* $d(x,y) = d(y,x)$, *for all* $x,y \in X$,

*(iii)* $d(x,y) \leq s[d(x,y)+d(y,z)]$, *for some real* $s \geq 1$, *for all* $x,y,z \in X$.

*Then, $d$ is called a b-metric on $X$ and $(X,d)$ is called a b-metric space.*

**Lemma 1.2.** *[7] Let $(X,d)$ be a metric space and $\{x_n\}$ be a sequence in $X$ such that*

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

*If $\{x_n\}$ is not a Cauchy sequence in $(X,d)$, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$. For all $k > 0$, corresponding to $m_k$, we can choose $n_k$ to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$, $d(x_{m_k}, x_{n_k - 1}) < \varepsilon$ and*

*(1)* $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k+1}) = \varepsilon$,

*(2)* $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \varepsilon$,

*(3)* $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon$,

*(4)* $\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \varepsilon$.

**Lemma 1.3.** *[8] Let $(X,d)$ be a b-metric space for some real $s \geq 1$ and $\{x_n\}$ be a sequence in $X$ such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

*If $\{x_n\}$ is not a b-Cauchy sequence in $(X,d)$, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$, $d(x_{m_k}, x_{n_k-1}) < \varepsilon$ and*

*(1)* $\varepsilon \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\varepsilon$,

*(2)* $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq s^2\varepsilon$,

*(3)* $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq s^2\varepsilon$,

*(4)* $\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\varepsilon$.

**Definition 1.4.** *[9] Let $X$ be a nonempty set and $T, g : X \longrightarrow X$ be mappings.*

*(1) A point $x \in X$ is called a fixed point of the mapping $T$ if $Tx = x$.*

*(2) A point $x \in X$ is called a coincidence point of the mappings $T$ and $g$ if $Tx = gx$.*

*(3) A point $x \in X$ is called a common fixed point of the mappings $T$ and $g$ if $Tx = gx = x$.*

**Definition 1.5.** *[9] Let $T, g : X \longrightarrow X$ be mappings on a b-metric space $(X,d)$. If*

$$\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0,$$

*for all $\{x_n\} \subseteq X$ such that the $\{gx_n\}$ and $\{Tx_n\}$ sequences are convergent and have the same limit points, then $T$ and $g$ are called compatible.*

**Remark 1.6.** *[10] If $T$ and $g$ commuting (that is, $Tgx = gTx$ for all $x \in X$), then $T$ and $g$ are compatible.*

**Definition 1.7.** *[4] Let $T, g : X \longrightarrow X$ be functions and $\{x_n\} \subseteq X$. The sequence $\{x_n\}$ is a Picard-Jungck sequence with a pair of $(T,g)$ if $gx_{n+1} = Tx$, for each $n \geq 0$*

**Definition 1.8.** *[11] Let $F : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be a continuous function and satisfy the following conditions:*

*(a)* $F(s,t) \leq s$;

*(b)* $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$*; for all* $s,t \in [0,\infty)$.

*Then, $F$ is called a C-class function.*

We denote $C$-class functions as $\mathtt{C}$.

**Definition 1.9.** *[4] Let $F : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be a function. There exists a $C_F \geq 0$ such that* *(a)* $F(s,t) > C_F \Rightarrow s > t$;

*(b)* $F(t,t) \leq C_F$ , $\forall s,t \in [0,\infty)$.

*Then, $F$ has property $C_F$.*

**Definition 1.10.** *[5] Let $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be a function satisfying the following conditions:*

*($\zeta a$)* $\zeta(0,0) = 0$;

*($\zeta b$)* $\zeta(t,s) < F(t,s)$, *for all $s,t > 0$; the function $F : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ is the element of $\mathcal{C}$ with property $C_F$.*

*($\zeta c$) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n,s_n) < C_F$.*

*Then, it is called a $C_F$-simulation function.*

We denote the class of all $C_F$-simulation functions as $Z_F$.

**Definition 1.11.** *Let $(X,d)$ be a b-metric space for some real $s \geq 1$ and $f,g : X \longrightarrow X$ be mappings. $\zeta$ is an element of $Z_F$ such that*

$$\zeta(s^4 d(Tx,Ty),(gx,gy)) \geq C_F, \tag{1.1}$$

*for all $x,y \in X$ with $gx \neq gy$. Then, $T$ is called a $(Z_{F,b},g)$-contraction.*

## 2. Main results

In this section, we introduce our main results.

**Remark 2.1.**

*(1) By axiom $(\zeta_b)$, it is clear that a simulation function must verify $\zeta(r,s^4 r) < C_F$ for all $r > 0$.*

*(2) Furthermore, $\zeta$ is the elements of $Z_F$ such that*

$$d(Tx,Ty) \leq s^4 d(Tx,Ty) < d(gx,gy), \tag{2.1}$$

*for all $x,y \in X$ with $gx \neq gy$. $T$ is a $(Z_{F,b},g)$-contraction.*

*To prove, assume that $gx \neq gy$. Then, $d(gx,gy) > 0$. If $Tx = Ty$, then $0 = d(Tx,Ty) = s^4 d(Tx,Ty) < d(gx,gy)$. On the contrary case, if $Tx \neq Ty$, then $0 < d(Tx,Ty)$ , by property $(\zeta_b)$ and (1.1), we have that*

$$C_F \leq \zeta(s^4 d(Tx,Ty),d(gx,gy)) < F(d(gx,gy),s^4 d(Tx,Ty))$$

*so (2.1) holds. In other words $d(Tx,Ty) \leq s^4 d(Tx,Ty) < d(gx,gy)$ is obtained.*

**Lemma 2.2.** *If $T$ is a $(Z_{F,b},g)$-contraction in a b-metric space $(X,d)$ and $x,y \in X$ are coincidence points of $T$ and $g$, then $Tx = gx = gy = Ty$. In particular, the following conditions hold.*

*(1) If $T$ (or $g$) is injective within the entire set of coincidence points of $T$ and $g$, then $T$ and $g$ have a single coincidence point at most.*

*(2) If $T$ and $g$ have a common fixed point, it is unique.*

*Proof.* To prove, assume that $gx \neq gy$. Then, $d(gx,gy) > 0$. Using (1.1) the following is obtained

$$C_F \leq \zeta(s^4 d(Tx,Ty),d(gx,gy)) = \zeta(s^4 d(gx,gy),d(gx,gy)).$$

Due to the item (1) of Remark 2.1, contradiction is obtained. In this case, our assumption is incorrect. Therefore, if $x$ and $y$ are coincidence points of $T$ and $g$, then $Tx = gx = gy = Ty$. The proof is completed. □

**Theorem 2.3.** *Let $T$ be a $(Z_{F,b},g)$-contraction in b-metrik space $(X,d)$. Suppose that there is a Picard-Jungck sequence $\{x_n\}$ of $(T,g)$. In addition, at least one of the following conditions holds.*

*(a) $(g(X),d)$ (or $(T(X),d)$) is complete.*

*(b) $(X,d)$ is complete, $T$ and $g$ are b-continuous and compatible.*

*(c) $(X,d)$ is complete, $T$ and $g$ are b-continuous and commuting.*

*$T$ and $g$ have at least one coincidence point. Furthermore, either the sequence $\{gx_n\}$ contains a coincidence point of $T$ and $g$, or at least one of the following conditions holds.*

*In case (a), the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point of $v \in X$ is a coincident point of $T$ and $g$ such that $gv = u$.*

*In cases (b) and (c), the sequence $\{gx_n\}$ is convergent to a coincidence point of $T$ and $g$.*

*In addition, if $x,y \in X$ are the coincidence points of $T$ and $g$, then $Tx = gx = gy = Ty$. If $T$ (or $g$) is injective within the entire set of coincidence points of $T$ and $g$, then $T$ and $g$ have a single coincidence point at most.*

*Proof.* The proof is completed if $\{x_n\}$ contains a coincidence point of $T$ and $g$. Suppose that $\{x_n\}$ does not contain any coincidence points of $T$ and $g$, for all $n \geq 0$; that is,

$$gx_n \neq Tx_n = gx_{n+1}.$$

In this case, we have

$$d(gx_n, gx_{n+1}) > 0 \tag{2.2}$$

for all $n \geq 0$.

Now, the evidence will be examined in three cases.

Step 1. Using ($\zeta$b) and (1.1), $s \geq 1$ and for all $n \geq 0$,

$$\begin{aligned} C_F &\leq \zeta(s^4(d(Tx_n, Tx_{n+1})), d(gx_n, gx_{n+1})) \\ &= \zeta(s^4 d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})) \\ &< F(d(gx_n, gx_{n+1}), s^4 d(gx_{n+1}, gx_{n+2})), \end{aligned} \tag{2.3}$$

for all $n \leq 0$, $0 < d(gx_{n+1}, gx_{n+2}) \leq s^4 d(gx_{n+1}, gx_{n+2}) < d(gx_n, gx_{n+1})$. Similarly, we can prove that $d(gx_{n+2}, gx_{n+3}) < d(gx_{n+1}, gx_{n+2})$. Therefore, $\{d(gx_n, gx_{n+1})\}$ is sub-zero, non-increasing and convergent.

Let $r > 0$ and $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = r$. Using axiom ($\zeta$c) to the sequences $\{t_n = d(gx_{n+1}, gx_{n+2})\}$ and $\{s_n = d(gx_n, gx_{n+1})\}$ with $t_n < s_n$,

$$C_F \leq \limsup_{n\to\infty} \zeta(s^4 d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1}) = \limsup_{n\to\infty} \zeta(s^4 t_n, s_n) < C_F.$$

Due to the with (2.3),

$$C_F \leq \limsup_{n\to\infty} \zeta(s^4 d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})),$$

for all $n \geq 0$, a contradiction is obtained. In this case, our assumption is incorrect. Therefore, we have $r = 0$; that is,

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0,$$

holds.

Step 2. Suppose that the sequence $\{gx_n\}$ is not a b-Cauchy sequence in $(X,d)$. Then, there exits an $\varepsilon > 0$ and sequences of positive integers $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ with $n(k) > m(k) \geq k$ such that $d(gx_{m(k)}, gx_{n(k)}) > \varepsilon$, $d(gx_{m(k)}, gx_{n(k)-1}) < \varepsilon$. $T$, using ($\zeta$b) axiom and $(Z_{F,b},g)$ contraction,we have

$$\begin{aligned} C_F &\leq \zeta(s^4(d(Tx_{m(k)}, Tx_{n(k)})), d(gx_{m(k)}, gx_{n(k)})) \\ &= \zeta(s^4 d(gx_{m(k)+1}, gx_{n(k)+1}), d(gx_{m(k)}, gx_{n(k)})) \\ &< F(d(gx_{m(k)}, gx_{n(k)}), s^4 d(gx_{m(k)+1}, gx_{n(k)+1})). \end{aligned}$$

It now consists of two different situations.

Case (i): $s = 1$.

In this case, $(X,d)$ is a metric space. By Lemma 1.2 there exits $\varepsilon > 0$ and sequence of positive integers $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ such that $n(k) > m(k) \geq k$ with $d(gx_{m(k)}, gx_{n(k)}) > \varepsilon$, $d(gx_{m(k)}, gx_{n(k-1)}) < \varepsilon$ and satisfying (1)-(4) of Lemma 1.2 and using ($\zeta$c), $\{t_n = d(gx_{m(k)+1}, gx_{n(k)+1})\}$ and $\{s_n = d(gx_{m(k)}, gx_{n(k)})\}$, we have

$$\begin{aligned} C_F &\leq \limsup_{n\to\infty} \zeta(d(gx_{m(k)+1}, gx_{n(k)+1}), d(gx_{m(k)}, gx_{n(k)})) \\ &< F(d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)+1}, gx_{n(k)+1})) \\ &< C_F \end{aligned}$$

which is a contradiction.

Case (ii): $s > 1$.

In this case, $(X,d)$ is a $b$-metric space. By Lemma 1.3 there exist $\varepsilon > 0$ and sequences of positive integers $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ such that $n(k) > m(k) \geq k$ with $d(gx_{m(k)}, gx_{n(k)}) > \varepsilon$, $d(gx_{m(k)}, gx_{n(k-1)}) < \varepsilon$ and satisfying (1)-(4) of Lemma 1.3, we have

$$\begin{aligned} C_F &\leq \limsup_{n\to\infty} \zeta(s^4 d(gx_{m(k)+1}, gx_{n(k)+1}), d(gx_{m(k)}, gx_{n(k)})) \\ &< F(d(gx_{m(k)}, gx_{n(k)}), s^4 d(gx_{m(k)+1}, gx_{n(k)+1})) \\ &< C_F \end{aligned}$$

which is a contradiction.

Consequently, by (i) and (ii), we have $\{gx_n\}$, is a $b$-Cauchy sequence in $(X,d)$.

Step 3. By assumptions (a), (b), (c), we will prove that $T$ and $g$ have a coincidence point.

Case (a): Suppose that $(g(X))$ (or $(T(X),d)$) is complete. We also found that the sequence $\{gx_n\}$ is a b-Cauchy sequence. In case for all $n \geq 0$, $gx_{n+1} = Tx_n \in T(X) \subseteq g(X)$, taking into account these, $u \in g(X)$, that is,

$$\lim_{n\to\infty} d(gx_n, u) = 0.$$

Since $Tx_n = gx_{n+1}$, for all $n$ we have,

$$\lim_{n\to\infty} d(Tx_n, u) = 0. \tag{2.4}$$

Let $v \in X$ be any point such that $gv = u$. Suppose that $v$ is not a coincidence point of $T$ and $g$, then $gv = u \neq Tv$. In this case, we have $\delta = d(Tv, gv) > 0$. Using (2.4), $n_0 \in N$ be such that $d(gx_n, gv) < \delta$ for all $n \geq n_0$. This means that $d(gx_n, gv) < \delta = d(Tv, gv)$, for all $n \geq n_0$.

In particular, $gx_n \neq Tv$ for all $n \geq n_0$, then

$$d(Tx_n, Tv) = d(gx_{n+1}, gv) > 0, \text{ for all } n \geq n_0. \tag{2.5}$$

On the other hand, if $gx_n = gv$ for all $n \geq n_1$, it contradicts the condition (2.2) for $\exists\, n_1 \in N$.

Therefore, the sequence $\{gx_n\}$ has a subsequence $\{gx_{\delta(n)}\}$ with

$$gx_{\delta(n)} \neq gv. \tag{2.6}$$

Now, let $n_2 \in N$ such that $\delta(n_2) \geq n_0$. Therefore, for all $n \geq n_2$, by (2.5) and (2.6), $d(gx_{\delta(n)}, gv) > 0$ and $d(Tx_{\delta(n)}, Tv) > 0$. Using($\zeta$b),

$$\begin{aligned} C_F &\leq \zeta(s^4 d(Tx_{\delta(n)}, Tv), d(gx_{\delta(n)}, gv))) \\ &< F(d(gx_{\delta(n)}, gv), s^4 d(Tx_{\delta(n)}, Tv)) \end{aligned}$$

this means that;

$$0 \leq d(Tx_{\delta(n)}, Tv) \leq s^4 d(Tx_{\delta(n)}, Tv) < d(gx_{\delta(n)}, gv) = d(gx_{\delta(n)}, u).$$

By $\lim_{n\to\infty} d(gx_{\delta(n)},u)=0$, $\lim_{n\to\infty} d(Tx_{\delta(n)},Tv)=0$. However, $\{Tx_{\delta(n)}\}=\{gx_{\delta(n)+1}\}$ is a supsequence of $\{gx_n\}$ and converges to $gv$. Due to the uniqueness of the limit, we have $gv=Tv$. This contradicts our assumption. Then, $u=gv=Tv$. In other words, $v$ is a coincidence point of $T$ and $g$.

Case (b): Suppose that $(X,d)$ is complete. $T$ and $g$ are continuous and compatible. In this case the sequence $\{gx_n\}$ is $\{gx_n\}\longrightarrow u\in X$, since $(X,d)$ is a $b$-Cauchy sequence on the complete $b$-metric space. Since $T$ is continuous, $\{ggx_n\}\longrightarrow gu$. Since $g$ is continuous, $\{Tgx_n\}\longrightarrow Tu$. Moreover, $T$ and $g$ are compatible, $\{Tx_n=gx_{n+1}\}$ and $\{gx_n\}$ have the same limit points, we deduce that

$$d(Tu,gu)=\lim_{n\to\infty} d(Tgx_n,ggx_{n+1})=\lim_{n\to\infty} d(Tgx_n,gTx_n)=0.$$

Therefore, $u$ is a coincidence point of $T$ and $g$.

Case (c): Suppose that $(X,d)$ is complete and $T$ and $g$ are continuous and commuting. In this case, if $T$ and $g$ are commuting, then $T$ and $g$ will be compatible which is the same with case (b). ☐

**Example 2.4.** *Let $X=[0,1]$ and $d:X\times X\longrightarrow[0,\infty)$ be defined as*

$$d(x,y)=\begin{cases}0, & x=y,\\ (x-y)^2, & x\neq y,\end{cases}$$

*Then, $d$ is a b-metric with coefficient $s=2$ but it is not a metric. Consider the mappings $T,g:X\longrightarrow X$ defined by $Tx=x+3$ and $gx=5x+1$ for all $x\in X$. In order to solve the non-linear equation*

$$x+3=5x+1$$

*Theorem 2.3 can be applied using the simulation function $\zeta(t,s)=s-t$.*

$$\begin{aligned}\zeta(s^4d(Tx,Ty),d(gx,gy)) &= d(gx,gy)-s^4d(Tx,Ty)\\ &=(5x+1-5y-1)^2-2^4.(x+3-y-3)^2\\ &=25(x-y)^2-16(x-y)^2\\ &=9(x-y)^2\\ &\geq 0.\end{aligned}$$

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# Practice of the Incomplete $p$-Ramification Over a Number Field – History of Abelian $p$-Ramification

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**Abstract**
The theory of $p$-ramification, regarding the Galois group of the maximal pro-$p$-extension of a number field $K$, unramified outside $p$ and $\infty$, is well known including numerical experiments with PARI/GP programs. The case of "incomplete $p$-ramification" (i.e., when the set $S$ of ramified places is a strict subset of the set $P$ of the $p$-places) is, on the contrary, mostly unknown in a theoretical point of view. We give, in a first part, a way to compute, for any $S \subseteq P$, the structure of the Galois group of the maximal $S$-ramified abelian pro-$p$-extension $H_{K,S}$ of any field $K$ given by means of an irreducible polynomial. We publish PARI/GP programs usable without any special prerequisites. Then, in an Appendix, we recall the "story" of abelian $S$-ramification restricting ourselves to elementary aspects in order to precise much basic contributions and references, often disregarded, which may be used by specialists of other domains of number theory. Indeed, the torsion $\mathscr{T}_{K,S}$ of $\mathrm{Gal}(H_{K,S}/K)$ (even if $S = P$) is a fundamental obstruction in many problems. All relationships involving $S$-ramification, as Iwasawa's theory, Galois cohomology, $p$-adic $L$-functions, elliptic curves, algebraic geometry, would merit special developments, which is not the purpose of this text.

**Keywords:** Abelian $S$-ramification, Class field theory, Class groups, Leopoldt's conjecture, $p$-adic regulators, Pro-$p$-groups, Units, $\mathbb{Z}_p$-extensions
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## 1. Introduction and basic results

The numerous references that we mention allow the reader to have a chronological overview of the contributions. Many results have been collected in our book (Edit. 2005) quoted [1].

### 1.1 Notion of Galois $S$-ramification

Let $p \geq 2$ be a prime number and let $K$ be a number field; we denote by $P := \{\mathfrak{p} \text{ prime}, \ \mathfrak{p} \,|\, p\}$ the set of $p$-places of $K$ and by $S$ an arbitrary set of finite places (later we shall assume $S \subseteq P$).

A main problem in Galois theory above $K$ is to study the Galois group $\mathscr{G}_{K,S}$ of the maximal pro-$p$-extension of $K$ which is $S$-ramified in the ordinary sense (i.e., unramified outside $S$ and non-complexified (= totally split) at the real infinite places of $K$ when $p = 2$).

As we will recall it in detail, in Section A.2, the study of $\mathscr{G}_{K,S}$ goes back to fundamental contributions of Serre [2], Šafarevič [3], Brumer [4], and has been largely extended, from the 1980's, in much works considering $S$-ramification (eventually with decomposition of another set $\Sigma$ of finite and infinite places).

The analogous theory for a local base field has also a long history that we shall not consider in this article.

### 1.2 Main cohomological invariants

For complete current information about the "cohomology of number fields", see the book of Neukirch–Schmidt–Wingberg [5, Chapter X].

When $S = P$, the $\mathbb{F}_p$-dimension of $\mathrm{H}^1(\mathscr{G}_{K,P}, \mathbb{Z}/p\mathbb{Z})$, which gives the minimal number of generators of $\mathscr{G}_{K,P}$, is the $p$-rank [1] of the abelianization:

$$\mathscr{A}_{K,P} := \mathscr{G}_{K,P}^{\mathrm{ab}} := \mathscr{G}_{K,P}/[\mathscr{G}_{K,P}, \mathscr{G}_{K,P}].$$

Denote by $(r_1, r_2)$ the signature of $K$ (whence $r_1 + 2r_2 = [K : \mathbb{Q}]$); then, the $\mathbb{F}_p$-dimension of $\mathrm{H}^2(\mathscr{G}_{K,P}, \mathbb{Z}/p\mathbb{Z})$, which gives the minimal number of relations between these generators, fulfills the identity:

$$\mathrm{rk}_p(\mathrm{H}^1(\mathscr{G}_{K,P}, \mathbb{Z}/p\mathbb{Z})) = \mathrm{rk}_p(\mathrm{H}^2(\mathscr{G}_{K,P}, \mathbb{Z}/p\mathbb{Z})) + r_2 + 1,$$

giving, for the torsion group $\mathscr{T}_{K,P}$ of $\mathscr{A}_{K,P}$ *under Leopoldt's conjecture*:

$$\mathrm{rk}_p(\mathscr{T}_{K,P}) = \mathrm{rk}_p(\mathrm{H}^2(\mathscr{G}_{K,P}, \mathbb{Z}/p\mathbb{Z})).$$

### 1.3 Class field theory

In the general case for $S$ (possibly containing tame places and not all the $p$-places) we may write:

$$\mathscr{A}_{K,S} = \Gamma_{K,S} \bigoplus \mathscr{T}_{K,S}, \text{ with } \Gamma_{K,S} \simeq \mathbb{Z}_p^{\widetilde{r}_{K,S}}, \tag{1.1}$$

where $\mathscr{T}_{K,S} := \mathrm{tor}_{\mathbb{Z}_p}(\mathscr{A}_{K,S})$ and $\widetilde{r}_{K,S} \ge 0$.

Without any $p$-adic assumption on the group of global units of $K$, we still have $\mathrm{rk}_p(\mathrm{H}^1(\mathscr{G}_{K,S}, \mathbb{Z}/p\mathbb{Z})) = \mathrm{rk}_p(\mathscr{A}_{K,S})$, but $\widetilde{r}_{K,S}$ (called the $\mathbb{Z}_p$-rank of $\mathscr{A}_{K,S}$) is more difficult when $S \subsetneq P$; however, $\mathrm{rk}_p(\mathscr{A}_{K,S}) = \widetilde{r}_{K,S} + \mathrm{rk}_p(\mathscr{T}_{K,S})$ is computable in complete generality with the invariants of class field theory for $K$ as follows (Šafarevič formula):

Let $K_{(S)}^\times$ be the subgroup of $K^\times$ of elements prime to $S$ and for any $\mathfrak{p} \in S$, let $K_\mathfrak{p}$ be the completion of $K$ at $\mathfrak{p}$; then:

$$\mathrm{rk}_p(\mathscr{A}_{K,S}) = \mathrm{rk}_p\big(V_{K,S}/K_{(S)}^{\times p}\big) + \sum_{\mathfrak{p} \in S \cap P} [K_\mathfrak{p} : \mathbb{Q}_p] + \sum_{\mathfrak{p} \in S} \delta_\mathfrak{p} - \delta_K - (r_1 + r_2 - 1), \tag{1.2}$$

where $V_{K,S} := \big\{\alpha \in K_{(S)}^\times,\ (\alpha) = \mathfrak{a}^p \text{ for an ideal } \mathfrak{a} \text{ of } K\big\}$, $\delta_\mathfrak{p} = 1$ or $0$ according as $K_\mathfrak{p}$ contains $\mu_p$ or not, and $\delta_K = 1$ or $0$ according as $K$ contains $\mu_p$ or not. Thus, from the relation (1.1):

$$\mathrm{rk}_p(\mathscr{T}_{K,S}) = \mathrm{rk}_p(\mathscr{A}_{K,S}) - \widetilde{r}_{K,S} = \mathrm{rk}_p\big(V_{K,S}/K_{(S)}^{\times p}\big) + \Big[\sum_{\mathfrak{p} \in S \cap P} [K_\mathfrak{p} : \mathbb{Q}_p] - \widetilde{r}_{K,S}\Big] + \sum_{\mathfrak{p} \in S} \delta_\mathfrak{p} - \delta_K - (r_1 + r_2 - 1),$$

where $\widetilde{r}_{K,S}$ fulfills the following formula:

$$\sum_{\mathfrak{p} \in S \cap P} [K_\mathfrak{p} : \mathbb{Q}_p] - \widetilde{r}_{K,S} = \dim_{\mathbb{Q}_p}\big(\mathbb{Q}_p \log_{S \cap P}(E_K)\big), \tag{1.3}$$

where $E_K$ is the group of global units of $K$ and $\log_{S \cap P} := \big(\log_\mathfrak{p}\big)_{\mathfrak{p} \in S \cap P}$ the family of $p$-adic logarithms over $S \cap P$ with values in $\bigoplus_{\mathfrak{p} \in S \cap P} K_\mathfrak{p}$. Note that for $S = P$,

$$r_{K,P} := \dim_{\mathbb{Q}_p}\big(\mathbb{Q}_p \log_P(E_K)\big),$$

is the $p$-adic rank of $E_K$ (i.e., the $\mathbb{Z}_p$-rank of the closure of the image $\iota_P(E_K)$ of $E_K$ in the group of local principal units $U_{K,P}$, where $\iota_P$ is the diagonal embedding; see § 2.1).

The Šafarevič and reflection formulas, generalized with decomposition, may be obtained via [1, Exercise II.5.4.1] or other classical references.

In general, $\widetilde{r}_{K,S}$ is non-obvious and varies from 0 to $r_2 + 1$ (see Wingberg [6, 7], Yamagishi [8], Maire [9, 10, 11], Labute [12], [13], Vogel [14] for some results and cases where $\mathscr{G}_{K,S}$ may be free with less than $r_2 + 1$ generators and our forthcoming numerical results showing that many $\mathbb{Z}_p$-ranks can occur).

[1] As usual, the $p$-rank of an abelian group $A$ is the $\mathbb{F}_p$-dimension of $A/A^p$.

For $S = P$ we obtain $\widetilde{r}_{K,P} = r_2 + 1$, under the Leopoldt conjecture, giving (since $\sum_{\mathfrak{p}\in P}[K_\mathfrak{p} : \mathbb{Q}_p] = r_1 + 2\,r_2$):

$$\mathrm{rk}_p(\mathscr{T}_{K,P}) = \mathrm{rk}_p(V_{K,P}/K_P^{\times p}) + \sum_{\mathfrak{p}\in P} \delta_\mathfrak{p} - \delta_K.$$

If $S = \emptyset$ then $\mathscr{A}_{K,S} = \mathscr{T}_{K,S} =: C\ell_K$, the $p$-class group of $K$ (ordinary sense).

**Remark 1.1.** *We shall not consider $S$-ramification with $S = P \cup T$, when $T$ is a finite set of tame places, because of the following exact sequence, under the Leopoldt conjecture (Neumann [15], Nguyen Quang Do [16, Corollary 4.3], [1, Theorem III.4.1.5]), where the $F_\mathfrak{l}$ are the residue fields:*

$$1 \longrightarrow \bigoplus_{\mathfrak{l}\in T}(F_\mathfrak{l}^\times \otimes \mathbb{Z}_p) \longrightarrow \mathscr{T}_{K,P\cup T} \longrightarrow \mathscr{T}_{K,P} \longrightarrow 1.$$

For some specialized applications (about number fields, elliptic curves, representation theory, Galois cohomology, Iwasawa's theory, $p$-adic $L$-functions) and some recent conjectures, one needs to study and compute the above $S$-invariants when $S$ is a subset of $P$ and $K/\mathbb{Q}$ not necessarily Galois. Even if $K/\mathbb{Q}$ is Galois, the Galois group does not necessarily operate on $S$. So the classical algebraic considerations (cohomology, Iwasawa's theory) largely collapse.

So the most tricky invariants in "incomplete $P$-ramification" are

$$\mathscr{T}_{K,S} \quad \text{and} \quad \widetilde{r}_{K,S} = \mathrm{rk}_p(\mathscr{A}_{K,S}) - \mathrm{rk}_p(\mathscr{T}_{K,S}) = \sum_{\mathfrak{p}\in S}[K_\mathfrak{p} : \mathbb{Q}_p] - \dim_{\mathbb{Q}_p}\big(\mathbb{Q}_p \log_S(E_K)\big).$$

Of course, they highly depend on the decomposition of the prime $p$ in the Galois closure of $K$ and probably of specific $p$-adic properties of units; but it remains the class field theory framework above the base field $K$.

## 2. General $p$-adic context of $S$-ramification

Consider a number field $K$ and a given prime $p \geq 2$. Let $S$ be a subset of the set $P$ of the $p$-places of $K$ and let $H_{K,S}$ be the maximal *abelian* $S$-ramified pro-$p$-extension of $K$; this field contains a (maximal) compositum $\widetilde{K}^S$ of $\mathbb{Z}_p$-extensions of $K$ and always the $p$-Hilbert class field $H_K := H_{K,\emptyset}$ of $K$.

These definitions are given in the ordinary sense when $p = 2$ (so that the real infinite places of $K$ are not complexified in the class fields considered; in other words they are totally split).

### 2.1 Fundamental exact sequences

Let $U_{K,S} := \bigoplus_{\mathfrak{p}\in S} U_\mathfrak{p}$, be the product of the groups of principal local units of $K_\mathfrak{p}$, $\mathfrak{p} \in S$, and let $\overline{E}_K^S$ be the closure of the image $\iota_S(E_K)$ of $E_K$ in $U_{K,S}$.

We denote by $W_{K,S} = \bigoplus_{\mathfrak{p}\in S} \mu_{K_\mathfrak{p}}$ the torsion group of the $\mathbb{Z}_p$-module $U_{K,S}$.

If $K/\mathbb{Q}$ is Galois and $S \subsetneq P$, $U_{K,S}$ is not necessarily a Galois module, which increases the difficulties.

The following $p$-adic result is valid without any assumption on $K$ and $S \subseteq P$:

**Lemma 2.1.** *We have the exact sequence:*

$$1 \to W_{K,S}/\mathrm{tor}_{\mathbb{Z}_p}(\overline{E}_K^S) \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}\big(U_{K,S}/\overline{E}_K^S\big) \xrightarrow{\log_S} \mathrm{tor}_{\mathbb{Z}_p}\big(\log_S(U_{K,S})/\log_S(\overline{E}_K^S)\big) \to 0.$$

*Proof.* The surjectivity comes from the fact that if $u \in U_{K,S}$ is such that $p^n \log_S(u) = \log_S(\overline{\varepsilon})$, $\overline{\varepsilon} \in \overline{E}_K^S$, then $u^{p^n} = \overline{\varepsilon}\cdot\xi$ for $\xi \in W_{K,S}$, hence there exists $m \geq n$ such that $u^{p^m} \in \overline{E}_K^S$, whence $u$ gives a preimage in $\mathrm{tor}_{\mathbb{Z}_p}\big(U_{K,S}/\overline{E}_K^S\big)$. If $u \in U_{K,S}$ is such that $\log_S(u) \in \log_S(\overline{E}_K^S)$, then $u = \overline{\varepsilon}\cdot\xi$ as above, giving the kernel equal to $\overline{E}_K^S \cdot W_{K,S}/\overline{E}_K^S = W_{K,S}/\mathrm{tor}_{\mathbb{Z}_p}(\overline{E}_K^S)$. □

Put:

$$\mathscr{W}_{K,S} := W_{K,S}/\mathrm{tor}_{\mathbb{Z}_p}(\overline{E}_K^S) \quad \& \quad \mathscr{R}_{K,S} := \mathrm{tor}_{\mathbb{Z}_p}\big(\log_S(U_{K,S})/\log_S(\overline{E}_K^S)\big).$$

Then the exact sequence of Lemma 2.1 becomes:

$$1 \longrightarrow \mathscr{W}_{K,S} \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}\big(U_{K,S}/\overline{E}_K^S\big) \xrightarrow{\log_S} \mathscr{R}_{K,S} \longrightarrow 0. \tag{2.1}$$

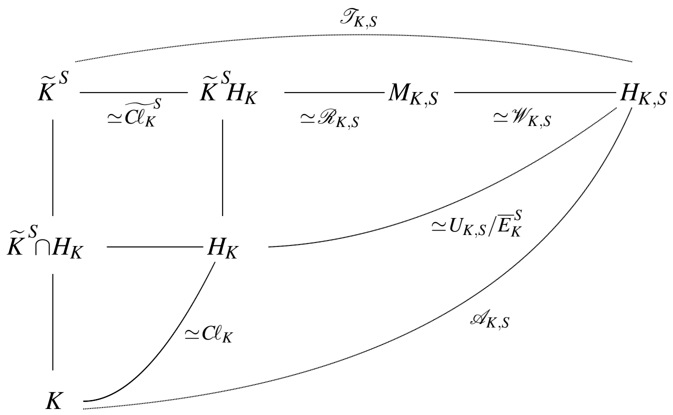
**Lemma 2.2.** *Let $\mu_K$ be the group of roots of unity of $p$-power order of $K$. Under the Leopoldt conjecture for $p$ in $K$ we have* $\mathrm{tor}_{\mathbb{Z}_p}(\overline{E}_K^P) = \iota_P(\mu_K)$*; thus, in that case,* $\mathscr{W}_{K,P} = W_{K,P}/\iota_P(\mu_K)$.

*Proof.* From Jaulent [17, Définition 2.11, Proposition 2.12] or [1, Theorem III.3.6.2 (vi)]. □

Note that for $S \subsetneq P$, we do not know if $\mathrm{tor}_{\mathbb{Z}_p}(\overline{E}_K^S)$ may be larger than $\iota_S(\mu_K)$ (as subgroups of $W_{K,S}$), even under the Leopoldt conjecture.

## 2.2 Diagram of $S$-ramification

Consider the following diagram under the Leopoldt conjecture for $p$ in $K$. By definition, $\mathscr{T}_{K,S} = \mathrm{tor}_{\mathbb{Z}_p}\big(\mathscr{A}_{K,S}\big)$ is the Galois group $\mathrm{Gal}(H_{K,S}/\widetilde{K}^S)$; let $\widetilde{C\ell_K}^S$ be the subgroup of $C\ell_K$ corresponding to $\mathrm{Gal}(H_K/\widetilde{K}^S \cap H_K)$ by class field theory.




Then from the schema we get:

$$\#\mathscr{T}_{K,S} = \big[H_K : \widetilde{K}^S \cap H_K\big] \cdot \#\mathrm{tor}_{\mathbb{Z}_p}\big(U_{K,S}/\overline{E}_K^S\big) = \#\widetilde{C\ell_K}^S \cdot \#\mathscr{R}_{K,S} \cdot \#\mathscr{W}_{K,S}. \tag{2.2}$$

Of course, for $p \geq p_0$ (explicit), $\#\mathscr{W}_{K,S} = \widetilde{C\ell_K}^S = 1$, whence $\mathscr{T}_{K,S} = \mathscr{R}_{K,S}$.

**Remark 2.3.** *When $S = P$, we have* $\mathrm{Gal}(H_{K,P}/H_K) \simeq U_{K,P}/\overline{E}_K^P$*, in which the image of $\mathscr{W}_{K,P}$ fixes $M_{K,P} =: H_K^{\mathrm{bp}}$, the Bertrandias–Payan field,* $\mathrm{Gal}(H_K^{\mathrm{bp}}/\widetilde{K}^P)$ *being the Bertrandias–Payan module as named by Nguyen Quang Do from [18] on the $p$-cyclic embedding problem. Then* $\mathscr{R}_{K,P} \simeq \mathrm{Gal}(H_K^{\mathrm{bp}}/\widetilde{K}^P H_K)$*. This "normalized regulator" $\mathscr{R}_{K,P}$ (as a $p$-group or as a $p$-power) is closely related to the classical $p$-adic regulator of $K$ (see [19, Proposition 5.2]).*

## 2.3 Local computations

Recall the following local computation:

**Theorem 2.4.** *[1, Theorem I.4.5 & Corollary I.4.5.4, ordinary sense]. For $\mathfrak{p} \mid p$ in $K$ and $j \geq 1$, let $U_{\mathfrak{p}}^j$ be the group of local units $1 + \overline{\mathfrak{p}}^j$, where $\overline{\mathfrak{p}}$ is the maximal ideal of the ring of integers of $K_{\mathfrak{p}}$. For $S \subseteq P$, denote by $\mathfrak{m}(S)$ the modulus $\prod_{\mathfrak{p}\in S} \mathfrak{p}^{e_{\mathfrak{p}}}$, where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$ in $K/\mathbb{Q}$.*

*For a modulus of the form $\mathfrak{m}(S)^n$, $n \geq 0$, let $C\ell_K(\mathfrak{m}(S)^n)$ be the corresponding ray class group (ordinary sense). Then for $m \geq n \geq 0$, we have:*

$$0 \leq \mathrm{rk}_p(C\ell_K(\mathfrak{m}(S)^m)) - \mathrm{rk}_p(C\ell_K(\mathfrak{m}(S)^n)) \leq \sum_{\mathfrak{p}\in S} \mathrm{rk}_p\big((U_{\mathfrak{p}}^1)^p\, U_{\mathfrak{p}}^{n\cdot e_{\mathfrak{p}}}/(U_{\mathfrak{p}}^1)^p\, U_{\mathfrak{p}}^{m\cdot e_{\mathfrak{p}}}\big).$$

**Corollary 2.5.** *[20, Theorem 2.1 & Corollary 2.2] We have:*

$$\mathrm{rk}_p(C\ell_K(\mathfrak{m}(S)^m)) = \mathrm{rk}_p(C\ell_K(\mathfrak{m}(S)^n)) = \mathrm{rk}_p(\mathscr{A}_{K,S}), \text{ for all } m \geq n \geq n_0,$$

*where $n_0 = 3$ for $p = 2$ and $n_0 = 2$ for $p > 2$. Thus $\mathscr{T}_{K,S} = 1$ if and only if $\mathrm{rk}_p(C\ell_K(\mathfrak{m}(S)^{n_0})) = \widetilde{r}_{K,S}$ ($\mathbb{Z}_p$-rank of $\mathscr{A}_{K,S}$).*

*Proof.* It is sufficient to get, for some fixed $n \geq 0$:

$$(U_{\mathfrak{p}}^1)^p\, U_{\mathfrak{p}}^{n\cdot e_{\mathfrak{p}}} = (U_{\mathfrak{p}}^1)^p, \text{ for all } \mathfrak{p} \in S,$$

hence $U_{\mathfrak{p}}^{n\cdot e_{\mathfrak{p}}} \subseteq (U_{\mathfrak{p}}^1)^p$ for all $\mathfrak{p} \in S$; indeed, we then have:

$$\mathrm{rk}_p(C\ell_K(\mathfrak{m}(S)^n)) = \mathrm{rk}_p(C\ell_K(\mathfrak{m}(S)^m)) = \widetilde{r}_{K,S} + \mathrm{rk}_p(\mathscr{T}_{K,S}) \text{ as } m \to \infty,$$

giving $\mathrm{rk}_p(C\ell_K(\mathfrak{m}(S)^n)) = \widetilde{r}_{K,S} + \mathrm{rk}_p(\mathscr{T}_{K,S})$ for such $n$.

The condition $U_{\mathfrak{p}}^{n\cdot e_{\mathfrak{p}}} \subseteq (U_{\mathfrak{p}}^1)^p$ is fulfilled as soon as $n \cdot e_{\mathfrak{p}} > \frac{p\cdot e_{\mathfrak{p}}}{p-1}$, whence $n > \frac{p}{p-1}$ (Fesenko–Vostokov [21, Chapter I, § 5.8, Corollary 2]) giving the value of $n_0$; furthermore, $C\ell_K(\mathfrak{m}(S)^{n_0})$ gives the $p$-rank of $\mathscr{T}_{K,S}$ as soon as the $\mathbb{Z}_p$-rank $\widetilde{r}_{K,S}$ is known. □

## 2.4 Practical computation of $\widetilde{r}_{K,S}$

Let $S \subseteq P$. From (1.3), we have: $\widetilde{r}_{K,S} = \sum_{\mathfrak{p}\in S}[K_{\mathfrak{p}} : \mathbb{Q}_p] - r_{K,S}$, where $r_{K,S} := \dim_{\mathbb{Q}_p}\big(\mathbb{Q}_p \log_S(E_K)\big)$.

(i) In [9, 10] Maire has given, in the relative Galois case, some results about $r_{K,S}$ depending on Schanuel's conjecture and the use of the representation $\mathbb{Q}_p\log_S(E_K)$ from the results of Jaulent [22].

(ii) In the Galois case, this rank has been studied by Nelson [23] giving formulas (or lower bounds) under the $p$-adic Schanuel conjecture.

(iii) We have proposed, in [1, III, § 4 (f)], a conjecture and a calculation process in the general non-Galois case using a Galois descent from the Galois closure $N$ of $K$ and the family of decomposition groups of the places of $N$ above $p$ and $\infty$. If $K/\mathbb{Q}$ is Galois then (with $\Sigma := P \setminus S$):

$$\mathrm{rk}_{\mathbb{Z}_p}\big(\mathrm{Gal}(\widetilde{K}^P/\widetilde{K}^S)\big) = \sum_{\mathfrak{p}\in\Sigma}[K_{\mathfrak{p}} : \mathbb{Q}_p] - \dim_{\mathbb{Q}_p}\big(\mathbb{Q}_p\log_P(\mathscr{E}_{K,S})\big),$$

where:

$$\mathscr{E}_{K,S} := \big\{\varepsilon \in E_K \otimes \mathbb{Z}_p,\ \iota_{\mathfrak{p}}(\varepsilon) = 1,\ \forall \mathfrak{p} \in S\big\} \quad \& \quad \iota_{\mathfrak{p}} : E_K \otimes \mathbb{Z}_p \to U_{\mathfrak{p}}^1.$$

But all these similar approaches are difficult for programming and not so obvious for random $K$ and $S$ because of conjectural aspects; so we shall preferably give extensive computations via PARI/GP [24] since ray class fields are well computed. But it remains the problem of justification of the "computing" of $\widetilde{r}_{K,S}$, when no theoretical value is known (see another explicit numerical method in [1, § III.5, Theorem 5.2]).

We conclude by the following comments:

**Remark 2.6.** *If $\mathscr{T}_{K,P} = 1$ (i.e., the field $K$ is called $p$-rational as proposed by Movahhedi in [25, 26]), this does not imply $\mathscr{T}_{K,S} = 1$ for $S \subsetneq P$ (the numerical examples will show many cases). In the opposite situation, we may have $\mathscr{T}_{K,P} \neq 1$, but often $\mathscr{T}_{K,S} = 1$ for $S \subsetneq P$.*

*This intricate aspects have been studied by Maire [11, Section 3] in which he introduces the "$S$-cohomologcal condition" $\mathrm{H}^2(\mathscr{G}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ (knowing that $\mathscr{G}_{K,S}$ is a free prop-group if and only if $\mathrm{H}^2(\mathscr{G}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p)$ and $\mathscr{T}_{K,S}$ are trivial) and that of "$S$-arithmetical condition" ($E_K \otimes \mathbb{Z}_p \to U_{K,S}$ injective), and compare them, which of course coincide for $S = P$; we know that the $S$-arithmetical condition implies the $S$-cohomologcal one.*

*We shall speak of $S$-rationality, when $\mathscr{T}_{K,S} = 1$ for $S \subseteq P$, even if this may be rather ambiguous when $S \subsetneq P$ because of the above observations; one must understand this as a "free $S$-ramification" over $K$ (i.e., giving a free abelian $S$-ramified pro-$p$-extension $H_{K,S}/K$). This is also justified by the fact that many variants of the definition have been given, as those of Jaulent–Sauzet [27, 28], Bourbon–Jaulent [29], where are defined and studied the case of singleton $S = \{\mathfrak{p}\}$ or that of the "2-birationality" of quadratic extensions of totally real fields when $S = \{\mathfrak{p}, \mathfrak{p}'\}$.*

# 3. Algorithmic approach of $S$-ramification

The principle is to consider a modulus $\mathfrak{m}_S := \prod_{\mathfrak{p}\in S}\mathfrak{p}^{\lambda_{\mathfrak{p}}}$, $S \subseteq P$, with $\lambda_{\mathfrak{p}} \gg 0$ for all $\mathfrak{p} \in S$ to "read" the structure of $\mathscr{A}_{K,S}$ on the ray class group $C\ell_K(\mathfrak{m}_S)$. The practice shows that the more convenient modulus is of the form:

$$\Big(\prod_{\mathfrak{p}\in S}\mathfrak{p}^{e_{\mathfrak{p}}}\Big)^n,$$

where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$ in $K/\mathbb{Q}$ and $n \gg 0$. Of course, this modulus is $(p^n)$ only for $S = P$; so we must use the ideal decomposition of $p$ in $K$, given by PARI/GP, and compute everywhere with ideals.

## 3.1 Main program computing $\mathscr{T}_{K,S}$ and $\widetilde{r}_{K,S}$

### 3.1.1 The PARI/GP program

```
==========================================================================================
{P=x^3+197*x^2+718*x+508;if(polisirreducible(P)==0,break);print(P);bp=2;Bp=5000;
n0=6;K=bnfinit(P,1);forprime(p=bp,Bp,n=n0+floor(30/p);print();print("p=",p);
F=idealfactor(K,p);d=matsize(F)[1];F1=component(F,1);for(j=1,d,print(F1[j]));
for(z=2^d,2^(d+1)-1,bin=binary(z);mod=List;for(j=1,d,listput(mod,bin[j+1],j));
M=1;for(j=1,d,ch=mod[j];if(ch==1,F1j=F1[j];ej=F1j[3];F1j=idealpow(K,F1j,ej);
M=idealmul(K,M,F1j)));Idn=idealpow(K,M,n);Kpn=bnrinit(K,Idn); \\ ray class field
Hpn=Kpn.cyc;L=List;e=matsize(Hpn)[2];R=0;for(k=1,e,c=Hpn[e-k+1];
w=valuation(c,p);if(w>0,R=R+1;listinsert(L,p^w,1)));
print("S=",mod," rk(A_S)=",R," A_S=",L)))}
==========================================================================================
```

### 3.1.2 Instructions for use and illustrations

See the Note at the end of Section A.8. The reader has only to copy and past the verbatim of the program and to use a "terminal session via Sage", on his or her computer, or a cell in the page `http://pari.math.u-bordeaux.fr/gp.html`

The programs in this article can be directly copied and pasted at:

`https://www.dropbox.com/s/1srmksbr2ujf40i/Incomplete%20p-ramification.pdf?dl=0`

It is assumed that the irreducible monic polynomial $\mathsf{P}$ defining $K$ is given and that the interval $[\mathsf{bp},\mathsf{Bp}]$ of tested primes $p$ is also given by the user.

(i) The program computes the decomposition of $p$ into $\mathsf{d}$ prime ideals; for instance, the following data gives, for $\mathsf{P}=\mathsf{x}^3+197*\mathsf{x}^2+718*\mathsf{x}+508$ and $p=2$, the decomposition $(p)=\mathfrak{p}\mathfrak{p}'$ in $\mathbb{Q}(x)$, using $\mathsf{idealfactor(K,p)}$:

```
[2, [-65, 0, 1]~, 1, 1, [0, 0, -1]~]
[2, [0, 0, 1]~, 1, 2, [0, 1, 0]~]
```

Recall that for an ideal as $[2,[0,0,1]^\sim,1,2,[0,1,0]^\sim]$, the 3th component is its ramification index, the 4th component is its residue degree. For the computation of the modulus $\mathfrak{m}_S$ (to be considered at the power $n$), we replace each prime ideal $\mathfrak{p}\in S$ by $\mathfrak{p}^{e_\mathfrak{p}}$ using the function $\mathsf{idealpow}$.

(ii) For each modulus $\mathfrak{m}_S=\prod_{\mathfrak{p}\in S}\mathfrak{p}^{e_\mathfrak{p}\cdot n}$, the program gives $\mathrm{rk}_p(\mathscr{A}_{K,S})$ and the $\mathbb{Z}$-structure of $\mathscr{A}_{K,S}/\mathscr{A}_{K,S}^{p^N}$, for $N$ of the order of $n$, under the form:

$$\mathscr{A}_{K,S}=[a_1,\ldots,a_r;\ b_1,\ldots,b_t],$$

where the coefficients $a_1,\ldots,a_r$ increase (resp. the coefficients $b_1,\ldots,b_t$ stabilize) as the exponent $n$ increses, so in the non-ambiguous cases, $b_1,\ldots,b_t$ give the group-invariants of $\mathscr{T}_{K,S}$ and $r$ is the $p$-rank $\widetilde{r}_{K,S}$ of $\mathrm{Gal}(\widetilde{K}^S/K)$.

Of course, if the rank $\widetilde{r}_{K,S}$ is not certain, we can not, in a mathematical point of view, deduce the structure of $\mathscr{T}_{K,S}$; but in practice the information is correct since one can always verify, with the program, the stabilization of the invariants $b_j$ whereas the $a_i$ increase linearly to infinity.

(iii) The symbolic data $S=[\delta_1,\ldots,\delta_d]$, $\delta_i\in\{0,1\}$, indicates that the $S$-modulus considered is:

$$\mathfrak{m}_S=\Big(\prod_{i=1}^{d}\mathfrak{p}_i^{e_{\mathfrak{p}_i}\cdot\delta_i}\Big)^n.$$

We have chosen $\mathsf{n}=\mathsf{floor}\big(\mathsf{n}_0+\frac{30}{\mathsf{p}}\big)$ to get small values when $p\gg 0$ but larger ones for small $p$ (especially $p=2$ giving possibly huge $\#\mathscr{T}_{K,S}$). The parameter $n_0$ may be increased at will (here $n_0=6$).

There are $2^{\#S}$ distinct sets $S$ parametrized with the binary writing of the integers $z\in[0,2^d-1]$.

For $S=[0,\ldots,0]$ one obtains the structure of the $p$-class group $\mathcal{C}\ell_K$.

(iv) We illustrate the program with an example where $K$ (a totally real cubic field) is not $S$-rational for some small $p$ and some $S\subseteq P$; but in almost all cases, $K$ is $S$-rational.

**Remark 3.1.** *We do not compute the Galois group associated to the given polynomial, nor the discriminant or the fundamental units; otherwise, the reader has only to add if necessary the instructions:*

```
print("Galois :",polgalois(P));
print("Discriminant: ",factor(component (component(K,7), 3)));
print("Fundamental system of units: ",component(component(K,8),5));
```

*giving, for the Galois group and the discriminant:*

Galoisgroup $= [6, -1, 1, \text{"S3"}]$ *in the PARI/GP notation*[2] *and* Discriminant $= [769, 1; 1390573, 1]$).

```
P=x^3 + 197*x^2 + 718*x + 508
p=2
[2, [-65, 0, 1]~, 1, 1, [0, 0, -1]~]
[2, [0, 0, 1]~, 1, 2, [0, 1, 0]~]
S=[0, 0] rk(A_S)=0 A_S=[]
S=[0, 1] rk(A_S)=1 A_S=[4]
S=[1, 0] rk(A_S)=0 A_S=[]
S=[1, 1] rk(A_S)=3 A_S=[274877906944, 4, 2]
p=3
[3, [3, 0, 0]~, 1, 3, 1]
S=[0] rk(A_S)=0 A_S=[]
S=[1] rk(A_S)=2 A_S=[22876792454961, 3]
p=5
[5, [-68, 0, 1]~, 1, 1, [-1, 2, -1]~]
[5, [12589, 2, -196]~, 1, 2, [2, 0, 1]~]
S=[0, 0] rk(A_S)=0 A_S=[]
S=[0, 1] rk(A_S)=1 A_S=[390625]
S=[1, 0] rk(A_S)=0 A_S=[]
S=[1, 1] rk(A_S)=2 A_S=[19073486328125, 390625]
p=7
[7, [-65, 0, 1]~, 1, 1, [3, 2, 1]~]
[7, [12519, 2, -195]~, 1, 2, [-2, 0, 1]~]
S=[0, 0] rk(A_S)=0 A_S=[]
S=[0, 1] rk(A_S)=1 A_S=[7]
S=[1, 0] rk(A_S)=0 A_S=[]
S=[1, 1] rk(A_S)=2 A_S=[33232930569601, 7]
p=11
[11, [11, 0, 0]~, 1, 3, 1]
S=[0] rk(A_S)=0 A_S=[]
S=[1] rk(A_S)=2 A_S=[3138428376721, 11]
p=13
[13, [13, 0, 0]~, 1, 3, 1]
S=[0] rk(A_S)=0 A_S=[]
S=[1] rk(A_S)=1 A_S=[1792160394037]
(...)
p=127
[127, [-66, 0, 1]~, 1, 1, [-16, 2, 2]~]
[127, [16240, 2, -252]~, 1, 2, [61, 0, 1]~]
S=[0, 0] rk(A_S)=0 A_S=[]
S=[0, 1] rk(A_S)=1 A_S=[127]
S=[1, 0] rk(A_S)=0 A_S=[]
S=[1, 1] rk(A_S)=2 A_S=[532875860165503, 127]
p=1571
[1571, [275, 0, 1]~, 1, 1, [-418, 2, -339]~]
[1571, [21576, 2, -339]~, 1, 2, [275, 0, 1]~]
S=[0, 0] rk(A_S)=0 A_S=[]
S=[0, 1] rk(A_S)=1 A_S=[1571]
S=[1, 0] rk(A_S)=0 A_S=[]
S=[1, 1] rk(A_S)=2 A_S=[2361746580786556 1078891, 1571]
p=1759
[1759, [1759, 0, 0]~, 1, 3, 1]
S=[0, 0] rk(A_S)=0 A_S=[]
S=[1] rk(A_S)=2 A_S=[5210277760467996 3122719, 1759]
p=3371
[3371, [-295, 0, 1]~, 1, 1, [-1597, 2, 231]~]
[3371, [-121, 0, 1]~, 1, 1, [355, 2, 57]~]
[3371, [415, 0, 1]~, 1, 1, [38, 2, -479]~]
S=[0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1] rk(A_S)=0 A_S=[]
S=[0, 1, 0] rk(A_S)=0 A_S=[]
S=[0, 1, 1] rk(A_S)=1 A_S=[3371]
S=[1, 0, 0] rk(A_S)=0 A_S=[]
S=[1, 0, 1] rk(A_S)=1 A_S=[3371]
S=[1, 1, 0] rk(A_S)=1 A_S=[3371]
S=[1, 1, 1] rk(A_S)=2 A_S=[4946650964538063853923491, 3371]
```

[2]See: http://galoisdb.math.upb.de/home

If, for the remarquable case $p = 5$, one has some doubt, one increases $n$, which gives (for $n = 50$):

```
[5, [-68, 0, 1]~, 1, 1, [-1, 2, -1]~]
[5, [12589, 2, -196]~, 1, 2, [2, 0, 1]~]
S=[0, 0] rk(A_S)=0 A_S=[]
S=[0, 1] rk(A_S)=1 A_S=[390625]
S=[1, 0] rk(A_S)=0 A_S=[]
S=[1, 1] rk(A_S)=2 A_S=[177635683940025046467781066894531 25, 390625]
```

Whence $\mathcal{T}_{K,S} \simeq \mathbb{Z}/5^8\mathbb{Z}$ for $S_1 = \{\mathfrak{p}\}$ (for the prime of residue degree 2) and $S_2 = P$. Note that once the substantial computation of $\mathsf{K} = \mathsf{bnfinit(P,1)}$ (giving all the basic information about the field) is done, very large values of $n$ do not increase much the execution time; so any skeptical user can make $n \to \infty$ to see that only the data 390625 remains constant.

(v) In [30, § 9.1] we have used some special families of polynomials (e.g., Lecacheux–Washington ones) in which we can force the $p$-adic regulator to be $p$-adicaly close to 0 at will; but we must take the parameter $n$ in proportion, even if here the $\mathbb{Z}_p$-ranks of the $\mathscr{A}_{K,S}$ are obvious, since $K$ is totally real, giving finite groups except for $S = P$ where $\mathrm{rk}_{\mathbb{Z}_p}(\mathscr{A}_{K,P}) = 1$:

```
P=x^3-134480895*x^2-263169*x-1
p=2
[2, [0, 0, 1]~, 1, 1, [1, 0, 1]~]
[2, [0, 1, 0]~, 1, 1, [1, 1, 0]~]
[2, [2, 1, 1]~, 1, 1, [1, 1, 1]~]
S=[0, 0, 0] rk(A_S)=6 A_S=[16, 16, 2, 2, 2, 2]
S=[0, 0, 1] rk(A_S)=6 A_S=[512, 16, 8, 2, 2, 2]
S=[0, 1, 0] rk(A_S)=6 A_S=[512, 16, 8, 2, 2, 2]
S=[0, 1, 1] rk(A_S)=6 A_S=[1024, 512, 8, 8, 2, 2]
S=[1, 0, 0] rk(A_S)=6 A_S=[512, 16, 8, 2, 2, 2]
S=[1, 0, 1] rk(A_S)=6 A_S=[1024, 512, 8, 8, 2, 2]
S=[1, 1, 0] rk(A_S)=6 A_S=[1024, 512, 8, 8, 2, 2]
S=[1, 1, 1] rk(A_S)=7 A_S=[9444732965739290427392, 1024, 1024, 8, 8, 2, 2]

x^3-7625984944841*x^2-387459856*x-1
p=3
[3, [1, -1, -1]~, 1, 1, [0, 1, 1]~]
[3, [2, 1, 0]~, 1, 1, [1, 1, 0]~]
[3, [2541994975055, -19683, 1]~, 1, 1, [-1, 0, -1]~]
S=[0, 0, 0] rk(A_S)=4 A_S=[27, 9, 3, 3]
S=[0, 0, 1] rk(A_S)=4 A_S=[177147, 9, 3, 3]
S=[0, 1, 0] rk(A_S)=4 A_S=[177147, 9, 3, 3]
S=[0, 1, 1] rk(A_S)=4 A_S=[177147, 59049, 3, 3]
S=[1, 0, 0] rk(A_S)=4 A_S=[177147, 9, 3, 3]
S=[1, 0, 1] rk(A_S)=4 A_S=[177147, 59049, 3, 3]
S=[1, 1, 0] rk(A_S)=4 A_S=[177147, 59049, 3, 3]
S=[1, 1, 1] rk(A_S)=5 A_S=[834385168331080533771857328695283, 177147, 59049, 3, 3]
P=x^3-1628427439432947*x^2-13841522500*x-1
p=7
[7, [1, -3, -3]~, 1, 1, [0, 1, 1]~]
[7, [4, 3, 0]~, 1, 1, [1, 1, 0]~]
[7, [542809146438439, -117649, 1]~, 1, 1, [2, 0, 2]~]
S=[0, 0, 0] rk(A_S)=2 A_S=[7, 7]
S=[0, 0, 1] rk(A_S)=2 A_S=[117649, 7]
S=[0, 1, 0] rk(A_S)=2 A_S=[117649, 7]
S=[0, 1, 1] rk(A_S)=3 A_S=[117649, 16807, 7]
S=[1, 0, 0] rk(A_S)=2 A_S=[117649, 7]
S=[1, 0, 1] rk(A_S)=3 A_S=[117649, 16807, 7]
S=[1, 1, 0] rk(A_S)=3 A_S=[117649, 16807, 7]
S=[1, 1, 1] rk(A_S)=4 A_S=[3219905755813179726837607, 117649, 16807, 7]
```

### 3.1.3 Example with $p$ totally split in degree 5

For $P = x^5 - 5$, $n_0 = 8$, and $p = 31$ (totally split) one finds one case of non $S$–rationality:
$\mathsf{S} = [1,0,0,0,1]$ $\mathsf{rk(A_S)} = 1$ $\mathsf{A_S} = [961]$, i.e., $\widetilde{r}_{K,S} = 0$, $\mathcal{T}_{K,S} \simeq \mathbb{Z}/31^2\mathbb{Z}$:

```
[31, [-14, 1, 0, 0, 0]~, 1, 1, [7, -15, 10, 14, 1]~]
[31, [-7, 1, 0, 0, 0]~, 1, 1, [14, 2, -13, 7, 1]~]
[31, [3, 1, 0, 0, 0]~, 1, 1, [-12, 4, 9, -3, 1]~]
[31, [6, 1, 0, 0, 0]~, 1, 1, [-6, 1, 5, -6, 1]~]
[31, [12, 1, 0, 0, 0]~, 1, 1, [-3, 8, -11, -12, 1]~]
```

```
S=[0, 0, 0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 0, 0, 1] rk(A_S)=0 A_S=[]
S=[0, 0, 0, 1, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 0, 1, 1] rk(A_S)=0 A_S=[]
S=[0, 0, 1, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1, 0, 1] rk(A_S)=0 A_S=[]
S=[0, 0, 1, 1, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1, 1, 1] rk(A_S)=1 A_S=[27512614111]
S=[0, 1, 0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 1, 0, 0, 1] rk(A_S)=0 A_S=[]
S=[0, 1, 0, 1, 0] rk(A_S)=0 A_S=[]
S=[0, 1, 0, 1, 1] rk(A_S)=1 A_S=[27512614111]
S=[0, 1, 1, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 1, 1, 0, 1] rk(A_S)=1 A_S=[27512614111]
S=[0, 1, 1, 1, 0] rk(A_S)=1 A_S=[27512614111]
S=[0, 1, 1, 1, 1] rk(A_S)=2 A_S=[27512614111, 27512614111]
S=[1, 0, 0, 0, 0] rk(A_S)=0 A_S=[]
S=[1, 0, 0, 0, 1] rk(A_S)=1 A_S=[961]
S=[1, 0, 0, 1, 0] rk(A_S)=0 A_S=[]
S=[1, 0, 0, 1, 1] rk(A_S)=1 A_S=[27512614111]
S=[1, 0, 1, 0, 0] rk(A_S)=0 A_S=[]
S=[1, 0, 1, 0, 1] rk(A_S)=1 A_S=[27512614111]
S=[1, 0, 1, 1, 0] rk(A_S)=1 A_S=[27512614111]
S=[1, 0, 1, 1, 1] rk(A_S)=2 A_S=[27512614111, 27512614111]
S=[1, 1, 0, 0, 0] rk(A_S)=0 A_S=[]
S=[1, 1, 0, 0, 1] rk(A_S)=1 A_S=[27512614111]
S=[1, 1, 0, 1, 0] rk(A_S)=1 A_S=[27512614111]
S=[1, 1, 0, 1, 1] rk(A_S)=2 A_S=[27512614111, 27512614111]
S=[1, 1, 1, 0, 0] rk(A_S)=1 A_S=[27512614111]
S=[1, 1, 1, 0, 1] rk(A_S)=2 A_S=[27512614111, 27512614111]
S=[1, 1, 1, 1, 0] rk(A_S)=2 A_S=[27512614111, 27512614111]
S=[1, 1, 1, 1, 1] rk(A_S)=3 A_S=[27512614111, 27512614111, 27512614111]
```

### 3.1.4 Example with $p$ totally split in degree 7

For the polynomial $P = x^7 - 7$ and $p = 43$, one finds two cases:

```
[43, [-18, 1, 0, 0, 0, 0, 0]~, 1, 1, [-2, 19, 13, -16, -20, 18, 1]~]
[43, [-7, 1, 0, 0, 0, 0, 0]~, 1, 1, [1, -6, -7, -1, 6, 7, 1]~]
[43, [9, 1, 0, 0, 0, 0, 0]~, 1, 1, [4, -10, -18, 2, -5, -9, 1]~]
[43, [13, 1, 0, 0, 0, 0, 0]~, 1, 1, [16, 12, 9, -4, -3, -13, 1]~]
[43, [14, 1, 0, 0, 0, 0, 0]~, 1, 1, [21, 20, 17, 8, -19, -14, 1]~]
[43, [15, 1, 0, 0, 0, 0, 0]~, 1, 1, [11, 5, 14, -21, 10, -15, 1]~]
[43, [17, 1, 0, 0, 0, 0, 0]~, 1, 1, [-8, 3, 15, -11, -12, -17, 1]~]
(...)
S=[0, 1, 0, 1, 0, 0, 1] rk(A_S)=1 A_S=[43]
S=[1, 1, 0, 0, 1, 0, 0] rk(A_S)=1 A_S=[43]
```

i.e., $\widetilde{r}_{K,S} = 0$ and $\mathscr{T}_{K,S} \simeq \mathbb{Z}/43\mathbb{Z}$ for the two above cases. For the other modulus, $\mathscr{T}_{K,S} = 1$.

### 3.1.5 Example with a field discovered by Jaulent–Sauzet

In [27], some numerical examples of $\{\mathfrak{l}\}(= \{\mathfrak{p}\})$-rational fields, which are not $p$-rational, are given; of course this corresponds to a suitable choice of $S = \{\mathfrak{p}\}$ and we give the case of the field defined by the polynomial:

$$P = x^{10} + 19x^8 + 8x^7 + 130x^6 + 16x^5 + 166x^4 - 888x^3 - 15x^2 + 432x + 243$$

for $p = 3$:

```
[3, [-1, 1, 0, 0, 1, 1, -1, 0, 0, -1]~, 2, 1, [2, 0, 2, 1, 2, 0, 1, 1, 2, 1]~]
[3, [-1, 1, 0, 1, 1, 0, -1, 0, 0, -1]~, 2, 1, [2, 0, 1, 2, 1, 2, 1, 1, 2, 1]~]
[3, [-5, 14, -4, -2, 5, 5, 13, -13, 2, 6]~, 2, 3, [0, 1, 1, 1, -1, -1, -1, -1, -1, 1]~]

S=[0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1] rk(A_S)=2 A_S=[14348907,14348907]
S=[0, 1, 0] rk(A_S)=0 A_S=[]
S=[0, 1, 1] rk(A_S)=5 A_S=[14348907,14348907,14348907,14348907, 3]
S=[1, 0, 0] rk(A_S)=0 A_S=[]
S=[1, 0, 1] rk(A_S)=5 A_S=[14348907,14348907,14348907,14348907, 3]
S=[1, 1, 0] rk(A_S)=1 A_S=[27]
S=[1, 1, 1] rk(A_S)=8 A_S=[14348907,14348907,14348907,14348907,14348907,14348907, 3, 3]
```

which is indeed $\{\mathfrak{p}\}$-rational for each prime ideal $\mathfrak{p}$, but the field is not 3-rational since $\mathscr{T}_{K,P} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Note the case $\mathscr{A}_{K,S} = \mathscr{T}_{K,S} \simeq \mathbb{Z}/27\mathbb{Z}$.

Many other numerical examples are available in [27, § 3.c].

### 3.1.6 Abelian fields with $\mathscr{T}_{K,S} = 1$ but $\mathscr{T}_{K,P} \neq 1$

We consider for this the cyclotomic field $\mathbb{Q}(\mu_{24})$. The following program may be used for any abelian field given by polcyclo(N) or polsubcyclo(N, d) giving the suitable polynomials of degree $d$ dividing $\varphi(N)$:

```
{P=polcyclo(24);bp=2;Bp=500;n0=8;K=bnfinit(P,1);forprime(p=bp,Bp,
n=n0+floor(30/p);print();print("p=",p);F=idealfactor(K,p);d=matsize(F)[1];
F1=component(F,1);for(j=1,d,print(F1[j]));for(z=2^d,2*2^d-1,bin=binary(z);
mod=List;for(j=1,d,listput(mod,bin[j+1],j));M=1;for(j=1,d,ch=mod[j];if(ch==1,F1j=F1[j];
ej=F1j[3];FF1j=idealpow(K,F1j,ej);M=idealmul(K,M, FF1j)));Idn=idealpow(K,M,n);
Kpn=bnrinit(K,Idn);Hpn=Kpn.cyc;L=List;e=matsize(Hpn)[2];R=0;
for(k=1,e,c=Hpn[e-k+1];w=valuation(c,p);if(w>0,R=R+1;listinsert(L,p^w,1)));
print("S=",mod," rk(A_S)=",R," A_S=",L)))}
```

```
p=3
[3, [-1, 0, -1, 0, 1, 0, 0, 0]~, 2, 2, [-1, -1, 1, 1, 1, 1, 0, 0]~]
[3, [-1, 0, 1, 0, 1, 0, 0, 0]~, 2, 2, [-1, -1, -1, -1, 1, 1, 0, 0]~]
S=[0, 0] rk(A_S)=0 A_S=[]
S=[0, 1] rk(A_S)=1 A_S=[22876792454961]
S=[1, 0] rk(A_S)=1 A_S=[22876792454961]
S=[1, 1] rk(A_S)=6 A_S=[68630377364883,22876792454961,22876792454961,22876792454961,22876792454961, 3]

p=7
[7, [-3, 0, -1, 0, 1, 0, 0, 0]~, 1, 2, [2, -3, -3, 1, -3, 1, 0, 0]~]
[7, [-3, 0, 1, 0, 1, 0, 0, 0]~, 1, 2, [2, -3, 3, -1, -3, 1, 0, 0]~]
[7, [2, 0, -2, 0, 1, 0, 0, 0]~, 1, 2, [-3, 2, -3, 2, 2, 1, 0, 0]~]
[7, [2, 0, 2, 0, 1, 0, 0, 0]~, 1, 2, [-3, 2, 3, -2, 2, 1, 0, 0]~]
S=[0, 0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 0, 1] rk(A_S)=0 A_S=[]
S=[0, 0, 1, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1, 1] rk(A_S)=2 A_S=[4747561509943, 7]
S=[0, 1, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 1, 0, 1] rk(A_S)=2 A_S=[4747561509943,4747561509943]
S=[0, 1, 1, 0] rk(A_S)=2 A_S=[4747561509943, 7]
S=[0, 1, 1, 1] rk(A_S)=4 A_S=[4747561509943,4747561509943,4747561509943, 7]
S=[1, 0, 0, 0] rk(A_S)=0 A_S=[]
S=[1, 0, 0, 1] rk(A_S)=2 A_S=[4747561509943, 7]
S=[1, 0, 1, 0] rk(A_S)=2 A_S=[4747561509943,4747561509943]
S=[1, 0, 1, 1] rk(A_S)=4 A_S=[4747561509943,4747561509943,4747561509943, 7]
S=[1, 1, 0, 0] rk(A_S)=2 A_S=[4747561509943, 7]
S=[1, 1, 0, 1] rk(A_S)=4 A_S=[4747561509943,4747561509943,4747561509943, 7]
S=[1, 1, 1, 0] rk(A_S)=4 A_S=[4747561509943,4747561509943,4747561509943, 7]
S=[1, 1, 1, 1] rk(A_S)=6 A_S=[4747561509943,4747561509943,4747561509943,4747561509943,4747561509943, 7]

p=13
[13, [-6, 0, 0, 0, 1, 0, 0, 0]~, 1, 2, [2, 6, 0, 0, -4, 1, 0, 0]~]
[13, [-2, 0, 0, 0, 1, 0, 0, 0]~, 1, 2, [6, 2, 0, 0, 3, 1, 0, 0]~]
[13, [2, 0, 0, 0, 1, 0, 0, 0]~, 1, 2, [-6, -2, 0, 0, 3, 1, 0, 0]~]
[13, [6, 0, 0, 0, 1, 0, 0, 0]~, 1, 2, [-2, -6, 0, 0, -4, 1, 0, 0]~]
S=[0, 0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 0, 1] rk(A_S)=0 A_S=[]
S=[0, 0, 1, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1, 1] rk(A_S)=2 A_S=[1792160394037,13]
S=[0, 1, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 1, 0, 1] rk(A_S)=2 A_S=[1792160394037,1792160394037]
S=[0, 1, 1, 0] rk(A_S)=2 A_S=[1792160394037,13]
S=[0, 1, 1, 1] rk(A_S)=4 A_S=[1792160394037,1792160394037,1792160394037,13]
S=[1, 0, 0, 0] rk(A_S)=0 A_S=[]
S=[1, 0, 0, 1] rk(A_S)=2 A_S=[1792160394037,13]
S=[1, 0, 1, 0] rk(A_S)=2 A_S=[1792160394037,1792160394037]
S=[1, 0, 1, 1] rk(A_S)=4 A_S=[1792160394037,1792160394037,1792160394037,13]
S=[1, 1, 0, 0] rk(A_S)=2 A_S=[1792160394037,13]
S=[1, 1, 0, 1] rk(A_S)=4 A_S=[1792160394037,1792160394037,1792160394037,13]
S=[1, 1, 1, 0] rk(A_S)=4 A_S=[1792160394037,1792160394037,1792160394037,13]
S=[1, 1, 1, 1] rk(A_S)=6 A_S=[1792160394037,1792160394037,1792160394037,1792160394037,1792160394037,13]
```

### 3.2 Experiments with the fields $K = \mathbb{Q}(\sqrt[p]{N})$, $N$ prime

These fields are studied in great detail by Lecouturier in [31, § 5] for their $p$-class groups and these fields have some remarkable properties. For instance if log is the discrete logarithm for $(\mathbb{Z}/p\mathbb{Z})^\times$ provided with a primitive root $g$, the expression $T = \sum_{k=1}^{(N-1)/2} k \cdot \log(k) \pmod p$ governs, under some conditions, the $p$-rank of $C\ell_K$ (from a result of Calegari–Emerton, after other similar results of Iimura, proved again in [31, Theorem 1.1]) and improved by Schaefer–Stubley[32].

So we shall give the general calculations, for all $S \subseteq P$, with that of $T$. We assume $N$ prime congruent to 1 modulo $p$, but the reader may suppress this condition. It seems that many interesting heuristics can be elaborated from the numerical results; we only give some examples (recall that the structure of the class group is given by the first data $S = \emptyset$):

```
{p=3;print("p=",p);n=8+floor(30/p);g=znprimroot(p);forprime(N=1,10^3,
if(Mod(N,p)!=1,next);P=x^p-N;print();print("P=",P);T=Mod(0,p);
for(k=1,(N-1)/2,if(Mod(k,p)==0,next);T=T+k*znlog(k,g));K=bnfinit(P,1);
F=idealfactor(K,p);d=matsize(F)[1];F1=component(F,1);
for(j=1,d,print(F1[j]));for(z=2^d,2*2^d-1,bin=binary(z);mod=List;
for(j=1,d,listput(mod,bin[j+1],j));M=1;for(j=1,d,ch=mod[j];
if(ch==1,F1j=F1[j];ej=F1j[3];F1j=idealpow(K,F1j,ej);
M=idealmul(K,M,F1j)));Idn=idealpow(K,M,n);Kpn=bnrinit(K,Idn);
Hpn=Kpn.cyc;L=List;e=matsize(Hpn)[2];R=0;
for(k=1,e,c=Hpn[e-k+1];w=valuation(c,p);if(w>0,R=R+1;
listinsert(L,p^w,1)));print("S=",mod," rk(A_S)=",R," A_S=",L)))}
```

```
p=3
P=x^3 - 7
[3, [-1, 1, 0]~, 3, 1, [1, 1, 1]~]
T=Mod(2,3) S=[0] rk(A_S)=1 A_S=[3]
T=Mod(2,3) S=[1] rk(A_S)=2 A_S=[387420489,387420489]
P=x^3 - 271
[3, [-2, 0, -1]~, 1, 1, [0, 0, 1]~]
[3, [-1, 1, 1]~, 2, 1, [2, 1, 0]~]
T=Mod(0,3) S=[0,0] rk(A_S)=1 A_S=[9]
T=Mod(0,3) S=[0,1] rk(A_S)=3 A_S=[129140163, 27, 3]
T=Mod(0,3) S=[1,0] rk(A_S)=2 A_S=[9, 3]
T=Mod(0,3) S=[1,1] rk(A_S)=4 A_S=[129140163,129140163, 27, 3]
P=x^3 - 523
[3, [0, 0, 1]~, 2, 1, [2, 1, 0]~]
[3, [1, 0, -1]~, 1, 1, [2, 1, 1]~]
T=Mod(0,3) S=[0,0] rk(A_S)=1 A_S=[9]
T=Mod(0,3) S=[0,1] rk(A_S)=2 A_S=[9, 3]
T=Mod(0,3) S=[1,0] rk(A_S)=3 A_S=[387420489, 9, 3]
T=Mod(0,3) S=[1,1] rk(A_S)=4 A_S=[387420489,129140163, 9, 3]

p=5
P=x^5 - 11
[5, [-1, 1, 0, 0, 0]~, 5, 1, [1, 1, 1, 1, 1]~]
T=Mod(4,5) S=[0] rk(A_S)=1 A_S=[5]
T=Mod(4,5) S=[1] rk(A_S)=3 A_S=[30517578125,6103515625,6103515625]
P=x^5 - 211
[5, [-1, 1, 0, 0, 0]~, 5, 1, [1, 1, 1, 1, 1]~]
T=Mod(4,5) S=[0] rk(A_S)=3 A_S=[5, 5, 5]
T=Mod(4,5) S=[1] rk(A_S)=5 A_S=[6103515625,6103515625,6103515625, 5, 5]
P=x^5 - 401
[5, [-1, 1, 0, 1, 0]~, 4, 1, [4, 3, 2, 0, 1]~]
[5, [1, 0, 0, -1, 0]~, 1, 1, [4, 3, 2, 1, 1]~]
T=Mod(0,5) S=[0,0] rk(A_S)=2 A_S=[5, 5]
T=Mod(0,5) S=[0,1] rk(A_S)=2 A_S=[25, 5]
T=Mod(0,5) S=[1,0] rk(A_S)=3 A_S=[6103515625,6103515625, 25]
T=Mod(0,5) S=[1,1] rk(A_S)=4 A_S=[6103515625,6103515625,1220703125, 25]

p=7
P=x^7 - 29
[7, [-1, 1, 0, 0, 0, 0, 0]~, 7, 1, [1, 1, 1, 1, 1, 1, 1]~]
T=Mod(6,7) S=[0] rk(A_S)=1 A_S=[7]
T=Mod(6,7) S=[1] rk(A_S)=4 A_S=[96889010407,13841287201,13841287201,13841287201]
P=x^7 - 197
[7, [0, 0, 0, 0, 0, 0, 1]~, 1, 1, [6, 5, 4, 3, 3, 2, 1]~]
[7, [1, 0, 0, 0, 0, 0, -1]~, 6, 1, [6, 5, 4, 3, 1, 2, 1]~]
```

```
T=Mod(0,7) S=[0,0] rk(A_S)=1 A_S=[7]
T=Mod(0,7) S=[0,1] rk(A_S)=4 A_S=[96889010407,13841287201, 1977326743, 49]
T=Mod(0,7) S=[1,0] rk(A_S)=1 A_S=[7]
T=Mod(0,7) S=[1,1] rk(A_S)=5 A_S=[96889010407,13841287201,1977326743,1977326743, 49]
P=x^7 - 337
[7, [-1, 1, 0, 0, 0, 0, 0]~, 7, 1, [1, 1, 1, 1, 1, 1, 1]~]
T=Mod(2,7) S=[0] rk(A_S)=2 A_S=[7, 7]
T=Mod(2,7) S=[1] rk(A_S)=5 A_S=[13841287201,13841287201,13841287201,13841287201, 7]

p=11
P=x^11 - 67
[11, [-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0]~, 11, 1, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]~]
T=Mod(8,11) S=[0] rk(A_S)=2 A_S=[11, 11]
T=Mod(8,11) S=[1] rk(A_S)=7 A_S=[285311670611,285311670611,25937424601,
                                          25937424601,25937424601,25937424601, 11]
P=x^11 - 727
[11, [-5, 0, 0, 0, 0, 0, 0, 0, 0, 0, -5]~, 1, 1, [10, 9, 8, 7, 6, 5, 4, 6, 3, 2, 1]~]
[11, [-5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5]~, 10, 1, [10, 9, 8, 7, 6, 5, 4, 4, 3, 2, 1]~]
T=Mod(0,11) S=[0,0] rk(A_S)=1 A_S=[11]
T=Mod(0,11) S=[0,1] rk(A_S)=6 A_S=[25937424601,25937424601,25937424601,25937424601,2357947691, 121]
T=Mod(0,11) S=[1,0] rk(A_S)=1 A_S=[11]
T=Mod(0,11) S=[1,1] rk(A_S)=7 A_S=[25937424601,25937424601,25937424601,
                                          25937424601,2357947691,2357947691,121]
p=13
P=x^13 - 53
[13, [-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]~, 13, 1, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]~]
T=Mod(11,13) S=[0] rk(A_S)=1 A_S=[13]
T=Mod(11,13) S=[1] rk(A_S)=7 A_S=[1792160394037,137858491849,137858491849,
                                 137858491849,137858491849,137858491849,137858491849]
P=x^13 - 677
[13, [-4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4]~, 12, 1, [12, 11, 10, 9, 8, 7, 6, 5, 5, 4, 3, 2, 1]~]
[13, [5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4]~, 1, 1,[12, 11, 10, 9, 8, 7, 6, 5, 2, 4, 3, 2, 1]~]
T=Mod(0,13) S=[0,0] rk(A_S)=1 A_S=[13]
T=Mod(0,13) S=[0,1] rk(A_S)=1 A_S=[13]
T=Mod(0,13) S=[1,0] rk(A_S)=7 A_S=[137858491849,137858491849,137858491849,
                                      137858491849,137858491849,10604499373, 169]
T=Mod(0,13) S=[1,1] rk(A_S)=8 A_S=[137858491849,137858491849,137858491849,
                             137858491849,137858491849,10604499373,10604499373, 169]
```

### 3.3 The fields $K=\mathbb{Q}\big(\sqrt{-\sqrt{-q}}\big)$ associated to elliptic curves

These fields, used by Coates–Li in [33, 34] to prove non-vanishing theorems for the central values at $s=1$ of the complex $L$-series of a family of elliptic curves studied by Gross (for any prime $q\equiv 7 \pmod 8$ and $p=2$), are particularly interesting.

Note once for all that the signature of $K$ is $[0,2]$, the Galois closure of $K$ is of degree 8 with Galois group $[8,-1,1,\text{"D(4)"}]$ and $D_K=2^m\,q^3$.

#### 3.3.1 Program for various $p$

In this part, we fix the prime number $q$ and compute the structure of $\mathscr{A}_{K,S}$ for all sets $S\subseteq P$. Recall that the parameter $n$ must be such that $p^n$ be much larger than the exponent of $\mathscr{T}_K$.

For instance, for $P=x^4+23$, we give the results for $p=3$ and $p=71$:

```
{q=23;P=x^4+q;print("P=",P);bp=2;Bp=500;n0=8;K=bnfinit(P,1);
forprime(p=bp,Bp,n=n0+floor(30/p);print();print("p=",p);
F=idealfactor(K,p);d=matsize(F)[1];F1=component(F,1);
for(j=1,d,print(F1[j]));for(z=2^d,2*2^d-1,bin=binary(z);mod=List;
for(j=1,d,listput(mod,bin[j+1],j));M=1;for(j=1,d,ch=component(mod,j);
if(ch==1,F1j=component(F1,j);ej=F1j[3];FF1j=idealpow(K,F1j,ej);
M=idealmul(K,M, FF1j)));Idn=idealpow(K,M,n);Kpn=bnrinit(K,Idn);
Hpn=Kpn.cyc;L=List;e=component(matsize(Hpn),2);R=0;
for(k=1,e,c=Hpn[e-k+1];w=valuation(c,p);if(w>0,R=R+1;
listinsert(L,p^w,1)));print("S=",mod," rk(A_S)=",R," A_S=",L)))}

P=x^4 + 23
p=3
[3, [-1, 1, 0, 0]~, 1, 1, [1, 0, 1, 1]~]
[3, [1, 1, 0, 0]~, 1, 1, [0, 0, 0, 1]~]
[3, [2, 0, 2, 0]~, 1, 2, [0, 0, -1, 0]~]
```

```
S=[0, 0, 0] rk(A_S)=1 A_S=[3]
S=[0, 0, 1] rk(A_S)=1 A_S=[68630377364883]
S=[0, 1, 0] rk(A_S)=1 A_S=[3]
S=[0, 1, 1] rk(A_S)=2 A_S=[68630377364883, 22876792454961]
S=[1, 0, 0] rk(A_S)=1 A_S=[3]
S=[1, 0, 1] rk(A_S)=2 A_S=[68630377364883, 22876792454961]
S=[1, 1, 0] rk(A_S)=1 A_S=[68630377364883]
S=[1, 1, 1] rk(A_S)=3 A_S=[68630377364883, 22876792454961, 22876792454961]
p=71
[71, [-32, 1, 0, 0]~, 1, 1, [0, 29, -5, 4]~]
[71, [32, 1, 0, 0]~, 1, 1, [4, 29, 9, 4]~]
[71, [31, 0, 2, 0]~, 1, 2, [-29, 0, 2, 0]~]
S=[0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1] rk(A_S)=1 A_S=[9095120158391]
S=[0, 1, 0] rk(A_S)=1 A_S=[71]
S=[0, 1, 1] rk(A_S)=2 A_S=[9095120158391, 9095120158391]
S=[1, 0, 0] rk(A_S)=1 A_S=[71]
S=[1, 0, 1] rk(A_S)=2 A_S=[9095120158391, 9095120158391]
S=[1, 1, 0] rk(A_S)=2 A_S=[9095120158391, 71]
S=[1, 1, 1] rk(A_S)=3 A_S=[9095120158391, 9095120158391, 9095120158391]
```

The user is invited to vary $n$ at will to certify the numerical results when the $p$-rank of $\mathscr{A}_{K,S}$ is unknown (i.e., when $S \subsetneq P$). In the above examples, some $\mathscr{T}_{K,S}$ are of order $p$ and the $\mathbb{Z}_p$-rank of $\mathscr{A}_{K,S}$ is 0 or 1.

### 3.3.2 Program for various $q$ and $p = 2$

The analogous program is the following ($n = 32$ is large enough):

```
{bq=3;Bq=100;p=2;n=32;forprime(q=bq,Bq,P=x^4+q;print();
print("q=",q," ",Mod(q,16));K=bnfinit(P,1);
F=idealfactor(K,p);d=matsize(F)[1];F1=component(F,1);
for(j=1,d,print(F1[j]));for(z=2^d,2*2^d-1,bin=binary(z);mod=List;
for(j=1,d,listput(mod,bin[j+1],j));M=1;for(j=1,d,ch=component(mod,j);
if(ch==1,F1j=component(F1,j);ej=F1j[3];FF1j=idealpow(K,F1j,ej);
M=idealmul(K,M, FF1j)));Idn=idealpow(K,M,n);Kpn=bnrinit(K,Idn);
Hpn=Kpn.cyc;L=List;e=component(matsize(Hpn),2);R=0;
for(k=1,e,c=Hpn[e-k+1];w=valuation(c,p);if(w>0,R=R+1;
listinsert(L,p^w,1)));print("S=",mod," rk(A_S)=",R," A_S=",L)))}
```

We give an example of each congruence class $q \pmod{16}$; for $q \equiv 7 \pmod{16}$, the decomposition of $(2)$ in $\mathbb{Q}(\sqrt{-q})$ is $(2) = \mathfrak{p} \cdot \mathfrak{p}^*$ where $e_\mathfrak{p} = 2$ in $K/\mathbb{Q}$:

```
q=17    Mod(1, 16)
[2, [1, 1, 0, 0]~, 4, 1, [1, 1, 1, 1]~]
S=[0] rk(A_S)=2 A_S=[8, 2]
S=[1] rk(A_S)=5 A_S=[4294967296, 2147483648, 2147483648, 8, 2]

q=3    Mod(3, 16)
[2, [1, 0, -1, 0]~, 2, 2, [1, 0, 1, 0]~]
S=[0] rk(A_S)=0 A_S=[]
S=[1] rk(A_S)=3 A_S=[4294967296, 2147483648, 1073741824]

q=5    Mod(5, 16)
[2, [1, 1, 0, 0]~, 4, 1, [1, 1, 1, 1]~]
S=[0] rk(A_S)=1 A_S=[4]
S=[1] rk(A_S)=3 A_S=[8589934592, 4294967296, 4294967296]

q=7    Mod(7, 16)
[2, [0, -1, 0, 1]~, 2, 1, [1, 0, 0, 1]~]
[2, [0, 1, 0, 0]~, 1, 2, [1, 1, 0, 0]~]
S=[0, 0] rk(A_S)=0 A_S=[]
S=[0, 1] rk(A_S)=2 A_S=[1073741824, 4]
S=[1, 0] rk(A_S)=1 A_S=[2147483648]
S=[1, 1] rk(A_S)=4 A_S=[2147483648, 2147483648, 1073741824, 2]

q=41    Mod(9, 16)
[2, [1, 1, 0, 0]~, 4, 1, [1, 1, 1, 1]~]
S=[0] rk(A_S)=2 A_S=[16, 2]
S=[1] rk(A_S)=4 A_S=[8589934592, 4294967296, 2147483648, 8]
```

```
q=11    Mod(11, 16)
[2, [1, 0, -1, 0]~, 2, 2, [1, 0, 1, 0]~]
S=[0] rk(A_S)=0 A_S=[]
S=[1] rk(A_S)=3 A_S=[4294967296, 2147483648, 1073741824]

q=13    Mod(13, 16)
[2, [1, 1, 0, 0]~, 4, 1, [1, 1, 1, 1]~]
S=[0] rk(A_S)=1 A_S=[4]
S=[1] rk(A_S)=3 A_S=[8589934592, 4294967296, 4294967296]

q=31    Mod(15, 16)
[2, [-1, 0, 0, 1]~, 1, 1, [0, 0, 0, 1]~]
[2, [0, 1, -1, 0]~, 2, 1, [1, 1, 0, 0]~]
[2, [2, 0, 1, 1]~, 1, 1, [1, 0, 1, 1]~]
S=[0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1] rk(A_S)=1 A_S=[4]
S=[0, 1, 0] rk(A_S)=2 A_S=[2147483648, 4]
S=[0, 1, 1] rk(A_S)=3 A_S=[2147483648, 1073741824, 8]
S=[1, 0, 0] rk(A_S)=1 A_S=[4]
S=[1, 0, 1] rk(A_S)=3 A_S=[1073741824, 4, 2]
S=[1, 1, 0] rk(A_S)=3 A_S=[2147483648, 1073741824, 8]
S=[1, 1, 1] rk(A_S)=5 A_S=[2147483648, 1073741824, 1073741824, 8, 2]
```

**Remark 3.2.** *A more complete table shows some rules:*

*(i) For* $q \equiv 3 \pmod 8$*,* $\mathscr{T}_{K,S} = 1$ *for* $S = \emptyset$ *and* $S = P = \{\mathfrak{p}\}$*;*

*(ii) For* $q \equiv 5 \pmod 8$*,* $\mathscr{T}_{K,\emptyset} = C\ell_K \simeq \mathbb{Z}/4\mathbb{Z}$ *and* $\mathscr{T}_{K,P} = 1$ *for* $P = \{\mathfrak{p}\}$ *(which means that the* 2*-Hilbert class field of K is contained in the compositum of the* $\mathbb{Z}_2$*-extensions of K);*

*(iii) For* $q \equiv 7 \pmod{16}$*, for* $S = \{\mathfrak{p}\}$ *with* $e_{\mathfrak{p}} = 2$*, we get* $\mathscr{T}_{K,S} \simeq \mathbb{Z}/4\mathbb{Z}$ *and for* $S = \{\mathfrak{p}^*\}$ *with* $e_{\mathfrak{p}^*} = 1$*, we get* $\mathscr{T}_{K,S} = 1$*; then* $\mathscr{T}_{K,P} \simeq \mathbb{Z}/2\mathbb{Z}$*.*

*These properties may be proved easily and are left to the reader as exercises on the* $\mathrm{Log}_S$*-function (Definition A.4): consider first the arithmetic of the subfield* $k = \mathbb{Q}(\sqrt{-q})$ *and use fixed point formulas (A.5) in* $K/k$*.*

*(iv) For* $q \equiv 15 \pmod{16}$*, the results do not follow any obvious rule and offers some interesting examples as the following ones:*

```
q=5503
[2, [-1, 0, 0, 1]~, 1, 1, [0, 0, 0, 1]~]
[2, [0, 1, -1, 0]~, 2, 1, [1, 1, 0, 0]~]
[2, [2, 0, 1, 1]~, 1, 1, [1, 0, 1, 1]~]
S=[0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1] rk(A_S)=1 A_S=[512]
S=[0, 1, 0] rk(A_S)=2 A_S=[2147483648, 8]
S=[0, 1, 1] rk(A_S)=3 A_S=[2147483648, 1073741824, 16]
S=[1, 0, 0] rk(A_S)=1 A_S=[512]
S=[1, 0, 1] rk(A_S)=3 A_S=[1073741824, 512, 2]
S=[1, 1, 0] rk(A_S)=3 A_S=[2147483648, 1073741824, 16]
S=[1, 1, 1] rk(A_S)=5 A_S=[2147483648, 1073741824, 1073741824, 16, 2]

q=8191
[2, [-1, 0, 0, 1]~, 1, 1, [0, 0, 0, 1]~]
[2, [0, 1, -1, 0]~, 2, 1, [1, 1, 0, 0]~]
[2, [2, 0, 1, 1]~, 1, 1, [1, 0, 1, 1]~]
S=[0, 0, 0] rk(A_S)=0 A_S=[]
S=[0, 0, 1] rk(A_S)=1 A_S=[64]
S=[0, 1, 0] rk(A_S)=2 A_S=[2147483648, 64]
S=[0, 1, 1] rk(A_S)=3 A_S=[2147483648, 1073741824, 128]
S=[1, 0, 0] rk(A_S)=1 A_S=[64]
S=[1, 0, 1] rk(A_S)=3 A_S=[1073741824, 64, 2]
S=[1, 1, 0] rk(A_S)=3 A_S=[2147483648, 1073741824, 128]
S=[1, 1, 1] rk(A_S)=5 A_S=[2147483648, 1073741824, 1073741824, 128, 2]

q=123551
[2, [-1, 0, 0, 1]~, 1, 1, [0, 0, 0, 1]~]
[2, [0, 1, -1, 0]~, 2, 1, [1, 1, 0, 0]~]
[2, [2, 0, 1, 1]~, 1, 1, [1, 0, 1, 1]~]
S=List([0, 0, 0]) rk(A_S)=0 A_S=List([])
```

```
S=List([0, 0, 1]) rk(A_S)=1 A_S=List([16])
S=List([0, 1, 0]) rk(A_S)=2 A_S=List([2147483648, 16])
S=List([0, 1, 1]) rk(A_S)=3 A_S=List([2147483648, 1073741824, 32])
S=List([1, 0, 0]) rk(A_S)=1 A_S=List([16])
S=List([1, 0, 1]) rk(A_S)=3 A_S=List([1073741824, 16, 2])
S=List([1, 1, 0]) rk(A_S)=3 A_S=List([2147483648, 1073741824, 32])
S=List([1, 1, 1]) rk(A_S)=5 A_S=List([2147483648, 1073741824, 1073741824, 32, 2])
```

# 1. Appendix: History of abelian $p$-ramification

## A.1 Motivations

We intend, in this detailed survey, to give a maximum of practical information and results about the torsion groups $\mathscr{T}_{K,S}$ that we have numerically computed in the first part of the paper with a PARI/GP program. Since all the invariants, associated with $\mathscr{T}_{K,S}$, need numerical computations for a better understanding, we choose the more suitable technical presentation (the main philosophical remark is that *they are all equivalent*).

For convenience, we indicate both the original historical contributions and the corresponding results processed systematically in our book [1].

We will not detail the immense domains of pro-$p$-groups and Galois cohomology, whose main purpose is for instance the existence of infinite towers of $S$-ramified extensions and the Fontaine–Mazur conjecture studied by various schools of mathematicians (for this, see, e.g., [5, § 10]), nor the analytic aspects as the non-vanishing at $s = 1$ of complex $L$-series associated to elliptic curves . . . Similarly, we shall not consider the context of Iwasawa's theory because this efficient tool does not exempt from having the "basic" arithmetical properties of the corresponding objects.

Note that the solutions of the analogous problems of $S$-ramification over local fields are not sufficient for a "globalization" over a number field $K$ as remarked by Nguyen Quang Do in [35, § 9]. Indeed, the global theory depends on Leopoldt's conjecture (usually assumed) and the torsion groups $\mathscr{T}_{K,S}$ are, in some sense, refinements of this conjecture.

So we will focus, mainly, on class field theory and on these specific deep $p$-adic properties or conjectures which are, in our opinion, the main obstructions for many contemporary researches.

We will not give the most general statements but restrict ourselves to the case of $S$-ramification, $S \subseteq P$, without decomposition of finite or infinite places (indeed, in these more elaborate cases, the formalism is identical and may be found in our book). Since the properties of $S$-ramification may be used by many researchers working on different subjects, we will try to explain the numerous steps of its progress. This must be understood for practical information and will be an opportunity to clarify the vocabulary and the main contributions.

We apologize for the probable lack of references (and citation of their authors).

## A.2 Prehistory

The origin of interest for $S$-ramification theory over a number field is probably a paper of Brumer [4], following Serre's book [2] and seems also due to a lecture by Šafarevič (1963) showing the importance of the subject. In [3], Šafarevič gives the cohomological characteristics of the group $\mathscr{G}_{K,S}$ (number of generators and relations, cohomological dimension . . .).

Recall at this step the Golod–Šafarevič theorem (1964), named soon after the theorem of Golod–Šafarevič–Gaschütz–Vinberg, saying that if a pro-$p$-group $\mathscr{G}$ is finite, then $r(\mathscr{G}) > \frac{1}{4}\,(d(\mathscr{G}))^2$ where $d(\mathscr{G})$ (resp. $r(\mathscr{G})$) is the minimal number of generators (resp. relations) for the presentation of $\mathscr{G}$. All of this was developed in Koch's book [36] from the works of many German mathematicians and is amply improved in [5] (see also in Hajir–Maire [37, 38] a good introduction on the subject and some of its developments [39], [40, 41], [42, 43]).

More precisely, in [3, Théorème I], Šafarevič gives, for any number field $K$ and any set of places $S$, the main formula (1.2) that we recall:

### A.2.1 Šafarevič formula

The $p$-rank of the $\mathbb{Z}_p$-module $\mathscr{A}_{K,S}$ (giving the minimal number of generators $\dim_{\mathbb{F}_p}(\mathrm{H}^1(\mathscr{G}_{K,S}, \mathbb{Z}/p\mathbb{Z}))$ of $\mathscr{G}_{K,S}$) is:

$$\mathrm{rk}_p(\mathscr{A}_{K,S}) = \mathrm{rk}_p\big(V_{K,S}/K_{(S)}^{\times p}\big) + \sum_{\mathfrak{p}\in S\cap P}[K_{\mathfrak{p}} : \mathbb{Q}_p] + \sum_{\mathfrak{p}\in S}\delta_{\mathfrak{p}} - \delta_K - (r_1 + r_2 - 1), \tag{A.1}$$

where:

$$K_{(S)}^{\times} := \big\{\alpha \in K^{\times}, \alpha \text{ prime to } S\big\}, \quad V_{K,S} := \big\{\alpha \in K_{(S)}^{\times},\ (\alpha) = \mathfrak{a}^p\big\},$$

then $\delta_{\mathfrak{p}} = 1$ or 0 according as the completion $K_{\mathfrak{p}}$ contains $\mu_p$ or not, and $\delta_K = 1$ or 0 according as $K$ contains $\mu_p$ or not.

Of course, $\dim_{\mathbb{F}_p}(\mathrm{H}^2(\mathscr{G}_{K,S}, \mathbb{Z}/p\mathbb{Z}))$, giving the minimal number of relations, is easily obtained only when $P \subseteq S$ (equal to $\mathrm{rk}_p(\mathscr{T}_{K,S})$ under Leopoldt's conjecture), which shall explain the forthcoming studies about this:
[5], [6, 7], [8], [11], [12], [14], [36], [44], Haberland [45], [46], El Habibi–Ziane [47] . . . .

### A.2.2 Kubota formalism

Mention that Kubota [48] begins the study of the structure of the dual $\mathscr{A}^*_{K,S}$ of $\mathscr{A}_{K,S}$, study which is based on the Grunwald–Wang theorem and which leads to a characterization of this group in terms of its fundamental invariants called, following Kaplansky, the "Ulm invariants".

Then in [49], Miki uses this formalism, about $\ell(= p)$-ramification, then class field theory, Iwasawa's theory, in direction of Leopoldt's conjecture. Some statements, equivalent to some results that we shall recall in this survey (as well as the notion of $p$-rationality and its main properties), should be mentioned in his paper, despite the difficulty of translating vocabulary and technique.

## A.3 Main developments after the pioneering works

The computation of $\mathrm{rk}_p(\mathscr{T}_{K,P})$, from Kummer theory, is already given by Bertrandias–Payan [18], then in [50, Théorèmes I.2, I.3, Corollaire 1] and by many authors, for instance by means of cohomological techniques (e.g., [26, Proposition 3]).

This will give reflection formulas.

### A.3.1 Reflection and rank formulas

From [51, Chapitre III, § 10] or [1, § II.5.4.1][Gr2003]. Using the Šafarevič formula and Kummer theory when $K$ contains the group $\mu_p$ of $p$th roots of unity, and writing (for $S \subseteq P$):

$$P = S \cup \Sigma \;\text{ with }\; S \cap \Sigma = \emptyset,$$

one obtains the reflection theorem in its simplest form:

$$\mathrm{rk}_p(\mathscr{A}^{\Sigma}_{K,S}) - \mathrm{rk}_p(\mathscr{A}^{S\,\mathrm{res}}_{K,\Sigma}) = \#S - \#\Sigma + \sum_{\mathfrak{p} \in S} [K_{\mathfrak{p}} : \mathbb{Q}_p] - r_1 - r_2, \tag{A.2}$$

where $\mathscr{A}^{\Sigma}_{K,S}$ is the Galois group of the maximal abelian pro-$p$-extension of $K$ in $H_{K,S}$, which is $\Sigma \cup \{\infty\}$-split (i.e., in which all the places of $\Sigma \cup \{\infty\}$ split completely), and similarly for the definition of $\mathscr{A}^{S\mathrm{res}}_{K,\Sigma}$, in the restricted sense for $p = 2$ (i.e., only $S$-split); in other words, the mention of $\{\infty\}$ is implicit in the upper script to give the ordinary sense when $p = 2$.

The case $S = P$ leads to the following well-known result:

**Theorem A.1.** *[1, Proposition III.4.2.2]. Let $K$ be any number field fulfilling the Leopoldt conjecture for the prime number $p$. Let $K' := K(\mu_p)$, $P'$ be the set of $p$-places above $P$ in $K'$, and let $P^{\mathrm{dec}}$ be the set of $p$-places of $K$ totaly split in $K'$. Let $\omega$ be the Teichmüller character and denote by $\mathrm{rk}_\omega$ the $p$-rank of an isotypic $\omega$-component for $\mathrm{Gal}(K'/K)$; then:*

$$\mathrm{rk}_p(\mathscr{T}_{K,P}) = \mathrm{rk}_\omega(\mathcal{C}\ell^{P'\mathrm{res}}_{K'}) + \#P^{\mathrm{dec}} - \delta_K,$$

*where $\mathcal{C}\ell^{P'\mathrm{res}}_{K'}$ is the quotient of the $p$-class group $\mathcal{C}\ell^{\mathrm{res}}_{K'}$ by the subgroup generated by the classes of $P'$ (in the restricted sense for $p = 2$) and where $\delta_K = 1$ or 0 according as $K$ contains $\mu_p$ or not. Whence the following properties:*

*(i) If $\mu_p \subset K$, we then have $\mathrm{rk}_p(\mathscr{T}_{K,P}) = \mathrm{rk}_p(\mathcal{C}\ell^{P\mathrm{res}}_K) + \#P - 1$.*

*(ii) We have $\mathscr{T}_{K,P} = 1$ if and only if:*

- *$\mu_p \not\subset K$ (so $p \neq 2$): then $P^{\mathrm{dec}} = \emptyset$ and the $\omega$-component of $\mathcal{C}\ell_{K'}$ is trivial;*
- *$\mu_p \subset K$: $p$ does not split in $K/\mathbb{Q}$ and the unique $\mathfrak{p} \in P$ generates $\mathcal{C}\ell^{\mathrm{res}}_K$.*

**Example A.2.** *For $K = \mathbb{Q}(\mu_p) =: \mathbb{Q}(\zeta_p)$, $p \neq 2$, taking $\Sigma = \emptyset$ and $S = P$:*

$$\mathrm{rk}_p(\mathscr{A}_{K,P}) - \mathrm{rk}_p(\mathscr{A}^{P}_{K,\emptyset}) = 1 + p - 1 - \tfrac{p-1}{2} = \tfrac{p+1}{2}.$$

*Since $\mathscr{A}^{P}_{K,\emptyset} = \mathcal{C}\ell_K/\langle c\ell_K(\mathfrak{p})\rangle$, with $\mathfrak{p} = (1-\zeta_p)$, and $\mathscr{A}_{K,P} \simeq \mathbb{Z}_p^{\frac{p+1}{2}} \bigoplus \mathscr{T}_{K,P}$, this yields:*

$$\mathrm{rk}_p(\mathscr{T}_{K,P}) = \mathrm{rk}_p(\mathcal{C}\ell_K), \tag{A.3}$$

*as well as the writing $\mathrm{rk}_p(\mathscr{T}^{\pm}_{K,P}) = \mathrm{rk}_p(\mathcal{C}\ell^{\mp}_K)$ (for analogous equalities with pairs of isotopic components associated by means of the mirror involution, and the consequences for Vandiver's conjecture, see [52]).*

If the condition $S \cup \Sigma = P$ is not fulfilled, we have (still assuming $\mu_p \subset K$) the reflection formula:

$$\mathrm{rk}_p(\mathscr{A}^{\Sigma}_{K,S}) - \mathrm{rk}_p(C\ell^{S\mathrm{res}}_K(\mathfrak{m}^*)) = \#S - \#\Sigma + \sum_{\mathfrak{p}\in S}[K_\mathfrak{p} : \mathbb{Q}_p] - r_1 - r_2, \ \text{ with } \mathfrak{m}^* := \prod_{\mathfrak{p}\in\Sigma}\mathfrak{p}^{pe_\mathfrak{p}+1} \cdot \prod_{\mathfrak{p}\in P\setminus S\cup\Sigma}\mathfrak{p}^{pe_\mathfrak{p}} \tag{A.4}$$

where $C\ell^{S\mathrm{res}}_K(\mathfrak{m}^*)$ is the $S$-split $p$-ray class group of modulus $\mathfrak{m}^*$ (see [1, Exercise II.5.4.1, proof of (iii)] and (iv) for the case $p = 2$). Note that $C\ell^{S\mathrm{res}}_K(\mathfrak{m}^*)$ is isomorphic to a quotient of $\mathscr{A}^{S\mathrm{res}}_{K,P\setminus S}$.

Finaly, if $K$ does not contain $\mu_p$, but assuming $P = S \cup \Sigma$ with $S \cap \Sigma = \emptyset$, the general formula is:

$$\mathrm{rk}_p(\mathscr{T}^{\Sigma}_{K,S}) = \mathrm{rk}_\omega(\mathscr{A}^{S'\,\mathrm{res}}_{K',\Sigma'}) + \sum_{\mathfrak{p}\in S}\delta_\mathfrak{p} - \delta_K - \#\Sigma - \big(r_1 + r_2 - 1 - r^{\Sigma}_{K,S}\big), \tag{A.5}$$

where:

$$r^{\Sigma}_{K,S} = \sum_{\mathfrak{p}\in S}[K_\mathfrak{p} : \mathbb{Q}_p] - \widetilde{r}^{\Sigma}_{K,S};$$

here, $\widetilde{r}^{\Sigma}_{K,S} \le r_2 + 1$ is the $\mathbb{Z}_p$-rank of $\mathbb{Z}_p\log_S(I_{K,S})$ modulo $\mathbb{Q}_p\log_S(E^{\Sigma}_K)$ dealing with the group $E^{\Sigma}_K$ of $\Sigma$-units of $K$ (see also [11], [14] for some applications).

One can restrict some of the above equalities to $p$-class groups, giving only inequalities on the $p$-ranks (Hecke theorem (1910), Scholz theorem (1932), Leopoldt Spiegelungssatz (1958), Armitage–Fröhlich–Serre, Oriat, for $p = 2$.

For reflection theorems and formulas with characters, see [1, II.5.4, Theorem II.5.4.5)] from the computations of [51, Ch. I, Theorem 5.18] where $p$-rank formulas link $p$-class groups and torsion groups as in Theorem A.1 (this context is used by Ellenberg–Venkatesh in [53] for the $\varepsilon$-conjecture on $p$-class groups).

For the annihilation of the Galois module $\mathscr{T}_{K,P}$, of real abelian extensions $K/\mathbb{Q}$, in relation with the construction of $p$-adic $L$-functions and reflection principle, see [54] and its bibliography. There is probably *equivalent information* whatever the process (algebraic or analytic), as shown by Oriat in [55]. This logical aspect should deserve further investigation.

### A.3.2 Regulators and $p$-adic residues of the $\zeta_p$-functions

We continue the story with the $p$-adic analytic computations of the residue of the $p$-adic $\zeta$-function at $s = 1$ of real abelian fields $K$ by Amice–Fresnel [56], from Kubota–Leopoldt $L_p$-functions (1964), by Coates [57], Serre [58] introducing $p$-adic pseudo-measures, then by Colmez [59] in full generality, via the formula:

$$\frac{1}{2^{[K:\mathbb{Q}]-1}}\lim_{s\to 1}(s-1)\,\zeta_{K,p}(s) = \frac{R_p\,h\,E_p(1)}{\sqrt{D}},$$

where $R_p$ is the classical $p$-adic regulator, $h$ the class number, $D$ the discriminant of $K$ and $E_p(1)$ the eulerian factor $\prod_{\mathfrak{p}|p}(1 - \mathrm{N}\mathfrak{p}^{-1})$. For totally real fields, the normalised $p$-adic regulator $\mathscr{R}_{K,P}$, in the formula (2.2), is given (under Leopoldt's conjecture) by the expression [19, Proposition 5.2]:

$$\#\mathscr{R}_{K,P} \sim \frac{1}{2}\cdot\frac{\big(\mathbb{Z}_p : \log(\mathrm{N}_{K/\mathbb{Q}}(U_{K,P}))\big)}{\#\mathscr{W}_{K,P}\cdot\prod_{\mathfrak{p}|p}\mathrm{N}\mathfrak{p}}\cdot\frac{R_p}{\sqrt{D}},$$

where $\sim$ means equality up to a $p$-adic unit factor; whence:

$$\tfrac{1}{2^{[K:\mathbb{Q}]-1}}\lim_{s\to 1}(s-1)\,\zeta_{K,p}(s) = \tfrac{1}{p\,[K\cap\mathbb{Q}^{\mathrm{c}}:\mathbb{Q}]}\,\#\mathscr{T}_{K,P},$$

where $\mathbb{Q}^{\mathrm{c}}$ is the $\mathbb{Z}_p$-cyclotomic extension of $K$. In [120], Hatada uses the link between the $p$-adic valuation of $\zeta_K(2-p)$ and that of $\mathscr{R}_{K,P}$ to study the $p$-rationality of some totally real number fields; he studies the case of quadratic fields with general Fibonacci sequences (from the fundamental unit), a method that will be rediscovered by some authors to characterize the $p$-rationality.

Mention the relative version of the Coates formula in the totally real case:

**Theorem A.3.** *[50, Théorème III.3]. Let $L/K$ be an abelian extension of totally real number fields fulfilling the Leopoldt conjecture. Let $\mathscr{N}_{L/K}$ be the group of local norms and let $C\ell^{\mathrm{gen}}_{L/K} := \mathrm{Gal}(H^{\mathrm{ab}}_L/LH_K)$ be the $p$-genus group in $L/K$; the superscript $^*$ denotes $\mathrm{Ker}(\mathrm{N}_{L/K})$. Then:*

$$\#\mathscr{T}_{L,P} \sim \frac{\#\mathscr{T}_{K,P}}{\big[L\cap H_{K,P} : L\cap K^{\mathrm{c}}\big]}\times\frac{\prod_{\mathfrak{l}\nmid p}e_{\mathfrak{l},p}}{[L : L\cap H_{K,P}]}\times\#C\ell^{\mathrm{gen}}_{L/K}\times\big(E_K\cap\mathscr{N}_{L/K} : \mathrm{N}_{L/K}(E_L)\big)$$
$$\times\big(\log_P(U^*_{L,P}) : \log_P(\overline{E}^*_L)\big)\times\big(\mathrm{tor}_{\mathbb{Z}_p}(U^*_{L,P}) : \mu^*_p\big),$$

*where $\mu^*_p = 1$ for $p \neq 2$ and $\#\mu^*_2 = \gcd(2,[L : K])$.*

### A.3.3 Cohomological interpretation

In [16], Nguyen Quang Do gives the cohomological interpretation of the dual of $\mathscr{T}_{K,P}$: $\mathscr{T}_{K,P}^* \simeq \mathrm{H}^2(\mathscr{G}_{K,P}, \mathbb{Z}_p)$, considered as the first of the mysterious non positive twists $\mathrm{H}^2(\mathscr{G}_{K,P}, \mathbb{Z}_p(i))$ of the motivic cohomology; for concrete results of genus type about the corresponding case of motivic tame kernels, see Assim–Movahhedi [60] and its important bibliography which would deserve to be in part among our references, despite it is beyond our goals.

It is indeed well known that $\mathrm{H}^2(\mathscr{G}_{K,P}, \mathbb{Z}_p)$ does appear as a tricky obstruction in many questions of Galois theory over number fields, whatever the technical approach. For $\mathrm{H}^2(\mathscr{G}_{K,S}, \mathbb{Z}_p)$, see [1, Appendix][Gr2003].

But considering the two "equivalent" invariants $\mathrm{H}^2(\mathscr{G}_{K,P}, \mathbb{Z}_p)$ and $\mathscr{T}_{K,P}$, only the last one may be used, with arithmetic or analytic tools, to obtain numerical experiments and to understand the true intrinsic $p$-adic difficulties.

### A.3.4 Principal Conjectures and Theorems

Considering the invariants $C\ell_K$ and $\mathscr{T}_{K,P}$ as fundamental objects, we have given, for the abelian fields $K$, the conjectural behavior of their isotopic $\chi$-components for irreducible $p$-adic characters $\chi$ in [61]; the proofs of these conjectures and of some improvements in Iwasawa's theory are well known and the reader may refer to the illuminating paper of Ribet [62] (available at `https://www.dropbox.com/s/1uir9crhidorejy/smf.Ribet.pdf?dl=0`) about the so-called "Principal Theorem" stemming from Bernoulli–Kummer–Herbrand then Ribet–Mazur–Wiles–Thaine–Rubin–Kolyvagin–Greither works on cyclotomy and $p$-adic $L$-functions, as a prelude of wide generalizations in the same spirit.

## A.4 Basic $p$-adic properties of $\mathscr{A}_{K,P}$ & $\mathscr{T}_{K,P}$

During the 1980's, we have written in [50, 63, 64] [3] the main properties of the groups $\mathscr{T}_{K,P}$ with their behaviour in any extension $L/K$ and proved (assuming Leopoldt's conjecture in the Galois closure of $L$) that the transfer maps:

$$\mathscr{A}_{K,P} \longrightarrow \mathscr{A}_{L,P} \ \ \& \ \ \mathscr{T}_{K,P} \longrightarrow \mathscr{T}_{L,P}$$

are always injective [50, Théorème I.1]; which has major consequences for the arithmetic of number fields (e.g., non-capitulation in an extension contrary to class groups). Of course, this property has been obtained soon after by Jaulent, Nguyen Quang Do and others with different techniques.

### A.4.1 The $p$-adic $\mathrm{Log}_S$-functions

**Definition A.4.** *[63, § 2, Théorème 2.1], [1, § III.2.2]. Let $I_{K,P}$ be the group of prime to $p$ ideals of $K$. We define the logarithm function:*

$$\mathrm{Log}_P : I_{K,P} \longrightarrow \Big(\bigoplus_{\mathfrak{p}\in P} K_{\mathfrak{p}}\Big)\Big/ \mathbb{Q}_p \mathrm{log}_P(E_K)$$

*as follows. For any ideal $\mathfrak{a} \in I_{K,P}$ let $m$ be such that $\mathfrak{a}^m =: (\alpha)$, $\alpha \in K^\times$, then:*

$$\mathrm{Log}_P(\mathfrak{a}) := \frac{1}{m}\mathrm{log}_P(\alpha) \pmod{\mathbb{Q}_p \mathrm{log}_P(E_K)}.$$

The main property of $\mathrm{Log}_P$ is that for any ideal $\mathfrak{a} \in I_{K,P}$, $\mathrm{Log}_P(\mathfrak{a})$ defines the Artin symbol in the compositum $\widetilde{K}^P$ of the $\mathbb{Z}_p$-extensions of $K$ by means of the canonical exact sequence:

$$1 \to \mathscr{T}_{K,P} \longrightarrow \mathscr{A}_{K,P} \xrightarrow{\mathrm{Log}_P} \mathrm{Log}_P(I_{K,P}) \simeq \mathrm{Gal}(\widetilde{K}^P/K) \to 1,$$

which may be generalized with arbitrary $S \subseteq P$:

$$1 \to \mathscr{T}_{K,S} \longrightarrow \mathscr{A}_{K,S} \xrightarrow{\mathrm{Log}_S} \mathrm{Log}_S(I_{K,S}) \simeq \mathrm{Gal}(\widetilde{K}^S/K) \to 1,$$

with an obvious definition of $\mathrm{Log}_S(\mathfrak{a})$ in $\bigoplus_{\mathfrak{p}\in S} K_{\mathfrak{p}}$ modulo $\mathbb{Q}_p \mathrm{log}_S(E_K)$.

This formalism is equivalent to that given by the theory of pro-$p$-groups (here $\mathscr{G}_{K,P}$), but may yield numerical computations as follows:

[3] [64] is only available at: `https://www.dropbox.com/s/fusia63znk0kcky/Lectures1982.pdf?dl=0`

The formula for $\#\mathscr{T}_{K,S}$, $S \subseteq P$, is the following [65, Theorems III.2.5], [1, Corollary III.2.6.1] (under Leopoldt's conjecture):

$$\#\mathscr{T}_{K,S} = \#\mathscr{W}_{K,S} \times \#\mathscr{R}_{K,S} \times \frac{\#C\ell_K}{\big(\mathbb{Z}_p\mathrm{Log}_S(I_{K,S}) : \mathbb{Z}_p\mathrm{Log}_S(P_{K,S})\big)}, \tag{A.6}$$

where $P_{K,S}$ is the group of principal ideals prime to $S$, so that $\mathbb{Z}_p\mathrm{Log}_S(P_{K,S})$ depends obviously on $\log_S(U_{K,S})$ modulo $\mathbb{Q}_p\log_S(E_K)$. When $S \subsetneq P$, $\mathscr{W}_{K,S}$ is not necessarily equal to $\mathrm{tor}_{\mathbb{Z}_p}(U_{K,S})/\iota_S(\mu_K)$ (cf. Lemmas 2.1, 2.2).

The denominator in (A.6) gives the degree $[\widetilde{K}^S \cap H_K : K]$ and the quotient gives $\#\widetilde{C\ell}_K^S$.

For $S = P$, the $\mathrm{Log}_P$-function allows, when $\mu_p \subset K$, the numerical determination of the initial Kummer radical contained in $\widetilde{K}^P$ [66], [67].

### A.4.2 Fixed point formula

Then we have obtained a fixed point formula for $S = P$ which, contrary to Chevalley's formula for class groups in cyclic extensions [68], does exist whatever the Galois extension $L/K$ ([63, § 5], [69, Section 2 (c)], [65, Proposition 6], [25, Appendice I], [70, Appendice]):

**Theorem A.5.** *[1, § IV.3, Theorem 3.3]. Let $L/K$ be a Galois extension of number fields and $G := \mathrm{Gal}(L/K)$. Let $p$ be a prime number; we assume that $L$ satisfies the Leopoldt conjecture for $p$. Then:*

$$\#\mathscr{T}_{L,P}^G = \#\mathscr{T}_{K,P} \times \frac{\prod_{\mathfrak{l} \nmid p} e_{\mathfrak{l},p}}{\Big(\sum_{\mathfrak{l} \nmid p} \frac{1}{e_{\mathfrak{l},p}} \mathbb{Z}_p\mathrm{Log}_P(\mathfrak{l}) + \mathbb{Z}_p\mathrm{Log}_P(I_{K,P}) : \mathbb{Z}_p\mathrm{Log}_P(I_{K,P})\Big)},$$

*where $e_{\mathfrak{l},p}$ is the $p$-part of the ramification index of $\mathfrak{l}$ in $L/K$.*

**Remark A.6.** *Contrary to the computation of $\mathrm{tor}_{\mathbb{Z}_p}(U_{K,P}/\overline{E}_K^P)$, that of the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p\log_P(E_K)$ does not need the knowledge of the group of units $E_K$; it only depends of Leopoldt's conjecture (assumed) and its $\mathbb{Q}_p$-dimension is $r_1 + r_2 - 1$; the case of $\mathbb{Q}_p\log_S(E_K)$ is more mysterious.*

The case of totally real fields is easier since the Log-function trivializes because we have $\bigoplus_{\mathfrak{p} \in P} K_{\mathfrak{p}} = \mathbb{Q}_p\log_P(E_K) \bigoplus \mathbb{Q}_p$, which allows explicit computations [50, Théorème III.1]:

**Corollary A.7.** *[1, Exercise IV.3.3.1]. In the case of a totally real number field $L$, the above formula becomes (under Leopoldt's conjecture): $\#\mathscr{T}_{L,P}^G = \#\mathscr{T}_{K,P} \cdot p^{\rho - r} \cdot \prod_{\mathfrak{l} \nmid p} e_{\mathfrak{l},p}$, where $p^r \sim [L : K]$ and $\rho$ only depends on the decomposition of the ramified primes $\ell \nmid p$ in $L/K$.*

### A.4.3 $p$-primitive ramification

The fixed point formula of Theorem A.5 allows to characterize the case where $\#\mathscr{T}_{L,P} = 1$ in a $p$-extension $L/K$:

**Corollary A.8.** *Let $L/K$ be any finite $p$-extension. Then $\mathscr{T}_{L,P} = 1$ if and only if the two following conditions are fulfilled (under Leopoldt's conjecture):*

*(i)* $\mathscr{T}_{K,P} = 1$;

*(ii)* $\Big(\sum_{\mathfrak{l} \nmid p} \frac{1}{e_{\mathfrak{l},p}} \mathbb{Z}_p\mathrm{Log}_P(\mathfrak{l}) + \mathbb{Z}_p\mathrm{Log}_P(I_{K,P}) : \mathbb{Z}_p\mathrm{Log}_P(I_{K,P})\Big) = \prod_{\mathfrak{l} \nmid p} e_{\mathfrak{l},p}$.

**Definition A.9.** *[1, § IV.3, (b)]. When the condition (ii) is fulfilled, we say that the $p$-extension $L/K$ is $p$-primitively ramified and that the set $T$ of tame places $\mathfrak{l}$, ramified in $L/K$, is primitive [65, Ch. III, Definition & Remark], which is equivalent (in terms of Frobenius automorphisms) to:*

$$\mathrm{Gal}(\widetilde{K}^P/K) \simeq \mathscr{A}_{K,P}/\mathscr{T}_{K,P} = \bigoplus_{\mathfrak{l} \in T} \Big\langle \Big(\frac{\widetilde{K}^P/K}{\mathfrak{l}}\Big) \Big\rangle. \tag{A.7}$$

Of course, any $P$-ramified extension is $p$-primitively ramified.

Then in [65, Ch. III, § 2, Theorem 2 & Corollary] are characterized, for $p = 2$ and $p = 3$, the abelian $p$-extensions $K$ of $\mathbb{Q}$ such that $\mathscr{T}_{K,P} = 1$. This is connected with the "regular kernel" of $K$ which, from results of Tate, follows similar properties which have been explained in a joint work with Jaulent [71] and developed in Jaulent–Nguyen Quang Do [72]. We can state:

**Theorem A.10.** *[1, Theorem III.4.2.5, Theorem IV.3.5]. Let $K$ be any number field.The following properties are equivalent:*

*(i) $K$ satisfies the Leopoldt conjecture at $p$ and $\mathscr{T}_{K,P} = 1$;*

*(ii) $\mathscr{A}_{K,P} := \mathscr{G}_{K,P}^{\rm ab} = {\rm Gal}(H_{K,P}/K) \simeq \mathbb{Z}_p^{r_2+1}$,*

*(iii) the Galois group $\mathscr{G}_{K,P}$ is a free pro-$p$-group on $r_2+1$ generators, which is equivalent to fulfill the following four conditions:*

- *$K$ satisfies the Leopoldt conjecture at $p$,*
- $C\ell_K \simeq \mathbb{Z}_p{\rm Log}_P(I_{K,P})\big/\big({\rm log}_P(U_{K,P}) + \mathbb{Q}_p{\rm log}_P(E_K)\big)$,
- ${\rm tor}_{\mathbb{Z}_p}(U_{K,P}) = \mu_p(K)$,
- $\mathbb{Z}_p{\rm log}_P(E_K)$ *is a direct summand in* ${\rm log}_P(U_{K,P})$.

## A.5 New formalisms and use of pro-$p$-group theory

### A.5.1 Infinitesimal arithmetic

From [69, 67, 73, 17]. At the same time, in his Thesis, Jaulent defines the *infinitesimal arithmetic* in a number field proving, in a nice conceptual framework, generalizations of our previous results, especially in the new context of *logarithmic classes* [73, 74], adding Iwasawa theory results, study of the $p$-regularity (replacing $\mathscr{T}_{K,P}$ by the tame kernel ${\rm K}_2(Z_K)$ of the ring of integers of $K$), and genus theory.

The same technical context of $\ell\,(=p)$-adic class field theory and a logarithmic class field theory was developed later in much papers, including computational methods of Bourbon–Jaulent [29]. He studies in [73] the logarithmic class group $\widetilde{C\ell}_K$ (do not confuse with $\widetilde{C\ell_K}^P$) whose finiteness is equivalent to the Gross (or Gross–Kuz'min) conjecture [75], [76] (a survey is given in [1, § III.7]); see also some comments in [77, 78].

Some properties of capitulation of generalized ray class groups and of $\widetilde{C\ell}_K$ are given in [79, 80, 81, 82].

### A.5.2 Pro-$p$-group theory version

Shortly after, at the end of the 1980's, in his thesis, Movahhedi [25, 26] gives a wide study of the abelian $p$-ramification theory, using mainly the properties of the pro-$p$-group $\mathscr{G}_{K,S}$ and deduces again most of the previous items, then he gives the main structural and cohomological properties of $\mathscr{G}_{K,P}$ and the classical characterization of the triviality of $\mathscr{T}_{K,P}$. He proposes for this to speak of "$p$-rational fields" [26, Definition 1], that is to say the number fields $K$ such that Leopoldt's conjecture holds for $p$ and $\mathscr{T}_{K,P} = 1$ (cf. Theorem A.10); this was inspired by the fact that $\mathbb{Q}$ is (obviously) $p$-rational for all $p$. This vocabulary has been adopted by the arithmeticians.

Then Movahhedi gives properties of $p$-rational extensions $L/K$ and the reciprocal of our result characterizing the $p$-rationality in a $p$-extension $L/K$, in other words the "going up" of the $p$-rationality:

**Theorem A.11.** *[25, Théorème 3, § 3]. Let $L/K$ be a $p$-extension of number fields. The field $L$ is $p$-rational if and only if $K$ is $p$-rational and the set $T$ of tame primes, ramified in $L/K$, is $p$-primitive in $K$. Moreover, under these conditions, the extension $T(L)$ of $T$ to $L$ is $p$-primitive.*

This implies that if $K$ is $p$-rational and $T$ $p$-primitive, then any $T$-ramified $p$-extension $L/K$ fulfills the Leopoldt conjecture and $T(L)$ is $p$-primitive (a particular case was given in [50, Théorème III.4] for totally real fields).

**Remark A.12.** *In practice, in research papers, one assumes in general an universal Leopoldt conjecture, so that the above statement becomes:*

*$L$ is $p$-rational if and only if $K$ is $p$-rational and $T$ is $p$-primitive*

*(equivalent to use the fixed point formula of Theorem A.5 and Corollary A.8).*

In the 1990's, the classical results on $p$-ramification, $p$-rationality, and $p$-regularity about the triviality of the tame kernel ${\rm K}_2(Z_K)$, are amply illustrated in various directions by Movahhedi, Nguyen Quang Do, Jaulent (see Movahhedi [26], Movahhedi–Nguyen Quang Do [70], Berger–Gras [83], Jaulent–Nguyen Quang Do [72], Jaulent–Sauzet [27] and Jaulent [17]): pro-$p$-group theory with explicit determination of a system of generators and relations for $\mathscr{G}_{K,S}$, Galois cohomology, Iwasawa's theory, Leopoldt and Gross conjectures.

Recall that in [77, Scolie, p. 112] Jaulent shows that, when $\mu_p \subset K$ the nullity of the $p$-Hilbert kernel ${\rm H}_2(L)\otimes\mathbb{Z}_p$ implies Leopoldt and Gross conjectures. Moreover [17] deals with ramification and decomposition.

Under the assumptions: $\mu_p \subset K$, ${\rm H}_2(L)\otimes\mathbb{Z}_p = 0$, for the Hilbert kernel, and the existence of $\mathfrak{p}_0 \in S$ such that $\mu_{K_{\mathfrak{p}_0}} = \mu_K$, some results in [84], after [25] and [70] on the primitive reciprocity laws, in the framework of $p$-rationality, describe (by means of generators and relations) the Galois group $\mathscr{G}_{K,S}$.

#### A.5.3 Links between these invariants and Iwasawa's theory

Despite the fact that we limit ourselves to arithmetical invariants of the base field (which is always possible), we give a short overview on the Iwasawa context and we indicate the main references for the reader.[4]

The base field invariants concerned are (in the case $S = P$), the torsion group $\mathscr{T}_{K,P}$, the $p$-Hilbert kernel $\mathrm{H}_2(K)\otimes\mathbb{Z}_p$, and the logarithmic class group $\widetilde{C\ell}_K$.

Let $K_\infty := K(\mu_{p^\infty})$, $\Gamma := \mathrm{Gal}(K_\infty/K) =: \langle\gamma\rangle$, $X$ the Galois group of the maximal abelian pro-$p$-extension of $K_\infty$, non-ramified and in which all places totally split. For a field $k$, we put $\mu_{p^\infty}(k) = \mu_{p^\infty}\cap k^\times$. For any module $M$ over the Iwasawa algebra, denote by $M(i)$ the $i$th twist on which $\Gamma$ acts by $\gamma\cdot m := \kappa^i(\gamma)\cdot m^\gamma$, where $\kappa$ is the cyclotomic character.

Then the interpretation of the above invariants, in the Iwasawa framework is given, in part, by the following two results:

**Theorem A.13.** *[16, Theorem 4.2]. Assuming the Leopoldt conjecture for p in K, one has the following exact sequence* $1\to\mu_{p^\infty}(K)\longrightarrow\bigoplus_{\mathfrak{p}\mid p}\mu_{p^\infty}(K_\mathfrak{p})\longrightarrow\mathscr{T}_{K,P}\longrightarrow\mathrm{Hom}_\Gamma(X,\mu_{p^\infty})\to 1$. *Then we have the following relation:*

$$\mathrm{Hom}_\Gamma(X,\mu_{p^\infty}) = \mathrm{Hom}_\Gamma(X(-1),\mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}(X(-1)_\Gamma,\mathbb{Q}_p/\mathbb{Z}_p)\simeq\mathrm{Gal}\big(H_K^{\mathrm{bp}}/\widetilde{K}^P\big)$$

*(see Remark 2.3), while (in relation with the paper of Federer–Gross–Sinnott [75]):*

$$X_\Gamma\simeq\widetilde{C\ell}_K. \tag{A.8}$$

The relation (A.8) is given in [67], then in [85, 17].

The considerable advantage of $\widetilde{C\ell}_K$, introduced in [73], is that it only involves some specific and explicit notions of classes and units of the base field $K$ and is then likely to be numerically calculated (Belabas–Jaulent [86]).

When $i$ varies, similar results may be interpreted by means of higher K-groups [87]. The main $\mathrm{K}_2$-theoretic interpretation is given as follows:

**Theorem A.14.** *[16, Theorem 5.6]. One has:* $(\mathrm{H}_2(K)\otimes\mathbb{Z}_p)^* = \mathrm{Ker}_P^2(\mathbb{Q}_p/\mathbb{Z}_p(-1))$*; if K contains* $\mu_{p^e}$, $e\geq 1$*, one obtains the perfect duality:* $\mathrm{Gal}\big(H_K^{\mathrm{bp}}/\widetilde{K}^P\big)[p^e]\times\big(\mathrm{H}_2(K)/p^e\mathrm{H}_2(K)\big)(-1)\longrightarrow\mu_{p^e}$*, where* $T[p^e] := \{x\in T,\ p^e\cdot x = 0\}$ *for a* $\mathbb{Z}_p$*-module T, and where* $\mathrm{Ker}_P^2$ *is the kernel of the localization homomorphism* $\mathrm{H}_2(\mathscr{G}_{K,P})\longrightarrow\bigoplus_{\mathfrak{p}\mid p}\mathrm{H}_2(\mathscr{G}_{K,\mathfrak{p}})$.

This result of duality does appear in [67, 85]. If $\mu_p\subset K$, the nullity of $\mathrm{H}_2(K)\otimes\mathbb{Z}_p$ is equivalent to that of $\widetilde{C\ell}_K$, which makes the link with the above Scolie [77, Scolie, p. 112] of Jaulent. For relations between logarithmic classes and higher K-groups, mention the work of Jaulent–Michel [88] and that of Hutchinson [89].

#### A.5.4 Synthesis 2003–2005

Because our Crelle papers, were written in french, whence largely ignored, all the results and consequences, that we have given in [61, 50, 63, 64, 66, 65, 51], were widely developed and improved in [1] where a systematic and general use of ramification and decomposition is considered, the infinite places playing a specific role (decomposition or complexification).

Furthermore, [1, Theorem V.2.4 and Corollary V.2.4.2] give a characterization (with explicit governing fields) of the existence of degree $p$ cyclic extensions of $K$ with given ramification and decomposition. This criteria has been used by Hajir–Maire and Hajir–Maire–Ramakrishna in several of their papers for results on $S$-ramified pro-$p$-groups (see, e.g., [90, Theorem 5.3], [91, Remark 2.2.] for the most recent publications).

### A.6 Present theoretical and algorithmic aspects

One may say that there is no important progress for $p$-rationality, for itself, since $p$-rational fields are in some sense the "simplest fields" in a $p$-adic sense, but that the significance of the $p$-adic properties of the groups $\mathscr{T}_{K,S}$, in much domains of number theory, has given a great lot of heuristics, conjectures, computations; so we shall now describe some of these aspects with some illustrations (it is not possible to be comprehensive since the concerned literature becomes enormous).

[4] This Subsection, describing the two different (but equivalent) techniques, is close to personal communications of Jean-François Jaulent and Thong Nguyen Quang Do (up to the notations and some comments). We thank them also for some remarks and corrections about this subsection.

### A.6.1 Absolute abelian Galois group $A_K$ of $K$

Let $K^{\rm ab}$ be the maximal abelian pro-extension of $K$. In [92], [93], Angelakis–Stevenhagen and Angelakis, after some work by Kubota [48] and Onabe [94], provide a direct computation of the *profinite group* $A_K := {\rm Gal}(K^{\rm ab}/K)$ for imaginary quadratic fields $K$, and use it to obtain many different $K$ that all have the same "minimal" absolute abelian Galois group, which is in some sense a condition of minimality of the groups $\mathscr{T}_{K,P}$ for all primes $p$. They obtain for instance, among other results and numerical illustrations:

**Theorem A.15.** *[92, Theorem 4.1 & Section 7]. An imaginary quadratic field $K \neq \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ of class number 1 has absolute abelian Galois group isomorphic to $\widehat{\mathbb{Z}}^2 \times \prod_{n\geq 1} \mathbb{Z}/n\mathbb{Z}$.*

This corresponds to the fact that such fields are $p$-rational for all $p$ (up to the factors $\mathscr{W}_{K,P}$ for $p = 2,3$). Then the generalization to an arbitrary $K$ involves the $\mathscr{T}_{K,P}$ for all primes $p$, giving:

**Theorem A.16.** *[95, Theorem 2.1 & Corollary 2.1]. Let $K^{\rm ab}$ be the maximal Abelian pro-extension of $K$. Let $\mathscr{H}_K$ be the compositum, over $p$, of the maximal $P$-ramified Abelian pro-$p$-extensions $H_{K,P}$ of $K$. Under the Leopoldt conjecture, there exists an Abelian extension $L_K$ of $K$ such that ${\rm Gal}(L_K/K) \simeq \prod_p \mathscr{T}_{K,P}$ and such that $\mathscr{H}_K$ is the direct compositum over $K$ of $L_K$ and the maximal $\widehat{\mathbb{Z}}$-extension of $K$, and such that we have the non-canonical isomorphism (for some explicit integers $\delta$ and $w$):*

$$
{\rm Gal}\big(K^{\rm ab}/L_K\big) = \widehat{\mathbb{Z}}^{r_2+1} \times {\rm Gal}\big(K^{\rm ab}/\mathscr{H}_K\big) \simeq \widehat{\mathbb{Z}}^{r_2+1} \times \prod_{n\geq 1}\Big((\mathbb{Z}/2\mathbb{Z})^{\delta} \times \mathbb{Z}/wn\mathbb{Z}\Big).
$$

Angelakis–Stevenhagen conjecture in [92, Conjecture 7.1] the infiniteness of imaginary quadratic fields $K$ such that $A_K \simeq \widehat{\mathbb{Z}}^2 \times \prod_{n\geq 1} \mathbb{Z}/n\mathbb{Z}$.

Note that when the $p$-class group of $K$ is non-trivial, $K$ is $p$-rational if and only if $C\ell_K$ is cyclic and the $p$-Hilbert class field $H_K$ is contained in $\widetilde{K}^P$ (assuming $\mathscr{W}_{K,P} = 1$).

Whence the importance of fields $K$ being $p$-rational for all $p$ (or more precisely such that $\mathscr{T}_{K,P} = \mathscr{W}_{K,P}$ for all $p$); it is an easier problem only for $\mathbb{Q}$ and imaginary quadratic fields, but dreadfully difficult when $K$ contains units of infinite order since it is an analogous question as for Fermat's quotients of algebraic numbers (various heuristics and conjectures in [96]), or values of $L$-functions which intervene as in Coates–Li [33, 34], Goren [97], and more or less, in many papers as Boeckle–Guiraud–Kalyanswamy–Khare [98] when the normalized $p$-adic regulator is a unit. We have conjectured that, in any given number field $K$, $\mathscr{T}_{K,P} = 1$ for almost all $p$.

### A.6.2 Greenberg's conjecture on Iwasawa's $\lambda$, $\mu$

For a totally real number field $K$, consider (under the Leopoldt conjecture) the cyclotomic $\mathbb{Z}_p$-extension $K^{\rm c}$ of $K$. Then Greenberg has conjectured in [99] that the Iwasawa's invariants $\lambda$ and $\mu$ are zero.

Equivalent formulations of this conjecture have been given, as in [100] for an encompassing approach covering the necessary and sufficient conditions considered by Greenberg in two particular cases (we give up to provide a complete bibliography), but we must mention that the two invariants $\mathscr{T}_{K,P}$ and $\widetilde{C\ell}_K$ (the logarithmic class group of Jaulent) are in some sense "governing invariants" for the Greenberg conjecture (in a theoretical and numerical viewpoint) and explain the $p$-adic obstructions for a standard proof in the framework of Iwasawa's theory; for instance, as soon as $\mathscr{T}_{K,P} = 1$ or $\widetilde{C\ell}_K = 1$, Greenberg's conjecture is true for trivial reasons. For this, see [101, Théorèmes 3.4, 4.8, 6.3] and [102] about $\mathscr{T}_{K,P}$, then the interpretation by Jaulent with the group of universal norms [103] and the following criterion (under the Gross-Leopoldt conjecture):

**Theorem A.17.** *[104, Théorème 7, § 1.4]. The totally real number field $K$ fulfills the conjecture of Greenberg if and only if its logarithmic class group $\widetilde{C\ell}_K$ capitulates in the cyclotomic $\mathbb{Z}_p$-extension $K^{\rm c}$ of $K$.*

If Greenberg's conjecture is true (which is no doubt), such general condition of capitulation is very reassuring since we recall that, on the contrary, the group $\mathscr{T}_{K,P}$ *never capitulates*. Moreover the property of capitulation (well known in Hilbert's class fields) is more general for generalized ray class groups and, especially, is possible in *absolute abelian extensions* as shown in many papers including [105], Bosca [106], then [81, 82].

This result may be deduced in the framework of Iwasawa's theory recalled in the § A.5.3 [100, Théorème 2.1].

Unfortunately, at the time of writing this text, no proof of Greenberg's conjecture does exist, despite some unsuccessful attempts in [107, 108] (to understand the key-points of the $p$-adic obstruction to be analyzed and possibly completed, see [104, § 3.4, Remark] and [102, § 6.2, Diagram]).

### A.6.3 Galois representations with open image

For constructions by Greenberg, in [109], of continuous Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{Z}_p)$ with open image, the $p$-rational fields play a great role, and the first obvious case is that of $p$-regular cyclotomic fields $\mathbb{Q}(\mu_p)$ which are $p$-rational (yet reported by [3], [65], and generalized by introducing in [71] the notion of $p$-regularity of number fields that we do not develop in this paper, for short, but which behaves as $p$-rationality; see a survey in [72]).

Then, an interesting typical conjecture is the following:

**Conjecture A.18.** *[109, Conjecture 4.2.1]. For any odd prime $p$ and for any $t \geq 1$, there exists a $p$-rational field $K$ such that $\mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^t$.*

Numerical examples and statistics have been given for various $p$ and $t$; see [109] and (Robert Bradshaw) the 3-rationality of:

$$K = \mathbb{Q}(\sqrt{13}, \sqrt{145}, \sqrt{209}, \sqrt{269}, \sqrt{373}, \sqrt{-1}).$$

Some PARI/GP programs are given in Pitoun–Varescon [110, 111], and [20, § 5.2] showing the 3-rationality of:

$$K = \mathbb{Q}(\sqrt{-2}, \sqrt{-5}, \sqrt{7}, \sqrt{17}, \sqrt{-19}, \sqrt{59}).$$

For fixed $p$ (e.g., $p = 3$), the probability of $p$-rationality decreases dramatically when $t \to \infty$; indeed, if $\mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^t$, $K$ is $p$-rational if and only if the $2^t - 1$ quadratic subfields $k$ of $K$ are $p$-rational whose probability is of the order of $\big(\frac{1}{p}\big)^{2^t-1}$ assuming that class groups and units of each $k$ are random and largely independent regarding the $p$-adic properties.

### A.6.4 Order of magnitude of $\mathscr{T}_{K,P}$ and conjectures

We have conjectured in [96, Conjecture 8.11] that for a fixed number field $K$, $\mathscr{T}_{K,P} = 1$ for all $p \gg 0$. Moreover, all numerical calculations show that the non-$p$-rationality constitutes an exception.

In another direction, fixing $p$ and taking $K$ in some given infinite family $\mathscr{K}$ (e.g., real fields of given degree $d$) we have given extensive numerical computations in direction of the following "$p$-adic Brauer–Siegel" property:

**Conjecture A.19.** *[30, Conjecture 8.1]. There exists a constant $\mathscr{C}_p(\mathscr{K})$ such that:*

$$v_p(\#\mathscr{T}_{K,p}) \leq \mathscr{C}_p(\mathscr{K}) \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)},$$

*for all $K \in \mathscr{K}$, where $\log_\infty$ is the usual complex logarithm.*

Thus there are two questions about $C_p(K) := \frac{v_p(\#\mathscr{T}_{K,P}) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}$ and the quantities $\mathscr{C}_p := \sup_K(C_p(K))$, $\mathscr{C}_K := \sup_p(C_p(K))$:

(i) The existence of $\mathscr{C}_K < \infty$, for a given $K$, only says that the Conjecture "$\mathscr{T}_{K,P} = 1\ \forall p \gg 0$" is true for the field $K$; for this field, we get $\limsup_p(C_p(K)) = 0$.

(ii) If $\mathscr{C}_p < \infty$ does exist for a given $p$, we have an universal $p$-adic analog of Brauer–Siegel theorem (the above Conjecture A.19).

These questions being out of reach, many results give, on the contrary, the infintness of primes $p$ yielding the $p$-rationality of a field $K$, in general under the $abc$ conjecture, following the method given by Silverman [112], Graves–Murty [113], Boeckle–Guiraud–Kalyanswamy–Khare [98], Maire–Rougnant [114]; for instance:

**Theorem A.20.** *[114, Corollary to Theorem A]. Let $K$ be a real quadratic field or an imaginary $S_3$-extension. If the generalized abc-conjecture holds for $K$, then $\#\big\{p \leq x,\ K \text{ is } p\text{-rational}\big\} \geq c \cdot \log(x)$ as $x \to \infty$, for some constant $c > 0$ depending on $K$.*

This shows the awesome distance between the two aspects of the problem; indeed, for $K = \mathbb{Q}(\sqrt{5})$, no prime number (up to $p < 10^{14}$ from Elsenhans–Jahnel: `https://oeis.org/A060305`) is known giving $\mathscr{T}_{K,P} \neq 1$.

In another viewpoint, as in [115] (after some works of Hartung, Horie, Naito) and [116], it is shown, using modular forms, the infiniteness of $p$-rational real quadratic fields for $p = 3$ and $p = 5$.

### A.6.5 Fermat curves

To study Fermat curves of exponent $p$, one uses the base field $K = \mathbb{Q}(\mu_p)$ and works in some Kummer extensions; for instance:

(i) Shu [117] gives general formulae of the root numbers of the Jacobian varieties of the Fermat curves $X^p + Y^p = \delta$, where $\delta$ is an integer, and studies their distribution. In this article the Vandiver conjecture or the regularity of $p$ implies some precise properties of the Selmer groups of these Jacobian varieties.

(ii) Davis–Pries [118] work in $P$-ramified Kummer extensions of $K$ with $P = \{\mathfrak{p} = (1-\zeta_p)\}$, as follows. Let $L \subset H_{K,P}$ be defined by:

$$L = K\big(\sqrt[p]{\zeta_p}, \sqrt[p]{1-\zeta_p}, \cdots, \sqrt[p]{1-\zeta_p^r}\big),\ r = \tfrac{p-1}{2},$$

The Kummer radical of $L$ is also generated by the real cyclotomic units and the numbers $\zeta_p$, $1-\zeta_p$. In the same way as previously, non-Vandiver's conjecture or non-regularity for $p$ are crucial obstructions.

Under the *Vandiver conjecture*, this radical is of $p$-rank $r+1$ since it is then given by $E_K \cdot \langle 1-\zeta_p \rangle$ modulo $K^{\times p}$.

Under the *regularity of* $p$, we get $\mathscr{T}_{K,P} = 1$ (reflection theorem (A.3)) and $L$ is the maximal $p$-elementary subextension of $H_{K,P}$; $L/K$ being $p$-ramified, whence $p$-primitively ramified (§ A.4.3), this gives the $p$-rationality of $L$.

Let $E$ be the maximal $p$-elementary subextension of $H_{L,P}$; since $\mathscr{T}_{L,P} = 1$ with $E/L$ $p$-ramified, we then have $\mathscr{T}_{E,P} = 1$ and $\mathrm{rk}_p(\mathrm{Gal}(E/L)) = r \cdot p^{r+1} + 1$. One can deduce that $C\ell_L = C\ell_E = 1$ since $E/K$ is totally ramified at $\mathfrak{p}$ (Theorem A.1 and Chevalley's formula in any successive $p$-cyclic extensions in $E/K$).

In simple cases as $p = 37$, where $\#C\ell_K = p$ and where $H_K \subseteq L$ in which $p$ splits, the formula of Theorem A.1 gives $\mathrm{rk}_p(\mathscr{T}_{L,P}) = \mathrm{rk}_p(C\ell_L^P) + p - 1$, whence $\mathrm{rk}_p(\mathrm{Gal}(E/L)) = r \cdot p^{r+1} + 2r + 1 + \mathrm{rk}_p(C\ell_L^P)$ depending on $C\ell_L^P$, a priori unknown.

The purpose of [118] is to get information on $\mathrm{H}^1(\mathrm{Gal}(E/K), M)$ for some $\mathrm{Gal}(E/K)$-modules $M$, subquotients of the relative homology $H_1(U, Y; \mathbb{F}_p)$ of the Fermat curve, where $U$ is the affine curve $x^p + y^p = 1$ and $Y$ the set of $2p$ cusps where $xy = 0$. They completely elucidate the case $p = 3$.

## A.7 Computational references and numerical tables

Many references may be cited:

The first table for the computation of $\#\mathscr{T}_{K,P}$ for imaginary quadratic fields is that of Charifi [119], using formula (A.6). In Hatada [120, 121] the computations correspond to statistics about the values (modulo $p$) of the normalized regulator $\mathscr{R}_{K,P}$ of real fields as $K = \mathbb{Q}(\sqrt{5})$ by the way of Fibonacci numbers and values at $2-p$ of zeta-functions as we have mentioned in § A.3.2. He obtains for instance that $\mathbb{Q}(\sqrt{2})$ is $p$-rational for all $p \leq 20000$, except $p = 13, 31$ (our program gives the next exception $p = 1546463$ up to $10^8$).

A precise study of $p$-rationality of imaginary quadratic fields is given by Angelakis–Stevenhagen in [92, Section 7].

A wide study of $\mathscr{T}_{K,P}$, with tables and publication of PARI/GP programs, is done by Pitoun [110, Chapitre 4], but these more conceptual programs are not so easy to manage by the reader. Then some statistical results with tables are given by Pitoun–Varescon in [111].

In [122] Hofmann–Zhang compute the valuation of the (usual) $p$-adic regulators of cyclic cubic fields with discriminant up to $10^{16}$, for $3 \leq p \leq 100$, and observe the distribution of these valuations.

About the conjecture of Greenberg [109] Kraft–Schoof [123] have computed such Iwasawa's invariants and confirm the conjecture for $p = 3$ and conductors $f$ of real quadratic fields $f \not\equiv 1 \pmod 3$ up to $10^4$. In [20], some heuristics on the conjecture and numerical examples are given with programs; then we illustrate the following conjecture of Hajir–Maire [43, Conjecture 4.16]:

*Given a prime $p$ and an integer $m \geq 1$, co-prime to $p$, there exist a totally imaginary field $K_0$ and a degree $m$ cyclic extension $K/K_0$ such that $K$ is $p$-rational; it is conjectured that the statement is true taking for $K_0$ an imaginary $p$-rational quadratic field.*

In [124, Table 1, § 2], Barbulescu–Ray give explicit $p$-rational large compositum of quadratic fields. We may cite some works by Bouazzaoui [125], El Habibi–Ziane [47] based on $p$-rationality of quadratic fields.

In the similar context of $p$-ramification, a new PARI/GP package allows the computation of the logarithmic class group $\widetilde{C\ell}_K$ of a number field by Belabas–Jaulent [86] that we can illustrate as follows where the invariants $[\mathsf{Y}, \mathsf{X}, \mathsf{Z}]$ are linked by the exact sequence:

$$1 \to X \longrightarrow Y := \widetilde{C\ell}_K \longrightarrow Z := C\ell_K^P := C\ell_K / \langle c\ell_K(P) \rangle \to 1.$$

```
{P=x^2+3;bp=2;Bp=10^8;K=bnfinit(P,1);print("P=",P);
forprime(p=bp,Bp,H=bnflog(K,p);if(H!=[[],[],[]],print("p=",p,"  ",H)))}

P=x^2 + 3
p=13       [[13], [13], []]
p=181      [[181], [181], []]
p=2521     [[2521], [2521], []]
p=76543    [[76543], [76543], []]
p=489061   [[489061], [489061], []]
p=6811741  [[6811741], [6811741], []]

P=x^2 + 5
p=5881     [[5881], [5881], []]
```

These are the only solutions for $p < 10^8$. More computations would give heuristics to see if the analogous conjecture: "$\widetilde{C\ell}_K = 1$ for all $p \gg 0$", is credible or not since the rarefaction of non-trivial cases is similar to that of the groups $\mathscr{T}_{K,P}$.

The case of real quadratic fields is studied in [101, § 5.2] with a table and in [104, § 2.4], after the work of Ozaki–Taya [126] and others.

In another direction, the paper [127] of Maire–Rougnant gives examples of triviality of isotopic components of the torsion groups $\mathscr{T}_{K,P}$; more precisely the fields $K$ are cyclic extensions of $\mathbb{Q}$ of degrees 3 and 4 (from polynomials of Balady, Lecacheux, Balady–Washington) and $S_3$-extensions of $\mathbb{Q}$.

In [30], are given numerous programs to test some heuristics and conjectures about the order of magnitude of the groups $\mathscr{T}_{K,P}$ in totally real number fields in a Brauer–Siegel framework.

## A.8 Conclusion and open questions

In all the aspects of $p$-rationality that we have developed (theoretical and computational), some interesting applications are done today, including for instance, for the most recent ones, [43] by Hajir–Maire on the $\mu$-invariant in Iwasawa's theory, then [90] by Hajir–Maire–Ramakrishna, showing the existence of $p$-rational fields having large $p$-rank of the class group, or [91] about the existence of a solvable number field $L$, $P$-ramified, whose $p$-Hilbert class field tower is infinite. See the bibliographies of these articles to expand the list of contributions.

Of course it is not possible to evoke all the studied families of pro-$p$-groups having some logical links with $S$-ramification (with more general sets $S$ regarding $P$) as, for instance, that of "mild groups" introduced by Labute [12] (and [13] for the case $p = 2$) dealing with the numbers of generators $d(G)$ and of relations $r(G)$ and defined as follows:

*A class of finitely presented pro-$p$-groups $G$ of cohomological dimension 2 such that $r(G) \geq d(G)$ and $d(G) \geq 2$ arbitrary.*

Many articles where then published giving an overview of the wide variety of such groups as the following short excerpt of a result of Schmidt about global fields [46, Theorem 1.1]:

*Let $S, T, \mathscr{M}$ be pairwise disjoint sets of places of $K$, where $S$ and $T$ are finite and $\mathscr{M}$ has Dirichlet density 0. Then there exists a finite set of places $S_0$ of $K$ which is disjoint from $S \cup T \cup \mathscr{M}$ and such that the group $\mathscr{G}^T_{K,S\cup S_0}$ has cohomological dimension 2.*

But let's go back to the basic abelian invariants, asking some open questions:

(i) We know the fixed point formula in a $p$-extension $L/K$ (under the conjecture of Leopoldt), but, even in a $p$-cyclic extension with Galois group $G$, and contrary to the case of $p$-class groups (as done in [128] after a very long history), we do not know how to compute the filtration $(M_i)_{i\geq 0}$, of $M := \mathscr{T}_{K,P}$, defined inductively by:

$$M_0 = 1 \text{ and } M_{i+1}/M_i := (M/M_i)^G, \text{ for all } i \geq 0.$$

(ii) The explicit computation of the $p$-rank, $\widetilde{r}_{K,S}$ (1.3), of $\mathscr{A}_{K,S}/\mathscr{T}_{K,S}$ for $S \subseteq P$, is available only in favorable Galois cases with an algebraic reasoning on the canonical representation $\mathbb{Q}_p \log_S(E_K)$ given by the Herbrand theorem on units under Leopoldt's conjecture (see § 2.4).

(iii) In the definition of $\mathscr{W}_{K,S} := W_{K,S}/\mathrm{tor}_{\mathbb{Z}_p}(\overline{E}^S_K)$, we do not know how to compute $\mathrm{tor}_{\mathbb{Z}_p}(\overline{E}^S_K) \supseteq \iota_S(\mu_K)$ when $S \subsetneq P$. We ignore, in a $p$-adic framework, if Leopoldt's conjecture is sufficient to obtain the responses apart from a Galois context.

A reasonable conjecture is that $\mathrm{tor}_{\mathbb{Z}_p}(\overline{E}^S_K) = \iota_S(\mu_K)$ whatever $K$, $p$ and $S$; but this must be deepened.

We hope that our programs in § 3.1.1 may help to give heuristics about this.

## Note

In the programs in verbatim text, one must replace the symbol of power (in aˆb) by the corresponding PARI/GP symbol (which is nothing else than that of the computer keyboard); otherwise the program does not work (this is due to the character font used by some Journals). The good print for the programs is also available at:

`https://www.dropbox.com/s/1srmksbr2ujf40i/Incomplete%20p-ramification.pdf?dl=0`

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# A Review on the Solutions of Difference Equations via Integer Sequences such as Fibonacci Numbers and Tribonacci Numbers

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**Abstract**
In this review article, we study the recent investigations on the forms of solutions of systems difference equations and difference equations in terms of well-known integer sequences such as Fibonacci numbers, Padovan numbers. We focus on the papers given some interesting relationships both between the exact solutions of difference equations and the integer sequences and between the equilibrium points of difference equations and golden ratio.

**Keywords:** Difference equations, Equilibrium point, Fibonacci number, Solutions, Tribonacci numbers
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## 1. Introduction

Difference equations and systems of difference equations are great importance in the field of mathematics as well as in other sciences. The applications of the theory of difference equations appear as discrete mathematical models of many phenomena such as in biology, economics, ecology, control theory, physics, engineering, population dynamics and so forth. Recently, there has been a growing interest in the study of finding closed-form solutions of difference equations and systems of difference equations. Some of the forms of solutions of these equations are representable via well-known integer sequences such as Fibonacci numbers, Lucas numbers, Pell numbers and Padovan numbers.

Now, we give information about integer sequences that establish a large part of our study.

- The Fibonacci sequence is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{1.1}$$

with initial conditions $F_0 = 0$, $F_1 = 1$. Also, it is obtained to extend the Fibonacci sequence backward as

$$F_{-n} = (-1)^{n+1} F_n.$$

The characteristic equation of (1.1) is $x^2 - x - 1 = 0$ such that the roots

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ (golden ratio) and } \beta = \frac{1-\sqrt{5}}{2}.$$

Also, there exists the following limit

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\alpha,$$

where $F_n$ is $n$th Fibonacci number.

- The Padovan sequence is defined by

$$P_n=P_{n-2}+P_{n-3},\ \ n\in\mathbb{N} \tag{1.2}$$

with initial conditions $P_{-2}=0$, $P_{-1}=0$, $P_0=1$.
The characteristic equation of (1.2) is $x^3-x-1=0$ such that the roots

$$\begin{aligned} p &= \frac{r^2+12}{6r} \\ q &= -\frac{r^2+12}{12r}-i\frac{\sqrt{3}}{2}\left(\frac{r}{6}-\frac{2}{r}\right) \\ t &= -\frac{r^2+12}{12r}+i\frac{\sqrt{3}}{2}\left(\frac{r}{6}-\frac{2}{r}\right) \end{aligned}$$

where $r=\sqrt[3]{108+12\sqrt{69}}$ and the unique real root is $p$ named as plastic number. Also, there exists the following limit

$$\lim_{n\to\infty}\frac{P_{n+1}}{P_n}=p,$$

where $P_n$ is $n$th Padovan number.

- Horadam sequence, a generalization of Fibonacci sequence, $(W_n(a,b;p,q))_{n\geq 0}$ or simply $(W_n)_{n\geq 0}$ is defined by

$$W_n=pW_{n-1}+qW_{n-2},\ \ W_0=a, W_1=b, \quad n\geq 2, \tag{1.3}$$

where $a$, $b$, $p$ and $q$ are arbitrary real numbers.
The characteristic equation of (1.3) is $x^2-px-q=0$ such that the roots

$$\lambda=\frac{p+\sqrt{p^2+4q}}{2} \text{ and } \mu=\frac{p-\sqrt{p^2+4q}}{2}.$$

Also, there exists the following limit

$$\lim_{n\to\infty}\frac{W_{n+1}}{W_n}=\lambda,$$

where $W_n$ is $n$th Horadam number.

- The generalized Padovan sequence, an extension of the padovan sequence, is defined by

$$S_n=pS_{n-2}+qS_{n-3},\ \ n\in\mathbb{N} \tag{1.4}$$

with initial conditions $S_{-2}=0$, $S_{-1}=0$, $S_0=1$, where $p$ and $q$ are arbitrary real numbers.
The characteristic equation of (1.4) is $x^3-px-q=0$ such that the roots

$$\begin{aligned} \phi &= \frac{R^2+12p}{6R} \\ \varphi &= -\frac{R^2+12p}{12R}+i\frac{\sqrt{3}}{2}\left(\frac{R}{6}-\frac{2p}{R}\right) \\ \psi &= -\frac{R^2+12p}{12R}-i\frac{\sqrt{3}}{2}\left(\frac{R}{6}-\frac{2p}{R}\right) \end{aligned}$$

where $R = \sqrt[3]{108q + 12\sqrt{-12p^3 + 81q^2}}$. Also, there exists the following limit

$$\lim_{n\to\infty} \frac{S_{n+1}}{S_n} = \phi,$$

where $S_n$ is $n$th generalized Padovan number.

- Generalized Tribonacci sequence is defined by

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}, \quad n \geq 3 \tag{1.5}$$

with initial conditions $V_0 = a, V_1 = b, V_2 = c$ and $r, s, t$ are real numbers.
The characteristic equation is $x^3 - rx^2 - sx - t = 0$, whose roots are

$$\begin{aligned}\alpha &= \alpha(r,s,t) = \frac{r}{3} + A + B\\ \beta &= \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B\\ \gamma &= \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B\end{aligned}$$

where

$$\begin{aligned}A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3}\\ \Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4},\\ \omega &= \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).\end{aligned}$$

- When $r = 1$, $s = 1$, $t = 1$ and $a = 0$, $b = 1$, $c = 1$ in (1.5), Tribonacci sequence is defined by

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \tag{1.6}$$

with initial conditions $T_0 = 0$, $T_1 = 1$, $T_2 = 1$. Also, it can be extended the Tribonacci sequence backward (negative subscripts) as

$$T_{-n} = T_{-n+3} - T_{-n+2} - T_{-n+1}.$$

The characteristic equation of (1.6) is

$$x^3 - x^2 - x - 1 = 0$$

such that the roots

$$\begin{aligned}\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}\\ \beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3}\\ \gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}\end{aligned}$$

where $\alpha$ is called Tribonacci constant and

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

is a primitive cube root of unity.
Furthermore, there exist the following limit

$$\lim_{n\to\infty} \frac{T_{n+r}}{T_n} = \alpha^r,$$

where $r \in \mathbb{Z}$ and $T_n$ is the $n$th Tribonacci number.

- Lucas sequence is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \ L_1 = 1.$$

- Pell sequence is defined by

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \ P_1 = 1.$$

- Pell-Lucas sequence is defined by

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 2, \ P_1 = 2.$$

- Jacobsthal sequence is defined by

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \ J_1 = 1.$$

- Jacobsthal-Lucas sequence is defined by

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 2, \ J_1 = 1.$$

- Perrin sequence is defined by

$$P_n = P_{n-2} + P_{n-3}, \quad P_0 = 3, \ P_1 = 0, \ P_2 = 2.$$

## 2. Literature review

In [1], Tollu et al. considered the following difference equations

$$x_{n+1} = \frac{1}{1+x_n}, \ y_{n+1} = \frac{1}{-1+y_n}, \ n = 0, 1, ..., \tag{2.1}$$

such that their solutions are associated with Fibonacci numbers, where initial conditions are $x_0 \in \mathbb{R} - \left\{-\frac{F_{m+1}}{F_m}\right\}_{m=1}^{\infty}$ and $y_0 \in \mathbb{R} - \left\{-\frac{F_{m+1}}{F_m}\right\}_{m=1}^{\infty}$ and $F_m$ is the $m$th Fibonacci number.

They investigated the some relationships both between Fibonacci numbers and solutions of equations (2.1) and between the golden ratio and equilibrium points of equations (2.1). Then, they proved that: the solutions of equations (2.1) are given by

$$x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0}, \ y_n = \frac{F_{-n} + F_{-(n-1)}y_0}{F_{-(n+1)} + F_{-n}y_0},$$

where $F_n$ is the $n$th Fibonacci number, and the nontrival solutions of equations (2.1) converge to $-\beta$ and $\beta$, so that $\beta$ is conjugate to the golden ratio.

Next, Rabago [2] presented a theoretical explanation in deriving the closed-form solution of Eq. (2.1) which Tollu et al. studied in [1] and provided another approach in proving Sroysang's conjecture (2013).

Then, in [3], Yazlik et al. studied the following rational difference equation systems

$$x_{n+1} = \frac{x_{n-1} \pm 1}{y_n x_{n-1}}, \ y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}}, \ n = 0, 1, ..., \tag{2.2}$$

such that their solutions associated with Padovan numbers. In their study, they obtained that the forms of solutions of system (2.2) are as follows

$$x_n = \begin{cases} \mp\frac{P_n x_{-1} y_0 \mp P_{n+1} x_{-1} + P_{n-1}}{P_{n-1} x_{-1} y_0 \mp P_n x_{-1} + P_{n-2}}, & \text{if } n \text{ is odd} \\ \mp\frac{P_n y_{-1} x_0 \mp P_{n+1} y_{-1} + P_{n-1}}{P_{n-1} y_{-1} x_0 \mp P_n y_{-1} + P_{n-2}}, & \text{if } n \text{ is even} \end{cases}$$

$$y_n = \begin{cases} \mp\frac{P_n y_{-1} x_0 \mp P_{n+1} y_{-1} + P_{n-1}}{P_{n-1} y_{-1} x_0 \mp P_n y_{-1} + P_{n-2}}, & \text{if } n \text{ is odd} \\ \mp\frac{P_n x_{-1} y_0 \mp P_{n+1} x_{-1} + P_{n-1}}{P_{n-1} x_{-1} y_0 \mp P_n x_{-1} + P_{n-2}}, & \text{if } n \text{ is even} \end{cases}$$

where $P_n$ is the $n$th Padovan number. Also, they demonstrated that every solutions of the systems (2.2) converge to point $(p,p)$ and $(-p,-p)$, where $p$ is the plastic number.

Tollu et al. [4] considered the following four Riccati difference equations

$$x_{n+1}=\frac{1+x_n}{x_n},\;\; y_{n+1}=\frac{1-y_n}{y_n},\;\; u_{n+1}=\frac{1}{u_n+1},\;\; v_{n+1}=\frac{1}{v_n-1}, \tag{2.3}$$

in which the initial conditions are real numbers. They derived the formulae for the solutions of equations (2.3) are given by

$$\begin{aligned} x_n &= \frac{F_{n+1}x_0+F_n}{F_nx_0+F_{n-1}},\\ y_n &= \frac{F_{-(n+1)}y_0+F_{-n}}{F_{-n}y_0+F_{-(n-1)}},\\ u_n &= \frac{F_n+F_{n-1}u_0}{F_{n+1}+F_nu_0},\\ v_n &= \frac{F_{-n}+F_{-(n-1)}v_0}{F_{-(n+1)}+F_{-n}v_0}, \end{aligned}$$

where $F_n$ is $n$th Fibonacci number, $F_{-n}$ is $n$th negative Fibonacci number. In addition to, they stated the asymptotic behaviors of the solutions of these equations and introduced that every solutions of these equations converge to their positive or negative equilibrium points.

Also, they in [5] studied the systems of difference equations

$$x_{n+1}=\frac{1+p_n}{q_n},\;\; y_{n+1}=\frac{1+r_n}{s_n},\;\; n\in\mathbb{N}_0,$$

where each of the sequences $p_n$, $q_n$, $r_n$ and $s_n$ is some of the sequences $x_n$ or $y_n$ by their own. They solved fourteen systems out of sixteen possible systems. In particularly, the representation formulae of solutions of twelve systems were stated via Fibonacci numbers. Also, for ten systems, they expressed that the solutions of these systems tend to the unique point $(\alpha,\alpha)$ where $\alpha$ is the golden ratio.

In [6], Halim concerned with the following systems of rational difference equations

$$x_{n+1}=\frac{1}{1+y_n},\;\; y_{n+1}=\frac{1}{1+x_n},\;\; n=0,1,..., \tag{2.4}$$

and

$$x_{n+1}=\frac{1}{1-y_n},\;\; y_{n+1}=\frac{1}{1-x_n},\;\; n=0,1,..., \tag{2.5}$$

initial conditions are arbitrary nonzero real numbers. He determined the form of solutions of system (2.4) as given below

$$\begin{aligned} x_{2n-1}&=\frac{F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0}, & x_{2n}&=\frac{F_{2n}+F_{2n-1}x_0}{F_{2n+1}+F_{2n}x_0},\\ y_{2n-1}&=\frac{F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0}, & y_{2n}&=\frac{F_{2n}+F_{2n-1}y_0}{F_{2n+1}+F_{2n}y_0}, \end{aligned}$$

and proved that the equilibrium point $E$ of system (2.4) is globally asymptotically stable, where $E=\left(\frac{-1+\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\right)=\left(\frac{1}{\alpha},\frac{1}{\alpha}\right)$, where $\alpha$ is the golden ratio. Furthermore, he established the solutions of system (2.5) are periodic with period six and are unstable.

In [7], Bacani and Rabago studied the behavior of solutions of the following nonlinear difference equations

$$x_{n+1}=\frac{q}{p+x_n^{v}}\;\text{ and }\; y_{n+1}=\frac{q}{-p+y_n^{v}}, \tag{2.6}$$

where $p$, $q\in\mathbb{R}^+$ and $v\in\mathbb{N}$. They proved that the solutions of equations (2.6) in case $v=1$ are as follows

$$\begin{aligned} x_n &= \frac{qW_n+x_0qW_{n-1}}{W_{n+1}+x_0W_n},\\ y_n &= \frac{qW_{-n}+y_0qW_{-(n-1)}}{W_{-(n+1)}+y_0W_{-n}}, \end{aligned}$$

where $W_n$ is the $n$th Horadam number.

In [8], Halim and Bayram investigated the solutions, stability character, and asymptotic behavior of the difference equation

$$x_{n+1} = \frac{\alpha}{\beta + \gamma x_{n-k}}, \ n \in \mathbb{N}_0,$$

where the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonzero real numbers, such that its solutions are associated to Horadam numbers, which are generalized Fibonacci numbers. Firstly, they had the difference equation

$$x_{n+1} = \frac{q}{p + x_{n-k}}, \tag{2.7}$$

by putting $q = \frac{\alpha}{\gamma}$ and $p = \frac{\beta}{\gamma}$. Then, they proved that the forms of the solutions of difference equation (2.7) are as follows

$$x_{(k+1)n+i} = \frac{W_{n+1} + W_n x_{i-(k+1)}}{W_{n+2} + W_{n+1} x_{i-(k+1)}} q, \ i = 1, 2, ..., k+1,$$

where $W_n$ is the $n$th Horadam number. Also, they obtained that the equilibrium point $E$ of difference equation (2.7) is globally asymptotically stable, where $E = \frac{-p+\sqrt{p^2+4q}}{2}$.

Then, in [9] Halim considered the system of difference equations

$$x_{n+1} = \frac{1}{1 + y_{n-2}}, \ y_{n+1} = \frac{1}{1 + x_{n-2}}, \ n = 0, 1, ..., \tag{2.8}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial conditions $x_{-2}$, $x_{-1}$, $x_0$, $y_{-2}$, $y_{-1}$, and $y_0$ are real numbers. He presented the relationship between Fibonacci numbers and the solutions of system (2.8), i.e., the form of the solutions of system (2.8) are given by

$$\begin{aligned} x_{6n+i} &= \tfrac{F_{2n+1}+F_{2n}y_{i-3}}{F_{2n+2}+F_{2n+1}y_{i-3}}, \quad i = 1, 2, 3, \\ y_{6n+i} &= \tfrac{F_{2n+1}+F_{2n}x_{i-3}}{F_{2n+2}+F_{2n+1}x_{i-3}}, \quad i = 1, 2, 3, \\ x_{6n+i} &= \tfrac{F_{2n+2}+F_{2n+1}x_{i-6}}{F_{2n+3}+F_{2n+2}x_{i-6}}, \quad i = 4, 5, 6, \\ y_{6n+i} &= \tfrac{F_{2n+2}+F_{2n+1}y_{i-6}}{F_{2n+3}+F_{2n+2}y_{i-6}}, \quad i = 4, 5, 6, \end{aligned}$$

where $F_n$ is the $n$th Fibonacci number. Otherwise, he showed that the equilibrium point $E$ of system (2.8) is globally asymptotically stable, where $E = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$.

El-Dessoky in [10] dealt with the following difference equation

$$x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}, \ n = 0, 1, ..., \tag{2.9}$$

where the parameters $\alpha$, $\beta$, $\gamma$ and $a$ and the initial conditions $x_{-t}$, $x_{-t+1}$, ,$x_{-1}$ and $x_0$ where $t = \max\{l, k\}$ are positive real numbers. He introduced the explicit formula of solutions of some special cases of Eq. (2.9) via Fibonacci numbers and also, discussed the global behavior of solutions of Eq. (2.9).

In [11], Halim and Rabago studied the systems of difference equaions

$$x_{n+1} = \frac{1}{\pm 1 \pm y_{n-k}}, \ y_{n+1} = \frac{1}{\pm 1 \pm x_{n-k}}, \ n, k \in \mathbb{N}_0,$$

where the initial conditions $x_{-k}$, $x_{-k+1}$, ..., $x_0$, $y_{-k}$, $y_{-k+1}$, ..., $y_0$ are nonzero real numbers.

Initially, they examined the form and behavior of solutions of system of difference equations

$$x_{n+1} = \frac{1}{1 + y_{n-k}}, \ y_{n+1} = \frac{1}{1 + x_{n-k}}. \tag{2.10}$$

Therefore, they determined that the exact solutions of system (2.10) are as follows

$$\begin{aligned} x_{2(k+1)n+i} &= \tfrac{F_{2n+1}+F_{2n}y_{i-(k+1)}}{F_{2n+2}+F_{2n+1}y_{i-(k+1)}}, \quad i = 1, 2, ..., k+1, \\ y_{2(k+1)n+i} &= \tfrac{F_{2n+1}+F_{2n}x_{i-(k+1)}}{F_{2n+2}+F_{2n+1}x_{i-(k+1)}}, \quad i = 1, 2, ..., k+1, \\ x_{2(k+1)n+i} &= \tfrac{F_{2n+2}+F_{2n+1}x_{i-(2k+2)}}{F_{2n+3}+F_{2n+2}x_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2, \\ y_{2(k+1)n+i} &= \tfrac{F_{2n+2}+F_{2n+1}y_{i-(2k+2)}}{F_{2n+3}+F_{2n+2}y_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2, \end{aligned}$$

and the equilibrium point of system (2.10) is globally asymptotically stable. In addition, the authors gave some results for other systems.

Then, in [12], the authors studied the rational difference equation

$$x_{n+1}=\frac{\alpha x_{n-1}+\beta}{\gamma x_n x_{n-1}},\ n\in\mathbb{N}_0, \tag{2.11}$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\alpha,\ \beta,\ \gamma\in\mathbb{R}^+$ and the initial conditions nonzero real numbers and also investigated the two-dimensional case of the this equation given by

$$x_{n+1}=\frac{\alpha x_{n-1}+\beta}{\gamma y_n x_{n-1}},\ y_{n+1}=\frac{\alpha y_{n-1}+\beta}{\gamma x_n y_{n-1}},\ n\in\mathbb{N}_0. \tag{2.12}$$

Firstly, they reduced the difference equation (2.11) to the difference equation

$$x_{n+1}=\frac{p x_{n-1}+q}{x_n x_{n-1}} \tag{2.13}$$

by using changes variables $p=\frac{\alpha}{\gamma}$ and $q=\frac{\beta}{\gamma}$. Then, they presented that the closed-form solution of difference equation (2.13) is given by

$$x_n=\frac{S_{n+1}x_{-1}+S_n x_0 x_{-1}+qS_{n-1}}{S_n x_{-1}+S_{n-1}x_0 x_{-1}+qS_{n-2}},$$

where $S_n$ is the $n$th generalized Padovan number and the equilibrium point of Eq (2.13) is globally asymptotically stable.

Later, they reduced the system of difference equation (2.12) to the system

$$x_{n+1}=\frac{p x_{n-1}+q}{y_n x_{n-1}},\ y_{n+1}=\frac{p y_{n-1}+q}{x_n y_{n-1}} \tag{2.14}$$

by using changes variables $p=\frac{\alpha}{\gamma}$ and $q=\frac{\beta}{\gamma}$. Then, they presented that the closed-form solutions of system (2.14) are given by

$$x_n = \begin{cases} \frac{S_{n+1}y_{-1}+S_n x_0 y_{-1}+qS_{n-1}}{S_n y_{-1}+S_{n-1}x_0 y_{-1}+qS_{n-2}}, & \text{if } n \text{ is even,} \\ \frac{S_{n+1}x_{-1}+S_n y_0 x_{-1}+qS_{n-1}}{S_n x_{-1}+S_{n-1}y_0 x_{-1}+qS_{n-2}}, & \text{if } n \text{ is odd,} \end{cases}$$

$$y_n = \begin{cases} \frac{S_{n+1}x_{-1}+S_n y_0 x_{-1}+qS_{n-1}}{S_n x_{-1}+S_{n-1}y_0 x_{-1}+qS_{n-2}}, & \text{if } n \text{ is even,} \\ \frac{S_{n+1}y_{-1}+S_n x_0 y_{-1}+qS_{n-1}}{S_n y_{-1}+S_{n-1}x_0 y_{-1}+qS_{n-2}}, & \text{if } n \text{ is odd,} \end{cases}$$

and the equilibrium point of the system (2.14) is global attractor.

Then, in [13], Stevic et al. the following nonlinear second-order difference equation

$$x_{n+1}=a+\frac{b}{x_n}+\frac{c}{x_n x_{n-1}},\ n\in\mathbb{N}_0, \tag{2.15}$$

in which parameters $a$, $b$, $c$ and the initial values $x_{-1}$ and $x_0$ are complex numbers such that $c\neq 0$. Next, they used the following change of variables

$$x_n=\frac{y_n}{y_{n-1}},$$

and obtained the following third-order linear difference equation with constant coefficients

$$y_{n+1}=ay_n+by_{n-1}+cy_{n-2}.$$

After, they introduced that the representation formula of every solution of Eq. (2.15) is

$$x_n=\frac{(s_{n+1}-as_n)x_{-1}+s_n x_0 x_{-1}+cs_{n-1}}{(s_n-as_{n-1})x_{-1}+s_{n-1}x_0 x_{-1}+cs_{n-2}},$$

where $s_n$ is the $n$th generalized Padovan number. Note that, Eq. (2.11) is a special case of Eq. (2.15) such that $a=0$.

Alotaibi et al. in [14] considered the following systems of difference equations

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \ y_{n+1} = \frac{x_n x_{n-2}}{\pm y_{n-1} \pm x_{n-2}}, \ n = 0, 1, ..., \tag{2.16}$$

where the initial conditions $x_{-2}$, $x_{-1}$, $x_0$, $y_{-2}$, $y_{-1}$, $y_0$ are arbitrary positive real numbers. They analyzed the solutions of the systems (2.16) such that their solutions are associated with Fibonacci numbers.

In [15], El-Dessoky et al. examined the following difference equation

$$y_{n+1} = \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}}, \ n = 0, 1, ..., \tag{2.17}$$

where $\alpha$, $\beta$, $A$, and $B$ are real numbers and the initial values $y_{-4}$, $y_{-3}$, $y_{-2}, y_{-1}$and $y_0$ are positive real numbers. They presented the solutions of Eq. (2.17) in terms of Fibonacci numbers according to some special cases of the parameters $\alpha$, $\beta$, $A$, and $B$.

Then, in [16], Matsunaga and Suzuki studied the following system of rational difference equations

$$x_{n+1} = \frac{a y_n + b}{c y_n + d}, \ y_{n+1} = \frac{a x_n + b}{c x_n + d}, \ n = 0, 1, ..., \tag{2.18}$$

where the parameters $a$, $b$, $c$, $d$ and the initial values $x_0$, $y_0$ are real numbers. They obtained that the explicit solutions of system (2.18) are as follows

$$\begin{aligned} x_{2n-1} &= \frac{(a y_0 + b) G_{2n-1} + (bc - ad) y_0 G_{2n-2}}{G_{2n} + (c y_0 - a) G_{2n-1}}, \ x_{2n} = \frac{(a x_0 + b) G_{2n} + (bc - ad) x_0 G_{2n-1}}{G_{2n+1} + (c x_0 - a) G_{2n}}, \\ y_{2n-1} &= \frac{(a x_0 + b) G_{2n-1} + (bc - ad) x_0 G_{2n-2}}{G_{2n} + (c x_0 - a) G_{2n-1}}, \ y_{2n} = \frac{(a y_0 + b) G_{2n} + (bc - ad) y_0 G_{2n-1}}{G_{2n+1} + (c y_0 - a) G_{2n}}, \end{aligned}$$

where $G_n$ is a generalized Fibonacci sequence defined by

$$G_{n+2} = (a + d) G_{n+1} + (bc - ad) G_n,$$

with $G_0 = 0$ and $G_1 = 1$. Moreover, they presented that every solution of system (2.18) converges to its equilibrium points.

In [17], Öcalan and Duman considered the following nonlinear recursive difference equation

$$x_{n+1} = \frac{x_{n-1}}{x_n}, \ n = 0, 1, ..., \tag{2.19}$$

with any nonzero initial values $x_{-1}$ and $x_0$. Then, they extended their all results to solutions of the following nonlinear recursive equations

$$x_{n+1} = \left( \frac{x_{n-1}}{x_n} \right)^p, \ p > 0 \text{ and } n = 0, 1, ..., \tag{2.20}$$

with any nonzero initial values $x_{-1}$ and $x_0$. Later, they obtained that the exact solution of Eq. (2.19) is

$$x_n = \begin{cases} \frac{x_{-1}^{f_{n-1}}}{x_0^{f_n}} & \text{if } n = 1, 3, 5, ..., \\ \frac{x_0^{f_n}}{x_{-1}^{f_{n-1}}} & \text{if } n = 2, 4, 6, ..., \end{cases}$$

where $f_n$ is the $n$th Fibonacci number. Under the special case of initial values, they determined that there exist non-oscillatory positive solutions of Eq. (2.19), which converge monotonically to the equilibrium point 1.

Furthermore, they given that the exact solution of Eq. (2.20) is

$$x_n = \begin{cases} \frac{x_{-1}^{g_{n-1}(p)}}{x_0^{f_n(p)}} & \text{if } n = 1, 3, 5, ..., \\ \frac{x_0^{f_n(p)}}{x_{-1}^{f_{n-1}(p)}} & \text{if } n = 2, 4, 6, ..., \end{cases}$$

where $f_n(p)$ and $g_n(p)$ are the $n$th Fibonacci-type number. And also, under the special case of initial values, they demonstrated that there exist non-oscillatory positive solutions of Eq. (2.20), which converge monotonically to the equilibrium point 1 and the Eq. (2.20) has unbounded solutions.

Next, Akrour et al. [18] studied the following system of difference equations

$$x_{n+1}=\frac{ay_nx_{n-1}+bx_{n-1}+c}{y_nx_{n-1}},\ y_{n+1}=\frac{ax_ny_{n-1}+by_{n-1}+c}{x_ny_{n-1}},\ n=0,1,...,$$

where the parameters $a$, $b$, $c$ are arbitrary real numbers with $c\neq 0$ and the initial values $x_{-1}$, $x_0$, $y_{-1}$ and $y_0$ are arbitrary nonzero real numbers. They examined that the explicit solutions of system (2.10) are given by

$$\begin{aligned}
x_{2n+1} &= \frac{cJ_{2n+1}+(J_{2n+3}-aJ_{2n+2})x_{-1}+J_{2n+2}x_{-1}y_0}{cJ_{2n}+(J_{2n+2}-aJ_{2n+1})x_{-1}+J_{2n+1}x_{-1}y_0},\\
x_{2n+2} &= \frac{cJ_{2n+2}+(J_{2n+4}-aJ_{2n+3})y_{-1}+J_{2n+3}x_0y_{-1}}{cJ_{2n+1}+(J_{2n+3}-aJ_{2n+2})y_{-1}+J_{2n+2}x_0y_{-1}},\\
y_{2n+1} &= \frac{cJ_{2n+1}+(J_{2n+3}-aJ_{2n+2})y_{-1}+J_{2n+2}x_0y_{-1}}{cJ_{2n}+(J_{2n+2}-aJ_{2n+1})y_{-1}+J_{2n+1}x_0y_{-1}},\\
y_{2n+2} &= \frac{cJ_{2n+2}+(J_{2n+4}-aJ_{2n+3})x_{-1}+J_{2n+3}x_{-1}y_0}{cJ_{2n+1}+(J_{2n+3}-aJ_{2n+2})x_{-1}+J_{2n+2}x_{-1}y_0},
\end{aligned}$$

where $J_n$ is defined by the recurrent relation

$$J_{n+3}=aJ_{n+2}+bJ_{n+1}+cJ_n,\ n\in\mathbb{N},$$

such that $J_0=0$, $J_1=1$, $J_2=a$.

Then, Okumuş and Soykan in [19] considered the following four difference equations

$$x_{n+1}=\frac{1}{x_n(x_{n-1}-1)-1},\quad n=0,1,...,$$

$$x_{n+1}=\frac{1}{x_n(x_{n-1}+1)+1},\quad n=0,1,..., \tag{2.21}$$

$$x_{n+1}=\frac{-1}{x_n(x_{n-1}-1)+1},\quad n=0,1,...,$$

$$x_{n+1}=\frac{-1}{x_n(x_{n-1}+1)-1},\quad n=0,1,...,$$

and determined the solutions of these difference equations are associated to Tribonacci numbers. For example, the solutions of Eq.(2.21) are

$$x_n=\frac{T_{n-1}x_{-1}x_0+(T_{n+1}-T_n)x_0+T_n}{T_nx_{-1}x_0+(T_{n-1}+T_n)x_0+T_{n+1}},$$

where the initial conditions $x_{-1},x_0\in\mathbb{R}-F_2$, with $F_2$ is the forbidden set of Eq.(2.21) given by

$$F_2=\cup_{n=-1}^{\infty}\{(x_{-1},x_0):T_nx_{-1}x_0+(T_{n-1}+T_n)x_0+T_{n+1}=0\},$$

and for the others see [19].

Also, in [21], they examined the following systems of difference equations

$$x_{n+1}=\frac{\pm 1}{y_n(x_{n-1}\pm 1)+1},\ y_{n+1}=\frac{\pm 1}{x_n(y_{n-1}\pm 1)+1},\ n=0,1,...,$$

and proved the exact solutions of these systems of difference equations via Tribonacci numbers. E.g. the form of solutions $\{x_n,y_n\}_{n=-1}^{\infty}$ of one of these systems is given by

$$\begin{aligned}
x_{2n-1} &= \frac{T_{2n-2}x_{-1}y_0+(T_{2n}-T_{2n-1})y_0+T_{2n-1}}{T_{2n-1}x_{-1}y_0+(T_{2n-2}+T_{2n-1})y_0+T_{2n}},\\
x_{2n} &= \frac{T_{2n-1}y_{-1}x_0+(T_{2n+1}-T_{2n})x_0+T_{2n}}{T_{2n}y_{-1}x_0+(T_{2n-1}+T_{2n})x_0+T_{2n+1}},\\
y_{2n-1} &= \frac{T_{2n-2}y_{-1}x_0+(T_{2n}-T_{2n-1})x_0+T_{2n-1}}{T_{2n-1}y_{-1}x_0+(T_{2n-2}+T_{2n-1})x_0+T_{2n}},\\
y_{2n} &= \frac{T_{2n-1}x_{-1}y_0+(T_{2n+1}-T_{2n})y_0+T_{2n}}{T_{2n}x_{-1}y_0+(T_{2n-1}+T_{2n})y_0+T_{2n+1}},
\end{aligned}$$

where the initial conditions $x_{-1}$, $y_{-1}$, $x_0$, $y_0 \in \mathbb{R} - F_1$, with $F_1$ is the forbidden set of system given by

$$F_1 = \cup_{n=-1}^{\infty} \{(x_{-1}, y_{-1}, x_0, y_0) : A_n = 0 \text{ or } B_n = 0 \text{ or } C_n = 0 \text{ or } D_n = 0\}$$

where

$$\begin{aligned} A_n &= T_{2n-1}x_{-1}y_0 + (T_{2n-2} + T_{2n-1})\, y_0 + T_{2n}, \\ B_n &= T_{2n}y_{-1}x_0 + (T_{2n-1} + T_{2n})\, x_0 + T_{2n+1}, \\ C_n &= T_{2n-1}y_{-1}x_0 + (T_{2n-2} + T_{2n-1})\, x_0 + T_{2n}, \\ D_n &= T_{2n}x_{-1}y_0 + (T_{2n-1} + T_{2n})\, y_0 + T_{2n+1}. \end{aligned}$$

Next, they in [22] studied the following difference equation

$$x_{n+1} = \frac{\gamma}{x_n (x_{n-1} + \alpha) + \beta}, \quad n = 0, 1, ..., \tag{2.22}$$

where the parameters $\alpha$, $\beta$ and $\gamma$ are nonnegative real numbers with $\gamma \neq 0$ and the initial values $x_{-1}$ and $x_0$ are arbitrary nonzero real numbers. They examined that the exact solutions of Eq.(2.22) is given by

$$x_n = \frac{tV_{n-1}x_{-1}x_0 + (V_{n+1} - rV_n)\, x_0 + V_n}{tV_n x_{-1}x_0 + (V_{n+2} - rV_{n+1})\, x_0 + V_{n+1}},$$

where $V_n$ is defined by the recurrent relation

$$V_{n+3} = rV_{n+2} + sV_{n+1} + tV_n, \ n \in \mathbb{N},$$

such that $V_0 = 0$, $V_1 = 1$, $V_2 = r$.

Besides these studies, for related studies on solving difference equations and systems of difference equations and investigating the asymptotic behavior of their solutions, see [20, 23-38].

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# Paranorm Ideal Convergent Fibonacci Difference Sequence Spaces

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**Abstract**
In this paper we introduce some new sequence spaces $c_0^I(\hat{F},p)$, $c^I(\hat{F},p)$ and $\ell_\infty^I(\hat{F},p)$ for $p=(p_n)$, a sequence of positive real numbers. In addition, we study some topological and algebraic properties on these spaces. Lastly, we examine some inclusion relations on these spaces.

**Keywords:** Fibonacci difference matrix, $I$–Cauchy, $I$–convergence, Paranormed space
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## 1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers, respectively. We denote

$$\omega = \{x=(x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the vector space of all real or complex sequences. Any vector subspace of $\omega$ is called a sequence space.

**Definition 1.1.** *Let $X$ be a linear space. A function $g : X \to \mathbb{R}$, is called paranorm if for all $x,y \in X$,*

*(i)* $g(x) \geq 0$ *for all* $x \in X$,

*(ii)* $g(-x) = g(x)$,

*(iii)* $g(x+y) \leq g(x)+g(y), \forall x,y \in X$,

*(iv)* $(c_n)$ *is a sequence of scalars with* $c_n \to c(n \to \infty)$ *and* $(x_n)$ *is a sequence of vetors with* $g(x_n - x) \to 0$ *as* $(n \to \infty)$*, then* $g(x_n c_n - xc) \to 0$ *as* $(n \to \infty)$.

A paranorm $g$ which $g(x) = 0$ implies that $x = \theta$ is called a total paranorm and the pair $(X,g)$ is called a totally paranormed space. The concept of paranorm is related to the linear metric spaces given by some total paranorm [1]. The notion of paranormed sequence was studied at the initial stage by Nakano[2] and Simons [3]. Later on it was investigated by Maddox [4, 5] and others [6]. Tripathy and Hazarika [7] generalized the sequence spaces of Maddox to introduced the new idea of paranorm $I$–convergent sequence spaces $c_0^I(p)$, $c^I(p)$, $\ell_\infty^I(p)$ and $\ell_\infty(p)$ where $p=(p_n)$ is the sequence of strictly positive real numbers.

Initially, as a generalization of statistical convergence which was first introduced by Fast [8] and Steinhaus [9] for real and complex sequences, the notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko et al.[10].

Recall that a family of sets $I \subseteq 2^{\mathbb{N}}$ is called an ideal if (i) for each $A, B \in I \Rightarrow A \cup B \in I$, (ii) for each $A \in I, B \subseteq A \Rightarrow B \in I$. An ideal $I$ is said to be admissible if $I \neq 2^{\mathbb{N}}$ and contains every finite subset of $\mathbb{N}$ and $I$ is said to be maximal if there cannot exist any non–trivial ideal $J \neq I$ containing $I$ as a subset. For each ideal $I$ there is a filter $\mathscr{F}(I)$ which corresponds to $I$ ( filter associated with ideal $I$), defined by $\mathscr{F}(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$. The notion of $I$–convergence defined in [10] as the sequence $(x_n) \in \omega$ is said to be $I$–convergent to a number $L \in \mathbb{C}$ if, for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to $I$. And we write $I\!-\!\lim x_n = L$. In case $L = 0$ then $(x_n) \in \omega$ is said to be $I$–null. Where $I$ assumed to be admissible. Some notions for usual convergence have been extended with respect to the admissible ideal in $\mathbb{N}$, such as the notions of bounded and Cauchy sequence extended to $I$–bounded and $I$–Cauchy defined in [11], respectively, as follows: A sequence $(x_n) \in \omega$ is said to be $I$–Cauchy if, for every $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that the set $\{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\}$ belongs to $I$. A sequence $(x_n) \in \omega$ is said to be $I$–bounded if there exists $K > 0$, such that, the set $\{n \in \mathbb{N} : |x_n| > K\}$ belongs to $I$. Throughout the paper, $c^I$, $c_0^I$ and $\ell_\infty^I$ represent the $I$–convergent, $I$–null and $I$–bounded sequence spaces, respectively. Further, details on ideal convergence see, [12, 13, 14, 15, 16, 17] and their references.

Let $\lambda$ and $\mu$ be two arbitrary sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. By the sequence space $\lambda_A$ defined by $\lambda_A := \{x = (x_k) \in \omega : Ax \in \lambda\}$, we denote the domain of the matrix $A$ in the space $\lambda$, the sequence $Ax = \{A_n(x)\}$ for all $x \in \lambda$, the $A$–transform of $x$, is in $\mu$ defined by $A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$, for each $n \in \mathbb{N}$. By $(\lambda, \mu)$, we denote the class of all matrices $A$ such that $\lambda \subseteq \mu_A$. Many researchers have addressed this approach to constructing a new sequence space by means of the matrix domain of a particular limitation method; see, for instance, [18, 19, 20, 21, 22, 23]. Recently, by using the sequence of Fibonacci numbers $\{f_n\}_{n=0}^{\infty}$ defined by the linear recurrence equalities $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}, n \geq 2$, Kara [24] defined the Fibonacci band matrix $\hat{F} = (f_{nk})$ as follows:

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n} & , (k = n-1) \\ \frac{f_n}{f_{n+1}} & , (k = n) \\ 0 & , 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, and introduced some new difference sequence spaces by means of the matrix $\hat{F}$. Where the notion of difference sequence spaces was firstly introduced by Kizmaz[25] for more detail [26, 27, 28, 29, 30]. Afterward, Kara and Demiriz [24] introduced the paranormed sequence spaces $c_0(\hat{F}, p)$, $c(\hat{F}, p)$ and $\ell_\infty(\hat{F}, p)$ related to the matrix domain of $\hat{F}$. i.e.,

$$c_0(\hat{F}, p) = \left\{ x = (x_n) \in \omega : \lim_{n \to \infty} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^{p_n} = 0 \right\}$$

$$c(\hat{F}, p) = \left\{ x = (x_n) \in \omega : \exists L \in \mathbb{C} \ni \lim_{n \to \infty} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^{p_n} = L \right\}$$

$$\ell_\infty(\hat{F}, p) = \left\{ x = (x_n) \in \omega : \sup_{n \in \mathbb{N}} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^{p_n} < \infty \right\}.$$

Lately, by combining the definitions of Fibonacci difference matrix $\hat{F}$ and the notion of ideal convergence, Khan et al.[13] introduced the sequence spaces $c_0^I(\hat{F})$, $c^I(\hat{F})$, and $\ell_\infty^I(\hat{F})$ defined as the set of all sequences whose $\hat{F}$–transforms are in the spaces $c_0^I$, $c^I$ and $\ell_\infty^I$, respectively, defined as follows:

$$\lambda_{\hat{F}} = \{x = (x_k) \in \omega : \hat{F}_n(x) \in \lambda\} \text{ for } \lambda = \{c_0^I, c^I, \ell_\infty^I\},$$

where the sequence $\hat{F}_n(x)$ is frequently used as the $\hat{F}$–transform of the sequence $x = (x_n)$ defined by

$$\hat{F}_n(x) = \begin{cases} \frac{f_0}{f_1} x_0 = x_0 & , n = 0 \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} & , n \geq 1 \quad \text{for all } n \in \mathbb{N}. \end{cases} \tag{1.1}$$

In this paper, by using the Fibonacci difference matrix $\hat{F}$ and same technique we introduce the paranorm ideal convergent Fibonacci difference sequence spaces $c_0^I(\hat{F}, p)$, $c^I(\hat{F}, p)$, and $\ell_\infty^I(\hat{F}, p)$ related to the matrix domain of $\hat{F}$ in the sequence spaces $c_0^I(p)$, $c^I(p)$ and $\ell_\infty^I(p)$. Further, we study some topological and algebraic properties on these spaces and examine some inclusion relations concerning these spaces.

**Definition 1.2.** *[13] Let $x = (x_n)$ and $z = (z_n)$ be two sequences. We say that $x_n = z_n$ for almost all n relative to I (in short a.a.n.r.I) if the set $\{n \in \mathbb{N} : x_n \neq z_n\} \in I$.*

**Definition 1.3.** *[31] A sequence space E is said to be symmetric, if $(x_{\pi(n)}) \in E$ whenever $(x_n) \in E$ where $\pi(n)$ is a permutation on $\mathbb{N}$.*

**Definition 1.4.** *[31] A sequence space E is said to be solid or normal, if $(\alpha_n x_n) \in E$ whenever $(x_n) \in E$ and for any sequence of scalars $(\alpha_n) \in \omega$ with $|\alpha_n| < 1$, for every $n \in \mathbb{N}$.*

**Definition 1.5.** *[31] Let $K = \{n_i \in \mathbb{N} : n_1 < n_2 < \dots\} \subseteq \mathbb{N}$ and E be a sequence space. A K–step space of E is a sequence space*

$$\lambda_K^E = \{(x_{n_i}) \in \omega : (x_n) \in E\}.$$

*A canonical pre–image of a sequence $(x_{n_i}) \in \lambda_K^E$ is a sequence $(y_n) \in \omega$ defined as follows:*

$$y_n = \begin{cases} x_n & \text{, if } n \in K \\ 0 & \text{, otherwise.} \end{cases}$$

*A canonical pre–image of a step space $\lambda_K^E$ is a set of canonical pre–images of all elements in $\lambda_K^E$. i.e., y is in canonical pre–image of $\lambda_K^E$ iff y is canonical pre–image of some element $x \in \lambda_K^E$.*

**Definition 1.6.** *[31] A sequence space E is said to be monotone, if it contains the canonical pre–images of its step space.*

**Lemma 1.7.** *[31] Every solid space is monotone.*

**Lemma 1.8.** *[ [31], Lemma 2.5] Let $K \in \mathscr{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

**Lemma 1.9.** *[ Lascarides [32], Proposition 1] Let $h = \inf p_n$, $H = \sup_n p_n$. Then the following conditions are equivalent:*

**(i)** $H < \infty$ *and* $h > 0$,

**(ii)** $c_0(p) = c_0$ *or* $\ell_\infty(p) = \ell_\infty$,

**(iii)** $\ell_\infty\{p\} = \ell_\infty(p)$,

**(iv)** $c_0\{p\} = c_0(p)$,

**(v)** $\ell\{p\} = \ell(p)$.

## 2. Main results

In this section, we introduce the paranormed sequence spaces $c_0^I(\hat{F}, p)$, $c^I(\hat{F}, p)$ and $\ell_\infty^I(\hat{F}, p)$ related to the matrix domain of $\hat{F}$ in the sequence spaces $c_0^I(p)$, $c^I(p)$ and $\ell_\infty^I(p)$. Further, we study some inclusion theorems and study some topological and algebraic properties on these resulting. We assume throughout this section that the sequences $x = (x_n)$ and $(\hat{F}_n(x))$ are connected by relation (1.1) and $p = (p_n)$ be a sequence of positive real numbers and $I$ is an admissible ideal of subset of $\mathbb{N}$. We define

$$c_0^I(\hat{F}, p) := \left\{x = (x_n) \in \omega : \left\{n \in \mathbb{N} : |\hat{F}_n(x)|^{p_n} \geq \varepsilon\right\} \in I\right\},$$

$$c^I(\hat{F}, p) := \left\{x = (x_n) \in \omega : \left\{n \in \mathbb{N} : |\hat{F}_n(x) - L|^{p_n} \geq \varepsilon, \text{ for some } L \in \mathbb{C}\right\} \in I\right\},$$

$$\ell_\infty^I(\hat{F}, p) := \left\{x = (x_n) \in \omega : \exists K > 0 \text{ s.t } \left\{n \in \mathbb{N} : |\hat{F}_n(x)|^{p_n} > K\right\} \in I\right\}.$$

We write

$$m_0^I(\hat{F}, p) := c_0^I(\hat{F}, p) \cap \ell_\infty(\hat{F}, p),$$

and

$$m^I(\hat{F}, p) := c^I(\hat{F}, p) \cap \ell_\infty(\hat{F}, p).$$

**Theorem 2.1.** *The sequence spaces $c^I(\hat{F},p)$, $c_0^I(\hat{F},p)$, $\ell_\infty^I(\hat{F},p)$, $m_0^I(\hat{F},p)$ and $m^I(\hat{F},p)$ are linear spaces.*

*Proof.* Let $x=(x_n)$, $y=(y_n)$ be two arbitrary elements of the space $c^I(\hat{F},p)$ and $\alpha,\beta$ be scalars. Now, since $x,y\in c^I(\hat{F},p)$, then for given $\varepsilon>0$, we have

$$\left\{n\in\mathbb{N}:|\hat{F}_n(x)-L_1|^{p_n}\geq\frac{\varepsilon}{2},\text{ for same } L_1\in\mathbb{C}\right\}\in I,$$

and

$$\left\{n\in\mathbb{N}:|\hat{F}_n(y)-L_2|^{p_n}\geq\frac{\varepsilon}{2},\text{ for same } L_2\in\mathbb{C}\right\}\in I.$$

Now, let

$$A_x=\left\{n\in\mathbb{N}:|\hat{F}_n(x)-L_1|^{p_n}<\frac{\varepsilon}{2M_1}\right\}\in\mathscr{F}(I),$$

$$A_y=\left\{n\in\mathbb{N}:|\hat{F}_n(y)-L_2|^{p_n}<\frac{\varepsilon}{2M_2}\right\}\in\mathscr{F}(I),$$

be such that $A_x^c,A_y^c\in I$, where $M_1=D\cdot\max\{1,\sup_n|\alpha|^{p_n}\}$, $M_2=D\cdot\max\{1,\sup_n|\beta|^{p_n}\}$ and $D=\max\{1,2^{H-1}\}$ and $H=\sup_n p_n\geq 0$. Then

$$\{n\in\mathbb{N}:|(\alpha\hat{F}_n(x)+\beta\hat{F}_n(y))-(\alpha L_1+\beta L_2)|^{p_n}<\varepsilon\}\supseteq\left\{\left\{n\in\mathbb{N}:|\alpha|^{p_n}|\hat{F}_n(x)-L_1|^{p_n}<\frac{\varepsilon}{2M_1}|\alpha|^{p_n}D\right\}\right.$$
$$\left.\cap\left\{n\in\mathbb{N}:|\beta|^{pn}|\hat{F}_n(x)-L_2|^{p_n}<\frac{\varepsilon}{2M_2}|\beta|^{p_n}D\right\}\right\}. \tag{2.1}$$

Thus, the set on the right hand side of equation (2.1) belongs to $\mathscr{F}(I)$. By definition of filter associated with an ideal the complement of the set on the left hand side of (2.1) belongs to $I$. This implies that $(\alpha x+\beta y)\in c^I(\hat{F},p)$. Hence, $c^I(\hat{F},p)$ is a linear space. The proof for other spaces will follow similarly. □

**Theorem 2.2.** *The classes of sequences $m^I(\hat{F},p)$ and $m_0^I(\hat{F},p)$ are paranormed spaces, paranormed by $g(x_n)=\sup_n|x_n|^{\frac{p_n}{M}}$, where $M=\max\{1,\sup_n p_n\}$.*

*Proof.* The proof of the result is easy, so omitted. □

**Theorem 2.3.** *The set $m^I(\hat{F},p)$ is closed subspace of $\ell_\infty(\hat{F},p)$.*

*Proof.* Let $(x_n^{(m)})$ is a Cauchy sequence in $m^I(\hat{F},p)$ such that $(x^{(m)})\to x$. We show that $x\in m^I(\hat{F},p)$. Since $(x_n^{(m)})\in m^I(\hat{F},p)$, then there exists $(a_m)$, and for every $\varepsilon>0$ such that

$$\{n\in\mathbb{N}:|\hat{F}_n^{(m)}(x)-a_m|^{p_n}\geq\varepsilon\}\in I.$$

We need to show that

**(i)** $(a_m)$ converges to $a$.

**(ii)** If $A=\{n\in\mathbb{N}:|\hat{F}_n(x)-a|^{p_n}<\varepsilon\}$, then $A^c\in I$.

(i) Since $(x_n^{(m)})$ be a Cauchy sequence in $m^I(\hat{F},p)$ then for a given $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\sup_n|\hat{F}_n^{(m)}(x)-\hat{F}_n^{(r)}(x)|^{\frac{p_n}{M}}<\frac{\varepsilon}{3},\text{ for all } m,r\geq n_0.$$

For a given $\varepsilon>0$, we have

$$B_{mr}=\{n\in\mathbb{N}:|\hat{F}_n^{(m)}(x)-\hat{F}_n^{(r)}(x)|^{p_n}<\left(\frac{\varepsilon}{3}\right)^M\},$$

$$B_r=\{n\in\mathbb{N}:|n\in\mathbb{N}:|\hat{F}_n^{(r)}(x)-a_r|^{p_n}<\left(\frac{\varepsilon}{3}\right)^M\},$$

$$B_m = \{n \in \mathbb{N} : |\hat{F}_n^{(m)}(x) - a_m|^{p_n} < \left(\frac{\varepsilon}{3}\right)^M\}.$$

Then $B^c_{mr}$, $B^c_r$, $B^c_m \in I$. Let $B^c = B^c_{mr} \cup B^c_m \cup B^c_r$, where

$$B = \{n \in \mathbb{N} : |a_m - a_r|^{p_n} < \varepsilon\}.$$

Then $B^c \in I$. We choose $n_0 \in B^c$, then for each $m, r \geq n_0$ we have

$$\{n \in \mathbb{N} : |a_m - a_r|^{p_n} < \varepsilon\} \supseteq \Bigg[\left\{n \in \mathbb{N} : |a_m - \hat{F}_n^{(m)}(x)|^{p_n} < \left(\frac{\varepsilon}{3}\right)^M\right\} \cap \left\{n \in \mathbb{N} : |\hat{F}_n^{(m)}(x) - \hat{F}_n^{(r)}(x)|^{p_n} < \left(\frac{\varepsilon}{3}\right)^M\right\} \cap \left\{n \in \mathbb{N} : |\hat{F}_n^{(r)}(x) - a_r|^{p_n} < \left(\frac{\varepsilon}{3}\right)^M\right\}\Bigg].$$

Then $(a_m)$ is a Cauchy sequence in $\mathbb{C}$. So, there exists a scalar $a \in \mathbb{C}$ such that $a_m \to a$, as $m \to \infty$.
(ii) For the next step, let $0 < \delta < 1$ be given. Then, we show that if

$$A = \{n \in \mathbb{N} : |\hat{F}_n(x) - a|^{p_n} < \delta\}$$

then $A^c \in I$. Since $x^{(m)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that,

$$A_1 = \{n \in \mathbb{N} : |\hat{F}_n^{(q_0)}(x) - \hat{F}_n(x)|^{p_n} < (\frac{\delta}{3D})^M\} \tag{2.2}$$

implies $A^c_1 \in I$. The numbers $q_0$ can be so chosen that together with (2.2) , we have

$$A_2 = \{n \in \mathbb{N} : |a_{q_0} - a|^{p_n} < (\frac{\delta}{3D})^M\}$$

such that $A^c_2 \in I$. Since $\{n \in \mathbb{N} : |\hat{F}_n^{(q_0)}(x) - a_{q_0}|^{p_n} \geq \delta\} \in I$, then, we have a subset $A_3$ of $\mathbb{N}$ such that $A^c_3 \in I$, where

$$A_3 = \left\{n \in \mathbb{N} : |\hat{F}_n^{(q_0)}(x) - a_{q_0}|^{p_n} < (\frac{\delta}{3D})^M\right\}.$$

Let $A^c = A^c_1 \cup A^c_2 \cup A^c_3$, where $A = \{n \in \mathbb{N} : |\hat{F}_n(x) - a|^{p_n} < \delta\}$. Therefore, for each $n \in A^c$, we have

$$\{n \in \mathbb{N} : |\hat{F}_n(x)) - a|^{p_n} < \delta\} \supseteq \Bigg[\left\{n \in \mathbb{N} : |\hat{F}_n(x) - \hat{F}_n^{(q_0)}(x)|^{p_n} < \left(\frac{\delta}{3D}\right)^M\right\} \cap \left\{n \in \mathbb{N} : |\hat{F}_n^{(q_0)}(x) - a_{q_0}|^{p_n} < \left(\frac{\delta}{3D}\right)^M\right\} \cap \left\{n \in \mathbb{N} : |a_{q_0} - a|^{p_n} < \left(\frac{\delta}{3D}\right)^M\right\}\Bigg].$$

Then the result follows. □

**Corollary 2.4.** *The set $m^I_0(\hat{F}, p)$ is closed subspace of $\ell_\infty(\hat{F}, p)$.*

Since the inclusions $m^I(\hat{F}, p) \subset \ell_\infty(\hat{F}, p)$ and $m^I_0(\hat{F}, p) \subset \ell_\infty(\hat{F}, p)$ are strict, so in view of last theorem, we have the following result.

**Theorem 2.5.** *The spaces $m^I(\hat{F}, p)$ and $m^I_0(\hat{F}, p)$ are nowhere dense subsets of $\ell_\infty(\hat{F}, p)$.*

**Theorem 2.6.** *The spaces $c^I_0(\hat{F}, p)$ and $m^I_0(\hat{F}, p)$ are solid and monotone.*

*Proof.* We shall prove the result for $c_0^I(\hat{F},p)$. The other result follows similarly. Let $x=(x_k)\in c_0^I(\hat{F},p)$ and $\alpha=(\alpha_n)$ be a sequence of scalars with $|\alpha|\le 1$, for all $n\in\mathbb{N}$. Since $|\alpha|^{p_n}\le \max\{1,|\alpha_n|^{p_n}\}\le 1$, for all $n\in\mathbb{N}$, we have

$$|\hat{F}_n(\alpha x)|^{p_n}\le|\alpha\hat{F}_n(x)|^{p_n}\le|\hat{F}_n(x)|^{p_n} \text{ for all } n\in\mathbb{N}.$$

From this we have

$$\{n\in\mathbb{N}:|\hat{F}_n(\alpha x)|^{p_n}\ge\varepsilon\}\subseteq\{n\in\mathbb{N}:|\hat{F}_n(x)|^{p_n}\ge\varepsilon\}\in I$$

which implies

$$\{n\in\mathbb{N}:|\hat{F}_n(\alpha x)|^{p_n}\ge\varepsilon\}\in I.$$

Therefore, $(\alpha x_n)\in c_0^I(\hat{F},p)$. Hence, the space $c_0^I(\hat{F},p)$ is solid, and hence, by Lemma 1.7 the space $c_0^I(\hat{F},p)$ is monotone. ☐

**Theorem 2.7.** *The spaces $c^I(\hat{F},p)$, $m^I(\hat{F},p)$ are neither monotone nor solid in general.*

*Proof.* Here we give a counter example for establishment of this result.

**Example 2.8.** *Let $I=I_f=\{A\subseteq\mathbb{N}: A \text{ is finite }\}$. Let $p_n=1$ if $n$ is even and $p_n=2$ if $n$ is odd. Consider the $K$–step spaces $E_K(\hat{F},p)$ of $E(\hat{F},p)$ defined as follows: Let $x=(x_n)\in E(\hat{F},p)$ and $y=(y_n)\in E_K(\hat{F},p)$ be such that*

$$\hat{F}_n(y)=\begin{cases}\hat{F}_n(x) & \text{, if } n \text{ is even}\\ 0 & \text{, otherwise.}\end{cases}$$

*Consider the sequence $x=(x_n)\in\omega$ such that $\hat{F}_n(x)=\frac{1}{n}$, for all $n\in\mathbb{N}$. Then $(x_n)\in E(\hat{F},p)$, but its $K^{th}$–step space pre–image does not belong to $E(\hat{F},p)$, where $E=c^I$ and $m^I$. Thus $c^I(\hat{F},p)$ and $m^I(\hat{F},p)$ are not monotone and hence by Lemma1.7 the spaces $c^I(\hat{F},p)$ and $m^I(\hat{F},p)$ are not solid.*

☐

**Theorem 2.9.** *Let $(p_n)$ and $(q_n)$ be two sequences of positive real numbers. Then $m_0^I(\hat{F},q)\subseteq m_0^I(\hat{F},p)$, if and only if $\lim_{n\in A} inf\,\frac{p_n}{q_n}>0$, where $A\subseteq\mathbb{N}$ such that $A\in\mathscr{F}(I)$.*

*Proof.* Let $\lim_{n\in A} inf\,\frac{p_n}{q_n}>0$ and $(x_n)\in m_0^I(\hat{F},q)$. Then there exists $\beta>0$ such that $p_n>\beta q_n$, for all sufficiently large $n\in A$. Since $(x_n)\in m_0^I(\hat{F},q)$, for a given $\varepsilon>0$, we have

$$B=\{n\in\mathbb{N}:|\hat{F}_n(x)|^{q_n}\ge\varepsilon\}\in I. \tag{2.3}$$

Let $G=A^c\cup B$. Then $G\in I$. Then for all sufficiently large $n\in G$,

$$\{n\in\mathbb{N}:|\hat{F}_n(x)|^{p_n}\ge\varepsilon\}\subseteq\{n\in\mathbb{N}:|\hat{F}_n(x)|^{\beta q_n}\ge\varepsilon\}\in I.$$

Therefore, $(x_n)\in m_0^I(\hat{F},p)$. The converse part of the result follows obviously. ☐

**Corollary 2.10.** *Let $(p_n)$ and $(q_n)$ be two sequences of positive real numbers. Then $m_0^I(\hat{F},p)=m_0^I(\hat{F},q)$ and only if $\liminf_{n\in A} p_nq_n>0$, where $A\subseteq\mathbb{N}$ such that $A\in\mathscr{F}(I)$.*

**Theorem 2.11.** *If $I$ neither maximal nor $I=I_f$, then the space $H(\hat{F},p)$ are not symmetric, where $H=c_0^I,c^I,m_0^I$, and $m^I$.*

*Proof.* We prove the result with the help of the following example.

**Example 2.12.** *Let $I=I_c=\{A\subseteq\mathbb{N}:\sum_{n\in A}n^{-1}<\infty\}$, (see [33]). Let*

$$A=\{n:n=s^2 \text{ or } t^3, \text{ for } s,t\in\mathbb{N}\}=\{n\in\mathbb{N}:n=s^2, \text{ for } n\in\mathbb{N}\}\cup\{n\in\mathbb{N}:n=t^3,t\in\mathbb{N}\},$$

*then*

$$\sum_{n\in A}n^{-1}<\infty.$$

*Let*

$$p_n = \begin{cases} 1 & \text{, if } n \text{ is even ,} \\ 2 & \text{, if } n \text{ is odd .} \end{cases}$$

*Consider the sequence* $x = (x_n)$ *such that*

$$\hat{F}_n(x) = \begin{cases} n^{-1} & \text{, if } n = t^3, t \in \mathbb{N}, \\ 0 & \text{, otherwise.} \end{cases}$$

*Consider the rearrangement* $\hat{F}_n(y)$ *of* $\hat{F}_n(x)$ *defined by*

$$\hat{F}_n(y) = (\hat{F}_1(x), \hat{F}_3(x), \hat{F}_3(x), \hat{F}_8(x), \hat{F}_4(x), \hat{F}_5(x), \hat{F}_{27}(x), \hat{F}_6(x), \hat{F}_7(x), \hat{F}_{64}(x), \hat{F}_8(x), \hat{F}_9(x), \dots).$$

*Then* $(y_n) \notin H(\hat{F}, p)$*, but* $(x_n) \in H(\hat{F}, p)$*, where* $H = c_0^I, c^I, m_0^I$*, and* $m^I$*.*

□

**Theorem 2.13.** *The spaces* $m_0^I(\hat{F}, p)$ *and* $m^I(\hat{F}, p)$ *are not separable.*

*Proof.* Let $A = \{m_1 < m_2 < \dots\}$ be an infinite subset of $\mathbb{N}$ such that $A \in I$. Let

$$p_n = \begin{cases} 1, & \text{if } n \in A; \\ 2, & \text{otherwise.} \end{cases}$$

Let $P = \{(\hat{F}_n(x)) : \hat{F}_n(x) = 0 \text{ or } 1, \text{if } n \in A; \hat{F}_n(x) = 0, \text{otherwise}\}$. Since $A$ is infinite, so $P$ is uncountable. Consider the class of open balls $B_1 = \{B(\hat{F}_n(z), \frac{1}{2}) : \hat{F}_n(z) \in P\}$. Let $C_1$ be an open cover of $m_0^I(\hat{F}, P)$ or $m^I(\hat{F}, p)$ containing $B_1$. Since $B_1$ is uncountable, so $C_1$ cannot be reduced to a countable subcover for $m_0^I(\hat{F}, p)$ as well as $m^I(\hat{F}, p)$. Thus, $m_0^I(\hat{F}, p)$ and $m^I(\hat{F}, p)$ are not separable. □

**Theorem 2.14.** *Let* $H = \sup_n p_n < \infty$ *and* $I$ *be a maximal admissible ideal. Then the following are equivalent:*

**(a)** $(x_n) \in c^I(\hat{F}, p)$,

**(b)** *There exists* $(y_n) \in c(\hat{F}, p)$ *such that* $x_n = y_n$*, for a.a.n.r.I,*

**(c)** *There exists* $(y_n) \in c(\hat{F}, p)$ *and* $(z_n) \in c_0^I(\hat{F}, p)$ *such that* $x_n = y_n + z_n$ *for all* $n \in \mathbb{N}$ *and* $\{n \in \mathbb{N} : |\hat{F}_n(x) - L|^{p_n} \geq \varepsilon\} \in I$*.*

**(d)** *There exists a subset* $K = \{n_i : i \in \mathbb{N}, n_1 < n_2 < n_3 < \dots\}$ *of* $\mathbb{N}$*, such that* $K \in \mathscr{F}(I)$ *and* $\lim_{n\to\infty} |\hat{F}_{n_i}(x) - L|^{p_{n_i}} = 0$.

*Proof.* **(a) implies (b).** Let $x = (x_n) \in c^I(\hat{F}, p)$, then for any $\varepsilon > 0$, there exists a number $L \in \mathbb{C}$ such that

$$\{n \in \mathbb{N} : |\hat{F}_n(x) - L|^{p_n} \geq \varepsilon\} \in I.$$

Let $(m_t)$ be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{n \leq m_t : |\hat{F}_n(x) - L|^{p_n} \geq t^{-1}\} \in I.$$

Define a sequence $y = (y_n)$ as $y_n = x_n$ for all $n \leq m_1$. For $m_t < n < m_{t+1}$, for $t \in \mathbb{N}$,

$$y_n = \begin{cases} x_n, & \text{if } |\hat{F}_n(x) - L|^{p_n} < t^{-1} \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_n) \in c(\hat{F}, p)$ and from the following inclusion

$$\{n \leq m_t : x_n \neq y_n\} \subseteq \{n \in \mathbb{N} : |\hat{F}_n(x) - L| \geq \varepsilon\} \in I$$

we get $x_n = y_n$ for *a.a.n.r.I*.

**(b) implies (c).** For $x = (x_n) \in c^I(\hat{F}, p)$ there exists $y = (y_n) \in c(\hat{F}, p)$ such that $x_n = y_n$ for $a.a.n.r.I$. Let $K = \{n \in \mathbb{N} : x_n \neq y_n\}$, then $K \in I$. Define a sequence $z = (z_n)$ as follows:

$$z_n = \begin{cases} x_n - y_n, & \text{if } n \in K \\ 0, & \text{otherwise.} \end{cases}$$

Then $(z_n) \in c_0^I(\hat{F}, p)$ and so $(y_n) \in c(\hat{F}, p)$.

**(c) implies (d).** Suppose (c) holds. Let $\varepsilon > 0$ be given. Let $P = \{n \in \mathbb{N} : |\hat{F}_n(x)|^{p_n} \geq \varepsilon\} \in I$, and

$$K = P^c = \{(n_i \in \mathbb{N} : i \in \mathbb{N}, n_1 < n_2 < n_3 < \dots\} \in \mathscr{F}(I).$$

Then we have

$$\lim_{i \to \infty} |\hat{F}_{n_i}(x) - L|^{p_{n_i}} = 0.$$

**(d) implies (a).** Let $\varepsilon > 0$ be given and suppose that (c) holds. Then for any $\varepsilon > 0$, and by Lemma 1.9 we have

$$\{n \in \mathbb{N} : |\hat{F}_n(x) - L|^{p_n} \geq \varepsilon\} \subseteq K^c \cup \{n \in K : |\hat{F}_n(x) - L|^{p_n} \geq \varepsilon\}.$$

Thus $(x_n) \in c^I(\hat{F}, p)$.

□

**Theorem 2.15.** *The sequence spaces:*

**(i)** *$c^I(\hat{F}, p)$ and $\ell_\infty(\hat{F}, p)$ overlap but neither one contains the other,*

**(ii)** *$c_0^I(\hat{F}, p)$ and $\ell_\infty(\hat{F}, p)$ overlap but neither one contains the other.*

*Proof.* **(i)** We prove that $c^I(\hat{F}, p)$ and $\ell_\infty(\hat{F}, p)$ are not disjoint. Consider the sequence $x = (x_n) \in \omega$ such that $\hat{F}_n(x) = \frac{1}{n}$ for $n \in \mathbb{N}$. Then, $x \in c^I(\hat{F}, p)$ but $x \in \ell_\infty(\hat{F}, p)$. Next, define the sequence $x = (x_n) \in \omega$ such that

$$\hat{F}_n(x) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is square} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $x \in c^I(\hat{F}, p)$ but $x \notin \ell_\infty(\hat{F}, p)$. Next, choose the sequence $x = (x_n) \in \omega$ such that

$$\hat{F}_n(x) = \begin{cases} n, & \text{if is even} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(x) \in \ell_\infty(\hat{F}, p)$ but $x \notin c^I(\hat{F}, p)$.

**(ii)** The proof is similar to proof of part one.

□

## 3. Conclusion

In this paper, we defined some new paranorm ideal convergent Fibonacci difference sequence spaces $c_0^I(\hat{F}, p)$, $c^I(\hat{F}, p)$ and $\ell_\infty^I(\hat{F}, p)$ as the sets of all sequences are in the space $c_0^I(p)$, $c^I(p)$ and $\ell_\infty^I(p)$ respectively. Furthermore, we studied some topological and algebraic properties of these spaces such as solidity, monotonicity and overlap. Also, we provided an example to show that these, new sequence spaces are not symmetric and show that the sets $m_0^I(\hat{F}, p)$ and $m^I(\hat{F}, p)$ are not separable.

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# The Univalent Function Created by the Meromorphic Functions Where Defined on the Period Lattice

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**Abstract**
The function $\xi(z)$ is obtained from the logarithmic derivative function $\sigma(z)$. The elliptic function $\wp(z)$ is also derived from the $\xi(z)$ function. The function $\wp(z)$ is a function of double periodic and meromorphic function on lattices region. The function $\wp(z)$ is also double function. The function $\varphi(z)$ meromorphic and univalent function was obtained by the serial expansion of the function $\wp(z)$. The function $\varphi(z)$ obtained here is shown to be a convex function.

**Keywords:** Convex function, Elliptic function, Latices, Meromorphic function
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## 1. Introduction

We begin this important paper by introducing some important functions and some important classes.

**Definition 1.1.** $\mathsf{A}$ *get the subset of complex numbers* $\mathbb{C}$ *. If* $\mathsf{A}$ *is a group according to the collection process, then* $\mathsf{A}$ *in called a module defined on the ring of integers* $\mathbb{Z}$*.*

**Definition 1.2.** *If the module* $\mathsf{A}$ *does not have a stack point in the finite plane, then this module* $\mathsf{A}$ *is called a lattice. Lattices can be divided into three groups as follows.*

*i. Zero dimensional lattices;*

$$W_m = \{m\omega : m = 0 \in \mathbb{Z}, \omega \neq 0 \in \mathbb{C}\}$$

*ii. One dimensional lattices;*

$$W_m = \{m\omega_1 : m \neq 0 \in \mathbb{Z}, \omega \neq 0 \in \mathbb{C}\}$$

*iii. Two dimensional lattices;*

$$W_{m,n} = \{m\omega_1 + n\omega_2 : m \neq 0, n \neq 0 \in \mathbb{Z}, \omega_1 \neq 0, \omega_2 \neq 0 \in \mathbb{C}\}$$

**Lemma 1.3.** *The function* $\xi(z)$ *is absolute and uniform convergence [1].*

*Proof.*

$$\xi(z) = \frac{1}{z} + \sum_{m,n\neq(0,0)} \left(\frac{1}{z-W} + \frac{1}{W} + \frac{z}{W^2}\right)$$

where

$$\begin{aligned}
\sum_{m,n\neq(0,0)} &= \sum_m\sum_n \left|\frac{1}{z-W_{mn}} + \frac{1}{W_{mn}} - \frac{z}{(W_{mn})^2}\right| = \left|\frac{(W_{mn})^2 + (z-W_{mn})W_{mn} + (1-W_{mn})z}{(z-W_{mn})(W_{mn})^2}\right| = \left|\frac{z}{(z-W_{mn})(W_{mn})^2}\right| \\
&= \left|\frac{z}{(1-\frac{z}{W_{mn}})(W_{mn})^2}\right| \leq \frac{|z|}{(1-\frac{|z|}{|W_{mn}|})\,|W_{mn}|^2} < \frac{2\,|z|}{|W_{mn}|^2}.
\end{aligned}$$

For all m,n such that $|W| > 2|z|$ the series under consideration in therefore absolutely and convergent.Thus, function $\xi(z)$ has a simple pole at point $z = W$. In that case, $\xi(z)$ is meromorphic. On the other hand it is clear that $\xi(z)$ in the odd function so $\xi(z) = -\xi(-z)$. □

**Theorem 1.4.** *The function $\xi(z)$ has following the power series for point $z = 0$ .*

$$\xi(z) = \frac{1}{z} - \frac{A_2}{3} - \frac{A_4}{5} - \ldots = \frac{1}{z} - \sum_{k\geq 2} \frac{A_{2k-2}}{2k-1} z^{2k-1}$$

*Proof.* Let

$$\xi(z) = \frac{1}{z} + \sum_{m,n\neq(0,0)} \left(\frac{1}{z-W} + \frac{1}{W} + \frac{z}{W^2}\right)$$

$$\xi(z) = \frac{1}{z} + \sum_{m,n\neq(0,0)} \left(\frac{1}{-W(1-\frac{z}{W})} + \frac{1}{W} + \frac{z}{W^2}\right)$$

then

$$\begin{aligned}
\xi(z) &= \frac{1}{z} + \sum_{m,n\neq(0,0)} \left[-\frac{1}{W}(1 + \frac{z}{W} + (\frac{z}{W})^2 + \ldots + \frac{1}{W} + \frac{z}{W^2}\right] \\
&= \frac{1}{z} + \sum_{m,n\neq(0,0)} \frac{1}{-\Delta_{mn}} \left[1 + \frac{z}{\Delta_{mn}} + \left(\frac{z}{\Delta_{mn}}\right)^2 + \ldots + \frac{1}{\Delta_{mn}} + \left(\frac{z}{(\Delta_{mn})^2}\right)\right] \\
&= \frac{1}{z} - \sum_{m,n\neq(0,0)} \frac{1}{-\Delta_{mn}} \left[\frac{z^2}{(\Delta_{mn})^3} + \frac{z^3}{(\Delta_{mn})^4} + \frac{z^4}{(\Delta_{mn})^5} + \ldots\right] \\
&= \frac{1}{z} + \sum_{m,n\neq(0,0)} \frac{1}{-W} \left[\frac{z^2}{W^3} + \frac{z^3}{W^4} + \frac{z^4}{W^5} + \ldots\right] \\
&= \frac{1}{z} - \sum_{m,n\neq(0,0)} \sum_{k=2} \frac{1}{W^{k+1}} z^k = \frac{1}{z} - \sum_{k=2} A_{k+1}.z^k \\
&= \frac{1}{z} - \sum_{k\geq 2} (z^2 + z^3 + z^4 + \ldots).A_{k+1}
\end{aligned}$$

where $A_{k+1} = \sum_{m,n\neq(0,0)}$ .

Coefficients of toms $z^{2k}$ in evidently zero for k=1,2,3, since the functions $\xi(z)$ is an odd function, ie equality is as follows

$$\xi(z) = \frac{1}{z} - \frac{A_2}{3} - \frac{A_4}{5} - \ldots = \frac{1}{z} - \sum_{k\geq 2} \frac{A_{2k-2}}{2k-1} z^{2k-1}.$$

□

**Definition 1.5.** *Weierstrass's function $\wp(z)$ is defined by the double series as*

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n \neq (0,0)} \left[ \frac{1}{(z-w)^2} + \frac{1}{W^2} \right]$$

$-\frac{d}{dz}\xi(z) = \wp(z)$ *equality can be seen here. That is to say $\wp(z)$ is double function [1].*

The function $\wp(z)$ is meromorphic function in the complex plan ($|z| < 1$) with second order poles at the lattices points $z = W$ . It is in double periodic with periods $\omega_1$ and $\omega_2$ . This mean that $\wp(z)$ satisfies. Considering the following equality $\wp(z) = \frac{1}{z^2} + \sum_{k \geq 2} A_{2k-2}.z^{2k-2}$ for $\frac{1}{z} - \sum_{k \geq 2} \frac{A_{2k-2}}{2k-1} z^{2k-1}$ where $-\frac{d}{dz}\xi(z) = \wp(z)$. The funtions $\wp(z)$ is a meromorphic and elliptic funtion which has $z = W$ second order pole points.

**Theorem 1.6.** *The series $\wp(z)$ is absolutely and uniformly convergent for every $z = W$ .*

*Proof.*

$$\left| \frac{1}{(z-W)^2} - \frac{1}{W^2} \right| = \left| \frac{W^2 - (z-W)^2}{(z-W)^2.W^2} \right| = \left| \frac{(2W-z).z}{(z-W)^2.W^2} \right| \leq \frac{|z|.\left(2|W| + \frac{|W|}{2}\right)}{\frac{1}{4}W^2W^2} = \frac{10|z|}{|W|^3}$$

where $|z| < \frac{1}{2}|W|$. Thus,

$$\sum_{m,n \neq (0,0)} \left| \frac{1}{(z-W)^2} - \frac{1}{W^2} \right| = \sum_{m,n \neq (0,0)} \frac{10|z|}{W^2}.$$

The function $\wp(z)$ is meromorphic region $|z| < 1$ whether the function $\wp(z)$ is not analytical region $|z| < 1$. If we get consecutive derivatives from the equation as

$$\wp(z) = \frac{1}{z^2} + \sum_{k \geq 2} A_{2k-2}.z^{2k-2}$$

$$\wp'(z) = -\frac{1.2}{z^3} + \sum_{k \geq 2} (2k-2).A_{2k-2}.z^{2k-3}$$

$$\wp''(z) = \frac{1.2.3}{z^4} + \sum_{k \geq 2} (2k-2).(2k-3).A_{2k-2}.z^{2k-4} \vdots$$

$$\wp^n(z) = (-1)^n \frac{(n+1)!}{z^{n+2}} + \sum_{k \geq 2} (2k-2).(2k-3)...(2k-(n+1)).A_{2k-2}.z^{2k-(n+1)}.$$

In that case

$$\wp^{2n-1}(z) = -\frac{(2n)!}{z^{2n+1}} + \sum_{k \geq 2} (2k-2).(2k-3)...(2k-2n).A_{2k-2}.z^{(2k-2n)}$$

$$\wp^{2n-1}(z) = -\frac{(2n)!}{z^{2n+1}} + \sum_{k \geq 2} (2k-2).(2k-3)...(2k-2n).A_{2k-2}.z^{(2k-2n)}$$

$$\wp^{2n-2}(z) = \frac{(n-1)!}{z^{2n+1}} + \sum_{k \geq 2} (2k-2).(2k-3)...(2k-(2n-1)).A_{2k-2}.z^{(2k-(2n-1))}$$

□

**Theorem 1.7.** *If $\alpha_i$ and $\beta_i$ $(i = 1,2,........,r)$ be the zeros and poles respectively of an elliptic function $f(z)$ in a cell, then*

$$\sum_{i=1}^{r}\alpha_i \equiv \sum_{i=1}^{r}\beta_i \qquad (mod.2\omega_1, 2\omega_2)$$

*where every zero or pole is counted as many times as the multiplicity indicates.*

*Proof.* We have

$$\begin{aligned}
\sum_{i=1}^{r}\alpha_i - \sum_{i=1}^{r}\beta_i &= \frac{1}{2\pi i}\int_P \frac{zf'(z)}{f(z)}dz \qquad \text{(P is any suitably chosen contour)}\\
&= \frac{1}{2\pi i}\left[\int_{z_0}^{z_0+2\omega_1}\frac{zf'(z)}{f(z)}dz + \int_{z_0+2\omega_1}^{z_0+2\omega_1+2\omega_2}\frac{zf'(z)}{f(z)}dz + \int_{z_0+2\omega_1+2\omega_2}^{z_0+2\omega_2}\frac{zf'(z)}{f(z)}dz + \int_{z_0+2\omega_2}^{z_0}\frac{zf'(z)}{f(z)}dz\right]\\
&= \frac{1}{2\pi i}\left[\int_{z_0}^{z_0+2\omega_1}(z-(z+2\omega_2))\frac{f'(z)}{f(z)}dz + \int_{z_0}^{z_0+2\omega_2}(z+2\omega_1-z)\frac{f'(z)}{f(z)}dz\right]\\
&= \frac{1}{2\pi i}\left[2\omega_1\int_{z_0}^{z_0+2\omega_2}\frac{f'(z)}{f(z)}dz - 2\omega_2\int_{z_0}^{z_0+2\omega_1}\frac{f'(z)}{f(z)}dz\right]\\
&= \frac{1}{2\pi i}\left\{2\omega_1\left[logf(z)\right]_{z_0}^{z_0+2\omega_2} - 2\omega_2\left[logf(z)\right]_{z_0}^{z_0+2\omega_1}\right\} = \frac{1}{2\pi i}(4\pi im\omega_1 - 4\pi in\omega_2) = (m2\omega_1 + 2n\omega_2) \qquad (n=-n).
\end{aligned}$$

Hence we conclude

$$\sum_{i=1}^{r}\alpha_i \equiv \sum_{i=1}^{r}\beta_i \qquad (mod.2\omega_1, 2\omega_2)[1].$$

□

**Theorem 1.8.** *The sum, difference, product and the quotient of any two co-periodic elliptic functions are also elliptic function of the same period.*

*Proof.* Since $f_i(z+2\omega) = f_i(z)$, where $2\omega = 2\omega_1$ and $2\omega_2$ $(i = 1,2)$ therefore

$$f_1(z+2\omega) \pm f_2(z+2\omega) = f_1(z) \pm f_2(z)$$

$$f_1(z+2\omega).f_2(z+2\omega) = f_1(z).f_2(z)$$

$$f_1(z+2\omega)/f_2(z+2\omega) = f_1(z)/f_2(z).$$

Again since the set of all meromorphic functions forms a field and $f_1(z) \pm f_2(z)$, $f_1(z).f_2(z)$ and $f_1(z)/f_2(z)$ are meromorphic and periodic with periods $2\omega_1$ and $2\omega_2$. So they are elliptic functions with the same periods [1]. □

**Theorem 1.9.** *Let $f(z)$ be regular and univalent in the closed disk $D : |z| \leq R$. Then $f(z)$ maps $D$ onto a convex domain if and only if*

$$Re\left[1 + \frac{zf'(z)}{f(z)}\right] \geq 0, \qquad for\ z\ on \qquad D : |z| \leq R.$$

*Suppose further that $f(0) = 0$. Then $f(z)$ maps $D$ onto a region that is starlike with respect to $w = 0$ if and only if*

$$Re\left[\frac{zf'(z)}{f(z)}\right] \geq 0, \qquad for\ z\ on \qquad D : |z| \leq R.$$

We must assume that $f(z)$ is univalent (or replace this with some order condition) or we fall into error. Indeed, suppose that $f(z)=z^2$. Then the inequality becomes for starlike $2\geq 0$ and also for convex domain becomes $2\geq 0$ . $f(z)=z^2$ is not really a convex or starlike domain. The concepts of convexity and starlikeness can be extended to multi-sheeted regions, and indeed these extensions have been thoroughly explored, but for the present we consider only plane regions. We observe that if $f(z)$ is univalent in D, then $f'(z)\neq 0$ in and hence the expression on the left is a harmonic function in D and takes its minimum on the boundary D . Thus, if $f(z)$ maps D onto a closed convex curve, then for each $r<R$, $f(z)$ maps D onto a convex curve, and hence maps D onto a convex domain. The same type of reasoning can be applied because if $f(z)$ is in $S$, then the singularity at $z=0$ is a removable singularity [2].

**Theorem 1.10.** *The function $\wp(z)$ and the function $\xi(z)$ have the following equality*

$$\frac{\wp^{(2n-1)}(z_1)}{\wp^{(2n-2)}(z_1)-\wp^{(2n-2)}(z_2)}=2\xi(z_2-z_1)-2n(\xi(z_1)-\xi(z_2)).$$

**Lemma 1.11.** *The sum, difference, product and quotient of any co-periodic elliptic functions are also elliptic function of the same period.*

**Lemma 1.12.** *If the elliptic function $f(z)$ has simple pole at and only at the points $\beta_1,\beta_2,\beta_3,...,\beta_n$ in cell with residues $A_1,A_2,A_3,...,A_n$, then*

$$\wp(z)=A_0+\sum_{r=1}^{s}(z-r).A_r,$$

*where $A_0$ is a constant. It is in the fact that a constant $A_0$ in zero. In that case, the function*

$$\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}$$

*is an elliptical function with poles at $z_2,-z_2$. 0 with residues 1, 1, -2n respectively. If the last equation is written in place of z, then the following equation is found*

$$\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}=A_0+\xi(z-z_2)+\xi(z-z_2)-2n\xi(z).$$

*If in the above equation z is written instead of (–z) then $\wp$ is an even function and $\xi(z)$ is an odd function*

$$-\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}=A_0-\xi(z+z_2)-\xi(z-z_2)+2n\xi(z).$$

$$\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}=-A_0+\xi(z+z_2)+\xi(z-z_2)-2n\xi(z)$$

*equations are obtained. If $A_0=0$ and $z_1$ are written instead of z then the following equation is continue*

$$\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}=\xi(z_1+z_2)+\xi(z_1-z_2)-2n\xi(z_1).$$

*The function $\varphi(z)$ defined as follows*

$$\varphi(z)=\wp(z)+\frac{z^3-1}{z^2}=z+\sum_{k\geq 2}A_{2k-2}.z^{2k-2}=z+A_2z^2+A_4z^4+...$$

*The function $\varphi(z)$ is an analytical function for every $z\in|z|<1$. Also because of its $\varphi(0)=0$ and $\varphi'(0)=1$,this function is class A.*

## 2. Main Theorem

**Theorem 2.1.** *The function $\varphi(z)$ is an univalent function.*

*Proof.* If $\varphi(z_1)-\varphi(z_2)=0$, then

$$\varphi(z_1)-\varphi(z_2)=z_1+\sum_{k\geq 2}A_{2k-2}.z_1^{2k-2}-z_2-\sum_{k\geq 2}A_{2k-2}.z_2^{2k-2}=0$$

$$(z_1-z_2)\left(1+\sum_{k\geq 2}A_{2k-2}(z_1^{2k-3}-z_1^{2k-4}z_2+...+z_2^{2k-3})\right)=0$$

$$1+\sum_{k\geq 2}A_{2k-2}(z_1^{2k-3}-z_1^{2k-4}z_2+...+z_2^{2k-3})\neq 0$$

$z_1-z_2=0$ be must because $1+\sum_{k\geq 2}A_{2k-2}(z_1^{2k-3}-z_1^{2k-4}z_2+...+z_2^{2k-3})\neq 0$ for every $z\in |z|<1$.

Thus, the function $\varphi(z)$ is in class $S$. The subclass of $S$ consisting of the convex functions is defined by $K$ , and $S^*$ denotes the subclass of starlike functions. Thus $K\subset S^*\subset S$ [3].

We can do this proof in another way as follows: $|z|<1$ is clear that there is convex region.

Note that $\varphi(z_1)-\varphi(z_2)=\int_{z_1}^{z_2}\varphi'(\eta)d\eta$.

If

$\eta=tz_2+(1-t)z_1$, $0\leq t0\leq 1$, then $z_1-\varphi(z_2)=\int_0^1\varphi'(tz_2+(1-t)z_1)d\eta$.

Because,

$\eta=(tz_2+(1-t)z_1)\in |z|<1$ and $Re\varphi'(z)=Re\varphi'(tz_2+(1-t)z_1)>0$.

Thus

$\varphi'(\eta)=\varphi'(tz_2+(1-t)z_1)\neq 0$. Therefore, if $z_1-z_2\neq 0$, then $\varphi(z_1)-\varphi(z_2)\neq 0$. This means that $\varphi(z)$ is univalent in $|z|<1$ . On the other hand,

$$Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right)=Re\left(\frac{1+4A_2z+14A_4z^3+36A_6z^5+...}{1+2A_2z+4A_4z^3+6A_6z^5+8A_8z^7+...}\right)=Re(1+2A_2z-4A_2A_2z^2+(10A_4+8A_2A_2A_2)z^3+...)>0$$

since for every $z\in |z|<1$.

□

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# Ostrowski and Trapezoid Type Inequalities for the Generalized $k$-$g$-Fractional Integrals of Functions with Bounded Variation

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**Abstract**
In this paper we establish some Ostrowski and trapezoid type inequalities for the $k$-$g$-fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for a general exponential fractional integral are also given.

**Keywords:** Functions of bounded variation, Generalized Riemann-Liouville fractional integrals, Hadamard fractional integrals, Ostrowski type inequalities, Trapezoid inequalities
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## 1. Introduction

Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function $k$ is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geq 1$, then $k$ is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *$k$-$g$-left-sided fractional integral* of $f$ by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.1}$$

and the *$k$-$g$-right-sided fractional integral* of $f$ by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b). \tag{1.2}$$

If we take $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, where $\Gamma$ is the *Gamma function*, then

$$S_{k,g,a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x [g(x) - g(t)]^{\alpha-1} g'(t) f(t)\, dt$$
$$=: I^{\alpha}_{a+,g} f(x),\ a < x \le b$$

and

$$S_{k,g,b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b [g(t) - g(x)]^{\alpha-1} g'(t) f(t)\, dt \tag{1.3}$$
$$=: I^{\alpha}_{b-,g} f(x),\ a \le x < b,$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function $f$ with respect to another function $g$ on $[a,b]$ as defined in [1, p. 100]

For $g(t) = t$ in (1.3) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [1, p. 111]

$$H^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t},\ 0 \le a < x \le b$$

and

$$H^{\alpha}_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t},\ 0 \le a < x < b.$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R^{\alpha}_{a+} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\, dt}{(x-t)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x \le b$$

and

$$R^{\alpha}_{b-} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\, dt}{(t-x)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x < b.$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"$\beta$-Exponential fractional integrals"*

$$E^{\alpha}_{a+,\beta} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x [\exp(\beta x) - \exp(\beta t)]^{\alpha-1} \exp(\beta t) f(t)\, dt,$$

for $a < x \le b$ and

$$E^{\alpha}_{b-,\beta} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t)\, dt,$$

for $a \le x < b$.

If we take $g(t) = t$ in (1.1) and (1.2), then we can consider the following *k-fractional integrals*

$$S_{k,a+} f(x) = \int_a^x k(x-t) f(t)\, dt,\ x \in (a,b] \tag{1.4}$$

and

$$S_{k,b-} f(x) = \int_x^b k(t-x) f(t)\, dt,\ x \in [a,b). \tag{1.5}$$

In [2], Raina studied a class of functions defined formally by

$$\mathscr{F}^{\sigma}_{\rho,\lambda}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k,\ \ |x| < R,\ \text{with } R > 0 \tag{1.6}$$

for $\rho$, $\lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.6), Raina defined the following left-sided fractional integral operator

$$\mathscr{J}^{\sigma}_{\rho,\lambda,a+;w}f(x) := \int_a^x (x-t)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left(w(x-t)^{\rho}\right)f(t)\,dt,\ x>a \tag{1.7}$$

where $\rho$, $\lambda > 0$, $w \in \mathbb{R}$ and $f$ is such that the integral on the right side exists.

In [3], the right-sided fractional operator was also introduced as

$$\mathscr{J}^{\sigma}_{\rho,\lambda,b-;w}f(x) := \int_x^b (t-x)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left(w(t-x)^{\rho}\right)f(t)\,dt,\ x<b \tag{1.8}$$

where $\rho$, $\lambda > 0$, $w \in \mathbb{R}$ and $f$ is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.7) and (1.8) from (1.4) and (1.5).

In [4], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathscr{T}^{\alpha}_{a+}f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\}f(t)\,dt,\ x>a \tag{1.9}$$

and

$$\mathscr{T}^{\alpha}_{b-}f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\}f(t)\,dt,\ x<b \tag{1.10}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $t\in\mathbb{R}$ we re-obtain the definitions of (1.9) and (1.10) from (1.4) and (1.5).

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. We can define the more general exponential fractional integrals

$$\mathscr{T}^{\alpha}_{g,a+}f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\}g'(t)f(t)\,dt,\ x>a$$

and

$$\mathscr{T}^{\alpha}_{g,b-}f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\}g'(t)f(t)\,dt,\ x<b$$

where $\alpha \in (0,1)$.

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathscr{L}^{\alpha}_{g,a+}f(x) := \int_a^x (g(x)-g(t))^{\alpha-1}\ln(g(x)-g(t))\,g'(t)f(t)\,dt,$$

for $0<a<x\le b$ and

$$\mathscr{L}^{\alpha}_{g,b-}f(x) := \int_x^b (g(t)-g(x))^{\alpha-1}\ln(g(t)-g(x))\,g'(t)f(t)\,dt,$$

for $0<a\le x<b$, where $\alpha>0$. These are obtained from (1.4) and (1.5) for the kernel $k(t) = t^{\alpha-1}\ln t$, $t>0$.

For $\alpha = 1$ we get

$$\mathscr{L}_{g,a+}f(x) := \int_a^x \ln(g(x)-g(t))\,g'(t)f(t)\,dt,\ 0<a<x\le b$$

and

$$\mathscr{L}_{g,b-}f(x) := \int_x^b \ln(g(t)-g(x))\,g'(t)f(t)\,dt,\ 0<a\le x<b.$$

For $g(t) = t$, we have the simple forms

$$\mathscr{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t)\, dt,\ 0 < a < x \le b,$$

$$\mathscr{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t)\, dt,\ 0 < a \le x < b,$$

$$\mathscr{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t)\, dt,\ 0 < a < x \le b$$

and

$$\mathscr{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t)\, dt,\ 0 < a \le x < b.$$

In the recent paper [5] we obtained the following Ostrowski and trapezoid type inequalities for the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function $f$ with respect to another function $g$ on $[a,b]$.

**Theorem 1.1.** *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Also let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Then we have*

$$\begin{aligned}
&\left| I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} \left[ (g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right] f(x) \right| \\
&\le \frac{1}{\Gamma(\alpha)} \left[ \int_a^x (g(t)-g(a))^{\alpha-1} g'(t) \bigvee_t^x (f)\, dt + \int_x^b (g(b)-g(t))^{\alpha-1} g'(t) \bigvee_x^t (f)\, dt \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \left[ (g(x)-g(a))^{\alpha} \bigvee_a^x (f) + (g(b)-g(x))^{\alpha} \bigvee_x^b (f) \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[ \frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\[1ex] \left( (g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \left( (\bigvee_a^x (f))^q + \left( \bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[1ex] \left[ \frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \left( (g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right) \end{cases}
\end{aligned}$$

*and*

$$\begin{aligned}
&\left| I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} \left[ (g(x)-g(a))^{\alpha} f(a) + (g(b)-g(x))^{\alpha} f(b) \right] \right| \\
&\le \frac{1}{\Gamma(\alpha)} \left[ \int_a^x (g(x)-g(t))^{\alpha-1} g'(t) \bigvee_a^t (f)\, dt + \int_x^b (g(t)-g(x))^{\alpha-1} g'(t) \bigvee_t^b (f)\, dt \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \left[ (g(x)-g(a))^{\alpha} \bigvee_a^x (f) + (g(b)-g(x))^{\alpha} \bigvee_x^b (f) \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[ \frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\[1ex] \left( (g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \left( (\bigvee_a^x (f))^q + \left( \bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[1ex] \left[ \frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \left( (g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right) \end{cases}
\end{aligned}$$

*for any $x \in (a,b)$.*

For applications to the classical Riemann-Liouville fractional integrals, Hadamard fractional integrals and Harmonic fractional integrals see [5].

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [6]-[21], [22]-[32] and the references therein.

Motivated by the above results, in this paper we establish some Ostrowski and trapezoid type inequalities for the $k$-$g$-fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for a general exponential fractional integral are also given.

## 2. Some identities for the operator $S_{k,g,a+,b-}$

For $k$ and $g$ as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned}
&S_{k,g,a+,b-}f(x)\\
&:=\frac{1}{2}\left[S_{k,g,a+}f(x)+S_{k,g,b-}f(x)\right]\\
&=\frac{1}{2}\left[\int_a^x k(g(x)-g(t))g'(t)f(t)\,dt+\int_x^b k(g(t)-g(x))g'(t)f(t)\,dt\right]
\end{aligned}$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

The following two parameters representation for the operator $S_{k,g,a+,b-}$ holds:

**Lemma 2.1.** *With the above assumptions for $k$, $g$ and $f$ we have*

$$\begin{aligned}
S_{k,g,a+,b-}f(x)=&\frac{1}{2}\left[\gamma K(g(b)-g(x))+\lambda K(g(x)-g(a))\right]\\
&+\frac{1}{2}\int_a^x k(g(x)-g(t))g'(t)\left[f(t)-\lambda\right]dt\\
&+\frac{1}{2}\int_x^b k(g(t)-g(x))g'(t)\left[f(t)-\gamma\right]dt
\end{aligned}\tag{2.1}$$

*for any $\lambda$, $\gamma\in\mathbb{C}$.*

*Proof.* We have, by taking the derivative over $t$ and using the chain rule, that

$$[K(g(x)-g(t))]'=K'(g(x)-g(t))(g(x)-g(t))'=-k(g(x)-g(t))g'(t)$$

for $t\in(a,x)$ and

$$[K(g(t)-g(x))]'=K'(g(t)-g(x))(g(t)-g(x))'=k(g(t)-g(x))g'(t)$$

for $t\in(x,b)$.

Therefore, for any $\lambda$, $\gamma\in\mathbb{C}$ we have

$$\begin{aligned}
&\int_a^x k(g(x)-g(t))g'(t)\left[f(t)-\lambda\right]dt\\
&=\int_a^x k(g(x)-g(t))g'(t)f(t)\,dt-\lambda\int_a^x k(g(x)-g(t))g'(t)\,dt\\
&=S_{k,g,a+}f(x)+\lambda\int_a^x[K(g(x)-g(t))]'\,dt\\
&=S_{k,g,a+}f(x)+\lambda\left[K(g(x)-g(t))\right]\Big|_a^x=S_{k,g,a+}f(x)-\lambda K(g(x)-g(a))
\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}
&\int_x^b k(g(t)-g(x))g'(t)\left[f(t)-\gamma\right]dt\\
&=\int_x^b k(g(t)-g(x))g'(t)f(t)\,dt-\gamma\int_x^b k(g(t)-g(x))g'(t)\,dt\\
&=S_{k,g,b-}f(x)-\gamma\int_x^b[K(g(t)-g(x))]'\,dt\\
&=S_{k,g,b-}f(x)-\gamma\left[K(g(t)-g(x))\right]\Big|_x^b=S_{k,g,b-}f(x)-\gamma K(g(b)-g(x))
\end{aligned}\tag{2.3}$$

for $x \in (a,b)$.

If we add the equalities (2.2) and (2.3) and divide by 2 then we get the desired result (2.1). □

**Corollary 2.2.** *With the above assumptions for $k$, $g$ and $f$ we have the Ostrowski type identity*

$$\begin{aligned} S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \\ &+\frac{1}{2}\int_a^x k(g(x)-g(t))g'(t)\left[f(t)-f(x)\right]dt \\ &+\frac{1}{2}\int_x^b k(g(t)-g(x))g'(t)\left[f(t)-f(x)\right]dt \end{aligned} \tag{2.4}$$

*and the trapezoid type identity*

$$\begin{aligned} S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right] \\ &+\frac{1}{2}\int_a^x k(g(x)-g(t))g'(t)\left[f(t)-f(a)\right]dt \\ &+\frac{1}{2}\int_x^b k(g(t)-g(x))g'(t)\left[f(t)-f(b)\right]dt \end{aligned} \tag{2.5}$$

*for any $x \in (a,b)$.*

For $x = \frac{a+b}{2}$ we can consider

$$\begin{aligned} M_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) \\ &= \frac{1}{2}\int_a^{\frac{a+b}{2}} k\left(g\left(\frac{a+b}{2}\right)-g(t)\right)g'(t)f(t)\,dt \\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g(t)-g\left(\frac{a+b}{2}\right)\right)g'(t)f(t)\,dt. \end{aligned}$$

By (2.4) we have the representation

$$\begin{aligned} &M_{k,g,a+,b-}f \\ &= \frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]f\left(\frac{a+b}{2}\right) \\ &+\frac{1}{2}\int_a^{\frac{a+b}{2}} k\left(g\left(\frac{a+b}{2}\right)-g(t)\right)g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt \\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g(t)-g\left(\frac{a+b}{2}\right)\right)g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt \end{aligned}$$

and (2.5) we have

$$\begin{aligned} &M_{k,g,a+,b-}f \\ &= \frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)f(b)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)f(a)\right] \\ &+\frac{1}{2}\int_a^{\frac{a+b}{2}} k\left(g\left(\frac{a+b}{2}\right)-g(t)\right)g'(t)\left[f(t)-f(a)\right]dt \\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g(t)-g\left(\frac{a+b}{2}\right)\right)g'(t)\left[f(t)-f(b)\right]dt. \end{aligned}$$

If $g$ is a function which maps an interval $I$ of the real line to the real numbers, and is both continuous and injective then we can define the *$g$-mean of two numbers $a$, $b \in I$* as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Using the $g$-mean of two numbers we can introduce

$$\begin{aligned}P_{k,g,a+,b-}f&:=S_{k,g,a+,b-}f(M_g(a,b))\\&=\frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)f(t)\,dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)f(t)\,dt.\end{aligned}$$

Using (2.4) and (2.5) we have the representations

$$\begin{aligned}&P_{k,g,a+,b-}f\\&=K\left(\frac{g(b)-g(a)}{2}\right)f(M_g(a,b))\\&+\frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)\left[f(t)-f(M_g(a,b))\right]dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)\left[f(t)-f(M_g(a,b))\right]dt\end{aligned}$$

and

$$\begin{aligned}&P_{k,g,a+,b-}f\\&=K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2}\\&+\frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)\left[f(t)-f(a)\right]dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)\left[f(t)-f(b)\right]dt.\end{aligned}$$

## 3. Some identities for the dual operator $\breve{S}_{k,g,a+,b-}$

Observe that

$$S_{k,g,x+}f(b)=\int_x^b k(g(b)-g(t))\,g'(t)f(t)\,dt,\ x\in[a,b)$$

and

$$S_{k,g,x-}f(a)=\int_a^x k(g(t)-g(a))\,g'(t)f(t)\,dt,\ x\in(a,b].$$

Define also the mixed operator

$$\begin{aligned}&\breve{S}_{k,g,a+,b-}f(x)\\&:=\frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right]\\&=\frac{1}{2}\left[\int_x^b k(g(b)-g(t))\,g'(t)f(t)\,dt+\int_a^x k(g(t)-g(a))\,g'(t)f(t)\,dt\right]\end{aligned}$$

for any $x\in(a,b)$.

**Lemma 3.1.** *With the above assumptions for $k$, $g$ and $f$ we have*

$$\begin{aligned}\check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\lambda K(g(b)-g(x))+\gamma K(g(x)-g(a))\right] \\ &+\frac{1}{2}\int_a^x k(g(t)-g(a))g'(t)[f(t)-\gamma]\,dt \\ &+\frac{1}{2}\int_x^b k(g(b)-g(t))g'(t)[f(t)-\lambda]\,dt\end{aligned} \tag{3.1}$$

*for any $\lambda$, $\gamma \in \mathbb{C}$.*

*Proof.* We have, by taking the derivative over $t$ and using the chain rule, that

$$[K(g(b)-g(t))]' = K'(g(b)-g(t))(g(b)-g(t))' = -k(g(b)-g(t))g'(t)$$

for $t \in (x,b)$ and

$$[K(g(t)-g(a))]' = K'(g(t)-g(a))(g(t)-g(a))' = k(g(t)-g(a))g'(t)$$

for $t \in (a,x)$.

For any $\lambda$, $\gamma \in \mathbb{C}$ we have

$$\begin{aligned}&\int_x^b k(g(b)-g(t))g'(t)[f(t)-\lambda]\,dt \\ &= \int_x^b k(g(b)-g(t))g'(t)f(t)\,dt - \lambda\int_x^b k(g(b)-g(t))g'(t)\,dt \\ &= S_{k,g,x+}f(b)+\lambda\int_x^b [K(g(b)-g(t))]'\,dt \\ &= S_{k,g,x+}f(b)-\lambda K(g(b)-g(x))\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\int_a^x k(g(t)-g(a))g'(t)[f(t)-\gamma]\,dt \\ &= \int_a^x k(g(t)-g(a))g'(t)f(t)\,dt - \gamma\int_a^x k(g(t)-g(a))g'(t)\,dt \\ &= \int_a^x k(g(t)-g(a))g'(t)f(t)\,dt - \gamma\int_a^x [K(g(t)-g(a))]'\,dt \\ &= \int_a^x k(g(t)-g(a))g'(t)f(t)\,dt - \gamma K(g(x)-g(a))\end{aligned} \tag{3.3}$$

for $x \in (a,b)$.

If we add the equalities (3.2) and (3.3) and divide by 2 then we get the desired result (3.1). □

**Corollary 3.2.** *With the assumptions of Lemma 3.1 we have the Ostrowski type identity*

$$\begin{aligned}\check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \\ &+\frac{1}{2}\int_a^x k(g(t)-g(a))g'(t)[f(t)-f(x)]\,dt \\ &+\frac{1}{2}\int_x^b k(g(b)-g(t))g'(t)[f(t)-f(x)]\,dt\end{aligned} \tag{3.4}$$

*and the trapezoid identity*

$$\begin{aligned}\check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right] \\ &+\frac{1}{2}\int_a^x k(g(t)-g(a))g'(t)[f(t)-f(a)]\,dt \\ &+\frac{1}{2}\int_x^b k(g(b)-g(t))g'(t)[f(t)-f(b)]\,dt\end{aligned} \tag{3.5}$$

*for $x \in (a,b)$.*

For $x=\frac{a+b}{2}$ we can consider

$$\begin{aligned}\check{M}_{k,g,a+,b-}f &:= \check{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)\\ &= \frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt\\ &+\frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt.\end{aligned}$$

Using the equalities (3.4) and (3.5), we have

$$\begin{aligned}&\check{M}_{k,g,a+,b-}f\\ &=\frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right]f\left(\frac{a+b}{2}\right)\\ &+\frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(\frac{a+b}{2}\right)\right]dt\\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(\frac{a+b}{2}\right)\right]dt\end{aligned}$$

and

$$\begin{aligned}&\check{M}_{k,g,a+,b-}f\\ &=\frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)f\left(b\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)f\left(a\right)\right]\\ &+\frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(a\right)\right]dt\\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(b\right)\right]dt.\end{aligned}$$

Using the $g$-mean of two numbers we can introduce

$$\begin{aligned}\check{P}_{k,g,a+,b-}f &:= \check{S}_{k,g,a+,b-}f\left(M_g\left(a,b\right)\right)\\ &= \frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt.\end{aligned}$$

Using the equalities (3.4) and (3.5), we have

$$\begin{aligned}\check{P}_{k,g,a+,b-}f &= K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)f\left(M_g\left(a,b\right)\right)\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(M_g\left(a,b\right)\right)\right]dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(M_g\left(a,b\right)\right)\right]dt\end{aligned}$$

and

$$\begin{aligned}\check{P}_{k,g,a+,b-}f &= K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\frac{f\left(b\right)+f\left(a\right)}{2}\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(a\right)\right]dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(b\right)\right]dt.\end{aligned}$$

## 4. Trapezoid functional $T_{k,g,a+,b-}$

We can also introduce the functional

$$T_{k,g,a+,b-}f := \frac{1}{2}\left[S_{k,g,a+}f(b) + S_{k,g,b-}f(a)\right] = \frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t) f(t)\,dt.$$

We have:

**Lemma 4.1.** *With the assumption of Lemma 2.1, we have*

$$T_{k,g,a+,b-}f = K(g(b)-g(a))\,\delta + \frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\left[f(t)-\delta\right] dt \tag{4.1}$$

*for any $\delta \in \mathbb{C}$.*

*Proof.* Observe that

$$\begin{aligned}
&\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\,dt \\
&= \int_a^b k(g(b)-g(t))\,g'(t)\,dt + \int_a^b k(g(t)-g(a))\,g'(t)\,dt \\
&= -\int_a^b \left[K(g(b)-g(t))\right]' dt + \int_a^b \left[K(g(t)-g(a))\right]' dt \\
&= -K(g(b)-g(t))\big|_a^b + K(g(t)-g(a))\big|_a^b \\
&= K(g(b)-g(a)) + K(g(b)-g(a)) = 2K(g(b)-g(a)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\left[f(t)-\delta\right] dt \\
&= \frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t) f(t)\,dt \\
&- \frac{1}{2}\delta\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\,dt \\
&= T_{k,g,a+,b-}f - \delta K(g(b)-g(a)),
\end{aligned}$$

which proves the desired equality (4.1). □

**Corollary 4.2.** *With the assumptions of Lemma 4.1 we have the Ostrowski type identity*

$$\begin{aligned}
&T_{k,g,a+,b-}f \\
&= K(g(b)-g(a))\,f(x) \\
&+ \frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\left[f(t)-f(x)\right] dt
\end{aligned} \tag{4.2}$$

*for any $x \in [a,b]$ and the trapezoid identity*

$$\begin{aligned}
&T_{k,g,a+,b-}f \\
&= K(g(b)-g(a))\,\frac{f(a)+f(b)}{2} \\
&+ \frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\left[f(t) - \frac{f(a)+f(b)}{2}\right] dt.
\end{aligned} \tag{4.3}$$

We observe that for $x=\frac{a+b}{2}$ we obtain from (4.2) that

$$\begin{aligned} & T_{k,g,a+,b-}f \\ & = K(g(b)-g(a)) f\left(\frac{a+b}{2}\right) \\ & +\frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))] g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right] dt. \end{aligned}$$

## 5. Inequalities for functions of bounded variation

We considered the cumulative function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases} \int_0^t k(s)\,ds \text{ if } 0<t, \\ 0 \text{ if } t=0. \end{cases}$$

We also define the function $\mathbf{K}:[0,\infty)\to[0,\infty)$ by

$$\mathbf{K}(t):=\begin{cases} \int_0^t |k(s)|\,ds \text{ if } 0<t, \\ 0 \text{ if } t=0. \end{cases}$$

We observe that if $k$ takes nonnegative values on $(0,\infty)$, as it does in some of the examples in Introduction, then $\mathbf{K}(t)=K(t)$ for $t\in[0,\infty)$.

**Theorem 5.1.** *Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Then we have the Ostrowski type inequality*

$$\begin{aligned} & \left|S_{k,g,a+,b-}f(x)-\frac{1}{2}[K(g(b)-g(x))+K(g(x)-g(a))]f(x)\right| \\ & \le \frac{1}{2}\left[\int_x^b |k(g(t)-g(x))| \bigvee_x^t (f) g'(t)\,dt+\int_a^x |k(g(x)-g(t))| \bigvee_t^x (f) g'(t)\,dt\right] \\ & \le \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b (f) \; +\mathbf{K}(g(x)-g(a))\bigvee_a^x (f)\right] \\ & \le \frac{1}{2}\begin{cases} \max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b(f); \\ [\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))]^{1/p}\left((\bigvee_a^x(f))^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ [\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right] \end{cases} \end{aligned} \tag{5.1}$$

*and the trapezoid type inequality*

$$\begin{aligned}
&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right| \\
&\leq \frac{1}{2}\left[\int_a^x |k(g(x)-g(t))| \bigvee_a^t (f) g'(t)\,dt + \int_x^b |k(g(t)-g(x))| \bigvee_t^b (f) g'(t)\,dt\right] \\
&\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b (f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x (f)\right] \\
&\leq \frac{1}{2}\begin{cases}
\max\left\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b (f); \\[2ex]
\left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p} \\
\times\left(\left(\bigvee_a^x (f)\right)^q+\left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right] \\
\times\left[\frac{1}{2}\bigvee_a^b (f)+\frac{1}{2}\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right]
\end{cases}
\end{aligned} \tag{5.2}$$

*for any* $x\in(a,b)$.

*Proof.* Using the equality (2.4) we have

$$\begin{aligned}
&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\leq \frac{1}{2}\left|\int_a^x k(g(x)-g(t))g'(t)[f(t)-f(x)]\,dt\right| \\
&+\frac{1}{2}\left|\int_x^b k(g(t)-g(x))g'(t)[f(t)-f(x)]\,dt\right| \\
&\leq \frac{1}{2}\int_a^x \left|k(g(x)-g(t))g'(t)[f(t)-f(x)]\right|dt \\
&+\frac{1}{2}\int_x^b \left|k(g(t)-g(x))g'(t)[f(t)-f(x)]\right|dt \\
&=\frac{1}{2}\int_a^x |k(g(x)-g(t))|\,|f(x)-f(t)|\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(t)-g(x))|\,|f(t)-f(x)|\,g'(t)\,dt \\
&=: B(x)
\end{aligned}$$

for $x\in(a,b)$.

Since $f$ is of bounded variation, then

$$|f(x)-f(t)|\leq\bigvee_t^x (f)\leq\bigvee_a^x (f)\ \text{ for } a<t\leq x\leq b$$

and

$$|f(t)-f(x)|\leq\bigvee_x^t (f)\leq\bigvee_x^b (f)\ \text{ for } a\leq x\leq t<b.$$

Therefore

$$\begin{aligned} B(x) &\leq \frac{1}{2}\int_a^x |k(g(x)-g(t))| \bigvee_t^x (f) g'(t)\,dt \\ &+ \frac{1}{2}\int_x^b |k(g(t)-g(x))| \bigvee_x^t (f) g'(t)\,dt \\ &\leq \frac{1}{2}\bigvee_a^x (f) \int_a^x |k(g(x)-g(t))| g'(t)\,dt \\ &+ \frac{1}{2}\bigvee_x^b (f) \int_x^b |k(g(t)-g(x))| g'(t)\,dt \\ &=: C(x) \end{aligned}$$

for $x \in (a,b)$.

We have, by taking the derivative over $t$ and using the chain rule, that

$$[\mathbf{K}(g(x)-g(t))]' = \mathbf{K}'(g(x)-g(t))(g(x)-g(t))' = -|k(g(x)-g(t))| g'(t)$$

for $t \in (a,x)$ and

$$[\mathbf{K}(g(t)-g(x))]' = \mathbf{K}'(g(t)-g(x))(g(t)-g(x))' = |k(g(t)-g(x))| g'(t)$$

for $t \in (x,b)$.

Then

$$\int_a^x |k(g(x)-g(t))| g'(t)\,dt = -\int_a^x [\mathbf{K}(g(x)-g(t))]'\,dt = \mathbf{K}(g(x)-g(a))$$

and

$$\int_x^b |k(g(t)-g(x))| g'(t)\,dt = \int_x^b [\mathbf{K}(g(t)-g(x))]'\,dt = \mathbf{K}(g(b)-g(x))$$

giving that

$$C(x) = \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b (f) + \mathbf{K}(g(x)-g(a))\bigvee_a^x (f)\right],$$

for $x \in (a,b)$, which proves the first and the second inequality in (5.1).

The last part of (4.1 is obvious by making use of the elementary Hölder type inequalities for positive real numbers $c$, $d$, $m$, $n \geq 0$

$$mc + nd \leq \begin{cases} \max\{m,n\}(c+d); \\ \\ (m^p+n^p)^{1/p}(c^q+d^q)^{1/q} \text{ with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

Further, by the identity (2.5) we have, as above,

$$\begin{aligned}
&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right|\\
&\leq \frac{1}{2}\int_a^x |k(g(x)-g(t))|\,|f(t)-f(a)|\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(t)-g(x))|\,|f(t)-f(b)|\,g'(t)\,dt\\
&\leq \frac{1}{2}\int_a^x |k(g(x)-g(t))|\bigvee_a^t(f)\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(t)-g(x))|\bigvee_t^b(f)\,g'(t)\,dt\\
&\leq \frac{1}{2}\bigvee_a^x(f)\int_a^x |k(g(x)-g(t))|\,g'(t)\,dt\\
&+\frac{1}{2}\bigvee_x^b(f)\int_x^b |k(g(t)-g(x))|\,g'(t)\,dt\\
&=\frac{1}{2}\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\frac{1}{2}\mathbf{K}(g(b)-g(x))\bigvee_x^b(f),
\end{aligned}$$

which proves (5.2). □

The following particular case for the functional

$$\begin{aligned}
P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f(M_g(a,b))\\
&=\frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(b)+g(a)}{2}-g(t)\right)g'(t)\,f(t)\,dt\\
&+\frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t)-\frac{g(b)+g(a)}{2}\right)g'(t)\,f(t)\,dt.
\end{aligned}$$

is of interest:

**Corollary 5.2.** *With the assumptions of Theorem 5.1 we have*

$$\begin{aligned}
\left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)f(M_g(a,b))\right| &\leq \frac{1}{2}\int_{M_g(a,b)}^b \left|k\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right|\bigvee_{M_g(a,b)}^t(f)\,g'(t)\,dt\\
&+\frac{1}{2}\int_a^{M_g(a,b)}\left|k\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right|\bigvee_t^{M_g(a,b)}(f)\,g'(t)\,dt\\
&\leq \frac{1}{2}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_b^b(f)
\end{aligned}\tag{5.3}$$

*and*

$$\begin{aligned}
\left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2}\right| &\leq \frac{1}{2}\int_a^{M_g(a,b)}\left|k\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right|\bigvee_a^t(f)\,g'(t)\,dt\\
&+\frac{1}{2}\int_{M_g(a,b)}^b\left|k\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right|\bigvee_t^b(f)\,g'(t)\,dt\\
&\leq \frac{1}{2}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_b^b(f).
\end{aligned}\tag{5.4}$$

We have:

**Theorem 5.3.** *With the assumptions of Theorem 5.1 we have the Ostrowski type inequality*

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))| \bigvee_t^x (f)\, g'(t)\,dt+\frac{1}{2}\int_x^b |k(g(b)-g(t))| \bigvee_x^t (f)\, g'(t)\,dt \\
&\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b (f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x (f)\right] \\
&\leq \frac{1}{2}\begin{cases}
\max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b(f); \\
\\
\left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]
\end{cases}
\end{aligned} \tag{5.5}$$

*and the trapezoid inequality*

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right| \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))| \bigvee_a^t (f)\, g'(t)\,dt+\frac{1}{2}\int_x^b |k(g(b)-g(t))| \bigvee_t^b (f)\, g'(t)\,dt \\
&\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b (f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x (f)\right] \\
&\leq \frac{1}{2}\begin{cases}
\max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b(f); \\
\\
\left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]
\end{cases}
\end{aligned} \tag{5.6}$$

*for any* $x\in(a,b)$.

*Proof.* Using the identity (3.4) we have

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\,|f(t)-f(x)|\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\,|f(t)-f(x)|\,g'(t)\,dt \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\bigvee_t^x(f)\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\bigvee_x^t(f)\,g'(t)\,dt \\
&\leq \frac{1}{2}\bigvee_a^x(f)\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt \\
&+\frac{1}{2}\bigvee_x^b(f)\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt \\
&=\frac{1}{2}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right],
\end{aligned}$$

for any $x\in(a,b)$, which proves (5.5).

By the identity (3.5) we have

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right| \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\,|f(t)-f(a)|\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\,|f(b)-f(t)|\,g'(t)\,dt \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\bigvee_a^t(f)\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\bigvee_t^b(f)\,g'(t)\,dt \\
&\leq \frac{1}{2}\bigvee_a^x(f)\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt \\
&+\frac{1}{2}\bigvee_x^b(f)\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt \\
&=\frac{1}{2}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right]
\end{aligned}$$

for any $x\in(a,b)$, which proves (5.6). □

Also, we have the particular inequalities for

$$\begin{aligned}
\breve{P}_{k,g,a+,b-}f &:= \breve{S}_{k,g,a+,b-}f(M_g(a,b)) \\
&=\frac{1}{2}\int_{M_g(a,b)}^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt \\
&+\frac{1}{2}\int_a^{M_g(a,b)} k(g(t)-g(a))\,g'(t)\,f(t)\,dt.
\end{aligned}$$

**Corollary 5.4.** *With the assumptions of Theorem 5.1 we have*

$$\left|\breve{P}_{k,g,a+,b-}f - K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2}\right| \le \frac{1}{2}\int_a^{M_g(a,b)} |k(g(t)-g(a))| \bigvee_t^{M_g(a,b)}(f)\, g'(t)\,dt + \frac{1}{2}\int_{M_g(a,b)}^b |k(g(b)-g(t))| \bigvee_{M_g(a,b)}^t (f)\, g'(t)\,dt \le \frac{1}{2}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_b^b(f)$$

*and*

$$\left|\breve{P}_{k,g,a+,b-}f - K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2}\right| \le \frac{1}{2}\int_a^{M_g(a,b)} |k(g(t)-g(a))| \bigvee_a^t(f)\, g'(t)\,dt + \frac{1}{2}\int_{M_g(a,b)}^b |k(g(b)-g(t))| \bigvee_t^b (f)\, g'(t)\,dt \le \frac{1}{2}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_b^b(f).$$

Finally, we have the following result for the trapezoid functional

$$T_{k,g,a+,b-}f := \frac{1}{2}\left[S_{k,g,a+}f(b) + S_{k,g,b-}f(a)\right] = \frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t) f(t)\,dt.$$

**Theorem 5.5.** *With the assumptions of Theorem 5.1 we have the trapezoid type inequality*

$$\left|T_{k,g,a+,b-}f - K(g(b)-g(a))\frac{f(a)+f(b)}{2}\right| \le \frac{1}{2}\mathbf{K}(g(b)-g(a))\bigvee_a^b(f). \tag{5.7}$$

*Proof.* From the identity (4.3) we have

$$\left|T_{k,g,a+,b-}f - K(g(b)-g(a))\frac{f(a)+f(b)}{2}\right| \le \frac{1}{2}\int_a^b |k(g(b)-g(t)) + k(g(t)-g(a))| \left|f(t) - \frac{f(a)+f(b)}{2}\right| g'(t)\,dt \le \frac{1}{2}\int_a^b \left[|k(g(b)-g(t))| + |k(g(t)-g(a))|\right] \left|f(t) - \frac{f(a)+f(b)}{2}\right| g'(t)\,dt =: D.$$

Since $f : [a,b] \to \mathbb{C}$ is of bounded variation, then for any $t \in [a,b]$ we have

$$\left|f(t) - \frac{f(a)+f(b)}{2}\right| = \left|\frac{f(t)-f(a)+f(t)-f(b)}{2}\right| \le \frac{1}{2}\left[|f(t)-f(a)| + |f(b)-f(t)|\right] \le \frac{1}{2}\bigvee_a^b(f).$$

Therefore

$$D \le \frac{1}{4}\bigvee_a^b(f)\int_a^b \left[|k(g(b)-g(t))| + |k(g(t)-g(a))|\right] g'(t)\,dt = \frac{1}{4}\bigvee_a^b(f)\left[\mathbf{K}(g(b)-g(a)) + \mathbf{K}(g(b)-g(a))\right] = \frac{1}{2}\mathbf{K}(g(b)-g(a))\bigvee_a^b(f),$$

which proves the desired result (5.7). □

## 6. Example for an exponential kernel

The above inequalities may be written for all the particular fractional integrals introduced in the introduction.

If we take, for instance $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, where $\Gamma$ is the *Gamma function*, then we recapture the results for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function $f$ with respect to another function $g$ on $[a,b]$ as outlined in [5].

For $\alpha$, $\beta \in \mathbb{R}$ we consider the kernel $k(t) := \exp[(\alpha + \beta i)t]$, $t \in \mathbb{R}$. We have

$$K(t) = \frac{\exp[(\alpha+\beta i)t] - 1}{(\alpha+\beta i)}, \text{ if } t \in \mathbb{R}$$

for $\alpha$, $\beta \neq 0$.

Also, we have

$$|k(s)| := |\exp[(\alpha+\beta i)s]| = \exp(\alpha s) \text{ for } s \in \mathbb{R}$$

and

$$\mathbf{K}(t) = \int_0^t \exp(\alpha s)\, ds = \frac{\exp(\alpha t) - 1}{\alpha} \text{ if } 0 < t,$$

for $\alpha \neq 0$.

Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. We have

$$\begin{aligned}\mathscr{E}^{\alpha+\beta i}_{g,a+,b-} f(x) &= \frac{1}{2}\int_a^x \exp[(\alpha+\beta i)(g(x) - g(t))]\, g'(t) f(t)\, dt \\ &+ \frac{1}{2}\int_x^b \exp[(\alpha+\beta i)(g(t) - g(x))]\, g'(t) f(t)\, dt\end{aligned}$$

for $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on $(a,b)$, having a continuous derivative $h'$ on $(a,b)$, then we can consider the following operator as well

$$\begin{aligned}&\kappa^{\alpha+\beta i}_{h,a+,b-} f(x) \\ &:= \mathscr{E}^{\alpha+\beta i}_{\ln h,a+,b-} f(x) \\ &= \frac{1}{2}\left[\int_a^x \left(\frac{h(x)}{h(t)}\right)^{\alpha+\beta i} \frac{h'(t)}{h(t)} f(t)\, dt + \int_x^b \left(\frac{h(t)}{h(x)}\right)^{\alpha+\beta i} \frac{h'(t)}{h(t)} f(t)\, dt\right],\end{aligned}$$

for $x \in (a,b)$.

By using the inequality (5.1) we have for $x \in (a,b)$ that

$$
\begin{aligned}
&\Bigg|\mathscr{E}_{g,a+,b-}^{\alpha+\beta i} f(x) \\
&-\frac{1}{2}\left[\frac{\exp[(\alpha+\beta i)(g(b)-g(x))]-1+\exp[(\alpha+\beta i)(g(x)-g(a))]-1}{(\alpha+\beta i)}\right] f(x)\Bigg| \\
&\leq \frac{1}{2}\left[\int_x^b \exp(\alpha(g(t)-g(x))) g'(t) \bigvee_x^t (f)\, dt+\int_a^x \exp(\alpha(g(x)-g(t))) g'(t) \bigvee_t^x (f)\, dt\right] \\
&\leq \frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha} \bigvee_x^b (f)\ +\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha} \bigvee_a^x (f)\right] \\
&\leq \frac{1}{2}\begin{cases}
\max\left\{\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}, \frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right\} \bigvee_a^b (f); \\[2ex]
\left[\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p+\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p\right]^{1/p}\left(\left(\bigvee_a^x (f)\right)^q+\left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left[\frac{\exp(\alpha(g(b)-g(x)))-1+\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b (f)+\frac{1}{2}\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right]
\end{cases}
\end{aligned}
$$

for $\alpha,\, \beta \in \mathbb{R}$ with $\alpha \neq 0$.

By using the inequality (5.2) we also have for $x \in (a,b)$ that

$$
\begin{aligned}
&\left|\mathscr{E}_{g,a+,b-}^{\alpha+\beta i} f(x)-\frac{1}{2}\left[\frac{(\exp[(\alpha+\beta i)(g(b)-g(x))]-1) f(b)+(\exp[(\alpha+\beta i)(g(x)-g(a))]-1) f(a)}{(\alpha+\beta i)}\right]\right| \\
&\leq \frac{1}{2}\left[\int_a^x \exp(\alpha(g(t)-g(x))) g'(t) \bigvee_a^t (f)\, dt+\int_x^b \exp(\alpha(g(x)-g(t))) g'(t) \bigvee_t^b (f)\, dt\right] \\
&\leq \frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha} \bigvee_x^b (f)\ +\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha} \bigvee_a^x (f)\right] \\
&\leq \frac{1}{2}\begin{cases}
\max\left\{\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}, \frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right\} \bigvee_a^b (f); \\[2ex]
\left[\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p+\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p\right]^{1/p} \\
\times\left(\left(\bigvee_a^x (f)\right)^q+\left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left[\frac{\exp(\alpha(g(b)-g(x)))-1+\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right] \\
\times\left[\frac{1}{2}\bigvee_a^b (f)+\frac{1}{2}\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right]
\end{cases}
\end{aligned}
$$

for $\alpha,\, \beta \in \mathbb{R}$ with $\alpha \neq 0$.

If we denote

$$
\begin{aligned}
\overline{\mathscr{E}}_{g,a+,b-}^{\alpha+\beta i} f &:= \mathscr{E}_{g,a+,b-}^{\alpha+\beta i} f(M_g(a,b)) \\
&= \frac{1}{2}\int_a^x \exp\left[(\alpha+\beta i)\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right] g'(t) f(t)\, dt \\
&\quad+\frac{1}{2}\int_x^b \exp\left[(\alpha+\beta i)\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right] g'(t) f(t)\, dt
\end{aligned}
$$

then by (5.3) and (5.4) we have the simpler results

$$\begin{aligned}\left|\overline{\mathscr{E}}^{\alpha+\beta i}_{g,a+,b-}f-\frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}f(M_g(a,b))\right| &\le \frac{1}{2}\int_{M_g(a,b)}^{b}\exp\left(\alpha\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right)g'(t)\bigvee_{M_g(a,b)}^{t}(f)\,dt\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)}\exp\left(\alpha\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right)g'(t)\bigvee_{t}^{M_g(a,b)}(f)\,dt\\ &\le \frac{1}{2}\frac{\exp\left(\alpha\left(\frac{g(b)-g(a)}{2}\right)\right)-1}{\alpha}\bigvee_{b}^{b}(f) \end{aligned} \tag{6.1}$$

and

$$\begin{aligned}\left|\overline{\mathscr{E}}^{\alpha+\beta i}_{g,a+,b-}f-\frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}\frac{f(b)+f(a)}{2}\right| &\le \frac{1}{2}\int_{a}^{M_g(a,b)}\exp\left(\alpha\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right)g'(t)\bigvee_{a}^{t}(f)\,dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^{b}\exp\left(\alpha\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right)g'(t)\bigvee_{t}^{b}(f)\,dt\\ &\le \frac{1}{2}\frac{\exp\left(\alpha\left(\frac{g(b)-g(a)}{2}\right)\right)-1}{\alpha}\bigvee_{b}^{b}(f). \end{aligned} \tag{6.2}$$

In particular, if we take in (6.1) and (6.2) $g=\ln t$, $t\in[a,b]\subset(0,\infty)$, then by using the notation $G(\gamma,\delta):=\sqrt{\gamma\delta}$ for the *geometric mean* of the positive real numbers $\gamma$, $\delta>0$ we have

$$\begin{aligned}\left|\bar{\kappa}^{\alpha+\beta i}_{a+,b-}f-\frac{\left(\frac{b}{a}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}f(G(a,b))\right| &\le \frac{1}{2}\int_{G(a,b)}^{b}\left(\frac{t}{G(a,b)}\right)^{\alpha}\frac{1}{t}\bigvee_{G(a,b)}^{t}(f)\,dt+\frac{1}{2}\int_{a}^{G(a,b)}\left(\frac{G(a,b)}{t}\right)^{\alpha}\frac{1}{t}\bigvee_{t}^{G(a,b)}(f)\,dt\\ &\le \frac{1}{2}\frac{\left(\frac{b}{a}\right)^{\alpha}-1}{\alpha}\bigvee_{b}^{b}(f)\end{aligned}$$

and

$$\begin{aligned}\left|\bar{\kappa}^{\alpha+\beta i}_{a+,b-}f-\frac{\left(\frac{b}{a}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\frac{f(b)+f(a)}{2}\right| &\le \frac{1}{2}\int_{G(a,b)}^{b}\left(\frac{G(a,b)}{t}\right)^{\alpha}\frac{1}{t}\bigvee_{t}^{b}(f)\,dt+\frac{1}{2}\int_{a}^{G(a,b)}\left(\frac{t}{G(a,b)}\right)^{\alpha}\frac{1}{t}\bigvee_{a}^{t}(f)\,dt\\ &\le \frac{1}{2}\frac{\left(\frac{b}{a}\right)^{\alpha}-1}{\alpha}\bigvee_{b}^{b}(f),\end{aligned}$$

where

$$\bar{\kappa}^{\alpha+\beta i}_{a+,b-}f:=\frac{1}{2}\int_{G(a,b)}^{b}\left(\frac{t}{G(a,b)}\right)^{\alpha+\beta i}\frac{1}{t}f(t)\,dt+\frac{1}{2}\int_{a}^{G(a,b)}\left(\frac{G(a,b)}{t}\right)^{\alpha+\beta i}\frac{1}{t}f(t)\,dt.$$

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# Some New Characterizations of Symplectic Curve in 4-Dimensional Symplectic Space

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**Abstract**
It is well known that there exist characterizations for curve in Euclidean space. Also, a lot of authors extended this characterizations for Minkowski space and obtained very different results.
In this paper, we introduce the geometric properties of Symplectic Curve in 4-Dimensional Symplectic Space which given by [1, 2]. Later we obtained the conditions for Symplectic Curve to lie on some subspaces of 4-Dimensional Symplectic Space and we give some characterizations and theorems for these curves.

**Keywords:** Symplectic curve, Symplectic space
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## 1. Introduction

Symplectic geometry provided a language for clasical mechanics. Through its recent huge development, it conquered an independent and rich territory, as a central branch of differantial geometry and topology. On the other hand Symplectic geometry is mathematical apparatus of such areas of Physics as classical mechanics, geometrical optics and thermodynamics. İn order words Symplectic geometry arose from the study of optics and mechanics.

The study of local symplectic invariants of curves in affine symplectic geometry was initialed by Chern and Wang [3]. The issue has however remained silent for many years, before being taken up on several occasions in recent literature [4, 5, 6, 7]. Recently, Kamran et al. [1] developed the results and obtained explicitly the symplectic invariants and Frenet frames for curves in affine symplectic geometry by successively differentiating the tangent vector of the curve and using the non-degenerate inner product associated to the symplectic form. Frenet frame and Frenet formulae of curves by using the equivariant moving frame method due to Fels and Olver [8, 9].

In the present paper, firstly we give a short view of the basis of symplectic curves in the 4-dimensional symplectic space and secondly we investigate the conditions for symplectic curves to lie on some subspaces of $\mathbb{R}^4$and we give some characterizations and theorems for symplectic curves.

## 2. Preliminaries

Let us give brief related to symplectic space. One can found a brief account of the symplectic space in [1, 2]. The symplectic space $Sim = (\mathbb{R}^4, \Omega)$ is the vector space $\mathbb{R}^4$ endowed with the standard symplectic form $\Omega$, given in global Darboux coordinates by $\Omega = \sum_{i=1}^{2} dx_i \wedge dy_i$. Each tangent space is endowed with symplectic inner product defined in canonical basis by

$$\langle u,v\rangle = \Omega(u,v) = x_1\eta_1 + x_2\eta_2 - y_1\xi_1 - y_2\xi_2$$

where $u = (x_1,x_2,y_1,y_2)$ and $v = (\xi_1,\xi_2,\eta_1,\eta_2)$.

A symplectic frame is a smooth section of the bundle of linear frames over $\mathbb{R}^4$ which assigns to every point $z \in \mathbb{R}^4$ an ordered basis of tangent vectors $a_1,a_2,a_3,a_4$ with the property that

$$\langle a_i,a_j\rangle = \langle a_{2+i},a_{2+j}\rangle = 0, 1 \le i,j \le 2,$$

$$\langle a_i,a_{2+j}\rangle = 0,\ 1 \le i \ne j \le 2,$$

$$\langle a_i,a_{2+i}\rangle = 1, 1 \le i \le 2.$$

Let $z(t) : \mathbb{R} \to \mathbb{R}^4$ denotes a local parametrized curve. In our notation, we allow $z$ to be defined on an open interval of $\mathbb{R}$. As it is customary in classical mechanics, we use the notation $\dot{z}$ to denote differentiation with respect to the parameter t:

$$\dot{z} = \frac{dz}{dt}.$$

**Definition 2.1.** *A curve $z(t)$ is said to be symplectic regular if it satisfies the following non-degeneracy condition*

$$\langle \dot{z}\ ,\ \ddot{z}\rangle \ne 0\ ,\ \textit{for all } t \in \mathbb{R}. \tag{2.1}$$

*With no loss of generality, the left-hand side of (2.1) may be assumed positive.*

**Definition 2.2.** *Let $t_0 \in \mathbb{R}$, then the symplectic arc length $s$ of a symplectic regular curve starting at $t_0$ is defined by*

$$s(t) = \int_{t_0}^{t} \langle \dot{z}\ ,\ \dddot{z}\rangle^{1/3} dt\ \ \textit{for } t \ge t_0. \tag{2.2}$$

Taking the extrerior differential of (2.2) we obtain the symplectic arc length element as

$$ds = \langle \dot{z}\ ,\ \dddot{z}\rangle^{1/3} dt.$$

Dually, the arc length derivative operator is

$$D = \frac{d}{ds} = \langle \dot{z}\ ,\ \dddot{z}\rangle^{-1/3} \frac{d}{dt}. \tag{2.3}$$

In the following, primes are used to denote differentiation with respect to the symplectic arc length derivative operator (2.3)

$$z' = \frac{dz}{ds}.$$

**Definition 2.3.** *A symplectic regular curve is parametrized by symplectic arc length if*

$$\langle \dot{z};\dddot{z}\rangle = 1,$$

*for all $t \in \mathbb{R}$.*

Let $z(s)$ be a symplectic regular curve in $Sim = (\mathbb{R}^4,\Omega)$. In this case there exists only one Frenet frame $\{a_1(s),a_2(s),a_3(s),a_4(s)\}$ for which $z(s)$ is a symplectic regular curve with Frenet equations

$$a_1'(s) = a_3(s)$$
$$a_2'(s) = H_2(s)a_4(s)$$
$$a_3'(s) = k_1(s)a_1(s) + a_2(s)$$
$$a_4'(s) = a_1(s) + k_2(s)a_2(s)$$

where $H_2(s) = constant(\ne 0), k_1(s), k_2(s)$ are symplectic curvatures of $z(s)$ [1].

## 3. The characterizations of symplectic curve in 4-dimensional symplectic space

In this section we will investigate some characterizations of symplectic curve to lie on some subspaces of 4-Dimensional Symplectic Space. In the following, we use notations and concepts from [10], unless otherwise stated

**Case 1 :** We will investigate the conditions under which symplectic curve z lies on the subspace spanned by $\{a_3(s), a_4(s)\}$. In this case we can write

$$z(s) = \lambda(s)a_3(s) + \mu(s)a_4(s) \tag{3.1}$$

for some differentiablae functions $\lambda$and $\mu$ of the parameter s. Diffrentiating (3.1) with respect to s

$$z^{'}(s) = \lambda^{'}(s)a_3(s) + \mu^{'}(s)a_4(s) + \lambda(s)a_3^{'}(s) + \mu(s)a_4^{'}(s)$$

and by using Frenet equations we find that

$$z^{'}(s) = (\lambda(s)k_1(s) + \mu(s))\,a_1(s) + (\lambda(s) + \mu(s)k_2(s))\,a_2(s) + \lambda^{'}(s)a_3(s) + \mu^{'}(s)a_4(s)$$

Since $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ is a Frenet frame we have the following equations.

$$\lambda^{'}(s) = 0$$

$$\mu^{'}(s) = 0$$

$$\lambda(s)k_1(s) + \mu(s) = 1$$

$$\lambda(s) + \mu(s)k_2(s) = 0$$

From $\lambda^{'}(s) = 0$ and $\mu^{'}(s) = 0$ we find that $\lambda(s) = c_1$ and $\mu(s) = c_2$.Thus we have the following theorem.

**Theorem 3.1.** *A symplectic curve z in $R^4$ lies on the subspace spanned by $\{a_3(s), a_4(s)\}$ if and only if it is in the form*

$$z(s) = c_1a_3(s) + c_2a_4(s)$$

*where $k_1 = const., k_2 = const.$ and $k_1(s) = \frac{1}{k_2(s)}$ , $c_1, c_2$ are constants.*

**Case 2:** We will investigate the conditions under which symplectic curve z lies on the subspace spanned by $\{a_1(s), a_3(s), a_4(s)\}$. In this case we can write

$$z(s) = \lambda(s)a_1(s) + \mu(s)a_3(s) + \gamma(s)a_4(s) \tag{3.2}$$

for some differentiable functions $\lambda, \mu$ and $\gamma$ of the parameter s. Differentiating (3.2) with respect to s

$$z^{'}(s) = \lambda^{'}(s)a_1(s) + \mu^{'}(s)a_3(s) + \gamma^{'}(s)a_4(s) + \lambda(s)a_1^{'}(s) + \mu(s)a_3^{'}(s) + \gamma(s)a_4^{'}(s)$$

and by using Frenet equations we find that

$$\begin{aligned} z^{'}(s) &= \left(\lambda^{'}(s) + \mu(s)k_1(s) + \gamma(s)\right)a_1(s) + (\mu(s) + \gamma(s)k_2(s))\,a_2(s) \\ &\quad + \left(\mu^{'}(s) + \lambda(s)\right)a_3(s) + \gamma^{'}(s)a_4(s) \end{aligned}$$

Since $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ is a Frenet frame we have the following equations.

$$\lambda^{'}(s) + \mu(s)k_1(s) + \gamma(s) = 1$$

$$\mu(s) + \gamma(s)k_2(s) = 0$$

$$\mu^{'}(s) + \lambda(s) = 0$$

$$\gamma^{'}(s) = 0$$

We obtain that $\gamma(s) = c, \mu(s) = -ck_2(s)$ and $\lambda(s) = ck_2^{\prime}(s)$. Thus we have the following theorem.

**Theorem 3.2.** *A symplectic curve z in $R^4$ lies on the subspace spanned by $\{a_1(s), a_3(s), a_4(s)\}$ if and only if it is in the form*

$$z(s) = ck_2'(s)a_1(s) - ck_2(s)a_3(s) + ca_4(s)$$

*where $k_2(s) \neq 0$, c is constant.*

**Case 3 :** We will investigate the conditions under which symplectic curve z lies on the subspace spanned by $\{a_2(s), a_3(s), a_4(s)\}$. In this case we can write

$$z(s) = \lambda(s)a_2(s) + \mu(s)a_3(s) + \gamma(s)a_4(s) \tag{3.3}$$

for some differentiable functions $\lambda, \mu$ and $\gamma$ of the parameter s. Differentiating (3.3) with respect to s

$$z'(s) = \lambda'(s)a_2(s) + \mu'(s)a_3(s) + \gamma'(s)a_4(s) + \lambda(s)a_2'(s) + \mu(s)a_3'(s) + \gamma(s)a_4'(s)$$

and by using Frenet equations we find that

$$\begin{aligned} z'(s) &= (\mu(s)k_1(s) + \gamma(s))\, a_1(s) + \Big(\lambda'(s) + \mu(s) + \gamma(s)k_2(s)\Big)\, a_2(s) \\ &\quad + \mu'(s)a_3(s) + \Big(\gamma'(s) + \lambda(s)H_2(s)\Big)\, a_4(s) \end{aligned}$$

Since $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ is a Frenet frame we have the following equations.

$$\mu(s)k_1(s) + \gamma(s) = 1$$
$$\lambda'(s) + \mu(s) + \gamma(s)k_2(s) = 0$$
$$\mu'(s) = 0$$
$$\gamma'(s) + \lambda(s)H_2(s) = 0$$

Here $\mu(s) = c_1$ and by using equations $\mu(s)k_1(s) + \gamma(s) = 1$ and $\lambda'(s) + \mu(s) + \gamma(s)k_2(s) = 0$, we find $\gamma(s) = 1 - c_1K_1(s)$ and $\lambda(s) = \int [c - (1 - c_1k_1(s))\, k_2(s)ds] + c_2$

$$\gamma(s) = \int \left[\left[-\int (-c - (1 - k_1)k_2)\, ds + c_2\right] H_2(s)\right] ds + c_3$$

Thus we have the following theorem.

**Theorem 3.3.** *A symplectic curve z in $R^4$ lies on the subspace spanned by $\{a_2(s), a_3(s), a_4(s)\}$ if and only if it is in the form*

$$z(s) = -\frac{c_1}{H_2(s)}k_1'(s)a_2(s) + c_1a_3(s) + (1 - c_1k_1(s))\, a_4(s)$$

*or*

$$z(s) = \left[\int [c_1(k_1(s)k_2(s) - 1) - k_2(s)]\, ds + c_2\right] a_2(s) + c_1a_3(s) + (1 - c_1k_1(s))\, a_4(s)$$

*where $c_1$,$c_2$ are constants.*

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