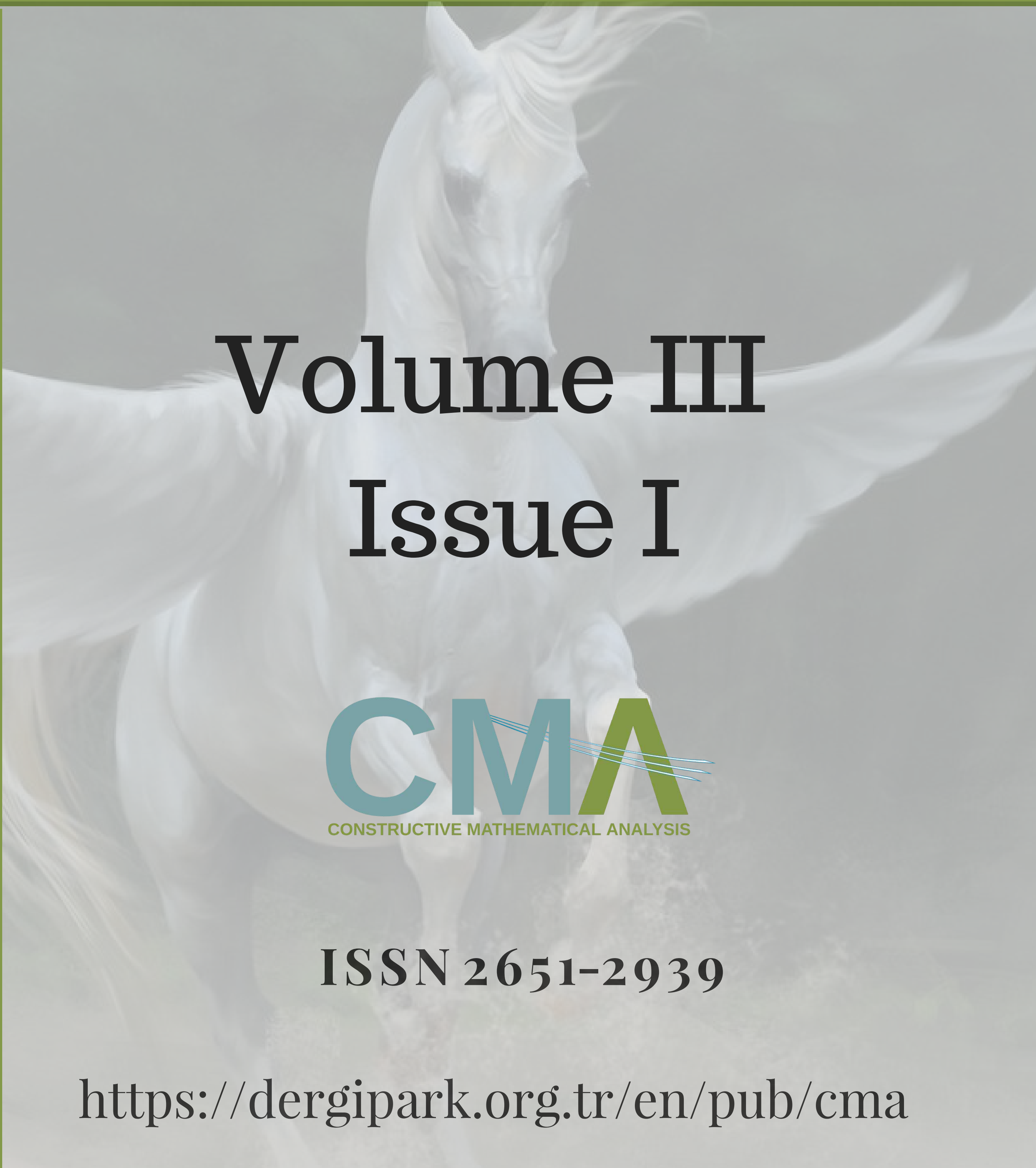
CONSTRUCTIVE MATHEMATICAL ANALYSIS



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CONSTRUCTIVE MATHEMATICAL ANALYSIS

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Contents

The Issue dedicated to

Prof Dr Anthony To-Ming Lau

University of Alberta, Canada

Professor Anthony To-Ming Lau is distinguished faculty member of Department of Mathematics and Statistics, University of Alberta, Canada.

Professor Lau has made significant contributions to harmonic analysis and functional Analysis, especially, amenability, semigroups and fixed point theory. He has published more than 150 research articles and monographs.

In 2018, Professor Lau received the prestigious prize "David Borwein Distinguished Career Award".

When the journal "Constructive Mathematical Analysis" was founded, Professor Lau was on of them who agreed to contribute to the journal as a Editorial board member.

Constructive Mathematical Analysis Editorial Team has pleasure and honor for dedicating the issue to Professor Anthony To-Ming Lau for his great contributions to the Mathematical community and Constructive Mathematical Analysis.

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Research Article

A New Asymptotic Series and Estimates Related to Euler Mascheroni Constant

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ABSTRACT. In this article, we give a new asymptotic series for a sequence (q_n) that converges to Euler-Mascheroni's constant with the convergence speed as n^{-4}. We present and prove a theorem about how to get the sequence (q_n). Using this asymptotic series, we establish the lower and upper bounds for the sequence (q_n).

Keywords: Euler-Mascheroni's constant, asymptotic series, inequalities.

2010 Mathematics Subject Classification: 26D15, 41A25, 34E05.

1. INTRODUCTION

One of the famous constants in mathematics is the Euler-Mascheroni's constant $\gamma = 0,57721566490153286...$. It is defined as the limit of the sequence:

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n$$

in honor of the Swiss mathematician Leonhard Euler (1707-1783) and the Italian mathematician Lorenzo Mascheroni (1750-1800), who studied the Euler-Mascheroni's constant γ. The sequence $(\gamma_n)_{n\geq 1}$ and the constant γ have many applications in several branches of mathematics as probability, analysis, special functions and number theory. The sequence $(\gamma_n)_{n\geq 1}$ converges very slowly to the constant γ, with the convergence speed as n^{-1}. In the beginning, Tims and Tyrell [18], and then Young [19] got the lower and upper bounds for the sequence $(\gamma_n)_{n\geq 1}$ as the following:

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}$$

with the convergence speed as n^{-1}. Many authors [2, 3, 6, 7, 10, 12–17] interested in obtaining sequences that converge very fast to the limit γ. One of them is DeTemple [6], who introduced the sequence

$$R_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right)$$

that converges to the limit γ as n^{-2}. Then Mortici [12] has introduced the sequence

$$t_n = 1 + \frac{1}{2} + ... + \frac{1}{n-1} + \frac{1}{2n} - \frac{1}{2}\ln\left(n^2 - \frac{1}{6}\right) \tag{1.1}$$

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in order to obtain a faster convergence to the limit γ with the convergence speed as n^{-4} and the following limit:

$$\lim_{n\to\infty} n^4 (t_n - \gamma) = \frac{11}{720}.$$

Then, Cristea [4] has showed in 2014, the following double inequality

$$\frac{11}{720n^4} - \frac{29}{9072n^6} < t_n - \gamma < \frac{11}{720n^4}$$

for all integers $n \geq 1$ and has got the following asymptotic series for the sequence (t_n) given in (1.1)

$$t_n = \gamma + \sum_{k=2}^{\infty} \frac{1}{2k} \left\{ \frac{1}{6^k} - B_{2k} \right\} \frac{1}{n^{2k}}$$

or

$$t_n = \gamma + \frac{11}{720n^4} - \frac{29}{9072n^6} + \frac{221}{51\,840n^8} - \frac{6469}{855\,360n^{10}} + \cdots$$

Cristea and Mortici [5] have introduced the sequence

$$s_n = 1 + \frac{1}{2} + ... + \frac{1}{n-2} + \frac{13}{12(n-1)} + \frac{5}{12n} - \ln n \tag{1.2}$$

that converges to the limit γ with the convergence speed as n^{-3} and have demonstrated the following double inequality

$$\frac{1}{12n^3} + \frac{11}{120n^4} < s_n - \gamma < \frac{1}{12n^3} + \frac{13}{120n^4}.$$

Then, X. Hu, D. Lu, X. Wang [9] have presented the following sequence:

$$r^3_{n,2} = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln n - \frac{1}{2} \ln \left(1 + \frac{1}{n - \frac{n}{3n+1}} \right)$$

that converges to the limit γ with the convergence speed as n^{-4}, with the following approximation:

$$\frac{1}{180(n+1)^4} < \gamma - r^3_{n,2} < \frac{1}{180n^4}.$$

The aim of the paper is to introduce a new sequence (q_n) that converges very fast to the limit γ and to establish the lower and upper bounds for this sequence. Motivated by Mortici [12] and Hu [9], we introduce new sequence

$$q_n(a,b,c) = 1 + \frac{1}{2} + ... + \frac{1}{n-2} + \frac{an+b}{n(n-1)} - \frac{1}{3} \ln \left(n^3 + c \right), \tag{1.3}$$

where a, b, c are real parameters and for $a = \frac{3}{2}, b = -\frac{5}{12}, c = \frac{1}{4}$ the new sequence given by

$$q_n = q_n(\frac{3}{2}, -\frac{5}{12}, \frac{1}{4}) = 1 + \frac{1}{2} + ... + \frac{1}{n-2} + \frac{13}{12(n-1)} + \frac{5}{12n} - \frac{1}{3} \ln \left(n^3 + \frac{1}{4} \right) \tag{1.4}$$

converges to the limit γ with the convergence speed as n^{-4}. We will show the following double inequality

$$\frac{11}{120n^4} + \frac{1}{12n^5} + \frac{181}{2016n^6} < q_n - \gamma < \frac{11}{120n^4} + \frac{1}{12n^5} + \frac{182}{2016n^6}$$

for all integers $n \geq 2$ in the left side inequality and for all integers $n \geq 225$ in the right side inequality. We will also construct the asymptotic series

$$q_n = \gamma + \frac{11}{120n^4} + \frac{1}{12n^5} + \frac{181}{2016n^6} + \frac{1}{12n^7} + \cdots$$

for the sequence (q_n) (1.4).

2. THE RESULTS

We consider the sequence $(q_n(a,b,c))$ given by (1.3). To obtain the best real parameters a,b,c, for which the sequence $(q_n(a,b,c))$ converges to γ with the highest convergence speed, we prove the following theorem:

Theorem 2.1. *(i) If $a \neq \frac{3}{2}, b \neq -\frac{5}{12}$ and $c \neq \frac{1}{4}$ then the sequence $(q_n(a,b,c))_{n\geq 1}$ has the convergence speed as n^{-1}.*

(ii) If $a = \frac{3}{2}, b \neq -\frac{5}{12}$ and $c \neq \frac{1}{4}$ then the sequence $(q_n(a,b,c))_{n\geq 1}$ has the convergence speed as n^{-2}.

(iii) If $a = \frac{3}{2}, b = -\frac{5}{12}$ and $c \neq \frac{1}{4}$ then the sequence $(q_n(a,b,c))_{n\geq 1}$ has the convergence speed as n^{-3}.

(iv) If $a = \frac{3}{2}, b = -\frac{5}{12}$ and $c = \frac{1}{4}$ then the sequence $(q_n(a,b,c))_{n\geq 1}$ has the convergence speed as n^{-4}.

We will use the following:

Lemma 2.1. *If the sequence $(x_n)_{n\geq 1}$ converges to x and if there exists the limit*

$$\lim_{n\to\infty} n^k \left(x_n - x_{n+1}\right) = l \in \mathbb{R}$$

with $k > 1$, then there exists the limit

$$\lim_{n\to\infty} n^{k-1} \left(x_n - x\right) = \frac{l}{k-1}.$$

For the proof see [11]. This lemma is a form of Cesaro-Stolz's lemma. We utilize it in the construction of the asymptotics series and in order to estimate the convergence speed.

Proof. We compute the difference

$$q_n(a,b,c) - q_{n+1}(a,b,c) = \frac{an+b}{n(n-1)} - \frac{1}{n-1} - \frac{an+a+b}{n(n+1)}$$
$$-\frac{1}{3}\ln\left(n^3+c\right) + \frac{1}{3}\ln\left((n+1)^3+c\right).$$

Using a computer program as Maple, we get

$$q_n(a,b,c) - q_{n+1}(a,b,c) = \left(a - \frac{3}{2}\right)\frac{1}{n^2} + \left(a + 2b - \frac{2}{3}\right)\frac{1}{n^3} + \left(a - c - \frac{5}{4}\right)\frac{1}{n^4}$$

$$+\left(a + 2b + 2c - \frac{4}{5}\right)\frac{1}{n^5} + O\left(\frac{1}{n^6}\right). \tag{2.5}$$

(i) If $a - \frac{3}{2} \neq 0$, then

$$\lim_{n\to\infty} n^2\left(q_n(a,b,c) - q_{n+1}(a,b,c)\right) = \left(a - \frac{3}{2}\right) \neq 0$$

and Lemma 2.1 says that

$$\lim_{n\to\infty} n\left(q_n(a,b,c) - \gamma\right) = \left(a - \frac{3}{2}\right) \neq 0.$$

We get that the sequence $(q_n(a,b,c))_{n\geq 1}$ has the convergence speed as n^{-1}.
(ii) If $a=\frac{3}{2}, b\neq -\frac{5}{12}$ and $c\neq \frac{1}{4}$ then the relation (2.5) is written as

$$q_n(a,b,c)-q_{n+1}(a,b,c)=\left(2b+\frac{5}{6}\right)\frac{1}{n^3}+\left(\frac{1}{4}-c\right)\frac{1}{n^4}$$

$$+\left(\frac{7}{10}+2b+2c\right)\frac{1}{n^5}+O\left(\frac{1}{n^6}\right). \tag{2.6}$$

If $b\neq -\frac{5}{12}$, then from the relation (2.6), we get

$$\lim_{n\to\infty} n^3\left(q_n(a,b,c)-q_{n+1}(a,b,c)\right)=\left(2b+\frac{5}{6}\right)\neq 0$$

and Lemma 2.1 says that

$$\lim_{n\to\infty} n^2\left(q_n(a,b,c)-\gamma\right)=\frac{1}{2}\left(2b+\frac{5}{6}\right)\neq 0.$$

We obtain that the sequence $\left(q_n(\frac{3}{2},b,c)\right)_{n\geq 1}$ has the convergence speed as n^{-2}.
(iii) If $a=\frac{3}{2}, b=-\frac{5}{12}$ and $c\neq\frac{1}{4}$ then the relation (2.5) is written as

$$q_n(a,b,c)-q_{n+1}(a,b,c)=\left(\frac{1}{4}-c\right)\frac{1}{n^4}+\left(-\frac{2}{15}+2c\right)\frac{1}{n^5}+O\left(\frac{1}{n^6}\right). \tag{2.7}$$

Then from the relation (2.7), we get

$$\lim_{n\to\infty} n^4\left(q_n(a,b,c)-q_{n+1}(a,b,c)\right)=\left(\frac{1}{4}-c\right)\neq 0$$

and Lemma 2.1 says that

$$\lim_{n\to\infty} n^3\left(q_n(a,b,c)-\gamma\right)=\frac{1}{3}\left(\frac{1}{4}-c\right)\neq 0.$$

We get that the sequence $\left(q_n(\frac{3}{2},-\frac{5}{12},c)\right)_{n\geq 1}$ has the convergence speed as n^{-3}.
(iv) If $a=\frac{3}{2}, b=-\frac{5}{12}$,and $c=\frac{1}{4}$ then the relation (2.5) is written as

$$q_n(a,b,c)-q_{n+1}(a,b,c)=\frac{11}{30n^5}+O\left(\frac{1}{n^6}\right) \tag{2.8}$$

and Lemma 2.1 says that

$$\lim_{n\to\infty} n^4\left(q_n(a,b,c)-\gamma\right)=\frac{11}{120}.$$

We get that the sequence $\left(q_n(\frac{3}{2},-\frac{5}{12},\frac{1}{4})\right)_{n\geq 1}$ has the convergence speed as n^{-4}. □

We notice that (2.8) gives us the approximation

$$q_n-\gamma\approx\frac{11}{120n^4}\quad \text{as } n\to\infty.$$

We give the following theorem related to the estimates of (q_n) given in (1.4):

Theorem 2.2. *We have the following double inequality for all integers $n\geq 2$ in the left side inequality and for all integers $n\geq 225$ in the right side inequality:*

$$\frac{11}{120n^4}+\frac{1}{12n^5}+\frac{181}{2016n^6}<q_n-\gamma<\frac{11}{120n^4}+\frac{1}{12n^5}+\frac{182}{2016n^6}.$$

Proof. We consider the following sequences

$$a_n = (q_n - \gamma) - \left(\frac{11}{120n^4} + \frac{1}{12n^5} + \frac{181}{2016n^6}\right)$$

and

$$b_n = (q_n - \gamma) - \left(\frac{11}{120n^4} + \frac{1}{12n^5} + \frac{182}{2016n^6}\right)$$

that converges to zero. To prove that $a_n > 0$ and $b_n < 0$, it suffices to show that $(a_n)_{n\geq 1}$ is strictly decreasing and $(b_n)_{n\geq 1}$ is strictly increasing. Let $f_1(n) = a_{n+1} - a_n$ and $f_2(n) = b_{n+1} - b_n$, where

$$\begin{aligned} f_1(x) &= \frac{8}{12x} + \frac{5}{12(x+1)} - \frac{1}{12(x-1)} + \frac{1}{3}\ln\left(x^3 + \frac{1}{4}\right) - \frac{1}{3}\ln\left((x+1)^3 + \frac{1}{4}\right) \\ &\quad -\left(\frac{11}{120(x+1)^4} - \frac{11}{120x^4}\right) - \left(\frac{1}{12(x+1)^5} - \frac{1}{12x^5}\right) - \left(\frac{181}{2016(x+1)^6} - \frac{181}{2016x^6}\right) \end{aligned}$$

and

$$\begin{aligned} f_2(x) &= \frac{8}{12x} + \frac{5}{12(x+1)} - \frac{1}{12(x-1)} + \frac{1}{3}\ln\left(x^3 + \frac{1}{4}\right) - \frac{1}{3}\ln\left((x+1)^3 + \frac{1}{4}\right) \\ &\quad -\left(\frac{11}{120(x+1)^4} - \frac{11}{120x^4}\right) - \left(\frac{1}{12(x+1)^5} - \frac{1}{12x^5}\right) - \left(\frac{182}{2016(x+1)^6} - \frac{182}{2016x^6}\right). \end{aligned}$$

We get

$$f_1'(x) = \frac{P(x-2)}{1680(x+1)^7(x-1)^2(4x^3+1)^1(12x+12x^2+4x^3+5)^1 x^5} > 0 \tag{2.9}$$

for all real numbers $x \geq 2$ and

$$f_2'(x) = -\frac{Q(x-225)}{120(x+1)^7(x-1)^2(12x+12x^2+4x^3+5)^1(4x^3+1)^1 x^7} < 0 \tag{2.10}$$

for all real numbers $x \geq 225$, where

$$\begin{aligned} P(x) &= 8615781393 + 48322358\,535x + 124\,451770884x^2 + 195088765300x^3 \\ &\quad +207843366162x^4 + 159018283386x^5 + 89932803430x^6 + 38082594545x^7 \\ &\quad +12078804629x^8 + 2834912752x^9 + 478671564x^{10} + 55071128x^{11} \\ &\quad +3869824x^{12} + 125440x^{13} \end{aligned}$$

and

$$\begin{aligned} Q(x) = {} & 22876\,348962124636919596278035200 \\ & +1561258918341618251050908\,15353280x \\ & +8964689205792820697567513156375x^2 \\ & +2382989135830296264858888825003x^3 \\ & +387400193922908539529960913x^4 \\ & +429535098002548661658 09975x^5 \\ & +34295429808865868353 7331x^6 \\ & +2028513740325127816093x^7 \\ & +89992142959018 01973x^8 \\ & +29943893833882652x^9 \\ & +73805584698144x^{10} \\ & +130981721712x^{11} \\ & +158491784x^{12} \\ & +117200x^{13} \\ & +40x^{14} \end{aligned}$$

are two polynomials with positive integers coefficients for all real numbers $x \geq 2$ and respectively for all real numbers $x \geq 225$. Then, from (2.9), we have f_1 is strictly increasing on $[2, \infty)$ and from (2.10), we have f_2 is strictly decreasing on $[225, \infty)$. It follows that from $f_1(\infty) = f_2(\infty) = 0$, we have $f_1 < 0$ on $[2, \infty)$ and $f_2 > 0$ on $[225, \infty)$. Thus, $(a_n)_{n \geq 2}$ is strictly decreasing and $(b_n)_{n \geq 225}$ is strictly increasing. This concludes the proof. $\square$

We can get the asymptotic series of the sequence (q_n), using the sequence (h_n)

$$h_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$$

harmonic sum in terms of digamma function ψ

$$h_n = \gamma + \frac{1}{n} + \psi(n),$$

with the digamma function defined by

$$\psi(x) = \frac{d}{dx}(\ln \Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

See, e.g., [1, p. 258, Rel. 6.3.2]. We have the following asymptotic expansion for the digamma function ψ that

$$\psi(x) = \ln x - \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kx^{2k}},$$

where B_j is the jth Bernoulli numbers given by

$$\frac{1}{e^t - 1} + \frac{1}{2} - \frac{1}{t} = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{t^{2j}}{(2j)!} B_j.$$

We will demonstrate the following theorem related to the asymptotic expansion of q_n :

Theorem 2.3. *We get the following asymptotic expansion of* (q_n) *as* $n \to \infty$:

$$q_n = \gamma + \frac{1}{12n(n-1)} - \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \frac{(-1)^{k-1}}{3 \cdot 4^k n^{3k}} + \frac{B_{2k}}{2n^{2k}} \right\}.$$

Proof. We get

$$\begin{aligned} q_n &= h_n - \frac{1}{n} + \frac{1}{12(n-1)} + \frac{5}{12n} - \frac{1}{3} \ln \left(n^3 + \frac{1}{4} \right) \\ &= \gamma + \psi(n) + \frac{1}{12(n-1)} + \frac{5}{12n} - \frac{1}{3} \ln \left(n^3 + \frac{1}{4} \right) \\ &= \gamma + \psi(n) - \ln n + \frac{1}{12(n-1)} + \frac{5}{12n} - \frac{1}{3} \ln \left(1 + \frac{1}{4n^3} \right) \\ &= \gamma + \frac{1}{12(n-1)} - \frac{1}{2n} + \frac{5}{12n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}} - \frac{1}{3} \ln \left(1 + \frac{1}{4n^3} \right) \\ &= \gamma + \frac{1}{12n(n-1)} - \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \frac{(-1)^{k-1}}{3 \cdot 4^k n^{3k}} + \frac{B_{2k}}{2n^{2k}} \right\}. \end{aligned}$$

Using the binomial theorem given in [8], we get

$$\frac{1}{12n(n-1)} = \frac{1}{12n^2\left(1 - \frac{1}{n}\right)} = \frac{1}{12n^2} + \frac{1}{12n^3} + \frac{1}{12n^4} + \frac{1}{12n^5} + \cdots$$

We get an explicite form as

$$q_n = \gamma + \frac{11}{120n^4} + \frac{1}{12n^5} + \frac{181}{2016n^6} + \frac{1}{12n^7} + \cdots \tag{2.11}$$

We notice that the three terms of the asymptotic series (2.11) were used for the estimate of q_n. We give the table with the above sequences:

n	$\lvert t_n - \gamma \rvert$	$\lvert s_n - \gamma \rvert$	$\lvert r_{n,2}^3 - \gamma \rvert$	$\lvert q_n - \gamma \rvert$
250	$1.309\,35 \times 10^{-17}$	$4.266\,67 \times 10^{-12}$	$2.252\,98 \times 10^{-14}$	$2.031\,75 \times 10^{-18}$
500	$2.045\,86 \times 10^{-19}$	$2.666\,67 \times 10^{-13}$	$7.075\,70 \times 10^{-16}$	$3.174\,6 \times 10^{-20}$
1000	$3.196\,65 \times 10^{-21}$	$1.666\,67 \times 10^{-14}$	$2.216\,68 \times 10^{-17}$	$4.960\,32 \times 10^{-22}$
10000	$3.196\,65 \times 10^{-27}$	$1.666\,67 \times 10^{-18}$	$2.221\,67 \times 10^{-22}$	$4.960\,32 \times 10^{-28}$
50000	$2.045\,86 \times 10^{-31}$	$2.666\,67 \times 10^{-21}$	$7.110\,76 \times 10^{-26}$	$3.174\,6 \times 10^{-32}$

Using the values from the above table, we conclude the superiority of the sequence $(q_n)_{n \ge 225}$ over Mortici's sequence $(t_n)_{n \ge 225}$, Lu's sequence $\left(r_{n,2}^3\right)_{n \ge 225}$, Cristea and Mortici's sequence $(s_n)_{n \ge 225}$. □

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Research Article

Growth Estimates for Analytic Vector-Valued Functions in the Unit Ball Having Bounded L-index in Joint Variables

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ABSTRACT. Our results concern growth estimates for vector-valued functions of $\mathbb{L}$-index in joint variables which are analytic in the unit ball. There are deduced analogs of known growth estimates obtained early for functions analytic in the unit ball. Our estimates contain logarithm of sup-norm instead of logarithm modulus of the function. They describe the behavior of logarithm of norm of analytic vector-valued function on a skeleton in a bidisc by behavior of the function $\mathbf{L}$. These estimates are sharp in a general case. The presented results are based on bidisc exhaustion of a unit ball.

Keywords: Bounded index, bounded $\mathbf{L}$-index in joint variables, analytic function, unit ball, growth estimates, maximum norm.

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1. INTRODUCTION

In this paper, we consider vector-valued functions of bounded $\mathbf{L}$-index in joint variables which are analytic in the unit ball. This paper is a continuation of investigations initiated in [1, 2, 3]. There was proposed the definition of $\mathbf{L}$-index boundedness in joint variables and obtained some criteria of $\mathbf{L}$-index boundedness in joint variables for vector-valued analytic functions in the unit ball.

Here, we pose the following goal: *to obtain growth estimates of analytic functions having bounded* $\mathbf{L}$*-index in joint variables.* It is important because functions of bounded index has many applications in analytic theory of linear differential equations. Moreover, vector-valued entire functions of bounded index in joint variables have applications to some system of partial differential equations [19]. Therefore, combination of sufficient conditions of L-index boundedness for analytic solutions of the system with growth estimates of functions from this class will give a priori estimates of growth for all analytical solutions of the system.

Other applications of concept of bounded index in analytic theory of differential equations were considered for various function classes: entire functions of bounded L-index in direction [12], entire functions of bounded $\mathbf{L}$-index in joint variables [15], analytic functions in the unit ball having bounded $\mathbf{L}$-index in joint variables [4], entire bivariate vector-valued function of bounded index [19].

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2. Notations, definitions and auxiliary propositions

We need some standard notations (for example see [5, 4, 6]). Let $\mathbb{R}_+ = [0; +\infty)$, $\mathbf{0} = (0,0) \in \mathbb{R}^2_+$, $\mathbf{1} = (1,1) \in \mathbb{R}^2_+$, $R = (r_1, r_2) \in \mathbb{R}^2_+$, $|(z,\omega)| = \sqrt{|z|^2 + |\omega|^2}$. For $A = (a_1, a_2) \in \mathbb{R}^2$, $B = (b_1, b_2) \in \mathbb{R}^2$, we will use formal notations without violation of the existence of these expressions: $AB = (a_1b_1, a_2b_2)$, $A/B = (a_1/b_1, a_2/b_2)$, $A^B = (a_1^{b_1}, a_2^{b_2})$, and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, 2\}$; the relation $A \le B$ is defined in the similar way. For $K = (k_1, k_2) \in \mathbb{Z}^2_+$, let us denote $K! = k_1! \cdot k_2!$. Addition, multiplication by scalar and conjugation in $\mathbb{C}^2$ is defined componentwise. For $z \in \mathbb{C}^2$, $w \in \mathbb{C}^2$ we define $\langle z, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2$, where $\overline{w}_1, \overline{w}_2$ is the complex conjugate of w_1, w_2.

The bidisc $\{(z,\omega) \in \mathbb{C}^2 : |z - z_0| < r_1, |\omega - \omega_0| < r_2\}$ is denoted by $\mathbb{D}^2((z_0,\omega_0), R)$, its skeleton $\{(z,\omega) \in \mathbb{C}^2 : |z - z_0| = r_1, |\omega - \omega_0| = r_2\}$ is denoted by $\mathbb{T}^2((z_0,\omega_0), R)$, the closed polydisc $\{(z,\omega) \in \mathbb{C}^2 : |z - z_0| \le r_1, |\omega - \omega_0| \le r_2\}$ is denoted by $\mathbb{D}^2[(z_0,\omega_0), R]$, $\mathbb{D}^2 = \mathbb{D}^2(\mathbf{0}; \mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The open ball $\{(z,\omega) \in \mathbb{C}^2 : \sqrt{|z - z_0|^2 + |\omega - \omega_0|^2} < r\}$ is denoted by $\mathbb{B}^2((z_0,\omega_0), r)$, the sphere $\{(z,\omega) \in \mathbb{C}^2 : \sqrt{|z - z_0|^2 + |\omega - \omega_0|^2} = r\}$ is denoted by $\mathbb{S}^2((z_0,\omega_0), r)$, and the closed ball $\{z \in \mathbb{C}^2 : \sqrt{|z - z_0|^2 + |\omega_0 - \omega_0|^2} \le r\}$ is denoted by $\mathbb{B}^2[(z_0,\omega_0), r]$, $\mathbb{B}^2 = \mathbb{B}^2(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C} : |z| < 1\}$.

Let $F(z,\omega) = (f_1(z,\omega), f_2(z,\omega))$ be an analytic vector-function in $\mathbb{B}^2$. Then at a point $(a, b) \in \mathbb{B}^2$, the function $F(z,\omega)$ has a bivariate Taylor expansion:

$$F(z,\omega) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} C_{kl}(z-a)^k(\omega-b)^m,$$

where $C_{km} = \frac{1}{k!m!}\left(\frac{\partial^{k+m} f_1(z,\omega)}{\partial z^k \partial\omega^m}, \frac{\partial^{k+m} f_2(z,\omega)}{\partial z^k \partial\omega^m}\right)\Big|_{z=a,\omega=b} = \frac{1}{k!m!}F^{(k,m)}(a,b)$.

Let $\mathbf{L}(z,\omega) = (l_1(z,\omega), l_2(z,\omega))$, where $l_j(z,\omega) : \mathbb{B}^2 \to \mathbb{R}^2_+$ is a positive continuous function such that

$$\forall (z,\omega) \in \mathbb{B}^2 : \ \ l_j(z,\omega) > \frac{\beta}{1 - \sqrt{|z|^2 + |\omega|^2}}, \tag{2.1}$$

$j \in \{1, 2\}$, where $\beta > \sqrt{2}$ is a some constant.

The norm for the vector-function $F : \mathbb{B}^2 \to \mathbb{C}^2$ is defined as the sup-norm:

$$\|F(z,\omega)\| = \max_{1\le j\le 2}\{|f_j(z,\omega)|\}.$$

We write

$$F^{(i,j)}(z,\omega) = \frac{\partial^{i+j}F(z,\omega)}{\partial z^i \partial\omega^j} = \left(\frac{\partial^{i+j} f_1(z,\omega)}{\partial z^i \partial\omega^j}, \frac{\partial^{i+j} f_2(z,\omega)}{\partial z^i \partial\omega^j}\right).$$

An analytic vector-function $F : \mathbb{B}^2 \to \mathbb{C}^2$ is said to be of bounded $\mathbf{L}$-index (in joint variables), if there exists $n_0 \in \mathbb{Z}_+$ such that

$$\forall (z,\omega) \in \mathbb{B}^2 \ \ \forall (i,j) \in \mathbb{Z}^2_+ :$$

$$\frac{\|F^{(i,j)}(z,\omega)\|}{i!j!l_1^i(z,\omega)l_2^j(z,\omega)} \le \max\left\{\frac{\|F^{(k,m)}(z,\omega)\|}{k!m!l_1^k(z,\omega)l_2^m(z,\omega)} : k, m \in \mathbb{Z}_+, k+m \le n_0\right\}. \tag{2.2}$$

The least such integer n_0 is called the $\mathbf{L}$-index in joint variables of the vector-function F and is denoted by $N(F, \mathbf{L}, \mathbb{B}^2)$. The concept of boundedness of $\mathbf{L}$-index in joint variables were considered for other classes of analytic functions. They are differed domains of analyticity: the unit ball [5, 4, 11, 13], the polydisc [8, 10], the Cartesian product of the unit disc and complex plane [9], n-dimensional complex space [7, 11, 14]. Vector-valued functions of one and several complex variables having bounded index were considered in [18, 20, 17, 23, 21, 19].

The function class $Q(\mathbb{B}^2)$ is defined as following: $\forall R\in\mathbb{R}^2_+, |R|\le\beta, j\in\{1,2\}$:

$$0<\lambda_{1,j}(R)\le\lambda_{2,j}(R)<\infty,$$

where

$$\lambda_{1,j}(R)=\inf_{(z_0,\omega_0)\in\mathbb{B}^2}\inf\left\{\frac{l_j(z,\omega)}{l_j(z_0,\omega_0)}:(z,\omega)\in\mathbb{D}^2[(z_0,\omega_0),R/\mathbf{L}(z_0,\omega_0)]\right\}, \tag{2.3}$$

$$\lambda_{2,j}(R)=\sup_{(z_0,\omega_0)\in\mathbb{B}^2}\sup\left\{\frac{l_j(z,\omega)}{l_j(z_0,\omega_0)}:(z,\omega)\in\mathbb{D}^2[(z_0,\omega_0),R/\mathbf{L}(z_0,\omega_0)]\right\}. \tag{2.4}$$

We need some propositions from [1, 2].

For an analytic vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$, we put

$$M(R,(z_0,\omega_0),F)=\max\left\{\|F(z,\omega)\|:(z,\omega)\in\mathbb{T}^2((z_0,\omega_0),R)\right\},$$

where $(z_0,\omega_0)\in\mathbb{B}^2$, $R\in\mathbb{R}^2_+$. Then

$$M(R,(z_0,\omega_0),F)=\max\left\{\|F(z,\omega)\|:(z,\omega)\in\mathbb{D}^2((z_0,\omega_0),R)\right\},$$

because the maximum modulus of the analytic vector-function in a closed bidisc is attained on its skeleton.

To prove an growth estimates, we need the following theorem. The theorem gives sufficient conditions by the estimate of maximum modulus on the skeleton of bidisc.

Theorem 2.1 ([2])**.** *Let $\mathbf{L}\in Q(\mathbb{B}^2)$. If analytic vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$ has bounded $\mathbf{L}$-index in joint variables, then for all $R',R''\in\mathbb{R}^2_+$, $R'<R''$, $|R''|\le\beta$ there exists $p_1=p_1(R',R'')\ge 1$ such that for every $(z_0,\omega_0)\in\mathbb{B}^2$ inequality*

$$M\left(\frac{R''}{\mathbf{L}(z_0,\omega_0)},(z_0,\omega_0),F\right)\le p_1 M\left(\frac{R'}{\mathbf{L}(z_0,\omega_0)},(z_0,\omega_0),F\right) \tag{2.5}$$

holds.

3. Growth estimates of analytic vector-valued functions in the unit ball

We put $[0,2\pi]^2=[0,2\pi]\times[0,2\pi]$. For $R=(r_1,r_2)\in\mathbb{R}^2_+$, $\Theta=(\theta_1,\theta_2)\in[0,2\pi]^2$, $A=(a_1,a_2)\in\mathbb{C}^2$, we will write

$$Re^{i\Theta}=(r_1e^{i\theta_1},r_2e^{i\theta_2}),\qquad \arg A=(\arg a_1,\arg a_2).$$

Denote by $K(\mathbb{B}^2)$ the class of positive continuous vector-valued functions $\mathbf{L}=(l_1,l_2)$, where every $l_j:\mathbb{B}^2\to\mathbb{R}_+$ obeys inequality (2.1) and there exists $c\ge 1$ such that for all $R\in\mathbb{R}^2_+$ with $|R|<1$ and $j\in\{1,2\}$,

$$\max_{\Theta_1,\Theta_2\in[0,2\pi]^2}\frac{l_j(Re^{i\Theta_2})}{l_j(Re^{i\Theta_1})}\le c.$$

In the case $\mathbf{L}(z,w)=(l_1(|z|,|w|),l_2(|z|,|w|))$, we have that $\mathbf{L}\in K(\mathbb{B}^2)$. Put $\boldsymbol{\beta}=\left(\frac{\beta}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)$.

Theorem 3.2. *Let* $\mathbf{L} \in Q(\mathbb{B}^2) \bigcap K(\mathbb{B}^2)$, $\beta > c\sqrt{2}$. *If an analytic vector-function* $F : \mathbb{B}^2 \to \mathbb{C}^2$ *has bounded* $\mathbf{L}$*-index in joint variables, then*

$$\ln \max\{|F(z,w)| \colon (z,w) \in \mathbb{T}^2(\mathbf{0}, R)\} =$$

$$= O\Big(\min \Big\{ \min_{\Theta \in [0,2\pi]^2} \Big(\int_0^{r_1} l_1(te^{i\theta_1}, r_2 e^{i\theta_2}) dt + \int_0^{r_2} l_2(r_1^0, t) dt \Big);$$

$$\min_{\Theta \in [0,2\pi]^2} \Big(\int_0^{r_1} l_1(te^{i\theta_1}, r_2 e^{i\theta_2}) dt + \int_0^{r_2} l_2(r_1^0, t) dt \Big) \Big\} \Big), \tag{3.6}$$

with $|R| \to 1 - 0$, $R^0 = (r_1^0, r_2^0)$ *is a fixed radius.*

Proof. Let $R > 0$, $|R| > 1$, $\Theta \in [0, 2\pi]^2$, and a point $(z^*, w^*) \in \mathbb{T}^2 \left(\mathbf{0}, R + \frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})} \right)$ be such that

$$\|F(z^*, w^*)\| = \max \left\{ \|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0}, R + \frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}) \right\}.$$

We put $z_0 = \frac{z^* r_1}{R + \boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})}$, $w_0 = \frac{w^* r_2}{R + \boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})}$. Thus,

$$|z_0 - z^*| = \left| \frac{z^* r_1}{r_1 + \frac{\beta}{c\sqrt{2} l_1(Re^{i\Theta})}} - z^* \right| = \left| \frac{z^* \beta / (c\sqrt{2} l_1(Re^{i\Theta}))}{r_1 + \frac{\beta}{c\sqrt{2} l_1(Re^{i\Theta})}} \right| = \frac{\beta}{c\sqrt{2} l_1(Re^{i\Theta})},$$

$$|w_0 - w^*| = \left| \frac{w^* r_2}{r_2 + \frac{\beta}{c\sqrt{2} l_2(Re^{i\Theta})}} - w^* \right| = \left| \frac{w^* \beta / (c\sqrt{2} l_2(Re^{i\Theta}))}{r_2 + \frac{\beta}{c\sqrt{2} l_2(Re^{i\Theta})}} \right| = \frac{\beta}{c\sqrt{2} l_2(Re^{i\Theta})},$$

$$\mathbf{L}(z_0, w_0) = \mathbf{L} \left(\frac{z^* r_1}{R + \boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})}, \frac{w^* r_2}{R + \boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})} \right) =$$

$$= \mathbf{L} \left(\frac{(R + \boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})) r_1 e^{i \arg z^*}}{R + \boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})}, \frac{(R + \boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})) r_2 e^{i \arg w^*}}{R + \boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})} \right) =$$

$$= \mathbf{L}(r_1 e^{i \arg z^*}, r_2 e^{i \arg w^*}).$$

Since $\mathbf{L} \in K(\mathbb{B}^2)$, we have

$$c\mathbf{L}(z_0, w_0) = c\mathbf{L}(r_1 e^{i \arg z^*}, r_2 e^{i \arg w^*}) \geq \mathbf{L}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \geq \frac{1}{c} \mathbf{L}(z_0, w_0).$$

We will consider two skeletons $\mathbb{T}^2 \left((z_0, w_0), \frac{\mathbf{e}}{\mathbf{L}(z_0, w_0)} \right)$ and $\mathbb{T}^2 \left((z_0, w_0), \frac{\boldsymbol{\beta}}{\mathbf{L}(z_0, w_0)} \right)$. By Theorem 2.1, there exist $p_1 = p_1 \left(\frac{\mathbf{e}}{c}, c\boldsymbol{\beta} \right) \geq 1$ such that (2.5) is true for $R' = \frac{\mathbf{e}}{c}$, $R'' = c\boldsymbol{\beta}$:

$$\max \left\{ \|F(z,w)\| : (z,w) \in \mathbb{T}^2 \left(\mathbf{0}, R + \frac{\boldsymbol{\beta}}{\mathbf{L}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})} \right) \right\} \leq$$

$$\leq \left\{ \|F(z,w)\| : (z,w) \in \mathbb{T}^2 \left((z_0, w_0), \frac{\boldsymbol{\beta}}{\mathbf{L}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})} \right) \right\} \leq$$

$$\leq \left\{ \|F(z,w)\| : (z,w) \in \mathbb{T}^2 \left((z_0, w_0), \frac{c\boldsymbol{\beta}}{\mathbf{L}(z_0, w_0)} \right) \right\} \leq$$

$$\leq p_1 \left\{ \|F(z,w)\| : (z,w) \in \mathbb{T}^2 \left((z_0, w_0), \frac{\mathbf{e}}{c\mathbf{L}(z_0, w_0)} \right) \right\} \leq$$

$$\leq p_1 \left\{ \|F(z,w)\| : (z,w) \in \mathbb{T}^2 \left(\mathbf{0}, R + \frac{\mathbf{e}}{\mathbf{L}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})} \right) \right\}. \tag{3.7}$$

The function $\ln^+ \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0},R)\}$ is convex relative $\ln r_1$, $\ln r_2$. Therefore,

$$\ln^+ \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0},R)\}- \tag{3.8}$$
$$-\ln^+ \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0},R+(r_1^0-r_1)e_1)\} = \int_{r_1^0}^{r_1} \frac{A_1(t,r_2)}{t}dt,$$

$$\ln^+ \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0},R)\}- \tag{3.9}$$
$$-\ln^+ \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0},R+(r_2^0-r_2)e_2)\} = \int_{r_2^0}^{r_2} \frac{A_2(r_1,t)}{t}dt$$

for each $0 < r_j^0 < r_j$, $j\{1,2\}$, where functions $A_1(t,r_2)$, $A_2(r_1,t)$ are positive non-decreasing t. Then from (3.7), we obtain

$$\ln p_1 \ge \ln\max\left\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2\left(\mathbf{0}, R+\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}-$$
$$-\ln\max\left\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2\left(\mathbf{0}, R+\frac{\mathbf{e}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}=$$
$$=\ln\max\left\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2\left(\mathbf{0}, R+\frac{\mathbf{e}+(\frac{\beta}{\sqrt{2}c}-1)\mathbf{e}_1}{\mathbf{L}(Re^{i\Theta})}\right)\right\}-$$
$$-\ln\max\left\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2\left(\mathbf{0}, R+\frac{\mathbf{e}+(\frac{\beta}{\sqrt{2}c}-1)\mathbf{e}_2}{\mathbf{L}(Re^{i\Theta})}\right)\right\}=$$
$$=\int_{r_1+1/l_1(Re^{i\Theta})}^{r_1+\beta/(c\sqrt{2}l_1(Re^{i\Theta}))} \frac{1}{t}A_1\left(t, r_2+\frac{\beta}{c\sqrt{2}l_2(Re^{i\Theta})}\right)dt+$$
$$+\int_{r_2+1/l_2(Re^{i\Theta})}^{r_2+\beta/(c\sqrt{2}l_2(Re^{i\Theta}))} \frac{1}{t}A_2\left(r_1+\frac{\beta}{c\sqrt{2}l_1(Re^{i\Theta},t)}\right)dt \ge$$
$$\ge \ln\left(1+\frac{\frac{\beta}{\sqrt{2}c}-1}{r_1l_1(Re^{i\Theta})+1}\right)A_1\left(r_1, r_2+\frac{1}{l_2(Re^{i\Theta})}\right)+\ln\left(1+\frac{\frac{\beta}{\sqrt{2}c}-1}{r_2l_2(Re^{i\Theta})+1}\right)\times$$
$$\times A_2\left(r_1+\frac{1}{l_1(Re^{i\Theta})}, r_2\right). \tag{3.10}$$

Then, we have $r_jl_j(Re^{i\Theta}) \longrightarrow +\infty$ with $|R| \longrightarrow 1-0$. We obtain, for $j \in \{1,2\}$ and $r_j \ge r_j^0$:

$$\ln\left(1+\frac{\frac{\beta}{\sqrt{2}c}-1}{r_jl_j(Re^{i\Theta})+1}\right) \sim \frac{\frac{\beta}{\sqrt{2}c}-1}{r_jl_j(Re^{i\Theta})+1} \ge \frac{\frac{\beta}{\sqrt{2}c}-1}{2r_jl_j(Re^{i\Theta})}.$$

Thus, (3.10) implies that

$$A_1\left(r_1, r_2+\frac{\beta}{c\sqrt{2}}l_2(Re^{i\Theta})\right) \le \frac{2\ln p_1}{\frac{\beta}{c\sqrt{2}}-1}r_1l_1(Re^{i\Theta}),$$

$$A_2\left(r_1\frac{\beta}{c\sqrt{2}l_1(Re^{i\Theta})}, r_2\right) \le \frac{2\ln p_1}{\frac{\beta}{c\sqrt{2}}-1}r_2l_2(Re^{i\Theta}).$$

Let $R^0=(r_1^0,r_2^0)$, where r_j^0 it is chosen higher. From inequalities (3.8) and (3.9) , it follows that

$$\ln\max\left\{\|F(z,w)\|:(z,w)\in\mathbb{T}^2(\mathbf{0},R)\right\}=$$
$$=\ln\max\left\{\|F(z,w)\|:(z,w)\in\mathbb{T}^2(\mathbf{0},R+(r_1^0-r_1)\mathbf{e}_1)\right\}+\int_{r_1^0}^{r_1}\frac{A_1(t,r_2)}{t}dt=$$
$$=\ln\max\left\{\|F(z,w)\|:(z,w)\in\mathbb{T}^2(\mathbf{0},R+(r_1^0-r_1)\mathbf{e}_1+(r_2^0-r_2)\mathbf{e}_2)\right\}+$$
$$+\int_{r_1^0}^{r_1}\frac{A_1(t,r_2)}{t}dt+\int_{r_2^0}^{r_2}\frac{A_2(r_1^0,t)}{t}dt=$$
$$=\ln\max\left\{\|F(z,w)\|:(z,w)\in\mathbb{T}^2(\mathbf{0},R^0)\right\}+\frac{2\ln p_1}{\frac{\beta}{\sqrt{2}c}-1}\times$$
$$\times\left(\int_0^{r_1}l_1(te^{i\theta_1},r_2e^{i\theta_2})dt+\int_0^{r_2}l_2(r_1^0e^{i\theta_1},te^{i\theta_2})dt\right)\le$$
$$\le(1+O(1))\frac{2\ln p_1}{\frac{\beta}{c\sqrt{2}}-1}\left(\int_0^{r_1}l_1(te^{i\theta_1},r_2e^{i\theta_2})dt+\int_0^{r_2}l_2(r_1^0e^{i\theta_1},te^{i\theta_2})dt\right).$$

Function $\ln\max\{\|F(z,w)\|:(z,w)\in\mathbb{T}^2(\mathbf{0},R)\}$ is independent of Θ . We obtain

$$\ln\max\{\|F(z,w)\|:(z,w)\in\mathbb{T}^2(\mathbf{0},R)\}=$$
$$=O(\min_{\Theta\in[0,2\pi]^2}\left(\int_0^{r_1}l_1(te^{i\theta_1},r_2e^{i\theta_2})dt+\int_0^{r_2}l_2(r_1^0e^{i\theta_1},te^{i\theta_2})dt\right),$$

with $|R|\longrightarrow 1-0$. Theorem is proved. $\square$

Corollary 3.1. *If $\mathbf{L}\in Q(\mathbb{B}^2)\bigcap K(\mathbb{B}^2)$, $\min_{\Theta\in[0,2\pi]^2}l_j(Re^{i\Theta})$ is non-decreasing in each variable r_k, $k\in\{1,2\}$, $j\in\{1,2\}$, $k\ne j$, an analytic vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$ has bounded L-index in joint variables, then*

$$\ln\max\{\|F(z,w)\|:(z,w)\in\mathbb{T}^2(\mathbf{0},R)\}=O\left(\min_{\Theta\in[0,2\pi]^2}\sum_{j=1}^2\int_0^{r_j}l_j(R^{(j)}e^{i\Theta})dt\right),$$

as $|R|\longrightarrow 1-0$, with $R^{(1)}=(t,r_2)$, $R^{(2)}=(r_1,t)$.

We denote $a^+=\max\{a,0\}$, $u_j(t)=u_j(t,R,\Theta)=l_j\left(\frac{tR}{r^*}e^{i\Theta}\right)$, with $a\in\mathbb{R}$, $t\in\mathbb{R}_+$, $j\in\{1,2\}$, $r^*=\max_{1\le j\le 2}r_j\ne 0$ and $\frac{t}{r^*}|R|<1$.

Theorem 3.3. *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable r_k, $k\in\{1,2\}$, $|R|<1$, $\Theta\in[0,2\pi]^2$. If the function L obeys inequality (2.1) and an analytic vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$ has bounded **L**-index in joint variables, then for each $\Theta\in[0,2\pi]^2$ and for all $R\in\mathbb{R}^2_+$, $|R|<1$ and $(s,p)\in\mathbb{Z}^2$,*

$$\ln\max\left\{\frac{\|F^{(s,p)}(Re^{i\Theta})\|}{s!p!l_1^s(Re^{i\Theta})l_2^p(Re^{i\Theta})}:s+p\le N\right\}\le$$
$$\le\ln\max\left\{\frac{\|F^{(s,p)}(\mathbf{0})\|}{s!p!l_1^s(\mathbf{0})l_2^p(\mathbf{0})}:s+p\le N\right\}+$$
$$\int_0^{r^*}\left(\max_{s+p\le N}\left\{(s+1)l_1\left(\frac{\tau}{r^*}Re^{i\Theta}\right)+(p+1)l_2\left(\frac{\tau}{r^*}Re^{i\Theta}\right)\right\}+\right.$$
$$\left.+\max_{s+p\le N}\left\{\frac{s(-u_1'(\tau))^+}{l_1(\frac{\tau}{r^*}Re^{i\Theta})}+\frac{p(-u_2'(\tau))^+}{l_2(\frac{\tau}{r^*}Re^{i\Theta})}\right\}\right)d\tau. \tag{3.11}$$

Proof. Let $R \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, $\Theta \in [0, 2\pi]^2$. We put $a_j = \frac{r_j}{r^*}$, $j \in \{1, 2\}$ and $A = (a_1, a_2)$. Consider the function

$$g(t) = \max\left\{ \frac{\|F^{(s,p)}(Ae^{i\Theta})\|}{s!p!l_1^s(Ae^{i\Theta})l_2^p(Ae^{i\Theta})} : s + p \le N \right\}, \tag{3.12}$$

where $At = (a_1 t, a_2 t)$, $Ate^{i\Theta} = (a_1 t^{i\theta_1}, a_2 t^{i\theta_2})$.

Since the function

$$\frac{\|F^{(s,p)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})\|}{s!p!l_1^s(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})l_2^p(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})}$$

is continuously differentiable function of real variable $t \in [0; +\infty)$, outside the zero set of function $\|F^{(s,p)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})\|$, then $g(t)$ is also a continuously differentiable function on $[0, \frac{r^*}{|R|})$ except for a countable set of points.
Hence, in view of $\frac{d}{dr}|g(r)| \le |g'(r)|$, which holds everywhere except $r = t$, where $g(t) = 0$ we obtain that:

$$\begin{gathered}
\frac{d}{dt}\left(\frac{\|F^{(s,p)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})\|}{s!p!l_1^s(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})l_2^p(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})} \right) = \\
= \frac{1}{s!p!l_1^s(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})l_2^p(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})} \frac{d}{dt}\|F^{(s,p)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})\| + \|F^{(s,p)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})\| \times \\
\times \frac{d}{dt} \frac{1}{s!p!l_1^s(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})l_2^p(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})} \le \frac{1}{s!p!l_1^s(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})l_2^p(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})} \times \\
\times \left(\|F^{(s+1,p)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2}) a_1 e^{i\theta_1}\| + \|F^{(s,p+1)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2}) a_2 e^{i\theta_2}\| \right) - \\
- \frac{\|F^{(s,p)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})\|}{s!p!l_1^s(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})l_2^p(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})} \left(\frac{s u_1'(t)}{l_1(Ate^{i\Theta})} + \frac{p u_2'(t)}{l_2(Ate^{i\Theta})} \right) \le \\
\le \frac{\|F^{(s+1,p)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})\|}{(s+1)!p!l_1^{s+1}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})l_2^p(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})} a_1 (s+1) l_1(a_1 e^{i\theta_1}, a_2 e^{i\theta_2}) + \\
+ \frac{\|F^{(s,p+1)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})\|}{s!(p+1)!l_1^s(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})l_2^{p+1}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})} a_2 (p+1) l_2(a_1 e^{i\theta_1}, a_2 e^{i\theta_2}) + \\
+ \frac{\|F^{(s,p)}(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})\|}{s!p!l_1^s(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})l_2^p(a_1 e^{i\theta_1}, a_2 e^{i\theta_2})} \left(\frac{s(-u_1'(t))^+}{l_1(Ate^{i\Theta})} + \frac{p(-u_2'(t))^+}{l_2(Ate^{i\Theta})} \right).
\end{gathered} \tag{3.13}$$

For absolutely continuous functions h_1, h_2 and $h(x) := \max\{h_j(z,w) : 1 \le j \le 2\}$, one has $h'(x) \le \max\{h_j'(z,w) : 1 \le j \le 2\}$, $x \in [a,b]$. The function g is absolutely continuous. Therefore, (3.13) implies that

$$g'(t) \le \max\left\{\frac{d}{dt}\left(\frac{\|F^{(s,p)}(a_1e^{i\theta_1}, a_2e^{i\theta_2})\|}{s!p!l_1^s(a_1e^{i\theta_1}, a_2e^{i\theta_2})l_2^p(a_1e^{i\theta_1}, a_2e^{i\theta_2})}\right) : s+p \le N\right\} \le$$
$$\le \max_{s+p\le N}\left\{\frac{a_1(s+1)l_1(Ae^{i\Theta})\|F^{(s+1,p)}(Ae^{i\Theta})\|}{(s+1)!p!l_1^{s+1}(Ae^{i\Theta})l_2^p(Ae^{i\Theta})}+\right.$$
$$+\frac{a_2(p+1)l_2(Ae^{i\Theta})\|F^{(s,p+1)}(Ae^{i\Theta})\|}{s!(p+1)!l_1^s(Ae^{i\Theta})l_2^{p+1}(Ae^{i\Theta})}+$$
$$\left.+\frac{\|F^{(s,p)}(Ae^{i\Theta})\|}{s!p!l_1^s(Ae^{i\Theta})l_2^p(Ae^{i\Theta})}\left(\frac{s(-u_1'(t))^+}{l_1(Ae^{i\Theta})}+\frac{p(-u_2'(t))^+}{l_2(Ae^{i\Theta})}\right)\right\} \le$$
$$\le g(t)(\max_{s+p\le N}\{a_1(s+1)l_1(Ae^{i\Theta})+a_2(p+1)l_2(Ae^{i\Theta})\}+$$
$$+\max_{s+p\le N}\left\{\frac{s(-u_1'(t))^+}{l_1(Ae^{i\Theta})}+\frac{p(-u_2'(t))^+}{l_2(Ae^{i\Theta})}\right\}) =$$
$$= g(t)(\beta(t)+\gamma(t)),$$

with

$$\beta(t) = \max_{s+p\le N}\left\{a_1(s+1)l_1(Ae^{i\Theta})+a_2(p+1)l_2(Ae^{i\Theta})\right\},$$
$$\gamma(t) = \max_{s+p\le N}\left\{\frac{s(-u_1'(t))^+}{l_1(Ae^{i\Theta})}+\frac{p(-u_2'(t))^+}{l_2(Ae^{i\Theta})}\right\}.$$

Then, $\frac{d}{dt}\ln g(t) \le \beta(t)+\gamma(t)$ and

$$g(t) \le g(0)\exp\int_0^t(\beta(\tau)+\gamma(\tau))d\tau, \tag{3.14}$$

because $g(0)\neq 0$. But, one has $r^*A = R$. It follows from (3.14) and (3.12) that

$$\ln\max\left\{\frac{\|F^{(s,p)}(Re^{i\Theta})\|}{s!p!l_1^s(Re^{i\Theta})l_2^p(Re^{i\Theta})} : s+p\le N\right\} \le \ln\max\left\{\frac{\|F^{(s,p)}(\mathbf{0})\|}{s!p!l_1^s(\mathbf{0})l_2^p(\mathbf{0})} : s+p\le N\right\}+$$
$$+\int_0^{r^*}\max_{s+p\le N}\left\{a_1(s+1)l_1(A\tau e^{i\Theta})+a_2(p+1)l_2(A\tau e^{i\Theta})\right\}d\tau+$$
$$+\int_0^{r^*}\max_{s+p\le N}\left\{\frac{s(-u_1'(\tau))^+}{l_1(A\tau e^{i\Theta})}+\frac{p(-u_2'(\tau))^+}{l_2(A\tau e^{i\Theta})}\right\}d\tau.$$

Inequality (3.14) is true. $\square$

Proposition 3.1. *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable r_k, $k\in\{1,2\}$, $|R|<1$, $\Theta\in[0,2\pi]^2$. If the function $\mathbf{L}$ obeys inequality (2.1) and an analytic vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$ has bounded $\mathbf{L}$-index $N=N(F,\mathbf{L},\mathbb{B}^2)$ in joint variables and there exists $C>0$ such that the function $\mathbf{L}$ satisfies the inequality*

$$\sup_{|R|<1}\max_{t\in[0,r_*]}\max_{\Theta\in[0,2\pi]^2}\max_{1\le j\le 2}\frac{(-(u_j(t,R,\Theta))_t')^+}{\frac{r_j}{r^*}l_j^2(\frac{t}{r^*}Re^{i\Theta})}\le C \tag{3.15}$$

and

$$\overline{\lim}_{|R|\longrightarrow 1-0} \frac{\ln \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0},R)\}}{\max_{\Theta\in[0,2\pi]^2} \int_0^1 \langle R, \mathbf{L}(\tau, Re^{i\Theta})\rangle d\tau} \leq (C+1)N+1. \tag{3.16}$$

Proof. If the function $\mathbf{L}$ satisfies inequality (2.1), then

$$\max_{\Theta\in[0,2\pi]^2} \int_0^1 \langle R, \mathbf{L}(\tau Re^{i\Theta})\rangle d\tau \longrightarrow +\infty, as |R| \longrightarrow 1-0. \tag{3.17}$$

We put $\widetilde{\beta}(t) = \sum_{j=1}^2 a_j l_j(Ate^{i\Theta})$. If in addition (3.15) holds, then for some s^*, p^*, $s^*+p^* \leq N$ and $\widetilde{s}$, $\widetilde{p}$, $\widetilde{s}+\widetilde{p} \leq N$,

$$\frac{\gamma(t)}{\widetilde{\beta}(t)} = \frac{\frac{s^*(-u_1'(t))^+}{l_1(Ate^{i\Theta})} + \frac{p^*(-u_2'(t))^+}{l_2(Ate^{i\Theta})}}{\sum_{j=1}^2 a_j l_j(Ate^{i\Theta})} \leq s^* \frac{(-u_1'(t))^+}{a_1 l_1^2(Ate^{i\Theta})} + p^* \frac{(-u_2'(t))^+}{a_2 l_2^2(Ate^{i\Theta})} \leq$$
$$\leq (s^*+p^*)C \leq NC$$

and

$$\frac{\beta(t)}{\widetilde{\beta}(t)} = \frac{a_1(\widetilde{s}+1)l_1(Ate^{i\Theta}) + a_2(p^*+1)l_2(Ate^{i\Theta})}{\sum_{j=1}^2 a_j l_j(Ate^{i\Theta})} = 1 + \frac{a_1\widetilde{s}l_1(Ate^{i\Theta})}{a_1 l_1(Ate^{i\Theta})} +$$
$$+ \frac{a_2\widetilde{p}l_2(Ate^{i\Theta})}{a_2 l_2(Ate^{i\Theta})} \leq 1 + \widetilde{s} + \widetilde{p} \leq 1+N.$$

But, $\|F(Ate^{i\Theta})\| \leq g(t) \leq g(0)\exp\int_0^t (\beta(\tau)+\gamma(\tau))d\tau$ and $r^*A = R$. Put $t = r^*$. In view of (3.17), we have

$$\ln \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0},R)\} = \ln \max_{\Theta\in[0,2\pi]^2} \|F(Re^{i\Theta})\| \leq$$
$$\leq \ln \max_{\Theta\in[0,2\pi]^2} g(r^*) \leq \ln g(0) + \max_{\Theta\in[0,2\pi]^2} \int_0^{r^*} (\beta(\tau)+\gamma(\tau))d\tau \leq$$
$$\leq \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^2} \int_0^{r^*} (\widetilde{\beta}(\tau))d\tau =$$
$$= \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^2} \int_0^{r^*} \sum_{j=1}^2 a_j l_j(A\tau e^{i\Theta})d\tau =$$
$$= \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^2} \int_0^{r^*} \sum_{j=1}^2 \frac{r_j}{r} l_j(\frac{\tau}{r^*} Re^{i\Theta})d\tau =$$
$$= \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^2} \int_0^1 \sum_{j=1}^2 r_j l_j(\tau Re^{i\Theta})d\tau.$$

Then, (3.16) is true. The Proposition 3.1 is proved. □

Proposition 3.2. *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable $r_k, k \in \{1,2\}$, $|R| < 1$, $\Theta \in [0,2\pi]^2$. If the function $\mathbf{L}$ obeys inequality (2.1) and an analytic vector-function $F : \mathbb{B}^2 \to \mathbb{C}^2$ has bounded $\mathbf{L}$-index $N = N(F,\mathbf{L})$ in joint variables and*

$$r^*(-(u_j(t,R,\Theta))_t'\big|_{t=r^*})^+/(r_j l_j^2(Re^{i\Theta})) \longrightarrow 0 \tag{3.18}$$

for all $\Theta \in [0, 2\pi]^2$, $j \in \{1, 2\}$, *with* $|R| \longrightarrow 1 - 0$ *, then*

$$\overline{\lim}_{|R|\longrightarrow 1-0} \frac{\ln \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0}, R)\}}{\max_{\Theta\in[0,2\pi]^2} \int_0^1 \langle R, \mathbf{L}(\tau, Re^{i\Theta})\rangle d\tau} \le N + 1. \tag{3.19}$$

If $\mathbf{L}(z, w) = \mathbf{L}(r_1, r_2) = \mathbf{L}(R)$, then (3.18) can be rewritten in another form.

Corollary 3.2. *Let* $\mathbf{L}(R)$ *be a positive continuously differentiable function in each variable* r_k*,* $k \in \{1, 2\}$*,* $|R| < 1$*. If the function* $\mathbf{L}$ *obeys inequality* (2.1) *and an analytic vector-function* $F : \mathbb{B}^2 \to \mathbb{C}^2$ *has bounded* L*-index* $N = N(F, \mathbf{L})$ *in joint variables and for each* $j \in \{1, 2\}$

$$\frac{\langle R, \nabla l_j(R)\rangle}{r_j l_j^2(R)} \longrightarrow 0,$$

with $|R| \longrightarrow 1 - 0$*, then*

$$\overline{\lim}_{|R|\longrightarrow 1-0} \frac{\ln \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0}, R)\}}{\int_0^1 \langle R, \mathbf{L}(\tau R)\rangle d\tau} \le N + 1,$$

where $\nabla l_j(R) = \left(\frac{\partial l_1(R)}{\partial r_1}, \frac{\partial l_2(R)}{\partial r_2}\right)$.

The main result in this section is following:

Theorem 3.4. *Let* $\mathbf{L}(R) = (l_1(R), l_2(R))$*,* $l_j(R)$ *be a positive continuously differentiable non-decreasing function in each variable* r_k*,* $k \in \{1, 2\}$*,*$|R| < 1$*. If the function* L *obeys inequality* (1) *and an analytic vector-function* $F : \mathbb{B}^2 \to \mathbb{C}^2$ *has bounded* L*-index* $N = N(F, \mathbf{L})$ *in joint variables, then*

$$\overline{\lim}_{|R|\longrightarrow 1-0} \frac{\ln \max\{\|F(z,w)\| : (z,w) \in \mathbb{T}^2(\mathbf{0}, R)\}}{\int_0^1 \langle R, \mathbf{L}(\tau R)\rangle d\tau} \le N + 1.$$

Proof. Note that $\mathbf{L}(Re^{i\Theta}) \equiv \mathbf{L}(R)$ in this theorem. Since $l_j(R)$ is a positive continuously differentiable non-decreasing function and $u_j(t) = u_j(t, R) = l_j\left(\frac{tR}{r^*}\right)$, one has $(u_j(t, R))'_t \le 0$. Therefore, we obtain $r^*(-(u_j(t, R))'_t\big|_{t=r^*})^+/(r_j l_j^2(R)) = 0$. Thus, condition (3.18) is satisfied. Thus, the theorem is a direct consequence of the Proposition 3.2. □

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Research Article

Decay of Fourier Transforms and Generalized Besov Spaces

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ABSTRACT. A characterization of the generalized Lipschitz and Besov spaces in terms of decay of Fourier transforms is given. In particular, necessary and sufficient conditions of Titchmarsh type are obtained. The method is based on two-sided estimate for the rate of approximation of a β-admissible family of multipliers operators in terms of decay properties of Fourier transforms.

Keywords: Generalized Besov spaces, Fourier transforms, Titchmarsh type theorem.

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1. INTRODUCTION

The study of decay of Fourier transform / Fourier coefficients is one of the classical topics in Fourier analysis. Classical inequalities as Hardy-Littlewood and Haurdorsff-Young (see [29]) give us the basic decay of Fourier transforms. Titchmarsh showed ([29]) that the decay of Fourier transform can be improved for univariate functions satisfying a Lipschitz condition defined by smoothness. His result reads as follows.

Theorem 1.1. ([29, Theorem 85]) *Let $f \in L^2$ and $\widehat{f}$ its Fourier transform. The following conditions are equivalent*

$$\int_{-\infty}^{\infty} |f(x+h) - f(x-h)|^2 dx = O(h^{2\alpha}) \quad as\ h \to 0^+ \quad (0 < \alpha < 1)$$

and

$$\int_{1/h \leq |x|} [\widehat{f}(x)]^2 dx = O(h^{2\alpha}) \quad as\ h \to 0^+.$$

Extensions of the Titchmarsh theorem were obtained by several authors ([19, 20, 21, 33]) and can be extended to higher dimensional Euclidean spaces ([7, 34]) replacing the majorant function $\varphi(h) = h^\alpha$ in the Lipschitz condition by a regularly varying one ([4, 16]). The problem concerning about Fourier series on $\mathbb{T}$ can be found in [24, 25] while for Fourier transforms in [31]. The problem in $L^p(\mathbb{R}^d)$ for Fourier series can be seen in [13, 18] and for Fourier transforms we suggest [6, 8, 13] and references quoted there.

In this paper, we provide a further extension of Theorem 1.1 for functions in $L^p(\mathbb{R}^d)$ and an abstract Lipschitz condition, see Theorem 1.3 below. In particular, for $p = 2$, $d = 1$ and $\varphi(t) = t^\alpha$, $t \in (0, \infty)$, $0 < \alpha < 1$, our achievement recovers Theorem 1.1, due Lemma 2.2. In order to present this generalized version of the result, we need to establish a two-sided estimate for the rate of approximation of an admissible family of multipliers operators in terms of decay

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properties of Fourier transforms. This extends the known results proved in [13] for $d \geq 2$ and for the combination of multivariate averages.

For $d \geq 1$ the Fourier transform $\widehat{f}$ of a function f, in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{i\xi \cdot x} dx, \quad x \in \mathbb{R}^d.$$

We write $L^p(\mathbb{R}^d) := (L^p(\mathbb{R}^d), \|\cdot\|_p)$ for the usual Banach spaces of p-integrable functions ($1 \leq p \leq \infty$).

We deal with a family of multipliers operators ([23]) $\{T_t\}_{t>0}$ on $L^p(\mathbb{R}^d)$ with its multiplier family $\{\eta_t\}_{t>0}$ generated by dilations of a measurable function $\eta : (0, \infty) \longrightarrow \mathbb{R}$, i.e.,

$$T_t(f)\hat{\ }(\xi) = \eta_t(|\xi|)\widehat{f}(\xi),$$

where $\eta_t(|\xi|) := \eta(t|\xi|)$, for all $\xi \in \mathbb{R}^d$ and $t > 0$. If there exists $\gamma > 0$ such that

$$[\min(1, ts)]^{2\gamma} \asymp |1 - \eta_t(s)|, \quad \text{for all } t > 0, \tag{1.1}$$

then we say that $\{T_t\}_{t>0}$ *is a γ-admissible family of multipliers operators on* $L^p(\mathbb{R}^d)$. A well-known admissible family of multipliers operators , on $L^p(\mathbb{R}^d)$ for $d \geq 2$, includes the classical spherical mean operator and its combinations (see [2, 9, 13] and references quoted there).

We will employ generalized Lipschitz (and Besov) classes defined in terms of the rate of approximation of an admissible family of multipliers operators. The main point of the definition resides on the majorant function (defined ahead) and not on the fractional choice of orders of admissibility for the families of multipliers operators above. Indeed, no new Lipschitz/Besov classes are given just by considering fractional orders admissible family of multipliers operators, due condition (1.1) and Marchaud-type inequalities (see [10, 22, 30] and references quoted there).

In order to state the main theorems of the paper, we need to introduce some more definition. A majorant function in this paper is always a nondecreasing measurable function $\varphi : (0, \infty) \longrightarrow \mathbb{R}_+$ such that

$$\lim_{t \to 0_+} \varphi(t) \to 0,$$

and

$$\int_0^t \frac{\varphi(u)}{u} du \lesssim \varphi(t) \quad \text{for all } t > 0. \tag{1.2}$$

We denote by M the collection of all majorant functions. For $\beta > 0$, we define the following subset of M

$$\Omega_\beta := \left\{ \varphi \in M : \int_t^\infty \frac{\varphi(u)}{u^{\beta+1}} du \lesssim \frac{\varphi(t)}{t^\beta}, \, t > 0 \right\}.$$

The family Ω_β can be defined in terms of the almost monotonicity property.

A function $\varphi : (0, \infty) \longrightarrow \mathbb{R}_+$ is β*-almost decreasing* ([4, p. 72]) if it satisfies the condition:

$$\frac{\varphi(u_2)}{u_2^\beta} \lesssim \frac{\varphi(u_1)}{u_1^\beta}, \quad \text{for any } u_1 \leq u_2.$$

For $\beta > 0$, we write

$$\Omega'_\beta := \{\varphi \in M : \text{there exists } 0 < \epsilon < \beta \text{ such that } \varphi \text{ is } (\beta - \epsilon)\text{-almost decreasing}\}.$$

$A(t) \asymp B(t)$ stands for $B(t) \lesssim A(t)$ and $A(t) \lesssim B(t)$, where $A(t) \lesssim B(t)$ means that $A(t) \leq c\, B(t)$, for some constant $c > 0$ not depending upon t.

Simple calculations and Bari-Stechkin Lemma ([1], see also [26, p.754]) are enough to prove that the classes Ω'_β and Ω_β coincide:

$$\Omega_\beta = \Omega'_\beta, \quad \text{for each } \beta > 0. \tag{1.3}$$

Obviously,

$$\bigcup_{0<\alpha<\beta} \Omega_\alpha = \Omega_\beta, \quad \text{for any } \beta > 0.$$

In fact, for any $0 < \alpha < \beta$ we have $\Omega_\alpha \subset \Omega_\beta$. In order to verify equality above, is enough to prove that for a given $\varphi \in \Omega_\beta$ there exists $0 < \alpha < \beta$ such that $\varphi \in \Omega_\alpha$. If $\varphi \in \Omega_\beta$, then (1.3) implies that φ is $(\beta - \epsilon)$-almost decreasing, for some $0 < \epsilon < \beta$. It means that for any $t \leq s$, it holds

$$\frac{\varphi(s)}{s^{\beta-\epsilon/2}} \lesssim \frac{\varphi(t)}{t^{\beta-\epsilon} s^{\epsilon/2}}.$$

Integrating both sides of inequality above, we obtain

$$\int_t^\infty \frac{\varphi(s)}{s^{\beta-\epsilon/2+1}} ds \lesssim \frac{\varphi(t)}{t^{\beta-\epsilon}} \int_t^\infty s^{-\epsilon/2-1} ds = 2/\epsilon \frac{\varphi(t)}{t^{\beta-\epsilon/2}}.$$

Thus, $\varphi \in \Omega_{\beta-\epsilon/2}$.

An interesting subclass of Ω_β is given via the following definition. A function $f : (0,\infty) \longrightarrow \mathbb{R}_+$ is *regularly varying* ([16]) with index $\alpha \in \mathbb{R}$ if for any $\lambda > 0$, it holds $f(\lambda x)/f(x) \to \lambda^\alpha$ as $x \to \infty$. We write RV_α for the set of all regularly varying functions with index α. It is not hard to see that if $\varphi \in \mathrm{RV}_\alpha$, then it can be represented as $\varphi(x) = x^\alpha \varsigma(x)$, $x \in (0,\infty)$, where ς is a regularly varying function with index zero (i.e., a slowly varying function). More than that the Representation Theorem ([4, p. 17]) gives a characterization for all regularly varying functions.

We observe that $\mathrm{RV}_\alpha \subsetneq \Omega_\beta$, for all $0 < \alpha < \beta$. This fact follows from basic theory of regularly varying functions, the needed details can be found in [4, p. 68–72]. Due to this, the following functions belong to Ω_β,

$$t^\alpha \ln(1+t), \quad (t\ln(1+t))^\alpha, \quad t^\alpha \ln(\ln(e+t)), \quad t^\alpha \exp\left[\frac{\ln t}{\ln(\ln t)}\right]$$

and

$$t^\alpha \exp[(\log t)^{\alpha_1} (\log_2 t)^{\alpha_2} \ldots (\log_n t)^{\alpha_n}],$$

where $\alpha_i \in (0,1)$, $i = 1, 2, \ldots, n$, for all $0 < \alpha < \beta$. The usual majorant function employed in the Titchmarsh theorem $\varphi(t) = t^\alpha$, belongs to Ω_β if and only if $0 < \alpha < \beta$.

Definition 1.2. For $\varphi \in \Omega_{2\beta}$, we define the *generalized Lipschitz class* in $L^p(\mathbb{R}^d)$ by

$$\mathrm{Lip}\,(p,\beta,\varphi) = \left\{ f \in L^p(\mathbb{R}^d) : \|T_t(f) - f\|_p = O(\varphi(t)) \ \text{ as } \ t \to 0^+ \right\}, \quad 1 \leq p \leq \infty, \tag{1.4}$$

where $\{T_t\}_t$ is a β-admissible family of multipliers operators.

Necessary and sufficient conditions of Titchmarsh type for the generalized Lipschitz class read as follow.

Theorem 1.3. *Let $\{T_t\}_{t>0}$ be a β-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$ and $\varphi \in \Omega_{2\beta}$.*
(A) Let $1 < p \leq 2$ and $p \leq q \leq p'$. If $f \in Lip\,(p,\beta,\varphi)$, then

$$\left(\int_{t \leq |\xi| \leq 2t} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi \right)^{1/q} = O\left(\varphi(t^{-1})\right), \quad \textit{as} \ \ t \to \infty. \tag{1.5}$$

(B) Let $2 \le p < \infty$, $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$ and $p' \le q \le p$. If

$$\left(\int_{t\le|\xi|\le 2t}\left[|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} = O\left(\varphi(t^{-1})\right), \quad \text{as } t\to\infty,$$

then $f \in Lip\,(p,\beta,\varphi)$.

In order to define the generalized Besov spaces, we need to restrict our majorant classes as follows. For $0 < q, \gamma < \infty$, we write

$$\Omega_\gamma^q := \left\{\varphi \in \Omega_\gamma : \int_0^1 \frac{1}{[\varphi(t^{-1})]^q}\frac{dt}{t} < \infty\right\}.$$

Definition 1.4. For $0 < q < \infty$ and $\varphi \in \Omega^q_{2\beta}$, we define the *generalized Besov space* $B^{\varphi}_{p,q}(\mathbb{R}^d)$ by

$$B^{\varphi}_{p,q}(\mathbb{R}^d) = \left\{f \in L^p(\mathbb{R}^d) : |f|_{B^{\varphi}_{p,q}} := \int_0^1 \left(\frac{\|T_t(f)-f\|_p}{\varphi(t)}\right)^q \frac{dt}{t} < \infty\right\}. \tag{1.6}$$

For $q = \infty$ and $\varphi \in \Omega_\gamma$,

$$B^{\varphi}_{p,\infty}(\mathbb{R}^d) := \left\{f \in L^p(\mathbb{R}^d) : |f|_{B^{\varphi}_{p,\infty}} := \sup_{t>0}\left\{\frac{\|T_t(f)-f\|_p}{\varphi(t)}\right\} < \infty\right\}.$$

As usual, if $q < \infty$, we endow $B^{\varphi}_{p,q}$ with the norm $\|\cdot\|_{B^{\varphi}_{p,q}} := \left(\|\cdot\|_p^q + |\cdot|_{B^{\varphi}_{p,q}}\right)^{1/q}$, otherwise $\|\cdot\|_{B^{\varphi}_{p,\infty}} := \|\cdot\|_p + |\cdot|_{B^{\varphi}_{p,\infty}}$. In particular, for $q = \infty$, these spaces are the generalized Lipschitz ones. The Besov spaces here seem to depend upon a majorant function and an admissible family of multipliers operators, but, as usual, that is not true. As a matter of fact, this is a topic of investigation [14].

The following gives us necessary and sufficient conditions in terms of decay properties of Fourier transforms for functions in the generalized Besov spaces.

Theorem 1.5. *Let $\{T_t\}_{t>0}$ be a β-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$ and $\varphi \in \Omega^q_{2\beta}$.*
(A) Let $1 < p \le 2$ and $p \le q \le p'$. If $f \in B^{\varphi}_{p,q}(\mathbb{R}^d)$, then

$$\int_0^\infty \int_{t\le|\xi|\le 2t}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\,\frac{dt}{t} < \infty. \tag{1.7}$$

(B) Let $2 \le p < \infty$, $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$ and $p' \le q \le p$. If

$$\int_0^\infty \int_{t\le|\xi|\le 2t}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\,\frac{dt}{t} < \infty, \tag{1.8}$$

then $f \in B^{\varphi}_{p,q}(\mathbb{R}^d)$.

For the particular choice $\varphi(t) = t^\alpha$, $0 < \alpha < \ell$ for some $\ell \in \mathbb{N}$, and the ℓ-th family of combinations of multivariate averages on $\mathbb{R}^d$, for $d \ge 2$, spaces $B^{\varphi}_{p,q}(\mathbb{R}^d) \cap \widehat{GM}^d_p$ became the ones characterized in [13, Section 7] ($\widehat{GM}^d_p$ is defined ahead).

The paper is organized as follows. In Section 2, we present a two-sided estimate for the rate of approximation of an β-admissible family of multipliers operators in terms of decay properties of Fourier transforms. This estimate plays a crucial role in the proof of Theorem 1.3, presented in this section. The inverse Fourier-Hankel transform of certain radial functions is applied in order to show the necessity of the condition concerning the majorant functions in order to prove Theorem 1.3. Section 3 is regarded to the proof of Theorem 1.5. Finally, in Section

4, we present the concept of general monotonicity of functions (GM_p^d class) and we outline how to make assumptions in Theorems 1.3 and 1.5 less restrictive. As a corollary, we prove a pointwise inequality for Fourier transforms of functions in $\widehat{GM}_p^d$, that is, a Riemann-Lebesgue type inequality.

2. Proof of Theorem 1.3

The rate of approximation of an admissible family of multipliers operators can be estimated in terms of decay properties of Fourier transforms as follows. For $d \geq 2$, the following result can be seen as a corollary of [13, Theorem 2.1, p. 1289] and the ideas of the proof are included below for completeness.

Proposition 2.1. *Let $\{T_t\}_{t>0}$ be a γ-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$ and $f \in L^p(\mathbb{R}^d)$.*

(A) Let $1 < p \leq 2$. If $p \leq q \leq p'$, then $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$ and

$$\left(\int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{2\gamma} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} \lesssim \|T_t(f) - f\|_p.$$

(B) Let $2 \leq p < \infty$. If $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$ and $p' \leq q \leq p$, then

$$\|T_t(f) - f\|_p \lesssim \left(\int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{2\gamma} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q}.$$

The proof of proposition above is a simple adaptation of the proof of [13, Theorem 2.1, p. 1289], since the main arguments completely fit here. An application of Pitt's inequality (see [3]) combined to the admissibility condition on the family of multipliers operators finishes the proof.

For $d \geq 2$, Theorem 2.1 in [13] is easily recovered from Proposition 2.1 for $\gamma = \ell$ a natural number and the combinations of multivariate averages family as the admissible one. The latter has a generalized version as follows. All the facts mentioned below can be found in [15]. Let $r > 0$, a real number. For each $t > 0$, we write

$$V_{r,t}(f)(x) := \frac{-2}{\binom{2r}{r}} \sum_{k=1}^{\infty} (-1)^k \binom{2r}{r-k} V_{kt}(f)(x), \quad f \in L^p(\mathbb{R}^d), \quad x \in \mathbb{R}^d, \tag{2.1}$$

where $\{V_t\}_t$ is the usual family of spherical mean operator on $L^p(\mathbb{R}^d)$, and for r and s real numbers,

$$\binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)}, \quad \text{for } s \notin \mathbb{Z}_-, \quad \binom{r}{0} = r \quad \text{and} \quad \binom{r}{s} = 0, \quad \text{for } s \in \mathbb{Z}_-.$$

The operator defined by (2.1) is bounded on $L^p(\mathbb{R}^d)$ and for $r = \ell$ a natural number the family $\{V_{r,t}\}_t$ becomes the combination of multivariate averages $\{V_{\ell,t}\}_t$ given in [9]. If $m_{r,t}$ stands for the multiplier of $V_{r,t}$, for each $t > 0$, then

$$1 - m_t^r(|\xi|) = 1 - m^r(t|\xi|) := \frac{2^{2r+1}\Gamma((m+1)/2)}{\binom{2r}{r}\Gamma(m/2)\Gamma(1/2)} \int_0^1 (\sin(t|\xi|s/2))^{2r} (1-s^2)^{(d-1)/2} ds, \quad \xi \in \mathbb{R}^d.$$

In this case, $\{V_{r,t}\}_t$ is a r-admissible family of multipliers operators, since

$$\min(1, s)^{2r} \asymp 1 - m_{r,t}(s) = 1 - m_r(ts), \quad s > 0.$$

Proof of Theorem 1.3 makes use of the next lemma.

Lemma 2.2. *Let $\varphi \in M$, $f \in L^p(\mathbb{R}^d)$ and $1 < p, q < \infty$. The following two conditions are equivalent:*

$$\left(\int_{1/t \le |\xi| \le 2/t} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi \right)^{1/q} \lesssim \varphi(t), \quad t > 0 \tag{2.2}$$

and

$$\left(\int_{1/t \le |\xi|} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi \right)^{1/q} \lesssim \varphi(t), \quad t > 0. \tag{2.3}$$

Proof. It is easy to see that (2.3) implies (2.2). Assuming that (2.2) holds, we write the integral in the left-hand side of inequality (2.3) in terms of the radial part (see [32]) of the integrating function, as follows

$$I(t) := \int_{1/t}^{\infty} r^{dq(1-1/p-1/q)} \left(\int_{S^{d-1}} |\widehat{f}(r\omega)|^q d\omega \right) r^{(d-1)} dr, \quad t > 0,$$

where S^{d-1} is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ centered at origin endowed with σ_{d-1} the induced Lebesgue measure (if $d = 1$ we skip this step). It is easily seen that

$$I(t) \lesssim \int_{1/t}^{\infty} r^{dq(1-1/p-1/q)} \left[\int_r^{2r} \left(\int_{S^{d-1}} |\widehat{f}(\rho\omega)|^q d\omega \right) d\rho \right] r^{(d-1)} \frac{dr}{r}.$$

If $r \le \rho \le 2r$, then $r^{dq(1-1/p-1/q)} \lesssim \rho^{dq(1-1/p-1/q)}$, and due to inequality (2.2) we arrive at

$$I(t) \lesssim \int_{1/t}^{\infty} \frac{[\varphi(r^{-1})]^q}{r} dr = \int_0^t \frac{[\varphi(u)]^q}{u} du.$$

In order to finish the proof, it is enough to observe that

$$\int_0^t \frac{[\varphi(u)]^q}{u} du \lesssim [\varphi(t)]^q \quad \text{and} \quad \int_0^t \frac{[\varphi(u)]}{u} du \lesssim \varphi(t), \quad t > 0,$$

are equivalent (see [26]) and the later is the condition (1.2) for $\varphi \in M$. $\square$

Proof. of Theorem 1.3. The proof of part (A) is a trivial application of Proposition 2.1, part (A). In order to prove part (B), we apply Proposition 2.1, part (B), and we obtain

$$\|T_t(f) - f\|_p^q \lesssim \int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{2\beta} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi.$$

Denoting by $I_q^\beta(f)$ the right-hand side of inequality above, we have

$$\|T_t(f) - f\|_p^q \lesssim I_q^\beta(f),$$

where

$$I_q^\beta(f) = \int_{|\xi| \ge 1/t} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi + t^{2q\beta} \int_{|\xi| < 1/t} |\xi|^{2q\beta} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi.$$

Due to Lemma 2.2, the proof will be completed if the following holds

$$t^{2q\beta} \int_{|\xi| < 1/t} \left[|\xi|^{2\beta} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi = O\left(\varphi(t)\right)^q, \quad \text{as } t \to 0^+. \tag{2.4}$$

We first consider the case $d \geq 2$ and we employ an adaption of the Titchmarsh proof in [29, Theorem 84]. For $t > 0$, denote

$$I_q^{\beta<}(f) := \int_{|\xi|<1/t} \left[|\xi|^{2\beta}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi.$$

The following inequality holds

$$I_q^{\beta<}(f) \leq \int_{|\tau|<1/t} |\tau|^{2q\beta} h(\tau)|\tau|^{q(d-1)} d\tau,$$

where

$$h(\tau) := \int_{S^{d-1}} \left[|\tau\omega|^{d(1-1/p-1/q)}|\widehat{f}(\tau\omega)|\right]^q d\sigma_{d-1}(\omega), \quad -1/t < \tau < 1/t.$$

By writing

$$\int_{|\tau|<1/t} |\tau|^{2q\beta} h(\tau)|\tau|^{q(d-1)} d\tau := I_q^{\beta^-}(h,t) + I_q^{\beta^+}(h,t), \tag{2.5}$$

where

$$I_q^{\beta^-}(h,t) := \int_{-1/t}^{0} (-\tau)^{2q\beta} \int_{S^{d-1}} \left[|\tau\omega|^{d(1-1/p-1/q)}|\widehat{f}(\tau\omega)|(-\tau)^{(d-1)}\right]^q d\sigma_d(\omega) d\tau$$

and

$$I_q^{\beta^+}(h,t) := \int_{0}^{1/t} \tau^{2q\beta} \int_{S^{d-1}} \left[|\tau\omega|^{d(1-1/p-1/q)}|\widehat{f}(\tau\omega)|\tau^{(d-1)}\right]^q d\sigma_d(\omega) d\tau, \quad t > 0,$$

it is sufficient to show that both $I_q^{\beta^-}(h,t)$ and $I_q^{\beta^+}(h,t)$ are $O\left(t^{-2q\beta}\left(\varphi(t)\right)^q\right)$ as $t \to 0^+$.

We define

$$\phi_+(t) = \int_{1/t}^{+\infty} h(\tau)\tau^{q(d-1)} d\tau, \quad t > 0,$$

and observe that

$$\lim_{t\to 0^+} t^{2q\beta}\phi_+(t^{-1}) = 0. \tag{2.6}$$

In fact, we have

$$\lim_{t\to 0^+} t^{2q\beta}\phi_+(t^{-1}) \lesssim \lim_{t\to 0^+} \left(t^{2\beta}\varphi(t)\right)^q = \lim_{t\to\infty} \left(\frac{\varphi(t^{-1})}{t^{2\beta}}\right)^q.$$

Equality (1.3) implies that there exists $0 < \epsilon < 2\beta$ such that φ is $(2\beta - \epsilon)$-almost decreasing. This leads us to

$$\lim_{t\to\infty} \left(\frac{\varphi(t^{-1})}{t^{2\beta}}\right)^q = \lim_{t\to\infty} \left(\frac{\varphi(t^{-1})}{t^{2\beta-\epsilon}}\right)^q \frac{1}{t^{q\epsilon}} \lesssim (\varphi(1))^q \lim_{t\to\infty} \frac{1}{t^{q\epsilon}} = 0,$$

and (2.6) holds.

Note that $\phi_+'(\tau) = -h(\tau^{-1})\tau^{-q(d-1)/2}$, $0 < \tau < 1/t$, and

$$I_q^{\beta^+}(h,t) = \int_0^{1/t} -\tau^{2q\beta}\phi_+'(\tau^{-1}) d\tau, \quad t > 0,$$

thus integration by parts and (2.6) imply

$$\begin{aligned} I_q^{\beta^+}(h,t) &= \left(-\tau^{2q\beta}\phi_+(\tau^{-1})\right)_0^{1/t} + 2q\beta \int_0^{1/t} \tau^{2q\beta-1}\phi_+(\tau^{-1})d\tau \\ &= -t^{-2q\beta}\phi_+(t) + 2q\beta \int_0^{1/t} \tau^{2q\beta-1}\phi_+(\tau^{-1})d\tau \\ &\leq 2q\beta \int_0^{1/t} \tau^{2q\beta-1}\phi_+(\tau^{-1})d\tau, \quad t>0. \end{aligned}$$

Since $\phi_+((\cdot)^{-1})$ is a nondecreasing function on $(0,\infty)$, it follows

$$I_q^{\beta^+}(h,t) \leq 2q\beta\phi_+(t) \int_0^{1/t} \tau^{2q\beta-1}d\tau = \phi_+(t)t^{-2q\beta}, \quad t>0. \tag{2.7}$$

Handling $I_q^{\beta^-}(h,t)$ as above, by defining

$$\phi_-(t) = \int_{-\infty}^{-1/t} h(\tau)(-\tau)^{q(d-1)}d\tau, \quad t>0,$$

we get

$$I_q^{\beta^-}(h,t) \leq t^{-2q\beta}\phi_-(t) + 2q\beta\phi_-(t)\int_{-1/t}^{0}(-\tau)^{2q\beta-1}d\tau = 2t^{-2q\beta}\phi_-(t), \quad t>0. \tag{2.8}$$

Combining inequalities (2.5), (2.7) and (2.8) with our assumptions (i.e. $\phi_+(t) = O\left(\varphi(t)\right)^q$ and $\phi_-(t) = O\left(\varphi(t)\right)^q$, as $t\to 0^+$), we reach to

$$\|T_t(f)-f\|_p \lesssim \left(\int_{\mathbb{R}^d}\left[\min(1,t|\xi|)^{2\beta}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} = O\left(\varphi(t)\right), \quad \text{as } t\to 0^+.$$

Thus, $f\in \mathrm{Lip}\,(p,\beta,\varphi)$.

For $d=1$, the same proof presented above can be rewritten with minor adjustments as follows. For $t>0$, denote

$$I_q^{\beta<}(f) := \int_{|\xi|<1/t}\left[|\xi|^{2\beta}|\xi|^{(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi = I_q^{\beta^-}(f,t) + I_q^{\beta^+}(f,t),$$

where

$$I_q^{\beta^-}(f,t) := \int_{-1/t}^{0}\left[\xi^{2\beta}|\xi|^{(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi$$

and

$$I_q^{\beta^+}(f,t) := \int_0^{1/t}\left[\xi^{2\beta}|\xi|^{(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi, \quad t>0,$$

it is sufficient to show that both $I_q^{\beta^-}(f,t)$ and $I_q^{\beta^+}(f,t)$ are $O\left(t^{-2q\beta}\left(\varphi(t)\right)^q\right)$ as $t\to 0^+$.

It is not hard to see that if

$$g(t) = \int_{|s|<1/t}|s|^{q(1-1/p-1/q)}|\widehat{f}(s)|^q ds, \quad t>0,$$

then

$$I_q^{\beta^-}(f,t) = \int_{-1/t}^{0} s^{2q\beta}g'(s^{-1})ds, \quad \text{and} \quad I_q^{\beta^+}(f,t) = \int_0^{1/t} s^{2q\beta}g'(s^{-1})ds, \quad t>0,$$

Also, we observe that the same reasoning applied in order to prove equality (2.6) fits here and we have

$$\lim_{t\to 0^+} t^{2q\beta} g(t^{-1}) = 0. \tag{2.9}$$

Thus integration by parts and (2.9) imply

$$\begin{aligned} I_q^{\beta^+}(f,t) &= -t^{-2q\beta}g(t) + 2q\beta \int_0^{1/t} s^{2q\beta-1} g(s^{-1}) d\,s \\ &\leq 2q\beta \int_0^{1/t} s^{2q\beta-1} g(s^{-1}) d\,s, \quad t > 0. \end{aligned}$$

Since $g((\cdot)^{-1})$ is a nondecreasing function on $(0,\infty)$, it follows

$$I_q^{\beta^+}(f,t) \leq 2q\beta g(t) \int_0^{1/t} s^{2q\beta-1} d\,s = g(t)t^{-2q\beta}, \quad t > 0. \tag{2.10}$$

Handling $I_q^{\beta^-}(f,t)$ similarly as above, we reach to

$$I_q^{\beta^-}(f,t) \leq t^{-2q\beta} g(t) + 2q\beta g(t) \int_{-1/t}^{0} (-s)^{2q\beta-1} d\,s = 2t^{-2q\beta} g(t), \quad t > 0. \tag{2.11}$$

Combining inequalities (2.10) and (2.11) with our assumption ($g(t) = O\left(\varphi(t)\right)^q$ as $t \to 0^+$), we obtain

$$\|T_t(f) - f\|_p \lesssim \left(\int_{\mathbb{R}} \left[\min(1, t|\xi|)^{2\beta} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)|\right]^q d\,\xi\right)^{1/q} = O\left(\varphi(t)\right), \quad \text{as } \ t \to 0^+,$$

and therefore $f \in \text{Lip}\,(p,\beta,\varphi)$. □

Corollary 2.3. *If $\varphi \in \Omega_{2\beta}$, then $f \in Lip\,(2,\beta,\varphi)$ if and only if*

$$\left(\int_{t\leq|\xi|\leq 2t} |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} = O\left(\varphi(t^{-1})\right), \quad \textit{as} \ \ t \to \infty.$$

Remark 2.4. We have defined the class Ω_β by the collection of all $\varphi \in M$ satisfying the following

$$\int_t^\infty \frac{\varphi(u)}{u^{\beta+1}} du \lesssim \frac{\varphi(t)}{t^\beta}. \tag{2.12}$$

Inequality (2.12) is necessary in order to have Theorem 1.3, part (B), true. Let $\varphi \in M$ a function that does not fulfill (2.12), then Theorem 1.3, part (A), still holds true. However, the same does not hold for part (B).

We consider the case $d \geq 2$, similarly we can deal with $d = 1$. Let $2 \leq p < \infty$ and $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ in $L^p(\mathbb{R}^d)$ given in terms of the inverse Fourier-Hankel transform of $|\xi|^{-(2\beta+1/p')}$, $\xi \in \mathbb{R}\setminus\{0\}$, that is,

$$f(x) = \frac{\sigma_d}{(2\pi)^d} \int_0^\infty \frac{j_{d/2-1}(xs)}{|x|^{2\beta+1/p'}} s^{d-1} ds,$$

where σ_d is the volume of the unit sphere in $\mathbb{R}^d$ and $j_\alpha(\,\cdot\,)$ denotes the normalize Bessel function (see [11]).

If $\varphi(t) := t^{2\beta}$, then $\varphi \in M$ but φ does not meet condition (2.12). Also, it is clear that

$$\int_{1/t\leq|\xi|} |\widehat{f}(\xi)|^{p'} d\xi = 2\int_{1/t}^{+\infty} \frac{1}{|\xi|^{2\beta p'+1}} d\xi = O([\varphi(t)]^{p'})$$

or, equivalently,

$$\left(\int_{1/t\le|\xi|\le 2/t} |\widehat{f}(\xi)|^{p'}\,d\xi\right)^{1/p'} = O(\varphi(t)).$$

It means that for $q = p'$, the function f fits into assumptions of Theorem 1.3, part (B). Also, we have

$$t^{2p'\beta}\int_{1/t<|\xi|} |\xi|^{2p'\beta}|\widehat{f}(\xi)|^{p'}\,d\,\xi = t^{2p'\beta}\int_{1/t<|\xi|} |\xi|^{-1}d\,\xi = +\infty, \quad \text{for all } t>0,$$

and therefore, $f \notin \mathrm{Lip}(p,\beta,\varphi)$.

3. Proof of Theorem 1.5

In this section, we only work with $d \ge 2$. For $d = 1$, the result was proved in [13] for the usual fractional moduli of smoothness ([5, 22]). If one wants to consider the admissible family of multipliers operators instead the fractional moduli of smoothness, for this case, with small adjustments the same proof presented in [13, p. 1310] fits here.

Proof. of Theorem 1.5.We rewrite the integral in the left-hand side of inequality (1.7), as $I_1 + I_2$, where

$$I_1 := \int_0^{1/2}\int_{t\le|\xi|\le 2t}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\frac{dt}{t}$$

and

$$I_2 := \int_{1/2}^{\infty}\int_{t\le|\xi|\le 2t}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\frac{dt}{t}.$$

Since φ is non-decreasing, for any $t \le |\xi| \le 2t$ it holds $\varphi(t^{-1}/2) \le \varphi(|\xi|^{-1})$ and we have

$$I_1 \lesssim \int_0^{1/2}\frac{1}{[\varphi(t^{-1}/2)]^q}\left(\int_{t\le|\xi|\le 2t}\left[|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)\frac{ds}{s}.$$

The change of variables $t = s/2$ leads us to

$$\begin{aligned} I_1 &\lesssim \int_0^{1}\frac{1}{[\varphi(s^{-1})]^q}\left(\int_{s/2\le|\xi|\le s}\left[|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)\frac{dt}{t}\\ &\lesssim \|(\cdot)^{d(1-1/p-1/q)}\widehat{f}(\cdot)\|_q^q. \end{aligned}$$

For I_2, the change of variables $t = s^{-1}/2$ implies

$$I_2 = \int_0^{1}\int_{1/2s\le|\xi|\le 1/s}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\frac{ds}{s}.$$

We note that if $0 < s \leq 1$ and $1/2s \leq |\xi| \leq 1/s$, then $\varphi(s) \leq \varphi(|\xi|^{-1})$ and $s|\xi| \leq 1$. Combining these inequalities to Propositon 2.1, part (A), we have

$$\begin{aligned} I_2 &\lesssim \int_0^1 \int_{1/2s\leq|\xi|\leq 1/s} \left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})} \right)^q d\xi \frac{ds}{s} \\ &\leq \int_0^1 \int_{1/2s\leq|\xi|\leq 1/s} \left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(s)} \right)^q d\xi \frac{ds}{s} \\ &\lesssim \int_0^1 \frac{\|T_s(f)-f\|_p^q}{[\varphi(s)]^q}\frac{ds}{s} = \int_0^1 \frac{\|T_s(f)-f\|_p^q}{[\varphi(s)]^q}\frac{ds}{s} \leq \|f\|^q_{B^{\varphi}_{p,q}}. \end{aligned}$$

Thus the first part of the theorem is proved.

To prove the second part, with an application of Proposition 2.1, part (B), we arrive at

$$\frac{\|T_t(f)-f\|_p^q}{[\varphi(t)]^q} \lesssim \int_{\mathbb{R}^d} I_t(\xi)d\xi = \int_0^\infty I_{t,0}(r)r^{(d-1)}dr \quad \text{for all} \quad t>0, \tag{3.1}$$

where

$$I_t(\xi) := \frac{\min(1,t|\xi|)^{2q\beta}}{[\varphi(t)]^q}|\xi|^{dq(1-1/p-1/q)}|\widehat{f}(\xi)|^q, \quad \xi\in\mathbb{R}^d,$$

and $I_{t,0}$ denotes its radial part. Integrating both sides of inequality (3.1) and defining

$$J_1 + J_2 := \int_0^1 \left(\int_0^1 I_0^t(r)r^{(d-1)}dr \right)\frac{dt}{t} + \int_0^1 \left(\int_1^{1/t} I_0^t(r)r^{(d-1)}dr \right)\frac{dt}{t}$$

and

$$J_3 := \int_0^1 \left(\int_{1/t}^\infty I_0^t(r)r^{(d-1)}dr \right)\frac{dt}{t},$$

we just need to conclude that $J_i < \infty$, $i = 1, 2, 3$.

In order to estimate J_1, we apply the $(2\beta-\epsilon)$-almost decreasingness property to φ, to obtain

$$\begin{aligned} J_1 &= \int_0^1 \frac{t^{2q\beta}}{[\varphi(t)]^q}\left[\int_0^1 r^{dq(1-1/p-1/q)+2q\beta} \left(\int_{S^{d-1}} |\widehat{f}(r\omega)|^q d\omega \right) r^{(d-1)}dr \right]\frac{dt}{t} \\ &\lesssim \int_0^1 t^{\epsilon q}\left[\int_0^1 r^{dq(1-1/p-1/q)} \left(\int_{S^{d-1}} |\widehat{f}(r\omega)|^q d\omega \right) r^{(d-1)}dr \right]\frac{dt}{t} \\ &\leq \|(\cdot)^{d(1-1/p-1/q)}\widehat{f}(\cdot)\|_q^q \int_0^1 t^{\epsilon q-1}dt < \infty. \end{aligned}$$

Moving on to the estimate for $J_2 + J_3$, we first write J_2 explicitly as follows

$$J_2 = \int_0^1 \frac{t^{2\beta q}}{[\varphi(t)]^q}\left[\int_1^{1/t} r^{dq(1-1/p-1/q)+2q\beta} \left(\int_{S^{d-1}} |\widehat{f}(r\omega)|^q d\omega \right) r^{(d-1)}dr \right]\frac{ds}{s}.$$

Since φ is $(2\beta-\epsilon)$-almost decreasing, we have

$$\frac{\varphi(r^{-1})}{r^{-2\beta+\epsilon}} \lesssim \frac{\varphi(t)}{t^{2\beta-\epsilon}}, \quad \text{for } 1\leq r\leq 1/t,$$

which leads us to

$$\frac{t^{2\beta}}{\varphi(t)} \lesssim \frac{r^{-2\beta+\epsilon}t^{\epsilon}}{\varphi(r^{-1})}, \quad \text{for } 1\leq r\leq 1/t.$$

Consequently,

$$J_2 \lesssim \int_0^1 t^{\epsilon q}\left[\int_1^{1/t} r^{dq(1-1/p-1/q)+q\epsilon}\left(\int_{S^{d-1}}\frac{|\widehat{f}(r\omega)|^q}{[\varphi(r^{-1})]^q}d\omega\right) r^{(d-1)}dr\right]\frac{ds}{s}.$$

Now, the change of variables $t=s^{-1}$ in the right-hand side of inequality above gives us

$$\begin{aligned} J_2 &\lesssim \int_1^{\infty} s^{-q\epsilon}\left[\int_1^{s} r^{dq(1-1/p-1/q)+q\epsilon}\left(\int_{S^{d-1}}\frac{|\widehat{f}(r\omega)|^q}{[\varphi(r^{-1})]^q}d\omega\right) r^{(d-1)}dr\right]\frac{ds}{s}\\ &\lesssim \int_1^{\infty} s^{-q\epsilon}\left\{\int_1^{s} r^{q\epsilon-1}\left[\int_r^{2r} u^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}}\frac{|\widehat{f}(u\,\omega)|^q}{[\varphi(u^{-1})]^q}d\omega\right) u^{(d-1)}du\right]dr\right\}\frac{ds}{s}. \end{aligned}$$

For J_3, the change of variable $t^{-1}=s$ implies

$$\begin{aligned} J_3 &= \int_0^1 \frac{1}{[\varphi(t)]^q}\left[\int_{1/t}^{\infty} r^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}}|\widehat{f}(r\omega)|^q d\omega\right) r^{(d-1)}dr\right]\frac{dt}{t}\\ &= \int_1^{\infty} \frac{1}{[\varphi(s^{-1})]^q}\left[\int_{s}^{\infty} r^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}}|\widehat{f}(r\omega)|^q d\omega\right) r^{(d-1)}dr\right]\frac{ds}{s}. \end{aligned}$$

Observing that, for all $1\le s\le r<\infty$, the inequality $\varphi(r^{-1})\le\varphi(s^{-1})$ holds, we obtain

$$\begin{aligned} J_3 &\lesssim \int_1^{\infty}\left[\int_{s}^{\infty} r^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}}\frac{|\widehat{f}(r\omega)|^q}{[\varphi(r^{-1})]^q}d\omega\right) r^{(d-1)}dr\right]\frac{ds}{s}\\ &\lesssim \int_1^{\infty}\left\{\int_s^{\infty} r^{-1}\left[\int_r^{2r} u^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}}\frac{|\widehat{f}(u\,\omega)|^q}{[\varphi(u^{-1})]^q}d\omega\right) u^{(d-1)}du\right]dr\right\}\frac{ds}{s}. \end{aligned}$$

Finally, taking in account the estimates for J_2 and J_3, Hardy's inequalities [23, p. 272] imply

$$\begin{aligned} J_2+J_3 &\lesssim \int_0^{\infty}\left[\int_r^{2r} u^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}}\frac{|\widehat{f}(u\,\omega)|^q}{[\varphi(u^{-1})]^q}d\omega\right) u^{(d-1)}du\right]\frac{dr}{r}\\ &= \int_0^{\infty}\int_{r\le|\xi|\le 2r}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\,\frac{dr}{r}<\infty, \end{aligned}$$

and $f\in B^{\varphi}_{p,q}(\mathbb{R}^d)$. The theorem is proved. $\square$

We close this section with a direct consequence of Theorem 1.5.

Corollary 3.1. *If $\varphi\in\Omega^q_{2\beta}$, then $f\in B^{\varphi}_{2,2}(\mathbb{R}^d)$ if and only if*

$$\int_0^{\infty}\int_{t\le|\xi|\le 2t}\left(\frac{|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^2 d\xi\,\frac{dt}{t}<\infty.$$

4. $\widehat{GM}^d_p$ Class: Riemann-Lebesgue Type inequality and Final Remarks

From now on, we will work with GM-classes (general monotone classes) of functions. This concept was firstly introduced in [27], where also the main properties were established.

A locally bounded variation function $g:(0,\infty)\longrightarrow\mathbb{R}$, vanishing at infinity and such that for some $c>0$ (only depending on g) satisfies

$$\int_t^{\infty}|d\,g(s)|\lesssim\int_{t/c}^{\infty}\frac{|g(s)|}{s}ds<\infty,\quad \text{for all } t>0, \tag{4.1}$$

is called *general monotone* (see [17, 25, 28]) and we write $g \in GM$. In addition, if g satisfies the following condition

$$\int_0^1 s^{d-1}|g(s)|ds + \int_1^\infty s^{(d-1)/2}|dg(s)| < \infty,$$

for $d \geq 1$ an integer number, then we write $g \in GM^d$ (see [12, 13] and references quoted there for details).

In this section, we write f_0 for the radial part of a given f from $\mathbb{R}^d$. We consider the following collection of functions defined in terms of the inverse Fourier-Hankel transform:

(4.2)
$$\widehat{GM}_p^d := \left\{ f \in L^p(\mathbb{R}^d) : f \text{ is radial, } f_0(t) = \frac{\sigma_{d-1}}{(2\pi)^d} \int_0^\infty s^{d-1} F_0(s) j_{d/2-1}(ts) ds, \ F_0 \in GM^d \right\}.$$

For $d \geq 2$ and $1 \leq p < 2d/(d+1)$, the collection above contains all radial positive-definite functions $f(x) = f_0(|x|)$, $x \in \mathbb{R}^d$, such that its Fourier transforms F_0 lies in GM^d. For $d = 1$, the same conclusion holds if $p = 1$ (see [13, p. 1293] and [17] for more examples).

Conditions in Theorem 2.1 can be considerably relaxed if we consider the class $\widehat{GM}_p^d$ as showed in [13, Theorem 4.1]. Following the path designed by the authors in [13], conditions of Theorem 2.1 are extended as follows.

Proposition 4.1. *Let $\{T_t\}_{t>0}$ be a β-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$ and $f \in \widehat{GM}_p^d$.*

(A) Let $1 < p \leq q < \infty$. If $\widehat{f}$ is nonnegative, then

$$\left(\int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{2\beta} |\xi|^{d(1-1/p-1/q)} \widehat{f}(\xi) \right]^q d\xi \right)^{1/q} \lesssim \|T_t(f) - f\|_p. \tag{4.3}$$

(B) Let $1 < q \leq p < \infty$ with $2d/(d+1) < p$. If $|\cdot|^{d(1-1/p-1/q)} \widehat{f}(\cdot) \in L^q$, then

$$\|T_t(f) - f\|_p \lesssim \left(\int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{2\beta} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi \right)^{1/q}. \tag{4.4}$$

Due to [13, Theorem 4.1, p. 1293] is not hard to see that the basics facts (besides several calculations) needed in order to repeat that proof in here are the following: $[\min(1, t(\cdot))]^{2\beta} F_0(\cdot)$ must be in GM^d, $h := f - T_t(f)$ must be radial and its radial part given by $h_0(s) = [1 - \eta_t(s)]F_0(s)$, $s \in (0, \infty)$. It is clear that all these facts hold true under assumptions made in Proposition 4.1, that is why the details of the proof were omitted.

Proposition 4.2. *Let $\{T_t\}_{t>0}$ be a β-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$, $1 < p \leq q < \infty$, and $\varphi \in \Omega_{2\beta}$. If $f \in Lip\,(p, \beta, \varphi) \cap \widehat{GM}_p^d$ and $\widehat{f}$ is nonnegative, then*

$$\left(\int_{t \leq |\xi| \leq 2t} \left[|\xi|^{d(1-1/p-1/q)} \widehat{f}(\xi) \right]^q d\xi \right)^{1/q} = O\left(\varphi(t^{-1})\right). \tag{4.5}$$

Additionally, if $2d/(d+1) < q$, $f \in \widehat{GM}_q^d$, $|\cdot|^{d(1-1/p-1/q)} \widehat{f}(\cdot) \in L^p(\mathbb{R}^d)$ and

$$\left(\int_{t \leq |\xi| \leq 2t} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^p d\xi \right)^{1/p} = O\left(\varphi(t^{-1})\right), \tag{4.6}$$

then $f \in Lip\,(q, \beta, \varphi)$.

The proof of (4.5) is a direct application of Theorem 4.1, part (A). While (4.6) follows from the proof of Theorem 1.3, but instead of applying Proposition 2.1, we need to apply Proposition 4.1, part (B). For $p = q$, the proposition above becomes the following.

Corollary 4.3. *Let* $2d/(d+1) < p$ *and* $f \in \widehat{GM}_p^d$ *such that* $\widehat{f}$ *is non-negative and* $|\cdot|^{d(1-2/p)}\widehat{f}(\cdot) \in L^p(\mathbb{R}^d)$. *Then* $f \in Lip\,(p, \beta, \varphi)$ *if and only if*

$$\left(\int_{t \le |\xi| \le 2t} \left[|\xi|^{d(1-2/p)} \widehat{f}(\xi) \right]^p d\xi \right)^{1/p} = O\left(\varphi(t^{-1}) \right).$$

Another consequence of Proposition 4.1 is a pointwise estimate for the Fourier transforms of functions in $\widehat{GM}_p^d$ satisfying the Lipschitz condition. The Riemann-Lebesgue type inequality is the content of the next result.

Corollary 4.4. *Let* $1 < p \le q < \infty$ *and* $\varphi \in \Omega_{2\beta}$. *If* $f \in \widehat{GM}_p^d \cap Lip\,(p, \beta, \varphi)$ *is such that* $\widehat{f}$ *is nonnegative, then*

$$\widehat{f}(\xi) = O\left(|\xi|^{-d/q'} \varphi(|\xi|^{-1}) \right), \quad \text{as } |\xi| \to \infty.$$

Proof. Observe that for $f \in \widehat{GM}_p^d$, if its Fourier transform $\widehat{f}$ is written as F_0, then it satisfies inequality (4.1) and it holds

$$F_0(t) \lesssim \int_{t/c}^{\infty} \frac{F_0(s)}{s} ds, \quad \text{for all } t > 0.$$

An application of Hölder inequality leads us to

$$F_0(t) \lesssim t^{-d/q'} \left(\int_{t/c}^{\infty} s^{qd-d-1} [F_0(s)]^q ds \right)^{1/q}, \quad \text{for all } t > 0.$$

Finally, Proposition (4.2) implies

$$\left(\int_{t/c}^{\infty} s^{qd-d-1} [F_0(s)]^q ds \right)^{1/q} = O\left(\varphi(t^{-1}) \right), \quad \text{as } t \to \infty,$$

and the proof follows. □

A version of Theorem 1.5 for $\widehat{GM}_p^d$ class also has a more relaxed condition version.

Proposition 4.5. *Let* $\{T_t\}_{t>0}$ *be a* β*-admissible family of multipliers operators on* $L^p(\mathbb{R}^d)$, $\varphi \in \Omega_{2\beta}^q$ *and* $f \in \widehat{GM}_p^d$.

(A) Let $1 < p \le q < \infty$. *If* $f \in B_{p,q}^{\varphi}(\mathbb{R}^d)$ *is such that* $\widehat{f}$ *is nonnegative, then*

$$\int_0^{\infty} t^{d(q-1)} \left(\frac{F_0(t)}{\varphi(t^{-1})} \right)^q \frac{dt}{t} < \infty.$$

(B) Let $1 < q \le p < \infty$ *with* $2d/(d+1) < p$. *If* $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$, *and*

$$\int_0^{\infty} t^{d(q-1)} \left(\frac{|F_0(t)|}{\varphi(t^{-1})} \right)^q \frac{dt}{t} < \infty,$$

then $f \in B_{p,q}^{\varphi}(\mathbb{R}^d)$.

The proof is a simple adaptation of the proofs of Theorem 1.5 above and Theorem 7.3 in [13, p. 1310]. For $p = q$, we obtain the following.

Corollary 4.6. *Let* $2d/(d+1) < p$, $f \in \widehat{GM}_p^d$ *such that* $\widehat{f}$ *is nonnegative and* $|\cdot|^{d(1-2/p)}\widehat{f}(\cdot) \in L^p(\mathbb{R}^d)$. *Then,* $f \in B^{\varphi}_{p,q}(\mathbb{R}^d)$ *if and only if*

$$\int_0^\infty t^{d(q-1)} \left(\frac{F_0(t)}{\varphi(t^{-1})}\right)^q \frac{dt}{t} < \infty.$$

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Research Article

Fekete-Szegö Problem for Certain Subclass of Analytic Functions with Complex Order Defined by $q-$Analogue of Ruscheweyh Operator

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ABSTRACT. In this paper, we study Fekete-Szegö problem for certain subclass of analytic functions with complex order in the open unit disk by applying the $q-$analogue of Ruscheweyh operator in conjunction with the principle of subordination between analytic functions.

Keywords: Analytic functions, univalent functions, $q-$derivative operator, $q-$analogue of Ruscheweyh operator, Fekete-Szego problem, subordination.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written as $f \prec g$ in $\mathbb{U}$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function ω, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence holds (see [12] and [7]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For function $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the $q-$derivative of a function f is defined by (see [10, 9] and [6])

$$D_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} & , z \neq 0 \\ f^{'}(0) & , z = 0 \end{cases} \tag{1.2}$$

provided that $f^{'}(0)$ exists and $D_q^2 f(z) = D_q(D_q f(z))$. We note from (1.2) that

$$\lim_{q\to 1^-} D_q f(z) = f^{'}(z) \quad \textit{and} \quad \lim_{q\to 1^-} D_q^2 f(z) = f^{''}(z).$$

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It is readily deduced from (1.1) and (1.2) that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \ a_k z^{k-1}, \tag{1.3}$$

where

$$[k]_q = \frac{q^k - 1}{q - 1}. \tag{1.4}$$

Aldweby and Darus [1] defined $q-$analogue of Ruscheweyh operator $\mathcal{R}_q^{\delta} : \mathcal{A} \to \mathcal{A}$ as follows:

$$\mathcal{R}_q^{\delta} f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\delta-1]_q!}{[\delta]_q! \, [k-1]_q!} a_k z^k \quad (\delta \geq -1),$$

where $[i]_q!$ is given by

$$[i]_q! = \begin{cases} [i]_q [i-1]_q \ldots [1]_q & , i \in \mathbb{N} = \{1, 2, 3, \ldots\} \\ 1 & , i = 0 \end{cases}.$$

We note that

$$\mathcal{R}_q^0 f(z) = f(z) \quad \textit{and} \quad \mathcal{R}_q^1 f(z) = z D_q f(z).$$

From the definition of $\mathcal{R}_q^{\delta}$ we observe that if $q \to 1^-$, we have

$$\lim_{q \to 1} \mathcal{R}_q^{\delta} f(z) = \mathcal{R}^{\delta} f(z) = z + \sum_{k=2}^{\infty} \frac{(k+\delta-1)!}{\delta! \, (k-1)!} a_k z^k,$$

where $\mathcal{R}^{\delta}$ is Ruscheweyh differential operator defined by Ruscheweyh [16].

It is easy to check that

$$z D_q \left(\mathcal{R}_q^{\delta} f(z) \right) = \left(1 + \frac{[\delta]_q}{q^{\delta}} \right) \mathcal{R}_q^{\delta+1} f(z) - \frac{[\delta]_q}{q^{\delta}} \mathcal{R}_q^{\delta} f(z). \tag{1.5}$$

If $q \to 1^-$, the equality (1.5) implies

$$z \left(\mathcal{R}^{\delta} f(z) \right)' = (1+\delta) \mathcal{R}^{\delta+1} f(z) - \delta \mathcal{R}^{\delta} f(z)$$

which is the well known recurrence formula for Ruscheweyh differential operator.

By making use of the $q-$analogue of Ruscheweyh operator $\mathcal{R}_q^{\delta}$ and the principle of subordination, we now introduce the following subclass of analytic functions of complex order.

Definition 1.1. *Let $\mathcal{P}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\Re\phi(z) > 0$ for $z \in \mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_{q,b}^{\delta}(\gamma, \phi)$ if it satisfies the following subordination condition:*

$$1 + \frac{1}{b} \left[\frac{(1-\gamma) z D_q \mathcal{R}_q^{\delta} f(z) + \gamma z D_q (z D_q \mathcal{R}_q^{\delta} f(z))}{(1-\gamma) \mathcal{R}_q^{\delta} f(z) + \gamma z D_q \mathcal{R}_q^{\delta} f(z)} - 1 \right] \prec \phi(z) \ \ (b \in \mathbb{C}^*). \tag{1.6}$$

We note that:

(i) $\lim_{q \to 1^-} \mathcal{K}_{q,b}^0 (\gamma, \phi) = \mathcal{K}_b (\gamma, \phi) \ (b \in \mathbb{C}^*)$

$$= \left\{ f \in \mathcal{A} : 1 + \frac{1}{b} \left[\frac{z f'(z) + \gamma z^2 f''(z)}{(1-\gamma) f(z) + \gamma z f'(z)} - 1 \right] \prec \phi(z) \right\},$$

(ii) $\mathcal{K}^{0}_{q,(1-\alpha)e^{-i\theta}\cos\theta}(0,\phi)=\mathcal{S}^{\theta}_{q}(\alpha;\phi)\left(|\theta|\le\frac{\pi}{2},0\le\alpha<1\right)$

$$=\left\{f\in\mathcal{A}:\frac{e^{i\theta}\frac{zD_qf(z)}{f(z)}-\alpha\cos\theta-i\sin\theta}{(1-\alpha)\cos\theta}\prec\phi(z)\right\},$$

(iii) $\mathcal{K}^{0}_{q,(1-\alpha)e^{-i\theta}\cos\theta}(1,\phi)=\mathcal{C}^{\theta}_{q}(\alpha;\phi)\left(|\theta|\le\frac{\pi}{2},0\le\alpha<1\right)$

$$=\left\{f\in\mathcal{A}:\frac{e^{i\theta}\frac{D_q(zD_qf(z))}{D_qf(z)}-\alpha\cos\theta-i\sin\theta}{(1-\alpha)\cos\theta}\prec\phi(z)\right\},$$

(iv) $\mathcal{K}^{\delta}_{q,1}(0,\phi)=\mathcal{S}^{\delta}_{q}(\phi)$ and $\mathcal{K}^{\delta}_{q,1}(1,\phi)=\mathcal{C}^{\delta}_{q}(\phi)$ (Alweby and Darus [3]),
(v) $\mathcal{K}^{0}_{q,b}(0,\phi)=\mathcal{S}_{q,b}(\phi)$ and $\mathcal{K}^{0}_{q,b}(1,\phi)=\mathcal{C}_{q,b}(\phi)$ (Seoudy and Aouf [18]),
(vi) $\mathcal{K}^{0}_{q,1}(0,\phi)=\mathcal{S}_{q}(\phi)$ and $\mathcal{K}^{0}_{q,1}(1,\phi)=\mathcal{C}_{q}(\phi)$ (Alweby and Darus [2]),
(vii) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}(0,\phi)=\mathcal{S}_{b}(\phi)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}(1,\phi)=\mathcal{C}_{b}(\phi)$ (Ravichandran et al. [15]),
(viii) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,1}(0,\phi)=\mathcal{S}^{*}(\phi)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,1}(1,\phi)=\mathcal{C}(\phi)$ (Ma and Minda [11]),
(ix) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}\left(0,\frac{1+(1-2\alpha)z}{1-z}\right)=\mathcal{S}^{*}_{\alpha}(b)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}\left(1,\frac{1+(1-2\alpha)z}{1-z}\right)=\mathcal{C}_{\alpha}(b)$ $(0\le\alpha<1)$ (Frasin [8]),
(x) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}\left(0,\frac{1+z}{1-z}\right)=\mathcal{S}^{*}(b)$ (Nasr and Aouf [14]),
(xi) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}\left(1,\frac{1+z}{1-z}\right)=\mathcal{C}(b)$ $(b\in\mathbb{C}^{*})$ (Nasr and Aouf [13] and Wiatrowski [19]),
(xii) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,1-\alpha}\left(0,\frac{1+z}{1-z}\right)=\mathcal{S}^{*}(\alpha)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,1-\alpha}\left(1,\frac{1+z}{1-z}\right)=\mathcal{C}(\alpha)$ $(0\le\alpha<1)$ (Robertson [17]),
(xiii) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,be^{-i\theta}\cos\theta}\left(0,\frac{1+z}{1-z}\right)=\mathcal{S}^{\theta}(b)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,be^{-i\theta}\cos\theta}\left(1,\frac{1+z}{1-z}\right)=\mathcal{C}^{\theta}(b)$ $(|\theta|<\frac{\pi}{2})$ (Al-Oboudi and Haidan [4] and Aouf et al. [5]).

In order to establish our main results, we need the following lemma.

Lemma 1.1. [11] *If $p(z)=1+c_1z+c_2z^2+\ldots$ is a function with positive real part in $\mathbb{U}$ and μ is a complex number, then*

$$\left|c_2-\mu c_1^2\right|\le 2\max\{1;|2\mu-1|\}.$$

The result is sharp for the functions given by

$$p(z)=\frac{1+z^2}{1-z^2}\ \text{ and }\ p(z)=\frac{1+z}{1-z}.$$

Lemma 1.2. [11] *If $p(z)=1+c_1z+c_2z^2+\ldots$ is an analytic function with a positive real part in $\mathbb{U}$, then*

$$\left|c_2-\nu c_1^2\right|\le\begin{cases}-4\nu+2 & if\ \ \nu\le0\\ 2 & if\ \ 0\le\nu\le1,\\ 4\nu-2 & if\ \ \nu\ge1\end{cases}$$

when $\upsilon<0$ or $\nu>1$, the equality holds if and only if $p(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0<\nu<1$, then the equality holds if and only if $p(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $\nu=0$, the equality holds if and only if

$$p(z)=\left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z}\qquad(0\le\lambda\le1)$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \leq 2 \qquad \left(0 \leq \nu \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu) \left|c_1\right|^2 \leq 2 \qquad \left(\frac{1}{2} \leq \nu \leq 1\right).$$

In the present paper, we obtain the Fekete-Szegö inequalities for the class $\mathcal{K}_{q,b}(\gamma, \phi)$. The motivation of this paper is to generalize previously results. Unless otherwise mentioned, we assume throughout this paper that the function $0 < q < 1$, $b \in \mathbb{C}^*$, $0 \leq \gamma \leq 1$, $\phi \in \mathcal{P}$, $[k]_q$ is given by (1.4) and $z \in \mathbb{U}$.

Theorem 1.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 \neq 0$. If f given by (1.1) belongs to the class $\mathcal{K}_{q,b}(\gamma, \phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|bB_1|}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q} \max\left\{1; \left|\frac{B_2}{B_1} + \left(1 - \frac{[1+\gamma q(q+1)]\,[\delta+2]_q}{(1+\gamma q)^2\,[\delta+1]_q}\mu\right)\frac{B_1 b}{q}\right|\right\}. \tag{1.7}$$

The result is sharp.

Proof. If $f \in \mathcal{K}^{\delta}_{q,b}(\gamma, \phi)$, then there is a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$1 + \frac{1}{b}\left[\frac{(1-\gamma)\, zD_q\mathcal{R}_q^{\delta} f(z) + \gamma z D_q(zD_q\mathcal{R}_q^{\delta} f(z))}{(1-\gamma)\,\mathcal{R}_q^{\delta} f(z) + \gamma z D_q \mathcal{R}_q^{\delta} f(z)} - 1\right] = \phi(\omega(z)). \tag{1.8}$$

Define the function $p(z)$ by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \ldots . \tag{1.9}$$

Since ω is a Schwarz function, we see that $\Re p(z) > 0$ and $p(0) = 1$. Therefore,

$$\begin{aligned}
\phi(\omega(z)) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right) \\
&= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \ldots\right]\right) \\
&= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \ldots . \qquad (1.10)
\end{aligned}$$

Now, by substituting (1.10) in (1.8), we have

$$\begin{aligned}
& 1 + \frac{1}{b}\left[\frac{(1-\gamma)\, zD_q\mathcal{R}_q^{\delta} f(z) + \gamma z D_q(zD_q\mathcal{R}_q^{\delta} f(z))}{(1-\gamma)\,\mathcal{R}_q^{\delta} f(z) + \gamma z D_q \mathcal{R}_q^{\delta} f(z)} - 1\right] \\
= \; & 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \ldots .
\end{aligned}$$

From the above equation, we obtain

$$\frac{1}{b} q(1+\gamma q)[\delta+1]_q a_2 = \frac{B_1 c_1}{2}$$

and

$$\frac{q}{b}\left([1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q a_3-(1+\gamma q)^2\left([\delta+1]_q\right)^2 a_2^2\right)=\frac{B_1c_2}{2}-\frac{B_1c_1^2}{4}+\frac{B_2c_1^2}{4}$$

or, equivalently,

$$a_2=\frac{B_1c_1b}{2q(1+\gamma q)[\delta+1]_q}$$

and

$$a_3=\frac{bB_1}{2[1+\gamma q(q+1)]q[\delta+2]_q[\delta+1]_q}\left\{c_2-\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1b}{q}\right]c_1^2\right\}.$$

Therefore, we have

$$a_3-\mu a_2^2=\frac{bB_1}{2q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q}\left\{c_2-\nu c_1^2\right\}, \tag{1.11}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1b}{q}\left(1-\frac{[1+\gamma q(q+1)][\delta+2]_q\mu}{(1+\gamma q)^2[\delta+1]_q}\right)\right]. \tag{1.12}$$

Our result now follows from Lemma 1.1. The result is sharp for the functions

$$1+\frac{1}{b}\left[\frac{(1-\gamma)zD_q\mathcal{R}_q^\delta f(z)+\gamma zD_q(zD_q\mathcal{R}_q^\delta f(z))}{(1-\gamma)\mathcal{R}_q^\delta f(z)+\gamma zD_q\mathcal{R}_q^\delta f(z)}-1\right]=\phi\left(z^2\right)$$

and

$$1+\frac{1}{b}\left[\frac{(1-\gamma)zD_q\mathcal{R}_q^\delta f(z)+\gamma zD_q(zD_q\mathcal{R}_q^\delta f(z))}{(1-\gamma)\mathcal{R}_q^\delta f(z)+\gamma zD_q\mathcal{R}_q^\delta f(z)}-1\right]=\phi(z).$$

This completes the proof of Theorem 1.1. □

Taking $\gamma=0$ and $b=1$ in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 6].

Corollary 1.1. *Let $\phi(z)=1+B_1z+B_2z^2+\ldots$ with $B_1\neq 0$. If f given by (1.1) belongs to the class $\mathcal{S}_q^\delta(\phi)$, then*

$$\left|a_3-\mu a_2^2\right|\leq\frac{|B_1|}{q[\delta+2]_q[\delta+1]_q}\max\left\{1;\left|\frac{B_2}{B_1}+\left(1-\frac{[\delta+2]_q}{[\delta+1]_q}\mu\right)\frac{B_1}{q}\right|\right\}.$$

The result is sharp.

Taking $\gamma=b=1$ in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 7].

Corollary 1.2. *Let $\phi(z)=1+B_1z+B_2z^2+\ldots$ with $B_1\neq 0$. If f given by (1.1) belongs to the class $\mathcal{K}_q^\delta(\phi)$, then*

$$\left|a_3-\mu a_2^2\right|\leq\frac{|B_1|}{q[1+q(q+1)][\delta+2]_q[\delta+1]_q}\max\left\{1;\left|\frac{B_2}{B_1}+\left(1-\frac{[1+q(q+1)][\delta+2]_q}{[\delta+1]_q(1+q)^2}\mu\right)\frac{B_1b}{q}\right|\right\}.$$

The result is sharp.

Taking $\gamma=\delta=0$ and $b=1$ in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [2, Theorem 2.1].

Corollary 1.3. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 \neq 0$. If f given by (1.1) belongs to the class $\mathcal{S}_q(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \tfrac{|B_1|}{q(q+1)} \max\left\{1; \left|\tfrac{B_2}{B_1} + (1-(q+1)\mu)\tfrac{B_1}{q}\right|\right\}.$$

The result is sharp.

Taking $\gamma = b = 1$ and $\delta = 0$ in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [2, Theorem 2.2].

Corollary 1.4. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 \neq 0$. If f given by (1.1) belongs to the class $\mathcal{K}_q(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \tfrac{|B_1|}{q(q+1)[1+q(q+1)]} \max\left\{1; \left|\tfrac{B_2}{B_1} + \left(1 - \tfrac{[1+q(q+1)]}{(1+q)}\mu\right)\tfrac{B_1}{q}\right|\right\}.$$

The result is sharp.

Taking $\gamma = \delta = 0$ and $q \to 1^-$ in Theorem 1.1, we obtain the following corollary which improves the result of Ravichandran et al. [15, Theorem 4.1].

Corollary 1.5. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 \neq 0$. If f given by (1.1) belongs to the class $\mathcal{S}_b(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|B_1 b|}{2} \max\left\{1; \left|\frac{B_2}{B_1} + (1-2\mu) B_1 b\right|\right\}.$$

The result is sharp.

Theorem 1.2. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\sigma_1 = \frac{(1+\gamma q)^2 [\delta+1]_q \left[bB_1^2 + q(B_2 - B_1)\right]}{[1+\gamma q(q+1)][\delta+2]_q bB_1^2}, \tag{1.13}$$

$$\sigma_2 = \frac{(1+\gamma q)^2 [\delta+1]_q \left[bB_1^2 + q(B_2 + B_1)\right]}{[1+\gamma q(q+1)][\delta+2]_q bB_1^2}, \tag{1.14}$$

$$\sigma_3 = \frac{(1+\gamma q)^2 [\delta+1]_q \left(bB_1^2 + qB_2\right)}{[1+\gamma q(q+1)][\delta+2]_q bB_1^2}. \tag{1.15}$$

If f given by (1.1) belongs to the class $\mathcal{K}_{q,b}^{\delta}(\gamma,\phi)$ with $b > 0$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{b}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q}\left[B_2 + \frac{B_1^2 b}{q}\left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q}\mu\right)\right] & , \mu \leq \sigma_1 \\ \frac{bB_1}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q} & , \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{b}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q}\left[-B_2 - \frac{B_1^2 b}{q}\left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q}\mu\right)\right] & , \mu \geq \sigma_2 \end{cases}. \tag{1.16}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \tfrac{q(1+\gamma q)^2[\delta+1]_q}{[1+\gamma q(q+1)][\delta+2]_q B_1^2 b}\left[B_1 - B_2 - \tfrac{B_1^2 b}{q}\left(1 - \tfrac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q}\mu\right)\right]|a_2|^2$$

$$\leq \frac{bB_1}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q}. \tag{1.17}$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \tfrac{q(1+\gamma q)^2[\delta+1]_q}{[1+\gamma q(q+1)][\delta+2]_q B_1^2 b}\left[B_1 + B_2 + \frac{B_1^2 b}{q}\left(1 - \tfrac{[1+\gamma q(q+1)][\delta+2]_q \mu}{(1+\gamma q)^2[\delta+1]_q}\right)\right]|a_2|^2$$

$$\leq \frac{bB_1}{q\left[1+\gamma q\left(q+1\right)\right]\left[\delta+2\right]_q\left[\delta+1\right]_q}. \tag{1.18}$$

The result is sharp.

Proof. Applying Lemma 1.2 to (1.11) and (1.12), we can obtain our results asserted by Theorem 1.2. □

Taking $\gamma = 0$ and $b = 1$ in Theorem 1.2, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 10].

Corollary 1.6. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\begin{aligned} \chi_1 &= \frac{[\delta+1]_q\left[B_1^2+q\left(B_2-B_1\right)\right]}{[\delta+2]_q B_1^2}, \\ \chi_2 &= \frac{[\delta+1]_q\left[B_1^2+q\left(B_2+B_1\right)\right]}{[\delta+2]_q B_1^2}, \\ \chi_3 &= \frac{[\delta+1]_q\left(B_1^2+qB_2\right)}{[\delta+2]_q B_1^2}. \end{aligned}$$

If f given by (1.1) belongs to the class $\mathcal{S}_q^\delta(\phi)$, then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{1}{q[\delta+2]_q[\delta+1]_q}\left[B_2+\frac{B_1^2}{q}\left(1-\frac{[\delta+2]_q}{[\delta+1]_q}\mu\right)\right] & ,\mu \leq \chi_1 \\ \frac{B_1}{q[\delta+2]_q[\delta+1]_q} & ,\chi_1 \leq \mu \leq \chi_2 \\ \frac{1}{q[\delta+2]_q[\delta+1]_q}\left[-B_2-\frac{B_1^2}{q}\left(1-\frac{[\delta+2]_q}{[\delta+1]_q}\mu\right)\right] & ,\mu \geq \chi_2 \end{cases}.$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|a_3-\mu a_2^2\right| + \frac{q[\delta+1]_q}{[\delta+2]_q B_1^2}\left[B_1-B_2-\frac{B_1^2}{q}\left(1-\frac{[\delta+2]_q}{[\delta+1]_q}\mu\right)\right]\left|a_2\right|^2 \leq \frac{B_1}{q[\delta+2]_q[\delta+1]_q}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|a_3-\mu a_2^2\right| + \frac{q[\delta+1]_q}{[\delta+2]_q B_1^2}\left[B_1+B_2+\frac{B_1^2}{q}\left(1-\frac{[\delta+2]_q\mu}{[\delta+1]_q}\right)\right]\left|a_2\right|^2 \leq \frac{B_1}{q[\delta+2]_q[\delta+1]_q}.$$

The result is sharp.

Taking $\gamma = b = 1$ in Theorem 1.2, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 11].

Corollary 1.7. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\begin{aligned} \varkappa_1 &= \frac{[2]_q^2[\delta+1]_q\left[B_1^2+q\left(B_2-B_1\right)\right]}{[3]_q[\delta+2]_q B_1^2}, \\ \varkappa_2 &= \frac{[2]_q^2[\delta+1]_q\left[B_1^2+q\left(B_2\mu+B_1\right)\right]}{[3]_q[\delta+2]_q B_1^2}, \\ \varkappa_3 &= \frac{[2]_q^2[\delta+1]_q\left(B_1^2+qB_2\right)}{[3]_q[\delta+2]_q B_1^2}. \end{aligned}$$

If f given by (1.1) belongs to the class $\mathcal{K}_q^{\delta}(\phi)$, then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{1}{q[3]_q[\delta+2]_q[\delta+1]_q}\left[B_2+\frac{B_1^2}{q}\left(1-\frac{[3]_q[\delta+2]_q}{[2]_q^2[\delta+1]_q}\mu\right)\right] & ,\mu\leq \varkappa_1 \\ \frac{B_1}{q[3]_q[\delta+2]_q[\delta+1]_q} & ,\varkappa_1\leq\mu\leq\varkappa_2 \\ \frac{1}{q[3]_q[\delta+2]_q[\delta+1]_q}\left[-B_2-\frac{B_1^2}{q}\left(1-\frac{[3]_q[\delta+2]_q}{[2]_q^2[\delta+1]_q}\mu\right)\right] & ,\mu\geq\varkappa_2 \end{cases}.$$

Further, if $\varkappa_1 \leq \mu \leq \varkappa_3$, then

$$\left|a_3-\mu a_2^2\right|+\frac{q[2]_q^2[\delta+1]_q}{[3]_q[\delta+2]_qB_1^2}\left[B_1-B_2-\frac{B_1^2}{q}\left(1-\frac{[3]_q[\delta+2]_q}{[2]_q^2[\delta+1]_q}\mu\right)\right]|a_2|^2\leq\frac{B_1}{q[3]_q[\delta+2]_q[\delta+1]_q}.$$

If $\varkappa_3 \leq \mu \leq \varkappa_2$, then

$$\left|a_3-\mu a_2^2\right|+\frac{q[2]_q^2[\delta+1]_q}{[3]_q[\delta+2]_q\,B_1^2}\left[B_1+B_2+\frac{B_1^2}{q}\left(1-\frac{[3]_q[\delta+2]_q}{[2]_q^2[\delta+1]_q}\mu\right)\right]|a_2|^2\leq\frac{B_1}{q[3]_q[\delta+2]_q[\delta+1]_q}.$$

The result is sharp.

Remark 1.1. *Putting $\delta=\gamma=0$ in Theorems 1.1 and 1.2, respectively, we deduce the corresponding results derived by Seoudy and Aouf* [18, Theorems 1 and 3, respectively].

Remark 1.2. *Putting $\delta=0$ and $\gamma=1$ in Theorems 1.1 and 1.2, respectively, we deduce the corresponding results derived by Seoudy and Aouf* [18, Theorems 2 and 4, respectively].

Remark 1.3. *For different choices of the parameters b,δ,q,γ and ϕ in Theorems 1.1 and 1.2, we can deduce some results for the classes $\mathcal{K}_b(\gamma,\phi)$, $\mathcal{S}_q^{\theta}(\alpha;\phi)$, $\mathcal{C}_q^{\theta}(\alpha;\phi)$, $\mathcal{S}_q(\phi)$, $\mathcal{C}_q(\phi)$, $\mathcal{S}_b(\phi)$, $\mathcal{C}_b(\phi)$, $\mathcal{S}^*(\phi)$, $\mathcal{C}(\phi)$, $\mathcal{S}_\alpha^*(b)$, $\mathcal{C}_\alpha(b)$, $\mathcal{S}^*(b)$, $\mathcal{C}(b)$, $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $\mathcal{S}^{\theta}(b)$ and $\mathcal{C}^{\theta}(b)$ which are defined in Section 1.*

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Research Article

A Fixed-Point Problem with Mixed-Type Contractive Condition

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ABSTRACT. We consider a fixed-point problem for mappings involving a mixed-type contractive condition in the setting of metric spaces. Precisely, we establish the existence and uniqueness of fixed point using the recent notions of F-contraction and (H, φ)-contraction.

Keywords: Fixed point, metric space, mixed-type contractive condition.

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1. INTRODUCTION

Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. In this paper, we study the following fixed-point problem:

$$(P_\tau) \qquad \begin{cases} \text{Find } z \in X \text{ such that } Tz = z, \\ \tau + F(H(d(Tx, Ty), \varphi(Tx), \varphi(Ty))) \leq F(H(d(x, y), \varphi(x), \varphi(y))), \quad \tau > 0, \end{cases}$$

for all $x, y \in X$ such that $H(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) > 0$.

This problem is determined by using three functions, namely $F : \mathbb{R}^+ \to \mathbb{R}$, $H : [0, +\infty[^3 \to [0, +\infty[$ and $\varphi : X \to [0, +\infty[$, with suitable properties (properly stated in Section 2).

Existence results of solutions for different fixed-point problems were proved by many authors. Here, we mention Banach [1] (the pioneering paper on contractions), Wardowski [12] (F-contractions, where F belongs to an appropriate family of functions, namely $\mathcal{F}$ in the sequel), Reem-Reich-Zaslavski [6] (contractive nonself-mappings), Reich-Zaslavski [7] (Matkowski contractions), Reich-Zaslavski [8] (Rakotch contractions), Jleli-Samet-Vetro [2] ((H, φ)-contractions, where H belongs to an appropriate family of functions, namely $\mathcal{H}$ in the sequel). Also, we recall the comprehensive book of Rus-Petruşel-Petruşel [9], and some results establishing the existence and uniqueness of fixed points that are zeros of a given function (see Samet-Vetro-Vetro [10] and Vetro-Vetro [11]). Finally, we quote the important results of Anthony To-Ming Lau and coworkers, who in a series of remarkable papers discussed the fixed-point property of mappings (see, for example, [3, 4, 5] and the references therein).

In this paper, we establish two existence and uniqueness results using a new type of contractive condition working on the classical metric space. In particular, we show that under appropriate assumptions these fixed points are zeros of given functions. Also, we give an example to support the new contractive condition. Precisely, the main result of our paper is the following existence and uniqueness theorem for problem (P_τ):

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Theorem 1.1. *If $\varphi : X \to [0, +\infty[$ is a lower semicontinuous function, then problem (P_τ) admits a unique solution z such that $\varphi(z) = 0$.*

2. Preliminaries

Let (X, d) be a metric space and let $T : X \to X$ be a mapping. We introduce the notation and notions needed in the sequel of this paper. For convenience of the reader, we start with basic facts concerning F-contractions and (H, φ)-contractions.

Definition 2.1. *Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:*

- (F_1) *F is nondecreasing,*
- (F_2) *for every sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to+\infty} \alpha_n = 0$ if and only if $\lim_{n\to+\infty} F(\alpha_n) = -\infty$,*
- (F_3) *there exists $k \in]0, 1[$ such that $\lim_{\alpha\to0^+} \alpha^k F(\alpha) = 0$.*

Now, the mapping T is said to be an F-contraction if there exists $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)) \quad \text{for all } x, y \in X,\ d(Tx, Ty) \neq 0. \tag{2.1}$$

From (2.1), since $\tau > 0$, we infer that

$$F(d(Tx, Ty)) < F(d(x, y)) \quad \text{for all } x, y \in X,\ Tx \neq Ty.$$

Using the property (F_1) of the function F, we deduce that

$$d(Tx, Ty) < d(x, y) \quad \text{for all } x, y \in X,\ Tx \neq Ty.$$

So, each F-contraction is a continuous mapping. Using this notion, Wardowski (see [12]) established the following significant result.

Theorem 2.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $z \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}$ is convergent to z.*

The functions from $\mathbb{R}^+$ to $\mathbb{R}$ defined by

- (i) $F(t) = \ln t$ for all $t \in \mathbb{R}^+$,
- (ii) $F(t) = t + \ln t$ for all $t \in \mathbb{R}^+$

are classical examples of functions belonging to $\mathcal{F}$.

In [2], Jleli *et al.* introduced a family $\mathcal{H}$ of functions $H : [0, +\infty[^3 \to [0, +\infty[$ satisfying the following conditions:

- (H_1) $\max\{a, b\} \leq H(a, b, c)$ for all $a, b, c \in [0, +\infty[$,
- (H_2) $H(0, 0, 0) = 0$,
- (H_3) H is continuous.

Some examples of functions belonging to $\mathcal{H}$ are given as follows:

- (i) $H(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$,
- (ii) $H(a, b, c) = \max\{a, b\} + c$ for all $a, b, c \in [0, +\infty[$,
- (iii) $H(a, b, c) = a + b + ab + c$ for all $a, b, c \in [0, +\infty[$.

Using a function $H \in \mathcal{H}$, the authors of [2] introduced the following notion of (H, φ)-contraction.

Definition 2.2. *Let (X, d) be a metric space, $\varphi : X \to [0, +\infty[$ be a given function and $H \in \mathcal{H}$. Then, $T : X \to X$ is called a (H, φ)-contraction with respect to the metric d if and only if*

$$H(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq kH(d(x, y), \varphi(x), \varphi(y)) \quad \textit{for all } x, y \in X,$$

for some constant $k \in]0, 1[$.

Now, we set

$$Z_\varphi := \{x \in X : \varphi(x) = 0\},$$
$$F_T := \{x \in X : Tx = x\}.$$

Furthermore, we say that T is a φ-Picard operator if and only if the following condition holds:

$$F_T \cap Z_\varphi = \{z\} \text{ and } T^n x \to z, \text{ as } n \to +\infty, \text{ for each } x \in X.$$

Consequently, we recall the following theorem of [2].

Theorem 2.3. *Let (X, d) be a complete metric space, $\varphi : X \to [0, +\infty[$ be a given function and $H \in \mathcal{H}$. Suppose that the following conditions hold:*

(A_1) *φ is lower semi-continuous,*
(A_2) *$T : X \to X$ is a (H, φ)-contraction with respect to the metric d.*

Then

(i) *$F_T \subset Z_\varphi$,*
(ii) *T is a φ-Picard operator,*
(iii) *for all $x \in X$ and for all $n \in \mathbb{N}$, we have*

$$d(T^n x, z) \leq \frac{k^n}{1-k} H(d(Tx, x), \varphi(Tx), \varphi(x)),$$

where $\{z\} = F_T \cap Z_\varphi = F_T$.

3. Main Results

Let $X \neq \emptyset$, $T : X \to X$, $x_0 \in X$ and $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Then, we call $\{x_n\}$ a sequence of Picard starting at x_0. In this section, we state and prove our results (Theorems 1.1 and 3.5), using a new mixed-type contraction. Precisely, we establish the existence and uniqueness of fixed point that are zeros of a given function.

Definition 3.3. *Let (X, d) be a metric space and let $T : X \to X$ be a mapping. The mapping T is called an F-H-contraction if there exist a function $F \in \mathcal{F}$, a function $H \in \mathcal{H}$, a real number $\tau > 0$ and a function $\varphi : X \to [0, +\infty[$ such that*

$$\tau + F(H(d(Tx, Ty), \varphi(Tx), \varphi(Ty))) \leq F(H(d(x, y), \varphi(x), \varphi(y))) \tag{3.2}$$

for all $x, y \in X$ with $H(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) > 0$.

We remark that every F-contraction is an F-H-contraction if we choose $H \in \mathcal{H}$ defined by $H(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$, and $\varphi : X \to [0, +\infty[$ defined by $\varphi(x) = 0$ for all $x \in X$. The following is an example of an F-H-contraction that is not an F-contraction, so that the new definition is a proper extension of the previous one.

Example 3.1. *Let $X = [0, 1]$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Consider the mapping $T : X \to X$ defined by*

$$Tx = \begin{cases} \dfrac{x}{2} & \text{if } x \in [0, 1[, \\[2ex] \dfrac{3}{4} & \text{if } x = 1. \end{cases}$$

Clearly, T is not an F-contraction since it is not continuous. Now, T is an F-H-contraction with respect to the functions $F \in \mathcal{F}$ defined by $F(t) = \ln t$ for all $t > 0$, $H \in \mathcal{H}$ defined by $H(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$, and $\varphi : X \to [0, +\infty[$ defined by $\varphi(x) = x$ for all $x \in X$.

Indeed, for all $x, y \in X$ *with* $0 < x \leq y < 1$ *or* $0 = x < y < 1$*, we have*

$$F(H(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty))) = \ln y,$$
$$F(H(d(x,y)+\varphi(x)+\varphi(y))) = \ln 2y,$$

and for all $x \in [0,1]$ *and* $y = 1$*, we have*

$$F(H(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty))) = \ln \frac{3}{2},$$
$$F(H(d(x,x)+\varphi(x)+\varphi(x))) = \ln 2.$$

Consequently, for every $0 < \tau < \ln \frac{4}{3}$*, we infer that*

$$\tau + F(H(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty))) \leq F(H(d(x,y)+\varphi(x)+\varphi(y)))$$

for all $x, y \in X$ *with* $H(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)) > 0$*, that is,* T *is an* F-H*-contraction.*

We establish the following auxiliary lemma.

Lemma 3.1. *Let* (X,d) *be a metric space and let* $T : X \to X$ *be an* F-H*-contraction with respect to the functions* $F \in \mathcal{F}$*,* $H \in \mathcal{H}$*,* $\varphi : X \to [0,+\infty[$ *and the real number* $\tau > 0$*. If* $\{x_n\}$ *is a sequence of Picard starting at* $x_0 \in X$*, then*

$$\lim_{n\to+\infty} H(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n)) = 0, \tag{3.3}$$

and hence

$$\lim_{n\to+\infty} d(x_{n-1},x_n) = 0 \quad \textit{and} \quad \lim_{n\to+\infty} \varphi(x_n) = 0. \tag{3.4}$$

Proof. Let x_0 be an arbitrary point in X and let $\{x_n\}$ be a sequence of Picard starting at $x_0 \in X$. Firstly, we assume that there exists $k \in \mathbb{N}$ such that $x_{k-1} = x_k$, then $x_n = x_k$ for all $n \geq k$. We claim that $H(d(x_{k-1},x_k),\varphi(x_{k-1}),\varphi(x_k)) = 0$.

Assume the contrary, that is, suppose $H(d(x_{k-1},x_k),\varphi(x_{k-1}),\varphi(x_k)) > 0$. We remark that

$$H(d(x_k,x_{k+1}),\varphi(x_k),\varphi(x_{k+1})) = H(d(x_{k-1},x_k),\varphi(x_{k-1}),\varphi(x_k)) > 0.$$

Using (3.2) with $x = x_{k-1}$ and $y = x_k$, we get

$$\begin{aligned}\tau + F(H(d(Tx_{k-1},Tx_k)),\varphi(Tx_{k-1}),\varphi(Tx_k))) &= \tau + F(H(0,\varphi(x_{k-1}),\varphi(x_{k-1})))\\ &\leq F(H(0,\varphi(x_{k-1}),\varphi(x_{k-1}))),\end{aligned}$$

which is a contradiction, since $\tau > 0$. So, $H(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n)) = 0$ for all $n \in \mathbb{N}$ with $n \geq k$. This ensures that (3.3) holds and, by the property (H_1) of the function H, (3.4) holds too.

Then, it is not restrictive to suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. By the property (H_1) of the function H, we obtain that

$$H(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n)) \geq d(x_{n-1},x_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Using (3.2), with $x = x_{n-1}$ and $y = x_n$, we deduce that

$$\tau + F(H(d(Tx_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(Tx_n))) \leq F(H(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n)))$$

for all $n \in \mathbb{N}$. The above inequality shows that

$$F(H(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1}))) < F(H(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n)))$$

for all $n \in \mathbb{N}$. Then, the property (F_1) of the function F implies that the sequence

$$\{H(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n))\}$$

is a decreasing sequence of positive real numbers. So, there exists some $l \geq 0$ such that

$$\lim_{n\to+\infty} H(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)) = l.$$

If $l = 0$, then the property (H_1) of the function H gives us

$$\lim_{n\to+\infty} d(x_{n-1}, x_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \varphi(x_{n-1}) = 0.$$

Now, suppose $l > 0$. Using (3.2), with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} F(H(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1}))) &\leq F(H(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))) - \tau \\ &\leq F(H(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) - n\tau \end{aligned}$$

for all $n \in \mathbb{N}$. From the previous inequality, passing to the limit as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} F(H(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1}))) = -\infty$$

and, using the property (F_2) of the function F, we get

$$\lim_{n\to+\infty} H(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = 0,$$

which leads to contradiction and hence $l = 0$. So, (3.3) and (3.4) hold. □

Remark 3.1. *Note that in the proof of Lemma 3.1, we use only the conditions* (F_1) *and* (F_2).

Now, we are ready to give the proof of Theorem 1.1. For reader convenience, we restate Theorem 1.1 in a classical fixed-point form.

Theorem 3.4. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be an* F-H*-contraction with respect to the functions* $F \in \mathcal{F}$, $H \in \mathcal{H}$, *the real number* $\tau > 0$ *and a lower semicontinuous function* $\varphi : X \to [0, +\infty[$ *such that* (3.2) *holds, that is,*

$$\tau + F(H(d(Tx, Ty), \varphi(Tx), \varphi(Ty))) \leq F(H(d(x, y), \varphi(x), \varphi(y)))$$

for all $x, y \in X$ *with* $H(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) > 0$. *Then,* T *has a unique fixed point* z *such that* $\varphi(z) = 0$.

Proof. We start with the proof of fixed-point uniqueness. Arguing by contradiction, we suppose that there exist $z, w \in X$ such that $z = Tz$, $w = Tw$ and $z \neq w$ (that is, T admits two distinct fixed points). The hypothesis $z \neq w$ ensures, by the property (H_1) of the function H, that

$$H(d(Tz, Tw), \varphi(Tz), \varphi(Tw)) \geq d(Tz, Tw) = d(z, w) > 0.$$

Now, using (3.2), with $x = z$ and $y = w$, we get that

$$\begin{aligned} \tau + F(H(d(Tz, Tw), \varphi(Tz), \varphi(Tw))) &= \tau + F(H(d(z, w), \varphi(z), \varphi(w))) \\ &\leq F(H(d(z, w), \varphi(z), \varphi(w))). \end{aligned}$$

Clearly, this is a contradiction, and hence we have, $w = z$. So, we obtain the claim.

The next step is to establish the existence of a fixed point. We consider a point $x_0 \in X$. Let $\{x_n\}$ be a sequence of Picard starting at x_0. We stress that if $x_{k-1} = x_k$ for some $k \in \mathbb{N}$, then $z = x_{k-1} = x_k = Tx_{k-1} = Tz$, that is, z is a fixed point of T such that $\varphi(z) = 0$. In fact, by Lemma 3.1, $H(d(x_{k-1}, x_k), \varphi(x_{k-1}), \varphi(x_k)) = 0$ and by the property (H_1) of the function H, we have $\varphi(z) = 0$. So, we can suppose that $x_{n-1} \neq x_n$ for every $n \in \mathbb{N}$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence. By Lemma 3.1, we say that

$$0 < h_{n-1} = H(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)) \to 0 \quad \text{as } n \to +\infty.$$

The property (F_3) of the function F ensures that there exists $k \in]0,1[$ such that $h_n^k F(h_n) \to 0$ as $n \to +\infty$. Using (3.2), with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} F(H(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1}))) &\leq F(H(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))) - \tau \\ &\leq F(H(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) - n\tau \end{aligned}$$

for all $n \in \mathbb{N}$, that is,

$$F(h_n) \leq F(h_{n-1}) - \tau \leq \cdots \leq F(h_0) - n\tau \quad \text{for all } n \in \mathbb{N}.$$

From

$$0 = \lim_{n \to +\infty} h_n^k F(h_n) \leq \lim_{n \to +\infty} h_n^k (F(h_0) - n\tau) \leq 0,$$

we deduce that

$$\lim_{n \to +\infty} h_n^k n = 0.$$

This ensures that the series $\sum_{n=1}^{+\infty} h_n$ is convergent. By the property (H_1) of the function H also the series $\sum_{n=1}^{+\infty} d(x_n, x_{n+1})$ is convergent, and hence $\{x_n\}$ is a Cauchy sequence. Now, since (X, d) is complete, there exists some $z \in X$ such that

$$\lim_{n \to +\infty} x_n = z.$$

By (3.4), taking into account that φ is a lower semicontinuous function, we get

$$0 \leq \varphi(z) \leq \liminf_{n \to +\infty} \varphi(x_n) = 0,$$

that is, $\varphi(z) = 0$. We assert that z is a fixed point of T. Clearly, z is a fixed point of T if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = z$ or $Tx_{n_k} = Tz$, for all $k \in \mathbb{N}$. Otherwise, we can assume that $x_n \neq z$ and $Tx_n \neq Tz$ for all $n \in \mathbb{N}$. So, using (3.2) with $x = x_n$ and $y = z$, we deduce that

$$\tau + F(H(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz))) \leq F(H(d(x_n, z), \varphi(x_n), \varphi(z))).$$

Since $\tau > 0$, this inequality leads to

$$H(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)) < H(d(x_n, z), \varphi(x_n), \varphi(z)) \quad \text{for all } n \in \mathbb{N},$$

and so

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + H(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)) \\ &< d(z, x_{n+1}) + H(d(x_n, z), \varphi(x_n), \varphi(z)) \end{aligned}$$

for all $n \in \mathbb{N}$.

Finally, letting $n \to +\infty$ in the above calculations and taking into account that H is continuous in $(0,0,0)$, we deduce that $d(z, Tz) \leq H(0,0,0) = 0$, that is, $z = Tz$. □

Imposing that F is a continuous function and relaxing the hypothesis (F_3), we establish the following result.

Theorem 3.5. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Assume that there exists a continuous function F that satisfies the conditions (F_1) and (F_2), a function $H \in \mathcal{H}$, a real number $\tau > 0$ and a lower semicontinuous function $\varphi : X \to [0, +\infty[$ such that (3.2) holds, that is,*

$$\tau + F(H(d(Tx, Ty), \varphi(Tx), \varphi(Ty))) \leq F(H(d(x, y), \varphi(x), \varphi(y)))$$

for all $x, y \in X$ with $H(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) > 0$. Then T, has a unique fixed point z such that $\varphi(z) = 0$.

Proof. Following the similar arguments as in the proof of Theorem 3.4, we obtain easily the uniqueness of the fixed point. In order to establish the existence of a fixed point, we consider a point $x_0 \in X$. Let $\{x_n\}$ be a sequence of Picard starting at x_0. Clearly if $x_{k-1} = x_k$ for some $k \in \mathbb{N}$, then $z = x_{k-1} = x_k = Tx_{k-1} = Tz$, that is, z is a fixed point of T such that $\varphi(z) = 0$ (see the proof of Theorem 3.4), and so, we have already done.

So, we can suppose that $x_{n-1} \neq x_n$ for every $n \in \mathbb{N}$. We claim that $\{x_n\}$ is a Cauchy sequence. We assume for way of contradiction that $\{x_n\}$ is not a Cauchy sequence. Then, there exist a positive real number ε and two sequences $\{m_k\}$ and $\{n_k\}$ such that

$$n_k > m_k \geq k \text{ and } d(x_{m_k}, x_{n_k}) \geq \varepsilon > d(x_{m_k}, x_{n_k-1}) \text{ for all } k \in \mathbb{N}.$$

By Lemma 3.1 and Remark 3.1, we say that $d(x_{n-1}, x_n) \to 0$, $\varphi(x_n) \to 0$ as $n \to +\infty$. This implies

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \varepsilon.$$

Now, the hypothesis that $d(x_{m_k}, x_{n_k}) > \varepsilon$ ensures that

$$H(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})) > 0 \quad \text{for all } k \in \mathbb{N}.$$

So, taking into account that H is a continuous function, we have

$$\begin{aligned}
&\lim_{k\to+\infty} H(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1})) \\
&= \lim_{k\to+\infty} H(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})) \\
&= H(\varepsilon, 0, 0) > 0.
\end{aligned}$$

Using again (3.2), with $x = x_{m_k-1}$ and $y = x_{n_k-1}$, we deduce that

$$\begin{aligned}
&\tau + F(H(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k}))) \\
&\leq F(H(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1})))
\end{aligned}$$

for all $k \in \mathbb{N}$. Letting $k \to +\infty$ in the previous inequality, since the function F is continuous, we get

$$\tau + F(H(\varepsilon, 0, 0))) \leq F(H(\varepsilon, 0, 0))),$$

which leads to contradiction. It follows that $\{x_n\}$ is a Cauchy sequence.

Now, since (X, d) is complete, there exists some $z \in X$ such that

$$\lim_{n\to+\infty} x_n = z.$$

By (3.4), taking into account that φ is a lower semicontinuous function, we get

$$0 \leq \varphi(z) \leq \liminf_{n\to+\infty} \varphi(x_n) = 0,$$

that is, $\varphi(z) = 0$. We assert that z is a fixed point of T. Clearly, z is a fixed point of T if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = z$ or $Tx_{n_k} = Tz$, for all $k \in \mathbb{N}$. Otherwise, we can assume that $x_n \neq z$ and $Tx_n \neq Tz$ for all $n \in \mathbb{N}$. Then, the property (H_1) of the function H ensures that $H(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)) > 0$ for all $n \in \mathbb{N}$. So, using (3.2), with $x = x_n$ and $y = z$, we deduce that

$$\tau + F(H(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz))) \leq F(H(d(x_n, z), \varphi(x_n), \varphi(z))) \text{ for all } n \in \mathbb{N}.$$

Since $\tau > 0$, we conclude that

$$H(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)) < H(d(x_n, z), \varphi(x_n), \varphi(z)) \quad \text{for all } n \in \mathbb{N},$$

and so

$$\begin{aligned} d(z,Tz) &\le d(z,x_{n+1}) + d(Tx_n,Tz) \\ &\le d(z,x_{n+1}) + H(d(Tx_n,Tz),\varphi(Tx_n),\varphi(Tz)) \\ &< d(z,x_{n+1}) + H(d(x_n,z),\varphi(x_n),\varphi(z)) \end{aligned}$$

for all $n \in \mathbb{N}$. Finally, letting $n \to +\infty$ and taking into account that H is continuous in $(0,0,0)$, we deduce that $d(z,Tz) \le H(0,0,0) = 0$, that is, $z = Tz$. $\square$

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