Universal Journal of Mathematics and Applications

VOLUME III
ISSUE I

ISSN 2619-9653

http://dergipark.gov.tr/ujma

VOLUME III ISSUE I
ISSN 2619-9653

March 2020
http://dergipark.gov.tr/ujma

UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

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Universal Journal of Mathematics and Applications, 3 (1) (2020) 1-11
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.637876

Summing Formulas for Generalized Tribonacci Numbers

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Article Info

Keywords: *Tribonacci numbers, Padovan numbers, Perrin numbers, Sum formulas, Summing formulas*
2010 AMS: *11B37, 11B39, 11B83*
Received: *24 October 2019*
Accepted: *11 December 2019*
Available online: *25 March 2020*

Abstract

In this paper, closed forms of the summation formulas for generalized Tribonacci numbers are presented. Then, some previous results are recovered as particular cases of the present results. As special cases, we give summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin, Narayana and some other third order linear recurrance sequences. All the summing formulas of well known recurrence sequences which we deal with are linear except the cases Pell-Padovan and Padovan-Perrin.

1. Introduction

In this work, we investigate linear summation formulas of generalized Tribonacci numbers. Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [11, 12], see also [9]. For linear sums of Fibonacci, Tribonacci, Tetranacci, Pentanacci and Hexanacci numbers, see [10,24], [8,16], [21, 31], [22], and [23] respectively. First, in this section, we present some background about generalized Tribonacci numbers. The generalized Tribonacci sequence $\{W_n(W_0,W_1,W_2;r,s,t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \ \ n\geq 3 \tag{1.1}$$

where W_0,W_1,W_2 are arbitrary complex numbers and r,s,t are real numbers. The generalized Tribonacci sequence has been studied by many authors, see for example [2,3,5,7,14,15,17,18,19,26,27,28,29,30].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1,2,3,...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

If we set $r = s = t = 1$ and $W_0 = 0, W_1 = 1, W_2 = 1$ then $\{W_n\}$ is the well-known Tribonacci sequence and if we set $r = s = t = 1$ and $W_0 = 3, W_1 = 1, W_2 = 3$ then $\{W_n\}$ is the well-known Tribonacci-Lucas sequence.

In fact, the generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Tribonacci-Lucas, Padovan (Cordonnier), Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas. In literature, for example, the following names and notations (see Table 1) are used for the special case of r,s,t and initial values.

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Sequences (Numbers)	Notation	OEIS [20]
Tribonacci	$\{T_n\} = \{W_n(0,1,1;1,1,1)\}$	A000073, A057597
Tribonacci-Lucas	$\{K_n\} = \{W_n(3,1,3;1,1,1)\}$	A001644, A073145
third order Pell	$\{P_n^{(3)}\} = \{W_n(0,1,2;2,1,1)\}$	A077939, A077978
third order Pell-Lucas	$\{Q_n^{(3)}\} = \{W_n(3,2,6;2,1,1)\}$	A276225, A276228
third order modified Pell	$\{E_n^{(3)}\} = \{W_n(0,1,1;2,1,1)\}$	A077997, A078049
Padovan (Cordonnier)	$\{P_n\} = \{W_n(1,1,1;0,1,1)\}$	A000931
Perrin (Padovan-Lucas)	$\{E_n\} = \{W_n(3,0,2;0,1,1)\}$	A001608, A078712
Padovan-Perrin	$\{S_n\} = \{W_n(0,0,1;0,1,1)\}$	A000931, A176971
Pell-Padovan	$\{R_n\} = \{W_n(1,1,1;0,2,1)\}$	A066983, A128587
Pell-Perrin	$\{C_n\} = \{W_n(3,0,2;0,2,1)\}$	-
Jacobsthal-Padovan	$\{Q_n\} = \{W_n(1,1,1;0,1,2)\}$	A159284
Jacobsthal-Perrin (-Lucas)	$\{D_n\} = \{W_n(3,0,2;0,1,2)\}$	A072328
Narayana	$\{N_n\} = \{W_n(0,1,1;1,0,1)\}$	A078012
third order Jacobsthal	$\{J_n^{(3)}\} = \{W_n(0,1,1;1,1,2)\}$	A077947
third order Jacobsthal-Lucas	$\{j_n^{(3)}\} = \{W_n(2,1,5;1,1,2)\}$	A226308

Table 1: A few special case of generalized Tribonacci sequences

Note that the sequence $\{C_n\}$ is't in the database of http://oeis.org [20], yet.

2. Sum formulas of Generalized Tribonacci Numbers with Positive Subscripts

The following Theorem presents some linear summing formulas of generalized Tribonacci numbers with positive subscripts.

Theorem 2.1. *For $n \geq 0$, we have the following formulas:*

(a) *(Sum of the generalized Tribonacci numbers) If $r+s+t-1 \neq 0$, then*

$$\sum_{k=0}^{n} W_k = \frac{W_{n+3}+(1-r)W_{n+2}+(1-r-s)W_{n+1}-W_2+(r-1)W_1+(r+s-1)W_0}{r+s+t-1}.$$

(b) *If $2s+2rt+r^2-s^2+t^2-1=(r+s+t-1)(r-s+t+1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{(-s+1)W_{2n+2}+(t+rs)W_{2n+1}+(t^2+rt)W_{2n}+(-1+s)W_2+(-t-rs)W_1+(-1+r^2-s^2+rt+2s)W_0}{(r+s+t-1)(r-s+t+1)}$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{(r+t)W_{2n+2}+(s-s^2+t^2+rt)W_{2n+1}+(t-st)W_{2n}+(-r-t)W_2+(-1+s+r^2+rt)W_1+(-t+st)W_0}{(r-s+t+1)(r+s+t-1)}.$$

(c) *If $r+t \neq 0$, $s=1$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{1}{r+t}(W_{2n+1}+tW_{2n}-W_1+rW_0)$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{1}{r+t}(W_{2n+2}+tW_{2n+1}-W_2+rW_1).$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}$$

i.e.

$$tW_{n-3} = W_n - rW_{n-1} - sW_{n-2}$$

we obtain

$$\begin{aligned}
tW_0 &= W_3 - rW_2 - sW_1 \\
tW_1 &= W_4 - rW_3 - sW_2 \\
tW_2 &= W_5 - rW_4 - sW_3 \\
&\vdots \\
tW_{n-1} &= W_{n+2} - rW_{n+1} - sW_n \\
tW_n &= W_{n+3} - rW_{n+2} - sW_{n+1}.
\end{aligned}$$

If we add the equations by side by, we get

$$\sum_{k=0}^{n} W_k = \frac{W_{n+3}+(1-r)W_{n+2}+(1-r-s)W_{n+1}-W_2+(r-1)W_1+(r+s-1)W_0}{r+s+t-1}.$$

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3}$$

we obtain

$$\begin{aligned} rW_3 &= W_4 - sW_2 - tW_1 \\ rW_5 &= W_6 - sW_4 - tW_3 \\ &\vdots \\ rW_{2n+1} &= W_{2n+2} - sW_{2n} - tW_{2n-1}. \\ rW_{2n+3} &= W_{2n+4} - sW_{2n+2} - tW_{2n+1} \end{aligned}$$

Now, if we add the above equations by side by, we get

$$r(-W_1+\sum_{k=0}^{n} W_{2k+1}) = (W_{2n+2}-W_2-W_0+\sum_{k=0}^{n} W_{2k}) - s(-W_0+\sum_{k=0}^{n} W_{2k}) - t(-W_{2n+1}+\sum_{k=0}^{n} W_{2k+1}). \tag{2.1}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3}$$

we write the following obvious equations;

$$\begin{aligned} rW_2 &= W_3 - sW_1 - tW_0 \\ rW_4 &= W_5 - sW_3 - tW_2 \\ rW_6 &= W_7 - sW_5 - tW_4 \\ &\vdots \\ rW_{2n} &= W_{2n+1} - sW_{2n-1} - tW_{2n-2} \\ rW_{2n+2} &= W_{2n+3} - sW_{2n+1} - tW_{2n}. \end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$r(-W_0+\sum_{k=0}^{n} W_{2k}) = (-W_1+\sum_{k=0}^{n} W_{2k+1}) - s(-W_{2n+1}+\sum_{k=0}^{n} W_{2k+1}) - t(-W_{2n}+\sum_{k=0}^{n} W_{2k}). \tag{2.2}$$

Then, solving the system (2.1)-(2.2), the required results of (b) and (c) follow.

For another proof (using mathematical induction) of the formula in Theorem 2.1 (a), see [4].
Taking $r=s=t=1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.2. *If $r=s=t=1$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+3}-W_{n+1}-W_2+W_0)$.
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{2}(W_{2n+1}+W_{2n}-W_1+W_0)$.
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}(W_{2n+2}+W_{2n+1}-W_2+W_1)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0=0, T_1=1, T_2=1$).

Corollary 2.3. *[8,16]For $n\geq 0$, Tribonacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+3}-T_{n+1}-1)$.
(b) $\sum_{k=0}^{n} T_{2k} = \frac{1}{2}(T_{2n+1}+T_{2n}-1)$.
(c) $\sum_{k=0}^{n} T_{2k+1} = \frac{1}{2}(T_{2n+2}+T_{2n+1})$.

Taking $W_n = K_n$ with $K_0=3, K_1=1, K_2=3$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Tribonacci-Lucas numbers.

Corollary 2.4. *[8,16]For $n \geq 0$, Tribonacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} K_k = \frac{1}{2}(K_{n+3} - K_{n+1})$.
(b) $\sum_{k=0}^{n} K_{2k} = \frac{1}{2}(K_{2n+1} + K_{2n} + 2)$.
(c) $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{2}(K_{2n+2} + K_{2n+1} - 2)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.5. *[25]If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{3}\left(W_{n+3} - W_{n+2} - 2W_{n+1} - W_2 + W_1 + 2W_0\right)$.
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}\left(W_{2n+1} + W_{2n} - W_1 + 2W_0\right)$.
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}\left(W_{2n+2} + W_{2n+1} - W_2 + 2W_1\right)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of third-order Pell numbers (take $W_n = P_n^{(3)}$ with $P_0^{(3)} = 0, P_1^{(3)} = 1, P_2^{(3)} = 2$).

Corollary 2.6. *[25]For $n \geq 0$, third-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} P_k^{(3)} = \frac{1}{3}(P_{n+3}^{(3)} - P_{n+2}^{(3)} - 2P_{n+1}^{(3)} - 1)$.
(b) $\sum_{k=0}^{n} P_{2k}^{(3)} = \frac{1}{3}(P_{2n+1}^{(3)} + P_{2n}^{(3)} - 1)$.
(c) $\sum_{k=0}^{n} P_{2k+1}^{(3)} = \frac{1}{3}(P_{2n+2}^{(3)} + P_{2n+1}^{(3)})$.

Taking $W_n = Q_n^{(3)}$ with $Q_0^{(3)} = 3, Q_1^{(3)} = 2, Q_2^{(3)} = 6$ in the last Proposition, we have the following Corollary which presents linear sum formulas of third-order Pell-Lucas numbers.

Corollary 2.7. *[25]For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} Q_k^{(3)} = \frac{1}{3}(Q_{n+3}^{(3)} - Q_{n+2}^{(3)} - 2Q_{n+1}^{(3)} + 2)$.
(b) $\sum_{k=0}^{n} Q_{2k}^{(3)} = \frac{1}{3}(Q_{2n+1}^{(3)} + Q_{2n}^{(3)} + 4)$.
(c) $\sum_{k=0}^{n} Q_{2k+1}^{(3)} = \frac{1}{3}(Q_{2n+2}^{(3)} + Q_{2n+1}^{(3)} - 2)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of third-order modified Pell numbers (take $W_n = E_n^{(3)}$ with $E_0^{(3)} = 0, E_1^{(3)} = 1, E_2^{(3)} = 1$).

Corollary 2.8. *[25]For $n \geq 0$, third-order modified Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} E_k^{(3)} = \frac{1}{3}(E_{n+3}^{(3)} - E_{n+2}^{(3)} - 2E_{n+1}^{(3)})$.
(b) $\sum_{k=0}^{n} E_{2k}^{(3)} = \frac{1}{3}(E_{2n+1}^{(3)} + E_{2n}^{(3)} - 1)$.
(c) $\sum_{k=0}^{n} E_{2k+1}^{(3)} = \frac{1}{3}(E_{2n+2}^{(3)} + E_{2n+1}^{(3)} + 1)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.9. *If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = W_{n+3} + W_{n+2} - W_2 - W_1$.
(b) $\sum_{k=0}^{n} W_{2k} = W_{2n+1} + W_{2n} - W_1$.
(c) $\sum_{k=0}^{n} W_{2k+1} = W_{2n+2} + W_{2n+1} - W_2$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P = 1, P_2 = 1$).

Corollary 2.10. *[1] For $n \geq 0$, Padovan numbers have the following properties.*

(a) $\sum_{k=0}^{n} P_k = P_{n+3} + P_{n+2} - 2$.
(b) $\sum_{k=0}^{n} P_{2k} = P_{2n+1} + P_{2n} - 1$.
(c) $\sum_{k=0}^{n} P_{2k+1} = P_{2n+2} + P_{2n+1} - 1$.

Taking $W_n = E_n$ with $E_0 = 3, E_2 = 0, E_2 = 2$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Perrin numbers.

Corollary 2.11. *[1] For $n \geq 0$, Perrin numbers have the following properties.*

(a) $\sum_{k=0}^{n} E_k = E_{n+3} + E_{n+2} - 2$.
(b) $\sum_{k=0}^{n} E_{2k} = E_{2n+1} + E_{2n}$.
(c) $\sum_{k=0}^{n} E_{2k+1} = E_{2n+2} + E_{2n+1} - 2$.

Taking $W_n = S_n$ with $S_0 = 0, S_2 = 0, S_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Padovan-Perrin numbers.

Corollary 2.12. *For $n \geq 0$, Padovan-Perrin numbers have the following properties.*

(a) $\sum_{k=0}^{n} S_k = S_{n+3} + S_{n+2} - 1.$
(b) $\sum_{k=0}^{n} S_{2k} = S_{2n+1} + S_{2n}.$
(c) $\sum_{k=0}^{n} S_{2k+1} = S_{2n+2} + S_{2n+1} - 1.$

If $r=0,s=2,t=1$ then $(r-s+t+1)=0$ so we can't use Theorem 2.1 (b). In other words, the method of the proof Theorem 2.1 (b) can't be used to find $\sum_{k=0}^{n} W_{2k}$ and $\sum_{k=0}^{n} W_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 2.13. *If $r=0,s=2,t=1$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{2}\left(W_{n+3}+W_{n+2}-W_{n+1}-W_2-W_1+W_0\right).$
(b) $\sum_{k=0}^{n} W_{2k} = W_{2n+1}+(W_2-W_1-W_0)n+W_0-W_1.$
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}\left(W_{2n+3}+W_{2n+2}-W_{2n+1}+2n(-W_2+W_1+W_0)-W_2+W_1-W_0\right).$

Proof.

(a) Taking $r=0,s=2,t=1$ in Theorem 2.1 (a) we obtain (a).
(b) and (c) Using the recurrence relation

$$W_n = 2W_{n-2}+W_{n-3}$$

we obtain

$$\sum_{k=0}^{0} W_{2k} = W_0$$
$$\sum_{k=0}^{1} W_{2k} = W_0+W_2 = W_3+W_2-2W_1$$
$$\sum_{k=0}^{2} W_{2k} = W_0+W_2+W_4 = W_5+2W_2-3W_1-W_0$$
$$\vdots$$
$$\sum_{k=0}^{n} W_{2k} = W_{2n+1}+(W_2-W_1-W_0)n+W_0-W_1.$$

This result can be also proved by mathematical induction. Note that from (a) we get

$$\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}\left(W_{2n+3}+W_{2n+2}+W_{2n+1}-W_2-W_1+W_0\right)-\sum_{k=0}^{n} W_{2k}.$$

Now, (c) follows from the last equation.

From the above Theorem we have the following Corollary which gives sum formulas of Pell-Padovan numbers (take $W_n=R_n$ with $R_0=1,R_1=1,R_2=1$).

Corollary 2.14. *For $n\geq 0$, Pell-Padovan numbers have the following property:*

(a) $\sum_{k=0}^{n} R_k = \frac{1}{2}\left(R_{n+3}+R_{n+2}-R_{n+1}-1\right).$
(b) $\sum_{k=0}^{n} R_{2k} = R_{2n+1}-n.$
(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{2}\left(R_{2n+3}+R_{2n+2}-R_{2n+1}+2n-1\right).$

Taking $W_n=C_n$ with $C_0=3,C_1=0,C_2=2$ in the last Theorem, we have the following Corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 2.15. *For $n\geq 0$, Pell-Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n} C_k = \frac{1}{2}\left(C_{n+3}+C_{n+2}-C_{n+1}+1\right).$
(b) $\sum_{k=0}^{n} C_{2k} = C_{2n+1}-n+3.$
(c) $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{2}\left(C_{2n+3}+C_{2n+2}-C_{2n+1}+2n-5\right).$

Taking $r=0,s=1,t=2$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.16. *If $r=0,s=1,t=2$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{2}\left(W_{n+3}+W_{n+2}-W_2-W_1\right).$
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{2}\left(W_{2n+1}+2W_{2n}-W_1\right).$
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}\left(W_{2n+2}+2W_{2n+1}-W_2\right).$

Taking $W_n=Q_n$ with $Q_0=1,Q_1=1,Q_2=1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Jacobsthal-Padovan numbers.

Corollary 2.17. *For $n\geq 0$, Jacobsthal-Padovan numbers have the following properties.*

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{2}\left(Q_{n+3}+Q_{n+2}-2\right).$

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{2}\left(Q_{2n+1} + 2Q_{2n} - 1\right).$
(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{2}\left(Q_{2n+2} + 2Q_{2n+1} - 1\right).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of Jacobsthal-Perrin numbers (take $W_n = D_n$ with $D_0 = 3, D_1 = 0, D_2 = 2$).

Corollary 2.18. *For $n \geq 0$, Jacobsthal-Perrin numbers have the following properties.*

(a) $\sum_{k=0}^{n} D_k = \frac{1}{2}\left(D_{n+3} + D_{n+2} - 2\right).$
(b) $\sum_{k=0}^{n} D_{2k} = \frac{1}{2}\left(D_{2n+1} + 2D_{2n}\right).$
(c) $\sum_{k=0}^{n} D_{2k+1} = \frac{1}{2}\left(D_{2n+2} + 2D_{2n+1} - 2\right).$

Taking $r = 1, s = 0, t = 1$ in Theorem 2.1 (a) and (c), we obtain the following Proposition.

Proposition 2.19. *If $r = 1, s = 0, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = W_{n+3} - W_2.$
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(W_{2n+2} + W_{2n+1} + 2W_{2n} - W_2 - W_1 + W_0).$
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(2W_{2n+2} + 2W_{2n+1} + W_{2n} - 2W_2 + W_1 - W_0).$

From the last Proposition, we have the following Corollary which presents linear sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 2.20. *For $n \geq 0$, Narayana numbers have the following properties.*

(a) $\sum_{k=0}^{n} N_k = N_{n+3} - 1.$
(b) $\sum_{k=0}^{n} N_{2k} = \frac{1}{3}(N_{2n+2} + N_{2n+1} + 2N_{2n} - 2).$
(c) $\sum_{k=0}^{n} N_{2k+1} = \frac{1}{3}(2N_{2n+2} + 2N_{2n+1} + N_{2n} - 1).$

Taking $r = 1, s = 1, t = 2$ in Theorem 2.1 (a) and (c), we obtain the following Proposition.

Proposition 2.21. *If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{3}(W_{n+3} - W_{n+1} - W_2 + W_0).$
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(W_{2n+1} + 2W_{2n} - W_1 + W_0).$
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(W_{2n+2} + 2W_{2n+1} - W_2 + W_1).$

Taking $W_n = J_n^{(3)}$ with $J_0^{(3)} = 0, J_1^{(3)} = 1, J_2^{(3)} = 1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of third order Jacobsthal numbers.

Corollary 2.22. *For $n \geq 0$, third order Jacobsthal numbers have the following properties.*

(a) *[6]* $\sum_{k=0}^{n} J_k^{(3)} = \frac{1}{3}(J_{n+3}^{(3)} - J_{n+1}^{(3)} - 1).$
(b) $\sum_{k=0}^{n} J_{2k}^{(3)} = \frac{1}{3}(J_{2n+1}^{(3)} + 2J_{2n}^{(3)} - 1).$
(c) $\sum_{k=0}^{n} J_{2k+1}^{(3)} = \frac{1}{3}(J_{2n+2}^{(3)} + 2J_{2n+1}^{(3)}).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of third order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0^{(3)} = 2, j_1^{(3)} = 1, j_2^{(3)} = 5$).

Corollary 2.23. *For $n \geq 0$, third order Jacobsthal-Lucas numbers have the following properties.*

(a) *[6]* $\sum_{k=0}^{n} j_k^{(3)} = \frac{1}{3}(j_{n+3}^{(3)} - j_{n+1}^{(3)} - 3).$
(b) $\sum_{k=0}^{n} j_{2k}^{(3)} = \frac{1}{3}(j_{2n+1}^{(3)} + 2j_{2n}^{(3)} + 1).$
(c) $\sum_{k=0}^{n} j_{2k+1}^{(3)} = \frac{1}{3}(j_{2n+2}^{(3)} + 2j_{2n+1}^{(3)} - 4).$

3. Sum formulas of Generalized Tribonacci Numbers with Negative Subscripts

The following Theorem presents some linear summing formulas (identities) of generalized Tribonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$, we have the following formulas:*

(a) *(Sum of the generalized Tribonacci numbers with negative indices) If $r + s + t - 1 \neq 0$, then*

$$\sum_{k=1}^{n} W_{-k} = \frac{-(r+s+t)W_{-n-1} - (s+t)W_{-n-2} - tW_{-n-3} + W_2 + (1-r)W_1 + (1-r-s)W_0}{r+s+t-1}.$$

(b) *If $(r+s+t-1)(r-s+t+1) \neq 0$ then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{-(r+t)W_{-2n+1} + (r^2+rt+s-1)W_{-2n} + (st-t)W_{-2n-1} + (1-s)W_2 + (t+rs)W_1 + (1-rt-2s-r^2+s^2)W_0}{(r+s+t-1)(r-s+t+1)}$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{(s-1)W_{-2n+1} - (t+rs)W_{-2n} - (t^2+rt)W_{-2n-1} + (r+t)W_2 + (1-r^2-rt-s)W_1 + (t-st)W_0}{(r+s+t-1)(r-s+t+1)}.$$

(c) *If* $(r+s+t-1)(r-s+t+1)\neq 0\wedge r+t=0\wedge s\neq 1$ *then*

$$\sum_{k=1}^{n}W_{-2k}=\frac{-W_{-2n}-tW_{-2n-1}+W_{2}+tW_{1}+(1-s)W_{0}}{s-1}$$

and

$$\sum_{k=1}^{n}W_{-2k+1}=\frac{1}{s-1}\left(-W_{-2n+1}-tW_{-2n}+W_{1}+tW_{0}\right).$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_{-n+3}=rW_{-n+2}+sW_{-n+1}+tW_{-n}\Rightarrow W_{-n}=-\frac{s}{t}W_{-(n-1)}-\frac{r}{t}W_{-(n-2)}+\frac{1}{t}W_{-(n-3)}$$

i.e.

$$tW_{-n}=W_{-n+3}-rW_{-n+2}-sW_{-n+1}$$

or

$$W_{-n}=\frac{1}{t}W_{-n+3}-\frac{r}{t}W_{-n+2}-\frac{s}{t}W_{-n+1}$$

we obtain

$$\begin{aligned}
tW_{-n}&=W_{-n+3}-rW_{-n+2}-sW_{-n+1}\\
tW_{-n+1}&=W_{-n+4}-rW_{-n+3}-sW_{-n+2}\\
tW_{-n+2}&=W_{-n+5}-rW_{-n+4}-sW_{-n+3}\\
&\vdots\\
tW_{-2}&=W_{1}-r\times W_{0}-s\times W_{-1}\\
tW_{-1}&=W_{2}-r\times W_{1}-s\times W_{0}.
\end{aligned}$$

If we add the above equations by side by, we get

$$\sum_{k=1}^{n}W_{-k}=\frac{-(rW_{-n-1}+s(W_{-n-1}+W_{-n-2})+t(W_{-n-1}+W_{-n-2}+W_{-n-3})-W_{2}+(r-1)W_{1}+(r+s-1)W_{0})}{r+s+t-1}.$$

(b) and (c) Using the recurrence relation

$$W_{-n+3}=rW_{-n+2}+sW_{-n+1}+tW_{-n}$$

i.e.

$$sW_{-n+1}=W_{-n+3}-rW_{-n+2}-tW_{-n}$$

we obtain

$$\begin{aligned}
sW_{-2n+1}&=W_{-2n+3}-rW_{-2n+2}-tW_{-2n}\\
sW_{-2n+3}&=W_{-2n+5}-rW_{-2n+4}-tW_{-2n+2}\\
&\vdots\\
sW_{-3}&=W_{-1}-rW_{-2}-tW_{-4}\\
sW_{-1}&=W_{1}-rW_{0}-tW_{-2}.
\end{aligned}$$

If we add the equations by side by, we get

$$s\sum_{k=1}^{n}W_{-2k+1}=(-W_{-2n+1}+W_{1}+\sum_{k=1}^{n}W_{-2k+1})-r(-W_{-2n}+W_{0}+\sum_{k=1}^{n}W_{-2k})-t(\sum_{k=1}^{n}W_{-2k}).\tag{3.1}$$

Similarly, using the recurrence relation

$$W_{-n+3}=rW_{-n+2}+sW_{-n+1}+tW_{-n}$$

i.e.

$$sW_{-n+1}=W_{-n+3}-rW_{-n+2}-tW_{-n}$$

we obtain

$$\begin{aligned}
sW_{-2n} &= W_{-2n+2} - rW_{-2n+1} - tW_{-2n-1} \\
sW_{-2n+2} &= W_{-2n+4} - rW_{-2n+3} - tW_{-2n+1} \\
&\vdots \\
sW_{-6} &= W_{-4} - rW_{-5} - tW_{-7} \\
sW_{-4} &= W_{-2} - rW_{-3} - tW_{-5} \\
sW_{-2} &= W_{0} - rW_{-1} - tW_{-3}.
\end{aligned}$$

If we add the above equations by side by, we get

$$s\sum_{k=1}^{n} W_{-2k} = (-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) - r(\sum_{k=1}^{n} W_{-2k+1}) - t(W_{-2n-1} - W_{-1} + \sum_{k=1}^{n} W_{-2k+1}).$$

Since

$$W_{-1} = (-\frac{s}{t}W_0 - \frac{r}{t}W_1 + \frac{1}{t}W_2).$$

it follows that

$$s\sum_{k=1}^{n} W_{-2k} = (-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) - r(\sum_{k=1}^{n} W_{-2k+1}) - t(W_{-2n-1} - (-\frac{s}{t}W_0 - \frac{r}{t}W_1 + \frac{1}{t}W_2) + \sum_{k=1}^{n} W_{-2k+1}). \tag{3.2}$$

Then, solving system (3.1)-(3.2) the required results of (b) and (c) follow.

Note that (c) of the above theorem can be written as follows: If $r+t = 0 \wedge s \neq 1$ then

$$\sum_{k=1}^{n} W_{-2k} = \frac{-W_{-2n} + rW_{-2n-1} + W_2 - rW_1 + (1-s)W_0}{s-1}$$

and

$$\sum_{k=1}^{n} W_{-2k} = \frac{-W_{-2n} + rW_{-2n-1} + W_2 - rW_1 + (1-s)W_0}{s-1}.$$

Next, we present several sum formulas (identities).
Taking $r = s = t = 1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.2. *If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-3W_{-n-1} - 2W_{-n-2} - W_{-n-3} + W_2 - W_0)$.
(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{2}(-W_{-2n+1} + W_{-2n} + W_1 - W_0)$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{2}(-W_{-2n} - W_{-2n-1} + W_2 - W_1)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 3.3. *For $n \geq 1$, Tribonacci numbers have the following properties.*

(a) *[13]* $\sum_{k=1}^{n} T_{-k} = \frac{1}{2}(-3T_{-n-1} - 2T_{-n-2} - T_{-n-3} + 1)$.
(b) $\sum_{k=1}^{n} T_{-2k} = \frac{1}{2}(-T_{-2n+1} + T_{-2n} + 1)$.
(c) $\sum_{k=1}^{n} T_{-2k+1} = \frac{1}{2}(-T_{-2n} - T_{-2n-1})$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above Proposition, we have the following Corollary which gives linear sum formulas of Tribonacci-Lucas numbers.

Corollary 3.4. *For $n \geq 1$, Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} K_{-k} = \frac{1}{2}(-3K_{-n-1} - 2K_{-n-2} - K_{-n-3})$.
(b) $\sum_{k=1}^{n} K_{-2k} = \frac{1}{2}(-K_{-2n+1} + K_{-2n} - 2)$.
(c) $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{2}(-K_{-2n} - K_{-2n-1} + 2)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.5. *If $r = 2, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{3}(-4W_{-n-1} - 2W_{-n-2} - W_{-n-3} + W_2 - W_1 - 2W_0)$.
(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}(-W_{-2n+1} + 2W_{-2n} + W_1 - 2W_0)$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-W_{-2n} - W_{-2n-1} + W_2 - 2W_1)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of third-order Pell numbers (take $W_n = P_n^{(3)}$ with $P_0^{(3)} = 0, P_1^{(3)} = 1, P_2^{(3)} = 2$).

Corollary 3.6. *For $n \geq 1$, third-order Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} P_{-k}^{(3)} = \frac{1}{3}(-4P_{-n-1}^{(3)} - 2P_{-n-2}^{(3)} - P_{-n-3}^{(3)} + 1)$.
(b) $\sum_{k=1}^{n} P_{-2k}^{(3)} = \frac{1}{3}(-P_{-2n+1}^{(3)} + 2P_{-2n}^{(3)} + 1)$.
(c) $\sum_{k=1}^{n} P_{-2k+1}^{(3)} = \frac{1}{3}(-P_{-2n}^{(3)} - P_{-2n-1}^{(3)})$.

Taking $W_n = Q_n^{(3)}$ with $Q_0^{(3)} = 3, Q_1^{(3)} = 2, Q_2^{(3)} = 6$ in the last Proposition, we have the following Corollary which gives linear sum formulas of third-order Pell-Lucas numbers.

Corollary 3.7. *For $n \geq 1$, third-order Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} Q_{-k}^{(3)} = \frac{1}{3}(-4Q_{-n-1}^{(3)} - 2Q_{-n-2}^{(3)} - Q_{-n-3}^{(3)} - 2)$.
(b) $\sum_{k=1}^{n} Q_{-2k}^{(3)} = \frac{1}{3}(-Q_{-2n+1}^{(3)} + 2Q_{-2n}^{(3)} - 4)$.
(c) $\sum_{k=1}^{n} Q_{-2k+1}^{(3)} = \frac{1}{3}(-Q_{-2n}^{(3)} - Q_{-2n-1}^{(3)} + 2)$.

From the last Proposition, we have the following Corollary which presents linear sum formulas of third-order modified Pell numbers (take $W_n = E_n^{(3)}$ with $E_0^{(3)} = 0, E_1^{(3)} = 1, E_2^{(3)} = 1$).

Corollary 3.8. *For $n \geq 1$, third-order modified Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} E_{-k}^{(3)} = \frac{1}{3}(-4E_{-n-1}^{(3)} - 2E_{-n-2}^{(3)} - E_{-n-3}^{(3)})$.
(b) $\sum_{k=1}^{n} E_{-2k}^{(3)} = \frac{1}{3}(-E_{-2n+1}^{(3)} + 2E_{-2n}^{(3)} + 1)$.
(c) $\sum_{k=1}^{n} E_{-2k+1}^{(3)} = \frac{1}{3}(-E_{-2n}^{(3)} - E_{-2n-1}^{(3)} - 1)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.9. *If $r = 0, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = -2W_{-n-1} - 2W_{-n-2} - W_{-n-3} + W_2 + W_1$.
(b) $\sum_{k=1}^{n} W_{-2k} = -W_{-2n+1} + W_1$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = -W_{-2n} - W_{-2n-1} + W_2$.

Taking $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Padovan numbers.

Corollary 3.10. *For $n \geq 1$, Padovan numbers have the following properties.*

(a) $\sum_{k=1}^{n} P_{-k} = -2P_{-n-1} - 2P_{-n-2} - P_{-n-3} + 2$.
(b) $\sum_{k=1}^{n} P_{-2k} = -P_{-2n+1} + 1$.
(c) $\sum_{k=1}^{n} P_{-2k+1} = -P_{-2n} - P_{-2n-1} + 1$.

From the last Proposition, we have the following Corollary which presents linear sum formulas of Perrin numbers (take $W_n = E_n$ with $E_0 = 3, E = 0, E_2 = 2$).

Corollary 3.11. *For $n \geq 1$, Perrin numbers have the following properties.*

(a) $\sum_{k=1}^{n} E_{-k} = -2E_{-n-1} - 2E_{-n-2} - E_{-n-3} + 2$.
(b) $\sum_{k=1}^{n} E_{-2k} = -E_{-2n+1}$.
(c) $\sum_{k=1}^{n} E_{-2k+1} = -E_{-2n} - E_{-2n-1} + 2$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Padovan-Perrin numbers.

Corollary 3.12. *For $n \geq 1$, Padovan-Perrin numbers have the following properties.*

(a) $\sum_{k=1}^{n} S_{-k} = -2S_{-n-1} - 2S_{-n-2} - S_{-n-3} + 1$.
(b) $\sum_{k=1}^{n} S_{-2k} = -S_{-2n+1}$.
(c) $\sum_{k=1}^{n} S_{-2k+1} = -S_{-2n} - S_{-2n-1} + 1$.

If $r = 0, s = 2, t = 1$ then $(r + s + t - 1)(r - s + t + 1) = 0$ so we can't use Theorem 3.1 (b) and (c). In other words, the method of the proof Theorem 3.1 (b) and (c) can't be used to find $\sum_{k=0}^{n} W_{2k}$ and $\sum_{k=0}^{n} W_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 3.13. *If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-3W_{-n-1} - 3W_{-n-2} - W_{-n-3} + W_2 + W_1 - W_0)$.
(b) $\sum_{k=1}^{n} W_{-2k} = -W_{-2n+1} + W_{-2n} + (W_1 - W_0) + (W_2 - W_1 - W_0)n$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{2}(W_{-2n+1} - 3W_{-2n} - W_{-2n-1} + (W_2 - W_1 + W_0) + 2(-W_2 + W_1 + W_0)n)$.

Proof.

(a) Taking $r=0, s=2, t=1$ in Theorem 3.1 (a) we obtain (a).
(b) and (c) Proof can be done as in the proof of Theorem 2.13. Induction also can be used for the proof.

From the last Theorem, we have the following Corollary which gives sum formula of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R = 1, R_2 = 1$).

Corollary 3.14. *For $n \geq 1$, Pell-Padovan numbers have the following property:*

(a) $\sum_{k=1}^{n} R_{-k} = \frac{1}{2}\left(-3R_{-n-1} - 3R_{-n-2} - R_{-n-3} + 1\right)$.
(b) $\sum_{k=1}^{n} R_{-2k} = -R_{-2n+1} + R_{-2n} - n$.
(c) $\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{2}(R_{-2n+1} - 3R_{-2n} - R_{-2n-1} + 1 + 2n)$.

Taking $W_n = C_n$ with $C_0 = 3, C = 0, C_2 = 2$ in the last Theorem, we have the following Corollary which gives sum formulas of Pell-Perrin numbers.

Corollary 3.15. *For $n \geq 1$, Pell-Perrin numbers have the following property:*

(a) $\sum_{k=1}^{n} C_{-k} = \frac{1}{2}\left(-3C_{-n-1} - 3C_{-n-2} - C_{-n-3} - 1\right)$
(b) $\sum_{k=1}^{n} C_{-2k} = -C_{-2n+1} + C_{-2n} - 3 - n$
(c) $\sum_{k=1}^{n} C_{-2k+1} = \frac{1}{2}(C_{-2n+1} - 3C_{-2n} - C_{-2n-1} + 5 + 2n)$

Taking $r=0, s=1, t=2$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.16. *If $r=0, s=1, t=2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}\left(-3W_{-n-1} - 3W_{-n-2} - 2W_{-n-3} + W_2 + W_1\right)$.
(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{2}\left(-W_{-2n+1} + W_1\right)$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{2}\left(-W_{-2n} - 2W_{-2n-1} + W_2\right)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

Corollary 3.17. *For $n \geq 1$, Jacobsthal-Padovan numbers have the following properties.*

(a) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{2}\left(-3Q_{-n-1} - 3Q_{-n-2} - 2Q_{-n-3} + 2\right)$.
(b) $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{2}\left(-Q_{-2n+1} + 1\right)$.
(c) $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{2}\left(-Q_{-2n} - 2Q_{-2n-1} + 1\right)$.

Taking $W_n = D_n$ with $D_0 = 3, D_1 = 0, D_2 = 2$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Jacobsthal-Perrin numbers.

Corollary 3.18. *For $n \geq 1$, Jacobsthal-Perrin numbers have the following properties.*

(a) $\sum_{k=1}^{n} D_{-k} = \frac{1}{2}\left(-3D_{-n-1} - 3D_{-n-2} - 2D_{-n-3} + 2\right)$.
(b) $\sum_{k=1}^{n} D_{-2k} = \frac{-1}{2} D_{-2n+1}$.
(c) $\sum_{k=1}^{n} D_{-2k+1} = \frac{1}{2}\left(-D_{-2n} - 2D_{-2n-1} + 2\right)$.

Taking $r=1, s=0, t=1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 3.19. *If $r=1, s=0, t=1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = -2W_{-n-1} - W_{-n-2} - W_{-n-3} + W_2$.
(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}\left(-2W_{-2n+1} + W_{-2n} - W_{-2n-1} + W_2 + W_1 - W_0\right)$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}\left(-W_{-2n+1} - W_{-2n} - 2W_{-2n-1} + 2W_2 - W_1 + W_0\right)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 3.20. *For $n \geq 1$, Narayana numbers have the following properties.*

(a) $\sum_{k=1}^{n} N_{-k} = -2N_{-n-1} - N_{-n-2} - N_{-n-3} + 1$.
(b) $\sum_{k=1}^{n} N_{-2k} = \frac{1}{3}\left(-2N_{-2n+1} + N_{-2n} - N_{-2n-1} + 2\right)$.
(c) $\sum_{k=1}^{n} N_{-2k+1} = \frac{1}{3}\left(-N_{-2n+1} - N_{-2n} - 2N_{-2n-1} + 1\right)$.

Taking $r=1, s=1, t=2$ in Theorem 3.1, we obtain the following Proposition.

Proposition 3.21. *If $r=1, s=1, t=2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{3}(-4W_{-n-1} - 3W_{-n-2} - 2W_{-n-3} + W_2 - W_0)$.
(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}\left(-W_{-2n+1} + W_{-2n} + W_1 - W_0\right)$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}\left(-W_{-2n} - 2W_{-2n-1} + W_2 - W_1\right)$.

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of third order Jacobsthal numbers.

Corollary 3.22. *For $n \geq 1$, third order Jacobsthal numbers have the following properties.*

(a) $\sum_{k=1}^{n} J_{-k}^{(3)} = \frac{1}{3}(-4J_{-n-1}^{(3)} - 3J_{-n-2}^{(3)} - 2J_{-n-3}^{(3)} + 1)$.
(b) $\sum_{k=1}^{n} J_{-2k}^{(3)} = \frac{1}{3}(-J_{-2n+1}^{(3)} + J_{-2n}^{(3)} + 1)$.
(c) $\sum_{k=1}^{n} J_{-2k+1}^{(3)} = \frac{1}{3}(-J_{-2n}^{(3)} - 2J_{-2n-1}^{(3)})$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of third order Jacobsthal-Lucas numbers (take $W_n = j_n^{(3)}$ with $j_0^{(3)} = 2, j_1^{(3)} = 1, j_2^{(3)} = 5$).

Corollary 3.23. *For $n \geq 1$, third order Jacobsthal-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} j_{-k}^{(3)} = \frac{1}{3}(-4j_{-n-1}^{(3)} - 3j_{-n-2}^{(3)} - 2j_{-n-3}^{(3)} + 3)$.
(b) $\sum_{k=1}^{n} j_{-2k}^{(3)} = \frac{1}{3}(-j_{-2n+1}^{(3)} + j_{-2n}^{(3)} - 1)$.
(c) $\sum_{k=1}^{n} j_{-2k+1}^{(3)} = \frac{1}{3}(-j_{-2n}^{(3)} - 2j_{-2n-1}^{(3)} + 4)$.

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Universal Journal of Mathematics and Applications, 3 (1) (2020) 12-18
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.634491

Homotopy Analysis Method for the Time-Fractional Boussinesq Equation

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Article Info

Keywords: *Homotopy analysis method, Time-fractional Boussinesq equation*
2010 AMS: *26A33, 35A35, 34A08*
Received: *18 October 2019*
Accepted: *10 February 2020*
Available online: *25 February 2020*

Abstract

In this paper, the exact and approximate analytical solutions to the time-fractional Boussinesq equation are constructed using the homotopy analysis method. Several examples about the fourth-order and sixth-order time-fractional Boussinesq equations show the flexibility and efficiency of the method. Furthermore, by choosing an appropriate value for the auxiliary parameter h, we can obtain the N-term approximate solution with improved accuracy.

1. Introduction

In recent years, the time-fractional differential equations have attracted a large amount of attention due to their broad applications in physics, biology, hydrology and engineering [9]. In particular, the time-fractional Boussinesq equation, as a generalization of the Boussinesq equation, can be used to describe the surface water waves with a long memory property. Roughly speaking, two types of methods have been used to solve the fractional differential equation, including analytical [5, 6, 12, 13, 16, 18–20] and numerical methods [1, 3, 7, 8, 11, 14, 17, 21–24]. As far as the analytical methods for the time-fractional Boussinesq equations are concerned, the authors in [5] applied the modified Kudryashov method to solve the nonlinear conformable time-fractional Boussinesq equations. The authors in [6] used the fractional Lie group method to solve the time-fractional Boussinesq equation. Xu et al. [16] also proposed an iterative method to construct the analytical solution. Recently, a Fourier spectral method [24] was developed to obtain the numerical solutions of the equation.

In this paper, we consider the time-fractional Boussinesq equation, which is defined by replacing the integer-order time derivatives with the fractional-order time derivatives. The fractional derivatives are in the Caputo's sense, so that the same initial conditions for the Boussinesq equation with integer-order derivatives can be imposed for the time-fractional equation. Furthermore, we apply the homotopy analysis method [10] to obtain the exact and approximate solution of the time-fractional Boussinesq equation. By solving various examples using the homotopy analysis method, we can show the flexibility and efficiency of the method. A key component of the method is the selection of the auxiliary parameter h. Numerical results provide some insights on how to choose h to obtain the N-term approximate solutions with improved accuracy.

The remaining of the paper is as follows: in section 2, we introduce the notations and basic properties of the fractional calculus, and describe the homotopy analysis method for the general nonlinear partial differential equations. In section 3, several examples about the fourth-order and sixth-order time-fractional Boussinesq equations are presented to demonstrate the performance of the homotopy analysis method. Numerical results about the N-term approximate solutions with various h and α are also discussed to provide the guideline of choosing the parameters.

2. Homotopy Analysis Method

In this section, we introduce the basic properties of fractional calculus and the homotopy analysis method for the general nonlinear partial differential equations.

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2.1. Preliminaries and Notations

In the literature, there are many definitions of fractional derivatives, including the Riemann-Liouville, Caputo, Grünwald-Letnikov, Caputo-Fabrizio and Atangana-Baleanu derivative [2,4,15]. Among all the aforementioned definitions, the Caputo derivative is one of the most widely used definitions for the time-fractional partial differential equations models. It is defined based on the Riemann-Liouville fractional integral. Let $\alpha > 0$ and n be the smallest integer that is greater than or equal to α, then for any locally integrable function f, its Riemann-Liouville (RL) fractional integral of order α is given by

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \tag{2.1}$$

Here Γ is the Gamma function. The RL fractional integral (2.1) is a generalization of the n-fold integral of f. For $f(t) = t^{\beta}$ with $\beta \geq 0$, we can show that $J^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$. The α^{th}-order Caputo fractional derivative of smooth enough f can then be defined as ${}_0D_t^{\alpha} f(t) := J^{n-\alpha}(f^{(n)}(t))$, where $f^{(n)}(t)$ is the n^{th} order derivative of f with respect to t and $\alpha \in (n-1, n]$. That is,

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-1-\alpha} f^{(n)}(s) ds. \tag{2.2}$$

The Caputo fractional derivative and the Riemann-Liouville fractional integral satisfy the following equality for smooth enough function f:

$$J^{\alpha}({}_0D_t^{\alpha} f(t)) = f(t) - \sum_{m=0}^{n-1} f^{(m)}(0+) \frac{t^m}{m!}, \quad t > 0, \tag{2.3}$$

where $f^{(m)}(0+)$ is the right-hand limit of $f^{(m)}(x)$ when x approaches zero from the right. Some exact solutions of the time-fractional differential equations can be represented using the Mittag-Leffler function [15], which is defined by the following series

$$E_{\alpha}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(1+m\alpha)}, \quad \text{for } \alpha > 0. \tag{2.4}$$

2.2. Homotopy Analysis Method

Suppose we consider the nonlinear partial differential equation

$$\mathscr{N}(u(x,t)) = 0, \tag{2.5}$$

where $\mathscr{N}$ in general is a nonlinear operator, and $u(x,t)$ is the exact solution to the equation (2.5). Let $\mathscr{L}$ be a linear operator that is a part of $\mathscr{N}$, then we consider the following equation

$$(1-q)\mathscr{L}(u(x,t) - u_0(x,t)) = qh\mathscr{N}(u(x,t)). \tag{2.6}$$

Here $q \in [0,1]$, h is a non-zero auxiliary parameter and $u_0(x,t)$ is an initial guess of the solution. We can see that when $q = 0$, equation (2.6) becomes $\mathscr{L}(u(x,t) - u_0(x,t)) = 0$. Therefore, $u_0(x,t)$ is the solution to (2.6) when $q = 0$, and $u(x,t)$ is the solution when $q = 1$. Since there is a solution to equation (2.6) for every given value of $q \in [0,1]$, we can replace $u(x,t)$ in (2.6) with $U(x,t;q)$. That is,

$$(1-q)\mathscr{L}(U(x,t;q) - u_0(x,t)) = qh\mathscr{N}(U(x,t;q)). \tag{2.7}$$

We further assume that $U(x,t;q)$ can be written as $U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m$, and we also assume that all the series are convergent. Thus (2.7) leads to

$$(1-q)\mathscr{L}\left(\sum_{m=1}^{\infty} u_m(x,t) q^m\right) = qh\mathscr{N}\left(u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m\right). \tag{2.8}$$

We then use the Taylor expansion of equation (2.8) and let the corresponding coefficients of q^m on both sides of the equation to be the same for all $m \geq 1$, and get

$$\mathscr{L}(u_1(x,t)) = h\mathscr{N}(u_0(x,t)), \tag{2.9}$$

$$\mathscr{L}(u_m(x,t) - u_{m-1}(x,t)) = h\frac{1}{(m-1)!} \frac{\partial^{m-1}\mathscr{N}(u_0(x,t))}{\partial q^{m-1}}\Big|_{q=0}, \quad m \geq 2. \tag{2.10}$$

If one can apply the inverse operator of $\mathscr{L}$ (denoted by $\mathscr{L}^{-1}$) on both sides of (2.9) and (2.10), then $u_m(x,t)$ can be calculated iteratively for $m \geq 1$. Finally, the analytical solution of equation (2.5) can be represented by

$$U(x,t;1) = \sum_{m=0}^{\infty} u_m(x,t), \tag{2.11}$$

assuming the series above converges.

3. Application to the Time-Fractional Boussinesq Equation

In this section, we construct the analytical and approximate analytical solutions of the time-fractional Boussinesq equation using the homotopy analysis method.

3.1. Example 1

We consider the following fourth-order time-fractional Boussinesq equation

$$\begin{cases} {}_0D_t^{\alpha}u = \beta u_{xxxx} + \gamma u_{xx} + \theta(u^2)_{xx} - 4\theta u^2, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = e^x, \quad u_t(x,0) = 0, \end{cases}$$

where $\alpha \in (1,2]$, u is a function of x and t, the coefficients $\beta > 0$, $\gamma > 0$ and θ are constants. To solve the initial value problem using the homotopy analysis method, we let $\mathscr{L}(u(x,t)) = {}_0D_t^{\alpha}u$ and $\mathscr{N}(u(x,t)) = {}_0D_t^{\alpha}u - \beta u_{xxxx} - \gamma u_{xx} - \theta(u^2)_{xx} + 4\theta u^2$ for sufficiently smooth $u(x,t)$. We apply the homotopy analysis method, and let $u_0 = e^x$. Thus (2.9) leads to

$$ {}_0D_t^{\alpha}u_1 = h\left({}_0D_t^{\alpha}u_0 - \beta(u_0)_{xxxx} - \gamma(u_0)_{xx} - \theta((u_0)^2)_{xx} + 4\theta(u_0)^2\right) = -h(\beta+\gamma)e^x. \tag{3.1}$$

We then apply J^{α} on both sides of (3.1) to get

$$u_1(x,t) = -\frac{\beta+\gamma}{\Gamma(1+\alpha)}he^x t^{\alpha}. \tag{3.2}$$

Similarly, (2.10) leads to

$$ {}_0D_t^{\alpha}(u_2 - u_1) = h({}_0D_t^{\alpha}u_1) - h\beta(u_1)_{xxxx} - h\gamma(u_1)_{xx} - h\theta\sum_{j=0}^{1}(u_j u_{1-j})_{xx} + 4h\theta\sum_{j=0}^{1}u_j u_{1-j} = h\,{}_0D_t^{\alpha}u_1 + \frac{(\beta+\gamma)^2}{\Gamma(1+\alpha)}h^2 e^x t^{\alpha}. \tag{3.3}$$

Again we apply J^{α} to (3.3) to get

$$u_2(x,t) = -\frac{\beta+\gamma}{\Gamma(1+\alpha)}h(1+h)e^x t^{\alpha} + \frac{(\beta+\gamma)^2}{\Gamma(1+2\alpha)}h^2 e^x t^{2\alpha}. \tag{3.4}$$

For general $m \geq 1$, we can obtain

$$u_m(x,t) = (1+h)u_{m-1} - h\beta J^{\alpha}((u_{m-1})_{xxxx}) - h\gamma J^{\alpha}((u_{m-1})_{xx}) - h\theta J^{\alpha}\left(\sum_{j=0}^{m-1}(u_j u_{m-1-j})_{xx}\right) + 4h\theta J^{\alpha}\left(\sum_{j=0}^{m-1}u_j u_{m-1-j}\right). \tag{3.5}$$

Using mathematical induction, we can show that

$$u_m(x,t) = e^x\sum_{j=0}^{m-1}(-1)^{j+1}\binom{m-1}{j}\frac{h^{j+1}(1+h)^{m-1-j}}{\Gamma(1+\alpha+j\alpha)}t^{(j+1)\alpha}(\beta+\gamma)^{j+1}. \tag{3.6}$$

Therefore, the analytical solution can be represented by

$$u(x,t) = e^x + e^x\sum_{m=1}^{\infty}\sum_{j=0}^{m-1}(-1)^{j+1}\binom{m-1}{j}\frac{h^{j+1}(1+h)^{m-1-j}}{\Gamma(1+\alpha+j\alpha)}t^{(j+1)\alpha}(\beta+\gamma)^{j+1}. \tag{3.7}$$

When $h = -1$, equation (3.7) becomes $u(x,t) = e^x\sum_{m=0}^{\infty}\frac{t^{m\alpha}(\beta+\gamma)^m}{\Gamma(1+m\alpha)}$, which is convergent for all $\beta,\gamma > 0$, $\alpha \in (1,2]$, $x \in \mathbf{R}$ and $t \geq 0$. Other choices of the parameter h will determine the rate of convergence. The N^{th} order approximation using the homotopy analysis method is given by

$$U^N(x,t) = e^x + e^x\sum_{m=1}^{N}\sum_{j=0}^{m-1}(-1)^{j+1}\binom{m-1}{j}\frac{h^{j+1}(1+h)^{m-1-j}}{\Gamma(1+\alpha+j\alpha)}t^{(j+1)\alpha}(\beta+\gamma)^{j+1}. \tag{3.8}$$

To investigate the convergence of the approximation solution for various values of h, we compute the L^{∞} error for $x \in [-1,1]$ at $t = 1$ and it is shown in Table 1.

N	$h=-1.2$	$h=-1.1053$	$h=-1$	$h=-0.8$	$h=-0.6$
4	1.0005e-02	1.4127e-06	6.7082e-03	1.7332e-01	7.0676e-01
8	1.4976e-05	9.8654e-08	6.2465e-08	1.7657e-03	5.2175e-02
12	1.1419e-08	2.3393e-11	4.2633e-14	1.2178e-05	3.1777e-03
16	8.0469e-11	1.0658e-14	0	6.7143e-08	1.7199e-04
20	6.9278e-14	3.5527e-15	0	3.1885e-10	8.5779e-06

Table 1: $\|U^N(\cdot,T) - u_{exact}(\cdot,T)\|_{\infty,x\in[-1,1]}$ at $T = 1$ for various approximation order N and auxiliary parameter h when $\alpha = 1.5$.

We can see from Table 1 that at $T = 1$, $h = -1.1053$ leads to the most accurate solution (with the L^{∞} error being 1.4127×10^{-6}) among all the 4-term approximations, which implies that the 4-term approximation with suitable choice of h can be very accurate. If we consider larger values of N, we can observe that the approximate solution with $h = -1$ becomes more accurate than the other values of h when $N \geq 8$. We then fix the values of N and h, i.e., $N = 10$ and $h = -1.1053$, and investigate the time evolution of the approximate analytical solutions for various t and α. The results are shown in Table 2. We observe that the 10-term approximate solutions with $h = -1.1053$ are very accurate.

T	$\alpha = 1.2$		$\alpha = 1.6$		$\alpha = 2.0$	
	Solution	Error	Solution	Error	Solution	Error
0.1	1.1200	2.6379e-12	1.0354	1.3352e-12	1.0100	1.3947e-12
0.2	1.2932	1.0809e-11	1.1096	7.7398e-12	1.0403	2.6135e-12
0.3	1.5111	3.6375e-11	1.2150	7.4181e-12	1.0914	2.4134e-12
0.4	1.7779	7.7804e-12	1.3516	2.8288e-11	1.1643	1.5979e-11
0.5	2.1011	1.9902e-10	1.5211	6.9401e-11	1.2606	2.9834e-11
0.6	2.4906	6.8443e-12	1.7268	2.4791e-11	1.3821	2.3717e-11
0.7	2.9584	9.4368e-10	1.9733	1.7407e-10	1.5313	2.6883e-11
0.8	3.5195	1.1823e-09	2.2661	4.4433e-10	1.7112	1.3323e-10
0.9	4.1916	2.1797e-09	2.6120	4.8104e-10	1.9254	2.7052e-10
1.0	4.9961	9.3082e-09	3.0193	1.4490e-10	2.1782	3.5943e-10

Table 2: $|U^N(x=0,T) - u_{exact}(x=0,T)|$ at $T = 0.1, 0.2, \ldots, 1$ and $\alpha = 1.2, 1.6$ and 2 when $N = 10$ and $h = -1.1053$.

3.2. Example 2

We then consider the fourth-order time-fractional Boussinesq equation in two dimensions.

$$\begin{cases} {}_0D_t^\alpha u = \beta_1 u_{xxxx} + \beta_2 u_{yyyy} + \gamma_1 u_{xx} + \gamma_2 u_{yy} + \theta_1 (u^2)_{xx} + \theta_2 (u^2)_{yy} - 4(\theta_1+\theta_2)u^2, & -\infty < x,y < \infty,\ t>0, \\ u(x,y,0) = e^{x+y}, \quad u_t(x,y,0) = 0. \end{cases}$$

Here the solution u is a function of x, y and t. We let $\mathscr{L}(u) = {}_0D_t^\alpha u$ and $\mathscr{N}(u) = {}_0D_t^\alpha u - \beta_1 u_{xxxx} - \beta_2 u_{yyyy} - \gamma_1 u_{xx} - \gamma_2 u_{yy} - \theta_1 (u^2)_{xx} - \theta_2 (u^2)_{yy} + 4(\theta_1+\theta_2)u^2$. Therefore,

$$\begin{aligned} {}_0D_t^\alpha u_1 &= h\left({}_0D_t^\alpha u_0 - \beta_1 (u_0)_{xxxx} - \beta_2 (u_0)_{yyyy} - \gamma_1 (u_0)_{xx} - \gamma_2 (u_0)_{yy}\right) - h\left(-\theta_1 (u_0)^2_{xx} - \theta_2 (u_0)^2_{yy} + 4(\theta_1+\theta_2)(u_0)^2\right) \\ &= -h(\beta_1+\beta_2+\gamma_1+\gamma_2)e^{x+y}. \end{aligned} \tag{3.9}$$

Here we have chosen $u_0 = e^{x+y}$. Therefore, we can obtain u_1 by applying the operator J^α on both sides of (3.9). That is,

$$u_1(x,y,t) = -h\frac{(\beta_1+\beta_2+\gamma_1+\gamma_2)t^\alpha}{\Gamma(1+\alpha)}e^{x+y}. \tag{3.10}$$

For $m \geq 2$, we get

$$\begin{aligned} {}_0D_t^\alpha (u_m - u_{m-1}) = h\left[{}_0D_t^\alpha u_{m-1} - \beta_1 \frac{\partial^4 u_{m-1}}{\partial x^4} - \beta_2 \frac{\partial^4 u_{m-1}}{\partial y^4} - \gamma_1 \frac{\partial^2 u_{m-1}}{\partial x^2} - \gamma_2 \frac{\partial^2 u_{m-1}}{\partial y^2}\right] - h\theta_1 \sum_{j=0}^{m-1} \frac{\partial^2 (u_j u_{m-1-j})}{\partial x^2} \\ - h\theta_2 \sum_{j=0}^{m-1} \frac{\partial^2 (u_j u_{m-1-j})}{\partial y^2} + 4h(\theta_1+\theta_2) \sum_{j=0}^{m-1} u_j u_{m-1-j}. \end{aligned} \tag{3.11}$$

We can show that when $m = 2$, there is

$${}_0D_t^\alpha (u_2 - u_1) = h\,{}_0D_t^\alpha u_1 + h^2 \frac{(\beta_1+\beta_2+\gamma_1+\gamma_2)^2 t^\alpha}{\Gamma(1+\alpha)}e^{x+y}. \tag{3.12}$$

Thus, (3.10) and (3.12) lead to

$$u_2(x,y,t) = -(1+h)h\frac{(\beta_1+\beta_2+\gamma_1+\gamma_2)t^\alpha}{\Gamma(1+\alpha)}e^{x+y} + h^2\frac{(\beta_1+\beta_2+\gamma_1+\gamma_2)^2 t^{2\alpha}}{\Gamma(1+2\alpha)}e^{x+y}.$$

Similarly, we can further derive the following formulations of $u_m(x,y,t)$ for $m = 3$:

$$u_3(x,y,t) = -(1+h)^2 hC_1 t^\alpha e^{x+y} + 2(1+h)h^2 C_2 t^{2\alpha} e^{x+y} - h^3 C_3 t^{3\alpha} e^{x+y},$$

where

$$C_n = \frac{(\beta_1+\beta_2+\gamma_1+\gamma_2)^n}{\Gamma(1+n\alpha)}, \quad n = 1,2,\ldots$$

In general, we can obtain

$$u_m(x,y,t) = e^{x+y}\sum_{j=0}^{m-1}(-1)^{j+1}\binom{m-1}{j}C_{j+1}h^{j+1}(1+h)^{m-1-j}t^{(j+1)\alpha}, \tag{3.13}$$

for $m \geq 1$. Thus the analytical solution is given by $u(x,y,t) = \sum_{m=0}^{\infty} u_m(x,y,t)$ where $u_m(x,y,t)$ is defined by equation (3.13). The 10-term approximate solution and its point-wise absolute error are plotted in Figure 3.1. It shows that the maximum absolute error occurs at $x = y = 1$, and the overall error is at the magnitude of 10^{-8}, which indicates the accuracy of the method.

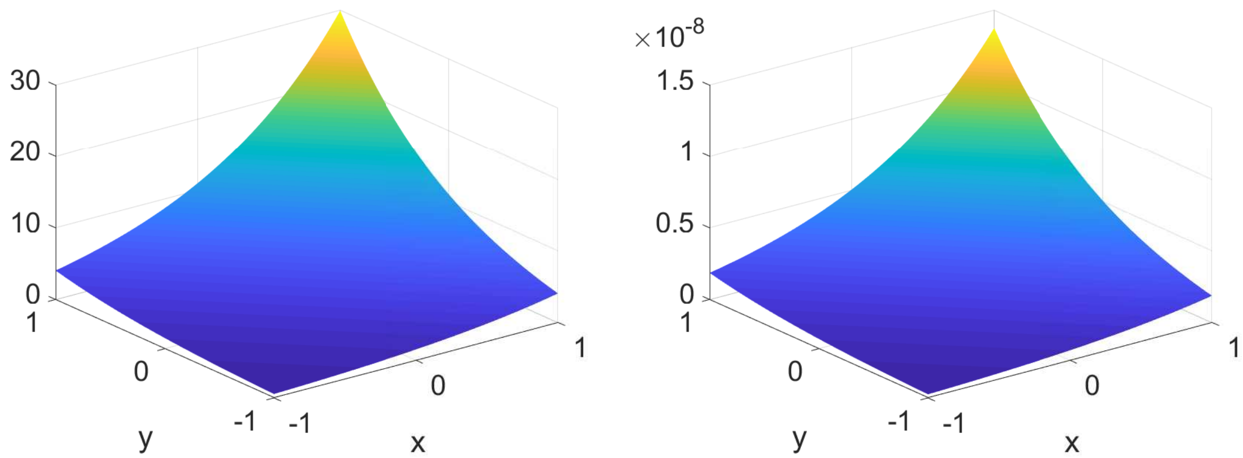


Figure 3.1: The 10-term approximate solution in example 2 and its absolute error. Here $\alpha = 1.7$, $\beta_1 = \gamma_2 = 1$, $\beta_2 = \gamma_1 = 0.5$ and $T = 1$. Left: the approximate solution at $T = 1$. Right: the absolute error of the approximate solution at $T = 1$.

3.3. Example 3

We then consider the time-fractional Boussinesq equation with the sixth-order spatial derivative:

$$\begin{cases} {}_0D_t^\alpha u = u_{xxxxxx} + u_{xxxx} + u_{xx} + \theta(u^2)_{xx} - 4\theta u^2, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = e^x, \quad u_t(x,0) = 0. \end{cases}$$

From the process of homotopy analysis method, we have

$$(1-q)\mathscr{L}(U(x,t;q) - u_0(x,t)) = qh\mathscr{N}(U(x,t;q)), \tag{3.14}$$

where $\mathscr{L}(U(x,t;q)) = {}_0D_t^\alpha U$, $\mathscr{N}(U(x,t;q)) = {}_0D_t^\alpha U - U_{xxxxxx} - U_{xxxx} - U_{xx} - \theta(U^2)_{xx} + 4\theta U^2$ and $u_0(x,t) = e^x$. We then differentiate (3.14) with respect to q for m times, and let $q = 0$ to get

$$ {}_0D_t^\alpha(u_1(x,t)) = h\mathscr{N}(u_0(x,t)), \tag{3.15}$$
$$ {}_0D_t^\alpha(u_m(x,t)) = {}_0D_t^\alpha(u_{m-1}(x,t)) + h\mathscr{R}(u_1, u_2, \dots, u_{m-1}). \tag{3.16}$$

Here we have assumed that $U(x,t;q) = \sum_{m=0}^\infty u_m(x,t)q^m$ is convergent for $q \in [0,1]$. The operator $\mathscr{R}$ is given by

$$\mathscr{R}(u_1, u_2, \dots, u_{m-1}) = {}_0D_t^\alpha u_{m-1} - (u_{m-1})_{xxxxxx} - (u_{m-1})_{xxxx} - (u_{m-1})_{xx} - \theta \sum_{j=0}^{m-1}(u_j u_{m-1-j})_{xx} + 4\theta \sum_{j=0}^{m-1}(u_j u_{m-1-j}). \tag{3.17}$$

If we apply J^α to both sides of (3.15) and (3.16), we can obtain

$$u_1(x,t) = hJ^\alpha \mathscr{N}(u_0(x,t)), \tag{3.18}$$
$$u_m(x,t) = u_{m-1}(x,t) + hJ^\alpha \mathscr{R}(u_1, u_2, \dots, u_{m-1}), \quad m \geq 2. \tag{3.19}$$

We can calculate the next three terms, i.e., $m = 1, 2$ and 3, as follows

$$u_1(x,t) = -3h\frac{t^\alpha e^x}{\Gamma(1+\alpha)}, \tag{3.20}$$

$$u_2(x,t) = -3(1+h)h\frac{t^\alpha e^x}{\Gamma(1+\alpha)} + 9h^2\frac{t^{2\alpha} e^x}{\Gamma(1+2\alpha)}, \tag{3.21}$$

$$u_3(x,t) = -3(1+h)^2h\frac{t^\alpha e^x}{\Gamma(1+\alpha)} + 18(1+h)h^2\frac{t^{2\alpha} e^x}{\Gamma(1+2\alpha)} - 27h^3\frac{t^{3\alpha} e^x}{\Gamma(1+3\alpha)}. \tag{3.22}$$

To derive the general formulation of $u_m(x,t)$, we first let $u_m(x,t)$ be

$$u_m(x,t) = \sum_{j=1}^m C_{m,j} h^j (1+h)^{m-j} \frac{t^{j\alpha} e^x}{\Gamma(1+j\alpha)}, \tag{3.23}$$

where $C_{m,j}$ is the undetermined coefficient in $u_m(x,t)$. From (3.20)-(3.22), we know that $C_{1,1} = C_{2,1} = C_{3,1} = -3$, $C_{2,2} = 9$, $C_{3,2} = 18$ and $C_{3,3} = -27$. Thus we only need to calculate $C_{m,j}$ for $m \geq 4$ and $j \in [1,m]$. Using equation (3.19), one can show that

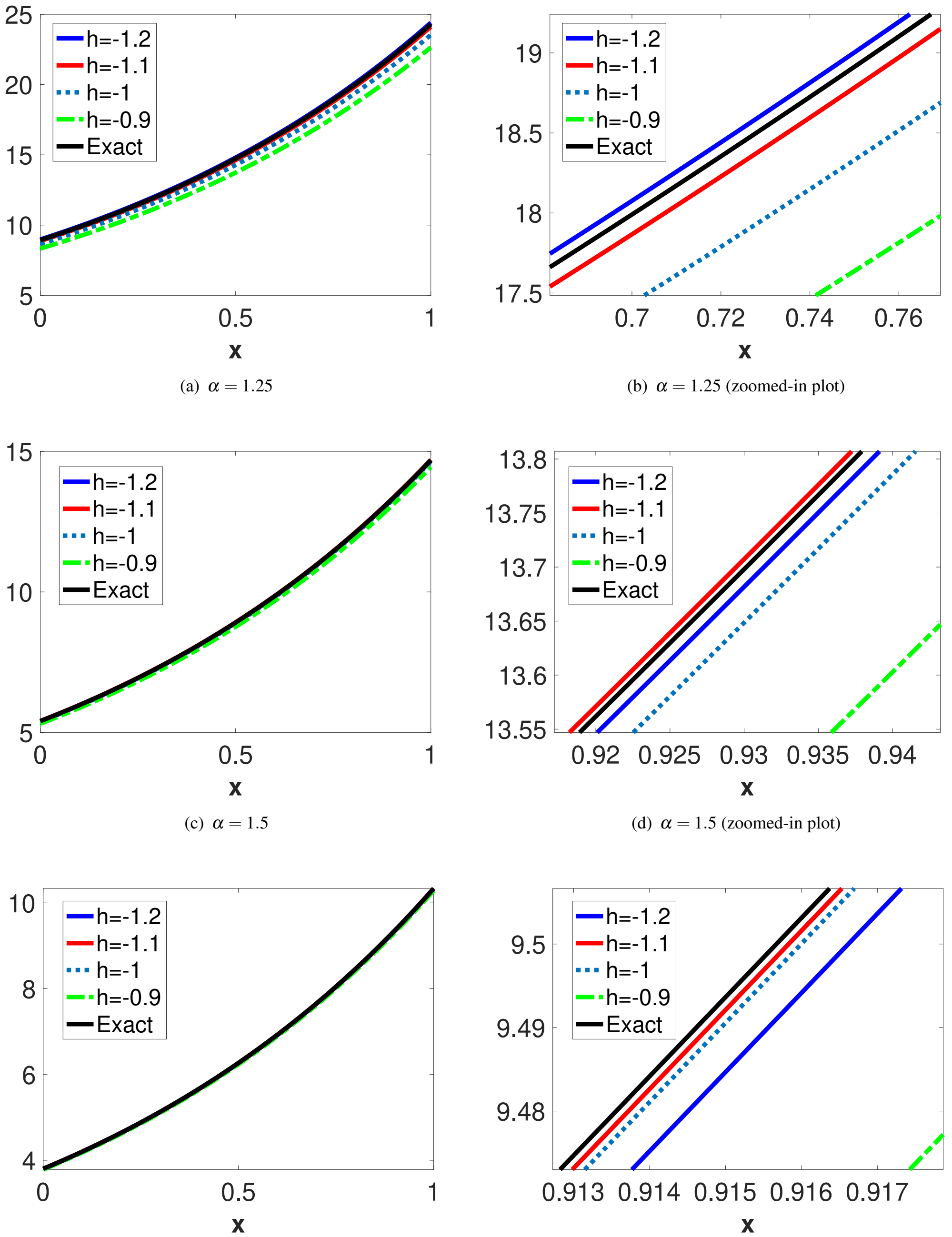
$$u_m(x,t) = (1+h)u_{m-1}(x,t) - 3hJ^\alpha u_{m-1}(x,t). \tag{3.24}$$

We further apply (3.23) to (3.24) to get

$$\begin{aligned} u_m(x,t) &= \sum_{j=1}^{m-1} C_{m-1,j} h^j (1+h)^{m-j} \frac{t^{j\alpha} e^x}{\Gamma(1+j\alpha)} - \sum_{j=1}^{m-1} 3C_{m-1,j} h^{j+1} (1+h)^{m-1-j} \frac{t^{(j+1)\alpha} e^x}{\Gamma(1+(j+1)\alpha)} \\ &= C_{m-1,1} h (1+h)^{m-1} \frac{t^{\alpha} e^x}{\Gamma(1+\alpha)} + \sum_{j=2}^{m-1} (C_{m-1,j} - 3C_{m-1,j-1}) h^j (1+h)^{m-j} \frac{t^{j\alpha} e^x}{\Gamma(1+j\alpha)} - 3C_{m-1,m-1} h^{m-1} (1+h) \frac{t^{m\alpha} e^x}{\Gamma(1+m\alpha)}. \end{aligned}$$

Therefore, we can derive the following difference equations for $m \geq 3$:

$$\begin{cases} C_{m,1} = C_{m-1,1}, \\ C_{m,j} = C_{m-1,j} - 3C_{m-1,j-1}, \quad \text{for } j \in [2, m-1], \\ C_{m,m} = -3C_{m-1,m-1}. \end{cases} \tag{3.25}$$



(a) $\alpha = 1.25$ (b) $\alpha = 1.25$ (zoomed-in plot)

(c) $\alpha = 1.5$ (d) $\alpha = 1.5$ (zoomed-in plot)

(e) $\alpha = 1.75$ (f) $\alpha = 1.75$ (zoomed-in plot)

Figure 3.2: Comparison of the 4-term approximate and exact solutions at the final time $T = 1$ in example 3. Here $\alpha = 1.25, 1.5$ and 1.75. In (a)-(f), the blue, red, dark blue, green and black lines represent the approximate solutions with $h = -1.2, -1.1, -1, -0.9$ and the exact solution, respectively. (a): solutions when $\alpha = 1.25$. (b): zoomed-in plot of solutions when $\alpha = 1.25$. (c): solutions when $\alpha = 1.5$. (d): zoomed-in plot of solutions when $\alpha = 1.5$. (e): solutions when $\alpha = 1.75$. (f): zoomed-in plot of solutions when $\alpha = 1.75$.

From the first equation in (3.25), we get $C_{m,1} = C_{3,1} = -3$. The third equation in (3.25) leads to $C_{m,m} = (-3)^{m-2}C_{2,2} = (-3)^m$. For $j = 2$ in the second equation of (3.25), we have $C_{m,2} = C_{m-1,2} - 3C_{m-1,1} = C_{m-1,2} + 9$. So we can derive $C_{m,2} = C_{2,2} + 9(m-2) = 9(m-1)$. Similarly, we can also show that $C_{m,3} = -\frac{27}{2}(m-1)(m-2)$ and $C_{m,4} = \frac{27}{2}(m^3 - 3m^2 + 2m - 18)$. The analytical formula of $C_{m,j}$ with $5 \le j \le m-1$ can be calculated in the similar manner, but we omit the detail here. The exact solution to the original problem is given by $\sum_{m=0}^{\infty} u_m(x,t)$.

If we take $h = -1$, then according to equation (3.23), $u_m(x,t) = C_{m,m}h^m \frac{t^{m\alpha}e^x}{\Gamma(1+m\alpha)} = \frac{3^m t^{m\alpha}e^x}{\Gamma(1+m\alpha)}$. Therefore, the solution becomes $u(x,t) = e^x E_\alpha(3t^\alpha)$, where E_α is the Mittag-Leffler function defined in (2.4). For this example, we compute the 4-term approximate solutions with various values of $\alpha \in (1,2)$ and $h < 0$, and plot the solutions in Figure 3.2. The top row in Figure 3.2 shows the 4-term approximate solutions for $h = -1.2, -1.1, -1$ and -0.9 when $\alpha = 1.25$. We observe that the approximation solution with $h = -0.9$ is the least accurate solution, followed by the solution with $h = -1$. The 4-term approximation solution with $h = -1.2$ is slightly more accurate than that with $h = -1.1$. We can see that with appropriate choice of h, a 4-term approximation solution can be very accurate. When $\alpha = 1.5$, similar results can be observed except that in this case the approximation solution with $h = -1.1$ is more accurate than that with $h = -1.2$. The bottom row in Figure 3.2 shows that when $\alpha = 1.75$, $h = -1.1$ leads to the most accurate approximate solution. Based on the discussion above, we see that the parameter h provides the flexibility of obtaining accurate solutions.

4. Conclusion

In this paper, we derive the analytical and the approximate analytical solutions of the one and two-dimensional time-fractional Boussinesq equation using the homotopy analysis method. The homotopy analysis method is a semi-analytical technique to solve the differential equations by representing the solution in the form of series with an auxiliary parameter h. We have demonstrated that the iterative method in [16] is a special case of the homotopy analysis method with $h = -1$. Numerical results show that the truncated N-term approximate solution with an appropriate value of N can be used as an accurate approximation of the analytical solution. In particular, we find that $N = 10$ with a suitable choice of h leads to accurate approximate solutions when the final time is less than or equal to 1, and we can obtain the approximate solutions with single-precision accuracy in the sense of L^∞ norm. Moreover, the accuracy can be further improved by selecting the optimal value of h, which is the advantage of using this method over using the iterative method with $h = -1$.

Acknowledgement

The author would like to thank the editor and the anonymous reviewers for their constructive comments, which help to significantly improve the quality of the paper.

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Universal Journal of Mathematics and Applications, 3 (1) (2020) 19-27
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: *www.dergipark.gov.tr/ujma*
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.622495

Some Integrals Connected with a New Quadruple Hypergeometric Series

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Article Info

Keywords: *Beta and Gamma functions, Integrals of Euler type, Integrals of Laplace type, Quadruple hypergeometric series.*
2010 AMS: *33C20, 33C65.*
Received: *20 September 2019*
Accepted: *30 December 2019*
Available online: *25 March 2020*

Abstract

Hypergeometric function of four variables was introduced by Bin-Saad and Younis. In the present paper a new integral representations of of Euler-type and Laplace-type involving double and triple hypergeometric series for these functions are derived.

1. Introduction

In mathematics, there are various special functions that are used in numerous applications [8, 13, 17, 20]. In addition, some special functions have also been shown to have applications in diverse areas as statistical physics, quantum physics, quantum mechanics, fluid dynamics, acoustics, electrical current, heat conduction, astronomy, economics [1, 7, 15, 21]. Hypergeometric functions have a large variety of applications in many areas of mathematics such as in algebraic geometry, Lie algebras, difference equations, group theory, representation theory, partition theory and Hodge theory [1–6, 9–12, 16, 22]. Moreover, multiple hypergeometric functions can be used to solve physical and chemical problems in many areas of applied mathematics [1, 14, 19]. In the present study we aim to obtain certain integral representations of Euler-type and Laplace-type involving new quadruple hypergeometric series namely by $X_i^{(4)}$ ($i = 11, 12, 13, 14, 15$).
Recall the Gaussian hypergeometric function defined by [19]

$$ {}_2F_1\left(a,b;c;x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad (|x| < 1), $$

where $(a)_n$ is the well-known Pochhammer symbol given by

$$ (a)_n := \begin{cases} 1, & (n=0) \\ a(a+1)...(a+n-1), & (n \in \mathbb{N} := \{1,2,...\}) \end{cases} = \frac{\Gamma(a+n)}{\Gamma(a)}, $$

$\Gamma(a)$ is the well-known Gamma function defined by

$$ \Gamma(a) = \int_0^{\infty} e^{-t} t^{a-1} dt, \quad (Re(a) > 0). \tag{1.1} $$

The Appell series F_1, F_2 and the Horn's series H_3 of two variables are defined as follows [19]:

$$ F_1\left(a,b,c;d;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad (\max\ \{|x|,|y|\} < 1), $$

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$$F_2(a,b,c;d,e;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \qquad (|x|+|y|<1)$$

and

$$H_3(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \qquad \left(|x|<r,|y|<s,r+\left(s-\frac{1}{2}\right)^2=\frac{1}{4}\right).$$

The Exton's triple hypergeometric functions X_5,X_6,X_7 and X_{14} are given by [10]

$$X_5(a,b,c;d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p}{(d)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \qquad \left(r<\frac{1}{4}\wedge\max\{s,t\}<\frac{1}{2}+\frac{1}{2}\sqrt{(1-4r)},|x|\le r,|y|\le s,|z|\le t\right),$$

$$X_6(a,b,c;d,e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p}{(d)_{m+n}(e)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \left(t+2\sqrt{r}<1\wedge s<\frac{1}{2}(1-t)+\frac{1}{2}\sqrt{(1-t)^2-4r},|x|\le r,|y|\le s,|z|\le t\right),$$

$$X_7(a,b,c;d,e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p}{(d)_m(e)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \left(s<1\wedge t<1\wedge r<\frac{1}{4}\min\{(1-s)^2,(1-t)^2\},|x|\le r,|y|\le s,|z|\le t\right)$$

and

$$X_{14}(a,b,c;d,e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+n}(e)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \left(r<\frac{1}{4}\wedge t<1\wedge s<(1-t)\left[\frac{1}{2}+\frac{1}{2}\sqrt{(1-4r)}\right],|x|\le r,|y|\le s,|z|\le t\right).$$

The following Srivastava's function of three variables H_A is defined in [19] as

$$H_A(a,b,c;d,e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n}(b)_{n+p}(c)_{p+m}}{(d)_{m+n}(e)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \qquad (r<1\wedge s<1\wedge t<(1-r)(1-s),|x|\le r,|y|\le s,|z|\le t).$$

Lauricella hypergeometric function of four variables $F_C^{(4)}$ [19] which is defined by

$$F_C^{(4)}(a,b;c_1,c_2,c_3,c_4;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \qquad (\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}+\sqrt{|u|}<1).$$

More recently, Bin-Saad et al. [2–5] introduced new hypergeometric series of four variables namely by $X_1^{(4)},X_2^{(4)},...,X_{10}^{(4)}$ and investigated their certain properties including integral representations, symbolic representations, generating functions, etc. Motivated largely by the aforementioned works of Bin-Saad et al. [4] and [5], we defined further quadruple hypergeometric functions as follows:

$$X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.2}$$

$$X_{12}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{n+p}(c_2)_m(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.3}$$

$$X_{13}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n}(c_2)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.4}$$

$$X_{14}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n+p}(c_2)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.5}$$

$$X_{15}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c)_{m+n+p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{1.6}$$

The structure of this article is as follows. In Section 2, we give several Euler-type integrals involving the new quadruple series $X_i^{(4)},(\ i=11,12,13,14,15)$. Certain integral representations of Laplace-type for our series rae given in section 3.

2. Integral representations of Euler-Type

This section gives various integral representations of Euler-Type for the series $X_{11}^{(4)}, X_{12}^{(4)}, ..., X_{15}^{(4)}$ in terms of the classical Gauss hypergeometric function ${}_2F_1$, the Appell's double hypergeometric functions F_1 and F_2, Horn's function H_3 of two variables, the Srivastava's triple series H_A, the Exton's hypergeometric series of three variables X_5, X_6, X_7 and X_{14}, and the quadruple series $X_{11}^{(4)}, X_{12}^{(4)}$ and $F_C^{(4)}$ as follows:

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{2\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a_1-\frac{1}{2}}(cos^2\alpha)^{c_2-a_1-\frac{1}{2}}\left(1-z\,sin^2\alpha\right)^{-a_3} \\ &\times X_{14}\left(1+a_1-c_2,a_2,a_3;c_1,c_3;x\,tan^4\alpha,-y\,tan^2\alpha,\frac{u}{(1-z\,sin^2\alpha)}\right)d\alpha \\ &(Re(a_1)>0, Re(c_2-a_1)>0), \end{aligned} \tag{2.1}$$

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{\Gamma(c_3)(S-T)^{a_2}(R-T)^{c_3-a_2}}{\Gamma(a_2)\Gamma(c_3-a_2)(S-R)^{c_3-a_3-1}}\int_R^S(\alpha-R)^{a_2-1}(S-\alpha)^{c_3-a_2-1} \\ &\times(\alpha-T)^{a_3-c_3}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u)\right]^{-a_3} \\ &\times X_6(a_1,1+a_2-c_3,a_3;c_1,c_2;x,\lambda_1 y,\lambda_2 z)\,d\alpha \\ &\left(\lambda_1=-\frac{(S-T)(\alpha-R)}{(R-T)(S-\alpha)},\lambda_2=\frac{(S-R)(\alpha-T)}{[(S-R)(\alpha-T)-(S-T)(\alpha-R)u)]}\right), \\ &(Re(a_2)>0, Re(c_3-a_2)>0, T<R<S), \end{aligned} \tag{2.2}$$

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{\Gamma(c_1)(1+M)^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^1\alpha^{a_1-1}(1+M\alpha)^{1+a_1+a_2-2c_1} \\ &\times\left[(1-\alpha)(1+M\alpha)+(1+M)^2\alpha^2 x\right]^{c_1-a_1-1}\left[(1+M\alpha)-(1+M)\alpha y\right]^{-a_2} \\ &\times F_2(a_3,1+a_1-c_1,a_2;c_2,c_3;\lambda_1 z,\lambda_2 u)\,d\alpha \\ &\left(\lambda_1=-\frac{(1+M)\alpha(1+M\alpha)}{[(1-\alpha)(1+M\alpha)+(1+M)^2\alpha^2 x]},\ \lambda_2=\frac{(1+M\alpha)}{[(1+M\alpha)-(1+M)\alpha y]}\right), \\ &(\Re(a_1)>0, \Re(c_1-a_1)>0, M>-1), \end{aligned} \tag{2.3}$$

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{\Gamma(c_2)\Gamma(c_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_2-a_1)\Gamma(c_3-a_2)}\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{c_2-a_1-1}\beta^{a_2-1} \\ &\times(1-\beta)^{c_3-a_2-1}(1-\alpha z-\beta u)^{-a_3}H_3(1+a_1-c_2,1+a_2-c_3;c_1; \\ &\left.\frac{\alpha^2 x}{(1-\alpha)^2},\frac{\alpha\beta y}{(1-\alpha)(1-\beta)}\right)d\alpha d\beta \\ &(Re(a_1)>0, Re(a_2)>0, Re(c_2-a_1)>0, Re(c_3-a_2)>0), \end{aligned} \tag{2.4}$$

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1-a_1)\Gamma(c_2-a_3)}\int_0^\infty\int_0^\infty\alpha^{a_1-1}(1+\alpha)^{1+a_1+a_2-2c_1}\beta^{a_3-1} \\ &\times(1+\beta)^{1+a_1-c_1-c_2}\left[(1+\alpha)(1+\beta)+\alpha^2(1+\beta)x+\alpha(1+\alpha)\beta z\right]^{c_1-a_1-1} \\ &\times[(1+\alpha)-\alpha y]^{-a_2}\,{}_2F_1\left(a_2,1+a_3-c_2;c_3;-\frac{(1+\alpha)\beta u}{[(1+\alpha)-\alpha y]}\right)d\alpha d\beta \\ &(Re(a_1)>0, Re(a_3)>0, Re(c_1-a_1)>0, Re(c_2-a_3)>0), \end{aligned} \tag{2.5}$$

$$X_{12}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(a_1+a_2+a_3)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a)\Gamma(c_1-a)}\int_0^1\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}$$
$$\times(1-\beta)^{a_3-1}\gamma^{a-1}(1-\gamma)^{c_1-a-1}F_C^{(4)}\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};c_2,\right.$$
$$\left.a,c_1-a,c_3;\lambda_1 x,\lambda_2 y,\lambda_3 z,\lambda_4 u\right)d\alpha d\beta d\gamma$$
$$\left(\lambda_1=4\alpha^2\beta^2,\lambda_2=4\alpha\beta^2\gamma(1-\alpha),\lambda_3=4\alpha\beta(1-\beta)(1-\gamma),\right.$$
$$\left.\lambda_4=4(1-\alpha)\beta(1-\beta)\right),$$
$$\left(Re(a_i)>0,i=(1,2,3),Re(a)>0,Re(c_1-a)>0\right), \tag{2.6}$$

$$X_{12}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u\right)=\frac{2\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a_1-\frac{1}{2}}(cos^2\alpha)^{c_2-a_1-\frac{1}{2}}$$
$$\times\left(1+x\,sin^2\alpha\,tan^2\alpha\right)^{c_2-a_1-1}H_A\left(a_3,a_2,1+a_1-c_2;c_3,c_1;u,\lambda y,\lambda z\right)d\alpha$$
$$\left(\lambda=-\frac{tan^2\alpha}{\left(1+x\,sin^2\alpha\,tan^2\alpha\right)}\right),$$
$$\left(Re(a_1)>0,Re(c_2-a_1)>0\right), \tag{2.7}$$

$$X_{12}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)}\int_0^{\infty}(e^{-\alpha})^{a_1}\left[(1-e^{-\alpha})+x\,e^{-2\alpha}\right]^{c_2-a_1-1}$$
$$\times H_A\left(a_3,a_2,1+a_1-c_2;c_3,c_1;u,\lambda y,\lambda z\right)d\alpha$$
$$\left(\lambda=-\frac{e^{-\alpha}}{\left[(1-e^{-\alpha})+x\,e^{-2\alpha}\right]}\right), \tag{2.8}$$
$$\left(Re(a_1)>0,Re(c_2-a_1)>0\right),$$

$$X_{12}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(c_3)(S-T)^{a_2}(R-T)^{c_3-a_2}}{\Gamma(a_2)\Gamma(c_3-a_3)(S-R)^{c_3-a_3-1}}\int_R^S(\alpha-R)^{a_2-1}(S-\alpha)^{c_3-a_2-1}$$
$$\times(\alpha-T)^{a_3-c_3}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u)\right]^{-a_3}$$
$$\times X_7\left(a_1,1+a_2-c_3,a_3;c_2,c_1;x,\lambda_1 y,\lambda_2 z\right)d\alpha$$
$$\left(\lambda_1=-\frac{(S-T)(\alpha-R)}{(R-T)(S-\alpha)},\lambda_2=\frac{(S-R)(\alpha-T)}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u)\right]}\right),$$
$$\left(Re(a_2)>0,Re(c_3-a_2)>0,T<R<S\right), \tag{2.9}$$

$$X_{12}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(c_1}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^1\alpha^{a_1-1}(1-\alpha)^{c_1-a_1-1}(1-\alpha y)^{-a_2}$$
$$\times(1-\alpha z)^{-a_3}\,{}_2F_1\left(\frac{1+a_1-c_1}{2},\frac{a_1-c_1}{2}+1;c_2;\frac{4\alpha^2x}{(1-\alpha)^2}\right) \tag{2.10}$$
$$\times{}_2F_1\left(a_2,a_3;c_3;\frac{u}{(1-\alpha y)(1-\alpha z)}\right)d\alpha$$
$$\left(Re(a_1)>0,Re(c_1-a_1)>0\right),$$

$$
\begin{aligned}
X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right) = {} & \frac{\Gamma(a_1+a_2+a_3)\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a)\Gamma(b)\Gamma(c_1-a)\Gamma(c_2-b)}\int_0^1\int_0^1\int_0^1\int_0^1 \alpha^{a_1-1} \\
& \times (1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a_3-1}\gamma^{a-1}(1-\gamma)^{c_1-a-1}\zeta^{b-1}(\zeta-1)^{c_2-b-1} \\
& \times F_C^{(4)}\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};a,c_1-a,b,c_2-b;\lambda_1 x,\lambda_2 y,\right. \\
& \left.\lambda_3 z,\lambda_4 u\right)d\alpha d\beta d\gamma d\zeta \\
& \left(\lambda_1=4\alpha^2\beta^2\gamma,\lambda_2=4\alpha\beta^2(1-\alpha)(1-\gamma),\lambda_3=4\alpha\beta\zeta(1-\beta),\right. \\
& \left.\lambda_4=4(1-\alpha)\beta(1-\beta)(1-\zeta)\right), \\
& (Re(a_i)>0,i=(1,2,3),Re(a)>0,Re(b)>0, \\
& Re(c_1-a)>0,Re(c_2-b)>0),
\end{aligned} \tag{2.11}
$$

$$
\begin{aligned}
X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right) = {} & \frac{2\Gamma(c_1)(1+M)^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha\right)^{c_1-a_1-\frac{1}{2}} \\
& \times\left[\left(1+Msin^2\alpha\right)+(1+M)^2xsin^2\alpha\, tan^2\alpha\right]^{c_1-a_1-1} \\
& \times\left(1+Msin^2\alpha\right)^{1+a_1+a_2-2c_1}\left[\left(1+Msin^2\alpha\right)-(1+M)ysin^2\alpha\right]^{-a_2} \\
& \times F_1\left(a_3,1+a_1-c_1,a_2;c_2;\lambda_1 z,\lambda_2 u\right)d\alpha \\
& \left(\lambda_1=-\frac{(1+M)\left(1+Msin^2\alpha\right)tan^2\alpha}{\left[\left(1+Msin^2\alpha\right)+(1+M)^2xsin^2\alpha\, tan^2\alpha\right]},\right. \\
& \left.\lambda_2=\frac{\left(1+Msin^2\alpha\right)}{\left[\left(1+Msin^2\alpha\right)-(1+M)ysin^2\alpha\right]}\right), \\
& (Re(a_1)>0,Re(c_1-a_1)>0,M>-1),
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right) = {} & \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^\infty \alpha^{a_1-1}(1+\alpha)^{1+a_1+a_2-2c_1}\left[(1+\alpha)+\alpha^2x\right]^{c_1-a_1-1} \\
& \times[(1+\alpha)-\alpha y]^{-a_2}F_1\left(a_3,1+a_1-c_1,a_2;c_2;-\frac{\alpha(1+\alpha)z}{[(1+\alpha)+\alpha^2x]},\right. \\
& \left.\frac{(1+\alpha)u}{[(1+\alpha)-\alpha y]}\right)d\alpha \\
& (Re(a_1)>0,Re(c_1-a_1)>0),
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right) = {} & \frac{\Gamma(c_2)(S-T)^{a_3}(R-T)^{c_2-a_3}}{\Gamma(a_3)\Gamma(c_2-a_3)(S-R)^{c_2-a_1-a_2-1}}\int_R^S(\alpha-R)^{a_3-1}(S-\alpha)^{c_2-a_3-1} \\
& \times(\alpha-T)^{a_1+a_2-c_2}[(S-R)(\alpha-T)-(S-T)(\alpha-R)z]^{-a_1} \\
& \times[(S-R)(\alpha-T)-(S-T)(\alpha-R)u]^{-a_2}H_3\left(a_1,a_2;c_1;\lambda_1 x,\lambda_2 y\right)d\alpha \\
& \left(\lambda_1=\frac{(S-R)^2(\alpha-T)^2}{[(S-R)(\alpha-T)-(S-T)(\alpha-R)z]^2},\right. \\
& \lambda_2=\frac{(S-R)^2(\alpha-T)^2}{[(S-R)(\alpha-T)-(S-T)(\alpha-R)z]} \\
& \left.\times\frac{1}{[(S-R)(\alpha-T)-(S-T)(\alpha-R)u]}\right), \\
& (Re(a_3)>0,Re(c_2-a_3)>0,T<R<S),
\end{aligned} \tag{2.14}
$$

$$\begin{aligned} X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right) &= \frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}\int_0^1 \alpha^{a_3-1}(1-\alpha)^{c_2-a_3-1}(1-\alpha z)^{-a_1} \\ &\times(1-\alpha u)^{-a_2} H_3\left(a_1,a_2;c_1;\frac{x}{(1-\alpha z)^2},\frac{y}{(1-\alpha z)(1-\alpha u)}\right)d\alpha \\ &(Re(a_3)>0, Re(c_2-a_3)>0), \end{aligned} \tag{2.15}$$

$$\begin{aligned} X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right) &= \frac{\Gamma(c_1)}{\Gamma(a)\Gamma(c_1-a)}\int_0^\infty \frac{\alpha^{a-1}}{(1+\alpha)^{c_1}} X_{11}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,\right. \\ &\left. a_3;a,a,c_1-a,c_2;\frac{\alpha x}{(1+\alpha)},\frac{\alpha y}{(1+\alpha)},\frac{z}{(1+\alpha)},u\right)d\alpha, \\ &(Re(a)>0, Re(c_2-a)>0), \end{aligned} \tag{2.16}$$

$$\begin{aligned} X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right) &= \frac{2\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}\int_0^{\frac{\pi}{2}} (sin^2\alpha)^{a_3-\frac{1}{2}}(cos^2\alpha)^{c_2-a_3-\frac{1}{2}}\left(1-u\,sin^2\alpha\right)^{-a_2} \\ &\times X_5\left(a_1,a_2,1+a_3-c_2;c_1;x,\frac{y}{(1-u\,sin^2\alpha)},-z\,tan^2\alpha\right)d\alpha \\ &(Re(a_3)>0, Re(c_2-a_3)>0), \end{aligned} \tag{2.17}$$

$$\begin{aligned} X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right) &= \frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}\int_0^\infty (e^{-\alpha})^{a_3}(1-e^{-\alpha})^{c_2-a_3-1}\left(1-ue^{-\alpha}\right)^{-a_2} \\ &\times X_5\left(a_1,a_2,1+a_3-c_2;c_1;x,\frac{y}{(1-ue^{-\alpha})},-\frac{ze^{-\alpha}}{(1-ue^{-\alpha})}\right)d\alpha \\ &(Re(a_3)>0, Re(c_2-a_3)>0), \end{aligned} \tag{2.18}$$

$$\begin{aligned} X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right) &= \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^1 \alpha^{a_1-1}\left[(1-\alpha)+\alpha^2 x\right]^{c_1-a_1-1} \\ &\times(1-\alpha y)^{-a_2}(1-\alpha z)^{-a_3}\,{}_2F_1\left(a_2,a_3;c_2;\frac{u}{(1-\alpha y)(1-\alpha z)}\right)d\alpha \\ &(Re(a_1)>0, Re(c_1-a_1)>0), \end{aligned} \tag{2.19}$$

$$\begin{aligned} X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right) &= \frac{2\Gamma(c_1)M^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^{\frac{\pi}{2}} \left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{1+a_1+a_2+a_3-2c_1} \\ &\times\left(cos^2\alpha\right)^{c_1-a_1-\frac{1}{2}}\left[\left(cos^2\alpha+Msin^2\alpha\right)+M^2xsin^2\alpha\,tan^2\alpha\right]^{c_1-a_1-1} \\ &\times\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mysin^2\alpha\right]^{-a_2}\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mzsin^2\alpha\right]^{-a_3} \\ &\times{}_2F_1\left(a_2,a_3;c_2;\lambda u\right)d\alpha \\ &\left(\lambda=\frac{\left(cos^2\alpha+Msin^2\alpha\right)^2}{\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mysin^2\alpha\right]\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mzsin^2\alpha\right]}\right) \\ &(Re(a_1)>0, Re(c_1-a_1)>0, M>0), \end{aligned} \tag{2.20}$$

$$\begin{aligned} X_{15}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u\right) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}\int_0^\infty\int_0^\infty \frac{\alpha^{a-1}}{(1+\alpha)^c}\frac{\beta^{b-1}}{(1+\beta)^{c-a}} \\ &\times X_{12}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;b,a,a,c-a-b;\lambda_1 x,\lambda_2 y,\lambda_2 z,\lambda_3 u\right) \\ &\times d\alpha d\beta \\ &\left(\lambda_1=\frac{\beta}{(1+\alpha)(1+\beta)},\lambda_2=\frac{\alpha}{(1+\alpha)},\lambda_3=\frac{1}{(1+\alpha)(1+\beta)}\right), \\ &(Re(a)>0, Re(b)>0, Re(c-a)>0, Re(c-a-b)>0), \end{aligned} \tag{2.21}$$

$$
\begin{aligned}
X_{15}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u\right) = {} & \frac{\Gamma(c)(1+M)^{a_2}}{\Gamma(a_2)\Gamma(c-a_2)} \int_0^1 \alpha^{a_2-1}(1-\alpha)^{c-a_2-1}(1+M\alpha)^{a_1+a_3-c} \\
& \times [(1+M\alpha)-(1+M)\alpha y]^{-a_1} [(1+M\alpha)-(1+M)\alpha u]^{-a_3} \\
& \times H_3\left(a_1,a_3;c-a_2;\lambda_1 x,\lambda_2 z\right) d\alpha \\
& \left(\lambda_1 = \frac{(1-\alpha)(1+M\alpha)}{[(1+M\alpha)-(1+M)\alpha y]^2},\right. \\
& \left.\lambda_2 = \frac{(1-\alpha)(1+M\alpha)}{[(1+M\alpha)-(1+M)\alpha y][(1+M\alpha)-(1+M)\alpha u]}\right), \\
& \left(Re(a_2)>0, Re(c-a_2)>0, M>-1\right),
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
X_{15}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u\right) = {} & \frac{2\Gamma(c)}{\Gamma(a_2)\Gamma(c-a_2)} \int_0^{\frac{\pi}{2}} (sin^2\alpha)^{a_2-\frac{1}{2}}(cos^2\alpha)^{c-a_2-\frac{1}{2}} \left(1-y\,sin^2\alpha\right)^{-a_1} \\
& \times \left(1-u\,sin^2\alpha\right)^{-a_3} H_3\left(a_1,a_3;c-a_2;\frac{xcos^2\alpha}{\left(1-y\,sin^2\alpha\right)^2},\right. \\
& \left.\frac{zcos^2\alpha}{\left(1-y\,sin^2\alpha\right)\left(1-u\,sin^2\alpha\right)}\right) d\alpha \\
& \left(Re(a_2)>0, Re(c-a_2)>0\right),
\end{aligned} \tag{2.23}
$$

$$
\begin{aligned}
X_{15}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u\right) = {} & \frac{\Gamma(c)}{\Gamma(a_2)\Gamma(c-a_2)} \int_0^{\infty} (e^{-\alpha})^{a_1+a_2+a_3}(1-e^{-\alpha})^{c-a_2-1}(e^{\alpha}-y)^{-a_1} \\
& \times (e^{\alpha}-u)^{-a_3} H_3\left(a_1,a_3;c-a_2;\frac{(1-e^{-\alpha})x}{(1-ye^{-\alpha})^2},\frac{(1-e^{-\alpha})z}{(1-ye^{-\alpha})(1-ue^{-\alpha})}\right) d\alpha \\
& \left(Re(a_2)>0, Re(c-a_2)>0\right),
\end{aligned} \tag{2.24}
$$

$$
\begin{aligned}
X_{15}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u\right) = {} & \frac{\Gamma(c)}{\Gamma(a_3)\Gamma(c-a_3)} \int_0^{\infty} (e^{-\alpha})^{a_3}(1-e^{-\alpha})^{c-a_3-1}\left(1-ze^{-\alpha}\right)^{-a_1} \\
& \times \left(1-ue^{-\alpha}\right)^{-a_2} H_3\left(a_1,a_2;c-a_3;\frac{e^{\alpha}(e^{\alpha}-1)x}{(e^{\alpha}-z)^2},\frac{e^{\alpha}(e^{\alpha}-1)y}{(e^{\alpha}-z)(e^{\alpha}-u)}\right) d\alpha \\
& \left(Re(a_3)>0, Re(c-a_3)>0\right).
\end{aligned} \tag{2.25}
$$

Proof. Once substituting the series definition of the special function in each integrand and then, changing the order of the integral and the summation, and finally taking into account the following integral representations of the Beta function and their various associated Eulerian integrals [8, 18, 20], we derive each of the integral representations from (2.1) to (2.25).

$$
B(a,b) = \begin{cases} \displaystyle\int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}\,dt & (Re(a)>0, Re(b)>0), \\ \displaystyle\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} & (a,b\in\mathbb{C}\setminus\mathbb{Z}_0^-), \end{cases}
$$

$$
B(a,b) = \int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}\,d\alpha = \int_0^{\infty} (e^{-\alpha})^{a}(1-e^{-\alpha})^{b-1}\,d\alpha, \qquad (Re(a)>0, Re(b)>0),
$$

$$
B(a,b) = 2\int_0^{\frac{\pi}{2}} (sin\alpha)^{2a-1}(cos\alpha)^{2b-1}\,d\alpha = \int_0^{\infty} \frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}\,d\alpha, \qquad (Re(a)>0, Re(b)>0),
$$

$$
B(a,b) = \frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}} \int_R^S \frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\,d\alpha = (M+1)^a \int_0^1 \frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)}\,d\alpha,
$$
$$
(T<R<S, M>-1, Re(a)>0, Re(b)>0).
$$

$\square$

3. Integral representations of Laplace-Type

In this section, we introduce Laplace integral representations of the hypergeometric series of four variables (1.2) to (1.6) by

$$X_{11}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u\right)$$
$$=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}\,t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+sty\right)\Psi_2\left(a_3;c_2,c_3;sz,tu\right)dsdt, \tag{3.1}$$
$$\left(Re(a_1)>0,Re(a_2)>0\right),$$

$$X_{12}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u\right)$$
$$=\frac{1}{\Gamma(a_2)\Gamma(a_3)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_2-1}\,t^{a_3-1}\mathsf{H}_7\left(a_1;c_2,c_1;x,sy+tz\right){}_0F_1\left(-;c_3;stu\right)dsdt, \tag{3.2}$$
$$\left(Re(a_2)>0,Re(a_3)>0\right),$$

$$X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right)$$
$$=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}\,t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+sty\right){}_1F_1\left(a_3;c_2;sz+tu\right)dsdt, \tag{3.3}$$
$$\left(Re(a_1)>0,Re(a_2)>0\right),$$

$$X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right)$$
$$=\frac{1}{\Gamma(a_2)\Gamma(a_3)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_2-1}\,t^{a_3-1}\mathsf{H}_6\left(a_1;c_1;x,sy+tz\right){}_0F_1\left(-;c_2;stu\right)dsdt, \tag{3.4}$$
$$\left(Re(a_2)>0,Re(a_3)>0\right),$$

$$X_{15}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u\right)$$
$$=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}\,t^{a_2-1}\,v^{a_3-1}{}_0F_1\left(-;c;s^2x+sty+svz+tuv\right)dsdtdv, \tag{3.5}$$
$$\left(Re(a_1)>0,Re(a_2)>0,Re(a_3)>0\right).$$

where ${}_0F_1, {}_1F_1, \mathsf{H}_6, \mathsf{H}_7$ and Ψ_2 are the confluent hypergeometric functions defined by

$${}_0F_1\left(-;c;x\right)=\sum_{m=0}^\infty\frac{1}{(c)_m}\frac{x^m}{m!},\quad\left(|x|<\infty\right),$$

$${}_1F_1\left(a;c;x\right)=\sum_{m=0}^\infty\frac{(a)_m}{(c)_m}\frac{x^m}{m!},\quad\left(|x|<\infty\right),$$

$$\mathsf{H}_6\left(a;c;x,y\right)=\sum_{m,n=0}^\infty\frac{(a)_{2m+n}}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\left(|x|<\frac{1}{4},\quad|y|<\infty\right),$$

$$\mathsf{H}_7\left(a;b,c;x,y\right)=\sum_{m,n=0}^\infty\frac{(a)_{2m+n}}{(b)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!},\left(|x|<\frac{1}{4},\quad|y|<\infty\right)$$

and

$$\Psi_2\left(a;b,c;x,y\right)=\sum_{m,n=0}^\infty\frac{(a)_{m+n}}{(b)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!},\left(|x|<\infty,\quad|y|<\infty\right).$$

Proof. It is noted that each of the integral representations (3.2) to (3.5) can be proved mainly by expressing the series definition of the involved the confluent hypergeometric functions in each integrand and changing the order of the integral sign and the summation, and finally using the formula (1.1). □

4. Conclusion

Concerning the hypergeometric functions of four variables, we mainly introduced five of them. On the basis of the definitions of the four variables hypergeometric functions, we succeed in establishing 25 integral formulas of Euler-type and five integral formulas of Laplace-type involving hypergeometric functions of two and three variables and confluent hypergeometric functions of two variables.

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Universal Journal of Mathematics and Applications, 3 (1) (2020) 28-32
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.609715

Fixed Point Theorems in b-Rectangular Metric Spaces

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Article Info

Keywords: *Almost contraction, b-Rectangular metric, Fixed point*
2010 AMS: *47H10,54H25*
Received: *23 August 2019*
Accepted: *5 February 2020*
Available online: *25 March 2020*

Abstract

The concept of b-rectangular metric space is introduced as a generalization of b-metric space and rectangular (generalized) metric space. In this paper, we introduce generalized almost contraction for two mappings and prove common fixed point theorems in b-rectangular metric spaces.

1. Introduction and Preliminaries

Banach contraction principle is one of the earlier and main results in fixed point theory. Banach contraction principle was proved in complete metric spaces. In last years, many generalizations of the concept of metric spaces are defined and some fixed point theorems was proved in these spaces. In particular, $b-$metric spaces was introduced by Bakhtin [2] and Czerwik [6] as a generalization of metric spaces. They proved Banach contraction principle in b-metric spaces. Since then, some authors proved fixed point theorems in b-metric spaces [11], [13], [18], [19], [20], [21]. Another generalization of metric spaces is generalized metric spaces (g.m.s.) or rectangular metric spaces (r.m.s.). Branciari [5] introduced the concept of generalized metric space by replacing the triangle inequality by a more general inequality - by the rectangular inequality. Thereafter, many authors initiated and studied many existing fixed point theorems in such spaces [7], [8], [12], [15]. Also, the concept of $b-$rectangular metric space is introduced as a generalization of $b-$metric space and rectangular (generalized) metric space by Geoge et al. [11]. Also see [9], [10], [14] , [16], [17], [22].

Definition 1.1. *[2], [6] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is a b-metric on X if, for all $x,y,z \in X$, the following conditions hold:*

(b1) $d(x,y) = 0$ *if and only if* $x = y$,
(b2) $d(x,y) = d(y,x)$,
(b3) $d(x,z) \leq s[d(x,y) + d(y,z)]$ *(b-triangular inequality).*

In this case, the pair (X,d) is called a b-metric space.

Definition 1.2. *[5] Let X be a nonempty set, and let $d : X \times X \to [0,\infty)$ be a mapping such that for all $x,y \in X$ and all distinct points $u,v \in X$, each distinct from x and y :*

(r1) $d(x,y) = 0$ *if and only if* $x = y$,
(r2) $d(x,y) = d(y,x)$,
(r3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ *(rectangular inequality).*

Then (X,d) is called rectangular or generalized metric space.

Definition 1.3. *[11] Let X be a nonempty set, $s \geq 1$ be a given real number and let $d : X \times X \to [0,\infty)$ be a mapping such that for all $x,y \in X$ and distinct points $u,v \in X$, each distinct from x and y :*

(br1) $d(x,y) = 0$ *if and only if* $x = y$,
(br2) $d(x,y) = d(y,x)$,
(br3) $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ *(b-rectangular inequality).*

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Then (X,d) is called a b-rectangular metric space or a $b-$generalized metric space (b-g.m.s.).

Note that every metric space is a rectangular metric space (g.m.s) and every rectangular metric space is a rectangular $b-$metric space (with coefficient $s=1$). However the converse is not necessarily true. Also, every metric space is a b-metric space and every b-metric space is a $b-$rectangular metric space (not necessarily with the same coefficient) [11].

Definition 1.4. *[11] (X,d) be a $b-$rectangular metric space and $\{x_n\}$ be a sequence in X and $x \in X$. Then*
(i) *The sequence $\{x_n\}$ is said to be convergent in (X,d) and converges to x, if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by*

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty.$$

(ii) *The sequence $\{x_n\}$ is said to be $b-$rectangular-Cauchy in (X,d) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d\left(x_n, x_{n+p}\right) < \varepsilon$ for all $n > n_0, p > 0$ or equivalently,*

$$\lim_{n\to\infty} d\left(x_{n+p}, x_n\right) = 0 \text{ for all } p > 0.$$

(iii) *(X,d) is said to be complete if every $b-$rectangular$-$Cauchy sequence in (X,d) converges to an element of X.*

Note that, limit of a sequence in a rectangular $b-$metric space is not necessarily unique.

Example 1.5. *[11] Let $X = A \cup B$, where $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and B is the set of all positive integers. Define $d : XxX \to [0,\infty)$ such that $d(x,y) = d(y,x)$ for all $x, y \in X$ and*

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 2\alpha \text{ if } x, y \in A \\ \frac{\alpha}{2n} \text{ if } x \in A \text{ and } y \in \{2,3\} \\ \alpha \text{ otherwise} \end{cases}$$

where $\alpha > 0$ is a constant. Then (X,d) is a b-rectangular metric space with coefficient $s = 2 > 1$. However we have the following:
1. *(X,d) is not a rectangular metric space, as $d\left(\frac{1}{2},\frac{1}{3}\right) = 2\alpha > \frac{17}{12} = d\left(\frac{1}{2},4\right) + d(4,3) + d\left(3,\frac{1}{3}\right)$ and hence not a metric space.*
2. *There does not exist $s > 1$ satisfying $d(x,y) \le s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$, and so (X,d) is not a $b-$metric space.*
3. *$B_{\frac{\alpha}{2}}\left(\frac{1}{2}\right) = \left\{2,3,\frac{1}{2}\right\}$ and there does not exist any open ball with centre 2 and contained in $B_{\frac{\alpha}{2}}\left(\frac{1}{2}\right)$. So $B_{\frac{\alpha}{2}}\left(\frac{1}{2}\right)$ is not an open set.*
4. *The sequence $\left\{\frac{1}{n}\right\}$ converges to 2 and 3 in b-rectangular metric space and so limit is not unique. Also $d\left(\frac{1}{n},\frac{1}{n+p}\right) = 2\alpha \nrightarrow 0$ as $n \to \infty$, therefore $\left\{\frac{1}{n}\right\}$ is not a b-rectangular$-$Cauchy sequence in b-rectangular metric space.*
5. *There does not exist any $r_1, r_2 > 0$ such that $B_{r_1}(2) \cap B_{r_2}(3) = \varnothing$ and (X,d) is not Hausdoff.*

In this paper, we prove some fixed point theorems for mappings satisfying almost contractive condition in b-rectangular metric spaces. Berinde [3] defined the notion of a weak contraction mapping which is more general than a contraction mapping. Afterward, many authors have studied this problem and obtained significant results [1], [4], [21], [23].

2. Fixed point theorems

Theorem 2.1. *Let (X,d) be a complete b-rectangular metric space with $s > 1$, and let $f, g : X \to X$ be two self maps satisfying*

$$d(fx, gy) \le \delta M(x,y) + LN(x,y) \tag{2.1}$$

for all $x, y \in X$, where $\delta \in \left[0, \frac{1}{s}\right)$ and $L \ge 0$ and

$$M(x,y) = \max\{d(x,y), d(x,fx), d(y,gy)\}$$

$$N(x,y) = \min\{d(x,fx), d(y,gy), d(x,gy), d(y,fx)\}.$$

Then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$ in X as $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n \ge 1$. Suppose that there is some $n \ge 1$ such that $x_n = x_{n+1}$. If $n = 2k$, then $x_{2k} = x_{2k+1}$ and from (2.1),

$$d(x_{2k+1}, x_{2k+2}) = d(fx_{2k}, gx_{2k+1}) \le \delta M(x_{2k}, x_{2k+1}) + LN(x_{2k}, x_{2k+1})$$

where

$$\begin{aligned} M(x_{2k}, x_{2k+1}) &= \max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, fx_{2k}), d(x_{2k+1}, gx_{2k+1})\} \\ &= \max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} \\ &= \max\{0, 0, d(x_{2k+1}, x_{2k+2})\} \end{aligned}$$

and

$$\begin{aligned} N(x_{2k}, x_{2k+1}) &= \min\{d(x_{2k}, fx_{2k}), d(x_{2k+1}, gx_{2k+1}), d(x_{2k}, gx_{2k+1}), d(x_{2k+1}, fx_{2k})\} \\ &= \min\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2}), d(x_{2k}, x_{2k+2}), d(x_{2k+1}, x_{2k+1})\} \\ &= 0 \end{aligned}$$

Thus we have

$$d\left(x_{2k+1},x_{2k+2}\right)\leq \delta d\left(x_{2k+1},x_{2k+2}\right)$$

which is a contradiction with $\delta\in\left[0,\frac{1}{s}\right)$. Therefore $x_{2k+1}=x_{2k+2}$. Hence we have $x_{2k}=x_{2k+1}=x_{2k+2}$. It means that $x_{2k}=fx_{2k}=gx_{2k}$,i.e. x_{2k} is a common fixed point of f and g.
If $n=2k+1$, then using same arguments, it can be shown that x_{2k+1} is a common fixed point of f and g.
Now suppose $x_n\neq x_{n+1}$ for all $n\geq 1$.

$$d\left(x_{2n+1},x_{2n+2}\right)=d\left(fx_{2n},gx_{2n+1}\right)\leq \delta M\left(x_{2n},x_{2n+1}\right)+LN\left(x_{2n},x_{2n+1}\right) \tag{2.2}$$

where

$$\begin{aligned} M\left(x_{2n},x_{2n+1}\right)&=\max\left\{d\left(x_{2n},x_{2n+1}\right),d\left(x_{2n},fx_{2n}\right),d\left(x_{2n+1},gx_{2n+1}\right)\right\}\\ &=\max\left\{d\left(x_{2n},x_{2n+1}\right),d\left(x_{2n},x_{2n+1}\right),d\left(x_{2n+1},x_{2n+2}\right)\right\}\\ &=\max\left\{d\left(x_{2n},x_{2n+1}\right),d\left(x_{2n+1},x_{2n+2}\right)\right\}\end{aligned}$$

and

$$\begin{aligned} N\left(x_{2n},x_{2n+1}\right)&=\min\left\{d\left(x_{2n},fx_{2n}\right),d\left(x_{2n+1},gx_{2n+1}\right),d\left(x_{2n},gx_{2n+1}\right),d\left(x_{2n+1},fx_{2n}\right)\right\}\\ &=\min\left\{d\left(x_{2n},x_{2n+1}\right),d\left(x_{2n+1},x_{2n+2}\right),d\left(x_{2n},x_{2n+2}\right),0\right\}\\ &=0.\end{aligned}$$

If $M\left(x_{2n},x_{2n+1}\right)=d\left(x_{2n+1},x_{2n+2}\right)$, then by (2.2)

$$d\left(x_{2n+1},x_{2n+2}\right)\leq \delta d\left(x_{2n+1},x_{2n+2}\right)$$

which is a contradiction. Thus $M\left(x_{2n},x_{2n+1}\right)=d\left(x_{2n},x_{2n+1}\right)$ and from (2.2)

$$d\left(x_{2n+1},x_{2n+2}\right)\leq \delta d\left(x_{2n},x_{2n+1}\right).$$

Similarly it can be proved that

$$d\left(x_{2n+3},x_{2n+2}\right)\leq \delta d\left(x_{2n+2},x_{2n+1}\right).$$

So

$$d\left(x_{n+1},x_n\right)\leq \delta d\left(x_n,x_{n-1}\right)\leq \delta^n d\left(x_1,x_0\right).$$

for all $n\geq 1$. Similarly, we can show $d\left(x_{n+2},x_n\right)\leq \delta^n d\left(x_2,x_0\right)$.
We can show that $\{x_n\}$ is a $b-$rectangular$-$Cauchy sequence. Using b-rectangular inequality and $x_n\neq x_{n+1}$ for all $n\geq 1$ and $d_n=d\left(x_n,x_{n+1}\right)$, $d_n^*=d\left(x_n,x_{n+2}\right)$

$$\begin{aligned} d\left(x_n,x_{n+2m+1}\right)&\leq s\left[d\left(x_n,x_{n+1}\right)+d\left(x_{n+1},x_{n+2}\right)+d\left(x_{n+2},x_{n+2m+1}\right)\right]\\ &\leq s\left[d_n+d_{n+1}\right]+s^2\left[d_{n+2}+d_{n+3}\right]+s^3\left[d_{n+4}+d_{n+5}\right]\\ &+\ldots+s^{m+1}d_{n+2m}\\ &\leq s\left[\delta^n d_0+\delta^{n+1}d_0\right]+s^2\left[\delta^{n+2}d_0+\delta^{n+3}d_0\right]\\ &+s^3\left[\delta^{n+4}d_0+\delta^{n+5}d_0\right]+\ldots+s^m\delta^{n+2m}d_0\\ &\leq s\delta^n\left[1+s\delta^2+s^2\delta^4+\ldots+s^m\delta^{2m}\right]d_0+s\delta^{n+1}\left[1+s\delta^2+s^2\delta^4+\ldots+s^m\delta^{2m}\right]d_0\\ &=\frac{1+\delta}{1-s\delta^2}s\delta^n d_0 \quad (s\delta^2<1).\end{aligned}$$

Hence,

$$d\left(x_n,x_{n+2m+1}\right)\leq \frac{1+\delta}{1-s\delta^2}s\delta^n d_0. \tag{2.3}$$

Similarly, we can show

$$d\left(x_n,x_{n+2m}\right)\leq \frac{1+\delta}{1-s\delta^2}s\delta^n d_0+\delta^{n-2}d_0^*. \tag{2.4}$$

Thus form (2.3) and (2.4), we obtain that

$$\lim_{n\to\infty} d\left(x_n,x_{n+p}\right)=0,$$

for all $p=1,2,3,\ldots$. Hence $\{x_n\}$ is a b-rectangular-Cauchy sequence in (X,d). By completeness of (X,d), there exists $r\in X$ such that $x_n=fx_{n-1}\to r$ as $n\to\infty$.

Now we prove that $fr = r$. By $b-$rectangular inequality ,

$$\begin{aligned}\frac{1}{s}d(fr,r) &\leq d(fr,gx_n)+d(gx_n,x_n)+d(x_n,r)\\ &\leq \delta M(r,x_n)+LN(r,x_n)+d(x_{n+1},x_n)+d(x_n,r)\end{aligned}$$

where

$$M(r,x_n)=\max\{d(r,x_n),d(r,fr),d(x_n,gx_n)\}\to d(r,fr),$$

as $n\to\infty$ and

$$N(r,x_n)=\min\{d(r,fr),d(x_n,gx_n),d(r,gx_n),d(x_n,fr)\}\to 0,$$

as $n\to\infty$.
Hence, taking the limit as $n\to\infty$, we obtain

$$\frac{1}{s}d(fr,r)\leq \delta d(r,fr)+L.0+0+0$$

that is $fr = r$. Hence r is a fixed point of f.
Now we show $gr = r$. Suppose $r \neq gr$, by (2.1)

$$d(r,gr)=d(fr,gr)\leq \delta M(r,r)+LN(r,r)$$

where

$$\begin{aligned}M(r,r) &= \max\{d(r,r),d(r,fr),d(r,gr)\}\\ &= \max\{0,0,d(r,gr)\}\\ &= d(r,gr)\end{aligned}$$

and

$$\begin{aligned}N(r,r) &= \min\{d(r,fr),d(r,gr),d(r,gr),d(r,fr)\}\\ &= 0.\end{aligned}$$

By (2.1),

$$d(r,gr)\leq \delta d(r,gr)$$

which is a contradiction. Thus $gr = r$.
Now we show that uniqueness, Suppose r and t are different common fixed points of f and g. By (2.1),

$$d(r,t)=d(fr,gt)\leq \delta M(r,t)+LN(r,t) \tag{2.5}$$

where

$$\begin{aligned}M(r,t) &= \max\{d(r,t),d(r,fr),d(t,gt)\}\\ &= d(r,t)\end{aligned}$$

and

$$\begin{aligned}N(r,t) &= \min\{d(r,fr),d(t,gt),d(r,gt),d(t,fr)\}\\ &= 0.\end{aligned}$$

Thus from (2.5),

$$d(r,t)\leq \delta d(r,t)$$

So $d(r,t)=0$, i.e. $r = t$. □

Example 2.2. *Let $X = A\cup B$, where $A=\left\{\frac{1}{n}: n\in\{2,3,4,5\}\right\}$ and $B=[1,2]$. Define $d: X\times X\to[0,\infty)$ such that $d(x,y)=d(y,x)$ for all $x,y\in X$ and*

$$\begin{aligned}&d\left(\frac{1}{2},\frac{1}{3}\right)=d\left(\frac{1}{4},\frac{1}{5}\right)=0.03; d\left(\frac{1}{2},\frac{1}{5}\right)=d\left(\frac{1}{3},\frac{1}{4}\right)=0.02;\\ &d\left(\frac{1}{2},\frac{1}{4}\right)=d\left(\frac{1}{5},\frac{1}{3}\right)=0.06;\ d(x,y)=|x-y|^2 \text{ otherwise.}\end{aligned}$$

Then (X,d) is a complete b-rectangular metric space with coefficient $s = 3 > 1$. But (X,d) is neither a metric space nor a a rectangular metric space. Let $f,g: X\to X$ be defined as

$$f(x)=\begin{cases}\frac{1}{4}, & x\in A\\ \frac{1}{5}, & x\in B\end{cases},\quad g(x)=\begin{cases}\frac{1}{4}, & x\in A\\ \frac{1}{6}, & x\in B\end{cases}$$

Then f and g satisfy all conditions of Theorem 2.1 with $\delta=\frac{1}{4}$ and $L\geq 0$ and $x=\frac{1}{4}$ is a unique fixed point of f and g.

Corollary 2.3. *Let (X,d) be a b-rectangular metric space with $s>1$, and let $f,g:X\to X$ be self maps satisfying*

$$d(fx,gy)\leq \delta d(x,y)+L\min\{d(x,fx),d(y,gy),d(x,gy),d(y,fx)\}$$

for all $x,y\in X$, where $\delta\in\left[0,\frac{1}{s}\right)$ and $L\geq 0$. Then f and g have a unique fixed point.

Corollary 2.4. *Let (X,d) be a complete $b-$rectangular metric space with $s>1$, and let $f:X\to X$ be a self map satisfying*

$$d(fx,fy)\leq \delta M(x,y)+LN(x,y)$$

for all $x,y\in X$, where $\delta\in\left[0,\frac{1}{s}\right)$ and $L\geq 0$ and

$$M(x,y)=\max\{d(x,y),d(x,fx),d(y,fy)\}$$

$$N(x,y)=\min\{d(x,fx),d(y,fy),d(x,fy),d(y,fx)\}$$

then f has a unique fixed point.

Corollary 2.5. *Let (X,d) be a $b-$rectangular metric space with $s>1$, and let $f:X\to X$ be a self map satisfying*

$$d(fx,fy)\leq \delta d(x,y)+L\min\{d(x,fx),d(y,fy),d(x,fy),d(y,fx)\}$$

for all $x,y\in X$, where $\delta\in\left[0,\frac{1}{s}\right)$ and $L\geq 0$. Then f has a unique fixed point.

3. Conclusion

The development of the field of fixed point theory depends on the generalization of the Banach Contraction principle on complete metric spaces. This generalization or extension comes up by either introducing new types of contractions or by working on a more general structured space such as b-rectangular metric spaces. In this article, we have proven some fixed point theorems for almost contraction on b-rectangular metric spaces and hence our results generalize many existing results in the literature.

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Universal Journal of Mathematics and Applications, 3 (1) (2020) 33-37
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.615166

Parameter Reduction Method for Pythagorean Fuzzy Soft Sets

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Article Info

Keywords: *Parameter reduction, Pythagorean fuzzy soft set.*
2010 AMS: *03E72, 68T37*
Received: *4 September 2019*
Accepted: *14 October 2019*
Available online: *25 March 2020*

Abstract

The aim of this paper is to give new parameter reduction methods according to Pythagorean fuzzy soft sets. The reason for the definition of these methods is to help decision-makers facilitate their decision-making processes. The algorithm of the first method defined is related to the selection of some parameters. The second method determines the parameters with less deviation than the other parameters. Further, numerical examples related to the new algorithms are examined.

1. Introduction

Fuzzy Sets (FSs) put forward by Zadeh [1] has influenced deeply all the scientific fields since the publication of his paper. It is seen that this concept, which is very important for real-life situations, had not enough solution to some problems in time. New quests for such problems have been coming up. Atanassov [2] initiated Intuitionistic fuzzy sets (IFSs) for such cases.

The SS theory, which contributed to the solution of uncertainties in non-parametric problems, was initiated by Molodtsov [3]. SS theory is an inherent extension of the FS theory and can, therefore, be easily applied to all branches of science and technology. This theory deals with a concentration of approximate illustration of objects. In the approximate illustration, there exists a set of predicate values and a set of approximate values. This theory is suitable and simply operative in performance due to the nonentity of restrictions on the approximate illustrations.

Yager [4] offered a new FS called Pythagorean fuzzy set(PFS). PFS has fascinated the care of great deal researchers in a little while time. The formulation of the negation for IFSs and PFSs is examined by Yager [5]. In [6], PF subsets and its relationship with IF subsets were debated and some set operations on PF subsets were defined. Peng et al. [7], given the definition of the Pythagorean fuzzy soft set(PFSS), investigated its properties. Kirisci [8] introduce Pythagorean fuzzy parametrized soft set(PFPSS) and examine some characteristics, operations. In addition, the answer of the decision-making(DM) problem with PFPSS and other related notions is presented in [8].

2. Preliminaries

Give the sets $\mathscr{U}$, P, $\rho(\mathscr{U})$ as an initial universe, parameters, the power set of $\mathscr{U}$ respectively, and $S \subset P$. Give the mapping $m : X \to \rho(\mathscr{U})$. Therefore, u_X is said to a soft set(SS) over U [3].

Choose the set $\mathscr{U} = \{x_i\}_{i=1}^{n}$. Let $\{S(j)\}_{j=1}^{k}$ be a set of parameters.
Let $\{\cup_{j=1}^{k} S(j)\} \subseteq P$ and every parameter set $S(k)$ indicate the kth class of parameters and the elements of $S(k)$ represents a exclusive attribute set.

The set

$$A = \{\langle x, u_A(x), v_A(x)\rangle : x \in \mathscr{U}\}$$

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is said to be an intuitionistic fuzzy set(IFS) A on $\mathscr{U}$, [2] where, $u_A, v_A : \mathscr{U} \to [0,1]$ such that $0 \leq u_A(x) + v_A(x) \leq 1$ for any $x \in \mathscr{U}$. The degree of indeterminacy $w_A = 1 - u_A(x) - v_A(x)$.

An Pythagorean fuzzy set(PFS) φ over $\mathscr{U}$ is given by

$$\varphi = \{\langle x, u_\varphi(x), v_\varphi(x)\rangle : x \in \mathscr{U}\},$$

where $u_\varphi, v_\varphi : \mathscr{U} \to [0,1]$ denote the degree of membership and the degree of non-membership of $x \in \mathscr{U}$ to φ, respectively, such that $0 \leq (u_\varphi(x))^2 + (v_\varphi(x))^2 \leq 1$ [4,5]. $\mathscr{I}_\varphi = \sqrt{1 - (u_\varphi(x)^2 - (v_\varphi(x))^2}$ represent the degree of indeterminacy.

Then, $\varphi(S)$ is called Pythagorean Fuzzy Soft Set(PFSS) on $\mathscr{U}$, if $\varphi(S)$ is mapping given by $\varphi(S) : P \to \rho(\mathscr{U})$ [7].

Remark 2.1. *It is easy to check that PFSSs generalize both IFSs and SSs. That is, all IF degrees are part of the PF degrees. In actual DM problems, the PFSS characterizes a larger membership space than the IFSS. That is, the PFSS a higher capability than the IFSS to model uncertainty in real DM problems.*

Let Pythagorean fuzzy numbers (PFNs) are denoted by $E = (u_S, v_S)$ [9]. Choose three PFNs $\theta = E(u,v), \theta_1 = \langle u_1, v_1\rangle, \theta_2 = \langle u_2, v_2\rangle$. We can give some basic operations as follows [4]:

- $\bar{\theta} = \langle u, v\rangle$;
- $\theta_1 \vee \theta_2 = \langle \max\{u_1, u_2\}, \min\{v_1, v_2\}\rangle$;
- $\theta_1 \wedge \theta_2 = \langle \min\{u_1, u_2\}, \max\{v_1, v_2\}\rangle$;
- $\theta_1 \oplus \theta_2 = \langle \sqrt{u_1^2 + u_2 2 - u_1^2 u_2^2}, v_1 v_2\rangle$;
- $\theta_1 \otimes \theta_2 = \langle u_1 u_2, \sqrt{v_1^2 + v_2 2 - v_1^2 v_2^2}\rangle$;
- $\alpha.\theta = \langle \sqrt{1 - (1-u^2)^\alpha}, v^\alpha\rangle$;
- $\theta^\alpha = \langle u^\alpha, \sqrt{1 - (1-v^2)^\alpha}, \rangle$;

for $\alpha > 0$.

Maji et al. [10] firstly gave IFSS. In [11], Q-IFSS defined and basic properties are investigated. Broumi et al. [12] are given new definitions for IFSS such as concentration, dilatation and normalization. In [13], first Zadeh's containment, IF conjunction, IF disjunction of two IFSSs are defined and some basic properties are examined. In [14], three new operations based on Second Zadeh's containment, conjunction and disjunction operations have been defined and studied. Maji [15] has been extended IFSS with new operations. In [16], a new approach to IFSS was presented with rough set for DM problems. In this study, we adopt the PFSS from idea of Ghosh and Das [17]. Kirisci [18] compared IFPSS and Riesz summability methods using medical real dataset.

3. Comparison of IFSs and PFSs

IFS, offered by Atanassov [2] is an extension of FS Theory [1]. IFS is characterized by a membership degree and a non-membership degree and therefore can indicate the fuzzy character of data in more detail comprehensively. The prominent characteristic of IFS is that it assigns to each element a membership degree and a non-membership degree with their sum equal to or less than 1. However, in some practical DM process, the sum of the membership degree and the non-membership degree to which an alternative satisfying a criterion provided by a decision maker may be bigger than 1, but their square sum is equal to or less than 1.

Therefore, Yager [4] proposed PFS characterized by a membership degree and a non-membership degree, which satisfies the condition that the square sum of its membership degree and non-membership degree is less than or equal to 1. Yager [19] gave an example to state this situation: a decision maker gives his support for membership of an alternative is $\frac{\sqrt{3}}{2}$ and his against membership is $\frac{1}{2}$. Owing to the sum of two values is bigger than 1, they are not available for IFS, but they are available for PFS since $(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 \leq 1$. Obviously, PFS is more capable than IFS to model the vagueness in the practical multicriteria decision-making problems.

The main difference between PFNs and IFNs is their corresponding constraint conditions, which can be easily shown in Figure 3.1. Here, we observe that intuitionistic membership grades are all points under the line $x + y \leq 1$ and the Pythagorean membership grades are all points with $x^2 + y^2 \leq 1$.

One important implication of this is that it allows the use of the PFSs in situations in which we cannot use IFSs. An example of this would be a case in which a user indicates that their support for membership of x is $\frac{\sqrt{3}}{2}$ and their support against membership is $\frac{1}{2}$. As we noted these values are not allowable for intuitionistic membership grades but allowable as Pythagorean membership grades. Thus in this case, rather then requiring the user to change their information to satisfy the constraints of the IFS we can use a PFS.

4. Parameter Reduction Method

Take a complete lattice $(\mathscr{L}, \leq_\mathscr{L})$, such that $\mathscr{L} = \{(x,y) : x, y \in [0,1], x^2 + y^2 < 1\}$. The corresponding partial order $\leq_\mathscr{L}$ id defined by $(x,y) \leq_\mathscr{L} (a,b) \quad x \leq a, \quad y \geq b, \forall (x,y), (a,b) \in \mathscr{L}$. Any ordered pair $(x,y) \in \mathscr{L}$ is said to be Pythagorean fuzzy value(PFV) or Pyhtagorean

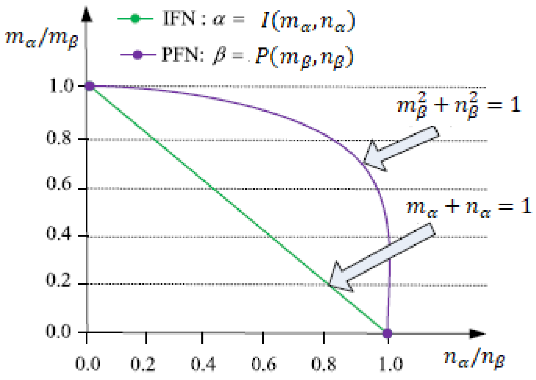

Figure 3.1: The PFNs and the IFNs

fuzzy number(PFN) [20].

Let $E=(u_E, v_E)$ be a Pythagorean fuzzy number (PFN). The mapping $\mathscr{SF}_E : L \rightarrow [-1,1]$ is called *score function*, if $\mathscr{SF}_E = u_E^2 - v_E^2$ for all $E \in L$ [8], [9].

For any two PFNs E, F

$$E \prec F, if \mathscr{SF}(E) < \mathscr{SF}(F),$$
$$E \succ F, if \mathscr{SF}(E) > \mathscr{SF}(F),$$
$$E \sim F, if \mathscr{SF}(E) = \mathscr{SF}(F).$$

Algorithm 1:

For each i, j, score values(SVs) of each of the entries of the PFSS $\varphi(S)$ denoted by $\phi = \mathscr{SF}_E$. Define the aggregated score as $\psi = \overline{\mathscr{SF}_E}(x_i) = \sum_{j=1}^{m} \mathscr{SF}_E(x_j)$. Consider $\lambda = \{x_i^*\}_{i=1}^r \subset P$ and $\mu = \{x_i^{**}\}_{i=1}^r \subset P$.

i. Calculate ϕ
ii. Compute ψ.
iii. Select the sets $\lambda, \mu \subset P$, where $\lambda, \mu \neq \emptyset$, $\lambda \cap \mu = \emptyset$, $\lambda \cup \mu \neq E$.
iv. Compute $\psi_{\lambda \subset P}$, $\psi_{\mu \subset P}$ $(\forall x_i \in \mathscr{U})$.
v. Choose the reduction parameter set of the PFSS ϕ as $P - (\lambda \cup \mu)$, if $\psi_{\lambda \subset P} - \psi_{\mu \subset P} = 0$.

Example 4.1. *Take the set of blouses $U = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. Parameters can be defined as p_1:bright, p_2:cheap, p_3:costly, p_4:very costly, p_5:colourful, p_6:cotton, p_7:polystyrene, p_8:long sleeve. The tabular representation of PFFS φ is given Table 1.*

Table 2 shows score of each entry and object for the PFSS φ (for $i = 1, 2, \cdots, 6$).

The order of the blouses is found as $y_5, y_6, y_4, y_1, y_2, y_3$. Now we choose the sets λ, μ such as $\lambda, \mu \subset P$ and $\lambda \cup \mu \neq P$. Consider the set $\lambda = \{p_1, p_4\}$ and $\mu = \{p_5, p_7\}$. It can be easily seen that reduced parameter set $P^* = \{p_2, p_3, p_6, p_8\}$ is obtained, when $\psi(y_i)$ of each object y_i are computed. The table which include the parameter set P^* is given Table 3. The object ordering in Table 2 and Table 3 appears to be the same.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
y_1	(0.5, 0.3)	(0.4, 0.3)	(0.5, 0.4)	(0.1, 0.6)	(0.3, 0.5)	(0.4, 0.2)	(0.6, 0.1)	(0.1, 0.5)
y_2	(0.2, 0.5)	(0.4, 0.1)	(0.5, 0.2)	(0.1, 0.2)	(0.5, 0.2)	(0.2, 0.3)	(0.2, 0.1)	(0.4, 0.6)
y_3	(0.4, 0.6)	(0.2, 0.5)	(0.4, 0.3)	(0.5, 0.3)	(0.6, 0.4)	(0.3, 0.4)	(0.3, 0.5)	(0.5, 0.4)
y_4	(0.5, 0.4)	(0.3, 0.5)	(0.3, 0.6)	(0.6, 0.2)	(0.2, 0.6)	(0.6, 0.2)	(0.4, 0.5)	(0.4, 0.3)
y_5	(0.3, 0.6)	(0.4, 0.2)	(0.6, 0.3)	(0.6, 0.3)	(0.5, 0.3)	(0.7, 0.1)	(0.3, 0.5)	(0.5, 0.5)
y_6	(0.2, 0.5)	(0.6, 0.2)	(0.6, 0.1)	(0.5, 0.2)	(0.1, 0.7)	(0.3, 0.5)	(0.7, 0.1)	(0.4, 0.5)

Table 1: Tabular representation for PFSS φ

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	ψ
y_1	0.16	0.07	0.09	-0.35	-0.16	0.12	0.35	-0.24	0.04
y_2	-0.21	0.15	0.21	-0.03	0.21	-0.05	0.03	-0.2	0.11
y_3	-0.2	-0.21	0.07	0.16	0.2	-0.07	-0.16	0.09	-0.12
y_4	0.09	-0.16	-0.27	0.32	-0.32	0.32	-0.09	0.07	-0.04
y_5	-0.27	0.12	0.27	0.27	0.16	0.48	-0.16	0	0.87
y_6	-0.21	0.32	0.35	0.21	-0.48	-0.16	0.48	-0.09	0.42

Table 2: Score of each object and entry for the PFSS φ

	p_2	p_3	p_6	p_8	ψ
y_1	0.07	0.09	0.12	-0.24	0.04
y_2	0.15	0.21	-0.05	-0.2	0.11
y_3	-0.21	0.07	-0.07	0.09	-0.12
y_4	-0.16	-0.27	0.32	0.07	-0.04
y_5	0.12	0.27	0.48	0	0.87
y_6	0.32	0.35	-0.16	-0.09	0.42

Table 3: Reduced parameter set

	p_1	p_2	p_3	p_4	p_5
y_1	(0.65, 0.42)	(0.7, 0.4)	(0.55, 0.12)	(0.9, 0.15)	(0.85, 0.2)
y_2	(0.45, 0.75)	(0.6, 0.4)	(0.33, 0.88)	(0.5, 0.5)	(0.52,0.5)
y_3	(0.6, 0.3)	(0.8, 0.05)	(0.7, 0.5)	(0.6, 0.3)	(0.7, 0.45)
y_4	(0,35, 0.65)	(0.4, 0.2)	(0.25, 0.75)	(0.4, 0.5)	(0.55, 0.45)

Table 4: Tabular representation of PFSS φ

	p_1	p_2	p_3	p_4	p_5	ψ
y_1	0.2461	0.33	0.2881	0.7875	0.6825	2.3342
y_2	-0.36	0.2	-0.6655	0	0.0204	-0.8051
y_3	0.27	0.6375	0.24	0.27	0.2875	1.705
y_4	-0.3	0.12	-0.5	-0.09	0.1	-0.67

Table 5: Score values for the PFSS φ

	p_1	p_3	p_4	p_5	ψ
y_1	0.2461	0.2881	0.7875	0.6825	2.0042
y_2	-0.36	-0.6655	0	0.0204	-1.0051
y_3	0.27	0.24	0.27	0.2875	1.0675
y_4	-0.3	-0.5	-0.09	0.1	-0.79

Table 6: The set of reduced parameter set and score values

Algorithm 2:

For $i \neq j$, the maximum score deviation denoted by $\sigma = \min\{|\overline{\mathscr{S}\mathscr{F}_N}(a_i) - \overline{\mathscr{S}\mathscr{F}_N}(a_j)|\}$, for $i, j = 1, 2, \cdots n$.

i. Calculate ϕ
ii. Compute ψ.
iii. Calculate σ with $P = \{p_1, p_2, \cdots, p_m\}$.
iv. Select the parameter $p_k \in P$ for the following situations for $i, j = 1, 2, \cdots, m$
   a) For $j \neq k$, $\mathscr{S}\mathscr{F}_N(a_i, p_k) \leq \mathscr{S}\mathscr{F}_N(a_i, p_j)$ $(\forall i)$.
   b) $\forall i$ and for $j \neq k$, $\max|\mathscr{S}\mathscr{F}_N(a_i, p_k) - \mathscr{S}\mathscr{F}_N(a_i, p_j)| < \sigma$.
v. Take the maximal number of parameter set A which fulfill the preceding step.
vi. Calculate the set $P - A$. This set is a reduced version of the parameters of PFSS.

Example 4.2. *Consider the information as given in Table 4. Firstly, compute SVs of* y_i, $(i = 1,2,3,4)$. *Obtained SVs are corresponding to the parameters P, which is defined as* $\psi = \overline{\mathscr{S}\mathscr{F}_N}(a_i) = \sum_{j=1}^{m} \mathscr{S}\mathscr{F}_N(a_i, p_j)$.*Therefore the objects are ordered with SVs and accuracy values. The order is shown as* $\{h_1, h_3, h_4, h_2\}$.

The maximum deviation is computed as $\sigma = \min\{3.1393, 0.6292, 3.0042, -2.5101, -0.1351, 2.375\} = -2.5101$. *We consider the parameter* $p_2 \in P$, *where* $\mathscr{S}\mathscr{F}_N(a_i, p_k) \leq \mathscr{S}\mathscr{F}_N(a_i, p_j)$ *for all* i, j *and* $\max|\mathscr{S}\mathscr{F}_N(a_i, p_k) - \mathscr{S}\mathscr{F}_N(a_i, p_j)| < \sigma$ *for all* i, j *(Table 5). Thus the reduction of the parameter* p_2 *will have no effect in the ordering of objects, which is shown in Table 6. So, we get the reduced set of parameters as* $P - \{p_2\}$.

5. Conclusion

Parameter reduction is primarily a method used to avoid redundant parameters. The results obtained with the minimum subset of parameters are the same as the entire set of parameters. Parameter reduction is essentially based on this idea. It is proposed new parameter reduction methods for PFSSs. Examples are given to illustrate the applicability of these methods.

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Universal Journal of Mathematics and Applications, 3 (1) (2020) 38-43
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.571525

Better Results for Trigonometrically Convex Functions via Hölder-İşcan and Improved Power-Mean Inequalities

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Article Info

Keywords: *Convex function, Trigonometrically convex function, Hölder-İşcan integral inequality, Improved power-mean integral inequality*
2010 AMS: *26A51, 26D10, 26D15.*
Received: *29 May 2019*
Accepted: *25 February 2020*
Available online: *25 March 2020*

Abstract

In this paper, using Hölder-İşcan and improved power-mean integral inequalities and together with an integral identity, we obtain Hadamard type inequalities for functions whose second derivatives in absolute value at certain power are trigonometrically convex functions. In addition, we prove that our results give better approach than previous results.

1. Introduction

Throughout the paper I is a non-empty interval in $\mathbb{R}$.

Definition 1.1. *A function $f : I \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

is valid for all $x,y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convexity theory has appeared as a powerful technique to study a wide class of related problems in pure and applied sciences. See articles [2,4,9–12] and the references therein.
Let $f : [a,b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality (for more information, see [5]). Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [2,3,13,15].

Definition 1.2 ([14]). *Let $h : (0,1) \subset J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h,I)$, if f is non-negative and for all $x,y \in I$, $\alpha \in (0,1)$ we have*

$$f(\alpha x+(1-\alpha)y) \le h(\alpha)f(x)+h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h,I)$.

Definition 1.3 ([7]). *A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex function on interval $[a,b]$, if for each $x,y \in [a,b]$ and $t \in [0,1]$,*

$$f(tx+(1-t)y) \le \left(\sin\frac{\pi t}{2}\right)f(x)+\left(\cos\frac{\pi t}{2}\right)f(y). \tag{1.1}$$

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Denoted by $TC(I)$ the class of all trigonometrically convex functions on interval I. Every non-negative convex function is trigonometrically convex and every trigonometrically convex function is h-convex with $h(t)=\frac{\pi t}{2}$.
A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 1.4 (Hölder-İşcan Integral Inequality [6])**.** *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on interval $[a,b]$ then*

$$\int_a^b |f(x)g(x)|\,dx \leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.$$
$$\left.\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.$$

Improved power-mean integral inequality as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.5 (Improved power-mean integral inequality [8])**.** *Let $q\geq 1$. If f and g are real functions defined on $[a,b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|\,dx \leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.$$

Definition 1.6. *(Beta Function) The Beta function denoted by $\beta(a,b)$ is defined by*

$$\beta(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt,\ \ a,b>0.$$

2. Main results

In this section, using Hölder-İşcan integral inequality and improved power-mean integral inequality and an integral identity, author obtain a generalization of Hermite-Hadamard type inequalities for functions whose second derivatives in absolute value at certain power are trigonometrically convex functions.
In order to establish some inequalities of Hermite-Hadamard type integral inequalities for trigonometrically convex functions, we will use the following lemma. This lemma can be easily obtained by taking partial integration in the lemma in [1] .

Lemma 2.1. *The following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{(b-a)^2}{2}\int_0^1 (t-t^2)f''(ta+(1-t)b)\,dt$$

Theorem 2.2. *Let $f:I\to\mathbb{R}$ be a continuously two times differentiable function, let $a<b$ in I. If the mapping $|f''|$ is trigonometrically convex function on interval $[a,b]$, then the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq (b-a)^2\,\frac{16-4\pi}{\pi^3}A\left(|f''(a)|,|f''(b)|\right)$$

holds for $t\in[0,1]$, where A is the arithmetic mean and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1 and inequality

$$|f''(ta+(1-t)b)|\leq\left(\sin\frac{\pi t}{2}\right)|f''(a)|+\left(\cos\frac{\pi t}{2}\right)|f''(b)|,$$

we obtain

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)^2}{2}\int_0^1 |t|\,|1-t|\,|f''(ta+(1-t)b)|\,dt\\
&\leq \frac{(b-a)^2}{2}\left(\int_0^1 t(1-t)\,|f''(ta+(1-t)b)|\,dt\right)\\
&\leq \frac{(b-a)^2}{2}\left(\int_0^1 t(1-t)\left[\left(\sin\frac{\pi t}{2}\right)|f''(a)|+\left(\cos\frac{\pi t}{2}\right)|f''(b)|\right]dt\right)\\
&= \frac{(b-a)^2}{2}\left[\left(\frac{16-4\pi}{\pi^3}\right)|f''(a)|+\left(\frac{16-4\pi}{\pi^3}\right)|f''(b)|\right]\\
&= (b-a)^2\,\frac{16-4\pi}{\pi^3}A\left(|f''(a)|,|f''(b)|\right)
\end{aligned}$$

where

$$\int_0^1 t(1-t)\sin\frac{\pi t}{2}dt=\int_0^1 t(1-t)\cos\frac{\pi t}{2}dt=\frac{16-4\pi}{\pi^3}.$$

This completes the proof of the theorem. □

Theorem 2.3. *Let $f: I \to \mathbb{R}$ be a continuously two times differentiable function, let $a < b$ in I and assume that $q > 1$. If the mapping $|f''|^q$ is trigonometrically convex function on interval $[a,b]$, then the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\beta^{\frac{1}{p}}(p+1,p+1)A^{\frac{1}{q}}\left(|f''(a)|^q,|f''(b)|^q\right) \tag{2.1}$$

holds for $t \in [0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, Hölder integral inequality and inequality

$$|f''(ta+(1-t)b)|^q \le \left(\sin\frac{\pi t}{2}\right)|f''(a)|^q + \left(\cos\frac{\pi t}{2}\right)|f''(b)|^q$$

which is the trigonometrically concexity of $|f''|^q$, we obtain

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{(b-a)^2}{2}\int_0^1 |t||1-t|\left|f''(ta+(1-t)b)\right|dt \\
&\le \frac{(b-a)^2}{2}\left(\int_0^1 t^p(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{(b-a)^2}{2}\left(\int_0^1 t^p(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\sin\frac{\pi t}{2}\right)|f''(a)|^q+\left(\cos\frac{\pi t}{2}\right)|f''(b)|^q\right]dt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left(|f''(a)|^q\frac{2}{\pi}+|f''(b)|^q\frac{2}{\pi}\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{2}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\beta^{\frac{1}{p}}(p+1,p+1)A^{\frac{1}{q}}\left(|f''(a)|^q,|f''(b)|^q\right)
\end{aligned}$$

where

$$\int_0^1 t^p(1-t)^p dt = \beta(p+1,p+1).$$

This completes the proof of the theorem. □

Theorem 2.4. *Let $f: I \to \mathbb{R}$ be a continuously two times differentiable function, let $a < b$ in I and assume that $q > 1$. If the mapping $|f''|^q$ is trigonometrically convex function on interval $[a,b]$, then the following inequality*

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+2)\left[\left(\frac{2}{\pi}-\frac{4}{\pi^2}\right)|f''(a)|^q+\left(\frac{4}{\pi^2}\right)|f''(b)|^q\right]^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+2,p+1)\left[\frac{4}{\pi^2}|f''(a)|^q+\left(\frac{2}{\pi}-\frac{4}{\pi^2}\right)|f''(b)|^q\right]^{\frac{1}{q}}
\end{aligned} \tag{2.2}$$

holds for $t \in [0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, Hölder-İşcan integral inequality and inequality

$$|f''(ta+(1-t)b)|^q \le \left(\sin\frac{\pi t}{2}\right)|f''(a)|^q + \left(\cos\frac{\pi t}{2}\right)|f''(b)|^q$$

which is the trigonometrically concexity of $|f''|^q$, we obtain

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{(b-a)^2}{2}\int_0^1 |t||1-t|\left|f''(ta+(1-t)b)\right|dt \\
&\le \frac{(b-a)^2}{2}\left(\int_0^1 (1-t)t^p(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)|f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\left(\int_0^1 tt^p(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t|f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{(b-a)^2}{2}\left(\int_0^1 (1-t)t^p(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left[\left(\sin\frac{\pi t}{2}\right)|f''(a)|^q+\left(\cos\frac{\pi t}{2}\right)|f''(b)|^q\right]dt\right)^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\left(\int_0^1 tt^p(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left[\left(\sin\frac{\pi t}{2}\right)|f''(a)|^q+\left(\cos\frac{\pi t}{2}\right)|f''(b)|^q\right]dt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+2)\left[\left(\frac{2}{\pi}-\frac{4}{\pi^2}\right)|f''(a)|^q+\left(\frac{4}{\pi^2}\right)|f''(b)|^q\right]^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+2,p+1)\left[\frac{4}{\pi^2}|f'(a)|^q+\left(\frac{2}{\pi}-\frac{4}{\pi^2}\right)|f'(b)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

where

$$\int_0^1 t^p (1-t)^{p+1} dt = \beta(p+1,p+2),$$
$$\int_0^1 t^{p+1} (1-t)^p dt = \beta(p+2,p+1)$$
$$\int_0^1 (1-t)\sin\frac{\pi t}{2} dt = \int_0^1 t\cos\frac{\pi t}{2} dt = \frac{2}{\pi} - \frac{4}{\pi^2},$$
$$\int_0^1 (1-t)\cos\frac{\pi t}{2} dt = \int_0^1 t\sin\frac{\pi t}{2} dt = \frac{4}{\pi^2}.$$

This completes the proof of the theorem. □

Remark 2.5. *The inequality (2.2) is better than the inequality (2.1).*

Proof. By using the properties

$$\beta(p+1,p+2) = \beta(p+2,p+1)$$
$$\beta(p+1,p+2) = \beta(p+1,p+1)\frac{p+1}{2(p+1)}$$

and the concavity of the function $h:[0,\infty)\to\mathbb{R}, h(x)=x^s, 0<s\le 1$, that is, if we use the property

$$\frac{u^s+v^s}{2} \le \left(\frac{u+v}{2}\right)^s$$

we can write the right hand-side of the inequality (2.1) as follow:

$$\begin{aligned}
&\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+2)\left[\left(\frac{2}{\pi}-\frac{4}{\pi^2}\right)|f''(a)|^q + \left(\frac{4}{\pi^2}\right)|f''(b)|^q\right]^{\frac{1}{q}} + \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+2,p+1)\left[\frac{4}{\pi^2}|f''(a)|^q + \left(\frac{2}{\pi}-\frac{4}{\pi^2}\right)|f''(b)|^q\right]^{\frac{1}{q}} \\
\le\ & 2\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+2)\left[\frac{\frac{2}{\pi}|f''(a)|^q + \frac{2}{\pi}|f''(b)|^q}{2}\right]^{\frac{1}{q}} \\
=\ & 2\frac{(b-a)^2}{2}\left[\beta(p+1,p+1)\frac{p+1}{2(p+1)}\right]^{\frac{1}{p}}\left[\frac{\frac{2}{\pi}|f''(a)|^q + \frac{2}{\pi}|f''(b)|^q}{2}\right]^{\frac{1}{q}} \\
=\ & \frac{(b-a)^2}{2}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\beta^{\frac{1}{p}}(p+1,p+1)A^{\frac{1}{q}}\left(|f''(a)|^q, |f''(b)|^q\right),
\end{aligned}$$

which is the required result. This completes the proof of the Remark. □

Theorem 2.6. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a continuously two times differentiable function, let $a<b$ in I and assume that $q\ge 1$. If the mapping $|f''|^q$ is trigonometrically convex function on interval $[a,b]$, then the following inequality holds for $t\in[0,1]$:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{8(4-\pi)}{\pi^3}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f''(a)|^q, |f''(b)|^q\right) \tag{2.3}$$

Proof. From Lemma 2.1, power-mean integral inequality and trigonometrically convexity of $|f''|^q$, we have

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{(b-a)^2}{2}\int_0^1 |t||1-t|\left|f''(ta+(1-t)b)\right|dt \\
&\le \frac{(b-a)^2}{2}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left[\left(\sin\frac{\pi t}{2}\right)|f''(a)|^q + \left(\cos\frac{\pi t}{2}\right)|f''(b)|^q\right]dt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(|f''(a)|^q\int_0^1 t(1-t)\sin\frac{\pi t}{2}dt + |f''(b)|^q\int_0^1 t(1-t)\cos\frac{\pi t}{2}dt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{4(4-\pi)}{\pi^3}|f''(a)|^q + \frac{4(4-\pi)}{\pi^3}|f''(b)|^q\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{8(4-\pi)}{\pi^3}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f''(a)|^q, |f''(b)|^q\right)
\end{aligned}$$

where

$$\int_0^1 t(1-t)dt = \frac{1}{6},$$
$$\int_0^1 t(1-t)\sin\frac{\pi t}{2}dt = \int_0^1 t(1-t)\cos\frac{\pi t}{2}dt = \frac{4(4-\pi)}{\pi^3}$$

This completes the proof of the theorem. □

Corollary 2.7. *Under the assumption of Theorem 2.6 with $q=1$, we get the following the inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{2}\left(\frac{8(4-\pi)}{\pi^3}\right)A\left(\left|f''(a)\right|,\left|f''(b)\right|\right)$$

Theorem 2.8. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a continuously two times differentiable function, let $a<b$ in I and assume that $q\geq 1$. If the mapping $|f''|^q$ is trigonometrically convex function on interval $[a,b]$, then the following inequality holds for $t\in[0,1]$:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|&\leq\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{32(\pi-3)}{\pi^4}\left|f''(a)\right|^q+\frac{4\left(24-4\pi-\pi^2\right)}{\pi^4}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\\&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{4\left(24-4\pi-\pi^2\right)}{\pi^4}\left|f''(a)\right|^q+\frac{32(\pi-3)}{\pi^4}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\end{aligned}\tag{2.4}$$

Proof. From Lemma 2.1, improved power-mean integral inequality and trigonometrically convexity of $|f''|^q$, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|&\leq\frac{(b-a)^2}{2}\int_0^1|t||1-t|\left|f''(ta+(1-t)b)\right|dt\\&\leq\frac{(b-a)^2}{2}\left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\&+\frac{(b-a)^2}{2}\left(\int_0^1 t^2(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2(1-t)\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\&\leq\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2\left[\left(\sin\frac{\pi t}{2}\right)\left|f''(a)\right|^q+\left(\cos\frac{\pi t}{2}\right)\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\\&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2(1-t)\left[\left(\sin\frac{\pi t}{2}\right)\left|f''(a)\right|^q+\left(\cos\frac{\pi t}{2}\right)\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\\&=\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\left|f''(a)\right|^q\int_0^1 t(1-t)^2\sin\frac{\pi t}{2}dt+\left|f''(b)\right|^q\int_0^1 t(1-t)^2\cos\frac{\pi t}{2}dt\right)^{\frac{1}{q}}\\&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\left|f''(a)\right|^q\int_0^1 t^2(1-t)\sin\frac{\pi t}{2}dt+\left|f''(b)\right|^q\int_0^1 t^2(1-t)\cos\frac{\pi t}{2}dt\right)^{\frac{1}{q}}\\&=\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{32(\pi-3)}{\pi^4}\left|f''(a)\right|^q+\frac{4\left(24-4\pi-\pi^2\right)}{\pi^4}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\\&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{4\left(24-4\pi-\pi^2\right)}{\pi^4}\left|f''(a)\right|^q+\frac{32(\pi-3)}{\pi^4}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\end{aligned}$$

where

$$\int_0^1 t(1-t)^2dt=\int_0^1 t^2(1-t)dt=\frac{1}{12},$$

$$\int_0^1 t(1-t)^2\sin\frac{\pi t}{2}dt=\int_0^1 t^2(1-t)\cos\frac{\pi t}{2}dt=\frac{32(\pi-3)}{\pi^4}$$

$$\int_0^1 t(1-t)^2\cos\frac{\pi t}{2}dt=\int_0^1 t^2(1-t)\sin\frac{\pi t}{2}dt=\frac{4}{\pi^4}\left(24-4\pi-\pi^2\right)$$

This completes the proof of the theorem. □

Remark 2.9. *The inequality (2.4) is better than the inequality (2.3).*

Proof. By using concavity of the function $h:[0,\infty)\to\mathbb{R}, h(x)=x^s, 0<s\leq 1$, we can write the right hand-side of the inequality (2.4) as follow:

$$\begin{aligned}&\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{32(\pi-3)}{\pi^4}\left|f''(a)\right|^q+\frac{4\left(24-4\pi-\pi^2\right)}{\pi^4}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\\&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{4\left(24-4\pi-\pi^2\right)}{\pi^4}\left|f''(a)\right|^q+\frac{32(\pi-3)}{\pi^4}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\\&\leq 2\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{4(4-\pi)}{\pi^3}\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}\\&=2\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{4(4-\pi)}{\pi^3}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right)\\&=\frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{8(4-\pi)}{\pi^3}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right),\end{aligned}$$

which is the required result. This completes the proof of the Remark. □

Corollary 2.10. *Under the assumption of Theorem 2.8 with $q = 1$, we get the following the inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2}\frac{8(4-\pi)}{\pi^3}A\left(\left|f''(a)\right|,\left|f''(b)\right|\right).$$

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