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Research Article

Strong Converse Inequalities and Quantitative Voronovskaya-Type Theorems for Trigonometric Fejér Sums

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ABSTRACT. Let σ_n denotes the classical Fejér operator for trigonometric expansions. For a fixed even integer r, we characterize the rate of convergence of the iterative operators $(I-\sigma_n)^r(f)$ in terms of the modulus of continuity of order r (with specific constants) in all $\mathbb{L}^p$ spaces $1 \leq p \leq \infty$. In particular, the constants depend not on p. Moreover, we present a quantitative version of the Voronovskaya-type theorems for the operators $(I-\sigma_n)^r(f)$.

Keywords: Fejér operators, iterative combinations, rate of convergence, direct and converse results, quantitative Voronovskaya-type theorems.

2010 Mathematics Subject Classification: 42A10, 42A20, 41A25, 41A27.

1. INTRODUCTION

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic, continuous functions f defined on the real line $\mathbb{R}$ with the sup norm

$$\|f\|_\infty = \max_{x\in[-\pi,\pi]} |\, f(x)\,| \,.$$

For $1 \leq p < \infty$, the Banach space $\mathbb{L}^p$ consists of all 2π-periodic, p-th power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\, f(x)\,|^p \, dx\right)^{1/p}.$$

In order to simplify, we write $X^p = \mathbb{L}^p$ for $1 \leq p < \infty$ and $X^\infty = C_{2\pi}$. As usual, for $r \in \mathbb{N}$, by W_p^r we mean the family of all functions $f \in X^p$ such that $f, \dots, D^{r-1}(f)$ are absolutely continuous and $D^r(f) \in X^p$. Here $D(f) = D^1(f) = f'$ and $D^{r+1}(f) = D(D^r(f))$.

For $f \in X^1$, the *conjugate function* is defined by

$$\widetilde{f}(x) = -\frac{1}{2\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{\tan(t/2)}dt = -\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_\varepsilon^\pi \frac{f(x+t)-f(x-t)}{\tan(t/2)}dt,$$

whenever the limit exists. It is know that if $f \in X^p$ with $1 < p < \infty$, then $\widetilde{f} \in X^p$, and that is not the case for $p = 1$ and $p = \infty$.

For $r \in \mathbb{N}$, function $f \in X^p$, and $h > 0$, the usual modulus of smoothness of order r of f is defined by

$$\omega_r(f,t)_p = \sup_{|h|\leq t} \|(I-T_h)^r(f)\|_p, \tag{1.1}$$

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where $T_h(f,x) = f(x+h)$ is the translation operator. We also use the notations $\Delta_h^r f(x) = (I - T_h)^r(f)$.

Let $\mathbb{T}_n$ denote the family of all real trigonometric polynomials of degree not greater than n. For $1 \le p \le \infty$ and $f \in X^p$, the best approximation of f by elements of $\mathbb{T}_n$ is defined by

$$E_{n,p}(f) = \inf_{T \in \mathbb{T}_n} \|f - T\|_p.$$

Recall that for $f \in \mathbb{L}^1$ and $k \in \mathbb{N}_0$, the Fourier coefficients are defined by

$$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt \quad \text{and} \quad b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt,$$

and the (formal) Fourier series is given by

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty}(a_k(f)\cos(kx) + b_k(f)\sin(kx)) = \sum_{k=0}^{\infty} A_k(f,x).$$

For $n \in \mathbb{N}$ and $f \in X^1$, the Fejér sum of order n is defined by

$$\sigma_n(f,x) = \sum_{k=0}^{n}\Big(1 - \frac{k}{n+1}\Big)A_k(f,x).$$

We also consider the conjugate operators

$$\widetilde{\sigma}_n(f,x) = \sum_{k=1}^{n}\Big(1 - \frac{k}{n+1}\Big)B_k(f,x),$$

where $B_k(f,x) = -b_k(f)\cos(kx) + a_k(f)\sin(kx)$.

In what follows the following notations are used: $\sigma_n^0 = I$ (identity operator), $\sigma_n^1(f) = \sigma_n(f)$ and $\sigma_n^{r+1} = \sigma_n(\sigma_n^r(f))$.

In [3], it was proved that if $1 \le p \le \infty$, $n > 1$, and $f \in X^p$, then

$$\frac{1}{2}\|f - \sigma_n(f)\|_p \le \widetilde{K}\Big(f, \frac{3}{n}\Big)_p \le 4\|f - \sigma_n(f)\|_p, \tag{1.2}$$

where

$$\widetilde{K}(f,t)_p = \inf\left\{\|f-g\|_p + t\|D(\widetilde{g})\|_p \;:\; g \in X^p,\, \widetilde{g} \in AC, D(\widetilde{g}) \in X^p\right\},$$

here $D(h) = h'$. Moreover, for $1 < p < \infty$, it was given a constant C_p such that, for each $f \in X^p$,

$$\frac{1}{2C_p}\|f - \sigma_n(f)\|_p \le \omega_1\Big(f, \frac{1}{n}\Big)_p \le 8C_p\|f - \sigma_n(f)\|_p. \tag{1.3}$$

On the other hand, in recent times there have been some interests in studying quantitative Voronovskaya-type theorems, but almost all the papers concern with positive linear operators in spaces of non-periodic functions. For instance; see [1], [2], [11], [12] and the references therein. The methods used in those papers are not useful in dealing with periodic functions for two reasons (at least). First, they use different kinds of Taylor's formula and second, in the non-periodical case do not appear conjugate functions.

It is known that the Voronovskaya-type theorems are related with the saturation class of some families of operators. We have noticed that in the case of trigonometric polynomial approximation process the Voronovskaya-type theorems depend on particular properties of the operators. In this paper we consider the Fejér operators (other approximation methods will be studied in forthcoming papers).

In this paper, two different kind of results are presented. First, we extend equation (1.2) by proving that, for an even integer r, $f \in X^p$ and $n \in \mathbb{N}$,

$$C_1 \omega_r\Big(f, \frac{\pi}{n}\Big)_p \le \|(I-\sigma_n)^r(f)\|_p \le C_2\, \omega_r\Big(f, \frac{\pi}{n}\Big)_p.$$

Second, we verify a quantitative Voronovskaya-type relation in the following form: $1 \le p \le \infty$, r is an even integer, $f \in W_p^r$, and $n \ge r$, then

$$\Big\|(n+1)^r (I-\sigma_n)^r(f) - (-1)^{r/2} D^r(f)\Big\|_p \le C_3 E_{n,p}(D^r(f)).$$

Here C_1, C_2 and C_3 are positive constants which will be given in an explicit form. These tasks will be accomplished in Sections 2 and 4, where other results are included. In order to present the estimate in Section 4 with specific constants, we need some results related with simultaneous approximation that will be proved in Section 3.

2. Strong inequalities for combinations of Fejér operators

First, we recall some known results that will be needed later.

Proposition 2.1. (i) *If $r \in \mathbb{N}$, $t > 0$ and $f \in X^p$, then $\omega_r(f,t)_p \le 2^r \|f\|_p$.*

(ii) ([15, page 103]) *If $r \in \mathbb{N}$ and $f \in W_p^r$, then*

$$\omega_r(f,t) \le t^r \|D^r(f)\|_p. \tag{2.4}$$

(iii) ([15, page 103]) *If $r, q \in \mathbb{N}$ and $f \in X^p$, then*

$$\omega_r(f,qt) \le q^r \omega_r(f,qt).$$

For a proof of (2.5), see [15, page 215]. The first inequality in (2.6) is a consequence of (2.5) (take $h = \pi/n$) and Proposition 2.1, but for a direct proof see [15, page 208].

Theorem 2.1. *If $1 \le p \le \infty$, $r, n \in \mathbb{N}$ and $T_n \in \mathbb{T}_n$, then*

$$\|T_n^{(r)}\|_p \le \Big(\frac{n}{2\sin(nh/2)}\Big)^r \|\Delta_h^r T_n\|_p, \tag{2.5}$$

for any $h \in (0, 2\pi/n)$. Moreover

$$\|D^r(T_n)\|_p \le n^r\,\|T\|_p \qquad \text{and} \qquad \|D^r(\widetilde{T}_n)\|_p \le n^r\,\|T_n\|_p. \tag{2.6}$$

The first inequality in (2.6) is easily derived from (2.5) by taking $h = \pi/n$ and using the assertion (i) in Proposition 2.1. Unfortunately, the second inequality is not included in some books, but a proof can be seen in [14, page 135] (notice that it is sufficient to verify the assertion for $r = 1$).

Proposition 2.2. (see [5, Cor. 1.2.4]) *If $1 \le p \le \infty$, $n \in \mathbb{N}$, and $f \in X^p$, then*

$$\|\sigma_n(f)\|_p \le \|f\|_p.$$

Let us show some algebraic relations related with iterates of Fejér operators.

Proposition 2.3. *For each $n, r \in \mathbb{N}$ and $f \in \mathbb{L}^1$ one has*

$$\sigma_n^r(f,x) = \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big)^r A_k(f,x)$$

and

$$(I-\sigma_n)^r(f) = f - \sum_{k=0}^{n} \Big(1 - \frac{k^r}{(n+1)^r}\Big) A_k(f).$$

Proof. By the orthogonality relations we know that, for $0 \leq k \leq n$, $A_k(\sigma_n(f)) = (1 - k/(n+1))A_k(f)$. For instance,

$$a_k(\sigma_n(f)) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_n(f,t)\cos(kt)dt = \sum_{k=0}^{n}\Big(1 - \frac{k}{n+1}\Big)\frac{1}{\pi}\int_{-\pi}^{\pi} A_k(f,t)\cos(kt)dt$$
$$= \Big(1 - \frac{k}{n+1}\Big)\frac{1}{\pi}\int_{-\pi}^{\pi} a_k(f)\cos^2(kt)dt = \Big(1 - \frac{k}{n+1}\Big)a_k(f).$$

With similar arguments, we can verified that $b_k(\sigma_n(f)) = (1 - k/(n+1))b_k(f)$.

Hence

$$\sigma_n^2(f) = \sum_{k=0}^{n}\Big(1 - \frac{k}{n+1}\Big)A_k(\sigma_n(f)) = \sum_{k=0}^{n}\Big(1 - \frac{k}{n+1}\Big)^2 A_k(f).$$

For other values of r, the proof follows by induction.

On the other hand,

$$(I - \sigma_n)^r(f) = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\sigma_n^j(f) = f + \sum_{j=1}^{r}(-1)^j\binom{r}{j}\sigma_n^j(f)$$
$$= f + \sum_{j=1}^{r}(-1)^j\binom{r}{j}\sum_{k=0}^{n}\Big(1 - \frac{k}{n+1}\Big)^j A_k(f)$$
$$= f + \sum_{k=0}^{n}\Big(\sum_{j=1}^{r}\binom{r}{j}\Big(\frac{k}{n+1} - 1\Big)^j\Big)A_k(f)$$
$$= f - \sum_{k=0}^{n}A_k(f) + \sum_{k=0}^{n}\Big(\sum_{j=0}^{r}\binom{r}{j}\Big(\frac{k}{n+1} - 1\Big)^j\Big)A_k(f)$$
$$= f - \sum_{k=0}^{n}A_k(f) + \sum_{k=0}^{n}\Big(\Big(\frac{k}{n+1} - 1 + 1\Big)^r\Big)A_k(f)$$
$$= f - \sum_{k=0}^{n}\Big(1 - \frac{k^r}{(n+1)^r}\Big)A_k(f). \qquad \square$$

Proposition 2.4. *For each $n, r \in \mathbb{N}$ and $T \in \mathbb{T}_n$ one has*

$$(I - \sigma_n)^r(T) = \begin{cases} \dfrac{(-1)^{r/2}}{(n+1)^r}D^r(T), & r \textit{ even}, \\[2ex] \dfrac{(-1)^{(r-1)/2}}{(n+1)^r}D^r(\widetilde{T}), & r \textit{ odd}. \end{cases}$$

Proof. It is easy to see that, for each polynomial $T \in \mathbb{T}_n$

$$T - \sigma_n(T) = \frac{D(\widetilde{T})}{n+1} \quad \text{and} \quad (I - \sigma_n)^2(T) = -\frac{1}{(n+1)^2}D^2(T). \tag{2.7}$$

Since $(I - \sigma_n)^r$ is a linear operator, we can consider only the case $T(x) = a\cos(kx) + b\sin(kx)$, where $a, b \in \mathbb{R}$ and $1 \leq k \leq n$. In such a case from Proposition 2.3, we know that $(I - \sigma_n)^r(T) = k^r T/(n+1)^r$.

If r is even, by induction one has

$$(I - \sigma_n)^{r+2}(T) = \frac{(-1)^{r/2}}{(n+1)^r}(I - \sigma_n)^2(D^r(T)) = \frac{(-1)^{(r+2)/2}}{(n+1)^r}D^{r+2}(T).$$

If $r > 1$ is odd, taking into account (2.7) one has

$$(I-\sigma_n)^r(T) = (I-\sigma_n)^{r-1}((I-\sigma_n)(T)) = \frac{1}{n+1}(I-\sigma_n)^{r-1}(D(\widetilde{T}))$$
$$= \frac{(-1)^{(r-1)/2}}{(n+1)^r}D^{r-1}(D(\widetilde{T})) = \frac{(-1)^{(r-1)/2}}{(n+1)^r}D^r(\widetilde{T}). \qquad \square$$

Theorem 2.2. *If $1 \le p \le \infty$, $f \in X^p$ and $n, r \in \mathbb{N}$, with r even, then*

$$\frac{1}{2^r + \pi^r(2^r+1)}\omega_r\left(f, \frac{\pi}{n+1}\right)_p \le \|(I-\sigma_n)^r(f)\|_p$$
$$\le \left(1 + 8r(6+\ln r)\right)\omega_r\left(f, \frac{\pi}{n+1}\right)_p.$$

Proof. (a) First inequality. If $f \in X^p$, fix $T \in \mathbb{T}_n$ such that $\|f-T\|_p = E_{n,p}(f)$. Since $(I-\sigma_n)^r(f) - f \in \mathbb{T}_n$, one has

$$E_{n,p}(f) = \|f-T\|_p \le \|f - ((I-\sigma_n)(f) - f)\|_p = \|(I-\sigma_n)(f)\|_p.$$

Thus, it follows from Propositions 2.1 and 2.4 that

$$\omega_r\left(f, \frac{\pi}{n+1}\right)_p \le 2^r\|f-T\|_p + \omega_r\left(T, \frac{\pi}{n+1}\right)_p \le 2^r\|f-T\|_p + \frac{\pi^r}{(n+1)^r}\|D^r(T)\|_p$$
$$= 2^r\|f-T\|_p + \pi^r\|(I-\sigma_n)^r(T)\|_p$$
$$\le 2^r\|f-T\|_p + \pi^r\left(2^r\|f-T\|_p + \|(I-\sigma_n)^r(f)\|_p\right)$$
$$\le \left(2^r + \pi(2^r+1)\right)\|(I-\sigma_n)^r(f)\|_p.$$

(b) Second inequality. Let T be given as in part (a). Using Propositions 2.1, 2.4 and 2.3 and equation (2.5) (with $h = \pi/n$), one has

$$\|(I-\sigma_n)^r(f)\|_p \le \|(I-\sigma_n)^r(f-T)\|_p + \|(I-\sigma_n)^r(T)\|_p$$
$$\le 2^r E_{n,p}(f) + \frac{1}{(n+1)^r}\|D^r(T)\|_p \le 2^r E_{n,p}(f) + \frac{1}{(n+1)^r}\left(\frac{n}{2}\right)^r\omega_r\left(T, \frac{\pi}{n}\right)_p$$
$$= (2^r+1)E_{n,p}(f) + \frac{1}{2^r}\omega_r\left(f, \frac{\pi}{n}\right)_p \le (2^r+1)E_{n,p}(f) + \omega_r\left(f, \frac{\pi}{n+1}\right)_p.$$

From [9, Theorem 6.1], we know that

$$E_{n,p}(f) \le \sqrt{r}\left(2\ln(r) + 12\right)\frac{([r/2]!)^2}{r!}\omega_r\left(f, \frac{\pi}{n+1}\right)_p.$$

It is known that (see [13]), for each $k \in \mathbb{N}$,

$$\frac{(k!)^2}{(2k)!} < \frac{\sqrt{\pi(k+1/2)}}{2^{2k}} = \frac{\sqrt{\pi(2k+1)}}{2^{2k}\sqrt{2}}.$$

Thus, if $r = 2k$, then

$$E_{n,p}(f) \le 2\sqrt{r}\left(\ln(r)+6\right)\frac{\sqrt{r\pi}}{2^r}\omega_r\left(f, \frac{\pi}{n+1}\right)_p \le \frac{8r}{2^{r+1}}\left(6+\ln r\right)\omega_r\left(f, \frac{\pi}{n+1}\right)_p$$

and

$$\|(I-\sigma_n)^r(f)\|_p \le \left(1 + 8r(6+\ln r)\right)\omega_r\left(f, \frac{\pi}{n+1}\right)_p. \qquad \square$$

Remark 2.1. In [8], Ditzian and Ivanov said that the investigation of the rate of convergence of $(I-\sigma_n)^r(f)$ to 0 can be handled via some Riesz means, but no details were given. Anyway, the estimate in [8] for the typical means were presented in terms of a K-functional. Our approach is direct, and do not use K-functionals.

3. Simultaneous approximation

Let us recall a Favard theorem.

Theorem 3.3. ([15, p. 289,]) *If* $1 \le p \le \infty$, $r, n \in \mathbb{N}$ *and* $g \in W^r_p$, *then*

$$E_{n,p}(g) \le \frac{\pi}{2(n+1)^r} E_{n,p}(D^r(g)) \tag{3.8}$$

and

$$E_{n,p}(\widetilde{g}) \le \frac{\pi}{2(n+1)^r} E_{n,p}(D^r(g)). \tag{3.9}$$

Theorem 3.3 is not written in the usual form of the Favard theorem. But, as Czipszer and Freud noticed [6, page 37], the inequalities presented above can be deduced from the original Favard ones.

It seems that the main results in simultaneous approximation by trigonometric polynomials are due to Czipszer and Freud [6]. They presented the arguments for continuous function, but (as some authors usually do) they explained that all the results hold in $\mathbb{L}^p$ spaces [6, pages 49-51]. Previously, Freud [10] announced the estimate

$$\|D(f) - D(T_n)\|_\infty \le C_r\Big(n^r \|f - T_n\|_\infty + E_{n,\infty}(D^r(f))\Big), \qquad f \in C^r_{2\pi},$$

where $T_n \in \mathbb{T}_n$ is the polynomial of the best approximation for f.

Here, we prefer to include a complete proof of Theorem 3.4 for several reasons. First, in [6] several details are omitted for X^p with $1 \le p < \infty$ and no information is included concerning the constants. It follows from our proof that the constants in Theorem 3.4 are not the best possible, but in application to the analysis of Fejér operators they provide *reasonable* estimates.

For the proofs, we need some auxiliary operators. The original idea goes back to de la Vallée Poussin [7].

Definition 3.1. *Fix* $r \in \mathbb{N}$. *For* $n \in \mathbb{N}$, $n \ge r$, *and* $f \in X^1$ *define*

$$C_{n,r}(f,x) = \frac{1}{[n(1+1/r)] - n} \sum_{k=n}^{[n(1+1/r)]-1} S_k(f,x).$$

Taking into account that $\widetilde{S}_n(f,x) = \sum_{k=1}^n B_k(f,x)$, we also set

$$\widetilde{C}_{n,r}(f,x) = \frac{1}{[n(1+1/r)] - n} \sum_{k=n}^{[n(1+1/r)]-1} \widetilde{S}_k(f,x).$$

We collect some properties of the sums $C_{n,r}$. The results are taken from [6, page 46].

Proposition 3.5. *Fix* $r \in \mathbb{N}$ *and set* $q = 1 + 1/r$. *If* $f \in \mathbb{L}^1$, $n \in \mathbb{N}$, *and*

$$I_{n,r}(x) = \frac{1}{[n/r]} \frac{\sin([n(2+1/r)]t/2)\sin([n/r]t/2)}{\sin^2(t/2)},$$

then $C_{n,r}(f) \in \mathbb{T}_{[nq]-1}$ *and*

$$C_{n,r}(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t) I_{n,r}(t)dt.$$

Proposition 3.6. *If $r, n \in \mathbb{N}$, $n \geq r$, $f, \widetilde{f} \in \mathbb{L}^1$, $g \in W_1^1$ and $T_n \in \mathbb{T}_n$, then*

$$C_{n,r}(\widetilde{f}, x) = \widetilde{C}_{n,r}(f, x), \qquad C_{n,r}(g', x) = \frac{d}{dx} C_{n,r}(g, x)$$

and $C_{n,r}(T_n, x) = T_n(x)$.

The next result gives an estimate of the norm of the operator $C_{n,r}$.

Proposition 3.7. *If $r, n \in \mathbb{N}$, and $n \geq r$, then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mid I_{n,r}(t) \mid dt \leq 3 + \frac{1}{r} + \ln(2r).$$

Proof. First, we write

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mid I_{n,r}(t) \mid dt = \frac{2}{\pi} \int_{0}^{\pi/2} \mid I_{n,r}(2t) \mid dt.$$

We split the interval $[0, \pi/2]$ by considering $A_1 = [0, \pi/(2n)]$,

$$A_2 = [\pi/(2n), \pi/(2[n/r])] \qquad \text{and} \qquad A_3 = [\pi/(2[n/r]), \pi/2].$$

Notice that

$$1 \leq n/r \leq n \qquad \text{and} \qquad 1 \leq [n/r] \leq n.$$

Taking into account that $\mid \sin(nx) \mid \leq n \sin x$, we obtain

$$\frac{2}{\pi} \int_{A_1} \mid I_{n,r}(2t) \mid dt \leq \frac{2}{\pi [n/r]} [n(2 + 1/r)][n/r] \int_{0}^{\pi/(2n)} dt \leq 2 + 1/r.$$

On the other hand, since $2x \leq \pi \sin x$, one has

$$\frac{2}{\pi} \int_{A_2} \mid I_{n,r}(2t) \mid dt \leq \frac{2}{\pi [n/r]} \frac{\pi}{2} [n/r] \int_{\pi/(2n)}^{\pi/(2[n/r])} \frac{dt}{t}$$
$$= \ln \frac{n}{[n/r]} < \ln 2 + \ln r,$$

because if $n \geq \max\{2, r\}$ and $n = qr + \theta$, with $q \in \mathbb{N}$ and $\theta \in [0, 1)$, then $2\theta < 2 \leq n = qr + \theta$. Hence, $qr > \theta$ and $n = qr + \theta < 2rq = 2r[n/r]$.

Finally,

$$\frac{2}{\pi} \int_{A_3} \mid I_{n,r}(2t) \mid dt \leq \frac{2}{\pi [n/r]} \left(\frac{\pi}{2}\right)^2 \int_{\pi/(2[n/r])}^{\pi/2} \frac{dt}{t^2}$$
$$\leq \frac{2}{\pi [n/r]} \left(\frac{\pi}{2}\right)^2 \frac{2[n/r]}{\pi} = 1.$$

Therefore,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mid I_{n,r}(t) \mid dt \leq 3 + \frac{1}{r} + \ln(2r). \qquad \square$$

Proposition 3.8. *If $f \in X^p$ $(1 \leq p \leq \infty)$, and $n, r \in \mathbb{N}$, $n \geq \max\{2, r\}$, then*

$$\|f - C_{n,r}(f)\|_p \leq \Big(4 + \frac{1}{r} + \ln(2r)\Big) E_{n,p}(f).$$

Proof. If $T \in \mathbb{T}_n$ and $\|f - T\|_p = E_{n,p}(f)$, then

$$\|f - C_{n,r}(f)\|_p = \|f - T - C_{n,r}(f - T)\|_p$$
$$\leq (1 + \|C_{n,r}\|_1) \|f - T\|_p \leq \Big(4 + \frac{1}{r} + \ln(2r)\Big) E_{n,p}(f). \qquad \square$$

Theorem 3.4. *Assume* $1 \leq p \leq \infty$, $r, n \in \mathbb{N}$ *and* $n \geq \max\{2, r\}$.
(i) *If* $g \in W_p^r$, $T \in \mathbb{T}_n$, *and* $\|g - T\|_p = E_{n,p}(g)$, *then*

$$\|D^r(g) - D^r(T)\|_p \leq \left(4 + \frac{1}{r} + \ln(2r)\right)\left(1 + \frac{e\pi}{2}\right) E_{n,p}(D^r(g)).$$

(ii) *If* $g, \widetilde{g} \in W_p^r$, $T \in \mathbb{T}_n$, *and* $\|g - T\|_p = E_{n,p}(g)$, *then*

$$\|D^r(\widetilde{g}) - D^r(\widetilde{T})\|_p \leq \left(4 + \frac{1}{r} + \ln(2r)\right)\left(E_{n,p}(D^r(\widetilde{g})) + \frac{e\pi}{2} E_{n,p}(D^r(g))\right).$$

Proof. (i) Let $C_{n,r}$ be given as in Definition 3.1. Notice that

$$([n(1 + 1/r)] - 1)^r < n^r \left(1 + \frac{1}{r}\right)^r < en^r, \qquad n \geq r.$$

From Propositions 3.7, 3.8 and 3.6, we obtain the following inequalities

$$\|C_{n,r}(g) - T\|_p = \|C_{n,r}(g - T)\|_p \leq \left(4 + \frac{1}{r} + \ln(2r)\right)\|g - T\|_p$$

$$\leq \left(4 + \frac{1}{r} + \ln(2r)\right)\frac{\pi}{2(n+1)^r} E_{n,p}(g^{(r)}),$$

$$\|D^r(g) - C_{n,r}(D^r(g))\|_p \leq \left(4 + \frac{1}{r} + \ln(2r)\right) E_{n,p}(D^r(g)),$$

and

$$\|D^r(C_{n,r}(g)) - D^r(T)\|_p \leq ([n(1 + 1/r)] - 1)^r \|C_{n,r}(g) - T\|_p$$

$$\leq \frac{e\pi}{2}\left(4 + \frac{1}{r} + \ln(2r)\right) E_{n,p}(D^r(g)).$$

Therefore,

$$\|D^r(g) - D^r(T)\|_p \leq \|D^r(g) - C_{n,r}(D^r(g))\|_p + \|D^r(C_{n,r}(g)) - D^r(T)\|_p$$

$$\leq \left(4 + \frac{1}{r} + \ln(2r)\right)\left(1 + \frac{e\pi}{2}\right) E_{n,p}(D^r(g)).$$

(ii) For the conjugate function, we consider the relations

$$C_{n,r}(D^r(\widetilde{g})) = D^r(\widetilde{C}_{n,r}(g))$$

and

$$\|D^r(\widetilde{g}) - D^r(\widetilde{T})\|_p \leq \|D^r(\widetilde{g}) - C_{n,r}(D^r(\widetilde{g}))\|_p + \|D^r(\widetilde{C}_{n,r}(g) - \widetilde{T})\|_p$$

$$\leq \left(4 + \frac{1}{r} + \ln(2r)\right) E_{n,p}(D^r(\widetilde{g})) + en^r \|C_{n,r}(g) - T\|_p$$

$$\leq \left(4 + \frac{1}{r} + \ln(2r)\right)\left(E_{n,p}(D^r(\widetilde{g})) + \frac{e\pi}{2} E_{n,p}(D^r(g))\right),$$

where we have used Theorem 2.1. $\square$

4. Voronovskaya-type theorems

Zamansky [16, Th. 14] proved that if $f \in C^1_{2\pi}$, then

$$\lim_{n\to\infty} \|(n+1)\Big(\widetilde{\sigma}_n(f) - \widetilde{f}\Big) - D(f)\|_\infty = 0.$$

Later he verified that [17, Theorem 1], if $E_n(f)_\infty \le C/n^r$, $r > 1$, then

$$\lim_{n\to\infty} \|n(\sigma_n(f) - f) + D(\widetilde{f})\|_\infty = 0.$$

The extension of the last inequality to $\mathbb{L}^p$ spaces, $1 < p < \infty$, was given by Butzer and Görlich in [4, page 386]. The result is presented in Corollary 4.1 below. We remark that the proof presented below is simpler than the ones of Zamansky and it provides the rate of convergence. We consider first the Fejér operators and later the iterative combination.

As usual, we set

$$S_n(f,x) = \sum_{k=0}^{n} A_k(f,x)$$

for the partial sums of the Fourier series of f.

Theorem 4.5. *If $1 \le p \le \infty$, $f, \widetilde{f} \in W^1_p$, and $n > 1$, then*

$$\Big\|(n+1)(\sigma_n(f) - f) + D(\widetilde{f})\Big\|_p \le (1+3e)\pi\, E_{n,p}(D(f)) + 6\, E_{n,p}(D(\widetilde{f}))$$

and

$$\Big\|(n+1)(\widetilde{\sigma}_n(f) - \widetilde{f}) - D(f)\Big\| \le (1+3e)\pi E_{n,p}(D(\widetilde{f})) + 6\, E_{n,p}(D(f)).$$

Proof. If $T_n \in \mathbb{T}_n$ satisfies $E_{n,p}(f) = \|f - T_n\|_p$, then taking into account (2.7), Theorem 3.4 and equation (3.8) one has

$$\begin{aligned}
\Big\|\sigma_n(f) - f + \frac{D(\widetilde{f})}{n+1}\Big\|_p &= \Big\|\sigma_n(f - T_n) - (f - T_n) + \frac{D(\widetilde{f}) - D(\widetilde{T}_n)}{n+1}\Big\|_p \\
&\le 2E_{n,p}(f) + \frac{5+\ln(2)}{n+1}\Big(E_{n,p}(D(\widetilde{f})) + \frac{e\pi}{2}\, E_{n,p}(D(f))\Big) \\
&\le \frac{\pi}{n+1} E_{n,p}(D(f)) + \frac{5+\ln(2)}{n+1}\Big(E_{n,p}(D(\widetilde{f})) + \frac{e\pi}{2}\, E_{n,p}(D(f))\Big) \\
&\le \Big(1+3e\Big)\frac{\pi}{(n+1)} E_{n,p}(D(f)) + \frac{6}{n+1}\, E_{n,p}(D(\widetilde{f})).
\end{aligned}$$

Set $g = \widetilde{f}$. Let us verify that $\widetilde{\sigma}_n(f) = \sigma_n(g)$ a.e. and $D(\widetilde{g})(x) = -D(f)(x)$ a.e. In fact, if

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k(f)\sin(kx)),$$

then

$$\widetilde{f}(x) \sim \sum_{k=1}^{\infty}(-b_k \cos(kx) + a_k(f)\sin(kx)).$$

Thus,

$$\widetilde{\sigma}_n(f,x) = \sum_{k=1}^{n}\Big(1 - \frac{k}{n+1}\Big)(-b_k \cos(kx) + a_k(f)\sin(kx)) = \sigma_n(\widetilde{f},x).$$

On the other hand,

$$D(f)(x) \sim \sum_{k=1}^{\infty} k(b_k \cos(kx) - a_k(f)\sin(kx)),$$

$$\widetilde{g}(x) \sim -\sum_{k=1}^{\infty}(a_k \cos(kx) + b_k(f)\sin(kx))$$

and

$$D(\widetilde{g})(x) \sim -\sum_{k=1}^{\infty} k(-b_k \cos(kx) + a_k(f)\sin(kx)) \sim -D(f)(x).$$

Therefore

$$\widetilde{\sigma}_n(f) - \widetilde{f} - \frac{D(f)}{n+1} = \sigma_n(g) - g + \frac{D(\widetilde{g})}{n+1}.$$

Since $g = \widetilde{f} \in W_p^1$ and $\widetilde{g} = -f \in W_p^1$, one has

$$\Big|\Big|(n+1)(\widetilde{\sigma}_n(f) - \widetilde{f}) + D(f)\Big|\Big| \leq \pi(1+3e)\, E_{n,p}(D(g)) + 6\, E_{n,p}(D(\widetilde{g}))$$

$$= \pi(1+3e)\, E_{n,p}(D(\widetilde{f})) + 6\, E_{n,p}(D(f))$$

and this proves the result. □

Corollary 4.1 is a simple consequence of the previous result. Recall that, for $1 < p < \infty$, the element $\widetilde{f}$ always exists.

Corollary 4.1. *If $1 < p < \infty$ and $f \in W_p^1$, then*

$$\lim_{n\to\infty} \Big|\Big|(n+1)(\sigma_n(f) - f) - D(\widetilde{f})\Big|\Big|_p = 0.$$

A result similar to Theorem 4.5 can be proved for the linear combination $(I - \sigma_n)^r$ of Fejér operators. Here, we only consider the case of even r.

Theorem 4.6. *If $1 \leq p \leq \infty$, $r \in \mathbb{N}$ is even, $f \in W_p^r$, and $n \geq r$, then*

$$\Big|\Big|(n+1)^r(I-\sigma_n)^r(f) - (-1)^{r/2}D^r(f)\Big|\Big|_p \leq \Big(2^{r+1} + 7(5+\ln(2r))\Big)E_{n,p}(D^r(f)).$$

Proof. If $T \in \mathbb{T}_n$ is chosen from the condition $E_{n,p}(f) = \|f - T\|_p$, from Proposition 2.4 one has

$$(I-\sigma_n)^r(f) - \frac{(-1)^{r/2}}{(n+1)^r}D^r(f) = (I-\sigma_n)^r(f-T) - \frac{(-1)^{r/2}}{(n+1)^r}D^r(f-T),$$

and it follows from (3.8) and Theorem 3.4 (here the condition $n \geq r$ is needed) that

$$\|(I-\sigma_n)^r(f) - \frac{(-1)^{r/2}}{(n+1)^2}D^r f\|_p \leq 2^r E_{n,p}(f) + \frac{1}{(n+1)^r}\|D^r(f-T)\|_p$$

$$\leq \frac{2^{r-1}\pi}{(n+1)^r}E_{n,p}(D^r(f)) + \left(4 + \frac{1}{r} + \ln(2r)\right)\left(1 + \frac{e\pi}{2}\right)\frac{1}{(n+1)^r}E_{n,p}(D^r(f))$$

$$\leq \frac{2^{r-1}\pi}{(n+1)^r}E_{n,p}(D^r(f)) + \Big(5 + \ln(2r)\Big)\frac{(1+2e)}{(n+1)^r}E_{n,p}(D^r(f)).$$ □

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Research Article

The Product of Two Functions Using Positive Linear Operators

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ABSTRACT. In this paper, we estimate the speed of convergence of the difference $L_n(fg) - (L_n f) \cdot (L_n g)$ towards 0, where (L_n) are positive linear operators used in the approximation of continuous functions. We also study in what conditions the formula $L_n'(fg) - fL_n'g - gL_n'f \to 0$ holds true.

Keywords: Positive linear operators, exponential type operators, Voronovskaya formula, Chebyshev-Grüss functional, Baskakov operators, Jain operators, Balász operators.

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1. Introduction

In the last period of time, it was investigated the following difference

$$L_n(fg)(x) - (L_n f)(x) \cdot (L_n g)(x),$$

a generalization to positive linear operators of an expression appearing in the classical inequalities of Chebyshev [15] and Grüss [24]. Starting with the papers [10, 19] and [7, 33], these celebrated inequalities were extended to the case of positive linear functionals and positive linear operators. Bounds for this difference were given using different methods (see [22, 20, 21]). Asymptotic results of Voronovskaya type for this Chebyshev-Grüss quantity were obtained in [18, 6, 16, 35, 9, 30] for different operators.

In this paper, we give a quantitative result of Voronovskaya type for the Chebyshev-Grüss expression for a large class of positive linear operators and for a large class of continuous functions. Our results, presented in Section 3, do not need as in [9, 30] the hypothesis of the existence of the second derivatives of the functions involved.

In Section 4, we study in what conditions do the differentiation formula $L_n'(fg) - fL_n'g - gL_n'f$ converges to zero. We generalize the result of Impens and Gavrea [27], which was given for Bernstein type operators and for functions defined on a compact interval. Using another approach, we extend the result to larger class of positive linear operators and to a larger class of continuous functions, including bounded and unbounded functions. We also give a Voronovskaya type result for the differential formula just mentioned.

In Section 2, we present a class of positive linear operators which is defined using a Chebyshev-Grüss expression. This class, which was introduced in [26], contains the Bernstein type operators [31], but is much larger, including also positive linear operators which do not preserve linear functions. Some examples are given in the final section of the paper as applications of the results obtained.

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2. Properties of the operators defined by a Chebyshev-Grüss quantity

For the value of $L_n f$ in $x \in I$, we use the notations

$$(L_n f)(x) = L_n(f)(x) = L_n(f, x) = L_n(f(t), x)$$

interchangeably.

Consider a sequence of positive linear operators (L_n) which preserve the constants and which is defined by the following relation involving a Chebyshev-Grüss type expression

$$b_n \left[L_n(tf(t), x) - L_n(t, x) \cdot L_n(f(t), x)\right] = b(x) \left(L_n f\right)'(x) \tag{2.1}$$

for every $x \in I$, where $I \subset \mathbb{R}$ is an interval, for every $n \in \mathbb{N}$ and for every f for which $L_n f$ and $(L_n f)'$ exist, where $b(x)$ is a positive function which is differentiable on I and (b_n) is a sequence of positive real numbers such that $b_n \to \infty$. Concerning the domain of definition of the operators L_n, we will give explanations in the next section.

Remark 2.1. *If the operators L_n preserve the linear functions, then the condition (2.1) can be written*

$$\left(L_n f\right)'(x) = \frac{b_n}{b(x)} \cdot L_n((e_1 - x)f, x),$$

which is satisfied by the class of so called exponential operators (see [31] and [28]), in particular Bernstein polynomials, the operators of Szász-Mirakyan, Baskakov, Post-Widder and Gauss-Weierstrass. Condition (2.1) characterizes a more general class of operators, which do not necessarily preserve linear functions. Other examples will be given at the end of the paper. A relation equivalent with (2.1) is

$$b_n \left[L_n((t-x)f(t), x) - L_n(t-x, x) \cdot L_n(f(t), x)\right] = b(x) \left(L_n f\right)'(x),$$

a relation obtained in [36] for a particular kind of operators.

Remark 2.2. *If we consider a function $f = g(t, x)$ which depends on x and t and which has a partial derivative with respect to x in every point (t, x), then, condition (2.1) can be written*

$$\begin{aligned} &b_n \left[L_n((t-x)g(t,x), x) - L_n(t-x, x) \cdot L_n(g(t,x), x)\right] \\ &\quad = b(x) \left[\left(L_n(g(t,x), x)\right)' - L_n\left(\frac{\partial g}{\partial x}(t,x), x\right)\right], \end{aligned}$$

where the operator L_n acts on the variable t. In particular, for $f = (t-x)^k$, $k \geq 1$ we obtain

$$b_n \cdot \left[\mu_{n,k+1}(x) - \mu_{n,1}(x)\mu_{n,k}(x)\right] = b(x) \left[\mu'_{n,k}(x) + k\mu_{n,k-1}(x)\right], \tag{2.2}$$

where $\mu_{n,k}(x) = L_n((t-x)^k, x)$ are the central moments of order k for the operator L_n. This recurrence expresses all the central moments in terms of only one function, namely $\mu_{n,1}$, since the value of $\mu_{n,0}$ is known: $\mu_{n,0}(x) = 1$.

Let us suppose that

$$\lim_{n\to\infty} a_n \cdot \frac{d^i}{dx^i}\mu_{n,1}(x) = a^{(i)}(x) \tag{2.3}$$

is true for every $x \in I$, and $i = 0, 1, 2, \ldots$, where $a(x)$ is an infinitely differentiable function on I and (a_n) is an increasing and unbounded sequence of positive real numbers.

Lemma 2.1. *If the sequence (b_n/a_n) converges to the real number $c \geq 0$, then for every integer $\ell \geq 0$ we have*

$$\lim_{n\to\infty} b_n^{\ell} \cdot \mu_{n,2\ell}(x) = \left(b(x)\right)^{\ell} \cdot (2\ell-1)!! \tag{2.4}$$

$$\lim_{n\to\infty} b_n^{\ell+1} \cdot \mu_{n,2\ell+1}(x) = \left(b(x)\right)^{\ell} (2\ell)!! \sum_{i=0}^{\ell} \frac{(2i-1)!!}{(2i)!!} [ib'(x) + c\cdot a(x)] \tag{2.5}$$

uniformly for x in a compact interval contained in I. We have used the notations

$$(2\ell-1)!! = 1\cdot 3 \dots (2\ell-1) \text{ and } (2\ell)!! = 2\cdot 4 \cdots (2\ell), \quad \ell \geq 1$$

and for $\ell = 0$ the value is 1.

Proof. The proof will be omitted since it is similar to the one found in Lemma 2 [26]. □

3. Quantitative Voronovskaya-type result for Chebyshev-Grüss expression

In this section, we are concerned with the asymptotic behaviour of the Grüss-Chebyshev expression, which will be denoted

$$T_n(f,g)(x) = L_n(fg)(x) - (L_n f)(x)\cdot(L_n g)(x).$$

We will prove that $b_n \cdot T_n(f,g)(x) \to b(x)f'(x)g'(x)$ and we will estimate the speed of this convergence. We show that such a result is valid for unbounded functions, too. In order to do this, let us introduce some notations.

Let $\theta : [0,\infty) \to \mathbb{R}$ be a uniformly continuous and monotonic function, let I be an interval $I \subset \mathbb{R}$ and let $\alpha \geq 0$ be a real number. We denote by $C_{\theta,\alpha}$ the space of continuous functions $f \in C(I)$ with the property that exists $M > 0$ such that $|f(x)| \leq Me^{\alpha\theta(|x|)}$, for every $x \in I$. Because of the symmetry and to simplify the notation, we consider in the following that $I \subset [0,\infty)$. This space $C_{\theta,\alpha}$ can be endowed with the norm

$$\|f\|_{\theta,\alpha} = \sup_{x\in I} e^{-\alpha\theta(x)}|f(x)|.$$

Lemma 3.2. *Consider a sequence of positive linear operators $L_n : C_{\theta,\alpha} \to C(I)$ such that*

$$\lim_{n\to\infty} L_n(e^{\alpha\theta(t)}, x) = e^{\alpha\theta(x)}. \tag{3.6}$$

Then, there is a positive function $M_\alpha(x)$ not depending on n such that

$$L_n\left(\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right), x\right) \leq M_\alpha(x), \quad n \geq n_0.$$

Proof. For $x \in I$, there is n_0 such that $\left|L_n(e^{\alpha\theta(t)}, x) - e^{\alpha\theta(x)}\right| \leq 1$ for every $n \geq n_0$. We obtain

$$L_n(\max(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}), x) \leq L_n(e^{\alpha\theta(t)} + e^{\alpha\theta(x)}, x) \leq 1 + 2e^{\alpha\theta(x)},$$

for every $n \geq n_0$. □

We will use the following weighted modulus

$$\omega_{\theta,\alpha}(f,\delta) = \sup_{\substack{x,t\in I \\ |t-x|\leq\delta}} \frac{|f(t)-f(x)|}{\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)},$$

which is suitable for functions from the space $C_{\theta,\alpha}$ (see [25]).

Theorem 3.1. *Let $f, g \in C_{\theta,\alpha}$ be continuously differentiable functions such that $f'(x)e^{-\alpha\theta(x)}$ and $g'(x)e^{-\alpha\theta(x)}$ are uniformly continuous on I. Let $L_n : C_{\theta,\alpha} \to C^1(I)$ be a sequence of positive linear operators preserving constant functions and having the properties (2.1), (2.3) and (3.6). Then, for some $n_0 \in \mathbb{N}$ and for every $n \geq n_0$ and $x \in I$, we have*

$$\begin{aligned}
&|b_n[L_n(fg)(x) - (L_nf)(x)\cdot(L_ng)(x)] - b(x)f'(x)g'(x)| \\
&\quad \leq \frac{M(x)}{a_n}\,|b(x)f'(x)g'(x)| + M(x)\,\omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right) \\
&\quad\quad + M(x)\left(|f'(x)|\,\omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right) + |g'(x)|\,\omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right)\right),
\end{aligned}$$

where $M(x) > 0$ does not depend on n and f.

Proof. Using the Taylor formula of the first order with Lagrange remainder, we obtain

$$\begin{aligned}
f(t) &= f(x) + f'(x)\cdot(t-x) + R_f, \quad R_f = (f'(c_1) - f'(x))\cdot(t-x) \\
g(t) &= g(x) + g'(x)\cdot(t-x) + R_g, \quad R_g = (g'(c_2) - g'(x))\cdot(t-x),
\end{aligned}$$

with c_1, c_2 between t and x. We multiply the relations and we apply the operators L_n. We get

$$\begin{aligned}
L_n(fg)(x) &= f(x)g(x)\mu_{n,0}(x) + [f(x)g'(x) + g(x)f'(x)]\mu_{n,1}(x) + L_n(R_fR_g)(x) \\
&\quad + f'(x)g'(x)\mu_{n,2}(x) + f(x)L_n(R_g)(x) + g(x)L_n(R_f)(x) \\
&\quad + f'(x)L_n((e_1 - xe_0)R_g)(x) + g'(x)L_n((e_1 - xe_0)R_f)(x).
\end{aligned}$$

We also have

$$\begin{aligned}
L_nf(x) &= f(x)\mu_{n,0}(x) + f'(x)\mu_{n,1}(x) + L_n(R_f)(x), \\
L_ng(x) &= g(x)\mu_{n,0}(x) + g'(x)\mu_{n,1}(x) + L_n(R_g)(x).
\end{aligned}$$

We get

$$\begin{aligned}
L_n(fg)(x) - (L_nf)(x)\cdot(L_ng)(x) &= f'(x)g'(x)[\mu_{n,2}(x) - \mu_{n,1}^2(x)] \\
&\quad + f'(x)[L_n((t-x)R_g)(x) - \mu_{n,1}(x)L_n(R_g)(x)] \\
&\quad + g'(x)[L_n((t-x)R_f)(x) - \mu_{n,1}(x)L_n(R_f)(x)] \\
&\quad + L_n(R_fR_g)(x) - L_n(R_f)(x)\cdot L_n(R_g)(x).
\end{aligned}$$

Because $b_n[\mu_{n,2}(x) - \mu_{n,1}^2(x)] = b(x)[1 + \mu'_{n,1}(x)]$, we have

$$\left|b_n[\mu_{n,2}(x) - \mu_{n,1}^2(x)] - b(x)\right| = \left|b(x)\mu'_{n,1}(x)\right| = \frac{\left|b(x)a_n\mu'_{n,1}(x)\right|}{a_n}.$$

We evaluate now the remainder from the Taylor formula using the modulus of continuity $\omega_{\theta,\alpha}$. From

$$\begin{aligned}
|R_f| &= |t-x|\cdot|f'(c_1) - f'(x)| \\
&\leq \max(e^{\alpha\theta(t)}, e^{\alpha\theta(x)})\,|t-x|\left(1 + \frac{|t-x|}{\delta}\right)\omega_{\theta,\alpha}(f', \delta),
\end{aligned}$$

we obtain

$$|(L_nR_f)(x)| \leq \left(A_{n,1}(x) + \sqrt{b_n}A_{n,2}(x)\right)\omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right),$$

where

$$A_{n,k}(x) = L_n\left(\max\left(e^{\alpha t}, e^{\alpha x}\right)|t-x|^k, x\right). \tag{3.7}$$

Because $L_n(e^{\alpha\theta(t)}, x)$ converges pointwise to $e^{\alpha\theta(x)}$ we have

$$L_n\left(\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right), x\right) \le M_\alpha(x), \quad n \ge n_0.$$

From Lemma 2.1, we have

$$L_n\left(|t-x|^{2k}, x\right) = \frac{1}{b_n^k} \cdot b_n^k \mu_{n,2k}(x) \le \frac{C_k(x)}{b_n^k}, \quad n \ge n_0.$$

Using the Cauchy-Schwarz inequality for positive linear operators we obtain for $k = 1, 2$

$$A_{n,k}(x) \le \sqrt{L_n\left(\max\left(e^{2\alpha t}, e^{2\alpha x}\right), x\right)} \cdot \sqrt{L_n\left(|t-x|^{2k}, x\right)} \le \frac{\sqrt{M_{2\alpha}(x) C_k(x)}}{\sqrt{b_n^k}}.$$

In conclusion,

$$|(L_n R_f)(x)| \le \frac{M_{0,2}(x)}{\sqrt{b_n}} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right).$$

Similarly,

$$|L_n((e_1 - xe_0)R_f, x)| \le L_n\left(|t-x|\,|R_f|, x\right) \le \frac{M_{0,3}(x)}{b_n} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right).$$

So, using $|\mu_{n,1}(x)| \le \sqrt{\mu_{n,2}(x)} \le \sqrt{C_2(x)}/\sqrt{b_n}$, we obtain

$$\begin{aligned}
&|L_n((e_1 - xe_0)R_f)(x) - \mu_{n,1}(x) L_n(R_f)(x)| \\
&\quad \le |L_n((e_1 - xe_0)R_f, x)| + |\mu_{n,1}(x)| \cdot |(L_n R_f)(x)| \\
&\quad \le \frac{M_{0,4}(x)}{b_n} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right).
\end{aligned}$$

Let us notice that if we replace f with g in the previous inequalities they hold true, too.

To evaluate the term $L_n(R_f R_g)(x) - L_n(R_f)(x) \cdot L_n(R_g)(x)$, let us observe that

$$\begin{aligned}
&|L_n(R_f R_g)(x)| \le L_n(|R_f|\,|R_g|, x) \\
&\le L_n\left(e^{2\alpha\max(\theta(t),\theta(x))}\,|t-x|^2\left(1 + \frac{|t-x|}{\delta_n}\right)^2, x\right) \cdot \omega_{\theta,\alpha}(f', \delta_n) \cdot \omega_{\theta,\alpha}(g', \delta_n) \\
&\le 2\left[A_{n,2}(x) + b_n A_{n,4}(x)\right] \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right) \omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right) \\
&\le \frac{M_{0,5}(x)}{b_n} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right) \omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right).
\end{aligned}$$

We have used the inequality $(1+u)^2 \le 2(1+u^2)$. We obtain

$$\begin{aligned}
&|L_n(R_f R_g)(x) - L_n(R_f)(x) \cdot L_n(R_g)(x)| \\
&\quad \le |L_n(R_f R_g)(x)| + |L_n(R_f)(x)| \cdot |L_n(R_g)(x)| \\
&\quad \le \frac{M_{0,5}(x) + M_{0,2}^2(x)}{b_n} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right) \omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right).
\end{aligned}$$

Choosing an appropriate expression $M(x) > 0$ not depending on n and f the proof is complete. □

Remark 3.3. *Because $1/a_n$ and $1/\sqrt{b_n}$ converge to zero when n tends to infinity and $f'(x)e^{-\alpha\theta(x)}$ and $g'(x)e^{-\alpha\theta(x)}$ are uniformly continuous on I, we have*

$$\lim_{n\to\infty} b_n[L_n(fg)(x) - (L_n f)(x) \cdot (L_n g)(x)] = b(x) f'(x) g'(x). \tag{3.8}$$

Similar results were obtained in [18, 6, 16, 35, 9, 30] *for functions for which the second derivative exists. But, there is no need to suppose the existence of the second derivative of f and g.*

Remark 3.4. *Theorem 3.1 is true even for operators L_n for which (2.1) cannot be proved. Indeed, it is only necessary that the following limits exist for a fixed x*

$$\lim_{n\to\infty} a_n \left(b_n[\mu_{n,2}(x) - \mu_{n,1}^2(x)] - b(x)\right) \text{ and } \lim_{n\to\infty} b_n^\ell \cdot \mu_{n,2\ell}(x),\ \ell = 1,2,3,4,$$

where $b(x)$ is the limit of $b_n \cdot \mu_{n,2}(x)$.

For example, let us consider the Jain operators [29]

$$P_n^{\beta_n}(f,x) = \sum_{k=0}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-nx-k\beta_n} \cdot f\left(\frac{k}{n}\right),$$

where (β_n) is a sequence of positive real numbers from $[0,1)$ converging to zero. It is known [17] *that*

$$P_n^{\beta_n}(t-x,x) = \frac{x}{1-\beta_n} - x = \frac{x\beta_n}{1-\beta_n},$$

so we choose $a_n = 1/\beta_n$ and condition (2.3) is satisfied with $a(x) = x$. We also have

$$P_n^{\beta_n}((t-x)^2,x) = \frac{x^2\beta_n^2}{(1-\beta_n)^2} + \frac{x}{n(1-\beta_n)^3}.$$

Choosing $b_n = n$ and supposing that $b_n/a_n = n\beta_n$ is convergent to the real number $c \geq 0$, we obtain

$$b(x) = \lim_{n\to\infty} nP_n^{\beta_n}((t-x)^2,x) = x.$$

After some computations, we obtain

$$\lim_{n\to\infty} a_n \left(b_n[\mu_{n,2}(x) - \mu_{n,1}^2(x)] - b(x)\right) = x.$$

The central moment of order 4 is (see [17]*)*

$$\begin{aligned} P_n^{\beta_n}((t-x)^4,x) = {} & \frac{x^4\beta_n^4}{(1-\beta_n)^4} + \frac{6x^3\beta_n^2}{n(1-\beta_n)^5} \\ & + \frac{x^2(-24\beta_n^5 + 12\beta_n + 48\beta_n^3 - 28\beta_n^2 + 4\beta_n + 3)}{n^2(1-\beta_n)^6} \\ & + \frac{x(105\beta_n^5 - 14\beta_n^4 - 2\beta_n^3 + 12\beta_n^2 + 8\beta_n + 1)}{n^3(1-\beta_n)^7}. \end{aligned}$$

We obtain $n^2P_n^{\beta_n}((t-x)^4,x) \to 3x^2$. For the central moments of order 6 and 8, we consider the significant terms from the formulas given in [23] *and obtain*

$$\lim_{n\to\infty} n^3P_n^{\beta_n}((t-x)^6,x) = 15x^3 \text{ and } \lim_{n\to\infty} n^4P_n^{\beta_n}((t-x)^8,x) = 105x^4.$$

The result of Theorem 3.1 is valid for $P_n^{\beta_n}$ in polynomial weighted space $C_{\theta,\alpha}$ with $\theta(x) = \ln x$, $x \in I = (0,\infty)$ (see [2]*).*

4. Voronovskaya-type result for a differentiation formula for positive linear operators

In [27], it is proved that the expression $L_n'(fg) - fL_n'g - gL_n'f$ converges to zero for exponential type operators under suitable conditions for the functions f and g. The result was proved for those operators $L_n : C(I) \to C(J)$, where I, J are compact intervals. We extend the result to noncompact intervals and to unbounded functions.

Theorem 4.2. *Let $f, g \in C_{\theta,\alpha}$ such that*

$$\omega_{\theta,\alpha}(f,\delta) \cdot \omega_{\theta,\alpha}(g,\delta) = o(\delta) \quad (\delta \to 0+). \tag{4.9}$$

Let $L_n : C_{\theta,\alpha} \to C^1(I)$ be a sequence of positive linear operators preserving constant functions and having the properties (2.1), (2.3) and (3.6). Then, for every $x \in I$

$$L_n'(fg)(x) - f(x)(L_n g)'(x) - g(x)(L_n f)'(x) \to 0.$$

Proof. Let us denote

$$\Delta_n(x) = L_n'(fg)(x) - f(x)(L_n g)'(x) - g(x)(L_n f)'(x).$$

Using (2.1), we obtain the following relation

$$\begin{aligned} \Delta_n(x) = {} & \frac{b_n}{b(x)} \cdot L_n((t-x)(f(t)-f(x))(g(t)-g(x)), x) \\ & - \frac{b_n}{b(x)} \cdot L_n(t-x, x) \cdot L_n((f(t)-f(x))(g(t)-g(x)), x). \end{aligned} \tag{4.10}$$

As in the proof of Theorem 3.1, because

$$|f(t)-f(x)| \le \max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)\left(1 + \frac{|t-x|}{\delta}\right)\omega_{\theta,\alpha}(f,\delta)$$

$$|g(t)-g(x)| \le \max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)\left(1 + \frac{|t-x|}{\delta}\right)\omega_{\theta,\alpha}(g,\delta),$$

we have

$$\begin{aligned} & |L_n((t-x)(f(t)-f(x))(g(t)-g(x)), x)| \\ & \quad \le 2\left(A_{n,1}(x) + b_n A_{n,3}(x)\right)\omega_{\theta,\alpha}\left(f, \frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g, \frac{1}{\sqrt{b_n}}\right) \\ & \quad \le \frac{M_{1,1}(x)}{\sqrt{b_n}} \cdot \omega_{\theta,\alpha}\left(f, \frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g, \frac{1}{\sqrt{b_n}}\right) \end{aligned}$$

and

$$\begin{aligned} & |L_n((f(t)-f(x))(g(t)-g(x)), x)| \\ & \quad \le 2\left(A_{n,0}(x) + b_n A_{n,2}(x)\right)\omega_{\theta,\alpha}\left(f, \frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g, \frac{1}{\sqrt{b_n}}\right) \\ & \quad \le M_{1,2}(x) \cdot \omega_{\theta,\alpha}\left(f, \frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g, \frac{1}{\sqrt{b_n}}\right). \end{aligned}$$

Because $L_n(t-x,x) \le \sqrt{\mu_{n,2}(x)} \le \frac{\sqrt{C_2(x)}}{\sqrt{b_n}}$, we finally obtain

$$|\Delta_n(x)| \le M(x)\sqrt{b_n} \cdot \omega_{\theta,\alpha}\left(f, \frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g, \frac{1}{\sqrt{b_n}}\right), \quad n \ge n_0,$$

for some $M(x)$ not depending on n and f. The condition (4.9) proves that Δ_n converges to zero for every $x \in I$. □

Remark 4.5. *We have the following evaluation of the modulus $\omega_{\theta,\alpha}$ (see relation (1) from [25])*

$$\omega_{\theta,\alpha}(f,\delta) \le (1 - e^{-\alpha\omega(\theta,\delta)})\,\|f\|_{\theta,\alpha} + \omega(f/w,\delta) \le \alpha\omega(\theta,\delta)\,\|f\|_{\theta,\alpha} + \omega(f/w,\delta),$$

where $w(x) = e^{\alpha\theta(x)}$ and ω is the usual modulus of continuity (the modulus $\omega_{\theta,\alpha}$ for $\alpha = 0$).

If $\theta \in \mathrm{Lip}_a(I)$*,* $f/w \in \mathrm{Lip}_b(I)$ *and* $g/w \in \mathrm{Lip}_c(I)$ *then, (4.9) is true if and only if*

$$a + a > 1, \quad a + b > 1, \quad a + c > 1 \quad \text{and} \quad b + c > 1.$$

Indeed, a function h *belongs to* $\mathrm{Lip}_\alpha(I)$ *if and only if there is a constant* $C_h > 0$ *such that* $\omega(f, \delta) \leq C_h \delta^\alpha$*. So,*

$$\omega_{\theta,\alpha}(f, \delta) \cdot \omega_{\theta,\alpha}(g, \delta) \leq (C_1\delta^a + C_2\delta^b)(C_1\delta^a + C_3\delta^c) = o(\delta) \quad (\delta \to 0+).$$

Remark 4.6. *Theorem 4.2 remains true even if* L_n *does not satisfy a condition like (2.1). We only need that the sequence of functions* $b_n^\ell \cdot \mu_{n,2\ell}(x)$ *converges pointwise for* $\ell = 1, 2$ *and* 3*.*

Theorem 4.3. *Let* $f, g \in C_{\theta,\alpha}$ *be two twice differentiable functions such that* $f''(x)e^{-\alpha\theta(x)}$ *and* $g''(x)e^{-\alpha\theta(x)}$ *are uniformly continuous on* I*. We suppose that* (b_n/a_n) *is convergent to* $c \geq 0$*. Let* $L_n : C_{\theta,\alpha} \to C^1(I)$ *be a sequence of positive linear operators preserving constant functions and having the properties (2.1), (2.3) and (3.6). Then, for every* $x \in I$

$$\lim_{n\to\infty} b_n \left[L_n'(fg)(x) - f(x)(L_n g)'(x) - g(x)(L_n f)'(x)\right]$$
$$= [b'(x) + 2c\, a(x)]f'(x)g'(x) + \frac{3b(x)}{2}[f'(x)g''(x) + f''(x)g'(x)].$$

Proof. We use Taylor's formula

$$h(t) = h(x) + h'(x) \cdot (t - x) + \frac{h''(x)}{2} \cdot (t - x)^2 + R_h,$$

for the functions f and g, where $R_h = (h''(c) - h''(x)) \cdot (t - x)^2/2$, with some c between t and x. We replace these formulas in the expression of Δ_n (see relation (4.10)) and after some computations, we obtain

$$b_n\Delta_n(x) = f'(x)g'(x)\frac{b_n^2}{b(x)}[\mu_{n,3}(x) - \mu_{n,1}(x)\mu_{n,2}(x)]$$
$$+ [f'(x)g''(x) + f''(x)g'(x)] \cdot \frac{b_n^2}{2b(x)}[\mu_{n,4}(x) - \mu_{n,1}(x)\mu_{n,3}(x)] + \frac{b_n^2 R}{b(x)},$$

where

$$R = \frac{1}{4}f''(x)g''(x)[\mu_{n,5}(x) - \mu_{n,1}(x)\mu_{n,5}(x)]$$
$$+ f'(x) \cdot E_{n,1}(g) + \frac{1}{2}f''(x) \cdot E_{n,2}(g) + g'(x) \cdot E_{n,1}(f) + \frac{1}{2}g''(x) \cdot E_{n,2}(f)$$
$$+ L_n(R_f \cdot R_g \cdot (t - x), x) - \mu_{n,1}(x) \cdot L_n(R_f \cdot R_g, x)$$

and

$$E_{n,k}(f) = L_n(R_f \cdot (t - x)^{k+1}, x) - \mu_{n,1}(x) \cdot L_n(R_f \cdot (t - x)^k, x).$$

We have

$$\frac{b_n^2}{b(x)}[\mu_{n,3}(x) - \mu_{n,1}(x)\mu_{n,2}(x)] = b_n\mu_{n,2}'(x) + 2b_n\mu_{n,1}(x) \to b'(x) + 2ca(x)$$

and

$$\frac{b_n^2}{2b(x)}[\mu_{n,4}(x) - \mu_{n,1}(x)\mu_{n,3}(x)] = \frac{b_n}{2}\mu_{n,3}'(x) + \frac{3b_n}{2}\mu_{n,2}(x) \to 0 + \frac{3b(x)}{2}.$$

We also have $b_n^2 R \to 0$, but since the computations are similar to those in the proof of Theorem 3.1, we omit the details. $\square$

Remark 4.7. *Let $f, g \in C_{\theta,\alpha}$ be two twice differentiable functions such that $f''(x)e^{-\alpha\theta(x)}$ and $g''(x)e^{-\alpha\theta(x)}$ are uniformly continuous on I. It can be proved in a similar way that*

$$\lim_{n\to\infty} b_n \left[L'_n(fg)(x) - (L_n f)(x)(L_n g)'(x) - (L_n g)(x)(L_n f)'(x)\right]$$
$$= b'(x)f'(x)g'(x) + b(x)[f'(x)g''(x) + f''(x)g'(x)].$$

This is just relation (3.8), where both terms have been differentiated.

5. Applications

We give a couple of examples of applications.

Example 5.1. *Consider the following Baskakov operators of Stancu type*

$$(L_n^{[\alpha,\beta,c]} f)(x) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \cdot f\left(\frac{k+\alpha}{n+\beta}\right), \quad n \geq 1$$

where α and β are real numbers such that $0 \leq \alpha \leq \beta$ and

$$p_{n,k}^{[c]}(x) = (-1)^k \binom{-n/c}{k} (cx)^k (1+cx)^{-\frac{n}{c}-k}, \quad c \neq 0$$
$$p_{n,k}^{[0]}(x) = \lim_{c\to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!} e^{-nx},$$

where $x \in [0, \infty)$ for $c \geq 0$ and $x \in [0, -1/c]$ for $c < 0$.

These operators are a particular example of the more general operators considered in [11]. For $\alpha = \beta = 0$, some properties of the operators were given in [1, 14] (see also [5, 32] and the references therein).

These operators preserve the constants and

$$L_n^{[\alpha,\beta,c]}(t, x) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \cdot \frac{k+\alpha}{n+\beta} = \frac{nx+\alpha}{n+\beta}.$$

We deduce that

$$(n+\beta) \cdot \left(L_n^{[\alpha,\beta,c]}(t, x) - x\right) = \alpha - \beta x,$$

which proves (2.3) for $a_n = n + \beta$ and $a(x) = \alpha - \beta x$.

We also have

$$\frac{\mathrm{d}}{\mathrm{d}x} p_{n,k}^{[c]}(x) = p_{n,k}^{[c]}(x) \cdot \frac{k-nx}{x(1+cx)} = \frac{n+\beta}{x(1+cx)} p_{n,k}^{[c]}(x) \cdot \left(\frac{k+\alpha}{n+\beta} - L_n^{[\alpha,\beta,c]}(t, x)\right).$$

Multiplying this equality with $f((k+\alpha)/(n+\beta))$ and summing up for k from 0 to infinity, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(L_n^{[\alpha,\beta,c]} f)(x) = \frac{n+\beta}{x(1+cx)} \left[L_n^{[\alpha,\beta,c]}(tf(t), x) - L_n^{[\alpha,\beta,c]}(t, x) \cdot L_n^{[\alpha,\beta,c]}(f, x)\right],$$

which is (2.1) for $b_n = n + \beta$ and $b(x) = x(1 + cx)$.

The results of Theorems 1,2 and 3 are valid for functions in the exponential space $C_{\theta,\alpha}$ for $\theta(x) = x$, because

$$L_n^{[\alpha,\beta,c]}(e^{\alpha t}, x) = \left(1 + cx - cxe^{\frac{\alpha}{n}}\right)^{-\frac{n}{c}} \to e^{\alpha x} \quad (n \to \infty).$$

Example 5.2. *Consider the Balász operators*

$$R_n(f, x) = \frac{1}{(1+\alpha_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (\alpha_n x)^k \cdot f\left(\frac{k}{\beta_n}\right), n \geq 1$$

introduced in [12] and studied in [13, 34, 4, 8, 3] for some particular cases of the sequences (α_n) and (β_n) of positive real numbers.

The operators R_n preserve the constants and

$$R_n(t,x)=\frac{n\alpha_n x}{\beta_n(1+\alpha_n x)} \text{ and } R_n(t^2,x)=\frac{n\alpha_n x+n^2\alpha_n^2x^2}{\beta_n^2(1+\alpha_n x)^2}.$$

We must have $R_n(t,x)\to x$ and $R_n(t^2,x)\to x^2$, so we choose (α_n) and (β_n) such that $\alpha_n\to 0$ and the sequence (β_n) such that $\beta_n\to\infty$ and $n\alpha_n/\beta_n\to 1$. The central moment of order 1 is

$$R_n(t-x,x)=\frac{(n\alpha_n-\beta_n)x-\alpha_n\beta_n x^2}{\beta_n(1+\alpha_n x)}.$$

We further impose that $n\alpha_n-\beta_n\to 0$ and $\alpha_n\beta_n\to c$, $c\geq 0$. With these conditions, we can choose $a_n=\beta_n$ and obtain $\beta_n R_n(t-x,x)=-cx^2$.

Let us prove that R_n satisfy (2.1). Because

$$\left(\frac{(\alpha_n x)^k}{(1+\alpha_n x)^n}\right)'=\frac{(\alpha_n x)^k}{(1+\alpha_n x)^n}\cdot\left(\frac{k}{x}-\frac{n\alpha_n}{1+\alpha_n x}\right),$$

we obtain

$$\begin{aligned}(R_n(f,x))'&=\frac{\beta_n}{x}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha_n x)^k}{(1+\alpha_n x)^n}\cdot\left(\frac{k}{\beta_n}-\frac{n\alpha_n x}{\beta_n(1+\alpha_n x)}\right)\cdot f\left(\frac{k}{\beta_n}\right)\\&=\frac{\beta_n}{x}R_n((t-R_n(t,x))f(t),x),\end{aligned}$$

which proves (2.1) with $b_n=\beta_n$ and $b(x)=x$.

We take $I=(0,\infty)$ and $\theta(x)=x$. The results of Theorem 1, 2 and 3 are valid for the operators R_n in the exponential weighted space $C_{\theta,\alpha}$, because for a fixed $x\geq 0$

$$R_n(e^{\alpha t},x)=\left(\frac{1+\alpha_n x e^{\frac{\alpha}{\beta_n}}}{1+\alpha_n x}\right)^n\to e^{\alpha x}.$$

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Research Article

On a Family of Hypergeometric Sobolev Orthogonal Polynomials on the Unit Circle

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ABSTRACT. In this paper, we study the following family of hypergeometric polynomials: $y_n(x) = \frac{(-1)^\rho}{n!} x^n {}_2F_0(-n, \rho; -; -\frac{1}{x})$, depending on a parameter $\rho \in \mathbb{N}$. Differential equations of orders $\rho+1$ and 2 for these polynomials are given. A recurrence relation for y_n is derived as well. Polynomials y_n are Sobolev orthogonal polynomials on the unit circle with an explicit matrix measure.

Keywords: Sobolev orthogonal polynomials, hypergeometric polynomials, unit circle, differential equation, recurrence relation.

2010 Mathematics Subject Classification: 33C45, 42C05.

1. INTRODUCTION

The theories of orthogonal polynomials (OP) on the real line and on the unit circle have many similarities as well as considerable differences [19, 6, 16, 17]. For a long time, they have been developed side by side by efforts of numerous mathematicians. The theory of Sobolev orthogonal polynomials is a much more *terra incognita* [11, 18, 10, 8]. In this theory, one can also see that some ideas come from the real line to the unit circle. Examples of such ideas are adding of Dirac deltas to the classical inner products and considering of coherent pairs of measures (see, e.g., [3, 4] and references therein). In the present paper, we shall follow the same line: we shall develop the ideas from [20] to get some new hypergeometric polynomials and study their properties.

Let μ be a probability measure on $\mathbb{T}$ with an infinite support. We assume that μ is defined on a σ-algebra $\mathfrak{A}$ which contains $\mathfrak{B}(\mathbb{T})$. Denote by p_n orthogonal polynomials on the unit circle (OPUC) with respect to μ ($\deg p_n = n$, but the positivity of leading coefficients is not assumed):

$$\int_{\mathbb{T}} p_n(z)\overline{p_m(z)} d\mu = A_n \delta_{n,m}, \qquad A_n > 0,\ n, m \in \mathbb{Z}_+. \tag{1.1}$$

Fix an arbitrary positive integer ρ. Consider the following differential equation:

$$(e^{-x} y(x))^{(\rho)} = e^{-x} p_n(x), \qquad n \in \mathbb{Z}_+. \tag{1.2}$$

Expanding the derivative by the Leibniz formula and canceling e^{-x}, we get:

$$\sum_{k=0}^{\rho} (-1)^{\rho-k} \binom{\rho}{k} y^{(k)}(x) = p_n(x), \qquad n \in \mathbb{Z}_+. \tag{1.3}$$

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Condition A. *Suppose that for each $n \in \mathbb{Z}_+$, there exists a n-th degree polynomial solution $y = y_n(x)$ of (1.3).*

If Condition A is satisfied, then y_n are Sobolev orthogonal polynomials on the unit circle (SOPUC):

$$\int_{\mathbb{T}} \left(y_n(z), y_n'(z), ..., y_n^{(\rho)}(z) \right) M \overline{\begin{pmatrix} y_m(z) \\ y_m'(z) \\ \vdots \\ y_m^{(\rho)}(z) \end{pmatrix}} d\mu = A_n \delta_{n,m}, \qquad n, m \in \mathbb{Z}_+, \tag{1.4}$$

where

$$M = \left((-1)^{l+j} \begin{pmatrix} \rho \\ l \end{pmatrix} \begin{pmatrix} \rho \\ j \end{pmatrix} \right)_{l,j=0}^{\rho}. \tag{1.5}$$

Recall that the classical orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ on $\mathbb{R}$ (namely, one of the following systems: Jacobi OP, Laguerre OP, Hermite OP) are eigenfunctions of a second-order linear differential operator L (see, e.g., [9]):

$$L p_n(x) = \lambda_n p_n(x), \qquad n = 0, 1, 2, \ldots . \tag{1.6}$$

On the other hand, the vector $\vec{p}(x) = (p_0(x), p_1(x), ...)^T$ is an eigenvector of the corresponding Jacobi matrix J:

$$J\vec{p}(x) = x\vec{p}(x). \tag{1.7}$$

In the case of OPUC, we can not give any property of form (1.6) with some linear differential operator. In this case and, more generally, in the case of SOPUC the notion of operator pencils seems to be appropriate. Operator pencils appeared in the theory of biorthogonal rational functions, see [7, 21].

By operator pencils or operator polynomials one means polynomials of complex variable λ whose coefficients are linear bounded operators acting in a Banach space X ([15, 12]):

$$L(\lambda) = \sum_{j=0}^{m} \lambda^j A_j, \tag{1.8}$$

where $A_j : X \to X$ ($j = 0, ..., m$). In the case $m = 1$ ($m = 2$), the pencil is called linear (respectively quadratic). Operator pencils with differential operators A_j appear in many physical problems, see ([13]) and references therein.

The following problem seems to be a suitable framework to study classical type SOPUC.

Problem 1. *To describe all SOPUC $\{y_n(z)\}_{n=0}^{\infty}$, satisfying the following two properties:*

(a) *Polynomials $y_n(z)$ satisfy the following differential equation:*

$$R y_n(z) = \lambda_n S y_n(z), \qquad n = 0, 1, 2, ..., \tag{1.9}$$

where R, S are linear differential operators of finite orders, having polynomial coefficients not depending on n; $\lambda_n \in \mathbb{C}$;

(b) *Polynomials $y_n(z)$ satisfy the following difference equation:*

$$L\vec{y}(z) = zM\vec{y}(z), \quad \vec{y}(z) = (y_0(z), y_1(z), ...)^T, \tag{1.10}$$

where L, M are semi-infinite complex banded (i.e. having a finite number of non-zero diagonals) matrices.

Relation (1.9) means that $y_n(z)$ are eigenvalues of the operator pencil $R - \lambda S$, while relation (1.10) shows that vectors of $y_n(z)$ are eigenvalues of the operator pencil $L - zM$. For example, consider the following case: $y_n(x) = z^n$. They satisfy the differential equation:

$$z(z^n)' = nz^n, \qquad n \in \mathbb{Z}_+,$$

and they obey (1.10) with L being the identity semi-infinite matrix, M being the semi-infinite matrix with all 1 on the first subdiagonal and 0 on other places.

Let us briefly describe the content of the paper. At first, we consider the following case: $p_n(x) = x^n$, and μ being the normalized arc length measure on $\mathbb{T}$. Equating coefficients of the same powers on the both sides of equation (1.3) one obtains a linear system of equations for the coefficients of an unknown polynomial $y(x)$ (the same idea was used in [2]). However, for large values of ρ it is not easy to get a convenient expression for solutions, without huge determinants or recurrences. In this case, equation (1.2) turned out to be useful. It gives a possibility to express $y_n(x)$ for $\rho = 1$ in terms of the incomplete gamma function. A step-by-step analysis for $\rho = 1, 2, ...,$ allows to obtain an explicit representation of $y_n(x)$. Explicit representations, differential equations and orthogonality relations for y_n will be given by Theorem 2.1. As a corollary, we obtain a solution to (1.3) in the general case (Corollary 2.1). Using Fasenmeier's method ([14]) for the reversed polynomials $y_n^*(x) = x^n y_n\left(\frac{1}{x}\right)$, we shall derive recurrence relations for $y_n(x)$ (Theorem 2.2) as well. Thus, we provide an example of SOPUC which satisfies conditions of Problem 1.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. The superscript T means the transpose of a (finite or infinite) vector. Set $\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}$. By $\mathfrak{B}(\mathbb{T})$, we mean the set of all Borel subsets of $\mathbb{T}$. By $\mathbb{P}$, we denote the set of all polynomials with complex coefficients. For a complex number c, we denote $(c)_0 = 1$, $(c)_k = c(c+1)...(c+k-1)$, $k \in \mathbb{N}$ (*the shifted factorial or Pochhammer symbol*). The generalized hypergeometric function is denoted by

$${}_mF_n(a_1, ..., a_m; b_1, ..., b_n; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k ... (a_m)_k}{(b_1)_k ... (b_n)_k} \frac{x^k}{k!},$$

where $m, n \in \mathbb{N}$, $a_j, b_l \in \mathbb{C}$.

2. Some Sobolev orthogonal polynomials on $\mathbb{T}$

As it was stated in the Introduction, in what follows we shall consider the following case: $p_n(x) = x^n$, and $\mu = \mu_0$ being the (probability) normalized arc length measure on $\mathbb{T}$, which may be identified with the Lebesgue measure on $[0, 2\pi)$. Rewrite equations (1.2), (1.3) for this case:

$$(e^{-x} y_n(x))^{(\rho)} = e^{-x} x^n, \qquad n \in \mathbb{Z}_+; \tag{2.11}$$

$$\sum_{k=0}^{\rho} (-1)^{\rho-k} \binom{\rho}{k} y_n^{(k)}(x) = x^n, \qquad n \in \mathbb{Z}_+. \tag{2.12}$$

We start with the case $\rho = 1$. In this case, equation (2.12) has the following form:

$$y_n'(x) - y_n(x) = x^n, \qquad n \in \mathbb{Z}_+. \tag{2.13}$$

Fix an arbitrary $n \in \mathbb{Z}_+$. We shall seek for a solution of the required form:

$$y_n(x) = \sum_{k=0}^{n} \mu_{n,k} x^k, \qquad \mu_{n,k} \in \mathbb{C}. \tag{2.14}$$

Substitute for y_n into (2.13) to get

$$\sum_{k=0}^{n-1}\{(k+1)\mu_{n,k+1}-\mu_{n,k}\}x^k-\mu_{n,n}x^n=x^n.$$

Comparing the coefficients of the same powers on the both sides, we obtain that

$$\mu_{n,n}=-1,\quad \mu_{n,k}=(k+1)\mu_{n,k+1},\quad k=n-1,n-2,...,0. \tag{2.15}$$

It can be verified by the induction argument that

$$\mu_{n,j}=(-n)_{n-j}(-1)^{n-j-1}=-\frac{n!}{j!},\qquad j=0,1,...,n.$$

Thus,

$$y_n(x)=-n!\sum_{k=0}^{n}\frac{x^k}{k!} \tag{2.16}$$

is a solution of (2.13). In the case $\rho>1$, it is not easy to solve the corresponding recurrence relation for the coefficients and we shall proceed in another way.

Observe that

$$y_n(x)=-e^x\Gamma(n+1,x), \tag{2.17}$$

where

$$\Gamma(\alpha,x)=\int_x^\infty e^{-t}t^{\alpha-1}dt,\qquad \alpha>0,$$

is *the complementary incomplete gamma function* ([1]). In fact, integrating (2.11) (with $\rho=1$) from a to b ($a,b\in\mathbb{R}$), we get

$$e^{-b}y_n(b)-e^{-a}y_n(a)=\int_a^b e^{-x}x^n dx.$$

Taking limit when $b\to+\infty$, we get

$$y_n(a)=-e^a\int_a^\infty e^{-x}x^n dx, \tag{2.18}$$

and relation (2.17) follows.

Suppose that we have constructed a polynomial solution (of the required form) $y_n(\rho;x)=y_n(x)$ of equation (2.11) for some positive integer ρ. Let us show how to get a polynomial solution $y_n(\rho+1;x)$ of equation (2.11) with $\rho+1$. Notice that, we do not state the uniqueness of such solutions for $\rho>2$. We shall need the following auxiliary equation:

$$(e^{-x}y_n(\rho+1;x))'=e^{-x}y_n(\rho;x),\qquad n\in\mathbb{Z}_+, \tag{2.19}$$

with an unknown $y_n(\rho+1;x)$. Equation (2.19) has a unique n-th degree polynomial solution. This can be verified comparing the coefficients of polynomials, in the same way as for equation (2.13). It is not easy to solve the corresponding recurrence relation in this case, but the existence and the uniqueness of a n-th degree polynomial solution is obvious.

Integrating relation (2.19) from t to b, we get

$$e^{-b}y_n(\rho+1;b)-e^{-t}y_n(\rho+1;t)=\int_t^b e^{-x}y_n(\rho;x)dx.$$

Taking limit when $b\to+\infty$, we get

$$y_n(\rho+1;t)=-e^t\int_t^\infty e^{-x}y_n(\rho;x)dx. \tag{2.20}$$

By (2.11), (2.19) we may write:

$$e^{-x}x^n = (e^{-x}y_n(\rho;x))^{(\rho)} = (e^{-x}y_n(\rho+1;x))^{(\rho+1)}.$$

Therefore, $y_n(\rho+1;\cdot)$ given by (2.20) is a required polynomial solution of (2.11) for $\rho+1$.

Equation (2.20) shows how to construct polynomial solutions step-by-step for $\rho = 1, 2, \ldots$. However, we are interested to get an explicit representation for every $y_n(\rho;x)$. Let

$$y_n(\rho;x) = \sum_{j=0}^{n} d_j(\rho)\frac{x^j}{j!}, \qquad n \in \mathbb{Z}_+,\ \rho \in \mathbb{N}, \tag{2.21}$$

with some unknown complex numbers $d_j(\rho)$. By (2.20), (2.18), (2.16) we may write

$$y_n(\rho+1;t) = -e^t\sum_{j=0}^{n} d_j(\rho)\frac{1}{j!}\int_t^\infty e^{-x}x^j dx = \sum_{j=0}^{n} d_j(\rho)\frac{1}{j!}y_j(1;t)$$

$$= -\sum_{j=0}^{n}\sum_{k=0}^{j} d_j(\rho)\frac{t^k}{k!}, \qquad n \in \mathbb{Z}_+,\ \rho \in \mathbb{N}. \tag{2.22}$$

Changing the order of summation in (2.22), we write:

$$y_n(\rho+1;t) = -\sum_{k=0}^{n}\sum_{j=k}^{n} d_j(\rho)\frac{t^k}{k!}.$$

Therefore,

$$d_k(\rho+1) = -\sum_{j=k}^{n} d_j(\rho), \qquad k = 0, 1, \ldots, n;\ \rho \in \mathbb{N}. \tag{2.23}$$

Relation (2.23) can be written in a matrix form for the vectors of coefficients $\vec{d}(\rho) := (d_0(\rho), \ldots, d_n(\rho))^T$, and a $(n+1)\times(n+1)$ upper-diagonal Toeplitz matrix T, having all nonzero elements equal to 1:

$$\vec{d}(\rho+1) = -T\vec{d}(\rho), \qquad \rho \in \mathbb{N}. \tag{2.24}$$

Therefore,

$$\vec{d}(\rho) = (-1)^\rho T^\rho (0, \ldots, 0, 1)^T, \qquad \rho \in \mathbb{N}. \tag{2.25}$$

Applying the Riesz calculus for evaluating T^ρ, one obtains the following solution:

$$d_k(\rho) = (-1)^\rho \binom{n-k+\rho-1}{n-k}, \qquad k = 0, 1, \ldots, n;\ n \in \mathbb{Z}_+,\ \rho \in \mathbb{N}. \tag{2.26}$$

We shall omit the details of calculating the resolvent $(T-\lambda E)^{-1}$. We only notice that, it was convenient to subtract the subsequent rows when solving the linear system of equations $(T-\lambda E)f = (0, \ldots, 0, 1)^T$. It can be directly verified that the resulting expression (2.26) obeys (2.23), by using the Pascal identity and the induction argument.

Thus, we have obtained the following representation for y_n:

$$y_n(\rho;x) = (-1)^\rho\sum_{j=0}^{n}\binom{n-j+\rho-1}{n-j}\frac{x^j}{j!}, \qquad n \in \mathbb{Z}_+,\ \rho \in \mathbb{N}. \tag{2.27}$$

Theorem 2.1. *Let $y_n(\rho;x)$ be polynomials given by relation (2.27) ($\rho \in \mathbb{N}$, $n \in \mathbb{Z}_+$). They have the following properties:*

(a) *Polynomials $y_n(\rho;x)$ admit the following representation:*

$$y_n(\rho;x) = \frac{(-1)^\rho}{n!} x^n {}_2F_0\left(-n,\rho;-;-\frac{1}{x}\right), \quad n\in\mathbb{Z}_+,\ \rho\in\mathbb{N};\ x\in\mathbb{C}\backslash\{0\}. \tag{2.28}$$

(b) *Polynomials $y(x)=y_n(\rho;x)$ satisfy the following differential equation:*

$$x\sum_{k=0}^{\rho}(-1)^{\rho-k}\begin{pmatrix}\rho\\k\end{pmatrix} y^{(k+1)}(x) - n\sum_{k=0}^{\rho}(-1)^{\rho-k}\begin{pmatrix}\rho\\k\end{pmatrix} y^{(k)}(x) = 0. \tag{2.29}$$

(c) *Polynomials $y(x)=y_n(\rho;x)$ obey the following differential equation:*

$$xy''(x) - (x+\rho-1)y'(x) - n\left[y'(x)-y(x)\right] = 0. \tag{2.30}$$

(d) *Polynomials $y_n(x)=y_n(\rho;x)$ are Sobolev orthogonal polynomials on $\mathbb{T}$:*

$$\int_{\mathbb{T}} \left(y_n(z), y_n'(z), ..., y_n^{(\rho)}(z)\right) M \overline{\begin{pmatrix} y_m(z)\\ y_m'(z)\\ \vdots\\ y_m^{(\rho)}(z)\end{pmatrix}} d\mu_0 = \delta_{n,m}, \qquad n,m\in\mathbb{Z}_+, \tag{2.31}$$

where M is given by (1.5).

Proof. (a): It is readily checked that the reversed polynomial for y_n is given by

$$y_n^*(\rho;x) = \frac{(-1)^\rho}{n!} {}_2F_0\left(-n,\rho;-;-x\right), \qquad n\in\mathbb{Z}_+,\ \rho\in\mathbb{N},$$

and relation (2.28) follows.
(b): Substitute for x^n from (2.12) into the following equality:

$$x(x^n)' = nx^n.$$

(c): Hypergeometric polynomials

$$u = u_n(z) := {}_2F_0\left(-n,\rho;-;z\right), \qquad n\in\mathbb{Z}_+,\ \rho\in\mathbb{N}, \tag{2.32}$$

satisfy the following differential equation:

$$z(-n+\theta)(\rho+\theta)u - \theta u = 0, \tag{2.33}$$

where $\theta = z\frac{d}{dz}$. The differential equation for the generalized hypergeometric function ${}_pF_q$ is usually written when $p,q\geq 1$. However, the arguments in [14, p. 75] can be applied in the case $q=0$ as well. Then for $z\neq 0$, we may write

$$z^2u''(z) + (\rho+1)zu'(z) - n(zu'(z)+\rho u(z)) - u'(z) = 0. \tag{2.34}$$

Observe that

$$u_n(z) = \frac{n!}{(-1)^{n+\rho}} z^n y_n(\rho;-\frac{1}{z}). \tag{2.35}$$

Calculating the derivatives u_n', u_n'' and inserting them into relation (2.34), after some algebraic simplifications, we get relation (2.30).
(d): This follows from our motivation and relation (1.4) in the Introduction. $\square$

Corollary 2.1. *Let μ be a probability measure on $\mathbb{T}$ with an infinite support. Denote by p_n orthogonal polynomials on $\mathbb{T}$ with respect to μ (the positivity of leading coefficients is not assumed) which satisfy (1.1). Let*

$$p_n(x) = \sum_{j=0}^{n} \xi_{n,j} x^j, \qquad \xi_{n,j} \in \mathbb{C},\ \xi_{n,n} \neq 0;\ n \in \mathbb{Z}_+. \tag{2.36}$$

Polynomials

$$\widehat{y}_n(\rho; x) = \sum_{j=0}^{n} \xi_{n,j} \frac{(-1)^\rho}{j!} x^j {}_2F_0 \left(-j, \rho; -; -\frac{1}{x} \right), \qquad n \in \mathbb{Z}_+, \tag{2.37}$$

are solutions to equation (1.3). Therefore, $\{\widehat{y}_n(\rho; x)\}_{n=0}^{\infty}$ are Sobolev orthogonal polynomials satisfying relation (1.4).

Proof. Since

$$D := \sum_{k=0}^{\rho} (-1)^{\rho-k} \binom{\rho}{k} \frac{d^k}{dx^k}$$

is a linear operator on polynomials, then we may write:

$$D \sum_{j=0}^{n} \xi_{n,j} \frac{(-1)^\rho}{j!} x^j {}_2F_0 \left(-j, \rho; -; -\frac{1}{x} \right) = D \sum_{j=0}^{n} \xi_{n,j} y_j(\rho; x)$$

$$= \sum_{j=0}^{n} \xi_{n,j} D y_j(\rho; x) = \sum_{j=0}^{n} \xi_{n,j} x^j = p_n(x).$$

Therefore, Condition A is satisfied and we conclude that $\widehat{y}_n$ are Sobolev orthogonal polynomials. □

Corollary 2.1 shows that one can take a system of OPUC with an explicit representation and construct a system of SOPUC by (2.37). For example, one can use the circular Jacobi orthogonal polynomials, see Example 8.2.5 and formulas (8.2.21), (8.2.22) in [6, pp. 229-230].

In order to obtain a recurrence relation for polynomials $y_n(\rho; x)$, we shall apply Fasenmeier's method ([14]) to hypergeometric polynomials $u_n(z)$ from (2.32). In the following considerations, we shall admit for ρ to be not only positive integer values but $\rho > 0$ as well. We shall express $u_n, u_{n-1}, u_{n-2}, z u_n(z), z u_{n-1}(z)$, using $u_{n+1}(z)$. Choose and fix an arbitrary integer n greater than or equal to 2. We may write

$$u_{n+1}(z) = \sum_{k=0}^{\infty} (-n-1)_k (\rho)_k \frac{z^k}{k!} = \sum_{k=0}^{\infty} \varepsilon_{n+1}(k),$$

where $\varepsilon_{n+1}(k) = \varepsilon_{n+1}(z; \rho; k) := (-n-1)_k (\rho)_k \frac{z^k}{k!}$. Using

$$(-n)_k = (-n-1)_k \frac{(n+1-k)}{(n+1)}, \qquad k \in \mathbb{Z}_+,$$

$$(-n+1)_k = (-n-1)_k \frac{(n+1-k)(n-k)}{(n+1)n}, \qquad k \in \mathbb{Z}_+,$$

and similar relations, we obtain that

$$u_n(z) = \sum_{k=0}^{\infty} \varepsilon_{n+1}(k) \frac{(n+1-k)}{(n+1)}, \tag{2.38}$$

$$u_{n-1}(z) = \sum_{k=0}^{\infty} \varepsilon_{n+1}(k) \frac{(n+1-k)(n-k)}{(n+1)n}, \tag{2.39}$$

$$u_{n-2}(z) = \sum_{k=0}^{\infty} \varepsilon_{n+1}(k) \frac{(n+1-k)(n-k)(n-1-k)}{(n+1)n(n-1)}, \tag{2.40}$$

$$zu_n(z) = \sum_{k=0}^{\infty} \varepsilon_{n+1}(k) \frac{(-k)}{(n+1)(\rho+k-1)}, \quad \rho \neq 1, \tag{2.41}$$

$$zu_{n-1}(z) = \sum_{k=0}^{\infty} \varepsilon_{n+1}(k) \frac{(-k)(n+1-k)}{n(n+1)(\rho+k-1)}, \quad \rho \neq 1. \tag{2.42}$$

We now assume that $\rho \neq 1$. Consider the following expression $R_n(z)$:

$$R_n(z) := \varphi_1 u_{n-1}(z) + \varphi_2 u_n(z) + \varphi_3 u_{n+1}(z) + \varphi_4 z u_n(z) +$$
$$+ \varphi_5 z u_{n-1}(z) + \varphi_6 u_{n-2}(z), \qquad \varphi_k \in \mathbb{C}. \tag{2.43}$$

We intend to choose parameters φ_k (depending on the chosen n) in such a way that $R_n(z) = 0$, $\forall z \in \mathbb{C}$. Substitute above expressions for $u_{n-2}, u_{n-1}, u_n, zu_n, zu_{n-1}$ into (2.43) to get

$$R_n(z) = \sum_{k=0}^{\infty} \varepsilon_{n+1}(k) \frac{1}{(n-1)n(n+1)(\rho+k-1)} I_{n,k},$$

where

$$I_{n,k} = \varphi_1(n-k)(n+1-k)(n-1)(\rho+k-1) + \varphi_2(n+1-k)(n-1)n(\rho+k-1) +$$
$$+ \varphi_3(n-1)n(n+1)(\rho+k-1) + \varphi_4(-1)k(n-1)n + \varphi_5(n+1-k)(-1)k(n-1) +$$
$$+ \varphi_6(n+1-k)(n-k)(n-1-k)(\rho+k-1). \tag{2.44}$$

Observe that $I_{n,k}$ is a polynomial of degree ≤ 4. Therefore, we may check that $I_{n,k} = 0$ for some distinct five values of k to get $R_n(z) \equiv 0$. This is a crucial point in the Fasenmeier's method.

We choose $k = -\rho + 1; n+1; n; n-1; 0$. After some obvious simplifications, we get the following five equations:

$$\varphi_5 = -\frac{n\varphi_4}{n+\rho}, \tag{2.45}$$

$$\varphi_3 = \frac{\varphi_4}{n+\rho}, \tag{2.46}$$

$$\varphi_2(\rho+n-1) + \varphi_3(n+1)(\rho+n-1) + \varphi_4(-1)n - \varphi_5 = 0, \tag{2.47}$$

$$\varphi_1 2(\rho+n-2) + \varphi_2 2n(\rho+n-2) + \varphi_3 n(n+1)(\rho+n-2) -$$
$$- \varphi_4(n-1)n - \varphi_5 2(n-1) = 0, \tag{2.48}$$

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_6 = 0. \tag{2.49}$$

Set $\varphi_4 = n + \rho$. Then,

$$\varphi_5 = -n, \quad \varphi_3 = 1.$$

By (2.47), we get

$$\varphi_2 = -1.$$

By (2.48), we obtain that $\varphi_1 = 0$. Finally, by (2.49) we conclude that $\varphi_6 = 0$, as well. Consequently, polynomials $u_n(z)$ satisfy the following relation:

$$u_{n+1}(z) - u_n(z) + (n+\rho)zu_n(z) - nzu_{n-1}(z) = 0,$$

$$n = 2, 3, ...; \rho > 0, \rho \neq 1. \tag{2.50}$$

It is directly checked that, relation (2.50) holds for the values $n = 0, 1$, if we set $u_{-1} := 0$. Moreover, we can take limit when $\rho \to 1$, to prove that relation (2.50) holds for $\rho = 1$; $n \in \mathbb{Z}_+$:

$$u_{n+1}(z) - u_n(z) + (n+\rho)zu_n(z) - nzu_{n-1}(z) = 0, \quad n \in \mathbb{Z}_+; \rho > 0. \tag{2.51}$$

Theorem 2.2. *Let $y_n(x) = y_n(\rho; x)$ be polynomials from relation (2.27) with $\rho \in \mathbb{N}$. They satisfy the following recurrence relation:*

$$(n+1)y_{n+1}(\rho; x) - (n+\rho)y_n(\rho; x) = x\left(y_n(\rho; x) - y_{n-1}(\rho; x)\right), \quad n \in \mathbb{Z}_+, \tag{2.52}$$

where $y_{-1}(\rho; x) := 0$

Proof. Use relations (2.35) and (2.50). □

Relation (2.52) can be written in the following matrix form:

$$\widehat{L}\vec{y}(\rho; x) = x\widehat{M}\vec{y}(\rho; x), \tag{2.53}$$

where $\vec{y}(\rho; x) = (y_0(\rho; x), y_1(\rho; x), ...)^T$. The semi-infinite matrix $\widehat{M}$ is two-diagonal, having all 1 on the main diagonal, and all -1 on the first sub-diagonal. The semi-infinite matrix $\widehat{L}$ is also two-diagonal, but having $(-\rho, -\rho-1, -\rho-2, ...)$ on the main diagonal, and $(1, 2, 3, ...)$ on the first upper diagonal.

Denote by $\widehat{L}_n$ ($\widehat{M}_n$) the $(n+1)\times(n+1)$ matrix standing on the intersection of the first $(n+1)$ rows and the first $(n+1)$ columns of $\widehat{L}$ (respectively $\widehat{M}$). Let x^* be an arbitrary root of $y_{n+1}(\rho; x)$. Then

$$\widehat{L}_n\vec{y}(\rho; x^*; n) = x^*\widehat{M}\vec{y}(\rho; x^*; n), \tag{2.54}$$

where $\vec{y}(\rho; x^*; n) = (y_0(\rho; x^*), y_1(\rho; x^*), ..., y_n(\rho; x^*))^T$. Thus, there is a link between the analytic theory of polynomials (the location of zeros) and the matrix theory (generalized eigenvalue problems, see [5]).

Notice that, monic polynomials $\widetilde{y}_n(\rho; x)$ are given by

$$\widetilde{y}_n(\rho; x) = \frac{n!}{(-1)^\rho} y_n(\rho; x), \quad n \in \mathbb{Z}_+, \ \rho \in \mathbb{N}.$$

The recurrence relation (2.52) takes the following form (we shifted the indices):

$$\widetilde{y}_n(\rho; x) = (x + n - 1 + \rho)\widetilde{y}_{n-1}(\rho; x) - (n-1)x\widetilde{y}_{n-2}(\rho; x), \qquad n \in \mathbb{N}. \tag{2.55}$$

Relation (2.55) is a particular case of a general recurrence relation (2.1) on page 5 in [7]. This general recurrence relation is related to R_I-fractions and biorthogonal rational functions [7, Theorem 2.1].

Finally, observe that $y_1(\rho; x) = (-1)^\rho(x+\rho)$. Thus, $y_1(1; x)$ has its root on the unit circle, while the roots of $y_1(\rho; x)$, for $\rho > 1$, are outside $\mathbb{T}$. Consequently, polynomials $y_n(\rho; x)$ are not orthogonal on the unit circle with respect to a scalar measure.

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Research Article

Gneiting Class, Semi-Metric Spaces and Isometric Embeddings

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ABSTRACT. This paper revisits the Gneiting class of positive definite kernels originally proposed as a class of covariance functions for space-time processes. Under the framework of quasi-metric spaces and isometric embeddings, the paper proposes a general and unifying framework that encompasses results provided by earlier literature. Our results allow to study the positive definiteness of the Gneiting class over products of either Euclidean spaces or high dimensional spheres and quasi-metric spaces. In turn, Gneiting's theorem is proved here by a direct construction, eluding Fourier inversion (the so-called Gneiting's lemma) and convergence arguments that are required by Gneiting to preserve an integrability assumption.

Keywords: Gneiting class, positive definite, completely monotone, isometric embedding, quasi-metric spaces.

2010 Mathematics Subject Classification: 42A82, 43A35.

1. INTRODUCTION

Positive definite kernels have a long history that traces back to many branches of pure and applied mathematics, as well as to statistics, machine learning, computer science and other applied sciences. Positive definite and radially symmetric kernels on metric spaces have been introduced in the seminal papers [22, 23].

There has been a growing interest in the last twenty years for positive definiteness over product spaces. The main motivation stems from stochastic processes defined continuously over subsets of the type $X \times Y$, where X is a subset of the d-dimensional Euclidean space $\mathbb{R}^d$, and Y is either the whole real line or the set of integers $\mathbb{Z}$, and represents time. The nomenclature *space-time covariance functions* is commonly accepted for kernels that are positive definite over such product spaces, and the reader is referred to [8] for a review. A wealth of literature is available for the case $X = \mathbb{R}^d$, and the reader is referred to [2, 7, 16, 18, 21] and to [6] for relevant contributions. Recently, much attention has been put on the case $X = S^{d-1}$, the unit sphere embedded in $\mathbb{R}^d$. A characterization theorem for this case (including the Hilbert sphere S^∞) is available thanks to [4]. Other contributions can be found in [10, 15] and recently in [27].

This paper considers quasi-metric spaces, that is, pairs (X, σ) where X is a non-empty set and σ is a *quasi-distance*, that is, a function $\sigma : X \times X \to [0, \infty)$ satisfying $\sigma(x, x') = \sigma(x', x)$, $x, x' \in X$, and $\sigma(x, x) = 0$, $x \in X$. A *semi-metric space* (X, σ) is a quasi-metric space if in addition to the previous properties one has σ satisfying the triangle inequality. Further, if $\sigma(x, x') > 0$ when $x \neq x'$, the semi-metric space (X, σ) becomes a metric space.

Normed spaces and inner product spaces are typical examples of quasi-metric spaces with quasi-distance given by $\sigma(x, x') = \|x - x'\|$, $x, x' \in X$, where $\| \cdot \|$ is the norm of the space. The notion of semi-metric spaces is usually preferred when one deals with isometric embeddings.

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A quasi-metric space (X,σ) is *isometrically embeddable* in a Hilbert space $(\mathbb{H},\langle\cdot,\cdot\rangle)$ if there exists a mapping $i: X\to\mathbb{H}$ such that

$$\langle i(x)-i(x'), i(x)-i(x')\rangle = \sigma(x,x')^2, \quad x,x'\in X.$$

This notion is explored in [28] and discussed in [3].

Let E be a nonempty set. A mapping $\varphi: E\times E\to\mathbb{R}$ is called *positive definite* if

$$\sum_{k,l=1}^{N} c_k c_l \varphi(x_k,x_l)\geq 0,$$

for any collection $\{c_k: k=1,\ldots,N\}\subset\mathbb{R}$ and any $\{x_1,\ldots,x_k\}\subset E$. If E is a quasi-metric space (X,σ), the positive definite function φ on E is usually demanded to be metric-dependent in the sense that

$$\varphi(x,x') = f(\sigma(x,x')), \quad x,x'\in X,$$

where f is a continuous function. Obviously, the domain of f is understood to be the *diameter set* of X, that is,

$$D_X^\sigma = \{\sigma(x,x') : x,x'\in X\},$$

while continuity on a semi-metric space is defined the same way it is so in a metric-space. Depending on E and its metric structure, one may find convenient characterizations for the positive definiteness of a function on E. One case that is somehow related to the present work involves the case where $E=\mathbb{R}^d$ without any metric structure but the function φ being translation-invariant, that is,

$$\varphi(x,y) = f(x-y), \quad x,y\in\mathbb{R}^d,$$

for some continuous function $f:\mathbb{R}^d\to\mathbb{R}$. In this case, a result of Bochner ([1]) shows that φ is positive definite if and only if f is the Fourier transform of a finite and positive Borel measure μ, i.e.,

$$f(x) = \int_{\mathbb{R}^d} e^{i\,x\,\cdot\,w} d\mu(w), \quad x\in\mathbb{R}^d, \tag{1.1}$$

with $\cdot$ denoting the dot product in $\mathbb{R}^d$.

For two quasi-metric spaces (X,σ) and (Y,ν), we denote by $PD(X\times Y,\sigma,\nu)$ the class of continuous functions $\varphi: D_X^\sigma\times D_Y^\nu\to\mathbb{R}$ such that the composite kernel

$$((x,y),(x',y'))\mapsto\varphi(\sigma(x,x'),\nu(y,y')), \quad (x,y),\ (x',y')\in X\times Y,$$

is positive definite on $X\times Y$. Analogously, we write $PD(X,\sigma)$ for the class of functions $\varphi: D_X^\sigma\to\mathbb{R}$ being continuous and such that the kernel $(x,x')\mapsto\varphi(\sigma(x,x'))$ is positive definite.

Next, let us recall the notion of complete monotonicity. A function $f:(0,\infty)\to\mathbb{R}$ is called *completely monotone* if it is infinitely often differentiable over $(0,\infty)$ and $(-1)^n f^{(n)}(t)\geq 0$ for all $t>0$ and all $n=0,1,\ldots$. In this paper, we will assume all completely monotone functions are bounded so that they have a continuous extension to $[0,\infty)$. In particular, $f(0)<\infty$. A nonnegative function $f:(0,\infty)\to\mathbb{R}$ having a completely monotone derivative is called a *Bernstein function*. A Bernstein function can be continuously extended to $[0,\infty)$. Additional information on completely monotone and Bernstein functions can be found in [20].

This paper deals with a class $\{G_\alpha:\alpha>0\}$ of continuous functions, where

$$G_\alpha(t,u) = \frac{1}{h(u)^\alpha} f\left(\frac{t}{h(u)}\right), \quad t,u\geq 0, \tag{1.2}$$

with f and h strictly positive and continuous. In principle, both functions f and h are defined over $[0,\infty)$, but they might be restricted to suitable subsets of $[0,\infty)$. Such a class has been especially popular in space-time geostatistics for the following reason: for d and l positive integers, f being bounded and completely monotone and h such that $\exp(-ch) \in PD(\mathbb{R}^l, \|\cdot\|)$ for all $c > 0$, where $\|\cdot\|$ stands for the Euclidean norm, sufficient conditions for G_α to belong to the class $PD(\mathbb{R}^d \times \mathbb{R}^\ell, \|\cdot\|^2, \|\cdot\|^2)$ were given in [7]. The resulting class $\{G_\alpha : \alpha \geq d/2\}$ is usually called *Gneiting's class*. For f being bounded completely monotone, necessary and sufficient conditions on h have been provided by [30] in order that G_α belong to $PD(\mathbb{R}^d \times Y, \|\cdot\|^2, \|\cdot\|_Y)$, where $(Y, \|\cdot\|_Y)$ is a normed linear space. Porcu et al. ([15]) presented sufficient conditions for G_α to belong to the class $PD\left(\mathbb{R}^d \times S^m, \|\cdot\|^2, \theta_m\right)$, where θ_m is the geodesic distance over S^m. Sufficient conditions for G_α to belong to the class $PD(S^m \times \mathbb{R}^d, \theta_m, \|\cdot\|^2)$ for all d and m were shown recently in [27]. In [21], Schlather has considered diagonalized versions $\tilde{G}_\alpha(t) := G_\alpha(t,t)$, $t \geq 0$, of G_α. Minor modifications of G_α within the class $PD(\mathbb{R}^d \times \mathbb{R}, \|\cdot\|^2, |\cdot|)$ have been proposed by [6] and [17]. Finally, [16] has considered the class $PD(\prod_{k=1}^m \mathbb{R}^k, \|\cdot\|^2, \ldots, \|\cdot\|^2)$ on the basis of a generalization of the function G_α.

The previous paragraph cannot be detached from a classical result proved by I. J. Schoenberg ([23]) involving conditionally negative definite functions. Recall that for a quasi-metric space (X, σ), a continuous function $f : D_X^\sigma \to \mathbb{R}$ is *conditionally negative definite* on X, and we write $f \in CND(X, \sigma)$, if for $n \geq 1$ and points $x_1, \ldots, x_N$ in X, it holds

$$\sum_{j,k=1}^{N} c_j c_k f(\sigma(x_j, x_k)) \leq 0,$$

for all real numbers $c_1, \ldots, c_n$ satisfying $\sum_{j=1}^N c_j = 0$. If (X, σ) is a quasi-metric space, a function $h : D_X^\sigma \to \mathbb{R}$ belongs to $CND(X, \sigma)$ if and only if all the functions $u \in D_X^\sigma \mapsto \exp(-sh(u))$, $s > 0$, belong to $PD(X, \sigma)$. In particular, some of results described in the previous paragraph can be re-established with the CND nomenclature.

Given the existing results, it is natural to ask for results that allow for a very general version as well as for a unifying framework. The plan of this paper is the following. Section 2 provides the necessary background, some preliminary results, a general abstract result that produces functions in $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$, where (X, σ) is quasi-metric and examples. In particular, the results imply an alternative proof of the original Gneiting's result that does not involve convergence arguments. Section 3 contains expanded versions of Gneiting's result and adaptations to the case, where one of the spaces is (S^m, θ_m).

2. Preliminary Findings

Positive definite functions of the type (1.1) that are additionally radially symmetric are characterized as those functions f belonging to the class $PD(\mathbb{R}^d, \|\cdot\|)$. Define the function $\Omega_d : [0, \infty) \to \mathbb{R}$ through $\Omega_d(0) = 1$ and the identity

$$\Omega_d(t) = \Gamma(d/2)\left(\frac{2}{t}\right)^{(d-2)/2} J_{(d-2)/2}(t), \quad t > 0, \tag{2.3}$$

where J_ν is the Bessel function of first kind and order ν. As showed in [23], the continuous function $f : [0, \infty) \to \mathbb{R}$ with $f(0) = 1$ belongs to the class $PD(\mathbb{R}^d, \|\cdot\|)$ if and only if

$$f(t) = \int_{[0,\infty)} \Omega_d(rt) d\mu(r), \quad t \geq 0,$$

where μ is a probability measure. Arguments in [5] show that Ω_d is the characteristic function of a random vector being uniformly distributed over the unit spherical shell S^{d-1} embedded in $\mathbb{R}^d$. Also, a convergence argument in [23] reveals that the class $\bigcap_d PD(\mathbb{R}^d, \|\cdot\|)$ is uniquely determined through scale mixtures of the type

$$f(t) = \int_{[0,\infty)} e^{-rt^2} d\mu(r), \quad t \geq 0,$$

with μ as before. This fact has a striking connection with completely monotone functions. By Bernstein–Widder's theorem ([26]), a continuous function $f : [0, \infty) \to \mathbb{R}$ restricts to a completely monotone if and only if it is the Laplace transform of a positive and bounded measure μ:

$$f(t) = \int_{[0,\infty)} e^{-rt} d\mu(r), \quad t \geq 0. \tag{2.4}$$

In particular, this shows that $f \in PD(\mathbb{R}^d, \|\cdot\|)$ if and only if $t \in (0, \infty) \mapsto f(\sqrt{t})$ is completely monotone.

For d a positive integer, let L^1_{d-1} denote the class of real measurable functions g on $[0, \infty)$ for which $\int_0^\infty |g(r)|r^{d-1}dr < \infty$. The *Fourier-Bessel transform* $\mathcal{F}_d(g)$ *of order* $(d-2)/2$ of a function $g \in L^1_{d-1}$ is defined by

$$\mathcal{F}_d(g)(t) = \int_0^\infty g(r)\Omega_d(tr)r^{d-1}dr, \quad t \in [0, \infty). \tag{2.5}$$

It is well-known that $\mathcal{F}_d$ maps continuously and injectively L^1_{d-1} into the set $C_0([0, \infty))$ of continuous functions on $[0, \infty)$ vanishing at infinity ([25, chapter 5]). On the other hand, the fact that $t \in [0, \infty) \mapsto \Omega_d(tr)$, $r > 0$, belongs to $PD(\mathbb{R}^d, \|\cdot\|)$, implies that the following elementary result holds.

Proposition 2.1. *If* $g : [0, \infty) \to [0, \infty)$ *belongs to* L^1_{d-1}*, then* $\mathcal{F}_d(g)$ *belongs to* $PD(\mathbb{R}^d, \|\cdot\|)$.

A generalization of Proposition 2.1 is stated below and will turn to be very useful for the findings following subsequently.

Theorem 2.2. *Let* (X, σ) *be a quasi-metric space. Let* $g : [0, \infty) \times D^\sigma_X \to \mathbb{R}$ *satisfy the following assertions:*

(i) $g(\cdot, u)$ *belongs to* L^1_{d-1} *for any fixed* $u \in D^\sigma_X$*;*
(ii) $g(r, \cdot)$ *belongs to* $PD(X, \sigma)$ *for any fixed* $r \geq 0$.

If the mapping $(t, u) \in [0, \infty) \times D^\sigma_X \mapsto \mathcal{F}_d(g(\cdot, u))(t)$ *is continuous on* $[0, \infty) \times D^\sigma_X$*, then it belongs to the class* $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$.

Proof. Using Equation (2.5), the function $\mathcal{F}_d(g(\cdot, u))$ can be written as

$$\mathcal{F}_d(g(\cdot, u))(t) = \int_0^\infty g(r, u)\Omega_d(tr)r^{d-1}dr, \quad (t, u) \in [0, \infty) \times D^\sigma_X,$$

which, in concert with Schur product theorem ([11, p. 455]), completes the proof. □

An implication of Bernstein-Widder's theorem is stated below. More details can be found in [14].

Proposition 2.3. *Let* (X, σ) *be a quasi-metric space. If* f *is bounded and completely monotone and* h *is a nonnegative valued function in* $CND(X, \sigma)$*, then* $f \circ h$ *belongs to* $PD(X, \sigma)$.

Proposition 2.3 is very useful to discuss the following important example.

Example 2.4. Let (X,σ) be a quasi-metric space and d a positive integer. Let h be a nonnegative valued function in $CND(X,\sigma)$. Then, we claim that

$$(t,u) \in [0,\infty) \times D_Y^\sigma \mapsto \frac{e^{-t\sqrt{h(u)}}}{\sqrt{h(u)}}$$

belongs to $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$. To show it, let $v > 0$. We first recall the identity ([9, p. 678])

$$\int_0^\infty r^{d/2}(r^2+v^2)^{-(d+1)/2} J_{(d-2)/2}(tr)dr = \frac{\sqrt{\pi}t^{(d-2)/2}}{2^{d/2}ve^{vt}\Gamma((d+1)/2)}, \quad v,t>0.$$

Resorting to Equation (2.3), and rearranging terms, we obtain

$$\int_0^\infty (r^2+v^2)^{-(d+1)/2}\Omega_d(tr)r^{d-1}dr = \frac{\Gamma(d/2)\sqrt{\pi}}{2ve^{vt}\Gamma((d+1)/2)}, \quad v,t>0.$$

Since the function on the right hand side is continuous, by letting $t \to 0^+$ we have that the identity above holds for $t = 0$ as well. Since, for r fixed, $v \in (0,\infty) \mapsto (r^2+v)^{-(d+1)/2}$ is bounded and completely monotone, if h is a nonnegative valued function in (X,σ), then Proposition 2.3 shows that $u \in D_X^\sigma \mapsto (r^2+h(u))^{-(d+1)/2}$ belongs to $PD(X,\sigma)$. After ignoring positive constants, we can invoke Theorem 2.2 to show our claim.

It might be interesting to note that this example does not belong to the Gneiting class G_α.

We now rephrase Theorem 2.2 according to the language of Fourier transforms. For an absolutely integrable function F in $\mathbb{R}^d$, its Fourier transform $\widehat{F}$ is given by the formula

$$\widehat{F}(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} F(y)e^{-i\,x\,\cdot\,y}dy, \quad x \in \mathbb{R}^d.$$

It is well known that if F is radial, that is, $F(x) = f(\|x\|)$, for some function $f : [0,\infty) \to \mathbb{R}$, then $\widehat{F}$ is radial as well [5]. For a function $G : \mathbb{R}^d \times D_X^\sigma \to \mathbb{R}$, we write $\widehat{G}(\cdot,u)$ to denote the Fourier transform of $x \in \mathbb{R}^d \mapsto G(x,u)$, for a fixed u, whenever it exists. If $G(\cdot,u)$ is radial in the first variable, that is,

$$G(x,u) = g(\|x\|,u), \quad x \in \mathbb{R}^d,$$

for some g and $\widehat{G}(\cdot,u)$ exists, then we may also write $\widehat{G}(x,u) = \widehat{g}(\|x\|,u)$, for some function $\widehat{g}(\cdot,u)$. This notation appears below.

Theorem 2.5. *Let (X,σ) be a quasi-metric space. Let $G : \mathbb{R}^d \times D_X^\sigma \to \mathbb{R}$ be radial in the first variable and assume the following assumptions hold:*

(i) *$g(\cdot,u)$ belongs to L^1_{d-1} for any fixed $u \in D_X^\sigma$;*
(ii) *$g(r,\cdot)$ belongs to $PD(X,\sigma)$ for any fixed $r \geq 0$.*

If the mapping $(t,u) \in [0,\infty) \times D_X^\sigma \mapsto \widehat{g}(t,u)$ is continuous on $[0,\infty) \times D_X^\sigma$, then it belongs to the class $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$.

Proof. Theorem 5.26 in [29] shows that if $g(\cdot,u) \in L^1_{d-1}$ for any $u \in D_X^\sigma$, then $g(\|\cdot\|,u)$ is absolutely integrable in $\mathbb{R}^d$. In particular, $\widehat{G}(\cdot,u)$ is well defined for any fixed u. Invoking again Theorem 5.26 in [29], we have that

$$\widehat{G}(x,u) = \frac{2^{-(d-2)/2}}{\Gamma(d/2)}\mathcal{F}_d(g(\cdot,u))(\|x\|), \quad x \in \mathbb{R}^d; u \in D_X^\sigma.$$

An application of Theorem 2.2 completes the proof. □

An illustration of Theorem 2.5 follows.

Proposition 2.6. *Let (X, σ) be a quasi-metric space and h a nonnegative valued function in the class $CND(X, \sigma)$. For $s > 0$, let H_s be the function defined through*

$$H_s(t, u) = \frac{e^{-st^2/h(u)}}{h(u)^{d/2}}, \quad (t, u) \in [0, \infty) \times D_X^\sigma. \tag{2.6}$$

Then H_s belongs to $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$ for all s.

Proof. We start by invoking the well-known identity ([1, p.13])

$$e^{-\|v\|^2/2} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\, v\, \cdot\, w} e^{-\|w\|^2/2} dw, \quad v \in \mathbb{R}^d.$$

Elementary Fourier inversion allows to write

$$2^{d/2} \frac{e^{-\|v\|^2/\xi h(u)}}{\xi^{d/2} h(u)^{d/2}} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\, v\, \cdot\, w} G(\|w\|, u) dw, \quad v \in \mathbb{R}^d; \xi > 0,$$

where $G : [0, \infty) \times D_X^\sigma \to \mathbb{R}$ is given through the identity

$$G(x, u) = e^{-sh(u)x^2/4}, \quad x \geq 0; u \in D_X^\sigma.$$

Since G is radial in the first argument, we can now write

$$g(r, u) = e^{-sh(u)r^2/4}, \quad r \geq 0; u \in D_X^\sigma.$$

Proposition 2.3 shows that $g(r, \cdot)$ satisfies Assumption *(ii)* in Theorem 2.5 for all $r \geq 0$. Assumptions (i) holds trivially, while for $\xi > 0$ fixed, the function

$$\widehat{g}(t, u) = 2^{d/2} \frac{e^{-t^2/\xi h(u)}}{\xi^{d/2} h(u)^{d/2}}, \quad (t, u) \in [0, \infty) \times D_X^\sigma,$$

is continuous. Theorem 2.5 shows that

$$(t, u) \in [0, \infty) \times D_X^\sigma \mapsto 2^{d/2} \frac{e^{-t^2/\xi h(u)}}{\xi^{d/2} h(u)^{d/2}},$$

belongs to the class $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$ for all positive ξ. A change of variable of the type $\xi = 1/s$ and the fact that we can ignore multiplicative positive constants complete the proof. $\square$

Remark 2.7. The function H_s provides a way to prove Gneiting's theorem [7] by direct construction, without resorting to Fourier transform techniques which in turn require integrability assumptions and the application of a convergence argument.

Remark 2.8. Proposition 2.6 can be also proved by invoking Theorem 2.2, in concert with the identity

$$\int_0^\infty e^{-h(u)r^2/4s} \Omega_d(tr) r^{d-1} dr = 2^{d-1} s^{d/2} \Gamma(d/2) \frac{e^{-st^2/h(u)}}{h(u)^{d/2}}, \quad s > 0; t \geq 0; u \in D_X^\sigma,$$

that is derived from the equality (see [9, p.706])

$$\int_0^\infty r^{d/2} e^{-vr^2} J_{(q-2)/2}(tr) dr = t^{(d-2)/2} \frac{e^{-t^2/4v}}{(2v)^{q/2}}, \quad t, v > 0.$$

Example 2.9. We use Formula 69 in Chapter 25 of [19]:

$$\int_0^\infty r^{(d+4)/2} e^{-vr^2} J_{(d-2)/2}(tr)dr = \frac{t^{(d-2)/2}}{2^{d/2}v^{(d+2)/2}} \left(\frac{d}{2} - \frac{t^2}{4v}\right) e^{-t^2/4v}, \quad t, v > 0.$$

Simple algebra manipulation as in the previous example leads to

$$\int_0^\infty r^2 e^{-vr^2} \Omega_d(tr) r^{d-1} dr = \frac{\Gamma(d/2)}{2} \left(\frac{d}{2} - \frac{t^2}{4v}\right) \frac{e^{-t^2/4v}}{v^{1+d/2}}, \quad v > 0; t \geq 0.$$

Replacing v by $h(u)/(4s)$, with $h \in CND(X, \sigma)$, yields that

$$\int_0^\infty r^2 e^{-h(u)r^2/4s} \Omega_d(tr) r^{d-1} dr = \Gamma(d/2) 2^{d+1} s^{1+d/2} \left(\frac{d}{2} - \frac{st^2}{h(u)}\right) \frac{e^{-st^2/h(u)}}{h(u)^{1+d/2}},$$

for $s, t > 0$ and $u \in D_X^\sigma$. Theorem 2.2 now shows that

$$(t, u) \in [0, \infty) \times D_X^\sigma \mapsto s^{1+d/2} \left(\frac{d}{2} - \frac{st^2}{h(u)}\right) \frac{e^{-st^2/h(u)}}{h(u)^{1+d/2}}$$

belongs to $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$. We can also integrate with respect to s in order to see that

$$(t, u) \in [0, \infty) \times D_X^\sigma \mapsto \frac{1}{h(u)^{1+d/2}} \int_0^\infty s^{1+d/2} \left(\frac{d}{2} - \frac{st^2}{h(u)}\right) e^{-st^2/h(u)} d\mu(s)$$

also belongs to $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$ as longs as μ is a convenient measure on $[0, \infty)$. Again, this example does not belong to the Gneiting class G_α.

3. Gneiting Class: Results

The following result is another implication of Bernstein-Widder's theorem. As in Proposition 2.3, it can be extracted from [14].

Proposition 3.1. *Let (X, σ) be a quasi-metric space. Let g be a Bernstein function and h a nonnegative valued function in $CND(X, \sigma)$. Then, $\exp(-c(g \circ h))$ belongs to $PD(X, \sigma)$ for all $c > 0$.*

We are now ready to state and prove one of our main contributions.

Theorem 3.2. *Let d be a positive integer and (X, σ) a quasi-metric space. Let G_α be the function defined at Equation (1.2) with f being completely monotone. If $a \in (0, 1]$ and $\alpha \geq d/2$, then the following assertions are true:*

- (i) *G_α belongs to $PD(\mathbb{R}^d \times X, \|\cdot\|^{2a}, \sigma)$ provided h is a nonnegative valued function in the class $CND(X, \sigma)$;*
- (ii) *G_α belongs to $PD(\mathbb{R}^d \times X, \|\cdot\|^{2a}, \sigma)$ provided $h := g \circ h_1$, where g is a positive Bernstein function and h_1 is a nonnegative function in $CND(X, \sigma)$;*
- (iii) *G_α belongs to $PD(\mathbb{R}^d \times X, \|\cdot\|^{2a}, \sigma^{2b})$ provided $b \in (0, 1]$, h is a positive Bernstein function, and (X, σ) is isometrically embeddable in a Hilbert space.*

Proof. Let us show Assertion (i) by invoking Proposition 2.6, which shows that H_s in Equation (2.6) defines an element of the class $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$ for all $s > 0$. This in turn shows, in concert with Bernstein–Widder's theorem that

$$G_{d/2}(t^2, u) = \frac{1}{h(u)^{d/2}} \int_0^\infty e^{-st^2/h(u)} d\mu(s) = \int_0^\infty H_s(t, u) d\mu(s)$$

is also a member of $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$, and thus $G_{d/2}(t,u) \in PD(\mathbb{R}^d \times X, \|\cdot\|^2, \sigma)$. We now observe that if h is a nonnegative valued function in $CND(X,\sigma)$, then the Laplace transform identity

$$\frac{1}{x} = \int_0^\infty e^{-sx} ds, \quad x > 0,$$

shows that $1/h \in PD(X,\sigma)$, while the identity

$$x^a = \left(\int_0^\infty \frac{1-e^{-s^2}}{s^{1+2a}} ds \right)^{-1} \int_0^\infty \frac{1-e^{-s^2 x}}{s^{1+2a}} ds, \qquad x \geq 0; a \in (0,1),$$

implies that $h^a \in PD(X,\sigma)$ for $a \in (0,1)$. To proceed, for $\alpha \geq d/2$, write

$$G_\alpha(t,u) = \frac{1}{h(u)^{\alpha-d/2}} G_{d/2}(t,u), \quad t,u \geq 0,$$

and notice that Proposition 2.3 shows that

$$u \in [0,\infty) \mapsto \frac{1}{h(u)^{\alpha-d/2}}$$

belongs to $PD(X,\sigma)$. Since it is an easy matter to verify that

$$(t,u) \in [0,\infty)^2 \mapsto \frac{1}{h(u)^{\alpha-d/2}}$$

belongs to $PD(\mathbb{R}^d \times X, \|\cdot\|, \sigma)$, we may invoke the Schur product theorem in order to deduce that $G_\alpha \in PD(\mathbb{R}^d \times X, \|\cdot\|^2, \sigma)$, for $\alpha \geq d/2$. Finally, for any Hilbert space $\mathbb{H}$, it is well-known that the semi-metric space $(\mathbb{H}, \|\cdot\|^a)$ is isometrically embeddable into $(\mathbb{H}, \|\cdot\|)$ itself. Therefore, we have that $G_\alpha \in PD(\mathbb{R}^d \times X, \|\cdot\|^{2a}, \sigma)$. Assertion (ii) follows from (i) in concert with Proposition 3.1. If (X,σ) is isometrically embeddable in a Hilbert space, then the function $h_1(u) = u^2$ belongs $CND(X,\sigma)$. Consequently, so does h_1^b, for $b \in (0,1]$. Thus, Assertion (iii) follows from (ii). □

In the last two results in the paper, we will employ the previous results in order to obtain positive definite functions on $S^m \times X$, where X is a quasi-metric space.

Theorem 3.3. *Let m be a positive integer. Let G_α be the function defined at Equation (1.2) with f being bounded and completely monotone. Then, the following assertions hold:*

(i) *G_α belongs to $PD(S^m \times X, (2-2\cos\theta_m)^a, \sigma)$ provided (X,σ) is a quasi -metric space, h is a nonnegative valued function in $CND(X,\sigma)$, $\alpha \geq (m+1)/2$ and $a \in (0,1]$;*

(ii) *G_α belongs to $PD(S^m \times S^l, (2-2\cos\theta_m)^a, \theta_l^b)$ provided $l \geq 1$, $\alpha \geq (m+1)/2$, $a,b \in (0,1]$ and h is a Bernstein function.*

Proof. Assertion (i) follows from the obvious identity

$$\|x-y\|^2 = 2 - 2\cos\theta_m(x,y), \quad x,y \in S^m,$$

in concert with Theorem 3.2-(i). As for Assertion (ii), we first notice that $(S^l, \theta_l^{1/2})$ is isometrically embeddable in $(S^\infty, \theta_\infty^{1/2})$. On the other hand, arguments in [24] show that $(S^\infty, \theta_\infty^{1/2})$ is isometrically embeddable in a Hilbert space. Thus, the assertion follows from Assertion (i) and Theorem 3.2-(iii). □

It becomes natural to ask whether Theorem 3.3 still holds when the metric $(2-2\cos\theta_m)^a$ is replaced with the geodesic θ_m. The answer seems to rely on a suitable choice of the completely monotone function f in the definition of G_α. Below, we show that it is true whenever $\alpha, s > 0$, $f(t) = (s+t)^{-\alpha}$ and h belongs $CND(X,\sigma)$. Indeed, it suffices to observe that

$$\frac{\Gamma(\alpha)}{(t+sh(u))^\alpha} = \int_0^\infty e^{-tx}e^{-sh(u)x}x^{\alpha-1}dx.$$

Applying Proposition 2.3 once again, it is now seen that

$$(t,u) \in [0,\pi]\times D_X^\sigma \mapsto \frac{\Gamma(\alpha)}{(t+sh(u))^\alpha} \tag{3.7}$$

belongs to $PD(S^m \times X, \theta_m, \sigma)$. Finally, one needs to observe that

$$\frac{1}{(t+sh(u))^\alpha} = \frac{1}{h(u)^\alpha} f\left(\frac{t}{h(u)}\right) = G_\alpha(t,u), \quad (t,u)\in[0,\pi]\times D_X^\sigma.$$

The elaborations above suggest that a special class of completely monotonic functions might turn to be useful for the result that follows. Following [12], we call a function $f:[0,\infty)\to\mathbb{R}$ a generalized Stieltjes function of order λ if

$$f(x) = C + \int_0^\infty e^{-xr}r^{\lambda-1}\phi(r)dr,$$

for some completely monotone function ϕ and some $C \geq 0$.

Theorem 3.4. *Let m be a positive integer and (X,σ) a quasi-metric space. Let G_α be the function defined at Equation (1.2) with f a generalized Stieltjes function of order $\lambda > 0$. Then, G_α belongs to $PD(S^m\times X,\theta_m,\sigma)$ provided $\alpha \geq \lambda$ and h is a nonnegative valued function in $CND(X,\sigma)$.*

Proof. According to [20, p. 16], we can write

$$\begin{aligned} G_{\epsilon+\lambda}(t,u) &= \frac{1}{h(u)^{\epsilon+\lambda}}\left[A + \int_0^\infty \frac{h(u)^\lambda}{(rh(u)+t)^\lambda}dr\right] \\ &= \frac{A}{h(u)^{\epsilon+\lambda}} + \frac{1}{h(u)^\epsilon}\int_0^\infty \frac{1}{(rh(u)+t)^\lambda}dr, \quad t\geq 0; u\in D_X^\sigma; \epsilon \geq 0, \end{aligned}$$

for some positive constant A and a convenient positive measure μ on $[0,\infty)$. Since the function in Equation (3.7) belongs to $PD(S^m\times X,\theta_m,\sigma)$, when h is a nonnegative valued function in $CND(X,\sigma)$, the same is true for

$$(t,u)\in[0,\pi]\times D_X^\sigma \mapsto \int_0^\infty \frac{1}{(rh(u)+t)^\lambda}dr.$$

Invoking Proposition 2.3 and taking into account that the class $PD(S^m\times X,\theta_m,\sigma)$ is a convex cone, it follows that $(t,u)\in[0,\pi]\times D_X^\sigma \mapsto G_{\epsilon+\lambda}(t,u)$ belongs to $PD(S^m\times X,\theta_m,\sigma)$. The proof is completed. □

Deeper results providing generalizations of Gneiting's result via generalized Stieltjes functions were obtained recently in [13]. In particular, it provides concrete examples of completely monotone functions f that lead to classes $\{G_\alpha : \alpha > 0\}$ of strictly positive definite functions.

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Research Article

A Note on the Stability of Some Functional Equations on Certain Groupoids

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ABSTRACT. In this paper, we show that the stability of Cauchy set-valued functional equations and of Jensen set-valued functional equations can be derived from the stability of the corresponding equations in single-valued version.

Keywords: Stability, Cauchy set-valued functional equation, Jensen set-valued functional equation, fixed point method.

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1. INTRODUCTION

In the sequel, $\mathbb{N}, \mathbb{N}_0, \mathbb{R}$ and $\mathbb{R}_+$ denote the set of positive integers, the set of nonnegative integers, the set of real numbers, and the set of nonnegative real numbers, respectively. For two nonempty sets X and Y, Y^X denotes the class of all functions from X into Y.

Given a groupoid $(G, \circ)$, the binary operation $\circ$ is *square-symmetric* if

$$2(a \circ b) = 2a \circ 2b \quad \text{for all } a, b \in G.$$

Here $2a := a \circ a$ for all $a \in G$. It is clear that every commutative semigroup is a square-symmetric groupoid but the converse is not true. (In fact, let $G := \mathbb{N}$ be equipped with a binary operation $a \circ b := a + 2b$ for all $a, b \in G$. Then $(G, \circ)$ is a square-symmetric groupoid and $\circ$ is not associative.) Recall that a groupoid $(G, \circ)$ is *divisible* if for each $a \in G$ there exists a unique element $a' \in G$ such that $2a' = a$. For convenience, we will write $\frac{a}{2} := a'$ or $\frac{1}{2}a := a'$. To simplify the notation, for each a in a groupoid $G := (G, \circ)$ and each $n \in \mathbb{N}_0$, we write $2^0 a := a$ and $2^{n+1}a := 2(2^n a)$. If, in addition, G is divisible, then we also write $\frac{a}{2^0} := a$ and $\frac{a}{2^{n+1}} := \frac{1}{2}\left(\frac{a}{2^n}\right)$ all $n \in \mathbb{N}_0$.

Lemma 1.1. *Suppose that $(G, \circ)$ is a square-symmetric groupoid. Then, the following assertions are true.*

(1) $2^n(a \circ b) = 2^n a \circ 2^n b$ *for all $a, b \in G$ and all $n \in \mathbb{N}_0$.*
(2) $\frac{1}{2^n}(a \circ b) = \frac{a}{2^n} \circ \frac{b}{2^n}$ *for all $a, b \in G$ and all $n \in \mathbb{N}_0$ provided that G is divisible.*

From now on, we assume that:

- $G := (G, \circ)$ and $H := (H, *)$ are square-symmetric groupoids;
- (H, d) is a complete metric space such that $*$ is *continuous*, that is, $\lim_{k\to\infty} d(u_k * v_k, u * v) = 0$ whenever $\lim_{k\to\infty} d(u_k, u) = \lim_{k\to\infty} d(v_k, v) = 0$;
- $\varphi : G \times G \to \mathbb{R}_+$ is a function.

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To study the stability of a Cauchy functional equation, we define the following class:

$$\mathscr{C}(G,H,\varphi) := \Big\{ f \in H^G : d(f(x \circ y), f(x) * f(y)) \le \varphi(x,y) \text{ for all } x,y \in G \Big\}.$$

Now, we introduce the stability concept of a functional equation as follows.

Definition 1.2. The Cauchy functional equation from G into H is said to be φ-*stable* if the class $\mathscr{C}(G,H,\varphi)$ satisfies the following property:

> *there exists a function $\Phi \in \mathbb{R}_+^G$ such that for every $f \in \mathscr{C}(G,H,\varphi)$ there exists a unique function $F \in \mathscr{C}(G,H,\mathbf{0})$, that is, $F(x \circ y) = F(x) * F(y)$ for all $x,y \in G$, such that*
> $$d(F(x), f(x)) \le \Phi(x) \quad \textit{for all } x \in G.$$

In this case, we also say that the class $\mathscr{C}(G,H,\varphi)$ is φ-*stable with respect to* Φ.

In 1940, Ulam [17] proposed a problem concerning the stability of the Cauchy functional equation from a group G into a metric group H. A year later, Hyers [10] was the first mathematician who answered Ulam's problem if G and H are Banach spaces. Many generalizations of Hyers' result have been studied [1, 8, 9, 15]. Inspired by the notion of square-symmetry, Páles et al. [12] and Kim [11] proved some stability results of the Cauchy functional equation from square-symmetric groupoids into metric square-symmetric groupoids based on the control function proposed by Găvruţa [9].

We now recall the following conditions given by Kim [11]. A triplet (G,H,φ) satisfies

***Condition (K1)*:** if the following two conditions hold:
- (K1a) $H := (H,*)$ is divisible;
- (K1b) there exists a real number $\gamma > 0$ such that $d(\frac{u}{2}, \frac{v}{2}) \le \gamma d(u,v)$ for all $u,v \in H$ and $\widetilde{\varphi}(x,y) := \sum_{k=0}^{\infty} \gamma^k \varphi(2^k x, 2^k y) < \infty$ for all $x,y \in G$;

***Condition (K2)*:** if the following two conditions hold:
- (K2a) $G := (G,\circ)$ is divisible;
- (K2b) there exists a real number $\gamma > 0$ such that $d(2u,2v) \le \gamma d(u,v)$ for all $u,v \in H$ and $\widetilde{\varphi}(x,y) := \sum_{k=1}^{\infty} \gamma^k \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) < \infty$ for all $x,y \in G$.

The following stability result was proved by Kim [11].

Theorem K. *If (G,H,φ) satisfies either Condition (K1) or Condition (K2), then the Cauchy functional equation from G into H is φ-stable. Moreover, the class $\mathscr{C}(G,H,\varphi)$ is φ-stable with respect to Φ, where*

$$\Phi(x) := \begin{cases} \gamma\widetilde{\varphi}(x,x) & \textit{if } (G,H,\varphi) \textit{ satisfies Condition (K1);} \\ \frac{1}{\gamma}\widetilde{\varphi}(x,x) & \textit{if } (G,H,\varphi) \textit{ satisfies Condition (K2);} \end{cases}$$

for all $x \in X$.

The authors were informed by the referee that Theorem K with Condition (K1) is related to the result of Forti [7].

Remark 1.3. According to Theorem K, for each $f \in \mathscr{C}(G,H,\varphi)$, the function $F \in \mathscr{C}(G,H,\mathbf{0})$ is uniquely determined by

$$F(x) = \begin{cases} \lim_{k\to\infty} \frac{1}{2^k} f(2^k x) & \text{if } (G,H,\varphi) \text{ satisfies Condition (K1);} \\ \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right) & \text{if } (G,H,\varphi) \text{ satisfies Condition (K2);} \end{cases}$$

for all $x \in X$.

A concept of set-valued functions in Banach spaces have been developed in the last decades. The result concerning the set-valued functional equation (a functional equation whose solutions are set-valued functions) seems to be pioneered by Aumann [3] and Debreu [2]. We recall the following: Suppose that $X := (X, \|\cdot\|)$ is a real normed space. We define

$$\mathcal{P}(X) := \{A : A \text{ is a subset of } X\};$$
$$\mathrm{BCC}(X) := \{A \in \mathcal{P}(X) \backslash \{\varnothing\} : A \text{ is bounded, closed, and convex}\}.$$

For each $A, B \in \mathrm{BCC}(X)$ and $\lambda \in \mathbb{R}$, we define:

- $A \oplus B := \mathrm{cl}\{a + b : a \in A \text{ and } b \in B\}$ (cl := the closure);
- $\lambda A := \{\lambda a : a \in A\}$;
- $\mathcal{H}(A, B) := \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\}$;
- $\operatorname{diam} A := \sup_{a,a'\in A} \|a - a'\|$.

It is worth mentioning that in the definition of $\oplus$ the closure is needed because $\{a + b : a \in A \text{ and } b \in B\}$ does not necessarily belong to $\mathrm{BCC}(X)$ if $A, B \in \mathrm{BCC}(X)$ [14, Theorem 2.1].

By using the fixed point alternative method proposed by Diaz and Magolis [6], Park et al. [13] proved some stability results of the Cauchy functional equations from a real normed space X into a complete metric commutative semigroup $(\mathrm{BCC}(Y), \oplus)$, where Y is a Banach space.

In this paper, we first show that the results of Park et al. [13] for the Cauchy functional equation from a real normed space X into $(\mathrm{BCC}(Y), \oplus)$ (where Y is a Banach space) is a consequence of Theorem K. In addition, inspired by the work of Kim [11], we prove the stability results of the Jensen functional equations by using a modified Brzdęk's fixed point theorem [16]. We also point out that the stability result of Jensen set-valued functional equations can be derived from that of the corresponding on certain groupoids. Roughly speaking, we obtain *the stability result of the Cauchy "set-valued" functional equation and that of the Jensen "set-valued" functional equation from the corresponding results of the "single-valued" version*.

2. Main Results

2.1. Stability of Cauchy set-valued functional equations via that of Cauchy single-valued functional equations. We first recall the following properties.

Lemma 2.1. *Let X be a real normed space. Suppose that $A, B, C, D \in \mathrm{BCC}(X)$ and $\lambda, \mu \in \mathbb{R}_+$. Then, following assertions are true.*

(a) $A \oplus B \in \mathrm{BCC}(X)$ *and* $\lambda A \in \mathrm{BCC}(X)$.
(b) $\lambda A \oplus \lambda B = \lambda(A \oplus B)$ *and* $(\lambda + \mu)A = \lambda A \oplus \mu A$.
(c) $\mathcal{H}(A \oplus C, B \oplus D) \leq \mathcal{H}(A, B) + \mathcal{H}(C, D)$.
(d) $|\operatorname{diam} A - \operatorname{diam} B| \leq 2\mathcal{H}(A, B)$.
(e) $\mathcal{H}(\lambda A, \lambda B) = \lambda\mathcal{H}(A, B)$.
(f) *If X is a Banach space, then $(\mathrm{BCC}(X), \oplus, \mathcal{H})$ is a complete metric divisible commutative semigroup (see [5]).*

We obtain the following proposition as a consequence of Lemma 2.1(d).

Proposition 2.2. *Suppose that X, Y are two real normed spaces and $f : X \to \mathrm{BCC}(Y)$ is a function. Then, the following assertions are true.*

(i) *If $\lim_{k\to\infty} \operatorname{diam} f\left(2^k x\right)/2^k = 0$ for all $x \in X$, then $F(x) := \lim_{k\to\infty} f(2^k x)/2^k$ is single-valued for all $x \in X$.*
(ii) *If $\lim_{k\to\infty} \operatorname{diam} 2^k f\left(x/2^k\right) = 0$ for all $x \in X$, then $F(x) := \lim_{k\to\infty} 2^k f\left(x/2^k\right)$ is single-valued .*

By using Lemma 2.1(f), we obtain the main results of Park et al. [13, Theorems 2.2 and 2.3] as a consequence of Theorem K.

Corollary 2.3. *Suppose that X is a real normed space and Y is a real Banach space. If there exists $L \in (0,1)$ and one of the following conditions is satisfied:*

(1) $\varphi(x,y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$ *for all $x, y \in X$;*
(2) $\varphi(x,y) \leq \frac{L}{2}\varphi(2x,2y)$ *for all $x, y \in X$,*

then the Cauchy functional equation is φ-stable in the class of set-valued functions $\mathscr{C}(G,H,\varphi)$. Moreover, the class $\mathscr{C}(G,H,\varphi)$ is φ-stable with respect to Φ, where

$$\Phi(x) := \begin{cases} \frac{1}{2-2L}\varphi(x,0) & \text{if (1) holds;} \\ \frac{L}{2-2L}\varphi(x,0) & \text{if (2) holds;} \end{cases}$$

for all $x \in X$. In addition, if $f \in \mathscr{C}(X, \mathrm{BCC}(Y), \varphi)$ and there exist positive real numbers M and α such that: $\operatorname{diam} f(x) \leq M\|x\|^\alpha$ *for all $x \in X$, where $\alpha \in (0,1)$ if* (1) *holds; or $\alpha \in (1,\infty)$ if* (2) *holds, then the set-valued function F given by Remark 1.3 is single-valued.*

2.2. **Stability of the Jensen functional equations and some results for Jensen "multi-valued" functional equations.** Throughout this subsection, we assume that a square-symmetric groupoid $G := (G, \circ)$ has an identity e_G. We define the following two classes of functions (we remark that in the second class the divisibility of G is required):

$$\mathscr{J}_1(G,H,\varphi) := \left\{ f \in H^G : \begin{array}{c} d\left(2f\left(x \circ y\right), f(2x) * f(2y)\right) \leq \varphi(x,y) \text{ for all } x,y \in G \text{ and} \\ f(e_G) * f(x) = f(x) = f(x) * f(e_G) \text{ for all } x \in G \end{array} \right\},$$

$$\mathscr{J}_2(G,H,\varphi) := \left\{ f \in H^G : \begin{array}{c} d\left(2f\left(\dfrac{x \circ y}{2}\right), f(x) * f(y)\right) \leq \varphi(x,y) \text{ for all } x,y \in G \text{ and} \\ f(e_G) * f(x) = f(x) = f(x) * f(e_G) \text{ for all } x \in G \end{array} \right\}.$$

We now introduce the stability notion of the *Jensen functional equations* as follows.

Definition 2.4. The Jensen functional equation of the first (second, resp.) kind from G into H is said to be *φ-stable* if the class $\mathscr{J}_1(G,H,\varphi)$ ($\mathscr{J}_2(G,H,\varphi)$, resp.) satisfies the following property:

> *there exist a function $\Phi \in \mathbb{R}_+^G$ such that for every $f \in \mathscr{J}_1(G,H,\varphi)$ ($f \in \mathscr{J}_2(G,H,\varphi)$, resp.) there exists a unique function $F \in \mathscr{J}_1(G,H,\mathbf{0})$ ($F \in \mathscr{J}_2(G,H,\mathbf{0})$, resp.), that is, $2F(x \circ y) = F(2x) * F(2y)$ for all $x,y \in G$ ($2F(\frac{x \circ y}{2}) = F(x) * F(y)$ for all $x,y \in G$, resp.), such that*
>
> $$d(F(x), f(x)) \leq \Phi(x) \quad \text{for all } x \in G.$$

In this case, we say that the class $\mathscr{J}_1(G,H,\varphi)$ ($\mathscr{J}_2(G,H,\varphi)$, resp.) is *$\varphi$-stable with respect to Φ*.

Brzdęk [4] proved a fixed point theorem for the stability result of the Cauchy functional equation on a commutative semigroup. Later, Saejung and Senasukh [16] modified Brzdęk's fixed point theorem and also proved some stability results of some functional equations on restricted domains. The proof of the following fixed point theorem is similar to the one given by Saejung and Senasukh [16], so it is omitted.

Theorem 2.5. *Suppose that X is a nonempty set and (Y,d) is a complete metric space. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\theta : X \to X$ are mappings and there exists $\alpha \in \mathbb{R}^+$ such that*

$$d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) \leq \alpha d(\xi(\theta(x)), \mu(\theta(x))) \quad \text{for all } \xi, \mu \in Y^X \text{and all } x \in X.$$

If there exists a mapping $\varphi : X \to Y$ such that

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} d((\mathcal{T}^k\varphi)(x), (\mathcal{T}^{k+1}\varphi)(x)) < \infty \quad \text{for all } x \in X, \tag{2.1}$$

then $\mathcal{T}$ has a fixed point $\psi \in Y^X$ such that

$$d(\psi(x), \varphi(x)) \leq \varepsilon^*(x) \quad \text{for all } x \in X.$$

Moreover, $\psi(x) = \lim_{k\to\infty}(\mathcal{T}^k\varphi)(x)$ for all $x \in X$.

As a consequence of our fixed point theorem (Theorem 2.5), we first obtain the following stability result of the Jensen functional equation of the first kind.

Theorem 2.6. *If (G, H, φ) satisfies either Condition (K1) or Condition (K2), then the Jensen functional equation from G into H is φ-stable. Moreover, the class $\mathscr{J}_1(G, H, \varphi)$ is φ-stable with respect to Φ, where*

$$\Phi(x) := \begin{cases} \gamma\widetilde{\varphi}(x, e_G) & \text{if } (G, H, \varphi) \text{ satisfies Condition (K1);} \\ \frac{1}{\gamma}\widetilde{\varphi}(x, e_G) & \text{if } (G, H, \varphi) \text{ satisfies Condition (K2);} \end{cases}$$

for all $x \in X$.

Proof. We first suppose that (G, H, φ) satisfies Condition (K1). Let $f \in \mathscr{J}_1(G, H, \varphi)$ be given. Set $X := (G, +)$ and $Y := (H, *, d)$. Define mappings $\mathcal{T} : Y^X \to Y^X$ and $\theta : X \to X$ by

$$(\mathcal{T}\xi)(x) := \frac{1}{2}\xi(2x) \quad \text{and} \quad \theta(x) := 2x$$

for all $\xi \in Y^X$ and all $x \in X$. We also let $\alpha := \gamma$. For each $\xi, \mu \in Y^X$ and each $x \in X$, we see that

$$d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) \leq \gamma d\left(\xi\left(2x\right), \mu\left(2x\right)\right) = \alpha d(\xi(\theta(x)), \mu(\theta)(x)).$$

We show that the following inequality holds for all $n \in \mathbb{N}_0$;

$$d\left(2(\mathcal{T}^n f)\left(x \circ y\right), (\mathcal{T}^n f)\left(2x\right) * (\mathcal{T}^n f)\left(2y\right)\right) \leq \gamma^n\varphi\left(2^n x, 2^n y\right) \quad \text{for all } x, y \in X. \tag{2.2}$$

The case $n = 0$ is trivially true. Suppose that (2.2) is valid for some $k \in \mathbb{N}_0$. Let $x, y \in X$. Because of Condition (K1), we have

$$\begin{aligned} & d\left(2(\mathcal{T}^{k+1} f)\left(x \circ y\right), (\mathcal{T}^{k+1} f)\left(2x\right) * (\mathcal{T}^{k+1} f)\left(2y\right)\right) \\ &= d\left(\frac{1}{2}(2(\mathcal{T}^k f))\left(2x \circ 2y\right), \frac{1}{2}(\mathcal{T}^k f)\left(2(2x)\right) * \frac{1}{2}(\mathcal{T}^k f)\left(2(2y)\right)\right) \\ &\leq \gamma(\gamma^k\varphi(2^k(2x), 2^k(2y)) = \gamma^{k+1}\varphi(2^{k+1}x, 2^{k+1}y). \end{aligned}$$

This shows that (2.2) is valid for all $n \in \mathbb{N}_0$ for all $x \in X$.

Let $k \in \mathbb{N}_0$ and $x \in X$ be given. We have

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} d\left((\mathcal{T}^k f)(x), (\mathcal{T}^{k+1} f)(x)\right) \leq \gamma \sum_{k=0}^{\infty} \gamma^k\varphi\left(2^k x, e_G\right) < \infty.$$

It follows from Theorem 2.5 that $\mathcal{T}$ has a fixed point $F : X \to Y$ such that

$$d(F(x), f(x)) \leq \varepsilon^*(x) \leq \gamma\widetilde{\varphi}(x, e_G) \quad \text{for all } x \in X.$$

Moreover, $F(x) = \lim_{k\to\infty}(\mathcal{T}^k f)(x) = \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$ for all $x \in X$. The continuity of $*$ and (2.2) imply that

$$2F\left(x \circ y\right) = F(2x) * F(2y) \quad \text{for all } x, y \in X.$$

We finally prove the uniqueness part. To prove this, suppose that there exist a function $\widetilde{F} \in \mathscr{J}_1(G, H, \mathbf{0})$ and a constant $C > 0$ such that

$$d(\widetilde{F}(x), f(x)) \leq C\widetilde{\varphi}(x, e_G) \quad \text{for all } x \in X.$$

For each $k \in \mathbb{N}_0$ and each $x \in X$, we first note that

$$F(x) = \frac{1}{2^k}F\left(2^k x\right) \quad \text{and} \quad \widetilde{F}(x) = \frac{1}{2^k}\widetilde{F}\left(2^k x\right).$$

We consider

$$\begin{aligned} d(F(x), \widetilde{F}(x)) &\leq d\left(\frac{1}{2^k}F\left(2^k x\right), \frac{1}{2^k}f\left(2^k x\right)\right) + d\left(\frac{1}{2^k}\widetilde{F}\left(2^k x\right), \frac{1}{2^k}f\left(2^k x\right)\right) \\ &\leq \gamma^k(\gamma + C)\widetilde{\varphi}(2^k x, e_G) = (\gamma + C)\sum_{i=k}^{\infty}\gamma^i\varphi\left(2^i x, e_G\right). \end{aligned}$$

By letting $k \to \infty$, we get $\widetilde{F}(x) = F(x)$.

With the similar method, we can prove the result if (G, H, φ) satisfies Condition (K2). □

Remark 2.7. According to Theorem 2.6, for each $f \in \mathscr{J}_1(G, H, \varphi)$ the function $F \in \mathscr{J}_1(G, H, \mathbf{0})$ is uniquely determined by f as in Remark 1.3.

The following two examples show that the bound $\Phi(x)$ in Theorem 2.6 is *optimal* in some particular cases.

Example 2.8. Let $G := (\mathbb{R}_+, +)$ and $H := (\mathbb{R}, +, d)$, where $d(x, y) := |x - y|$ for all $x, y \in H$, and $f : G \to H$ be defined by $f(x) := \sqrt{x}$ for all $x \in G$. We see that

$$d(2f(x \circ y), f(2x) * f(2y)) = |2\sqrt{x + y} - \sqrt{2x} - \sqrt{2y}| := \varphi(x, y) \quad \text{for all } x, y \in G.$$

Then $f \in \mathscr{J}_1(G, H, \varphi)$ and (G, H, φ) satisfies Condition (K1) with $\gamma = \frac{1}{2}$. For each $x \in G$, we also note that

$$\widetilde{\varphi}(x, 0) := \sum_{k=0}^{\infty}\frac{1}{2^k}\varphi\left(2^k x, 0\right) = \sum_{k=0}^{\infty}2^{-k/2}\varphi(x, 0) = (2 + \sqrt{2})\varphi(x, 0).$$

Note that $F(x) = 0$ and $\Phi(x) = \gamma\widetilde{\varphi}(x, 0) = \sqrt{x} = f(x)$ for all $x \in G$.

Example 2.9. Let $G := (\mathbb{R}, +)$, $H := (\mathbb{R}, +, d)$ (where d is defined as in Example 2.8), and $f : G \to H$ be defined by $f(x) := x^2$ for all $x \in G$. We see that

$$d(2f(x \circ y), f(2x) * f(2y)) = 2(x - y)^2 := \varphi(x, y) \quad \text{for all } x, y \in G.$$

Then $f \in \mathscr{J}_2(G, H, \varphi)$ and (G, H, φ) satisfies Condition (K2) with $\gamma = 2$. For each $x \in G$, we note that

$$\widetilde{\varphi}(x, 0) = \sum_{k=1}^{\infty}2^k\varphi\left(\frac{x}{2^k}, 0\right) = \sum_{k=1}^{\infty}2^{-k}\varphi(x, 0) = \varphi(x, 0).$$

Note that $F(x) = 0$ and $\Phi(x) = \frac{1}{\lambda}\widetilde{\varphi}(x, 0) = \frac{1}{2}\varphi(x, 0) = x^2 = f(x)$ for all $x \in G$.

By using Theorem 2.6, we obtain the following stability result given by Park et. al. [13, Theorems 3.4 and 3.5] as a consequence.

Corollary 2.10. *Suppose that X is a real normed space and Y is a real Banach space. If there exists $L \in (0, 1)$ and one of the following conditions is satisfied:*

(1) $\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$ *for all $x, y \in X$;*

(2) $\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y)$ *for all $x, y \in X$,*

then the Jensen functional equation is φ-stable in the class of set-valued functions $\mathscr{J}_2(X, \mathrm{BCC}(Y), \varphi)$. Moreover, the class $\mathscr{J}_2(X, \mathrm{BCC}(Y), \varphi)$ is φ-stable with respect to Φ, where

$$\Phi(x) := \begin{cases} \frac{L}{1-L}\varphi(x,0) & \text{if (1) holds;} \\ \frac{1}{1-L}\varphi(x,0) & \text{if (2) holds;} \end{cases}$$

for all $x \in X$. In addition, if $f \in \mathscr{J}_2(X, \mathrm{BCC}(Y), \varphi)$ and there exist positive real numbers M and α such that: $\operatorname{diam} f(x) \leq M\|x\|^\alpha$ *for all $x \in X$, where $\alpha \in (0,1)$ if* (1) *holds; or $\alpha \in (1,\infty)$ if* (2) *holds, then the set-valued function F given by Remark 1.3 is single-valued.*

Proof. Let $(G, \circ) := (X, +)$ and $(H, *, d) := (\mathrm{BCC}(Y), \oplus, \mathcal{H})$. We define $\psi : G \times G \to \mathbb{R}_+$ by $\psi(x,y) := \varphi(2x, 2y)$ for all $x, y \in G$. It is easy to see that

- if (1) holds, then $\psi(2^k x, 2^k y) \leq (2L)^{k+1}\varphi(x,y)$ for all $x, y \in G$ and all $k \in \mathbb{N}_0$;
- if (2) holds, then $\psi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) \leq (L/2)^{k-1}\varphi(x,y)$ for all $x, y \in G$ and all $k \in \mathbb{N}_0$.

It follows that (G, H, ψ) satisfies Condition (Ki) if (i) holds where i=1, 2. It follows from Theorem 2.6 that the class $\mathscr{J}_2(X, \mathrm{BCC}(Y), \varphi)$ is φ-stable with Φ. $\square$

Final remark: The stability of set-valued functional equations inherited by that of the single-valued corresponding equations on appropriate structures.

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