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Smarandache Curves of the Evolute Curve According to Sabban Frame

Süleyman Şenyurt^{1*}, Yasin Altun¹

Abstract

The aim of this paper is to define Smarandache curves according to the Sabban frame belonging to the unit Darboux vector of spherical indicatrix curve of the evolute curve. Also, we calculate the geodesic curvatures of these curves. Finally, the results are expressed depending on the involute curve.

Keywords: Darboux vector, Involute curve, Evolute curve, Geodesic curvature, Sabban frame, Smarandache curves.

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1. Introduction and Preliminaries

In the theory of curves in the Euclidean space, one of the interesting problems is the characterization of a regular curve. It is known that the shape and size of a regular curve can be determined by using its curvature and torsion. Another approach to the solution of the problem is to consider the relationship between the corresponding Frenet vectors of two curves. For example, involute and evolute curves arise from this relationship. By definition, if the position vector of a curve is composed by the Frenet frame's vectors of another curve, then the curve is called a Smarandache curve [6]. Special Smarandache curves studied by some authors [1, 2, 5, 6, 7, 8], and related reference therein [9, 10, 11].

Let $\alpha : I \rightarrow E^3$ be a unit speed curve denoted by the moving Frenet apparatus of $\{T, N, B, \kappa, \tau\}$. The Frenet formulae is given by [3]

$$T'(s) = \kappa(s)N(s), N'(s) = -\kappa(s)T(s) + \tau(s)B(s), B'(s) = -\tau(s)N(s). \quad (1.1)$$

The Darboux vector defined by

$$W = \tau T + \kappa B. \quad (1.2)$$

The unit Darboux vector is given by

$$C = \sin \omega T + \cos \omega B$$

where

$$\sin \omega = \frac{\tau}{\|W\|}, \cos \omega = \frac{\kappa}{\|W\|}, \angle(W, B) = \omega, [12]$$

Let $\alpha : I \rightarrow \mathbb{E}^3$ be a unit speed curve and $\alpha_1 : I \rightarrow \mathbb{E}^3$ be a C^2 -differentiable curve. If the tangent vector of the curve α is orthogonal to the tangent vector of the α_1 , then α_1 is called evolute of the α .

If the curve α_1 is evolute of α , then we can write,

$$\alpha_1(s) = \alpha(s) + \rho(s)N(s) - \rho(s) \tan(\varphi(s) + c) B(s)$$

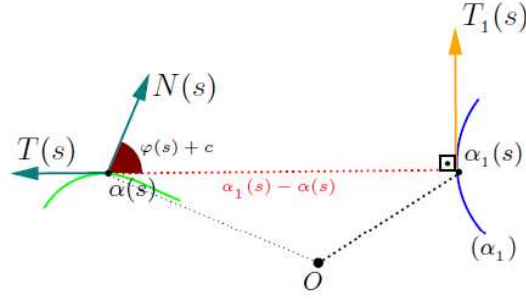


Figure 1.1. Evolute curve

where $c \in R$, $\rho = \frac{1}{\kappa}$ and $\varphi(s) = \int_0^s \tau(s) ds$. Also the relations between the Frenet vectors of evolute involute curves are given as [3]

$$\begin{cases} T_1(s) = \cos(\varphi + c) N(s) - \sin(\varphi + c) B(s) \\ N_1(s) = -T(s) \\ B_1(s) = \sin(\varphi + c) N + \cos(\varphi + c) B, \end{cases} \quad (1.3)$$

and for the curvatures we have

$$\kappa_1(s) = \frac{\kappa^3 \cos^3(\varphi + c)}{\kappa \tau \sin(\varphi + c) - \kappa' \cos(\varphi + c)}, \quad \tau_1(s) = \frac{-\kappa^3 \sin(\varphi + c) \cos^2(\varphi + c)}{\kappa \tau \sin(\varphi + c) - \kappa' \cos(\varphi + c)}. \quad (1.4)$$

Similarly (1.2), we can write of Darboux vector of evoluto curve

$$W_1 = \tau_1 T_1 + \kappa_1 B_1.$$

Let $\gamma: I \rightarrow S^2$ be a unit speed spherical curve. We can write

$$\gamma(s) = \gamma(s), \quad t(s) = \gamma'(s), \quad d(s) = \gamma(s) \wedge t(s), \quad [5] \quad (1.5)$$

here the set $\{\gamma(s), t(s), d(s)\}$ denotes the Sabban frame of γ on S^2 . It follows that we also have the equations,

$$\gamma'(s) = t(s), \quad t'(s) = -\gamma(s) + \kappa_g(s) d(s), \quad d'(s) = -\kappa_g(s) t(s), \quad [5]. \quad (1.6)$$

where κ_g is the geodesic curvature of the curve γ on S^2 whose curvature is

$$\kappa_g(s) = \langle t'(s), d(s) \rangle [4, 5]. \quad (1.7)$$

2. Smarandache curves of the evolute curve according to Sabban frame

Let C_1 be the unit Darboux vector of the evolute curve α_1 and let (C_1) be the unit speed spherical curve on S^2 . Then we can write

$$C_1 = \sin \omega_1 T_1 + \cos \omega_1 B_1, \quad T_{C_1} = \cos \omega_1 T_1 - \sin \omega_1 B_1, \quad C_1 \wedge T_{C_1} = N_1. \quad (2.1)$$

where $\angle(W_1, B_1) = \omega_1$. It follows from the equation (1.6) that of (C_1) are

$$C_1' = T_{C_1}, \quad T_{C_1}' = -C_1 + \frac{\|W_1\|}{\omega_1'} C_1 \wedge T_{C_1}, \quad (C_1 \wedge T_{C_1})' = -\frac{\|W_1\|}{\omega_1'} T_{C_1}. \quad (2.2)$$

From the equation (1.7), we have the following geodesic curvature of (C_1)

$$\kappa_g = \langle T_{C_1}', C_1 \wedge T_{C_1} \rangle \implies \kappa_g = \frac{\|W_1\|}{\omega_1'}.$$

Definition 2.1. Let C_1 be a spherical curve on S^2 and let C_1, T_{C_1} be the unit vectors of (C_1) . In this case $\beta_1(s)$ can be defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}}(C_1 + T_{C_1}). \quad (2.3)$$

In other words, substituting the equation (2.1) into equation (2.3) we write

$$\beta_1(s) = \frac{1}{\sqrt{2}} \left((\sin \omega_1 + \cos \omega_1) T_1 + (\cos \omega_1 - \sin \omega_1) B_1 \right). \tag{2.4}$$

Theorem 2.2. *The geodesic curvature of $\beta_1(s)$ -Smarandache curve is given by*

$$\kappa_g^{\beta_1} = \frac{1}{\left(2 + \frac{1}{\eta^2}\right)^{\frac{5}{2}}} \left(\frac{1}{\eta} \bar{\lambda}_1 - \frac{1}{\eta} \bar{\lambda}_2 + 2\bar{\lambda}_3 \right)$$

where

$$\eta = \frac{\kappa \tau^2 \sin(\varphi + c) - \kappa' \tau \cos(\varphi + c)}{\kappa^3 \cos^2(\varphi + c)}, \bar{\lambda}_1 = -2 - \frac{1}{\eta^2} + \frac{1}{\eta'} \frac{1}{\eta}, \bar{\lambda}_2 = -2 - 3 \frac{1}{\eta^2} - \frac{1}{\eta^4} - \frac{1}{\eta'} \frac{1}{\eta}, \bar{\lambda}_3 = 2 \frac{1}{\eta} + \frac{1}{\eta^3} + \frac{1}{\eta'}.$$

Proof: By differentiating (2.3) we can write

$$T_{\beta_1} = \frac{\omega_1' (\cos \omega_1 - \sin \omega_1)}{\sqrt{2\omega_1'^2 + \|W_1\|^2}} T_1 + \frac{\|W_1\|}{\sqrt{2\omega_1'^2 + \|W_1\|^2}} N_1 - \frac{\omega_1' (\cos \omega_1 + \sin \omega_1)}{\sqrt{2\omega_1'^2 + \|W_1\|^2}} B_1. \tag{2.5}$$

Considering the equations (2.4) and (2.5) we get

$$\beta_1 \wedge T_{\beta_1} = \frac{\|W_1\| (\cos \omega_1 + \sin \omega_1)}{\sqrt{2\|W_1\|^2 + 4(\omega_1')^2}} T_1 - \frac{\omega_1'}{\sqrt{2\|W_1\|^2 + 4(\omega_1')^2}} N_1 + \frac{\|W_1\| (\cos \omega_1 - \sin \omega_1)}{\sqrt{2\|W_1\|^2 + 4(\omega_1')^2}} B_1. \tag{2.6}$$

it follows by differentiating (2.5), with the coefficients

$$\mathfrak{K}_1 = -2 - \left(\frac{\|W_1\|}{\omega_1'}\right)^2 + \left(\frac{\|W_1\|}{\omega_1'}\right)' \left(\frac{\|W_1\|}{\omega_1'}\right), \mathfrak{K}_2 = -2 - 3 \left(\frac{\|W_1\|}{\omega_1'}\right)^2 - \left(\frac{\|W_1\|}{\omega_1'}\right)^4 - \left(\frac{\|W_1\|}{\omega_1'}\right)' \left(\frac{\|W_1\|}{\omega_1'}\right), \mathfrak{K}_3 = 2 \left(\frac{\|W_1\|}{\omega_1'}\right) + \left(\frac{\|W_1\|}{\omega_1'}\right)^3 + \left(\frac{\|W_1\|}{\omega_1'}\right)' \tag{2.7}$$

we can find out,

$$T'_{\beta_1} = \frac{(\omega_1')^4 \sqrt{2} (\mathfrak{K}_1 \sin \omega_1 + \mathfrak{K}_2 \cos \omega_1)}{\left(\|W_1\|^2 + (\omega_1')^2\right)^2} T_1 + \frac{\mathfrak{K}_3 (\omega_1')^4 \sqrt{2}}{\left(\|W_1\|^2 + (\omega_1')^2\right)^2} N_1 + \frac{(\omega_1')^4 \sqrt{2} (\mathfrak{K}_1 \cos \omega_1 - \mathfrak{K}_2 \sin \omega_1)}{\left(\|W_1\|^2 + (\omega_1')^2\right)^2} B_1. \tag{2.8}$$

From the equation (2.6) and (2.8), geodesic curvature $\kappa_g^{\beta_1}$ for the evolute curve β_1 is

$$\kappa_g^{\beta_1} = \langle T'_{\beta_1}, \beta_1 \wedge T_{\beta_1} \rangle = \frac{1}{\left(2 + \left(\frac{\|W_1\|}{\omega_1'}\right)^2\right)^{\frac{5}{2}}} \left(\frac{\|W_1\|}{\omega_1'} \mathfrak{K}_1 - \frac{\|W_1\|}{\omega_1'} \mathfrak{K}_2 + 2\mathfrak{K}_3 \right).$$

From the equation (1.3) and (1.4) Sabban apparatus of the β_1 -Smarandache curve for involute curves are

$$\beta_1(s) = \frac{1}{\sqrt{2}} (N + B), T_{\beta_1} = \frac{1}{\sqrt{1 + 2\eta^2}} (-T + \eta N - \eta B),$$

$$\beta_1 \wedge T_{\beta_1} = \frac{1}{\sqrt{2 + 4\eta^2}} (-2\eta T - \eta N + B), T'_{\beta_1} = \frac{\sqrt{2}}{(2\eta^2 + 1)^2} (-\eta^4 \bar{\lambda}_3 T + \eta^4 \bar{\lambda}_2 N + \eta^4 \bar{\lambda}_1 B),$$

hence the proof is completed.

Definition 2.3. *Let C_1 be a spherical curve on S^2 and let $C_1, C_1 \wedge T_{C_1}$ be the unit vectors of (C_1) . In this case $\beta_2(s)$ can be defined by*

$$\beta_2(s) = \frac{1}{\sqrt{2}} (C_1 + C_1 \wedge T_{C_1}). \tag{2.9}$$

From the equation (1.3), (1.4) and (2.1) we can write

$$\beta_2 = \frac{1}{\sqrt{2}} (-T + B). \tag{2.10}$$

Theorem 2.4. *The geodesic curvature of $\beta_2(s)$ -Smarandache curve is given by*

$$\kappa_g^{\beta_2} = \frac{1 + \eta}{\eta - 1}.$$

Proof: By differentiating (2.10), we can write

$$T_{\beta_2} = N. \tag{2.11}$$

Considering the equations (2.10) and (2.11) it is easily seen that

$$\beta_2 \wedge T_{\beta_2} = \frac{1}{\sqrt{2}}(-T - B)$$

and then differentiating (2.11) we can write

$$T'_{\beta_2} = \frac{\sqrt{2}}{\eta - 1}(-T - \eta B).$$

It follows that $\kappa_g^{\beta_2}$ geodesic curvature for β_2 is

$$\kappa_g^{\beta_2} = \frac{1 + \eta}{\eta - 1}.$$

Definition 2.5. Let C_1 be a spherical curve on S^2 and let T_{C_1} , $C_1 \wedge T_{C_1}$ be the unit vectors of (C_1) . In this case $\beta_3(s)$ can be defined by

$$\beta_3(s) = \frac{1}{\sqrt{2}}(T_{C_1} + C_1 \wedge T_{C_1}). \tag{2.12}$$

From the equations (2.1), (1.3) and (1.4) we can write

$$\beta_3 = \frac{1}{\sqrt{2}}(-T + N). \tag{2.13}$$

Theorem 2.6. The geodesic curvature of $\beta_3(s)$ -Smarandache curve is given by

$$\kappa_g^{\beta_3} = \frac{1}{\left(2 + \frac{1}{\eta^2}\right)^{\frac{5}{2}}} \left(2 \frac{1}{\eta} \bar{\sigma}_1 - \bar{\sigma}_2 + \bar{\sigma}_3\right). \tag{2.14}$$

Proof: By differentiating (2.13) we have

$$T_{\beta_3} = \frac{1}{\sqrt{2 + \eta^2}}(-T - N - \eta B). \tag{2.15}$$

Considering the equations (2.13) and (2.15) it is easily seen that

$$\beta_3 \wedge T_{\beta_3} = \frac{1}{\sqrt{2\eta^2 + 4}}(-\eta T - \eta N + 2B).$$

By differentiating (2.15) with the coefficients

$$\bar{\sigma}_1 = \frac{1}{\eta} + 2\frac{1}{\eta^3} + 2\frac{1}{\eta'}\frac{1}{\eta}, \bar{\sigma}_2 = -1 - 3\frac{1}{\eta^2} - 2\frac{1}{\eta^4} - \frac{1}{\eta'}, \bar{\sigma}_3 = -\frac{1}{\eta^2} - 2\frac{1}{\eta^4} + \frac{1}{\eta'}$$

we get

$$T'_{\beta_3} = \frac{\sqrt{2}}{(\eta^2 + 2)^2}(-\eta^4 \bar{\sigma}_3 T + \eta^4 \bar{\sigma}_2 N + \eta^4 \bar{\sigma}_1 B).$$

By this way, geodesic curvature $\kappa_g^{\beta_3}$ for the involute curve of β_3 is given by

$$\kappa_g^{\beta_3} = \frac{1}{\left(2 + \frac{1}{\eta^2}\right)^{\frac{5}{2}}} \left(2 \frac{1}{\eta} \bar{\sigma}_1 - \bar{\sigma}_2 + \bar{\sigma}_3\right). \tag{2.16}$$

Definition 2.7. Let C_1 be a spherical curve on S^2 and let T_{C_1} , $C_1 \wedge T_{C_1}$ be the unit vectors of (C_1) . In this case $\beta_4(s)$ can be defined by

$$\beta_4(s) = \frac{1}{\sqrt{3}}(C_1 + T_{C_1} + C_1 \wedge T_{C_1}). \tag{2.17}$$

From the equations (1.3), (1.4) and (2.1) we can write that

$$\beta_4(s) = \frac{1}{\sqrt{3}} \left(-T + N + B \right). \tag{2.18}$$

Theorem 2.8. The geodesic curvature of $\beta_4(s)$ -Smarandache curve is given by

$$\kappa_g^{\beta_4} = \frac{(2\frac{1}{\eta} - 1)\bar{\psi}_1 + (-1 - \frac{1}{\eta})\bar{\psi}_2 + (2 - \frac{1}{\eta})\bar{\psi}_3}{4\sqrt{2}\left(1 - \frac{1}{\eta} + \frac{1}{\eta^2}\right)^{\frac{5}{2}}}.$$

Proof: By differentiating (2.18), we get

$$T_{\beta_4} = \frac{1}{\sqrt{2(1 - \eta + \eta^2)}} \left(-T + (\eta - 1)N - \eta B \right). \tag{2.19}$$

Considering the equations (2.18) and (2.19) it is easily seen that

$$\beta_4 \wedge T_{\beta_4} = \frac{1 - 2\eta}{\sqrt{6 - 6\eta + 6\eta^2}} T - \frac{\eta + 1}{\sqrt{6 - 6\eta + 6\eta^2}} N + \frac{2 - \eta}{\sqrt{6 - 6\eta + 6\eta^2}} B.$$

By differentiating (2.19), with the coefficients

$$\bar{\psi}_1 = -2 + 4\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} + 2\frac{1}{\eta'} \left(2\frac{1}{\eta} - 1 \right), \bar{\psi}_2 = -2 + 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} - 2\frac{1}{\eta^4} - \frac{1}{\eta'} \left(1 + \frac{1}{\eta} \right), \bar{\psi}_3 = 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 4\frac{1}{\eta^3} - 2\frac{1}{\eta^4} + \frac{1}{\eta'} \left(2 - \frac{1}{\eta} \right)$$

we can write that

$$T'_{\beta_4} = \frac{1}{4\left(1 + \eta + \eta^2\right)^2} \left(-\bar{\psi}_3 \eta^4 T + \bar{\psi}_2 \eta^4 N + \bar{\psi}_1 \eta^4 B \right)$$

then we get geodesic curvature $\kappa_g^{\beta_4}$, for the involute curve $\beta_4(s_{\beta_4})$ as

$$\kappa_g^{\beta_4} = \frac{(2\frac{1}{\eta} - 1)\bar{\psi}_1 + (-1 - \frac{1}{\eta})\bar{\psi}_2 + (2 - \frac{1}{\eta})\bar{\psi}_3}{4\sqrt{2}\left(1 - \frac{1}{\eta} + \frac{1}{\eta^2}\right)^{\frac{5}{2}}}.$$

Example. Let us consider the unit speed spherical curve

$$\alpha(t) = \left(\frac{2}{5} \sin(2t) - \frac{1}{40} \sin(8t), -\frac{2}{5} \cos(2t) + \frac{1}{40} \cos(8t), \frac{4}{15} \sin(3t) \right)$$

and evolute of this curve,

$$\alpha_1(t) = \left(\frac{1}{40} \frac{16 \sin(2t) \sin(3t) \cos(1) - \sin(8t) \sin(3t) \cos(1) + 8 \cos(5t) \cos(1)}{\sin(3t) \cos(1)} - \frac{1}{40} \frac{8 \sin(1) \sin(2t) - 2 \sin(1) \sin(8t)}{\sin(3t) \cos(1)}, \right. \\ \left. - \frac{1}{40} \frac{16 \cos(2t) \sin(3t) \cos(1) - \cos(8t) \sin(3t) \cos(1)}{\sin(3t) \cos(1)} + \frac{1}{40} \frac{8 \sin(5t) \cos(1) - 8 \sin(1) \cos(2t) - 2 \sin(1) \cos(8t)}{\sin(3t) \cos(1)}, \right. \\ \left. \frac{1}{60} \frac{7 \cos(1) - 16 \cos(1) (\cos(3t))^2 + 12 \sin(1) \sin(3t)}{\sin(3t) \cos(1)} \right)$$

and Frenet vectors of the evolute curve α_1 ,

$$T_1 = \left(\frac{64}{5} \cos(1) (\cos(t))^5 - 16 \cos(1) (\cos(t))^3 + 4 \cos(1) \cos(t) + \frac{128}{5} \sin(1) \sin(t) (\cos(t))^7 - \frac{192}{5} \sin(1) \sin(t) (\cos(t))^5 \right. \\ \left. + 16 \sin(1) \sin(t) (\cos(t))^3, \frac{64}{5} \cos(1) \sin(t) (\cos(t))^4 - \frac{48}{5} \sin(t) (\cos(t))^2 \cos(1) + \frac{4}{5} \cos(1) \sin(t) + \frac{24}{5} \sin(1) (\cos(t))^2 + \frac{3}{5} \sin(1) \right. \\ \left. - \frac{128}{5} \sin(1) (\cos(t))^8 + \frac{256}{5} \sin(1) (\cos(t))^6 - 32 \sin(1) (\cos(t))^4, -\frac{3}{5} \cos(1) - \frac{16}{5} \sin(1) \sin(t) (\cos(t))^2 + \frac{4}{5} \sin(1) \sin(t) \right), \\ N_1 = \left(-\frac{4}{5} \cos(2t) + \frac{1}{5} \cos(8t), -\frac{4}{5} \sin(2t) + \frac{1}{5} \sin(8t), -\frac{4}{5} \cos(3t) \right), \\ B_1 = \left(-\frac{128}{5} \sin(t) (\cos(t))^7 \cos(1) + \frac{192}{5} \sin(t) (\cos(t))^5 \cos(1) - 16 \sin(t) (\cos(t))^3 \cos(1) + \frac{64}{5} (\cos(t))^5 \sin(1) \right. \\ \left. - 16 (\cos(t))^3 \sin(1) + 4 \cos(t) \sin(1) - \frac{24}{5} (\cos(t))^2 \cos(1) - 3/5 \cos(1) + \frac{128}{5} (\cos(t))^8 \cos(1) \right. \\ \left. - \frac{256}{5} \cos(1) (\cos(t))^6 + 32 (\cos(t))^4 \cos(1) + \frac{64}{5} \sin(t) (\cos(t))^4 \sin(1) \right. \\ \left. - \frac{48}{5} \sin(1) \sin(t) (\cos(t))^2 + 4/5 \sin(1) \sin(t), \frac{16}{5} \sin(t) (\cos(t))^2 \cos(1) - 4/5 \cos(1) \sin(t) - 3/5 \sin(1) \right)$$

where $c = 1$ and $t = 0$ to 2π .

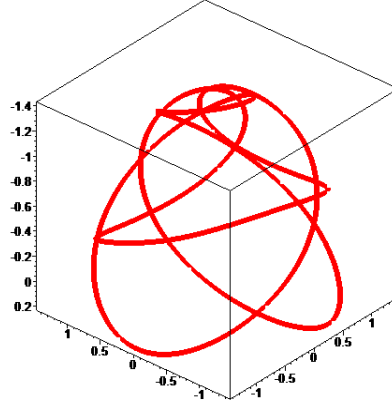


Figure 2.1. $C_1T_{C_1}$ -Smarandache curve (β_1 -curve)

$$\beta_1(t) = \left(-\frac{128}{5} \sin(t) (\cos(t))^7 + \frac{192}{5} \sin(t) (\cos(t))^5 - 16 \sin(t) (\cos(t))^3 + \frac{64}{5} (\cos(t))^5 - 16 (\cos(t))^3 + 4 \cos(t), \right. \\ \left. -\frac{24}{5} (\cos(t))^2 - \frac{3}{5} + \frac{128}{5} (\cos(t))^8 - \frac{256}{5} (\cos(t))^6 + 32 (\cos(t))^4 + \frac{64}{5} \sin(t) (\cos(t))^4 - \frac{48}{5} \sin(t) (\cos(t))^2 + \frac{4}{5} \sin(t), \right. \\ \left. \frac{16}{5} \sin(t) (\cos(t))^2 - \frac{4}{5} \sin(t) - \frac{3}{5} \right)$$

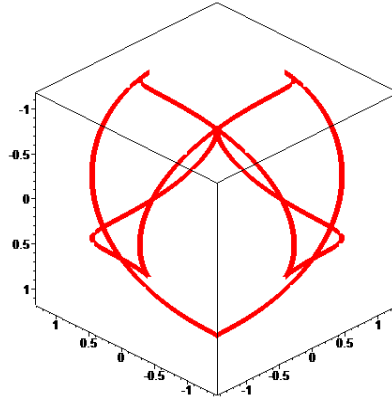


Figure 2.2. $C_1(C_1 \wedge T_{C_1})$ -Smarandache curve (β_2 -curve)

$$\beta_2(t) = \left(-8 (\cos(t))^2 + 1 + \frac{128}{5} (\cos(t))^8 - \frac{256}{5} (\cos(t))^6 + 32 (\cos(t))^4 - \frac{128}{5} \sin(t) (\cos(t))^7 + \frac{192}{5} \sin(t) (\cos(t))^5 - 16 \sin(t) (\cos(t))^3, \right. \\ \left. -\frac{16}{5} \sin(t) \cos(t) + \frac{128}{5} \sin(t) (\cos(t))^7 - \frac{192}{5} \sin(t) (\cos(t))^5 + 16 \sin(t) (\cos(t))^3 - \frac{24}{5} (\cos(t))^2 - \frac{3}{5} + \frac{128}{5} (\cos(t))^8 \right. \\ \left. -\frac{256}{5} (\cos(t))^6 + 32 (\cos(t))^4, -\frac{16}{5} (\cos(t))^3 + \frac{12}{5} \cos(t) + \frac{16}{5} \sin(t) (\cos(t))^2 - \frac{4}{5} \sin(t) \right)$$

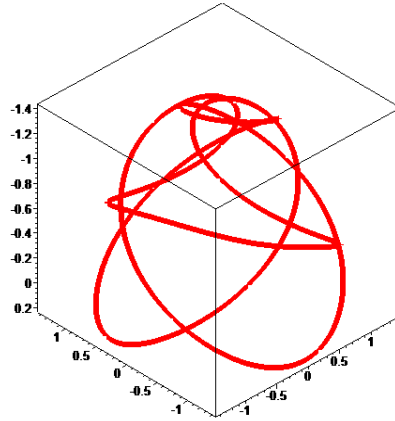


Figure 2.3. $T_{C_1}(C_1 \wedge T_{C_1})$ -Smarandache curve (β_3 -curve)

$$\begin{aligned} \beta_3(t) = & \left(-8(\cos(t))^2 + 1 + \frac{128}{5}(\cos(t))^8 - \frac{256}{5}(\cos(t))^6 + 32(\cos(t))^4 + \frac{64}{5}(\cos(t))^5 - 16(\cos(t))^3 + 4\cos(t), \right. \\ & -\frac{16}{5}\sin(t)\cos(t) + \frac{128}{5}\sin(t)(\cos(t))^7 - \frac{192}{5}\sin(t)(\cos(t))^5 + 16\sin(t)(\cos(t))^3 + \frac{64}{5}\sin(t)(\cos(t))^4 \\ & \left. -\frac{48}{5}\sin(t)(\cos(t))^2 + \frac{4}{5}\sin(t), -\frac{16}{5}(\cos(t))^3 + \frac{12}{5}\cos(t) - \frac{3}{5} \right) \end{aligned}$$

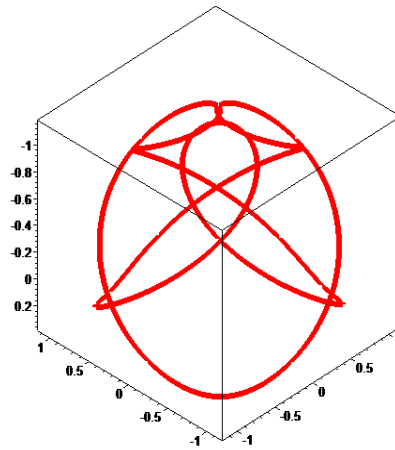


Figure 2.4. $C_1 T_{C_1}(C_1 \wedge T_{C_1})$ -Smarandache curve (β_4 -curve)

$$\begin{aligned} \beta_4(t) = & \left(-\frac{256}{15}\sin(t)(\cos(t))^7 + \frac{128}{5}\sin(t)(\cos(t))^5 - \frac{32}{3}\sin(t)(\cos(t))^3 + \frac{128}{15}(\cos(t))^5 - \frac{32}{3}(\cos(t))^3 + \frac{8}{3}\cos(t) - \frac{16}{3}(\cos(t))^2 + \frac{2}{3} \right. \\ & + \frac{256}{15}(\cos(t))^8 - \frac{512}{15}(\cos(t))^6 + \frac{64}{3}(\cos(t))^4, -\frac{16}{5}(\cos(t))^2 - \frac{2}{5} + \frac{256}{15}(\cos(t))^8 - \frac{512}{15}(\cos(t))^6 + \frac{64}{3}(\cos(t))^4 \\ & + \frac{128}{15}\sin(t)(\cos(t))^4 - \frac{32}{5}\sin(t)(\cos(t))^2 + \frac{8}{15}\sin(t) - \frac{32}{15}\sin(t)\cos(t) + \frac{256}{15}\sin(t)(\cos(t))^7 - \frac{128}{5}\sin(t)(\cos(t))^5 \\ & \left. + \frac{32}{3}\sin(t)(\cos(t))^3, \frac{32}{15}\sin(t)(\cos(t))^2 - \frac{8}{15}\sin(t) - \frac{2}{5} - \frac{32}{15}(\cos(t))^3 + \frac{8}{5}\cos(t) \right) \end{aligned}$$

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Analytic Families of Self-Adjoint Compact Operators Which Commute with Their Derivative

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Abstract

Spectral properties of analytic families of compact operators on a Hilbert space are studied. The results obtained are then used to establish that an analytic family of self-adjoint compact operators on a Hilbert space \mathcal{H} , which commute with their derivative, must be functionally commutative.

Keywords: Compact operator, Spectral decomposition, Analytic projection, Analytic eigenvalue, Functional commutativity, Analytic operator-valued function.

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1. Preliminaries

In [6], Stuart Goff studied analytic hermitian function matrices which commute with their derivative on some real interval I , i.e., $A(t)A'(t) = A'(t)A(t)$ for all $t \in I$. He obtained as a main result that these matrices are functionally commutative on I , i.e.,

$$A(s)A(t) = A(t)A(s)$$

for all $s, t \in I$ [[6], Theorem 3.6].

Subsequently, in [5], Jean-Claude Evard while studying the nonlinear differential equation

$$A(t) \frac{dA(t)}{dt} = \frac{dA(t)}{dt} A(t), t \in \Omega,$$

where Ω is an open interval in \mathbb{R} and A is a differentiable map from Ω into the \mathbb{C} -Banach space M_n of all $n \times n$ matrices $(\alpha_{i,j})$, with $\alpha_{i,j} \in \mathbb{C}$ for $i, j \in \{1, \dots, n\}$, was led to consider the more general problem where Ω is an open connected subset of a Banach space on \mathbb{R} or \mathbb{C} . In his paper Evard generalized Goff's theorem in ([5], Theorem 4.3) and summarizes the history and motivations behind the problem on matrix functions commuting with their derivative from 1950 to 1982. It also suggests further paths of investigations such as the one of interest to us, indeed our main result, Theorem 3, extends the final dimensional result of Goff [6] to the infinite-dimensional situation of compact self-adjoint operators on a Hilbert space.

In this paper, we study analytic families of compact self-adjoint operators, on a complex Hilbert space, which commute with their derivative on some real interval I . Our main result establishes that these operators must be functionally commutative on I , that is,

$$A(s)A(t) = A(t)A(s)$$

for all $s, t \in I$, extending the main result of [6] and [5] from the case of matrices to the infinite dimensional situation of operators on a Hilbert space. Indeed, Stuart Goff studied the case of Hermitian function matrices which commute with their derivative in [6] and Jean-Claude Evard extended Goff's result to matrix functions defined on an open subset of a normed space in [5].

We recall that $\mathcal{B}(\mathcal{H})$ denotes the Banach algebra of all bounded operators on the complex Hilbert space \mathcal{H} and by definition the *spectrum* of T , denoted by $\text{Sp}T$, is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible in $\mathcal{B}(\mathcal{H})$. In this paper $A(t)$ will denote an analytic family of compact self-adjoint operators on a complex Hilbert space and defined on a real interval $I = (a, b)$ with $a < 0 < b$. Let λ be an eigenvalue of $A(0)$. In fact, more generally $A(t)$ can be an analytic family of bounded self-adjoint operators on \mathcal{H} with the property that λ is an isolated point of the spectrum $\text{Sp}A(0)$, and such that the λ -eigenspace of $A(0)$ is finite-dimensional. Let D be a closed disk centered at λ such that $\text{Sp}A(0) \cap D = \{\lambda\}$. It follows that, for t sufficiently small, $\text{Sp}A(t) \cap \gamma = \emptyset$ where $\gamma = \partial D$ is the boundary of D . For such t , we have the orthogonal Riesz projections

$$P = \frac{1}{2\pi i} \int_{\gamma} (\xi I - A)^{-1} d\xi$$

with range $\mathcal{H}(t)$, depending analytically on t , such that $P(0)$ is the orthogonal projection of \mathcal{H} onto the λ -eigenspace of $A(0)$.

We have the following important result in the general setting of a Banach algebra. Let f be an analytic function from a domain D of \mathbb{C} into the algebra of compact operators $\mathcal{K}(X)$ on a Banach space X and let $\lambda_0 \in D$, $\alpha_0 \in \text{Sp}f(\lambda_0)$ with $\alpha_0 \neq 0$. Suppose α_0 is an eigenvalue of multiplicity one, or equivalently that the Riesz projection associated to the null space $\mathcal{N}(f(\lambda_0)1 - \alpha_0 I)$ has rank one. Then there exist $r, \delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies that $\text{Sp}(f(\lambda)) \cap B(\alpha_0, r)$ contains only one eigenvalue $\alpha(\lambda)$. What can be said about α ? In this particular case it is known that α is holomorphic on $B(\lambda_0, \delta)$. A proof of the next theorem is given in [1], pp 59-60.

Theorem 1.1 (Holomorphic Variation of Isolated Spectral Values). *Let f be an analytic function from a domain D of \mathbb{C} into a Banach algebra \mathcal{U} . Suppose there exists $\lambda_0 \in D$, $\alpha_0 \in \text{Sp}f(\lambda_0)$ and $r, \delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies that $\lambda \in D$ and that $\text{Sp}(f(\lambda)) \cap B(\alpha_0, r)$ contains only one point $\alpha(\lambda)$. Then α is holomorphic in a neighbourhood of λ_0 .*

A more general discussion about the behaviour of isolated parts of the spectrum is dealt with in [2], Chapter 10. Another source of interest on this topic is [8], Chapter XII, where a proof of the analyticity of discrete eigenvalues in the nondegenerate case for analytic families of operators is given. We recall that a point $\lambda \in \text{Sp}(A)$ is called discrete if λ is isolated and its associated Riesz projection P_λ is finite-dimensional; if P_λ is one-dimensional, we say that λ is a nondegenerate eigenvalue. These results are contained in the Kato-Rellich theorem ([8], Chapter III, p.15). If $A(z)$ is an operator depending analytically on a complex parameter z near $z = 0$ and σ_0 is a component of $\text{Sp}(A(0))$, then the Riesz projection $P(z)$ will still be defined for sufficiently small z , and will represent an idempotent depending analytically on z . Our main result will be based essentially on the possibility of decomposing an operator like $A(z)$ into a sum $\sum \mu_k P_k(z)$, where the $P_k(z)$ are mutually orthogonal analytic projections, i.e. $(P_k(z)P_l(z) = 0$ for $k \neq l)$, such that $A(z)P_k(z) = P_k(z)A(z)$. It is a spectral decomposition with the added condition of analyticity. For a more complete and comprehensive modern reference for spectral theory we refer the reader to [3], Chapter VIII.

2. Self-Adjoint Compact Operators on a Hilbert Space

It is well known that self-adjoint $n \times n$ matrices can be diagonalized, i.e. can be written as $\sum_{\alpha=1}^k \lambda_\alpha P_\alpha$ where the P_α are self-adjoint orthogonal projections and the λ_α are real numbers. This result can be extended to self-adjoint compact operators on a Hilbert space. A proof of the next theorem is given in [1], pp 25-26.

Theorem 2.1 (Spectral Theorem for Self-Adjoint Compact Operators on a Hilbert Space). *Let \mathcal{H} be a Hilbert space and let T be a self-adjoint compact operator on \mathcal{H} . Let $\{\lambda_k\}_{k \geq 1}$ be the discrete set of nonzero eigenvalues of T . Also let $E_0 = \mathcal{N}(T)$ and $E_k = \mathcal{N}(T - \lambda_k I)$, for $k \geq 1$. Then we have the following properties:*

- (i) *for $k \geq 0$ the closed subspaces E_k are orthogonal and their Hilbertian direct sum is \mathcal{H} . Moreover, if P_k denotes the self-adjoint projection on E_k we have $TP_k = P_k T$ for all k ,*
- (ii) *the series $\sum_{k \geq 1} \lambda_k P_k$ converges in norm in $\mathcal{B}(\mathcal{H})$ and we have*

$$T = \sum_{k \geq 1} \lambda_k P_k.$$

3. Analytic self-adjoint compact operators on a Hilbert space

The proof of the following lemma is suggested by the argument used in the proof of Theorem 3.5 of [5]. Here $z \in I$ where I is an interval in \mathbb{R} . As in [5], we say that $A(t)$ commutes with its derivative on a real interval I if it satisfies the commutation equation

$$A(t) \frac{dA(t)}{dt} = \frac{dA(t)}{dt} A(t), t \in I,$$

that we write simply as

$$A(t)A'(t) = A'(t)A(t), t \in I.$$

Lemma 3.1. *Let $A(t)$ be an analytic family of self-adjoint compact operators on a Hilbert space \mathcal{H} which commute with its derivative. Then the projections associated to the eigenvalues of $A(t)$ commute with their derivative.*

Proof. Let $t \in I$ such that $A'(t)A(t) = A(t)A'(t)$ and

$$\text{Sp}A(t) \subset \bigcup_{n=1}^{\infty} (\mathbb{C} - \Gamma_i),$$

where Γ_i is a simple contour which does not meet $\text{Sp}A(t)$. Then as in the proof of Theorem 3.5 of [5],

$$P_k(t) = \frac{1}{2\pi i} \int_{\Gamma_k} (z - A(t))^{-1} dz,$$

commutes with its derivative

$$P'_k(t) = \frac{1}{2\pi i} \int_{\Gamma_k} (z - A(t))^{-1} (-A'(t))(z - A(t))^{-1} dz,$$

because

$$A(t)A'(t) = A'(t)A(t)$$

and Γ_k is compact (so differentiation inside the integral sign is justified). □

Evard proved in [4] that if $P(t)$ commute with its derivative, its range $\text{ran}(P(t))$ is not only invariant under their derivative, but also constant. Indeed he proved in Theorem 6 of the same paper that the family $P(t)$ itself is constant.

Lemma 3.2. *If a family of projections $P(t)$ commutes with its derivative on an interval $I \in \mathbb{R}$, then $P(t)$ is constant.*

Proof. Since $P^2(t) = P(t)$, it follows by differentiation of the two sides that

$$P'(t)P(t) + P(t)P'(t) = P'(t).$$

Now by hypothesis

$$P'(t)P(t) = P(t)P'(t)$$

so we get

$$2P'(t)P(t) = P'(t),$$

which by multiplication by $P(t)$ yields

$$2P'(t)P(t) = P'(t)P(t).$$

Hence, $P'(t)P(t) = 0$. Going back to the relation

$$2P'(t)P(t) = P'(t),$$

we conclude that $P'(t) = 0$, which means $P(t)$ is constant. □

Using the previous results, we establish our main result in the next theorem.

Theorem 3.3. *Let $A(t)$ be an analytic family of compact self-adjoint operators on a Hilbert space H . Suppose that $A(t)$ commutes with its derivative for all $t \in I \subset \mathbb{R}$. Then $A(t)$ is functionally commutative, i.e. $A(s)A(t) = A(t)A(s)$ for all $s, t \in I$.*

Proof. By Theorem 2, any compact self-adjoint operator on a Hilbert space admits a spectral decomposition, so we can write,

$$A(t) = \sum_{k=1}^{\infty} \lambda_k(t) P_k(t)$$

where $\{\lambda_k\} \subset \text{Sp}A(t)$ and $P_k^2(t) = P_k(t)$. Moreover by Lemma 2, the projections $P_k(t)$ commute with their derivative, and by Lemma 3 they are constant. Hence,

$$A(t) = \sum_{i=1}^{\infty} \lambda_i(t) P_i$$

where $P_i^2 = P_i$ are constant projections. Consequently we get,

$$A(s) = \sum_{i=1}^{\infty} \mu_i(s) P_i$$

and

$$A(s)A(t) = A(t)A(s) \text{ for all } s, t \in I.$$

□

Note. If T is a compact operator, then its point spectrum is nonempty and countable, which may not hold for noncompact (normal) operators. But this is not the main role played by compact operators in the Spectral Theorem - we can deal with an uncountable weighted sum of projections. What is actually special with a compact operator is that a compact normal operator not only has a nonempty point spectrum but it has enough eigenspaces to span H . That makes the difference, since normal (noncompact) operators may have an empty point spectrum or it may have eigenspaces but not enough to span the whole space H . However, the Spectral Theorem survives the lack of compactness if the point spectrum is replaced with the whole spectrum (which is never empty). Such an approach for the general case of the Spectral Theorem (i.e. for normal, not necessarily compact operators) requires measure theory. This is a work that we intend to undertake in a future investigation.

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Reproducing Kernels for Hardy and Bergman Spaces of the Upper Half Plane

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Abstract

Using invertible isometries between Hardy and Bergman spaces of the unit disk \mathbb{D} and the corresponding spaces of the upper half plane \mathbb{U} , we determine explicitly the reproducing kernels for the Hardy and Bergman spaces of \mathbb{U} . As a consequence, we obtain the duality relations for the reflexive Hardy and Bergman spaces of the half plane \mathbb{U} .

Keywords: Bergman projection, Duality, Hardy and Bergman spaces, Reproducing kernel, Szegő projection.

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1. Introduction and Preliminary results

Let \mathbb{C} be the complex plane. The set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, is called the (open) unit disc. Let dA denote the area measure on \mathbb{D} , and for $\alpha \in \mathbb{R}$, $\alpha > -1$, we define a positive Borel measure dm_α on \mathbb{D} by $dm_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$. On the other hand, the set $\mathbb{U} = \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ denotes the upper half of the complex plane \mathbb{C} , and where $\Im(\omega)$ stands for the imaginary part of ω . Also, $\Re(\omega)$ shall denote the real part of the complex number ω . For $\alpha > -1$, we define a weighted measure on \mathbb{U} by $d\mu_\alpha(\omega) = (\Im(\omega))^\alpha dA(\omega)$. The Cayley transform $\psi(z) := \frac{i(1+z)}{1-z}$ maps the unit disc \mathbb{D} conformally onto the upper half-plane \mathbb{U} with inverse $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$ mapping \mathbb{U} conformally onto \mathbb{D} .

For an open subset Ω of \mathbb{C} , let $\mathcal{H}(\Omega)$ denote the Fréchet space of analytic functions $f : \Omega \rightarrow \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of Ω . Let $\text{Aut}(\Omega) \subset \mathcal{H}(\Omega)$ denote the group of biholomorphic maps $f : \Omega \rightarrow \Omega$. For $1 \leq p < \infty$, the Hardy spaces of the upper half plane, $H^p(\mathbb{U})$, are defined as

$$H^p(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{H^p(\mathbb{U})} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{1/p} < \infty \right\},$$

while the Hardy spaces of the unit disc, $H^p(\mathbb{D})$, by

$$H^p(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{H^p(\mathbb{D})}^p := \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty \right\}.$$

We note that every function $f \in H^p(\mathbb{U})$ (or $H^p(\mathbb{D})$) has non-tangential boundary values almost everywhere on $\partial\mathbb{U}$ (or $\partial\mathbb{D}$). In particular, H^p -functions may be identified with their boundary values and with this convention,

$$\|f\|_{H^p(\mathbb{U})} = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{\frac{1}{p}},$$

and respectively,

$$\|f\|_{H^p(\mathbb{D})} = \left(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

On the other hand, for $1 \leq p < \infty$, $\alpha > -1$, the weighted Bergman spaces of the upper half plane, $L_a^p(\mathbb{U}, \mu_\alpha)$, are defined by

$$L_a^p(\mathbb{U}, \mu_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{L_a^p(\mathbb{U}, \mu_\alpha)} = \left(\int_{\mathbb{U}} |f(z)|^p d\mu_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\},$$

while the corresponding spaces of the disc, $L_a^p(\mathbb{D}, m_\alpha)$, by

$$L_a^p(\mathbb{D}, m_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L_a^p(\mathbb{D}, m_\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p dm_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\}.$$

In particular, $L_a^p(\mathbb{U}, \mu_\alpha) = L^p(\mathbb{U}, \mu_\alpha) \cap \mathcal{H}(\mathbb{U})$ and $L_a^p(\mathbb{D}, m_\alpha) = L^p(\mathbb{D}, m_\alpha) \cap \mathcal{H}(\mathbb{D})$, where $L^p(\mathbb{U}, \mu_\alpha)$ or simply $L^p(\mu_\alpha)$ ($L^p(\mathbb{D}, m_\alpha)$ or simply $L^p(m_\alpha)$) denotes the classical Lebesgue spaces associated with the weighted measure μ_α , and respectively m_α . It is important to note that the case $\alpha = 0$ yields the (unweighted) Bergman spaces.

As noted in [1] in the case of the disc, the Hardy space $H^p(\mathbb{U})$ behaves in many ways as the limiting case of $L_a^p(\mathbb{U}, \mu_\alpha)$ as $\alpha \rightarrow -1^+$. Therefore, we shall let X denote either the Hardy space $H^p(\mathbb{U})$ or the weighted Bergman space $L_a^p(\mathbb{U}, \mu_\alpha)$, and we associate with each X , a parameter $\gamma = \frac{\alpha+2}{p}$, where $\alpha = -1$ in the case that $X = H^p(\mathbb{U})$. Also, we shall let $X(\mathbb{D})$ denote the corresponding spaces of analytic functions of the unit disc \mathbb{D} . Therefore, we formulate the growth conditions for Hardy and Bergman spaces simultaneously in the next results; while known, we provide much simpler proofs. But first we give the following result which gives the isometries between the spaces X and $X(\mathbb{D})$.

Proposition 1.1. *Let $f \in X$, and define $S_\psi f = (\psi')^\gamma f \circ \psi$. Then $S_\psi : X \rightarrow X(\mathbb{D})$ is continuous with inverse $S_{\psi^{-1}} g = ((\psi^{-1})')^\gamma g \circ \psi^{-1}$. In fact, if $X = H^p(\mathbb{U})$, then S_ψ is an isometry, and, in the case $X = L_a^p(\mathbb{U}, \mu_\alpha)$, $\|S_\psi f\|_{L_a^p(\mathbb{D}, m_\alpha)} = 2^{\alpha/p} \|f\|_{L_a^p(\mathbb{U}, \mu_\alpha)}$.*

Moreover, $S_{\psi^{-1}}$ is an isometry on $H^p(\mathbb{D})$, and if $X(\mathbb{D}) = L_a^p(\mathbb{D}, m_\alpha)$, then

$$\|S_{\psi^{-1}} g\|_{L_a^p(\mathbb{U}, \mu_\alpha)} = 2^{-\alpha/p} \|g\|_{L_a^p(\mathbb{D}, m_\alpha)}.$$

In particular, $S_\psi^{-1} = S_{\psi^{-1}}$ in the setting of Bergman spaces as well as Hardy spaces.

Proof. First, we suppose that $X = L_a^p(\mathbb{U}, \mu_\alpha)$. Let $f \in L_a^p(\mathbb{U}, \mu_\alpha)$, then change of variables yields

$$\begin{aligned} \|f\|_{L_a^p(\mu_\alpha)} &= \int_{\mathbb{U}} |f(\omega)|^p (\Im(\omega))^\alpha dA(\omega) \\ &= \int_{\mathbb{D}} |f(\psi(z))|^p (\Im(\psi(z)))^\alpha |\psi'(z)|^2 dA(z), \end{aligned}$$

and $\Im(\psi(z)) = \frac{(1-|z|^2)}{2} |\psi'(z)|$. Thus $\|f\|_{L_a^p(\mu_\alpha)}^p = 2^{-\alpha} \|S_\psi f\|_{L_a^p(m_\alpha)}^p$.

For the case $X = H^p(\mathbb{U})$, we may identify $f \in X$ with its boundary values. Then change of variables yields

$$\begin{aligned} \|f\|_{H^p(\mathbb{U})}^p &= \int_{\mathbb{R}} |f(x)|^p dx = \int_{\partial\mathbb{D}} |f(\psi(z))|^p |\psi'(z)| dm(z) \\ &= \int_{\partial\mathbb{D}} |(\psi'(z))^\gamma (f \circ \psi)(z)|^p dm(z), \end{aligned}$$

where $dm(e^{i\theta}) = d\theta$ denotes arc-length measure on $\partial\mathbb{D}$. Thus $\|S_\psi f\|_{H^p(\mathbb{D})} = \|f\|_{H^p(\mathbb{U})}$.

Similarly, if $g \in L_a^p(\mathbb{D}, m_\alpha)$, then again by change of variables, we obtain

$$\begin{aligned} \|g\|_{L_a^p(m_\alpha)}^p &= \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^\alpha dA(z) \\ &= \int_{\mathbb{U}} |g(\psi^{-1}(\omega))| (1-|\psi^{-1}(\omega)|^2)^\alpha |(\psi^{-1})'|^2 dA(\omega), \end{aligned}$$

where $(1 - |\psi^{-1}(\omega)|^2) = 2|(\psi^{-1})'(\omega)|\Im(\omega)$. Thus

$$\|g\|_{L_a^p(m_\alpha)}^p = 2^\alpha \int_{\mathbb{U}} |(\psi^{-1})'|^{\alpha+2} |g \circ \psi^{-1}|^p |\Im(\omega)|^\alpha dA(\omega) = 2^\alpha \|S_{\psi^{-1}}g\|_{L_a^p(\mu_\alpha)}^p.$$

If $g \in H^p(\mathbb{D})$, then

$$\begin{aligned} \|g\|_{H^p(\mathbb{D})}^p &= \int_{\partial\mathbb{D}} |g(z)|^p dm(z) = \int_{\mathbb{R}} |g(\psi^{-1}(x))|^p |(\psi^{-1})'| dx \\ &= \|S_{\psi^{-1}}g\|_{H^p(\mathbb{R})}^p. \end{aligned}$$

□

Lemma 1.2. *Let $X(\mathbb{D})$ denote either $H^p(\mathbb{D})$ or $L_a^p(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$. Let $\gamma = \frac{\alpha+2}{p}$ ($\alpha = -1$ in case $X(\mathbb{D}) = H^p(\mathbb{D})$). Then there exists a constant $C = C_{X(\mathbb{D})}$ such that for every $f \in X(\mathbb{D})$ and $z \in \mathbb{D}$,*

$$|f(z)| \leq \frac{C\|f\|_{X(\mathbb{D})}}{(1-|z|^2)^\gamma}. \quad (1.1)$$

Proof. We begin by showing that $|f(0)| \leq C\|f\|$. Let $f \in H^p(\mathbb{D})$. Then $\forall r, 0 < r < 1$, the mean value property implies that $f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$. Thus $|f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta$ and Jensen's inequality implies

$$|f(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \|f\|_{H^p(\mathbb{D})}^p.$$

Similarly, if $f \in L_a^p(m_\alpha)$, then $\forall r, 0 < r < 1$, $|f(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$. Thus

$$|f(0)|^p \int_0^1 (1-r^2)^\alpha 2r dr \leq \int_0^1 (1-r^2)^\alpha 2r dr \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta = \|f\|_{L_a^p(m_\alpha)}^p.$$

If $a \in \mathbb{D}$, let $\phi_a(z) = \frac{a-z}{1-\bar{a}z} \in \text{Aut}(\mathbb{D})$, where $\text{Aut}(\mathbb{D})$ denotes the group of automorphisms of \mathbb{D} . Then $S_{\phi_a}f := (\phi_a')^\gamma f \circ \phi_a$ is an isometry on $X(\mathbb{D})$.

Indeed, in the Hardy space case,

$$\|S_{\phi_a}f\|_{H^p(\mathbb{D})}^p = \int_0^{2\pi} |f(\phi_a(e^{i\theta}))|^p |\phi_a'(e^{i\theta})|^\gamma d\theta = \int_0^{2\pi} |f(e^{it})|^p dt.$$

In the Bergman space case, we note that, by the Schwarz-Pick Lemma [2, Lemma I.1.2], $(1-|z|^2)|\phi_a'(z)| = 1 - |\phi_a(z)|^2 \forall z \in \mathbb{D}$, and therefore a change of variables argument implies

$$\begin{aligned} \|S_{\phi_a}f\|_{L_a^p(m_\alpha)}^p &= \int_{\mathbb{D}} |f(\phi_a(z))|^p ((1-|z|^2)|\phi_a'(z)|)^\alpha |\phi_a'(z)| dA(z) \\ &= \int_{\mathbb{D}} |f(\omega)|^p (1-|\omega|^2)^\alpha dA(\omega) = \|f\|_{L_a^p(m_\alpha)}^p. \end{aligned}$$

Thus if $a \in \mathbb{D}$, $|\phi_a'(0)|^\gamma |f(a)| = |S_{\phi_a}f(0)| \leq C\|f\|$ or $|f(a)| \leq \frac{C\|f\|}{(1-|a|^2)^\gamma}$, as claimed. □

The following is an immediate consequence of the above Lemmas,

Corollary 1.3. *Let X denote either $H^p(\mathbb{U})$ or $L_a^p(\mu_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$. Let $\gamma = \frac{\alpha+2}{p}$ ($\alpha = -1$ in case $X = H^p(\mathbb{U})$). Then there exists a constant $C = C_X$ such that for every $f \in X$ and $z \in \mathbb{U}$,*

$$|f(z)| \leq \frac{C\|f\|_X}{|\Im(z)|^\gamma}. \quad (1.2)$$

Proof. Let $g = S_\psi f$. Then by Proposition 1.1 above, $\|g\|_{H^p} = \|f\|_{H^p}$ and $\|g\|_{L_a^p(m_\alpha)} = 2^{-\frac{\alpha}{p}} \|f\|_{L_a^p(\mu_\alpha)}$. Now, if $a = \psi^{-1}(z)$, then $(1-|a|^2)|\psi'(a)| = 2\Im(z)$ and

$$|\psi'(a)|^\gamma |f(z)| \leq |g(a)| \leq \frac{C\|f\|}{(1-|a|^2)^\gamma},$$

implying that $|f(z)| \leq \frac{C\|f\|}{(2\Im(z))^\gamma}$. □

As consequence of the growth conditions given by equations (1.1) and (1.2), both Hardy and Bergman spaces (X and $X(\mathbb{D})$) are Banach spaces. In fact, if $p = 2$, it turns out that these spaces are Hilbert spaces, and moreover, the Bergman space $L_a^p(\cdot)$ is a closed subspace of the classical Lebesgue space $L^p(\cdot)$. For a detailed theory of Hardy spaces, we refer to [2, 3, 4], while for Bergman spaces, see [4, 5, 6, 7, 8].

The reproducing kernels for Hardy and Bergman spaces of the unit disk \mathbb{D} are well known in literature. See for instance [3] for Hardy spaces, and [4, 8] for Bergman spaces. The corresponding reproducing kernels for the Hardy and Bergman spaces of the upper half plane \mathbb{U} is not well captured in literature. In general, the theory of analytic spaces of the upper half plane is much less complete compared to the unit disk setting. In this paper, we determine explicitly the reproducing kernels on \mathbb{U} for the two spaces. As a result, we also give the corresponding projections and consequently, establish some known duality properties of these spaces. These results are part of my Ph.D. dissertation [9].

2. Reproducing Kernels on the upper half plane

Let \mathcal{H} denote a Hilbert space of functions defined on an open set $\Omega \subset \mathbb{C}$. We call a reproducing kernel for \mathcal{H} , a complex function $K : \Omega \times \Omega \rightarrow \mathbb{C}$ such that, if we put $K_\omega(z) := K(z, \omega)$, then the following two properties hold:

1. for every $\omega \in \Omega$, the function K_ω belongs to \mathcal{H} , and
2. for all $f \in \mathcal{H}$ and $\omega \in \Omega$, we have

$$f(\omega) = \langle f, K_\omega \rangle.$$

It is clear that the above two properties imply that such a kernel K satisfies the identity $K(z, \omega) = \overline{K(\omega, z)}$ for all $z, \omega \in \Omega$. Indeed

$$\begin{aligned} K(z, \omega) &= K_\omega(z) = \langle K_\omega, K_z \rangle \\ &= \overline{\langle K_z, K_\omega \rangle} = \overline{K_z(\omega)} = \overline{K(\omega, z)}. \end{aligned}$$

The growth condition estimates (for X and $X(\mathbb{D})$) given by equations (1.1) and (1.2) and the Riesz representation theorem for \mathcal{H}^* , imply that $H^2(\cdot)$ and $L_a^2(\cdot)$ are reproducing kernel Hilbert spaces.

2.1 Bergman spaces of the upper half plane

For the Bergman spaces (weighted), the reproducing kernel is also called the Bergman kernel (weighted). In particular, for the weighted Bergman spaces, we denote the weighted Bergman kernels by $K_{\alpha, \mathbb{D}}$ and $K_{\alpha, \mathbb{U}}$ on \mathbb{D} and \mathbb{U} , respectively. If the setting \mathbb{D} or \mathbb{U} is understood, we simply write K_α .

For Bergman spaces of the unit disc, $L_a^p(\mathbb{D}, m_\alpha)$, $\alpha > -1$, the weighted Bergman kernel has been computed in [8] and is given by

$$K_{\alpha, \mathbb{D}}(z, \omega) = \frac{1}{(1 - z\bar{\omega})^{2+\alpha}}. \quad (2.1)$$

The function K_α on the unit disc \mathbb{D} has been exhaustively studied in literature and some of its properties, for example, boundedness have far reaching consequences in the theory of analytic functions in $L_a^p(\mathbb{D}, m_\alpha)$. For a comprehensive theory of Bergman kernels and hence projections on \mathbb{D} , see for instance [4] or [8], and references therein.

In this section, we shall present the corresponding theory of the Bergman kernels and projections on the upper half-plane \mathbb{U} . In the next result, we compute explicitly the weighted Bergman kernel on the upper half-plane \mathbb{U} , which we denote by $K_\alpha = K_{\alpha, \mathbb{U}}$, and is acting on the Hilbert space $L_a^2(\mathbb{U}, \mu_\alpha)$.

Theorem 2.1. *If $\alpha > -1$, then the weighted Bergman kernel of $L_a^2(\mathbb{U}, \mu_\alpha)$ is given by*

$$K_\alpha(z, \omega) = \frac{2^\alpha}{[-i(z - \bar{\omega})]^{2+\alpha}}. \quad (2.2)$$

Proof. Let $K_{\alpha, \mathbb{U}}$, $K_{\alpha, \mathbb{D}}$ be the weighted Bergman kernels of $L_a^2(\mathbb{U}, \mu_\alpha)$ and $L_a^2(\mathbb{D}, m_\alpha)$ respectively. Then $K_{\alpha, \mathbb{D}}$ is given by equation (2.1). We need to compute $K_{\alpha, \mathbb{U}}$. The Cayley transform $\psi(z) = \frac{i(1+z)}{1-z}$ maps \mathbb{U} conformally onto \mathbb{D} with inverse, $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$. It follows from Proposition 1.1 that $Tf(\xi) := 2^{-\frac{\alpha}{2}} (\psi'(\xi))^{1+\frac{\alpha}{2}} f(\psi(\xi))$ is an isometric surjective isomorphism

of $L_a^2(\mathbb{U}, \mu_\alpha)$ onto $L_a^2(\mathbb{D}, m_\alpha)$. Since $L_a^2(\cdot)$ is a Hilbert space, it then follows that T is unitary, that is, $T^* = T^{-1}$. For every $\xi \in \mathbb{D}$, and by writing $K_{\alpha, \mathbb{D}, \xi} = K_{\alpha, \mathbb{D}}(\cdot, \xi)$ simply as $K_{\mathbb{D}, \xi}$, we have using the definition of K_α ,

$$\begin{aligned} Tf(\xi) &= 2^{-\frac{\alpha}{2}} (\psi'(\xi))^{1+\frac{\alpha}{2}} f(\psi(\xi)) = 2^{-\frac{\alpha}{2}} \frac{(2i)^{1+\frac{\alpha}{2}}}{(1-\xi)^{2+\alpha}} f(\psi(\xi)), \\ &= \langle Tf, K_{\mathbb{D}, \xi} \rangle_{L_a^2(\mathbb{D}, m_\alpha)} = \langle f, T^{-1}K_{\mathbb{D}, \xi} \rangle_{L_a^2(\mathbb{U}, \mu_\alpha)}. \end{aligned}$$

We now compute $T^{-1}K_{\mathbb{D}, \xi}$. For $z \in \mathbb{U}$, we have

$$(T^{-1}K_{\mathbb{D}, \xi})(z) = 2^{\frac{\alpha}{2}} ((\psi^{-1}(z))')^{1+\frac{\alpha}{2}} K_{\mathbb{D}, \xi}(\psi^{-1}(z)).$$

But by equation (2.1),

$$K_{\mathbb{D}, \xi}(\psi^{-1}(z)) = \frac{1}{(1 - \frac{z-i}{z+i}\bar{\xi})^{2+\alpha}} = \frac{(z+i)^{2+\alpha}}{(1-\bar{\xi})^{2+\alpha}[z - \bar{\psi}(\xi)]}.$$

Therefore,

$$\begin{aligned} 2^{-\frac{\alpha}{2}} (\psi'(\xi))^{1+\frac{\alpha}{2}} f(\psi(\xi)) &= Tf(\xi) = \langle f, T^{-1}K_{\mathbb{D}, \xi} \rangle_{L_a^2(\mathbb{U}, \mu_\alpha)} \\ &= 2^{\frac{\alpha}{2}} \frac{(-2i)^{1+\frac{\alpha}{2}}}{(1-\bar{\xi})^{2+\alpha}} \left\langle f, \frac{1}{(z - \bar{\psi}(\xi))^{2+\alpha}} \right\rangle, \end{aligned}$$

which implies that $f(\psi(\xi)) = 2^\alpha \left\langle f, \frac{1}{[-i(z - \bar{\psi}(\xi))]^{2+\alpha}} \right\rangle$, and thus,

$$f(\omega) = \left\langle f, \frac{2^\alpha}{[-i(z - \bar{\omega})]^{2+\alpha}} \right\rangle, \quad \omega \in \mathbb{U}.$$

In particular, if we write $\psi(\xi) = \omega$, then

$$K_{\mathbb{U}, \omega}(z) = \frac{2^\alpha}{[-i(z - \bar{\omega})]^{2+\alpha}}, \quad \text{as desired.}$$

□

It is important to note that a similar formula has been obtained through the use of Paley-Weiner theorem which in itself involves Fourier transform, see [5]. We consider the method applied in this paper to be more direct and simple. Since $L_a^2(\mathbb{U}, \mu_\alpha)$ is a closed subspace of the Hilbert space $L^2(\mathbb{U}, \mu_\alpha)$, there exists an orthogonal projection $P_\alpha : L^2(\mathbb{U}, \mu_\alpha) \rightarrow L_a^2(\mathbb{U}, \mu_\alpha)$ which we shall call the weighted Bergman projection on $L^2(\mathbb{U}, \mu_\alpha)$.

Proposition 2.2. *The weighted Bergman projection P_α from $L^2(\mathbb{U}, \mu_\alpha)$ onto the subspace $L_a^2(\mathbb{U}, \mu_\alpha)$ is given explicitly by*

$$P_\alpha f(z) = \int_{\mathbb{U}} K_\alpha(z, \omega) f(\omega) d\mu_\alpha(\omega), \quad (2.3)$$

where K_α is the weighted Bergman kernel on the half plane given by equation (2.2).

Proof. Indeed, by the reproducing property of $K_\alpha(z, \omega)$ and the self - adjointness of P_α on $L^2(\mathbb{U}, \mu_\alpha)$, we have

$$\begin{aligned} P_\alpha f(z) &= \langle P_\alpha f, K_\alpha(\cdot, z) \rangle_{L_a^2(\mu_\alpha)} = \langle f, P_\alpha K_\alpha(\cdot, z) \rangle_{L_a^2(\mu_\alpha)} \\ &= \langle f, K_\alpha(\cdot, z) \rangle_{L_a^2(\mu_\alpha)} = \int_{\mathbb{U}} K_\alpha(z, \omega) f(\omega) d\mu_\alpha(\omega), \quad \text{as claimed.} \end{aligned}$$

□

At this point, it is natural to ask whether the Bergman projection P_α extends in some meaningful way to $L_a^p(\mathbb{U}, \mu_\alpha)$ for the case $p \neq 2$, and in that case, whether the reproducing property of $K_\alpha(z, \omega)$, (that is, $P_\alpha F = F$) holds in $L_a^p(\mathbb{U}, \mu_\alpha)$. These questions were posed in [5]. In this section, we address these questions but first we prove some elementary results that will be useful in the sequel.

Proposition 2.3. Let $X = H^p(\mathbb{U})$ or $L_a^p(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$. Let $a > 0$, $b \in \mathbb{R}$ and define $Tf(z) = f(az + b)$ for every $f \in X$, then $\|T\| \leq a^{-\gamma}$.

Proof. If $f \in L_a^p(\mathbb{U}, \mu_\alpha)$, then

$$\begin{aligned} \|Tf\|_p^p &= \int_{\mathbb{U}} |f(az + b)|^p \frac{(\Im(az + b))^\alpha}{a^{\alpha+2}} |a|^2 dA(z) \\ &= \int_{a\mathbb{U}} |f(\omega)|^p (\Im(\omega))^\alpha dA(\omega) a^{-(\alpha+2)}, \end{aligned}$$

and if $f \in H^p(\mathbb{U})$, then

$$\begin{aligned} \|Tf\|_p^p &= \sup_{y>0} \int_{-\infty}^{\infty} |f(ax + iay + b)|^p dx = \sup_{t>0} \int_{-\infty}^{\infty} |f(ax + b + it)|^p dx \\ &= \frac{1}{a} \sup_{t>0} \int_{-\infty}^{\infty} |f(s + it)|^p ds = \frac{1}{a} \|f\|_p^p. \end{aligned}$$

□

The next two Lemmas give examples of analytic functions and the conditions they must satisfy to belong to the spaces X and $X(\mathbb{D})$.

Lemma 2.4. Let $X(\mathbb{D})$ denote one of the spaces $H^p(\mathbb{D})$ or $L_a^p(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$ ($\alpha = -1$ if $X(\mathbb{D}) = H^p(\mathbb{D})$), and let $\gamma = (\alpha + 2)/p$. Then for $\eta \in \mathbb{C}$,

$$(e^{i\theta} - z)^\eta \in X(\mathbb{D}) \text{ if and only if } \Re \eta > -\gamma.$$

Proof. We first consider the Bergman space case, that is $X(\mathbb{D}) = L_a^p(\mathbb{D}, m_\alpha)$.

Recall $(e^{i\theta} - z)^\eta \in L_a^p(\mathbb{D}, m_\alpha) \Leftrightarrow \int_{\mathbb{D}} |(e^{i\theta} - z)^\eta|^p dm_\alpha(z) < \infty$. Now,

$$\begin{aligned} \int_{\mathbb{D}} |(e^{i\theta} - z)^\eta|^p dm_\alpha(z) &= \int_{\mathbb{D}} |(e^{i\theta} - z)^\eta|^p (1 - |z|^2)^\alpha dA(z) \\ &= \int_{\mathbb{D}} |1 - ze^{-i\theta}|^{p\Re(\eta)} (1 - |z|^2)^\alpha dA(z) \\ &= \int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha}{|1 - ze^{-i\theta}|^{-p\Re(\eta)}} dA(z). \end{aligned} \tag{2.4}$$

It then follows immediately from [8, Lemma 3.10] that equation (2.4) is bounded if and only if $-p\Re(\eta) - \alpha - 2 < 0$, that is, $\Re(\eta) > -\frac{\alpha+2}{p}$, as desired.

For $X(\mathbb{D}) = H^p(\mathbb{D})$, we use the fact that functions in $H^p(\mathbb{D})$ can be identified with their boundary values. Fix $\theta \in \mathbb{R}$ and let $f(z) = (e^{i\theta} - z)^\eta$. Then f has boundary values $f(e^{it}) = (e^{i\theta} - e^{it})^\eta$ and $f \in H^p(\mathbb{D})$ is equivalent to

$$\begin{aligned} \int_{-\pi}^{\pi} |f(e^{it})|^p dt &= \int_{\theta-\pi}^{\theta+\pi} \left| (1 - e^{i(t-\theta)})^\eta \right|^p dt \\ &= \int_{-\pi}^{\pi} 2^{p\Re(\eta)} |\sin(t/2)|^{p\Re(\eta)} dt < \infty. \end{aligned} \tag{2.5}$$

But the equation (2.5) holds if and only if $p\Re(\eta) > -1$, as claimed. □

Lemma 2.5. Let X denote one of the spaces $H^p(\mathbb{U})$ or $L_a^p(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$ ($\alpha = -1$ if $X = H^p(\mathbb{U})$), and let $\gamma = (\alpha + 2)/p$. If $c \in \mathbb{R}$ and $\lambda, \nu \in \mathbb{C}$, then

1. $f(\omega) = (\omega - c)^\lambda (\omega + i)^\nu \in X$ if and only if $\Re(\lambda + \nu) < -\gamma < \Re(\lambda)$. In particular, $(\omega - c)^\lambda \notin X$ for any $\lambda \in \mathbb{C}$, and $(\omega + i)^\nu \in X$ if and only if $\Re \nu < -\gamma$.
2. $f(\omega) = e^{i\omega} / \omega^c \in X$ if and only if $1/p < c < \gamma$. In particular, $e^{i\omega} / \omega^c \notin H^p(\mathbb{U})$ for any $c \in \mathbb{R}$.

Proof. See [10, Lemma 3.2]. □

We now give the following proposition,

Proposition 2.6. *Let $X = H^p(\mathbb{U})$ or $L_a^p(\mathbb{U}, \mu_\alpha)$. If $v > 0$, then*

$$(\omega + iv)^\nu \in X \text{ if and only if } \nu > -\gamma.$$

Proof. If $f(\omega) = (\omega + iv)^\nu$, then

$$f(\omega) = v^\nu \left(\frac{1}{v} \omega + i \right)^\nu = v^\nu Tg(\omega),$$

where $Th(z) = h(\frac{1}{v}z)$ and $g(\omega) = (\omega + i)^\nu$. Now Proposition 2.3 and Lemma 2.5 immediately yield the desired result. \square

To begin addressing the questions mentioned earlier in this section concerning the extension of the Bergman kernel K_α to the cases $p \neq 2$, we give the following direct consequence of the above proposition.

Corollary 2.7. *For fixed $\omega \in \mathbb{U}$, the Bergman projection P_α belongs to $L_a^q(\mathbb{U}, \mu_\alpha)$ if and only if $1 < q \leq \infty$.*

Proof. If $K_\alpha(z, \omega) = \frac{2^\alpha}{(-i(z-\bar{\omega}))^{\alpha+2}}$, ($z, \omega \in \mathbb{U}$), then for fixed $\omega \in \mathbb{U}$,

$$K_\alpha(z, \omega) = 2^\alpha i^{\alpha+2} ((z - \Re(\omega)) + i\Im(\omega))^{-(\alpha+2)}.$$

Therefore by Proposition 2.6, $K_\alpha(\cdot, \omega) \in L_a^q(\mathbb{U}, \mu_\alpha)$ if and only if $-(\alpha+2) < -(\alpha+2)/q$, which is equivalent to $q > 1$. Moreover, if $z = x + iy$, $y > 0$, we have

$$|K_\alpha(z, \omega)| \leq \frac{2^\alpha}{(\Im(\omega))^{\alpha+2}},$$

implying that $K_\alpha(\cdot, \omega) \in L_a^\infty(\mathbb{U}, \mu_\alpha)$. \square

We can now prove the following result;

Proposition 2.8. *Let $1 \leq p < \infty$, then for each $f \in L_a^p(\mathbb{U}, \mu_\alpha)$,*

$$f(z) = \int_{\mathbb{U}} f(\omega) K_\alpha(z, \omega) d\mu_\alpha(\omega).$$

Proof. If $f \in L_a^p(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$, then by Corollary 2.7, $K_\alpha(\cdot, z) \in L_a^q(\mu_\alpha)$, $\frac{1}{p} + \frac{1}{q} = 1$ and so Hölder's inequality implies that

$$f \mapsto \int_{\mathbb{U}} f(\omega) K_\alpha(z, \omega) d\mu_\alpha(\omega) = \langle f, K_\alpha(\cdot, z) \rangle$$

is continuous; moreover, by the reproducing property,

$$f(z) = \int_{\mathbb{U}} f(\omega) K_\alpha(z, \omega) d\mu_\alpha(\omega) \text{ for all } f \in L_a^p(\mu_\alpha) \cap L_a^2(\mu_\alpha).$$

Since $L_a^p(\mu_\alpha) \cap L_a^2(\mu_\alpha)$ is dense in $L_a^p(\mu_\alpha)$, we're done. \square

The following theorem characterizes when P_α is a bounded projection from $L^p(\mathbb{U}, \mu_\alpha)$ onto $L_a^p(\mathbb{U}, \mu_\alpha)$, see D. Békollé, et.al. [5] for the details.

Theorem 2.9. *The Bergman projection*

$$P_\alpha f(z) := \int_{\mathbb{U}} f(\omega) K_\alpha(z, \omega) d\mu_\alpha(\omega), \quad \alpha > -1,$$

is a bounded projection from $L^p(\mathbb{U}, \mu_\alpha)$ onto $L_a^p(\mathbb{U}, \mu_\alpha)$ if and only if $1 < p < \infty$.

An immediate consequence of the boundedness of the Bergman projection P_α on $L_a^p(\mathbb{U}, \mu_\alpha)$ is the duality of Bergman spaces $L_a^p(\mathbb{U}, \mu_\alpha)$ which we give in the following result,

Corollary 2.10. Let $1 < p < \infty$ and q be conjugate to p in the sense that $\frac{1}{p} + \frac{1}{q} = 1$. Let $(L_a^p(\mathbb{U}, \mu_\alpha))^*$ be the dual space of $L_a^p(\mathbb{U}, \mu_\alpha)$, then

$$(L_a^p(\mathbb{U}, \mu_\alpha))^* \approx L_a^q(\mathbb{U}, \mu_\alpha), \quad \alpha > -1, \quad (2.6)$$

under the sesquilinear pairing

$$\langle f, g \rangle = \int_{\mathbb{U}} f(\omega) \overline{g(\omega)} d\mu_\alpha \quad (f \in L_a^p(\mu_\alpha), g \in L_a^q(\mu_\alpha)). \quad (2.7)$$

Proof. The classical duality between L^p -spaces gives

$$(L^p(\mathbb{U}, \mu_\alpha))^* \approx L^q(\mathbb{U}, \mu_\alpha).$$

By Hahn-Banach extension theorem and the boundedness of the Bergman projection P_α for $1 < p < \infty$, (Theorem 2.9), we have

$$(L_a^p(\mathbb{U}, \mu_\alpha))^* = P_\alpha(L^p(\mathbb{U}, \mu_\alpha))^* \approx P_\alpha L^q(\mathbb{U}, \mu_\alpha) = L_a^q(\mathbb{U}, \mu_\alpha), \quad \text{as desired.}$$

□

It is important to take note that under the above duality pairing, see equation (2.7), the adjoint operator is conjugate linear. Moreover, $L_a^p(\mu_\alpha)$ spaces for $1 < p < \infty$ are reflexive and thus;

$$(L_a^q(\mu_\alpha))^* \approx (L_a^p(\mu_\alpha))^{**} \approx L_a^p(\mu_\alpha).$$

2.2 Hardy spaces of the upper half plane

The reproducing kernel for Hardy spaces is also called the Cauchy - Szegő kernel or simply the Szegő kernel, with the corresponding projection called the Cauchy - Szegő projection or simply the Szegő projection. We refer to [6] or [11, Chapter 8] for a good account of the theory of the Szegő kernel and projection on Hardy spaces. Recall that functions in $H^p(\mathbb{D})$ have boundary values almost everywhere in $L_a^p(\partial\mathbb{D})$ and that if $1 \leq p < \infty$, then $H^p(\mathbb{D}) = \text{cl}_{L^p(\partial\mathbb{D})} \mathbb{C}[z]$, where $\mathbb{C}[z]$ denotes analytic polynomials in z , and $\text{cl}_{L^p(\partial\mathbb{D})}$ is the $L^p(\partial\mathbb{D})$ -closure. In fact the Hilbert space $H^2(\mathbb{D})$ has orthonormal basis $(z^n)_{n \geq 0}$.

As noted in [6, Chapter 2], Cauchy's theorem implies that the reproducing kernel for $H^1(\mathbb{D})$ is given by

$$S_{\mathbb{D}}(z, \omega) = \frac{1}{1 - z\bar{\omega}} \quad (z \in \partial\mathbb{D}, \omega \in \mathbb{D}). \quad (2.8)$$

Therefore, the Cauchy-Szegő projection $P_{\mathbb{D}}$ is given by

$$\begin{aligned} P_{\mathbb{D}}\varphi(z) &= \langle \varphi, S(\omega, z) \rangle = \int_{\partial\mathbb{D}} \varphi(\omega) \overline{S_{\mathbb{D}}(\omega, z)} dm(\omega) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\varphi(e^{it})}{1 - e^{-it}z} dt, \end{aligned}$$

and satisfies $P_{\mathbb{D}}f = f$ for all $f \in H^1(\mathbb{D})$. The following theorem whose details can be found in [6, Chapter 2] characterizes the boundedness of the Szegő projection for the case when $p \neq 1$.

Theorem 2.11. If $1 < p < \infty$, then $P_{\mathbb{D}} : L^p(\partial\mathbb{D}) \rightarrow H^p(\mathbb{D})$ is bounded and surjective.

In the next theorem, we establish the corresponding Cauchy-Szegő kernel on the upper half plane.

Theorem 2.12. The Cauchy - Szegő kernel for $H^2(\mathbb{U})$ is given by

$$S_{\mathbb{U}}(z, \xi) = \frac{i}{z - \bar{\xi}}. \quad (2.9)$$

Proof. Let $T : H^p(\mathbb{U}) \rightarrow H^p(\mathbb{D})$ be given by $Tf(z) = (\psi'(z))^{1/p} f(\psi(z))$. Then by Proposition 1.1, T is surjective isometry. In particular, $T : H^2(\mathbb{U}) \rightarrow H^2(\mathbb{D})$ is unitary with $T^* = T^{-1}$, and $T^{-1}g(\omega) = ((\psi^{-1})'(\omega))^{1/2} g(\psi^{-1}(\omega))$, where ψ is the Cayley transform. We wish to compute the corresponding Szegő kernel on the upper half-plane, \mathbb{U} : Let $\xi \in \mathbb{U}$, $z = \psi^{-1}(\xi) \in \mathbb{D}$. Also, let $f \in H^2(\mathbb{U})$ and $g = Tf$. Then

$$\begin{aligned} g(z) &= \int_{\partial\mathbb{D}} g(\omega) S_{\mathbb{D}}(z, \omega) dm(\omega) = \langle g, S(\cdot, z) \rangle_{\partial\mathbb{D}} = \langle Tf, S_{\mathbb{D}, z} \rangle_{\partial\mathbb{D}} \\ &= \langle f, T^* S_{\mathbb{D}, z} \rangle_{\mathbb{R}}. \end{aligned}$$

But we have

$$g(z) = g(\psi^{-1}(\xi)) = Tf(\psi^{-1}(\xi)) = (\psi'(\psi^{-1}(\xi)))^{1/2} f(\xi) = \frac{\xi + i}{(2i)^{1/2}} f(\xi),$$

and

$$\begin{aligned} (T^*S_{\mathbb{D},z})(\omega) &= ((\psi^{-1})'(\omega))^{1/2} S_{\mathbb{D}}(\psi^{-1}(\omega), \psi^{-1}(\xi)) \\ &= \frac{(2i)^{1/2}}{(\omega + i)} \left(\frac{1}{1 - \frac{\omega-1}{\omega+i} \left(\frac{\xi+i}{\xi-i} \right)} \right) \\ &= \frac{(2i)^{1/2}(\bar{\xi} - i)}{-2i(\omega - \bar{\xi})}. \end{aligned}$$

Thus

$$\begin{aligned} f(\xi) &= \frac{(2i)^{1/2}}{(\xi + i)} \int_{\mathbb{R}} f(x) \overline{\left(\frac{(2i)^{1/2}(\bar{\xi} - i)}{2(-i)(x - \bar{\xi})} \right)} dx \\ &= \int_{\mathbb{R}} f(x) \overline{\left(\frac{(2i)^{1/2}(-2i)^{1/2}}{2(-i)(x - \bar{\xi})} \right)} dx. \end{aligned}$$

Therefore, for every $f \in H^2(\mathbb{U})$, $\xi \in \mathbb{U}$,

$$f(\xi) = \int_{\mathbb{R}} f(x) \overline{\left(\frac{i}{x - \bar{\xi}} \right)} dx.$$

Thus the Szegő kernel for $H^2(\mathbb{U})$ is given by

$$S_{\mathbb{U}}(z, \xi) = \frac{i}{z - \bar{\xi}}.$$

□

Corollary 2.13. *The Cauchy - Szegő projection P from $L^2(\mathbb{R})$ onto $H^2(\mathbb{U})$ is given explicitly by*

$$P\varphi(\xi) = \int_{\mathbb{R}} \varphi(x) S_{\mathbb{U}}(\xi, x) dx,$$

where $S_{\mathbb{U}}$ is the Cauchy - Szegő kernel given by equation (2.9).

Proof. Adopting the above notation, we have

$$\begin{aligned} P\varphi(\xi) &= \int_{\mathbb{R}} \varphi(x) \overline{S_{\mathbb{U}}(x, \xi)} dx \\ &= \int_{\mathbb{R}} \varphi(x) \left(\frac{-i}{x - \bar{\xi}} \right) dx \\ &= \int_{\mathbb{R}} \varphi(x) \frac{i}{\bar{\xi} - x} dx \\ &= \int_{\mathbb{R}} \varphi(x) S_{\mathbb{U}}(\xi, x) dx. \end{aligned}$$

□

Therefore, the upper half-plane analogue of Theorem 2.11 is the following,

Theorem 2.14. *If $f \in H^p(\mathbb{U})$, $1 \leq p < \infty$, then for every $\xi \in \mathbb{U}$,*

$$f(\xi) = \int_{\mathbb{R}} f(x) S_{\mathbb{U}}(\xi, x) dx,$$

and if $p > 1$, then the Szegő projection $P : L^p(\mathbb{R}) \rightarrow H^p(\mathbb{U})$ given by

$$P\varphi(\xi) = \int_{\mathbb{R}} \varphi(x) S_{\mathbb{U}}(\xi, x) dx$$

is bounded and surjective.

The boundedness of the Szegő projection P on $L^p(\mathbb{R})$ given by Theorem 2.14 immediately yields the following duality of Hardy spaces $H^p(\mathbb{U})$, for $1 < p < \infty$.

Corollary 2.15. *Let $1 < p < \infty$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $(H^p(\mathbb{U}))^*$ be the dual space of $H^p(\mathbb{U})$. Then*

$$(H^p(\mathbb{U}))^* \approx H^q(\mathbb{U}), \tag{2.10}$$

via the sesquilinear pairing

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx \quad (f \in H^p(\mathbb{U}), g \in H^q(\mathbb{U})). \tag{2.11}$$

Proof. It is well known that

$$(L^p(\mathbb{R}))^* \approx L^q(\mathbb{R}).$$

Now, the Hahn-Banach extension theorem together with the boundedness of the Szegő projection P for $1 < p < \infty$ will yield,

$$(H^p(\mathbb{U}))^* \approx (H^p(\mathbb{R}))^* = P(L^p(\mathbb{R}))^* \approx PL^q(\mathbb{R}) = H^q(\mathbb{R}) \approx H^q(\mathbb{U}).$$

□

Again, we take note that under the pairing in equation (2.11), the adjoint operator from $\mathcal{L}(X)$ to $\mathcal{L}(X^*)$ is also conjugate linear. Since the Hardy spaces $H^p(\mathbb{U})$, $1 < p < \infty$, are reflexive Banach spaces, it follows that

$$(H^q(\mathbb{U}))^* \approx (H^p(\mathbb{U}))^{**} \approx H^p(\mathbb{U}).$$

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Classifying the Metrics for Which Geodesic Flow on the Group $SO(n)$ is Algebraically Completely Integrable

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Abstract

The aim of this paper is to demonstrate the rich interaction between the Kowalewski-Painlevé analysis, the properties of algebraic completely integrable (a.c.i.) systems, the geometry of its Laurent series solutions, and the theory of Abelian varieties. We study the classification of metrics for which geodesic flow on the group $SO(n)$ is a.c.i. For $n = 3$, the geodesic flow on $SO(3)$ is always a.c.i., and can be regarded as the Euler rigid body motion. For $n = 4$, in the Adler-van Moerbeke's classification of metrics for which geodesic flow on $SO(4)$ is a.c.i., three cases come up; two are linearly equivalent to the Clebsch and Lyapunov-Steklov cases of rigid body motion in a perfect fluid, and there is a third new case namely the Kostant-Kirillov Hamiltonian flow on the dual of $so(4)$. Finally, as was shown by Haine, for $n \geq 5$ Manakov's metrics are the only left invariant diagonal metrics on $SO(n)$ for which the geodesic flow is a.c.i.

Keywords: Jacobians varieties, Prym varieties, Integrable systems, Topological structure of phase space, Methods of integration.

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1. Introduction

A dynamical system is algebraic completely integrable in the sense of Adler-van Moerbeke [3, 6] if it can be linearized on a complex algebraic torus (Abelian variety). The invariants (often called first integrals or constants) of the motion are polynomials and the phase space coordinates, or some algebraic functions of these, restricted to a complex invariant variety defined by putting these invariants equals to generic constants are meromorphic functions on an Abelian variety. These manifolds are described explicitly as being affine part of complex algebraic tori and the flows (run with complex time) generated by the constants of the motion can be solved by quadrature, that is to say their solutions can be expressed in terms of Abelian integrals.

Consider the group $SO(n)$ and its Lie algebra $so(n)$ paired with itself, via the customary inner product

$$\langle X, Y \rangle = -\frac{1}{2} \operatorname{tr}(X.Y), \quad X, Y \in so(n).$$

A left invariant metric on $SO(n)$ is defined by a non-singular symmetric linear map

$$\Lambda : so(n) \longrightarrow so(n), \quad X \longmapsto \Lambda.X,$$

and by the following inner product; given two vectors gX and gY in the tangent space $SO(n)$ at the point $g \in SO(n)$,

$$\langle gX, gY \rangle = \langle X, \Lambda^{-1}.Y \rangle.$$

The question of classifying the metrics for which geodesic flow on $SO(n)$ is algebraically completely integrable is difficult. For $SO(3)$, we know that the Euler rigid body motion [11] can be regarded as geodesic flow on $SO(3)$ and this problem is always algebraically completely integrable. For $SO(4)$, Adler and van Moerbeke [4] have shown that the geodesic flow on $SO(4)$ for the metric defined by the above quadratic form is algebraically completely integrable in the three cases described in the section 5 and these are the only ones that exist. In addition, these three cases come up: two are linearly equivalent to cases of rigid body motion in a perfect fluid studied there is a long time, respectively by Clebsch and Lyapunov-Steklov, and there is a third new case namely the Kostant-Kirillov Hamiltonian flow on the dual of $so(4)$. For $n \geq 5$, as was shown by Haine [14], Manakov's metrics are the only left invariant diagonal metrics on $SO(n)$ for which the geodesic flow is algebraically completely integrable.

The paper is organized as follows. In Section 2, we explain the notion of algebraically completely integrable systems. Section 3 deals with geodesic flow on $SO(3)$ and Euler rigid body motion. In this section, we show that the Euler rigid body motion is always algebraically completely integrable which can be regarded as geodesic flow on $SO(3)$, and the integration of the equations is done by means of elliptic Jacobi functions. In section 4, we briefly recall some results concerning the Clebsch and Lyapunov-Steklov cases of a solid in an ideal fluid. Section 5 deals with the Adler-van Moerbeke classification of algebraic integrable geodesic flow on the group $SO(4)$. This classification concerns Manakov geodesic flow on the group $SO(4)$ which is linearly equivalent to Clebsch rigid body motion in a perfect fluid, geodesic flow on the group $SO(4)$ which is linearly equivalent to Lyapunov-Steklov rigid body motion in a perfect fluid and geodesic flow on $SO(4)$ with a quartic invariant. Section 6 deals with the algebraic complete integrability of geodesic flow on $SO(n)$ for $n \geq 5$.

2. Algebraic complete integrability

The definition of the algebraic complete integrability of a Hamiltonian system varies according to the literature and is usually found (with some minor variants) in any modern text on integrable systems. The integrable systems that we will deal with here are complex integrable systems on an affine space \mathbb{C}^m , the algebra that we consider is just that of the polynomial functions and we focus on algebraic complete integrability in the sense of Adler-van Moerbeke. We will work with complexes instead of real ones. Concepts such as: Liouville integrability, involution, commutativity of vector fields and so on, can be defined as in the real case. On the other hand difficulties arise : we know that there are no compact holomorphic submanifolds in the complex space \mathbb{C}^m (maximum principle), therefore the complex tori that we can get in Arnold Liouville's theorem are not compact. The difficult problem of the compactification of invariant varieties therefore arises. In addition, the solutions of the system in question are not uniform (single-valued). First, we will recall some results, define and explain the concept of algebraic complete integrability of Hamiltonian systems in general (although for the problems studied in this paper, we will be concerned by the affine space \mathbb{C}^m).

Consider a Hamiltonian completely integrable system

$$X_H : \dot{z} = J \frac{\partial H}{\partial z} \equiv f(z), \quad z \in \mathbb{R}^m, \quad m = 2n + k, \quad J(z) \text{ polynomial in } z, \quad (2.1)$$

with $n + k$ functionally independent invariants H_1, \dots, H_{n+k} of which k invariants (Casimir functions) lead to zero vector fields

$$J \frac{\partial H_{n+j}}{\partial z}(z) = 0, \quad 1 \leq j \leq k,$$

the $n = (m - k)/2$ remaining ones are in involution (i.e., $\{H_i, H_j\} = 0$), which give rise to n commuting vector fields. According to the Arnold-Liouville theorem [7], if the invariant manifolds

$$\bigcap_{i=1}^{n+k} \{z \in \mathbb{R}^m : H_i(z) = c_i\},$$

are compact, then for most values of $c_i \in \mathbb{R}$, their connected components are diffeomorphic to real tori $\mathbb{R}^n / \text{Lattice}$ and the flows $g_t^{X_1}(x), \dots, g_t^{X_n}(x)$ defined by the vector fields X_{H_1}, \dots, X_{H_n} , are straight-line motions on these tori.

Consider now $z \in \mathbb{C}^m$ and $t \in \mathbb{C}$. Let $\Delta \subset \mathbb{C}^m$ be a non-empty Zariski open set. By the functional independence of the first integrals, the map (momentum mapping)

$$\varphi \equiv (H_1, \dots, H_{n+k}) : \mathbb{C}^m \longrightarrow \mathbb{C}^{n+k},$$

is a generic submersion (i.e., $dH_1(z), \dots, dH_{n+k}(z)$ are linearly independent) on Δ . Let

$$\Omega = \left\{ c = (c_i) \in \mathbb{C}^{n+k} : \exists z \in \varphi^{-1}(c) \text{ with } dH_1(z) \wedge \dots \wedge dH_{n+k}(z) = 0 \right\},$$

be the set of critical values of the map φ , i.e., $\Omega = \varphi(\mathbb{C}^m \setminus \Delta)$ and denote by $\overline{\Omega}$ the Zariski closure of Ω in \mathbb{C}^{n+k} .

Proposition 2.1. *The set defined by*

$$\Gamma = \left\{ z \in \mathbb{C}^m : \varphi(z) \in \mathbb{C}^{n+k} \setminus \overline{\Omega} \right\},$$

is everywhere dense in \mathbb{C}^m for the usual topology.

Proof. Indeed, it suffices to show that the set $\Gamma = \varphi^{-1}(\mathbb{C}^{n+k} \setminus \overline{\Omega})$, is a non-empty Zariski open set in \mathbb{C}^m . Since a polynomial mapping between affine algebraic sets is continuous for the Zariski topology, then the above set is indeed a Zariski open set in \mathbb{C}^m and it is nonempty. Suppose this one is empty, that is $\varphi(\mathbb{C}^m) \subset \overline{\Omega}$. Since the map φ is submersive on a non-empty open set of Zariski $\Delta \subset \mathbb{C}^m$, then $\varphi(\Delta)$ is open in \mathbb{C}^{n+k} . By Sard's theorem for varieties [33], $\mathbb{C}^{n+k} \setminus \overline{\Omega}$ is a non-empty Zariski open set and therefore everywhere dense for the usual topology in \mathbb{C}^{n+k} . So $\varphi(\Delta) \cap (\mathbb{C}^{n+k} \setminus \overline{\Omega}) \neq \emptyset$, which is absurd. This completes the proof. \square

Let M_c be the complex affine variety defined by

$$M_c \equiv \varphi^{-1}(c) = \bigcap_{i=1}^{n+k} \{z \in \mathbb{C}^m : H_i(z) = c_i\}, \quad (2.2)$$

for all $c \equiv (c_1, \dots, c_{n+k}) \in \mathbb{C}^{n+k} \setminus \overline{\Omega}$, the fiber M_c is smooth.

Definition 2.2. *The system (2.1) is algebraic complete integrable (a.c.i.) in the sense of Adler-van Moerbeke with Abelian functions z_i [3, 6, 26] when, for every $c \in \mathbb{C}^{n+k} \setminus \overline{\Omega}$, the fiber M_c (2.2) is the affine part of an Abelian variety (complex algebraic torus)*

$$\tilde{M}_c = T^n \simeq \mathbb{C}^n / L_c, \quad (L_c \text{ a lattice in } \mathbb{C}^n)$$

and moreover, the flows $g_{X_i}^t(z)$, $z \in M_c$, $t \in \mathbb{C}$, defined by the vector fields X_{H_i} , $1 \leq i \leq n$, are straight line on T^n , i.e.,

$$[g_{X_i}^t(z)]_j = f_j(p + t(k_1^i, \dots, k_n^i)),$$

where $f_j(t_1, \dots, t_n)$ are Abelian functions on T^n , $f_j(p) = z_j$, $1 \leq j \leq m$.

We will be concerned with a.c.i. systems that are irreducible i.e., when the generic Abelian variety is irreducible (that is, it does not contain a subtorus). The following remark is intended to present interrelated definitions, all involving a.c.i. systems. For comments on definitions, see [38].

Remark 2.3. 1) [34] *Let H be a smooth function on a $2n$ -dimensional symplectic manifold (M, ω) . The Hamiltonian system defined by the vector field X_H is a.c.i., if there exists a smooth algebraic variety \mathcal{M} , a co-symplectic structure $\tilde{\omega}$ which restricts to ω along M , i.e., $\tilde{\omega} \in \Lambda^2 T_{\mathcal{M}}$ and a morphism $h : \mathcal{M} \rightarrow \mathcal{U}$ where \mathcal{U} is a Zariski open subset of \mathbb{C}^n , all defined over the real field such that:*

(i) *h is a proper submersive whose components are in involution; that is,*

$$\{X_i \circ h, X_j \circ h\} \equiv \tilde{\omega}(d(X_i \circ h), d(X_j \circ h)) = 0,$$

X_i being coordinates on \mathbb{R}^n .

(ii) *M is a component of $\mathcal{M}_{\mathbb{R}}$, the $\tilde{\omega}$ on M is the $\tilde{\omega}$ on \mathcal{M} along M , and H is a C^∞ -function of $X_j \circ h|_M$.*

2) [6] *Let $(M, \{.,.\}, \varphi)$ be a complex integrable system where M is a non-singular affine variety and $\varphi = (H_1, \dots, H_s)$ is given by regular algebraic functions H_i . We say that this system is an a.c.i. system if for generic $c \in \mathbb{C}^s$ the fiber of $\varphi^{-1}(c)$ is an affine part of an Abelian variety and if the Hamiltonian vector fields X_{H_i} , are translation invariant, when restricted to these fibers.*

3) [9] *Let $(M, \{.,.\})$ be a smooth Poisson variety. An a.c.i. Hamiltonian system consists of a proper flat morphism $H : M \rightarrow B$ where B is a smooth variety such that, over the complement $B \setminus \Delta$ of some proper closed subvariety $\Delta \subset B$, the morphism H is a Lagrangian fibration whose fibers are isomorphic to Abelian varieties.*

4) [18] Let M be a $2n$ -dimensional complex manifold with a holomorphic symplectic structure ω , a holomorphic function $H : M \rightarrow \mathbb{C}$ and n holomorphic functions which pairwise are in involution and also with H , i.e.,

$$\{H_i, H_j\} = \{H_i, H\} = 0, \quad 1 \leq i, j \leq n.$$

Let B be an open dense subset of \mathbb{C}^n and

$$F : F^{-1}(B) \subset M \rightarrow B,$$

a submersive map. The Hamiltonian system defined by the vector field X_H is an a.c.i. system if there exists a bundle $\pi : A \rightarrow B$ of Abelian varieties, a divisor $D \subset A$, an isomorphism $\sigma : F^{-1}(B) \rightarrow A \setminus D$, and a vector field Y on $A \setminus D$ which restricts to a linear vector field on the fibers of π , so that the diagram :

$$\begin{array}{ccc} F^{-1}(B) & \xrightarrow{\sigma} & A \setminus D \\ F \searrow & & \swarrow \pi \\ & & B \end{array}$$

is commutative and such that the vector field X_H is σ -related to Y .

5) Hitchin [15] gave a large class of interesting integrable systems that are almost by construction algebraic completely integrable and showed that the cotangent bundle of moduli spaces of stable vector bundles on Riemann surfaces carry the structure of integrable systems and are indeed a.c.i.

It will be interesting to insist a little more on the interpretation of the algebraic complete integrability of a Hamiltonian system, especially in the sense of Adler-van Moerbeke.

Remark 2.4. a) The complete algebraic integrability in the sense of Adler-van Moerbeke in the case where $M = \mathbb{C}^m$ means that :

(i) the system (2.1) with polynomial right hand possesses $n+k$ independent polynomial invariants $H_1 \equiv H, H_2, \dots, H_{n+k}$ of which k invariants lead to zero vector fields, the $n = (m-k)/2$ remaining ones are in involution, which give rise to n commuting vector fields. For generic c_i , the invariant manifolds

$$\bigcap_{i=1}^{n+k} \{z \in \mathbb{R}^m : H_i = c_i\},$$

are assumed compact, connected and therefore real tori by the Arnold-Liouville theorem.

(ii) the invariant manifolds, thought of as affine varieties in \mathbb{C}^m (non-compact), can be completed into complex algebraic tori, i.e.,

$$\bigcap_{i=1}^{n+k} \{z \in \mathbb{C}^m : H_i(z) = c_i\} = T^n \setminus \left\{ \mathcal{D} \equiv \left(\begin{array}{l} \text{one or several codimension} \\ \text{one subvarieties} \end{array} \right) \right\},$$

where the tori $T^n = \mathbb{C}^n / \text{Lattice} = \text{complex algebraic torus (Abelian variety)}$, depend on the c 's. In the natural coordinates (t_1, \dots, t_n) of these tori, the Hamiltonian flows (run with complex time) defined by the vector fields generated by the constants of the motion are straight-line motions and the coordinates $z_i = z_i(t_1, \dots, t_n)$ are meromorphic in (t_1, \dots, t_n) .

b) It must be realized that the existence of polynomial first integrals (invariants) for a Hamiltonian system does not necessarily imply the complete integrability of this system. For example, the Hamiltonian system where

$$H(x, y) = \frac{x^2}{2} + P(y), \quad (P(y) \text{ being a polynomial in } y),$$

will be algebraically completely integrable with Abelian (here elliptic) functions if and only if $P(y)$ is a polynomial of degree 3 or 4. Following Mumford [34], the commuting vector fields X_{H_1}, \dots, X_{H_n} define on the real torus $M_c \subset \mathbb{R}^{2n}$ defined by the intersection of the constants of the motion $H_1 = c_1, \dots, H_n = c_n$, an addition law

$$\oplus : M_c \times M_c \rightarrow M_c, \quad (x, y) \mapsto x \oplus y = g_{t+s}(p), \quad p \in M_c,$$

with $x = g_t(p), y = g_s(p), g_t(p) = g_{t_1}^{X_1} \dots g_{t_n}^{X_n}(p)$, where $g_{t_i}^{X_i}(p)$ denote the flows generated by X_{H_i} . Algebraic complete integrability means that this addition law is rational, that is:

$$(x \oplus y)_j = R_j(x_i, y_i, c),$$

where $R_j(x_i, y_i, c)$ is a rational function of all coordinates $x_i, y_i, 1 \leq i \leq n$. Putting $x = p, y = g_i^{X_i}(p)$, in this formula, we notice that on the real torus T_c , the flows $g_i^{X_i}(p)$ depend rationally on the initial condition p . Moreover, a Weierstrass theorem on functions admitting an addition theorem states that the coordinates x_i are restricted to the real torus:

$$\mathbb{R}^n / \text{Lattice} \longrightarrow M_c, \quad (t_1, \dots, t_n) \longmapsto x_i(t_1, \dots, t_n),$$

are Abelian functions. Geometrically, this means that the real torus $M_c \simeq \mathbb{R}^n / \text{Lattice}$ is the affine part of an algebraic complex torus (Abelian variety) $\mathbb{C}^n / \text{Lattice}$ and that the real functions $x_i(t_1, \dots, t_n), (t_i \in \mathbb{R})$, are the restrictions on this real torus of the meromorphic functions $x_i(t_1, \dots, t_n), (t_i \in \mathbb{C})$, of n complex variables, with $2n$ periods (n periods + n imaginary periods). In degenerate situations, some of these periods may be infinite, as for example in the case of a harmonic oscillator.

c) If the Hamiltonian flow (2.1) is a.c.i., it means that the variables z_i are meromorphic on the torus T^n and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathcal{D} \subset T^n$. By the a.c.i. definition, the flow (2.1) is a straight line motion in T^n and thus it must hit the divisor \mathcal{D} in at least one place. Moreover through every point of \mathcal{D} , there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equations must admit Laurent expansions which depend on the $n - 1$ parameters defining \mathcal{D} and the $n + k$ constants c_i defining the torus T^n , the total count is therefore $m - 1 = \dim(\text{phase space}) - 1$ parameters. The fact that a.c.i. systems possess $(m - 1)$ -dimensional families of Laurent solutions, was implicitly used by Kowalewski [21] in her classification of integrable rigid body motions (see also the Painlevé analysis for ordinary differential equations [35]). Such a necessary condition (see [5]) for a.c.i. can be formulated as in the theorem below.

Theorem 2.5. *If the Hamiltonian system (2.1) (with invariant tori not containing elliptic curves) is a.c.i., then each z_i blows up after a finite (complex) time, and for every z_i , there is a family of solutions*

$$z_i = \sum_{j=0}^{\infty} z_i^{(j)} t^{j-s_i}, \quad s_i \in \mathbb{Z}, \quad \text{some } s_i > 0, \quad (2.3)$$

depending on $\dim(\text{phase space}) - 1 = m - 1$, free parameters. Moreover, the system (2.1) possesses families of Laurent solutions depending on $m - 2, m - 3, \dots, m - n$ free parameters. The coefficients of each one of these solutions are rational functions on affine algebraic varieties of dimensions $m - 1, m - 2, m - 3, \dots, m - n$.

The question raised several years ago is whether this criterion is also sufficient. The main problem will be to complete the affine variety M_c (2.2) into an Abelian variety. A naive guess would be to take the natural compactification \overline{M}_c of M_c by projectivizing the equations. Indeed, this can never work for a general reason: an Abelian variety \tilde{M}_c of dimension bigger or equal than two is never a complete intersection, that is it can never be described in some projective space \mathbb{P}^n by $n - \dim \tilde{M}_c$ global polynomial homogeneous equations. In other words, if M_c is to be the affine part of an Abelian variety, \overline{M}_c must have a singularity somewhere along the locus at infinity. The trajectories of the vector fields (2.1) hit every point of the singular locus at infinity and ignore the smooth locus at infinity. In fact, the existence of meromorphic solutions to the differential equations (2.1) depending on some free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor. More precisely, around the points of hitting, the system of differential equations (2.1) admit a Laurent expansion solution depending on $m - 1$, free parameters and in order to regularize the flow at infinity, we use these parameters to blowing up the variety \overline{M}_c along the singular locus at infinity. The new complex variety obtained in this fashion is compact, smooth and has commuting vector fields on it; it is therefore an Abelian variety.

The system (2.1) with $k + n$ polynomial invariants has a coherent tree of Laurent solutions, when it has families of Laurent solutions in t , depending on $n - 1, n - 2, \dots, m - n$ free parameters. Adler and van Moerbeke [5] have shown that if the system possesses several families of $(n - 1)$ -dimensional Laurent solutions (principal Painlevé solutions) they must fit together in a coherent way and as we mentioned above, the system must possess $(n - 2)$ -, $(n - 3)$ -, ... dimensional Laurent solutions (lower Painlevé solutions), which are the gluing agents of the $(n - 1)$ -dimensional family. The gluing occurs via a rational change of coordinates in which the lower parameter solutions are seen to be genuine limits of the higher parameter solutions an which in turn appears due to a remarkable propriety of a.c.i. systems; they can be put into quadratic form both in the original variables and in their ratios (to see further). As a whole, the full set of Painlevé solutions glue together to form a fiber bundle with singular base. A partial converse to the above theorem can be formulated as follows [5]:

Theorem 2.6. *If the Hamiltonian system (2.1) satisfies the condition a)(i) in the remark 2.2 of a.c.i. and if it possesses a coherent tree of Laurent solutions, then the system is a.c.i. and there are no other $m - 1$ -dimensional Laurent solutions but those provided by the coherent set.*

We assume that the divisor is very ample and in addition projectively normal (see [12, 27] for definitions when needed). Consider a point $p \in \mathcal{D}$, a chart U_j around p on the torus and a function y_j in $\mathcal{L}(\mathcal{D})$ having a pole of maximal order at p . Then the vector $(1/y_j, y_1/y_j, \dots, y_N/y_j)$ provides a good system of coordinates in U_j . Then taking the derivative with regard to one of the flows

$$\left(\frac{y_i}{y_j}\right) \cdot = \frac{\dot{y}_i y_j - y_i \dot{y}_j}{y_j^2}, \quad 1 \leq j \leq N,$$

are finite on U_j as well. Therefore, since y_j^2 has a double pole along \mathcal{D} , the numerator must also have a double pole (at worst), i.e., $\dot{y}_i y_j - y_i \dot{y}_j \in \mathcal{L}(2\mathcal{D})$. Hence, when \mathcal{D} is projectively normal, we have that

$$\left(\frac{y_i}{y_j}\right) \cdot = \sum_{k,l} a_{k,l} \left(\frac{y_k}{y_j}\right) \left(\frac{y_l}{y_j}\right),$$

i.e., the ratios y_i/y_j form a closed system of coordinates under differentiation. At the bad points, the concept of projective normality play an important role: this enables one to show that y_i/y_j is a bona fide Taylor series starting from every point in a neighborhood of the point in question. Moreover, the Laurent solutions provide an effective tool for find the constants of the motion. For that, just search polynomials H_i of z , having the property that evaluated along all the Laurent solutions $z(t)$ they have no polar part. Indeed, since an invariant function of the flow does not blow up along a Laurent solution, the series obtained by substituting the formal solutions (2.3) into the invariants should, in particular, have no polar part. The polynomial functions $H_i(z(t))$ being holomorphic and bounded in every direction of a compact space, (i.e., bounded along all principle solutions), are thus constant by a Liouville type of argument. It thus an important ingredient in this argument to use all the generic solutions. To make these informal arguments rigorous is an outstanding question of the subject.

Assume Hamiltonian flows to be weight-homogeneous with a weight $s_i \in \mathbb{N}$, going with each variable z_i , i.e.,

$$f_i(\alpha^{s_1} z_1, \dots, \alpha^{s_m} z_m) = \alpha^{s_i+1} f_i(z_1, \dots, z_m), \quad \forall \alpha \in \mathbb{C}.$$

Observe that then the constants of the motion H can be chosen to be weight-homogeneous:

$$H(\alpha^{s_1} z_1, \dots, \alpha^{s_m} z_m) = \alpha^k H(z_1, \dots, z_m), \quad k \in \mathbb{Z}.$$

The study of the a.c.i. of Hamiltonian systems, includes several passages to prove rigorously. Here we mention the main passages, leaving the detail when studying the different problems in the following sections. We saw that if the flow is algebraically completely integrable, the differential equations (2.1) must admits Laurent series solutions (2.3) depending on $m-1$ free parameters. We must have $k_i = s_i$ and coefficients in the series must satisfy at the 0^{th} step non-linear equations,

$$f_i(z_1^{(0)}, \dots, z_m^{(0)}) + g_i z_i^{(0)} = 0, \quad 1 \leq i \leq m, \quad (2.4)$$

and at the k^{th} step, linear systems of equations :

$$(L - kI)z^{(k)} = \begin{cases} 0 & \text{for } k = 1 \\ \text{some polynomial in } z^{(1)}, \dots, z^{(k-1)} & \text{for } k > 1, \end{cases} \quad (2.5)$$

where

$$L = \text{Jacobian map of (2.4)} = \frac{\partial f}{\partial z} + gI \Big|_{z=z^{(0)}}.$$

If $m-1$ free parameters are to appear in the Laurent series, they must either come from the non-linear equations (2.4) or from the eigenvalue problem (2.5), i.e., L must have at least $m-1$ integer eigenvalues. These are much less conditions than expected, because of the fact that the homogeneity k of the constant H must be an eigenvalue of L . Moreover the formal series solutions are convergent as a consequence of the majorant method [6, 24]. Thus, the first step is to show the existence of the Laurent solutions, which requires an argument precisely every time k is an integer eigenvalue of L and therefore $L - kI$ is not invertible. One shows the existence of the remaining constants of the motion in involution so as to reach the number $n+k$. Then you have to prove that for given c_1, \dots, c_m , the set

$$\mathcal{D} \equiv \left\{ \begin{array}{l} z_i(t) = t^{-v_i} \left(z_i^{(0)} + z_i^{(1)} t + z_i^{(2)} t^2 + \dots \right), \quad 1 \leq i \leq m \\ \text{Laurent solutions such that : } H_j(z_i(t)) = c_j + \text{Taylor part} \end{array} \right\},$$

defines one or several $n - 1$ dimensional algebraic varieties ("Painlevé" divisor) having the property that $M_c \cup \mathcal{D}$, is a smooth compact, connected variety with n commuting vector fields independent at every point, i.e., a complex algebraic torus $\mathbb{C}^n / \text{Lattice}$. The flows $J \frac{\partial H_{k+i}}{\partial z}, \dots, J \frac{\partial H_{k+n}}{\partial z}$ are straight line motions on this torus. Let's point out and we'll see all this in more detail later, that having computed the space of functions $\mathcal{L}(\mathcal{D})$ with simple poles at worst along the expansions, it is often important to compute the space of functions $\mathcal{L}(k\mathcal{D})$ of functions having k -fold poles at worst along the expansions. These functions play a crucial role in the study of the procedure for embedding the invariant tori into projective space.

The idea of the Adler-van Moerbeke's proof [3] we shall give here is closely related to the geometric spirit of the Arnold-Liouville theorem [7]. Namely, a compact complex n -dimensional variety on which there exist n holomorphic commuting vector fields which are independent at every point is analytically isomorphic to a n -dimensional complex torus $\mathbb{C}^n / \text{Lattice}$ and the complex flows generated by the vector fields are straight lines on this complex torus.

Theorem 2.7. *Let*

$$\overline{\mathcal{A}} = \bigcap_i \{Z = (Z_0, Z_1, \dots, Z_n) \in \mathbb{P}^N(\mathbb{C}) : P_i(Z) = 0\},$$

be an irreducible variety defined by an intersection involving a large number of homogeneous polynomials P_i with smooth and irreducible affine part $\mathcal{A} = \overline{\mathcal{A}} \cap \{Z_0 \neq 0\}$. Put $\mathcal{A} \equiv \mathcal{A} \cup \mathcal{D}$, i.e., $\mathcal{D} = \overline{\mathcal{A}} \cap \{Z_0 = 0\}$ and consider the map

$$f : \overline{\mathcal{A}} \rightarrow \mathbb{P}^N(\mathbb{C}), \quad Z \mapsto f(Z).$$

Let

$$\widetilde{\mathcal{A}} = f(\overline{\mathcal{A}}) = \overline{f(\mathcal{A})}, \quad \mathcal{D} = \bigcup_{i=1}^r \mathcal{D}_i, \quad \mathcal{S} \equiv f(\mathcal{D}) = \bigcup_{i=1}^r f(\mathcal{D}_i) \equiv \bigcup_{i=1}^r \mathcal{S}_i.$$

where \mathcal{D}_i are codimension 1 subvarieties. Assume that:

- (i) f maps \mathcal{A} smoothly and 1-1 onto $f(\mathcal{A})$.
- (ii) There exist n holomorphic vector fields X_1, \dots, X_n on \mathcal{A} which commute and are independent at every point. One vector field, say X_k (where $1 \leq k \leq n$), extends holomorphically to a neighborhood of \mathcal{S}_k in the projective space $\mathbb{P}^N(\mathbb{C})$.
- (iii) For all $p \in \mathcal{S}_k$, the integral curve $f(t) \in \mathbb{P}^N(\mathbb{C})$ of the vector field X_k through $f(0) = p \in \mathcal{S}_k$ has the property that

$$\{f(t) : 0 < |t| < \varepsilon, t \in \mathbb{C}\} \subset f(\mathcal{A}).$$

This condition means that the orbits of X_k through \mathcal{S}_k go immediately into the affine part and in particular, the vector field X_k does not vanish on any point of \mathcal{S}_k .

Then

- a) \widetilde{M} is compact, connected and admits an embedding into $\mathbb{P}^N(\mathbb{C})$.
- b) $\widetilde{\mathcal{A}}$ is diffeomorphic to a n -dimensional complex torus. The vector fields X_1, \dots, X_n extend holomorphically and remain independent on $\widetilde{\mathcal{A}}$.
- c) $\widetilde{\mathcal{A}}$ is a Kähler variety.
- d) \widetilde{M} is a Hodge variety. In particular, \mathcal{A} is the affine part of an Abelian variety $\widetilde{\mathcal{A}}$.

Proof. a) A crucial step is to show that the orbits running through \mathcal{S}_k form a smooth variety Σ_p , $p \in \mathcal{S}_k$ such that $\Sigma_p \setminus \mathcal{S}_k \subseteq \mathcal{A}$. Let $p \in \mathcal{S}_k$, $\varepsilon > 0$ small enough, $g_{X_k}^t$ the flow generated by X_k on \mathcal{A} and $\{g_{X_k}^t : t \in \mathbb{C}, 0 < |t| < \varepsilon\}$, the orbit going through the point p . The vector field X_k is holomorphic in the neighborhood of any point $p \in \mathcal{S}_k$ and non-vanishing, by (ii) and (iii). Then the flow $g_{X_k}^t$ can be straightened out after a holomorphic change of coordinates. Let $\mathcal{H} \subset \mathbb{P}^N(\mathbb{C})$ be a hyperplane transversal to the direction of the flow at p and let Σ_p be the surface element formed by the divisor \mathcal{S}_k and the orbits going through p . Consider the segment of $\mathcal{S}' \equiv \mathcal{H} \cap \Sigma_p$ and so locally, we have $\Sigma_p = \mathcal{S}' \times \mathbb{C}$. We shall show that Σ_p is smooth. Note that \mathcal{S}' is smooth. Indeed, suppose that \mathcal{S}' is singular at 0, then Σ_p would be singular along the trajectory (t -axis) which go immediately into the affine $f(\mathcal{A})$, by condition (iii). Hence, the affine part would be singular which is impossible by condition (i). So, \mathcal{S}' is smooth and by the implicit function theorem, Σ_p is smooth too. Consider now the map

$$\overline{\mathcal{A}} \subset \mathbb{P}^m(\mathbb{C}) \rightarrow \mathbb{P}^N(\mathbb{C}), \quad Z \mapsto f(Z),$$

where $Z = (Z_0, Z_1, \dots, Z_n) \in \mathbb{P}^m(\mathbb{C})$ and $\widetilde{\mathcal{A}} = f(\overline{\mathcal{A}}) = \overline{f(\mathcal{A})}$. Recall that the flow exists in a full neighborhood of p in $\mathbb{P}^N(\mathbb{C})$ and it has been straightened out. Therefore, near $p \in \mathcal{S}_k$, we have $\Sigma_p = \widetilde{\mathcal{A}}$ and $\Sigma_p \setminus \mathcal{S}_k \subseteq \mathcal{A}$. Otherwise, there would exist an element $\Sigma'_p \subset \widetilde{\mathcal{A}}$ such that

$$\{g_{X_k}^t : t \in \mathbb{C}, 0 < |t| < \varepsilon\} = (\Sigma_p \cap \Sigma'_p) \setminus p \subset \mathcal{A},$$

by condition (iii). In other words, $\Sigma_p \cap \Sigma'_p = t$ -axis and hence \mathcal{A} would be singular along the t -axis which is impossible. Since the variety \mathcal{A} is irreducible and since the generic hyperplane section $\mathcal{H}_{gen.}$ of $\widetilde{\mathcal{A}}$ is also irreducible, all hyperplane sections are connected and hence \mathcal{D} is also connected. Now consider the graph $G_f \subset \mathbb{P}^m(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$ of the map f , which is irreducible together with $\widetilde{\mathcal{A}}$. It follows from the irreducibility of G_f that a generic hyperplane section $G_f \cap (\mathcal{H}_{gen.} \times \mathbb{P}^N(\mathbb{C}))$ is irreducible, hence the special hyperplane section $G_f \cap (\{Z_0 = 0\} \times \mathbb{P}^N(\mathbb{C}))$ is connected and therefore the projection map

$$Proj_{\mathbb{P}^N(\mathbb{C})}[G_f \cap (\{Z_0 = 0\} \times \mathbb{P}^N(\mathbb{C}))] = f(\mathcal{D}) \equiv \mathcal{S},$$

is connected. Hence, the variety

$$\widetilde{\mathcal{A}} = \mathcal{A} \cup (\cup_{p \in \mathcal{S}_k} \Sigma_p) = \mathcal{A} \cup \mathcal{S}_k \subseteq \mathbb{P}^N(\mathbb{C}),$$

is compact, connected and embeds smoothly into $\mathbb{P}^N(\mathbb{C})$ via f .

b) Let g^t be the flow generated by X_t on \mathcal{A} and let $p_1 \in \widetilde{\mathcal{A}} \setminus \mathcal{A}$. For $\varepsilon > 0$ and for all $t_1 \in \mathbb{C}$ such that $0 < |t_1| < \varepsilon$, note that $q \equiv g^{t_1}(p_1)$ is well defined and $g^{t_1}(p_1) \in f(\mathcal{A})$, using condition (iii). Let $U(q) \subseteq \mathcal{A}$ be a neighborhood of q and let

$$g^{t_2}(p_2) = g^{-t_1} \circ g^{t_2} \circ g^{t_1}(p_2), \quad \forall p_2 \in U(p_1) \equiv g^{-t_1}(U(q)),$$

which is well defined since by commutativity one can see that the right hand side is independent of t_1 :

$$g^{-(t_1+\varepsilon)} \circ g^{t_2} \circ g^{t_1+\varepsilon}(p_2) = g^{-(t_1+\varepsilon)} \circ g^{t_2} \circ g^{t_1} \circ g^\varepsilon(p_2) = g^{-(t_1+\varepsilon)} \circ g^\varepsilon \circ g^{t_2} \circ g^{t_1}(p_2) = g^{-t_1} \circ g^{t_2} \circ g^{t_1}(p_2).$$

Notice that $g^{t_2}(p_2)$ is a holomorphic function of p_2 and t_2 , because in $U(p_1)$ the function g^{t_1} is holomorphic and its image is away from \mathcal{S} , i.e., in the affine, g^{t_2} is holomorphic. The same argument applies to $g^{t_3}(p_3), \dots, g^{t_n}(p_n)$ where

$$g^{t_n}(p_n) = g^{-t_{n-1}} \circ g^{t_n} \circ g^{t_{n-1}}(p_n), \quad \forall p_n \in U(p_{n-1}) \equiv g^{-t_{n-1}}(U(q)).$$

Thus X_1, \dots, X_n have been holomorphically extended, remain independent and commuting on $\widetilde{\mathcal{A}}$. Therefore, we can show along the same lines as in the Arnold-Liouville theorem [7] that $\widetilde{\mathcal{A}}$ is a complex torus $\mathbb{C}^n / \text{lattice}$. And that will done, by considering the local diffeomorphism

$$\mathbb{C}^n \longrightarrow \widetilde{\mathcal{A}}, \quad t = (t_1, \dots, t_n) \longmapsto g^t p = g^{t_1} \circ \dots \circ g^{t_n}(p),$$

for a fixed origin $p \in f(\mathcal{A})$. The additive subgroup $L = \{t \in \mathbb{C}^n : g^t p = p\}$, is a lattice of \mathbb{C}^n (spanned by $2n$ vectors in \mathbb{C}^n , independent over \mathbb{R}), hence $\mathbb{C}^n / L \longrightarrow \widetilde{\mathcal{A}}$ is a biholomorphic diffeomorphism.

c) Let

$$ds^2 = \sum_{k=1}^n dt_k \otimes d\bar{t}_k,$$

be a hermitian metric on the complex variety $\widetilde{\mathcal{A}}$ and let ω its fundamental $(1, 1)$ -form. We have

$$\omega = -\frac{1}{2} \text{Im} ds^2 = \frac{\sqrt{-1}}{2} \sum_{k=1}^n dt_k \wedge d\bar{t}_k.$$

So we see that ω is closed and the metric ds^2 is kähler and consequently $\widetilde{\mathcal{A}}$ is a Kähler variety.

d) On the Kähler variety $\widetilde{\mathcal{A}}$ are defined periods of ω . If these periods are integers (possibly after multiplication by a number), we obtain a variety of Hodge. More specifically, integrals $\int_{\gamma_k} \omega$ of the form ω (where γ_k are cycles in $H_2(\widetilde{\mathcal{A}}, \mathbb{Z})$) determine the periods ω . As they are integers, then $\widetilde{\mathcal{A}}$ is a Hodge variety. The variety $\widetilde{\mathcal{A}}$ is equipped with n holomorphic vectors fields, independent and commuting. From a) and b) the variety $\widetilde{\mathcal{A}}$ is both a projective variety and a complex torus and hence an Abelian variety as a consequence of Chow theorem [12, 27]. Another proof is to use the result that we just show since every Hodge torus is Abelian, the converse is also true. Note also that by Moishezon's theorem [30], a compact complex kähler variety having as many independent meromorphic functions as its dimension is an Abelian variety. \square

3. Geodesic flow on the group $SO(3)$ and Euler rigid body motion

The Euler rigid body motion [11] (also called Euler-Poinsot [36] motion of the solid) can be regarded as geodesic flow on $SO(3)$. This problem expresses the free motion of a rigid body around a fixed point and the motion of the body is governed by the following equation

$$\dot{M} = [M, \Lambda M], \quad (3.1)$$

with

$$M = (M_{ij})_{1 \leq i, j \leq 3} \equiv \sum_{i=1}^3 m_i e_i \equiv \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so(3),$$

and

$$\Lambda M = \sum_{i=1}^3 \lambda_i m_i e_i \equiv \begin{pmatrix} 0 & -\lambda_3 m_3 & \lambda_2 m_2 \\ \lambda_3 m_3 & 0 & -\lambda_1 m_1 \\ -\lambda_2 m_2 & \lambda_1 m_1 & 0 \end{pmatrix} \in so(3), \quad \lambda_i \equiv \frac{1}{I_i},$$

where (m_1, m_2, m_3) is the angular momentum, and I_1, I_2, I_3 , the principal moments of inertia about the principal axes of inertia. Equation (3.1) is written explicitly in the form

$$\begin{aligned} \dot{m}_1 &= (\lambda_3 - \lambda_2) m_2 m_3, \\ \dot{m}_2 &= (\lambda_1 - \lambda_3) m_1 m_3, \\ \dot{m}_3 &= (\lambda_2 - \lambda_1) m_1 m_2, \end{aligned} \quad (3.2)$$

and can also be written as a Hamiltonian vector field

$$\dot{x} = J \frac{\partial H}{\partial x}, \quad x = (m_1, m_2, m_3)^\top,$$

with the Hamiltonian

$$H = \frac{1}{2} (\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2),$$

and

$$J = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so(3).$$

The system (3.2) has beside the energy $H_1 = H$, a trivial invariant H_2 , i.e., such that : $J \frac{\partial H_2}{\partial x} = 0$, implying

$$H_2 = \frac{1}{2} (m_1^2 + m_2^2 + m_3^2).$$

The system evolves on the intersection of the sphere $H_1 = c_1$ and the ellipsoid $H_2 = c_2$. In \mathbb{R}^3 , this intersection will be isomorphic to two circles (with $\frac{c_2}{\lambda_3} < c_1 < \frac{c_2}{\lambda_1}$). The system (3.2) is completely integrable and the vector $J \frac{\partial H}{\partial x}$ gives a flow on a variety

$$M_c = \bigcap_{i=1}^2 \{x \in \mathbb{R}^3 : H_i(x) = c_i\}, \quad (c = (c_1, c_2) \text{ is not a critical value}),$$

diffeomorphic to a real torus of dimension 1, that is to say a circle.

From the first integrals H_1 and H_2 , we express m_1 and m_3 as a function of m_2 . These expressions are then introduced into the second equation of the system (3.2) to obtain a differential equation in m_2 and $\frac{dm_2}{dt}$ only. In more detail, the following relationships are easily obtained from equations $H_1 = c_1$ and $H_2 = c_2$:

$$m_1^2 = \frac{2c_1 - r^2 \lambda_3 - (\lambda_2 - \lambda_3) m_2^2}{\lambda_1 - \lambda_3}, \quad (3.3)$$

$$m_3^2 = \frac{2c_2\lambda_1 - 2c_1 - (\lambda_1 - \lambda_2)m_2^2}{\lambda_1 - \lambda_3}. \quad (3.4)$$

By substituting these expressions in the second equation of the system (3.2), we obtain

$$\dot{m}_2 = \sqrt{(2c_1 - 2c_2\lambda_3 - (\lambda_2 - \lambda_3)m_2^2)(2c_2\lambda_1 - 2c_1 - (\lambda_1 - \lambda_2)m_2^2)}.$$

By integrating this equation, we obtain a function $t(m_2)$ in the form of an elliptic integral. To reduce this to the standard form, we can assume that $2c_2 > \frac{2c_1}{\lambda_2}$ (otherwise, it is enough to invert the indices 1 and 3 in all the previous formulas). We rewrite the previous equation, in the form

$$\frac{dm_2}{\sqrt{(2c_1 - 2c_2\lambda_3)(2c_2\lambda_1 - 2c_1)dt}} = \sqrt{\left(1 - \frac{(\lambda_2 - \lambda_3)m_2^2}{2c_1 - 2c_2\lambda_3}\right) \left(1 - \frac{(\lambda_1 - \lambda_2)m_2^2}{2c_2\lambda_1 - 2c_1}\right)}.$$

By setting

$$\tau = t\sqrt{(\lambda_2 - \lambda_3)(2c_2\lambda_1 - 2c_1)}, \quad s = m_2\sqrt{\frac{\lambda_2 - \lambda_3}{2c_1 - 2c_2\lambda_3}},$$

we obtain

$$\frac{ds}{d\tau} = \sqrt{(1 - s^2) \left(1 - \frac{(\lambda_1 - \lambda_2)(2c_1 - 2c_2\lambda_3)}{(\lambda_2 - \lambda_3)(2c_2\lambda_1 - 2c_1)}s^2\right)},$$

which suggests choosing elliptic functions as a module

$$k^2 = \frac{(\lambda_1 - \lambda_2)(2c_1 - 2c_2\lambda_3)}{(\lambda_2 - \lambda_3)(2c_2\lambda_1 - 2c_1)}.$$

Inequalities $\lambda_1 > \lambda_2 > \lambda_3$, $\frac{2c_1}{\lambda_1} < 2c_2 < \frac{2c_1}{\lambda_3}$ and $r^2 > \frac{2c_1}{\lambda_2}$ show that $0 < k^2 < 1$. So we get

$$\frac{ds}{d\tau} = \sqrt{(1 - s^2)(1 - k^2s^2)}.$$

This equation admits the solution (we choose the origin of the times such that $m_2 = 0$ for $t = 0$):

$$\tau = \int_0^s \frac{ds}{\sqrt{(1 - s^2)(1 - k^2s^2)}}.$$

It is the integral of a holomorphic differential on an elliptic curve

$$\mathcal{E} : w^2 = (1 - s^2)(1 - k^2s^2). \quad (3.5)$$

The inverse function $s(\tau)$ is one of Jacobi's elliptic functions [27] : $s = \mathbf{sn}\tau$, which also determines m_2 as a function of time, that is,

$$m_2 = \sqrt{\frac{2H_1 - r^2\lambda_3}{\lambda_2 - \lambda_3}} \cdot \mathbf{sn}\tau.$$

According to the equalities (3.3) and (3.4), we know that the functions m_1 and m_3 are expressed algebraically as a function of m_2 , so

$$\begin{aligned} m_1 &= \sqrt{\frac{2H_1 - r^2\lambda_3}{\lambda_1 - \lambda_3}} \cdot \sqrt{1 - \mathbf{sn}^2\tau}, \\ m_3 &= \sqrt{\frac{r^2\lambda_1 - 2H_1}{\lambda_1 - \lambda_3}} \cdot \sqrt{1 - k^2\mathbf{sn}^2\tau}. \end{aligned}$$

Given the definition of the other two elliptical functions [27] :

$$\operatorname{cn} \tau = \sqrt{1 - \operatorname{sn}^2 \tau}, \quad \operatorname{dn} \tau = \sqrt{1 - k^2 \operatorname{sn}^2 \tau},$$

and the fact that $\tau = t\sqrt{(\lambda_2 - \lambda_3)(2c_2\lambda_1 - 2c_1)}$, we finally get the following explicit formulas :

$$\begin{aligned} m_1 &= \sqrt{\frac{2c_1 - 2c_2\lambda_3}{\lambda_1 - \lambda_3}} \operatorname{cn}(t\sqrt{(\lambda_2 - \lambda_3)(2c_2\lambda_1 - 2H_1)}), \\ m_2 &= \sqrt{\frac{2c_1 - 2c_2\lambda_3}{\lambda_2 - \lambda_3}} \operatorname{sn}(t\sqrt{(\lambda_2 - \lambda_3)(2c_2\lambda_1 - 2c_1)}), \\ m_3 &= \sqrt{\frac{2c_2\lambda_1 - 2c_1}{\lambda_1 - \lambda_3}} \operatorname{dn}(t\sqrt{(\lambda_2 - \lambda_3)(2c_2\lambda_1 - 2c_1)}). \end{aligned}$$

In other words, the integration of the Euler equations is done by means of elliptic Jacobi functions. In fact, the Euler rigid body motion is always algebraically completely integrable. The two circles of the intersection M_c , (with $\frac{c_1}{\lambda_3} < c_2 < \frac{c_1}{\lambda_1}$, otherwise it is empty) forms the real part of a complex torus of dimension 1, defined by the elliptic curve \mathcal{E} (3.5). The complex intersection ($\subset \mathbb{C}^3$) is the affine part of an elliptic curve

$$\bar{M}_c = \{X \in \mathbb{P}^3(\mathbb{C}) : H_1(X) = c_1 X_0^2\} \cap \{X \in \mathbb{P}^3(\mathbb{C}) : H_2(X) = c_2 X_0^2\}.$$

We show that \bar{M}_c is isomorphic to the elliptic curve \mathcal{E} . In addition, the circle defined by $\{H_1 = c_1\} \cap \{H_2 = c_2\}$ extends to the complex torus $\mathbb{C}/\text{lattice}$ and the flow linearizes on this torus. If $p(t) = (m_1(t), m_2(t), m_3(t))$, is a solution of the system (3.2), the law connecting $p(t_1 + t_2)$ to $p(t_1)$ and $p(t_2)$ is the the addition law on the elliptic curve \mathcal{E} . From equations (3.2), the unique holomorphic differential on \bar{M}_c is given by

$$\omega = \frac{dm_1}{(\lambda_3 - \lambda_2)m_2m_3} = \frac{dm_2}{(\lambda_1 - \lambda_3)m_1m_3} = \frac{dm_3}{(\lambda_2 - \lambda_1)m_1m_2},$$

so $t = \int_{p(0)}^{p(t)} \omega$, $p(0) \in \bar{M}_c$. The system (3.2) is invariant by the transformations $t \rightarrow \alpha^{-1}t$, $m_1 \rightarrow \alpha m_1$, $m_2 \rightarrow \alpha m_2$, $m_3 \rightarrow \alpha m_3$.

We seek solutions of the system (3.2) or of equation (3.1) in the form of Laurent series

$$M(t) = t^{-1} \left(M^{(0)} + M^{(1)}t + M^{(2)}t^2 + \dots \right) = \sum_{j=0}^{\infty} M^{(j)}t^{j-1}, \quad (3.6)$$

depending on $\dim(\text{phase space}) - 1 = 2$ free parameters. Substituting (3.6) into equation (3.1), one obtains

$$\sum_{j=0}^{\infty} (j-1)M^{(j)}t^{j-2} = \sum_{j=0}^{\infty} \left(\sum_{i=0}^j [M^{(i)}, \Lambda M^{(j-i)}] \right) t^{j-2}.$$

Therefore,

$$(j-i)M^{(j)} = \sum_{i=0}^j [M^{(i)}, \Lambda M^{(j-i)}],$$

and we see that the coefficients $M^{(0)}, M^{(1)}, \dots$, satisfy the equations

$$M^{(0)} + [M^{(0)}, \Lambda M^{(0)}] = 0, \quad (3.7)$$

$$(L - kI)M^{(k)} = - \sum_{i=1}^{k-1} [M^{(i)}, \Lambda C^{(k-i)}] \quad k \geq 1,$$

where L is the linear operator $L : so(3) \rightarrow so(3)$ defined by

$$L(Y) = Y + [M^{(0)}, \Lambda Y] + [Y, \Lambda M^{(0)}] = \text{Jacobian of (3.7)}.$$

The matrix $M^{(0)}$ appearing in L is a solution of the nonlinear equation (3.7). A simple calculation shows that the matrix $(L - kI)$ is always invertible except for $k = 2$ and therefore its rank is equal to 1. This shows that the coefficient $M^{(2)}$ contains two free parameters and can be assimilated to c_1 and c_2 . In a detailed and explicit way, let us look for the solutions of the system (3.2) in the form of Laurent's series

$$\begin{aligned} m_1 &= \frac{1}{t} \left(m_1^{(0)} + m_1^{(1)}t + m_1^{(2)}t^2 + \dots \right), \\ m_2 &= \frac{1}{t} \left(m_2^{(0)} + m_2^{(1)}t + m_2^{(2)}t^2 + \dots \right), \\ m_3 &= \frac{1}{t} \left(m_3^{(0)} + m_3^{(1)}t + m_3^{(2)}t^2 + \dots \right), \end{aligned}$$

depending on $\dim(\text{phase space}) - 1 = 2$ free parameters. Substituting these equations into the system (3.2), we see that

1) the coefficients $m_1^{(0)}, m_2^{(0)}, m_3^{(0)}$, satisfy the equations

$$\begin{aligned} m_1^{(0)} + (\lambda_3 - \lambda_2)m_2^{(0)}m_3^{(0)} &= 0, \\ m_2^{(0)} + (\lambda_1 - \lambda_3)m_1^{(0)}m_3^{(0)} &= 0, \\ m_3^{(0)} + (\lambda_2 - \lambda_1)m_1^{(0)}m_2^{(0)} &= 0, \end{aligned}$$

whose solutions are

$$1^{st} \text{ case : } m_1^{(0)} = \frac{-1}{\sqrt{(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_3)}}, \quad m_2^{(0)} = \frac{1}{\sqrt{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}}, \quad m_3^{(0)} = \frac{1}{\sqrt{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)}}.$$

$$2^{nd} \text{ case : } m_1^{(0)} = \frac{1}{\sqrt{(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_3)}}, \quad m_2^{(0)} = \frac{1}{\sqrt{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}}, \quad m_3^{(0)} = \frac{-1}{\sqrt{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)}}.$$

$$3^{rd} \text{ case : } m_1^{(0)} = \frac{1}{\sqrt{(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_3)}}, \quad m_2^{(0)} = \frac{-1}{\sqrt{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}}, \quad m_3^{(0)} = \frac{1}{\sqrt{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)}}.$$

$$4^{th} \text{ case : } m_1^{(0)} = \frac{-1}{\sqrt{(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_3)}}, \quad m_2^{(0)} = \frac{-1}{\sqrt{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}}, \quad m_3^{(0)} = \frac{-1}{\sqrt{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)}}.$$

2) the coefficients $m_1^{(1)}, m_2^{(1)}, m_3^{(1)}$, satisfy equations

$$\begin{aligned} (\lambda_3 - \lambda_2)m_2^{(0)}m_3^{(1)} + (\lambda_3 - \lambda_2)m_2^{(1)}m_3^{(0)} &= 0, \\ (\lambda_1 - \lambda_3)m_1^{(0)}m_3^{(1)} + (\lambda_1 - \lambda_3)m_1^{(1)}m_3^{(0)} &= 0, \\ (\lambda_2 - \lambda_1)m_1^{(0)}m_2^{(1)} + (\lambda_2 - \lambda_1)m_1^{(1)}m_2^{(0)} &= 0, \end{aligned}$$

the solutions of which are in all cases : $m_1^{(1)} = m_2^{(1)} = m_3^{(1)} = 0$.

3) the coefficients $m_1^{(2)}, m_2^{(2)}, m_3^{(2)}$, satisfy equations

$$\begin{aligned} m_1^{(2)} - \lambda_3 m_2^{(0)} m_3^{(2)} - \lambda_3 m_2^{(1)} m_3^{(1)} - \lambda_3 m_2^{(2)} m_3^{(0)} \\ + \lambda_2 m_2^{(0)} m_3^{(2)} + \lambda_2 m_2^{(1)} m_3^{(1)} + \lambda_2 m_2^{(2)} m_3^{(0)} &= 0, \\ m_2^{(2)} - \lambda_1 m_1^{(0)} m_3^{(2)} - \lambda_1 m_1^{(1)} m_3^{(1)} - \lambda_1 m_1^{(2)} m_3^{(0)} \\ + \lambda_3 m_1^{(0)} m_3^{(2)} + \lambda_3 m_1^{(1)} m_3^{(1)} + \lambda_3 m_1^{(2)} m_3^{(0)} &= 0, \\ m_3^{(2)} - \lambda_2 m_1^{(0)} m_2^{(2)} - \lambda_2 m_1^{(1)} m_2^{(1)} - \lambda_2 m_1^{(2)} m_2^{(0)} \\ + \lambda_1 m_1^{(0)} m_2^{(2)} + \lambda_1 m_1^{(1)} m_2^{(1)} + \lambda_1 m_1^{(2)} m_2^{(0)} &= 0, \end{aligned}$$

whose solutions corresponding to the different cases are respectively,

$$1^{st} \text{ case : } m_1^{(2)} = \frac{\sqrt{(\lambda_3 - \lambda_2)}}{\sqrt{(\lambda_1 - \lambda_3)}} m_2^{(2)} + \frac{\sqrt{(\lambda_3 - \lambda_2)}}{\sqrt{(\lambda_2 - \lambda_1)}} m_3^{(2)}.$$

$$2^{nd} \text{ case : } m_1^{(2)} = -\frac{\sqrt{(\lambda_3 - \lambda_2)}}{\sqrt{(\lambda_1 - \lambda_3)}} m_2^{(2)} + \frac{\sqrt{(\lambda_3 - \lambda_2)}}{\sqrt{(\lambda_2 - \lambda_1)}} m_3^{(2)}.$$

$$3^{rd} \text{ case : } m_1^{(2)} = \frac{\sqrt{(\lambda_3 - \lambda_2)}}{\sqrt{(\lambda_1 - \lambda_3)}} m_2^{(2)} - \frac{\sqrt{(\lambda_3 - \lambda_2)}}{\sqrt{(\lambda_2 - \lambda_1)}} m_3^{(2)}.$$

$$4^{th} \text{ case : } m_1^{(2)} = -\frac{\sqrt{\lambda_3-\lambda_2}}{\sqrt{\lambda_1-\lambda_3}} m_2^{(2)} - \frac{\sqrt{\lambda_3-\lambda_2}}{\sqrt{\lambda_2-\lambda_1}} m_3^{(2)}.$$

where $m_2^{(2)}$ et $m_3^{(2)}$ are two free parameters.

Therefore, for the first case, we have

$$m_1 = \frac{-1}{t\sqrt{(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)}} + \left(\frac{\sqrt{(\lambda_3-\lambda_2)}}{\sqrt{(\lambda_1-\lambda_3)}} m_2^{(2)} + \frac{\sqrt{(\lambda_3-\lambda_2)}}{\sqrt{(\lambda_2-\lambda_1)}} m_3^{(2)} \right) t + \dots,$$

$$m_2 = \frac{1}{t\sqrt{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}} + m_2^{(2)} t + \dots,$$

$$m_3 = \frac{1}{t\sqrt{(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)}} + m_3^{(2)} t + \dots.$$

Substituting these developments into the first integrals H_1 and H_2 , we get

$$H_1 = 2\frac{\sqrt{\lambda_3-\lambda_2}}{\sqrt{\lambda_2-\lambda_1}} \left(\frac{1}{\lambda_3-\lambda_2} - \frac{1}{\lambda_1-\lambda_3} \right) m_2^{(2)} + 2\frac{\sqrt{\lambda_3-\lambda_2}}{\sqrt{\lambda_1-\lambda_3}} \left(\frac{1}{\lambda_3-\lambda_2} - \frac{1}{\lambda_2-\lambda_1} \right) m_3^{(2)},$$

$$H_2 = 2\frac{\sqrt{\lambda_3-\lambda_2}}{\sqrt{\lambda_2-\lambda_1}} \left(\frac{\lambda_2}{\lambda_3-\lambda_2} - \frac{\lambda_1}{\lambda_1-\lambda_3} \right) m_2^{(2)} + 2\frac{\sqrt{\lambda_3-\lambda_2}}{\sqrt{\lambda_1-\lambda_3}} \left(\frac{\lambda_3}{\lambda_3-\lambda_2} - \frac{\lambda_1}{\lambda_2-\lambda_1} \right) m_3^{(2)},$$

and we deduce the relations :

$$m_3^{(2)} = \frac{1}{6\sqrt{(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)}} \left((\lambda_3 - \lambda_2)(\lambda_1 H_1 - H_2) - (\lambda_1 - \lambda_3)(\lambda_2 H_1 - H_2) \right),$$

$$m_2^{(2)} = \frac{1}{6\sqrt{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}} \left((\lambda_2 - \lambda_1)(\lambda_3 H_1 - H_2) - (\lambda_3 - \lambda_2)(\lambda_1 H_1 - H_2) \right).$$

Obviously, similar expressions are obtained for the other cases.

We deduce from what precedes the following result:

Theorem 3.1. *The Euler rigid body motion is always algebraically completely integrable and can be regarded as geodesic flow on $SO(3)$. In addition, the integration of the equations is done by means of elliptic Jacobi functions.*

4. Clebsch and Lyapunov-Steklov cases of a solid in an ideal fluid

The equations of motion of a solid in an ideal fluid have the form (Kirchhoff's equations [17]) :

$$\begin{aligned} \dot{p}_1 &= p_2 \frac{\partial H}{\partial l_3} - p_3 \frac{\partial H}{\partial l_2}, \\ \dot{p}_2 &= p_3 \frac{\partial H}{\partial l_1} - p_1 \frac{\partial H}{\partial l_3}, \\ \dot{p}_3 &= p_1 \frac{\partial H}{\partial l_2} - p_2 \frac{\partial H}{\partial l_1}, \\ \dot{l}_1 &= p_2 \frac{\partial H}{\partial p_3} - p_3 \frac{\partial H}{\partial p_2} + l_2 \frac{\partial H}{\partial l_3} - l_3 \frac{\partial H}{\partial l_2}, \\ \dot{l}_2 &= p_3 \frac{\partial H}{\partial p_1} - p_1 \frac{\partial H}{\partial p_3} + l_3 \frac{\partial H}{\partial l_1} - l_1 \frac{\partial H}{\partial l_3}, \\ \dot{l}_3 &= p_1 \frac{\partial H}{\partial p_2} - p_2 \frac{\partial H}{\partial p_1} + l_1 \frac{\partial H}{\partial l_2} - l_2 \frac{\partial H}{\partial l_1}, \end{aligned} \tag{4.1}$$

where (p_1, p_2, p_3) is the velocity of a point fixed relatively to the solid, (l_1, l_2, l_3) the angular velocity of the body expressed with regard to a frame of reference also fixed relatively to the solid and H is the Hamiltonian. Equations (4.1) can be regarded as the equations of the geodesics of the right-invariant metric on the group $E(3) = SO(3) \times \mathbb{R}^3$ of motions of 3-dimensional Euclidean space \mathbb{R}^3 , generated by rotations and translations.

Equations (4.1) have the trivial first integrals (or invariants):

$$H_1 = H, \quad H_2 = \sum_{k=1}^3 p_k^2, \quad H_3 = \sum_{k=1}^3 p_k l_k. \tag{4.2}$$

We distinguish two integrable cases: the case of Clebsch [8] and the case of Lyapunov-Steklov [28, 37].

4.1 Clebsch's case

In Clebsch's case, we have

$$H_1 = H = \frac{1}{2} \sum_{k=1}^3 (a_k p_k^2 + b_k l_k^2), \quad (4.3)$$

with

$$\frac{a_2 - a_3}{b_1} + \frac{a_3 - a_1}{b_2} + \frac{a_1 - a_2}{b_3} = 0.$$

An additional integral is

$$H_4 = \frac{1}{2} \sum_{k=1}^3 (b_k p_k^2 + \rho l_k^2), \quad (4.4)$$

where the constant ρ satisfies the conditions

$$\rho = \frac{b_1(b_2 - b_3)}{a_2 - a_3} = \frac{b_2(b_3 - b_1)}{a_3 - a_1} = \frac{b_3(b_1 - b_2)}{a_1 - a_2}.$$

We shall study briefly Kötter's solution [19] by quadratures of the equations (4.1), in terms of genus 2 hyperelliptic integrals. In fact, the transformation to the separating coordinates s_1 and s_2 which leads to the quadratures in terms of hyperelliptic integrals is quite involved. Finding this transformation require a great deal of luck and ingenuity. After the substitution $b_k \rightarrow \rho b_k$, $1 \leq k \leq 3$, and after an appropriate linear combination of H_1 and H_2 , the equations (4.2), (4.3), (4.4) can be written in the form

$$\begin{aligned} p_1^2 + p_2^2 + p_3^2 &= A, \\ b_1 p_1^2 + b_2 p_2^2 + b_3 p_3^2 + l_1^2 + l_2^2 + l_3^2 &= B, \\ b_1 l_1^2 + b_2 l_2^2 + b_3 l_3^2 - b_2 b_3 p_1^2 - b_1 b_3 p_2^2 - b_1 b_2 p_3^2 &= C, \\ \rho l_1 + \rho l_2 + \rho l_3 &= D, \end{aligned}$$

where A, B, C, D are constants. Following [19, 10, 23], we introduce coordinates φ_k, ψ_k , $1 \leq k \leq 3$ by setting

$$\begin{aligned} \varphi_k &= p_k \left(\frac{\sqrt{\prod_{j=1}^3 (z_1 - b_j)}}{\sqrt{z_1 - b_k} \sqrt{\partial R / \partial z_1}} + \sqrt{-1} \frac{\sqrt{\prod_{j=1}^3 (z_2 - b_j)}}{\sqrt{z_2 - b_k} \sqrt{\partial R / \partial z_2}} \right) + l_k \left(\frac{\sqrt{z_1 - b_k}}{\sqrt{\partial R / \partial z_1}} + \sqrt{-1} \frac{\sqrt{z_2 - b_k}}{\sqrt{\partial R / \partial z_2}} \right), \\ \psi_k &= p_k \left(\frac{\sqrt{\prod_{j=1}^3 (z_1 - b_j)}}{\sqrt{z_1 - b_k} \sqrt{\partial R / \partial z_1}} - \sqrt{-1} \frac{\sqrt{\prod_{j=1}^3 (z_2 - b_j)}}{\sqrt{z_2 - b_k} \sqrt{\partial R / \partial z_2}} \right) + l_k \left(\frac{\sqrt{z_1 - b_k}}{\sqrt{\partial R / \partial z_1}} - \sqrt{-1} \frac{\sqrt{z_2 - b_k}}{\sqrt{\partial R / \partial z_2}} \right), \end{aligned}$$

where

$$R(z) = \prod_{i=1}^4 (z - z_i),$$

and z_1, z_2, z_3, z_4 are the roots of the equation

$$A^2 \left(z^2 - z \sum_{k=1}^3 b_k \right) + Bz - C + 2D \sqrt{\prod_{k=1}^3 (z - b_k)} = 0.$$

Let s_1 and s_2 be the roots of the equation

$$\frac{\psi_1^2}{v_1^2 - s} + \frac{\psi_2^2}{v_2^2 - s} + \frac{\psi_3^2}{v_3^2 - s} = 0,$$

where

$$v_k = \frac{\frac{\sqrt{z_3 - b_k}}{\sqrt{\partial R / \partial z_3}} + \sqrt{-1} \frac{\sqrt{z_4 - b_k}}{\sqrt{\partial R / \partial z_4}}}{\frac{\sqrt{z_1 - b_k}}{\sqrt{\partial R / \partial z_1}} + \sqrt{-1} \frac{\sqrt{z_2 - b_k}}{\sqrt{\partial R / \partial z_2}}}, \quad 1 \leq k \leq 3.$$

An expression of the original variables $p_1, p_2, p_3, l_1, l_2, l_3$ in terms of s_1 and s_2 can be found in [19]. After some algebraic manipulations, we obtain the following equations for s_1 and s_2 :

$$\begin{aligned}\frac{ds_1}{dt} &= \frac{(as_1 + b)\sqrt{P_5(s_1)}}{s_2 - s_1}, \\ \frac{ds_2}{dt} &= \frac{(as_2 + b)\sqrt{P_5(s_2)}}{s_1 - s_2},\end{aligned}$$

where a, b are constants and $P_5(s)$ is a polynomial of degree 5 of the form

$$P_5(s) = s(s - v_1^2)(s - v_2^2)(s - v_3^2)(s - v_1^2 v_2^2 v_3^2).$$

These equations can be integrated by the Abelian mapping

$$\mathcal{H} \rightarrow \text{Jac}(\mathcal{H}) = \mathbb{C}^2 / \Lambda, \quad P \mapsto \left(\int_{P_0}^P \theta_1, \int_{P_0}^P \theta_2 \right),$$

where the hyperelliptic curve \mathcal{H} of genus 2 is given by the equation $w^2 = P_5(s)$, Λ is the lattice generated by the vectors $n_1 + Mn_2, (n_1, n_2) \in \mathbb{Z}^2, M$ is the matrix of period of the curve \mathcal{H} , (θ_1, θ_2) is a canonical basis of holomorphic differentials on \mathcal{H} , i.e.,

$$\theta_1 = \frac{ds}{\sqrt{P_5(s)}}, \quad \theta_2 = \frac{sds}{\sqrt{P_5(s)}},$$

and P_0 is a fixed point. Consequently, we have

Theorem 4.1. *The system of differential equations (4.1) in the Clebsch's case can be integrated in terms of genus 2 hyperelliptic functions of time.*

4.2 Lyapunov-Steklov's case

In Lyapunov-Steklov's case, we have

$$H_1 = H = \frac{1}{2} \sum_{k=1}^3 (a_k p_k^2 + b_k l_k^2) + \sum_{k=1}^3 c_k p_k l_k,$$

with

$$\begin{aligned}a_1 &= A^2 b_1 (b_2 - b_3)^2 + B, & a_2 &= A^2 b_2 (b_3 - b_1)^2 + B, & a_3 &= A^2 b_3 (b_1 - b_2)^2 + B, \\ c_1 &= A b_2 b_3 + C, & c_2 &= A b_1 b_3 + C, & c_3 &= A b_1 b_2 + C,\end{aligned}$$

where A, B, C are constants. A fourth first integral is given by

$$H_4 = \frac{1}{2} \sum_{k=1}^3 (d_k p_k^2 + l_k^2) - A \sum_{k=1}^3 b_k p_k l_k,$$

where

$$d_1 = A^2 (b_2 - b_3)^2, \quad d_2 = A^2 (b_3 - b_1)^2, \quad d_3 = A^2 (b_1 - b_2)^2.$$

A long and delicate calculation [20] shows that in this case too, the integration is done using hyperelliptic functions of genus two.

5. The classification of algebraic integrable geodesic flow on $SO(4)$

In several problems, when studying the geodesic flow on $SO(4)$, it is more convenient to use the coordinates $u = (x_1, x_2, x_3)$ and $v = (x_4, x_5, x_6)$, they correspond to the decomposition $u \oplus v \in so(4) \simeq so(3) \oplus so(3)$. In these coordinates, the geodesic flow on the group $SO(4)$ can be written as

$$X_H : \dot{u} = u \wedge \frac{\partial H}{\partial u}, \quad \dot{v} = v \wedge \frac{\partial H}{\partial v},$$

for invariant metric defined by the quadratic form

$$H = \frac{1}{24} \sum_{i=1}^3 (3(3c_i + d_i)x_i^2 + (c_i + 3d_i)x_{i+3}^2 + 6(d_i - c_i)x_i x_{i+3}), \quad (5.1)$$

with coefficients

$$c_i = \frac{b_i}{a_i}, d_i = \frac{b_j - b_k}{a_j - a_k}, \sum_{i=1}^3 a_i = 0, \sum_{i=1}^3 b_i = 0,$$

and ijk permutations of 123. This geodesic flow has three quadratic invariants, namely, the Casimir functions $\|u\|^2$ and $\|v\|^2$, and the metric above, and one quartic invariant. The invariants $\|u\|^2$ and $\|v\|^2$ define the 4-dimensional non degenerate symplectic leaves of Hamiltonian structure, which therefore are parameterized by the values of $\|u\|^2$ and $\|v\|^2$. More precisely, in the classification [4, 6] of algebraic integrable geodesic flow on $SO(4)$, three cases come up; two are linearly equivalent to cases of rigid body motion in a perfect fluid studied last century, respectively by Clebsch and Lyapunov-Steklov, and there is a third new case namely the Kostant-Kirillov Hamiltonian flow on the dual of $so(4)$. The metric H is obviously written in the quadratic form

$$H = \frac{1}{2} \sum_{j=1}^6 \lambda_j x_j^2 + \sum_{j=1}^3 \mu_j x_j x_{j+3},$$

where $\lambda_1, \dots, \lambda_6, \mu_1, \mu_2, \mu_3 \in \mathbb{C}$ and $\lambda_{12}\lambda_{23}\lambda_{31}\lambda_{45}\lambda_{56}\lambda_{64}\mu_1\mu_2\mu_3 \neq 0$ with $\lambda_{jk} \equiv \lambda_j - \lambda_k$. Explicitly, the equations above are written

$$\begin{aligned} \frac{dx_1}{dt} &= \lambda_{32}x_2x_3 + \mu_3x_2x_6 - \mu_2x_3x_5, \\ \frac{dx_2}{dt} &= \lambda_{13}x_3x_1 + \mu_1x_3x_4 - \mu_3x_1x_5, \\ \frac{dx_3}{dt} &= \lambda_{21}x_1x_2 + \mu_2x_1x_5 - \mu_1x_2x_4, \\ \frac{dx_4}{dt} &= \lambda_{65}x_5x_6 + \mu_3x_3x_5 - \mu_2x_2x_6, \\ \frac{dx_5}{dt} &= \lambda_{46}x_6x_4 + \mu_1x_1x_6 - \mu_3x_3x_4, \\ \frac{dx_6}{dt} &= \lambda_{54}x_4x_5 + \mu_2x_2x_4 - \mu_1x_1x_5. \end{aligned}$$

The equations have besides the energy $H_1 = H$, two trivial constants of the motion

$$H_2 = x_1^2 + x_2^2 + x_3^2, H_3 = x_4^2 + x_5^2 + x_6^2.$$

Adler and van Moerbeke [4, 6] have shown that the geodesic flow on $SO(4)$ for the metric defined by the above quadratic form is algebraically completely integrable in the three cases described in the following subsections and these are the only ones that exist.

5.1 Manakov geodesic flow on the group $SO(4)$ and Clebsch rigid body motion in a perfect fluid

The geodesic flow for this metric takes the following commutator form (Euler-Arnold equations) :

$$\dot{X} = [X, \Lambda.X], \quad \cdot \equiv \frac{d}{dt} \quad (5.2)$$

where

$$X = (X_{ij})_{1 \leq i, j \leq 4} = \sum_{i=1}^6 x_i e_i = \begin{pmatrix} 0 & -x_3 & x_2 & -x_4 \\ x_3 & 0 & -x_1 & -x_5 \\ -x_2 & x_1 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{pmatrix} \in so(4),$$

and

$$\Lambda.X = (\lambda_{ij}X_{ij})_{1 \leq i, j \leq 4} = \sum_{i=1}^6 \lambda_i x_i e_i = \begin{pmatrix} 0 & -\lambda_3 x_3 & \lambda_2 x_2 & -\lambda_4 x_4 \\ \lambda_3 x_3 & 0 & -\lambda_1 x_1 & -\lambda_5 x_5 \\ -\lambda_2 x_2 & \lambda_1 x_1 & 0 & -\lambda_6 x_6 \\ \lambda_4 x_4 & \lambda_5 x_5 & \lambda_6 x_6 & 0 \end{pmatrix} \in so(4).$$

In view of the isomorphism between (\mathbb{R}^6, Λ) and $(so(4), [,])$ we write the system (5.2) as

$$\begin{aligned} \dot{x}_1 &= (\lambda_3 - \lambda_2)x_2x_3 + (\lambda_6 - \lambda_5)x_5x_6, \\ \dot{x}_2 &= (\lambda_1 - \lambda_3)x_1x_3 + (\lambda_4 - \lambda_4)x_4x_6, \\ \dot{x}_3 &= (\lambda_2 - \lambda_1)x_1x_2 + (\lambda_5 - \lambda_4)x_4x_5, \\ \dot{x}_4 &= (\lambda_3 - \lambda_5)x_3x_5 + (\lambda_6 - \lambda_2)x_2x_6, \\ \dot{x}_5 &= (\lambda_4 - \lambda_3)x_3x_4 + (\lambda_1 - \lambda_6)x_1x_6, \\ \dot{x}_6 &= (\lambda_2 - \lambda_4)x_2x_4 + (\lambda_5 - \lambda_1)x_1x_5. \end{aligned} \tag{5.3}$$

The quadratic form H is diagonal with regard to the customary $so(4)$ coordinates (Manakov metric [29]), i.e.,

$$2H = \sum_{\substack{j,k=1 \\ j < k}}^4 \Lambda_{jk} X_{jk}^2, \quad (X_{jk})_{1 \leq j, k \leq 4} \in so(4),$$

with

$$\Lambda_{jk} = \frac{\beta_j - \beta_k}{\alpha_j - \alpha_k}, \quad (\alpha_j, \beta_j \in \mathbb{C}, 1 \leq j \leq 4),$$

all Λ_{jk} distinct. The extra invariant H_4 is quadratic and we'll see how this was done independently by Haine [13] and Mumford (appendix in[2]) that the flow evolves on Abelian surfaces $\mathbb{C}^2/lattice \subseteq \mathbb{P}^7(\mathbb{C})$, having period matrix

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}, \quad \text{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0, \quad (a, b, c \in \mathbb{C}),$$

and also the linearization takes place on a Prym variety (For the definition and properties of Prym varieties, see for example [13, 25]). The periods of this Prym variety provide the exact periods of the motion in terms of Abelian integrals. The problem of the solid body in a fluid in the case of Clebsch is a particular case of this metric (see subsection 4.1).

Let's see in more detail the linearization of this problem [13, 6, 22]. Let $x \in \mathbb{C}^6$, $t \in \mathbb{C}$ and $Z \subset \mathbb{C}^6$ a non-empty Zariski open set. The momentum map

$$g : (H_1, \dots, H_4) : \mathbb{C}^6 \longrightarrow \mathbb{C}^4,$$

is submersive on Z , i.e., $dH_1(x), \dots, dH_4(x)$ are linearly independent on Z . Let

$$I = g(\mathbb{C}^6 \setminus Z) = \{c = (c_i) \in \mathbb{C}^4 : \exists x \in g^{-1}(c) \text{ with } dH_1(x) \wedge \dots \wedge dH_4(x) = 0\},$$

be the set of critical values of g and \bar{I} the Zariski closure of I in \mathbb{C}^4 . The non-empty Zariski open set Z can be chosen as the set

$$Z = \{x \in \mathbb{C}^6 : g(x) \in \mathbb{C}^4 \setminus \bar{I}\}.$$

The invariant variety defined by

$$M_c = g^{-1}(c) = \bigcap_{i=1}^4 \{x \in \mathbb{C}^6 : H_i(x) = c_i\},$$

is the fibre of a morphism from \mathbb{C}^6 to \mathbb{C}^4 , thus M_c is a smooth affine surface for generic $c = (c_1, \dots, c_4) \in \mathbb{C}^4$ and the main problem will be to complete M_c into an Abelian surface. Now, how does one find the compactification of M_c into an Abelian surface? This compactification is not trivial and the simplest one obtained as a closure :

$$\overline{M_c} = \bigcap_{i=1}^4 \{H_i(x) = c_i x_0^2\} \subset \mathbb{P}^6(\mathbb{C}),$$

i.e.,

$$\begin{aligned} x_1x_4 + x_2x_5 + x_3x_6 &= c_1x_0^2, \\ x_1^2 + x_2^2 + \dots + x_6^2 &= c_2x_0^2, \\ \lambda_1x_1^2 + \lambda_2x_2^2 + \dots + \lambda_6x_6^2 &= c_3x_0^2, \\ \mu_1x_1^2 + \mu_2x_2^2 + \dots + \mu_6x_6^2 &= c_4x_0^2, \end{aligned}$$

where $[x_0 : x_1 : \dots : x_6]$ are homogeneous coordinates on $\mathbb{P}^6(\mathbb{C})$, does not lead to this result (in the following we will not distinguish between x_1 as a homogeneous coordinates $[x_0 : x_1]$ and as an affine coordinate x_1/x_0). Indeed, an Abelian surface is not simply-connected and therefore cannot be projective complete intersection. In other words, if M_c is to be the affine part of an Abelian surface, M_c must have a singularity somewhere along the locus at infinity $C = \overline{M}_c \cap \{x_0 = 0\}$. A direct calculation shows that C is an ordinary double curve of M_c except at 16 ordinary pinch points of \overline{M}_c ; the variety \overline{M}_c has a local analytic equation $x^2 = yz^2$. The reduced curve C_r is a smooth elliptic curve. Now, it's only after blowing up \overline{M}_c along the curve C_r that one gets the desired Abelian surface.

Theorem 5.1. a) The divisor of poles of the functions x_1, x_2, \dots, x_6 is a Riemann surface \mathcal{D} of genus 9. For generic constants, the surface M_c is the affine part of an Abelian surface \widetilde{M}_c obtained by gluing to M_c the divisor \mathcal{D} .

b) The flow (5.3) evolves on an Abelian surface $\widetilde{M}_c \cong \mathbb{C}^2 / \text{lattice of polarization}$

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}, \text{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0.$$

c) The Abelian surface \widetilde{M}_c which completes the affine surface M_c is the Prym variety $\text{Prym}_\alpha(\Gamma)$ of the genus 3 Riemann surface Γ :

$$\Gamma : \begin{cases} w^2 = -c_1 (x_5^0 x_6^0)^2 - c_2 (x_6^0)^2 z - c_3 (x_5^0)^2 z + c_4 y \\ y^2 = z(\alpha^2 z - 1)(\beta^2 z + 1) \end{cases}$$

for the involution

$$\sigma : \Gamma \longrightarrow \Gamma, (w, y, z) \longmapsto (-w, y, z),$$

interchanging the two sheets of the double covering

$$\Gamma \longmapsto \Gamma_0, (w, y, z) \longmapsto (y, z),$$

where Γ_0 is the elliptic curve defined by

$$\Gamma_0 : y^2 = z(\alpha^2 z - 1)(\beta^2 z + 1).$$

Proof. a) Consider points at infinity which are limit points of trajectories of the flow. There is a Laurent decomposition of such asymptotic solutions,

$$X(t) = t^{-1} \left(X^{(0)} + X^{(1)}t + X^{(2)}t^2 + \dots \right) \quad (5.4)$$

which depend on $\dim(\text{phase space}) - 1 = 5$ free parameters. Putting (5.4) into (5.2), solving inductively for the $X^{(k)}$, one finds at the 0^{th} step a non-linear equation,

$$X^{(0)} + [X^{(0)}, \Lambda.X^{(0)}] = 0, \quad (5.5)$$

and at the k^{th} step, a linear system of equations

$$(L - kI) \left(X^{(k)} \right) = \begin{cases} 0 & \text{for } k = 1 \\ \text{quadratic polynomial in } X^{(1)}, \dots, X^{(k-1)} & \text{for } k \geq 2 \end{cases} \quad (5.6)$$

where L denotes the linear map

$$L(Y) = [Y, \Lambda.X^{(0)}] + [X^{(0)}, \Lambda.Y] + Y = \text{Jacobian map of (5.5)}.$$

One parameter appear at the 0^{th} step, i.e., in the resolution of (5.5) and the 4 remaining ones at the k^{th} step, $k = 1, \dots, 4$. Taking into account only solutions trajectories lying on the surface M_c , we obtain one-parameter families which are parameterized by a Riemann surface. To be precise we search for the set \mathcal{D} of Laurent solutions (5.4) restricted to the affine invariant surface M_c , i.e.,

$$\begin{aligned} \mathcal{D} &= \text{closure of the continuous components of} \\ &\quad \{\text{Laurent solutions } X(t) \text{ such that } H_i(X(t)) = c_i, 1 \leq i \leq 4\}, \\ &= \bigcap_{i=1}^4 \{t^0 - \text{coefficient of } H_i(X(t)) = c_i\}, \\ &= \text{a Riemann surface (algebraic curve) whose affine equation is} \\ &\quad \begin{cases} w^2 + c_1 \left(x_5^{(0)} x_6^{(0)}\right)^2 + c_2 \left(x_4^{(0)} x_6^{(0)}\right)^2 + c_3 \left(x_4^{(0)} x_5^{(0)}\right)^2 + c_4 x_4^{(0)} x_5^{(0)} x_6^{(0)}, \\ \equiv w^2 + F \left(x_4^{(0)}, x_5^{(0)}, x_6^{(0)}\right) = 0 \end{cases} \end{aligned} \quad (5.7)$$

where w is an arbitrary parameter and where $x_4^{(0)}, x_5^{(0)}, x_6^{(0)}$ parameterizes the elliptic curve

$$\mathcal{E} : \begin{cases} \left(x_4^{(0)}\right)^2 + \left(x_5^{(0)}\right)^2 + \left(x_6^{(0)}\right)^2 = 0 \\ \left(\beta x_5^{(0)} + \alpha x_6^{(0)}\right) \left(\beta x_5^{(0)} - \alpha x_6^{(0)}\right) = 1 \end{cases} \quad (5.8)$$

with (α, β) such that $\alpha^2 + \beta^2 + 1 = 0$. The Riemann surface \mathcal{D} is a two-sheeted ramified covering of the elliptic curve \mathcal{E} and it easy to check that the elliptic curve \mathcal{E} is exactly the reduced curve C_r . The branch points are defined by the 16 zeroes of $F \left(x_4^{(0)}, x_5^{(0)}, x_6^{(0)}\right)$ on \mathcal{E} . The Riemann surface \mathcal{D} is unramified at infinity and by Riemann-Hurwitz's formula,

$$2g(\mathcal{D}) - 2 = N(2g(\mathcal{E}) - 2) + R,$$

where N is the number of sheets and R the ramification index, the genus $g(\mathcal{D})$ of \mathcal{D} is 9. To show that M_c is the affine part of an Abelian surface \widetilde{M}_c with $\widetilde{M}_c \setminus M_c = \mathcal{D}$, we can use the method of Laurent's developments (see Haine [13]) or theorem 2.3. Here, by following Mumford (see appendix to [2]), we give an abstract algebro-geometrical proof that the four quadrics in this problem intersect in the affine part of an Abelian surface using Enriques classification of algebraic surfaces. For this, we will compute the invariants of \widetilde{M}_c and use Enriques classification of algebraic surfaces (see [12], p.590). We denote as usual by $K_{\widetilde{M}_c}$ the canonical bundle, $\chi(\mathcal{O}_{\widetilde{M}_c})$ the Euler characteristic and $q(\widetilde{M}_c)$ the irregularity of \widetilde{M}_c . Now if

$$\phi : \widetilde{M}_c \longrightarrow \overline{M}_c \subset \mathbb{P}^6(\mathbb{C})$$

is the normalization of \overline{M}_c , then the pullback map on sections

$$\phi^* : H^0(\overline{M}_c, \mathcal{O}_{\overline{M}_c}) \longrightarrow H^0(\widetilde{M}_c, \mathcal{O}_{\widetilde{M}_c}),$$

is an isomorphism and

$$K_{\widetilde{M}_c} = \widetilde{K}_{\overline{M}_c} - \mathcal{D}, \quad \widetilde{K}_{\overline{M}_c} = \phi^*(K_{\overline{M}_c})$$

and so for H a hyperplane in $\mathbb{P}^6(\mathbb{C})$,

$$K_{\widetilde{M}_c} = \phi^* \left(\overline{M}_c \cdot K_{\mathbb{P}^6(\mathbb{C})} + \left(\sum_{i=1}^4 \deg H_i \right) \cdot H \right) - \mathcal{D} = 0.$$

Also

$$\chi(\mathcal{O}_{\widetilde{M}_c}) = \chi(\phi_* \mathcal{O}_{\widetilde{M}_c} / \mathcal{O}_{\overline{M}_c}) + \chi(\mathcal{O}_{\overline{M}_c}) = \chi(\phi_* \mathcal{O}_{\mathcal{D}} / \mathcal{O}_{\mathcal{D}}) + \chi(\mathcal{O}_{\mathcal{D}}).$$

Recall that the Riemann surface \mathcal{D} (5.7) of genus 9, is a double cover ramified over 16 points of the elliptic curve \mathcal{E} (5.8). We shall use the Koszul complex to compute $\chi(\mathcal{O}_{\widetilde{M}_c})$. In the local ring at each point of $\mathbb{P}^6(\mathbb{C})$ the localizations of the 4 homogeneous polynomials H_i give a regular sequence, and the Koszul complex gives a canonical resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^6(\mathbb{C})}(-8) \rightarrow \mathcal{O}_{\mathbb{P}^6(\mathbb{C})}(-6)^4 \rightarrow \mathcal{O}_{\mathbb{P}^6(\mathbb{C})}(-4)^6 \rightarrow \mathcal{O}_{\mathbb{P}^6(\mathbb{C})}(-2)^4 \rightarrow \mathcal{O}_{\mathbb{P}^6(\mathbb{C})} \rightarrow \mathcal{O}_{\widetilde{M}_c} \rightarrow 0$$

Thus $\chi(\mathcal{O}_{\widetilde{M}_c}) = 8$, hence $\chi(\mathcal{O}_{\widetilde{M}_c}) = 0$ and $q(\widetilde{M}_c) = 2$. By Enriques-Kodaira's classification theorem, it follows that \widetilde{M}_c is an Abelian surface.

b) Let

$$L \equiv \left\{ f : f \text{ meromorphic on } \widetilde{M}_c, (f) + \mathcal{D} \geq 0 \right\}$$

be the set of meromorphic functions on \widetilde{M}_c with at worst a simple pole along \mathcal{D} and let

$$\chi(\mathcal{D}) = \dim H^0(\widetilde{M}_c, \mathcal{O}(\mathcal{D})) - \dim H^1(\widetilde{M}_c, \mathcal{O}(\mathcal{D})),$$

be the Euler characteristic of \mathcal{D} . The adjunction formula and the Riemann-Roch theorem for divisors on Abelian surfaces imply that

$$g(\mathcal{D}) = \frac{K_{\widetilde{M}_c} \cdot \mathcal{D} + \mathcal{D} \cdot \mathcal{D}}{2} + 1,$$

and

$$\chi(\mathcal{D}) = p_a(\widetilde{M}_c) + 1 + \frac{1}{2}(\mathcal{D} \cdot (\mathcal{D} - K_{\widetilde{M}_c})),$$

where $g(\mathcal{D})$ is the geometric genus of \mathcal{D} and $p_a(\widetilde{M}_c)$ is the arithmetic genus of \widetilde{M}_c . Since \widetilde{M}_c is an Abelian surface ($K_{\widetilde{M}_c} = 0, p_a(\widetilde{M}_c) = -1$),

$$g(\mathcal{D}) - 1 = \frac{\mathcal{D} \cdot \mathcal{D}}{2} = \chi(\mathcal{D}).$$

Using Kodaira-Serre duality [12, 27], Kodaira-Nakano vanishing theorem [12, 27] and a theorem on theta-functions [12, 27], it is easy to see that

$$g(\mathcal{D}) - 1 = \dim L(\mathcal{D}) (\equiv h^0(L)) = \delta_1 \delta_2, \quad (5.9)$$

where $\delta_1, \delta_2 \in \mathbb{N}$, are the elementary divisors of the polarization $c_1(L)$ of \widetilde{M}_c . Note that the natural reflection about the origin of \mathbb{C}^2 , is given by

$$\sigma \equiv -id : (x_0, x_1, \dots, x_6) \mapsto (-x_0, x_1, \dots, x_6),$$

and has 16 fixed points on \widetilde{M}_c , given by the 16 branch points on \mathcal{D} covering the 16 roots of the polynomial $F(x_4^0, x_5^0, x_6^0)$ (5.7). Since L is symmetric ($\sigma^*L \simeq L$), σ can be lifted to L as an involution $\tilde{\sigma}$ in two ways differing in sign and for each section (theta-function) $s \in H^0(L)$, we therefore have $\tilde{\sigma}s = \pm s$. Recall that a section $s \in H^0(L)$ is called even (resp. odd) if $\tilde{\sigma}s = +s$ (resp. $\tilde{\sigma}s = -s$). Under $\tilde{\sigma}$ the vector space $H^0(L)$ splits into an even and odd subspace $H^0(L) = H^0(L)^{even} \oplus H^0(L)^{odd}$ with $H^0(L)^{even}$ containing all the even sections and $H^0(L)^{odd}$ all odd ones. Using the inverse formula [12, 27], we see after a small computation that

$$\begin{aligned} \dim H^0(\mathcal{L})^{even} &= \frac{\delta_1 \delta_2}{2} + 2^{-1+\#\text{ even } \delta_k}, k = 1, 2 \\ \dim H^0(\mathcal{L})^{odd} &= \frac{\delta_1 \delta_2}{2} - 2^{-1+\#\text{ even } \delta_k}, k = 1, 2 \end{aligned} \quad (5.10)$$

Notice that $c_1(L) = \phi^*(H)$ and $(c_1(L))^2 = 16$ (since the degree of \widetilde{M}_c is 16). By the classification theory of ample line bundles on Abelian varieties, $\widetilde{M}_c \simeq \mathbb{C}^2/L_\Omega$ with period lattice given by the columns of the matrix

$$\Omega = \begin{pmatrix} \delta_1 & 0 & a & c \\ 0 & \delta_2 & c & b \end{pmatrix}, \text{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0,$$

according to (5.9), with

$$\delta_1 \delta_2 = h^0(L) = g(\mathcal{D}) - 1 = 8, \quad \delta_1 | \delta_2, \quad \delta_i \in \mathbb{N}^*.$$

Hence we have two possibilities : (i) $\delta_1 = 1, \delta_2 = 8$ and (ii) $\delta_1 = 2, \delta_2 = 4$. From formula (5.10), the corresponding line bundle L has in case (i), 5 even sections, 3 odd ones and in case (ii), 6 even sections, 2 odd ones. Now x_1, \dots, x_6 are 6 even sections, showing that case (ii) is the only alternative and the period matrix has the form

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}, \text{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0.$$

c) After substitution $z \equiv (x_4^0)^2$, the Riemann surface \mathcal{D} can also be seen as a four-sheeted unramified covering of another Riemann surface Γ , determined by the equation

$$\Gamma : G(w, z) \equiv \left(w^2 + c_1 (x_5^0 x_6^0)^2 + c_2 (x_6^0)^2 z + c_3 (x_5^0)^2 z \right)^2 - c_4^2 (x_5^0 x_6^0)^2 z = 0.$$

It is straightforward to verify that the equations (5.8) are equivalent to $(x_5^0)^2 = \beta^2 z + 1$ and $(x_6^0)^2 = \alpha^2 z - 1$. To compute the genus of Γ , we observe that the Riemann surface Γ is invariant under an involution

$$\sigma : \Gamma \longrightarrow \Gamma, \quad (w, z) \longmapsto (-w, z). \quad (5.11)$$

Consider a map

$$\rho : \Gamma \longrightarrow \Gamma_0 \equiv \Gamma / \sigma, \quad (w, y, z) \longmapsto (y, z),$$

of the Riemann surface Γ onto an elliptic curve $\Gamma_0 \equiv \Gamma / \sigma$, that is given by the equation

$$\Gamma_0 : y^2 = z(\alpha^2 z - 1)(\beta^2 z + 1). \quad (5.12)$$

The genus of the Riemann surface

$$\Gamma : \begin{cases} w^2 = -c_1 (x_5^0 x_6^0)^2 - c_2 (x_6^0)^2 z - c_3 (x_5^0)^2 z + c_4 y \\ y^2 = z(\alpha^2 z - 1)(\beta^2 z + 1) \end{cases} \quad (5.13)$$

is calculated by means of the map ρ . The latter is two-sheeted ramified covering of the elliptic curve Γ_0 with 4 branch points. Using the Riemann-Hurwitz formula, we obtain $g(\Gamma) = 3$.

I now will proceed to show that the Abelian surface \widetilde{M}_c can be identified as Prym variety $\text{Prym}_\sigma(\Gamma)$. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a basis of cycles in the Riemann surface Γ with the intersection indices $a_i o a_j = b_i o b_j = 0, a_i o b_j = \delta_{ij}$, such that

$$\sigma(a_1) = a_3, \quad \sigma(b_1) = b_3, \quad \sigma(a_2) = -a_2, \quad \sigma(b_2) = -b_2$$

for the involution σ (5.11). By the Poincaré residue formula, the 3 holomorphic 1-forms $\omega_0, \omega_1, \omega_2$ in Γ are the differentials

$$P(w, z) \frac{dz}{(\partial G / \partial w)(w, z)} \Big|_{G(w, z)=0} = P(w, z) \frac{dz}{4wy},$$

for P a polynomial of degree $\leq \deg G - 3 = 1$. Therefore

$$\omega_0 = \frac{dz}{y}, \quad \omega_1 = \frac{zdz}{wy}, \quad \omega_2 = \frac{dz}{wy},$$

form a basis of holomorphic differentials on Γ and obviously

$$\sigma^*(\omega_0) = \omega_0, \quad \sigma^*(\omega_k) = -\omega_k, \quad (k = 1, 2),$$

for the involution σ (5.11). It is well known that the period matrix Ω of $\text{Prym}_\sigma(\Gamma)$ can be written as follows

$$\Omega = \begin{pmatrix} 2 \int_{a_1} \omega_1 & \int_{a_2} \omega_1 & 2 \int_{b_1} \omega_1 & \int_{b_2} \omega_1 \\ 2 \int_{a_1} \omega_2 & \int_{a_2} \omega_2 & 2 \int_{b_1} \omega_2 & \int_{b_2} \omega_2 \end{pmatrix},$$

Let (dt_1, dt_2) be a basis of holomorphic 1-forms on \widetilde{M}_c such that $dt_j|_{\mathcal{D}} = \omega_j, (j = 1, 2)$,

$$L_{\Omega'} = \left\{ \sum_{k=1}^2 m_k \int_{a'_k} \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} + n_k \int_{b'_k} \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} : m_k, n_k \in \mathbb{Z} \right\},$$

the lattice associated to the period matrix

$$\Omega' = \begin{pmatrix} \int_{a'_1} dt_1 & \int_{a'_2} dt_1 & \int_{b'_1} dt_1 & \int_{b'_2} dt_1 \\ \int_{a'_1} dt_2 & \int_{a'_2} dt_2 & \int_{b'_1} dt_2 & \int_{b'_2} dt_2 \end{pmatrix},$$

where (a'_1, a'_2, b'_1, b'_2) is a basis of $H_1(\widetilde{M}_c, \mathbb{Z})$ and let

$$\widetilde{M}_c \longrightarrow \mathbb{C}^2 / L_{\Omega'} : p \longmapsto \int_{p_0}^p \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix},$$

be the uniformizing map. By the Lefschetz theorem on hyperplane section [12, 27], the map $H_1(\mathcal{D}, \mathbb{Z}) \longrightarrow H_1(\widetilde{M}_c, \mathbb{Z})$ induced by the inclusion $\mathcal{D} \hookrightarrow \widetilde{M}_c$ is surjective and consequently we can find 4 cycles a'_1, a'_2, b'_1, b'_2 on the Riemann surface \mathcal{D} such that

$$\Omega' = \begin{pmatrix} \int_{a'_1} \omega_1 & \int_{a'_2} \omega_1 & \int_{b'_3} \omega_1 & \int_{b'_4} \omega_1 \\ \int_{a'_1} \omega_2 & \int_{a'_2} \omega_2 & \int_{b'_3} \omega_2 & \int_{b'_4} \omega_2 \end{pmatrix},$$

and

$$L_{\Omega'} = \left\{ \sum_{k=1}^2 m_k \int_{a'_k} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + n_k \int_{b'_k} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} : m_k, n_k \in \mathbb{Z} \right\}.$$

Recalling that $F(x_4^0, x_5^0, x_6^0)$ (5.7) has 4 zeroes on Γ_0 (5.12) and 16 zeroes on \mathcal{E} (5.8), it follows that the 4 cycles a'_1, a'_2, b'_1, b'_2 on \mathcal{D} which we look for are $2a_1, a_2, 2b_1, b_2$ and they form a basis of $H_1(\widetilde{M}_c, \mathbb{Z})$ such that

$$\Omega' = \begin{pmatrix} 2 \int_{a_1} \omega_1 & \int_{a_2} \omega_1 & 2 \int_{b_1} \omega_1 & \int_{b_2} \omega_1 \\ 2 \int_{a_1} \omega_2 & \int_{a_2} \omega_2 & 2 \int_{b_1} \omega_2 & \int_{b_2} \omega_2 \end{pmatrix} = \Omega,$$

is a Riemann matrix. Thus, \widetilde{M}_c and $\text{Prym}_{\sigma}(\Gamma)$ are two Abelian varieties analytically isomorphic to the same complex torus $\mathbb{C}^2 / L_{\Omega}$. By Chow's theorem [12, 27], \widetilde{M}_c and $\text{Prym}_{\sigma}(\Gamma)$ are then algebraically isomorphic. This completes the proof of the theorem. \square

Remark 5.2. *It is well known that this problem has been solved via the Lax spectral curve technique by Adler and van Moerbeke [1]. Strange as it may seem, the use of the spectral curve method may not give the tori correctly, but perhaps with period doubling, in contrast with the statement that the correct tori would be obtained by the Kowalewski-Painlevé analysis. This indicated a need for caution in interpretation of the result for tori calculated from the Lax spectral curve technique. A striking example of this phenomenon appears in the problem studied in this subsection. We know from [1, 22], that the linearization of the Euler-Arnold equations obtained using the isospectral deformation method (Lax technique) takes place on the Prym variety $\text{Prym}_{\sigma}(\mathcal{E})$ of a genus 3 Riemann surface \mathcal{E} ; the latter is a double ramified cover of an elliptic curve \mathcal{E}_0 . Also, we have just seen from the asymptotic analysis of the Euler-Arnold equations, the affine variety M_c completes into an Abelian surface \widetilde{M}_c upon adding a Riemann surface \mathcal{D} of genus 9, which is a 4-fold unramified cover of another Riemann surface Γ of genus 3; the latter is a double ramified cover of an elliptic curve Γ_0 . The Abelian surface \widetilde{M}_c can also be identified as the Prym variety $\text{Prym}_{\sigma}(\Gamma)$ and the problem linearizes on $\text{Prym}_{\sigma}(\Gamma)$. From the fundamental exponential sequence*

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_{\widetilde{M}_c} \xrightarrow{\text{exp.}} \mathcal{O}_{\widetilde{M}_c}^* \rightarrow 0,$$

we get the map

$$\dots \rightarrow H^1(\widetilde{M}_c, \mathcal{O}_{\widetilde{M}_c}^*) \rightarrow H^2(\widetilde{M}_c, \mathbb{Z}) \rightarrow \dots$$

i.e., the first Chern class of a line bundle on \widetilde{M}_c . Recall that any line bundle with Chern class zero can be realized by constant multipliers. Therefore the group $\text{Pic}^0(\widetilde{M}_c)$ of holomorphic line bundles on \widetilde{M}_c with Chern class zero is given by

$$\text{Pic}^0(\widetilde{M}_c) = H^1(\widetilde{M}_c, \mathcal{O}_{\widetilde{M}_c}^*) / H^1(\widetilde{M}_c, \mathbb{Z}),$$

and is naturally isomorphic to the dual Abelian surface \widetilde{M}_c^{\vee} of \widetilde{M}_c (\vee means the dual Abelian surface). The relationship between \widetilde{M}_c and \widetilde{M}_c^{\vee} is symmetric like the relationship between two vectors spaces set up a bilinear pairing. It is interesting

to observe that the Abelian surfaces $\widetilde{M}_c = \text{Prym}_\sigma(\Gamma)$ obtained from the asymptotic analysis of the differential equations and $\text{Prym}_\sigma(\mathcal{C})$ obtained from the orbits in the Kac-Moody Lie algebra are not identical but only isogenous, i.e., one can be obtained from the other by doubling some periods and leaving other unchanged. The precise relation between these two Abelian surfaces is $\widetilde{M}_c = (\text{Prym}_\sigma(\mathcal{C}))^\vee$, i.e., they are dual of each other. The functions x_1, \dots, x_6 are themselves meromorphic on \widetilde{M}_c , while only their squares are on $\text{Prym}_\sigma(\mathcal{C})$. The relationship between the Riemann surfaces Γ and \mathcal{C} is quite intricate. As usual we let Θ the theta divisor on $\text{Jac}(\Gamma)$, we have

$$\text{Prym}_\sigma(\mathcal{C}) \setminus \Pi = \Theta \cap \text{Prym}_\sigma(\mathcal{C}) = \Gamma,$$

with Π a Zariski open set of $\text{Prym}_\sigma(\mathcal{C})$. Also

$$\Theta \cap \widetilde{M}_c = \mathcal{C},$$

where Θ is a translate of the theta divisor of $\text{Jac}(\mathcal{C})$ invariant under the involution σ . Moser [31] was aware of a similar situation in the context of the Jacobi's geodesic flow problem on ellipsoids.

5.2 Manakov geodesic flow on the group $SO(4)$ and Lyapunov-Steklov rigid body motion in a perfect fluid

The quadratic form H satisfies the conditions

$$(\mu_1^2, \mu_2^2, \mu_3^2) = \frac{\lambda_{12}\lambda_{23}\lambda_{31}\lambda_{45}\lambda_{56}\lambda_{64}}{(\lambda_{46}\lambda_{32} - \lambda_{65}\lambda_{13})^2} \left(\frac{(\lambda_{23} - \lambda_{56})^2}{\lambda_{23}\lambda_{56}}, \frac{(\lambda_{31} - \lambda_{64})^2}{\lambda_{31}\lambda_{64}}, \frac{(\lambda_{12} - \lambda_{45})^2}{\lambda_{12}\lambda_{45}} \right),$$

with the product $\mu_1\mu_2\mu_3$ being rational in $\lambda_1, \dots, \lambda_6$ and with the following sign specification

$$\mu_1\mu_2\mu_3 = \frac{\lambda_{12}\lambda_{23}\lambda_{31}\lambda_{45}\lambda_{56}\lambda_{64}}{(\lambda_{46}\lambda_{32} - \lambda_{65}\lambda_{13})^3} (\lambda_{12} - \lambda_{45})(\lambda_{23} - \lambda_{56})(\lambda_{31} - \lambda_{64}).$$

The problem of the solid body in a fluid in the case of Lyapunov-Steklov is a particular case of this metric (see subsection 4.2). The extra invariant H_4 is quadratic and the flow linearizes on 2-dimensional hyperelliptic Jacobians. More precisely

$$\bigcap_{j=1}^4 \{x \in \mathbb{C}^6 : Q_j(x) = c_j\} = \text{Jac}(\text{hyperelliptic curve } \mathcal{C} \text{ of genus } 2) \setminus \mathcal{D},$$

where \mathcal{D} is a divisor of genus 17, which contains 4 translates of the Θ -divisor in $\text{Jac}(\mathcal{C})$, each of which is isomorphic to \mathcal{C} . The hyperelliptic curve \mathcal{C} is a double cover of the curve \mathcal{C}_0 (isomorphic to $\mathbb{P}^1(\mathbb{C})$) defined as

$$\{t_1 : t_2 : t_3 : t_4\} \in \mathbb{P}^3(\mathbb{C}) \text{ such that } \sum t_j Q_j \text{ has rank } 3\}.$$

The periods of the motion are given by the periods of the hyperelliptic curve \mathcal{C} . When studying the differential systems in this case as well as the invariants via Kowalewski-Painlevé's analysis, it is advantageous to rewrite them in a simpler form in order to reduce the notations and thus avoid too much calculation, since it requires less variables (see [6] for more detail). We show that this geodesic flow X_1 and a commuting flow X_2 can be written respectively in the form

$$\begin{aligned} \dot{x}_1 &= x_2 x_6, \\ \dot{x}_2 &= \frac{1}{2} x_3 (x_1 + x_4), \\ \dot{x}_3 &= \frac{1}{2} x_2 (x_1 + x_4), \\ \dot{x}_4 &= x_3 x_5, \\ \dot{x}_5 &= x_3 x_4, \\ \dot{x}_6 &= x_1 x_2, \end{aligned}$$

and

$$\begin{aligned} \dot{x}_1 &= x_5 x_6, \\ \dot{x}_2 &= x_3 x_4, \\ \dot{x}_3 &= x_2 x_4, \\ \dot{x}_4 &= x_5 (2x_3 - x_6), \\ \dot{x}_5 &= x_4 (2x_3 - x_6), \\ \dot{x}_6 &= x_1 x_5. \end{aligned}$$

The Hamiltonian structure being determined by the Poisson bracket :

$$\{H, F\} = \left\langle \frac{\partial H}{\partial x}, J \frac{\partial F}{\partial x} \right\rangle = \sum_{i,j} J_{ij} \frac{\partial H}{\partial x_i} \frac{\partial F}{\partial x_j},$$

where

$$J = \begin{pmatrix} 0 & x_3 & x_2 & 0 & 0 & 2x_2 - x_5 \\ -x_3 & 0 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 0 & x_4 \\ -2x_2 + x_5 & 0 & 0 & -x_5 & -x_4 & 0 \end{pmatrix}.$$

The problem is to show, among other things, that the geodesic flow on $SO(4)$ for a left invariant in the case of the second metric is a weight homogeneous algebraic complete integrable system. Using the asymptotic expansion method as in the previous case (Kowalewski-Painlevé's analysis), we obtain the following results [6] :

Theorem 5.3. *This geodesic flow has four quadric invariants :*

$$\begin{aligned} H_1 &= -x_4^2 + x_5^2 = c_1, \\ H_2 &= -x_1^2 + x_6^2 = c_2, \\ H_3 &= -x_2^2 - x_3^2 = \frac{c_3}{4}, \\ H_4 &= -(x_1 - x_4)^2 + 2(x_2 - x_5)^2 + 2(x_3 - x_6)^2 = 4c_4, \end{aligned}$$

(with generic $(c_1, c_2, c_3, c_4) \in \mathbb{C}^4$), and it evolves on some hyperelliptic Jacobians. The hyperelliptic curve is a double cover of the curve of rank four quadrics (isomorphic to \mathbb{P}^1) :

$$\left\{ t \in \mathbb{P}^3 \text{ such that } t_1(H_1 - c_1 x_0^2) + t_2(H_2 - c_2 x_0^2) + t_3(H_3 - \frac{c_3}{4} x_0^2) + t_4(H_4 - 4c_4 x_0^2) \text{ has rank } 4 \right\},$$

ramified at the six points where the rank drops to 3. The system in question possesses Laurent solutions depending on 5 free parameters and the affine surface defined by the constants of motion can be completed into a torus T by adjoining a singular divisor \mathcal{D} . The latter consists of four copies $\mathcal{H}_1, \dots, \mathcal{H}_4$ of the genus two hyperelliptic curves. Analyze the points of intersection of these curves. All these curves are translates of the Θ -divisor by $\frac{1}{2}$ -periods and three of these curves form a very ample and projectively normal divisor which results in the embedding of the Jacobian in \mathbb{P}^8 and the functions having poles there form a closed system of quadratic equations under differentiations as well as their ratios. The line bundle $[\mathcal{D}]$ defines a polarization of type $(4, 4)$ on T and leads to an embedding in \mathbb{P}^{15} . The three flows X_1, X_2 and $2X_1 - X_2$ are doubly tangent to each of the four curves $\mathcal{H}_1, \dots, \mathcal{H}_4$ at four points P_1, \dots, P_4 and the sixteen half-periods on the torus are given by the total set of branch points of these hyperelliptic curves. The values of the constants of motion $c_1/c_4, c_2/c_4, c_3/c_4$ provide the three moduli for the full family of 2-dimensional hyperelliptic Jacobians.

5.3 Geodesic flow on $SO(4)$ with a quartic invariant

The form H satisfies

$$(\mu_1^4, \mu_2^4, \mu_3^4) = \lambda_{13} \lambda_{46} \lambda_{21} \lambda_{54} \lambda_{32} \lambda_{65} \left(\frac{1}{\lambda_{32} \lambda_{65}}, \frac{1}{\lambda_{13} \lambda_{46}}, \frac{1}{\lambda_{21} \lambda_{54}} \right).$$

The quantities ζ, ξ and η defined by

$$\zeta^2 \equiv \frac{\lambda_{46}}{\lambda_{13}}, \quad \xi^2 \equiv \frac{\lambda_{54}}{\lambda_{21}}, \quad \eta^2 \equiv \frac{\lambda_{65}}{\lambda_{32}},$$

satisfy the quadratic relations

$$\zeta \xi + \xi \eta + \eta \zeta + 1 = 0, \quad 3\xi \eta + \eta - \xi + 1 = 0.$$

The geodesic flow has a quartic invariant, evolves on Abelian surfaces $\tilde{A} \subseteq \mathbb{P}^{23}(\mathbb{C})$ having period matrix

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 12 & c & b \end{pmatrix}, \quad \text{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0, \quad (a, b, c \in \mathbb{C}),$$

and it will be expressed in terms of Abelian integrals. See for details [3, 6]. After the following linear change of coordinates, which is meaningful insofar $a \neq 0, -1, 1, -1/3, 1/3$,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} &= \sqrt{-1} \begin{pmatrix} a-1 & -1 \\ 3a+1 & 1 \end{pmatrix} \begin{pmatrix} (a-1)z_1 \\ (3a-1)(a+1)z_4 \end{pmatrix}, \\ \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} &= -\sqrt{-1} \begin{pmatrix} a+1 & -1 \\ 3a-1 & 1 \end{pmatrix} \begin{pmatrix} (a+1)z_2 \\ (3a+1)(a-1)z_5 \end{pmatrix}, \\ \begin{pmatrix} x_3 \\ x_6 \end{pmatrix} &= \sqrt{-1} \begin{pmatrix} a-1 & a+1 \\ 3a+1 & 3a-1 \end{pmatrix} \begin{pmatrix} (a-1)z_3 \\ (a+1)z_6 \end{pmatrix}, \end{aligned}$$

the geodesic flow takes (after rescaling time) on the simple form :

$$\begin{aligned} \dot{z}_1 &= z_3 z_5, \\ \dot{z}_2 &= z_4 z_6, \\ \dot{z}_3 &= \frac{1-a}{2} z_4 z_5 + z_1 z_5 + \frac{1+a}{2} z_1 z_2, \\ \dot{z}_4 &= \frac{2a}{3a-1} z_5 z_6 + \frac{a-1}{3a-1} z_2 z_3, \\ \dot{z}_5 &= \frac{2a}{3a+1} z_5 z_6 + \frac{a+1}{3a+1} z_2 z_3, \\ \dot{z}_6 &= \frac{1+a}{2} z_4 z_5 + z_2 z_4 + \frac{1-a}{2} z_1 z_2, \end{aligned} \tag{5.14}$$

with three quadratic invariants (in z) :

$$\begin{aligned} H_1 &\equiv aF_2 + \frac{1-a}{3a+1} F_7 = A_1, \\ H_2 &\equiv -aF_1 - \frac{a+1}{3a-1} = A_2, \\ H_3 &\equiv \frac{2F_6}{(3a-1)(3a+1)} - \frac{F_1}{3a+1} + \frac{F_2}{3a-1} = A_3, \end{aligned}$$

and a quartics invariant (in z) :

$$\begin{aligned} H_4 &\equiv -\frac{1-a}{3a+1} (F_1^2 + F_4^2) + \frac{1+a}{3a-1} (F_2^2 + F_5^2) + \frac{3(1-a^2)}{(3a-1)(3a+1)} (2F_1 F_2 - F_3^2) \\ &\quad + \frac{4(1+a)}{(3a-1)(3a+1)} F_2 (F_6 + F_8) + \frac{4(1-a)}{(3a-1)(3a+1)} F_1 (F_6 + F_7) = A_4, \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} F_1 &= z_4^2 - z_2 z_5, F_2 = z_5^2 - z_1 z_4, F_3 = z_1 z_2 - z_4 z_5, F_4 = \frac{2}{3a-1} (z_2 z_3 - z_5 z_6), \\ F_5 &= \frac{-2}{3a+1} (z_1 z_6 - z_3 z_4), F_6 = z_1 z_4 + z_2 z_5 - z_3 z_6, F_7 = z_1^2 - z_3^2 + z_1 z_4, F_8 = z_2^2 - z_6^2 + z_2 z_5. \end{aligned}$$

The geodesic flow in question admits one family of Laurent solutions,

$$z = \frac{\zeta}{t} \left(\mathbf{1} + UY^1 t + \frac{1}{\gamma Z^2 + \delta} \left(U^2 Y_0^2 + \sum_{i=1}^3 A_i Y_i^2 \right) t^2 + o(t^3) \right),$$

where $\mathbf{1} = (1, 1, \dots, 1)^\top$, Y^1, Y_0^2, Y_i^2 are appropriate vectors depending on Y, Z and a only, $\gamma \equiv 4a$, $\delta \equiv (a-1)(3a+1)$ and

$$\zeta = \text{diag} \left(\frac{Y^2}{Z}, \frac{Z^2}{Y}, -\frac{Y}{Z}, Z, Y, -\frac{Z}{Y} \right),$$

with $Y, Z \in \mathbb{C}$ such that $Y^2 + Z^2 = 1$. The 5-dimensional family of Laurent solutions depend on the parameters Z, U, A_1, A_2 and A_3 . The vectors Y_i^2 can be chosen such that $H_i(z(t)) = A_i$ for $i = 1, 2, 3$. Confining the 5-dimensional family of Laurent solutions to the invariant manifold

$$M_c = \bigcap_{i=1}^4 \{z : H_i(z) = A_i\},$$

yields a relation between the free parameters, defining a curve

$$\mathcal{D} : \begin{cases} (U, V, Y, Z) \text{ such that } Z^2 = V, Y^2 = 1 - V \text{ and} \\ P(U, V) = (U^2(1 - V)V(\alpha V + \beta))^2 - 2U^2(1 - V)VP(V) + Q(V) = 0, \end{cases}$$

where

$$\alpha = 16a^3, \beta = (a - 1)^3(3a + 1),$$

and

$$\begin{aligned} P(V) &= (\alpha V + \beta) [((3a^2 + 1)A_3 - A_1 - A_2)(V - 1) + A_1V - A_2(V - 1)] \\ &\quad - 2V(V - 1) [A_1(1 - a)^3(1 + 3a) + A_2(1 + a^3(1 - 3a) - A_3(1 - a^2)(1 - 9a^2))], \\ Q(V) &= [((3a^2 + 1)A_3 - A_1 - A_2)V(V - 1) - A_1V + A_2(V - 1)]^2 \\ &\quad + V(V - 1) [(4aV + (a - 1)(3a + 1))A_4 + 4A_1A_2 - (a - 1)(3a + 1)(a + 1)(3a - 1)A_3^2]. \end{aligned}$$

Note that

$$P^2(V) - (\alpha V + \beta)^2 Q(V) = V(1 - V)R(V), \quad (5.16)$$

with $R(V)$ being a cubic polynomial. The curve \mathcal{D} is an unramified 4 - 1 cover of the curve

$$\mathcal{C} : P(U, V) = 0.$$

In view of (5.16), the curve \mathcal{C} itself is a double cover of the hyperelliptic curve

$$\mathcal{H} : W^2 = V(1 - V)R(V),$$

of genus 2, ramified over four points where $Q(V) = 0$. Therefore \mathcal{C} has genus 5 and \mathcal{D} has genus 17. The curve \mathcal{D} must be thought of as being a very ample divisor on some Abelian surface \widetilde{M}_c , to be constructed according to the method described in the theorem 2.3. The curve \mathcal{D} , wrapped around \widetilde{M}_c intersects itself transversally in 8 points, adding 8 to the genus 17. Therefore the torus $\widetilde{M}_c \simeq \mathbb{C}^2/L_\Omega$ on which the geodesic flow linearizes, is defined by a period lattice Ω given by the columns of the matrix

$$\Omega = \begin{pmatrix} \delta_1 & 0 & a & c \\ 0 & \delta_2 & c & b \end{pmatrix}, \text{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0,$$

with

$$\delta_1 \delta_2 = g(\mathcal{D}) - 1 = 24, \quad \delta_1 | \delta_2, \quad \delta_i \in \mathbb{N}^*.$$

So we have the following two possibilities : (i) $\delta_1 = 1, \delta_2 = 24$ and (ii) $\delta_1 = 2, \delta_2 = 12$. The line bundle

$$L(\mathcal{D}) = \{1, z_1, \dots, z_6, f_1, \dots, f_5, g_1, \dots, g_8, h_1, \dots, h_4\},$$

is specified as follows

$$\begin{aligned} g_1 &= -2az_2f_2 - (1 - a)z_4f_3, & g_2 &= -2az_1f_1 - (1 + a)z_5f_3, & g_3 &= (1 - a)z_5f_4 + (1 + a)z_4f_5, \\ g_4 &= (1 + a)z_5f_5 + (1 - a)z_1f_4, & g_5 &= (1 - a)z_4f_4 + (1 + a)z_2f_5, & g_6 &= -(1 - a)z_3f_1 - (1 + a)z_6f_2, \\ g_7 &= 2az_5f_2 - (1 - a)z_1f_3, & g_8 &= -2az_4f_1 - (1 + a)z_2f_3, & h_1 &= 4a^2f_1f_2 + (1 - a^2)f_3, \\ h_2 &= 2a^2f_1f_5 - (1 - a)f_3f_4, & h_3 &= -2a^2f_2f_4 - (1 + a)f_3f_5, & h_4 &= 2a^2f_4f_5 + f_3((1 + a)f_2 - (1 - a)f_1). \end{aligned}$$

Now the reflection about the origin on the Abelian surface amounts to flipping the time for each linear flow on it, but since the flow $\frac{dz}{dt_1}$ given by (5.14) is quadratic and since the other flow $\frac{dz}{dt_2}$ (commuting with the first) is quartic (as it derives from the quartic Hamiltonian (5.15)), flipping the signs of t_1 and t_2 for each of the flows amounts to the flip $(z_1, \dots, z_6) \mapsto (-z_1, \dots, -z_6)$.

From formula (5.10), the above line bundle $L(\mathcal{D})$ has in case (i), 11 even sections, 13 odd ones and in case (ii), 10 even sections, 14 odd ones, showing that case (ii) is the only alternative and the period matrix has the form

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 12 & c & b \end{pmatrix}, \operatorname{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0.$$

Differentiating $\frac{1}{z_1}$ and $\frac{z_2}{z_1}$ with respect to t_1 (corresponding to the flow (5.14)) and t_2 (corresponding to the quartic flow generated by H_4 (5.15)), yield two differentials ω_1 and ω_2 defined on the curve \mathcal{C} :

$$\omega_1 = \frac{\varphi(V)dV}{U\sqrt{V(1-V)R(V)}} \omega_2 = \frac{dV}{U\sqrt{V(1-V)R(V)}},$$

where $\varphi(V)$ is a rational function in V having the form

$$\varphi(V) = \frac{4aV + (a-1)(3a+1)}{V(1-V)} [(\alpha V + \beta)U^2V(1-V) + (A_3(3a^2+1) - A_1 - A_2)(1-V)V - A_1V - A_2(1-V)].$$

The restriction of the differentials dt_1 and dt_2 to the curve \mathcal{D} are

$$\omega_1 = dt_1|_{\mathcal{D}} = \varphi(Z^2)\omega_2, \omega_2 = dt_2|_{\mathcal{D}} = \frac{dZ}{UY\sqrt{R}}.$$

Recall that \mathcal{C} is a double ramified cover of a hyperelliptic curve \mathcal{H} of genus 2, whose sheets are interchanged by the involution $(V, U) \mapsto (V, -U)$. Hence

$$\operatorname{Jac}(\mathcal{C}) = \operatorname{Prym}(\mathcal{C}/\mathcal{H}) \oplus \operatorname{Jac}(\mathcal{H}).$$

Since ω_1 and ω_2 are both odd differentials for that involution, the flows evolve on the 3-dimensional $\operatorname{Prym}(\mathcal{C}/\mathcal{H})$ and therefore $\widetilde{M}_c \subset \operatorname{Prym}(\mathcal{C}/\mathcal{H})$. This shows that $\operatorname{Prym}(\mathcal{C}/\mathcal{H})$ splits further, up to isogenies, into an elliptic curve \mathcal{E} and the 2-dimensional invariant torus \widetilde{M}_c :

$$\widetilde{M}_c \oplus \mathcal{E} = \operatorname{Prym}(\mathcal{C}/\mathcal{H}).$$

In summary, we have

Theorem 5.4. *The affine invariant surface M_c for the Adler-van Moerbeke geodesic flow completes into a generic Abelian surface \widetilde{M}_c of polarization (1,6), i.e., defined by a period matrix of the form*

$$\begin{pmatrix} 1 & 0 & a & c \\ 0 & 6 & c & b \end{pmatrix}, \operatorname{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0,$$

by adjoining at infinity a curve of genus 25, with 8 normal crossings and smooth version \mathcal{D} . There exists an elliptic curve \mathcal{E} such that M_c satisfies $\widetilde{M}_c \oplus \mathcal{E} = \operatorname{Prym}(\mathcal{C}/\mathcal{H})$. More precisely

$$\bigcap_{j=1}^4 \{x \in \mathbb{C}^6 : H_j(x) = c_j\} = \widetilde{M}_c \setminus \{a \text{ curve of genus 25 with 8 singular points}\}.$$

Put in a more geometrical language, the tori \widetilde{M}_c contain a very ample and projectively normal curve of geometric genus 25, with 8 normal crossings whose smooth version \mathcal{D} is a $4-1$ unramified cover of a curve \mathcal{C} of genus 5. The curve \mathcal{C} itself is a double cover ramified over 4 points of a genus 2 hyperelliptic curve \mathcal{H} . Moreover, the linearization takes place on a 2-dimensional subtorus of the 3-dimensional Prym variety $\operatorname{Prym}(\mathcal{C}/\mathcal{H})$ with

$$\operatorname{Prym}(\mathcal{C}/\mathcal{H}) = \widetilde{A} \oplus \mathcal{E},$$

where \mathcal{E} is an elliptic curve.

This situation provides a full description of the moduli for the Abelian surfaces of polarization (1,6). For more information, see [6].

6. The algebraic complete integrability of geodesic flow on $SO(n)$, $n \geq 5$

We have seen previously that if a system is algebraically completely integrable, then it must admit a family of meromorphic Laurent series depending on "dim(phase space) – 1" free parameters. Now, trying to generalize the result to the geodesic flow on $SO(n)$ for $n \geq 5$ using the same method leads to insurmountable calculations even with the help of a computer. Therefore, to answer the question concerning the classification of the metrics for which the geodesic flow on $SO(n)$ while taking into account these difficulties, it was necessary to resonate differently.

The geodesic flow for this problem takes the following commutator form (Euler-Arnold equations) :

$$\dot{X} = [X, \Lambda.X], \quad (6.1)$$

where

$$X = (x_{ij}) \equiv \sum_{i < j} x_{ij} e_{ij} \in so(n),$$

and $\Lambda.X = (\lambda_{ij} X_{ij})$, $\lambda_{ij} = \lambda_{ji}$. The quadratic form H is diagonal with regard to the customary $so(n)$ coordinates (Manakov metric [29]), with

$$\Lambda_{jk} = \frac{\beta_j - \beta_k}{\alpha_j - \alpha_k} \iff [X, \beta] + [\alpha, \Lambda.X] = 0, \forall X \in so(n), \quad (6.2)$$

with

$$\alpha = \text{diag}(\alpha_1, \dots, \alpha_n), \quad \beta = \text{diag}(\beta_1, \dots, \beta_n), \quad \prod_{i < j} (\alpha_i - \beta_j) \neq 0,$$

all Λ_{jk} distinct. Note that it turns out that the geodesic flow on $SO(n)$ admits a lot of invariant manifolds on which they reduce to geodesic flow on $SO(3)$ and the solutions of the differential equation with initial conditions on these manifolds are elliptic functions and this without any condition on the metric. Haine [14] has shown that looking at solutions near these special a priori known solutions and imposing these solutions to be single-valued functions of $t \in \mathbb{C}$, suffices to single out the left invariant diagonal metrics for which the geodesic flow is algebraically completely integrable. Under the non-degeneracy assumption on the diagonal metric Λ that all Λ_{ij} be distinct, the system (6.1) is algebraically completely integrable with Abelian functions x_{ij} if and only if the metric Λ satisfies (6.2) (Manakov's conditions).

Theorem 6.1. *For $n \geq 5$, Manakov's metrics are the only left invariant diagonal metrics on $SO(n)$ for which the geodesic flow is algebraically completely integrable.*

This criterion was first used, without proof by Lyapunov [28] (the proof is due to Haine [14]), who showed that the only integrable tops whose solutions have analytic properties belong to the classical known cases : Euler top, Lagrange top and Kowalewski top.

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Lower Bounds for the Blow up Time to a Coupled Nonlinear Hyperbolic Type Equations

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Abstract

The initial and Dirichlet boundary value problem of nonlinear hyperbolic type equations in a bounded domain is studied. We established a lower bounds for the blow up time.

Keywords: Hyperbolic type equations, Lower bounds, Nonlinear damping term.

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1. Introduction

In this paper, we deal with the lower bounds of the blow up time of solutions of the following hyperbolic type equations

$$\begin{cases} u_{tt} - \operatorname{div} \left(\rho \left(|\nabla u|^2 \right) \nabla u \right) - \Delta u_{tt} + |u_t|^{m-1} u_t = f_1(u, v), & (x, t) \in \Omega \times (0, T), \\ v_{tt} - \operatorname{div} \left(\rho \left(|\nabla v|^2 \right) \nabla v \right) - \Delta v_{tt} + |v_t|^{r-1} v_t = f_2(u, v), & (x, t) \in \Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega, \\ u(x, t) = v(x, t) = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ ($n = 1, 2, 3$) is a bounded domain with a sufficiently smooth boundary $\partial\Omega$; $m, r \geq 1$ are constants, and $f_i(u, v) : \mathbb{R}^2 \rightarrow \mathbb{R}$ ($i = 1, 2$) are functions which will be specified later. Also,

$$\rho(s) = b_1 + b_2 s^q, \quad q, b_1, b_2 \geq 0.$$

In the absence of the dispersion terms (Δu_{tt} and Δv_{tt}), eq. (1.1) reduces to the following system

$$\begin{cases} u_{tt} - \operatorname{div} \left(\rho \left(|\nabla u|^2 \right) \nabla u \right) - \Delta u_{tt} + |u_t|^{m-1} u_t = f_1(u, v), \\ v_{tt} - \operatorname{div} \left(\rho \left(|\nabla v|^2 \right) \nabla v \right) - \Delta v_{tt} + |v_t|^{r-1} v_t = f_2(u, v). \end{cases} \quad (1.2)$$

In [1], Wu et al. considered the global existence and the blow up of the solution of the problem (1.2). Later, Fei and Hongjun [2] improved the blow up result in [1]. Finally, in [3], Pişkin and Polat studied the existence, the decay and the blow up of the solutions for the problem (1.2).

The aim of this paper note is to derive a lower bound for the blow up time occurs. Before stating our main theorem, we give some notations, lemmas and theorems.

2. Preliminaries

In this paper, we denote $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$. Moreover, c_i ($i = 1, 2, \dots$) are arbitrary constants.

Let

$$f_1(u, v) = (p+1) \left[a|u+v|^{p-1}(u+v) + b|uv|^{\frac{p-1}{2}}v \right],$$

and

$$f_2(u, v) = (p+1) \left[a|u+v|^{p-1}(u+v) + b|uv|^{\frac{p-1}{2}}u \right],$$

where $a, b > 0$ are constant and p satisfies

$$\begin{cases} 1 < p & \text{if } n \leq 2, \\ 1 < p \leq \frac{n}{n-2} & \text{if } n > 2. \end{cases} \quad (2.1)$$

By a simple calculation, we have

$$uf_1(u, v) + vf_2(u, v) = (p+1)F(u, v), \quad (u, v) \in \mathbb{R}^2, \quad (2.2)$$

where

$$F(u, v) = \left[a|u+v|^{p+1} + 2b|uv|^{\frac{p+1}{2}} \right]. \quad (2.3)$$

We define

$$J(t) = \frac{1}{2} \left[b_1 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right) \right] + \frac{1}{2q+2} \left[b_2 \left(\|\nabla u\|_{2q+2}^{2q+2} + \|\nabla v\|_{2q+2}^{2q+2} \right) \right] - \int_{\Omega} F(u, v) dx, \quad (2.4)$$

and

$$I(t) = \left[b_1 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right) \right] + \left[b_2 \left(\|\nabla u\|_{2q+2}^{2q+2} + \|\nabla v\|_{2q+2}^{2q+2} \right) \right] - (p+1) \int_{\Omega} F(u, v) dx. \quad (2.5)$$

We also define the energy functional as follows

$$\begin{aligned} E(t) &= \frac{1}{2} \left(\|u_t\|^2 + \|v_t\|^2 \right) + \frac{1}{2} \left[b_1 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right) \right] \\ &\quad + \frac{1}{2q+2} \left[b_2 \left(\|\nabla u\|_{2q+2}^{2q+2} + \|\nabla v\|_{2q+2}^{2q+2} \right) \right] + \frac{1}{2} \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) - \int_{\Omega} F(u, v) dx. \end{aligned} \quad (2.6)$$

We also define

$$W_- = \left\{ (u, v) : (u, v) \in W_0^{1,2q+2}(\Omega) \times W_0^{1,2q+2}(\Omega), I(u, v) < 0 \right\}. \quad (2.7)$$

The next lemma shows that our energy functional (2.6) is a nonincreasing function along the solution of (1.1).

Lemma 2.1. *Energy functional is a nonincreasing function for $t \geq 0$ and*

$$E'(t) = - \left(\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} \right) \leq 0. \quad (2.8)$$

Proof. Multiplying the first equation in (1.1) by u_t and the second equation by v_t , integrating over Ω . Then integrating by parts, we get

$$E(t) - E(0) = - \int_0^t \left(\|u_{\tau}\|_{m+1}^{m+1} + \|v_{\tau}\|_{r+1}^{r+1} \right) d\tau \text{ for } t \geq 0 \quad (2.9)$$

Lemma 2.2. (Sobolev-Poincare inequality) [4]. Let

$$\begin{cases} 2 \leq p < \infty; & n = 1, 2, \\ 2 \leq p \leq \frac{2n}{n-2}; & n \geq 3 \end{cases}$$

then there is a constant $C_* = C_*(\Omega, p)$ such that

$$\|u\|_p \leq C_* \|\nabla u\|, \quad \forall u \in H_0^1(\Omega).$$

□

Lemma 2.3. [5, 6] There exist two positive constants c_1 and c_2 such that

$$\int_{\Omega} |f_1(u, v)|^2 dx \leq c_1 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^p$$

and

$$\int_{\Omega} |f_2(u, v)|^2 dx \leq c_2 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^p$$

are satisfied.

The local existence theorem which can be established combining the arguments of [3].

Theorem 2.4. (Existence-uniqueness). Assume that (2.1) holds. Then further that $u_0, v_0 \in W_0^{1,2q+2}(\Omega) \cap L^{p+1}(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$. Then the system (1.1) has a unique local solution

$$u, v \in C\left([0, T]; W_0^{1,2q+2}(\Omega) \cap L^{p+1}(\Omega)\right).$$

Theorem 2.5. [7]. Suppose that $r > \max\{p, q\}$ and $E(0) < 0$ hold. Then the solution u of the system blows up in finite time T^* .

3. Lower bound for blow up time

In this section, our aim is to determine a lower bound for blow up time of the system (1.1).

Theorem 3.1. Let $u_0, v_0 \in W_0^{1,2q+2}(\Omega) \cap L^{p+1}(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, $(u_0, v_0) \in W_-$, and $1 < p, q < r$. Assume that (2.1) holds. Then the solutions u of the problem (1.1) become unbounded at finite time T^* . Also, the lower bounds for the blow up time is given by

$$\int_{\psi(0)}^{\infty} \frac{d\psi(z)}{\psi(\tau) + E(0) + \frac{c_1+c_2}{2}(p+1)^p \psi^p(\tau)} \leq T^*.$$

Proof. We define

$$\psi(t) = \int_{\Omega} F(u, v) dx \tag{3.1}$$

By taking a derivative of (3.1), we get

$$\psi'(t) = \int_{\Omega} (u_t F_u + v_t F_v) dx \tag{3.2}$$

Thanks to Young's inequality, we have

$$\psi'(t) \leq \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) dx + \frac{1}{2} \int_{\Omega} (F_u^2 + F_v^2) dx.$$

By the Lemma 3, we get

$$\psi'(t) \leq \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) dx + \frac{c_1+c_2}{2} \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^p \tag{3.3}$$

Since $I(t) < 0$, we have

$$b_1 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right) + b_2 \left(\|\nabla u\|_{2q+2}^{2q+2} + \|\nabla v\|_{2q+2}^{2q+2} \right) \leq (p+1) \int_{\Omega} F(u, v) dx. \tag{3.4}$$

Inserting (3.4) into (3.3), we have

$$\begin{aligned} \psi'(t) &\leq \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) dx + \frac{c_1 + c_2}{2} \left((p+1) \int_{\Omega} F(u, v) dx \right)^p \\ &= \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) dx + \frac{c_1 + c_2}{2} (p+1)^p \left(\int_{\Omega} F(u, v) dx \right)^p \\ &= \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) dx + \frac{c_1 + c_2}{2} (p+1)^p \psi^p(t) \end{aligned} \tag{3.5}$$

By the definition $E(t)$, we get

$$\begin{aligned} &(q+1) \left(\|u_t\|^2 + \|v_t\|^2 \right) + (q+1)b_1 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right) \\ &+ b_2 \left(\|\nabla u\|_{2q+2}^{2q+2} + \|\nabla v\|_{2q+2}^{2q+2} \right) + (q+1) \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) \\ &= (2q+2)E(t) + (2q+2) \int_{\Omega} F(u, v) dx \\ &\leq (2q+2)E(0) + (2q+2)\psi(t) \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\psi'(t) \leq \psi(t) + E(0) + \frac{c_1 + c_2}{2} (p+1)^p \psi^p(t). \tag{3.7}$$

Applying Theorem 5, we have

$$\lim_{t \rightarrow T^*} \int_{\Omega} F(u, v) dx = \infty \tag{3.8}$$

According to (3.7), (3.8), we have

$$\int_{\psi(0)}^{\infty} \frac{d\psi(z)}{\psi(\tau) + E(0) + \frac{c_1 + c_2}{2} (p+1)^p \psi^p(\tau)} \leq T^*.$$

This completes the proof. □

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Semi-Invariant Submanifolds of Almost α -Cosymplectic f -Manifolds

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Abstract

In this paper, we have and study several properties of semi-invariant submanifolds of an almost α -cosymplectic f -manifold. We give an example and investigate the integrability conditions for the distributions involved in the definition of a semi-invariant submanifold of an almost α -cosymplectic f -manifold.

Keywords: Almost α -cosymplectic f -manifolds, Semi-invariant submanifolds, Integrability conditions

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1. Introduction

Contact geometry has been seen to underly many physical phenomena and be related to many other mathematical structures. Contact structures first appeared in the work of Sophus Lie [1] on partial differential equations. They reappeared in Gibbs' work on thermodynamics, Huygens' work on geometric optics and in Hamiltonian dynamics. ([2], [3], [4]).

On the other hand, the notion of CR -submanifold of a Kaehler manifold was introduced by Bejancu [5]. Later, semi-invariant (or contact CR -) submanifolds of a Sasakian manifold was studied by Shahid, Sharfuddin and Husain [6], Kobayashi [7], Matsumoto [8] and many others. Submanifolds of cosymplectic manifold have been studied by Ludden [9], A. Cabras, A. Ianus and G.H. Pitis [10].

Later, the subject was considered for Riemannian manifolds with an almost contact structure. In this sense A. Bejancu and N. Papaghiuc study semi-invariant submanifolds of a Sasakian manifold or Sasakian space form ([11],[12], [13], [14]) and C.L. Bejan, A., et.al. study them on cosymplectic manifolds in ([15], [16]). B. B. Sinha and R. N. Yadav studied the integrable conditions of distributions and the geometry of leaves on a semi-invariant submanifolds in a Kenmotsu manifold [17].

In 2014, Öztürk et.al. introduced and studied almost α -cosymplectic f -manifold [18] defined for any real number α which is defined a metric f -manifold with f -structure $(\varphi, \xi_i, \eta^i, g)$ satisfying the condition $d\eta^i = 0, d\Omega = 2\alpha\bar{\eta} \wedge \Omega$.

In this paper, we introduce properties of semi-invariant submanifolds of an almost α -cosymplectic f -manifold. In Section 2, we review basic formulas and definitions for almost α -cosymplectic f -manifolds. In Section 3, we define semi-invariant submanifolds of an almost α -cosymplectic f -manifold. We also present a way to build these submanifolds and give an example. In Section 4, we obtain some basic results for semi-invariant submanifolds of an almost α -cosymplectic f -manifold. In Section 5, we investigate the integrability of the distributions involved in the definition of a semi-invariant submanifold. In last section we focus mixed totally geodesic of semi-invariant submanifolds of an almost α -cosymplectic f -manifold.

2. Preliminaries

Let \tilde{M} be a real $(2n+s)$ -dimensional framed metric manifold [19] with a framed $(\varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$, that is, φ is a non-vanishing tensor field of type $(1,1)$ on \tilde{M} which satisfies $\varphi^3 + \varphi = 0$ and has constant rank $r = 2n$; ξ_1, \dots, ξ_s are s vector fields; η^1, \dots, η^s are 1-forms and g is a Riemannian metric on \tilde{M} such that

$$\varphi^2 = -I + \sum_{i=1}^s \eta^i \otimes \xi_i \quad (2.1)$$

$$\eta^i(\xi_j) = \delta_j^i, \quad \varphi(\xi_i) = 0, \quad \eta^i \circ \varphi = 0, \quad (2.2)$$

$$\eta^i(X) = g(X, \xi_i), \quad (2.3)$$

$$g(X, \varphi Y) + g(\varphi X, Y) = 0, \quad (2.4)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^s \eta^i(X) \eta^i(Y) \quad (2.5)$$

for all $X, Y \in \Gamma(T\tilde{M})$ and $i, j \in \{1, \dots, s\}$. In above case, we say that \tilde{M} is a metric f -manifold and its associated structure will be denoted by $\tilde{M}(\varphi, \xi_i, \eta^i, g)$ [19].

A 2-form Ω is defined by $\Omega(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(T\tilde{M})$, is called the fundamental 2-form. A framed metric structure is called normal [19] if

$$[\varphi, \varphi] + 2d\eta^i \otimes \xi_i = 0$$

where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to φ . Throughout this paper we denote by $\bar{\eta} = \eta^1 + \eta^2 + \dots + \eta^s$, $\bar{\xi} = \xi_1 + \xi_2 + \dots + \xi_s$ and $\bar{\delta}_i^j = \delta_i^1 + \delta_i^2 + \dots + \delta_i^s$.

Definition 2.1. Let $\tilde{M}(\varphi, \xi_i, \eta^i, g)$ be a $(2n+s)$ -dimensional a metric f -manifold for each η^i , $(1 \leq i \leq s)$ 1-forms and each 2-form Ω , if $d\eta^i = 0$ and $d\Omega = 2\alpha\bar{\eta} \wedge \Omega$ satisfy, then \tilde{M} is called almost α -cosymplectic f -manifold [18].

Let \tilde{M} be an almost α -cosymplectic f -manifold. Since the distribution D is integrable, we have $L_{\xi_i} \eta^j = 0$, $[\xi_i, \xi_j] \in D$ and $[X, \xi_j] \in D$ for any $X \in \Gamma(D)$. Then the Levi-Civita connection is given by [18]:

$$2g((\tilde{\nabla}_X \varphi)Y, Z) = 2\alpha g\left(\sum_{i=1}^s (g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X), Z\right) + g(N(Y, Z), \varphi X) \quad (2.6)$$

for any $X, Y \in \Gamma(T\tilde{M})$. Putting $X = \xi_i$ we obtain $\tilde{\nabla}_{\xi_i} \varphi = 0$ which implies $\tilde{\nabla}_{\xi_i} \xi_j \in D^\perp$ and then $\tilde{\nabla}_{\xi_i} \xi_j = \tilde{\nabla}_{\xi_j} \xi_i$, since $[\xi_i, \xi_j] = 0$. We put $A_i X = -\tilde{\nabla}_X \xi_i$ and $h_i = \frac{1}{2}(L_{\xi_i} \varphi)$, where L denotes the Lie derivative operator. If \tilde{M} is almost α -cosymplectic f -manifold with Kaehlerian leaves [20], we have

$$(\tilde{\nabla}_X \varphi)Y = \sum_{i=1}^s [-g(\varphi A_i X, Y)\xi_i + \eta^i(Y)\varphi A_i X]$$

or

$$(\tilde{\nabla}_X \varphi)Y = \sum_{i=1}^s [\alpha (g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X) + g(h_i X, Y)\xi_i - \eta^i(Y)h_i X]. \quad (2.7)$$

Proposition 2.2. ([18]) For any $i \in \{1, \dots, s\}$ the tensor field A_i is a symmetric operator such that

- (i) $A_i(\xi_j) = 0$, for any $j \in \{1, \dots, s\}$
- (ii) $A_i \circ \varphi + \varphi \circ A_i = -2\alpha\varphi$
- (iii) $tr(A_i) = -2\alpha n$
- (iv) $\tilde{\nabla}_X \xi_i = -\alpha\varphi^2 X - \varphi h_i X$.

Proposition 2.3. ([21]) For any $i \in \{1, \dots, s\}$ the tensor field h_i is a symmetric operator and satisfies

- (i) $h_i(\xi_j) = 0$, for any $j \in \{1, \dots, s\}$
- (ii) $h_i \circ \varphi + \varphi \circ h_i = 0$
- (iii) $tr h_i = 0$
- (iv) $tr(\varphi h_i) = 0$.

Let \tilde{M} be an almost α -cosymplectic f -manifold with respect to the curvature tensor field \tilde{R} of $\tilde{\nabla}$, the following formulas are proved in [18], for all $X, Y \in \Gamma(T\tilde{M})$, $i, j \in \{1, \dots, s\}$.

$$\begin{aligned} \tilde{R}(X, Y)\xi_i &= \alpha^2 \sum_{k=1}^s (\eta^k(Y)\varphi^2 X - \eta^k(X)\varphi^2 Y) \\ &- \alpha \sum_{k=1}^s (\eta^k(X)\varphi h_k Y - \eta^k(Y)\varphi h_k X) \\ &+ (\tilde{\nabla}_Y \varphi h_i)X - (\tilde{\nabla}_X \varphi h_i)Y, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \tilde{R}(X, \xi_j)\xi_i &= \sum_{k=1}^s \delta_j^k (\alpha^2 \varphi^2 X + \alpha \varphi h_k X) \\ &+ \alpha \varphi h_i X - h_i h_j X + \varphi(\tilde{\nabla}_{\xi_j} h_i)X \end{aligned} \tag{2.9}$$

$$\tilde{R}(\xi_j, X)\xi_i - \varphi \tilde{R}(\xi_j, \varphi X)\xi_i = 2(-\alpha^2 \varphi^2 X + h_i h_j X). \tag{2.10}$$

Moreover, by using the above formulas, in [18] it is obtained that

$$\tilde{S}(X, \xi_j) = -2n\alpha^2 \sum_{k=1}^s \eta^k(X) - (div \varphi h_i)X \tag{2.11}$$

$$\tilde{S}(\xi_i, \xi_j) = -2n\alpha^2 - tr(h_j h_i) \tag{2.12}$$

for all $X, Y \in \Gamma(T\tilde{M})$, $i, j \in \{1, \dots, s\}$, where \tilde{S} denote, the Ricci tensor field of the Riemannian connection.

From [18], we have the following result.

Proposition 2.4. Let \tilde{M} be an almost α -cosymplectic f -manifold and M be an integral manifold of D . Then

- (i) when $\alpha = 0$, M is totally geodesic if and only if all the operators h_i vanish;
- (ii) when $\alpha \neq 0$, M is totally umbilic if and only if all the operators h_i vanish.

3. Semi-Invariant Submanifolds of Almost α -Cosymplectic f -Manifolds

The submanifold M of the almost α -cosymplectic f -manifold \tilde{M} is said to be semi-invariant [22] if it is endowed with two pair of orthogonal distribution D, D^\perp satisfying the conditions

- (i) $TM = D \oplus D^\perp \oplus \{\xi_1, \xi_2, \dots, \xi_s\}$

(ii) the distribution D is invariant under φ , that is

$$\varphi D_x = D_x, \text{ for each } x \in M,$$

(iii) the distribution D^\perp is anti-invariant under φ , that is

$$\varphi D_x^\perp \subset T_x M^\perp \text{ for each } x \in M.$$

The distribution D (resp. D^\perp) is called the horizontal (resp. vertical) distribution. A semi-invariant submanifold M is said to be invariant (resp. anti-invariant) submanifold if we have $(D_x^\perp = 0)$ respectively $(D_x = 0)$ for each $x \in M$. We say that M is proper semi-invariant submanifold if it is a semi-invariant submanifold which is neither an invariant nor anti-invariant submanifold [22].

We denote by same symbol g both metrics on \tilde{M} and M . The projection morphism of TM to D and D^\perp are denoted by P and Q respectively. For any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$ we have

$$X = PX + QX + \sum_{i=1}^s \eta^i(X) \xi_i \quad (3.1)$$

$$\varphi N = CN + DN \quad (3.2)$$

and

$$h_i X = t_i X + f_i X \quad (3.3)$$

where CN and $t_i X$ (resp. DN and $f_i X$) denotes the tangential (resp. normal) of φN and $h_i X$, respectively.

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y) \quad (3.4)$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N \quad (3.5)$$

for any $X, Y \in \Gamma(TM)$ and $N \in TM^\perp$, where ∇ is the Levi-civita connection on M , ∇^\perp is the linear connection induced by $\tilde{\nabla}$ on the normal bundle TM^\perp , B is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N . Also we have

$$g(B(X, Y), N) = g(A_N X, Y) \quad (3.6)$$

for any $X, Y \in \Gamma(TM), N \in \Gamma(TM^\perp)$ [19].

We now give an example of semi-invariant submanifold of an almost α -cosymplectic f -manifold.

Example 3.1. Let us denote the standart coordinates of R^{2n+s} $(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s)$ and take $(2n+s)$ -dimensional manifold $\tilde{M} \subset R^{2n+s}$ defined by

$$\tilde{M} = \{(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s) \mid z_1, \dots, z_s \neq 0\}.$$

Consider following vector fields as a global basis of \tilde{M} :

$$X_i = e^{\sum_{i=1}^n z_i} \frac{\partial}{\partial x_i}, \quad Y_i = \frac{\partial}{\partial y_i}, \quad \xi_j = \frac{\partial}{\partial z_j}, \quad i = 1, \dots, n \quad j = 1, \dots, s.$$

The brackets of these vector fields are

$$[\xi_j, X_i] = e^{\sum_{i=1}^n z_i} \frac{\partial}{\partial x_i}, \quad [\xi_j, Y_i] = [X_k, X_i] = [X_i, Y_k] = [Y_i, Y_k] = 0$$

for any $i, k \in \{1, \dots, n\}$ and $j \in \{1, \dots, s\}$. One may easily verify that putting

$$\eta^j = dz_j, \quad g = \sum_{i=1}^n [e^{-2(z_1 + \dots + z_s)} dx_i^2 + dy_i^2] + \sum_{j=1}^s dz_j^2,$$

$$\varphi(\xi_j) = 0, \quad \varphi\left(\frac{\partial}{\partial x_i}\right) = e^{-(z_1+\dots+z_s)} \frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_i}\right) = -e^{(z_1+\dots+z_s)} \frac{\partial}{\partial x_i},$$

$(\varphi, \xi_i, \eta^i, g)$ is an almost contact metric f -structure on \tilde{M} . We shall check that $(\tilde{M}, \varphi, \xi_i, \eta^i, g)$ is an almost α -cosymplectic f -manifold. Obviously, $\eta^j = dz_j \Rightarrow d\eta^j = d^2z_j = 0$ from poicare metric we get $d\eta^j = 0$. To verify the condition $d\Phi = 2\alpha\bar{\eta} \wedge \Phi$, considering that all Φ_{ij} 's are zero except for $\Phi_{ii} = g\left(\frac{\partial}{\partial x_i}, \varphi\frac{\partial}{\partial y_i}\right) = -e^{-(z_1+\dots+z_s)}$ and hence

$$\Phi = -\frac{1}{e^{(z_1+\dots+z_s)}} \sum_{i=1}^n dx_i \wedge dy_i$$

holds. As a result, the exterior derivative $d\Phi$ is given by

$$\begin{aligned} d\Phi &= -e^{-(z_1+\dots+z_s)} \sum_{i=1}^n dx_i \wedge dy_i \wedge (dz_1 + \dots + dz_s) \\ d\Phi &= e^{-(z_1+\dots+z_s)} e^{(z_1+\dots+z_s)} \Phi \wedge (\eta^1 + \dots + \eta^s) \\ d\Phi &= \bar{\eta} \wedge \Phi = 2\left(\frac{1}{2}\right)\bar{\eta} \wedge \Phi. \end{aligned}$$

Since the Nijenhuis torsion of φ is not zero, the manifold is an almost $\left(\frac{1}{2}\right)$ -cosymplectic f -manifold.

Now, we definite the distributions

$$D = sp\{X_1, Y_1, X_2, Y_2, \dots, X_m, Y_m\}$$

and

$$D^\perp = sp\{X_{m+1}, X_{m+2}, \dots, X_{m+p}\} (m < n).$$

It is clear that $TM = D \oplus D^\perp \oplus \{\xi_1, \dots, \xi_s\}$, $\dim M = 2m + p + s$. Let

$$TM^\perp = \{Y_{m+1}, Y_{m+2}, \dots, Y_{m+p}, Y_{m+p+1}, \dots, Y_n, X_{m+p+1}, \dots, X_n\}$$

then we have $\varphi D = D$ and $\varphi D^\perp \subset TM^\perp$. Consequently, M is a semi-invariant submanifold of an almost $\frac{1}{2}$ -cosymplectic f -manifold.

4. Basic Lemmas

For any $X, Y \in \Gamma(TM)$, we put

$$u(X, Y) = \nabla_X \varphi P Y - A_{\varphi Q Y} X. \quad (4.1)$$

We start with proving the following lemma.

Lemma 4.1. *Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold with Kaehlerian leaves \tilde{M} . Then we have*

$$P(u(X, Y)) = \varphi P \nabla_X Y - \sum_{i=1}^s [\alpha \eta^i(Y) \varphi P X + \eta^i(Y) P t_i X] \quad (4.2)$$

$$Q(u(X, Y)) = Q C B(X, Y) - \sum_{i=1}^s \eta^i(Y) Q t_i X \quad (4.3)$$

$$\begin{aligned} B(X, \varphi P Y) + \nabla_X^\perp \varphi Q Y &= \varphi Q \nabla_X Y + D B(X, Y) \\ &\quad - \sum_{i=1}^s [\alpha \eta^i(Y) \varphi Q X - \eta^i(Y) f_i X] \end{aligned} \quad (4.4)$$

$$\begin{aligned} \eta^i(u(X, Y)) \xi_i &= \sum_{i=1}^s [\alpha g(\varphi P X, Y) \xi_i + g(h_i X, Y) \xi_i] \\ &\quad - \sum_{i,j=1}^s \eta^i(Y) \eta^j(t_i X) \xi_i. \end{aligned} \quad (4.5)$$

Proof. For $X, Y \in \Gamma(TM)$, putting (3.1), (3.2) and (3.3) in the equation (2.7) we get

$$\begin{aligned} (\tilde{\nabla}_X \varphi)Y &= \sum_{i=1}^s [\alpha(g(\varphi PX, Y)\xi_i - \eta^i(Y)\varphi PX - \eta^i(Y)\varphi QX) \\ &+ g(h_i X, Y)\xi_i - \eta^i(Y)h_i X] \\ &= \sum_{i=1}^s [\alpha(g(\varphi PX, Y)\xi_i - \eta^i(Y)\varphi PX - \eta^i(Y)\varphi QX) + g(h_i X, Y)\xi_i \\ &- \eta^i(Y)P_{t_i}X - \eta^i(Y)Q_{t_i}X - \eta^i(Y)\sum_{j=1}^s \eta^j(t_i X)\xi_j - \eta^i(Y)f_i X]. \end{aligned}$$

On the other hand, by using (3.1), (3.2), (3.4) and (3.5) we have

$$\begin{aligned} (\tilde{\nabla}_X \varphi)Y &= \tilde{\nabla}_X \varphi Y - \varphi \tilde{\nabla}_X Y \\ &= \tilde{\nabla}_X \varphi PY + \tilde{\nabla}_X \varphi QY - \varphi(\nabla_X Y + B(X, Y)) \\ &= \nabla_X \varphi PY + B(X, \varphi PY) - A_{\varphi QY}X + \nabla_X^\perp \varphi QY \\ &- \varphi P \nabla_X Y - \varphi Q \nabla_X Y - CB(X, Y) - DB(X, Y) \\ (\tilde{\nabla}_X \varphi)Y &= P \nabla_X \varphi PY + Q \nabla_X \varphi PY + \sum_{i=1}^s \eta^i(\nabla_X \varphi PY)\xi_i + B(X, \varphi PY) \\ &- PA_{\varphi QY}X - QA_{\varphi QY}X + \nabla_X^\perp \varphi QY - \sum_{i=1}^s \eta^i(A_{\varphi QY}X)\xi_i \\ &- \varphi P \nabla_X Y - \varphi Q \nabla_X Y - CB(X, Y) - DB(X, Y). \end{aligned}$$

Taking the components of D , ξ_i , D^\perp and TM^\perp in above equations, we have our assertion. \square

Lemma 4.2. *Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold with Kaehlerian leaves \tilde{M} . Then we have*

$$\varphi P(A_N X) + P(\nabla_X CN) = P(A_{DN} X) \quad (4.6)$$

$$Q((C\nabla_X^\perp N) + A_{DN} X - \nabla_X CN) = 0 \quad (4.7)$$

$$\eta(A_{DN} X - \nabla_X CN) = \alpha g(X, CN) + g(h_i X, N)\xi_i \quad (4.8)$$

$$B(X, CN) + \varphi Q(A_N X) + \nabla_X^\perp DN = D\nabla_X^\perp N \quad (4.9)$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$

Proof. By using the decompositions (3.1), (3.2) and the equations of Gauss and Weingarten in (2.7) we have

$$\begin{aligned} (\tilde{\nabla}_X \varphi)N &= \tilde{\nabla}_X \varphi N - \varphi \tilde{\nabla}_X N = \sum_{i=1}^s [\alpha g(\varphi X, N)\xi_i + g(h_i X, N)\xi_i] \\ \nabla_X CN + B(X, CN) - A_{DN} X + \nabla_X^\perp DN + \varphi A_N X - \varphi \nabla_X^\perp N &= \sum_{i=1}^s [\alpha g(\varphi X, N)\xi_i + g(h_i X, N)\xi_i] \\ &= P \nabla_X CN + Q \nabla_X CN + \sum_{i=1}^s \eta^i(\nabla_X CN)\xi_i + B(X, CN) - PA_{DN} X - QA_{DN} X - \sum_{i=1}^s (A_{DN} X)\xi_i \\ &+ \nabla_X^\perp DN + \varphi PA_N X + \varphi QA_N X - C\nabla_X^\perp N - D\nabla_X^\perp N \\ &= - \sum_{i=1}^s [\alpha g(X, CN)\xi_i + g(h_i X, N)\xi_i] \end{aligned}$$

Then (4.6)- (4.9) follows by taking the components on each of the vector bundle D , D^\perp , ξ_i and respectively TM^\perp . \square

Lemma 4.3. *Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold \tilde{M} . Then we have*

$$\nabla_X \xi_i = \alpha X - \varphi t_i X - C f_i X \quad \forall X \in \Gamma(D) \quad (4.10)$$

$$\nabla_X \xi_i = \alpha X - \varphi t_i X - C f_i X \quad \forall X \in \Gamma(D^\perp) \quad (4.11)$$

$$\nabla_{\xi_i} \xi_j = 0, \quad B(X, \xi_i) = -D f_i X. \quad (4.12)$$

Proof. For $X \in \Gamma(TM)$, using (3.2), (3.3) and (3.4) we obtain

$$\begin{aligned} \tilde{\nabla}_X \xi_i &= \nabla_X \xi_i + B(X, \xi_i) = -\alpha \varphi^2 X - \varphi h_i X \\ &= \alpha X - \alpha \sum_{i=1}^s \eta^i(X) \xi_i - \varphi h_i X \\ &= \alpha X - \alpha \sum_{i=1}^s \eta^i(X) \xi_i - \varphi t_i X - \varphi f_i X \\ &= \alpha X - \alpha \sum_{i=1}^s \eta^i(X) \xi_i - \varphi t_i X - C f_i X - D f_i X. \end{aligned} \quad (4.13)$$

Thus (4.10)-(4.12) follows from (4.13). \square

Lemma 4.4. *Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold with Kaehlerian leaves \tilde{M} . Then we have*

$$A_{\varphi X} Y = A_{\varphi Y} X \quad (4.14)$$

for all $X, Y \in \Gamma(D^\perp)$.

Proof. For all $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(TM)$, by using (3.4) and (3.6), we get

$$\begin{aligned} g(A_{\varphi X} Y, Z) &= g(B(Y, Z), \varphi X) = g(\tilde{\nabla}_Z Y, \varphi X) \\ &= -g(\varphi \tilde{\nabla}_Z Y, X) = -g(\tilde{\nabla}_Z \varphi Y - (\tilde{\nabla}_Z \varphi) Y, X) \\ &= -g(\tilde{\nabla}_Z \varphi Y, X) = g(\varphi Y, \tilde{\nabla}_Z X) \\ &= g(\varphi Y, B(Z, X)) = g(A_{\varphi Y} X, Z), \end{aligned}$$

which proves (4.14). \square

Lemma 4.5. *Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold \tilde{M} . Then we have,*

$$\nabla_{\xi_i} U \in \Gamma(D), \quad (4.15)$$

$$\nabla_{\xi_i} V \in \Gamma(D^\perp), \quad (4.16)$$

$$[U, \xi_i] \in \Gamma(D), \quad (4.17)$$

$$[V, \xi_i] \in \Gamma(D^\perp) \quad (4.18)$$

for any $i \in \{1, 2, \dots, s\}$, $U \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$.

Proof. For $U \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$,

$$g(\nabla_{\xi_i} U, \xi_j) = \xi_i g(U, \xi_j) - g(U, \nabla_{\xi_i} \xi_j) = 0$$

and

$$g(\nabla_{\xi_i} U, V) = \xi_i g(U, V) - g(U, \nabla_{\xi_i} V) = g(\varphi^2 U, \nabla_{\xi_i} V) = -g(\varphi U, \varphi \nabla_{\xi_i} V) = -g(\varphi U, \nabla_{\xi_i} \varphi V) = g(\nabla_{\xi_i} \varphi U, \varphi V) = 0,$$

so $\nabla_{\xi_i} U \in \Gamma(D)$. In a similiary way is deduced (4.16). On the other hand, using (4.10) and (4.11), we have

$$g([U, \xi_i], \xi_j) = g(\nabla_U \xi_i, -\nabla_{\xi_i} U, \xi_j) = 0$$

and

$$g([U, \xi_i], V) = g(\nabla_U \xi_i, V) - g(\nabla_{\xi_i} U, V) = 0.$$

Thus completes the proof. \square

Lemma 4.6. *Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold \tilde{M} . Then we have*

$$g(X, t_i Y) = g(t_i X, Y), \quad (4.19)$$

$$\varphi t_i X + t_i \varphi X + C f_i X = 0, \quad (4.20)$$

$$D f_i X + f_i \varphi X = 0 \quad (4.21)$$

for any $X, Y \in \Gamma(M)$.

Proof. Since h_i is symmetric, we get

$$g(X, h_i Y) = g(h_i X, Y)$$

$$g(X, t_i Y + f_i Y) = g(t_i X, Y) + g(f_i X, Y)$$

$$g(X, t_i Y) + g(X, f_i Y) = g(t_i X, Y) + g(f_i X, Y).$$

From above equation we get (4.19). By making use of proposition 2.3 and using (3.2), (3.3), we get

$$\varphi t_i X + t_i \varphi X + C f_i X + D f_i X + f_i \varphi X = 0. \quad (4.22)$$

Comparing the tangential and normal part of (4.22), we get (4.20) and (4.21), respectively. \square

5. Integrability of distribution on a semi-invariant submanifold in an almost α -cosymplectic f -manifold

Theorem 5.1. *Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold \tilde{M} . Then the distribution D is never integrable.*

Proof. For all $X, Y \in \Gamma(D)$, we have

$$\begin{aligned} g([X, Y], \xi_i) &= g(\nabla_X Y, \xi_i) - g(\nabla_Y X, \xi_i) \\ &= -g(Y, \nabla_X \xi_i) + g(X, \nabla_Y \xi_i) \\ &= -g(Y, \alpha X - \varphi t_i X - C f_i X) + g(X, \alpha Y - \varphi t_i Y - C f_i Y) \\ &= g(Y, \varphi t_i X) + g(Y, C f_i X) - g(X, \varphi t_i Y) - g(X, C f_i Y) \\ &= g(Y, \varphi t_i X + C f_i X) - g(X, \varphi t_i Y + C f_i Y) \\ &= -g(Y, t_i \varphi X) + g(X, t_i \varphi Y) \\ &= -g(t_i Y, \varphi X) + g(t_i X, \varphi Y) \\ &= -g(Y, t_i \varphi X) - g(\varphi t_i X, Y) \\ &= -g(Y, t_i \varphi X + \varphi t_i X) \\ &= g(Y, C f_i X) \neq 0. \end{aligned}$$

This follows the non-integrability of D . \square

Corollary 5.2. *The distribution $D \oplus D^\perp$ never involutive.*

Theorem 5.3. *Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold with Kaehlerian leaves \tilde{M} . The distribution $D \oplus \{\xi_1, \dots, \xi_s\}$ is integrable if and only if*

$$B(X, \varphi Y) = B(\varphi X, Y) \quad (5.1)$$

is satisfied.

Proof. From (4.4), the distribution $D \oplus \{\xi_1, \dots, \xi_s\}$ is integrable if and only if

$$B(X, \varphi Y) - B(Y, \varphi X) = \varphi Q[X, Y] = 0$$

is satisfied so, $B(X, \varphi Y) = B(Y, \varphi X)$. \square

Theorem 5.4. *Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold with Kaehlerian leaves \tilde{M} . Then the distribution D^\perp is integrable.*

Proof. From (4.1), we have for $X, Y \in \Gamma(D^\perp)$

$$U(X, Y) = -A_{\phi Q}YX$$

operating ϕ in (4.2) we get

$$P\nabla_X Y = \phi P(A_{\phi Y}X) \quad (5.2)$$

for any $X, Y \in \Gamma(D^\perp)$. By virtue of Lemma 4.4, (5.2) reduce to

$$P([X, Y]) = 0$$

which is prove that $[X, Y] \in \Gamma(D^\perp)$. \square

6. Mixed totally geodesic semi-invariant submanifolds

Definition 6.1. A semi-invariant submanifold M of an almost α -cosymplectic f -manifold \tilde{M} is called mixed totally geodesic if the second fundamental form satisfies $B(X, Y) = 0$ for any $X \in D$ and $Y \in D^\perp$ [5].

Theorem 6.2. Let M be a semi-invariant submanifold of almost α -cosymplectic f -manifold \tilde{M} . Then M is mixed totally geodesic submanifold of almost α -cosymplectic f -manifold \tilde{M} if and only if

$$A_V X \in \Gamma(D) \quad (\forall X \in \Gamma(D), V \in \Gamma(TM)^\perp) \quad (6.1)$$

and

$$A_V X \in \Gamma(D)^\perp \quad (\forall X \in \Gamma(D)^\perp, V \in \Gamma(TM)^\perp). \quad (6.2)$$

Proof. Consider $A_V X$, let $X \in \Gamma(D)$ and $V \in \Gamma(TM)^\perp$ and $Y \in \Gamma(D^\perp)$, then we have

$$\begin{aligned} g(B(X, Y), V) &= g(A_V X, Y) \\ &= 0 \Leftrightarrow A_V X \in \Gamma(D). \end{aligned}$$

On the other hand, if $A_V X \in \Gamma(D)$, we get

$$\begin{aligned} g(A_V X, V) &= g(B(X, Y), V) \\ &= 0 \Leftrightarrow B(X, Y) = 0. \end{aligned}$$

In a similar way is deduced (6.2). \square

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