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Energy-Momentum Distribution for Magnetically Charged Black Hole Metric

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Abstract

This work investigates the well known localization problem of energy and momentum. The purpose of this paper is two fold. First, we compute Einstein, Landau-Lifshitz and Bergmann's energy-momentum complexes for static spherically symmetric magnetically charged regular black hole spacetime in general relativity. We observe strong coincidences among the results obtained form the three descriptions. These resembling results from different energy-momentum prescriptions may offer some basis to explain a exclusive quantity which supports Virabhadra's viewpoint. Secondly, the problem is discussed in modified gravity. In particular, we use generalized Landau-Lifshitz prescription for the determination of energy-momentum with reference to $f(R)$ theory of gravity. We explicitly compute the energy-momentum complex for the static spherically symmetric magnetically consistent regular black hole metric for a well-known choice of the $f(R)$ gravity models.

1. Introduction

The energy-momentum localization in curved spacetimes is one of the most important issues since the emergence of general relativity (GR). This problem has no proper solution till date. Several attempts have been carried out by researchers to overcome this issue, using different tools and hypothesis. A unique tensorial definition of energy and momentum has been a focus of many findings in the GR. Energy-momentum tensor T_a^b is a second rank symmetric, localized and divergence-less quantity introduced in both, the special relativity and the classical mechanics. It gives the account of the energy and momentum matter source, and non-gravitational field sources. Given below equation defines the conservation law of energy and momentum

$$T^b_{a;\,b} = 0. \tag{1.1}$$

Unique definitions of energy and momentum exist in classical physics. However, ordinary derivatives transforms to covariant derivatives in GR. Thus, we get

$$T^b_{a;b} = \frac{1}{\sqrt{-g}}(\sqrt{-g}T^b_a)_{,b} - \Gamma^b_{ac}T^c_b = 0. \tag{1.2}$$

This conservation law was formulated by Einstein [1]. Eq. (1.2) shows that T_a^b does not satisfy Eq. (1.1) in the presence of gravitational field. The summation of these two terms (stress-energy tensor and a pseudo-tensor) remains divergence-less. The addition of a non-tensor quantity to justify the gravitational field energy was criticized by many researchers. Levi Civita argued on an alternate gravitational energy tensor. Penrose [2] introduced another concept of energy, known as quasi-local energy in order to find a feasible expression other than t_a^b (pseudo-tensor). Pauli criticized Einstein's work on energy-momentum distribution but Einstien argued that his energy-momentum complex (EMC) gave reasonable outcomes for the energy and momentum of isolated systems which obey the conservation laws.

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Later on, different prescriptions were suggested by many researchers including Landau-Lifshitz [3], Bergmann-Thomson [4], Tolman [5], Weinberg [6], Papapetrou [7], and Möller [8]. All of these works are coordinate dependent i.e one has to perform the computations in the quasi-cartesian coordinates except Möller's prescription which is coordinate free. Due to non-tensorial nature, these complexes are also called pseudo-tensors. Misner et al. [9] proved that spherical coordinate system can be used for the localization of energy. But later on, possibility of energy localization in any system was proved by Sarracino and Cooperstock [10] as they showed that the energy can be localized in any system, if it is localized in spherical systems. In 1990, Virabhadra claimed that energy-momentum complexes might provide intuitive conclusions just for the isolated systems total energy. Virabhadra and his co-authors [11] studied several spacetimes and found various energy distributions for such spacetimes. The issue of localization of energy-momentum in the GR gained a new viewpoint from the results elaborated by Xulu et al. [12]. Rosen [13] investigated the total energy of a closed homogenous isotropic universe using Einstein's EMC. Many efforts have been made to solve the problem of localization of energy and momentum [14]-[20]. Amir and Mirshekari [21] investigated energy-momentum distributions in static and non-static cosmic strings space-times using Einstein, Landau-Lifshitz, papapetrou, Bergmann-Thomson, Tolman, Møller, and Weinberg's prescriptions. They noted strong similarities between the findings. These results were thought to be the extension of Virabhadra's point of view that different energy-momentum prescriptions may provide some basis to explain an exclusive quantity. Xulu and his collaborators [12] investigated the matter source of black hole phantom metric with the help of Einstein's EMC. Sharif and Kanwal [22] evaluated energy-momentum distribution of Bell-Szekeres metric in the GR and teleparallel gravity with the help of Bergmann-Thomson, Einstein, Landau-Lifshitz, and Møller's prescriptions. Bergmann and Einstein's prescriptions for Bianchi type-V spacetime in the GR and teleparallel theory of gravity were investigated by Salti and Aydogdu [23]. Bianchi type-II universe was investigated by Aydogdu using Landau-Lifshitz and Einstein EMC in the GR [24]. Banerjee and Sen [25] calculated the total energy density for Bianchi type-I universe using Einstein's EMC. Sharif and Fatima [26] computed the energy-momentum distribution of the Weyl metrics, singularity-free cosmological model and non-null Einstein-Maxwell solution using different prescriptions.

Modified theories of gravity have enthused many researchers lately due to the collective motivation imminent from high-energy physics, astrophysics, and cosmology. Among several alternatives to the Einstein's theory of gravity, theories which comprise high order scalar curvature invariants, and explicitly the class of $f(R)$ theories, enjoys a long history. In the last seven years or so, there has been a novel incentive for their study, leading to numerous fascinating results in this modified gravity. Furthermore, it has proven to be an effective and consistent to the GR and it continues to behave so. This definitely happens to be the reason to consider this theory for our present work Mainly, most of the work in the literature on the problem of energy momentum localization is with in the framework of the GR. In particular, the $f(R)$ gravity seems an interesting and straight forward modification for the discussion. Multämaki et al. [27] were the pioneers to generalize Landau-Lifshitz EMC in $f(R)$ theories of gravity. However, they restricted the generalization for those spacetimes having constant scaler curvature. Sharif and Shamir evaluated the energy density of plane symmetric solutions for some popular choices of $f(R)$ models. They also investigated the energy distribution of cosmic string spacetime [28]. Similar work can be found in [29]-[32].

Black holes gained much importance since Schwarzschild era. Xulu computed the energy distribution of Melvins magnetic universe and a charged dilaton black hole [12]. Gao and Zhang [33] investigated the properties of a phantom black hole metric. The accretion process of phantom fluid onto a black hole was studied by Babichev et al. [34]. Fabris and Bronnikov [35] obtained some interesting results on investigating the physics of neutral phantom black holes. Ding et al. [36] studied the influence of phantom fields on strong gravitational lensing.

In this paper, we interest to investigate some EMCs for static spherically symmetric magnetically charged regular black hole metric. For this purpose, we choose Landau-Lifshitz, Einstein, and Bergmann's prescriptions. We also investigate the energy and momenta for generalized Landau-Lifshitz EMC in $f(R)$ theory. The sequence of this manuscript is given as: Section 2 gives a brief discussion on the static spherically symmetric magnetically charged regular black hole metric. Section 3 is devoted to discuss different EMCs. In particular Einstein, Landau-Lifshitz and Bergmann's EMCs have been discussed for the magnetically charged regular black hole metric in detail. Section 4 gives the generalized expressions for Landau-Lifshitz in context of $f(R)$ gravity. Specifically, we calculate the components of energy and momentum. The last section is devoted to the comments and conclusive remarks.

2. Magnetically charged regular black hole metric

We use here the static, spherically symmetric magnetically charged regular black hole metric [37],

$$ds^2 = w(r)dt^2 - \frac{dr^2}{w(r)} - r^2 d\Omega^2, \tag{2.1}$$

where metric function $w(r)$ is given

$$w(r) = 1 - \frac{2m(r)}{r},$$

with $m(r)$ is the mass function given by

$$\begin{aligned} m(r) &= \frac{q^{\frac{3}{2}}}{16\sqrt[4]{\gamma}}\left[\ln\frac{2\sqrt{\gamma}q - 2\sqrt[4]{\gamma}\sqrt{q}r + r^2}{2\sqrt{\gamma}q + 2\sqrt[4]{\gamma}\sqrt{q}r + r^2} + 2\arctan\left(1 + \frac{r}{\sqrt[4]{\gamma}\sqrt{q}}\right)\right. \\ &\quad \left. -2\arctan\left(1 - \frac{r}{\sqrt[4]{\gamma}\sqrt{q}}\right)\right], \end{aligned}$$

where q gives magnetic charge. To workout a black hole solution being consistent at $r = 0$, the parameter γ has to assume the value [37]

$$\gamma = \left(\frac{\pi q^{\frac{3}{2}}}{8M}\right)^4.$$

Thus the metric function $w(r)$ turns out to be

$$\begin{aligned} w(r) &= 1+\frac{M}{\pi r}\left[\ln\frac{32M^2r^2+8\pi Mq^2r+\pi^2q^4}{32M^2r^2-8\pi Mq^2r+\pi^2q^4}-2\arctan\left(\frac{8Mr}{\pi q^2}+1\right)\right. \\ &\left. +2\arctan\left(1-\frac{8Mr}{\pi q^2}\right)\right]. \end{aligned}$$

Here M is the mass parameter. Ding and collaborators. [36] explored that for $M=0$, the metric provides the Ellis wormhole geometries. For the computation of components of energy and momentum, the line element (2.1)should be transformed from spherical polar coordinates (t, r, θ, ϕ) to quasi-cartesian coordinates i.e. (t, x, y, z) applying the following transformations

$$\begin{aligned} x &= r\cos\phi\sin\theta, \\ y &= r\sin\phi\sin\theta, \\ z &= r\cos\theta. \end{aligned}$$

The line element just transformed reads

$$\begin{aligned} ds^2 &= w(r)dt^2-(dx^2+dy^2+dz^2)-\frac{1}{w(r)}\left(\frac{xdx+ydy+zdz}{r}\right)^2 \\ &+\left(\frac{xdx+ydy+zdz}{r}\right)^2. \end{aligned}$$

The corresponding determinant g of the tensor g_{ik} gives

$$g=-1.$$

We obtain seven non-zero independent contra-variant components of the symmetric metric tensor.

$$\begin{aligned} g^{00} &= \frac{1}{w}, \\ g^{11} &= \frac{x^2(1-w)-r^2}{r^2}, \\ g^{22} &= \frac{y^2(1-w)-r^2}{r^2}, \\ g^{33} &= \frac{z^2(1-w)-r^2}{r^2}, \\ g^{12} &= \frac{(1-w)xy}{r^2}, \\ g^{23} &= \frac{(1-w)yz}{r^2}, \\ g^{31} &= \frac{(1-w)xz}{r^2}. \end{aligned}$$

3. Energy-momentum prescriptions in the GR

Here, we discuss the three different EMCs in the GR. In particular we discuss Einstein, Landau-Lifshitz and Bergmann's EMCs.

3.1. Einstein energy-momentum prescription

The Einstein EMC is given as[38]

$$\Theta_a^b=\frac{1}{16\pi}h^{bc}{}_{a,c},$$

where

$$h^{bc}{}_a=-h^{cb}{}_a=\frac{g_{ad}}{\sqrt{-g}}\left[-g\left(g^{bd}g^{ce}-g^{cd}g^{be}\right)\right]_{,e}.$$

The energy, momentum components are denoted by Θ_0^0 and Θ_i^0 respectively. Θ_a^b satisfies the covarient local Einstein's conservation laws

$$\frac{\partial\Theta_a^b}{\partial x^b}=0.$$

The momentum four-vector (or the components of energy-momentum) is expressed as

$$P_a=\int\int\int\Theta_a^0dx^1dx^2dx^3. \tag{3.1}$$

P_i with $(i = 1,2,3)$ provides the components of momentum and the energy the energy is represented by P_0. Energy of physical system when the integration is taken over the hypersurface element by considering constant t is

$$E = \int\int\int \Theta_0^0 dx^1 dx^2 dx^3.$$

All the calculations are restricted to be carried out in quasi-cartesian coordinates. Applying Gauss's divergence theorem on the Eq. (3.1)leads to

$$P_a = \frac{1}{16\pi}\int\int h^{0b}{}_a \mu_b dS, \tag{3.2}$$

where $\mu_b = \frac{x_i}{r}$ is the normal unit vector directed outward over an infinitesimal surface element dS. For the purpose to attain energy, we acquire three independent components of h_a^{bc}

$$\begin{aligned} h^{01}{}_0 &= \frac{2x(1-w)}{r^2}, \\ h^{02}{}_0 &= \frac{2y(1-w)}{r^2}, \\ h^{03}{}_0 &= \frac{2z(1-w)}{r^2}. \end{aligned} \tag{3.3}$$

Using the Eq. (3.3) in Eq. (3.2), the energy distribution is obtained as

$$P_E(r) = \frac{r(1-w)}{4},$$

where $P_E(r)$ expresses the total energy (gravitational field plus matter) within radius r. Likewise, P_i gives the total momentum due to both gravitational field and matter. It is to be noted that

$$\begin{aligned} h_1^{01} = h_2^{01} = h_3^{01} = 0, \\ h_1^{02} = h_2^{02} = h_3^{02} = 0, \\ h_1^{03} = h_2^{03} = h_3^{03} = 0, \end{aligned}$$

suggesting that $P_x = P_y = P_z = 0$. It is worthwhile to note that our results agree with [12] when we take $q = 0$.

3.2. Landau-Lifshitz energy-momentum prescription

The EMC of Landau-Lifshitz is [38]

$$L^{ab} = (-g)(T^{ab} + t^{ab}) = \frac{1}{16\pi}\chi^{abcd}{}_{,cd},$$

where χ^{abcd} is defined as

$$\chi^{abcd} = -g(g^{ab}g^{cd} - g^{ac}g^{bd}).$$

The components of energy and momentum are expressed by L^{00} and L^{0i} respectively.

$$\frac{\partial L^{ab}}{\partial x^b} = 0. \tag{3.4}$$

Eq. (3.4) further gives the conservation law for the quantity

$$P^a = \int\int (-g)(T^{ab} + t^{ab})\mu_b dS.$$

The quantities t^{ab} vanish in the quasi-cartesian coordinates when there is no gravitational field, and the above relation takes the form

$$P^a = \int\int (-g)T^{ab}\mu_b dS,$$

which represents the four-momentum of the physical system. It gives the total four-momentum of the matter plus gravitational field. Thus, t^{ab} is refereed as the energy-momentum pseudo-tensor of the gravitational field. The energy-momentum components are described as a three-dimensional integral space, given as

$$P^a = \int\int\int Ł^{a0} dx^1 dx^2 dx^3. \tag{3.5}$$

Here also using Gauss's Theorem in Eq. (3.5), we get

$$P^a = \frac{1}{16\pi}\int\int \chi^{a0bd}_{,d}\mu_b dS, \tag{3.6}$$

After some lengthy calculations the following independent components of χ^{abcd} turn out to be

$$\begin{aligned}
\chi^{0011} &= \frac{r^2+x^2(w-1)}{wr^2},\\
\chi^{0022} &= \frac{r^2+y^2(w-1)}{wr^2},\\
\chi^{0033} &= \frac{r^2+z^2(w-1)}{wr^2},\\
\chi^{0012} &= \frac{-(w-1)}{x}ywr^2,\\
\chi^{0023} &= \frac{-(w-1)}{y}zwr^2,\\
\chi^{0031} &= \frac{-(w-1)}{x}zwr^2.
\end{aligned} \tag{3.7}$$

Using Eq. the (3.7) in Eq. (3.6), we obtain

$$P_L(r)=\frac{(1-w)r}{4w},$$

where $P_L(r)$ gives the total energy within radius r. Here we also get all the other components zero as expected from the geometry of a static metric.

3.3. Bergmann energy-momentun prescription

The Bergmann's EMC is given as[4]

$$B^{ab}=\frac{1}{16\pi}\beta^{abc}_{,c},$$

where β^{abc} is given as

$$\beta^{abc}=g^{ad}V^{bc}_d,$$

and V^{bc}_d is defined as

$$V^{bc}_d=-V^{cb}_d=\frac{g_{de}}{\sqrt{-g}}\left[-g\left(g^{be}g^{cf}-g^{ce}g^{bf}\right)\right]_{,f}.$$

Here B^{ab} approves the covariant local laws of conservation

$$\frac{\partial B^{ab}}{\partial x^b}=0.$$

The energy-momentum components are expressed as

$$P^a=\int\int\int B^{a0}dx^1dx^2dx^3. \tag{3.8}$$

Applying the Gauss's theorem in the Eq. (3.8), we obtain

$$P^a=\frac{1}{16\pi}\int\int\beta^{a0b}\mu_b dS. \tag{3.9}$$

Now, to obtain the energy distribution, we get just three components of B^{abc}

$$\begin{aligned}
B^{001} &= \frac{2x(1-w)}{wr^2},\\
B^{002} &= \frac{2y(1-w)}{wr^2},\\
B^{003} &= \frac{2z(1-w)}{wr^2}.
\end{aligned} \tag{3.10}$$

Using Eqs. (3.10) in the Eq. (3.9), the energy distribution is obtained as

$$P_B(r)=\frac{r(1-w)}{4w}.$$

All the other components of B^{abc} are vanished resulting in zero momentum. It is worthwhile to mention here that the energy-momentum distribution for magnetically charged regular black hole metric in both Landau-Lifshitz and Bergmann prescription is same. Now, we investigate the energy-momentum distribution in modified $f(R)$ gravity.

4. Energy-momentum distribution in $f(R)$ theory of gravity

Exploring the localization problem in alternative theories of gravity might be an intresting task. Researchers have worked on this localization issue in the teleparallel theory of gravity [39]-[41] expecting a positive solution of this complex problem. The energy-momentum distribution for particular space-times has been computed using different prescriptions by Sharif and Jamil [42]. No general conclusion was deduced form the results though they were consistent in some cases. Because of the cosmologically vibrant $f(R)$ model, the $f(R)$ theory has proven to be very attractive in the recent years. The generalization of Landau-Lifshitz EMC has been studied in the framework of this modified gravity [27].

4.1. Generalized Landau-Lifshitz energy-momentum complex

The $f(R)$ gravity is a generalised form of the GR. The $f(R)$ gravity is equipped with the following field equations

$$F(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}-\nabla_{\mu}\nabla_{\nu}F(R)+g_{\mu\nu}\Box F(R)=8\pi GT^{m}_{\mu\nu},$$

where $F(R)$ is the differentiation of $f(R)$ with respect to R. $T^{m}_{\mu\nu}$ defines the standard minimally coupled stress-energy tensor. The generalized Landau-Lifshitz EMC is given by [27]

$$T^{\mu\nu} = F(R_0)\tau^{\mu\nu}_{LL}+\frac{1}{6k^2}(F(R_0)R_0-f(R_0))\partial_{\lambda}(g^{\mu\nu}x^{\lambda}-g^{\mu\lambda}x^{\nu}). \tag{4.1}$$

where $\tau^{LL}_{\mu\nu}$ represents the Landau-Lifshitz EMC worked out in the GR and κ is the coupling constant. This is a generic expression validating any $f(R)$ theory of gravity when the scaler curvature of the chosen metric happens to be constant. The 00-component becomes

$$T^{00}=F(R_0)\tau^{00}_{LL}+\frac{1}{6k^2}(F(R_0)R_0-f(R_0))\partial_{\lambda}(g^{00}x^{\lambda}-g^{0\lambda}x^{0}), \tag{4.2}$$

where τ^{00}_{LL} (the Landau-Lifshitz EMC) may also be computed the summation of energy-momentum and energy-momentum pseudo tensor as

$$\tau^{00}_{LL}=(-g)(T^{00}+t^{00}_{LL}), \tag{4.3}$$

where t^{00}_{LL} is given by

$$\begin{aligned} t^{00}_{LL} &= \frac{1}{2k}[(2\Gamma^{\gamma}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta}-\Gamma^{\gamma}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}-\Gamma^{\gamma}_{\alpha\gamma}\Gamma^{\delta}_{\beta\delta})(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\nu}g^{\alpha\beta}) \\ &+ (g^{\mu\alpha}g^{\beta\gamma})(\Gamma^{\nu}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}+\Gamma^{\nu}_{\beta\gamma}\Gamma^{\delta}_{\alpha\delta}-\Gamma^{\nu}_{\gamma\delta}\Gamma^{\delta}_{\alpha\beta}-\Gamma^{\nu}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta}) \\ &+ (g^{\nu\alpha}g^{\beta\gamma})(\Gamma^{\mu}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}+\Gamma^{\mu}_{\beta\gamma}\Gamma^{\delta}_{\alpha\delta}-\Gamma^{\mu}_{\gamma\delta}\Gamma^{\delta}_{\alpha\beta}-\Gamma^{\mu}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta}) \\ &+ (g^{\alpha\beta}g^{\gamma\delta})(\Gamma^{\mu}_{\alpha\gamma}\Gamma^{\nu}_{\beta\delta}-\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\gamma\delta})]. \end{aligned}$$

It may be noted we need here the cartesian coordinates to apply the general formula in the Eq. (4.2). For convenience and without any loss of generality, we discuss the energy-momentum distribution of magnetically charged regular black hole metric in $f(R)$ gravity using polar coordinates.

4.2. Energy distribution of magnetically charged regular black hole metric

It is mentioned here that the value of scaler curvature for the metric (2.1) is

$$R=\frac{2}{r}\left(\frac{d^2m}{dr^2}-\frac{dm}{dr}\right).$$

On putting the value of $m(r)$ the Ricci scaler comes out to be constant i.e $R_0=0$. Hence, the static spherically symmetric magnetically charged regular black hole metric is an exact solution of any $f(R)$ theory, that satisfies the constant curvature condition. Now, as the Ricci scaler R remains constant, we may find the energy density for this $f(R)$ model by implementing the generalized Landau-Lifshitz EMC. Using the Eq. (4.2) comes out with the following 00-component

$$\tau^{00} = F(R_0)\tau^{00}_{LL}+\frac{1}{2\kappa}(F(R_0)R_0)-f(R_0)(rF(R)+3f(R)). \tag{4.4}$$

Using Eq. (4.3) and after some manipulations, we get

$$\begin{aligned} \tau^{00}_{LL} &= \frac{1}{2\kappa^2}\Bigg[\left(\frac{-3}{2}\frac{w'(r)^2}{w(r)^2}-rw'(r)-rw'(r)sin^2\theta-\frac{7}{r^2}+2w(r)-2\left(\frac{cos^2\theta}{sin^2\theta}\right)\right. \\ &+ \left. w(r)sin^2\theta+\frac{-3}{2}\frac{w'(r)}{rw(r)}-\frac{2}{r}\frac{cos\theta}{sin\theta}\right)\left(1+\frac{1}{r^2w(r)}+\frac{1}{r^2w(r)sin^2\theta}\right) \\ &+ \left(\frac{1}{w(r)^2}\right)\left(\frac{w'(r)^2}{2w'(r)}+\frac{2w'(r)}{rw(r)}\right)\Bigg]. \end{aligned} \tag{4.5}$$

Substituting Eq. (4.5) in Eq. (4.4), it follows that

$$\begin{aligned}
\tau^{00} &= \frac{1}{2\kappa^2}\Bigg[\Bigg(\frac{-3}{2}\frac{w'(r)^2}{w(r)^2} - rw'(r) - rw'(r)sin^2\theta - \frac{7}{r^2} + 2w(r) - 2\Big(\frac{cos^2\theta}{sin^2\theta}\Big)\\
&+ w(r)sin^2\theta + \frac{-3}{2}\frac{w'(r)}{rw(r)} - \frac{2}{r}\frac{cos\theta}{sin\theta}\Bigg)\Big(1+\frac{1}{r^2w(r)} + \frac{1}{r^2w(r)sin^2\theta}\Big)\\
&+ \Big(\frac{1}{w(r)^2}\Big)\Big(\frac{w'(r)^2}{2w'(r)} + \frac{2w'(r)}{rw(r)}\Big)\Bigg]F(R_0)\\
&+ \frac{1}{6k^2}(F(R_0)R_0 - f(R_0))\Big(\frac{-rw'(r)}{w(r)^2} + \frac{3}{w(r)}\Big).
\end{aligned}$$

Now we discuss an important case by choosing a $f(R)$ model. It is to be noted that we can not consider a model which becomes non-analytic at $R_0 = 0$. For example we can not choose $f(R) = R + \frac{c_2}{R}$. Thus the simplest and commonly used model is

$$f(R) = R + c_1R^2.$$

For this model, the corresponding 00-component of the Landau-Lifshitz EMC, reads as

$$\begin{aligned}
\tau^{00} &= \tau^{00}_{LL}(1) = \frac{1}{2\kappa^2}\Bigg[\Bigg(\frac{-3}{2}\frac{w'(r)^2}{w(r)^2} - rw'(r) - rw'(r)sin^2\theta - \frac{7}{r^2} + 2w(r) - 2\Big(\frac{cos^2\theta}{sin^2\theta}\Big)\\
&+ w(r)sin^2\theta + \frac{-3}{2}\frac{w'(r)}{rw(r)} - \frac{2}{r}\frac{cos\theta}{sin\theta}\Bigg)\Big(1+\frac{1}{r^2w(r)} + \frac{1}{r^2w(r)sin^2\theta}\Big)\\
&+ \Big(\frac{1}{w(r)^2}\Big)\Big(\frac{w'(r)^2}{2w'(r)} + \frac{2w'(r)}{rw(r)}\Big)\Bigg].
\end{aligned}$$

4.3. Momentum Of magnetically charged regular black hole metric

We also calculate the momenta of the magnetically charged regular black hole metric by implementing the same technique and relations. τ^{0i} represents the components of momentum. For momentum calculation, Eq. (4.1) becomes

$$T^{0i} = F(R_0)\tau^{0i}_{LL} + \frac{1}{6k^2}(F(R_0)R_0 - f(R_0))\partial_\lambda(g^{0i}x^\lambda - g^{0\lambda}x^i). \tag{4.6}$$

Using Eq. (4.6), the simplified momentum components for $i = 1,2,3$ are

$$\begin{aligned}
\tau^{01} &= \frac{1}{2\kappa^2}\Bigg[\Bigg(\frac{2w(r)w'(r)}{r} - \frac{2w'(r)}{rw(r)} + (2w(r)+3)cos^2\theta - \frac{w'(r)^2}{w(r)^2} - \frac{w'(r)^2}{2} -\\
&2\frac{cos^2\theta}{sin^2\theta}\Bigg)(-1) + \Bigg(-1 - \frac{1}{w(r)^2 - \frac{1}{r^2w(r)}} - \frac{1}{r^2w(r)sin^2\theta}\Bigg)\\
&\Big(\frac{w(r)w'(r)}{2r} + rw(r)sin\theta cos\theta\Big) + \Big(-1 + w(r)^2 + \frac{w(r)}{r^2} + \frac{w(r)}{r^2sin^2\theta}\Big)\\
&\Big(\frac{w'(r)^2}{2w(r)^2} - \frac{2w'(r)}{rw(r)}\Big)\Bigg],
\end{aligned}$$

$$\begin{aligned}
\tau^{02} &= \frac{1}{2\kappa^2}\Bigg[\Bigg(\frac{-w'(r)^2}{2} + \frac{2w(r)w'(r)}{r} - 2w(r) - 2f(r)sin^2\theta\\
&- \frac{w'(r)^2}{2w(r)}\frac{2}{r}\frac{cos\theta}{sin\theta} - cos^2\theta - \frac{2w'(r)}{rw(r)} - 2(\frac{cos^2\theta}{sin^2\theta})\Bigg)\Big(\frac{-1}{r^2w(r)}\Big)\\
&+ \Big(-1 - \frac{1}{w(r)^2} - \frac{1}{r^2w(r)} - \frac{1}{r^2w(r)sin^2\theta}\Big)\Big(\frac{-4sin\theta cos\theta}{r}\Big)\Bigg],
\end{aligned}$$

$$
\begin{aligned}
\tau^{03} &= \frac{1}{2\kappa^2}\Bigg[\Bigg(\frac{2w(r)w'(r)}{r}+\frac{2w'(r)}{rw(r)}-2w(r)-3w(r)sin^2\theta+4cos^2\theta \\
&- \frac{w'(r)^2}{2}-\frac{w'(r)^2}{2w(r)^2}-\frac{2}{r^2}-\frac{2cos^2\theta}{sin^2\theta}+\frac{w'(r)}{2w'(r)r}-\frac{2cos\theta}{rsin\theta}+\frac{w'(r)cos\theta}{2w(r)sin\theta}\Bigg) \\
&\Bigg(\frac{-1}{w(r)r^2sin^2\theta}\Bigg)+\Bigg(-1+\frac{1}{w(r)^2}-\frac{1}{w(r)r^2}-\frac{1}{w(r)r^2sin^2\theta}\Bigg) \\
&\Bigg(-w(r)sin^2\theta+\frac{2}{r^2}+\frac{2cos^2\theta}{sin^2\theta}+cos^2\theta\Bigg)+\Bigg(\frac{-1}{r^2w(r)sin^2\theta}-\frac{w(r)}{r^2sin^2\theta}\Bigg) \\
&\Bigg(\frac{3}{4}\frac{w'(r)^2}{w(r)^2}-\frac{2w'(r)}{rw(r)}\Bigg)\Bigg].
\end{aligned}
$$

5. Concluding remarks

Many investigations have been put forward to address the issue of energy-momentum as it is an important conserved quantity. Unfortunately, there does not exist a general definition of energy and momentum in the GR. In this work, we focus to investigate the well-known problem of localization of energy-momentum with reference to the GR by using these three EMCs and also give some analysis under the modified gravity. In particular, we calculate the energy and momentum distributions for a static spherically symmetric magnetically charged regular black hole metric using Einstein, Bergmann-Thomson and Landau-Lifshitz EMCs. We conclude that energy turns to be well-defined and finite in these prescriptions for the black hole metric. It is worth noting that for $w=1$ the final results of all these three prescriptions gives constant energy equal to zero. It is to be noted that the unique results are obtained for Einstein, Bergmann-Thomson and Landau-Lifshitz energy-momentum prescriptions. Extension of Virbhadra's viewpoint [43] (different energy-momentum prescriptions may provide some basis to define a unique quantity) is supported by the coincidences observed in the results of these prescriptions. It is worthwhile to mention here that our results agree with [12] when we ignore the magnetic charge, i.e. $q=0$.

We have also worked on the energy and momentum distributions of the same metric in the context of modified gravity. For this purpose, we choose $f(R)$ theory of gravity and the Landau-Lifshitz energy-momentum prescription. Inspired by the recent interesting $f(R)$ gravity models, we generalize the results obtained for Landau-Lifshitz prescription. Here, we limit ourselves to investigate the Landau-Lifshitz EMC using the constant curvature assumption. The obtained energy and momentum components are well-defined for a space $r>0$. It would be an attractive task to get more generalized results by evaluating the Landau-Lifshitz EMC for non-constant Ricci scaler. Extending other EMCs in the context of $f(R)$ gravity as well in other modified theories of gravity would also be interesting.

For the comparative analysis of Energy-Momentum Distribution for Magnetically Charged Black Hole Metric, we have noted some worthy works in modified $f(\mathscr{R})$ theories of gravity in [29], [27], and [28] under some specific assumptions of different models and parameters. It has been noted that our work about Einstein, Landau-Lifshitz and Bergmann's energy-momentum complexes for static spherically symmetric magnetically charged regular black hole metric is consistent and similar in many aspects. However, a very few dissimilarities wherever they have appeared might be because of some different choices of the models and the corresponding parameters.

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A Mathematical Note on the Evolutionary Competitiveness of the Trisexual Nematode Auanema Rhodensis

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Abstract

Trisexual species with female, male and self-fertilizing hermaphrodite sub-populations are rather exceptions in nature. Though, certain nematode/ worm species, like Auanema Rhodensis, have evolved that way. Applying Kendall-like non-logistic mating functions, we provide a series of reproduction models to holistically study the iterations between the sexes and shed light on the increased population stability/ survival strength compared to bisexual species or trisexual species with non-self-fertilizing hermaphrodites. Besides the increased survival strength, the survival of such trisexual species populations is, in contrast to usually known (bisexual) species populations, entirely linked to the relation between birth and death proportionality factors, and no population thresholds are required for survival. In that sense, while mathematically studying the complete equilibria and bifurcation landscape in terms of existence and (non-linear) stability, as well as the global dynamics of these models, we provide a comprehensive analysis of the reproduction dynamics of trisexual species.

1. Introduction

Recently, the trisexual nematode/ worm Auanema Rhodensis has drawn a lot of attention as an amazing example for reproductive mode evolution and, especially, for bending the typical rules of genetics, cf. Refs. [1, 2, 3, 4]. The nematode species Auanema Rhodensis is a member of the Auanema (Rhabditina) genus that evolved several trisexual species which have female, male and hermaphrodite members, cf. Ref. [5]. Auanema Rhodensis hermaphrodite's are self-fertilizing which is very uncommon in particular in the context of having a female and male sex as well. Moreover, the way Auanema Rhodensis handles the 'X'-chromosome is also not typical at all, see Ref. [4]: both the females and hermaphrodites are XX, whereas the males have a single X. The females produce eggs bearing one X-chromosome, males produce exclusively X-bearing sperms[1], and hermaphrodites produce XX-bearing sperm and eggs with no X. Hence, crossing females and males leads to either female or hermaphrodite offspring, males and hermaphrodites produce only male offspring, and the self-fertilizing hermaphrodites produce either female or hermaphrodite offspring, too. These reproduction dynamics are sketched in Fig. 1.1.

In Ref. [1] it is shown that XX-individuals of Auanema Rhodensis become hermaphrodites or females depending on whether they undergo the non-feeding juvenile stage, called dauer larva, or not. The actual reasons for undergoing or skipping the dauer larva stage seem to be still unknown, cf. Ref. [4].

Typically, and in alignment with the commonly accepted Mendel-rules of genetics, females are XX (producing X-bearing eggs) and males are XY (producing an equal split of the heterogametic sperm with an X- and a Y-chromosome/ -gamet), which typically leads to an (almost) equal ratio of male and female offspring. Moreover, when it comes to hermaphrodites, which are pretty common among invertebrates, typically one observes them being XX-type as well and producing one X-bearing egg and one X-bearing sperm such that it needs two of them to create offspring.

[1] Though, the cell-biological mechanism of meiosis suggest that an X-male must produce two sperm cells with a haploid set of chromosomes, i.e. an X-bearing sperm together with a non-X-bearing sperm. These non-X-bearing sperms are discarded later on.

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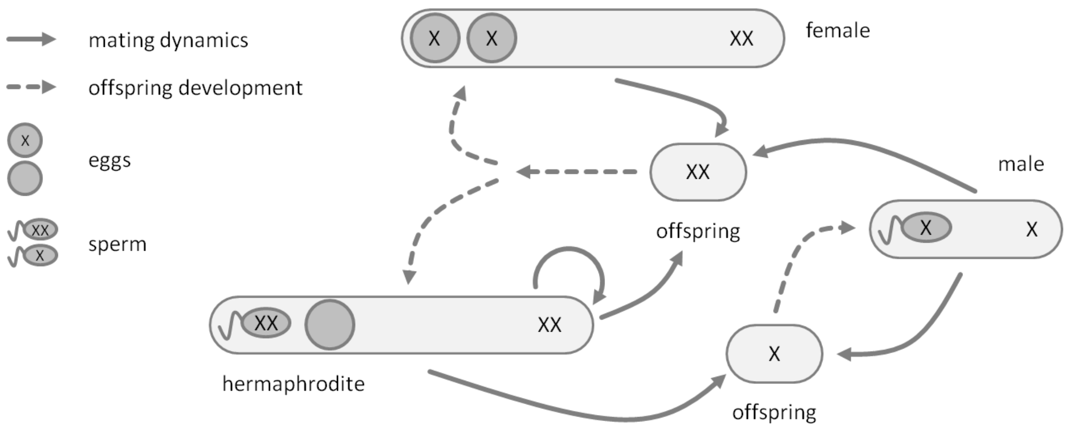


Figure 1.1: Sketch of the reproduction dynamics of the trisexual nematode Auanema Rhodensis and the interaction between the female, male and hermaphrodite species.

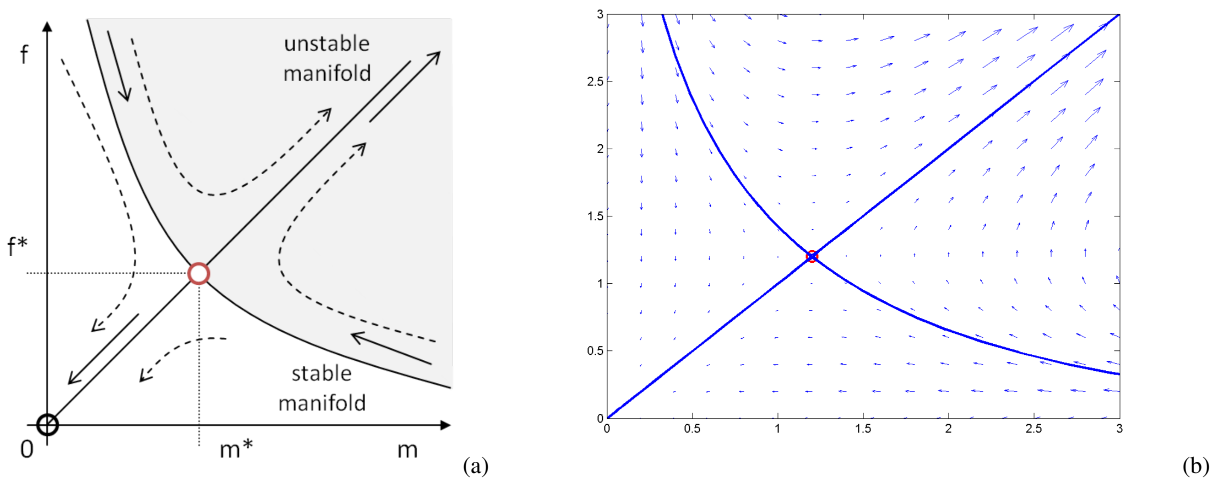


Figure 1.2: Sketch of the dynamics for a bisexual female-male population (a), and a corresponding simulation with $\alpha = 0.5$ (which leads to a symmetric behavior), $\beta = 1$, and $\delta = 0.6$ (b).

So the major question is whether the reproduction mode of Auanema Rhodensis has or has not some evolutionary competitive advantage compared to the typical reproduction mode found in non-self-fertilizing trisexual species?

Therefore, we set-up a series mathematical ordinary differential equation models for both the Auanema Rhodensis population as well as for a non-self-fertilizing trisexual population and compare the results of the corresponding mathematical discussions in terms of equilibria and invariant structures, their stability and domains of attraction. Hereby, our interest is on the early population growth dynamics to obtain insights on how fast a population can occupy a given biological niche and how few specimen are required to keep a population alive. We assume these traits (fast growth dynamics and population resilience at low numbers of individuals) as indicators of evolutionary fitness. Once the population has (nearly) occupied the habitat new effects, like competition for food or migration, take place, that are, by purpose, not considered here, as they occur due to population density and independent of how fast and resilient this population density is reached.

To get a first impression on how different the population dynamics of trisexual species will turn out to be compared to bisexual species, let us already in this introduction shortly discuss the time-evolution of a bisexual heterogametic population with XX-females and XY-males as just introduced, cf. Refs. [6, 7, 8, 9]. In accordance with the standard description of population dynamics, cf. Refs. [10, 11, 12, 13], we assume that the time-dependent female population $f : \mathbb{R}_0^+ \to \mathbb{R}$ and the time-dependent male population $m : \mathbb{R}_0^+ \to \mathbb{R}$ are completely mixed, that there are no spatial effects (i.e. we are in some kind of averaged picture) or limiting factors, like nutrients or overcrowding. Then, by suppressing the time arguments and using a Kendall-like non-logistic mating function, cf. Ref. [8], we obtain

$$\dot{f} = \beta \cdot \alpha \cdot f \cdot m - \delta \cdot f, \qquad \text{and} \qquad \dot{m} = \beta \cdot (1-\alpha) \cdot f \cdot m - \delta \cdot m,$$

where $\beta > 0$ is proportionality factor related to birth events and taking fertility, mating success, etc. into account, $\delta > 0$ is the sex-independent proportionality factor related to death events, and $\alpha \in (0,1)$ is a splitting factor of the offspring into females and males.

We rather immediately have for all admissible values of the parameters, that the non-negative quadrant with $f \geq 0$ and $m \geq 0$ is (positively and negatively) invariant, and that there are two equilibrium points: the origin $(f,m) = (0,0)$, i.e. the complete extinction point, and the point $(f^*,m^*) := \left(\frac{\delta}{\beta \cdot (1-\alpha)}, \frac{\delta}{\beta \cdot \alpha}\right)$. For all admissible values of the parameters the origin is asymptotically stable, which means that too small populations will eventually get extinct. Hence, one is typically interested in population threshold values at which the dynamic attraction of the origin is no longer present and where one can guarantee mathematical long-time survival of the complete population. The second equilibrium point (f^*,m^*) turns out to be locally an unstable saddle with one stable and one unstable manifold.

Self-Fertilizing Hermaphrodites		Non-Self-Fertilizing Hermaphrodites	
Model	Description	Model	Description
Appendix E1	• f + m → f • h + m → m • h → h	Appendix E2	• f + m → f • h + m → m • h + h → h
A	• f + m, h → f • h + m → m • h → h	B	• f + m, h + h → f • h + m → m • h + h → h
C	• f + m, h → f • h + m → m • h, f + m → h	D	• f + m, h + h → f • h + m → m • h + h, f + m → h
Appendix F1	• f + m → f • h + m → m • h, f + m → h	Appendix F2	• f + m → f • h + m → m • h + h, f + m → h

f: females, m: males, h: hermaphrodites

Figure 1.3: Overview of the different models that we discuss in this article for trisexual species with self-fertilizing and non-self-fertilizing hermaphrodites. A chain of symbols like $f+m,h\to f$ indicates that the offspring of both the crossing of females and males and the self-fertilization of hermaphrodites evolves to females.

Fig. 1.2 gives a sketch of the phase space dynamics and some results of a numerical simulation for this bisexual population model. Besides others, it illustrates that the stable manifold of the saddle point equilibrium acts as a separatrix between the region of extinction, i.e. the domain of attraction of the origin, and the region of survival/ (unbounded) growth (the gray area in the figure). Certainly, our model does not capture the whole picture of a population, as at some time limitations to growth will be present (modeled, e.g., via logistic functions), though we see that survival of a bisexual population requires a certain threshold on the population or threshold mixture between the sexes.

Let us return to the mathematical discussion of trisexual species. In Refs. [14, 15] a heterogametic species with the three sexes XX, XY, and YY is studied. It is suggested that, under the conditions discussed there, such trisexual systems are not stable over time, and are destined to converge to bisexual systems. In Ref. [16] numerical computations, based on a stochastic evolutionary algorithm, are performed that indicated that self-fertilizing strategies are rather stable in evolutionary settings and out-competed sexual ones. Though, changing environments seem to favor bisexual species. The reproduction of the three- or pluri-/ multisexual species studied therein are such that three or more members of a species have to meet in order to produce offspring.

In this article we augment this body of knowledge by studying the very specific situation of the self-fertilizing trisexual species Auanema Rhodensis with a bisexual reproduction between females and males as well as hermaphrodites and males, and a asexual reproduction of the hermaphrodites. Moreover, as already stated, we compare this situation with that of a generic trisexual species, where, instead of a asexual reproduction, two hermaphrodites are required to cross.

In particular, we analyze several three-dimensional models, cf. Fig. 1.3: The first one is a model for Auanema Rhodensis with its self-fertilizing hermaphrodites (Model A), and the second one is a model for a trisexual species with non-self-fertilizing hermaphrodites (Model B). Next, the option of female-male offspring to mature to hermaphrodites is examined. Model C applies the Auanema Rhodensis case with its self-fertilizing hermaphrodites, and Model C the case of a trisexual species with non-self-fertilizing hermaphrodites. Certain special cases are finally discussed in the appendices, like the situation that female-male offspring evolves to females and hermaphrodite offspring evolves to hermaphrodites only (Models E1 and E2), as well as both a part of female-male offspring and the whole of hermaphrodite offspring evolves to hermaphrodites and the second part of female-male offspring evolves to females (Models F1 and F2). In all these models the origin is an equilibrium point and for some parameter vales it is even unstable with a certain bifurcation scenario in Model A. Moreover, for specific values of the parameters a line of additional equilibria in the interior of the positive orthant exists.

Our key insights are that, compared to bisexual species populations, the introduction of self-fertilizing hermaphrodite members stabilizes a trisexual population. Moreover, instead of species thresholds for the female and male sub-populations, the survival of such a trisexual species populations is entirely linked to the relation between birth and death proportionality factors. I.e., no matter how small the population may become recovery is certain, provided the birth and death proportionality factors are favorable. (In contrast, a too small bi-sexual species population will for sure become extinct.)

The remainder of this article is structured as follows: In Section 2 we set-up a deterministic Model of the population dynamics of Auanema Rhodensis (Model A) and give a complete description of its equilibria in terms of existence and (non-linear) stability, as well as a description of the global dynamics of this model. Analogously, Section 3 mathematically studies Model B, Section 4 examines Model C, and Model D is analyzed in Section 5. The mathematical discussion of the somewhat restricted models A and B before that of the complete models C and D allows us to describe and develop the (standard) methods necessary for the complete model in the easier cases and then being able to refer to the previous steps later on for the required solution steps. Thus, at these later stages, we can focus on the specific and interesting additional challenges of the complete models. Finally, the main part of the text concludes with a resume in Section 6, where we summarize and interpret our results obtained by our models and their comparison with each other and bisexual species populations. As stated, the appendices study specific special cases to complete the discussion of the reproduction dynamics of trisexual species populations. First, in appendix A, we state and analyze mathematical models for species with neither hermaphrodites-producing females nor females-producing hermaphrodites, and second, in appendix B, we state and analyze mathematical models for species with hermaphrodites-producing females and no-females-producing hermaphrodites.

2. A first mathematical model for the Auanema Rhodensis population (Model A)

In a first approach to study the competitiveness of a Auanema Rhodensis population, let us assume that there is a complete bias towards female offspring after crossing female and male members of the population, and that with some splitting factor hermaphrodite offspring are females or hermaphrodites.

Following the description of the reproductive dynamics given in the introduction, let $f:\mathbb{R}_0^+\to\mathbb{R}$ denote the time-dependent female Auanema Rhodensis population, $m:\mathbb{R}_0^+\to\mathbb{R}$ the corresponding male population, and $h:\mathbb{R}_0^+\to\mathbb{R}$ the corresponding hermaphrodite population. Assuming complete mixing and spatial homogeneity of the Auanema Rhodensis population, then, according to the standard description of population dynamics, cf. Refs. [10, 11, 12, 13], the deterministic population dynamics of Auanema Rhodensis are given by

$$\dot{f} = \beta_1\cdot f\cdot m+(1-\alpha)\cdot\beta_3\cdot h-\delta\cdot f\,, \tag{2.1}$$

$$\dot{m} = \beta_2\cdot h\cdot m-\delta\cdot m\,, \tag{2.2}$$

$$\dot{h} = \alpha\cdot\beta_3\cdot h-\delta\cdot h\,, \tag{2.3}$$

where we suppressed the time arguments of the population functions $f(t)$, $m(t)$, and $h(t)$. Here, $\beta_1,\beta_2,\beta_3>0$ are proportionality factors related to birth events, $\delta>0$ is the sex-independent proportionality factor related to death events, and $\alpha\in(0,1)$ a splitting factor of the hermaphrodite offspring into females and hermaphrodites[2]. For further reference, we will denote this set of equation Eqs. (2.1)-(2.3) as Model A.

In view of later simulations, we see that Eq. (2.3) decouples from the system Eqs. (2.1)-(2.3) such that $h(t)=h_0\cdot\exp((\alpha\cdot\beta_3-\delta)\cdot t)$, $t\in\mathbb{R}_0^+$, where $h_0\geq 0$ is the initial hermaphrodite population at the initial time $t_0=0$.

2.1. A mathematical discussion of Model A's equilibrium points and their stability

For all admissible values of the parameters, we immediately have that the non-negative orthant $\mathscr{P}_0^+ := \{(f,m,h)\in\mathbb{R}^3 : f,m,h\geq 0\}$ is invariant, and that the complete population eventually becomes extinct for a vanishing hermaphrodite population.

Moreover, for all admissible values of the parameters, the origin $(f,m,h)=(0,0,0)$ is an equilibrium (complete extinction equilibrium), where the local linearized dynamics around the origin are determined by $\dot{x}=A_1x$ with $x=(f,m,h)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f\\ m\\ h\end{pmatrix} = \underbrace{\begin{pmatrix} -\delta & 0 & (1-\alpha)\cdot\beta_3\\ 0 & -\delta & 0\\ 0 & 0 & \alpha\cdot\beta_3-\delta\end{pmatrix}}_{=:A_1}\begin{pmatrix} f\\ m\\ h\end{pmatrix}+\begin{pmatrix}\beta_1\cdot f\cdot m\\ \beta_2\cdot h\cdot m\\ 0\end{pmatrix}.$$

Due to the triangular structure of A_1 the eigenvalues of A_1 can be read off as $-\delta<0$ with algebraic multiplicity two and corresponding 2-dimensional eigenspace spanned by $(1,0,0)$ and $(0,1,0)$, as well as $\alpha\cdot\beta_3-\delta$ with the corresponding eigenspace spanned by $(1-\alpha,0,\alpha)$. Next, the Theorem of Linearized Stability implies the following stability properties of the origin in the complete system Eqs. (2.1)-(2.3): First, for $\alpha\cdot\beta_3-\delta<0$ the origin is an asymptotically stable equilibrium, second, for $\alpha\cdot\beta_3-\delta>0$ the origin is a unstable equilibrium. Finally, taking the non-linear structure into account, we have that for $\alpha\beta_3-\delta=0$ the origin is a Lyapunov stable (though not asymptotically stable) hyperbolic equilibrium of the complete system Eqs. (2.1)-(2.3).

Hence, at $\alpha\cdot\beta_3-\delta=0$ a bifurcation takes place. Biologically, this bifurcation point corresponds to the situation where the hermaphrodite populations remains constant for all times (as the right-hand side of Eq. (2.2) vanishes). Moreover, for parameters that allow $\alpha\cdot\beta_3-\delta>0$ a survival of the population is guaranteed for all initial conditions with $h_0>0$ due to the unstable/ repulsive character of the origin and its unstable direction $(1-\alpha,0,\alpha)$, as we will see in more detail in Sections 2.2 and 2.3.

Besides the origin, for parameters with $\alpha\cdot\beta_3=\delta$, i.e. at the bifurcation point, every point of the form $(f,m,h)=(f_0,0,h_i)$ is an equilibrium as well, where $f_0 := f_0(h_i) := (1-\alpha)\cdot\beta_3\cdot h_i\cdot\delta^{-1}$, and $h_i\geq 0$ is a feasible initial value of Eq. (2.3) which reduces in the case of $\alpha\cdot\beta_3=\delta$ to $\dot{h}=0$. This gives rise to a line of equilibria

$$\Gamma_0 := \left\{(f_0,0,h_i)\in\mathscr{P}_0^+ : f_0=f_0(h_i)=(1-\alpha)\cdot\beta_3\cdot h_i\cdot\delta^{-1}\,, h_i>0\right\}$$

at the $\{m=0\}$-face of the non-negative orthant $\mathscr{P}_0^+$ that emerges from the origin.

In order to determine the stability of a member $(f_0,0,h_i)\in\Gamma_0$, we translate the coordinate system into $(f_0,0,h_i)$, i.e. $(f,m,h)\mapsto(f+f_0,m,h+h_i)$, and discuss the Taylor-expansion of the appropriately modified complete system Eqs. (2.1)-(2.3) about this new coordinate frame origin. Here, the correspondingly transformed equivalent dynamics read

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f\\ m\\ h\end{pmatrix} = \underbrace{\begin{pmatrix} -\delta & \beta_1\cdot f_0 & (1-\alpha)\cdot\beta_3\\ 0 & \beta_2\cdot h_i-\delta & 0\\ 0 & 0 & 0\end{pmatrix}}_{=:B_1}\begin{pmatrix} f\\ m\\ h\end{pmatrix}+\begin{pmatrix}\beta_1\cdot f\cdot m\\ \beta_2\cdot h\cdot m\\ 0\end{pmatrix}.$$

[2] According to the description of Auanema Rhodensis hermaphrodites as being similar to the females of the species in the biological literature, cf. [1, 2, 5, 3, 4], there seems to be pure self-fertilization of the hermaphrodites only, i.e. two hermaphrodites of this species do not match. In other words, biological insights suggest that hermaphrodites of the Auanema Rhodensis species have the same sexual organs as females of this species.

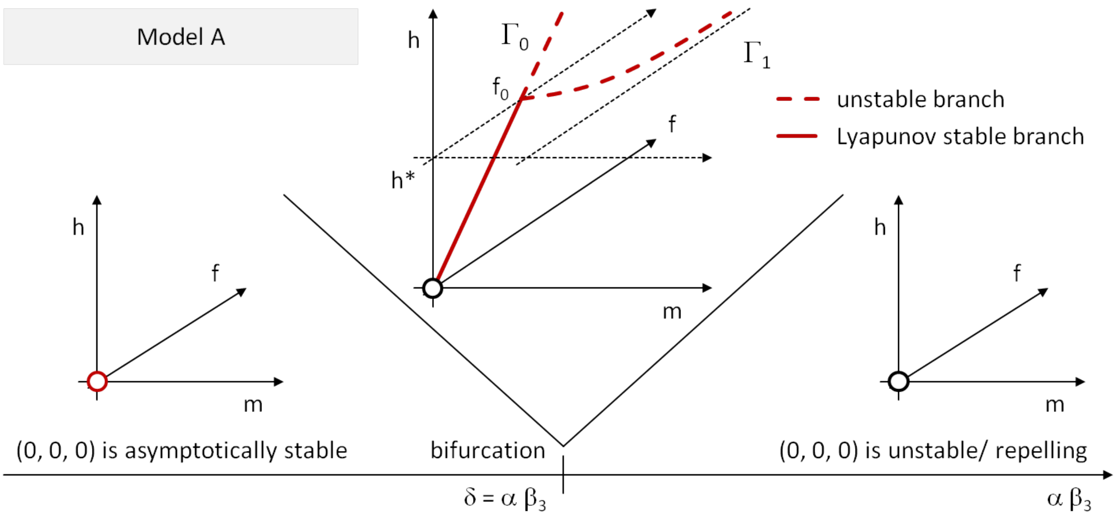


Figure 2.1: Sketch of the stability bifurcation of the extinction equilibrium at the origin and the occurrence of the lines Γ_0 and Γ_1 of additional equilibria at the bifurcation point $\alpha \cdot \beta_3^{-1} = \delta$.

The eigenvalues of B_1 are 0 with the corresponding eigenspace being spanned by $((1-\alpha),0,\delta)$, and $-\delta < 0$ with the corresponding eigenspace being spanned by $(1,0,0)$, as well as

$$\beta_2 \cdot h_i - \delta \begin{cases} < 0, & \text{for } h_i < h^*, \\ = 0, & \text{for } h_i = h^* := \delta \cdot \beta_2^{-1} = \alpha \cdot \beta_3 \cdot \beta_2^{-1}, \\ > 0, & \text{for } h_i > h^*, \end{cases}$$

with the corresponding eigenspace being spanned by $(\beta_1 \cdot f_0, \beta_2 \cdot h_i, 0)$. Hence, the elements of Γ_0 are Lyapunov stable equilibrium points for $0 \le h_i < h^*$ and unstable ones for $h^* \le h_i$.

Moreover, there are additional equilibrium values (f^*, m^*, h^*) in the interior of $\mathscr{P}_0^+$ only if the parameters are such that $\alpha \cdot \beta_3 = \delta$, i.e. again at the bifurcation point. From Eq. (2.2) we obtain $h^* = \delta \cdot \beta_2^{-1} = \alpha \cdot \beta_3 \cdot \beta_2^{-1}$. Next, from Eq. (2.1) we get, after some small algebraic manipulations[3],

$$0 < m^* = \frac{\alpha \cdot \beta_3}{\beta_1 \cdot \beta_2 \cdot f^*} \cdot (\beta_2 \cdot f^* - (1-\alpha) \cdot \beta_3) \overset{f^* \to \infty}{\longrightarrow} \alpha \cdot \beta_3 \cdot \beta_1^{-1} =: m_{\max},$$

and $f^* > f_0 = f_0(h^*) = (1-\alpha) \cdot \beta_3 \cdot \beta_2^{-1}$. In particular, these additional equilibria (f^*, m^*, h^*) thus form a one-dimensional family Γ_1 depending on the feasible values of f^*.

In order to determine the stability of a member $(f^*, m^*, h^*) \in \Gamma_1$, we progress as outlined previously for the members of Γ_0: We translate the coordinate system into (f^*, m^*, h^*), i.e. $(f,m,h) \mapsto (f+f^*, m+m^*, h+h^*)$, and discuss the Taylor-expansion of the appropriately modified complete system Eqs. (2.1)-(2.3) about this new coordinate frame origin. Here, the correspondingly transformed equivalent dynamics read

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f \\ m \\ h \end{pmatrix} = \underbrace{\begin{pmatrix} \beta_1 \cdot m^* - \delta & \beta_1 \cdot f^* & (1-\alpha)\cdot\beta_3 \\ 0 & 0 & \beta_2 \cdot m^* \\ 0 & 0 & 0 \end{pmatrix}}_{=:C_1} \begin{pmatrix} f \\ m \\ h \end{pmatrix} + \begin{pmatrix} \beta_1 \cdot f \cdot m \\ \beta_2 \cdot h \cdot m \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} \beta_1 \cdot m^* - \delta &= \beta_1 \cdot \tfrac{\alpha \cdot \beta_3}{\beta_1 \cdot \beta_2 \cdot f^*} \cdot (\beta_2 \cdot f^* - (1-\alpha)\cdot\beta_3) - \alpha \cdot \beta_3 \\ &= \tfrac{\alpha \cdot \beta_3}{\beta_2 \cdot f^*} \cdot (\beta_2 \cdot f^* - (1-\alpha)\cdot \beta_3 - \beta_2 \cdot f^*) < 0. \end{aligned}$$

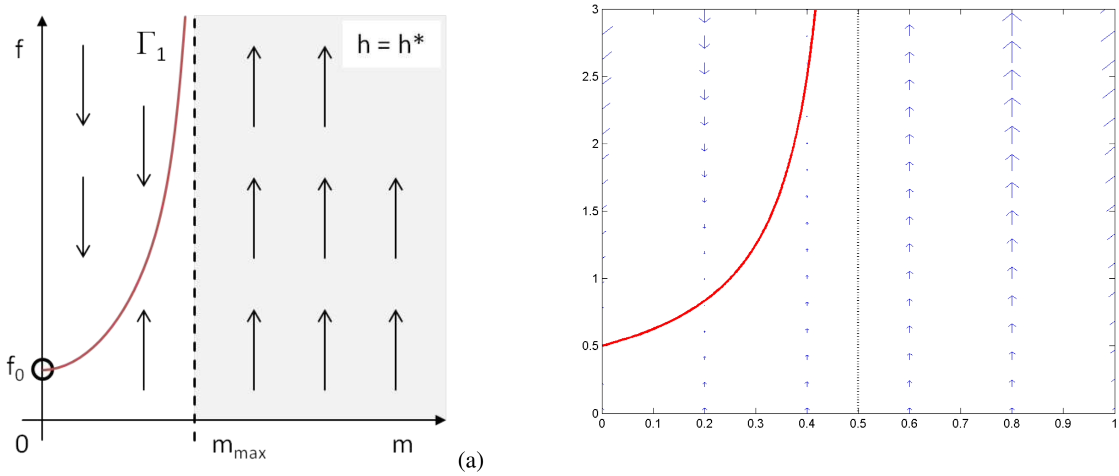
The eigenvalues of C_1 are $\beta_1 \cdot m^* - \delta < 0$ with the corresponding eigenspace being spanned by $(1,0,0)$, and the algebraically two-dimensional eigenvalue 0 with $(-\beta_1 \cdot f^*, \beta_1 \cdot m^* - \delta, 0)$ spanning the corresponding geometrically one-dimensional eigenspace. Taking the non-linear terms into account, we hence have that each element $(f^*, m^*, h^*) \in \Gamma_1$ is an unstable hyperbolic equilibrium of the complete system Eqs. (2.1)-(2.3).

Note, that further equilibria of Eqs. (2.1)-(2.3) do not exist and we have established a complete classification of the equilibria of our system including their stability. Fig. 2.1 summarizes our results on the types and stability properties of these equilibria as well as the underlying bifurcation scenarios.

2.2. The global dynamics at the bifurcation point

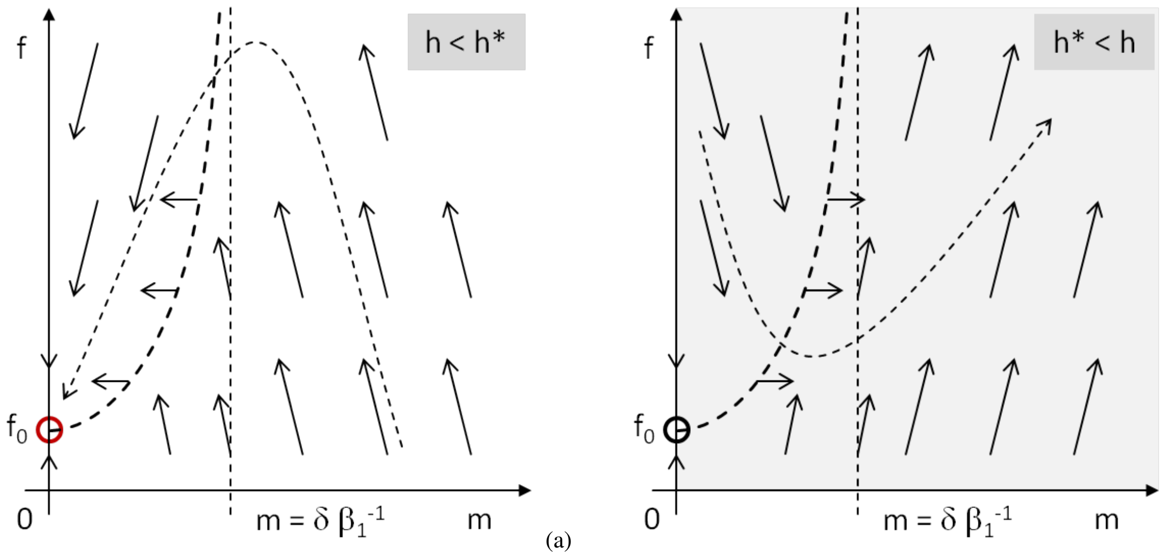
Once knowing what happens locally around the equilibria $(0,0,0)$, $(f_0,0,h_i) \in \Gamma_0$, and $(f^*, m^*, h^*) \in \Gamma_1$, we are interested in the global dynamics, i.e. the complete picture of the dynamics in the phase space $\mathscr{P}_0^+$. Let us start with parameters leading to the origin's bifurcation

[3] By virtue of Eq. (2.1), $f \neq 0$ holds at an equilibrium other than the origin.



(a) (b)

Figure 2.2: Sketch of the dynamics of Model A at the bifurcation parameters $\alpha \cdot \beta_3 = \delta$ with $h \equiv h^*$ (a), and a corresponding simulation with $\alpha = 0.5$, $\beta_1 = \beta_2 = \beta_3 = 1$, and $\delta = 0.5$ (b).



(a) (b)

Figure 2.3: Sketch of the dynamics of Model A at the bifurcation parameters $\alpha \cdot \beta_3 = \delta$ with $h < h^*$ (a), and $h > h^*$ (b).

point $\alpha \cdot \beta_3 = \delta$, and discuss f-m-planar slices with different constant values of $h = \text{const}$. For $\alpha \cdot \beta_3 = \delta$ the governing system Eqs. (2.1)-(2.3) reduces to

$$\dot{f}(t) = \beta_1 \cdot f(t) \cdot m(t) + (1-\alpha) \cdot \beta_3 \cdot h_i - \delta \cdot f(t)\,, \tag{2.4}$$

$$\dot{m}(t) = (\beta_2 \cdot h_i - \delta) \cdot m(t)\,, \tag{2.5}$$

together with $\dot{h}(t) = 0$, where $f(0) = f_i \geq 0$, $m(0) = m_i \geq 0$, and $h(0) = h_i \geq 0$. Here, Eq. (2.5) becomes $\dot{m}(t) = 0$ (i.e. $m(t) = m_i$) for $h_i = h^* = \delta \cdot \beta_2^{-1} = \alpha \cdot \beta_3 \cdot \beta_2^{-1}$, such that in this case the dynamics in the f-m-plane are governed by vectors parallel to the f-axis (the m-value is constant) pointing to $f = 0$ or to $f \to \infty$ corresponding to the sign determined in Eq. (2.4).

Next, for $h_i < h^*$ we have $\dot{m} < 0$ for all times and thus that the direction field is always pointing towards $m = 0$ and as well either towards $f = 0$, $f \to \infty$ or being stationary (for $\dot{f} = 0$) depending on the sign determined in Eq. (2.4). Finally, for $h_i > h^*$ we have $\dot{m} > 0$ for all times and thus that the direction field is always pointing towards $m \to \infty$ and as well either towards $f = 0$, $f \to \infty$ or being stationary (for $\dot{f} = 0$) depending on the sign determined in Eq. (2.4).

As some first illustration at the bifurcation point $\alpha \cdot \beta_3 = \delta$, let us assume (i) that $\alpha = 0.5$, i.e. hermaphrodites give birth to the same amount of females and hermaphrodites, and (ii) that the birth proportionality factors $\beta_1, \beta_2, \beta_3$ are equal. In this situation, we have $0 < m^* < h^* < f^*$. Fig. 2.2 uses these parameters to show a sketch of the slice of the dynamics for $h \equiv h^*$. Here, the direction of the evolution vector field, the line of equilibria Γ_1, and the region of survival/ (unbounded) growth (the gray area in the figure), are included, as well as the results of a corresponding simulations that takes that actual strength of the evolution vector field into account (length of the direction vectors). Analogously, Fig. 2.3 shows sketches of the typical dynamics for $h < h^*$ as well as for $h^* < h$. Again, the region of survival/ (unbounded) growth is displayed by the gray area in the figure. Altogether this allows us to compose Fig. 2.1, Fig. 2.2, and Fig. 2.3 together and to receive the complete picture of the governing dynamics at the bifurcation point.

What remains to discuss are the dynamics for values of the parameters with $\alpha \cdot \beta_3 \neq \delta$.

2.3. A global view outside of the bifurcation point

As we have already seen, if the hermaphrodites vanish then the extinction of the complete population happens for sure. By virtue of the decoupling of the hermaphrodite dynamics from those of females and males this already gives the essential indication for survival of the complete population.

If $0 < \alpha \cdot \beta_3 < \delta$ (and $h_i > 0$), then Eq. (2.3) leads to $\dot{h}(t) < 0$ and hence, independent of the initial values for the females and males, the complete population will eventually get extinct. In this case, the origin is a globally asymptotically stable equilibrium governing the dynamics.

If $\delta < \alpha \cdot \beta_3$ (and $h_i > 0$), then according to Eq. (2.3) the hermaphrodite population continues to grow (beyond any bounds) as $\dot{h}(t) > 0$, i.e. we are in the setting we denoted as a survival regime. Next considering Eq. (2.2), eventually a value of h is hence reached where $\beta_2 \cdot h > \delta$, and hence eventually the size of the male population will increase (beyond any bounds) together with h (presumed $m_i > 0$). In complete analogy, the size of the female population will eventually increase (beyond any bounds), even if their initial size vanishes ($f_i = 0$).

We thus see that in Model A the survival of Auanema Rhodensis is entirely linked to the relation between the birth ($\alpha \cdot \beta_3$) and death (δ) proportionality factors. This as well contrasts the dynamics of bisexual species where survival is linked to certain population thresholds and, due to the asymptotic stability of the origin, too small populations are determined to get extinct for all choices of the birth and death proportionality factors.

The results presented in Ref. [1] indicate that XX-individuals may be forced to develop into hermaphrodites instead of females (and vice versa) based on the juvenile stages they undergo. Hence, there is also some kind of (unusual) mechanism that increases the hermaphrodite population and may additionally stabilize the complete population. We will discuss this as Model C in Section 4.

Next, let us compare the our self-fertilizing hermaphrodite nematode Auanema Rhodensis (Model A) with a species that has non-self-fertilizing hermaphrodite in order to discuss competitive advantages of the one or other reproduction model.

3. Trisexual species with non-self-fertilizing hermaphrodites – Model B

In order to gain insights into the dynamics of trisexual species, this section mathematically analyzes a model equivalent to Model A though with non-self-fertilizing hermaphrodites. I.e., two members of the hermaphrodite species are required to produce offspring such that the crossing terms with a single h in Eqs. (2.1) and (2.3) are replaced by a term with h^2. Assuming again complete mixing and spatial homogeneity of the involved populations, then, according to the standard description of population dynamics, this leads to the following ordinary differential Model B[4]:

$$\dot{f} = \beta_1 \cdot f \cdot m + (1-\alpha) \cdot \beta_3 \cdot h^2 - \delta \cdot f\,, \tag{3.1}$$

$$\dot{m} = \beta_2 \cdot h \cdot m - \delta \cdot m\,, \tag{3.2}$$

$$\dot{h} = \alpha \cdot \beta_3 \cdot h^2 - \delta \cdot h\,. \tag{3.3}$$

As in Model A, we have for all admissible values of the parameters, that $\mathscr{P}_0^+$ is invariant, and that the population eventually becomes extinct for vanishing numbers of hermaphrodite population. Again, the dynamics of the hermaphrodite population $\dot{h} = h \cdot (\alpha \cdot \beta_3 \cdot h - \delta)$ decouples from the other governing equations, with a repulsive equilibrium at $h^* = \delta \cdot (\alpha \cdot \beta_3)^{-1}$ that constitutes a threshold value first of all for the hermaphrodite sub-population and hence for the total population as well.

The origin $(f,m,h) = (0,0,0) \in \mathscr{P}_0^+$ is an equilibrium (complete extinction equilibrium). In this case, the local linearized dynamics around the origin are determined by $\dot{x} = A_2 x$ with $x = (f,m,h)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f \\ m \\ h \end{pmatrix} = \underbrace{\begin{pmatrix} -\delta & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -\delta \end{pmatrix}}_{=:A_2} \begin{pmatrix} f \\ m \\ h \end{pmatrix} + \begin{pmatrix} \beta_1 \cdot f \cdot m + (1-\alpha) \cdot \beta_3 \cdot h^2 \\ \beta_2 \cdot h \cdot m \\ \alpha \cdot \beta_3 \cdot h^2 \end{pmatrix}.$$

Hence, the origin is an asymptotically stable equilibrium in model B for all admissible parameter values. Biologically, these are somewhat bad news as it tells us that once the population numbers are too small the population will get extinct for sure.

Moreover, there is a line $\Gamma_2 \subset \mathscr{P}_0^+$ of additional equilibria $(f^*, m^*, h^*) \in \Gamma_2$. From Eq. (3.3) we effortlessly get $h^* = \delta \cdot (\alpha \cdot \beta_3)^{-1}$, and by virtue of Eq. (3.2) we obtain that the line of equilibria Γ_2 exists only for parameter values where $\alpha \cdot \beta_3 = \beta_2$. Next, from Eq. (3.1) we get[5]

$$0 \leq m^* = \frac{\delta}{\alpha \cdot \beta_1 \cdot f^*} \cdot (\alpha \cdot f^* - (1-\alpha) \cdot h^*) \overset{f^* \to \infty}{\longrightarrow} \delta \cdot \beta_1^{-1} = m_{\max}\,,$$

where $f^* \geq (1-\alpha) \cdot \alpha^{-1} \cdot h^*$. After linearly translating the coordinate system into $(f^*, m^*, h^*) \in \Gamma_2$, the Taylor-expansion of the appropriately modified complete system Eqs. (3.1)-(3.3) about this new coordinate frame origin reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f \\ m \\ h \end{pmatrix} = \underbrace{\begin{pmatrix} \beta_1 \cdot m^* - \delta & \beta_1 \cdot f^* & 2 \cdot (1-\alpha) \cdot \beta_3 \cdot h^* \\ 0 & 0 & \beta_2 \cdot m^* \\ 0 & 0 & \delta \end{pmatrix}}_{=:B_2} \begin{pmatrix} f \\ m \\ h \end{pmatrix} + \begin{pmatrix} \beta_1 \cdot f \cdot m + (1-\alpha) \cdot \beta_3 \cdot h^2 \\ \beta_2 \cdot h \cdot m \\ \alpha \cdot \beta_3 \cdot h^2 \end{pmatrix}.$$

[4] Here, we are discussing the situation of non-self-fertilizing hermaphrodites that can additionally pair with males in order to have a model comparable to Model A. From a biological perspective it may make sense to assume that in this situation hermaphrodites have the same sexual organs as males and females and can thus also mate with females of the species. Considering, in terms of a comparison to the biological situation witnessed at Auanema Rhodensis, the gametic structure of female eggs and hermaphrodite sperms would, in such a pairing, result in XXX-offspring, that we can discard as not fit for survival.

[5] By virtue of Eq. (3.1), $f \neq 0$ holds at an equilibrium other than the origin.

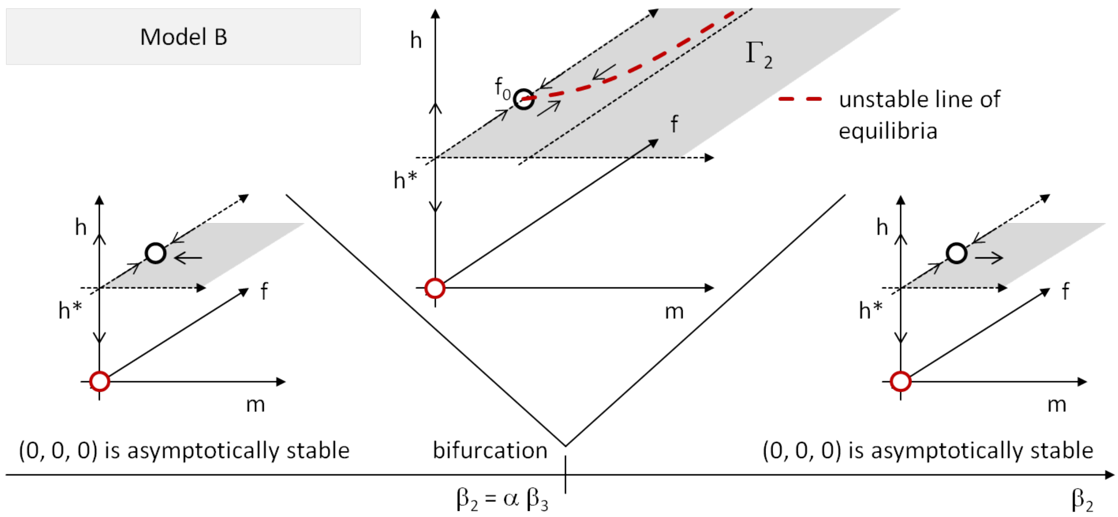


Figure 3.1: Sketch of the stability bifurcation of the equilibrium $(0,0,h^*)$ at and the occurrence of the line Γ_2 of additional equilibria at the bifurcation point $\alpha \cdot \beta_3 = \beta_2$.

The three distinct eigenvalues of B_2 are $\beta_1 \cdot m^* - \delta < 0$ with its corresponding eigenspace being spanned by $(1,0,0)$, 0 with its corresponding eigenspace being spanned by $(\beta_1 \cdot f^*, \beta_1 \cdot m^*, 0)$, and $\delta > 0$ with its corresponding eigenspace being spanned by $(0, \beta_2 \cdot m^*, \delta)$. Hence, Γ_2 is a line of unstable (flattened) saddle point equilibria.

Let $f(0) =: f_i$, $m(0) =: m_i$, and $h(0) =: h_i$. If $\alpha \cdot \beta_3 = \beta_2$ and $0 < h_i < h^*$, then Eq. (3.3) leads to $\dot{h}(t) < 0$ and hence, independent of the initial values for the females and males as well as the values of the other parameter, the complete population will eventually get extinct. On the other hand, if $\alpha \cdot \beta_3 = \beta_2$ and $h^* < h_i$ then according to Eq. (3.3) the hermaphrodite population continues to grow (beyond any bounds) as $\dot{h}(t) > 0$, i.e. we are in a survival regime. Next considering Eq. (3.1), eventually a value of h is hence reached where $\beta_2 \cdot h > \delta$, and hence eventually the size of the male population will increase (beyond any bounds) together with h (presumed $m_i > 0$). In complete analogy, the size of the female population will eventually increase (beyond any bounds), even if their initial size vanishes ($f_i = 0$). In particular, in the case $\alpha \cdot \beta_3 = \beta_2$ the affine plane $h = h^*$ serves as a 2-dimensional separatrix between the complete extinction regime and the survival regime.

What remains is to discuss the dynamics if $\alpha \cdot \beta_3 \neq \beta_2$. Let $f_0 := f(0)$, $m_0 := m(0)$, and $h_0 := h(0)$ denote the initial population sizes of the female, male, and hermaphrodite sub-populations. Besides the origin, the point $(f,m,h) = \left(\frac{1-\alpha}{\alpha} \cdot h^*, 0, h^*\right)$, with $h^* = \delta \cdot (\alpha \cdot \beta_3)^{-1}$, is an equilibrium for all admissible parameter values. Due to the instability in the h-direction, this additional equilibrium point is unstable.

First for $\beta_2 < \alpha \cdot \beta_3$, based on Eq. (3.3) if, $h_0 < h^* = \delta \cdot (\alpha \cdot \beta_3)^{-1}$, then, irrespective of the size of the female and male sub-populations, the hermaphrodite sub-population and hence the complete population eventually gets extinct. If $h_0 = h^* = \delta \cdot (\alpha \cdot \beta_3)^{-1} < \delta \cdot \beta_2^{-1}$, then the hermaphrodite sub-population stays constant for all times, i.e. $h(t) \equiv h^*$ for all $t \in \mathbb{R}_0^1$, whereas the male sub-population eventually gets extinct due to Eq. (3.2). Next, in this limit $h(t) \equiv h^*$ and $m \equiv 0$ Eq. (3.1) becomes $\dot{f} = \delta \cdot \left(\frac{(1-\alpha)\cdot\delta}{\alpha^2 \beta_2} - f\right)$ such that all initial conditions (f_0, m_0, h^*) in the affine plane $h = h^*$ converge towards the unstable equilibrium $(f,m,h) = \left(\frac{1-\alpha}{\alpha} \cdot h^*, 0, h^*\right)$. Finally, if $h_0 > h^* = \delta \cdot (\alpha \cdot \beta_3)^{-1}$, then the hermaphrodite sub-population and thus both, the male and the female sub-populations will eventually, grow beyond all bounds (survival regime). Hence, in the case $\beta_2 < \alpha \cdot \beta_3$ the affine plane $h = h^*$ with its in this plane attractive equilibrium $(f,m,h) = \left(\frac{1-\alpha}{\alpha} \cdot h^*, 0, h^*\right)$ serves as a separatrix between the complete extinction regime and the survival regime.

Second for $\beta_2 > \alpha \cdot \beta_3$, we analogously conclude complete extinction if $h_0 < h^* = \delta \cdot (\alpha \cdot \beta_3)^{-1}$. If $h_0 = h^* = \delta \cdot (\alpha \cdot \beta_3)^{-1} > \delta \cdot \beta_2^{-1}$, then the hermaphrodite sub-population stays constant for all times, i.e. $h(t) \equiv h^*$ for all $t \in \mathbb{R}_0^1$, whereas for positive initial conditions the male sub-population eventually grows beyond all bounds due to Eq. (3.2). Then, due to Eq. (3.1), the female sub-population grows beyond all bounds as well. For $m_0 = 0$ an initial condition $(f_0, 0, h^*)$ either is or eventually converges towards the equilibrium $(f,m,h) = \left(\frac{1-\alpha}{\alpha} \cdot h^*, 0, h^*\right)$. If $h_0 > h^* = \delta \cdot (\alpha \cdot \beta_3)^{-1}$, then as above, we can conclude that we are again in a survival regime for all initial conditions (f_0, m_0, h_0). Hence, in the case $\beta_2 > \alpha \cdot \beta_3$ the affine plane $h = h^*$ serves again as a separatrix between the complete extinction regime and the survival regime.

The complete bifurcation diagram is sketched in Fig. 3.1. Note that further equilibria of Eqs. (3.1)-(3.3) do not exist and we have thus indeed established a complete classification of the equilibria of our system including their stability.

In summary we can already conclude that survival/ proliferation of the species in model A depends on the relation between the birth and death parameters. On the other hand, independent of the feasible parameter values, the non-self-fertilizing exhibits the classical feature of bi-sexual species of population thresholds. In particular, in this case, a certain number of hermaphrodites.

4. Model C

As a significant extension of Model A, we now include the situation that a part $1-\gamma$ with $\gamma \in (0,1)$ of the female-male offspring may become hermaphrodites. Thus, in particular Eq. 2.3 is changed by an additional term. Assuming again complete mixing and spatial homogeneity of the involved populations, then, according to the standard description of population dynamics, this leads to the following ordinary differential model C:

$$\dot{f} \;=\; \gamma\cdot\beta_1\cdot f\cdot m+(1-\alpha)\cdot\beta_3\cdot h-\delta\cdot f\,, \tag{4.1}$$

$$\dot{m} \;=\; \beta_2\cdot h\cdot m-\delta\cdot m\,, \tag{4.2}$$

$$\dot{h} \;=\; \alpha\cdot\beta_3\cdot h+(1-\gamma)\cdot\beta_1\cdot f\cdot m-\delta\cdot h\,, \tag{4.3}$$

As in our previous models, the non-negative orthant $\mathscr{P}_0^+$ is invariant for all admissible values of the parameters. Contrary to the previous models, though, and as intended by introducing the transport term $(1-\gamma)\cdot\beta_1\cdot f\cdot m$ from female-male offspring towards the hermaphrodite sub-population, a vanishing hermaphrodite sub-population, this time, does not automatically imply extinction of the complete population.

Like in Model A, for all admissible values of the parameters, the origin $(f,m,h)=(0,0,0)$ is an equilibrium (complete extinction equilibrium), where the local linearized dynamics around the origin are determined by $\dot{x}=A_3x$ with $x=(f,m,h)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f\\ m\\ h\end{pmatrix} = \underbrace{\begin{pmatrix} -\delta & 0 & (1-\alpha)\cdot\beta_3\\ 0 & -\delta & 0\\ 0 & 0 & \alpha\cdot\beta_3-\delta\end{pmatrix}}_{=:A_3}\begin{pmatrix} f\\ m\\ h\end{pmatrix}+\begin{pmatrix}\gamma\cdot\beta_1\cdot f\cdot m\\ \beta_2\cdot h\cdot m\\ (1-\gamma)\cdot\beta_1\cdot f\cdot m\end{pmatrix}.$$

The eigenvalues of A_3 are $-\delta<0$ with algebraic multiplicity two and corresponding 2-dimensional eigenspace spanned by $(1,0,0)$ and $(0,1,0)$, as well as $\alpha\cdot\beta_3-\delta$ with the corresponding eigenspace spanned by $(1-\alpha,0,\alpha)$. For $\alpha\cdot\beta_3-\delta<0$ the origin is an asymptotically stable equilibrium, and, due to the non-linear part, for $\alpha\cdot\beta_3-\delta\geq 0$ the origin is a unstable equilibrium. Again, a bifurcation w.r.t. to the stability of the origin takes place at $\alpha\cdot\beta_3=\delta$.

Moreover, at this bifurcation point $\alpha\cdot\beta_3=\delta$ the line

$$\Gamma_3 \;:=\; \left\{(f_0,0,h_i)\in\mathscr{P}_0^+\;:\;f_0=f_0(h_i)=(1-\alpha)\cdot\beta_3\cdot h_i\cdot\delta^{-1}\,,\,h_i>0\right\}$$

is (as in Model A) a line of equilibria at the $\{m=0\}$-face of $\mathscr{P}_0^+$ that emerges from the origin. The correspondingly transformed equivalent dynamics about each member $(f_0,0,h_i)\in\Gamma_3$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f\\ m\\ h\end{pmatrix} \;=\; \underbrace{\begin{pmatrix} -\delta & \gamma\cdot\beta_1\cdot f_0 & (1-\alpha)\cdot\beta_3\\ 0 & \beta_2\cdot h_i-\delta & 0\\ 0 & (1-\gamma)\cdot\beta_1\cdot f_0 & 0\end{pmatrix}}_{=:B_3}\begin{pmatrix} f\\ m\\ h\end{pmatrix}+\begin{pmatrix}\gamma\cdot\beta_1\cdot f\cdot m\\ \beta_2\cdot h\cdot m\\ (1-\gamma)\cdot\beta_1\cdot f\cdot m\end{pmatrix},$$

and, in complete alignment to the eigenvalue structure of B_3, the eigenvalues of the rank two matrix B_3 are 0 with the corresponding eigenspace being spanned by $((1-\alpha),0,\delta)$, and $-\delta<0$ with the corresponding eigenspace being spanned by $(1,0,0)$, as well as

$$\beta_2\cdot h_i-\delta\;\begin{cases} <0\,, & \text{for } h_i<h^*\,,\\ =0\,, & \text{for } h_i=h^*:=\delta\cdot\beta_2^{-1}=\alpha\cdot\beta_3\cdot\beta_2^{-1}\,,\\ >0\,, & \text{for } h_i>h^*\,,\end{cases}$$

with the corresponding eigenspace being spanned by $(\beta_1\cdot f_0,\beta_2\cdot h_i,0)$. Hence, the elements of Γ_3 are Lyapunov stable equilibrium points for $0\leq h_i<h^*$ and unstable ones for $h^*\leq h_i$.

Finally, below the bifurcation point, i.e. for $\alpha\cdot\beta_3<\delta$, there is an additional equilibrium point $x^*:=(f^*,m^*,h^*)$ in the interior of $\mathscr{P}_0^+$: From Eq. (4.2) we first obtain $h^*:=\delta\cdot\beta_2^{-1}$. Plugging this into Eq. (4.3) leads to

$$\beta_1\cdot f\cdot m \;=\; \frac{\delta\cdot(\delta-\alpha\cdot\beta_3)}{\beta_2\cdot(1-\gamma)}\;>\;0\,,$$

which requires $\alpha\cdot\beta_3<\delta$. Substituting $\beta_1\cdot f\cdot m$ accordingly into Eq. (4.1) gives

$$f^*\;:=\;\frac{\gamma\cdot\beta_1\cdot(\delta-\alpha\cdot\beta_3)+(1-\gamma)\cdot(1-\alpha)\cdot\beta_3}{\beta_2\cdot(1-\gamma)}\;>\;0\,,$$

and

$$m^*\;:=\;\frac{\delta\cdot(\delta-\alpha\cdot\beta_3)}{\gamma\cdot\beta_1\cdot(\delta-\alpha\cdot\beta_3)+(1-\gamma)\cdot(1-\alpha)\cdot\beta_3}\;>\;0\,.$$

The correspondingly transformed equivalent dynamics about $x^*:=(f^*,m^*,h^*)$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f\\ m\\ h\end{pmatrix} \;=\; \underbrace{\begin{pmatrix} \gamma\cdot\beta_1\cdot m^*-\delta & \gamma\cdot\beta_1\cdot f^* & (1-\alpha)\cdot\beta_3\\ 0 & 0 & \beta_2\cdot m^*\\ (1-\gamma)\cdot\beta_1\cdot m^* & (1-\gamma)\cdot\beta_1\cdot f^* & \alpha\cdot\beta_3-\delta\end{pmatrix}}_{=:B_4}\begin{pmatrix} f\\ m\\ h\end{pmatrix}+\begin{pmatrix}\gamma\cdot\beta_1\cdot f\cdot m\\ \beta_2\cdot h\cdot m\\ (1-\gamma)\cdot\beta_1\cdot f\cdot m\end{pmatrix}.$$

Due to the complicated nature of f^* and m^* as well as the nearly complete occupation of B_4 with non-vanishing entries a direct computation of the eigenvalues and its corresponding eigenvectors is cumbersome. Therefore, we apply an indirect approach to determine the stability of the equilibrium (f^*,m^*,h^*). Let $B_4:=(b_{i,j})_{i,j=1,2,3}$, then the characteristic polynomial $\chi_{B_4}(\lambda)$ of B_4 reads

$$\chi_{B_4}(\lambda)\;=\;-\lambda^3+(b_{1,1}+b_{3,3})\cdot\lambda^2-(b_{1,1}\cdot b_{3,3}-b_{1,3}\cdot b_{3,1})\cdot\lambda+b_{1,2}\cdot b_{2,3}\cdot b_{3,1}\,.$$

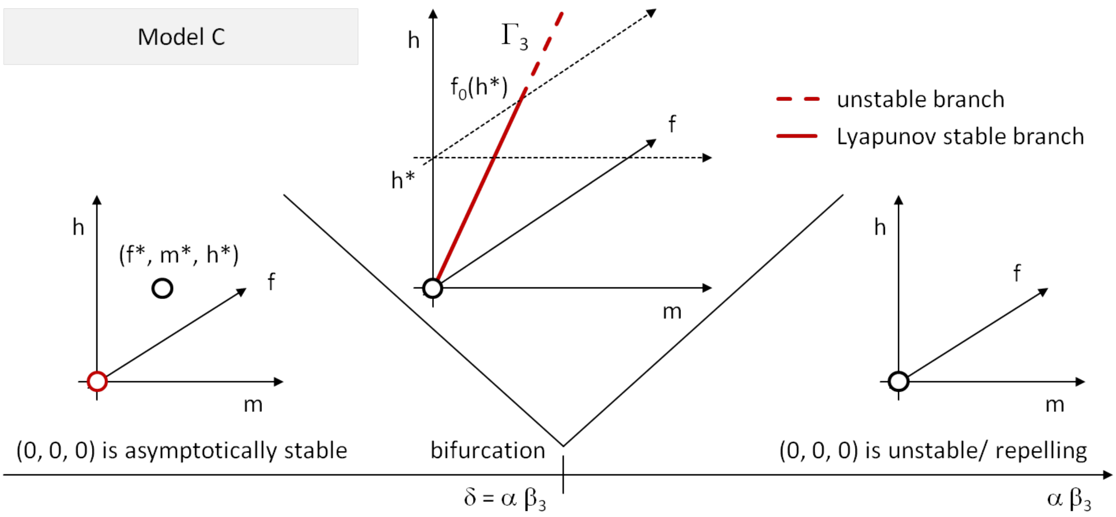


Figure 4.1: Sketch of the stability bifurcation of the extinction equilibrium at the origin at the bifurcation point $\alpha\cdot\beta_3^{-1}=\delta$, as well as of the transformation of the equilibrium (f^*,m^*,h^*) with the occurrence of the line of equilibria Γ_3.

As $b_{1,1}=\gamma\cdot\beta_1\cdot m^*-\delta<0$ Descartes's sign rule gives that there is one positive real positive root, such that the equilibrium (f^*,m^*,h^*) is unstable. For $\alpha\cdot\beta_3\uparrow\delta$ the m-component of the equilibrium (f^*,m^*,h^*) vanishes such that the interior point (f^*,m^*,h^*) moves with increasing values of $\alpha\cdot\beta_3$ towards the f-h-face of the first orthant, and eventually bifurcates to the line of equilibria Γ_3 at $\alpha\cdot\beta_3=\delta$.

The complete bifurcation diagram is sketched in Fig. 4.1. Note that no further equilibria of Eqs. (4.1)-(4.3) exist and we have thus indeed established a complete classification of the equilibria of our system including their stability.

In analogy to Model A we again have that the complete population survives/ proliferates for admissible parameter values $\delta<\alpha\beta_3$. In addition, model C offers some survival/ proliferation advantage at the bifurcation point.

5. Model D

Finally, we perform the same extension as from Model A to Model B and assume that the dynamics described by Model C are altered in view of a non-self-fertilizing hermaphrodite population. I.e., in particular the transport from female-male offspring towards the hermaphrodite sub-population is hence added. Assuming furthermore once again complete mixing and spatial homogeneity of the involved populations, then, according to the standard description of population dynamics, this leads to the following ordinary differential equation model D:

$$\dot{f} = \gamma\cdot\beta_1\cdot f\cdot m+(1-\alpha)\cdot\beta_3\cdot h^2-\delta\cdot f, \tag{5.1}$$
$$\dot{m} = \beta_2\cdot h\cdot m-\delta\cdot m, \tag{5.2}$$
$$\dot{h} = \alpha\cdot\beta_3\cdot h^2+(1-\gamma)\cdot\beta_1\cdot f\cdot m-\delta\cdot h, \tag{5.3}$$

Again, the non-negative orthant $\mathscr{P}_0^+$ is invariant for all admissible values of the parameters, and, as in Model C, a vanishing hermaphrodite population does not automatically imply extinction of the complete population compared to Model B.

Though, in complete analogy to Model B, the origin $(f,m,h)=(0,0,0)\in\mathscr{P}_0^+$ is an asymptotically stable equilibrium for all admissible parameter values (complete extinction equilibrium), as the local linearized dynamics about the origin are determined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f\\ m\\ h\end{pmatrix}=\begin{pmatrix}-\delta & 0 & 0\\ 0 & -\delta & 0\\ 0 & 0 & -\delta\end{pmatrix}\begin{pmatrix} f\\ m\\ h\end{pmatrix}+\begin{pmatrix}\gamma\cdot\beta_1\cdot f\cdot m+(1-\alpha)\cdot\beta_3\cdot h^2\\ \beta_2\cdot h\cdot m\\ (1-\gamma)\cdot\beta_1\cdot f\cdot m+\alpha\cdot\beta_3\cdot h^2\end{pmatrix}.$$

Next, in complete analogy to Model B, for all admissible values of the parameters the point $(f_0,0,h_0)$ is an equilibrium as well, where $h_0:=\delta\cdot(\alpha\cdot\beta_3)^{-1}$ and $f_0:=\delta\cdot(1-\alpha)\cdot(\alpha^2\cdot\beta_3)^{-1}=(1-\alpha)\cdot\alpha^{-1}\cdot h_0$. The local linearized dynamics about $(f_0,0,h_0)$ read as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f\\ m\\ h\end{pmatrix} = \underbrace{\begin{pmatrix}-\delta & \gamma\cdot\beta_1\cdot f_0 & 2\cdot(1-\alpha)\cdot h_0\\ 0 & \beta_2\cdot h_0-\delta & 0\\ 0 & (1-\gamma)\cdot\beta_1\cdot f_0 & 2\cdot\alpha\cdot\beta_3\cdot h_0-\delta\end{pmatrix}}_{=:B_4}\begin{pmatrix} f\\ m\\ h\end{pmatrix}+\begin{pmatrix}\gamma\cdot\beta_1\cdot f\cdot m+(1-\alpha)\cdot\beta_3\cdot h^2\\ \beta_2\cdot h\cdot m\\ (1-\gamma)\cdot\beta_1\cdot f\cdot m+\alpha\cdot\beta_3\cdot h^2\end{pmatrix},$$

where the systems matrix B_4 has three distinct eigenvalues of algebraic multiplicity one: $-\delta<0$, $2\cdot\alpha\cdot\beta_3\cdot h_0-\delta=\delta>0$, as well as $\beta_2\cdot h_0-\delta$ which is negative for $h_0<\delta\cdot\beta_2^{-1}$, vanishes for $h_0=h^*=\delta\cdot\beta_2^{-1}$, and is positive for $h_0<\delta\cdot\beta_2^{-1}$. Hence, the equilibrium $(f_0,0,h_0)$ is unstable and the stability of one of its eigen-manifolds changes at $h_0=h^*$.

Finally, for parameter values with $\beta_2>\alpha\cdot\beta_3$, there is a further equilibrium point (f^*,m^*,h^*) in the interior of $\mathscr{P}_0^+$: From Eq. (5.2) we get $h^*=\delta\cdot\beta_2^{-1}$, and plugging this into Eq. (5.2), we obtain

$$\beta_1\cdot f\cdot m = \frac{\delta^2\cdot(\beta_2-\alpha\cdot\beta_3)}{\beta_2^2\cdot(1-\gamma)} > 0,$$

which requires $\beta_2 > \alpha\cdot\beta_3$. Substituting $\beta_1\cdot f\cdot m$ accordingly into Eq. (5.1) gives

$$f^* \;=\; \frac{\delta\cdot(\gamma\cdot(\beta_2-\alpha\cdot\beta_3)+(1-\alpha)\cdot(1-\gamma)\cdot\beta_3)}{\beta_2^2\cdot(1-\gamma)} \;>\; 0\,,$$

and

$$m^* \;=\; \frac{\delta\cdot(\beta_2-\alpha\cdot\beta_3)}{\beta_1\cdot(\gamma\cdot(\beta_2-\alpha\cdot\beta_3)+(1-\alpha)\cdot(1-\gamma)\cdot\beta_3)} \;>\; 0\,.$$

Due to $\beta_2\cdot h^*-\delta=0$, the correspondingly transformed equivalent dynamics about (f^*,m^*,h^*) are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f\\ m\\ h\end{pmatrix} \;=\; \underbrace{\begin{pmatrix}\gamma\cdot\beta_1\cdot m^*-\delta & \gamma\cdot\beta_1\cdot f^* & 2\cdot(1-\alpha)\cdot h^*\\ 0 & 0 & \beta_2\cdot m^*\\ (1-\gamma)\cdot\beta_1\cdot m^* & (1-\gamma)\cdot\beta_1\cdot f^* & 2\cdot\alpha\cdot\beta_3\cdot h^*-\delta\end{pmatrix}}_{=:B_5}\begin{pmatrix} f\\ m\\ h\end{pmatrix}+\begin{pmatrix}\gamma\cdot\beta_1\cdot f\cdot m+(1-\alpha)\cdot\beta_3\cdot h^2\\ \beta_2\cdot h\cdot m\\ (1-\gamma)\cdot\beta_1\cdot f\cdot m+\alpha\cdot\beta_3\cdot h^2\end{pmatrix},$$

such that we can follow the same line of argumentation as in Model C. Let $B_5 := (b_{i,j})_{i,j=1,2,3}$, then its characteristic polynomial $\chi_{B_4}(\lambda)$ reads

$$\chi_{B_5}(\lambda) \;=\; -\lambda^3+(b_{1,1}+b_{3,3})\cdot\lambda^2-(b_{1,1}\cdot b_{3,3}-b_{1,3}\cdot b_{3,1})\cdot\lambda+b_{1,2}\cdot b_{2,3}\cdot b_{3,1}\,.$$

In complete analogy to Model C, we have that $b_{1,1}=\gamma\cdot\beta_1\cdot m^*-\delta<0$, and hence Descartes's sign rule implies that there is one positive real positive root, such that the equilibrium (f^*,m^*,h^*) is unstable. For $\alpha\cdot\beta_3\uparrow\beta_2$ the m-component of the equilibrium (f^*,m^*,h^*) vanishes such that the interior point (f^*,m^*,h^*) moves with increasing values of $\alpha\cdot\beta_3$ towards the f-h-face of the first orthant, and eventually coincides with the previously discussed equilibrium $(f^*=f_0,0,h^*=h_0)$ at $\alpha\cdot\beta_3=\beta_2$.

6. Resume

During the course of our mathematical discussion and comparison of the two models we can conclude, that models A and D show a survival/ proliferation advantage over models B and C, respectively, for small population numbers. Survival/ proliferation in models A and D depends only on the birth and death parameters of the hermaphrodite populations and not on certain threshold populations. In models B and D the origin (complete extinction equilibrium) is always asymptotically stable and hence supporting the extinction of the complete population, whereas models A and prevent, under some further conditions, the complete extinction due to the unstable/ repulsive nature of the origin.

Additionally, to numerical studies already carried out in the literature, cf. Ref. [16], we analytically showed the extremely robust survival/ proliferation properties of self-fertilizing species. Moreover, in contrast, to the usual heterogametic view of reproduction that predict convergence towards a bisexual species, cf. Refs. [14, 15], the homogametic reproduction strategy used by Aunema Rhodensis shows clear advantages in terms of a survival of all sub-species.

Thus, the mathematical discussion of the dynamics of self-fertilizing trisexual homogametic species adds a further piece to the puzzle of evolutionary competitiveness.

Appendix A: Mathematical Models for Species with Neither Hermaphrodites-Producing Females Nor Females-Producing Hermaphrodites

As discussed in the introduction, the two mathematical models in this appendix serve a reference and a completion of our study of the dynamics of trisexual species. They are rather simple in the mutual interaction of the female, male and hermaphrodite populations and their mathematical analysis is straight-forward utilizing the same lines of argumentation as applied in the main part of the article.

Model E_1: A Species with Self-Fertilizing Hermaphrodites

A mathematical model of a trisexual species with self-fertilizing hermaphrodites, like our Auanema Rhodensis, where there are no-hermaphrodites producing females and no-females producing hermaphrodites can be considered as a special limit case of Model A with $\alpha=1$ as

$$\dot f \;=\; (\beta_1\cdot m-\delta)\cdot f\,,\quad \dot m \;=\; (\beta_2\cdot h-\delta)\cdot m\,,\quad \text{and}\quad \dot h \;=\; (\beta_3-\delta)\cdot h\,,$$

where, as previously, $\beta_1,\beta_2,\beta_3>0$ are the proportionality factors related to birth events, and $\delta>0$ is the sex-independent proportionality factor related to death events. We will call this set of equations together with the corresponding initial conditions $f(0)=f_i\geq 0$, $m(0)=m_i\geq 0$ and $h(0)=h_i\geq 0$ as Model E_1.

The non-negative orthant $\mathscr{P}_0^+$ is invariant under the dynamics of Model E_1. Immediately, we see that the origin $(0,0,0)$ is an equilibrium point for all admissible values of the parameters, that is asymptotically stable for $\beta_3<\delta$, Lyapunov stable for $\beta_3=\delta$, and unstable for $\beta_3>\delta$. As in Model A, we have a bifurcation point at $\beta_3=\delta$.

In particular, for $\beta_3<\delta$ (or $\beta_3>\delta$) we have that $\dot h<0$ (or $\dot h>0$) for all times such that the hermaphrodite population eventually vanishes (or grows beyond all bounds). This implies, that at some finite time $\beta_2\cdot h-\delta<0$ (or $\beta_2\cdot h-\delta>0$) such that the male population eventually vanishes (or grows beyond all bounds). Consequently, at some finite time $\beta_1\cdot m-\delta<0$ (or $\beta_1\cdot m-\delta>0$) such that the female population eventually vanishes (or grows beyond all bounds). Hence, for $\beta_3<\delta$ (or $\beta_3>\delta$) the origin is a globally stable (or globally unstable) equilibrium. Note, that there are no further equilibrium points for $\beta_3\neq\delta$.

Finally, the dynamics for $\beta_3=\delta$ are sketched in Fig. 6.1 and a complete picture of the additional equilibria that occur at this set of parameters is given in the following proposition.

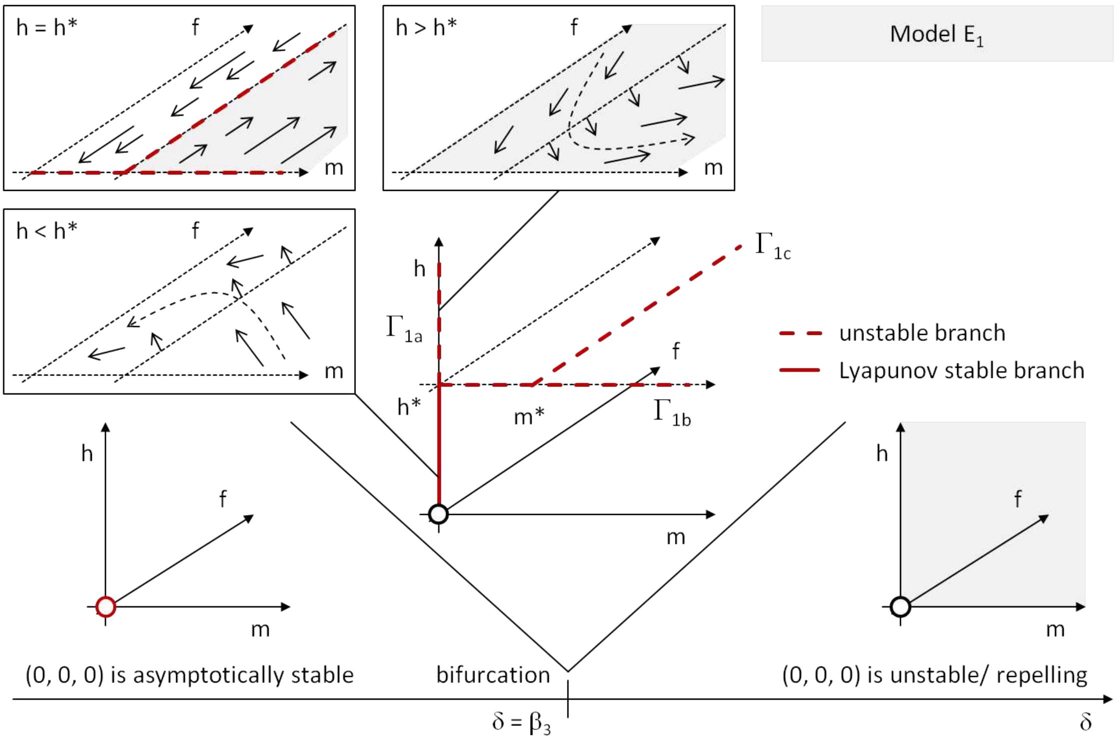


Figure 6.1: Sketch of the bifurcation scenario and some illustrative dynamics of Model E_1.

Theorem E.1: Additional Lines of Equilibria at the Bifurcation Point. *Let $h^* := \delta \cdot \beta_2^{-1}$ and $m^* := \delta \cdot \beta_1^{-1}$. For $\beta_3 = \delta$ and besides the origin there are three lines of equilibria*

- $\Gamma_{1a} := \left\{(0,0,h_i) \in \mathscr{P}_0^+ : h_i \geq 0\right\}$*, where each element is Lyapunov stable for $0 \leq h_i \leq h^*$ and unstable for $h^* < h_i$,*
- $\Gamma_{1b} := \left\{(0,m_i,h^*) \in \mathscr{P}_0^+ : m_i \geq 0\right\}$*, where each element is unstable, and*
- $\Gamma_{1c} := \left\{(f_i,m^*,h^*) \in \mathscr{P}_0^+ : f_i \geq 0\right\}$*, where each element is unstable.*

Moreover, there are no further equilibrium points for $\beta_3 = \delta$.

Proof: A short calculation leads to the existence of these lines of equilibria and that, besides the origin, there are no further ones. The local dynamics about each equilibrium point $(\tilde{f},\tilde{m},\tilde{h})$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f \\ m \\ h \end{pmatrix} = \underbrace{\begin{pmatrix} \beta_1 \cdot \tilde{m} - \delta & \beta_1 \cdot \tilde{f} & 0 \\ 0 & \beta_2 \cdot \tilde{h} - \delta & \beta_2 \cdot \tilde{m} \\ 0 & 0 & 0 \end{pmatrix}}_{=:E_1(\tilde{f},\tilde{m},\tilde{h})} \begin{pmatrix} f \\ m \\ h \end{pmatrix} + \begin{pmatrix} \beta_1 \cdot f \cdot m \\ \beta_2 \cdot h \cdot m \\ 0 \end{pmatrix}.$$

The eigenvalues of $E_1(0,0,h_i)$, with a specific value $h_i \geq 0$, are $\lambda_1 := 0$ with the corresponding eigenspace being spanned by $(0,0,1)$, $\lambda_2 := -\delta < 0$ with the corresponding eigenspace being spanned by $(1,0,0)$, and $\lambda_3 := \beta_2 \cdot h_i - \delta$ with the corresponding eigenspace being spanned by $(0,1,0)$. In particular, $\lambda_3 < 0$ if $0 \leq h_i < h^*$, $\lambda_3 = 0$ if $h_i = h^*$, and $\lambda_3 > 0$ if $h^* < h_i$.

The eigenvalues of $E_1(0,m_i,h^*)$, with a specific value $m_i \geq 0$, are $\lambda_1 := 0$ with algebraic multiplicity two and the corresponding one-dimensional eigenspace being spanned by $(0,1,0)$, and $\lambda_2 := \beta_1 \cdot m_i - \delta$ with the corresponding eigenspace being spanned by $(1,0,0)$. In particular, $\lambda_2 < 0$ if $0 \leq m_i < m^*$, $\lambda_2 = 0$ if $m_i = m^*$, and $\lambda_2 > 0$ if $m^* < m_i$.

The only eigenvalue of $E_1(f_i,m^*,h^*)$, with a specific value of $f_i \geq 0$, is $\lambda_1 := 0$ with algebraic multiplicity three and corresponding one-dimensional eigenspace being spanned by $(1,0,0)$. □

The global dynamics $\beta_3 = \delta$ are derived in complete analogy to Model A.

Model E_2: A Species with Non-Self-Fertilizing Hermaphrodites

Next, a mathematical model of a trisexual species with non-self-fertilizing hermaphrodites, where there are no-hermaphrodites-producing females and no-females producing hermaphrodites can be considered as a special limit case of Model B with $\alpha = 1$ as

$$\dot{f} = (\beta_1 \cdot m - \delta) \cdot f, \quad \dot{m} = (\beta_2 \cdot h - \delta) \cdot m, \quad \text{and} \quad \dot{h} = (\beta_3 \cdot h - \delta) \cdot h,$$

where, as previously, $\beta_1, \beta_2, \beta_3 > 0$ are the proportionality factors related to birth events, and $\delta > 0$ is the sex-independent proportionality factor related to death events. We will call this set of equations together with the corresponding initial conditions $f(0) = f_i \geq 0$, $m(0) = m_i \geq 0$ and $h(0) = h_i \geq 0$ as Model E_2.

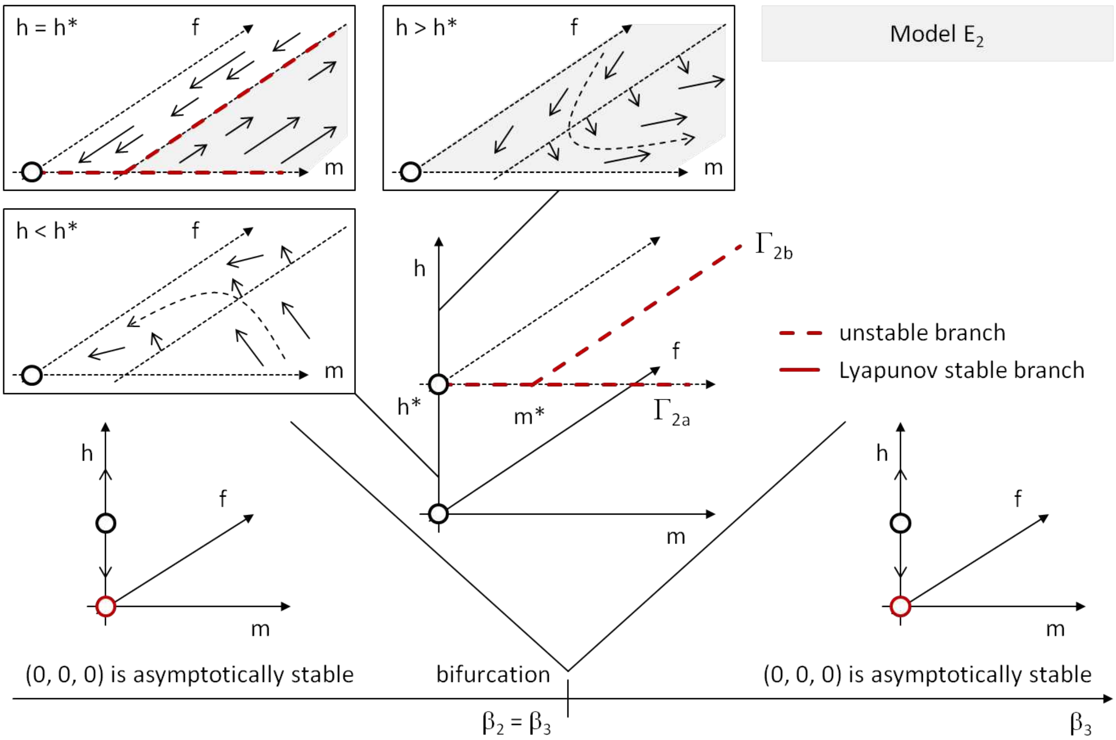


Figure 6.2: Sketch of the bifurcation scenario and some illustrative dynamics of Model E_2.

The non-negative orthant $\mathscr{P}_0^+$ is invariant under the dynamics of Model E_1, and for all values of the parameters the origin $(0,0,0)$ is an asymptotically stable equilibrium point with $\mathrm{diag}(-\delta,-\delta,-\delta)$ as the matrix of linearization about the origin. Moreover, for all values of the parameters, the point $(0,0,h^*)$, with $h^* := \delta\cdot\beta_3^{-1}$, is a further equilibrium point. The matrix of linearization about $(0,0,h^*)$ is $\mathrm{diag}(-\delta,\delta\cdot(\beta_2\cdot\beta_3^{-1}-1),\delta)$ such that this equilibrium is unstable though exhibits a change of stability at $\beta_2=\beta_1$ where a geometrically two-dimensional (asymptotically) stable sub-space bifurcates into a (asymptotically) stable one and an unstable one.

Finally, the dynamics for Model E_2 are sketched in Fig. 6.2 and a complete picture of the additional equilibria that occur at $\beta_3=\beta_2$ is given in the following proposition.

Theorem E.2: Additional Lines of Equilibria. *Let $h^* := \delta\cdot\beta_3^{-1}$ and $m^* := \delta\cdot\beta_1^{-1}$. For $\beta_3=\beta_2$ and besides the origin there are two lines of equilibria*

- $\Gamma_{2a} := \left\{(0,m_i,h^*)\in\mathscr{P}_0^+ : m_i\geq 0\right\}$*, where each element is unstable, and*
- $\Gamma_{2b} := \left\{(f_i,m^*,h^*)\in\mathscr{P}_0^+ : f_i\geq 0\right\}$*, where each element is unstable.*

Moreover, there are no further equilibrium points for $\beta_3=\beta_2$.

Proof: A short calculation leads to the existence of these lines of equilibria and that, besides the origin, there are no further ones. The local dynamics about each equilibrium point $(\tilde f,\tilde m,\tilde h)$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f\\ m\\ h\end{pmatrix} = \underbrace{\begin{pmatrix}\beta_1\cdot\tilde m-\delta & \beta_1\cdot\tilde f & 0\\ 0 & 0 & \beta_2\cdot\tilde m\\ 0 & 0 & 0\end{pmatrix}}_{=:E_2(\tilde f,\tilde m,\tilde h)}\begin{pmatrix} f\\ m\\ h\end{pmatrix} + \begin{pmatrix}\beta_1\cdot f\cdot m\\ \beta_2\cdot h\cdot m\\ \beta_3\cdot h^2\end{pmatrix}.$$

The eigenvalues of $E_2(0,m_i,h^*)$, with a specific value $m_i\geq 0$, are $\lambda_1 := 0$ with algebraic multiplicity two and the corresponding one-dimensional eigenspace being spanned by $(0,1,0)$, and $\lambda_2 := \beta_1\cdot m_i-\delta$ with the corresponding eigenspace being spanned by $(1,0,0)$. In particular, $\lambda_2<0$ if $0\leq m_i<m^*$, $\lambda_2=0$ if $m_i=m^*$, and $\lambda_2>0$ if $m^*<m_i$.

The only eigenvalue of $E_2(f_i,m^*,h^*)$, with a specific value of $f_i\geq 0$, is $\lambda_1 := 0$ with algebraic multiplicity three and corresponding one-dimensional eigenspace being spanned by $(1,0,0)$. □

The global dynamics $\beta_3=\delta$ are derived in complete analogy to Model B.

Appendix B: Mathematical Models for Species with Hermaphrodites-Producing Females and No-Females-Producing Hermaphrodites

The two rather simple mathematical models in this appendix close the remaining gap in our discussion by studying species with hermaphrodites-producing females and no-females-producing hermaphrodites, i.e. models where female XX-offspring can evolve to both females and hermaphrodites, whereas hermaphrodite XX-offspring can only evolve to hermaphrodites.

Model F_1: A Species with Self-Fertilizing Hermaphrodites

Analogue to our previous modeling the driving biological dynamics in the setting with self-fertilizing hermaphrodites, like for Auanema Rhodensis, with hermaphrodites-producing females and no-females-producing hermaphrodites read as

$$\dot{f} \;=\; (\gamma \cdot \beta_1 \cdot m - \delta) \cdot f\,, \quad \text{and} \quad \dot{m} \;=\; (\beta_2 \cdot h - \delta) \cdot m\,,$$

as well as

$$\dot{h} \;=\; \beta_3 \cdot h + (1-\gamma) \cdot \beta_1 \cdot m \cdot f - \delta \cdot h\,,$$

where $\beta_1, \beta_2, \beta_3 > 0$ are proportionality factors related to birth events, $\delta > 0$ is the sex-independent proportionality factor related to death events, and γ the transport factor from female offspring towards the female sub-population (correspondingly, the factor $1-\gamma$ is the transport factor from female offspring towards the hermaphrodites sub-population). We will call this set of equations together with the corresponding initial conditions $f(0) = f_i \geq 0$, $m(0) = m_i \geq 0$ and $h(0) = h_i \geq 0$ as Model F_1.

The non-negative orthant $\mathscr{P}_0^+$ is invariant under the dynamics of Model F_1, and for all admissible values of the parameters the origin $(0,0,0)$ is an equilibrium with $\operatorname{diag}(-\delta, -\delta, \beta_3 - \delta)$ as the matrix of linearization about it. In particular, the origin is asymptotically stable for $\beta_3 < \delta$, Lyapunov stable for $\beta_3 = \delta$, and unstable for $\beta_3 > \delta$. As in Model A, we have a bifurcation point at $\beta_3 = \delta$.

Let $h^* := \delta \cdot \beta_2^{-1}$. At $\beta_3 = \delta$ we have two additional lines of equilibria:

- $\Gamma_{3,a} := \{(0,0,h_i) \in \mathscr{P}_0^+ \,:\, h_i \geq 0\}$, where, analogous to the proof of Theorem E.1, each element is Lyapunov stable for $0 \leq h_i \leq h^*$ and unstable for $h^* < h_i$, and
- $\Gamma_{3,b} := \{(0,m_i,h^*) \in \mathscr{P}_0^+ \,:\, m_i \geq 0\}$, where, analogous to the proof of Theorem E.1, each element is unstable.

If $\beta_3 > \delta$ holds, besides the origin, another equilibrium exists at the point $(f,m,h) := (f^*,m^*,h^*)$, where $m^* := \delta \cdot (\gamma \cdot \beta_1)^{-1}$, $h^* := \delta \cdot \beta_2^{-1}$, and

$$f^* \;:=\; \frac{(\beta_3 - \delta) \cdot h^*}{(1-\gamma) \cdot \beta_1 \cdot m^*} \;=\; \frac{(\beta_3 - \delta) \cdot \gamma}{(1-\gamma) \cdot \beta_2} \;>\; 0\,.$$

The correspondingly transformed equivalent dynamics about this interior mixed species equilibrium (f^*,m^*,h^*) are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f \\ m \\ h \end{pmatrix} \;=\; \underbrace{\begin{pmatrix} \overbrace{\gamma \cdot \beta_1 \cdot m^* - \delta}^{=0} & \gamma \cdot \beta_1 \cdot f^* & 0 \\ 0 & 0 & \beta_2 \cdot m^* \\ (1-\gamma) \cdot \beta_1 \cdot m^* & (1-\gamma) \cdot \beta_1 \cdot f^* & \beta_3 - \delta \end{pmatrix}}_{=: F_1(f^*,m^*,h^*)} \begin{pmatrix} f \\ m \\ h \end{pmatrix} + \begin{pmatrix} \gamma \cdot \beta_1 \cdot f \cdot m \\ \beta_2 \cdot h \cdot m \\ (1-\gamma) \cdot \beta_1 \cdot f \cdot m \end{pmatrix}.$$

The three distinct eigenvalues of $F_1(f^*,m^*,h^*)$, each with geometric multiplicity 1, are $\lambda_1 := \delta \cdot \beta_2 \cdot (\gamma \cdot \beta_1)^{-1} > 0$, $\lambda_2 := (\beta_3 - \delta) \cdot \gamma^2 \cdot \beta_1\,((1-\gamma) \cdot \beta_2)^{-1}$, and $\lambda_3 = \delta \cdot (1-\gamma) \cdot \gamma^{-1} > 0$. Hence, the equilibrium (f^*,m^*,h^*) is unstable for all admissible parameter values.

Moreover, there are no further equilibrium points, and the global dynamics in Model F_1 are derived in complete analogy to Model A.

Model F_2: A Species with Non-Self-Fertilizing Hermaphrodites

The driving biological dynamics in the setting with non-self-fertilizing hermaphrodites together with hermaphrodites-producing females and no-females-producing hermaphrodites read as

$$\dot{f} \;=\; (\gamma \cdot \beta_1 \cdot m - \delta) \cdot f\,, \quad \text{and} \quad \dot{m} \;=\; (\beta_2 \cdot h - \delta) \cdot m\,,$$

as well as

$$\dot{h} \;=\; \beta_3 \cdot h^2 + (1-\gamma) \cdot \beta_1 \cdot m \cdot f - \delta \cdot h\,,$$

where $\beta_1, \beta_2, \beta_3 > 0$ are proportionality factors related to birth events, $\delta > 0$ is the sex-independent proportionality factor related to death events, and γ the transport factor from female offspring towards the female sub-population (correspondingly, the factor $1-\gamma$ is the transport factor from female offspring towards the hermaphrodites sub-population). We will call this set of equations together with the corresponding initial conditions $f(0) = f_i \geq 0$, $m(0) = m_i \geq 0$ and $h(0) = h_i \geq 0$ as Model F_2.

The non-negative orthant $\mathscr{P}_0^+$ is invariant under the dynamics of Model F_2, and for all admissible values of the parameters the origin $(0,0,0)$ is an asymptotically stable equilibrium with $\operatorname{diag}(-\delta, -\delta, -\delta)$ as the matrix of linearization about it.

Let $m^* := \delta \cdot (\gamma \cdot \beta_1)^{-1}$, $h^* := \delta \cdot \beta_2^{-1}$. For $0 < \beta_3 < \beta_2$, the point (f^*,m^*,h^*) is an additional mixed species equilibium, where

$$f^* \;=\; \frac{(\beta_2 - \beta_3) \cdot \gamma \cdot \delta}{(1-\gamma) \cdot \beta_2^2} \;>\; 0\,.$$

The correspondingly transformed equivalent dynamics about this interior mixed species equilibrium (f^*,m^*,h^*) are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f \\ m \\ h \end{pmatrix} \;=\; \underbrace{\begin{pmatrix} \overbrace{\gamma \cdot \beta_1 \cdot m^* - \delta}^{=0} & \gamma \cdot \beta_1 \cdot f^* & 0 \\ 0 & 0 & \beta_2 \cdot m^* \\ (1-\gamma) \cdot \beta_1 \cdot m^* & (1-\gamma) \cdot \beta_1 \cdot f^* & 2 \cdot \beta_3 \cdot h^* - \delta \end{pmatrix}}_{=: F_2(f^*,m^*,h^*)} \begin{pmatrix} f \\ m \\ h \end{pmatrix} + \begin{pmatrix} \gamma \cdot \beta_1 \cdot f \cdot m \\ \beta_2 \cdot h \cdot m \\ (1-\gamma) \cdot \beta_1 \cdot f \cdot m + \beta_3 \cdot h^2 \end{pmatrix},$$

such that, in complete analogy to Model F_1, this is an unstable equilibrium.

At $\beta_3 = \beta_2$, additional to the origin, this mixed species equilibrium bifurcates to a line of equilibria $\Gamma_{4,a} := \{(0, m_i, h^*) \in \mathscr{P}_0^+ : m_i > 0\}$ such that analogous to the proof of Theorem E.2 each member of this line is unstable.

Moreover, there are no further equilibrium points, and the global dynamics in Model F_2 are derived in complete analogy to Model A.

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Domination Edge Integrity of Corona Products of P_n with $P_m, C_m, K_{1,m}$

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Abstract

Vulnerability is the most important concept in analysis of communication networks to disruption. Any network can be modelled by graphs. So measures defined on graphs gives an idea in design. Integrity is one of the well-known vulnerability measures interested in remaining structure of a graph after any failure. Domination is also an another popular concept in network design. Nowadays new vulnerability measures take a great role in network design. Recently designers take into account of any failure not only on nodes also on links which have special properties. A new measure edge domination integrity of a connected and undirected graph was defined by E. Kılıç and A. Beşirik such as $DI'(G) = min\{\ |S| + m(G-S) : S\ \subseteq\ E(G)\}$ where $m(G-S)$ is the order of a maximum component of $G-S$ and S is an edge dominating set. In this paper some results concerning this parameter on corona products of graph structures $P_n \odot P_m$, $P_n \odot C_m$, $P_n \odot K_{1,m}$ are presented.

1. Introduction

A communication network can be modeled by a graph G where nodes are represented by vertices $V(G)$ and links are represented by edges such as $E(G)$ respectively. Any communication network can be considered to be highly vulnerable to any disruption on its nodes or links. All graphs considered in this paper are connected, undirected, do not contain loops and multiple edges. First simple vulnerability measures are connectivity or edge connectivity which shows how easily a graph can be broken apart [1]. Later on, it is observed that these measures are not enough to compare the stability of network structures which have the same order. Most network designers are interested in what happens in the remaining part of the network after failures such as, how many nodes or links are still connected to each other and what is the communication between remaining parts. Integrity and the edge integrity concepts are interested in these questions. Both types of integrity were introduced by C. A. Barefoot et al. [2] and W. Goddard and H.C. Swart [3] has great contributions for this area. Integrity or edge integrity have been widely studied on specific graph families and relationships with other parameters and bounds were obtained K. S. Bagga et al. have presented many results about edge integrity in [4].

The order of a graph G will generally be denoted by n. For a real number x; $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Domination is another important concept widely studied in graph theory. A subset S of V is called a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number $\gamma(G)$ (or γ for short) of G is the minimum cardinality taken over all dominating sets of G [5].
S. Mitchell and S.T. Hedetniemi [6] have introduced the concept of edge domination. A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ (or γ' for short) of G is the minimum cardinality taken over all edge dominating sets of G. Later on S.Arumugamm [7] did some contributions to topic.

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Domination and integrity were examined together and many new vulnerability measures were defined. Some of them are domination integrity [8] [9] , domination edge integrity [10], and total domination integrity [11].

The concept of domination edge integrity of a connected graph as a new vulnerability parameter was defined by E. Kılıç and A. Beşirik [10] as follows.

Definition 1.1. *The domination edge integrity of a connected graph G is denoted by $DI^{'}(G)$ and is defined by*

$$DI^{'}(G) = \min\{|S| + m(G-S) \ : S \text{ is an edge dominating set }\}$$

where $m(G-S)$ is the order of a maximum component of $G-S$.

Definition 1.2. *A subset S of $E(G)$ is a $DI^{'}$-set if $DI^{'}(G) = \min\{|S| + m(G-S) \ : S \subseteq E(G)\}$ where S is an edge dominating set of G.*

$DI^{'}$ values of P_n, C_n, $K_{1,n}$, $K_{m,n}$ were presented and some properties for domination edge integrity value of a connected graph were determined in [10].

2. DI' of corona products P_n with some graphs

$DI^{'}$ values of some resulting graphs after corona operation of P_n with P_m, C_m, $K_{1,m}$ are found as follows.

Definition 2.1. *The corona $G_1 \odot G_2$ is defined as G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i'th node of G_1 to every node in the i'th copy of G_2 [1].*

Proposition 2.2. *Let n be an integer, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$.*

Proof. There are 2 cases for integer n.

Case 1: Let n is an even integer. Then $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil = \frac{n}{2}$. Hence; $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = \frac{n}{2} + \frac{n}{2} = n$.

Case 2: Let n is an odd integer. Then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ (since n is odd, then $n-1$ is even) and $\lceil \frac{n}{2} \rceil = \frac{n-1}{2} + 1$. Hence, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = \frac{n-1}{2} + \frac{n-1}{2} + 1 = n$.

□

In proof of all theorems, for graph G of order m, edge dominating sets X_1 and X_2 are taken which satisfies, $m(P_n \odot G - X_1) = 2(m+1)$ and $m(P_n \odot G - X_2) = m+1$ respectively (Figure 2.1). There is no other possible selection of edge dominating sets which gives $DI^{'}$ to be minimum. If X_3 is taken to be another edge dominating set, cardinality of X_3 is greater than both X_1 and X_2. It is easy to observe from structure of corona product of P_n with G. And also $m(P_n \odot G - X_1) > 2(m+1)$ since more edges are added. This selection does not give a minimum result.

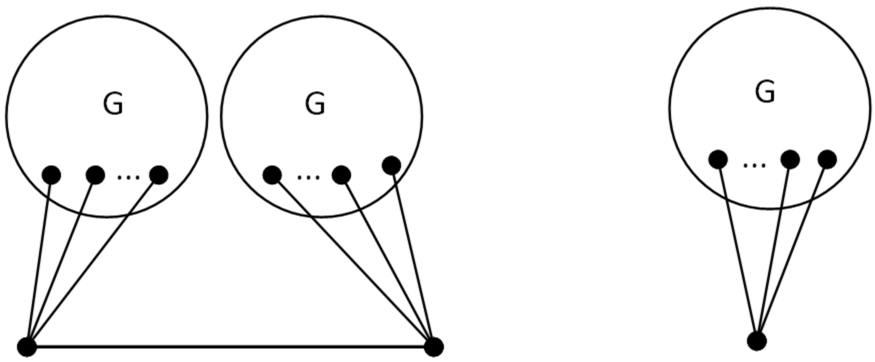

Figure 2.1: Maximum components of $(P_n \odot G) - X_1$ and $(P_n \odot G) - X_2$

For $n < 3$, $DI^{'}$ values of corona products of P_n with P_m, C_m, $K_{1,m}$ are obvious.

Theorem 2.3. *For $n \geq 3$ and $m \geq 2$, let n to be odd and*

$$A = \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + 2m + 3,$$

$$B = n + n. \left\lceil \frac{m-1}{3} \right\rceil + m.$$

Then, $DI^{'}(P_n \odot P_m)$ is obtained as follows,

$$DI^{'}(P_n \odot P_m) = \begin{cases} A, & if \ m+2 < \frac{n-1}{2} \\ B, & if \ m+2 > \frac{n-1}{2} \\ A = B, & if \ m+2 = \frac{n-1}{2} \end{cases}$$

Proof. Let $V(P_n)=\{v_1,v_2,...,v_n\}$ and $V(P_m)=\{u_1,u_2,...,u_m\}$ for path graph P_n and P_m. Let $E(P_n)=\{v_1v_2,v_2v_3,...,v_{n-1}v_n\}$ and $E(P_m)=\{u_1u_2,...,u_{m-1}u_m\}$.
For $n\geq 3$ and n is odd, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n\odot P_m$ and $m(P_n\odot P_m-X_1)=2(m+1)$. X_1 is obtained as follows.
Let $S_1=\{v_2v_3,v_4v_5,...,v_{n-1}v_n\}\subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1|=\left\lfloor\frac{n-1}{2}\right\rfloor$.
Let S_{2_i} is a minimum edge dominating set of ith copy of P_m and $|S_{2_i}|=\left\lceil\frac{m-1}{3}\right\rceil$ and $S_2=S_{2_1}\cup S_{2_2}\cup...\cup S_{2_n}$.
$S_1\cup S_2$ is not an edge dominating set of $P_n\odot P_m$ because some edges between v_1 and vertices of 1st copy of P_m are not dominated by any edges in $S_1\cup S_2$. So one of these edges (called e) is added $S_1\cup S_2$, $X_1=S_1\cup S_2\cup\{e\}$ is an edge dominating set of $P_n\odot P_m$. Therefore, $|X_1|=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+1$ and $m(P_n\odot P_m-X_1)=2(m+1)$. Thus,

$$DI^{'}(P_n\odot P_m)\leq|X_1|+m(P_n\odot P_m-X_1)=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+2m+3=DI^{'}(P_n\odot P_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n\odot P_m$ and $m(P_n\odot P_m-X_2)=m+1$. X_2 is obtained as follows.
Let $S_1^{'}=E(P_n)$. $S_1^{'}$ is an edge dominating set of P_n and $\left|S_1^{'}\right|=n-1$.
Let S_{2_i} is a minimum edge dominating set of ith copy of P_m and $|S_{2_i}|=\left\lceil\frac{m-1}{3}\right\rceil$ and $S_2=S_{2_1}\cup S_{2_2}\cup...\cup S_{2_n}$.
$X_2=S_1^{'}\cup S_2$ is an edge dominating set of $P_n\odot P_m$.Therefore, $|X_2|=n-1+n\left\lceil\frac{m-1}{3}\right\rceil$ and $m(P_n\odot P_m-X_2)=m+1$. Thus,

$$DI^{'}(P_n\odot P_m)\leq|X_2|+m(P_n\odot P_m-X_2)=n+n\left\lceil\frac{m-1}{3}\right\rceil+m=DI^{'}(P_n\odot P_m)_{X_2}.$$

Because of definition of $DI^{'}$, the relationship between $DI^{'}(P_n\odot P_m)_{X_1}$ and $DI^{'}(P_n\odot P_m)_{X_2}$ must be examined as follows.

1. If $m+2<\frac{n-1}{2}$, then we have

$$\begin{aligned}DI^{'}(P_n\odot P_m)_{X_1} &= \left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+2m+3=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+m+2+m+1\\ &< \frac{n-1}{2}+n\left\lceil\frac{m-1}{3}\right\rceil+\frac{n-1}{2}+m+1\\ &= n-1+n\left\lceil\frac{m-1}{3}\right\rceil+m+1=n+n\left\lceil\frac{m-1}{3}\right\rceil+m=DI^{'}(P_n\odot P_m)_{X_2}.\end{aligned}$$

Since $DI^{'}(P_n\odot P_m)_{X_1}<DI^{'}(P_n\odot P_m)_{X_2}$, then $DI^{'}(P_n\odot P_m)=DI^{'}(P_n\odot P_m)_{X_1}=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+2m+3$.

2. If $m+2>\frac{n-1}{2}$, then we have $DI^{'}(P_n\odot P_m)_{X_2}<DI^{'}(P_n\odot P_m)_{X_1}$. It can be proved in similar way as above. Therefore, $DI^{'}(P_n\odot P_m)=DI^{'}(P_n\odot P_m)_{X_2}=n+n\left\lceil\frac{m-1}{3}\right\rceil+m$.

3. If $m+2=\frac{n-1}{2}$, then we have $DI^{'}(P_n\odot P_m)_{X_1}=DI^{'}(P_n\odot P_m)_{X_2}$. Hence, $DI^{'}(P_n\odot P_m)=DI^{'}(P_n\odot P_m)_{X_1}=DI^{'}(P_n\odot P_m)_{X_2}$.

□

Theorem 2.4. *For $n\geq 4$ and $m\geq 2$, let n to be even and*

$$A=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+2m+4,$$

$$B=n+n\left\lceil\frac{m-1}{3}\right\rceil+m.$$

Then $DI^{'}(P_n\odot P_m)$ is obtained as follows,

$$DI^{'}(P_n\odot P_m)=\begin{cases}A, & if\ m+3<\left\lceil\frac{n-1}{2}\right\rceil\\ B, & if\ m+3>\left\lceil\frac{n-1}{2}\right\rceil\\ A=B, & if\ m+3=\left\lceil\frac{n-1}{2}\right\rceil\end{cases}$$

Proof. Let $V(P_n)=\{v_1,v_2,...,v_n\}$ and $V(P_m)=\{u_1,u_2,...,u_m\}$ for path graph P_n and P_m. Let $E(P_n)=\{v_1v_2,v_2v_3,...,v_{n-1}v_n\}$ and $E(P_m)=\{u_1u_2,...,u_{m-1}u_m\}$. For $n\geq 2$ and n is even, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n\odot P_m$ and $m(P_n\odot P_m-X_1)=2(m+1)$. X_1 is obtained as follows.

Let $S_1 = \{v_2v_3, v_4v_5, ..., v_{n-1}v_n\} \subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1| = \lfloor \frac{n-1}{2} \rfloor$.
Let S_{2_i} is a minimum edge dominating set of ith copy of P_m and $|S_{2_i}| = \lceil \frac{m-1}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup ... \cup S_{2_n}$.
$S_1 \cup S_2$ is not an edge dominating set of $P_n \odot P_m$ because some edges between v_1 and vertices of 1st copy of P_m and v_n and vertices of nth copy of P_m are not dominated by any edges in $S_1 \cup S_2$. So one of edges between v_1 and vertices of 1st copy of P_m (called e_1) and one of edges between v_n and vertices of nth copy of P_m (called e_2) are added $S_1 \cup S_2$, $X_1 = S_1 \cup S_2 \cup \{e_1, e_2\}$ is an edge dominating set of $P_n \odot P_m$. Therefore, $|X_1| = \lfloor \frac{n-1}{2} \rfloor + n \lceil \frac{m-1}{3} \rceil + 2$ and $m(P_n \odot P_m - X_1) = 2(m+1)$. Thus,

$$DI^{'}(P_n \odot P_m) \leq |X_1| + m(P_n \odot P_m - X_1) = \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + 2m + 4 = DI^{'}(P_n \odot P_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n \odot P_m$ and $m(P_n \odot P_m - X_2) = m+1$. X_2 is obtained as follows.
Let $S_1^{'} = E(P_n)$. $S_1^{'}$ is an edge dominating set of P_n and $\left|S_1^{'}\right| = n-1$.
Let S_{2_i} is a minimum edge dominating set of ith copy of P_m and $|S_{2_i}| = \lceil \frac{m-1}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup ... S_{2_n}$.
$X_2 = S_1^{'} \cup S_2$ is an edge dominating set of $P_n \odot P_m$.Therefore, $|X_2| = n - 1 + n \lceil \frac{m-1}{3} \rceil$ and $m(P_n \odot P_m - X_2) = m+1$. Thus,

$$DI^{'}(P_n \odot P_m) \leq |X_2| + m(P_n \odot P_m - X_2) = n + n \left\lceil \frac{m-1}{3} \right\rceil + m = DI^{'}(P_n \odot P_m)_{X_2}.$$

Because of definition of $DI^{'}$, the relationship between $DI^{'}(P_n \odot P_m)_{X_1}$ and $DI^{'}(P_n \odot P_m)_{X_2}$ must be examined as follows.

1. If $m+3 < \lceil \frac{n-1}{2} \rceil$, then we have

$$\begin{aligned} DI^{'}(P_n \odot P_m)_{X_1} &= \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + 2m + 4 = \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + m + 3 + m + 1 \\ &< \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil + m + 1, \text{ (by Proposition 2.2)} \\ &= n - 1 + n \left\lceil \frac{m-1}{3} \right\rceil + m + 1 = n + n \left\lceil \frac{m-1}{3} \right\rceil + m = DI^{'}(P_n \odot P_m)_{X_2}. \end{aligned}$$

Since $DI^{'}(P_n \odot P_m)_{X_1} < DI^{'}(P_n \odot P_m)_{X_2}$, then $DI^{'}(P_n \odot P_m) = DI^{'}(P_n \odot P_m)_{X_1} = \lfloor \frac{n-1}{2} \rfloor + n \lceil \frac{m-1}{3} \rceil + 2m + 4$.

2. If $m+3 > \lceil \frac{n-1}{2} \rceil$, then we have $DI^{'}(P_n \odot P_m)_{X_2} < DI^{'}(P_n \odot P_m)_{X_1}$. It can be proved in similar way as above. Therefore, $DI^{'}(P_n \odot P_m) = DI^{'}(P_n \odot P_m)_{X_2} = n + n \lceil \frac{m-1}{3} \rceil + m$.

3. If $m+3 = \lceil \frac{n-1}{2} \rceil$, then we have $DI^{'}(P_n \odot P_m)_{X_1} = DI^{'}(P_n \odot P_m)_{X_2}$. Hence, $DI^{'}(P_n \odot P_m) = DI^{'}(P_n \odot P_m)_{X_1} = DI^{'}(P_n \odot P_m)_{X_2}$.

□

Theorem 2.5. *For $n \geq 3$ and $m \geq 3$, let n to be odd and*

$$A = \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m}{3} \right\rceil + 2m + 3,$$

$$B = n + n \left\lceil \frac{m}{3} \right\rceil + m.$$

Then, $DI^{'}(P_n \odot C_m)$ is obtained as follows,

$$DI^{'}(P_n \odot C_m) = \begin{cases} A, & if\ m+2 < \frac{n-1}{2} \\ B, & if\ m+2 > \frac{n-1}{2} \\ A = B, & if\ m+2 = \frac{n-1}{2} \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $V(C_m) = \{u_1, u_2, ..., u_m\}$ for path graph P_n and cycle graph C_m. Let $E(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$ and $E(C_m) = \{u_1u_2, ..., u_{m-1}u_m, u_mu_1\}$.
For $n \geq 3$ and n is odd, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n \odot C_m$ and $m(P_n \odot C_m - X_1) = 2(m+1)$. X_1 is obtained as follows.
Let $S_1 = \{v_2v_3, v_4v_5, ..., v_{n-1}v_n\} \subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1| = \lfloor \frac{n-1}{2} \rfloor$.
Let S_{2_i} is a minimum edge dominating set of ith copy of C_m and $|S_{2_i}| = \lceil \frac{m}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup ... \cup S_{2_n}$.
$S_1 \cup S_2$ is not an edge dominating set of $P_n \odot C_m$ because some edges between v_1 and vertices of 1st copy of C_m are not dominated by any

edges in $S_1 \cup S_2$. So one of these edges (called e) is added $S_1 \cup S_2$, $X_1 = S_1 \cup S_2 \cup \{e\}$ is an edge dominating set of $P_n \odot C_m$. Therefore, $|X_1| = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 1$ and $m(P_n \odot C_m - X_1) = 2(m+1)$. Thus,

$$DI^{'}(P_n \odot C_m) \leq |X_1| + m(P_n \odot C_m - X_1) = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 3 = DI^{'}(P_n \odot C_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n \odot C_m$ and $m(P_n \odot C_m - X_2) = m+1$. X_2 is obtained as follows.
Let $S_1^{'} = E(P_n)$. $S_1^{'}$ is an edge dominating set of P_n and $\left|S_1^{'}\right| = n-1$.
Let S_{2_i} is a minimum edge dominating set of ith copy of C_m and $|S_{2_i}| = \left\lceil \frac{m}{3} \right\rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup \ldots \cup S_{2_n}$.
$X_2 = S_1^{'} \cup S_2$ is an edge dominating set of $P_n \odot C_m$.Therefore, $|X_2| = n-1+n\left\lceil \frac{m}{3} \right\rceil$ and $m(P_n \odot C_m - X_2) = m+1$. Thus,

$$DI^{'}(P_n \odot C_m) \leq |X_2| + m(P_n \odot C_m - X_2) = n + n\left\lceil \frac{m}{3} \right\rceil + m = DI^{'}(P_n \odot C_m)_{X_2}.$$

Because of definition of $DI^{'}$, the relationship between $DI^{'}(P_n \odot C_m)_{X_1}$ and $DI^{'}(P_n \odot C_m)_{X_2}$ must be examined as follows.

1. If $m+2 < \frac{n-1}{2}$, then we have

$$\begin{aligned} DI^{'}(P_n \odot C_m)_{X_1} &= \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 3 = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + m + 2 + m + 1 \\ &< \frac{n-1}{2} + n\left\lceil \frac{m}{3} \right\rceil + \frac{n-1}{2} + m + 1 \\ &= n - 1 + n\left\lceil \frac{m}{3} \right\rceil + m + 1 = n + n\left\lceil \frac{m}{3} \right\rceil + m = DI^{'}(P_n \odot C_m)_{X_2}. \end{aligned}$$

Since $DI^{'}(P_n \odot C_m)_{X_1} < DI^{'}(P_n \odot C_m)_{X_2}$, then $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_1} = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 3$.

2. If $m+2 > \frac{n-1}{2}$, then we have $DI^{'}(P_n \odot C_m)_{X_2} < DI^{'}(P_n \odot C_m)_{X_1}$. It can be proved in similar way as above. Therefore, $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_2} = n + n\left\lceil \frac{m}{3} \right\rceil + m$.

3. If $m+2 = \frac{n-1}{2}$, then we have $DI^{'}(P_n \odot C_m)_{X_1} = DI^{'}(P_n \odot C_m)_{X_2}$. Hence, $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_1} = DI^{'}(P_n \odot C_m)_{X_2}$.

□

Theorem 2.6. *For $n \geq 4$ and $m \geq 3$, let to n be even and*

$$A = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 4,$$

$$B = n + n\left\lceil \frac{m}{3} \right\rceil + m.$$

Then, $DI^{'}(P_n \odot C_m)$ is obtained as follows,

$$DI^{'}(P_n \odot C_m) = \begin{cases} A, & if\ m+3 < \left\lceil \frac{n-1}{2} \right\rceil \\ B, & if\ m+3 > \left\lceil \frac{n-1}{2} \right\rceil \\ A = B, & if\ m+3 = \left\lceil \frac{n-1}{2} \right\rceil \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(C_m) = \{u_1, u_2, \ldots, u_m\}$ for path graph P_n and cycle graph C_m. Let $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$ and $E(C_m) = \{u_1u_2, \ldots, u_{m-1}u_m, u_mu_1\}$.
For $n \geq 2$ and n is even, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n \odot C_m$ and $m(P_n \odot C_m - X_1) = 2(m+1)$. X_1 is obtained as follows.
Let $S_1 = \{v_2v_3, v_4v_5, \ldots, v_{n-1}v_n\} \subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1| = \left\lfloor \frac{n-1}{2} \right\rfloor$.
Let S_{2_i} is a minimum edge dominating set of ith copy of C_m and $|S_{2_i}| = \left\lceil \frac{m}{3} \right\rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup \ldots \cup S_{2_n}$.
$S_1 \cup S_2$ is not an edge dominating set of $P_n \odot C_m$ because some edges between v_1 and vertices of 1st copy of C_m and v_n and vertices of nth copy of C_m are not dominated by any edges in $S_1 \cup S_2$. So one of edges between v_1 and vertices of 1st copy of C_m (called e_1) and one of edges between v_n and vertices of nth copy of C_m (called e_2) are added $S_1 \cup S_2$, $X_1 = S_1 \cup S_2 \cup \{e_1, e_2\}$ is an edge dominating set of $P_n \odot C_m$. Therefore, $|X_1| = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2$ and $m(P_n \odot C_m - X_1) = 2(m+1)$. Thus,

$$DI^{'}(P_n \odot C_m) \leq |X_1| + m(P_n \odot C_m - X_1) = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 4 = DI^{'}(P_n \odot C_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n \odot C_m$ and $m(P_n \odot C_m - X_2) = m+1$. X_2 is obtained as follows.
Let $S_1^{'} = E(P_n)$. $S_1^{'}$ is an edge dominating set of P_n and $\left|S_1^{'}\right| = n-1$.
Let S_{2_i} is a minimum edge dominating set of ith copy of C_m and $|S_{2_i}| = \lceil \frac{m}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup ... \cup S_{2_n}$.
$X_2 = S_1^{'} \cup S_2$ is an edge dominating set of $P_n \odot C_m$. Therefore, $|X_2| = n-1+n\lceil \frac{m}{3} \rceil$ and $m(P_n \odot C_m - X_2) = m+1$. Thus,

$$DI^{'}(P_n \odot C_m) \leq |X_2| + m(P_n \odot C_m - X_2) = n + n\left\lceil \frac{m}{3} \right\rceil + m = DI^{'}(P_n \odot C_m)_{X_2}.$$

Because of definition of $DI^{'}$, the relationship between $DI^{'}(P_n \odot C_m)_{X_1}$ and $DI^{'}(P_n \odot C_m)_{X_2}$ must be examined as follows.

1. If $m+3 < \lceil \frac{n-1}{2} \rceil$, then we have

$$\begin{aligned} DI^{'}(P_n \odot C_m)_{X_1} &= \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m+4 = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + m+3+m+1 \\ &< \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil + m+1, \text{ (by Proposition 2.2)} \\ &= n-1+n\left\lceil \frac{m}{3} \right\rceil + m+1 = n+n\left\lceil \frac{m}{3} \right\rceil + m = DI^{'}(P_n \odot C_m)_{X_2}. \end{aligned}$$

Since $DI^{'}(P_n \odot P_m)_{X_1} < DI^{'}(P_n \odot P_m)_{X_2}$, then $DI^{'}(P_n \odot P_m) = DI^{'}(P_n \odot P_m)_{X_1} = \lfloor \frac{n-1}{2} \rfloor + n\lceil \frac{m-1}{3} \rceil + 2m+4$.

2. If $m+3 > \lceil \frac{n-1}{2} \rceil$, then we have $DI^{'}(P_n \odot C_m)_{X_2} < DI^{'}(P_n \odot C_m)_{X_1}$. It can be proved in similar way as above. Therefore, $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_2} = n+n\lceil \frac{m}{3} \rceil + m$.

3. If $m+3 = \lceil \frac{n-1}{2} \rceil$, then we have $DI^{'}(P_n \odot C_m)_{X_1} = DI^{'}(P_n \odot C_m)_{X_2}$. Hence, $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_1} = DI^{'}(P_n \odot C_m)_{X_2}$.

□

Theorem 2.7. *For $n \geq 3$, let $A = \lfloor \frac{n-1}{2} \rfloor + n + 2m + 4$ and $B = 2n + m + 1$. Then, $DI^{'}\left(P_n \odot K_{1,m}\right)$ is obtained as follows,*

$$DI^{'}\left(P_n \odot K_{1,m}\right) = \begin{cases} A, & if\ m+2 < \lceil \frac{n-1}{2} \rceil \\ B, & if\ m+2 > \lceil \frac{n-1}{2} \rceil \\ A=B, & if\ m+2 = \lceil \frac{n-1}{2} \rceil \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ for path graph P_n and for each ith copy of $K_{1,m}$ vertex set $V_i(K_{1,m}) = \{u_{1_i}, u_{2_i}, ..., u_{m_i}, u_{m+1_i}\}$.(u_{1_i} is central vertex of ith copy of $K_{1,m}$.) Let $E(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$ and $E_i(K_{1,m}) = \{u_{1_i}u_{2_i}, ..., u_{1_i}u_{m_i}, u_{1_i}u_{m+1_i}\}$.
For $n \geq 2$ and n is even, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n \odot K_{1,m}$ and $m(P_n \odot K_{1,m} - X_1) = 2(m+2)$. X_1 is obtained as follows.
Let $S_1 = \{v_2v_3, v_4v_5, ..., v_{n-1}v_n\} \subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1| = \lfloor \frac{n-1}{2} \rfloor$.
$S_2 = \{v_1u_{1_1}, v_2u_{1_2}, ..., v_nu_{1_n}\}$ is a minimum edge dominating set of $P_n \odot K_{1,m} - E(P_n)$ and $|S_2| = n$.
$S_1 \cup S_2$ is an edge dominating set of $P_n \odot K_{1,m}$. Therefore, $|X_1| = \lfloor \frac{n-1}{2} \rfloor + n$ and $m(P_n \odot P_m - X_1) = 2(m+2)$. Thus,

$$DI^{'}\left(P_n \odot K_{1,m}\right) \leq |X_1| + m(P_n \odot K_{1,m} - X_1) = \left\lfloor \frac{n-1}{2} \right\rfloor + n + 2m + 4 = DI^{'}(P_n \odot P_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n \odot K_{1,m}$ and $m(P_n \odot K_{1,m} - X_2) = m+2$. X_2 is obtained as follows.
Let $S_1^{'} = E(P_n)$. $S_1^{'}$ is an edge dominating set of P_n and $\left|S_1^{'}\right| = n-1$.
$S_2 = \{v_1u_{1_1}, v_2u_{1_2}, ..., v_nu_{1_n}\}$ is a minimum edge dominating set of $P_n \odot K_{1,m} - E(P_n)$ and $|S_2| = n$.
$X_2 = S_1^{'} \cup S_2$ is an edge dominating set of $P_n \odot K_{1,m}$. Therefore, $|X_2| = n-1+n = 2n-1$ and $m(P_n \odot K_{1,m} - X_2) = m+2$. Thus,

$$DI^{'}\left(P_n \odot K_{1,m}\right) \leq |X_2| + m(P_n \odot K_{1,m} - X_2) = 2n + m + 1 = DI^{'}\left(P_n \odot K_{1,m}\right)_{X_2}.$$

Because of definition of $DI^{'}$, the relationship between $DI^{'}\left(P_n \odot K_{1,m}\right)_{X_1}$ and $DI^{'}\left(P_n \odot K_{1,m}\right)_{X_2}$ must be examined as follows.

1. If $m+2 < \lceil \frac{n-1}{2} \rceil$, then we have

$$
\begin{aligned}
DI'\left(P_n \odot K_{1,m}\right)_{X_1} &= \left\lfloor \frac{n-1}{2} \right\rfloor + n + 2m + 4 = \left\lfloor \frac{n-1}{2} \right\rfloor + n + m + 2 + m + 2 \\
&< \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lceil \frac{n-1}{2} \right\rceil + m + 2, \text{ (by Proposition2.2)} \\
&= n - 1 + n + m + 2 = 2n + m + 1 = DI'\left(P_n \odot K_{1,m}\right)_{X_2}.
\end{aligned}
$$

Since $DI'\left(P_n \odot K_{1,m}\right)_{X_1} < DI'\left(P_n \odot K_{1,m}\right)_{X_2}$, then $DI'\left(P_n \odot K_{1,m}\right) = DI'\left(P_n \odot K_{1,m}\right)_{X_1} = \left\lfloor \frac{n-1}{2} \right\rfloor + n + 2m + 4$.

2. If $m+2 > \left\lceil \frac{n-1}{2} \right\rceil$, then we have $DI'\left(P_n \odot K_{1,m}\right)_{X_2} < DI'\left(P_n \odot K_{1,m}\right)_{X_1}$. It can be proved in similar way as above. Therefore, $DI'\left(P_n \odot K_{1,m}\right) = DI'\left(P_n \odot K_{1,m}\right)_{X_2} = 2n + m + 1$.

3. If $m+2 = \left\lceil \frac{n-1}{2} \right\rceil$, then we have $DI'\left(P_n \odot K_{1,m}\right)_{X_1} = DI'\left(P_n \odot K_{1,m}\right)_{X_2}$. Hence, $DI'\left(P_n \odot K_{1,m}\right) = DI'\left(P_n \odot K_{1,m}\right)_{X_1} = DI'\left(P_n \odot K_{1,m}\right)_{X_2}$.

□

3. Conclusion

Domination concept and integrity are very valuable measures for network designers. Due to the changes of network models and design styles based on demands, measures on nodes or links which have specific properties became more important. For example edge domination and integrity are some of these concepts related to these specific properties. Edge domination only gives an idea about the communication on graph model after any faiulers on edges (links) and integrity itself gives information about only the stability of graph model of network based on edges. But our new measure domination edge integrity [10] combines these two important concepts. Corona operation is one of those commonly used operations in network design which has applications. In this paper, domination edge integrity of some graphs under corona operation is examined such as $P_n \odot P_m$, $P_n \odot C_m$, $P_n \odot K_{1,n}$ and results are obtained. For future work we plan to extend our results on other important classes of graphs under corona operation and generalize the obtained results.

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f-Asymptotically $\mathscr{I}_{\sigma\theta}$-Equivalence of Real Sequences

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Abstract

In this manuscript, we present the ideas of asymptotically $[\mathscr{I}_{\sigma\theta}]$-equivalence, asymptotically $\mathscr{I}_{\sigma\theta}(f)$-equivalence, asymptotically $[\mathscr{I}_{\sigma\theta}(f)]$-equivalence and asymptotically $\mathscr{I}(S_{\sigma\theta})$-equivalence for real sequences. In addition to, investigate some connections among these new ideas and we give some inclusion theorems about them.

1. Introduction

Before starting article, we give the basic concepts and properties of statistical convergence, ideal convergence, invariant mean and invariant convergence, asymptotically equivalence and modulus function. Throughout this study, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers. Statistical convergence and ideal convergence have recently begun to attract interest in science and engineering as well as by mathematicians. The idea of convergence of a real sequence was extended to statistical convergence by Fast [1] and Schoenberg [2] independently, and then statistical convergence has been studied by many authors. Kostyrko et al. [3] firstly, introduced the notion of $\mathscr{I}$-convergence as a generalization of statistical convergence.

Invariant convergence has recently been gaining more and more interest among mathematicians working on summability theory. Several authors including Raimi [4], Schaefer [5], Mursaleen and Edely [6], Mursaleen [7], Savaş [8–10], Nuray and Savaş [11], Pancaroğlu and Nuray [12] studied on σ-convergent sequences and some properties of σ-convergence. The notion of lacunary strong σ-convergence was defined by Savaş [10]. Then, Savaş and Nuray [13] introduced the notion of σ-statistical convergence and also, defined lacunary σ-statistical convergence and examined some inclusion theorems with examples. After that, Nuray et al. [14] defined the notions of σ-uniform density of a subset A of $\mathbb{N}$, $\mathscr{I}_\sigma$-convergence and examined connections between $\mathscr{I}_\sigma$-convergence and σ-convergence and also, $\mathscr{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Also, Pancaroğlu and Nuray [12] studied statistical lacunary σ-summability. Recently, Ulusu and Nuray [15] investigated the concepts of lacunary $\mathscr{I}_\sigma$-convergence and lacunary $\mathscr{I}_\sigma$-Cauchy sequence of real numbers.

The concept of asymptotically equivalence and applications are of interest to scientists working on convergence types. Marouf [16] peresented ideas of asymptotically equivalence. Patterson and Savaş [17, 18] denoted the ideas of asymptotically lacunary statistically equivalence and asymptotically $\sigma\theta$-statistical equivalence of real sequences. Ulusu [19, 20] studied the notion asymptotically ideal invariant equivalence and asymptotically lacunary $\mathscr{I}_\sigma$-equivalence.

Modulus function and its various applications are used in many sub-disciplines in the field of mathematics. Nakano [21] denoted the notion f modulus function. Maddox [22], Pehlivan [23] and several authors using a modulus function f, define some new concepts and give some inclusion theorems with examples. Kumar and Sharma [24] using a modulus function f, investigated lacunary $\mathscr{I}$-equivalence of real sequences.

Now, let's give some basic and important concepts, lemma and properties that are related to our work subject and we will use in our article,

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by citing the authors we give in the references (see [3, 9, 10, 14–16, 21–26]).

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of positive integers). A continuous linear functional ψ on ℓ_∞, the space of bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies the following conditions:

1. $\psi(a_n) \geq 0$, when the sequence (a_n) has $a_n \geq 0$, for all n,
2. $\psi(e) = 1$, where $e = (1,1,1,\ldots)$ and
3. $\psi(a_{\sigma(n)}) = \psi(a_n)$ for all $(a_n) \in \ell_\infty$.

The mappings σ are supposed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $m, n \in \mathbb{N}^+$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus ψ extends the limit functional on c, the space of convergent sequences, in the sense that $\psi(a_n) = \lim a_n$ for all $(a_n) \in c$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Throughout this study, let $\theta = \{k_r\}$ be a lacunary sequence.

The concept of lacunary strong σ-convergence was defined as below:

$$L_\theta = \left\{ a = (a_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |a_{\sigma^k(m)} - K| = 0 \right\},$$

uniformly in $m = 1, 2, \ldots$.

If for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : |a_{\sigma^k(n)} - K| \geq \varepsilon \right\} \right| = 0,$$

uniformly in $n = 1, 2, \ldots$, then the sequence $a = (a_k)$ is $S_{\sigma\theta}$-convergent to K.

Let $\mathscr{I}$ be a family of subsets of $2^{\mathbb{N}}$. If the following conditions holds, then we named $\mathscr{I} \subseteq 2^{\mathbb{N}}$ an ideal:

(i) $\emptyset \in \mathscr{I}$,

(ii) For any $C, D \in \mathscr{I}$, we get $C \cup D \in \mathscr{I}$,

(iii) For any $C \in \mathscr{I}$ and any $D \subseteq C$, we get $D \in \mathscr{I}$.

An ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is named a non-trivial if $\mathbb{N} \notin \mathscr{I}$ and a non-trivial ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is named admissible if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$. Throughout this study, we let $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal.

Let $H \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| H \cap \left\{ \sigma(n), \sigma^2(n), \ldots, \sigma^m(n) \right\} \right| \text{ and } S_m = \max_n \left| H \cap \left\{ \sigma(n), \sigma^2(n), \ldots, \sigma^m(n) \right\} \right|.$$

If the limits

$$\underline{V}(H) = \lim_{m \to \infty} \frac{s_m}{m} \text{ and } \overline{V}(H) = \lim_{m \to \infty} \frac{S_m}{m}$$

exist, then they are named a lower σ-uniform density and an upper σ-uniform density of the set H, respectively. If $\underline{V}(H) = \overline{V}(H)$, then $V(H) = \underline{V}(H) = \overline{V}(H)$ is named the σ-uniform density of H.

Denote by $\mathscr{I}_\sigma$ the class of all $H \subseteq \mathbb{N}$ with $V(H) = 0$. Obviously, $\mathscr{I}_\sigma$ is an admissible ideal in $\mathbb{N}$.

A sequence $a = (a_k)$ is told to be $\mathscr{I}_\sigma$-convergent to K if for each $\varepsilon > 0$, the set $H_\varepsilon = \left\{ k : |a_k - K| \geq \varepsilon \right\}$ belongs to $\mathscr{I}_\sigma$, i.e., $V(H_\varepsilon) = 0$. It is denoted by $\mathscr{I}_\sigma - \lim_{k \to \infty} a_k = K$.

Let $\theta = \{k_r\}$ be a lacunary sequence, $H \subseteq \mathbb{N}$ and

$$s_r = \min_n \left\{ \left| H \cap \{ \sigma^m(n) : m \in I_r \} \right| \right\} \text{ and } S_r = \max_n \left\{ \left| H \cap \{ \sigma^m(n) : m \in I_r \} \right| \right\}.$$

If the limits

$$\underline{V_\theta}(H) = \lim_{r \to \infty} \frac{s_r}{h_r} \text{ and } \overline{V_\theta}(H) = \lim_{r \to \infty} \frac{S_r}{h_r}$$

exist, then they are named a lower lacunary σ-uniform density and an upper lacunary σ-uniform density of the set H, respectively. If $\underline{V_\theta}(H) = \overline{V_\theta}(H)$, then $V_\theta(H) = \underline{V_\theta}(H) = \overline{V_\theta}(H)$ is named the lacunary σ-uniform density of H.

Denoted by $\mathscr{I}_{\sigma\theta}$ the class of all $H \subseteq \mathbb{N}$ with $V_\theta(H) = 0$. Obviously, $\mathscr{I}_{\sigma\theta}$ is an admissible ideal in $\mathbb{N}$.

A sequence (a_k) is told to be lacunary $\mathscr{I}_\sigma$-convergent or $\mathscr{I}_{\sigma\theta}$-convergent to K if for each $\varepsilon > 0$, $H_\varepsilon = \left\{ k : |a_k - K| \geq \varepsilon \right\} \in \mathscr{I}_{\sigma\theta}$, i.e., $V_\theta(H_\varepsilon) = 0$. It is denoted by $\mathscr{I}_{\sigma\theta} - \lim_{k \to \infty} a_k = K$.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically equivalent if $\lim_{k\to\infty}\frac{a_k}{v_k}=1$, (denoted by $a\sim v$).

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly asymptotically lacunary invariant equivalent of multiple K if

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{a_{\sigma^k(m)}}{v_{\sigma^k(m)}}-K\right|=0,$$

uniformly in m (denoted by $a\overset{N_{\sigma\theta}^K}{\sim}v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically $N_{\sigma\theta}$-equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically lacunary invariant statistical equivalent of multiple K if for each $\varepsilon>0$,

$$\lim_r \frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{a_{\sigma^k(m)}}{v_{\sigma^k(m)}}-K\right|\geq\varepsilon\right\}\right|=0,$$

uniformly in m (denoted by $a\overset{S_{\sigma\theta}^K}{\sim}v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically lacunary invariant statistical equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly asymptotically lacunary $\mathscr{I}$-equivalent of multiple K provided that for each $\varepsilon>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\in\mathscr{I}$$

(denoted by $a\overset{\mathscr{I}(N_\theta^K)}{\sim}v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly asymptotically lacunary $\mathscr{I}$-equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically lacunary statistical $\mathscr{I}$-equivalent of multiple K if for each $\varepsilon>0$ and $\gamma>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\right|\geq\gamma\right\}\in\mathscr{I}$$

(denoted by $a\overset{\mathscr{I}(S_\theta^K)}{\sim}v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically lacunary $\mathscr{I}$-statistical equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically $\mathscr{I}_{\sigma\theta}$-equivalent of multiple K if for each $\varepsilon>0$,

$$\widetilde{H}_\varepsilon=\left\{k\in I_r:\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\in\mathscr{I}_{\sigma\theta},$$

i.e., $V_\theta(\widetilde{H}_\varepsilon)=0$. It is denoted by $a\overset{\mathscr{I}_{\sigma\theta}^K}{\sim}v$. If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically $\mathscr{I}_{\sigma\theta}$-equivalent.

A function $f:[0,\infty)\to[0,\infty)$ is called a modulus if

1. $f(t)=0$ if and if only if $t=0$,
2. $f(t+v)\leq f(t)+f(v)$,
3. f is increasing,
4. f is continuous from the right at 0.

A modulus may be unbounded (for example $f(t)=t^p$, $0<p<1$) or bounded (for example $f(t)=\frac{t}{t+1}$).

Throughout this study, let f be a modulus function.

Two non-negative $a=(a_k)$ and $v=(v_k)$ are told to be strongly f-asymptotically lacunary $\mathscr{I}$-equivalent of multiple K provided that for each $\varepsilon>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}$$

(denoted by $a\overset{\mathscr{I}^K(N_\theta^f)}{\sim}v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly f-asymptotically lacunary $\mathscr{I}$-equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly asymptotically $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\in\mathscr{I}_\sigma$$

(denoted by $a\overset{[\mathscr{I}_\sigma^K]}{\sim}v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly asymptotically $\mathscr{I}$-invariant equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be f-asymptotically $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$,

$$\left\{k\in\mathbb{N}: f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_\sigma$$

(denoted by $a \overset{\mathscr{I}_\sigma^K(f)}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply f-asymptotically $\mathscr{I}$-invariant equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly f-asymptotically $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$,

$$\left\{n\in\mathbb{N}: \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_\sigma$$

(denoted by $a \overset{[\mathscr{I}_\sigma^K(f)]}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly f-asymptotically $\mathscr{I}$-invariant equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically $\mathscr{I}$-invariant statistical equivalent of multiple K if for each $\varepsilon>0$ and each $\gamma>0$,

$$\left\{n\in\mathbb{N}: \frac{1}{n}\left|\left\{k\leq n: \left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\right|\geq\gamma\right\}\in\mathscr{I}_\sigma$$

(denoted by $a \overset{\mathscr{I}(S_\sigma^K)}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically $\mathscr{I}$-invariant statistical equivalent.

Lemma 1.1. *[23] Let $0<\lambda<1$. Then, we have $f(s)\leq 2f(1)\lambda^{-1}s$, for each $s\geq\lambda$.*

2. Main results

Now, we give the original definitions of our article and explain the theorems that are original, together with their proofs. Our theorems give many features and necessity relations between these new concepts.

Definition 2.1. *Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly asymptotically lacunary $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$*

$$\left\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\in\mathscr{I}_{\sigma\theta}$$

(denoted by $a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly asymptotically lacunary $\mathscr{I}$-invariant equivalent.

Definition 2.2. *Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be f-asymptotically lacunary $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$*

$$\left\{k\in I_r: f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_{\sigma\theta},$$

(denoted by $a \overset{\mathscr{I}_{\sigma\theta}^K(f)}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply f-asymptotically lacunary $\mathscr{I}$-invariant equivalent.

Definition 2.3. *Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly f-asymptotically lacunary $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$*

$$\left\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_{\sigma\theta}$$

(denoted by $a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly f-asymptotically lacunary $\mathscr{I}$-invariant equivalent.

Theorem 2.4. *For two non-negative sequences $a=(a_k)$ and $v=(v_k)$ we have $a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v \Rightarrow a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v$.*

Proof. Let $a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v$ and $\varepsilon>0$ be given. For $0\leq s\leq\lambda$, select $0<\lambda<1$ such that $f(s)<\varepsilon$. Then, we have

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{a_k}{v_k}-K\right|\right)=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{a_k}{v_k}-K\right|\leq\lambda}} f\left(\left|\frac{a_k}{v_k}-K\right|\right)+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{a_k}{v_k}-K\right|>\lambda}} f\left(\left|\frac{a_k}{v_k}-K\right|\right)$$

and so by Lemma 1.1

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{a_k}{v_k}-K\right|\right)<\varepsilon+\left(\frac{2f(1)}{\lambda}\right)\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{a_k}{v_k}-K\right|.$$

Thus, for each $\gamma > 0$ we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \gamma \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{a_k}{v_k} - K \right| \geq \frac{(\gamma - \varepsilon)\lambda}{2f(1)} \right\}.$$

Since $a \overset{[\mathscr{I}^K_{\sigma\theta}]}{\sim} v$, the next set and so the first set in the foregoing statement pertain to $\mathscr{I}_{\sigma\theta}$. This proves that

$$a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v.$$

□

Theorem 2.5. *If* $\lim_{s\to\infty} \frac{f(s)}{s} = \alpha > 0$*, then for two non-negative sequences* $a = (a_k)$ *and* $v = (v_k)$ *we have*

$$a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v \Leftrightarrow a \overset{[\mathscr{I}^K_{\sigma\theta}]}{\sim} v.$$

Proof. In Theorem 2.4, we showed that $a \overset{[\mathscr{I}^K_{\sigma\theta}]}{\sim} v \Rightarrow a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v$. Now, we must show that

$$a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v \Rightarrow a \overset{[\mathscr{I}^K_{\sigma\theta}]}{\sim} v.$$

For all $a \geq 0$, if we let $\lim_{a\to\infty} \frac{f(a)}{a} = \alpha > 0$, then we have $f(a) \geq \alpha a$. Assume that $a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v$. Since

$$\frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \frac{1}{h_r} \sum_{k \in I_r} \alpha \left(\left| \frac{a_k}{v_k} - K \right| \right) = \alpha \left(\frac{1}{h_r} \sum_{k \in I_r} \left| \frac{a_k}{v_k} - K \right| \right)$$

holds, hence for each $\varepsilon > 0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \alpha\varepsilon \right\}.$$

Since $a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v$, the next set and so the first set in the foregoing statement pertains to $\mathscr{I}_{\sigma\theta}$. This proves that

$$a \overset{[\mathscr{I}^K_{\sigma\theta}]}{\sim} v \Leftrightarrow a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v.$$

□

Definition 2.6. *Two non-negative sequences* $a = (a_k)$ *and* $v = (v_k)$ *are told to be asymptotically lacunary* $\mathscr{I}$*-invariant statistical equivalent of multiple* K *if for any* $\varepsilon > 0$ *and any* $\gamma > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathscr{I}_{\sigma\theta}$$

(denoted by $a \overset{\mathscr{I}(S^K_{\sigma\theta})}{\sim} v$*). If we let* $K = 1$*, then* $a = (a_k)$ *and* $v = (v_k)$ *are told to be simply asymptotically lacunary* $\mathscr{I}$*-invariant statistical equivalent.*

Theorem 2.7. *For two non-negative sequences* $a = (a_k)$ *and* $v = (v_k)$ *we have*

$$a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v \Rightarrow a \overset{\mathscr{I}(S^K_{\sigma\theta})}{\sim} v.$$

Proof. Granted that $a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v$ and $\varepsilon > 0$ be given. Since

$$\frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon}} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq f(\varepsilon) \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon \right\} \right|$$

holds, hence for each $\gamma > 0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \gamma f(\varepsilon) \right\}.$$

Since $a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v$, the next set and so the first set in the foregoing statement pertains to $\mathscr{I}_{\sigma\theta}$ and hence, $a \overset{\mathscr{I}(S^K_{\sigma\theta})}{\sim} v$. □

Theorem 2.8. *If* f *is bounded, then for two non-negative sequences* $a = (a_k)$ *and* $v = (v_k)$ *we have*

$$a \overset{\mathscr{I}(S^K_{\sigma\theta})}{\sim} v \Leftrightarrow a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v.$$

Proof. In Theorem 2.7, we showed that $a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v \Rightarrow a \overset{\mathscr{I}(S^K_{\sigma\theta})}{\sim} v$. Now, we must show that

$$a \overset{\mathscr{I}(S^K_{\sigma\theta})}{\sim} v \Rightarrow a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v.$$

Granted that f is bounded and $a \overset{\mathscr{I}(S^K_{\sigma\theta})}{\sim} v$. Hence, there exists a positive real number L such that $|f(a)| \leq L$, for all $a \geq 0$. Further using this fact, we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{a_k}{v_k}-K\right|\right) &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{a_k}{v_k}-K\right|\geq\varepsilon}} f\left(\left|\frac{a_k}{v_k}-K\right|\right)+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{a_k}{v_k}-K\right|<\varepsilon}} f\left(\left|\frac{a_k}{v_k}-K\right|\right) \\
&\leq \frac{L}{h_r}\left|\left\{k\in I_r : \left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\right|+f(\varepsilon).
\end{aligned}$$

This proves that $a \overset{[\mathscr{I}^K_{\sigma\theta}(f)]}{\sim} v$. □

3. Conclusion

In the present study, using modulus function and lacunary sequences, we investigated the types of asymptotically ideal invariant equivalence for real sequences and give theorems about some properties. These new concepts can be examine for set sequences.

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Modified Block-Pulse Functions Scheme for Solve of Two-Dimensional Stochastic Integral Equations

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Abstract

In this paper, two-dimensional modified block-pulse functions (2D-MBPFs) method is introduced for approximate solution of 2D-linear stochastic Volterra-Fredholm integral equations so the ordinary and stochastic operrational matrices of integration are utilized to reduce the computation of such equations into some algebraic equations. Convergence analysis of this method is discussed. Finally an illustrative example is given to show the accuracy of the proposed method so the results of it is compared with the block-pulse functions (BPFs) method.

1. Introduction

Mainly 2D-integral equations furnish the important implement for modeling the engineering and science problems [1, 2]. We have used the variant methods for solving 2D-linear stochastic integral equations in [3, 4, 5, 6, 7] that the BPFs method is one of these methods. The BPFs are very common in use, but it seems that their convergence is weak. Here the modified block-pulse functions (MBPFs) method is used for deriving approximation solution of 2D-linear stochastic Volterra-Fredholm integral equation of the second kind

$$g(x,y) = f(x,y) + \int_0^1 \int_0^1 V_1(x,y,s,t)g(s,t)dsdt + \int_0^y \int_0^x V_2(x,y,s,t)g(s,t)dsdt + \int_0^y \int_0^x V_3(x,y,s,t)g(s,t)dB(s)dB(t), \tag{1.1}$$

where $(x,y) \in [0,T_1) \times [0,T_2)$ and

$$s \leqslant x < t \leqslant y. \tag{1.2}$$

In (1.1), $g(x,y)$ is the unknown function and the condition (1.2) is necessary.
We organize the paper as follows:
The properties of 2D-MBPFs are introduced in the next section. In Section 3 we solve (1.1) by finding the ordinary and stochastic operational matrices. We depict the error analysis in Section 4. The certitude of the method is evinced by an example in Section 5. Eventually, we afford the brief conclusion in Section 6.

2. Two dimentional MBPFs

An $(n_1+1)\times(n_2+1)$-set of 2D-MBPFs $\omega_{a_1,a_2}(x,y)$ $(a_1 = 0,1,...,n_1)$; $(a_2 = 0,1,...,n_2)$ consists of $(n_1+1)\times(n_2+1)$ functions which are defined over district D by [8]

$$\omega_{a_1,a_2}(x,y) = \begin{cases} 1, & (x,y) \in D_{a_1,a_2} \\ 0, & otherwise, \end{cases} \tag{2.1}$$

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where

$$D_{a_1,a_2} = (x,y) : x \in I_{a_1,\varepsilon}, y \in I_{a_2,\varepsilon},$$

and

$$I_{a_1,\varepsilon} = \begin{cases} [0, k_1 - \varepsilon), & a_1 = 0 \\ [a_1 k_1 - \varepsilon, (a_1+1)k_1 - \varepsilon), & a_1 = 1(1)(n_1 - 1) \\ [1-\varepsilon, 1) & a_1 = n_1, \end{cases}$$

$$I_{a_2,\varepsilon} = \begin{cases} [0, k_2 - \varepsilon), & a_2 = 0 \\ [a_2 k_2 - \varepsilon, (a_2+1)k_2 - \varepsilon), & a_2 = 1(1)(n_2 - 1) \\ [1-\varepsilon, 1) & a_2 = n_1, \end{cases}$$

where n_1 and n_2 are arbitary positive integers and we have

$$k_1 = \frac{T_1}{n_1} \;\; , \;\; k_2 = \frac{T_2}{n_2}.$$

From (2.1), we can represent 2D-MBPFs as

$$\omega_{a_1,a_2}(x,y) = \omega_{a_1}(x)\omega_{a_2}(y),$$

where ω_{a_1} and ω_{a_2} are the one-dimensional MBPFs. Similar to the one-dimensional case, 2D-MBPFs have the elementary properties that are: disjointness, orthogonality and completeness. Also the set of 2D-MBPFs can be written as a vector $\Omega(x,y)$ of dimension $\varsigma_1 = (n_1+1)(n_2+1) \times 1$ as

$$\Omega(x,y) = [\omega_{0,0}(x,y), \dots, \omega_{0,n_2}(x,y), \dots, \omega_{n_1,0}(x,y), \dots, \omega_{n_1,n_2}(x,y)]^T, \tag{2.2}$$

where $(x,y) \in D$. For every ς_1-vector K from (2.2) we have

$$\Omega(x,y)\Omega^T(x,y)K = \tilde{K}\Omega(x,y), \tag{2.3}$$

where $\tilde{K} = diag(K)$ is a diagonal matrix of dimension $\varsigma_2 = (n_1+1)(n_2+1) \times (n_1+1)(n_2+1)$. Moreover, for every ς_2-matrix H we get

$$\Omega^T(x,y)H\Omega(x,y) = \hat{H}^T\Omega(x,y), \tag{2.4}$$

where $\hat{H}$ is an ς_1-vector with elements equal to the diagonal entries of matrix H.

2.1. Two dimensional MBPFs expansions

A function $f(x,y)$ defined over $L^2(D)$ can be expanded by the 2D-MBPFs as [8, 9]

$$f \simeq f_\varepsilon = \sum_{a_1=0}^{n_1} \sum_{a_2=0}^{n_2} f_{a_1,a_2}\omega_{a_1,a_2} = F^T\Omega,$$

where F is an ς_1-vector given by

$$F = [f_{0,0}, \dots, f_{0,n_2}, \dots, f_{n_1,0}, \dots, f_{n_1,n_2}]^T,$$

and Ω is defined in (2.2). The modified block-pulse coefficients, f_{a_1,a_2} , are obtained as

$$f_{a_1,a_2} = \frac{1}{\ell(I_{a_1,\varepsilon}) \times \ell(I_{a_2,\varepsilon})} \int_{I_{a_1,\varepsilon}} \int_{I_{a_2,\varepsilon}} f(x,y)dydx,$$

where $\ell(I_{a_1,\varepsilon})$ and $\ell(I_{a_2,\varepsilon})$ are length of intervals $I_{a_1,\varepsilon}$ and $I_{a_2,\varepsilon}$ respectively. Similarly for every function $f(x,y,s,t)$, we can write

$$f(x,y,s,t) \simeq \Omega(x,y)^T F_\varepsilon \Omega(s,t),$$

where F_ε is 2D-MBPF coefficient matrix of dimension ς_2.

2.2. Ordinary perational matrix of 2D-MBPFs

By the double integration of the vector Ω defined in (2.2) we have [8, 10, 11]

$$\int_0^y \int_0^x \Omega(s,t)dsdt \simeq P_\varepsilon \Omega(x,y) = [O_{\varepsilon,(n_1+1)\times(n_1+1)} \otimes O_{\varepsilon,(n_2+1)\times(n_2+1)}]\Omega(x,y), \tag{2.5}$$

where $(x,y) \in D$ and P_ε is ς_2-ordinary operational matrix of integration for 2D-MBPFs so O_ε is defined in [5]. In (2.5), $\otimes$ denotes the Kronecker product. By disjointness and orthogonality properties of 2D-MBPFs we have

$$\int_0^1 \int_0^1 \Omega(s,t)\Omega^T(s,t)dsdt = \begin{pmatrix} (k_1-\varepsilon)(k_2-\varepsilon) & 0 & 0 & \dots & 0 \\ 0 & k_1k_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & k_1k_2 & 0 \\ 0 & 0 & \dots & 0 & \varepsilon \end{pmatrix} = R_\varepsilon, \tag{2.6}$$

where R_ε is the ς_2-known matrix.

2.3. Stochastic operational matrix of 2D-MBPFs

Similarly we obtain

$$\int_0^y \int_0^x \Omega(s,t)dB(s)dB(t) \simeq P_{\varepsilon,s}\Omega(x,y) = [O_{\varepsilon,s,(n_1+1)\times(n_1+1)} \otimes O_{\varepsilon,s,(n_2+1)\times(n_2+1)}]\Omega(x,y), \tag{2.7}$$

where $P_{\varepsilon,s}$ is the ς_2-stochastic operational matrix of integration for 2D-MBPFs where $O_{\varepsilon,s}$ is defined in [3]. In the next sections, it is assumed that $T_1 = T_2 = 1$.

3. Method of solution

Now, we solve (1.1) using 2D-MBPFs. By applying 2D-MBPFs approximates for functions

$$f(x,y)\ ,\ V_1(x,y,s,t)\ ,\ V_2(x,y,s,t)\ ,\ V_3(x,y,s,t)\ ,\ g(x,y),$$

we have

$$f = F_\varepsilon^T \Omega, \tag{3.1}$$

$$V_1 = \Omega^T(x,y)\Gamma_\varepsilon \Omega(s,t), \tag{3.2}$$

$$V_2 = \Omega^T(x,y)\Delta_\varepsilon \Omega(s,t), \tag{3.3}$$

$$V_3 = \Omega^T(x,y)\Theta_\varepsilon \Omega(s,t), \tag{3.4}$$

and

$$g = G_\varepsilon^T \Omega, \tag{3.5}$$

where the vectors F_ε and G_ε and matrices Γ_ε, Δ_ε and Θ_ε are the MBPFs coefficients of f, g, V_1, V_2 and V_3 respectively. In (3.1), F_ε is ς_1-known vector, also in (3.2), (3.3) and (3.4), Γ_ε, Δ_ε and Θ_ε are ς_2-known matrices but in (3.5), G_ε is ς_1-unknown vector. In (1.1), To approximate Fredholm integral case from (3.2), (3.5) and using operational matrix R_ε from (2.6) we get

$$\begin{aligned}
\int_0^1 \int_0^1 V_1 g ds dt &= \int_0^1 \int_0^1 \Omega^T(x,y)\Gamma_\varepsilon \Omega(s,t)\Omega^T(s,t)G_\varepsilon ds dt \\
&= \Omega^T(x,y)\Gamma_\varepsilon \left(\int_0^1 \int_0^1 \Omega(s,t)\Omega^T(s,t) ds dt\right) G_\varepsilon \\
&= \Omega^T \Gamma_\varepsilon R_\varepsilon G_\varepsilon = (\Gamma_\varepsilon R_\varepsilon G_\varepsilon)^T \Omega = U_\varepsilon^T \Omega,
\end{aligned}$$

where U_ε is an ς_1-vector obtained from $\Gamma_\varepsilon R_\varepsilon G_\varepsilon$. Therefore for the approximation of the first 2D-integral we have

$$\int_0^1 \int_0^1 V_1 g ds dt \simeq U_\varepsilon^T \Omega. \tag{3.6}$$

In addition from (2.3), (3.3) and (3.5) we get [12]

$$\begin{aligned}
\int_0^y \int_0^x V_2 g ds dt \simeq \int_0^y \int_0^x \Omega^T(x,y)\Delta_\varepsilon \Omega(s,t)\Omega^T(s,t)G_\varepsilon ds dt &= \Omega^T(x,y)\Delta_\varepsilon \left(\int_0^y \int_0^x \Omega(s,t)\Omega^T(s,t)G_\varepsilon ds dt\right) \\
&= \Omega^T \Delta_\varepsilon \left(\int_0^y \int_0^x \tilde{G}_\varepsilon \Omega(s,t) ds dt\right) = \Omega^T \Delta_\varepsilon \tilde{G}_\varepsilon \left(\int_0^y \int_0^x \Omega(s,t) ds dt\right),
\end{aligned}$$

where from (2.5) we arrive

$$\int_0^y \int_0^x V_2 g ds dt \simeq \Omega^T \Delta_\varepsilon \tilde{G}_\varepsilon P_\varepsilon \Omega,$$

in which $\Delta_\varepsilon \tilde{G}_\varepsilon P_\varepsilon$ is an ς_2-matrix. From (2.4) we can write

$$\int_0^y \int_0^x V_2 g ds dt \simeq \hat{W}_\varepsilon^T \Omega, \tag{3.7}$$

where $\hat{W}_\varepsilon$ is an ς_1-vector with components equal to the diagonal entries of matrix $\Delta_\varepsilon \tilde{G}_\varepsilon P_\varepsilon$. Similarly from (2.3), (3.4) and (3.5) we conclude

$$\begin{aligned}
\int_0^y \int_0^x V_3 g dB(s)dB(t) &\simeq \int_0^y \int_0^x \Omega^T(x,y)\Theta_\varepsilon \Omega(s,t)\Omega^T(s,t)G_\varepsilon dB(s)dB(t) = \Omega^T(x,y)\Theta_\varepsilon \left(\int_0^y \int_0^x \Omega(s,t)\Omega^T(s,t)G_\varepsilon dB(s)dB(t)\right) \\
&= \Omega^T \Theta_\varepsilon \left(\int_0^y \int_0^x \tilde{G}_\varepsilon \Omega(s,t)dB(s)dB(t)\right) = \Omega^T \Theta_\varepsilon \tilde{G}_\varepsilon \left(\int_0^y \int_0^x \Omega(s,t)dB(s)dB(t)\right),
\end{aligned}$$

by using (2.7) we can arrive

$$\int_0^y \int_0^x V_3 g dB(s)dB(t) \simeq \Omega^T \Theta_\varepsilon \tilde{G}_\varepsilon P_{\varepsilon,s} \Omega,$$

in which $\Theta_\varepsilon \tilde{G}_\varepsilon P_{\varepsilon,s}$ is an ς_2-matrix. From (2.4) we can write

$$\int_0^y \int_0^x V_3 g dB(s) dB(t) \simeq \hat{W}_{\varepsilon,s}^T \Omega, \tag{3.8}$$

where $\hat{W}_{\varepsilon,s}$ is an ς_1-vector with components equal to the diagonal entries of matrix $\Theta_\varepsilon \tilde{G}_\varepsilon P_{\varepsilon,s}$. Applying (3.1), (3.5), (3.6), (3.7) and (3.8) in (1.1) give

$$G_\varepsilon^T \Omega \simeq F_\varepsilon^T \Omega + \hat{U}_\varepsilon^T \Omega + \hat{W}_\varepsilon^T \Omega + \hat{W}_{\varepsilon,s}^T \Omega. \tag{3.9}$$

By replacing $\simeq$ with $=$, in (3.9) we can get

$$G_\varepsilon - \hat{U}_\varepsilon - \hat{W}_\varepsilon - \hat{W}_{\varepsilon,s} = F_\varepsilon, \tag{3.10}$$

where after solving System (3.10), we can find G_ε and get

$$g = G_\varepsilon^T \Omega.$$

Then

$$g \simeq g_\varepsilon = \frac{1}{\mu} \sum_{i=0}^{\mu-1} g_{\varepsilon_i},$$

where $\varepsilon_i = \dfrac{ik}{\mu}$, $i = 0(1)(\mu-1)$ is the estimation of the solution of (1.1) and μ is a positive integer.

4. Error analysis

In this section, we show that the convergence order of the proposed method is $\dfrac{1}{\mu n}$ by introducing several theorems. For convenience, we put $n_1 = n_2 = n$, so $k_1 = k_2 = \dfrac{1}{n}$.

Theorem 4.1. *Suppose that h is a differentiable function from $S \subset R^2$ into R, and for every $t \in S$*

$$\|h'\|_2 \leq \xi,$$

where $\xi \in R$. Then

$$|h(d) - h(c)| \leq \xi |d - c|,$$

for all $c, d \in S$.

Proof. See [10]. □

Theorem 4.2. *Assume that*

$$F_{n,\varepsilon} = \sum_{a=0}^{n} \sum_{b=0}^{n} \sum_{c=0}^{n} \sum_{d=0}^{n} F_{a,b,c,d} \omega_{a,b,c,d},$$

and

$$F_{a,b,c,d} = \frac{1}{\ell(I_{a,\varepsilon})\ell(I_{b,\varepsilon})\ell(I_{c,\varepsilon})\ell(I_{d,\varepsilon})} \times \int_0^1 \int_0^1 \int_0^1 \int_0^1 F \omega_{a,b,c,d} dt ds dy dx,$$

where $a, b, c, d = 0, 1, \ldots, n$. Then the mean square error between F and $F_{n,\varepsilon}$ on $(x, y, s, t) \in D_{a,b,c,d}$ reaches its minimum, moreover we have

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 F^2 dt ds dy dx = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} F_{a,b,c,d}^2 \|\omega_{a,b,c,d}\|_2^2.$$

Proof. By using [11], we can easily prove this theorem. □

Theorem 4.3. *Assume f is continuous and differentiable over district $[-k, 1+k] \times [-k, 1+k]$ and $f_{n,\varepsilon}$; $\varepsilon_i = \dfrac{ik}{\mu}$ for $i = 0, 1, \ldots, \mu - 1$ are correspondingly $2D-MBPFs(\varepsilon_0) = 2D-BPFs$, $2D-MBPFs(\varepsilon_1)$, ..., $2D-MBPFs(\varepsilon_{\mu-1})$ expansions of f based on $(n+1)^2$ 2D-MBPFs over district $[0, 1) \times [0, 1)$ and $\bar{f}_{n,\mu}(x, y) = \dfrac{1}{\mu} \sum_{i=0}^{\mu-1} f_{n,\varepsilon_i}(x, y)$, then for sufficient large n we have*

$$\|e_\varepsilon\|_2 \leq \frac{\sqrt{2}N}{\mu n},$$

therefore

$$\|e_\varepsilon\|_2 = O(\frac{1}{\mu n}),$$

where N is bounded of $\|Df\|_2$.

Proof. See [13]. □

Theorem 4.4. *If F be an enough smooth function on $S=[0,1)^4$ with $\|F\|_2 \leq M$. Let*

$$\hat{F}_n = \hat{F}_{n,\varepsilon_0} = \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n} F_{a,b,c,d}\,\omega_{a,b,c,d},$$

be $4D-MBPFs(\varepsilon_0)=4D-BPFs$ expansion of F and

$$e = F - \hat{F}_n,$$

then

$$\|e_\varepsilon\|_2 = O(\frac{1}{n}).$$

Proof. We have

$$e_{a,b,c,d} = F - F_{a,b,c,d}\phi_{a,b,c,d} = F - F_{a,b,c,d},$$

where ϕ is the set of 4D-BPFs of dimension $n_1n_2n_3n_4$ and

$$S_{a,b,c,d} = \{\frac{a-1}{n} \leq x < \frac{a}{n}, \frac{b-1}{n} \leq y < \frac{b}{n}, \frac{c-1}{n} \leq s < \frac{c}{n}, \frac{d-1}{n} \leq t < \frac{d}{n}\},$$

and $(x,y,s,t) \in S_{a,b,c,d}$. By using the mean value theorem we get

$$\|e_{a,b,c,d}\|_2^2 = \int_{(a-1)/n}^{a/n}\int_{(b-1)/n}^{b/n}\int_{(c-1)/n}^{c/n}\int_{(d-1)/n}^{d/n} \left(F - F_{a,b,c,d}\right)^2 dtdsdydx$$

$$= \frac{1}{n^4}\left(F(\gamma_1,\gamma_2,\gamma_3,\gamma_4) - F_{a,b,c,d}\right)^2; \qquad (\gamma_1,\gamma_2,\gamma_3,\gamma_4) \in S_{a,b,c,d}. \tag{4.1}$$

We know

$$F_{a,b,c,d} = \frac{1}{k^4}\int_{(a-1)k}^{ak}\int_{(b-1)k}^{bk}\int_{(c-1)k}^{ck}\int_{(d-1)k}^{dk} F_{a,b,c,d}\,dtdsdydx,$$

therefore by using mean value theorem we have

$$F_{a,b,c,d} = n^4 \times \frac{1}{n^4} \times F(\theta_1,\theta_2,\theta_3,\theta_4); \qquad (\theta_1,\theta_2,\theta_3,\theta_4) \in S_{a,b,c,d}. \tag{4.2}$$

From Theorem 4.1 and involving (4.2) into (4.1) we obtain

$$\|e_{a,b,c,d}\|_2^2 = \frac{1}{n^4}\left(V(\gamma_1,\gamma_2,\gamma_3,\gamma_4) - V(\theta_1,\theta_2,\theta_3,\theta_4)\right)^2 \leq \frac{1}{n^4} \times 4k^2 \times M^2 = \frac{4M^2}{n^6}. \tag{4.3}$$

So

$$\begin{aligned} \|e\|_2^2 &= \int_0^1\int_0^1\int_0^1\int_0^1 e^2 dtdsdydx \\ &= \int_0^1\int_0^1\int_0^1\int_0^1 \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n} e_{abcd}^2 dtdsdydx + 2\sum_{a<a'}^{n}\sum_{b<b'}^{n}\sum_{c<c'}^{n}\sum_{d<d'}^{n}\int_0^1\int_0^1\int_0^1\int_0^1 e_{abcd} \times e_{a'b'c'd'} dtdsdydx. \end{aligned}$$

Since for $a<a', b<b', c<c'$ and $d<d'$ we have

$$S_{a,b,c,d} \cap S_{a',b',c',d'} = \varnothing,$$

where (4.3) give

$$\|e\|_2^2 = \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n} \|e_{abcd}\|_2^2 \leq n^4 \times \frac{4M^2}{n^6} = \frac{1}{n^2} \times 4M^2,$$

namely

$$\|e\|_2 = O(\frac{1}{n}).$$

□

Theorem 4.5. *Assume $F(x,y,s,t)$ is continuous and differentiable over district $[-k,1+k]\times[-k,1+k]\times[-k,1+k]\times[-k,1+k]$, moreover suppose $F_{n,\varepsilon_i}(x,y,s,t)$; $\varepsilon_i = \frac{ik}{\mu}$ for $i=0,1,\ldots,\mu-1$ are correspondingly $4D-MBPFs(\varepsilon_0)=4D-BPFs$, $4D-MBPFs(\varepsilon_1)$, ..., $4D-MBPFs(\varepsilon_{\mu-1})$ expansions of F based on $(n+1)^4$ 4D-MBPFs over district $[0,1)^4$ and*

$$\bar{F}_{n,\mu} = \frac{1}{\mu}\sum_{i=0}^{\mu-1} F_{n,\varepsilon_i},$$

then for sufficient large values n

$$\|F - \bar{F}_{n,\mu}\|_\infty \lesssim \frac{1}{\mu}\max_{\varepsilon_i}\|F - \bar{F}_{n,\varepsilon_i}\|_\infty.$$

Proof. We consider partial differentials
$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t},$$
in $D^4 = [\frac{i-1}{n}, \frac{i+1}{n})^4$ which are approximately equal to constants A_1, A_2, A_3 and A_4 respectively, where n is so large. Also we use function, $z = A_1x + A_2y + A_3s + A_4t + B$ instead of F in D^4. Now in the district $[\frac{i}{n}, \frac{i}{n} + \varepsilon_1)^4$ we have

$$\begin{aligned}
\bar{F}_{n,\mu}(x,y,s,t) &= \frac{1}{\mu}\sum_{j=1}^{\mu-1}\frac{1}{16}\times[(A_1+A_2+A_3+A_4)\rho_1+B+(A_1+A_2+A_3)\rho_1+A_4\rho_2+B+(A_1+A_2+A_4)\rho_1+A_3\rho_2+B+(A_1+A_2)\rho_1 \\
&+ (A_3+A_4)\rho_2+B+(A_1+A_3+A_4)\rho_1+A_2\rho_2+B+(A_1+A_3)\rho_1+(A_2+A_4)\rho_2+B+(A_1+A_4)\rho_1+(A_2+A_3)\rho_2+B+A_1\rho_1 \\
&+ (A_2+A_3+A_4)\rho_2+B+A_1\rho_2+(A_2+A_3+A_4)\rho_1+B+(A_1+A_4)\rho_2+(A_2+A_3)\rho_1+B+(A_1+A_3)\rho_2+(A_2+A_4)\rho_1+B \\
&+ (A_1+A_3+A_4)\rho_2+A_2\rho_1+B+(A_1+A_2)\rho_2+(A_3+A_4)\rho_1+B+(A_1+A_2+A_4)\rho_2+A_3\rho_1+B \\
&+ (A_1+A_2+A_3)\rho_2+A_4\rho_1+B+(A_1+A_2+A_3+A_4)\rho_2+B] \\
&= (A_1+A_2+A_3+A_4)(\frac{\frac{i}{n}+\frac{i+1}{n}}{2})+B-\frac{(A_1+A_2+A_3+A_4)k(\mu-1)}{2\mu}, \qquad (4.4)
\end{aligned}$$

where
$$\rho_1 = (\frac{i}{n} - \frac{jk}{\mu}),$$
and
$$\rho_2 = (\frac{i+1}{n} - \frac{jk}{\mu}).$$
Since $\frac{i+1}{n} = \frac{i}{n} + k$, we can reformulete (4.4) as
$$\bar{F}_{n,\mu} = (A_1+A_2+A_3+A_4)\frac{i}{n} + B + \frac{(A_1+A_2+A_3+A_4)k}{2\mu}.$$
Also we have
$$\max_{x,y,s,t\in[\frac{i}{n},\frac{i}{n}+\varepsilon_j)} |F-\bar{F}_{n,\mu}| \simeq \max_{x,y,s,t\in[\frac{i}{n},\frac{i}{n}+\varepsilon_j)} |A_1x+A_2y+A_3s+A_4t+B-\bar{F}_{n,\mu}| = \frac{(A_1+A_2+A_3+A_4)k}{2\mu}. \qquad (4.5)$$
Therefore, we get
$$\begin{aligned}
\max_{\substack{\varepsilon_i \\ x,y,s,t\in D}} \|F-F_{n,\varepsilon_i}\|_\infty \geq \max_{\substack{\varepsilon_i \\ x,y,s,t\in D'}} |F-F_{n,\varepsilon_i}| &\simeq |(A_1+A_2+A_3+A_4)\varpi_1+B-\frac{1}{16}[(A_1+A_2+A_3+A_4)\varpi_1+B \\
&+(A_1+A_2+A_3)\varpi_1+A_4\varpi_2+B+(A_1+A_2+A_4)\varpi_1+A_3\varpi_2+B+(A_1+A_2)\varpi_1+(A_3+A_4)\varpi_2+B+(A_1++A_3+A_4)\varpi_1+A_2\varpi_2+B \\
&+(A_1+A_3)\varpi_1+(A_2+A_4)\varpi_2+B+(A_1+A_4)\varpi_1+(A_2+A_3)\varpi_2+B+A_1\varpi_1+(A_2+A_3+A_4)\varpi_2+B+A_1\varpi_2+(A_2+A_3+A_4)\varpi_1+B \\
&+(A_1+A_4)\varpi_2+(A_2+A_3)\varpi_1+B+(A_1+A_3)\varpi_2+(A_2+A_4)\varpi_1+B+(A_1+A_3+A_4)\varpi_2+A_2\varpi_1+B+(A_1+A_2)\varpi_2+(A_3+A_4)\varpi_1+B \\
&+(A_1+A_2+A_4)\varpi_2+A_3\varpi_1+B+(A_1+A_2+A_3)\varpi_2+A_4\varpi_1+B+(A_1+A_2+A_3+A_4)\varpi_2+B]| \\
&= \frac{(A_1+A_2+A_3+A_4)}{2}k, \qquad (4.6)
\end{aligned}$$
where $\varpi_1 = \frac{i}{n}, \frac{i}{n}+k = \varpi_2$, $D = [\frac{i-1}{n}, \frac{i+1}{n})$ and $D' = [\frac{i}{n}, \frac{i}{n}+k)$. From (4.5) and (4.6) we get
$$\|F-\bar{F}_{n,\mu}\|_\infty \lesssim \frac{1}{\mu}\max_{\varepsilon_i}\|F-\bar{F}_{n,\varepsilon_i}\|_\infty.$$
□

Remark 4.6. *Let*
$$e_{n,\varepsilon} = F - \bar{F}_{n,\mu},$$
and
$$e_n = F - \bar{F}_n,$$
then from Theorem 4.2, Theorem 4.4 and Theorem 4.5 we have
$$\|e_{n,\varepsilon}\|_2 \leqslant \frac{2M}{\mu n},$$
also we can write
$$\lim_{n\to+\infty} F_{n,\varepsilon_i} = F.$$

Theorem 4.7. *If g be the exact solution of* (1.1) *and* $\hat{g}_{n,\mu}(x,y)$ *be the 2D-MBPFs approximate solution of it. Also*

(1) $\|g\|_2 \leq \alpha \quad , \quad (s,t) \in [0,1)^2$,

(2) $\|V_i\|_2 \leq \beta_i \ , \ i = 1,2,3 \ , \ (x,y,s,t) \in [0,1)^4$,

(3) $W_1(x,y) = \sup_{x\in[0,1)} x \times \sup_{y\in[0,1)} y$,

(4) $W_2(x,y) = \sup_{x\in[0,1)} |B(x)| \times \sup_{y\in[0,1)} |B(y)|$,

(5) $\left[\beta_1 + \beta_2 + \frac{2\beta_1 + 2\beta_2}{\mu n} + \left(\beta_3 + \frac{2\beta_3}{\mu n}\right) \times W_2(x,y)\right] < 1$,

then

$$\|g - \hat{g}_n\|_2 = O(\frac{1}{\mu n}).$$

Proof. From (1.1), we get

$$g - \hat{g}_{n,\mu} = f - \hat{f}_{n,\mu} + \int_0^1 \int_0^1 \left(V_1 g - \hat{V}_{1,n,\mu}\hat{g}_{n,\mu}\right) dsdt + \int_0^y \int_0^x \left(V_2 g - \hat{V}_{2,n,\mu}\hat{g}_{n,\mu}\right) dsdt$$
$$+ \int_0^y \int_0^x \left(V_3(x,y,s,t)g(s,t) - \hat{V}_{3,n,\mu}\hat{g}_{n,\mu}\right) dB(s)dB(t),$$

so the mean value theorem give

$$\|g - \hat{g}_{n,\mu}\|_2 \leq \|f - \hat{f}_{n,\mu}\|_2 + \|V_1 g - \hat{V}_{1,n,\mu}\hat{g}_{n,\mu}\|_2 + xy\|V_2 g - \hat{V}_{2,n,\mu}\hat{g}_{n,\mu}\|_2 + B(x)B(y)\|V_3 g - \hat{V}_{3,n,\mu}\hat{g}_{n,\mu}\|_2. \tag{4.7}$$

By using Remark 4.6 and two first hypothesises, we obtain

$$\begin{aligned} \|V_1 g - \hat{V}_{1,n,\mu}\hat{g}_{n,\mu}\|_2 &\leq \|V_1\|_2\|g - \hat{g}_{n,\mu}\|_2 + \|V_1 - \hat{V}_{1,n,\mu}\|_2\left(\|g - \hat{g}_{n,\mu}\|_2 + \|g\|_2\right) \\ &\leq \beta_1\|g - \hat{g}_{n,\mu}\|_2 + \frac{2\beta_1}{\mu n}\left(\|g - \hat{g}_{n,\mu}\|_2 + \alpha\right) = \left(\beta_1 + \frac{2\beta_1}{\mu n}\right)\|g - \hat{g}_{\mu n}\|_2 + \frac{2\beta_1}{\mu n}\alpha. \end{aligned} \tag{4.8}$$

Similarly we have

$$\|V_2 g - \hat{V}_{2,n,\mu}\hat{g}_{n,\mu}\|_2 = \left(\beta_2 + \frac{2\beta_2}{\mu n}\right)\|g - \hat{g}_{n,\mu}\|_2 + \frac{2\beta_2}{\mu n}\alpha, \tag{4.9}$$

and

$$\|V_3 g - \hat{V}_{3,n,\mu}\hat{g}_{n,\mu}\|_2 = \left(\beta_3 + \frac{2\beta_3}{\mu n}\right)\|g - \hat{g}_{n,\mu}\|_2 + \frac{2\beta_3}{\mu n}\alpha. \tag{4.10}$$

Substituting (4.8), (4.9) and (4.10) in (4.7) and Theorem 4.3 conclude

$$\|g - \hat{g}_{n,\mu}\|_2 \leq \frac{\sqrt{2}N}{\mu n} + \left[\left(\beta_1 + \frac{2\beta_1}{\mu n}\right)\|g - \hat{g}_{n,\mu}\|_2 + \frac{2\beta_1}{\mu n}\alpha\right] + xy\left[\left(\beta_2 + \frac{2\beta_2}{\mu n}\right)\|g - \hat{g}_{n,\mu}\|_2 + \frac{2\beta_2}{\mu n}\alpha\right]$$
$$+ B(x)B(y)\left[\left(\beta_3 + \frac{2\beta_3}{\mu n}\right)\|g - \hat{g}_{n,\mu}\|_2 + \frac{2\beta_3}{\mu n}\alpha\right].$$

By taking sup and Hypothesises 3 and 4, we have

$$\|g - \hat{g}_{n,\mu}\|_2 \leq \frac{\sqrt{2}N}{\mu n} + \left[\left(\beta_1 + \frac{2\beta_1}{\mu n}\right)\sup_{s\leq x\ ,\ t\leq y}\|g - \hat{g}_{n,\mu}\|_2 + \frac{2\beta_1}{\mu n}\alpha\right] + W_1(x,y)\left[\left(\beta_2 + \frac{2\beta_2}{\mu n}\right)\sup_{s\leq x\ ,\ t\leq y}\|g - \hat{g}_{n,\mu}\|_2 + \frac{2\beta_2}{\mu n}\alpha\right]$$
$$+ W_2(x,y)\left[\left(\beta_3 + \frac{2\beta_3}{\mu n}\right)\sup_{s\leq x\ ,\ t\leq y}\|g - \hat{g}_{n,\mu}\| + \frac{2\beta_3}{\mu n}\alpha\right],$$

so

$$\|g - \hat{g}_{n,\mu}\|_2 \leq \frac{\dfrac{\sqrt{2}N + 2\beta_1\alpha + 2\beta_2\alpha}{\mu n} + \dfrac{2\beta_3\alpha}{\mu n} \times W_2(x,y)}{1 - \left[\beta_1 + \beta_2 + \dfrac{2\beta_1 + 2\beta_2}{\mu n} + \left(\beta_3 + \dfrac{2\beta_3}{\mu n}\right) \times W_2(x,y)\right]},$$

and from the boundedness of Brownian motion we get

$$\|g - \hat{g}_{n,\mu}\|_2 = O(\frac{1}{\mu n}).$$

$\square$

n	μ	$\bar{g}$	$\bar{e}$	(L,U)
2	1 (BPFs)	0.488705	0.188705	$(0.487912, 0.489498)$
	3 (MBPFs)	0.397704	0.097704	$(0.390937, 0.404471)$
3	1 (BPFs)	0.327094	0.027093	$(0.326647, 0.327540)$
	3 (MBPFs)	0.316383	0.016382	$(0.304258, 0.328508)$
4	1 (BPFs)	0.246046	0.053945	$(0.245386, 0.246705)$
	3 (MBPFs)	0.582630	0.034799	$(0.222132, 0.343128)$
5	1 (BPFs)	0.400235	0.100235	$(0.399732, 0.400738)$
	3 (MBPFs)	0.360730	0.060730	$(0.356168, 0.365292)$

Table 1: Results in $(0.1, 0.2)$

n	μ	$\bar{g}$	$\bar{e}$	(L,U)
2	1 (BPFs)	0.996151	0.296151	$(0.987828, 1.004470)$
	3 (MBPFs)	0.906776	0.206776	$(0.870497, 0.943055)$
3	1 (BPFs)	0.997234	0.297234	$(0.976630, 1.017840)$
	3 (MBPFs)	0.880689	0.180689	$(0.833557, 0.927822)$
4	1 (BPFs)	0.792689	0.092689	$(0.788890, 0.796488)$
	3 (MBPFs)	0.751917	0.051917	$(0.733360, 0.770475)$
5	1 (BPFs)	0.799899	0.099899	$(0.799426, 0.800373)$
	3 (MBPFs)	0.782178	0.082178	$(0.776386, 0.787971)$

Table 2: Results in $(0, 0.7)$

5. Numerical example

We consider a numerical example to illustrate the efficiency of the MBPFs method. Consider the 2D-linear stochastic Volterra-Fredholm integral equation

$$g(x,y) = f(x,y) + \int_0^1 \int_0^1 (xyst) g(s,t) ds dt + \int_0^y \int_0^x (xyst) g(s,t) ds dt + \int_0^y \int_0^x (xyst) g(s,t) dB(s) dB(t),$$

where

$$f(x,y) = x + y - \frac{xy}{3} - \frac{x^3y^3(x+y)}{6} - (x^3y^2 + x^2y^3)B(x)B(y) + B(x)(x^3y + 2x^2y) \int_0^y B(t)dt + B(y)(y^3x + 2y^2x) \int_0^x B(s)ds,$$

with the exact solution

$$g(x,y) = x + y.$$

The solution mean $(\bar{g}(x,y))$, error mean $(\bar{e}(x,y))$ and %95 confidence interval (L,U) at arbitrary points $(0.1, 0.2)$ and $(0, 0.7)$ for some values of n and μ are shown in Table 1 and Table 2. In this tables by the comparison between the computed results by the presented method and the BPFs method we will see that in the MBPFs method we achieve the good accuracy by increasing μ. You can see three-dimensional graphs of this example in Fig. 5.1 and Fig. 5.2.

6. Conclusion

In this paper, we have successfully developed the 2D-MBPFs numerical method for approximate a solution for 2D-linear stochastic Volterra-Fredholm integral equations. The numerical results represent that $\bar{e}$ in new method is lesser from $\bar{e}$ in BPF method.

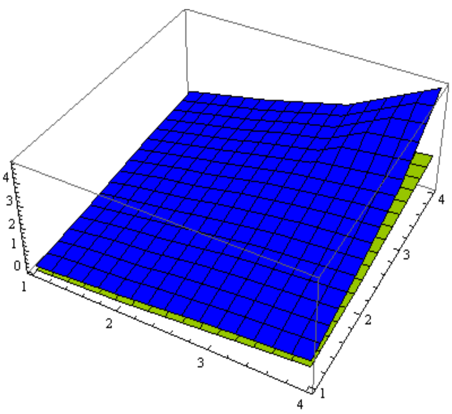

Figure 5.1: $(n = 3)$

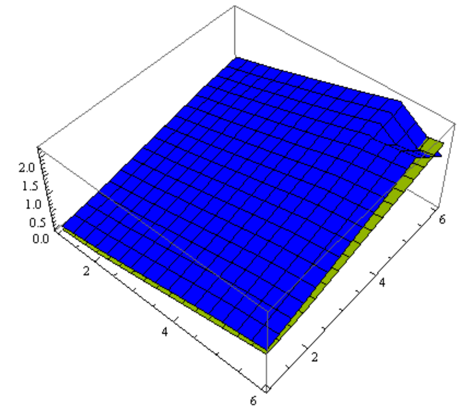

Figure 5.2: $(n = 5)$

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