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A NEW GENERALIZED-UPPER RECORD VALUES-G FAMILY OF LIFETIME DISTRIBUTIONS

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Abstract. A new family of lifetime distributions is introduced via distribution of the upper record values, the well-known concept in survival analysis and reliability engineering. Some important properties of the proposed model including quantile function, hazard function, order statistics are obtained in a general setting. A special case of this new family is proposed by considering the exponential and Weibull distribution as the parent distributions. In addition estimating unknown parameters of specialized distribution is examined from the perspective of the traditional statistics. A simulation study is presented to investigate the bias and mean square error of the maximum likelihood estimators. Moreover, one example of real data set is studied; point and interval estimations of all parameters are obtained by maximum likelihood and bootstrap (parametric and non-parametric) procedures. Finally, the superiority of the proposed model in terms of the parent exponential distribution over other known distributions is shown via the example of real observations.

1. Introduction

The statistical distribution theory has been widely explored by researchers in recent years. Given the fact that the data from our surrounding environment follow various statistical models, it is necessary to extract and develop appropriate high-quality models. In addition, sometimes it is necessary to provide applications from

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existing models. For more details, see the Samuel et al. (2018) and Ababneh et al. (2018).

Recently, Alzaatreh et al. (2013) have introduced a new model of lifetime distributions, which the researchers refer to its special case as *generalized*$-G$ distribution. It is based on the combination of one arbitrary CDF F of a continuous random variable X with the baseline CDF G. The integration form of new CDF H is stated as

$$H(x) = \frac{1}{F(1)} \int_{-\infty}^{G(x)} f(t)dt, \, x \in R, \tag{1}$$

where f is the corresponding density function of F and $F(1) = P(X \leq 1)$. This interesting method attracted the attention of some researchers. Generating new model based on this method resulted in creating very flexible statistical modeling.

The upper and lower record values, in a sequence of independent and identically distributed (iid) random variables $X_1, X_2, ...$, have applications in different areas of applied probability and reliability engineering. Let X_i 's have a common absolutely continuous distribution G with survival function $\bar{G}$. Define a sequence of record times $U(n), n = 1, 2, ...$, as follows:

$$U(n+1) = min\ \{j : j > U(n),\ X_j > X\},\ n \geq 1,$$

with $U(1) = 1$. Then, the sequence of upper record values $\{R^n, n \geq 1\}$ is defined by $R^n = X_U(n), n \geq 1$, where $R^1 = X_1$. The survival function of R^n is given by

$$\bar{G}_n^U(t) = \bar{G}(t) \sum_{x=0}^{n-1} \frac{[-\log \bar{G}(t)]^x}{x!},\ t \geq 0,\ n = 1, 2....$$

The corresponding CDF of the random variable R^n is

$$G_n^U(t) = 1 - \bar{G}(t) \sum_{x=0}^{n-1} \frac{[-\log \bar{G}(t)]^x}{x!},\ t \geq 0,\ n = 1, 2....$$

Here, we introduce a new family of lifetime distributions by compounding CDF of upper record values $G_n^U(t)$ of a parent distribution G and an arbitrary CDF F with PDF f.

This new model will be denote by $G - UR - G(or\ GURG)$ distribution. One of our main motivation to introduce this new category of distributions is to provide more flexibility for fitting real datasets in comparing with other well-known classic statistical distributions.

We first derive the fundamental and statistical properties of $GURG$ in a general setting and then we propose a special case of this model by considering Weibull distribution instead of the parent distribution G and exponential distribution instead of the parent distribution F for fixed value $n = 2$. It is referred to as $GURWE$ distribution. We provide a comprehensive discussion about the statistical and reliability properties of the new $GURWE$ model. Furthermore, we consider Maximum likelihood and bootstrap estimation procedures to estimate the unknown parameters of the new model for complete data set. In addition, the asymptotic confidence intervals and parametric and non-parametric bootstrap confidence intervals are calculated.

2. New general model and its properties

In this section, we provide the structure of our new model and some of its main properties in a general setting. Motivated by the idea of Alzaatreh et al. (2013), a new class of statistical distributions is proposed. The new model is constructed by implementing Alzaattreh idea to the upper record value distribution $G_n^U(t)$. Let the non-negative random variable X have CDF and PDF F and f, respectively. In view of (1), the CDF of new general class of lifetime distributions is defined as:

$$\begin{aligned} H(x,n) &= \frac{1}{F(1)}\int_0^{G_n^U(x)} f(t)dt \\ &= \frac{F(G_n^U(x))}{F(1)},\ x \geq 0,\ n=1,2... \end{aligned} \tag{2}$$

The (PDF) is

$$h(x,n) = \frac{g_n^U(x)}{F(1)}\ f(G_n^U(x)),\ x>0,\ n=1,2... \tag{3}$$

where $g_n^U(x)$ is the PDF of the $n-$ upper record value distribution and

$$g_n^U(x) = g(x)\frac{[-\log \bar{G}(x)]^{n-1}}{(n-1)!},\ x>0,\ n=1,2... \tag{4}$$

Using (2) and (4), the survival $\bar{H}(x,n)$ and the hazard rate $r(x,n)$ functions for $GURG$ distribution are given, respectively, by:

$$\bar{H}(x,n) = 1 - \frac{F(G_n^U(x))}{F(1)}$$

and

$$r(x,n) = \frac{g_n^U(x)\ f(G_n^U(x))}{F(1)-F(G_n^U(x))},\ x>0,\ n=1,2....$$

The pth quantile x_p of the $GURG$ distribution can be obtained from

$$x_p = G_n^{U^{-1}}\left(F^{-1}(F(1)p)\right),$$

where $G_n^{U^{-1}}$ is the inverse function of CDF G_n^U.

3. Special case based on the parent Weibull and Exponential Distributions

Let $G(x) = 1 - e^{-\alpha x^{\beta}}$, $F(x) = 1 - e^{-\lambda x}$ and $n = 2$. From (2) we have:

$$\begin{aligned} H(x) &= H(x, n = 2) \\ &= \frac{1}{F(1)} \int_{-\infty}^{G(x)+\bar{G}(x)\log\bar{G}(x)} f(t)dt \\ &= \frac{1}{1-e^{-\lambda}}[1 - e^{-\lambda\left(G(x)+\bar{G}(x)\log\bar{G}(x)\right)}] \\ &= \frac{1}{1-e^{-\lambda}} e^{-\lambda\left(1-e^{-\alpha x^{\beta}}(1+\alpha x^{\beta})\right)}, \quad x \geq 0. \end{aligned} \tag{5}$$

The corresponding PDF is :

$$h(x) = \frac{\alpha^2\beta\lambda}{1-e^{-\lambda}} x^{2\beta-1} e^{-\alpha x^{\beta}} e^{-\lambda(1-e^{-\alpha x^{\beta}}(1+\alpha x^{\beta}))},$$

where $x > 0$, $\alpha, \lambda, \beta > 0$.

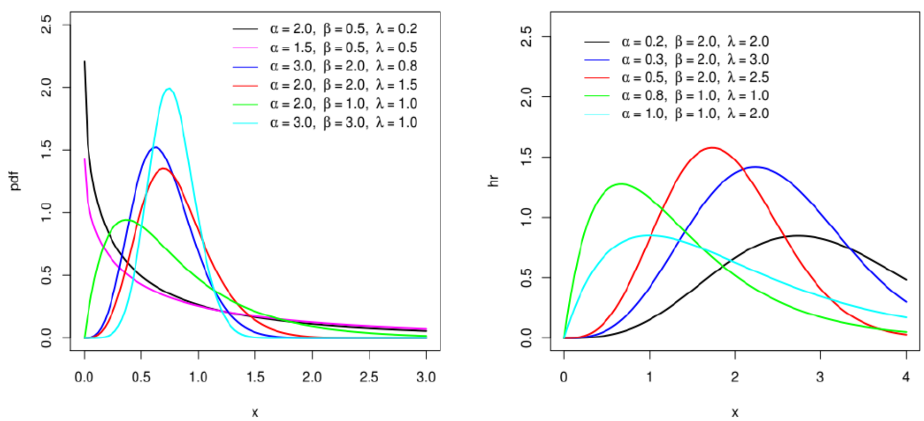

Figure 1. Plots of the $GUREW(\alpha, \beta, \lambda)$ density (left) and failure rate function (right) for selected values of α, β, λ.

The survival and hazard rate functions are

$$\bar{H}(x) = 1 - \frac{1}{1-e^{-\lambda}} \left(1 - e^{-\lambda\left(1-e^{-\alpha x^{\beta}}(1+\alpha x^{\beta})\right)}\right),$$

and

$$r(x) = \frac{h(x)}{\bar{H}(x)} = \frac{\frac{\alpha^2\beta\lambda}{1-e^{-\lambda}} x^{2\beta-1} e^{-\alpha x^{\beta}} e^{-\lambda(1-e^{-\alpha x^{\beta}}(1+\alpha x^{\beta}))}}{1 - \frac{1}{1-e^{-\lambda}}\left(1 - e^{-\lambda\left(1-e^{-\alpha x^{\beta}}(1+\alpha x^{\beta})\right)}\right)},$$

respectively.

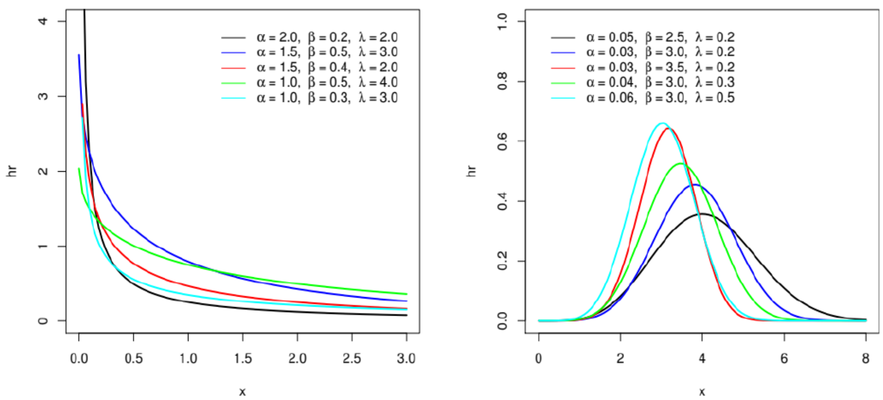

FIGURE 2. Plots of failure rate function for selected values of the parameters.

3.1. **Some properties of the *GUREW* distribution.** In this section, we obtain some properties of the *GUREW* distribution, such as quantiles, moments, moment generating function and order statistics distribution. The characterizations of *GUREW* distribution are presented in subsection 3.5.

3.2. **Quantiles.** For the *GUREW* distribution, the pth quantile x_p is the solution of $H(x_p) = p$, hence

$$x_p = \left(-\frac{1}{\alpha} - \frac{1}{\alpha} W_{-1}\left(-e^{-1}\left(1 + \frac{1}{\lambda}\log(1-(1-e^{-\lambda})p)\right)\right)\right)^{1/\beta}, \qquad 0 \leq p \leq 1,$$

which is the base of generating GUREW random variates, where W_{-1} denotes the negative branch of the Lambert function.

3.3. **Moments and Moment generating function.** In this subsection, moments and related measures including coefficients of variation, skewness and kurtosis are presented. Tables of values for the first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are also presented. The rth moment of the *GUREW* distribution, denoted by $\mu_r^{'}$, is

$$\mu_r^{'} = E(X^r) = \sum_{k=0}^{\infty}\sum_{t=0}^{k}\sum_{j=0}^{t} \frac{(-1)^k(-1)^t\binom{k}{t}\binom{t}{j}\alpha^{j+1}\lambda^{k+1}}{(1-e^{-\lambda})(t+1)} E_{X_W}[X^{r+(j+1)\beta}],$$

where $X_W \sim Weibull(\alpha(t+1), \beta)$ and $E_{X_W}[X^{r+(j+1)\beta}] = \frac{\Gamma\left(1+\frac{r+j\beta+j\beta}{\beta}\right)}{(\alpha(t+1))^{r+(j+1)\beta}}$. The variance, CV, CS, and CK are given by

$$\sigma^2 = \mu'^2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'^2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1}, \tag{6}$$

$$CS = \frac{E[(X-\mu)^3]}{[E(X-\mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}}, \tag{7}$$

and

$$CK = \frac{E[(X-\mu)^4]}{[E(X-\mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2}, \tag{8}$$

respectively. Table 1 lists the first six moments of the *GUREW* distribution for selected values of the parameters, when $\alpha = 3$. Table 2 lists the first six moments of the *GUREW* distribution for selected values of the parameters, when $\beta = 0.5$. These values can be determined numerically using R.

The moment generating function of the *GUREW* distribution is given by

$$E(e^{tX}) = \sum_{k=0}^{\infty}\sum_{t=0}^{k}\sum_{j=0}^{t} \frac{(-1)^t(-1)^k\,\lambda^{k+1}\binom{k}{t}\binom{t}{j}\Gamma(j+2)}{k!(t+1)^j\,(1-e^{-\lambda})} E_{X_G}[e^{tX^{1/\beta}}],$$

where $X_G \sim Gamma\,(j+2, \alpha(t+1))$.

TABLE 1. Moments of the *GUREW* distribution for selected parameter values when $\alpha = 3$.

μ'_r	$\beta=0.5,\ \lambda=0.5$	$\beta=0.5,\ \lambda=1.5$	$\beta=1.5,\ \lambda=0.5$	$\beta=1.5,\ \lambda=1.5$
μ'_1	0.5671076	0.3992706	0.6760747	0.5884041
μ'_2	1.179537	0.709571	0.5688418	0.4399216
μ'_3	5.38217	3.07405	0.5671076	0.3992693
μ'_4	42.7557	24.04752	0.6488483	0.4256478
μ'_5	521.4693	291.9992	0.832901	0.519231
μ'_6	9033.24	5051.822	1.179537	0.709571
SD	0.9262429	0.7417237	0.3343124	0.3061082
CV	1.6332755	1.8576967	0.4944904	0.5202346
CS	4.706703	5.762384	0.8405167	1.051078
CK	44.17227	65.22099	3.881779	4.575555

TABLE 2. Moments of the *GUREW* distribution for selected parameter values when $\beta = 0.5$.

μ_r'	$\alpha = 0.5$, $\lambda = 0.5$	$\alpha = 1$, $\lambda = 1$	$\alpha = 1.5$, $\lambda = 1.5$	$\alpha = 2$, $\lambda = 2$
μ_1'	20.33652	4.299195	1.597077	0.7470063
μ_2'	1478.852	74.72371	11.35314	2.721693
μ_3'	218312	2993.463	196.7392	25.74113
μ_4'	48809999	212208.3	6156.162	449.2581
μ_5'	13959548045	23124985	299005.4	12245.5
μ_6'	4.637651e+12	3540458803	20691289	476369.5
SD	32.638596	7.499375	2.966898	1.470943
CV	1.604925	1.744367	1.857705	1.969118
CS	4.168103	5.189165	5.762391	6.433476
CK	30.14692	53.11153	65.221	81.28218

3.4. **Order statistics.** Order statistics play an important role in probability and statistics. In this subsection, we present the distribution of the ith order statistic from the *GUREW* distribution. The *PDF* of the *ith* order statistic from the *GUREW PDF*, $f_{GUREW}(x)$, is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f_{GUREW}(x) \left[F_{GUREW}(x)\right]^{i-1} \left[1 - F_{GUREW}(x)\right]^{n-i}$$
$$= \frac{n!}{(i-1)!(n-i)!} f_{GUREW}(x) \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m \left[F_{GUREW}(x)\right]^{m+i-1}.$$

Using the binomial expansion

$$\left[1 - F_{GUREW}(x)\right]^{n-i} = \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m \left[F_{GUREW}(x)\right]^{m},$$

we have

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m \left[F_{GUREW}(x)\right]^{m+i-1} f_{GUREW}(x)$$

3.5. **Characterization Results.** This section is devoted to the characterizations of the *GUREW* distribution in different directions: (i) based on the ratio of two truncated moments; (ii) in terms of the reverse hazard function and (iii) based on the conditional expectation of certain function of the random variable. Note that (i) can be employed also when the cdf does not have a closed form. We would also like to mention that due to the nature of *GUREW* distribution, our characterizations may be the only possible ones. We present our characterizations $(i) - (iii)$ in three subsections.

3.5.1. *Characterizations based on two truncated moments.* This subsection deals with the characterizations of $GUREW$ distribution based on the ratio of two truncated moments. Our first characterization employs a theorem due to Glänzel (1987), see Theorem 1 of Appendix A . The result, however, holds also when the interval H is not closed, since the condition of the Theorem is on the interior of H.

Proposition 3.5.1. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x) = x^{-\beta} e^{\lambda\left(1-e^{-\alpha x^{\beta}}\left(1+\alpha x^{\beta}\right)\right)}$ and $q_2(x) = q_1(x)\, e^{-\alpha x^{\beta}}$ for $x > 0$. The random variable X has PDF (6) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x) = \frac{1}{2} e^{-\alpha x^{\beta}}, \quad x > 0.$$

Proof. Suppose the random variable X has PDF (6), then

$$(1 - F(x))\, E\left[q_1(X) \mid X \geq x\right] = \frac{\alpha\lambda}{1-e^{-\lambda}} e^{-\alpha x^{\beta}}, \quad x > 0,$$

and

$$(1 - F(x))\, E\left[q_2(X) \mid X \geq x\right] = \frac{\alpha\lambda}{2(1-e^{-\lambda})} e^{-2\alpha x^{\beta}}, \quad x > 0.$$

Further,

$$\xi(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-\alpha x^{\beta}} < 0 \ , \quad for \quad x > 0.$$

Conversely, if ξ is of the above form, then

$$s'(x) = \frac{\xi'(x)\, q_1(x)}{\xi(x)\, q_1(x) - q_2(x)} = \alpha\beta x^{\beta-1}, \quad x > 0,$$

and consequently

$$s(x) = \alpha x^{\beta}, \quad x > 0.$$

Now, according to Theorem 1, X has density (6).

Corollary 3.5.1. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition A.1. The random variable X has PDF (6) if and only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the following differential equation

$$\frac{\xi'(x)\, q_1(x)}{\xi(x)\, q_1(x) - q_2(x)} = \alpha\beta x^{\beta-1}, \quad x > 0.$$

Corollary 3.5.2. The general solution of the differential equation in Corollary 3.5.1 is

$$\xi(x) = e^{\alpha x^{\beta}} \left[-\int \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D \right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 3.5.1 with $D = 0$. Clearly, there are other triplets (q_1, q_2, ξ) which satisfy conditions of Theorem1.

3.5.2. *Characterization in terms of reverse hazard function.* The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in support\ of\ F.$$

In this subsection we present a characterization of GUREW distribution in terms of the reverse hazard function.

Proposition 3.5.2. Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has PDF (6) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r'_F(x) + \alpha\beta x^{\beta-1} r_F(x) = \alpha^2 \beta\lambda (2\beta - 1) x^{2(\beta-1)} e^{-\alpha x^\beta},\ x > 0.$$

Proof. If X has PDF (6) , the clearly the above differential equation holds. Now, if this equation holds, the

$$\frac{d}{dx}\left\{e^{\alpha x^\beta} r_F(x)\right\} = \alpha^2\beta\lambda \frac{d}{dx}\left\{x^{2\beta-1}\right\},\ x > 0.,$$

from which we obtain the reverse hazard function corresponding to the PDF (6).

3.5.3. *Characterization based on the conditional expectation of certain function of the random variable.* In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it here which can be used to characterize GUREW distribution.

Proposition 3.5.3. Let $X : \Omega \to (e, f)$ be a continuous random variable with *cdf* F . Let $\psi(x)$ be a differentiable function on (e, f) with $\lim_{x\to f^-} \psi(x) = 1$. Then for $\delta \neq 1$,

$$E[\psi(X) \mid X \leq x] = \delta\psi(x), \qquad x \in (e, f)$$

implies that

$$\psi(x) = (F(x))^{\frac{1}{\delta}-1}, \qquad x \in (e, f).$$

Remark 3.5.1. For $(e, f) = (0, \infty)$, $\psi(x) = \frac{1}{(1-e^{-\lambda})^{1/\lambda}} e^{-\left(1-e^{-\alpha x^\beta}\left(1+\alpha x^\beta\right)\right)}$ and $\delta = \frac{\lambda}{\lambda+1}$, Proposition 3.5.3 provides a characterization of GUREW.

4. Inference procedure

In this section, we consider estimation of the unknown parameters of the $GUREW(\alpha, \beta, \lambda)$ distribution via maximum likelihood method and bootstrap estimation.

4.1. **Maximum likelihood estimation.** Let $x_1, \ldots, x_n$ be a random sample from the *GUREW* distribution and $\Delta = (\alpha, \beta, \lambda)$ be the vector of parameters. The log-likelihood function is given by

$$\begin{aligned} L = L(\Delta) = & 2n \log \alpha + n \log \beta + n \log \frac{\lambda}{1 - e^{-\lambda}} + (2\beta - 1) \sum_{i=1}^{n} \log x_i \\ & - \alpha \sum_{i=1}^{n} x_i^{\beta} - \lambda \sum_{i=1}^{n} \left(1 - e^{-\alpha x_i^{\beta}(1+\alpha x_i^{\beta})}\right). \end{aligned} \tag{9}$$

The elements of the score vector are given by

$$\frac{dL}{d\alpha} = \frac{2n}{\alpha} - \sum_{i=1}^{n} x_i^{\beta} - \lambda\alpha \sum_{i=1}^{n} x_i^{2\beta} e^{-\alpha x_i^{\beta}} = 0,$$

$$\begin{aligned} \frac{dL}{d\beta} = & \frac{n}{\beta} + 2 \sum_{i=1}^{n} \log x_i - \alpha \sum_{i=1}^{n} x_i^{\beta} \log x_i \\ & - \alpha^2 \lambda \sum_{i=1}^{n} x_i^{2\beta} \log x_i \, e^{-\alpha x_i^{\beta}} = 0, \end{aligned}$$

and

$$\frac{dL}{d\lambda} = \frac{n}{\lambda} - \frac{n e^{-\lambda}}{1 - e^{-\lambda}} - \sum_{i=1}^{n} \left(1 - e^{-\alpha x_i^{\beta}(1+\alpha x_i^{\beta})}\right) = 0.$$

respectively.

The maximum likelihood estimate, $\hat{\Delta}$ of $\Delta = (\alpha, \beta, \lambda)$ is obtained by solving the nonlinear equations $\frac{dL}{d\alpha} = 0$, $\frac{dL}{d\beta} = 0$, $\frac{dL}{d\lambda} = 0$ simultaneously. These equations do not have closed forms so, the values of the parameters α,λ and β must be found using iterative methods. Therefore, the maximum likelihood estimate, $\hat{\Delta}$ of $\Delta = (\alpha, \beta, \lambda)$ can be determined using an iterative method such as the Newton-Raphson procedure.

4.2. **Bootstrap estimation.** The parameters of the fitted distribution can be estimated by parametric (resampling from the fitted distribution) or non-parametric (resampling with replacement from the original data set) bootstraps resampling (see Efron and Tibshirani, 1994). These two parametric and nonparametric bootstrap procedures are described as below.

Parametric bootstrap procedure:

(1) Estimate θ (vector of unknown parameters), say $\hat{\theta}$, by using the *MLE* procedure based on a random sample.
(2) Generate a bootstrap sample $\{X_1^*, \ldots, X_m^*\}$ using $\hat{\theta}$ and obtain the bootstrap estimate of θ, say $\widehat{\theta^*}$, from the bootstrap sample based on the *MLE* procedure.

(3) Repeat Step 2 $NBOOT$ times.
(4) Order $\widehat{\theta^*}_1, \ldots, \widehat{\theta^*}_{NBOOT}$ as $\widehat{\theta^*}_{(1)}, \ldots, \widehat{\theta^*}_{(NBOOT)}$. Then obtain γ-quantiles and $100(1-\alpha)\%$ confidence intervals for the parameters.

In the case of $GUREW$ distribution, the parametric bootstrap estimators (PBs) of α, β and λ, are $\hat{\alpha}_{PB}$, $\hat{\beta}_{PB}$ and $\hat{\lambda}_{PB}$, respectively.

Nonparametric bootstrap procedure

(1) Generate a bootstrap sample $\{X_1^*, \ldots, X_m^*\}$, with replacement from the original data set.
(2) Obtain the bootstrap estimate of θ with MLE procedure, say $\widehat{\theta^*}$, by using the bootstrap sample.
(3) Repeat Step 2 $NBOOT$ times.
(4) Order $\widehat{\theta^*}_1, \ldots, \widehat{\theta^*}_{NBOOT}$ as $\widehat{\theta^*}_{(1)}, \ldots, \widehat{\theta^*}_{(NBOOT)}$. Then obtain γ-quantiles and $100(1-\alpha)\%$ confidence intervals for the parameters.

In the case of $GUREW$ distribution, the nonparametric bootstrap estimators (NPBs) of α, β and λ, are $\hat{\alpha}_{NPB}, \hat{\beta}_{NPB}$ and $\hat{\lambda}_{NPB}$, respectively.

5. Algorithm and a simulation study

In this section, we give an algorithm for generating the random data $x_1, \ldots, x_n$ from the GUREW distribution and hence a simulation study is done to evaluate the performance of the MLEs.

5.1. **Algorithm.** Here, we obtain an algorithm for generating the random data $x_1, \ldots, x_n$ from the GUREW distribution as follows.

The algorithm is based on generating random data from the inverse CDF of the GUREW distribution.

- Generate $U_i \sim Uniform(0,1)$; $i = 1, \ldots, n$,
- set

$$X_i = \left(-\frac{1}{\alpha} - \frac{1}{\alpha} W_{-1}\left(-e^{-1}\left(1 + \frac{1}{\lambda}\log(1-(1-e^{-\lambda})U_i)\right)\right)\right)^{1/\beta},$$

where W_{-1} denote the negative branch of the Lambert function.

5.2. **Monte Carlo simulation study.** Here, we assess the performance of the MLE's of the parameters with respect to the sample size n for the $GUREW$ distribution. The assessment of the performance is based on a simulation study via Monte Carlo method. Let $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ be the MLEs of the parameters α, β and λ, respectively. We calculate the mean square error (MSE) and bias of the MLE's of the parameters α, β and λ based on the simulation results of 2000 independent replications. Results are summarized in Table 3 for different values of α, β and λ. From Table 3 the results verify that MSE of the MLE's of the parameters decrease

TABLE 3. MSEs and Average biases(values in parentheses) of the simulated estimates.

		$\alpha = 2$	$\beta = 0.5$	$\lambda = 0.5$
n	30	0.5782 (0.1286)	3.7220 (0.0262)	0.0082 (0.0070)
	50	0.5484 (0.1170)	3.7073 (-0.0058)	0.0057 (-0.0058)
	100	0.4796 (0.1137)	3.4564 (-0.0365)	0.0040 (-0.0117)
	200	0.3955 (0.0774)	3.1149 (0.0061)	0.0030 (-0.0134)
		$\alpha = 2$	$\beta = 1$	$\lambda = 1.5$
n	30	0.8754 (0.3646)	4.8570 (0.0593)	0.2866 (-0.5077)
	50	0.8185 (0.3408)	4.3000 (0.0907)	0.2953 (-0.5256)
	100	0.6055 (0.2053)	4.0039 (0.3363)	0.2927 (-0.5301)
	200	0.4345 (0.1305)	3.1229 (0.4160)	0.2856 (-0.5276)
		$\alpha = 1.5$	$\beta = 1.5$	$\lambda = 1.5$
n	30	0.6282 (0.3410)	4.6770 (-0.5706)	0.0650 (-0.0135)
	50	0.5480 (0.2777)	4.6123 (-0.3817)	0.0438 (-0.0450)
	100	0.4122 (0.1702)	4.1145 (-0.1196)	0.0284 (-0.0428)
	200	0.3085 (0.1111)	3.1749 (-0.0368)	0.0188 (-0.0417)
		$\alpha = 0.5$	$\beta = 2$	$\lambda = 2$
n	30	0.1439 (0.1702)	8.8891 (-0.6927)	0.1047 (-0.0097)
	50	0.1472 (0.1552)	5.5487 (-0.5219)	0.0773 (-0.0453)
	100	0.1072 (0.1136)	4.2505 (-0.2866)	0.0450 (-0.0544)
	200	0.0815 (0.0845)	3.2902 (-0.1761)	0.0296 (-0.0529)
		$\alpha = 1$	$\beta = 1$	$\lambda = 1$
n	30	0.2922 (0.1631)	4.0435 (-0.2354)	0.0297 (-0.0041)
	50	0.2635 (0.1397)	4.1884 (-0.1596)	0.0213 (-0.0189)
	100	0.2186 (0.0984)	3.6548 (0.0001)	0.0140 (-0.0253)
	200	0.1774 (0.0895)	3.1011 (-0.0176)	0.0100 (-0.0303)
		$\alpha = 0.5$	$\beta = 1.5$	$\lambda = 1.5$
n	30	0.1469 (0.1499)	4.5239 (-0.5508)	0.0641 (-0.0076)
	50	0.1378 (0.1378)	4.3793 (-0.3924)	0.0444 (-0.0400)
	100	0.1094 (0.0994)	3.8707 (-0.1563)	0.0289 (-0.0439)
	200	0.0675 (0.0696)	2.8043 (-0.1028)	0.0172 (-0.0396)

with respect to sample size n for all the parameters. So, the MLEs of α, β and λ are consistent estimators.

6. Practical data application

In this section, we present an application of the $GUREW$ distribution to a practical data set to illustrate its flexibility among a set of competitive models. In order to achieve this goal, we consider a real data set corresponding to the remission times (in months) of a random sample of 128 bladder cancer patients. These data were previously studied by Lee and Wang (2003). This data set consists of the following observations:

0.08 0.20 0.40 0.50 0.51 0.81 0.90 1.05 1.19 1.26 1.35 1.40 1.46 1.76 2.02 2.02 2.07 2.09 2.23 2.26 2.46 2.54 2.62 2.64 2.69 2.69 2.75 2.83 2.87 3.02 3.25 3.31 3.36 3.36 3.48 3.52 3.57 3.64 3.70 3.82 3.88 4.18 4.23 4.26 4.33 4.34 4.40 4.50 4.51 4.87 4.98 5.06 5.09 5.17 5.32 5.32 5.34 5.41 5.41 5.49 5.62 5.71 5.85 6.25 6.54 6.76 6.93 6.94 6.97 7.09 7.26 7.28 7.32 7.39 7.59 7.62 7.63 7.66 7.87 7.93 8.26 8.37 8.53 8.65 8.66 9.02 9.22 9.47 9.74 10.06 10.34 10.66 10.75 11.25 11.64 11.79 11.98 12.02 12.03 12.07 12.63 13.11 13.29 13.80 14.24 14.76 14.77 14.83 15.96 16.62 17.12 17.14 17.36 18.10 19.13 20.28 21.73 22.69 23.63 25.74 25.82 26.31 32.15 34.26 36.66 43.01 46.12 79.05

Graphical measure: The total time test (TTT) plot due to Aarset (1987) is an important graphical approach to verify whether the data can be applied to a specific distribution or not. According to Aarset (1987), the empirical version of the TTT plot is given by plotting $T(r/n) = [\sum_{i=1}^{r} y_{i:n} + (n-r)y_{r:n}]/\sum_{i=1}^{n} y_{i:n}$ against r/n, where $r = 1,\dots,n$ and $y_{i:n}(i = 1,\dots,n)$ are the order statistics of the sample. Aarset (1987) showed that the hazard function is constant if the TTT plot is graphically presented as a straight diagonal, the hazard function is increasing (or decreasing) if the TTT plot is concave (or convex). The hazard function is U-shaped if the TTT plot is convex and then concave, if not, the hazard function is unimodal. The TTT plots for data set is presented in Fig 3. These plots indicate that the empirical hazard rate functions of the data set is upside-down bathtub shapes. Therefore, the $GUREW$ distribution is appropriate to fit this data set.

6.1. **Bootstrap inference for $GUREW$ parameters.** In this section, we obtain point and 95% confidence interval (CI) estimation of the $GUREW$ parameters by parametric and non-parametric bootstrap methods. We provide results of bootstrap estimation in Table 4 for the complete data set. It is interesting to observe the joint distribution of the bootstrapped values in a scatter plot in order to understand the potential structural correlation between the parameters. The corresponding plots of the bootstrap estimation are shown in Fig 4.

6.2. **MLE inference and comparison with other models.** Now, we fit the $GUREW$ distribution to a data set and compare it with Lidley, Generalized Lindley (GL), Gamma Lindley (GaL), Power Lindley (PL), Exponential Lindley (EL), gamma, generalized exponential, exponential and Weibull distributions. Table 5 shows the $MLEs$ of the parameters, log-likelihood, Akaike information criterion (AIC), Cramrvon Mises(W^*), AndersonDarling (A^*) and $p-value(P)$ statistics

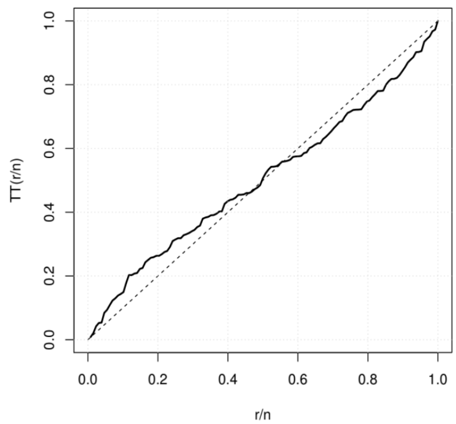

FIGURE 3. Scaled-TTT plot of the data set.

TABLE 4. Bootstrap point and interval estimation of the parameters α, β and λ.

	parametric bootstrap		non-parametric bootstrap	
	point estimation	CI	point estimation	CI
α	0.183	(0.055,0.336)	0.172	(0.052,0.313)
β	0.770	(0.602,0.913)	0.765	(0.578,0.906)
λ	3.703	(0.775,46.763)	3.898	(1.100,66.902)

for the data set. The $GUREW$ distribution provides the best fit for the data set as it shows the lowest AIC, A^* and W^* than other considered models. The relative histograms, fitted $GUREW$, Lindley, GL, GaL,EXP, PL, EL, gamma, generalized exponential and Weibull $PDFs$ for data are plotted in Fig 5. The plots of the empirical and fitted survival functions, $P-P$ plots and $Q-Q$ plots for the $GUREW$ and other fitted distributions are displayed in Fig 5 and Fig 6 respectively. These plots also support the results in Table 5. We compare the $GUREW$ model with a set of competitive models, namely:
(i) Lindley distribution (Lindley, 1958). The one-parameter Lindley density function is given by

$$f(x;\beta)=\frac{\beta^2}{1+\beta}(1+x)\,e^{-\beta x};\quad x>0,$$

where $\beta>0$.

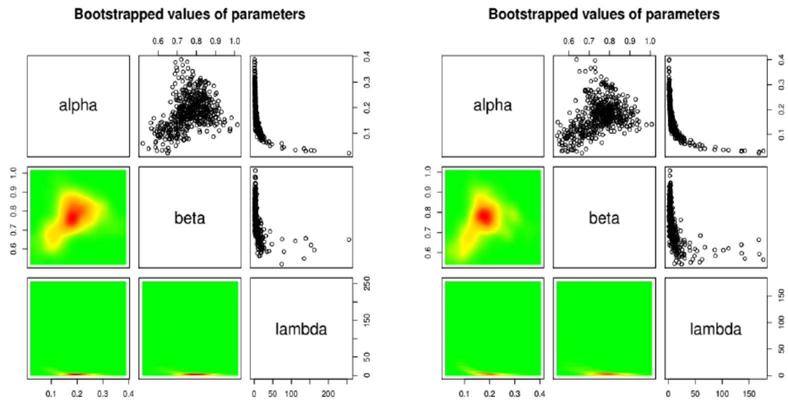

FIGURE 4. Parametric (left) and non-parametric (right) bootstrapped values of parameters of the $GUREW$ distribution for the real data.

(ii) Generalized Lindley distribution (GL) (Zakerzadeh and Dolati, 2009) . The three-parameter GL density function is given by

$$f(x;\theta,\alpha,\beta)=\frac{\theta^{\alpha+1}}{(\theta+\beta)\Gamma(\alpha+1)}x^{\alpha-1}\left(\alpha+\beta x\right)e^{-\theta x};\quad x>0,$$

where $\theta>0,\alpha>0$ and $\beta>0$.
(iii) Exponentiated Lindley distribution (EL) (Nadarajah et al., 2011). The two-parameter EL density function is given by

$$f(x;\theta,\alpha)=\frac{\alpha\theta^2}{(1+\theta)}(1+x)e^{-\theta x}\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha-1};\quad x>0,$$

where $\theta>0$ and $\alpha>0$.
(iv) Power Lindley distribution (PL) (Ghitany et al., 2013). The two-parameter PL density function is given by

$$f(x;\theta,\alpha)=\frac{\alpha\,\theta^2}{\theta+1}(1+x^\alpha)x^{\alpha-1}e^{-\theta x^\alpha};\quad x>0$$

where $\alpha>0$ and $\theta>0$.
(v) Gamma Lindley distribution (GaL) (Zeghdoudi and Nedjar. 2015). The two-parameter GaL density function is given by

$$f(x;\theta,\alpha)=\frac{\theta^2}{\alpha(1+\theta)}[(\alpha+\alpha\theta-\theta)x+1]e^{-\theta x};\quad x>0,$$

where $\theta>0$ and $\alpha>0$.
(vi) The two-parameter Weibull distribution is given by

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta}\,(\frac{x}{\beta})^{\alpha-1}\,e^{-(\frac{x}{\beta})^\alpha};\quad x>0$$

where $\alpha > 0$ and $\beta > 0$.
(vii) The two-parameter Gamma distribution is given by

$$f(x; \alpha, \theta) = \frac{1}{\theta^\alpha \, \Gamma(\alpha)} \, x^{\alpha-1} \, e^{-(x/\theta)}; \quad x > 0$$

where $\alpha > 0$ and $\theta > 0$ and $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \, e^{-t} \mathrm{d}t$.
(viii) The one parameter Exponential distribution is given by

$$f(x; \lambda) = \lambda \, e^{-\lambda x}$$

where $\lambda > 0$.
(ix) The two-parameter generalized exponential (GE) distribution is given by

$$f(x; \alpha, \lambda) = \alpha\lambda \, e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}; \quad x > 0$$

where $\alpha > 0$ and $\lambda > 0$.

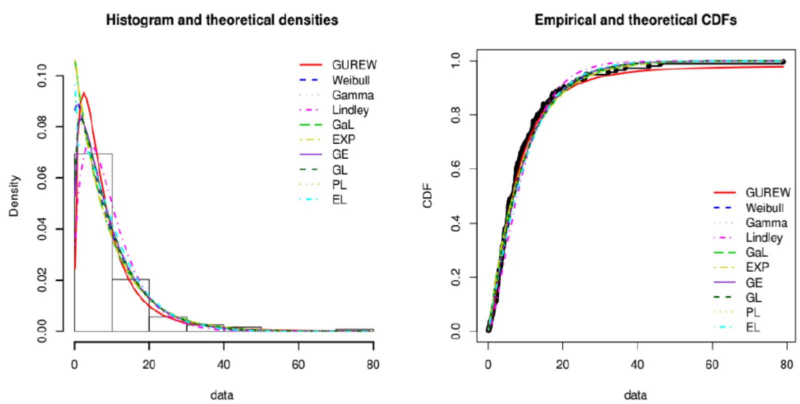

FIGURE 5. Estimated densities and Empirical and Estimated cdf for the data set.

7. CONCLUSION

In this article, a new model for the lifetime distributions is introduced and its main properties are discussed. A special submodel of this family is taken up by considering exponential distributions in place of the parent distribution F and Weibull distribution in place of the parent distribution G. We show that the proposed distribution has variability of hazard rate shapes such as increasing, decreasing and upside-down bathtub shapes. From a practical point of view, we show that the proposed distribution is more flexible than some commonly known statistical distributions for a given data set.

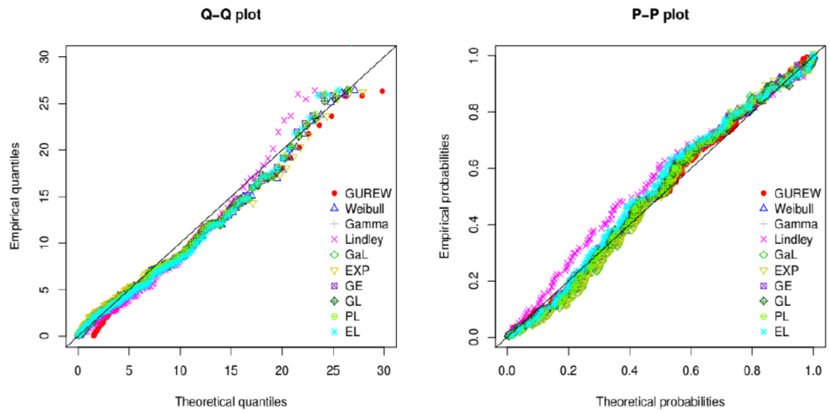

FIGURE 6. Q-Q and P-P plots for the data set.

TABLE 5. Parameter estimates (standard errors), log-likelihood values and goodness of fit measures

Model	$MLEs$ of parameters (s.e)	Log-likelihood	AIC	BIC	A^*	W^*	$K.S$	P
$GUREW$	$\hat{\alpha}=0.17(0.06)$ $\hat{\beta}=0.77\,(0.08)$ $\hat{\lambda}=3.94\,(3.02)$	-409.78	825.56	834.12	0.13	0.01	0.03	0.99
$Lindley$	$\hat{\beta}=0.19\,(0.01)$	-419.52	841.05	843.91	2.78	0.51	0.11	0.06
GL	$\hat{\theta}=1.25e-01\,(1.72e-02)$ $\hat{\alpha}=1.71e-01\,(1.30e-01)$ $\hat{\beta}=3.03e-05\,(8.38e+03)$	-413.36	832.73	841.29	0.77	0.13	0.07	0.49
PL	$\hat{\theta}=0.29\,(0.03)$ $\hat{\alpha}=0.83\,(0.04)$	-413.35	830.70	836.41	0.78	0.12	0.06	0.59
EL	$\hat{\theta}=0.16\,(0.01)$ $\hat{\alpha}=0.73\,(0.09)$	-416.28	836.57	842.27	1.32	0.24	0.09	0.21
GaL	$\hat{\theta}=0.10\,(0.02)$ $\hat{\alpha}=0.09\,(0.03)$	-414.34 GE	832.68	838.38	1.17	0.17	0.08	0.31
GE	$\hat{\alpha}=1.21\,(0.14)$ $\hat{\lambda}=0.69\,(0.09)$	-413.07	830.15	835.85	0.71	0.12	0.07	0.51
EXP	$\hat{\lambda}=0.10\,(0.009)$	-414.34	830.68	833.53	1.17	0.17	0.08	0.31
$Weibull$	$\hat{\alpha}=1.04\,(0.06)$ $\hat{\beta}=9.56\,(0.85)$	-414.08	832.17	837.87	0.95	0.15	0.06	0.55
$Gamma$	$\hat{\alpha}=1.17\,(0.13)$ $\hat{\theta}=0.12\,(0.01)$	-413.36	830.73	836.43	0.77	0.13	0.07	0.49

Appendix A

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H=[a,b]$ be an interval for some $d<b$ $(a=-\infty,\ b=\infty$ might as well be allowed$)$. Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and

let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(X\right)\ \mid X \geq x\right] = \mathbf{E}\left[q_1\left(X\right)\ \mid X \geq x\right] \xi\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1\left(H\right)$, $\xi \in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ , particularly

$$F\left(x\right) = \int_a^x C\left|\frac{\xi'\left(u\right)}{\xi\left(u\right) q_1\left(u\right) - q_2\left(u\right)}\right| \exp\left(-s\left(u\right)\right)\ du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\xi'\ q_1}{\xi q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

References

[1] Ababneh, M., Ebrahem, M. A. H., Designing degradation experiments using a Weibull distribution, J*ournal of Statistics and Management Systems*, 21(6) (2018), 971-983.

[2] Alzaatreh, A., Lee, C., Famoye, F., A new method for generating families of continuous distributions, *Metron*, 71(1)(2013), 63-79.

[3] Aarset, M. V., How to identify a bathtub hazard rate, *IEEE Transactions on Reliability*, 36(1) (1987), 106-108.

Efron, B., Tibshirani, R. J., An introduction to the bootstrap, CRC press, 1994.

[4] Glänzel, W., A characterization theorem based on truncated moments and its application to some distribution families, *Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Reidel, Dordrecht, Vol. B*, (1987), 75–84.

[5] Glänzel, W., Some consequences of a characterization theorem based on truncated moments, *Statistics: A Journal of Theoretical and Applied Statistics*, 21 (4) (1990), 613–618.

[6] Hamedani, G.G., On certain generalized gamma convolution distributions II, Technical Report No. 484, MSCS, Marquette University, 2013.

Lee, E. T., Wang, J., Statistical methods for survival data analysis, (Vol. 476), John Wiley & Sons, 2003.

[7] Lindley, D.V., Fiducial distributions and Bayes theorem, *Journal of the Royal Statistical Society Series B,* 20 (1958), 102-107.

[8] Murthy, D. P., Xie, M., Jiang, R., Weibull models, (Vol. 505). John Wiley & Sons, 2004.

[9] Nadarajah, S., Bakouch, H. S., Tahmasbi, R., A generalized Lindley distribution, *Sankhya B*, 73 (2) (2011), 331-359.

[10] Samuel, B. R., Balamurali, S., Aslam, M., Designing of repetitive group sampling plan under truncated life test based on generalized inverted exponential distribution, *Journal of Statistics and Management Systems*, 21(6) (2018), 955-970.

[11] Zeghdoudi, H., Nedjar, S., Gamma Lindley distribution and its application, *Journal of Applied Probability and Statistics*, 11 (1) (2015), 1-11.

[12] Zakerzadeh, H., Dolati, A., Generalized Lindley distribution, *Journal of Mathematical Extension*, 3 (2) (2009), 13-25.

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ALMOST CONTACT METRIC AND METALLIC RIEMANNIAN STRUCTURES

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Abstract. The metallic structure is a fascinating topic that continually generates new ideas. In this work, new metallic manifolds are constructed starting from both almost contact metric manifolds and we obtain some important notions like the metallic deformation. We give a concrete example to confirm this construction.

1. Introduction

Manifolds equipped with certain differential-geometric structures possess rich geometric structures and such manifolds and relations between them have been studied widely in differential geometry. Indeed, almost complex manifolds, almost contact manifolds and almost product manifolds and relations between such manifolds have been studied extensively by many authors.

The differential geometry of the Golden on Riemannian manifolds is a popular subject for mathematicians. In 2007, Hreţcanu [12] introduced the Golden structure on manifolds and in [14] the geometry of the golden structure on manifolds was studied. Now, Such manifolds have been studied by various authors (see [3, 5, 11, 15, 16]). Later, the author in [3] gave a set of techniques to construct many compatible well-known structures on a Riemannian manifold, starting from a Golden Riemannian manifold. And also he established in [4] an interesting class of almost Golden Riemannian manifolds such as the s-Golden manifolds.

As generalization of the Golden mean, the metallic means family appear in 1997 by Vera W. de Spinadel (see [10]) which contains the silver mean, the bronze mean, the copper mean and the nickel mean, etc. The metallic mean family plays an important role in establishing a relationship between mathematics and architecture.

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For example, silver and golden mean can be seen in the sacred art of India, Egypt, China, Turkey and different ancient civilizations. Now, there are also several recent works in this direction [13, 14, 7] and others. Recently, a new type of structure on a differentiable manifold is studied in [9] and the relation between metallic structure and almost quadratic φ-structure is considered in [17].

Here we show that there exists a correspondence between the metallic Riemannian structures and the almost contact metric structures.

This text is organized in the following way:

Section 2 is devoted to the background of the structures which will be used in the sequel.

In Section 3, starting from an almost contact metric structures we define metallic Riemannian structures and we investigate conditions for those structures being integrable and parallel then we give an example to confirm these latter properties.

In Section 4, we give the notion of metallic transformation and we use it for some questions of the characterization of certain geometric structures.

The Section 5 is devoted to give a generalization of the notion of metallic transformation which deduces the particular known cases.

In the last Section, we give an open question where we propose the first step to study the reverse, i.e. the construction of an almost contact metric structure starting from a metallic Riemannian structure.

2. Review of needed notions

In this section, we give a brief information for metallic Riemannian manifolds and almost contact metric manifolds. We note that throughout this paper all manifolds and bundles, along with sections and connections, are assumed to be of class C^∞.

Let (M, g) be a Riemannian manifold. We present metallic Riemannian manifolds following [14]. A (p, q)-metallic structure on M is a polynomial structure of second degree given by a $(1, 1)$-tensor field Φ which satisfies

$$\Phi^2 = p\Phi + qI, \tag{1}$$

where I is the identity transformation and p, q are fixed integers such that $x^2 - px - q = 0$ has a positive irrational root $\sigma_{p,q}$.

The number $\sigma_{p,q}$ is usually named a member of the metallic family. These numbers, denoted by:

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}, \tag{2}$$

are also called (p, q)-metallic numbers.

For example, we can talk about Golden structure if $p = 1$, $q = 1$ when the $\sigma_{1,1}$ is exactly the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$, or about the silver structure ($p = 2$, $q = 1$, $\sigma_{2,1} = 1 + \sqrt{2}$), the bronze structure ($p = 3$, $q = 1$, $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$) , the nickel structure ($p = 1$, $q = 3$, $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$) , the copper structure ($p = 1$, $q = 2$, $\sigma_{1,2} = 2$). The above numbers are closely related with different mathematical domains as dynamical systems, quasicristales, theory of Cantorial fractal-like micro-space-time.

For the Riemannian manifold (M, g) endowed with the (p, q)-metallic structure, we say that the metric g is Φ-compatible and that M is a Riemannian metallic manifold [14], if

$$g(\Phi X, Y) = g(X, \Phi Y), \tag{3}$$

for all X, Y vectors fiels on M. If we substitute ΦX into X in (3), equation (3) may also written as

$$g(\Phi X, \Phi Y) = g(\Phi^2 X, Y) = g((p\Phi + qI)X, Y) = pg(\Phi X, Y) + qg(X, Y).$$

Here, we can show that such a metric always exists on a manifold with a metallic structure Φ.

Proposition 1. *If (M, Φ) is a metallic manifold, then M admits a Riemannian metric g such that*

$$g(\Phi X, Y) = g(X, \Phi Y).$$

Proof. Let h be any Riemannian metric on M and define g by

$$g(X, Y) = h(\Phi X, \Phi Y) + qh(X, Y),$$

and check the details. □

Note from Proposition 3.2 of [14] that every almost product structure J induces two metallic structures on M given as follows:

$$\Phi_{1,2} = \frac{1}{2}\Big(pI \pm (2\sigma_{p,q} - p)J\Big) \tag{4}$$

is an almost product structure on M.
Conversely, every metallic structure Φ on M induces two almost product structures on M given as follows:

$$J_{1,2} = \pm\frac{2\Phi - pI}{2\sigma_{p,q} - p}. \tag{5}$$

For a metallic manifold (M, Φ, g) and the associated almost product J, it is easy to see that

$$g(JX, Y) = g(X, JY), \tag{6}$$

for every tangent vector fields X on M.

In order that the Golden structure Φ is integrable, it is necessary and sufficient that it is possible to introduce a torsion-free affine connection ∇ with respect to which the structure tensor Φ is covariantly constant. Also, we know that the integrability of Φ is equivalent to the vanishing of the Nijenhuis tensor N_Φ [14], where

$$N_\Phi(X, Y) = \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y]. \tag{7}$$

The link between the Nihenjuis tensors Φ and J is given by

$$N_J = \frac{4}{p^2 + 4q} N_\Phi, \tag{8}$$

which show that the metallic structure Φ is integrable if and ony if the associated almost product J is integrable.

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1,\ \varphi^2(X) = -X + \eta(X)\xi \quad \textit{and} \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{9}$$

for any vectors fields X,Y on M. In particular, in an almost contact metric manifold we also have

$$\varphi\xi = 0 \quad \textit{and} \quad \eta \circ \varphi = 0. \tag{10}$$

Such a manifold is said to be a contact metric manifold if

$$d\eta = \Omega, \tag{11}$$

where $\Omega(X,Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M.
On the other hand, the almost contact metric structure of M is said to be normal if

$$N_\varphi(X,Y) = [\varphi,\varphi](X,Y) + 2d\eta(X,Y)\xi = 0, \tag{12}$$

for any X and Y vectors fields on M, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

An almost contact metric structure (φ, ξ, η, g) on M is said to be:

$$\begin{cases} (a) : \textit{Sasaki} \ \Leftrightarrow \ \Omega = \mathrm{d}\eta \textit{ and } (\varphi,\xi,\eta) \textit{ is normal}, \\ (b) : \textit{Cosymplectic} \ \Leftrightarrow \ \mathrm{d}\Omega = d\eta = 0 \textit{ and } (\varphi,\xi,\eta) \textit{ is normal}, \\ (c) : \textit{Kenmotsu} \ \Leftrightarrow \ \mathrm{d}\eta = 0,\ \mathrm{d}\Omega = 2\Omega\wedge\eta \ \textit{ and } (\varphi,\xi,\eta) \textit{ is normal}. \end{cases} \tag{13}$$

where d denotes the exterior derivative.

In [20], the author proves that (φ, ξ, η, g) is trans-Sasakian structure if and only if (φ, ξ, η, g) is normal and

$$d\eta = \alpha\Omega, \qquad d\Omega = 2\beta\eta\wedge\Omega, \tag{14}$$

where $\alpha = \frac{1}{2n}\delta\Omega(\xi)$, $\beta = \frac{1}{2n}div\xi$ and δ is the codifferential of g.

A trans-Sasakian structure (φ, ξ, η, g) on M is said to be

$$\begin{cases} (a) : \ \alpha - \textit{Sasaki} \quad if \quad \beta = 0, \\ (b) : \ \beta - \textit{Kenmotsu} \quad if \quad \alpha = 0, \\ (c) : \ \textit{Cosymplectic} \quad if \quad \alpha = \beta = 0. \end{cases} \tag{15}$$

(see [6], [18] and [23]).

The relation between trans-Sasakian, α-Sasakian and, β-Kenmotsu structures was discussed by Marrero [19].

Proposition 2. *(Marrero* [19]*)*
A trans-Sasakian manifold of dimension ≥ 5 *is either* α*-Sasakian,* β*-Kenmotsu or cosymplectic.*

Proposition 3. *(Marrero* [19], *Proposition 4.2)*
Let $(M, \varphi, \xi, \eta, g)$ *be a 3-dimensional Sasakian manifold. If we take* $\overline{g} = fg + (1-f)\eta \otimes \eta$ *where* $f > 0$ *a non-constant function on* M *then, ,* $(M, \varphi, \xi, \eta, \overline{g})$ *is a trans-Sasakian structure of type* $\left(\frac{1}{f}, \frac{1}{2}\xi(\ln f)\right)$.

3. Induced Metallic structures by almost contact structures

In this section, starting from an almost contact metric structure we define a metallic Riemannian structure and we investigate conditions for those structures being integrable and parallel.

Theorem 4. *Every almost contact metric structure* (φ, ξ, η, g) *on a* $(2n+1)$*-dimensional Riemannian manifold* (M, g) *induces only two metallic structures on* (M, g)*, given as follows:*

$$\Phi_1 = \sigma_{p,q} I + (p - 2\sigma_{p,q})\eta \otimes \xi, \qquad \Phi_2 = \sigma^*_{p,q} I + (p - 2\sigma^*_{p,q})\eta \otimes \xi, \tag{16}$$

where ξ *is the unique eigenvector of* Φ_1 *and* Φ_2 *associated with* $\sigma^*_{p,q} = p - \sigma_{p,q}$ *and* $\sigma_{p,q}$ *respectively.*

Proof. We try to write the metallic structure Φ_i with $i \in \{1, 2\}$ defined on a $(2n+1)$-dimensional Riemannian manifold (M, g), using almost contact metric structure (φ, ξ, η, g), in the form $\Phi = a_i I + b_i \eta \otimes \xi$, where a_i and b_i are non-zero constant. Thus

$$\Phi^2 = a_i^2 I + b_i(2a_i + b_i)\eta \otimes \xi,$$

and using formula (1) with $\Phi_1 \xi = \sigma^*_{p,q}\xi$ and $\Phi_2 \xi = \sigma_{p,q}\xi$, we obtain the formulas (16). Moreover, we have

$$g(\Phi_i X, Y) = g(X, \Phi_i Y) \Leftrightarrow g(\varphi X, Y) = -g(X, \varphi Y),$$

for every $i \in \{1, 2\}$ and for every tangent vectors fields X and Y on M.
On the other hand, suppose that there exist another metallic structure on M induces by the almost contact metric structure (φ, ξ, η, g) denoted by Ψ and admits ξ as the unique eigenvector associated with $\sigma_{p,q}$ (resp. $\sigma^*_{p,q}$) then, we have

$$\Psi^2 = p\Psi + qI, \qquad \Psi\xi = \sigma_{p,q}\xi \quad (resp.\ \Psi\xi = \sigma^*_{p,q}\xi). \tag{17}$$

First, note that for all $i \in \{1, 2\}$ we have

$$\Phi_i \Psi = \Psi \Phi_i,$$

and using (1) and (17) we get

$$\Psi^2 - \Phi_i^2 = p\big(\Psi - \Phi_i\big), \qquad i = \overline{1, 2}$$

which gives

$$\Psi = p - \Phi_i \in \{\Phi_1, \Phi_2\}.$$

□

Remark 5. *Using the two formulas in (16) we note that*

$$\Phi_1 + \Phi_2 = pI.$$

Proposition 6. *If (M, Φ, g) is a metallic Riemannian manifold, then (M, Φ, G) is also a metallic Riemannian manifold, where G is a Riemannian metric given by:*

$$G(X, Y) = g(\varphi X, \varphi Y),$$

for all vectors fields X,Y on M.

Proof. Since the proof of the following proposition is obvious, we don't give the proof of it. □

Using formula (4), we get the following:

Proposition 7. *Every almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ induces four almost product structures on (M, g), given as follows:*

$$J_1 = I - 2\eta \otimes \xi, \qquad J_2 = I + 2\eta \otimes \xi, \tag{18}$$
$$J_3 = -I + 2\eta \otimes \xi, \qquad J_4 = -I - 2\eta \otimes \xi.$$

We note that through out this paper, we shall be setting

$$\Phi = \sigma_{p,q} I + (p - 2\sigma_{p,q})\, \eta \otimes \xi. \tag{19}$$

Observe that,

$$\sigma_{p,q}^2 = p\sigma_{p,q} + q, \qquad \sigma_{p,q} + \sigma_{p,q}^* = p \qquad \textit{and} \qquad \sigma_{p,q}.\sigma_{p,q}^* = -q.$$

We know that the metallic structure Φ is integrable (i.e. $N_\Phi = 0$) if and only if the almost product J is integrable (i.e. $N_J = 0$) with

$$N_J(X, Y) = [X, Y] + [JX, JY] - J[X, JY] - J[JX, Y].$$

So, for all X, Y vectors fields on M and using (18), we get

$$\frac{1}{8} N_J(X, Y) = \big(d\eta(X, Y) + \eta(X) d\eta(\xi, Y) + \eta(Y) d\eta(X, \xi)\big)\xi, \tag{20}$$

witch give the following theorem:

Theorem 8. *Let (M^{2n+1}, Φ, g) be a metallic Riemannian manifold induced by the almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$. Then Φ is integrable if and only if η is closed.*

Proof. Using the formula (20) with supposing that $d\eta = 0$, we get $N_J = 0$.
For the inverse, suppose that $N_J = 0$. From (20) we have

$$d\eta(X,Y) + \eta(X)d\eta(\xi,Y) + \eta(Y)d\eta(X,\xi) = 0, \tag{21}$$

taking $Y = \xi$ we obtain for all X vector field on M,

$$d\eta(X,\xi) = 0, \tag{22}$$

Applying (22) in (21) we get

$$d\eta(X,Y) = 0,$$

for all X and Y vectors fields tangent to M. □

Remark 9. *If (M,φ,ξ,η,g) is an almost cosymplectic or an almost Kenmotsu manifold then (M,Φ,g) is an integrable metallic Riemannian manifold but for the contact case it is never integrable.*

Lemma 10. *If (Φ,g) is a metallic Riemannian structure induced by an almost contact metric structure (φ,ξ,η,g) on M then we have*

$$\varphi\Phi = \Phi\varphi = \sigma_{p,q}\varphi. \tag{23}$$

Proof. Using formulas (19) and (10), the proof is direct. □

Proposition 11. *Let (M^{2n+1},Φ,g) be a metallic Riemannian manifold induced by the almost contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$. If ∇ is the Levi-Cevita connection then for all X and Y vectors fields tangent to M we have*

$$(\nabla_X\Phi)Y = (p - 2\sigma_{p,q})\big(g(\nabla_X\xi,Y)\xi + \eta(Y)\nabla_X\xi\big). \tag{24}$$

Proof. From

$$(\nabla_X\Phi)Y = \nabla_X\Phi Y - \Phi\nabla_X Y,$$

and using formula (19), the proof is direct. □

On the other hand, we know that the integrability of Φ is equivalent to the existence of a torsion-free affine connection with respect to which the equation $\nabla\Phi = 0$ holds. Now we shall introduce another possible sufficient condition of the integrability of metallic structures on Riemannian manifolds.

Proposition 12. *Let (M^{2n+1},Φ,g) be a metallic Riemannian manifold induced by the almost contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$. Then Φ is integrable (i.e. $\nabla\Phi = 0$) if and only if $\nabla_X\xi = 0$ for all X vector field on M where ∇ is the the Levi-Cevita connection of g.*

Proof. The necessity was observed above (see (24)). For the sufficiency, it suffices to replace Y by ξ in (24). □

Remark 13. *If (φ,ξ,η,g) is a cosymplectic structure then (Φ,g) is a parallel metallic Riemannian structure.*

Example 14. *For this example, we rely on our example in* [1]. *We denote the Cartesian coordinates in a 3-dimensional Euclidean space E^3 by (x, y, z) and define a symmetric tensor field g by*

$$g = \begin{pmatrix} \rho^2 + \tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix}$$

where ρ and τ are functions on E^3 such that $\rho \neq 0$ everywhere.
Further, we define an almost contact metric (φ, ξ, η) on E^3 by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\tau & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = (-\tau, 0, 1).$$

Using the formula (19) we get

$$\Phi = \begin{pmatrix} \sigma_{p,q} & 0 & 0 \\ 0 & \sigma_{p,q} & 0 \\ \tau(2\sigma_{p,q} - p) & 0 & 1 - \sigma_{p,q} \end{pmatrix},$$

where we can check that $\Phi^2 = p\Phi + qI$.
The fundamental 1-form η have the form,

$$\eta = dz - \tau dx,$$

and hence

$$d\eta = \tau_2 dx \wedge dy + \tau_3 dx \wedge dz.$$

With a straightforward computation, one can get

$$\nabla_{\partial x}\xi = \frac{1}{\rho^2}\begin{pmatrix} \rho\rho_3 + \tau\tau_3 \\ \tau_2 \\ \tau\rho\rho_3 + \tau\tau_3 \end{pmatrix}; \quad \nabla_{\partial y}\xi = \frac{1}{\rho^2}\begin{pmatrix} -\tau_2 \\ \rho_3 \\ -\tau\tau_2 \end{pmatrix}; \quad \nabla_{\partial z}\xi = \frac{1}{\rho^2}\begin{pmatrix} -\tau_3 \\ 0 \\ -\tau\tau_3 \end{pmatrix},$$

where $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\tau_i = \frac{\partial \tau}{\partial x_i}$.

On the other hand, according to the cases given in [1], *the structure (φ, ξ, η, g) is a:*

(1) *Cosymplectic when $\rho_3 = \tau_2 = \tau_3 = 0$,*
(2) *Kenmotsu when $\rho_3 = \rho$, $\tau_2 = 0$ and $\tau_3 = 0$.*

So,

(a) *If $\tau_2 = \tau_3 = 0$ (i.e. $d\eta = 0$) then the metallic structure Φ is integrable.*
(b) *If $\rho_3 = \tau_2 = \tau_3 = 0$ (i. e. $\nabla_X \xi = 0$) then the metallic structure Φ is parallel.*

4. Metallic transformation

Let (M^{2n+1}, Φ, g) be a metallic Riemannian manifold induced by the almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$.

We mean a change of structures tensors of the form

$$\tilde{\varphi} = \varphi, \qquad \tilde{\xi} = \frac{1}{p - \sigma_{p,q}}\xi, \qquad \tilde{\eta} = (p - \sigma_{p,q})\eta,$$
$$\tilde{g}(X,Y) = g(\Phi X, \Phi Y) = \sigma_{p,q}^2 g + p(p - 2\sigma_{p,q})\eta \otimes \eta.$$

Proposition 15. *If (φ, ξ, η, g) is an almost contact metric structure, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also an almost contact metric structure.*

Proof. Obvious (using formulas (9)). □

We refer to this construction as metallic deformation.

Theorem 16. *Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost contact metric manifold obtained as above, then it is:*

(a) *α-Sasaki with $\alpha = \frac{p-\sigma_{p,q}}{\sigma_{p,q}^2}$ if and only if $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold.*

(b) *β-Kenmotsu with $\beta = \frac{1}{p-\sigma_{p,q}}$ if and only if $(M, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold.*

(c) *Cosymplectic if and only if $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold.*

Proof. Let $(M, \varphi, \xi, \eta, g)$ be a trans-Sasakian manifold of type (α, β). The fundamental 1-form $\tilde{\eta}$ and the 2-forme $\tilde{\Omega}$ of the structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ defined as above have the forms,

$$\tilde{\eta} = (p - \sigma_{p,q})\eta \qquad \textit{and} \qquad \tilde{\Omega} = \sigma_{p,q}^2 \Omega,$$

where Ω is the 2-form of the almost contact metric structure (φ, ξ, η, g) and hence

$$\mathrm{d}\tilde{\eta} = (p - \sigma_{p,q})\mathrm{d}\eta \qquad \textit{and} \qquad \mathrm{d}\tilde{\Omega} = \sigma_{p,q}^2 \mathrm{d}\Omega, \tag{25}$$

using the formulas (14) we get

$$\mathrm{d}\tilde{\eta} = \frac{p - \sigma_{p,q}}{\sigma_{p,q}^2}\alpha\tilde{\Omega} \qquad \textit{and} \qquad \mathrm{d}\tilde{\Omega} = \frac{2\beta}{p - \sigma_{p,q}}\tilde{\eta} \wedge \tilde{\Omega}. \tag{26}$$

Knowing that the trans-Sasakian manifolds of type $(1,0)$, $(0,1)$ and $(0,0)$ are called Sasakian, Kenmotsu and cosymplectic manifolds respectively then the proof is completed. □

Remark 17. *Note that the metallic transformation preserve the structure cosymplectic for all two positive integers p and q and the Kenmotsu structure only for $q = p + 1$ but the Sasakian structure is never preserved.*

A straightforward computation yields the following proposition:

Proposition 18. *If (Φ, g) be a metallic Riemannian structure induced by the almost contact metric structure (φ, ξ, η, g), then the structure $(\widehat{\varphi}, \widehat{\xi}, \widehat{\eta}, \widehat{g})$ given by*

$$\widehat{\varphi} = \varphi, \qquad \widehat{\xi} = \frac{1}{(p - \sigma_{p,q})^n}\xi, \qquad \widehat{\eta} = (p - \sigma_{p,q})^n\eta,$$

$$\widehat{g}(X, Y) = g(\Phi^n X, \Phi^n Y) = \sigma_{p,q}^{2n} g + \big(\sigma_{p,q}^{2n} - (p - \sigma_{p,q})^{2n}\big)\eta \otimes \eta.$$

for any integer number n, is also an almost contact metric structure.

5. Generalized $\mathcal{D}$-homothetic transformation

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with $dimM = 2n + 1$. The equation $\eta = 0$ defines a $2n$-dimensional distribution $\mathcal{D}$ on M. By an $2n$-homothetic deformation or $\mathcal{D}$-homothetic deformation [22] we mean a change of structure tensors of the form

$$\overline{\varphi} = \varphi, \qquad \overline{\eta} = a\eta, \qquad \overline{\xi} = \frac{1}{a}\xi, \qquad \overline{g} = ag + a(a - 1)\eta \otimes \eta,$$

where a is a positive constant. If $(M, \varphi, \xi, \eta, g)$ is a contact metric structure with contact form η, then $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is also a contact metric structure [22].

This idea works equally well for almost contact metric structures. The deformation

$$\tilde{\varphi} = \varphi, \qquad \tilde{\xi} = \frac{1}{h}\xi, \qquad \tilde{\eta} = h\eta, \qquad \tilde{g}(X, Y) = f^2 g + (h^2 - f^2)\eta \otimes \eta,$$

is again an almost contact metric structure where f and h are two non-zero functions on M.

From the theorem (16), we can deduce the following proposition:

Proposition 19. *Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost contact metric manifold obtained as above, then it is:*

(a) *α-Sasaki with $\alpha = \frac{h}{f^2}$ and h is constant if and only if $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold.*
(b) *β-Kenmotsu with $\beta = \frac{1}{h}$ with f is constant if and only if $(M, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold.*
(c) *Cosymplectic where f, h are constant if and only if $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold.*

Special cases:

- For $h = \pm f$, we get the conformal transformation [21].
- For $h = f^2$ and $f = constant$, we get the deformation of Tanno [22].
- For $h = \pm 1$, we get the deformation of Marrero [19].
- For $f = \pm 1$, we get the $\mathcal{D}$-isometric [2].

6. Open problem

Finally, we propose the first steps to construct an almost contact metric structure from a metallic Riemannian structure.
Let (M^{2n+1}, Φ, g) be a metallic Riemannian manifold and ξ be the unique eigenvector of Φ associated with $\sigma^*_{p,q} = p - \sigma_{p,q}$ (resp. $\sigma_{p,q}$) which give $\Phi\xi = \sigma^*_{p,q}\xi$ (resp. $\Phi\xi = \sigma_{p,q}\xi$) and let η be the g-dual of ξ i.e. $\eta(X) = g(X, \xi)$ for all vector field X on M such that $\eta(\xi) = 1$.

Proposition 20. *The metallic structure Φ admits the following expression:*

$$\Phi = \sigma_{p,q}I + (p - 2\sigma_{p,q})\eta \otimes \xi, \qquad \big(resp. \quad \Phi = \sigma^*_{p,q}I + (p - 2\sigma^*_{p,q})\eta \otimes \xi\big), \tag{27}$$

Proof. We try to write the metallic structure Φ in the form $\Phi = aI + b\eta \otimes \xi$, where $a, b \in \mathbb{R}^*$. Thus

$$\Phi^2 = a^2 I + b(a + \sigma^*_{p,q})\eta \otimes \xi,$$

on the other hand, we have

$$p\Phi + qI = (ap + q)I + pb\, \eta \otimes \xi,$$

using formulas (1) we obtain the formulas (27). □

One can construct on M^{2n+1} an almost contact metric structure (φ, ξ, η, g) starting from a metallic Riemannian structure and study its nature taking into account the two parameters p and q.

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References

[1] Beldjilali, G., Belkhelfa, M. , Kählerian structures on D-homothetic bi-warping, *J. Geom. Symmetry Phys.* 42 (2016), 1-13.

[2] Beldjilali, G., Structures and $\mathcal{D}$-isometric warping, *HSJG*, 2(1) (2020), 21-29 .

[3] Beldjilali, G., Induced structures on Golden Riemannian manifolds., *Beitr Algebra Geom.*, 59 (2018), 761-777.

[4] Beldjilali, G., s-Golden Manifolds, *Mediterr. J. Math.*, (2019), https://doi.org/10.1007/s00009-019-1343-9.

[5] Beldjilali, G., A new class of Golden Riemannian manifold. *Int. Electron. J. Geom.*, 13(1) (2020), 1-8.

[6] Blair, D. E. , Riemannian Geometry of Contact and Symplectic Manifolds, *Progress in Mathematics*, Vol. 203, Birhauser, Boston, 2002

[7] Acet, B. E., Lightlike hypersurfaces of metallic semi-Riemannian manifolds, *Int. J. of Geo. Meth. in Mod. Phys.*, Vol. 15 (2018), 185-201 .

[8] Crasmareanu, M., Hretecanu, C.E., Golden differential geometry, *Chaos, Solitons & Fractals*, 38 (2008), 1124-1146. doi: 10.1016/j.chaos.2008.04.007.

[9] Debmath, P, Konar, A., A new type of structure on differentiable manifold, *Int. Electron. J. Geom.*, 4 (2011), 102-114.

[10] De Spinadel, V.W., The metallic means family and the multifractal, *spectra. Nonlinear Anal.*, 36(6) (1999), 721-745.

[11] Etayo, F., Santamaria, R. and Upadhyay, A., On the Geometry of Almost Golden Riemannian Manifolds, *Mediterr. J. Math.*,14 187 (2017). https://doi.org/10.1007/s00009-017-0991-x.

[12] Hreţcanu, C.E.: Submanifolds in Riemannian manifold with Golden structure. *Workshop on Finsler Geometry and its Applications*, Hungary, (2007).

[13] Hreţcanu C.E., and Blaga A.M., Hemi-slant submanifolds in metallic Riemannian manifolds, *Carpathian J.*, 35 (1) (2019), 59-68.

[14] Hreţcanu C.E., Crasmareanu M., Metallic structures on Riemannian Manifolds, *Rev Un Mat Argentina.* 54 (2013), 15-27.

[15] Gezer A., Cengiz N., Salimov A., On integrability of Golden Riemannian structures, *Turkish J. Math.*, 37 (2013), 693-703.

[16] Gezer A., Karaman C., Golden-Hessian structures, *Proceedings of the National Academy of Sciences*, India Section A: Physical Sciences, 86 (1) (2016), 41-46, .

[17] Gönül S., Küpeli Erken I., Yazla A. and Murathan C., A Neutral relation between metallic structure and almost quadratic φ-structure, *Turkish Journal of Mathematics*, 43 (2019), 268-278.

[18] Kenmotsu K., A class of almost contact Riemannian manifolds, *J. Tohoku Math.*, 24 (1972), 93-103.

[19] Marrero J. C., The local structure of trans-Sasakian manifolds, *Annali di Matematica Pura ed Applicata* 162(1) (1992), 77-86.

[20] Oubiña, J.A., New classes of almost contact metric structures, *Publicationes Mathematicae*, Debrecen, 32 (1985), 187-193.

[21] Sharfuddin, A., Husain, S. I., Almost contact structures induced by conformal transformation, *Publ. Inst. Math. Nouv.*, 32 (46) (1982), 155-159.

[22] Tanno S., The topology of contact Riemannian manifolds, *Illinois J. Math.*, 12 (1968), 700-717.

[23] Yano, K., Kon, M., Structures on Manifolds, *Series in Pure Math.*, Vol 3, World Sci.,1984.

https://communications.science.ankara.edu.tr

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A SUBCLASS OF PSEUDO-TYPE MEROMORPHIC BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, a new subclass of pseudo-type meromorphic bi-univalent functions is defined on $\triangle = \{z \mid : z \in C \text{ and } 1 < |z| < \infty\}$, we derive estimates on the initial coefficient $|b_0|$, $|b_1|$ and $|b_2|$. Relevant connections of the new results with various well-known results are indicated.

1. Introduction

Let A denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit open disk $U = \{z : z \in C, |z| < 1\}$. Also, let the class of univalent and normalized analytic function in the unit disc U be denoted by S with the normalization conditions

$$f(0) = 0 = f^{'}(0) - 1.$$

Furthermore, bi-univalency concept is extended to the class of meromorphic functions defined on $\triangle = \{z : z \in C, 1 < |z| < \infty\}$. For this aim, let Σ denote the class of meromorphic univalent functions g of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{2}$$

defined on the domain $\triangle$. It is well known that every function $g \in \Sigma$ has an inverse $g^{-1} = h$, defined by

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$$g^{-1}(g(z)) = z \quad (z \in \triangle),$$

and

$$g^{-1}(g(w)) = w \quad (M < |w| < \infty,\ M > 0),$$

where

$$g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \frac{b_1}{w} - \frac{b_1 b_0 + b_2}{w^2} - \frac{b_1^2 + b_1 b_0^2 + 2b_0 b_2 + b_3}{w^3} + \dots . \tag{3}$$

A simple computation shows that

$$\begin{aligned} w = g(h(w)) &= (b_0 + B_0) + w + \frac{b_1 + B_1}{w} + \frac{B_2 - b_1 B_0 + b_2}{w^2} \\ &+ \frac{B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3}{w^3} + \dots . \end{aligned} \tag{4}$$

Comparing the initial coefficients in (4), we find that

$$\begin{aligned} b_0 + B_0 = 0 \quad &\Rightarrow \quad B_0 = -b_0 \\ b_1 + B_1 = 0 \quad &\Rightarrow \quad B_1 = -b_1 \\ B_2 - b_1 B_0 + b_2 = 0 \quad &\Rightarrow B_2 = -(b_2 + b_1 b_0) \\ B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3 = 0 \quad &\Rightarrow B_3 = -(b_3 + 2b_0 b_1 + b_1 b_0^2 + b_1^2). \end{aligned}$$

A function $f \in \Sigma$ is said to be meromorphic bi-univalent if $f^{-1} \in \Sigma$. The family of all meromorphic bi-univalent functions is denoted by Σ'. Estimates on the coefficient of meromorphic univalent functions were investigated by some researchers recently; for example, Schiffer [11] obtained the estimate $|b_2| < \frac{3}{2}$ for meromorphic univalent functions $f \in S$ with $b_0 = 0$. Also, Duren [12] obtained the inequality $|b_2| < \frac{2}{n+1}$ for $f \in S$ with $b_k = 0, 1 \le k \le \frac{n}{2}$. Springer [8] used variational methods to prove that proved that

$$|B_3| < 1 \text{ and } |B_3 + \frac{1}{2}B_1^2| < \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \le \frac{(2n-2)!}{n!(n-1)!} \ (n = 1, 2, ...).$$

Later on, Kubota [16] has proved that the Springer conjecture is true for n = 3;4;5. Furthermore Schober [7] obtained sharp bounds for $|B_{2n-1}|$ if $1 \le n \le 7$. Recently. Kapoor and Mishra [5] found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order α in U^*.

Recently, some several researchers such as (see [1], [2], [3], [4], [6], [9], [13] [14]) introduced new subclasses of meromorphically bi-univalent functions and obtained estimates on the initial coefficients for functions belonging to these subclasses.

In 2013, Babalola [10] defined a new subclass λ-pseudo starlike function of order $0 \leq \beta < 1$ satisfying the analytic condition

$$\Re\left\{\frac{z(f(z)'^{\lambda}}{f(z)}\right\} > \beta \ (\lambda \geq 1, \ z \in U). \tag{5}$$

In particular, Babalola [10] proved that all λ-pseudo-starlike functions are Bazilevic of type $1 - \frac{1}{\lambda}$ and order $\beta^{\frac{1}{\lambda}}$ and are univalent in open unit disk U.

Motivated by the earlier work of ([9], [15]), in the present paper, we introduce a new subclasses of the class Σ' and the estimates for the coefficients $|b_0|, |b_1|$ and $|b_2|$ are investigated. Some new consequences of the new results are also pointed out.

2. Coefficient Bounds for the Function Class $\Sigma'_{h,p}(\lambda, \mu)$

We begin by introducing the function class $\Sigma'_{h,p}(\lambda, \mu)$ by means of the following definition.

Definition 2.1. Let the functions $h; p : \triangle \to C$ be analytic functions and

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots, \quad p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots,$$

such that

$$\min\{\Re(h(z)), \Re(p(z))\} > 0, z \in \triangle.$$

A function $g(z) \in \Sigma'$ given by (2) is said to be in the class $\Sigma'_{h,p}(\lambda, \mu)$ if the following conditions are satisfied:

$$g \in \ \Sigma' \text{and} \ \ 1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda\left(\frac{z(g(z)'^{\mu}}{g(z)}\right) - 1\right] \in h(\triangle),$$
$$(0 < \lambda \leq 1, \ \mu \geq 1, \ z \in \triangle), \tag{6}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda\left(\frac{w(h(w)'^{\mu}}{h(w)}\right) - 1 \in p(\triangle)\right]$$
$$(0 < \lambda \leq 1, \ \mu \geq 1, \ w \in \triangle), \tag{7}$$

where $g \in \ \Sigma'$ and $\gamma \in C \setminus \{0\}$ and the function h is given by (3).

Remark 2.1. There are many choices of h and p which would provide interesting subclasses of class $\Sigma'_{h,p}(\lambda, \mu)$.

(1) If we take

$$h(z) = p(z) = \left(\frac{1 + \frac{1}{z}}{1 - \frac{1}{z}}\right)^{\alpha} = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \cdots, \ (0 < \alpha \leq 1, \ z \in \triangle).$$

So it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f \in \Sigma'_\alpha(\lambda,\mu)$. Then

$$\left|\arg\left(1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu}+\lambda\left(\frac{z(g(z)'^{\mu}}{g(z)}\right)-1\right]\right)\right|<\frac{\alpha\pi}{2}$$

$$(0<\lambda\leq 1,\ 0<\alpha\leq 1,\ \mu\geq 1,\ z\in\triangle),$$

and

$$\left|\arg\left(1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu}+\beta\left(\frac{w(h(w)'^{\mu}}{h(w)}\right)-1\right]\right)\right|<\frac{\alpha\pi}{2}$$

$$(0<\lambda\leq 1,\ 0<\alpha\leq 1,\ \mu\geq 1,\ w\in\triangle),$$

where $g(z)\in\Sigma'$ and $\gamma\in C\setminus\{0\}$ and the function h is given by (3).

(2) If we take

$$h(z)=p(z)=\frac{1+\frac{1-2\beta}{z}}{1-\frac{1}{z}}=1+\frac{2(1-\beta)}{z}+\frac{2(1-\beta)}{z^2},\ (0\leq\beta<1,\ z\in\triangle).$$

So it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f\in\Sigma'_\beta(\lambda,\mu)$. Then

$$\Re\left(1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu}+\lambda\left(\frac{z(g(z)'^{\mu}}{g(z)}\right)-1\right]\right)>\beta$$

$$(0<\lambda\leq 1,\ 0\leq\beta<1,\ \mu\geq 1,\ z\in\triangle),$$

and

$$\Re\left(1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu}+\beta\left(\frac{w(h(w)'^{\mu}}{h(w)}\right)-1\right]\right)>\beta$$

$$(0<\lambda\leq 1,\ 0\leq\beta<1,\ \mu\geq 1,\ w\in\triangle),$$

where $g\in\ \Sigma'$ and $\gamma\in C\setminus\{0\}$ and the function h is given by (3).

Theorem 2.1. *Let $g(z)$ be given by (2) be in the class $\Sigma'_\alpha(\lambda,\mu)$. Then*

$$|b_0|\leq\min\left\{\sqrt{\frac{|\gamma|^2(|h_1|^2+|p_1|^2)}{2(\mu-\lambda\mu-\lambda)^2}},\sqrt{\frac{|\gamma|(|h_2|+|p_2|)}{|\mu(\mu-1)(1-\lambda)+2\lambda|}}\right\}\tag{8}$$

and

$$|b_1|\leq\min\{\frac{|\gamma|(|h_2|+|p_2|)}{|2(\mu(\mu-1)(1-\lambda)+2\lambda)|},$$

$$\frac{|\gamma|}{|(\mu-\lambda-2\lambda\mu)|}\left(\sqrt{\frac{|h_2|^2+|p_2|^2}{2}+\frac{[\mu(\mu-1)(1-\lambda)+2\lambda]^2[h_1^2+p_1^2]^2}{16(\mu-\lambda\mu-\lambda)^2}}\right)\},\tag{9}$$

and

$$|b_2|\quad\leq\quad\frac{|\gamma|}{2|(\mu-\lambda-3\lambda\mu)|}\left[\frac{(\mu(\mu-1)(\mu-2)(1-\lambda)-6\lambda)\gamma^2|p_1|^3}{3|(\mu-\lambda\mu-\lambda)^3|}\right.$$

$$+\frac{2\mu(\mu-1)(1-\lambda)+8\mu\lambda-2\mu+6\lambda}{2\mu(\mu-1)(1-\lambda)-(1-\lambda)\mu+5\lambda+4\lambda\mu}|h_3| + \frac{2\mu(\mu-1)(1-\lambda)+2\mu\lambda+4\lambda}{2\mu(\mu-1)(1-\lambda)-(1-\lambda)\mu+5\lambda+4\lambda\mu}|p_3|\Bigg] \tag{10}$$

Proof. Let $g \in \Sigma'_\alpha(\lambda, \mu)$. Then, by Definition 2.1 of meromorphically bi-univalent function class $\Sigma'_\alpha(\lambda, \mu)$, the conditions (6) and (7) can be rewritten as follows:

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu}+\lambda\left(\frac{z(g(z)'^{\mu}}{g(z)}\right)-1\right]=h(z) \quad (z\in\triangle) \tag{11}$$

and

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu}+\beta\left(\frac{w(h(w)'^{\mu}}{h(w)}\right)-1\right]=p(w), \quad (w\in\triangle) \tag{12}$$

respectively. Here, and in what follows,the functions $h(z) \in P$ and $p(w) \in P$ have the following forms:

$$h(z)=1+\frac{p_1}{z}+\frac{p_2}{z^2}+\frac{p_3}{z^3}+\cdots \quad (z\in\triangle) \tag{13}$$

and

$$p(w)=1+\frac{q_1}{w}+\frac{q_2}{w^2}+\frac{q_3}{w^3}+\cdots \quad (w\in\triangle) \tag{14}$$

upon substituting from (13) and (14) into (11) and (12), respectively, and equating the coefficients, we get

$$\frac{(\mu-\lambda\mu-\lambda)}{\gamma}b_0=h_1 \tag{15}$$

$$\frac{1}{2\gamma}\left[(\mu(\mu-1)(1-\lambda)+2\lambda)b_0^2+2(\mu-\lambda-2\lambda\mu)b_1\right]=h_2 \tag{16}$$

$$\frac{1}{6\gamma}[\mu(\mu-1)(\mu-2)(1-\lambda)-\lambda]b_0^3+\frac{1}{\gamma}[\mu(\mu-1)(1-\lambda)+2\lambda+\lambda\mu]b_0b_1 + \frac{1}{\gamma}[\mu-\lambda-3\mu\lambda]b_2=h_3 \tag{17}$$

$$-\frac{(\mu-\lambda\mu-\lambda)}{\gamma}b_0=p_1 \tag{18}$$

$$\frac{1}{2\gamma}\left[(\mu(\mu-1)(1-\lambda)+2\lambda)b_0^2+2(\lambda-\mu+2\lambda\mu)b_1\right]=p_2 \tag{19}$$

and

$$\frac{1}{6\gamma}[6\lambda-(\mu(\mu-1)(\mu-2)(1-\lambda))b_0^3+6(\mu(\mu-1)(1-\lambda) - \mu(1-\lambda)+3\lambda+3\lambda\mu)b_0b_1+6(\lambda-\mu+3\mu\lambda)b_2]=p_3. \tag{20}$$

From (15) and (18), we find that

$$h_1=-q_1 \tag{21}$$

and

$$2(\mu-\lambda\mu-\lambda)^2b_0^2=\gamma^2(h_1^2+p_1^2) \tag{22}$$

that is,
$$|b_0|^2 \leq \frac{|\gamma|^2(|h_1|^2+|p_1|^2)}{2(\mu-\lambda\mu-\lambda)^2}. \tag{23}$$
Adding (16) and (19), we get
$$[(\mu(\mu-1)(1-\lambda)+2\lambda)]\, b_0^2 = \gamma(h_2+p_2) \tag{24}$$
that is,
$$|b_0|^2 \leq \frac{|\gamma|(|h_2|+|p_2|)}{|\mu(\mu-1)(1-\lambda)+2\lambda|}. \tag{25}$$
From (23) and (25) we get the desired estimate on the coefficient $|b_0|$ as asserted in (8).

Next, in order to find the bound on $|b_0|$, by subtracting the equation (16) from the equation (19), we get
$$2(\mu(\mu-1)(1-\lambda)+2\lambda)b_1 = \gamma(h_2-p_2), \tag{26}$$
that is,
$$|b_1| \leq \frac{|\gamma|(|h_2|+|p_2|)}{|2(\mu(\mu-1)(1-\lambda)+2\lambda)|}. \tag{27}$$
By squaring and adding (16) and (19), using (22) in the computation leads to
$$b_1^2 = \frac{\gamma^2}{(\mu-\lambda-2\lambda\mu)^2}\left(\frac{h_2^2+p_2^2}{2} - \frac{[\mu(\mu-1)(1-\lambda)+2\lambda]^2[h_1^2+p_1^2]^2}{16(\mu-\lambda\mu-\lambda)^2}\right). \tag{28}$$
that is,
$$|b_1| \leq \frac{|\gamma|}{|(\mu-\lambda-2\lambda\mu)|}\left(\sqrt{\frac{|h_2|^2+|p_2|^2}{2} + \frac{[\mu(\mu-1)(1-\lambda)+2\lambda]^2[h_1^2+p_1^2]^2}{16(\mu-\lambda\mu-\lambda)^2}}\right). \tag{29}$$
From (26) and (28) we get the desired estimate on the coefficient $|b_1|$ as asserted in (9).

In order to find the estimate $|b_2|$, consider the sum of (17) and (20), we have
$$b_0b_1 = \frac{\gamma(h_3+p_3)}{2\mu(\mu-1)(1-\lambda)-(1-\lambda)\mu+5\lambda+4\lambda\mu}. \tag{30}$$
Subtracting (20) from (17) with $h_1 = -p_1$, we obtain
$$\frac{2(\mu-\lambda-3\lambda\mu)b_2}{\gamma} = h_3 - p_3 - \frac{(\mu-\lambda-3\lambda\mu)b_0b_1}{\gamma} - \frac{[\mu(\mu-1)(\mu-2)(1-\lambda)-6\lambda]b_0^3}{3\gamma}. \tag{31}$$
Using (21) and (30) in (31) give to
$$\begin{aligned} b_2 \quad = \quad & \frac{\gamma}{2(\mu-\lambda-3\lambda\mu)}\Bigg[\frac{(\mu(\mu-1)(\mu-2)(1-\lambda)-6\lambda)\gamma^2 p_1^3}{3(\mu-\lambda\mu-\lambda)^3} \\ & + \frac{2\mu(\mu-1)(1-\lambda)+8\mu\lambda-2\mu+6\lambda}{2\mu(\mu-1)(1-\lambda)-(1-\lambda)\mu+5\lambda+4\lambda\mu}h_3 \end{aligned}$$

$$-\frac{2\mu(\mu-1)(1-\lambda)+2\mu\lambda+4\lambda}{2\mu(\mu-1)(1-\lambda)-(1-\lambda)\mu+5\lambda+4\lambda\mu}p_3\Bigg]$$

This evidently completes the proof of Theorem 2.1. □

If we take $\lambda = 1$ in Theorem 2.1, we get the following Corollary.

Corollary 2.2. *Let $g(z)$ be given by (1.2) be in the class $\Sigma'_{\lambda,\beta}(\alpha)$. Then*

$$|b_0| \leq \min\left\{\sqrt{\frac{|\gamma|^2(|h_1|^2+|p_1|^2)}{2}}, \sqrt{\frac{|\gamma|(|h_2|+|p_2|)}{2}}\right\}, \tag{32}$$

$$|b_1| \leq \min\left\{\frac{|\gamma|(|h_2|+|p_2|)}{4}, \frac{|\gamma|}{|\mu+1|}\left(\sqrt{\frac{|h_2|^2+|p_2|^2}{2}+\frac{(|h_1|^2+|p_1|^2)^2}{4}}\right)\right\}. \tag{33}$$

and

$$|b_2| \leq \frac{|\gamma|}{2|(2\mu+1)|} \times \left[2\gamma^2|p_1|^3 + \frac{6(\mu+1)}{5+4\mu}|h_3| + \frac{2(\mu+2)}{5+4\mu}|p_3|\right]. \tag{34}$$

If we take

$$h(z) = p(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\alpha} = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \cdots, \ (0 < \alpha \leq 1, \ z \in \triangle),$$

and

$$h(z) = p(z) = \frac{1+\frac{1-2\beta}{z}}{1-\frac{1}{z}} = 1 + \frac{2(1-\mu)}{z} + \frac{2(1-\mu)}{z^2}, \ (0 < \mu \leq 1, \ z \in \triangle),$$

respectively, in the Theorem 2.1, we obtain the following results which is an improvement of estimates obtained by Srivastava et. at [9].

Corollary 2.3. *Let $g(z)$ be given by (2) be in the class $\Sigma'_{\lambda,\beta}(\alpha)$. Then*

$$|b_0| \leq 2\alpha \tag{35}$$

and

$$|b_1| \leq \frac{2\sqrt{5}\alpha^2}{\lambda+1}. \tag{36}$$

Corollary 2.4. *Let $g(z)$ be given by (2) be in the class $\Sigma'_{\lambda,\beta}(\mu)$. Then*

$$|b_0| \leq 2(1-\mu) \tag{37}$$

and

$$|b_1| \leq \frac{2(1-\mu)\sqrt{4\mu^2-8\mu+5}}{\lambda+1}. \tag{38}$$

Remark 2.2. For function $g \in \Sigma'_{h,p}(\lambda,\mu)$ given by (2) by taking $p(z) = h(z) = \frac{1+Az}{1+Bz}$ $-1 \leq B < A \leq 1$), we obtain the initial coefficient estimates $|b_0|$, $|b_1|$, and $|b_2|$ which leads to the results discussed in Theorem 2.2 of [15].

References

[1] Alamoush, A. G., Darus, M., Faber polynomial Coefficients estimates for a new subclass of meromorphic bi-univalent functions, *Advances in Inequalities and Applications*, 2016:3 (2016).

[2] Deniz, E., Certain subclasses of bi univalent functions satisfying subordinate conditions, *Journal of Classical Analysis,* 2(1) (2013), 49–60.

[3] Deniz, E., Yolcu, H. T., Faber polynomial coefficients for meromorphic bi-subordinate functions of complex order, *AIMS Mathematics,* 5(1) (2020), 640–649.

[4] Deniz, E., Jahangiri, J. M., Kina, S. K., Hamidi, S. G., Faber polynomial coefficients for generalized bi-subordinate functions of complex order, *Journal of Mathematical Inequalities,* 12(3) (2018), 645–653.

[5] Kapoor, G. P., Mishra, A. K., Coefficients estimates for inverses of starlike functions of positive order, J*ournal of Mathematical Analysis and Applications,* 329(2) (2007), 922–934.

[6] Çağlar, M., Deniz, E., Initial coefficients for a subclass of bi-univalent functions defined by Salagean differential operator, *Communications Faculty of Sciences University of Ankara Series A1-Mathematics and Statistics,* 66(1) (2017), 85-91.

[7] Schober, G., Coefficients of inverses of meromorphic univalent functions, Proceedings of the American Mathematical Society 67(1) (1977), 111-116.

[8] Springer, G., The Coefficients problem for schlicht mappings of the exterior of the unit circle, *Transactions of the American Mathematical Society,* 70 (1951), 421–450.

[9] Srivastava, H.M., Joshi, B.S., Joshi, S.S., Pawar, H., Coefficient estimates for certain subclasses of meromorphically bi-univalent functions, *Palestine Journal of Mathematics,* 5 (2016), 250–258.

[10] Babalola, K. O., On λ-pseudo-starlike functions, *Journal of Classical Analysis,* 3 (2013), 137–147.

[11] Schiffer, M., On an extremum problem of conformal representation, *Bulletin de la Socit Mathmatique de France,* 66 (1938), 48–55.

[12] Duren, P. L., Coefficients of meromorphic schlicht functions, *Proceedings of the American Mathematical Society,* 28 (1971), 169–172.

[13] Hamidi, S. G., Halim, S. A., Jahangiri, J. M., Coefficients estimates for a class of meromorphic bi-univalent functions, *Comptes Rendus Mathematique,* 351 (2013), 349–352.

[14] Hamidi, S. G., Janani, T., Murugusundaramoorthy, G., Jahangiri, J.M., Coefficient estimates for certain classes of meromorphic bi-univalent functions, *Comptes Rendus Mathematique,* 352 (2014), 277–282.

[15] Janani, T., Murugusundaramoorthy, G., Vijaya, K., New subclass of pseudo-type meromorphic bi-univalent functions of complex order, *Novi Sad Journal of Mathematics,* 48(1) (2018), 93–102.

[16] Kubota, Y., Coefficients of meromorphic univalent functions, *Kodai Mathematical Seminar Reports,* 28(2-3) (1977), 253–261.

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ON SOME PROPERTIES OF INTUITIONISTIC FUZZY SOFT BOUNDARY

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Abstract. The purpose of this paper is to initiate the concept of Intuitionistic Fuzzy(IF) soft boundary. We discuss and explore the characterizations and properties of IF soft boundary in general as well as in terms of IF soft interior and IF soft closure. Examples and counter examples are also presented to validate the discussed results.

1. Introduction

The notion of fuzzy sets was introduced by Zadeh [23]. After that several researches were conducted on the generalizations of the notion of fuzzy set. As a generalization of the notion of fuzzy set, intuitionistic fuzzy set (IFS) and intuitionistic L-fuzzy sets (ILFS) were initiated and explored by Atanassov [1-3] and [5].

In our daily life situations, we usually face complicated problems in different fields like economics, engineering, medical sciences, social sciences, etc. involving imprecise and uncertain data in nature. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all of these have their advantages as well as inherent limitations in dealing with uncertainties. One major problem shared by those theories is their incompatibility with the parameterization tools. To overcome these difficulties, Molodtsov [19] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the

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usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. This theory has proven useful in many different fields such as decision making [6][20], data analysis [4][24], forecasting [21] and so on. The topological structures of soft sets are studied and discussed in [7-13].

Maji et al. introduced the concept of intuitionistic fuzzy soft sets[16-18], which is a generalization of fuzzy soft sets[15] and standard soft sets. It is to be noted that the parameters may not always be crisp, rather they may be intuitionistic fuzzy in nature. The problems of object recognition have received paramount importance in recent years. The recognition problem may be viewed as multiobserver decision making problem, where the final identification of the object is based on the set of inputs from different observers who provide the overall object characterization in terms of diverse set of parameters. Different algebraic structures of IF soft sets are studied and explored in [22]. D. Coker [5] introduced and studied the concept of IF topological spaces. Z. Li et.al [14] initiated IF topological structures of IF soft sets. They explored the notions of IF soft open(closed) sets, IF soft interior(closure) and IF soft base in IF soft topological spaces.

In this paper, we initiate the concept of IF soft boundary. We discuss and explore the characterizations and properties of IF soft boundary in general as well as in terms of IF soft interior and IF soft closure. Examples and counter examples are also presented to validate the discussed results.

2. Preliminaries

First we recall some definitions and results which will use in the sequel.

Definition 1. [23] *A fuzzy set f on X is a mapping $f : X \to I = [0, 1]$. The value $f(x)$ represents the degree of membership of $x \in X$ in the fuzzy set f, for $x \in X$.*

Definition 2. [19] *Let X be an initial universe and E be a set of parameters. Let $P(X)$ denotes the power set of X and A be a non-empty subset of E. A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).*

Definition 3. [15] *Let I^X denotes the set of all fuzzy sets on X and $A \subseteq X$. A pair (f, A) is called a fuzzy soft set over X, where $f : X \to I^X$ is a function. That is, for each $a \in A$, $f(a) = f_a : X \to I$ is a fuzzy set on X .*

Definition 4. [2] *An intuitionistic fuzzy set A over the universe X is defined as:*
$A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$,

where $\mu_A : X \to [0, 1]$, $\nu_A : X \to [0, 1]$ with the property that $0 \le \mu_A(x) + \nu_A(x) \le 1$, for all $x \in X$. The values $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and nonmembership of x to A respectively .

Definition 5. [2] *Let* $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$ *and* $B = \{(x, \mu_B(x), \nu_B(x)); x \in X\}$ *are intuitionistic fuzzy set over the universe* X*. Then*
(1) $A^c = \{(x, \nu_A(x), \mu_A(x)); x \in X\}$.
(2) $A \subseteq B$ *if and only if* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$*, for all* $x \in X$.
(3)$A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$.
(4) $A \cap B = \{(x, min\{\mu_A(x), \mu_B(x)\}, max\{\nu_A(x), \nu_B(x)\} : x \in X\}$.
(5) $A \cup B = \{(x, max\{\mu_A(x), \mu_B(x)\}, min\{\nu_A(x), \nu_B(x)\} : x \in X\}$.

Definition 6. [2] *An intuitionistic fuzzy set* A *over the universe* X *is said to be intuitionistic fuzzy null set denoted as* $\tilde{0}$*, and is defined as:* $A = \{(x, 0, 1) : x \in X\}$.

Definition 7. [2] *An intuitionistic fuzzy set* A *over the universe* X *is said to be intuitionistic fuzzy absolute set denoted as* $\tilde{1}$*, and is defined as:* $A = \{(x, 1, 0) : x \in X\}$.

Definition 8. [17] *Let* X *be the initial universal set and* E *be the set of parameters. Let* IF^X *denotes the set of all intuitionistic fuzzy soft sets on* X *and* $A \subseteq X$*. A pair* (IF, A) *is called a IF fuzzy soft set over* X*, where* $f : A \to IF^X$ *is a function. That is, for each* $a \in A$*,* $f(a) = f_a : A \to IF^X$*, is an intuitionistic fuzzy set on* X *and is defined as:* $F(a) = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$.

From now on, for our convenience, we will represent the intuitionistic fuzzy soft set (IF, A) as IF soft set f_A. Now we give the example of intuitionistic fuzzy soft sets as:

Example 9. *Let* $(IF, A) = f_A$ *describe the character of the employees with respect to the given parameters, for finding the best employee of the financial year. Let the set of employees under consideration be* $X = \{x_1, x_2, x_3, x_4\}$*. Let* $E = \{$ *regular workload (r), conduct (c), field performances (g), sincerity(s), pleasing personality (p)*$\}$ *be the set of parameters framed to choose the best employee. Suppose the administrator Mr. X has the parameter set* $A = \{r, c, p\} \subseteq E$ *to choose the best employee. Then* f_A *be the IF soft set over* X*, defined as follows:*
$f(r)(x_1) = (0.8, 0.1)$, $f(r)(x_2) = (0.7, 0.5)$, $f(r)(x_3) = (0.9, 0.1)$, $f(r)(x_4) = (0.7, 0.2)$
$f(c)(x_1) = (0.6, 0.2)$, $f(c)(x_2) = (0.7, 0.1)$, $f(c)(x_3) = (0.5, 0.3)$, $f(c)(x_4) = (0.3, 0.6)$
$f(p)(x_1) = (0.6, 0.2)$, $f(p)(x_2) = (0.7, 0.1)$, $f(p)(x_3) = (0.5, 0.3)$, $f(p)(x_4) = (0.3, 0.6)$
That is,
$f_A = (IF, A) \tilde{\tilde{=}} \{F(r) = \{(x_1, 0.8, 0.1), (x_2, 0.7, 0.05), (x_3, 0.9, 0.1), (x_4, 0.7, 0.2)\},$
$F(c) = \{(x_1, 0.6, 0.2), (x_2, 0.7, 0.1), (x_3, 0.5, 0.3), (x_4, 0.3, 0.6)\},$
$F(p) = \{(x_1, 0.6, 0.2), (x_2, 0.7, 0.1), (x_3, 0.5, 0.3), (x_4, 0.3, 0.6)\}\}.$
The tabular representation of IF soft sets f_A *is:*

	x_1	x_2	x_3	x_4
r	(0.8, 0.1)	(0.7, 0.5)	(0.9, 0.1)	(0.7, 0.2)
c	(0.6, 0.2)	(0.7, 0.1)	(0.5, 0.3)	(0.3, 0.6)
p	(0.6, 0.2)	(0.7, 0.1)	(0.5, 0.3)	(0.3, 0.6)

In short, we will represent f_A as:
$f_A \tilde{\tilde{=}} \{\{x_{(0.8,0.1)}, x_{(0.7,0.05)}, x_{(0.9,0.1)}, x_{(0.7,0.2)}\}, \{x_{(0.6,0.2)}, x_{(0.7,0.1)}, x_{(0.5,0.3)}, x_{(0.3,0.6)}\}, \{x_{(0.6,0.2)}, x_{(0.7,0.1)}, x_{(0.5,0.3)}, x_{(,0.3,0.6)}\}\}$.

Definition 10. [17] *Two IF soft sets f_A and g_B over a common universe X, we say that f_A is a IF soft subset of g_B, if*
(1) $A \subseteq B$ and
(2) for all $a \in A$, $f_a \leq g_a$; implies f_a is a IF subset of g_a.
We denote it by $f_A \tilde{\tilde{\subseteq}} g_A$. f_A is said to be a IF soft super set of g_B, if g_B is a IF soft subset of f_A. We denote it by $f_A \tilde{\tilde{\supseteq}} g_B$.
Note that two IF soft sets f_A and g_B over a common universe X are said to be IF soft equal, if f_A is a IF soft subset of g_B and g_B is a If soft subset of f_A.

Definition 11. [17] *The union of two IF soft sets f_A and g_B over the common universe X is the IF soft set h_C, where $C = A \cup B$ and for all $c \in C$,*

$$h(c) = \begin{cases} f(c), & \text{if } c \in A - B \\ g(c), & \text{if } c \in B - A \\ f(c) \cup g(c), & \text{if } c \in A \cap B \end{cases}$$

We write $f_A \tilde{\tilde{\cup}} g_B = h_C$.

Definition 12. [17] *The intersection h_C of two IF soft sets f_A and g_B over a common universe X, denoted $f_A \tilde{\tilde{\cap}} g_B$, is defined as $C = A \cap B$, and $h(c) = f(c) \cap g(c)$, for all $c \in C$.*

Definition 13. [17] *The relative complement of a IF soft set f_A is the fuzzy soft set f_A^c, which is denoted by $(f_A)^c$ and where $f^c : A \to IF(X)$ is a IF set-valued function that is, for each $x \in A$, $f^c(A)$ is a IF set in X, whose membership function $f_a^c(x) = (f_a(x))^c$, for all $x \in A$. Here f_a^c is the membership function of $f^c(a)$.*

Definition 14. [14] *Let τ be the collection of IF soft sets over X, then τ is said to be a IF soft topology on X, if*
(1) $\tilde{\tilde{\Phi}}_A$, $\tilde{\tilde{X}}_A$ belong to τ.
(2) If $(f_A)_i \in \tau$, for all $i \in I$, then $\tilde{\tilde{\cup}}_{i \in I}(f_A)_i \in \tau$.
(3) For $f_a, g_b \in \tau$ implies that $f_a \tilde{\tilde{\cap}} g_b \in \tau$.
The triplet (X, τ, A) is called an IF soft topological space over X. Every member of τ is called IF soft open set. A IF soft set is called IF soft closed if and only if its complement is IF soft open.

Example 15. *Let $X = \{x_1, x_2\}$, $A = \{e_1, e_2\}$ and $\tau = \{\widetilde{\widetilde{\Phi_A}}, \widetilde{\widetilde{X_A}}, f_A, g_A, h_A, k_A\}$, where f_A, g_A, h_A, k_A are IF soft sets over X, defined as follows*

$f(e_1)(x_1) = (0.2, 0.8)$, $f(e_1)(x_2) = (0.6, 0.3)$,
$f(e_2)(x_1) = (0.2, 0.5)$, $f(e_2)(x_2) = (0.9, 0.1)$,
$g(e_1)(x_1) = (0.1, 0.8)$, $g(e_1)(x_2) = (0.6, 0.1)$,
$g(e_2)(x_1) = (0.2, 0.8)$, $g(e_2)(x_2) = (0.8, 0.1)$,
$h(e_1)(x_1) = (0.2, 0.8)$, $h(e_1)(x_2) = (0.6, 0.1)$,
$h(e_2)(x_1) = (0.2, 0.5)$, $h(e_2)(x_2) = (0.9, 0.1)$,
$k(e_1)(x_1) = (0.1, 0.8)$, $k(e_1)(x_2) = (0.6, 0.3)$,
$k(e_2)(x_1) = (0.2, 0.8)$, $k(e_2)(x_2) = (0.8, 0.1)$,

Then $\tau = \{\widetilde{\widetilde{\Phi}}_A, \widetilde{\widetilde{X}}_A, (\{x_{(0.2,0.8)}, x_{(0.6,0.3)}\}, \{x_{(0.2,0.5)}, x_{(0.9,0.1)}\}),$ $(\{x_{(0.1,0.8)}, x_{(0.6,0.1)}\}, \{x_{(0.2,0.8)}, x_{(0.8,0.1)}\}), (\{x_{(0.2,0.8)}, x_{(0.6,0.1)}\}, \{x_{(0.2,0.5)}, x_{(0.9,0.1)}\}),$ $(\{x_{(0.1,0.8)}, x_{(0.6,0.3)}\}, \{x_{(0.2,0.8)}, x_{(0.8,0.1)}\})\}$ *is an IF soft topology on* X *and hence* (X, τ, A) *is an IF soft topological space over* X.

Definition 16. [14] *Let* τ *be the collection of IF soft sets over* X. *Then*
(1) $\tilde{\tilde{\Phi}}_A$, $\tilde{\tilde{X}}_A$ *are IF soft closed sets over* X.
(2) The intersection of any number of IF soft closed sets is an IF soft closed set over X.
(3) The union of any two IF soft closed sets is an IF soft closed set over X.

Definition 17. [14] *Let* (X, τ, A) *be an IF soft topological space over* X *and* f_A *be an IF soft set over* X. *Then IF soft interior of IF soft set* f_A *over* X *is denoted by* $int(f_A)$ *and is defined as the union of all IF soft open sets contained in* f_A. *Thus* $int(f_A)$ *is the largest IF soft open set contained in* f_A.

Definition 18. [14] *Let* (X, τ, A) *be an IF soft topological space over* X *and* f_A *be an IF soft set over* X. *Then the IF soft closure of* f_A, *denoted by* $cl(f_A)$ *is the intersection of all IF soft closed super sets of* f_A. *Clearly* $cl(f_A)$ *is the smallest IF soft closed set over* X *which contains* f_A.

3. Properties of Intuitionistic Fuzz Soft Boundary

Definition 19. [14] *An IF soft set* f_A *over* X *is said to be a null IF soft set and is denoted by* $\tilde{\tilde{\phi}}$ *if and only if, for each* $e \in A$, $f_A(e) = \tilde{\tilde{0}}$, *where* $\tilde{\tilde{0}}$ *is the membership function of null IF set over* X, *which takes value* $(0, 1)$, *for all* $x \in X$.

Definition 20. [14] *An IF soft set* f_A *over* X *is said to be an absolute IF soft set and is denoted by* $\tilde{\tilde{X}}$ *if and only if, for each* $e \in A$, $f_A(e) = \tilde{\tilde{1}}$, *where* $\tilde{\tilde{1}}$ *is the membership function of absolute IF set over* X, *which takes value* $(1, 0)$, *for all* $x \in X$.

Now we define:

Definition 21. *The difference* h_C *of two IF soft sets* f_A *and* g_B *over* X, *denoted by* $f_A \tilde{\tilde{\setminus}} g_A$, *is defined as* $f_A \tilde{\tilde{\setminus}} g_B \tilde{\tilde{=}} f_A \tilde{\tilde{\cap}} (g_B)^c$.

Example 22. *Let f_A and g_A be two IF fuzzy soft set defined as:*
$f_A = (\{x_{(0.2,0.8)}, x_{(0.6,0.3)}\}, \{x_{(0.2,0.5)}, x_{(0.9,0.1)}\})$ *and*
$g_A \tilde{\tilde{=}} (\{x_{(0.1,0.8)}, x_{(0.6,0.1)}\}, \{x_{(0.2,0.8)}, x_{(0.8,0.1)}\})$. *Then*
$f_A \tilde{\tilde{\setminus}} g_B \tilde{\tilde{=}} f_A \tilde{\tilde{\cap}} (g_B)^c$
$\tilde{\tilde{=}} (\{x_{(0.2,0.8)}, x_{(0.6,0.3)}\}, \{x_{(0.2,0.5)}, x_{(0.9,0.1)}\}) \tilde{\tilde{\cap}} (\{x_{(0.8,0.1)}, x_{(0.1,0.6)}\}, \{x_{(0.8,0.2)}, x_{(0.1,0.8)}\})$
$\tilde{\tilde{=}} (\{x_{(0.2,0.8)}, x_{(0.1,0.6)}\}, \{x_{(0.2,0.5)}, x_{(0.1,0.8)}\})$.

Definition 23. *Let (X, τ, A) be an IF soft topological space over X and f_A be an IF soft set over X. Then the IF soft boundary of f_A, denoted by $bd(f_A)$ and is defined as, $bd(f_A) \tilde{\tilde{=}} cl(f_A) \tilde{\tilde{\cap}} cl((f_A)^c)$.*

Example 24. *In the above Example 15, the IF soft closed sets are*
$\widetilde{\widetilde{\Phi_A}} \tilde{\tilde{=}} \{x_{(0,1)}, x_{(0,1)}\}, \{x_{(0,1)}, x_{(0,1)}\})$, $\widetilde{\widetilde{X_A}} \tilde{\tilde{=}} (\{x_{(1,0)}, x_{(1,0)}\}, \{x_{(1,0)}, x_{(1,0)}\})$,
$(\{x_{(0.8,0.2)}, x_{(0.3,0.6)}\}, \{x_{(0.5,0.2)}, x_{(0.1,0.9)}\})$, $(\{x_{(0.8,0.1)}, x_{(0.1,0.6)}\}, \{x_{(0.8,0.2)}, x_{(0.1,0.8)}\})$,
$(\{x_{(0.8,0.2)}, x_{(0.1,0.6)}\}, \{x_{(0.5,0.2)}, x_{(0.1,0.9)}\})$, $(\{x_{(0.8,0.1)}, x_{(0.3,0.6)}\}, \{x_{(0.8,0.2)}, x_{(0.1,0.8)}\})$.
Let us take an IF soft set k_A as: $k_A \tilde{\tilde{=}} (\{x_{(0.6,0.3)}, x_{(0.1,0.8)}\}, \{x_{(0.3,0.4)}, x_{(0.1,0.9)}\})$.
Then $cl(k_A) \tilde{\tilde{=}} (\{x_{(0.8,0.2)}, x_{(0.1,0.6)}\}, \{x_{(0.5,0.2)}, x_{(0.1,0.9)}\})$. *Also*
$(k_A)^c \tilde{\tilde{=}} (\{x_{(0.3,0.6)}, x_{(0.8,0.1)}\}, \{x_{(0.4,0.3)}, x_{(0.9,0.1)}\})$ *and* $cl((k_A)^c) \tilde{\tilde{=}} \widetilde{\widetilde{\Phi_A}}$. *Thus,*
$bd(k_A) \tilde{\tilde{=}} cl(k_A) \cap cl((k_A)^c) \tilde{\tilde{=}} \widetilde{\widetilde{\Phi_A}}$.

Theorem 25. *Let f_A be an IF soft set of an IF soft topological space over X. Then the following hold:*
(1) $(bd(f_A))^c \tilde{\tilde{=}} int(f_A) \tilde{\tilde{\cup}} int(f_A^c)$.
(2) $cl(f_A) \tilde{\tilde{=}} int(f_A) \tilde{\tilde{\cup}} bd(f_A)$.
(3) $bd(f_A) \tilde{\tilde{=}} cl(f_A) \tilde{\tilde{\setminus}} int(f_A)$.
(4) $int(f_A) \tilde{\tilde{=}} f_A \tilde{\tilde{\setminus}} bd(f_A)$.
(5) $bd(cl(f_A)) \tilde{\tilde{\subseteq}} bd(f_A)$.
(6) $bd(f_A) \tilde{\tilde{\cap}} int(f_A) \tilde{\tilde{=}} \widetilde{\widetilde{\Phi_A}}$.
(7) $cl(int(f_A)) \tilde{\tilde{=}} f_A \tilde{\tilde{\setminus}} bd(f_A)$.

Proof. (1).

$$\begin{aligned} int(f_A) \tilde{\tilde{\cup}} int(f_A^c) &= ((int(f_A))^c)^c \tilde{\tilde{\cup}} ((int(f_A^c))^c)^c \\ &\tilde{\tilde{=}} [(int(f_A))^c \tilde{\tilde{\cap}} int(f_A^c)^c]^c \\ &\tilde{\tilde{=}} [cl(f_A^c) \tilde{\tilde{\cap}} cl(f_A)]^c \\ &\tilde{\tilde{=}} (bd(f_A))^c. \end{aligned}$$

(2).

$$\begin{array}{rcl} int(f_A)\tilde{\tilde{\cup}}bd(f_A) & \tilde{\tilde{\cong}} & int(f_A)\tilde{\tilde{\cup}}(cl(f_A)\tilde{\tilde{\cap}}cl(f_A^c)) \\ & \tilde{\tilde{\cong}} & [int(f_A)\tilde{\tilde{\cup}}cl(f_A)]\tilde{\tilde{\cap}}[int(f_A)\tilde{\tilde{\cup}}cl(f_A^c)] \\ & \tilde{\tilde{\cong}} & cl(f_A)\tilde{\tilde{\cap}}[int(f_A)\tilde{\tilde{\cup}}(int(f_A))^c] \\ & \tilde{\tilde{\cong}} & cl(f_A)\tilde{\tilde{\cap}}(int(f_A)\tilde{\tilde{\cup}}(int(f_A))^c) \\ & \tilde{\tilde{\cong}} & cl(f_A)\tilde{\tilde{\cap}}\widetilde{\widetilde{X_A}} \\ & \tilde{\tilde{\cong}} & cl(f_A). \end{array}$$

(3).

$$\begin{array}{rcl} bd(f_A) & \tilde{\tilde{\cong}} & cl(f_A)\tilde{\tilde{\cap}}cl(f_A^c) \\ & \tilde{\tilde{\cong}} & cl(f_A)\tilde{\tilde{\cap}}(int(f_A))^c \text{ (by Theorem 4.5(6)[14]).} \\ & \tilde{\tilde{\cong}} & cl(f_A)\tilde{\tilde{\setminus}}int(f_A) \end{array}$$

(4).

$$\begin{array}{rcl} f_A\tilde{\tilde{\setminus}}bd(f_A) & \tilde{\tilde{\cong}} & f_A\tilde{\tilde{\cap}}bd(f_A^c) \\ & \tilde{\tilde{\cong}} & f_A\tilde{\tilde{\cap}}(int(f_A)\tilde{\tilde{\cup}}int(f_A^c)) \text{ (by (1))} \\ & \tilde{\tilde{\cong}} & [f_A\tilde{\tilde{\cap}}int(f_A)]\tilde{\tilde{\cup}}[f_A\tilde{\tilde{\cap}}int(f_A^c)] \\ & \tilde{\tilde{\cong}} & int(f_A)\tilde{\tilde{\cup}}\widetilde{\widetilde{\Phi_A}} \\ & \tilde{\tilde{\cong}} & int(f_A). \end{array}$$

(5).

$$\begin{array}{rcl} bd(cl((f_A)) & \tilde{\tilde{\cong}} & cl(cl(f_A))\tilde{\tilde{\setminus}}int(cl(f_A)) \\ & \tilde{\tilde{\cong}} & cl(f_A)\tilde{\tilde{\setminus}}int(cl(f_A)) \\ & \tilde{\tilde{\subseteq}} & cl(f_A)\tilde{\tilde{\setminus}}int(f_A) \\ & \tilde{\tilde{\cong}} & bd(f_A). \end{array}$$

(6) follows form (3) and (7) follows directly by the definition of an IF soft boundary. □

Remark 26. *By (3) of above Theorem 25, it is clear that* $bd(f_A)\tilde{\tilde{\cong}}bd(f_A^c)$.

Theorem 27. *Let* f_A *be an IF soft set of an IF soft topological space over* X. *Then:*
(1) f_A *is an IF soft open set over* X *if and only if* $f_A\tilde{\tilde{\cap}}bd(f_A)\tilde{\tilde{\cong}}\widetilde{\widetilde{\Phi_A}}$.
(2) f_A *is an IF soft closed set over* X *if and only if* $bd(f_A)\tilde{\tilde{\subseteq}}f_A$.
(3) If g_A *be an IF soft closed(respt. open) set of an IF soft topological space with* $f_A\tilde{\tilde{\subseteq}}g_A$, *then* $bd(f_A)\tilde{\tilde{\subseteq}}g_A$ *(respt.* $bd(f_A)\tilde{\tilde{\subseteq}}(g_A)^c$*).*

Proof. (1). Let f_A be an IF soft open set over X. Then $int(f_A)\tilde{\tilde{\cong}}f_A$ implies $f_A\tilde{\tilde{\cap}}bd(f_A)\tilde{\tilde{\cong}}int(f_A)\tilde{\tilde{\cap}}bd(f_A)\tilde{\tilde{\cong}}\widetilde{\widetilde{\Phi_A}}$.
Conversely, let $f_A\tilde{\tilde{\cap}}bd(f_A)\tilde{\tilde{\cong}}\widetilde{\widetilde{\Phi_A}}$. Then $f_A\tilde{\tilde{\cap}}cl(f_A)\tilde{\tilde{\cap}}cl(f_A^c)\tilde{\tilde{\cong}}\widetilde{\widetilde{\Phi_A}}$ or $f_A\tilde{\tilde{\cap}}cl(f_A^c)\tilde{\tilde{\cong}}\widetilde{\widetilde{\Phi_A}}$ or

$cl(f_A^c)\tilde{\subseteq}f_A^c$, which implies f_A^c is an IF soft closed and hence f_A is an IF soft open set.
(2). Let f_A be an IF soft closed set over X. Then $cl(f_A)\tilde{\tilde{=}}f_A$. Now
$bd(f_A)\tilde{\tilde{=}}cl(f_A)\tilde{\tilde{\cap}}cl(f_A^c)\tilde{\subseteq}cl(f_A)\tilde{\tilde{=}}f_A$. That is, $bd(f_A)\tilde{\tilde{\subseteq}}f_A$.
Conversely, $bd(f_A)\tilde{\tilde{\subseteq}}f_A$. Then $bd(f_A)\tilde{\tilde{\cap}}f_A^c\tilde{\tilde{=}}\widetilde{\widetilde{\Phi_A}}$. Since $bd(f_A)\tilde{\tilde{=}}bd(f_A^c)\tilde{\tilde{=}}\widetilde{\widetilde{\Phi_A}}$, we have $bd(f_A^c)\tilde{\tilde{\cap}}f_A^c\tilde{\tilde{=}}\widetilde{\widetilde{\Phi_A}}$. By (1), f_A^c is IF soft open and hence f_A is IF soft closed.
(3). $f_A\tilde{\tilde{\subseteq}}g_A$ follows that $cl(f_A)\tilde{\tilde{\subseteq}}cl(g_A)$. Since g_A is IF soft closed, then we get, $bd(f_A)\tilde{\tilde{=}}cl(f_A)\tilde{\tilde{\cap}}cl((f_A)^c)\tilde{\tilde{\subseteq}}cl(g_A)\tilde{\tilde{\cap}}cl((f_A)^c)\tilde{\tilde{\subseteq}}cl(g_A)\tilde{\tilde{=}}g_A$. Similarly for the other inclusion. □

The following example shows that (1) and (2) are not true, if f_A is not IF soft open and IF soft closed respectively.

Example 28. *In the above Example 15, an IF soft closed sets are* $\widetilde{\widetilde{\Phi_A}}\tilde{\tilde{=}}\{x_{(0,1)},x_{(0,1)}\},\{x_{(0,1)},x_{(0,1)}\})$, $\widetilde{\widetilde{X_A}}\tilde{\tilde{=}}(\{x_{(1,0)},x_{(1,0)}\},\{x_{(1,0)},x_{(1,0)}\})$, $(\{x_{(0.8,0.2)},x_{(0.3,0.6)}\},\{x_{(0.5,0.2)},x_{(0.1,0.9)}\})$, $(\{x_{(0.8,0.1)},x_{(0.1,0.6)}\},\{x_{(0.8,0.2)},x_{(0.1,0.8)}\})$, $(\{x_{(0.8,0.2)},x_{(0.1,0.6)}\},\{x_{(0.5,0.2)},x_{(0.1,0.9)}\})$, $(\{x_{(0.8,0.1)},x_{(0.3,0.6)}\},\{x_{(0.8,0.2)},x_{(0.1,0.8)}\})$. *Let us take* $f_A\tilde{\tilde{=}}(\{x_{(0.6,0.1)},x_{(0.1,0.7)}\},\{x_{(0.7,0.3)},x_{(0.1,0.9)}\})$, *which is not IF soft open and not IF soft closed. Then* $cl(f_A)\tilde{\tilde{=}}(\{x_{(0.8,0.1)},x_{(0.1,0.6)}\},\{x_{(0.8,0.2)},x_{(0.1,0.8)}\})$. *Also* $(f_A)^c\tilde{\tilde{=}}(\{x_{(0.1,0.6)},x_{(0.7,0.1)}\},\{x_{(0.3,0.7)},x_{(0.9,0.1)}\})$ *and* $cl((f_A)^c)\tilde{\tilde{=}}\widetilde{\widetilde{X_A}}$. *Thus,* $bd(f_A)\tilde{\tilde{=}}cl(f_A)\cap cl((f_A)^c)\tilde{\tilde{=}}(\{x_{(0.8,0.1)},x_{(0.1,0.6)}\},\{x_{(0.8,0.2)},x_{(0.1,0.8)}\})$. *We observe that,* $f_A\tilde{\tilde{\cap}}bd(f_A)\tilde{\tilde{\neq}}\widetilde{\widetilde{\Phi_A}}$ *and* $bd(f_A)\tilde{\nsubseteq}f_A$.

The following example verify (3) of above Theorem 27.

Example 29. *In the above Example, let us take an IF fuzzy soft closed set* $g_A\tilde{\tilde{=}}(\{x_{(0.8,0.1)},x_{(0.3,0.6)}\},\{x_{(0.8,0.2)},x_{(0.1,0.8)}\})$ *and any IF soft set* $f_A\tilde{\tilde{=}}(\{x_{(0.6,0.1)},x_{(0.1,0.7)}\},\{x_{(0.7,0.3)},x_{(0.1,0.9)}\})$. *Then* $f_A\tilde{\tilde{\subseteq}}g_A$. *Clearly,* $bd(f_A)\tilde{\tilde{=}}(\{x_{(0.8,0.1)},x_{(0.1,0.6)}\},\{x_{(0.8,0.2)},x_{(0.1,0.8)}\})\tilde{\tilde{\subseteq}}g_A$.

Theorem 30. *Let f_A and g_B be an IF soft sets of an IF soft topological space over X. Then the following hold:*
(1) $bd([f_A\tilde{\tilde{\cup}}g_B])\tilde{\tilde{\subseteq}}[bd(f_A\tilde{\tilde{\cap}}(g_B^c))]\tilde{\tilde{\cup}}[bd(g_B)\tilde{\tilde{\cap}}cl(((f_A)^c))]$.
(2) $bd([f_A\tilde{\tilde{\cap}}g_B])\tilde{\tilde{\subseteq}}[bd(f_A)\tilde{\tilde{\cap}}cl(g_B)]\tilde{\tilde{\cup}}[bd(g_B)\tilde{\tilde{\cap}}cl((f_A))]$.

Proof. (1).

$$\begin{aligned}
bd((f_A\tilde{\tilde{\cup}}g_B)) &\tilde{\tilde{=}} cl((f_A\tilde{\tilde{\cup}}g_B))\tilde{\tilde{\cap}}cl(((f_A\tilde{\tilde{\cup}}g_B)^c))\\
&\tilde{\tilde{=}} (cl(f_A)\tilde{\tilde{\cup}}cl(g_B))\tilde{\tilde{\cap}}cl((f_A^c\tilde{\tilde{\cap}}g_B^c))\\
&\tilde{\tilde{\subseteq}} (cl(f_A)\tilde{\tilde{\cup}}cl(g_B))\tilde{\tilde{\cap}}[cl((f_A)^c)\tilde{\tilde{\cap}}cl((g_B)^c)]\\
&\tilde{\tilde{=}} (cl(f_A)\tilde{\tilde{\cap}}cl((f_A)^c))\tilde{\tilde{\cap}}(cl((g_B)^c)\tilde{\tilde{\cup}}cl(g_B))\tilde{\tilde{\cap}}[cl((f_A)^c)\tilde{\tilde{\cap}}cl((g_B)^c)]\\
&\tilde{\tilde{=}} (bd(f_A)\tilde{\tilde{\cap}}cl((g_B)^c))\tilde{\tilde{\cup}}(bd(g_B)\tilde{\tilde{\cap}}cl(f_A^c))\\
&\tilde{\tilde{\subseteq}} bd(f_A)\tilde{\tilde{\cup}}bd(g_B).
\end{aligned}$$

(2).

$$\begin{aligned}
bd([f_A\tilde{\tilde{\cap}}g_B]) &\tilde{\tilde{=}} cl((f_A\tilde{\tilde{\cap}}g_B))\tilde{\tilde{\cap}}cl((f_A\tilde{\tilde{\cap}}g_B)^c)\\
&\tilde{\tilde{\subseteq}} [cl(f_A)\tilde{\tilde{\cap}}cl(g_B)]\tilde{\tilde{\cap}}[cl((f_A^c\tilde{\tilde{\cup}}g_B^c))]\\
&= [cl(f_A)\tilde{\tilde{\cap}}cl(g_B)]\tilde{\tilde{\cap}}[cl((f_A)^c)\tilde{\tilde{\cup}}cl((g_B)^c)]\\
&\tilde{\tilde{=}} [(cl(f_A)\tilde{\tilde{\cap}}cl((g_B)))\tilde{\tilde{\cap}}cl((f_A)^c)]\tilde{\tilde{\cup}}[(cl((f_A))\tilde{\tilde{\cap}}cl((g_B)))\tilde{\tilde{\cap}}cl((g_B)^c)]\\
&\tilde{\tilde{=}} (bd((f_A))\tilde{\tilde{\cap}}bd((g_B)))\tilde{\tilde{\cup}}(cl(f_A)\tilde{\tilde{\cap}}bd(g_B)).
\end{aligned}$$

□

Corollary 31. *Let f_A and g_B be IF soft sets of an IF soft topological space over X. Then, $bd((f_A\tilde{\tilde{\cap}}g_B))\tilde{\tilde{\subseteq}}bd(f_A)\tilde{\tilde{\cap}}bd(g_B)$.*

Theorem 32. *Let f_A be an IF soft set of an IF soft topological space over X. Then we have: $bd((bd((bd(f_A)))))\tilde{\tilde{=}}bd((bd(f_A)))$.*

Proof.

$$\begin{aligned}
bd((bd((bd(f_A))))) &\tilde{\tilde{=}} cl((bd((bd(f_A)))))\tilde{\tilde{\cap}}cl(((bd((bd(f_A))))^c))\\
&= (bd((bd(f_A))))\tilde{\tilde{\cap}}cl(((bd((bd(f_A))))^c))
\end{aligned} \tag{1}$$

Now consider

$$\begin{aligned}
((bd((bd(f_A))))^c) &\tilde{\tilde{=}} [cl((bd(f_A)))\tilde{\tilde{\cap}}((bd(f_A))^c)]^c\\
&\tilde{\tilde{=}} [bd(f_A)\tilde{\tilde{\cap}}cl((bd(f_A))^c)]^c\\
&\tilde{\tilde{=}} (bd(f_A))^c\tilde{\tilde{\cup}}(cl((bd(f_A))^c))^c
\end{aligned}$$

Therefore

$$\begin{aligned}
cl(((bd((bd(f_A))))^c)) &\tilde{\tilde{=}} cl([cl(((bd(f_A)))^c)\tilde{\tilde{\cup}}(cl(((bd(f_A))^c)))^c])\\
&\tilde{\tilde{=}} cl((cl(((bd(f_A)))^c)))\tilde{\tilde{\cup}}cl(((cl(((bd(f_A))^c)))^c))\\
&\tilde{\tilde{=}} g_A\tilde{\tilde{\cup}}((cl(((bd(g_A)))^c)))^c)\tilde{\tilde{=}}\widetilde{\widetilde{X_A}}
\end{aligned} \tag{2}$$

where $g_A\tilde{\tilde{=}}cl((cl(((bd(f_A)))^c)))$. From (1) and (2), we have

$$bd((bd((bd(f_A)))))\tilde{\tilde{=}}bd((bd(f_A)))\tilde{\tilde{\cap}}\widetilde{\widetilde{X_A}}\tilde{\tilde{=}}bd((bd(f_A))).$$

□

Theorem 33. *Let f_A and g_A be a IF soft open sets of IF soft topological space over X. Then the following hold:*
(1) $(f_A \tilde{\tilde{\setminus}} int(g_A)) \tilde{\tilde{\subseteq}} int(f_A) \tilde{\tilde{\setminus}} int(g_A)$.
(2) $bd(int(f_A)) \tilde{\tilde{\subseteq}} bd(f_A)$.

Proof. (1).

$$\begin{aligned}
(1).\ (f_A \tilde{\tilde{\setminus}} int(g_A)) &\tilde{\tilde{=}} (f_A \tilde{\tilde{\cap}} int((g_A)^c)) \\
&\tilde{\tilde{=}} int(f_A) \tilde{\tilde{\cap}} int((g_A)^c) \\
&\tilde{\tilde{=}} int(f_A) \tilde{\tilde{\cap}} (cl((g_A)))^c \ (by\ Theorem\ 4.5(5)[14]) \\
&\tilde{\tilde{=}} int(f_A) \tilde{\tilde{\setminus}} cl(g_A) \\
&\tilde{\tilde{\subseteq}} int(f_A) \tilde{\tilde{\setminus}} int(g_A).
\end{aligned}$$

(2).

$$\begin{aligned}
(2).\ bd(int(f_A)) &\tilde{\tilde{=}} cl(int(f_A)) \tilde{\tilde{\cap}} cl(((int(f_A))^c)) \\
&\tilde{\tilde{\subseteq}} cl(int(f_A)) \tilde{\tilde{\cap}} cl((cl((f_A^c)))) \ (by\ Theorem\ 4.5(5)[14]) \\
&\tilde{\tilde{\subseteq}} cl(f_A) \tilde{\tilde{\cap}} cl((f_A^c)) \tilde{\tilde{=}} bd(f_A).
\end{aligned}$$

□

Theorem 34. *Let f_A be an IF soft set of an IF soft topological space over X. Then $bd(f_A) \tilde{\tilde{=}} \widetilde{\widetilde{\Phi}}_A$ if and only if f_A is an IF soft closed set and an IF soft open set.*

Proof. Suppose that $bd(f_A) \tilde{\tilde{=}} \tilde{\tilde{\Phi}}$.
(i) First we prove that f_A is an IF soft closed set. Consider

$$\begin{aligned}
&bd(f_A) \tilde{\tilde{=}} \widetilde{\widetilde{\Phi}}_A \Rightarrow cl(f_A) \tilde{\tilde{\cap}} cl((f_A^c)) \tilde{\tilde{=}} \widetilde{\widetilde{\Phi}}_A \\
&\Rightarrow cl(f_A) \tilde{\tilde{\subseteq}} (cl((f_A^c)))^c \tilde{\tilde{=}} int(f_A) \ (by\ Theorem\ 4.5(6)[14]) \\
&\Rightarrow cl(f_A) \tilde{\tilde{\subseteq}} f_A \Rightarrow cl(f_A) \tilde{\tilde{=}} f_A
\end{aligned}$$

This implies that f_A is an IF soft closed set.
(ii) Using (i), we now prove that f_A is an IF soft open set.
$bd(f_A) \tilde{\tilde{=}} \widetilde{\widetilde{\Phi}}_A \Rightarrow cl(f_A) \tilde{\tilde{\cap}} cl((f_A^c))$ or $f_A \tilde{\tilde{\cap}} (int(f_A))^c \tilde{\tilde{=}} \widetilde{\widetilde{\Phi}}_A \Rightarrow f_A \tilde{\tilde{\subseteq}} int(f_A) \Rightarrow int(f_A) \tilde{\tilde{=}} f_A$
This implies that f_A is an IF soft open set.
Conversely, suppose that f_A is an IF soft open and an IF soft closed set. Then

$$\begin{aligned}
bd(f_A) &\tilde{\tilde{=}} cl(f_A) \tilde{\tilde{\cap}} cl((f_A^c)) \\
&\tilde{\tilde{=}} cl(f_A) \tilde{\tilde{\cap}} (int(f_A))c \ (by\ Theorem\ 4.5(6)[14] \\
&\tilde{\tilde{=}} f_A \tilde{\tilde{\cap}} f_A^c \tilde{\tilde{=}} \widetilde{\widetilde{\Phi}}_A.
\end{aligned}$$

This completes the proof. □

The following example shows that the condition that f_A is IF soft open and IF soft closed is necessary in the above theorem.

Example 35. *In the above Example 28, let us take an IF soft set* $f_A \widetilde{\widetilde{=}} (\{x_{(0.7,0.1)}, x_{(0.1,0.6)}\}, \{x_{(0.6,0.5)}, x_{(0.1,0.9)}\})$*., which is not IF soft closed and IF soft open. Calculations show that*
$bd(f_A) \widetilde{\widetilde{=}} cl(f_A) \cap cl((f_A)^c) \widetilde{\widetilde{=}} (\{x_{(0.8,0.1)}, x_{(0.1,0.6)}\}, \{x_{(0.8,0.2)}, x_{(0.1,0.8)}\}) \widetilde{\widetilde{\neq}} \widetilde{\widetilde{\Phi}}_A$.

4. Conclusion

The importance of decision making problem in an imprecise environment is growing very significantly in recent years. The concept of intuitionistic fuzzy soft sets in a decision making problem and the problem is solved with the help of 'similarity measurement' technique. In this paper, we initiated the concept of IF soft boundary. We discussed and explored the characterizations and properties of IF soft boundary in general as well as in terms of IF soft interior and IF soft closure. Examples and counter examples are also presented to validate the discussed results. In future studies, we will study the further topological structures in IF soft sets. We will also explore applications of the topological structures of IF soft sets in medical diagnosis system, and other decision making problems. We hope that the addition of this concept and properties will be a good addition in the tool box of IF soft sets and will be helpful for the researchers working in this field.

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References

[1] Atanassov, K., lntuitionistic fuzzy sets, in: V. Sgurev, Ed., VII ITKR's Session, Sofia, June 1983, Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984.
[2] Atanassov, K., lntuitionistic fuzzy sets, *Fuzzy Sets and Systems,* 20 (1986), 87–96.
[3] Atanassov, K. and Stoeva S., Intuitionistic fuzzy sets, *Polish Symp. on Interval and Fuzzy Mathematics,* Poznan, (August 1983), 23–26.
[4] Chen, D., Tsang, E. C. C., Yeung, D. S., Wang, X., The parameterization reduction of soft sets and its applications, Comput. Math. Appl., 49 (2005), 757–763.
[5] Coker, D., An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Sets and Systems,* 88 (1997), 81–89.
[6] Feng, F., Jun, Y. B., Liu, X., Li, L., An adjustable approach to fuzzy soft set based decision making, *J. Comput. Appl. Math.*, 234 (1) (2010), 10–20.
[7] Hussain, S., A note on soft connectedness, *Journal of Egyptian Mathematical Society,* 23(1) (2015), 6–11.
[8] Hussain, S., Ahmad, B., Some properties of soft topological spaces, *Computers and Mathematics with Applications,* 62(11) (2011), 4058–4067.
[9] Hussain, S., Ahmad B., Soft separation axioms in soft topological spaces, *Hacettepe Journal of Mathematics and Statistics*, 44(3) (2015), 559–568.
[10] Hussain, S., On some soft functions, *Mathematical Science Letters*, 4(1)(2015), 55–61.

[11] Hussain, S., On some generalized soft mappings, *Hacettepe Journal of Mathematics and Statistics,* 45(3) (2016), 743–754.
[12] Hussain, S., Akiz, H. F., Alajlan, A. I., On algebraic properties of soft real points, *Moroccan J. of Pure and Appl. Anal.* (MJPAA), 4(1) (2018) 9–16.
[13] Hussain, S., Akiz, H. F., Alajlan, A. I., On soft real point matrices and their operations, *Fixed Point Theory and Applications*, 9 (2019), 1–20.
[14] Li, Z., Cui, R., On the topological structures of intuitionistic fuzzy soft sets, *Annals of fuzzy Mathematics and Informatics*, 5(1) (2013), 229–239.
[15] Maji, P. K., Biswas, R., Roy, A. R., Fuzzy soft sets, *J. Fuzzy Math.*, 9(3)(2001),589–602.
[16] Maji, P. K., Biswas, R., Roy, A. R., Intuitionistic fuzzy soft sets, *The Journal of Fuzzy Mathematics*, 9(3) (2001), 677–691.
[17] Maji, P. K., Biswas, R., Roy, A. R., On Intuitionistic fuzzy soft sets, *The Journal of Fuzzy Mathematics*, 12(3) (2004), 669–683.
[18] Maji, P. K., More on intuitionistic fuzzy soft sets, *LNCS,* 5908,(2006), 231–240.
[19] Molodtsov, D., Soft set theory: first results, *Computers and Mathematics with Applications,* 37 (1999)19–31.
[20] Roy, A. R., Maji, P. K., A fuzzy soft set theoretic approach to decision making problems, *J. Comput. Appl. Math.*, 203 (2007), 412–418.
[21] Xiao, Z., Gong, K., Zou, Y., A combined forecasting approach based on fuzzy soft sets, *J. Comput. Appl. Math.*, 228 (1) (2009), 326–333.
[22] Yin, Y., Li, H., Jun, Y. B., On algebraic structure of intuitionistic fuzzy soft sets, *Computers and Mathematics with Applications*, 64 (2012), 2896–2911.
[23] Zadeh, L. A., Fuzzy sets, *Information and Control,* 8(1996), 338–353.
[24] Zou, Y., Xiao, Z., Data analysis approaches of soft sets under incomplete information, *Knowledge-Based Syst.*, 21(8) (2008), 941-945.

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SPECIAL HELICES ON EQUIFORM DIFFERENTIAL GEOMETRY OF SPACELIKE CURVES IN MINKOWSKI SPACE-TIME

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Abstract. In this paper, we establish k-type helices for equiform differential geometry of spacelike curves in 4-dimensional Minkowski space E_1^4. Also we obtain (k, m)-type slant helices for equiform differential geometry of spacelike curves in Minkowski space-time.

1. Introduction

Helices, which are an important subject of the theory of curves in differential geometry, are studied by physicists, engineers and biologists. Helix (or general helix) is described as an in 3-dimensional Euclidean space (or Minkowski) tangent vector field forming a constant angle with a fixed direction of the curve. So, many authors were interested in helices to study it in Euclidean (or Minkowski) 3- and 4-space and they gave new characterizations for an helix. In the 4-dimensional Minkowski space k-type slant helices were defined in a study by Ali et al. [1]. In addition, M.Y. Yılmaz and M.Bektaş in [6] defined (k, m)-type slant helices in 4-dimensional Euclidean space.

In our study, we establish k-type helices and (k, m)-type slant helices for equiform differential geometry of spacelike curves in 4-dimensional Minkowski space E_1^4 and give some new characterizations for these helices.

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2. Geometric Preliminaries

Let $E^4 = \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in R\}$ be a 4-dimensional vector space. For any two vectors $x = (x_1, x_2, x_3, x_4),\ y = (y_1, y_2, y_3, y_4)$ in E^4, the pseudo scalar product of x and y is defined by $\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. We call $(E^4, \langle ., .\rangle)$ a Minkowski 4-space and denote it by E^4_1. We say that a vector x in $E^4_1 \setminus \{0\}$ is a spacelike vector, a lightlike vector or a timelike vector if $\langle x, x\rangle$ is positive, zero, negative respectively.

The norm of a vector $x \in E^4_1$ is defined by $\|x\| = \sqrt{|\langle x, x\rangle|}$. For any two vectors a, b in E^4_1, we say that a is pseudo-perpendicular to b if $\langle a, b\rangle = 0$. Let $\alpha : I \subset R \to E^4_1$ be an arbitrary curve in E^4_1, we say that a curve α is a spacelike curve if $\langle \dot\alpha(t), \dot\alpha(t)\rangle > 0$ for any $t \in I$. The arclength of a spacelike curve γ measured from $\alpha(t_0)$ $(t_0 \in I)$ is

$$s(t) = \int_{t_0}^{t} \|\dot\alpha(t)\| \mathrm{d}t. \tag{1}$$

Hence a parameter s is determined such that $\|\alpha'(s)\| = 1$, where $\alpha'(s) = d\alpha/ds$. Consequently, we say that a spacelike curve α is parameterized by arclength if $\|\alpha'(s)\| = 1$. Throughout the rest of this paper s is assumed arclength parameter. For any $x, y, z \in E^4_1$, we define a vector $x \times y \times z$ by

$$x \times y \times z = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{vmatrix}, \tag{2}$$

where $x_i = (x_i^1, x_i^2, x_i^3, x_i^4)$. Let $\alpha : I \longrightarrow E^4_1$ be a spacelike curve in E^4_1. Therefore we can construct a pseudo-orthogonal frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b_1}(s), \mathbf{b_2}(s)\}$, which satisfies the following Frenet-Serret type formula of E^4_1 along α.

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b_1} \\ \mathbf{b_2} \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \mu_1\kappa_1 & 0 & \mu_2\kappa_2 & 0 \\ 0 & \mu_3\kappa_2 & 0 & \mu_4\kappa_3 \\ 0 & 0 & \mu_5\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b_1} \\ \mathbf{b_2} \end{bmatrix}, \tag{3}$$

where κ_1, κ_2 and κ_3 are respectively, first, second and third curvature of the spacelike curve α and we have

$$\begin{aligned} &\kappa_1(s) = \|\alpha''(s)\|, \\ &\mathbf{n}(s) = \frac{\alpha''(s)}{\kappa_1(s)}, \\ &\mathbf{b_1}(s) = \frac{\mathbf{n}'(s) + \mu_1\kappa_1(s)\mathbf{t}(s)}{\|\mathbf{n}'(s) + \mu_1\kappa_1(s)\mathbf{t}(s)\|}, \end{aligned}$$

$$\mathbf{b_2}(s) = \mathbf{t}(s) \times \mathbf{n}(s) \times \mathbf{b_1}(s).$$

Denote by $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b_1}(s), \mathbf{b_2}(s)\}$ the moving Frenet frame along the spacelike curve α, where s is a pseudo arclength parameter [1,2,3,5,7].

3. Equiform Differential Geometry of Curves

3.1. **Spacelike Curves:**

Definition 3.1. *Unless otherwise stated, we use the same terminology such as [2,4]. Let* $\alpha : I \longrightarrow E_1^4$ *be* **a spacelike curve**. *We define the equiform parameter of* $\alpha(s)$ *by*

$$\sigma = \int \frac{ds}{\rho} = \int \kappa_1 ds, \tag{4}$$

where $\rho = \dfrac{1}{\kappa_1}$ *is the radius of curvature of the curve* α.

It follows

$$\frac{ds}{d\sigma} = \rho. \tag{5}$$

Let h be a homothety with the center in the origin and the coefficient λ. If we put $\alpha^* = h(\alpha)$ then it follows

$$s^* = \lambda s, \text{ and } \rho^* = \lambda\rho, \tag{6}$$

where s^* is the arclength parameter of α^* and ρ^* the radius of curvature of α^*. Hence α is an equiform invariant parameter of α.

Notation 3.1. Let us note that κ_1, κ_2 and κ_3 are not invariants of the homothety group, it follows $\kappa_1^* = \dfrac{1}{\lambda}\kappa_1$, $\kappa_2^* = \dfrac{1}{\lambda}\kappa_2$ and $\kappa_3^* = \dfrac{1}{\lambda}\kappa_3$. The vector

$$\mathbf{V}_1 = \frac{d\alpha(s)}{d\sigma}, \tag{7}$$

is called a tangent vector of the curve α in the equiform geometry. From (5) and (7), we get

$$\mathbf{V}_1 = \frac{d\alpha(s)}{d\sigma} = \rho\frac{d\alpha(s)}{ds} = \rho\mathbf{t}. \tag{8}$$

Furthermore, we define the tri-normals by

$$\mathbf{V}_2 = \rho\mathbf{n}, \quad \mathbf{V}_3 = \rho\mathbf{b_1}, \quad \mathbf{V}_4 = \rho\mathbf{b_2}. \tag{9}$$

It is easy to check that the tetrahedron $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$ is an equiform invariant tetrahedron of the curve α. Now, we will find the derivatives of these vectors with respect to σ using by (5), (7) and (9) as follows:

$$\mathbf{V}_1' = \frac{d}{d\sigma}(\mathbf{V}_1) = \rho\frac{d}{ds}(\rho\mathbf{t}) = \dot{\boldsymbol{\rho}}\mathbf{V_1} + \mathbf{V_2}$$

where the derivative with respect to the arclength s is denoted by a dot and respect to σ by a dash. Similarly, we obtain

$$\begin{aligned}
\mathbf{V}_2' &= \frac{d}{d\sigma}(\mathbf{V}_2) = \rho\frac{d}{ds}(\rho\mathbf{n}) = \boldsymbol{\mu_1}\mathbf{V_1} + \dot{\boldsymbol{\rho}}\mathbf{V_2} + \boldsymbol{\mu_2}\left(\frac{\boldsymbol{\kappa_2}}{\boldsymbol{\kappa_1}}\right)\mathbf{V_3},\\
\mathbf{V}_3' &= \frac{d}{d\sigma}(\mathbf{V}_3) = \rho\frac{d}{ds}(\rho\mathbf{b_1}) = \boldsymbol{\mu_3}\left(\frac{\boldsymbol{\kappa_2}}{\boldsymbol{\kappa_1}}\right)\mathbf{V_2} + \dot{\boldsymbol{\rho}}\mathbf{V_3} + \boldsymbol{\mu_4}\left(\frac{\boldsymbol{\kappa_3}}{\boldsymbol{\kappa_1}}\right)\mathbf{V_4},\\
\mathbf{V}_4' &= \frac{d}{d\sigma}(\mathbf{V}_4) = \rho\frac{d}{ds}(\rho\mathbf{b_2}) = \boldsymbol{\mu_5}\left(\frac{\boldsymbol{\kappa_3}}{\boldsymbol{\kappa_1}}\right)\mathbf{V_3} + \dot{\boldsymbol{\rho}}\mathbf{V_4}.
\end{aligned} \tag{10}$$

Definition 3.2. *The functions* $\mathbf{K}_i : I \longrightarrow R$ $(i = 1, 2, 3)$ *defined by*

$$\mathbf{K}_1 = \dot{\rho},\ \mathbf{K}_2 = \frac{\kappa_2}{\kappa_1},\ \mathbf{K}_3 = \frac{\kappa_3}{\kappa_1} \tag{11}$$

are called i^{th} *equiform curvatures of the curve* α*.*

These functions $\mathbf{K}_i$ are differential invariant of the group of equiform transformations, too. Therefore, the formulas analogous to famous the Frenet formulas in the equiform geometry of the Minkowski space E_1^4 have the following form:

$$\begin{aligned}
\mathbf{V}_1' &= \mathbf{K}_1\mathbf{V}_1 + \mathbf{V}_2,\\
\mathbf{V}_2' &= \mu_1\mathbf{V}_1 + \mathbf{K}_1\mathbf{V}_2 + \mu_2\mathbf{K}_2\mathbf{V}_3,\\
\mathbf{V}_3' &= \mu_3\mathbf{K}_2\mathbf{V}_2 + \mathbf{K}_1\mathbf{V}_3 + \mu_4\mathbf{K}_3\mathbf{V}_4,\\
\mathbf{V}_4' &= \mu_5\mathbf{K}_3\mathbf{V}_3 + \mathbf{K}_1\mathbf{V}_4.
\end{aligned} \tag{12}$$

Notation 3.2. The equiform parameter $\sigma = \int \kappa_1(s)ds$ for closed curves is called the total curvature, and it plays an important role in global differential geometry of the Euclidean space. Also, the functions $\frac{\kappa_2}{\kappa_1}$ and $\frac{\kappa_3}{\kappa_1}$ have been already known as conical curvatures and they also have interesting geometric interpretation.

Because of the equiform Frenet formulas (12), the following equalities regarding equiform curvatures can be given

$$\begin{aligned}
\mathbf{K}_1 &= \frac{1}{\rho^2}\langle \mathbf{V}_j', \mathbf{V}_j\rangle;\ (j = 1, 2, 3, 4),\\
\mathbf{K}_2 &= \frac{1}{\mu_2\rho^2}\langle \mathbf{V}_2', \mathbf{V}_3\rangle = \frac{1}{\mu_3\rho^2}\langle \mathbf{V}_3', \mathbf{V}_2\rangle,\\
\mathbf{K}_3 &= \frac{1}{\mu_4\rho^2}\langle \mathbf{V}_3', \mathbf{V}_4\rangle = \frac{1}{\mu_5\rho^2}\langle \mathbf{V}_4', \mathbf{V}_3\rangle.
\end{aligned} \tag{13}$$

Definition 3.3. *Let* α *be a spacelike curve in* E_1^4 *with equiform Frenet frame* $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$*. If there exists a non-zero constant vector field* U *in* E_1^4 *such that* $< \mathbf{V}_i, U >=$ *constant for* $1 \leq i \leq 4$*, then* α *is said to be a* $k-$*type slant helix and* U *is called the slope axis of* α*.*

Theorem 3.1. *Let α be a spacelike curve with Frenet formulas in equiform geometry of the Minkowski space E_1^4. Then, if the curve α is a 1-type helix (or general helix), then we have*

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1 c, \tag{14}$$

where c is a constant.

Proof. Assume that α is a 1-type helix. Then for a constant field U such that $\langle \mathbf{V}_1, U\rangle = c$ is a constant. Differentiating this equation with respect to σ, we get

$$\langle \mathbf{V}_1', U\rangle = 0,$$

and using equiform Frenet equations, we find

$$\mathbf{K}_1\langle \mathbf{V}_1, U\rangle + \langle \mathbf{V}_2, U\rangle = 0$$

and using $\langle \mathbf{V}_1, U\rangle = c$,

$$\mathbf{K}_1 c + \langle \mathbf{V}_2, U\rangle = 0. \tag{15}$$

From (15), it is written as follows:

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1 c,$$

thus, the proof is completed. □

Theorem 3.2. *Let α be a spacelike curve with Frenet formulas in equiform geometry of the Minkowski space E_1^4. Then, if the curve α is a 2-type helix, then we have*

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = -\mathbf{K}_1 c_1 \tag{16}$$

where c_1 is a constant.

Proof. If the curve α is a 2-type helix. Therefore for a constant field U such that $\langle \mathbf{V}_2, U\rangle = c_1$ is a constant. Differentiating this equation with respect to σ, we get

$$\langle \mathbf{V}_2', U\rangle = 0,$$

and using equiform Frenet equations, we have

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1\langle \mathbf{V}_2, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0$$

and using $\langle \mathbf{V}_2, U\rangle = c_1$, we find

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1 c_1 + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0. \tag{17}$$

From (17), we obtain

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = -\mathbf{K}_1 c_1.$$

The proof is completed. □

Theorem 3.3. *Let α be a spacelike curve with Frenet formulas in equiform geometry of the Minkowski space E_1^4. In that case, if the curve α is a 3-type helix, then we have*

$$\mu_3\mathbf{K}_2\langle \mathbf{V}_2, U\rangle + \mu_4\mathbf{K}_3\langle \mathbf{V}_4, U\rangle = -\mathbf{K}_1 c_2, \tag{18}$$

where c_2 is a constant.

Proof. If the curve α is a 3-type helix. Thus, for a constant field U such that

$$\langle \mathbf{V}_3, U \rangle = c_2 \tag{19}$$

is a constant. Differentiating this equation with respect to σ, we get

$$\langle \mathbf{V}_3', U \rangle = 0$$

and using equiform Frenet equations, we have

$$\mu_3 \mathbf{K}_2 \langle \mathbf{V}_2, U \rangle + \mathbf{K}_1 \langle \mathbf{V}_3, U \rangle + \mu_4 \mathbf{K}_3 \langle \mathbf{V}_4, U \rangle = 0, \tag{20}$$

and by setting (19) in (20), we can write

$$\mu_3 \mathbf{K}_2 \langle \mathbf{V}_2, U \rangle + \mathbf{K}_1 c_2 + \mu_4 \mathbf{K}_3 \langle \mathbf{V}_4, U \rangle = 0, \tag{21}$$

and from the last equation, we find

$$\mu_3 \mathbf{K}_2 \langle \mathbf{V}_2, U \rangle + \mu_4 \mathbf{K}_3 \langle \mathbf{V}_4, U \rangle = -\mathbf{K}_1 c_2,$$

the proof is completed. □

Theorem 3.4. *Let α be a spacelike curve with Frenet formulas in equiform geometry of the Minkowski space E_1^4. Then, if the curve α is a 4-type helix, in that case, we have*

$$\langle \mathbf{V}_3, U \rangle = -\frac{\mathbf{K}_1}{\mathbf{K}_3 \mu_5} c_3, \tag{22}$$

where c_3 is a constant.

Proof. If the curve α is a 4-type helix. Then for a constant field U such that

$$\langle \mathbf{V}_4, U \rangle = c_3, \tag{23}$$

is a constant. By differentiating of this last equation with respect to σ, we get

$$\langle \mathbf{V}_4', U \rangle = 0,$$

and using equiform Frenet equations, we obtain

$$\langle \mu_5 \mathbf{K}_3 \mathbf{V}_3 + \mathbf{K}_1 \mathbf{V}_4, U \rangle = 0. \tag{24}$$

From (24), we get

$$\mu_5 \mathbf{K}_3 \langle \mathbf{V}_3, U \rangle + \mathbf{K}_1 \langle \mathbf{V}_4, U \rangle = 0. \tag{25}$$

Substituting (23) in (25), we obtain

$$\langle \mathbf{V}_3, U \rangle = -\frac{\mathbf{K}_1}{\mathbf{K}_3 \mu_5} c_3.$$

The proof is completed. □

4. (k, m)-type slant helices in $\mathbf{E}_1^4$

In this section, we will define (k, m) type slant helices for spacelike curve with equiform Frenet frame in E_1^4 such as [6].

Definition 4.1. *Let α be a spacelike curve in E_1^4 with equiform Frenet frame $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$. We call α is a (k, m)- type slant helix if there exists a non-zero constant vector field $U \in E_1^4$ satisfies $\langle \mathbf{V}_k, U\rangle = c_1$ (c_1 is a constant) and $\langle \mathbf{V}_m, U\rangle = c_2$ (c_2 is a constant) for $1 \leq k, m \leq 4$, $k \neq m$. The constant vector U is on axis of α.*

Theorem 4.1. *If the curve α is a $(1, 2)$-type slant helix in E_1^4, then we have*

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mu_1 c_1 + \mathbf{K}_1 c_2}{\mu_2 \mathbf{K}_2},$$

and

$$\mathbf{K}_1 = -\frac{c_2}{c_1} \text{ is a constant.}$$

Proof. If the curve α is a $(1, 2)$-type slant helix in E_1^4, then for a constant field U. We can write

$$\langle \mathbf{V}_1, U\rangle = c_1, \tag{26}$$

and

$$\langle \mathbf{V}_2, U\rangle = c_2 \tag{27}$$

is a constant. Differentiating (26) and (27) with respect to σ, we have that

$$\langle \mathbf{V}_1', U\rangle = 0$$

and

$$\langle \mathbf{V}_2', U\rangle = 0.$$

Using equiform Frenet equations, the following equations can be obtained:

$$\mathbf{K}_1\langle \mathbf{V}_1, U\rangle + \langle \mathbf{V}_2, U\rangle = 0, \tag{28}$$

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1\langle \mathbf{V}_2, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0. \tag{29}$$

By setting (26) and (27) in (28), we find

$$\mathbf{K}_1 c_1 + c_2 = 0, \tag{30}$$

and substituting (26) and (27) in (28), we obtain

$$\mu_1 c_1 + \mathbf{K}_1 c_2 + \mu_2 \mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0. \tag{31}$$

Finally, we have the following equations:

$$\begin{aligned} \mathbf{K}_1 &= -\frac{c_2}{c_1}, \\ \langle \mathbf{V}_3, U\rangle &= -\frac{\mu_1 c_1 + \mathbf{K}_1 c_2}{\mu_2 \mathbf{K}_2}. \end{aligned}$$

The proof is completed. $\square$

Theorem 4.2. *If the curve α is a $(1,3)$-type slant helix in E_1^4, then there exists a constant such that*

$$\langle \mathbf{V}_4, U\rangle = \frac{\mu_3\mathbf{K}_2\mathbf{K}_1c_1 - \mathbf{K}_1c_3}{\mu_4\mathbf{K}_3}. \tag{32}$$

where c_1 and c_3 are constant.

Proof. If the curve α is a $(1,3)$-type slant helix in E_1^4, then for a constant field U. We can write

$$\langle \mathbf{V}_1, U\rangle = c_1 \tag{33}$$

and

$$\langle \mathbf{V}_3, U\rangle = c_3 \tag{34}$$

is a constant. Differentiating (33) and (34) with respect to σ, we get

$$\langle \mathbf{V}_1', U\rangle = 0,$$

and

$$\langle \mathbf{V}_3', U\rangle = 0.$$

Using equiform Frenet equations, we have

$$\mathbf{K}_1\langle \mathbf{V}_1, U\rangle + \langle \mathbf{V}_2, U\rangle = 0, \tag{35}$$

$$\mu_3\mathbf{K}_2\langle \mathbf{V}_2, U\rangle + \mathbf{K}_1\langle \mathbf{V}_3, U\rangle + \mu_4\mathbf{K}_3\langle \mathbf{V}_4, U\rangle = 0. \tag{36}$$

By setting (33) in (35), we obtain

$$\mathbf{K}_1c_1 + \langle \mathbf{V}_2, U\rangle = 0. \tag{37}$$

From (37), we find as follows:

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1c_1. \tag{38}$$

Substituting (34) and (38) in (36), we find

$$\langle \mathbf{V}_4, U\rangle = \frac{\mu_3\mathbf{K}_2\mathbf{K}_1c_1 - \mathbf{K}_1c_3}{\mu_4\mathbf{K}_3}.$$

The proof is completed. □

Theorem 4.3. *If the curve α is a $(1,4)$-type slant helix in E_1^4, then there exists a constant such that*

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1c_1$$

and

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mathbf{K}_1c_4}{\mu_5\mathbf{K}_3}.$$

Proof. If the curve α is a $(1,4)$-type slant helix in E_1^4, then for a constant field U. We can write

$$\langle \mathbf{V}_1, U\rangle = c_1 \tag{39}$$

and

$$\langle \mathbf{V}_4, U\rangle = c_4 \tag{40}$$

is a constant. Differentiating (39) and (40) with respect to σ, we get

$$\langle \mathbf{V}_1', U\rangle = 0,$$

and it follows

$$\langle \mathbf{V}_4', U\rangle = 0.$$

Using equiform Frenet equations, we have

$$\mathbf{K}_1\langle \mathbf{V}_1, U\rangle + \langle \mathbf{V}_2, U\rangle = 0, \tag{41}$$

and

$$\mu_5\mathbf{K}_3\langle \mathbf{V}_3, U\rangle + \mathbf{K}_1\langle \mathbf{V}_4, U\rangle = 0. \tag{42}$$

By setting (39) in (41), we obtain as below:

$$\mathbf{K}_1 c_1 + \langle \mathbf{V}_2, U\rangle = 0. \tag{43}$$

Substituting (40) in (42), we can write

$$\mu_5\mathbf{K}_3\langle \mathbf{V}_3, U\rangle + \mathbf{K}_1 c_4 = 0. \tag{44}$$

From (43) and (44), we get

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1 c_1,$$

and

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mathbf{K}_1 c_4}{\mu_5\mathbf{K}_3}.$$

The proof is completed. □

Theorem 4.4. *If the curve α is a $(2,3)$-type slant helix in E_1^4, then there exist constants such that*

$$\langle \mathbf{V}_1, U\rangle = -\frac{\mathbf{K}_1 c_2 + \mu_2\mathbf{K}_2 c_3}{\mu_1}, \tag{45}$$

and

$$\langle \mathbf{V}_4, U\rangle = -\frac{\mu_3\mathbf{K}_2 c_2 + \mathbf{K}_1 c_3}{\mu_4\mathbf{K}_3}. \tag{46}$$

Proof. If the curve α is a $(2,3)$-type slant helix in E_1^4, thus for a constant field U. We can write as below:

$$\langle \mathbf{V}_2, U\rangle = c_2 \tag{47}$$

is a constant and

$$\langle \mathbf{V}_3, U\rangle = c_3 \tag{48}$$

is a constant. Differentiating (47) and (48) with respect to σ, we get

$$\langle \mathbf{V}_2', U\rangle = 0,$$

and

$$\langle \mathbf{V}_3', U\rangle = 0.$$

Using equiform Frenet formulas, we have the following equations:

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1\langle \mathbf{V}_2, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0, \tag{49}$$

$$\mu_3\mathbf{K}_2\langle \mathbf{V}_2, U\rangle + \mathbf{K}_1\langle \mathbf{V}_3, U\rangle + \mu_4\mathbf{K}_3\langle \mathbf{V}_4, U\rangle = 0. \tag{50}$$

Substituting (47) and (48) in (49), we can write

$$\langle \mathbf{V}_1, U\rangle = -\frac{\mathbf{K}_1 c_2 + \mu_2\mathbf{K}_2 c_3}{\mu_1},$$

and by setting (47) and (48) in (50), we obtain

$$\langle \mathbf{V}_4, U\rangle = -\frac{\mu_3\mathbf{K}_2 c_2 + \mathbf{K}_1 c_3}{\mu_4\mathbf{K}_3}.$$

The proof is completed. □

Theorem 4.5. *If the curve α is a $(2,4)$-type slant helix in E_1^4, then there exists constant such that*

$$\langle \mathbf{V}_1, U\rangle = \frac{\mu_2\mathbf{K}_2\mathbf{K}_1 c_4 \; - \mu_5\mathbf{K}_3\mathbf{K}_1 c_2}{\mu_1\mu_5\mathbf{K}_3},$$

where c_2 and c_4 are constants.

Proof. If the curve α is a $(2,4)$-type slant helix in E_1^4, then for a constant field U. We can write the following equations:

$$\langle \mathbf{V}_2, U\rangle = c_2, \tag{51}$$

and

$$\langle \mathbf{V}_4, U\rangle = c_4 \tag{52}$$

is a constant. By differentiating (51) and (52) with respect to σ, we get the following equations:

$$\langle \mathbf{V}_2', U\rangle = 0$$

and

$$\langle \mathbf{V}_4', U\rangle = 0.$$

Using equiform Frenet equations, we have as below:

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1\langle \mathbf{V}_2, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0, \tag{53}$$

$$\mu_5\mathbf{K}_3\langle \mathbf{V}_3, U\rangle + \mathbf{K}_1\langle \mathbf{V}_4, U\rangle = 0. \tag{54}$$

Using (52), in the last equation, we get

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mathbf{K}_1 c_4}{\mu_5\mathbf{K}_3}. \tag{55}$$

Substituting (51) and (55) in (53), we obtain

$$\langle \mathbf{V}_1, U\rangle = \frac{\mu_2\mathbf{K}_2\mathbf{K}_1 c_4 \; - \mu_5\mathbf{K}_3\mathbf{K}_1 c_2}{\mu_1\mu_5\mathbf{K}_3},$$

the proof is completed. □

Theorem 4.6. *If the curve α is a $(3,4)$-type slant helix in E_1^4, then we have*

$$\langle\langle \mathbf{V}_2, U\rangle = \frac{\mathbf{K}_3}{\mathbf{K}_2}\frac{\left(\mu_5 c_3^2 - \mu_4 c_4^2\right)}{\mu_3 c_4} \tag{56}$$

and

$$\mathbf{K}_1 = -\mu_5 \mathbf{K}_3 \frac{c_3}{c_4}.$$

Proof. If the curve α is a $(3,4)$-type slant helix in E_1^4, then for a constant field U. We can write

$$\langle \mathbf{V}_3, U\rangle = c_3 \tag{57}$$

is a constant and

$$\langle \mathbf{V}_4, U\rangle = c_4 \tag{58}$$

is a constant. By differentiating (57) and (58) with respect to σ, we get

$$\langle \mathbf{V}_3', U\rangle = 0$$

and it follows

$$\langle \mathbf{V}_4', U\rangle = 0.$$

Using equiform Frenet formulas, we get as follows:

$$\mu_3 \mathbf{K}_2 \langle \mathbf{V}_2, U\rangle + \mathbf{K}_1 \langle \mathbf{V}_3, U\rangle + \mu_4 \mathbf{K}_3 \langle \mathbf{V}_4, U\rangle = 0, \tag{59}$$

$$\mu_5 \mathbf{K}_3 \langle \mathbf{V}_3, U\rangle + \mathbf{K}_1 \langle \mathbf{V}_4, U\rangle = 0. \tag{60}$$

By setting (57) and (58) in (60), we have the following equation:

$$\mathbf{K}_1 = -\mu_5 \mathbf{K}_3 \frac{c_3}{c_4}, \tag{61}$$

and substituting (57) and (58) in (59), we obtain

$$\langle \mathbf{V}_2, U\rangle = -\frac{\mathbf{K}_1 c_3}{\mu_3 \mathbf{K}_2} - \frac{\mu_4 \mathbf{K}_3 c_4}{\mu_3 \mathbf{K}_2}. \tag{62}$$

Using (61), in the last equation, we find

$$\langle \mathbf{V}_2, U\rangle = \frac{\mathbf{K}_3}{\mathbf{K}_2}\frac{\left(\mu_5 c_3^2 - \mu_4 c_4^2\right)}{\mu_3 c_4}. \tag{63}$$

The proof is completed. □

References

[1] Ali, A. T., López, R., Turgut, M., k-type partially null and pseudo null slant helices in Minkowski 4-space, *Math.Commun.,* 17 (2012), 93-103.

[2] Abdel-Aziz, H.S., Saad, M.K. Abdel-Salam, A.A., Equiform Differential Geometry of Curves in Minkowski Space-Time, arXiv.org/math/ arXiv:1501.02283.

[3] İlarslan, K., Nešović, E., Spacelike and Timelike Normal Curves in Minkowski Space-Time, *Publications de L'Institut Mathematique, Nouvelle série,* tome 85(99) (2009), 111-118.

[4] Aydın, M.E., Ergüt, M., The equiform differential geometry of curves in 4-dimensional Galilean space G4, *Stud. Univ. Babeş-Bolyai Math.,* 58(3) (2013), 399–406.

[5] Turgut, M., Yilmaz, S., Characterizations of Some Special Space-like Curves in Minkowski Space-time, *International J.Math. Combin.,* 2 (2008), 17-22.

[6] Yılmaz, M.Y., Bektaş, M., Slant helices of (k, m)-type in E^4 , *Acta Univ. Sapientiae, Mathematica,* 10(2) (2018), 395–401.

[7] Yilmaz, S., Turgut, M., On the characterizations of inclined curves in Minkowski space-time E_1^4, *International Mathematical Forum,* 3(16) (2008), 783-792.

https://communications.science.ankara.edu.tr

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ON NEW INTEGRAL INEQUALITIES USING MIXED CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. During the past two decades or so, fractional integral operators have been one of the most important tools in the development of inequalities theory. By this means, a lot generalized intergral inequalities involving various the fractional integral operators have been presented in the literature. Very recently, mixed conformable fractional integral operators has been introduced by T. Abdeljawad and with the help of these operators some new integral inequalities are obtained. The main aim of the paper is to establish some new Chebyshev type fractional integral inequalities by using mixed conformable fractional integral operators.

1. Introduction and Preliminaries

In the present paper, our work is based on a celebrated functional introduced by Chebyshev [4], which is defined by

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \quad (1)$$

where f and g are two integrable functions which are synchronous on $[a,b]$, i.e.

$$(f(x)-f(y))\,(g(x)-g(y))\geq 0$$

for any $x,y\in[a,b]$, then the Chebyshev inequality is given by $T(f,g)\leq 0$.

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The Chebyshev functional (1) has many applications in numerical quadrature, transform theory, probability, study of existence for solutions of differential equations, and in statistical problems. Moreover, under suitable assumptions (Chebyshev inequality, Grüss inequality, Minkowski inequality, Hermite-Hadamard inequality, Ostrowski inequality etc.), inequalities are playing a significant role in the field of mathematical sciences, particularly, in the theory of approximations.

A remarkably large number inequalities of above type involving the special fractional integral (such as the Riemann-Liouville, conformable, Erdélyi-Kober, Katugampola, Hadamard and Weyl types) have been investigated by many researchers and received considerable attention to it (see [8–10, 14, 16]).

Now, some fractional integral operators and Chebyshev type inequalities obtained with the help of these operators will be given in the following order:

Definition 1. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha \in R^{+}$ with $a \in R_0^{+}$ are defined, respectively, by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\, f(t)\, dt \quad (x > a)$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt \quad (x < b)$$

where Γ is the familiar Gamma function (see, e.g., [19, Section 1.1]). It is noted that $J_{a+}^{1} f(x)$ and $J_{b-}^{1} f(x)$ become the usual Riemann integrals.

Theorem 2. *[7] Let p be a positive function on $[0,\infty[$ and let f and g be two differentiable functions on $[0,\infty[$. If $f' \in L_r([0,\infty[)$, $g' \in L_s([0,\infty[)$, $r^{-1}+s^{-1}=1$, then for all $t>0$, $\alpha>0$, we have*

$$\begin{aligned} & 2\left|J^{\alpha}p(t)J^{\alpha}pfg(t) - J^{\alpha}pf(t)J^{\alpha}pg(t)\right| \\ \leq\ & \frac{||f'||_r ||g'||_s}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|\tau-\rho| p(\tau)p(\rho)d\tau d\rho \\ \leq\ & ||f'||_r ||g'||_s t\left(J^{\alpha}p(t)\right)^2. \end{aligned}$$

Theorem 3. *[7] Let p be a positive function on $[0,\infty[$ and let f and g be two differentiable functions on $[0,\infty[$. If $f' \in L_r([0,\infty[)$, $g' \in L_s([0,\infty[)$, $r^{-1}+s^{-1}=1$, then for all $t>0$, $\alpha,\beta>0$, we have*

$$\begin{aligned} & \left|J^{\alpha}p(t)J^{\beta}pfg(t) + J^{\beta}p(t)J^{\alpha}pfg(t) - J^{\alpha}pf(t)J^{\beta}pg(t) - J^{\beta}pf(t)J^{\alpha}pg(t)\right| \\ \leq\ & \frac{||f'||_r ||g'||_s}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|\tau-\rho| p(\tau)p(\rho)d\tau d\rho \\ \leq\ & ||f'||_r ||g'||_s t J^{\alpha}p(t)J^{\beta}p(t). \end{aligned}$$

Definition 4. *Let $\alpha>0$, $\mu>-1$, $\beta,\eta \in R$; then, a generalized fractional integral $I_t^{\alpha,\beta,\eta,\mu}$ (in terms of the Gauss hypergeometric function) of order α for a real-valued* *continuous function $f(t)$ is defined by [5] (see also [12])*

$$I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} = \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_0^t \tau^{\mu}(t-\tau)^{\alpha-1} {}_2F_1\left(\alpha+\beta+\mu-\eta;\alpha;1-\frac{\tau}{t}\right) f(\tau)d\tau, \tag{2}$$

where the function ${}_2F_1(-)$ appearing as a kernel for the operator (2) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)n(b)n}{(c)_n}\frac{t^n}{n!},$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)\dots(a+n-1),, \ \ (a)_0 = 1.$$

Theorem 5. *[2] Let p be a positive function and let f and g be two synchronous functions on $[0,\infty)$. If $f' \in L_r([0,\infty))$, $g' \in L_s([0,\infty))$, $r^{-1}+s^{-1}=1$, then (for all $t>0$, $\beta<1$, $\mu>-1$, $\alpha>\max\{0,-\beta-\mu\}$, $\beta-1<\eta<0$)*

$$\begin{aligned}
& 2\left|I_t^{\alpha,\beta,\eta,\mu}\{p(t)\}I_t^{\alpha,\beta,\eta,\mu}\{p(t)f(t)g(t)\} - I_t^{\alpha,\beta,\eta,\mu}\{p(t)f(t)\}I_t^{\alpha,\beta,\eta,\mu}\{p(t)g(t)\}\right| \\
\leq \ & \frac{t^{-2\alpha-2\beta-4\mu}||f'||_r||g'||_s}{\Gamma^2(\alpha)} \int_0^t\int_0^t \tau^{\mu}\rho^{\mu}(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} \\
& \times {}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)|\tau-\rho|d\tau d\rho \\
\leq \ & ||f'||_r||g'||_s t\left(I_t^{\alpha,\beta,\eta,\mu}\{p(t)\}\right)^2.
\end{aligned}$$

Theorem 6. *[2] Let p be a positive function and let f and g be two synchronous functions on $[0,\infty)$. If $f' \in L_r([0,\infty))$, $g' \in L_s([0,\infty))$, $r>1$ $r^{-1}+s^{-1}=1$, then*

$$\begin{aligned}
& \Big|I_t^{\alpha,\beta,\eta,\mu}\{p(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{p(t)f(t)g(t)\} + I_t^{\gamma,\delta,\zeta,\nu}\{p(t)\}I_t^{\alpha,\beta,\eta,\mu}\{p(t)f(t)g(t)\} \\
& -I_t^{\alpha,\beta,\eta,\mu}\{p(t)f(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{p(t)g(t)\} - I_t^{\gamma,\delta,\zeta,\nu}\{p(t)f(t)\}I_t^{\alpha,\beta,\eta,\mu}\{p(t)g(t)\}\Big| \\
\leq \ & \frac{t^{-\alpha-\beta-\gamma-\delta-2(\mu+\nu)}||f'||_r||g'||_s}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^t\int_0^t \tau^{\mu}\rho^{\mu}(t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1} \\
& \times {}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right) {}_2F_1\left(\gamma+\delta+\nu,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)|\tau-\rho|d\tau d\rho \\
\leq \ & ||f'||_r||g'||_s t I_t^{\gamma,\delta,\zeta,\nu}\{p(t)\}, I_t^{\alpha,\beta,\eta,\mu}\{p(t)\},
\end{aligned}$$

for all $t>0$, $\alpha>\max\{0,-\beta-\mu\}$, $\beta<1$, $\mu>-1$, $\beta-1<\eta<0$, $\gamma>\max\{0,-\delta-\nu\}$, $\delta<1$, $\nu>-1$, $\delta-1<\zeta<0$.

Definition 7. *[11] The Hadamard fractional integral of order $\alpha \in R^+$ of a function $f(t)$, for all $t>1$, is defined as*

$${}_HJ^{\alpha}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}.$$

Theorem 8. *[13] Let p be a positive function and let f and g be two differentiable functions on $[1,\infty)$. If $f' \in L_r([1,\infty))$, $g' \in L_s([1,\infty))$, $r>1$, $r^{-1}+s^{-1}=1$, then for all $t>1$ and $\alpha>0$,*

$$\begin{aligned} & 2\left|{}_HJ^{\alpha}\{p(t)\}\,{}_HJ^{\alpha}\{p(t)f(t)g(t)\} - {}_HJ^{\alpha}\{p(t)f(t)\}\,{}_HJ^{\alpha}\{p(t)g(t)\}\right| \\ \leq\ & \frac{||f'||_r||g'||_s}{\Gamma^2(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{p(\tau)p(\rho)}{\tau\rho}|\tau-\rho|d\tau d\rho \\ \leq\ & ||f'||_r||g'||_s t\left({}_HJ^{\alpha}\{p(t)\}\right)^2. \end{aligned}$$

Theorem 9. *[13] Let p be a positive function and let f and g be two differentiable functions on $[1,\infty)$. If $f' \in L_r([1,\infty))$, $g' \in L_s([1,\infty))$, $r>1$, $r^{-1}+s^{-1}=1$, then*

$$\begin{aligned} & \big|{}_HJ^{\alpha}\{p(t)\}\,{}_HJ^{\beta}\{p(t)f(t)g(t)\} + {}_HJ^{\beta}\{p(t)\}\,{}_HJ^{\alpha}\{p(t)f(t)g(t)\} \\ & -{}_HJ^{\alpha}\{p(t)f(t)\}\,{}_HJ^{\beta}\{p(t)g(t)\} - {}_HJ^{\beta}\{p(t)f(t)\}\,{}_HJ^{\alpha}\{p(t)g(t)\}\big| \\ \leq\ & \frac{||f'||_r||g'||_s}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{p(\tau)p(\rho)}{\tau\rho}|\tau-\rho|d\tau d\rho \\ \leq\ & ||f'||_r||g'||_s t\,{}_HJ^{\alpha}\{p(t)\}\,{}_HJ^{\beta}\{p(t)\}, \end{aligned}$$

for all $t>1$, $\alpha>0$ and $\beta>0$.

Definition 10. *[12] Let $\alpha>0$, $\beta>0$ and $\eta\in R$, then the Erdélyi-Kober fractional integral operators $I_{\beta}^{\eta,\alpha}$ of order α for a real-valued continuous function $f(t)$ is defined as*

$$\begin{aligned} I_{\beta}^{\eta,\alpha}\{f(t)\} &= \frac{t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t \tau^{\beta\eta}(t^{\beta}-\tau^{\beta})^{\alpha-1}f(\tau)d(\tau^{\beta}) \\ &= \frac{\beta t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t \tau^{\beta(\eta+1)-1}(t^{\beta}-\tau^{\beta})^{\alpha-1}f(\tau)d\tau. \end{aligned}$$

Theorem 11. *[2] Suppose that p be a positive function, f and g be differentiable functions on $[0,\infty)$, $f'\in L_r([0,\infty))$, $g'\in L_s([0,\infty))$ such that $r^{-1}+s^{-1}=1$ with $r>1$. Then for all $t>0$, $\alpha>0$, $\beta>0$, $\eta\in R$ and $\eta>-1$:*

$$\begin{aligned} & 2\left|I_{\beta}^{\eta,\alpha}\{p(t)\}I_{\beta}^{\eta,\alpha}\{p(t)f(t)g(t)\} - I_{\beta}^{\eta,\alpha}\{p(t)f(t)\}I_{\beta}^{\eta,\alpha}\{p(t)g(t)\}\right| \\ \leq\ & \frac{\beta^2 t^{-2\beta\eta+\alpha}||f'||_r||g'||_s}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^{\mu}\rho^{\mu}(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} \\ & \times{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)|\tau-\rho|d\tau d\rho \\ \leq\ & ||f'||_r||g'||_s t\left(I_{\beta}^{\eta,\alpha}\{p(t)\}\right)^2. \end{aligned}$$

Theorem 12. *[2] Suppose that p be a positive function, f and g be differentiable functions on $[0,\infty)$, $f'\in L_r([0,\infty))$, $g'\in L_s([0,\infty))$ such that $r>1$ and $r^{-1}+$*

$s^{-1}=1$. *Then for all* $t>0$ *the following inequality holds:*

$$\begin{aligned}
&\Big|I_{\beta}^{\eta,\alpha}\{p(t)\}I_{\delta}^{\zeta,\gamma}\{p(t)f(t)g(t)\}+I_{\delta}^{\zeta,\gamma}\{p(t)\}I_{\beta}^{\eta,\alpha}\{p(t)f(t)g(t)\}\\
&-I_{\beta}^{\eta,\alpha}\{p(t)f(t)\}I_{\delta}^{\zeta,\gamma}\{p(t)g(t)\}-I_{\delta}^{\zeta,\gamma}\{p(t)f(t)\}I_{\beta}^{\eta,\alpha}\{p(t)g(t)\}\Big|\\
\leq\ &\frac{\beta\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}||f'||_r||g'||_s}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t\tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^{\beta}-\tau^{\beta})^{\alpha-1}(t^{\delta}-\rho^{\delta})^{\gamma-1}\\
&\times p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\\
\leq\ &||f'||_r||g'||_s tI_{\beta}^{\eta,\alpha}\{p(t)\},I_{\delta}^{\zeta,\gamma}\{p(t)\},
\end{aligned}$$

where $\alpha,\beta,\gamma,\delta>0$, $\eta,\zeta\in R$ *and* $\eta>-1$ *and* $\zeta>-1$.

Definition 13. *[15] Let* $p\geq 0$, $q>0$, $\omega,\delta,\lambda,\sigma,c,\rho\in C$, $\Re(c)>0$, $\Re(\rho)>0$ *and* $\Re(\sigma)>0$. *Let* $f\in L[a,b]$ *and* $x\in[a,b]$. *Then the fractional integral operator* $(\epsilon_{a^+,\rho,\sigma}^{\omega,\delta,q,c}f)$ *defined by Rahman et al. is as the following:*

$$(\epsilon_{a^+,\rho,\sigma}^{\omega,\delta,q,c}f)(x)=\int_a^x(x-\tau)^{\sigma-1}E_{p,\sigma}^{\delta,q,c}(\omega(x-\tau)^{\rho};p)f(\tau)d\tau$$

where

$$E_{\rho,\sigma}^{\delta,q,c}(z;p)=\sum_{n=0}^{\infty}\frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{z^n}{n!}$$

and $B_p(x,y)$ *is an extension of Beta function defined in [15]*

$$B_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt\qquad x,y,p>0,$$

where $\Re(p)>0$, $\Re(x)>0$ *and* $\Re(y)>0$. *Also, here* B *is familiar Beta function as follows:*

$$B\left(a,b\right)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}\left(1-t\right)^{b-1}dt,\quad a,b>0.$$

Theorem 14. *[17] Let* t *be a positive function on* $[0,\infty)$ *and let* f *and* g *be two differentiable functions on* $[0,\infty)$. *If* $f'\in L_r([0,\infty))$, $g'\in L_s([0,\infty))$, $r^{-1}+s^{-1}=1$, *then for all* $x>0$, $\alpha,\beta>0$, *we have*

$$\begin{aligned}
&2\left|\left(\epsilon_{0^+,\alpha,\beta,\sigma}^{\omega,\delta,q,r,c}tfg\right)(x;p)\left(\epsilon_{0^+,\alpha,\beta,\sigma}^{\omega,\delta,q,r,c}t\right)(x;p)-\left(\epsilon_{0^+,\alpha,\beta,\sigma}^{\omega,\delta,q,r,c}tf\right)(x;p)\left(\epsilon_{0^+,\alpha,\beta,\sigma}^{\omega,\delta,q,r,c}tg\right)(x;p)\right|\\
\leq\ &||f'||_r||g'||_s\int_0^x\int_0^x(x-\tau)^{\beta-1}(x-\rho)^{\beta-1}|\tau-\rho|t(\tau)t(\rho)\\
&\times E_{0^+,\alpha,\beta,\sigma}^{\omega,\delta,q,r,c}\left(\omega(x-\tau)^{\alpha};p\right)E_{0^+,\alpha,\beta,\sigma}^{\omega,\delta,q,r,c}\left(\omega(x-\rho)^{\alpha};p\right)d\tau d\rho\\
\leq\ &||f'||_r||g'||_s x\left(\left(\epsilon_{0^+,\alpha,\beta,\sigma}^{\omega,\delta,q,r,c}t\right)(x;p)\right)^2.
\end{aligned}$$

Theorem 15. *[17] Let t be a positive function on $[0,\infty)$ and let f and g be two differentiable functions on $[0,\infty)$. If $f' \in L_r([0,\infty))$, $g' \in L_s([0,\infty))$, $r^{-1}+s^{-1}=1$, then for all $x>0$, $\alpha,\beta,\lambda,\theta>0$, we have*

$$\begin{aligned}
&\Big|\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t\right)(x;p)\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}tfg\right)(x;p)+\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t\right)(x;p)\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tfg\right)(x;p)\\
&-\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tf\right)(x;p)\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}tg\right)(x;p)-\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}tf\right)(x;p)\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tg\right)(x;p)\Big|\\
\leq\;& ||f'||_r||g'||_s\int_0^x\int_0^x(x-\tau)^{\beta-1}(x-\rho)^{\theta-1}|\tau-\rho|t(\tau)t(\rho)\\
&\times E^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}\left(\omega(x-\tau)^{\alpha};p\right)E^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}\left(\omega(x-\rho)^{\lambda};p\right)d\tau d\rho\\
\leq\;& ||f'||_r||g'||_s x\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t\right)(x;p)\left(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t\right)(x;p).
\end{aligned}$$

Definition 16. *[1] Let f be defined on $[a,b]$ and $\alpha\in C$, $Re(\alpha)>0$, $\rho>0$. Then*
(i) The mixed left conformable fractional integral of f is defined by

$$ {}^{b}_{a}\mathfrak{J}^{\alpha,\rho}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x f(s)\bigg(\frac{(b-s)^{\rho}-(b-x)^{\rho}}{\rho}\bigg)^{\alpha-1}(b-s)^{\rho-1}ds. \tag{3}$$

and
(ii) The mixed right conformable fractional integral of f is defined by

$$ {}^{a}\mathfrak{J}^{\alpha,\rho}_{b}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b f(s)\bigg(\frac{(s-a)^{\rho}-(x-a)^{\rho}}{\rho}\bigg)^{\alpha-1}(s-a)^{\rho-1}ds. \tag{4}$$

For recent results related to this operators, we refer the reader [1,6,18].

2. Main Results

We obtain in this section certain integral inequalities for the differentiable functions involving the mixed conformable fractional integral operator.

Theorem 17. *Let p be a positive function on $[0,\infty[$ and let f and g be two differentiable functions on $[0,\infty[$. If $f'\in L_r([0,\infty[)$, $g'\in L_s([0,\infty[)$, $r^{-1}+s^{-1}=1$, then for all $t>0$, $\alpha,\rho>0$, we have*

$$\begin{aligned}
&2\left|{}^{b}_{0}\mathfrak{J}^{\alpha,\rho}p(t)\,{}^{b}_{0}\mathfrak{J}^{\alpha,\rho}pfg(t)-{}^{b}_{0}\mathfrak{J}^{\alpha,\rho}fp(t)\,{}^{b}_{0}\mathfrak{J}^{\alpha,\rho}pg(t)\right|\\
\leq\;&\frac{||f'||_r||g'||_s}{\Gamma^2(\alpha)}\left[\int_0^t\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right.\\
&\left.\times\left(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|p(x)p(y)dxdy\right]\\
\leq\;&||f'||_r||g'||_s t\left({}^{b}_{0}\mathfrak{J}^{\alpha,\rho}p(t)\right)^2.
\end{aligned}\tag{5}$$

Proof. Let f and g be two functions satisfying the conditions of Theorem 17 and let p be a positive function on $[0,\infty[$.
Define

$$H(x,y) := (f(x)-f(y)(g(x)-g(y))); \;\; x,y\in(0,t),\, t>0. \tag{6}$$

Multiplying (6) by $\frac{1}{\Gamma(\alpha)}\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}p(x)$ and integrating the resulting identity with respect to x from 0 to t, we can write

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}p(x)H(x,y)dx\\ =\;& {}_0^b\mathfrak{J}^{\alpha,\rho}pfg(t)-f(y)\,{}_0^b\mathfrak{J}^{\alpha,\rho}pg(t)-g(y)\,{}_0^b\mathfrak{J}^{\alpha,\rho}pf(t)+f(y)g(y)\,{}_0^b\mathfrak{J}^{\alpha,\rho}p(t).\end{aligned} \tag{7}$$

Now, multiplying (7) by $\frac{1}{\Gamma(\alpha)}\left(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}p(y)$ and integrating the resulting identity with respect to y from 0 to t, we can write

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\\ &\times\left(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}p(x)p(y)H(x,y)dx\\ =\;& 2\left({}_0^b\mathfrak{J}^{\alpha,\rho}p(t)\,{}_0^b\mathfrak{J}^{\alpha,\rho}pfg(t)-{}_0^b\mathfrak{J}^{\alpha,\rho}pf(t)\,{}_0^b\mathfrak{J}^{\alpha,\rho}pg(t)\right).\end{aligned} \tag{8}$$

On the other hand, we have

$$H(x,y) := \int_x^y\int_x^y f'(u)g'(w)dudw. \tag{9}$$

Using Hölder inequality for double integral, we can write

$$|H(x,y)| \le \left|\int_x^y\int_x^y |f'(u)|^r dudw\right|^{r^{-1}}\left|\int_x^y\int_x^y |g'(w)|^s dudw\right|^{s^{-1}}. \tag{10}$$

Since

$$\left|\int_x^y\int_x^y |f'(u)|^r dudw\right|^{r^{-1}} = |x-y|^{r^{-1}}\left|\int_x^y |f'(u)|^r du\right|^{r^{-1}} \tag{11}$$

and

$$\left|\int_x^y\int_x^y |g'(w)|^s dudw\right|^{s^{-1}} = |x-y|^{s^{-1}}\left|\int_x^y |g'(w)|^s dw\right|^{s^{-1}}, \tag{12}$$

then, we can estimate H as follows:

$$|H(x,y)| \le |x-y|\left|\int_x^y |f'(u)|^r du\right|^{r^{-1}}\left|\int_x^y |g'(w)|^s du\right|^{s^{-1}}. \tag{13}$$

On the other hand, we have

$$\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}$$

$$
\begin{aligned}
&\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}p(x)p(y)|H(x,y)|dxdy \\
\leq\ & \frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1} \\
&\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1} \\
&\times p(x)p(y)|x-y|\left|\int_x^y|f'(u)|^r du\right|^{r^{-1}}\left|\int_x^y|g'(w)|^s dw\right|^{s^{-1}}dxdy. \qquad (14)
\end{aligned}
$$

Applying again Hölder inequality to right-hand side of (14), we can write

$$
\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1} \\
&\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}p(x)p(y)|H(x,y)|dxdy \\
\leq\ & \left[\frac{1}{\Gamma^r(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right. \qquad (15) \\
&\left.\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|p(x)p(y)\left|\int_x^y|f'(u)|^r du\right|dxdy\right]^{r^{-1}} \\
&\times\left[\frac{1}{\Gamma^s(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right. \\
&\left.\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|p(x)p(y)\left|\int_x^y|g'(w)|^s dw\right|dxdy\right]^{s^{-1}}.
\end{aligned}
$$

Now, using the fact that

$$
\left|\int_x^y|f'(u)|^r du\right|\leq||f'||_r^r,\quad \left|\int_x^y|g'(w)|^s dw\right|\leq||g'||_s^s\ , \qquad (16)
$$

we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1} \\
&\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}p(x)p(y)|H(x,y)|dxdy \\
\leq\ & \left[\frac{||f'||_r^r}{\Gamma^r(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right. \qquad (17)
\end{aligned}
$$

$$\times \left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|p(x)p(y)dxdy\Bigg]^{r^{-1}}$$

$$\times \left[\frac{||g'||_s^s}{\Gamma^s(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right.$$

$$\left.\times \left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|p(x)p(y)dxdy\right]^{s^{-1}}.$$

From (17), we get

$$\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\\
&\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}p(x)p(y)|H(x,y)|dxdy\\
\le\ &\left[\frac{||f'||_r||g'||_s}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right.\\
&\left.\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|p(x)p(y)dxdy\right]^{r^{-1}}.
\end{aligned}\tag{18}$$

Since $r^{-1}+s^{-1}=1$, then we have

$$\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\\
&\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}p(x)p(y)|H(x,y)|dxdy\\
\le\ &\frac{||f'||_r||g'||_s}{\Gamma^2(\alpha)}\left[\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right.\\
&\left.\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|p(x)p(y)dxdy\right].
\end{aligned}\tag{19}$$

By the relations (6) and (19) and using the properties of the modulus, we get the first inequality in Theorem 17 .
Now we shall prove the second inequality of Theorem 17, we have

$$0\le x\le t,\ 0\le y\le t.$$

Hence

$$0\le|x-y|\le t. \tag{20}$$

Therefore, we have

$$
\begin{aligned}
& \frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t \left(\frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right)^{\alpha-1} (b-x)^{\rho-1} \\
& \times \left(\frac{(b-y)^\rho - (b-t)^\rho}{\rho} \right)^{\alpha-1} (b-y)^{\rho-1} p(x)p(y)|H(x,y)|dxdy \\
\leq \; & \frac{||f'||_r ||g'||_s t}{\Gamma^2(\alpha)} \left[\int_0^t \int_0^t \left(\frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right)^{\alpha-1} (b-x)^{\rho-1} \right. \\
& \left. \times \left(\frac{(b-y)^\rho - (b-t)^\rho}{\rho} \right)^{\alpha-1} (b-y)^{\rho-1} p(x)p(y)dxdy \right] \\
= \; & ||f'||_r ||g'||_s t \left({}_0^b\mathfrak{J}^{\alpha,\rho} p(t) \right)^2 .
\end{aligned}
$$

Theorem 17 is thus proved. □

Theorem 18. *Let p be a positive function on $[0,\infty[$ and let f and g be two differentiable functions on $[0,\infty[$. If $f' \in L_r([0,\infty[)$, $g' \in L_s([0,\infty[)$, $r^{-1}+s^{-1}=1$, then for all $t>0$, $\alpha,\beta,\rho>0$, we have*

$$
\begin{aligned}
& \big| {}_0^b\mathfrak{J}^{\alpha,\rho} p(t) \, {}_0^b\mathfrak{J}^{\beta,\rho} pfg(t) + {}_0^b\mathfrak{J}^{\beta,\rho} p(t) \, {}_0^b\mathfrak{J}^{\alpha,\rho} pfg(t) \\
& - {}_0^b\mathfrak{J}^{\alpha,\rho} pf(t) \, {}_0^b\mathfrak{J}^{\beta,\rho} pg(t) - {}_0^b\mathfrak{J}^{\beta,\rho} pf(t) \, {}_0^b\mathfrak{J}^{\alpha,\rho} pg(t) \big| \\
\leq \; & \frac{||f'||_r ||g'||_s}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t \left(\frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right)^{\alpha-1} (b-x)^{\rho-1} \\
& \times \left(\frac{(b-y)^\rho - (b-t)^\rho}{\rho} \right)^{\beta-1} (b-y)^{\rho-1} |x-y| p(x)p(y)dxdy \\
\leq \; & ||f'||_r ||g'||_s t \, {}_0^b\mathfrak{J}^{\alpha,\rho} p(t) \, {}_0^b\mathfrak{J}^{\beta,\rho} p(t), \qquad (21)
\end{aligned}
$$

where $H(x,y)$ are the same as given in (6).

Proof. Using the identity (7), we can write

$$
\begin{aligned}
& \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_0^t \int_0^t \left(\frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right)^{\alpha-1} (b-x)^{\rho-1} \\
& \times \left(\frac{(b-y)^\rho - (b-t)^\rho}{\rho} \right)^{\alpha-1} (b-y)^{\rho-1} p(x)p(y)H(x,y)dxdy \\
= \; & {}_0^b\mathfrak{J}^{\alpha,\rho} p(t) \, {}_0^b\mathfrak{J}^{\beta,\rho} pfg(t) + {}_0^b\mathfrak{J}^{\beta,\rho} p(t) \, {}_0^b\mathfrak{J}^{\alpha,\rho} pfg(t) \\
& - {}_0^b\mathfrak{J}^{\alpha,\rho} pf(t) \, {}_0^b\mathfrak{J}^{\beta,\rho} pg(t) - {}_0^b\mathfrak{J}^{\beta,\rho} pf(t) \, {}_0^b\mathfrak{J}^{\alpha,\rho} pg(t). \qquad (22)
\end{aligned}
$$

From the relation (13), we can obtain the following estimation

$$
\frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right)^{\alpha-1} (b-x)^{\rho-1} p(x)|H(x,y)|dx
$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}|x-y|p(x)$$
$$\times\left|\int_x^y |f'(u)|^r du\right|^{r^{-1}}\left|\int_x^y |g'(w)|^s\right|^{s^{-1}} dx. \tag{23}$$

Therefore, we have

$$\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_0^t\int_0^t \left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}$$
$$\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\beta-1}(b-y)^{\rho-1}p(x)p(y)|H(x,y)|dxdy$$
$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t \left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}$$
$$\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\beta-1}(b-y)^{\rho-1}|x-y|p(x)p(y)$$
$$\times\left|\int_x^y |f'(u)|^r du\right|^{r^{-1}}\left|\int_x^y |g'(w)|^s\right|^{s^{-1}} dxdy. \tag{24}$$

Applying Hölder inequality for double integral to the right-hand side of (24), yields

$$\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_0^t\int_0^t \left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}$$
$$\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\beta-1}(b-y)^{\rho-1}p(x)p(y)|H(x,y)|dxdy$$
$$\leq \left[\frac{1}{\Gamma^r(\alpha)}\int_0^t\int_0^t \left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right.$$
$$\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\beta-1}(b-y)^{\rho-1}|x-y|p(x)p(y)$$
$$\left.\times\left|\int_x^y |f'(u)|^r du\right| dxdy\right]^{r^{-1}}$$
$$\times\left[\frac{1}{\Gamma^s(\beta)}\int_0^t\int_0^t \left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right.$$
$$\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\beta-1}(b-y)^{\rho-1}|x-y|p(x)p(y)$$
$$\left.\times\left|\int_x^y |g'(w)|^s dw\right| dxdy\right]^{s^{-1}}. \tag{25}$$

By (16) and (25), we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\\
&\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\beta-1}(b-y)^{\rho-1}p(x)p(y)|H(x,y)|dxdy\\
\leq\ &\frac{||f'||_r||g'||_s}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t\left(\frac{(b-x)^\rho-(b-t)^\rho}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\\
&\times\left(\frac{(b-y)^\rho-(b-t)^\rho}{\rho}\right)^{\beta-1}(b-y)^{\rho-1}|x-y|p(x)p(y)dxdy. \qquad (26)
\end{aligned}$$

Using (22) and (26) and the properties of modulus, we get the first inequality in (21). □

3. Remarks

Now, let us briefly consider some special cases of the main results. In Theorem 17 and Theorem 18, if we choose $\rho = 1$ and make use of the relationship (3), then the main results are reduced to Theorem 2 and Theorem 3 obtained by Dahmani *et al.* [7].

References

[1] Abdeljawad, T., Fractional operators with boundary points dependent kernels and integration by parts, *Discrete and Continuous Dynamical Systems Series S*, 13(3) (2020), 351-375.

[2] Baleanu, D., Purohit, S.D., Chebyshev Type Integral Inequalities Involving the Fractional Hypergeometric Operators, *Abstract and Applied Analysis*, 2014 (2014), Article ID 609160, 1–10.

[3] Baleanu, D., Purohit, S.D., Prajapati, J.C., Integral inequalities involving generalized Erdélyi-Kober fractional integral operators, *Open Mathematics*, 14 (2016), 89–99.

[4] Chebyshev, P.L., Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites, *Proc. Math. Soc. Charkov*, 2 (1882), 93–98.

[5] Curiel, L., Galué, L., A generalization of the integral operators involving the Gauss hypergeometric function, *Revista Técnica de la Facultad de Ingeniería Universidad del Zulia*, 19(1) (1996), 17–22.

[6] Çelik, B., Set, E., Akdemir, A.O., Mixed Conformable Fractional Grüss-Type Inequalities, 2nd International Conference on Life and Engineering Sciences (ICOLES 2019), Book of Abstracts, Istanbul, Turkey, 2019.

[7] Dahmani, Z., Mechouar, O. and Brahami, S., Certain inequalities related to the Chebyshev's functional involving a Riemann-Liouville operator, *Bull. Math. Anal. Appl.*, 3(4) (2011), 38–44.

[8] Ekinci, A., Özdemir, M.E., Some new integral Inequalities via Riemann Liouville integral operators, *Applied and Computational Mathematics*, 18(3) (2019), 288–295.

[9] Farid, G., Existence of an integral operator and its consequences in fractional and conformable integrals, *Open Journal of Mathematical Sciences,* 3 (2019), 210–216.

[10] Gürbüz, M., Özdemir, M.E., On Some Inequalities for Product of Different Kinds of Convex Functions, *Turkish Journal of Science,* 5(1) (2020), 23–27.

[11] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., Theory and Applications of Fractional Differential Equations, vol. 204 of North-Holland Mathematics Studies, Elsevier, Amsterdam, The Netherlands, 2006.

[12] Kiryakova, V., Generalized Fractional Calculus and Applications, vol. 301 of Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow, UK, 1994.

[13] Ntouyas, S.K., Purohit, S.D., Tariboon, J., Certain Chebyshev Type Integral Inequalities Involving Hadamard's Fractional Operators, *Abstract and Applied Analysis*, 2014 (2014), Article ID 249091, 1–7.

[14] Özdemir, M.E., Set, E., Akdemir, A.O., Sarıkaya, M.Z., Some new Chebyshev type inequalities for functions whose derivatives belongs to L_p spaces, *Afrika Matematika,* 26 (2015), 1609–1619.

[15] Rahman, G., Baleanu, D., Qurashi, M.A., Purohit, S.D., Mubeen, S., Arshad, M., The extended Mittag-Leffler function via fractional calculus, *J. Nonlinear Sci. Appl.*, 10 (2017), 4244–4253.

[16] Sarıkaya, M.Z., Alp, N., On Hermite-Hadamard-Fejer type integral inequalities for generalized convex functions via local fractional integrals, *Open Journal of Mathematical Sciences,* 3(1) (2019), 273–284.

[17] Set, E., Özdemir, M.E., Demirbaş, S., Chebyshev type inequalities involving extended generalized fractional integral operators, *AIMS Mathematics,* 5(4) (2020), 3573–3583.

[18] Set, E., New Inequalities of Chebyshev Type for Mixed Conformable Fractional Integral Operators, 2nd International Conference on Life and Engineering Sciences (ICOLES 2019), Book of Abstracts, Istanbul, Turkey, 2019.

[19] Srivastava, H.M., Choi, J., Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

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DIGITAL HAUSDORFF DISTANCE ON A CONNECTED DIGITAL IMAGE

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Abstract. A digital image X can be considered as a subset of $\mathbb{Z}^n$ together with an adjacency relation where $\mathbb{Z}$ is the set of the integers and n is a natural number. The aim of this study is to measure the closeness of two subsets of a connected digital image. To do this, we adapt the Hausdorff distance in the topological setting to its digital version. In this paper, we define a metric on a connected digital image by using the length of the shortest digital simple path. Then we use this metric to define the r-thickening of the subsets of a connected digital image and define the digital Hausdorff distance between them.

1. Introduction

Digital images can be considered to be subsets of $\mathbb{Z}^n$. We study digital images to analyze not only their features but also their correlations with the others. Investigating the features of the digital images by their topological properties would be fine but the problem here is that the digital images are not topological spaces, they are just sets. This problem can be achieved by imposing an adjacency relation on a digital image to adapt the topological concepts such as neighborhood, continuity, connectivity, homotopy, and contractibility to their digital versions.

The distance between two points in a connected digital image is obtained via *the shortest simple path metric* which is denoted by d^κ. Then the distance, called *a digital Hausdorff distance*, between two subsets of a connected digital image can be calculated via d^κ as follows: One starts with fattening each subset by taking the union of the neighborhoods with radius r of all its points. We call this new set the *r-thickening of a subset*. Then the distance between subsets is the minimum

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r such that the r-thickening of both include another. Like the Hausdorff distance in topological setting [20], the digital Hausdorff distance will measure how close the two subsets are to each other. Note that the distance may differ according to the adjacency relation defined on a digital image. We also investigate that assigning each point to its neighborhood with radius r leads to a strongly (hence a weakly) continuous multi-valued function from a connected digital image to itself (Theorem 6). We observe that the image of the r-thickening of a given subset under a digital continuous map is contained in the r-thickening of the image of that subset (Theorem 14). We also show that the digital Hausdorff distance between the image of two subsets of a connected digital image under a continuous map is less than or equal to the Hausdorff distance between these two subsets (Theorem 24).

2. Background

To study the features of the digital images, we start with an *adjacency relation* defined on the points of $\mathbb{Z}^n$ to adapt the fundamental concepts of topology. For an integer ℓ with $1 \leq \ell \leq n$, we say two distinct points $x = (x_1, x_2, \ldots, x_2)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{Z}^n$ are *c_ℓ-adjacent* [11] if

i): $|x_i - y_i| = 1$ for at most ℓ indices i and

ii): for all indices j such that $|x_j - y_j| \neq 1$, we have $x_j = y_j$.

It turns out that c_1-adjacency in $\mathbb{Z}$ is 2-adjacency, c_1 and c_2 adjacencies in $\mathbb{Z}^2$ are 4-adjacency and 8-adjacency, and c_1, c_2 and c_3-adjacencies in $\mathbb{Z}^3$ are 6-adjacency, 18-adjacency and 26-adjacency respectively.

A digital image X is a subset of $\mathbb{Z}^n$ for some natural number n together with an adjacency relation κ inherited from $\mathbb{Z}^n$ and represented by (X, κ).

The continuous functions between the digital images are defined in terms of adjacency relations. Let (X, κ) and (Y, λ) be digital images and $f : X \to Y$ be a function. Then we say that f is (κ, λ)-*continuous* iff $f(x) = f(x')$ or $f(x)$ and $f(x')$ is λ-adjacent whenever x and x' are κ-adjacent for $x, x' \in X$ [2].

Consider the digital interval

$$[a, b]_{\mathbb{Z}} = \{c \in \mathbb{Z} \ : \ a \leq c \leq b\}$$

for integers a, b with $a < b$ where 2-adjacency relation is assumed [1]. Then a κ-*path* from x to y in a digital image (X, κ) is a sequence $(x = x_0, x_1, x_2, \ldots, x_m = y)$ in X such that x_i is κ-adjacent to x_{i+1} for $m \geq 1$ and $0 \leq i \leq m - 1$ [26]. In that case, m denotes the *length* of this path. A κ-path can be also considered as a $(2, \kappa)$-continuous map $\alpha : [0, m]_{\mathbb{Z}} \to X$ such that $\alpha(0) = x$ and $\alpha(m) = y$ [16]. Note that such a κ-path α from x to y can be reversed and the resulting map $\overline{\alpha}$, which is explicitly defined by $\overline{\alpha}(t) := \alpha(m - t)$ for $t \in [0, m]_{\mathbb{Z}}$, is a κ-path from y to x. We say that a κ-path $(x = x_0, x_1, x_2, \ldots, x_{m-1} = y)$ from x to y is *simple*, provided x_i and x_j are κ-adjacent if and only if either $j = i+1$, $i \in [0, m-2]_{\mathbb{Z}}$ or $i = j+1$, $j \in [0, m-2]_{\mathbb{Z}}$ [11,12]. A digital image (X, κ) is said to be *κ-connected*, provided there exists a κ-path between any pair of elements in X [15].

The digital analogue of homotopy is given as follows [2,16]: Let (X, κ) and (Y, λ) be two digital images and $f, g : X \to Y$ be (κ, λ)-continuous maps. Then we say f and g are *digitally (κ, λ)-homotopic* if there is a positive integer m and a function

$$H : X \times [0, m]_{\mathbb{Z}} \to Y$$

such that

- **i):** $H(x, 0) = f(x)$ and $H(x, m) = g(x)$;
- **ii):** for each $t \in [0, m]_{\mathbb{Z}}$, the induced map $H_t : X \to Y$ defined by $H_t(x) = H(x, t)$ is (κ, λ)-continuous; and
- **iii):** for each $x \in X$, the induced map $H_x : [0, m]_{\mathbb{Z}} \to Y$ defined by $H_x(t) = H(x, t)$ for $t \in [0, m]_{\mathbb{Z}}$ is $(2, \lambda)$-continuous.

Such a function H is called a *digital (κ, λ)-homotopy* between f and g. We denote $f \simeq_{\kappa,\lambda} g$ if there exists a digital (κ, λ)-homotopy between them.

3. Results

3.1. **The shortest digital path.** Suppose X is a κ-connected digital image, $x \in X$ and A is a nonempty subset of X. Boxer defines $l_X^\kappa(x, A)$ to be the length of the shortest κ-path from x to any other point A in [3]. For our purpose, we would want a digital path to be simple. Let $\ell^\kappa(x, A)$ denote *the length of the shortest simple κ-path in X from x to any point of A*. In that case, if x_1 is another point in X then $\ell^\kappa(x, \{x_1\})$ turns into its original definition and denotes the length of the shortest simple κ-path from x to x_1 given in [11,12]. Since any digital path can be reversed, $\ell^\kappa(x_0, \{x_1\}) = \ell^\kappa(x_1, \{x_0\})$ for any pair of elements x_0, x_1 in X. Throughout the paper we assume $\ell^\kappa(x, \{x\}) = 0$.

Under the assumption $\ell^\kappa(x, \{x\}) = 0$, consider the function $d^\kappa : X \times X \to \mathbb{R}$ defined by

$$d^\kappa(x_0, x_1) := \ell^\kappa(x_0, \{x_1\}) \tag{1}$$

for x_0, x_1 in X. It's trivial that $d^\kappa(x_0, x_1) = 0$ iff $x_0 = x_1$ and we have $d^\kappa(x_0, x_1) = d^\kappa(x_1, x_0)$ since every digital simple κ-path from x_0 to x_1 can be reversed as we mentioned before.

d^κ also satisfies the triangle inequality

$$d^\kappa(x_0, x_1) \leq d^\kappa(x_0, x_2) + d^\kappa(x_2, x_1)$$

for $x_0, x_1, x_2 \in X$. Suppose $\alpha : [0, m]_{\mathbb{Z}} \to X$ and $\beta : [0, k]_{\mathbb{Z}} \to X$ are simple κ-paths with shortest lengths from x_0 to x_2 and x_2 to x_1 respectively. In that case, we have $\alpha(0) = x_0$, $\alpha(m) = x_2$, $\beta(0) = x_2$, and $\beta(k) = x_1$. Then $\gamma : [0, m+k]_{\mathbb{Z}} \to X$ defined by

$$\gamma(t) = \begin{cases} \alpha(t), & 0 \le t \le i \\ \beta(t-m), & i \le t \le m+k \end{cases}$$

is a κ-path from x_0 to x_1. Note that γ may not be simple. Since there always exists a simple κ-path from x_0 to x_1 with a shortest length, say i, then i should be less than or equal to $m+k$.

Hence d^κ is a metric on a connected digital image X. We refer to [13, 14] for further reading.

Definition 1. *We call the metric d^κ given in (1) the shortest simple κ-path metric on a connected digital image.*

Let X be a κ-connected digital image and $x_0 \in X$. The κ-neighborhood of x_0 in X with some radius $r \ge 0$ is given by [11]

$$\mathcal{B}_\kappa(x_0, r) = \{x \in X \ : \ d^\kappa(x_0, x) \le r\}.$$

Obviously, $B_\kappa(x_0, r)$ is a κ-connected subset of X and $\mathcal{B}_\kappa(x_0, 0) = \{x_0\}$.

Lemma 2. *Suppose (X, κ) is a digital image and $x_1, x_2 \in X$. If x_1 and x_2 are κ-adjacent, then $x_2 \in \mathcal{B}_\kappa(x_1, 1)$ and $x_1 \in \mathcal{B}_\kappa(x_2, 1)$.*

Proof. The shortest simple κ-path from x_1 and x_2 in X is a path $\alpha : [0,1]_{\mathbb{Z}} \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$. □

A digital image (X, κ) is said to be *κ-contractible* if its identity function is digitally (κ, κ)-homotopic to a constant map on X [1].

Theorem 3. *Suppose X is a κ-connected digital image and $x \in X$. Then $\mathcal{B}_\kappa(x, r)$ is κ-contractible for $r = 1, 2$ but $\mathcal{B}_\kappa(x, r)$ may not be κ-contractible for $r \ge 3$.*

Proof. To see the contractibility of $\mathcal{B}_\kappa(x, r)$ for $r = 1, 2$, we will construct a digital homotopy between the identity map on it and a constant map at x. We define the digital homotopy

$$H : \mathcal{B}_\kappa(x, r) \times [0,1]_{\mathbb{Z}} \to \mathcal{B}_\kappa(x, r)$$

by $H(y, 0) = y$ and $H(y, 1) = x$ for every $y \in \mathcal{B}_\kappa(x, r)$. Then H is the desired homotopy.

Consider the 8-connected digital image

$$\mathrm{MSC}_8 = \{(0,0), (1,1), (2,1), (3,0), (2,-1), (1,-1)\}$$

in $\mathbb{Z}^2$ illustrated in Figure 1. Observe that $\mathcal{B}_8((0,0), 3)$ is the entire image MSC_8 which is not 8-contractible [11]. □

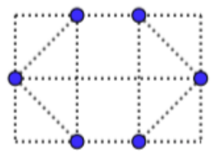

FIGURE 1. The digital image MSC_8 is 8-connected but not 8-contractible [11].

Two subsets A and B of a digital image (X, κ) are said to be κ-adjacent if there exist $a \in A$ and $b \in B$ such that $a = b$ or a and b are κ-adjacent [3].

Corollary 4. *If x and y are κ-adjacent in a κ-connected digital image (X, κ), then $\mathcal{B}_\kappa(x, r)$ and $\mathcal{B}_\kappa(y, r)$ are κ-adjacent sets for every nonnegative integer r.*

3.2. **The κ-neighborhoods as multi-valued functions.** Next consider a multi-valued function $F : X \multimap Y$ between two digital images (X, κ) and (Y, λ). That is, F maps each point of X to a subset of Y and for a subset A of X, $F(A) = \cup_{a \in A} F(a)$. The continuity notion for a multi-valued digital map is also defined (see [8–10]) but in this paper we only consider the other two continuity notions given in [27].

Definition 5. *[27] Suppose (X, κ) and (Y, λ) are two digital images and $F : X \multimap Y$ is a multi-valued map.*

- **i):** *F is said to be* weakly continuous*, provided whenever x_0 and x_1 are κ-adjacent elements in X, $F(x_0)$ and $F(x_2)$ are λ-adjacent subsets of Y.*
- **ii):** *F is said to be* strongly continuous*, provided whenever x_0 and x_1 are κ-adjacent elements in X, every point of $F(x_0)$ is λ-adjacent to some point in $F(x_1)$ and vice versa.*

Let (X, κ) be a κ-connected digital image and r be a nonnegative integer. Define the multi-valued function $F_r : X \multimap X$ by $F_r(x) = \mathcal{B}_\kappa(x, r)$ for $x \in X$. Then the following corollary is one of the immediate consequences of Lemma 2.

Theorem 6. *The multi-valued function F_r is strongly continuous.*

Proof. The multi-valued function F_0 turns into a single-valued function and it satisfies the strong continuity condition immediately. The proof is also trivial when $r = 1$: Let x_1 and x_2 be κ-adjacent in X. Since $x_1 \in \mathcal{B}_\kappa(x_2, 1)$, any point of $\mathcal{B}_\kappa(x_1, 1)$ is κ-adjacent to a some point of $\mathcal{B}_\kappa(x_2, 1)$ and vice versa. For $r \geq 2$, observe that any element $x \in \mathcal{B}_\kappa(x_1, r) \setminus \mathcal{B}_\kappa(x_2, r)$ is contained in $\mathcal{B}_\kappa(x_2, r+1)$ and this completes the proof. $\square$

Corollary 7. *F_r is weakly continuous.*

By Corollary 7, the map F_r is also a connectivity preserving function. Note that a multi-valued function $F : X \multimap Y$ is *connectivity preserving* iff F is weakly continuous and $F(x)$ is a λ-connected subset of the digital image (Y, λ) for every $x \in X$ [6].

Definition 8. *Suppose (X,κ) is a κ-connected digital image and A is a nonempty subset of X. For a nonnegative integer r, the r-thickening of A, $A^{(r,\kappa)}$, is given by*

$$A^{(r,\kappa)} = \cup_{a\in A}\mathcal{B}_\kappa(a,r)$$

Remark 9. $A^{(r,\kappa)} = F_r(A)$.

Suppose κ_1 and κ_2 are two adjacency relations on a set X. Then we say that κ_1 *dominates* κ_2, $\kappa_1 \geq_d \kappa_2$, if for $x_1, x_2 \in X$, if x_1 and x_2 are κ_1-adjacent then x_1 and x_2 are κ_2-adjacent [4]. Further if X is κ_1-(hence κ_2)-connected and A is a subset of X, then $A^{(r,\kappa_1)} \subseteq A^{(r,\kappa_2)}$.

Example 10. *Consider the 18-connected digital image* $\mathrm{MSS}_{18} = \{x_i\}_{i=0}^{9}$ *in* $\mathbb{Z}^3$ *[12] where*

$$x_0 = (0,0,0), x_1 = (1,1,0), x_2 = (0,1,-1), x_3 = (0,2,-1), x_4 = (1,2,0),$$
$$x_5 = (0,3,0), x_6 = (-1,2,0), x_7 = (0,2,1), x_8 = (0,1,1), x_9 = (-1,1,0)$$

(see Figure 2). Let $A = \{x_0, x_7\}$ and $B = \{x_5\}$. Then $A^{(1,18)} = \mathcal{B}_{18}(x_0,1) \cup \mathcal{B}_{18}(x_7,1) = X \setminus \{x_3\}$ and $B^{(2,18)} = X \setminus \{x_0\}$.

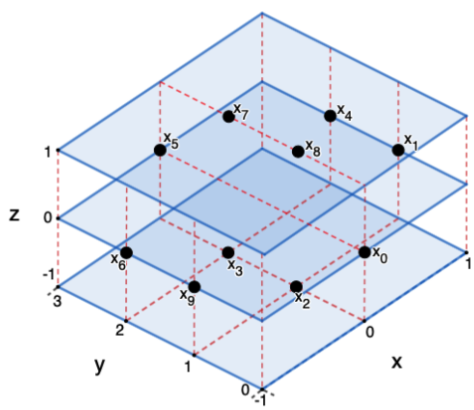

FIGURE 2. 18-connected digital image MSS_{18} [12].

Proposition 11. *Let (X,κ) be a κ-connected digital image and A be a subset of X. For two nonnegative integers r_1 and r_2, we have $(A^{(r_2,\kappa)})^{(r_1,\kappa)} = A^{(r_1+r_2,\kappa)}$.*

Proof. It is trivial that $A^{(r_1+r_2,\kappa)} \subseteq (A^{(r_2,\kappa)})^{(r_1,\kappa)}$. Let $a \in (A^{(r_2,\kappa)})^{(r_1,\kappa)}$. Then there is $b \in A^{(r_2,\kappa)}$ such that $d^\kappa(a,b) \leq r_1$ and $c \in A$ such that $d^\kappa(b,c) \leq r_2$. Therefore

$$d^\kappa(a,c) \leq d^\kappa(a,b) + d^\kappa(b,c) \leq r_1 + r_2$$

so $a \in A^{(r_1+r_2,\kappa)}$ and this completes the proof. □

The immediate consequence is the following Corollary.

Corollary 12. *Suppose (X,κ) is a κ-connected digital image and A and C are κ-adjacent subsets of X. Then there exists $a \in A$ such that $a \in C^{(1,\kappa)}$ or vice-versa.*

Proof. The proof follows from Lemma 2 □

Proposition 13. *Suppose (X,κ) is a κ-connected digital image and A is a subset of X. Then the following holds:*

i): *$X \setminus A$ and A are κ-adjacent,*
ii): *There exists $x \in X \setminus A$ such that $x \in A^{(1,\kappa)}$.*

Proof. **:**

i): If $X \setminus A$ and A were not κ-adjacent sets, this would mean that none of the elements of $X \setminus A$ would be κ-adjacent to any elements of A so that X would not be κ-connected.
ii): This follows from $i)$ and Corollary 12.

□

Theorem 14. *Suppose (X,κ) and (Y,λ) are digital images and X is κ-connected. If A is a subset of X and $f : X \to Y$ is (κ,λ)-continuous, then $f(A^{(r,\kappa)}) \subseteq f(A)^{(r,\lambda)}$ for every positive integer r.*

Proof. Let $y \in f(A^{(r,\kappa)})$. Then there exists an element $x \in A^{(r,\kappa)}$ such that $f(x) = y$. Let a be a point in A such that $\ell^\kappa(x,a) \leq r$; such a point a exists, since $x \in A^{(r,\kappa)}$. That is, the length of the shortest simple κ-path from x to a in X is less than or equal to r. We also have a λ-path between $y = f(x)$ and $f(a)$ by the continuity of f and the length of this path cannot be greater than $\ell^\kappa(x,a)$. If this λ-path is simple then we have $\ell^\lambda(y, f(a)) \leq r$. If it is not simple, we can reduce it to a simple λ-path from y to $f(a)$ so that the length of the reduced path cannot be greater than $\ell^\kappa(x,a)$. Hence $y \in f(A)^{(r,\lambda)}$. □

For a digital image (X,κ) and its nonempty subset Y, we say Y is *κ-dominating* in X [7] if for every $x \in X$, there exists $y \in Y$ such that $d^\kappa(x,y) \leq 1$. Unlike the definition of κ-dominating, in the following we consider two subsets of a digital image which need not be contained in one another and we give a notion of κ-monitoring. For more details on κ-dominating, see [5].

Definition 15. *Let (X,κ) be a digital image, A and B be the subsets of X. We say that A κ-monitors B if for any $b \in B$, there exists $a \in A$ such that $d^\kappa(a,b) \leq 1$.*

Suppose κ_1 and κ_2 are two adjacency relations on a set X such that $\kappa_2 \geq_d \kappa_1$ and A and B are subsets of X. If A κ_2-monitors B, then A also κ_1-monitors B.

Remark 16. *If A κ-monitors B, then A and B are κ-adjacent sets. However the converse may not be true.*

Lemma 17. *Let (X,κ) be a κ-connected digital image and A and B be the subsets of X. Then A κ-monitors B if and only if $B \subseteq A^{(1,\kappa)}$.*

Proof. Let A κ-monitors B and $b \in B$. Then there is $a \in A$ such that $d^\kappa(a,b) \leq 1$ so that $b \in A^{(1,\kappa)}$. This is always true for every element b in B, hence $B \subseteq A^{(1,\kappa)}$. On the contrary, let $B \subseteq A^{(1,\kappa)}$. This means that any element b in B is an element in $A^{(1,\kappa)}$ so that there is $a \in A$ such that $d^\kappa(a,b) \leq 1$. Therefore A κ-monitors B. □

Theorem 18. *Let (X,κ) and (Y,λ) be two digital images, X be κ-connected, $f : X \to Y$ be a (κ,λ)-continuous function, and A and B be subsets of X. If A κ-monitors B, then $f(A)$ λ-monitors $f(B)$.*

Proof. If A κ-monitors B, then $B \subseteq A^{(\kappa,1)}$ by Lemma 17. Applying f to that gives $f(B) \subseteq f(A^{(\kappa,1)})$. Since $f(A^{(\kappa,1)}) \subseteq f(A)^{(1,\lambda)}$ by Theorem 14, we have $f(B) \subseteq f(A)^{(1,\lambda)}$ so that $f(A)$ λ-monitors $f(B)$. □

The following corollary follows from the definition of a strongly continuous multi-valued function.

Corollary 19. *Let (X,κ) and (Y,λ) be two digital images, $x_0, x_1 \in X$ and $F : X \multimap Y$ be a strongly continuous multi-valued function. If x_0 and x_1 are κ-adjacent, then $F(x_1)$ λ-monitors $F(x_0)$ and vice versa.*

3.3. **Digital Hausdorff Distance.** We know that d^κ is a metric on a κ-connected digital image X. Next, we want to measure the distance between two nonempty subsets A and B of X. We call the distance digital Hausdorff distance between them. To do this, we will find the minimum r so that the r-thickening of each subset will contain another.

Definition 20. *The digital Hausdorff distance between two subsets A and B of a κ-connected digital image X is*

$$d_H^\kappa(A,B) = \min\{r \geq 0 \ : \ B \subseteq A^{(r,\kappa)} \text{ and } A \subseteq B^{(r,\kappa)}\} \tag{2}$$

Suppose (X,κ) is a κ-connected digital image and A and B are subsets of X. If κ_1 is another adjacency relation on X with $\kappa \geq_d \kappa_1$, then X is also κ_1-connected digital image and $d_H^\kappa(A,B) \geq d_H^{\kappa_1}(A,B)$ since the length of the shortest simple κ-path between any given pair in X might be greater than or equal to the length of the shortest simple κ_1-path between them.

Example 21. *Consider the 8-connected digital image $X = \{x_i\}_{i=1}^6$ in $\mathbb{Z}^2$ where*

$$\{x_1 = (0,0), x_2 = (0,-1), x_3 = (1,-1), x_4 = (1,0), x_5 = (2,1), x_6 = (3,1)\}$$

(see Figure 3) and let $A = \{x_1, x_2\}$ *and* $B = \{x_5, x_6\}$. *Then* $d_H^8(A, B) = 3$ *since* $A \subseteq B^{(2,8)}$, $B \subseteq A^{(3,8)}$, *and* $B \nsubseteq A^{(2,8)}$. *We also have*

$$d_H^8(F_1(A), F_1(B)) = 2$$

since $F_1(A) \subseteq F_1(B)^{(1,8)}$, $F_1(B) \subseteq F_1(A)^{(2,8)}$, *and* $F_1(B) \nsubseteq F_1(A)^{(1,8)}$ *where* $F_1 : X \multimap X$ *is a multimap defined by* $F_1(x) = \mathcal{B}_\kappa(x, 1)$ *for* $x \in X$.

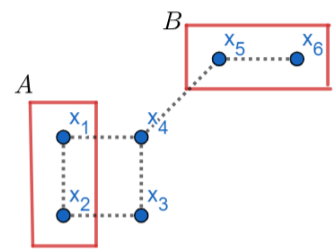

FIGURE 3. The 8-connected digital image X and the points in red rectangules from left to right are A and B.

Proposition 22. *Let* (X, κ) *be a* κ*-connected digital image and* A *be a subset of* X. *Then* $d_H^\kappa(A, A \cup \{x\}) = \ell^\kappa(x, A)$ *for* $x \in X$.

Proof. The proof follows from the fact that if $\ell^\kappa(x, A) = n$ then $x \in A^{(n,\kappa)}$. □

Proposition 23. *For a nonnegative integer* r,

$$d_H^\kappa(\mathcal{B}_\kappa(x_1, r), \mathcal{B}_\kappa(x_2, r)) \leq 1$$

whenever x_1 *is* κ*-adjacent to* x_2 *in a* κ*-connected digital image* (X, κ).

Proof. The proof follows from the fact that $\mathcal{B}_\kappa(x, r+1) = \mathcal{B}_\kappa(x, r)^{(1,\kappa)}$ for all $x \in X$ and every nonnegative integer r. □

Now we will prove the stability of the digital Hausdorff distance under a digital continuous map.

Theorem 24. *Let* (X, κ) *and* (Y, λ) *be digital images,* X *connected,* $A, B \subseteq X$. *If* $f : X \to Y$ *is* (κ, λ)*-continuous, then*

$$d_H^\lambda(f(A), f(B)) \leq d_H^\kappa(A, B).$$

Proof. Assume that $d_H^\kappa(A, B) = s$. Then by the definition of the digital Hausdorff distance, s is the minimum number such that $A \subseteq B^{(s,\kappa)}$ and $B \subseteq A^{(s,\kappa)}$. By these inclusions and Theorem 14, we have

$$f(A) \subseteq f(B^{(s,\kappa)}) \subseteq f(B)^{(s,\lambda)}$$
$$f(B) \subseteq f(A^{(s,\kappa)}) \subseteq f(A)^{(s,\lambda)}.$$

so that $f(A)$ and $f(B)$ can be covered by the s-thickening of $f(B)$ and $f(A)$ with respect to the adjacency λ respectively. Hence $d_H^{\lambda}(f(A), f(B)) \leq s$. $\square$

Example 25. *Consider the following two 4-connected digital images $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3\}$ in $\mathbb{Z}^2$ illustrated in Figure 4. Let $A = \{x_1, x_5\}$ and $B = \{x_2, x_3, x_4\}$ be the two subsets of X. If the digital $(4,4)$-continuous map $f : X \to Y$ is given with*

$$f(x_1) = f(x_4) = y_2, f(x_5) = y_3, \text{ and } f(x_2) = f(x_3) = y_1,$$

then $d_H^4(A, B) = 2$ and $d_H^4(f(A), f(B)) = 1$.

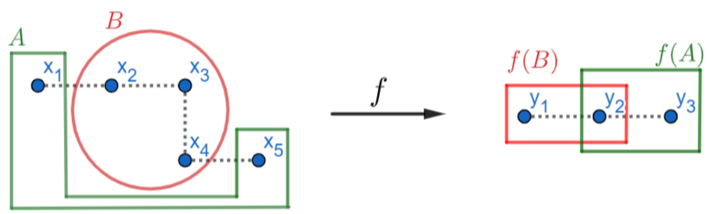

FIGURE 4. The Hausdorff distance is stable under a digital continuous map between digital images. In this example, we have $d_H^4(f(A), f(B)) \leq d_H^4(A, B)$.

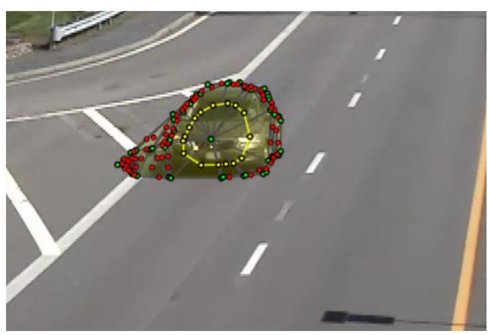

FIGURE 5. Triangulated traffic video frame shape.

4. Applications

This section briefly presents applications of Hausdorff Distance. The following two applications were suggested by James F. Peters [21].

4.1. **Zero-Shot Surface Shape Recognition.** This application of the proposed Hausdorff distance between sets focuses on a zero-shot recognition approach in the detection and classification of surface shapes recorded in video frames. Zero-shot classification of images with no training data is highly attractive, since it is less rigid than traditional classification techniques that rely on training data and, hence, build into the learning process unwanted *à priori* assumptions implicit in the training data. For more about this, see M. Molina and J. Sánchez [19], J. Lu and J. Li and Z. Yan and C. Zhang [18] and J.F. Peters [22].

Let sh$E$, bdy(sh$E$) and int(sh$E$) denote a surface shape in a video frame, shape boundary and shape interior, respectively. Also, let $p \in \text{int}(\text{sh}E)$ be the shape centroid and let $th > 0$ be a threshold. For each shape in a video frame, find the Hausdorff distance $D(p, \text{bdy}(\text{sh}E))$ between the centroid of sh$E$ and bdy(sh$E$) that is less than or equal to a threshold th, defined by

$$\overbrace{D(p, \text{bdy}(\text{sh}E)) = \inf\{\|p-q\| : q \in \text{bdy}(\text{sh}E)\} < th\}}^{\textbf{Hausdoff distance criterion}}.$$

This would be useful in finding video frames that contain shapes that have the required distance property relative to a fixed threshold th.

Example 26. *A sample traffic video frame that displays a triangulated vehicle shape shE is shown in Fig. 5. A green vertex p inside the yellow cycle on shE marks the location of the shape centroid. Vertices along the shape boundary are represented by red bullets. The Hausdorff distance $D(p, bdy(shE))$ would be computed and compared with other vehicle shapes in this video to construct a vehicle shape class.*

For a particular shape class, the members of the class satisfy the Hausdorf distance criterion.

4.2. **Descriptive Leader Uniform Topology.** A clusters form of proximity space-based uniform topology was introduced by S. Leader [17], elaborated in [24, 25]. This application uses the Hausdorff distance property from Application 4.1 as a feature for a set of video frame shapes equipped with a descriptive proximity mapping $\Phi : \text{sh}E \to \mathbb{R}$, which provides a basis for the formation of a descriptive Leader uniform class of shapes $\mathfrak{C}_\Phi(\text{sh}E)$ for each shape sh$E$ in the following way.

$$\begin{aligned}
th &= \text{selected threshold such that } th > 0.\\
fr &= \text{video frame.}\\
\text{sh}E \in fr &= \text{video frame shape.}\\
\text{bdy}(\text{sh}E) &= \text{boundary of sh}E.\\
D(p \in \text{sh}E, \text{bdy}(\text{sh}E)) &= inf\,\{\|p-q\| : q \in \text{bdy}(\text{sh}E)\} < th.\\
\Phi(\text{sh}E) &= D(p \in \text{sh}E, \text{bdy}(\text{sh}E)).\\
\mathfrak{C}_\Phi(\text{sh}E) &= \overbrace{\{\text{sh}E' : \|\Phi(\text{sh}E) - \Phi(\text{sh}E')\| < th\}}^{\textbf{Class of video frame shapes}}.
\end{aligned}$$

Then, for each given shape sh$E$ in a video, a shape shE' belongs to a class $\mathfrak{C}_\Phi(\mathrm{sh}E)$ of shapes relative to sh$E$, provided $\Phi(\mathrm{sh}E) = \Phi(\mathrm{sh}E')$ in video frames fr and fr' define a descriptive Leader uniform topology, i.e., a collection of shape classes in which nonempty disjoint sets of shapes are descriptively near each other [23, §4.16, p. 189].

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References

[1] Boxer, L., Digitally continuous functions, *Pattern Recognition Letters,* 15 (1994), 833-839.

[2] Boxer, L., A classical construction for the digital fundamental group, *Journal of Mathematical Imaging and Vision,* 10 (1999), 51-62.

[3] Boxer, L., Multivalued functions in digital topology, *Note di Matematica*, 37(2) (2017), 61-76.

[4] Boxer, L., Alternate product adjacencies in digital topology, *Applied General Topology*, 19(1) (2018), 21-53.

[5] Boxer, L., Ege, O., Karaca, I., Lopez, J., Louwsma, J., Digital fixed points, approximate fixed points, and universal functions, *Applied General Topology*, 17(2) (2016), 159-172.

[6] Boxer, L. Staecker, P. C., Connectivity preserving multivalued functions in digital topology, *Journal of Mathematical Imaging and Vision*, 55(3) (2016), 370-377.

[7] Chartrand, G., Lesniak, L., Graphs & digraphs, 2nd ed., Wadsworth, Inc., Belmont, Ca, 1986.

[8] Escribano, C., Giraldo, A., Sastre, M., Digitally continuous multivalued functions, in Discrete Geometry for Computer Imagery, *Lecture Notes in Computer Science,* v. 4992, 81-92, Springer, 2008.

[9] Escribano, C., Giraldo, A., Sastre, M., Digitally continuous multi-valued functions, morphological thinning algorithms, *Journal of Mathematical Imaging and Vision,* 42(1) (2012), 76-91.

[10] Giraldo, A., Sastre, M., On the composition of digitally continuous multivalued functions, *Journal of Mathematical Imaging and Vision,* 53 (2) (2015), 196-209.

[11] Han, S.-E., Non-product property of the digital fundamental group, *Information Sciences,* 171(1) (2005), 73-91.

[12] Han, S.-E., Digital fundamental group and Euler characteristic of a connected sum of digital closed surfaces, *Information Sciences,* 177 (2007), 3314-3326.

[13] Han, S.-E., Estimation of the complexity of a digital image from the viewpoint of fixed point theory, *Applied Mathematics and Computation*, 347 (2019), 236-248.

[14] Han, S.-E., Digital k-Contractibility of an n-Times Iterated Connected Sum of Simple Closed k-Surfaces and Almost Fixed Point Property, *Mathematics*, 8(345) (2020), 1-24, doi:10.3390/math8030345

[15] Herman, G.T., Oriented surfaces in digital spaces, *CVGIP: Graphical Models and Image Processing,* 55 (1993), 381-396.

[16] Khalimsky, E., Motion, deformation and homotopy in finite spaces, *In Proceedings IEEE International Conference on Systems, Man, and Cybernetics,* (1987), 227-234.

[17] Leader, S., On clusters in proximity spaces, *Fund. Math.,* 47 (1959), 205-213.

[18] Lu, J., Li, J., Yan, Z., Zhang, C., Zero-Shot Learning by Generating Pseudo Feature Representations, arXiv1703.06389v1, 2017, 1-9.

[19] Molina, M., Sánchez, J., Zero-Shot Learning with Partial Attributes, *Intelligent Computing Systems. ISICS 2018. Communications in Computer and Information Science*, vol 820, Springer Nature, Switzerland AG, (2018), 147–158, https://doi-org.uml.idm.oclc.org/10.1007/978-3-319-76261-6_12.

[20] Munkres, J., Topology (2nd ed.), pp. 280-281, Prentice Hall, 1999.

[21] Peters, J.F., personal communication.

[22] Peters, J.F., Computational geometry, topology and physics of digital images with applications. Shape complexes, optical vortex nerves and proximities, Springer Nature*,* Cham, Switzerland, 2020, xxv+440 pp., https://doi-org.uml.idm.oclc.org/10.1007/978-3-030-22192-8, Zbl07098311.

[23] Peters, J.F., Topology of Digital Images. Visual Pattern Discovery in Proximity Spaces, *Intelligent Systems Reference Library* 134 (2014), Springer Heidelberg, xxvii+ 433pp, https://doi-org.uml.idm.oclc.org/10.1007/978-3-642-53845-2, Zentralblatt MATH Zbl 1295 68010.

[24] Peters, J.F., Computational proximity. Excursions in the topology of digital images, *Intelligent Systems Reference Library* 102 (2016), Cham: Springer, xv + 411pp, https://doi-org.uml.idm.oclc.org/10.1007/978-3-319-30362-4, Zentralblatt MATH Zbl 1382.68008.

[25] Peters, J.F., Proximal Voronoï regions, convex polygons, & Leader uniform topology, *Adv. Math.: S.J.*, 4 (1) (2015), 1-5, Zbl 1335.65032.

[26] Rosenfeld, A., Continuous functions on digital images, *Pattern Recognition Letters,* 4 (1987), 177-184.

[27] Tsaur, R. and Smyth, M. B., "Continuous" multifunctions in discrete spaces with applications to fixed point theory, *In: Bertrand G., Imiya A., Klette R. (eds) Digital and Image Geometry, Lecture Notes in Computer Science,* vol 2243 Springer, Berlin, Heidelberg, 2001.

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A COMPARATIVE STUDY ON THE PERFORMANCE OF FREQUENTIST AND BAYESIAN ESTIMATION METHODS UNDER SEPARATION IN LOGISTIC REGRESSION

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Abstract. Separation is one of the most commonly encountered estimation problems in the context of logistic regression, which often occurs with small and medium sample sizes. The method of maximum likelihood (MLE; [8]) provides spuriously high parameter estimates and their standard errors under separation in logistic regression. Many researchers in social sciences utilize simple but ad-hoc solutions to overcome this issue, such as "doing nothing strategy", removing variable(s) from the model, and combining the levels of the categorical variable in the data causing separation etc. The limitations of these basic solutions have motivated researchers to use more appropriate and innovative estimation techniques to deal with the problem. However, the performance and comparison of these techniques have not been fully investigated yet. The main goal of this paper is to close this research gap by comparing the performance of frequentist and Bayesian estimation methods for coping with separation. A simulation study is performed to investigate the performance of asymptotic, bootstrap-based, and Bayesian estimation techniques with respect to bias, precision, and accuracy measures under separation. In line with the simulation study, a real-data example is used to illustrate how to utilize these methods to solve separation in logistic regression.

1. Introduction

The logistic regression is a well-founded analysis technique that can be utilized to determine the relationship between a dichotomous outcome and a set of categorical and/or continuous predictors. Although researchers in social sciences often do not encounter challenges in applying this technique to their data sets, complications may arise when a linear combination of predictors allocate the values of

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outcome, which is called the separation problem [1]. To illustrate the separation problem in logistic regression, consider the simplest scenario in which a dichotomous response is predicted by a continuous predictor. Suppose that the outcome has the values of $\mathrm{R} = \{0, 0, 0, 0, 0, 1, 1, 1, 1, 1\}$ and the predictor has the values of $P = \{2, 7, 3, 5, 6, 9, 14, 10, 12, 16\}$. In this case, the values of response are zero when the values of predictor are smaller than 8 and the values of response are 1 for the values of predictor greater than 8. This implies that the probability of observing zero or one is perfectly predicted (known as complete separation) and there is nothing left to be estimated. When separation occurs, the method of maximum likelihood (MLE; [8]) does not provide a reliable set of parameter estimates and their standard errors, which in turn cause to obtain undependable test statistics. Many researchers benefit from basic (but ad-hoc) solutions to overcome separation in logistic regression.

Since separation does not necessarily have a negative influence on all parameters in the model, some researchers do not pay special attention to this issue by simply and only reporting their results with respect to chi-square test statistics; although these statistics are only correct for non-problematic variables in the data. However, these variables often interact with problematic ones, and thus, the estimates and standard errors of these interactions should not be trusted either. Moreover, if the variable causing separation is categorical, then the estimates obtained for other variables in the model are not interpretable, since they are determined on the basis of the reference level of this categorical variable. Some researchers avoid these issues by removing the problematic variable(s) from the model. However, this approach is subject to two main drawbacks. First, discarding an important variable may end up with an inappropriate model specification, and consequently, a set of bias estimates for model parameters, which is known as the omitted variable bias [24]. Second, even if a predictor causing separation has an insignificant (or weakly significant) effect on the outcome, caution should be taken when eliminating this variable from the model, since it can be a confounder. That is, the relationship between this variable and the outcome may influence the outcome's associations with other variables in the model. Another common way of coping with this issue is combining the levels of variable causing separation, which is only applicable when this variable is categorical. This approach is also not recommended not only because collapsing categories alter the research question at hand, but also because it may cause the loss of information obtained from the data [1].

In response to these challenges, many researchers focus on more complicated but powerful data analysis techniques to deal with separation in logistic regression. Heinze and Schemper [14] compare the performance of Firth's penalized maximum likelihood estimation (PMLE; [7]) against the method of maximum likelihood [8], an imputation method using Bayesian logistic regression [3], and exact logistic regression [22]. This study is limited in the sense that it investigates the performance of only these four methods with respect to (only) bias measures. In the discussion of their study, they suggest the use of Firth's method to cope with separation in logistic regression. Moreover, they state that the separation problem may not only occur in the original sample, but it may also occur in bootstrap samples. However, they do not inspect the performance of Firth's method in the context of bootstrapping. Ohkura and Kamakura [28] utilized nonparametric bootstrapping in conjunction with Firth's method to compare the performance of their bootstrap-base test against Wald and Firth's tests under separation. However, the performance of Firth's method with nonparametric bootstrapping has not been compared against any Bayesian estimation method and the usual Firth's method with respect to bias, precision and accuracy measures. This study aims at filling this gap by investigating and comparing the performance of frequentist and Bayesian estimation methods with respect to bias, precision, and accuracy measures, respectively. Here, frequentist way of coping with separation is performed using Firth's method [7] and its counterpart with nonparametric bootstrapping [6]. The choice of prior distribution is a crucial point to solve separation in logistic regression using Bayesian methods. Thus, the Markov Chain Monte Carlo (MCMC) algorithms are utilized as Bayesian solutions to separation using seven different priors.

The outline of the paper is as follows. In Sections 2 and 3, the logistic regression and the separation problem in logistic regression are elaborated, respectively. In Section 4, three methods used to obtain the estimates of model parameters and their standard errors under separation are described. In Section 5, a simulation study is performed to investigate and compare the performance of these methods with respect to bias, precision, and accuracy measures. In Section 6, a real life example is presented to exemplify how to deal with separation using these estimation techniques in logistic regression. The paper will be concluded with a brief discussion.

2. Logistic Regression Modeling

The logistic regression is one of the most commonly used analysis techniques to predict a binary outcome (containing zeros and ones) in the context of generalized linear models [21]. The logistic regression model is defined as:

$$f(\pi_i) = x_i^T \beta \quad , \quad i = 1, 2, ..., N, \tag{1}$$

where $\pi_i = E(y_i)$ is the expected value of the binary outcome for the ith observation, $\beta = (\beta_0, \beta_1, ..., \beta_{P-1})^T \in \mathrm{IR}^{\mathrm{P}\times 1}$ is the vector of model parameters and $x_i^T = (1, x_{i1}, x_{i2}, ..., x_{i(P-1)}) \in \mathrm{IR}^{\mathrm{N}\times\mathrm{P}}$ is the design matrix containing ones in the first column as the coefficients of the intercept, β_0, and the values of the explanatory variables in the data, respectively. The logit link function, $f(\pi_i)$, relates the expected values of the outcome to the linear predictor, $x_i^T \beta$:

$$f(\pi_i) = log(\frac{\pi_i}{1 - \pi_i}), \tag{2}$$

where $\pi_i = \frac{exp(x_i^T \beta)}{1+exp(x_i^T \beta)}$, which is also known as the conditional probability of success.

Since the outcome containing 0's and 1's has a Bernoulli distribution with the probability of success π_i for the ith observation, the likelihood function of the data can be defined as follows:

$$L(\beta \mid y_1, y_2, ..., y_N) = \prod_{i=1}^{N} \pi_i^{y_i}(\ 1 - \pi_i)^{1-y_i}, \tag{3}$$

where $y_i \in \{0, 1\}$ for $i = 1, 2, ..., N$. The likelihood function above is not easy to differentiate, and thus, it is transformed from the original scale into the log scale:

$$log\ L(\beta \mid y_1, y_2, ..., y_N) = \sum_{i=1}^{N}\ y_i\ log\ (\pi_i) + (1 - y_i)\ log\ (\ 1 - \pi_i\ ). \tag{4}$$

The β's are estimated by maximizing the log likelihood function above using the method of maximum likelihood [8], so that the data at hand have the highest probability of being observed. This is done by differentiating the log likelihood function above with respect to the β's, setting the resulting functions to zeros and solving the equations for each of β's, respectively.

Since the maximum likelihood estimates of model parameters, the $\hat{\beta}$'s, and their standard errors do not involve closed-form solutions, they are obtained numerically. This can be achieved quickly and conveniently by utilizing computer-intensive iterative methods such as the Newton-Raphson algorithm [27]. However, there may be certain situations in which even the numerical methods fail to provide parameter estimates and their standard errors. In the next section, one of these situations called the separation problem will be elaborated.

3. Separation Problem

The logistic regression cannot always be easily used to predict a dichotomous outcome containing zeros and ones. One common issue that arises when estimating model parameters and their standard errors in the context of logistic regression causing (nearly) perfect allocation of the values of an outcome in the data at hand is called the (quasi) complete separation problem [1]. In a regular situation in which there is no problem of (quasi) complete separation, the expected probabilities of an outcome for a logistic regression model can take values between the numbers 0 and 1. In complete separation, since a linear function of predictor(s) perfectly predicts the outcome, the expected probabilities are either 0 or 1 (and not between these values). Similarly, in quasi complete separation, since the values of an outcome almost perfectly predicted, almost all expected probabilities (but not all of them) are either 0 or 1.

Figure 1 is created based on two empirical data sets given in the study of [33, p. 276], which shows the scatter plot of the values of an outcome against that of a linear predictor in the presence of complete and quasi-complete separation. As can be seen on the left panel of the figure for the first data, the values of the linear predictor perfectly separate the values of the outcome. Thus, only by observing the plot, we can make a perfect inference about the predicted values of the outcome. That is, the predicted values of the outcome take the value of zero when the linear predictor is smaller than zero and take the value of one when the linear predictor is larger than one. Similarly, as can be seen on the right panel of the figure for the second data, the values of the linear predictor nearly perfectly separate the values of the outcome, which is a sign of quasi-complete separation. In this case, the predicted values of the outcome take the value of zero, a value between zero and one (only for three observations) and the value of one, when the linear predictor is smaller than zero, equal to zero, and larger than zero, respectively. Next, it will be

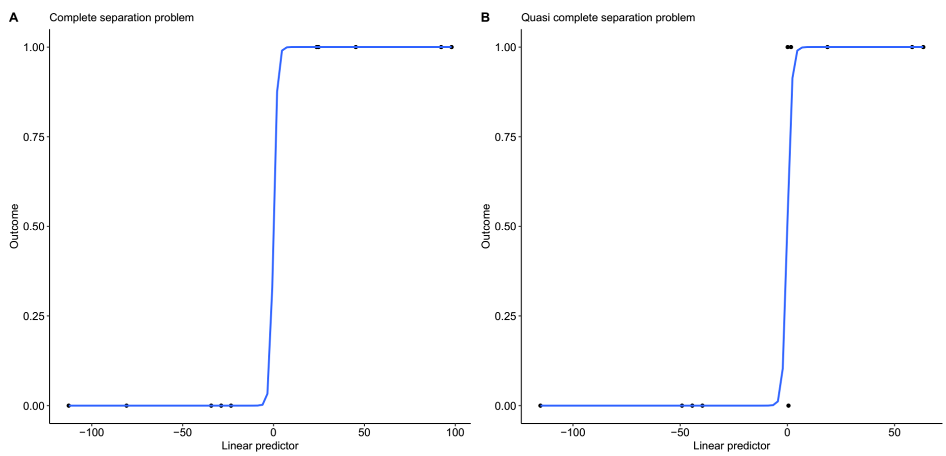

FIGURE 1. Illustrations of the (quasi) complete separation problem

elaborated how to remedy the adverse impacts of separation in estimating model parameters and their standard errors using three different estimation methods.

4. Estimation Methods

The separation [1] often occurs with small and medium sample sizes when estimating model parameters and their standard errors in logistic regression. The Newton-Raphson algorithm used to obtain MLEs does not converge for (some of) model parameters when the data suffer from separation. This nonconvergence causes spuriously high parameter estimates and standard errors [33, pp. 282-283] and results in unreliable test statistics and hypothesis testing. In response to this challenge, researchers have been paying attention to more appropriate estimation techniques than MLE to overcome separation in logistic regression. In the sequel, three of such advanced estimation methods will be elaborated, respectively.

Firth's method: Firth [7] proposed a method to improve the parameter estimates in logistic regression by reducing the bias occurs with small samples when using the method of maximum likelihood for estimation. Since Firth's method incorporates a penalizing factor into the log likelihood in (4), it is also known as the method of penalized maximum likelihood estimation. Firth's penalized log likelihood function is defined as:

$$L^*(\beta \mid y_1, y_2, ..., y_N) = L(\beta \mid y_1, y_2, ..., y_N) + \frac{1}{2}\ log\ |I(\beta)|, \tag{5}$$

where $I(\beta) = x_i^T W x_i$ is the information matrix and $W = \mathrm{diag}[\pi_\mathrm{i}(1 - \pi_\mathrm{i})]$ [35, p. 164]. Heinze and Schemper [14] have adopted the penalized log likelihood function above to overcome separation in the analysis of two cancer studies. Firth's method is flexible in the sense that it can be incorporated into nonparametric resampling techniques when estimating model parameters and their standard errors.

Firth's method with nonparametric bootstrapping: Nonparametric bootstrapping [6] is a resampling (with replacement) technique that can be used as an alternative to the method of maximum likelihood to obtain MLEs and their standard errors, when model assumptions are not satisfied (see [34], [15, p. 44]). Nonparametric bootstrapping uses the information given in the original sample to generate, for example, $B = 1000$ bootstrap samples, in each of which model parameters are estimated using the method of maximum likelihood. Subsequently, it calculates the averages and standard deviations of the bootstrap estimates across these samples to obtain the overall parameter estimates and their standard errors.

The usual nonparametric bootstrapping using the method of maximum likelihood for estimation in each bootstrap sample assumes that the original sample adequately represents the population of interest, which is often not a reasonable assumption for small samples. Thus, since separation usually occurs with small and medium samples, it is not recommended to use nonparametric bootstrapping in conjunction with MLEs under separation. Nonparametric bootstrapping can still be used for a small or medium sample in the context of logistic regression when the data suffer from separation. This can be done by replacing MLEs with PMLEs obtained using Firth's method in each bootstrap sample. The method of maximum likelihood and nonparametric bootstrapping with MLEs produce bias estimates with small samples [15], and thus, they should not be used to overcome separation in logistic regression. Bayesian methods are good alternatives to Firth's method and nonparametric bootstrapping with PMLEs to deal with separation in logistic regression.

Bayesian approach using MCMC algorithms: Bayesian estimation using Markov chain Monte Carlo (MCMC) algorithms benefits from prior knowledge on the distribution of model parameters and information in the data at hand to generate posterior samples, which are, in turn, utilized to obtain parameter estimates and their standard errors. The Metropolis Hastings [13, 23], Gibbs sampling [10], and Hamiltonian Monte Carlo (HMC; [2, 5, 26]) are three of the best known MCMC algorithms that can be used to obtain the estimates of model parameters and their standard errors for small samples in logistic regression. The HMC (also known as Hybrid Monte Carlo) and Gibbs sampling algorithms are used for Bayesian estimation in this paper using the R packages "rstanarm" [12], "runjags" [4], and "bayesreg" [19].

Rainey [29] suggests to utilize two priors when estimating model parameters using Bayesian approaches under separation in logistic regression, which are Jeffrey's invariant prior [16], [35] and a weakly informative Cauchy(0, 2.5) prior [9]. Bayesian approach using Jeffrey's prior is the same with Firth's penalized maximum likelihood estimation method, since the penalty part of the log likelihood function in (5), $\frac{1}{2}\, log\, |I(\beta)|$, is equal to the log of Jeffrey's prior in logistic regression [29]. Moreover, using weakly informative Cauchy(0, 2.5) prior to cope with separation in logistic regression is highly controversial. Ghosh, Li and Mitra [11] state that using a Cauchy(0, 2.5) prior imposes too much insufficient information into the analysis to overcome separation in logistic regression. They show that using Cauchy(0, 2.5) prior may cause spuriously high posterior means for parameters in the presence of separation in logistic regression and may not even enable researchers to obtain these means. Their results suggest to use weakly informative priors with lighter tails than that of Cauchy(0, 2.5) prior such as Normal and Student-t (df = 7) priors. Thus, in addition to Cauchy(0, 2.5) prior, a weakly informative Normal(0, 2.5) prior (the default prior for regression coefficients in rstanarm) and Student-t(0, 2.5, df = 7) prior will be utilized to obtain parameter estimates and their standard errors.

Mansournia, Geroldinger, Greenland, and Heinze [20] utilize Firth's method [7], Ridge logistic regression [31], lasso logistic regression [17], [30], and Bayesian estimation using weakly informative priors. The difference between the current study and the study in Mansournia et al. [20] is threefold. First, Mansournia et al. [20] utilize Bayesian estimation using only Cauchy(0, 2.5) and Log-F(1, 1) priors. As will be shown later in this paper, Bayesian estimation using these priors does not necessarily perform well in logistic regression under separation problem. Thus, the current study also uses Bayesian estimation via Normal(0, 2.5), Student-t(0, 2.5, df = 7), and Log-F(2, 2) priors. Second, Mansournia et al. [20] do not perform a simulation study to inspect the performance of methods used in their study, while the current study compares the performance of both frequentist and Bayesian estimation methods with respect to bias, precision, and accuracy measures. Third, Mansournia et al. [20] investigate the frequentist Ridge and Lasso logistic regressions to cope with separation. Researchers often need to determine the value of a penalizing parameter ($\lambda \geq 0$; also called the tuning or shrinkage parameter utilized on all the regression coefficients besides the intercept in the model) using, for example, cross-validation in order to employ these techniques to solve the problem. However, obtaining the tuning parameter λ is often a complicated and cumbersome task in logistic regression under separation. In many cases where the data suffer from the separation problem the tuning parameter can be estimated as very close to zero, which means that the penalized estimates are very close to the usual MLEs.

To remedy this, the current study does not inspect the usual Ridge and Lasso logistic regressions to solve the separation problem in logistic regression, but instead it utilizes their Bayesian counterparts, that is, Bayesian Ridge and Bayesian Lasso logistic regressions. Note that the tuning parameter λ is set to 1 in Bayesian Ridge logistic regression and $\lambda^2 \sim \text{Exp}(1)$ in Bayesian Lasso logistic regression for each regression coefficients in the model (see [19, p. 7]).

5. Simulation study

5.1. **Simulation Steps.** In this section, the performance of the methods on estimating model parameters will be compared to each other for the data sets containing separation in the context of logistic regression. The model used in the simulation is:

$$f(\pi_i) = \beta_0 + \beta_1 I_i + \beta_2 x_{i1} + \beta_3 x_{i2}, \tag{6}$$

where $f(\pi_i)$ is the logit link function in (2), β_0 is the intercept, β_1 is the coefficient of a dummy variable I_i and β_2 and β_3 are the coefficients of two continuous variables x_{i1} and x_{i2}, respectively, for $i = 1, 2, ..., N$. The simulation comprises the following steps:

(1) Set the entries in the vector of model parameters, $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$, equal to 1.
(2) Choose the sample size in the simulation as $N = 20, 50$, and 100.
(3) Generate the values of dummy variable I_i of size N, such that the probability of observing a success is 0.25.
(4) Generate the values of continuous variables x_{i1} and x_{i2} of size N from the standard normal distribution, such that their values are independent from each other and the values of dummy variable.
(5) By multiplying the values of the design matrix $x_i^T = (1, I_i, x_{i1}, x_{i2}) \in \mathbb{R}^{\text{N}\times\text{P}}$ and parameter vector $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T \in \mathbb{R}^{\text{P}\times 1}$, calculate the linear predictor part of the model, $x_i^T\beta$, where $N = 20, 50$, or 100 and $P = 4$.
(6) Calculate the probability of success for each observation, $\pi_i = \frac{exp(x_i^T\beta)}{1+exp(x_i^T\beta)}$ for $i = 1, 2, ..., N$.
(7) Generate the values of the response using the success probabilities, that is, $y_i \sim \text{Bernoulli}(\pi_i)$ for $i = 1, 2, ..., N$.
(8) Check the model fit to detect separation in the data using the R package "brglm2".
    (a) If there is no separation problem in the data, return to Step 3.
    (b) If there is a separation problem in the data, obtain the estimates of model parameters using each estimation method elaborated in the previous section.
(9) Repeat Steps 3-8 until having a set of parameter estimates for S = 1000 samples, each of which containing separation problem.

(10) Calculate the values of the bias, precision, and accuracy measures for each method using the estimates obtained for these samples.

Note that the measures of bias, precision, and accuracy need to be calculated for each method, which are the method of maximum likelihood, Firth's method (with and without nonparametric bootstrapping), and Bayesian approach using Normal(0, 2.5), Cauchy(0, 2.5), Student-t(0, 2.5, df = 7), Log-F(1, 1), Log-F(2, 2), Ridge and Lasso priors.

5.2. **Bias, precision, and accuracy measures for evaluating performance.** The performance of the methods will be compared to each other using the measures of bias, precision, and accuracy given in Walther and Moore [32]. These measures are defined as:

$$\begin{aligned} \mathrm{Bias_p} &= \frac{1}{S}\sum_{s=1}^{S}(\hat{\beta}_{sp} - \beta_p), \\ \mathrm{Precision_p} &= \frac{1}{S}\sum_{s=1}^{S}(\hat{\beta}_{sp} - \bar{\beta}_p)^2, \\ \mathrm{Accuracy_p} &= \frac{1}{S}\sum_{s=1}^{S}(\hat{\beta}_{sp} - \beta_p)^2, \end{aligned} \tag{7}$$

where $\bar{\beta}_p = \frac{1}{S}\sum_{s=1}^{S}\hat{\beta}_{sp}$ and $\beta_j = 1$ for $s = 1, 2, ..., 1000$ and $p = 0, 1, 2, 3$. The $\mathrm{Bias_p}$ is the mean of the differences between parameter β_p and its estimate across $S = 1000$ samples. Similarly, $\mathrm{Precision_p}$ is the mean of the squared differences between an estimate and its expected value (i.e., $\bar{\beta}_p$) in $S = 1000$ samples, which is calculated for each parameter, separately. The measure of accuracy for the pth parameter, $\mathrm{Accuracy_p}$, is the mean of the squared differences between parameter β_p and its estimates across $S = 1000$ samples, which is a combination of $\mathrm{Bias_p}$ and $\mathrm{Precision_p}$. Note that the term "bias" is directly related and the terms "precision" and "accuracy" are inversely related to their corresponding equations in (7). That is, a small value of $\mathrm{Bias_p}$ means a low bias, while small values of $\mathrm{Precision_p}$ and $\mathrm{Accuracy_p}$ imply high precision and accuracy when estimating model parameters.

Another accuracy measure that can be used to investigate the performance of methods on estimating model parameters is the mean squared error (MSE), representing the estimation error for each sample in the simulation. The MSE is the total mean squared error between all parameters and their estimates:

$$\mathrm{MSE} = \frac{1}{P}\sum_{p=0}^{P-1}(\beta_p - \hat{\beta}_p)^2, \tag{8}$$

where $P = 4$ is the number of parameters in the model. The mean of MSE values across $S = 1000$ simulation samples can be used to compare the overall performance of methods on estimating model parameters. A small value of MSE means a high overall accuracy when estimating model parameters.

5.3. **Simulation Results.** Table 1 displays Bias_p, $\text{Precision}_\text{p}$ and Accuracy_p values obtained from 1000 simulated data sets, each of which contains the separation problem. This table shows that the estimate of parameter β_1 often has a higher bias and a lower precision and accuracy than that of parameters β_2 and β_3, since dummy variables are more prone to suffer from separation than continuous variables. Because of the same reason, although increasing the sample size increases the precision when estimating each parameter, this reduces the bias and improves the accuracy only for parameters β_0, β_2, and β_3, but not for parameter β_1. It seems that Firth's penalized maximum likelihood estimation and Bayesian estimation using Log-F(2, 2) prior provide smaller biases and higher precision and accuracy measures when compared to other estimation methods. Similarly, these methods have smaller MSE values (higher overall accuracy measures) when compared to other methods (see Table 2). Moreover, both tables show that Bayesian estimation may not perform well with Ridge prior, since the corresponding estimates may have spuriously high precision and accuracy values (indicating low precision and accuracy for these estimates). However, the values in these tables are point estimates, and thus, a set of graphical visualizations are designed to facilitate the interpretation of the simulation results.

TABLE 1. Bias, precision, and accuracy measures for performance evaluation.

		PMLE				PMLE via NB				MCMC Normal(0, 2.5)			
Measure	N	β_0	β_1	β_2	β_3	β_0	β_1	β_2	β_3	β_0	β_1	β_2	β_3
Bias_p	20	0.12	0.28	0.09	0.07	0.30	0.09	0.32	0.29	0.89	0.74	0.53	0.50
	50	0.01	1.57	0.02	0.04	0.19	1.61	0.24	0.26	0.28	1.93	0.19	0.19
	100	0.01	2.21	-0.02	0.01	0.07	2.23	0.06	0.09	0.14	2.47	0.06	0.09
$\text{Precision}_\text{p}$	20	0.81	2.49	0.99	1.13	0.90	2.89	1.16	1.19	1.77	1.26	1.37	1.35
	50	0.31	0.98	0.39	0.54	0.44	1.10	0.55	0.64	0.38	0.32	0.34	0.32
	100	0.10	0.32	0.12	0.11	0.12	0.36	0.15	0.14	0.12	0.16	0.14	0.12
Accuracy_p	20	0.82	2.57	1.00	1.13	0.99	2.90	1.26	1.27	2.57	1.81	1.65	1.60
	50	0.31	3.46	0.39	0.54	0.48	3.70	0.61	0.71	0.46	4.03	0.38	0.36
	100	0.10	5.22	0.13	0.11	0.13	5.35	0.16	0.15	0.14	6.24	0.14	0.13

		MCMC Cauchy(0, 2.5)				MCMC Student-t(0, 2.5, df = 7)				MCMC Log-F(1, 1)			
Measure	N	β_0	β_1	β_2	β_3	β_0	β_1	β_2	β_3	β_0	β_1	β_2	β_3
Bias_p	20	1.23	2.02	0.97	0.93	0.95	0.91	0.61	0.57	0.63	1.23	0.51	0.47
	50	0.25	3.89	0.21	0.22	0.27	2.20	0.20	0.20	0.22	2.56	0.18	0.18
	100	0.12	4.85	0.06	0.09	0.14	2.80	0.07	0.09	0.12	3.23	0.06	0.09
$\text{Precision}_\text{p}$	20	4.09	6.08	4.38	4.90	2.07	1.59	1.67	1.66	1.13	1.85	1.35	1.35
	50	0.49	1.88	0.47	0.55	0.40	0.47	0.36	0.35	0.32	0.60	0.34	0.32
	100	0.12	0.88	0.14	0.13	0.12	0.24	0.14	0.12	0.12	0.38	0.14	0.12
Accuracy_p	20	5.60	10.15	5.33	5.76	2.98	2.41	2.04	1.98	1.53	3.35	1.62	1.57
	50	0.56	16.99	0.51	0.60	0.48	5.30	0.40	0.39	0.37	7.15	0.37	0.36
	100	0.14	24.39	0.15	0.14	0.14	8.08	0.15	0.13	0.13	10.81	0.14	0.13

		MCMC Log-F(2, 2)				Bayesian Ridge LR				Bayesian Lasso LR			
Measure	N	β_0	β_1	β_2	β_3	β_0	β_1	β_2	β_3	β_0	β_1	β_2	β_3
Bias_p	20	0.29	0.36	0.10	0.08	4.32	2.10	3.47	3.86	0.45	-0.38	-0.45	-0.43
	50	0.15	1.40	0.05	0.04	0.28	1.85	-0.01	0.05	0.18	0.96	-0.28	-0.27
	100	0.10	2.05	0.01	0.03	0.10	2.83	-0.03	-0.01	0.07	2.45	-0.13	-0.10
$\text{Precision}_\text{p}$	20	0.51	0.61	0.57	0.59	607.5	135.9	382.9	602.4	1.00	1.09	1.03	1.33
	50	0.23	0.32	0.22	0.21	8.32	3.65	5.29	13.38	1.07	4.34	1.54	2.04
	100	0.10	0.21	0.11	0.10	0.11	0.80	0.14	0.12	0.10	0.89	0.14	0.13
Accuracy_p	20	0.59	0.73	0.59	0.60	626.2	136.3	394.9	617.3	1.20	1.24	1.23	1.52
	50	0.25	2.27	0.23	0.21	8.40	7.07	5.29	13.38	1.10	5.25	1.62	2.11
	100	0.11	4.41	0.11	0.10	0.12	8.78	0.14	0.12	0.11	6.89	0.16	0.14

TABLE 2. The overall accuracy measure (MSE) for performance evaluation.

	PMLE	PMLE via NB	MCMC Normal(0, 2.5)
N	$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$	$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$	$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$
20	1.38	1.61	1.91
50	1.18	1.37	1.31
100	1.39	1.44	1.66
	MCMC Cauchy(0, 2.5)	MCMC Student-t(0, 2.5, df = 7)	MCMC Log-F(1, 1)
N	$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$	$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$	$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$
20	6.71	2.36	2.02
50	4.66	1.64	2.06
100	6.20	2.13	2.80
	MCMC Log-F(2, 2)	Bayesian Ridge LR	Bayesian Lasso LR
N	$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$	$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$	$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$
20	0.63	750.4	1.30
50	0.74	8.53	2.52
100	1.18	2.29	1.82

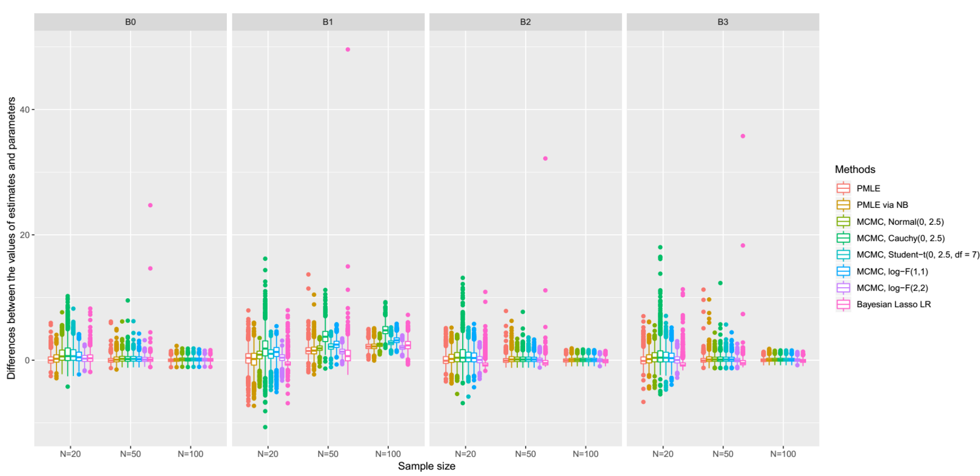

FIGURE 2. Boxplots used to interpret bias measures

Figures 2 and 3 display the differences between the values of estimates and parameters and the squared differences between the values of estimates and their expected values across the simulated data sets using varying sample sizes, which

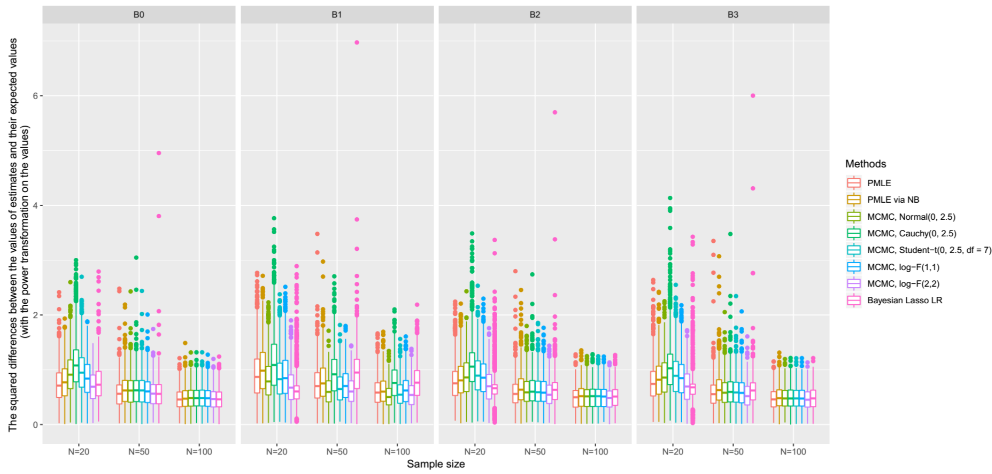

FIGURE 3. Boxplots used to interpret precision measures

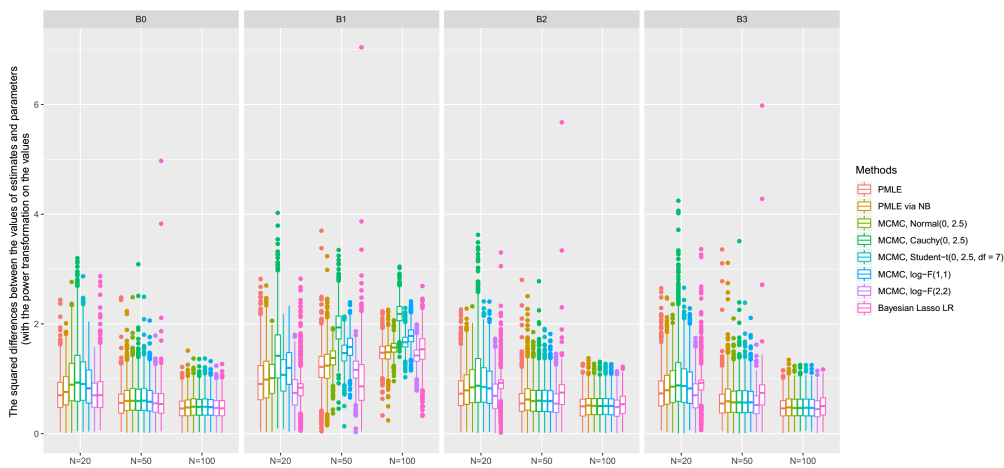

FIGURE 4. Boxplots used to interpret accuracy measures

are used to obtain the values of Bias_p and $\text{Precision}_\text{p}$ for each estimation method, respectively. [1] It seems that most of the methods perform well in terms of Bias_p and $\text{Precision}_\text{p}$ measures. However, Bias_p and $\text{Precision}_\text{p}$ measures of Bayesian

[1] The $\textbf{Precision}_\text{p}$, $\textbf{Accuracy}_\text{p}$ and MSE values are always positive and they spread over large scales. Thus, the $y = x^{\frac{1}{4}}$ transformation is utilized on these values to better visualize and compare the performance of the methods (see [18, p. 12]).

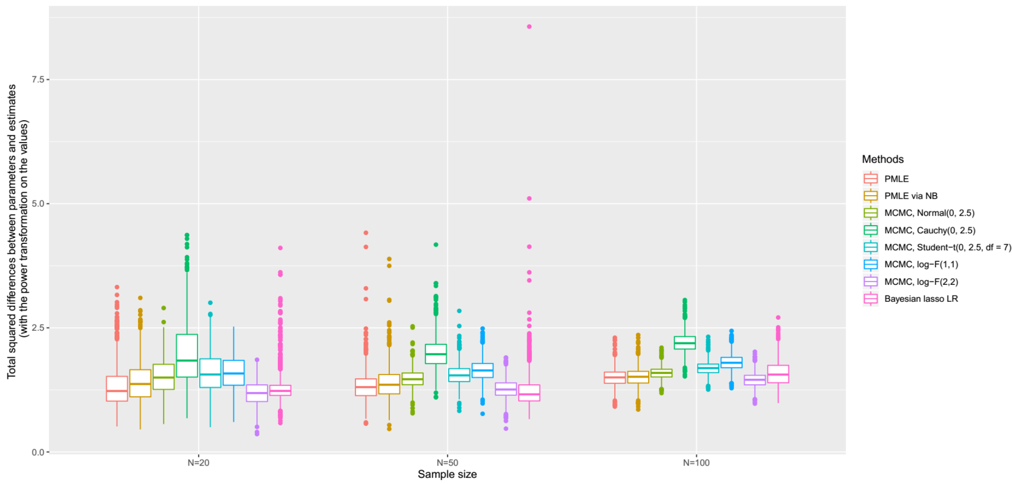

FIGURE 5. Boxplots used to interpret MSE values

estimation using Cauchy(0, 2.5) and Lasso priors have higher standard errors when compared to that of other methods under investigation. It seems that Bayesian estimation using log-F(2, 2) prior involves smaller amount of bias and have higher precision in estimating model parameters when compared to other methods. Note that the figures in the paper do not show the results for Bayesian estimation using Ridge prior, since this method produces spuriously high parameter estimates and their standard errors.

Figures 4 and 5 show the squared differences and the sums of squared differences between the values of estimates and parameters using varying sample sizes, which are utilized to obtain $\mathrm{Accuracy}_\mathrm{p}$ and MSE values, respectively. Increasing the sample size improves the accuracy for each parameter, and thus, the total accuracy when estimating model parameters using each method. The estimates obtained by using Bayesian estimation with Log-F(2, 2) prior often have higher (total) accuracy measures, and thus, lower $\mathrm{Accuracy}_\mathrm{p}$ and MSE values, when compared to other methods. Since nonparametric bootstrapping assumes an original sample that adequately represents the population of interest, the performance of Firth's method and Firth's method with nonparametric bootstrapping better resemble each other for large sample sizes (e.g., when $N = 100$). It seems that Bayesian approach with weakly informative Normal(0, 2.5) prior performs better than that with Student-t(0, 2.5, df = 7) or Log-F(1, 1) prior which in turn performs better than that with Cauchy(0, 2.5) prior. This result is in line with the suggestions made in Ghosh et al. [11], which state that Cauchy(0, 2.5) prior provides too much deficient information, and thus, instead of using this prior, Normal(0, 2.5) and Student-t(0, 2.5, df = 7) priors should be used when dealing with separation in logistic regression.

6. An example: Endometrial cancer data

A study in Heinze and Schemper [14] is used to illustrate how to analyze the data at hand under separation in logistic regression. In the study, the dichotomous outcome histology (HG: 0 = grade 0-II, 1 = grade III-IV) represents the histology of the endometrium by commonly accepted risk factors for endometrial cancer patients ($N = 79$). This outcome is predicted by the categorical variable neovasculization (NV: 0 = absent, 1 = present) and two continuous variables pulsatility index of arteria uterina (PI) and endometrium high (EH). The logistic regression model used to analyze the endometrial cancer data is:

$$f(\pi_i) = \beta_0 + \beta_1 \mathrm{NV}_i + \beta_2 \mathrm{PI}_i + \beta_3 \mathrm{EH}_i, \tag{9}$$

where $f(.)$ is the logit link function, β_0 is the intercept and β_1, β_2, and β_3 are the regression coefficients of variables NV, PI and EH, respectively, for $i = 1, 2, ..., 79$.

Since there is no observation in the endometrial cancer data for NV = 1 and HG = 0, the data suffer from quasi-complete separation, which has a detrimental effect on the estimate of parameter β_1 and its standard error when the estimation process is performed using the usual method of maximum likelihood. Therefore, Firth's method, Firth's method with nonparametric bootstrapping, Bayesian approach using Normal(0, 2.5), Cauchy(0, 2.5), Student-t(0, 2.5, df= 7), Log-F(1, 1), Log-F(2, 2), Ridge and Lasso priors are used to obtain parameter estimates and their standard errors (see Table 3).[2]

The estimates of parameters β_2 and β_3 across the methods are reasonably close to each other, while the estimates of parameters β_0 and β_1 across the methods may differ from each other. Figure 6 shows that the predicted probabilities of the outcome histology for some of the observations in the data are exactly equal to 1 (in the upper right corner of the plot), when using the method of maximum likelihood for estimation, which is a sign of the quasi-complete separation problem. Bayesian approach using the MCMC algorithm with Cauchy(0, 2.5) prior does not provide a convincing solution to the separation for endometrial cancer data, since some of the predicted probabilities of the outcome are (almost) equal to 1. The plots for other methods more closely resemble the regular logistic regression plot in which predicted probabilities are between the numbers 0 and 1.

Here, several diagnostics are introduced to inspect whether the MCMC algorithm produces adequate posterior samples for parameters when using weakly informative Normal(0, 2.5), Cauchy(0, 2.5), and Student-t(0, 2.5, df = 7) priors. The potential scale reducing factor ($\hat{R}$) and effective sample size (ESS) statistics for each parameter are used to determine whether the MCMC algorithm converges properly with high estimation accuracy. These statistics are obtained by inspecting multiple chains and dissimilarities between them (default number of chains is often 4). The $\hat{R}$ statistic shows whether the chains converge to the same area by exploring the

[2]For more details on obtaining the estimates of model parameters and their standard errors using R code for each estimation method see Supplementary material.

TABLE 3. Estimates and standard errors of the coefficients for the logistic regression.

	PMLE		PMLE via NB		MCMC Normal(0, 2.5)	
β	$\hat{\beta}$	$SE(\hat{\beta})$	$\hat{\beta}$	$SE(\hat{\beta})$	$\hat{\beta}$	$SE(\hat{\beta})$
β_0	3.77	1.49	4.69	2.45	4.52	1.55
β_1	2.93	1.55	3.25	1.10	3.33	1.39
β_2	-0.03	0.04	-0.05	0.07	-0.04	0.04
β_3	-2.60	0.78	-3.11	1.29	-3.09	0.82
	MCMC Cauchy(0, 2.5)		MCMC Student-t(0, 2.5, df =7)		MCMC Log-F(1, 1)	
β	$\hat{\beta}$	$SE(\hat{\beta})$	$\hat{\beta}$	$SE(\hat{\beta})$	$\hat{\beta}$	$SE(\hat{\beta})$
β_0	4.40	1.57	4.47	1.59	3.23	1.20
β_1	5.53	4.07	3.53	1.73	3.95	1.63
β_2	-0.04	0.04	-0.04	0.04	-0.02	0.04
β_3	-3.01	0.82	-3.08	0.85	-2.45	0.66
	MCMC Log-F(2, 2)		Bayesian Ridge LR		Bayesian Lasso LR	
β	$\hat{\beta}$	$SE(\hat{\beta})$	$\hat{\beta}$	$SE(\hat{\beta})$	$\hat{\beta}$	$SE(\hat{\beta})$
β_0	2.40	1.04	3.86	1.59	3.53	1.49
β_1	3.10	1.31	4.64	4.59	3.60	3.15
β_2	-0.01	0.03	-0.03	0.04	-0.02	0.03
β_3	-2.06	0.59	-2.71	0.84	-2.58	0.82

ratio of their within and between variances. A value of $\hat{R} < 1.1$ indicates good convergence of the chains for the corresponding parameter. A high value of the ESS statistic indicates low autocorrelation and high estimation accuracy within the chains, where ESS > 1000 is often considered to be an adequate sample size statistic for many social scientists [25]. Table 4 displays the values of $\hat{R}$ and ESS statistics obtained for each parameter, where the MCMC algorithm is used with Normal(0, 2.5), Cauchy(0, 2.5), and Student-t(0, 2.5, df $= 7$) priors, respectively. The use of MCMC algorithm with Normal(0, 2.5) and Student-t(0, 2.5, df $= 2.5$) priors results in good convergence of the chains (i.e., $\hat{R} = 1$ for each parameter) with low autocorrelation, and consequently, high estimation accuracy (i.e., ESS > 1000 for each parameter). Although the MCMC algorithm with Cauchy(0, 2.5) prior produces good convergence of the chains for each parameter, there is a high autocorrelation and a low estimation accuracy within parameter samples, especially when looking at the relationship between the outcome and dichotomous predictor NV (i.e., ESS $= 103$ for parameter β_1). Thus, the focus from now on will be particularly on parameter β_1 to visually inspect the difference between the MCMC

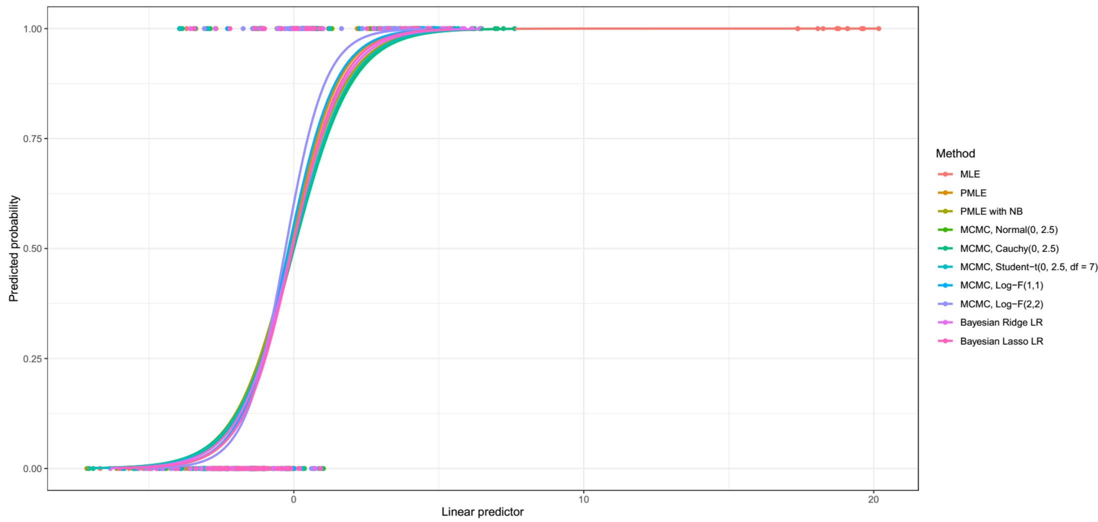

FIGURE 6. The values of linear predictor against predicted probabilities

algorithm with weakly informative Normal(0, 2.5), Cauchy(0, 2.5), and Student-t(0, 2.5, df = 7) priors.

TABLE 4. The $\hat{R}$ and ESS statistics for each parameter under Normal(0, 2.5), Cauchy(0, 2.5), and Student-t(0, 2.5, df = 7) priors.

	HMC Normal(0, 2.5)		HMC Cauchy(0, 2.5)		HMC Student-t(0, 2.5, df = 7)	
	$\hat{R}$	ESS	$\hat{R}$	ESS	$\hat{R}$	ESS
β_0	1.0	2620	1.0	1800	1.0	2303
β_1	1.0	2064	1.0	848	1.0	1752
β_2	1.0	3342	1.0	1988	1.0	3355
β_3	1.0	2280	1.0	1874	1.0	1933

Figure 7 shows the histograms of marginal posterior distribution, trace plot (chains separate), autocorrelation plot (combined chains) and log posterior for parameter β_1 under the three priors, respectively. A marginal posterior distribution is obtained for one single parameter by not taking other parameters in the model into account. The histograms show that the marginal posterior distribution of parameter β_1 is normal when using the normal prior and is close to be normal when using the Student-t prior with df = 7 degrees of freedom, for which the mean (solid

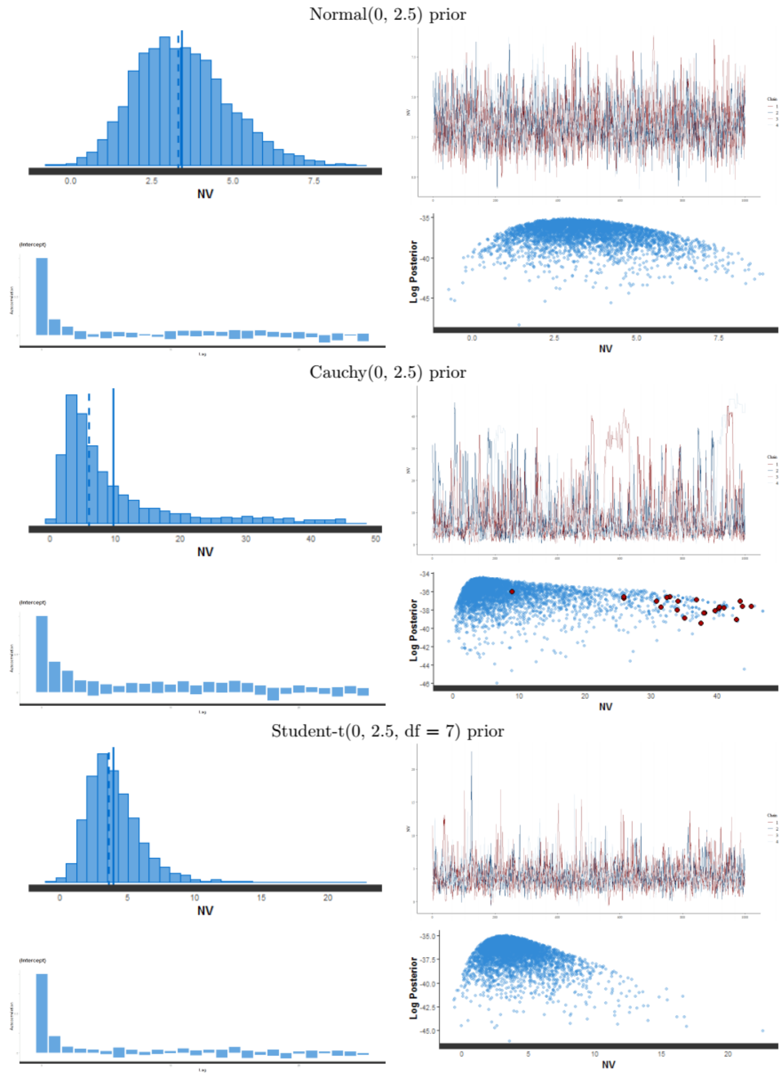

FIGURE 7. Marginal posterior distributions, trace and autocorrelation plots and log posteriors for parameter β_1 under weakly informative Normal(0, 2.5), Cauchy(0, 2.5), and Student-t(0, 2.5, df = 7) priors

line) and the median (dashed line) are (almost) equal to each other. The marginal posterior of parameter β_1 using the Cauchy prior has a right skewed (i.e., the mean to the right of the median) distribution. By default, the MCMC algorithm in rstanarm utilizes 2000 posterior samples of parameter β_1 for each chain (i.e., 8000 samples in total), half of which are used in a warm-up phase and discarded later on before showing diagnostics and making inference. Thus, each of the four trace plots above under the three priors is created by using 1000 posterior samples of parameter β_1. Based on these plots, the chains display adequate mixing under Normal(0, 2.5) and Student-t(0, 2.5, df = 7) priors, but they may exhibit consecutive periods in positive direction under Cauchy(0, 2.5) prior. Based on the autocorrelation plots, independently from the prior distribution of parameter β_1, the correlation between variable NV and its value at lag zero is one, since the latter represents the variable itself. The height of spike at lag zero is quickly reduced to zero (and fluctuated around zero afterwards) with increasing values of lags under Normal(0, 2.5) and Student-t(0, 2.5, df = 7) priors for parameter β_1, respectively, which is a sign against autocorrelation. However, when using Cauchy(0, 2.5) prior for parameter β_1, the decrease in the height of spike at lag zero is relatively slow (and does not fluctuate considerably around zero) compared to that using Normal(0, 2.5) and Student-t(0, 2.5, df = 7) priors, which is a sign of positive autocorrelation.

The marginal posterior distribution for β_1 is highly curved when using the MCMC algorithm with Cauchy(0, 2.5) prior. This causes many divergent transitions in the MCMC algorithm, which are shown by the red points in the log posterior scatter plot above. This is evidence of too large step size in the MCMC algorithm under Cauchy(0, 2.5) prior. In this case, the results of MCMC algorithm should not be trusted. The MCMC algorithm needs a smaller step size to avoid divergent transitions and to draw plausible samples from the marginal posterior distribution of β_1, which can easily be adjusted by increasing the default value of δ parameter in rstanarm (e.g., from 0.95 to 0.99). Table 5 shows the estimates of parameters and their standard errors and the values of $\hat{R}$ and ESS statistics, when using Cauchy(0, 2.5) prior with divergent ($\delta = 0.95$) and non-divergent ($\delta = 0.99$) transitions, respectively. Based on this table, decreasing the step size in the MCMC algorithm by increasing the value of δ from 0.95 to 0.99 does not have much influence on parameter estimates and their standard errors. Moreover, increasing the value of δ results in a non-convergence (i.e., $\hat{R} = 1.1$ for parameter β_1) and a decrease in estimation accuracy (i.e., ESS is only 35 for parameter β_1). Therefore, it is not recommended to use this prior to overcome separation in the endometrial cancer data.

7. Discussion

Researchers in social sciences commonly use simple data manipulation techniques to overcome separation in logistic regression. These solutions are often unsatisfactory and do not meet the expectations of researchers. Thus, many researchers have been paying attention to more convenient approaches for estimation, such as

TABLE 5. Estimates and standard errors and the $\hat{R}$ and ESS statistics under Cauchy(0, 2.5) prior with divergent and non-divergent transitions.

	Divergent transitions $\delta = 0.95$				Non-divergent transitions $\delta = 0.99$			
β	$\hat{\beta}$	$SE(\hat{\beta})$	$\hat{R}$	ESS	$\hat{\beta}$	$SE(\hat{\beta})$	$\hat{R}$	ESS
β_0	4.53	1.66	1.0	929	4.40	1.75	1.0	258
β_1	5.99	4.54	1.0	103	6.05	4.61	1.1	35
β_2	-0.04	0.04	1.0	1088	-0.04	0.04	1.0	1400
β_3	-3.08	0.89	1.0	988	-3.01	0.90	1.0	234

symptotic and bootstrap-based bias reduction methods and Bayesian methods using weakly informative priors. However, the performance of these methods have not been fully investigated yet with respect to bias, precision, and accuracy measures in the context of logistic regression.

In the simulation, three methods were used to obtain the estimates of model parameters and their standard errors: Firth's penalized maximum likelihood estimation, Firth's method with nonparametric bootstrapping, and Bayesian approach with seven different priors. In a concrete real life example, parameter estimation was performed using these three methods for the endometrial cancer data. Supplementary material contains the relevant R code for obtaining the estimates of model parameters and their standard errors for each estimation method presented in this paper. Results of the simulation study and the analysis of the endometrial cancer data have showed that although most of the methods perform well in coping with the consequences of separation problem in logistic regression, Bayesian estimation with Log-F(2, 2) prior performs better than other methods.

The choice of prior distribution in Bayesian approach plays an essential role to overcome separation in logistic regression. It was shown both by the simulation and real life example that Bayesian approach with Cauchy(0, 2.5) or Ridge prior does not provide a reliable solution to separation in logistic regression, since these priors incorporate too much detrimental information into the analysis. A more coherent weakly informative prior such as Normal(0, 2.5), Student-t(0, 2.5, df = 7), Log-F(1, 1), Log-F(2, 2), or Lasso prior should be utilized in place of Cauchy(0, 2.5) prior when dealing with separation in the data.

References

[1] Albert, A., Anderson, J. A., On the existence of maximum likelihood estimates in logistic regression models, *Biometrica*, 71 (1984), 1-10.

[2] Betancourt, M. J., Byrne, S., Livingstone, S., Girolami, M., The Geometric Foundations of Hamiltonian Monte Carlo. ArXiv e-prints 1410.5110, 2014.

[3] Clogg, C. C., Rubin, D. B., Schenker, N., Schultz, B., Weidman, L., Multiple imputation of industry and occupation codes in census public-use samples using Bayesian logistic regression, *Journal of the American Statistical Association*, 86 (1991), 68-78.

[4] Denwood, M. J., runjags: An R package providing interface utilities, model templates, parallel computing methods and additional distributions for MCMC models in JAGS, *Journal of Statistical Software*, 71 (2016), 1-25.

[5] Duane, S., Kennedy, A. D., Pendleton, B. J., Roweth, D., Hybrid Monte Carlo, *Physics Letters B*, 195 (1987), 216-222.

[6] Efron, B., Tibshirani, R. J. An introduction to the bootstrap, New York: Chapman & Hall, 1993.

[7] Firth, D., Bias reduction of maximum likelihood estimates, *Biometrika*, 80 (1993), 27-38.

[8] Fisher, R. A., On the mathematical foundations of theoretical statistics, *Philosophical Transactions of the Royal Society*, 222 (1922), 309-368.

[9] Gelman, A., Jakulin, A., Pittau, M. G., Su, Y., A weekly informative prior distribution for logistic and other regression models, *Annals of Applied Statistics*, 2 (2008), 1360-83.

[10] Geman, S. and Geman, D. Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 6 (1984), 721-741.

[11] Ghosh, J., Li., Y., Mitra, R., On the use of Cauchy prior distributions for Bayesian logistic regression, *International Society for Bayesian Analysis*, 13 (2018), 359-383.

[12] Goodrich, B., Gabry, J., Ali, I., Brilleman, S., rstanarm: Bayesian applied regression modeling via Stan, *R package version 2.17.4,* 2018.

[13] Hastings, W., Monte Carlo sampling methods using Markov chains and their applications, *Biometrika,* (1970), 57, 97-109.

[14] Heinze, G., Schemper, M., A solution to the problem of separation in logistic regression, *Statistics in Medicine*, 21 (2002), 2409-2419.

[15] Hox, J. J., Multilevel analysis: Techniques and applications (2nd ed.), New York, NY: Routledge, 2010.

[16] Jeffreys, H., An invariant form of the prior probability in estimation problems, *Proceedings of the Royal Society of London A*, 186 (1946), 453-61.

[17] Lokhorst, J., The lasso and generalised linear models, Honors Project, University of Adelaide, Adelaide, 1999.

[18] Maciejewski, R., Data representations, transformations, and statistics for visual reasoning, *Synthesis Lectures on Visualization*, 2 (2011), 1-85.

[19] Makalic, E., Schmidt, D. High-dimensional Bayesian regularised regression with the BayesReg package, 2016, arXiv:1611.06649.

[20] Mansournia, M. A., Geroldinger, A., Greenland, S., Heinze, G., Separation in logistic regression: Causes, consequences, and control, *American Journal of Epidemiology*, 187 (2018), 864-870.

[21] McCullagh, P., Nelder, J., Generalized linear models (2nd ed.), Boca Raton, FL: Chapman & Hall / CRC, 1989.

[22] Mehta, C. R., Patel, N. R., Exact logistic regression: theory and examples, *Statistics in Medicine*, 14 (1995), 2143-2160.

[23] Metropolis, N., Rosenbuth, A., Rosenbuth, M., Teller, A., Teller, E., Equations of state calculations by fast computing machines, *The Journal of Chemical Physics*, 21 (1953), 1087-1092.

[24] Mood, C., Logistic regression: Why we cannot do what we think we can do, and what we can do about it, *European Sociological Review*, 26 (2010), 67-82.

[25] Muth, C., Oravecz, Z., Gabry, J., User-friendly Bayesian regression modeling: A tutorial with rstanarm an shinystan, *The Quantitative Methods for Psychology,* 14 (2018), 99-119.

[26] Neal, R., MCMC using Hamiltonian Dynamics, *In Handbook of Markov Chain Monte Carlo* (S. Brooks, A. Gelman, G. L. Jones and X.-L. Meng, eds. CRC Press, New York.), 2013.

[27] Newton, I., Philosophiae naturalis principia mathematica, Colonia Allobrogum: sumptibus CI. et Ant. Philibert, 1760.

[28] Ohkura, M., Kamakura, T. Test for a regression parameter in a logistic regression model under the small sample size and the high event occurrence probability, *Japanese Applied Statistics (in Japanese)*, 40 (2011), 41-51.

[29] Rainey, C., Dealing with separation in logistic regression models. *Political Analysis*, 24 (2016), 339-355.

[30] Roth, V., The generalized lasso,*IEEE Transactions on Neural Networks*, 15 (2004), 16-28.

[31] Schaefer, R. L., Roi, L. D., Wolfe, R. A., A ridge logistic estimator, *Communications in Statistics - Theory and Methods*, 13 (1984), 99-113.

[32] Walther, B. A., Moore, J. L., The concepts of bias, precision and accuracy, and their use in testing the performance of species richness estimators, with a literature review of estimator performance, *Ecography*, 28 (2005), 815-829.

[33] Webb, M. C., Wilson, J. R., Chong, J., An analysis of quasi-complete binary data with logistic models: Applications to alcohol abuse data, *Journal of Data Science*, 2 (2004), 273-285.

[34] Yuan, K. H., Hayashi, K., Standard errors in covariance structure models: Asymptotic versus bootstrap. *British Journal of Mathematical and Statistical Psychology*, 59 (2006), 397-417.

[35] Zorn, C., A solution to separation in binary response models, *Political Analysis*, 13 (2005), 157-170.

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SOME IDENTITIES INVOLVING (p,q)-FIBONACCI AND LUCAS QUATERNIONS

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ABSTRACT. In this study, we firstly examined the Horadam quaternions defined and studied by Halici and Karataş in [4]. Then, we used the Binet's formula to show some properties of the (p,q)-Fibonacci and Lucas quaternions. We also give some important identities including these quaternions.

1. Introduction

Fibonacci and Lucas quaternions cover a wide range of interest in modern mathematics as they appear in the comprehensive works of [2, 4–6]. The Fibonacci quaternion $Q_{F,n}$ is the n-th term of the sequence where each term is the sum of the two previous terms beginning with the initial values $Q_{F,0} = i + j + 2k$ and $Q_{F,1} = 1 + i + 2j + 3k$. The well-known Fibonacci quaternion numbers are defined as

$$Q_{F,n} = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},\ n \geq 0, \tag{1}$$

where $i^2 = j^2 = k^2 = ijk = -1$. Similarly, Lucas quaternions are defined as $Q_{L,n} = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}$ for $n \geq 0$, where F_n and L_n are n-th Fibonacci and Lucas number, respectively.

Ipek [8] studied the (p,q)-Fibonacci quaternions $Q_{\mathcal{F},n}$ which is defined as

$$Q_{\mathcal{F},n} = pQ_{\mathcal{F},n-1} + qQ_{\mathcal{F},n-2},\ n \geq 2 \tag{2}$$

with initial conditions $Q_{\mathcal{F},0} = i + pj + (p^2 + q)k$, $Q_{\mathcal{F},1} = 1 + pi + (p^2 + q)j + (p^3 + 2pq)k$ and $p^2 + 4q > 0$. Note that the (p,q)-Fibonacci numbers are defined by $\mathcal{F}_n = p\mathcal{F}_{n-1} + q\mathcal{F}_{n-2}$, $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$. Then, if $p = q = 1$, we get the

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classical Fibonacci quaternion $Q_{F,n}$. If $p = 2q = 2$, we get the Pell quaternion $Q_{P,n} = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$, where P_n is the n-th Pell number.

Another important sequence is the (p,q)-Lucas sequence. This sequence is defined by the recurrence relation

$$\mathcal{L}_n = p\mathcal{L}_{n-1} + q\mathcal{L}_{n-2},\ \mathcal{L}_0 = 2,\ \mathcal{L}_1 = p. \tag{3}$$

The well-known Binet's formulas for (p,q)-Fibonacci and Lucas quaternion, see [8], are given by

$$Q_{\mathcal{F},n} = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta} \text{ and } Q_{\mathcal{L},n} = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n, \tag{4}$$

respectively. Here, α, β are roots of the characteristic equation $t^2 - pt - q = 0$, and $\underline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ and $\underline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$. We note that $\alpha + \beta = p$, $\alpha\beta = -q$ and $\alpha - \beta = \sqrt{p^2 + 4q}$.

The generalized of Fibonacci quaternion $Q_{w,n}$ is defined by Halici and Karataş in [4] as

$$Q_{w,0} = a + bi + (pb + qa)j + ((p^2 + q)b + pqa)k,$$

$$Q_{w,1} = b + (pb + qa)i + ((p^2 + q)b + pqa)j + ((p^3 + 2pq)b + q(p^2 + q)a)$$

and $Q_{w,n} = pQ_{w,n-1} + qQ_{w,n-2}$, for $n \geq 2$ which is called as the generalized Fibonacci quaternions. So, each term of the generalized Fibonacci sequence $\{Q_{w,n}\}_{n\geq 0}$ is called generalized Fibonacci quaternion.

The Binet formula for generalized Fibonacci quaternion $Q_{w,n}$, see [4], is given by

$$Q_{w,n} = \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha - \beta}, \tag{5}$$

where $A = b - a\beta$, $B = b - a\alpha$, α, β are roots of the characteristic equation $t^2 - pt - q = 0$, and $\underline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ and $\underline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$. If $a = 0$ and $b = 1$, we get the classical (p,q)-Fibonacci quaternion $Q_{\mathcal{F},n}$. If $a = 2$ and $b = p$, we get the (p,q)-Lucas quaternion $Q_{\mathcal{L},n}$.

In this paper, we study some properties of the (p,q)-Fibonacci quaternions, (p,q)-Lucas quaternions and the generalized Fibonacci quaternions.

2. Main results

There are three well-known identities for generalized Fibonacci numbers, namely, Catalan's, Cassini's, and d'Ocagne's identities. The proofs of these identities are based on Binet formulas. We can obtain these types of identities for generalized Fibonacci quaternions using the Binet formula for $Q_{w,n}$. Then, we require $\underline{\alpha}\underline{\beta}$ and $\underline{\beta}\underline{\alpha}$. These products are given in the following lemma.

Lemma 1. *We have*

$$\underline{\alpha}\underline{\beta} = Q_{\mathcal{L},0} - [q] - q\Delta\omega, \tag{6}$$

and

$$\underline{\beta}\underline{\alpha} = Q_{\mathcal{L},0} - [q] + q\Delta\omega, \tag{7}$$

where $\omega = qi + pj - k$, $[q] = 1 - q + q^2 - q^3$ *and* $\Delta = \alpha - \beta$.

Proof. From the definitions of $\underline{\alpha}$ and $\underline{\beta}$, and using $i^2 = j^2 = k^2 = -1$ and $ijk = -1$, we have

$$\begin{aligned}\underline{\alpha}\underline{\beta} &= 2 + (\alpha+\beta)i + (\alpha^2+\beta^2)j + (\alpha^3+\beta^3)k \\ &\quad - (1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3) + \alpha^2\beta^2(\beta-\alpha)i + \alpha\beta(\alpha^2-\beta^2)j + \alpha\beta(\beta-\alpha)k \\ &= 2 + pi + (p^2+2q)j + (p^3+3pq)k - (1 - q + q^2 - q^3) - q\Delta(qi + pj - k) \\ &= Q_{\mathcal{L},0} - [q] - q\Delta\omega,\end{aligned}$$

where $[q] = 1 - q + q^2 - q^3$ and $\omega = qi + pj - k$, and the final equation gives Eq. (6). The other identity can be computed similarly. □

The Lemma 1 gives us the following useful identity:

$$\underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha} = 2(Q_{\mathcal{L},0} - [q]). \tag{8}$$

The following theorem gives Catalan's identities for generalized Fibonacci quaternions.

Theorem 2. *For any integers* m *and* n *with* $m \geq n$, *we have*

$$Q_{w,m}^2 - Q_{w,m+n}Q_{w,m-n} = -AB(-q)^m\mathcal{F}_{-n}\left((Q_{\mathcal{L},0} - [q])\mathcal{F}_n - q\omega\mathcal{L}_n\right), \tag{9}$$

where $A = b - a\beta$, $B = b - a\alpha$, *and* $\mathcal{F}_n$, $\mathcal{L}_n$ *are the* n*-th* (p,q)*-Fibonacci and* (p,q)*-Lucas numbers, respectively.*

Proof. From the Binet formula for generalized Fibonacci quaternions $Q_{w,n}$ in (5) and $\Delta^2 = p^2 + 4q$, we have

$$\begin{aligned}&\Delta^2\left(Q_{w,m}^2 - Q_{w,m+n}Q_{w,m-n}\right) \\ &\quad = \left(A\underline{\alpha}\alpha^m - B\underline{\beta}\beta^m\right)^2 - \left(A\underline{\alpha}\alpha^{m+n} - B\underline{\beta}\beta^{m+n}\right)\left(A\underline{\alpha}\alpha^{m-n} - B\underline{\beta}\beta^{m-n}\right) \\ &\quad = AB(-q)^{m-n}\left(\underline{\alpha}\underline{\beta}\alpha^{2n} + \underline{\beta}\underline{\alpha}\beta^{2n} - (-q)^n\left(\underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha}\right)\right).\end{aligned}$$

We require Eqs. (6) and (7). Using this equations, we obtain

$$\begin{aligned}&Q_{w,m}^2 - Q_{w,m+n}Q_{w,m-n} \\ &\quad = \frac{AB(-q)^{m-n}}{\Delta^2}\left((Q_{\mathcal{L},0} - [q])(\alpha^{2n} + \beta^{2n} - 2(-q)^n) - q\Delta\omega(\alpha^{2n} - \beta^{2n})\right) \\ &\quad = \frac{AB(-q)^{m-n}}{\Delta^2}\left((Q_{\mathcal{L},0} - [q])(\mathcal{L}_{2n} - 2(-q)^n) - q\Delta^2\omega\mathcal{F}_{2n}\right).\end{aligned}$$

Using the identity $\Delta^2\mathcal{F}_n^2 = \mathcal{L}_{2n} - 2(-q)^n$ gives

$$Q_{w,m}^2 - Q_{w,m+n}Q_{w,m-n} = AB(-q)^{m-n}\left((Q_{\mathcal{L},0} - [q])\mathcal{F}_n^2 - q\omega\mathcal{F}_{2n}\right),$$

where $\mathcal{L}_n$, $\mathcal{F}_n$ are the n-th (p,q)-Lucas and (p,q)-Fibonacci numbers, respectively. With the help of the identities $\mathcal{F}_{2n} = \mathcal{F}_n\mathcal{L}_n$ and $\mathcal{F}_{-n} = -(-q)^n\mathcal{F}_n$, we have Eq. (9). The proof is completed. $\square$

Taking $n = 1$ in this theorem and using $\mathcal{F}_{-1} = \frac{1}{q}$, we obtain Cassini's identities for generalized Fibonacci quaternions. This result gives another version of the Corollary 3.6 in [10].

Corollary 3. *For any integer m, we have*

$$Q_{w,m}^2 - Q_{w,m+1}Q_{w,m-1} = AB(-q)^{m-1}\left(Q_{\mathcal{L},0} - [q] - pq\omega\right), \tag{10}$$

where $A = b - a\beta$, $B = b - a\alpha$ and $[q] = 1 - q + q^2 - q^3$.

The following theorem gives d'Ocagne's identities for generalized Fibonacci quaternions.

Theorem 4. *For any integers n and m with $n \geq m$, we have*

$$Q_{w,n}Q_{w,m+1} - Q_{w,n+1}Q_{w,m} = (-q)^m AB\left((Q_{\mathcal{L},0} - [q])\mathcal{F}_{n-m} - q\omega\mathcal{L}_{n-m}\right), \tag{11}$$

$\mathcal{F}_n$, $\mathcal{L}_n$ are the n-th (p,q)-Fibonacci and (p,q)-Lucas numbers, respectively.

Proof. Using the Binet formula for the generalized Fibonacci quaternions gives

$$\begin{aligned}
&\Delta^2(Q_{w,n}Q_{w,m+1} - Q_{w,n+1}Q_{w,m}) \\
&\qquad = \left(A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n\right)\left(A\underline{\alpha}\alpha^{m+1} - B\underline{\beta}\beta^{m+1}\right) \\
&\qquad\quad - \left(A\underline{\alpha}\alpha^{n+1} - B\underline{\beta}\beta^{n+1}\right)\left(A\underline{\alpha}\alpha^m - B\underline{\beta}\beta^m\right) \\
&\qquad = \Delta(-q)^m AB\left(\underline{\alpha}\underline{\beta}\alpha^{n-m} - \underline{\beta}\underline{\alpha}\beta^{n-m}\right).
\end{aligned}$$

We require the Eqs. (6) and (7). Substituting these into the previous equation, we have

$$\begin{aligned}
&Q_{w,n}Q_{w,m+1} - Q_{w,n+1}Q_{w,m} \\
&= \frac{1}{\Delta}(-q)^m AB\left((Q_{\mathcal{L},0} - [q])(\alpha^{n-m} - \beta^{n-m}) - q\Delta\omega(\alpha^{n-m} + \beta^{n-m})\right) \\
&= (-q)^m AB\left((Q_{\mathcal{L},0} - [q])\mathcal{F}_{n-m} - q\omega\mathcal{L}_{n-m}\right).
\end{aligned}$$

The second identity in the above equality, can be proved using $\mathcal{L}_{n-m} = \alpha^{n-m} + \beta^{n-m}$ and $\Delta\mathcal{F}_{n-m} = \alpha^{n-m} - \beta^{n-m}$. This proof is completed. $\square$

In particular, writing $m = n - 1$ in this theorem and using the identity $\mathcal{L}_1 = p$, we obtain Cassini's identities for generalized Fibonacci quaternions. Now, taking $m = n$ in this theorem and using the initial conditions $\mathcal{F}_0 = 0$ and $\mathcal{L}_0 = 2$, we obtain the next identity.

Corollary 5. *For any integer n, we have*

$$Q_{w,n}Q_{w,n+1} - Q_{w,n+1}Q_{w,n} = 2(-q)^{n+1}AB\omega, \tag{12}$$

where $A = b - a\beta$, $B = b - a\alpha$ and $\omega = qi + pj - k$.

Theorem 6. *For any integers n, r and s, we have*

$$Q_{\mathcal{L},n+r}Q_{\mathcal{F},n+s} - Q_{\mathcal{L},n+s}Q_{\mathcal{F},n+r} = 2(-q)^{n+r}\mathcal{F}_{s-r}(Q_{\mathcal{L},0} - [q]). \tag{13}$$

Proof. The Binet formulas for the (p,q)-Lucas and (p,q)-Fibonacci quaternions give

$$\begin{aligned}
&\Delta(Q_{\mathcal{L},n+r}Q_{\mathcal{F},n+s} - Q_{\mathcal{L},n+s}Q_{\mathcal{F},n+r}) \\
&\qquad = \left(\underline{\alpha}\alpha^{n+r} + \underline{\beta}\beta^{n+r}\right)\left(\underline{\alpha}\alpha^{n+s} - \underline{\beta}\beta^{n+s}\right) \\
&\qquad\ \ - \left(\underline{\alpha}\alpha^{n+s} + \underline{\beta}\beta^{n+s}\right)\left(\underline{\alpha}\alpha^{n+r} - \underline{\beta}\beta^{n+r}\right) \\
&\qquad = (\alpha\beta)^n(\alpha^s\beta^r - \alpha^r\beta^s)(\underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha}).
\end{aligned}$$

Using initial condition $Q_{\mathcal{L},0}$, we have

$$Q_{\mathcal{L},n+r}Q_{\mathcal{F},n+s} - Q_{\mathcal{L},n+s}Q_{\mathcal{F},n+r} = 2(-q)^{n+r}\mathcal{F}_{s-r}(Q_{\mathcal{L},0} - [q]).$$

□

After deriving these famous identities, we present some other identities for the generalized Fibonacci quaternions. In particular, when using the Binet formulas to obtain identities for the (p,q)-Fibonacci and (p,q)-Lucas quaternions, we require $\underline{\alpha}^2$ and $\underline{\beta}^2$. These products are given in the next lemma.

Lemma 7. *We have*

$$\underline{\alpha}^2 = (Q_{\mathcal{L},0} - r_{p,q}) + \Delta(Q_{\mathcal{F},0} - s_{p,q}), \tag{14}$$

and

$$\underline{\beta}^2 = (Q_{\mathcal{L},0} - r_{p,q}) - \Delta(Q_{\mathcal{F},0} - s_{p,q}), \tag{15}$$

where $\Delta = \alpha - \beta$, $r_{p,q} = 1 + \frac{p}{2}(\mathcal{F}_2 + \mathcal{F}_4 + \mathcal{F}_6) + q(\mathcal{F}_1 + \mathcal{F}_3 + \mathcal{F}_5)$, $s_{p,q} = \frac{1}{2}(\mathcal{F}_2 + \mathcal{F}_4 + \mathcal{F}_6)$ and $\mathcal{F}_n$ is the n-th (p,q)-Fibonacci number.

Proof. From the definitions of $\underline{\alpha}$ and $\underline{\beta}$, and using $i^2 = j^2 = k^2 = -1$, $ijk = -1$ and $\alpha^n = \mathcal{F}_n\alpha + q\mathcal{F}_{n-1}$ for $n \geq 1$, we have

$$\begin{aligned}
\underline{\alpha}^2 &= 2(1 + \alpha i + \alpha^2 j + \alpha^3 k) - (1 + \alpha^2 + \alpha^4 + \alpha^6) \\
&= 2 + pi + (p^2 + 2q)j + (p^3 + 3pq)k + \Delta(i + pj + (p^2 + q)k) \\
&\ \ - (1 + (\mathcal{F}_2\alpha + q\mathcal{F}_1) + (\mathcal{F}_4\alpha + q\mathcal{F}_3) + (\mathcal{F}_6\alpha + q\mathcal{F}_5)) \\
&= (Q_{\mathcal{L},0} - r_{p,q}) + \Delta(Q_{\mathcal{F},0} - s_{p,q}),
\end{aligned}$$

where $r_{p,q} = 1 + \frac{p}{2}(\mathcal{F}_2 + \mathcal{F}_4 + \mathcal{F}_6) + q(\mathcal{F}_1 + \mathcal{F}_3 + \mathcal{F}_5)$ and $s_{p,q} = \frac{1}{2}(\mathcal{F}_2 + \mathcal{F}_4 + \mathcal{F}_6)$ and the final equation gives Eq. (14). The other can be computed similarly. □

We present some interesting identities for (p,q)-Fibonacci, (p,q)-Lucas quaternions and generalized Fibonacci quaternions. A similar identity can be seen in Theorem 3.11 in [10].

Theorem 8. *For any integer n, we have*

$$Q_{\mathcal{L},n}^2 - Q_{\mathcal{F},n}^2 = \begin{pmatrix} \frac{\Delta^2-1}{\Delta^2}(Q_{\mathcal{L},0} - r_{p,q})\mathcal{L}_{2n} + (Q_{\mathcal{F},0} - s_{p,q})\mathcal{F}_{2n} \\ +2\frac{(\Delta^2+1)(-q)^n}{\Delta^2}(Q_{\mathcal{L},0} - [q]). \end{pmatrix} \tag{16}$$

Proof. Using the Binet formulas for the (p,q)-Fibonacci and (p,q)-Lucas quaternions, we obtain

$$\begin{aligned}\Delta^2(Q_{\mathcal{L},n}^2 - Q_{\mathcal{F},n}^2) &= \Delta^2\left(\underline{\alpha}\alpha^n + \underline{\beta}\beta^n\right)^2 - \left(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n\right)^2 \\ &= (\Delta^2-1)(\underline{\alpha}^2\alpha^{2n} + \underline{\beta}^2\beta^{2n}) + (\Delta^2+1)(\alpha\beta)^n(\underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha}).\end{aligned}$$

Substituting Eqs. (6) and (7) into the last equation, we have

$$\Delta^2(Q_{\mathcal{L},n}^2 - Q_{\mathcal{F},n}^2) = (\Delta^2-1)(\underline{\alpha}^2\alpha^{2n} + \underline{\beta}^2\beta^{2n}) + 2(\Delta^2+1)(\alpha\beta)^n(Q_{\mathcal{L},0} - [q]). \tag{17}$$

Then, using Eqs. (14) and (15), we obtain

$$\underline{\alpha}^2\alpha^{2n} + \underline{\beta}^2\beta^{2n} = (\alpha^{2n} + \beta^{2n})(Q_{\mathcal{L},0} - r_{p,q}) + \Delta(Q_{\mathcal{F},0} - s_{p,q})(\alpha^{2n} - \beta^{2n}). \tag{18}$$

Substituting Eq. (18) into Eq. (17) gives Eq. (16). □

Corollary 9. *For any integers n and m with $m \geq n$, we have*

$$Q_{\mathcal{F},n}Q_{w,m} - Q_{w,m}Q_{\mathcal{F},n} = 2(-q)^{n+1}\omega W_{m-n}, \tag{19}$$

where $\omega = qi + pj - k$ and $W_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta}$ is the n-th generalized Fibonacci number.

Proof. The Binet formulas for the (p,q)-Fibonacci and generalized Fibonacci quaternions give

$$\begin{aligned}&\Delta^2(Q_{\mathcal{F},n}Q_{w,m} - Q_{w,m}Q_{\mathcal{F},n}) \\ &\quad= \left(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n\right)\left(A\underline{\alpha}\alpha^m - B\underline{\beta}\beta^m\right) \\ &\quad\quad - \left(A\underline{\alpha}\alpha^m - B\underline{\beta}\beta^m\right)\left(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n\right) \\ &\quad= (A\alpha^m\beta^n - B\alpha^n\beta^m)(\underline{\alpha}\underline{\beta} - \underline{\beta}\underline{\alpha}).\end{aligned}$$

Using Eqs. (6) and (7), we have

$$\begin{aligned}Q_{\mathcal{F},n}Q_{w,m} - Q_{w,m}Q_{\mathcal{F},n} &= (\alpha\beta)^n(A\alpha^{m-n} - B\beta^{m-n})(\underline{\alpha}\underline{\beta} - \underline{\beta}\underline{\alpha}) \\ &= 2(-q)^{n+1}\omega W_{m-n},\end{aligned}$$

where $\omega = qi + pj - k$ and W_n is the n-th generalized Fibonacci number defined by $W_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta}$. □

Taking $m = n$ in this corollary and using $W_0 = a$, we obtain the next identity.

Corollary 10. *For any integer n, we have*

$$Q_{\mathcal{F},n}Q_{w,n} - Q_{w,n}Q_{\mathcal{F},n} = 2(-q)^{n+1}a\omega, \tag{20}$$

where $A = b - a\beta$, $B = b - a\alpha$ and $\omega = qi + pj - k$.

3. Conclusion

Sequences of numbers have been studied over several years, including the well known Horadam sequence and, consequently, on the Horadam quaternions studied in [4]. In this paper we have also contributed for the study of (p, q)-Fibonacci and Lucas quaternion sequence, deducing some of their identities using the Binet-style formula of Horadam quaternions. It is our intention to continue the study of this type of sequences, exploring some their applications in the science domain. For example, a new type of sequences in the complex algebra with the use of these numbers and their combinatorial properties.

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References

[1] Cerda-Morales, G., On generalized Fibonacci and Lucas numbers by matrix methods, *Hacettepe Journal of Mathematics and Statistics*, 42(2) (2013), 173–179.
[2] Halici, S., On Fibonacci quaternions, *Adv. Appl. Clifford Algebras,* 22 (2012), 321–327.
[3] Halici, S., On complex Fibonacci quaternions, *Adv. Appl. Clifford Algebras,* 23 (2013), 105–112.
[4] Halici, S., Karataş, A., On a generalization for Fibonacci quaternions, Chaos, *Solitons and Fractals,* 98 (2017), 178–182.
[5] Horadam, A. F., Complex Fibonacci numbers and Fibonacci quaternions, *Am. Math. Month.,* 70 (1963), 289–291.
[6] Horadam, A. F., Quaternion recurrence relations, *Ulam Quarterly,* 2 (1993), 23–33 .
[7] Horadam, A. F., Jacobsthal representation numbers, *Fibonacci Quarterly,* 34 (1996), 40–54.
[8] Ipek, A., On (p, q)-Fibonacci quaternions and their Binet formulas, generating functions and certain binomial sums, *Adv. Appl. Clifford Algebras,* 27 (2017), 1343–1351.
[9] Iyer, M. R., A note on Fibonacci quaternions, *Fibonacci Quaterly,* 7(3) (1969), 225–229.
[10] Patel, B.K., Ray, P.K., On the properties of (p, q)-Fibonacci and (p, q)-Lucas quaternions, *Mat. Rep.* 21(1) (2019), 15–25.

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SOME RESULTS AROUND QUADRATIC MAPS

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Abstract. This paper dedicated to study quadratic maps. We present some new operator equalities and inequalities by using quadratic map in the framework of $\mathcal{B}(\mathscr{H})$. Applications for particular case of interest are also provided.

1. Introduction and preliminaries

As customary, we reserve α for scalars and other capital letters denote general elements of the C^*-algebra $\mathcal{B}(\mathscr{H})$ of all bounded linear operator acting on Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. The absolute value of operator A is denoted by $|A| = (A^*A)^{\frac{1}{2}}$, where A^* stands for the adjoint of A. An operator A is called positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$. A linear map $\phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ is positive if $\phi(A) \geq 0$ whenever $A \geq 0$. More information on such maps can be found in [10, p. 18]. The study of linear maps on an algebra of bounded linear operators on a Hilbert space has been developed by many authors (see for instance [3, 5, 7, 13, 14]). Also, for a host of positive linear map inequalities, and for diverse applications of these inequalities, we refer to [8, 11, 15], and references therein. As is known to all, the linear property plays an important role to obtain this inequalities.

The motivation of this paper is to present some results concerning equalities and inequalities for maps without linear property on complex Hilbert spaces. In order to prove our main results, we need the following essential definitions. A map $\varphi : \mathcal{B}(\mathscr{H}) \times \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ is a *sesquilinear*, if satisfying the following conditions:

(1) $\varphi(\alpha A_1 + \beta A_2, B) = \alpha\varphi(A_1, B) + \beta\varphi(A_2, B)$;

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(2) $\varphi(A, \alpha B_1 + \beta B_2) = \overline{\alpha}\varphi(A, B_1) + \overline{\beta}\varphi(A, B_2)$;

for all $\alpha, \beta \in \mathbb{C}$ and $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathscr{H})$. A sesquilinear form φ is called *positive* if $\varphi(A, A) \geq 0$, for each $A \in \mathcal{B}(\mathscr{H})$. The sesquilinear form φ is said to be *symmetric* if $\varphi(A, B) = \varphi(B, A)$ for all $A, B \in \mathcal{B}(\mathscr{H})$. The map $\Phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ defined by $\Phi(A) = \varphi(A, A)$, is called the *quadratic map associated with* φ. It can be easily verified that the definition of quadratic map is different from positive linear map. In fact, by using a sesquilinear map we create a quadratic map, that is not necessarily linear and positive.

The paper is organized in the following way: After this Introduction, in Section 2 we deduce some equalities. The parallelogram law is recovered (see Theorem 2.1 and 2.2) and some other interesting operator equalities are established. Afterward, in Section 3, we get an extension of some well known inequalities such as, triangle (Theorem 3.1) inequality. Especially, Bohr's inequality is generalized to the context of quadratic map (see Theorem 3.4). Some results concerning this inequality are surveyed (see Corollary 3.5 and 3.6). In Section 4 before closing the paper, we give an application of our results in the previous sections. We show that our results are a generalization of some well known works due to Fujii [9] and Hirzallah [12].

2. Some equalities for quadratic maps

Here and throughout, Φ stands for the quadratic map. Our first main result in this section reads as follows.

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathscr{H})$. Then*

$$\Phi(A+B) + \Phi(A-B) = 2(\Phi(A) + \Phi(B)). \tag{2.1}$$

Proof. We observe that

$$\varphi(A+B, A+B) = \varphi(A, A) + \varphi(A, B) + \varphi(B, A) + \varphi(B, B). \tag{2.2}$$

Replace B by $-B$ in the above equality, we deduce

$$\varphi(A-B, A-B) = \varphi(A, A) - \varphi(A, B) - \varphi(B, A) + \varphi(B, B). \tag{2.3}$$

By adding (2.2) and (2.3), we obtain desired result (2.1). □

The following generalization of the parallelogram law holds.

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathscr{H})$ and $0 \neq t \in \mathbb{R}$. Then*

$$\Phi(A+B) + \frac{1}{t}\Phi(tA - B) = (1+t)\Phi(A) + \left(1 + \frac{1}{t}\right)\Phi(B). \tag{2.4}$$

Proof. We observe that

$$\begin{aligned}
&\Phi(A+B)+\frac{1}{t}\Phi(tA-B)\\
&\quad=\Phi(A)+\Phi(B)+\varphi(A,B)+\varphi(B,A)\\
&\qquad+t\Phi(A)+\frac{1}{t}\Phi(B)-\varphi(A,B)-\varphi(B,A)\\
&\quad=(1+t)\Phi(A)+\left(1+\frac{1}{t}\right)\Phi(B).
\end{aligned}$$

which proves the theorem. □

Remark 2.3. Assume that φ is a positive sesquilinear form. If $0<t\le 1$, then $\frac{1}{t}\ge 1$, so that the second term $\frac{1}{t}\Phi(tA-B)$ of the left side of the equality (2.4) is greater that $\Phi(tA-B)$. Hence we have

$$\Phi(A\mp B)+\Phi(tA\pm B)\le(1+t)\Phi(A)+\left(1+\frac{1}{t}\right)\Phi(B).$$

Similarly, if either $t\ge 1$ or $t<0$, then

$$\Phi(A\mp B)+\Phi(tA\pm B)\ge(1+t)\Phi(A)+\left(1+\frac{1}{t}\right)\Phi(B).$$

The following result can be regarded as an extension of the well-known Apollonius's identity (see, e.g., [2, Lemma 2.12]).

Theorem 2.4. *Let $A,B,C\in\mathcal{B}(\mathscr{H})$. Then*

$$\Phi(A-B)=2\Phi(C-A)+2\Phi(C-B)-4\Phi\left(C-\frac{A+B}{2}\right).\tag{2.5}$$

Proof. By Theorem 2.1, we have

$$\begin{aligned}
\Phi\left(C-\frac{A+B}{2}\right)&=\Phi\left(\frac{C}{2}-\frac{A}{2}+\frac{C}{2}-\frac{B}{2}\right)\\
&=2\left[\Phi\left(\frac{C}{2}-\frac{A}{2}\right)+\Phi\left(\frac{C}{2}-\frac{B}{2}\right)\right]-\Phi\left(\frac{B}{2}-\frac{A}{2}\right)\\
&=\frac{1}{2}\left[\Phi(C-A)+\Phi(C-B)\right]-\frac{1}{4}\Phi(B-A).
\end{aligned}$$

which is clearly equivalent to (2.5). □

The following result concerning the quadratic maps may be stated.

Theorem 2.5. *Let $A,B\in\mathcal{B}(\mathscr{H})$. Let φ be symmetric sesquilinear form and $\Phi(A)=\Phi(B)$. Then for each $\pm1,0\ne\alpha\in\mathbb{R}$ we have*

$$\Phi(A+\alpha B)=\Phi(B+\alpha A).\tag{2.6}$$

Proof. One can easily see that

$$\begin{aligned}\Phi(A+\alpha B) &= \Phi(A) + 2\alpha\varphi(A,B) + \alpha^2\Phi(B) \\ &= \Phi(B) + 2\alpha\varphi(B,A) + \alpha^2\Phi(A) \\ &= \Phi(B+\alpha A).\end{aligned}$$

Therefore we obtain the desired equality (2.6). □

Theorem 2.6. *Let $A, B, C \in \mathcal{B}(\mathscr{H})$ such that $A+B+C=0$, and let φ be a symmetric sesquilinear form and $\Phi(A) = \Phi(B)$. Then*

$$\Phi(A-C) = \Phi(B-C).$$

Proof. By easy computation we have

$$\begin{aligned}&\Phi(A-C) + \Phi(A-B) \\ &\quad = 2\Phi(A) + \Phi(C) + \Phi(B) - 2\varphi(A, B+C) \\ &\quad = 4\Phi(A) + \Phi(C) + \Phi(B).\end{aligned}$$

Also

$$\Phi(B-C) + \Phi(A-B) = 4\Phi(B) + \Phi(C) + \Phi(A).$$

Hence

$$\Phi(A-C) = \Phi(B-C).$$

□

The results in the following proposition is derived from the Theorem 2.6.

Proposition 2.7. *Let $A, B, C, D \in \mathcal{B}(\mathscr{H})$ such that $A+B+C+D=0$, and let φ be a symmetric sesquilinear form and $\Phi(A) = \Phi(B)$, $\Phi(C) = \Phi(D)$. Then*

$$\Phi(A-C) = \Phi(B-D),$$

and

$$\Phi(B-C) = \Phi(A-D).$$

Proof. It is easy to obtain that

$$\Phi(A-C) + \Phi(A-B) = 2\Phi(A) + \Phi(C) + \Phi(B) - 2\varphi(A, C+B),$$

$$\Phi(B-D) + \Phi(A-B) = 2\Phi(B) + \Phi(C) + \Phi(A) - 2\varphi(B, A+D).$$

Subtracting and using the hypothesis, this gives

$$\begin{aligned}\Phi(A-C) - \Phi(B-D) &= 2\varphi(B, A+D) - 2\varphi(A, C+B) \\ &= 2\varphi(B, A+D) + 2\varphi(A, A+D) \\ &= 2\varphi(A+B, A+D).\end{aligned}$$

Now

$$\varphi(A+B, A) = \Phi(A) + \varphi(B,A) = \Phi(B) + \varphi(A,B) = \varphi(A+B, B),$$

and

$$\varphi(A+B,D) = -\varphi(C+D,D) = -\varphi(C+D,C) = \varphi(A+B,C).$$

Therefore

$$\varphi(A+B,A+D) = \varphi(A+B,B+C) = -\varphi(A+B,A+D) = 0,$$

which implies that

$$\Phi(A-C) = \Phi(B-D).$$

We can easily also check that

$$\Phi(B-C) = \Phi(A-D).$$

□

3. Some inequalities for quadratic maps

The following simple result is of interest in itself as well:

Theorem 3.1. *Let $A, B \in \mathcal{B}(\mathscr{H})$ and Φ be a positive quadratic map associated with φ such that φ is symmetric. Then*

$$4\operatorname{Re}\varphi(A,B) \leq \Phi(A+B) \leq 2(\Phi(A)+\Phi(B)). \tag{3.1}$$

Proof. Since $\Phi(A-B) \geq 0$, then

$$\Phi(A) + 2\operatorname{Re}\varphi(A,B) + \Phi(B) \geq 4\operatorname{Re}\varphi(A,B),$$

therefore

$$\Phi(A+B) \geq 4\operatorname{Re}\varphi(A,B). \tag{3.2}$$

On the other hand

$$\Phi(A) + \Phi(B) \geq 2\operatorname{Re}\varphi(A,B),$$

then

$$2(\Phi(A)+\Phi(B)) \geq \Phi(A) + \Phi(B) + 2\operatorname{Re}\varphi(A,B),$$

so

$$2(\Phi(A)+\Phi(B)) \geq \Phi(A+B). \tag{3.3}$$

By (3.2) and (3.3) we deduce the desired result (3.1). □

It is worth to mention that the right side of inequality (3.1) is an extension of the triangle inequality.

Corollary 3.2. *Let $A, B, C \in \mathcal{B}(\mathscr{H})$ and Φ be a positive quadratic map associated with φ such that φ is symmetric. Then*

$$\Phi(A-C) \leq 2(\Phi(A-B)+\Phi(B-C)). \tag{3.4}$$

The forthcoming theorem gives an upper bound for $\Phi(A+B)$.

Theorem 3.3. *Let $A, B \in \mathcal{B}(\mathscr{H})$ and Φ be a positive quadratic map associated with φ such that φ is symmetric and $\Phi(A) = \Phi(B)$. Then for each $0 \neq \alpha \in \mathbb{R}$,*

$$\Phi(A+B) \leq \Phi\left(\alpha A + \alpha^{-1}B\right).$$

Proof. We know that for any real numbers $\alpha \neq 0$, $\left(\alpha - \alpha^{-1}\right)^2 \geq 0$ so $\alpha^2 + \alpha^{-2} \geq 2$. Using the fact that $\Phi(A) = \Phi(B)$, one has

$$\begin{aligned}\Phi\left(\alpha A + \alpha^{-1} B\right) &= \alpha^2 \Phi(A) + 2\varphi(A, B) + \alpha^{-2}\Phi(B) \\ &= \left(\alpha^2 + \alpha^{-2}\right)\left(\frac{\Phi(A) + \Phi(B)}{2}\right) + 2\varphi(A, B) \\ &\geq \Phi(A) + \Phi(B) + 2\varphi(A, B) \\ &= \Phi(A + B).\end{aligned}$$

This completes the proof of Theorem 3.3. □

Several authors discussed operator version of Bohr inequality (see for instance [4]). In the following, we give a unified version of Bohr inequality.

Theorem 3.4. *Let $A, B \in \mathcal{B}(\mathscr{H})$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p \leq q$, and let Φ be a positive quadratic map. Then*

$$\Phi(A - B) + \Phi((1 - p) A - B) \leq p\Phi(A) + q\Phi(B).$$

Proof. By easy computation observe that

$$\begin{aligned}&p\Phi(A) + q\Phi(B) - \Phi(A - B) - \Phi((1 - p) A - B) \\ &= (2 - p)(p - 1)\Phi(A) + (q - 2)\Phi(B) - (p - 2)(\varphi(A, B) + \varphi(B, A)) \\ &= (2 - p)(p - 1)\Phi(A) + \left(\frac{2 - p}{p - 1}\right)\Phi(B) + (2 - p)(\varphi(A, B) + \varphi(B, A)) \\ &= (2 - p)\Phi\left(\sqrt{p - 1} A + \frac{1}{\sqrt{p - 1}} B\right) \\ &\geq 0\end{aligned}$$

where the last inequality follows from the fact that $p \leq q$ and so the proof is complete. □

The following corollary is a natural consequence of the above result.

Corollary 3.5. *Let $A, B \in \mathcal{B}(\mathscr{H})$ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and let Φ be a positive quadratic map. Then*

$$\Phi(A - B) \leq p\Phi(A) + q\Phi(B).$$

The next results follows by applying Corollary 3.5, first to the operators A, B and second to the operators $A, -B$.

Corollary 3.6. *Let $A, B \in \mathcal{B}(\mathscr{H})$ and Φ be a positive quadratic map. Then for any $p > 1$,*

$$\pm(\varphi(A, B) + \varphi(B, A)) \leq (p - 1)\Phi(A) + \frac{1}{p - 1}\Phi(B).$$

4. Special case

For two bounded linear operators $A, B \in \mathcal{B}(\mathscr{H})$, we define the map $\varphi : \mathcal{B}(\mathscr{H}) \times \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$, with $\varphi(A, B) = B^*A$. This leads to $\varphi(A, A) = |A|^2$. It is obvious that $\varphi(A, B)$, is linear in the first variable and conjugate-linear in the second. For this we first observe from (2.1) the classic parallelogram law for operators.

$$|A + B|^2 + |A - B|^2 = 2\left(|A|^2 + |B|^2\right).$$

We have from (2.5), the following well known equality

$$|A - B|^2 = 2|C - A|^2 + 2|C - B|^2 - 4\left|C - \frac{A + B}{2}\right|^2.$$

The following generalization of parallelogram law is derived from inequality (2.4), which is obtained in [9, Theorem 4.1].

$$|A + B|^2 + \frac{1}{t}|tA - B|^2 = (1 - t)|A|^2 + \left(1 + \frac{1}{t}\right)|B|^2.$$

We also remark that, Corollary 2.3 is equivalent to [9, Theorem 3.1] by interchanging $\varphi(A, B) = B^*A$.

Also, Theorem 3.4 becomes

$$|A - B|^2 + |(1 - p)A - B|^2 \le p|A|^2 + q|B|^2.$$

This result was obtained in [12, Theorem 1].

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References

[1] Amir, D., Characterizations of inner product spaces, Birkhauser Verlag, 1986.

[2] Bauschke, H.H., Combettes, P.L., Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, New York, 2011.

[3] Bhatia, R., Sharma, R., Some inequalities for positive linear maps, *Linear Algebra Appl.,* 436(6) (2012), 1562–1571.

[4] Cheung, W.S., Pečarić, J., Bohr's inequalities for Hilbert space operators, *J. Math. Anal. Appl.,* 323(1) (2006), 403–412.

[5] Choi, M.D., Hadwin, D., Nordgren, E., Radjavi, H., Rosenthal, P., On positive linear maps preserving invertibility, *J. Funct. Anal.* 59(3) (1984), 462–469.

[6] Cvetkovski, Z., Inequalities: Theorems, Techniques and Selected Problems, Springer Science & Business Media, 2012.

[7] Evans, D.E., Positive linear maps on operator algebras, *Comm. Math. Phys.,* 48(1) (1976), 15–22.

[8] Fu, X., Some generalizations of operator inequalities, *J. Math. Inequal.,* 9(1) (2015), 101–105.

[9] Fujii, M., Zuo, H., Matrix order in Bohr inequality for operators, *Banach J. Math. Anal.,* 1 (2010), 21–27.

[10] Furuta, T., Mićić Hot, J., Pečarić, J., Seo, Y., Mond-Pečarić method in operator inequalities, Monographs in Inequalities, Zagreb, 2005.

[11] Gumus I.H., A note on a conjecture about Wielandt's inequality, *Linear Multilinear Algebra*, 63(9) (2015), 1909–1913.
[12] Hirzallah, O., Non-commutative operator Bohr inequality, *J. Math. Anal. Appl.,* 282(2) (2003), 578–583.
[13] Lin, M., On an operator Kantorovich inequality for positive linear maps, *J. Math. Anal. Appl.,* 402(1) (2013), 127–132.
[14] Størmer, E., Positive linear maps of operator algebras, *Acta Math.,* 110(1) (1963), 233–278.
[15] Zhang, P., More operator inequalities for positive linear maps, *Banach J. Math. Anal.,* 9(1) (2015), 166–172.

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INVARIANT SUBMANIFOLDS IN GOLDEN RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study invariant submanifolds of a golden Riemannian manifold with the aid of induced structures on them by the golden structure of the ambient manifold. We demonstrate that any invariant submanifold in a locally decomposable golden Riemannian manifold leaves invariant the locally decomposability of the ambient manifold. We give a necessary and sufficient condition for any submanifold in a golden Riemannian manifold to be invariant. We obtain some necessary conditions for the totally geodesicity of invariant submanifolds. Moreover, we find some facts on invariant submanifolds. Finally, we present an example of an invariant submanifold.

1. Introduction

The differential geometry of submanifolds has occupied an important place in natural and engineering sciences since some particular types of submanifolds have been used as a geometric tool to solve many problems concerning these disciplines. In particular, invariant submanifolds have a key role in applied mathematics and theoretical physics as a method, such as for determining non-linear normal modes in non-linear systems [1] and constructing the reduced description for dissipative systems of reaction kinetics [2]. When considered from this point of view, invariant submanifolds have a special meaning in differential geometry. Invariant submanifolds are one of typical classes among all submanifolds of an ambient manifold. It is well known that in general, an invariant submanifold inherits almost all properties of the ambient manifold. Therefore, invariant submanifolds are an active and fruitful research field playing a significant role in the development of modern differential

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geometry. Also, the papers related to invariant submanifolds have appeared in various ambient manifolds, such as almost contact Riemannian manifolds [3, 4], normal contact metric manifolds [5], Sasakian manifolds [6], almost product Riemannian manifolds [7], CR-manifolds [8] etc.

Recently, C^∞-differentiable manifolds endowed with golden structures, i.e., golden manifolds have become a popular topic in differential geometry. In [9], M. C. Crâşmăreanu and C. E. Hreţcanu have shown that there exists a close relationship between golden and almost product structures. In this sense, F. Etayo, R. Santamaría and A. Upadhyay have analyzed almost golden Riemannian manifolds by use of the corresponding almost product structures in [10], where the concept of a golden manifold was defined as a C^∞-differentiable manifold admitting an integrable golden structure. In [11], M. Gök, S. Keleş and E. Kılıç have examined the Schouten and Vrănceanu connections on golden manifolds. The different kind of classes of submanifolds in a golden Riemannian manifold have been defined according to the behaviour of their tangent bundles with respect to the action of the golden structure of the ambient manifold and studied by several geometers in [12, 13, 14, 15, 16]. Invariant submanifolds, which are one of important and known classes of submanifolds, have been investigated in a golden Riemannian manifold for the first time by C. E. Hreţcanu and M. C. Crâşmăreanu with the help of induced structures on them by the golden structure of the ambient manifold in [17] we can find their some fundamental properties. The authors have obtained a characterization for any submanifold in a golden Riemannian manifold to be invariant and proved that the Nijenhuis tensor of the induced structure vanishes identically on invariant submanifolds in the case that the ambient manifold is a locally decomposable golden Riemannian manifold. Also, an example of an invariant submanifold regarding a product of two spheres in an Euclidean space has been given in [18].

The main purpose of this paper is to examine invariant submanifolds of a golden Riemannian manifold by means of induced structures on them by the golden structure of the ambient manifold.

The paper has three sections and is organized as follows: Section 2 is devoted to preliminaries containing basic definitions, concepts, formulas, notations and results for golden Riemannian manifolds and their submanifolds. Section 3 deals with an investigation of invariant submanifolds of a golden Riemannian manifold. We prove that any invariant submanifold of a locally decomposable golden Riemannian manifold is also locally decomposable. We obtain a characterization for any submanifold in a golden Riemannian manifold to be invariant. We find some necessary conditions for any invariant submanifold to be totally geodesic. Also, we get other results on invariant submanifolds. Lastly, we construct an induced structure on a product of hyperspheres in an Euclidean space as an example of a golden Riemannian structure.

2. Preliminaries

In this section, we recall some basic facts on golden Riemannian manifolds and their submanifolds.

A non-trivial C^{∞}-tensor field f of type $(1,1)$ on a C^{∞}-differentiable manifold $\overline{M}$ is called a polynomial structure of degree n if it satisfies the algebraic equation

$$Q\left(x\right)=x^{n}+a_{n}x^{n-1}+\cdots+a_{2}x+a_{1}I=0, \tag{1}$$

where I is the identity $(1,1)$-tensor field on $\overline{M}$ and $f^{n-1}\left(p\right),f^{n-2}\left(p\right),\ldots,f\left(p\right),I$ are linearly independent for every point $p\in\overline{M}$. Also, the monic polynomial $Q\left(x\right)$ is named the structure polynomial [19].

A polynomial structure $\overline{\Phi}$ of degree 2 with the structure polynomial $Q\left(x\right)=x^{2}-x-1$ on a C^{∞}-differentiable real manifold $\overline{M}$ is called a golden structure. That is, the golden structure $\overline{\Phi}$ is a tensor field of type $(1,1)$ satisfying the algebraic equation

$$\overline{\Phi}^{2}=\overline{\Phi}+I. \tag{2}$$

In this case, we say that $\overline{M}$ is a golden manifold. We denote by $\Gamma\left(T\overline{M}\right)$ the Lie algebra of differentiable vector fields on $\overline{M}$. If there exists a Riemannian metric $\overline{g}$ on $\overline{M}$ endowed with a golden structure $\overline{\Phi}$ such that $\overline{g}$ and $\overline{\Phi}$ verify the relation

$$\overline{g}\left(\overline{\Phi}X,Y\right)=\overline{g}\left(X,\overline{\Phi}Y\right) \tag{3}$$

for any vector fields $X,Y\in\Gamma\left(T\overline{M}\right)$, then the pair $\left(\overline{g},\overline{\Phi}\right)$ is said to be a golden Riemannian structure and the triple $\left(\overline{M},\overline{g},\overline{\Phi}\right)$ is called a golden Riemannian manifold. The eigenvalues of the golden structure $\overline{\Phi}$ are $\phi=\frac{1+\sqrt{5}}{2}$ and $1-\phi=\frac{1-\sqrt{5}}{2}$ being the roots of the algebraic equation $x^{2}-x-1=0$, where the former is the golden ratio [9, 17, 18].

Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M},\overline{g},\overline{\Phi}\right)$. We denote by $T_{p}M$ and $T_{p}M^{\perp}$ its tangent and normal spaces at a point $p\in M$, respectively. Then the tangent space $T_{p}\overline{M}$ admits the decomposition

$$T_{p}\overline{M}=T_{p}M\oplus T_{p}M^{\perp}$$

for each point $p\in M$. The induced Riemannian metric on M is given by

$$g\left(X,Y\right)=\overline{g}\left(i_{*}X,i_{*}Y\right) \tag{4}$$

for any vector fields $X,Y\in\Gamma(TM)$, where i_{*} is the differential of the immersion $i:M\longrightarrow\overline{M}$. We consider a local orthonormal frame $\{N_{1},\ldots,N_{r}\}$ of the normal bundle $TM^{\perp}$. For every tangent vector field $X\in\Gamma(TM)$, the vector fields $\overline{\Phi}\left(i_{*}X\right)$ and $\overline{\Phi}\left(N_{\alpha}\right)$ on the ambient manifold $\overline{M}$ can be decomposed into tangential and normal components as follows:

$$\overline{\Phi}\left(i_{*}X\right)=i_{*}\left(\Phi\left(X\right)\right)+\sum_{\alpha=1}^{r}u_{\alpha}\left(X\right)N_{\alpha} \tag{5}$$

and

$$\overline{\Phi}\left(N_\alpha\right) = \varepsilon i_*\left(\xi_\alpha\right) + \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta, \varepsilon = \pm 1, \tag{6}$$

respectively, where Φ is a tensor field of type $(1,1)$ on M, ξ_α's are tangent vector fields on M, u_α's are differential 1-forms on M and $(a_{\alpha\beta})$ is a matrix of type $r \times r$ of real functions on M for any $\alpha, \beta \in \{1, \ldots, r\}$. Thus, we obtain a structure $\left(\Phi, g, u_\alpha, \varepsilon\xi_\alpha, \left(a_{\alpha\beta}\right)_{r\times r}\right)$ induced on M by the golden Riemannian structure $\left(\overline{g}, \overline{\Phi}\right)$. We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\overline{M}$ and M, respectively. Then the Gauss and Weingarten formulas of M in $\overline{M}$ are given, respectively, by

$$\overline{\nabla}_{i_*X} i_*Y = i_*\nabla_X Y + \sum_{\alpha=1}^{r} h_\alpha\left(X, Y\right) N_\alpha \tag{7}$$

and

$$\overline{\nabla}_{i_*X} N_\alpha = -i_* A_\alpha X + \sum_{\beta=1}^{r} l_{\alpha\beta}\left(X\right) N_\beta \tag{8}$$

for any vector fields $X, Y \in \Gamma(TM)$, where h_α's are the second fundamental tensors corresponding to N_α's, A_α's are the shape operators in the direction of N_α's and $l_{\alpha\beta}$'s are the 1-forms on M corresponding to the normal connection $\nabla^\perp$ for any $\alpha, \beta \in \{1, \ldots, r\}$. Also, the following relations hold:

$$h\left(X, Y\right) = \sum_{\alpha=1}^{r} h_\alpha\left(X, Y\right) N_\alpha, \tag{9}$$

$$h_\alpha\left(X, Y\right) = h_\alpha\left(Y, X\right), \tag{10}$$

$$h_\alpha\left(X, Y\right) = g\left(A_\alpha X, Y\right), \tag{11}$$

$$\nabla^\perp_X N_\alpha = \sum_{\beta=1}^{r} l_{\alpha\beta}\left(X\right) N_\beta \tag{12}$$

and

$$l_{\alpha\beta} = -l_{\beta\alpha} \tag{13}$$

for any vector fields $X, Y \in \Gamma(TM)$ [17].

As it is well known, the submanifold M is called totally geodesic if $h = 0$. Besides, the mean curvature vector H of M is defined by

$$H = \sum_{i=1}^{n} h\left(e_i, e_i\right),$$

where $\{e_1, \ldots, e_n\}$ is orthonormal basis of the tangent space T_pM at a point $p \in M$. If the mean curvature vector H vanishes identically, then M is said to be a minimal submanifold. If $h\left(X, Y\right) = g\left(X, Y\right) H$ for any vector fields $X, Y \in \Gamma(TM)$, then M is named a totally umbilical submanifold [20].

The triple $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$ is called a locally decomposable golden Riemannian manifold if the golden structure $\overline{\Phi}$ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$, i.e., the covariant derivative $\overline{\nabla}\,\overline{\Phi}$ is identically zero. Also, under the assumption that the induced structure is a golden structure, the same definition can be applied to the submanifold (M, g, Φ) in terms of the Levi-Civita connection ∇ of M.

3. Invariant Submanifolds of Golden Riemannian Manifolds

This section is mainly concerned with invariant submanifolds in golden Riemannian manifolds. We show that any invariant submanifold in a locally decomposable golden Riemannian manifold preserves the locally decomposability of the ambient manifold. We get an equivalent expression to the invariance of any submanifold in a golden Riemannian manifold. We give some necessary conditions for the totally geodesicity of invariant submanifolds. Besides, we obtain some results on invariant submanifolds.

As a beginning, we remember that the notion of an invariant submanifold in golden Riemannian manifolds. Any invariant submanifold M of a golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$ is submanifold such that the golden structure $\overline{\Phi}$ of the ambient manifold $\overline{M}$ carries each tangent vector of the submanifold M into its corresponding tangent space in the ambient manifold $\overline{M}$, in other words,

$$\overline{\Phi}\left(T_pM\right) \subseteq T_pM$$

for any point $p \in M$.

Let M be an n-dimensional invariant submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then we have $\xi_\alpha = 0$ and $u_\alpha = 0$ for any $\alpha \in \{1, \ldots, r\}$. Hence, (5) and (6) can be expressed in the following forms:

$$\overline{\Phi}\left(i_*X\right) = i_*\left(\Phi\left(X\right)\right) \tag{14}$$

and

$$\overline{\Phi}\left(N_\alpha\right) = \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta, \tag{15}$$

respectively.

Theorem 1. [18, Remark 3.1] *Let M be an n-dimensional invariant submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then the induced structure $\left(\Phi, g, u_\alpha = 0, \varepsilon\xi_\alpha = 0, \left(a_{\alpha\beta}\right)_{r\times r}\right)$ on M by the golden Riemannian structure $\left(\overline{g}, \overline{\Phi}\right)$ satisfies the following relations:*

$$\Phi^2\left(X\right) = \Phi\left(X\right) + X, \tag{16}$$

$$a_{\alpha\beta} = a_{\beta\alpha}, \tag{17}$$

$$\sum_{\gamma=1}^{r} a_{\alpha\gamma} a_{\beta\gamma} = \delta_{\alpha\beta} + a_{\alpha\beta}, \tag{18}$$

$$g(\Phi(X), Y) = g(X, \Phi(Y)) \tag{19}$$

and

$$g(\Phi(X), \Phi(Y)) = g(\Phi(X), Y) + g(X, Y) \tag{20}$$

for any vector fields $X, Y \in \Gamma(TM)$.

Theorem 2. [18, Theorem 3.2] *Let* M *be an* n*-dimensional submanifold of codimension* r*, isometrically immersed in an* m*-dimensional golden Riemannian manifold* $(\overline{M}, \overline{g}, \overline{\Phi})$*. Then* M *is an invariant submanifold if and only if the induced structure* (Φ, g) *on* M *is a golden Riemannian structure whenever* Φ *is non-trivial.*

Theorem 3. [17, Theorem 2.1] *Let* M *be an* n*-dimensional invariant submanifold of codimension* r*, isometrically immersed in an* m*-dimensional locally decomposable golden Riemannian manifold* $(\overline{M}, \overline{g}, \overline{\Phi})$*. Then the induced structure* $\left(\Phi, g, u_\alpha = 0, \varepsilon\xi_\alpha = 0, (a_{\alpha\beta})_{r\times r}\right)$ *on* M *by the golden Riemannian structure* $(\overline{g}, \overline{\Phi})$ *verifies the following relations:*

$$(\nabla_X \Phi) Y = 0, \tag{21}$$

$$h_\alpha(X, \Phi Y) = \sum_{\beta=1}^{r} h_\beta(X, Y) a_{\beta\alpha}, \tag{22}$$

$$\Phi(A_\alpha X) = \sum_{\beta=1}^{r} a_{\alpha\beta} A_\beta X \tag{23}$$

and

$$X(a_{\alpha\beta}) = \sum_{\gamma=1}^{r} l_{\alpha\gamma}(X) a_{\gamma\beta} + \sum_{\gamma=1}^{r} l_{\beta\gamma}(X) a_{\alpha\gamma} \tag{24}$$

for any vector fields $X, Y \in \Gamma(TM)$.

Theorem 4. *Let* M *be an* n*-dimensional invariant submanifold of codimension* r*, isometrically immersed in an* m*-dimensional locally decomposable golden Riemannian manifold* $(\overline{M}, \overline{g}, \overline{\Phi})$*. Then* M *is a locally decomposable golden Riemannian manifold whenever* Φ *is non-trivial.*

Proof. Taking into consideration Theorem 2, the proof is obvious from (21). □

Theorem 5. *Let* M *be an* n*-dimensional submanifold of codimension* r*, isometrically immersed in an* m*-dimensional golden Riemannian manifold* $(\overline{M}, \overline{g}, \overline{\Phi})$*. Then* M *is an invariant submanifold if and only if there exists a local orthonormal frame of the normal bundle* $TM^\perp$ *such that it consists of eigenvectors of the golden structure* $\overline{\Phi}$*.*

Proof. At first, we recall that it is possible to transform the local orthonormal frame $\{N_1, \ldots, N_r\}$ of the normal bundle $TM^{\perp}$ into another local orthonormal frame $\{N'_1, \ldots, N'_r\}$ such that $\xi'_\alpha = \sum_{\gamma=1}^{r} k^\gamma_\alpha \xi_\gamma$ and $a'_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$, where (k^γ_α) is an orthogonal matrix of type $r \times r$ and λ_a's are the eigenvalues of the matrix $(a_{\alpha\beta})_{r\times r}$ for any $\alpha, \beta \in \{1, \ldots, r\}$. If M is an invariant submanifold, then the tangent vector fields ξ'_α's are zero. Hence, we obtain from (15) that

$$\overline{\Phi}\left(N'_\alpha\right) = \lambda_\alpha N'_\alpha,\ \alpha = 1, \ldots, r,$$

which shows that the normal vector fields N'_α's are eigenvectors of the golden structure $\overline{\Phi}$. Conversely, we assume that $\overline{\Phi}\left(N'_\alpha\right) = \lambda_\alpha N'_\alpha$ for any $\alpha \in \{1, \ldots, r\}$. Then it follows from (15) that

$$\xi'_\alpha = 0,\ \alpha = 1, \ldots, r,$$

from which we conclude that the submanifold M is invariant. □

Theorem 6. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then M is a totally geodesic invariant submanifold if the following relations are satisfied:*

$$\overline{\Phi} i_* = \phi i_* \tag{25}$$

and

$$\overline{\Phi} N_\alpha = (1-\phi)\, N_\alpha,\ \alpha = 1, \ldots, r. \tag{26}$$

Proof. Using (25) and (26) in (5) and (6), respectively, we get

$$\Phi = \phi I \tag{27}$$

and

$$a_{\alpha\beta} = (1-\phi)\, \delta_{\alpha\beta} \tag{28}$$

for any $\alpha, \beta \in \{1, \ldots, r\}$. On the other hand, (25) and (26) mean that M is an invariant submanifold. Thus, in virtue of (27) and (28), it results by a simple computation from (23) that

$$\sqrt{5} A_\alpha = 0,\ \alpha = 1, \ldots, r,$$

which proves that the submanifold M is totally geodesic. As a result, the proof has been completed. □

Theorem 7. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then M is a totally geodesic invariant submanifold if the following relations are verified:*

$$\overline{\Phi} i_* = (1-\phi)\, i_* \tag{29}$$

and

$$\overline{\Phi} N_\alpha = \phi N_\alpha,\ \alpha = 1, \ldots, r. \tag{30}$$

Proof. Applying (29) and (30) to (5) and (6), respectively, we deduce

$$\Phi = (1-\phi)\, I \tag{31}$$

and

$$a_{\alpha\beta} = \phi\delta_{\alpha\beta} \tag{32}$$

for any $\alpha, \beta \in \{1, \dots, r\}$. On the other hand, it is clear from (29) and (30) that M is an invariant submanifold. Hence, taking into account (31) and (32), we obtain by a straightforward computation from (23) that

$$-\sqrt{5}A_\alpha = 0,\ \alpha = 1, \dots, r,$$

which implies that the submanifold M is totally geodesic. Consequently, the proof has been shown. □

Theorem 8. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then the second fundamental tensors h_θ's are zero for any $\theta \in \{1, \dots, t < r\}$ if the following relations are satisfied:*

$$\overline{\Phi} i_* = \phi i_*, \tag{33}$$

$$\overline{\Phi} N_\theta = (1-\phi)\, N_\theta,\ \theta = 1, \dots, t \tag{34}$$

and

$$\overline{\Phi} N_\mu = \phi N_\mu,\ \mu = t+1, \dots, r. \tag{35}$$

Proof. Taking account of (33), (34) and (35), in view of (5) and (6), we obtain

$$\Phi = \phi I, \tag{36}$$

$$a_{\theta\vartheta} = (1-\phi)\, \delta_{\theta\vartheta},\ \theta, \vartheta = 1, \dots, t \tag{37}$$

and

$$a_{\mu\nu} = \phi\delta_{\mu\nu},\ \mu, \nu = t+1, \dots, r. \tag{38}$$

On the other hand, it follows from (33), (34) and (35) that the submanifold M is invariant. Hence, by means of (36), (37) and (38), (23) takes the form

$$\sqrt{5}A_\theta = 0,\ \theta = 1, \dots, t < r,$$

from which we have

$$h_\theta = 0,\ \theta = 1, \dots, t < r.$$

□

Theorem 9. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then the second fundamental tensors h_θ's are zero for any $\theta \in \{1, \dots, t < r\}$ if the following relations are verified:*

$$\overline{\Phi} i_* = (1-\phi)\, i_*, \tag{39}$$

$$\overline{\Phi} N_\theta = \phi N_\theta,\ \theta = 1, \dots, t, \tag{40}$$

and

$$\overline{\Phi}N_{\mu} = (1-\phi)\, N_{\mu},\ \mu = t+1,\ldots,r. \tag{41}$$

Proof. By reason of (39), (40) and (41), we infer from (5) and (6) that

$$\Phi = (1-\phi)\, I, \tag{42}$$

$$a_{\theta\vartheta} = \phi\delta_{\theta\vartheta},\ \theta,\vartheta = 1,\ldots,t \tag{43}$$

and

$$a_{\mu\nu} = (1-\phi)\,\delta_{\mu\nu},\ \mu,\nu = t+1,\ldots,r. \tag{44}$$

On the other hand, it is obvious from (39), (40) and (41) that the submanifold M is invariant. Thus, using (42), (43) and (44), (23) is reduced to

$$-\sqrt{5}A_{\theta} = 0,\ \theta = 1,\ldots,t < r,$$

which implies

$$h_{\theta} = 0,\ \theta = 1,\ldots,t < r.$$

□

Theorem 10. *Let M be an n-dimensional totally umbilical invariant submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M},\overline{g},\overline{\Phi}\right)$. If*

$$\left\{tr(\Phi)\right\}^2 \neq n\left\{n + tr(\Phi)\right\},$$

or equivalently

$$\left\{tr(\Phi)\right\}^2 \neq \lambda^2 n^2,$$

then M is a totally geodesic submanifold, where λ is one of the eigenvalues of the golden structure $\overline{\Phi}$.

Proof. We denote by $\{e_1,\ldots,e_n\}$ an orthonormal basis of the tangent space T_pM at a point $p \in M$. Since the submanifold M is totally umbilical, there are constants σ_{α}'s such that $h_{\alpha} = \sigma_{\alpha} g$ for any $\alpha \in \{1,\ldots,r\}$. Then (22) is given by

$$\sigma_{\alpha} g\,(X,\Phi Y) = \sum_{\beta=1}^{r} a_{\beta\alpha}\sigma_{\beta} g\,(X,Y) \tag{45}$$

for any vector fields $X, Y \in \Gamma\,(TM)$. Putting $X_p = Y_p = e_i$ for any $i \in \{1,\ldots,n\}$ at the point $p \in M$ in (45), we have

$$\sigma_{\alpha} g\,(e_i,\Phi e_i) = g\,(e_i,e_i)\sum_{\beta=1}^{r} a_{\beta\alpha}\sigma_{\beta}. \tag{46}$$

Summing over i in (46), we get

$$\sum_{i=1}^{n} \sigma_{\alpha} g\,(e_i,\Phi e_i) = n\sum_{\beta=1}^{r} a_{\beta\alpha}\sigma_{\beta},$$

which implies

$$tr(\Phi)\sigma_\alpha = n\sum_{\beta=1}^{r} a_{\beta\alpha}\sigma_\beta. \tag{47}$$

Multiplying (47) by the matrix element $a_{\beta\alpha}$ and then summing over α, we obtain

$$tr(\Phi)\sum_{\alpha=1}^{r} a_{\beta\alpha}\sigma_\alpha = n\sum_{\gamma=1}^{r}\sum_{\alpha=1}^{r} a_{\beta\alpha}a_{\alpha\gamma}\sigma_\gamma. \tag{48}$$

Using (18), (48) takes the form

$$tr(\Phi)\sum_{\alpha=1}^{r} a_{\beta\alpha}\sigma_\alpha = n\sigma_\beta + n\sum_{\gamma=1}^{r} a_{\beta\gamma}\sigma_\gamma,$$

from which we have

$$\sigma_\beta = \frac{1}{n}\left(tr(\Phi) - n\right)\sum_{\alpha=1}^{r} a_{\beta\alpha}\sigma_\alpha. \tag{49}$$

Hence, substituting (49) into (47), we find

$$\left\{tr(\Phi)\left(tr(\Phi) - n\right) - n^2\right\}\sum_{\beta=1}^{r} a_{\alpha\beta}\sigma_\beta = 0. \tag{50}$$

On the other hand, on account of the fact that $\{tr(\Phi)\}^2 \neq n\{n + tr(\Phi)\}$, or equivalently $\{tr(\Phi)\}^2 \neq \lambda^2 n^2$ in the hypothesis, it follows from (50) that

$$\sum_{\beta=1}^{r} a_{\alpha\beta}\sigma_\beta = 0.$$

Therefore, we infer from (49) that

$$\sigma_\beta = 0,\ \beta = 1, \ldots, r,$$

which demonstrates that the submanifold M is totally geodesic. □

Now, we give an example.

Example 11. *Let* $\left(E^{2(p+q)}, \langle,\rangle\right)$ *be the* $2(p+q)$*-dimensional Euclidean space, where* p *and* q *are two positive natural numbers. Hereafter we use the following abbreviations for a point and a tangent vector in the Euclidean space* $E^{2(p+q)}$*, respectively:*

$$\left(x^i, y^i, z^j, w^j\right) = \left(x^1, \ldots, x^p, y^1, \ldots, y^p, z^1, \ldots, z^q, w^1, \ldots, w^q\right)$$

and

$$\left(X^i, Y^i, Z^j, W^j\right) = \left(X^1, \ldots, X^p, Y^1, \ldots, Y^p, Z^1, \ldots, Z^q, W^1, \ldots, W^q\right).$$

We consider a tensor field $\overline{\Phi}$ *of type* $(1,1)$ *defined by*

$$\overline{\Phi}\left(X^i, Y^i, Z^j, W^j\right) = \left(\phi X^i, \phi Y^i, (1-\phi)Z^j, (1-\phi)W^j\right)$$

for any tangent vector $\left(X^i, Y^i, Z^j, W^j\right) \in T_{(x^i,y^i,z^j,w^j)}E^{2(p+q)}$, *where* ϕ *and* $1-\phi$ *are the roots of the algebraic equation* $x^2 - x - 1 = 0$, *i.e.,* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $1-\phi = \frac{1-\sqrt{5}}{2}$. *In this case, it is easy to show that* $\left(\langle,\rangle, \overline{\Phi}\right)$ *is a golden Riemannian structure and* $\left(E^{2(p+q)}, \langle,\rangle, \overline{\Phi}\right)$ *is a golden Riemannian manifold.*

Because of the fact that $E^{2(p+q)} = E^p \times E^p \times E^q \times E^q$, *we have the following four hyperspheres:*

$$S^{p-1}(r_1) = \left\{ \left(x^1, \ldots, x^p\right) : \sum_{i=1}^{p} \left(x^i\right)^2 = r_1^2 \right\},$$

$$S^{p-1}(r_2) = \left\{ \left(y^1, \ldots, y^p\right) : \sum_{i=1}^{p} \left(y^i\right)^2 = r_2^2 \right\},$$

$$S^{q-1}(r_3) = \left\{ \left(z^1, \ldots, z^q\right) : \sum_{j=1}^{q} \left(z^j\right)^2 = r_3^2 \right\}$$

and

$$S^{q-1}(r_4) = \left\{ \left(w^1, \ldots, w^q\right) : \sum_{j=1}^{q} \left(w^j\right)^2 = r_4^2 \right\}.$$

We construct the product manifold $S^{p-1}(r_1) \times S^{p-1}(r_2) \times S^{q-1}(r_3) \times S^{q-1}(r_4)$ *in a similar way as in* [18]. *We denote it by* M *for simplicity. Its every point has the coordinates* $\left(x^i, y^i, z^j, w^j\right)$ *satisfying the equation*

$$\sum_{i=1}^{p} \left(x^i\right)^2 + \sum_{i=1}^{p} \left(y^i\right)^2 + \sum_{j=1}^{q} \left(z^j\right)^2 + \sum_{j=1}^{q} \left(w^j\right)^2 = R^2,$$

where $R^2 = r_1^2 + r_2^2 + r_3^2 + r_4^2$. *Then* M *is a submanifold of codimension* 4 *in the Euclidean space* $E^{2(p+q)}$ *and* M *is a submanifold of codimension* 3 *in the sphere* $S^{2(p+q)-1}(R)$. *Hence, there exist successive embeddings such that*

$$M \hookrightarrow S^{2(p+q)-1}(R) \hookrightarrow E^{2(p+q)}.$$

Also, its tangent space $T_{(x^i,y^i,z^j,w^j)}M$ *at a point* $\left(x^i, y^i, z^j, w^j\right)$ *is as follows:*

$$T_{(x^i,0^i,0^j,0^j)}S^{p-1}(r_1) \oplus T_{(0^i,y^i,0^j,0^j)}S^{p-1}(r_2) \oplus T_{(0^i,0^i,z^j,0^j)}S^{q-1}(r_3) \oplus T_{(0^i,0^i,0^j,w^j)}S^{q-1}(r_4).$$

As it is seen, any tangent vector $\left(X^i, Y^i, Z^j, W^j\right) \in T_{(x^i,y^i,z^j,w^j)}E^{2(p+q)}$ *belongs to* $T_{(x^i,y^i,z^j,w^j)}M$ *for every point* $\left(x^i, y^i, z^j, w^j\right) \in M$ *if and only if*

$$\sum_{i=1}^{p} x^i X^i = \sum_{i=1}^{p} y^i Y^i = \sum_{j=1}^{q} z^j Z^j = \sum_{j=1}^{q} w^j W^j = 0.$$

In addition, since $\left(X^i, Y^i, Z^j, W^j\right)$ *is a tangent vector on the sphere* $S^{2(p+q)-1}(R)$, *we have*

$$T_{(x^i,y^i,z^j,w^j)}M \subset T_{(x^i,y^i,z^j,w^j)}S^{2(p+q)-1}(R)$$

for every point $\left(x^i, y^i, z^j, w^j\right) \in M$.

Let us consider a local orthonormal basis $\{N_1, N_2, N_3, N_4\}$ *for the normal space* $T_{(x^i,y^i,z^j,w^j)}M^{\perp}$ *at a point* $\left(x^i, y^i, z^j, w^j\right)$. *Then we can choose the normal vectors* N_1, N_2, N_3 *and* N_4 *such that*

$$N_1 = \frac{1}{R}\left(x^i, y^i, z^j, w^j\right),$$

$$N_2 = \frac{1}{R}\left(\frac{r_2}{r_1}x^i, -\frac{r_1}{r_2}y^i, \frac{r_4}{r_3}z^j, -\frac{r_3}{r_4}w^j\right),$$

$$N_3 = \frac{1}{R}\left(\frac{r_3}{r_1}x^i, -\frac{r_4}{r_2}y^i, -\frac{r_1}{r_3}z^j, \frac{r_2}{r_4}w^j\right)$$

and

$$N_4 = \frac{1}{R}\left(\frac{r_4}{r_1}x^i, \frac{r_3}{r_2}y^i, -\frac{r_2}{r_3}z^j, -\frac{r_1}{r_4}w^j\right).$$

We identify i_*X *with* X *for any tangent vector* $X \in T_{(x^i,y^i,z^j,w^j)}M$. *From (6), we have*

$$\overline{\Phi}N_\alpha = \xi_\alpha + \sum_{\beta=1}^{4} a_{\alpha\beta}N_\beta \tag{51}$$

for any $\alpha \in \{1, 2, 3, 4\}$. *Also, we remark that*

$$a_{\alpha\beta} = \left\langle \overline{\Phi}N_\alpha, N_\beta \right\rangle$$

for any $\alpha, \beta \in \{1, 2, 3, 4\}$. *Then by straightforward computations, we obtain the elements of the matrix* $\mathcal{A} = (a_{\alpha\beta})_{4\times4}$ *as follows:*

$$a_{11} = a_{22} = \frac{1}{2R^2}\left(R^2 + \sqrt{5}\left(r_1^2 + r_2^2 - r_3^2 - r_4^2\right)\right),$$

$$a_{12} = a_{21} = a_{34} = a_{43} = 0,$$

$$a_{13} = a_{31} = -a_{24} = -a_{42} = \frac{\sqrt{5}}{R^2}\left(r_1r_3 - r_2r_4\right),$$

$$a_{14} = a_{41} = a_{23} = a_{32} = \frac{\sqrt{5}}{R^2}\left(r_1r_4 + r_2r_3\right),$$

$$a_{33} = a_{44} = \frac{1}{2R^2}\left(R^2 - \sqrt{5}\left(r_1^2 + r_2^2 - r_3^2 - r_4^2\right)\right).$$

Hence, using the matrix elements $a_{\alpha\beta}$*'s given above, it follows from (51) that*

$$\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0_{2(p+q)}. \tag{52}$$

In this case, we have

$$\overline{\Phi}\left(T_{(x^i,y^i,z^j,w^j)}M^{\perp}\right) \subseteq T_{(x^i,y^i,z^j,w^j)}M^{\perp}.$$

From (5), we can write the following relation:

$$\overline{\Phi}\left(X^i, Y^i, Z^j, W^j\right) = \Phi\left(X^i, Y^i, Z^j, W^j\right) + \sum_{\alpha=1}^{4} u_\alpha\left(X^i, Y^i, Z^j, W^j\right)N_\alpha. \tag{53}$$

We recall that

$$u_\alpha\left(X^i, Y^i, Z^j, W^j\right) = \varepsilon\left\langle\left(X^i, Y^i, Z^j, W^j\right), \xi_\alpha\right\rangle$$

for any $\alpha \in \{1,2,3,4\}$*, where* $\varepsilon = \pm 1$*. Then we get from (52) that*

$$u_1 = u_2 = u_3 = u_4 = 0. \tag{54}$$

Thus, we infer from (53) and (54) that

$$\overline{\Phi}\left(X^i, Y^i, Z^j, W^j\right) = \Phi\left(X^i, Y^i, Z^j, W^j\right).$$

In the circumstances, we have

$$\overline{\Phi}\left(T_{(x^i,y^i,z^j,w^j)}M\right) \subseteq T_{(x^i,y^i,z^j,w^j)}M$$

and

$$\Phi^2 = \Phi + I.$$

Consequently, we establish an induced structure $\left(\Phi, \langle,\rangle, \varepsilon\xi_\alpha = 0_{2(p+q)}, u_\alpha = 0, \mathcal{A}\right)$ *on the product of hyperspheres* M *by the golden Riemannian structure* $\left(\langle,\rangle, \overline{\Phi}\right)$ *on the Euclidean space* $E^{2(p+q)}$*. Moreover,* $(\Phi, \langle,\rangle)$ *is a golden Riemannian structure and* M *is an invariant submanifold in the Euclidean space* $E^{2(p+q)}$*.*

References

[1] Shaw, S., Pierre, C., Non-linear normal modes and invariant manifolds, *J. Sound Vib.,* 150(1) (1991), 170-173.

[2] Gorban, A. N., Karlin, I. V., Method of invariant manifold for chemical kinetics, *Chem. Eng. Sci.,* 58(21) (2003), 4751-4768.

[3] Yano, K., Ishihara, S., Invariant submanifolds of an almost contact manifold, *Kodai Math. Sem. Rep.,* 21(3) (1969), 350-364.

[4] Kenmotsu, K., A class of almost contact Riemannian manifolds, *Tohoku Math. J.,* 24(2) (1972), 93-103.

[5] Kon, M., Invariant submanifolds of normal contact metric manifolds, *Kodai Math. Sem. Rep.,* 25(3) (1973), 330-336.

[6] Kon, M., Invariant submanifolds in Sasakian manifolds, *Math. Ann.,* 219(3) (1976), 277-290.

[7] Adati, T., Submanifolds of an almost product Riemannian manifold, *Kodai Math. J.,* 4(2) (1981), 327-343.

[8] Takemura, Y., On the invariant submanifold of a CR-manifold, *Kodai Math. J.,* 5(3) (1982), 416-425.

[9] Crâşmăreanu, M. C., Hreţcanu, C. E., Golden differential geometry, *Chaos Solitons Fractals,* 38(5) (2008), 1229-1238.

[10] Etayo, F., Santamaría, R., Upadhyay, A., On the geometry of almost golden Riemannian manifolds, *Mediterr. J. Math.,* 14(5) (2017), Article ID 187, 14 pages.

[11] Gök, M., Keleş, S., Kılıç, E., Schouten and Vrănceanu connections on golden manifolds, *Int. Electron. J. Geom.,* 12(2) (2019), 169-181.

[12] Erdoğan, F. E., Yıldırım, C., On a study of the totally umbilical semi-invariant submanifolds of golden Riemannian manifolds, *J. Polytechnic,* 21(4) (2018), 967-970.

[13] Blaga, A. M., Hreţcanu, C. E., Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold, *Novi Sad J. Math.,* 48(2) (2018), 55-80.

[14] Hreţcanu, C. E., Blaga, A. M., Slant and semi-slant submanifolds in metallic Riemannian manifolds, *J. Funct. Spaces,* 2018 (2018), Article ID 2864263, 13 pages.

[15] Hreţcanu, C. E., Blaga, A. M., Hemi-slant submanifolds in metallic Riemannian manifolds, *Carpathian J. Math.,* 35(1) (2019), 59-68.

[16] Gök, M., Keleş, S., Kılıç, E., Some characterizations of semi-invariant submanifolds of golden Riemannian manifolds, *Mathematics,* 7(12) (2019), Article ID 1209, 12 pages.

[17] Hreţcanu, C. E., Crâşmăreanu, M. C., On some invariant submanifolds in a Riemannian manifold with golden structure, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat., (N.S.)* 53(suppl. 1) (2007), 199-211.

[18] Hreţcanu, C. E., Crâşmăreanu, M. C., Applications of the golden ratio on Riemannian manifolds, *Turk. J. Math.,* 33(2) (2009), 179-191.

[19] Goldberg, S. I., Yano, K., Polynomial structures on manifolds, *Kodai Math. Sem. Rep.,* 22(2) (1970), 199-218.

[20] Yano, K., Kon, M., Structures on Manifolds, World Scientific, Singapore, 1984.

https://communications.science.ankara.edu.tr

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LATTICE STRUCTURES OF AUTOMATA

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Abstract. Structures and the number of subautomata of a finite automaton are investigated. It is shown that the set of all subautomata of a finite automaton $\mathcal{A}$ is upper semilattice. We give conditions which allow us to determine whether for a finite upper semilattice $(L, \leq)$ there exists an automaton $\mathcal{A}$ such that the set of all subautomata of $\mathcal{A}$ under set inclusion is isomorphic to $(L, \leq)$. Examples illustrating the results are presented.

1. Introduction

With the advent of electronic computers in the 1950's, the study of simple formal models of computers such as automata was given a lot of attention. The aims were multiple: to understand the limitations of machines, to determine to what extent they might come to replace humans, and later to obtain efficient schemes to organize computations. One of the simplest models that quickly emerged is the finite automaton which, in algebraic terms, is basically the action of a finitely generated free semigroup on a finite set of states and thus leads to a finite semigroup of transformations of the states. From its very beginning, the theory of automata, especially the algebraic one, was based on numerous algebraic ideas and methods. The fact that automata without outputs, and hence the automata without outputs belonging to arbitrary automata, can be treated as algebras whose all fundamental operations are unary, that is as unary algebras. This makes possible to investigate automata from the aspect of Universal algebra and to use its ideas, methods and results [1, 3, 4, 5, 6, 8, 9, 10].

Here we deal with some important concepts of lattice theory and automata theory that will be used in the paper. Let $(P, \leq)$ be a poset and let $a, b \in P$ with $a \neq b$. Then a is called a predecessor of b, and b is called a successor of a if $a \leq c \leq b$ and

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$c \in P$ imply $c = a$ or $c = b$. We denote this relation by $< a, b >$. By $|D|$ we denote the cardinality of D. For every element a of P we set $|\{b \in P :< b, a >\}| = o(a)$, $o(P)$ denotes $\max\{o(a) : a \in P\}$ and $B(P) = \{a \in P : o(a) \leq 1\}$. Two posets $(P_1, \leq_1)$ and $(P_2, \leq_2)$ are said to be isomorphic, denoted by $(P_1, \leq_1) \cong (P_2, \leq_2)$, if there exists a bijection f of P_1 onto P_2 such that for any $a, b \in P_1$, $a \leq_1 b$ if and if only $f(a) \leq_2 f(b)$. A poset $(L, \leq)$ is said to be an upper semilattice if for any $x, y \in L$ there exists the least upper bound of x and y [1, 7, 8].

Let Σ be a nonempty finite set. Denote by Σ^* the free monoid over Σ and ε the empty string of Σ. A finite automaton is a triple $\mathcal{A} = (X, \Sigma, \lambda)$ where X and Σ are nonempty finite sets called a state set and alphabet, respectively and $\lambda : X \times \Sigma^* \to X$ is the transition function satisfying $\forall x \in X$, $\forall a, b \in \Sigma^*$, $\lambda(x, ab) = \lambda(\lambda(x, a), b)$ and $\lambda(x, \varepsilon) = x$. Define a relation $\sim$ on X by $\forall p, q \in X$, $p \sim q$ if and only $\lambda(p, u) = q$, $\lambda(q, v) = p$ for some $u, v \in \Sigma^*$. The relation $\sim$ is an equivalence relation. Let $p \in X$. We denote the equivalence class $\{q \in X : p \sim q\}$ by T_p. This subset T_p is called a layer of X. If $\rho(\mathcal{A}) = \{T_p : p \in X\}$, we define a partial order $\preccurlyeq$ on $\rho(\mathcal{A})$ as follows: For $p, q \in X$, $T_p \preccurlyeq T_q$ if and only if there exists $v \in \Sigma^*$ such that $\lambda(q, v) = p$. An automaton $\mathcal{B} = (X', \Sigma, \theta)$ is called a subautomaton of automaton $\mathcal{A} = (X, \Sigma, \lambda)$ if and only if $X' \subseteq X$ and $\theta = \lambda|_{X' \times \Sigma^*}$, i.e., θ is the restriction of λ to $X' \times \Sigma^*$. We denote the set of all of subautomata of $\mathcal{A}$ by $\sigma(\mathcal{A})$. Let $\mathcal{B}$, $\mathcal{C} \in \sigma(\mathcal{A})$. By $\mathcal{B} \sqsubseteq \mathcal{C}$, we mean that $\mathcal{B}$ is a subautomaton of $\mathcal{C}$. Then $\sqsubseteq$ is a partial order on $\sigma(\mathcal{A})$; hence $(\sigma(\mathcal{A}), \sqsubseteq)$ is a poset, see [7]. Moreover, $(\sigma(\mathcal{A}), \sqsubseteq)$ is a finite upper semilattice by [7, Proposition 2]. Throughout this paper, we shall assume unless otherwise stated, that posets, upper semilattices, lattices and automata are finite.

A directed graph is a graph that is a set of vertices connected by edges, where the edges have a direction associated with them. We define a directed graph on finite poset $(A, \leq)$, $G(A)$, with vertices as elements of A and for two distinct vertices a and b, we have edge (a, b) if and only if $< a, b >$ (for a vertex a, the in-degree of a, $deg^-(a)$, is the number of edges going to a).

2. Subautomaton

Our starting point is the following lemma.

Lemma 1. *Let $(L, \leq)$ be a finite poset. Then there exists an automaton $\mathcal{A} = (X, \Sigma, \lambda)$ such that $(\rho(\mathcal{A}), \preccurlyeq) \cong (L, \leq)$.*

Proof. We construct an automaton $\mathcal{A} = (X, \Sigma, \lambda)$ in the following way. Let $X = L$ and $\Sigma = \{a_1, a_2, \cdots, a_n\}$, where $n = o(L) + 1$ (so $n \geq 1$). Now by the same technique as in [7,Theorem 1], for each $l \in L$, consider $l_1, l_2, \cdots, l_{o(l)}$, all the predecessors of l. Define λ as follows: $\lambda(l, a_i) = l_i$ for $i = 1, 2, \cdots, o(l)$ and $\lambda(l, a_i) = l$ for $i = o(l) + 1, \cdots, n$. This gives $\lambda(l, w) \leq l$ for any $l \in L$ and $w \in \Sigma^*$. Thus $T_l = \{l\}$.

Define $f : L \to \rho(\mathcal{A}) = \{T_l : l \in L\}$ by $f(l) = T_l$. Clearly, f is a bijective mapping.

It remains to prove $l \leq l'$ and $l \neq l'$ if and only if $T_l \preceq T_{l'}$ and $T_l \neq T_{l'}$. Suppose that $l \leq l'$ and $l \neq l'$. Since L is finite set, there exist $x_1 = l, x_2, \cdots, x_m = l'$ of L such that $< x_{i-1}, x_i > (i = 2, \cdots, m)$. Therefor there is an element $a_{i-1} \in \Sigma$ such that $\lambda(x_i, a_{i-1}) = x_{i-1}$. Thus we have $T_{x_{i-1}} \preceq T_{x_i}$ and $T_{x_{i-1}} \neq T_{x_i}$. It follows $T_l \preceq T_{l'}$ and $T_l \neq T_{l'}$. The other implication is clear. So $(\rho(\mathcal{A}), \preccurlyeq) \cong (L, \leq)$. □

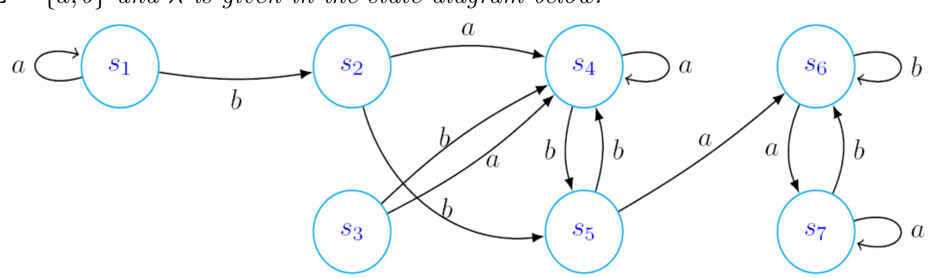
At this point we investigate structures and the number of subautomata of a finite automaton. The following proposition is a reformulation of [7, Theorem 2] and it gives a more explicit description of subautomaton of an automaton.

Proposition 2. *Let $\mathcal{A} = (X, \Sigma, \lambda)$ be an automaton. Then $\mathcal{B} = (X', \Sigma, \theta)$ is a subautomaton of $\mathcal{A}$ if and only if the following conditions are satisfied:*

(i) *The set $X'$is an union of layers of X.*
(ii) *If T_p and T_q are two layers of X with $T_p \subseteq X'$ and $T_q \preccurlyeq T_p$, then $T_q \subseteq X'$.*
(iii) $\theta = \lambda|_{X' \times \Sigma^*}$.

Proof. The sufficiency follows by (i), (ii) and (iii). Conversely, suppose that $\mathcal{B}$ is a subautomaton of $\mathcal{A}$. To see that (i), let $p \in X'$. Then $p \in T_p \subseteq \cup_{q \in X'} T_q$; so $X' \subseteq \cup_{q \in X'} T_q$. For the reverse inclusion, assume that $t \in \cup_{q \in X'} T_q$. Then $t \in T_p$ for some $p \in X'$; hence there exists $\omega \in \Sigma^*$ such that $\lambda(p, \omega) = t$. Now $\lambda(p, \omega) = \theta(p, \omega) = t$ gives $t \in X'$. Thus $X' = \cup_{q \in X'} T_q$. To prove that (ii), from $T_q \preccurlyeq T_p$ we conclude that there exists $\omega \in \Sigma^*$ such that $\lambda(p, \omega) = q$. Therefore $p \in T_p \subseteq X'$ gives $\lambda(p, \omega) = \theta(p, \omega) = q \in X'$. By an argument like that (i), we get $T_q \subseteq X'$. (iii) is clear. □

Example 3. *Consider automaton $\mathcal{D} = (X, \Sigma, \lambda)$, where $X = \{s_1, s_2, \cdots, s_7\}$, $\Sigma = \{a, b\}$ and λ is given in the state diagram below:*



By the definition of layer, we have layers: $T_1 = \{s_1\}$, $T_2 = \{s_2\}$, $T_3 = \{s_3\}$, $T_4 = \{s_4, s_5\}$, $T_5 = \{s_6, s_7\}$ are all of layers of $\mathcal{D}$. The following Figure describes relationship between $T_i, (1 \leq i \leq 7)$. Set $X_1' = T_5$, $X_2' = T_4 \cup X_1'$, $X_3' = T_2 \cup X_2'$, $X_4' = T_3 \cup X_2'$, $X_5' = X_4' \cup X_3'$, $X_6' = T_1 \cup X_3'$ and $X_7' = X_5' \cup T_1$. By Proposition 2, any subautomaton of $\mathcal{D}$ is of the form: $\mathcal{B}_i = (X_i', \Sigma, \theta_i)$ where $\theta_i = \lambda|_{X_i' \times \Sigma^*}$ for $i = 1, \ldots, 7$.

Definition 4. *A non-empty subset L of a poset $(P, \leq)$ is called a lower set, if for $a \in P$, $b \in L$ and $a \leq b$ implies $a \in L$. In particular, for any $a \in P$ one obtains the*

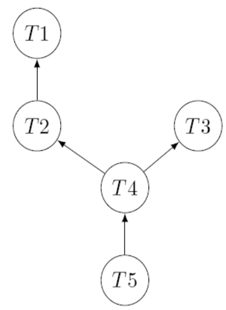


FIGURE 1. $G(\rho(D))$

principle lower set $< a >= \{t \in P : t \leq a\}$.

The set of all lower sets of a poset P is denoted by $LS(P)$. The following theorem shows that relationship between $(\sigma(\mathcal{A}), \sqsubseteq)$ and $(LS(\rho(\mathcal{A})), \subseteq)$.

Theorem 5. *Let $\mathcal{A}$ be any automaton. Then* $(\sigma(\mathcal{A}), \sqsubseteq) \cong (LS(\rho(\mathcal{A})), \subseteq)$.

Proof. By Proposition 2, $\mathcal{B}$ is a subautomaton of $\mathcal{A}$ if and only if $\rho(\mathcal{B}) \in LS(\rho(\mathcal{A}))$. We define the mapping $f : \sigma(\mathcal{A}) \to LS(\rho(\mathcal{A}))$ as follows: $f(\mathcal{B}) = \rho(\mathcal{B})$ for each subautomaton $\mathcal{B}$ of $\mathcal{A}$. An inspection will show that f is a poset isomorphism. □

Corollary 6. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two automata such that*

$$(\rho(\mathcal{A}_1), \preceq) \cong (\rho(\mathcal{A}_2), \preceq).$$

Then

$$(\sigma(\mathcal{A}_1), \sqsubseteq) \cong (\sigma(\mathcal{A}_2), \sqsubseteq).$$

Proof. Apply Theorem 5. □

Proposition 7. *Let L_1, L_2 and L be Lower sets of a poset $(P, \leq)$.*

(i) $< L_1, L_2 >$ *if and only if* $L_2 = L_1 \cup \{t\}$, *where t is a maximal element in* L_2.

(ii) *If* $deg^-(L) \neq 0$ *and* $D = \{t_i : t_i$ *is a maximal element in* $L\}$, *then* $|D| = deg^-(L)$. *Moreover, if* $deg^-(L) = 0$, *then L has an unique maximal element.*

(iii) *If either L_1, L_2 are minimal elements in $LS(P)$ or* $< L, L_1 >$ *and* $< L, L_2 >$, *then there exists* $L' \in LS(P)$ *with* $< L_1, L' >$ *and* $< L_2, L' >$.

Proof. (i) Assume that $< L_1, L_2 >$. Then $L_2 = L_1 \cup \{t_1, t_2, \ldots, t_n\}$ for some $t_i \notin L_1, (1 \leq i \leq n)$. If $n \geq 2$, then the set $H = \{t_1, t_2, \ldots, t_n\}$ has a minimal element, say t'. Set $L' = L_1 \cup \{t'\}$. By the definition of lower set, $L' \in LS(P)$ such that $L_1 \subsetneqq L' \subsetneqq L_2$ which is a contradiction. Thus

$L_2 = L_1 \cup \{t_1\}$. Now we show that t_1 is a maximal element in L_2. Otherwise, there exists an element t in L_2 such that $t_1 \leq t$ and $t_1 \neq t$. Therefore $t \in L_1$ (since $L_2 = L_1 \cup \{t_1\}$). So $t_1 \in L_1$ which is a contradiction . The other implication is clear.

(ii) Let $t_1, \cdots, t_k$ be the all of distinct maximal elements in L. If L_1 is a lower set of P with $< L_1, L >$, then there exists an unique element t_i (for some $1 \leq i \leq k$) such that $L = L_1 \cup \{t_i\}$ and $t_i \notin L_1$) by (i). Also, each lower set L' with $L' \subseteq L$ such that $|\{t_i : t_i \notin L'\}| = 1$ implies that $< L', L >$; hence $|D| = deg^-(L)$. Finally, If $deg^-(L) = 0$, then L has not a proper subset in $LS(P)$. Therefore $L = \{t\}$ for some element t in P, and the proof is complete.

(iii) Suppose that L_1, L_2 are minimal elements in $LS(P)$. Then $L_1 = \{t_1\}$ and $L_2 = \{t_2\}$. Set $L' = \{t_1, t_2\}$. Clearly, $< L_1, L' >$ and $< L_2, L' >$ by (i). Similarly, if $< L, L_1 >$ and $< L, L_2 >$, then we set $L' = L_1 \cup L_2$; thus $L' \in LS(P)$ gives $< L_1, L' >$ and $< L_2, L' >$ by (i).

□

Lemma 8. *Let P be a poset. Then P is a chain with $|P| = n$ if and only if $LS(P)$ is a chain with $|LS(P)| = n$.*

Proof. Let $LS(P) = \{L_1, \cdots, L_n\}$ with $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n$. Then we can take $L_1 = \{t_1\}$, $L_2 = \{t_1, t_2\}, \ldots,$ and $L_n = L = \{t_1, t_2, \ldots, t_n\}$ by Proposition 7 (i). We claim that $t_1 \leq t_2 \leq \ldots \leq t_n$. Assume to the contrary, let $t_k \nleq t_{k+1}$ for some k $(1 \leq k < n)$. Since t_{k+1} is a maximal element in L_{k+1} by Proposition 7 (i), we get $t_{k+1} \nleq t_k$. Set $L' = \{t_i : t_i \leq t_{k+1}\}$. Then $L' \in LS(P)$ with $L' \nsubseteq L_k$ and $L_k \nsubseteq L'$ that is a contradiction. Conversely, suppose $t_1 \leq t_2 \leq \ldots \leq t_n$. For each j $(1 \leq j \leq n)$, we set $L_j = \{t_1, t_2, \ldots, t_j\}$. Then by the definition of lower set and Proposition 7 (i), L_j $(1 \leq j \leq n)$ is an element of $LS(P)$ with $L_1 \subseteq L_2 \subseteq \ldots \subseteq L_n$. □

In view of the proof of the Proposition 7 and Lemma 8, we have the following corollary for automata.

Corollary 9. •

(a) *Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{C}$ be subautomata of $\mathcal{A} = (X, \Sigma, \lambda)$.*

(i) *$< \mathcal{A}_1, \mathcal{A}_2 >$ if and only if $\rho(\mathcal{A}_1) \subseteq \rho(\mathcal{A}_2)$ and $\rho(\mathcal{A}_2) = \rho(\mathcal{A}_1) \cup \{T\}$, where T is a maximal element in $\rho(\mathcal{A}_2)$.*

(ii) *If $deg^-(\mathcal{C}) \neq 0$ and $D = \{T_i : T_i$ is a maximal element in $\rho(\mathcal{C})\}$, then $|D| = deg^-(\mathcal{C})$. Moreover, if $deg^-(\mathcal{C}) = 0$, then C has an unique maximal layer.*

(iii) *If either $\mathcal{A}_1$, $\mathcal{A}_2$ are minimal elements in $\sigma(\mathcal{A})$ or $< \mathcal{C}, \mathcal{A}_1 >$ and $< \mathcal{C}, \mathcal{A}_2 >$, then there exists $\mathcal{A}' \in \sigma(\mathcal{A})$ with $< \mathcal{A}_1, \mathcal{A}' >$ and $< \mathcal{A}_2, \mathcal{A}' >$.*

(b) *Let $\mathcal{A} = (X, \Sigma, \lambda)$ be an automaton. Then $\rho(A)$ is a chain with $|\rho(A)| = n$ if and only if $\sigma(A)$ is a chain with $|\sigma(A)| = n$*

Definition 10. *A poset $(P, \leq)$ is called decomposable, if there exist proper subpoests $(P_1, \leq), \cdots, (P_n, \leq)$ of P such that $P = \cup_{i=1}^{n} P_i$, $P_i \cap P_j = \emptyset$ for $i \neq j$ $(1 \leq i, j \leq n)$ and for every couple $a_i \in P_i$, $b_j \in P_j$ be incomparable where $i \neq j$ $(1 \leq i, j \leq n)$. In this case, we say that P_i is a decomposition component of P $(1 \leq i \leq n)$.*

Proposition 11. *Let $(P_1, \leq)$ and $(P_2, \leq)$ be decomposition components of a poset $(P, \leq)$. Then L is a lower set of P if and only if it satisfies one of the following conditions:*

(i) *Either L is a lower set of P_1 or is a lower set of P_2.*
(ii) *There exist a lower set L_1 of P_1 and a lower set L_2 of P_2 such that $L = L_1 \cup L_2$.*

Proof. Let $L = \{t_1, \ldots, t_n\}$ be a lower set of P. If $L \in LS(P_1)$ or $L \in LS(P_2)$, then we are done. Otherwise, without loss of generality, we can assume that $L_1 = \{t_1, \cdots, t_k\} \subseteq P_1$ and $L_2 = \{t_{k+1}, \cdots, t_n\} \subseteq P_2$ $(1 \leq k < n)$. If $t \in L_i$, $t' \in L$ and $t' \leq t$ for $i = 1, 2$, then $t' \in L_i$ by Definition 10. Now $L_i \in LS(P_i)$, as required. □

Theorem 12. *Let $(P_1, \leq)$ and $(P_2, \leq)$ be decomposition components of a poset $(P, \leq)$. Then*

$$|LS(P)| = (|LS(P_1)| + 1)(|LS(P_2)| + 1) - 1$$

.

Proof. Assume that $|LS(P_1)| = m$ and $|LS(P_2)| = n$. Then the number of lower set that satisfies conditions (i) and (ii) in Proposition 11 are $n+m$ and nm, respectively, as required. □

Corollary 13. (i) *Let $(P_1, \leq), \cdots, (P_n, \leq)$ be decomposition components of a poset $(P, \leq)$. Then $|LS(P)| = \Pi_{i=1}^{n}(|LS(P_i)| + 1) - 1$.*
(ii) *Assume that $(P, \leq)$ is any poset and let p be a prime number such that $|Ls(P)| = p - 1$. Then $(P, \leq)$ is indecomposable.*

Proof. •

(i) The proof is straightforward by induction on n and Theorem 12.
(ii) Assume to the contrary, let there exist decomposition components $(P_1, \leq)$ and $(P_2, \leq)$ of a poset P. By assumption and (i), $(|LS(P_1)|+1)(|LS(P_2)|+1) = p$ that is a contradiction (because $|LS(P_i)| \geq 1$).

□

Definition 14. *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two finite posets with $P_1 \cap P_2 = \emptyset$. Then we can define the poset $(P_1 \cup P_2, \leq)$ as follows:*

(i) *For any $i = 1, 2$, $a, b \in P_i$, $a \leq b$ if $a \leq_i b$.*
(ii) *For any $a \in P_1$ and $b \in P_2$, $a \leq b$.*

Lemma 15. *Let $(P_1, \leq_{p_1}) \cong (P'_1, \leq_{p'_1})$ and $(P_2, \leq_{p_2}) \cong (P'_2, \leq_{p'_2})$ with $P_1 \cap P'_1 = P_2 \cap P'_2 = \emptyset$. Then $(P_1 \cup P_2, \leq) \cong (P'_1 \cup P'_2, \leq)$.*

Proof. The proof is straightforward by Definition 14. □

Proposition 16. *Let* $(P_1, \leq_1)$ *and* $(P_2, \leq_2)$ *be two posets with* $P_1 \cap P_2 = \emptyset$ *and let* $(P_1 \cup P_2, \leq)$*. Then the following hold:*

(i) $(LS(P_1 \cup P_2), \subseteq)$ *is poset isomorphic to* $(LS(P_1) \cup LS(P_2), \subseteq)$*. Moreover,* $|LS(P_1 \cup P_2)| = |LS(P_1)| + |LS(P_2)|$

(ii) *If* $(P_1, \leq_1)$ *is an upper semilattice, then* $(B(P_1) \cup B(P_2), \subseteq) = (B(P_1 \cup P_2), \subseteq)$*.*

Proof. •

(i) Define the mapping $f : LS(P_1) \cup LS(P_2) \to LS(P_1 \cup P_2)$ as follows: If $L \in LS(P_1)$, then $L \in LS(P_1 \cup P_2)$; so we set $f(L) = L$. If $L \in LS(P_2)$, then we set $f(L) = P_1 \cup L$. It is easy to see that f is a poset isomorphism.

(ii) It suffices to show that $B(P_1 \cup P_2) = B(P_1) \cup B(P_2)$. Clearly, $B(P_1 \cup P_2) \subseteq B(P_1) \cup B(P_2)$. For the reverse inclusion, let $l \in P_1$ and $deg^{-}(l) \leqslant 1$. Then indegree l in $P_1 \cup P_2$ is equal to 1 or 0. Moreover, if $l \in P_2$ and $deg^{-}(l) = 1$, then indegree l in $P_1 \cup P_2$ is equal to 1. Also, if $l \in P_2$ and $deg^{-}(l) = 0$, then we have only $< 1, l >$ in $P_1 \cup P_2$ (1 is the greatest element in P_1); hence indegree l in $P_1 \cup P_2$ is equal to 1. So l in $B(P_1 \cup P_2)$, and we have equality.

□

Corollary 17. (i) *Let* $(P, \leq)$ *be a poset and* $t \in P$ *be a maximum element of* P*. Then* $|LS(P)| = |LS(P_1)| + 1$ *where* $P_1 = P \setminus \{t\}$ *is subposet of* P*.*

(ii) *Let* $(P, \leq)$ *be a poset and* $t \in P$ *be a minimum element of* P*. Then* $|LS(P)| = |LS(P_1)| + 1$ *where* $P_1 = P \setminus \{t\}$ *is subposet of* P*.*

Proof. Apply Proposition 16 (i). □

According to the above results, computation $|\sigma(A)|$ become easier.

Example 18. *In the Example 3, consider subposets* $P_1 = (\{T_4, T_5\}, \preceq)$ *and* $P_2 = (\{T_1, T_2, T_3\}, \preceq)$*. Then* $(\rho(\mathcal{D}), \preceq) \cong (P_1 \cup P_2, \preceq)$*. We have* $|LS(P_1)| = 2$ *and* $|LS(P_2)| = 5$ *by Lemma 8 and Theorem 12 , then* $|\sigma(D)| = 7$ *by Proposition 16 (i) .*

The following theorem gives estimate for the number of lower set of a poset.

Theorem 19. *Let* $(P, \leq)$ *is a poset with* $|P| = n$ *and* $t_1, \cdots, t_m$ *be the all of minimal element of* P *. Then the following inequality is valid:*

$$2^m - 1 \leq |LS(P)| \leq 2^{n-1} + 2^{m-1} - 1$$

Proof. Clearly, every nonempty subset of $A = \{t_1, \cdots, t_m\}$ is a lower set of P. we know the number of the all non-empty subsets of A is equal to $2^m - 1$. So $|LS(P)| \geq 2^m - 1$. It remains to prove the other side unequal. It easily seen that every lower set is equal to union the number of principle lower sets. Assume that a, b be two maximal elements of P . Then $< a > \cup < b >$ is a lower set of P which is distinct from $< a >$ and $< b >$. Therefor, if P has more maximal elements, then there exist more lower sets. Now, if each element of P is maximal element, then $n = m$. It follows $|LS(P)| = 2^m - 1$; Hence we are done. otherwise, we consider poset $P = \{t_1, \cdots, t_m, l_1, \cdots, l_{n-m}\}$ where $t_1 \leq l_i$ and each couple t_j, l_i are incomparable for $i = 1, \cdots, n-m$, $j = 2, \cdots, m$. In this case P has $n-1$ maximal elements. Also $P_1 = \{t_1, l_1, \cdots, l_{n-m}\}$ and $P_2 = \{t_2, \cdots, t_m\}$ are decomposition components of P . Thus $|LS(P_1)| = 2^{n-m}$ by Corollary 16(ii) and Corollary 13 (i) and $|LS(P_2)| = 2^{m-1} - 1$ by Corollary 13(i). Now the assertion follows from Theorem 12 . □

Definition 20. *An automaton $\mathcal{A} = (X, \Sigma, \lambda)$ is called decomposable, if there exist proper subautomata $\mathcal{A}_1 = (X_1, \Sigma, \lambda_1), \cdots, \mathcal{A}_n = (X_n, \Sigma, \lambda_n)$ of $\mathcal{A}$ such that $X = \cup_{i=1}^{n} X_i$ and $X_i \cap X_j = \emptyset$ for $i \neq j$ $(1 \leq i, j \leq n)$. In this case, we say that $\mathcal{A}_i$ is a decomposition component of $\mathcal{A}$ $(1 \leq i \leq n)$.*

The next Theorem follows from Proposition 11, Corollary 13, Proposition 16 and Theorem 19.

Theorem 21. (i) *Let $\mathcal{B} = (X_{\mathcal{B}}, \Sigma, \lambda_{\mathcal{B}})$ and $\mathcal{C} = (X_{\mathcal{C}}, \Sigma, \lambda_{\mathcal{C}})$ be decomposition components of an automaton $\mathcal{A} = (X, \Sigma, \lambda)$. Then $\mathcal{A}' = (X', \Sigma, \lambda')$ is a subautomaton of $\mathcal{A}$ if and only if it satisfies one of the following conditions:*
   1. *Either $\mathcal{A}'$ is a subautomaton of $\mathcal{B}$ or is a subautomaton of $\mathcal{C}$.*
   2. *There exist a subautomaton $\mathcal{B}' = (X_1', \Sigma, \lambda_1)$ of $\mathcal{B}$ and a subautomaton $\mathcal{C}' = (X_2', \Sigma, \lambda_2)$ of $\mathcal{C}$ such that $X' = X_1' \cup X_2'$.*

(ii) (1) *Let $\mathcal{A}_1 = (X_1, \Sigma, \lambda_1), \cdots, \mathcal{A}_n = (X_n, \Sigma, \lambda_n)$ be decomposition components of an automaton $\mathcal{A} = (X, \Sigma, \lambda)$. Then $|\sigma(A)| = \Pi_{i=1}^{n}(|\sigma(\ \mathcal{A}_i)| + 1) - 1$.*
   (2) *Assume that $\mathcal{A}$ is any automaton and let p be a prime number such that $|\sigma(\mathcal{A}))| = p - 1$. Then $\mathcal{A}$ is indecomposable.*

(iii) *Let $\mathcal{A}$ be an automaton and $(P_1, \preceq)$ and $(P_2, \preceq)$ be subposets of $(\rho(\mathcal{A}), \preceq)$ such that $P_1 \cap P_2 = \emptyset$ and $(\rho(\mathcal{A}), \preceq) \cong (P_1 \cup P_2, \preceq)$. Then $|\sigma(A)| = |LS(P_1)| + |LS(P_2)|$*

(iv) *Let $\mathcal{A}$ be an automaton , $|(\rho(\mathcal{A})| = n$ and $|\{T \in (\rho(\mathcal{A})$: T is a minimal layer$\}| = m$. Then Then the following inequality is valid:*

$$2^m - 1 \leq |\sigma(\mathcal{A})| \leq 2^{n-1} + 2^{m-1} - 1$$

The remaining of this paper is dedicated to the following question: For which posets L does there exist an automaton $\mathcal{A}$ such that $\sigma(\mathcal{A})$ is isomorphic to L?.

Theorem 22. *If $(P, \leq)$ is a poset, then $(P, \leq) \cong (B(LS(P)), \subseteq)$.*

Proof. We define the mapping $f : P \to B(LS(P))$ as follows: If $t \in P$, then we set $f(t) = \{t' : t' \in P, t' \leq t\}$. It is clear that $f(t) \in LS(P)$. Since t is an unique maximum element in $f(t)$, Proposition 7 (ii) gives $deg^{-}(f(t)) \leq 1$; so $f(t) \in B(LS(P))$. An inspection will show that f is a poset isomorphism. $\square$

Theorem 23. *If $\mathcal{A} = (X, \Sigma, \lambda)$ is an automaton, then*

$$(B(\sigma(\mathcal{A})), \sqsubseteq) \cong (\rho(\mathcal{A}), \preceq).$$

Proof. By Theorem 5, we have $(B(\sigma(\mathcal{A})), \sqsubseteq) \cong (B(LS(\rho(\mathcal{A})), \subseteq))$; hence $(B(\sigma(\mathcal{A})), \sqsubseteq) \cong (\rho(\mathcal{A}), \preceq)$ by Theorem 22. $\square$

Theorem 24. *An upper semilattice $(L, \leq)$ is isomorphic to an upper semilattice of subautomata of an automaton if and only if $(LS(B(L)), \subseteq) \cong (L, \leq)$.*

Proof. If $(L, \leq)$ is an upper semilattice, then $(B(L), \leq)$ is a poset; hence there exists an automaton $\mathcal{A}$ such that $(B(L), \leq) \cong (\rho(\mathcal{A}), \preceq)$ by Lemma 1 which implies that $(LS(B(L)), \subseteq) \cong (LS(\rho(\mathcal{A})), \subseteq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ by Theorem 5. If $(L, \leq) \cong (LS(B(L)), \subseteq)$, then we are done. If $(L, \leq) \ncong (LS(B(L)), \subseteq)$, then we show that there is not any automaton $\mathcal{C}$ such that $(L, \leq) \cong (\sigma(\mathcal{C}), \sqsubseteq)$. Assume to the contrary, let $(L, \leq) \cong (\sigma(\mathcal{C}), \sqsubseteq)$ for some automaton $\mathcal{C}$. Since $(B(\sigma(\mathcal{C})), \sqsubseteq) \cong (B(L), \leq)$, it follows that $(B(L), \leq) \cong (\rho(\mathcal{C}), \preceq)$ by Theorem 23, so $(\rho(\mathcal{C}), \preceq) \cong (\rho(\mathcal{A}), \preceq)$; hence $(\sigma(\mathcal{C}), \sqsubseteq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ by Corollary 6. Thus $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq) \cong (LS(B(L)), \subseteq)$ that is a contradiction. $\square$

Example 25. (i) *Let L be an upper semilattice as described in Figure 2. Then there is not any automaton $\mathcal{A}$ such that $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$. If it is, then we conclude that the graph G in Figure 3 corresponds to $G(B(L))$, so $|LS(B(L))| = (2+1)(2+1) - 1 = 8$ by Lemma 8 and Theorem 12, but $|L| = 6$. Thus $(LS(B(L)), \subseteq) \ncong (L, \leq)$.*

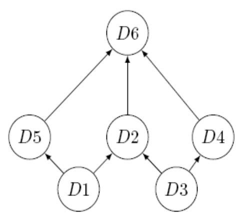


FIGURE 2. L

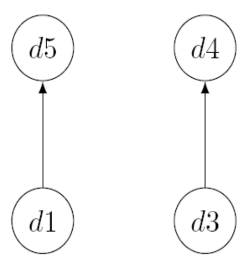


FIGURE 3. G

(ii) *Let L_1 be an upper semilattice as described in Figure 4. Then there exists an automaton $\mathcal{A}$ such that $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ (The graph G_1 in Figure 5 corresponds to $G(B(L))$. An inspection will show that $(LS(B(L_1)), \subseteq) \cong (L_1, \leq)$).*

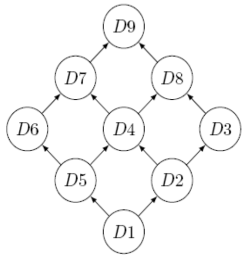


FIGURE 4. L_1

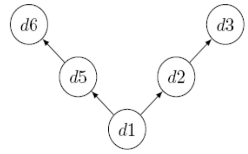


FIGURE 5. G_1

Theorem 26. *Let $(L_1, \leq_1), (L_2, \leq_2)$ and $(L, \leq)$ be upper semilattices such that $L_1 \cap L_2 = \emptyset$ and $(L, \leq) \cong (L_1 \cup L_2, \leq)$. Then there exists an automaton $\mathcal{A}$ such that $(\sigma(\mathcal{A}), \sqsubseteq) \cong (L, \leq)$ if and only if there exist automata $\mathcal{B}$ and $\mathcal{C}$ such that $(\sigma(\mathcal{B}), \sqsubseteq) \cong (L_1, \leq_1)$and $(\sigma(\mathcal{C}), \sqsubseteq) \cong (L_2, \leq_2)$.*

Proof. Apply Proposition 16 and Theorem 24 . □

Example 27. *Let L be an upper semilattice as described in Figure 6. Then $L_1 = \{B_1, B_2, B_3, B_4, B_5, B_6\}$ and $L_2 = \{B_7, B_8, B_9, B_{10}, B_{11}, B_{12}\}$ are subupper semilattices satisfy conditions of Theorem 26. Also there is not any automaton $\mathcal{B}$ such that $(L_1, \leq) \cong (\sigma(\mathcal{B}), \sqsubseteq)$ (see Example 25 (i)). Thus there is not any automaton $\mathcal{A}$ such that $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$.*

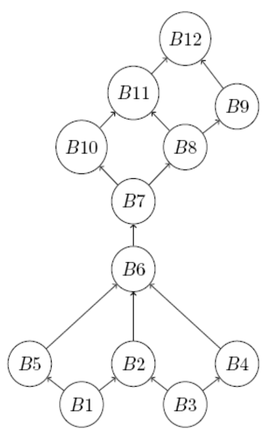


FIGURE 6. L

Example 28. *let L be an upper semilattice as described in Figure 7. Then $L_1 = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}, B_{11}\}$ and*
$L_2 = \{D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9\}$ are subupper semilattices satisfy conditions of Theorem 26. There exists an automaton $\mathcal{C}$ such that $(L_2, \leq) \cong (\sigma(\mathcal{C}), \sqsubseteq)$ (see Example 25 (ii)). Similarly, there exists an automaton $\mathcal{B}$ such that $(L_1, \leq) \cong$

$(\sigma(\mathcal{B})\ \sqsubseteq)$. Hence there exists an automaton $\mathcal{A}$ such that $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ By Theorem 26.

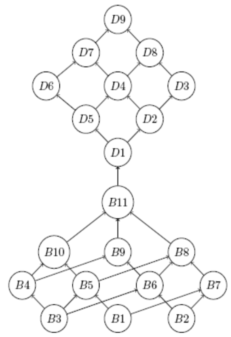


FIGURE 7. L

With the help of results proved by Sališ̆, we find another way to detect whether there exists some automaton $\mathcal{A}$ such that $(\sigma(\mathcal{A}), \sqsubseteq)$ isomorphic to a given finite upper semilattice $(L, \leq)$. Sališ̆ consider $= (\emptyset, \Sigma, \lambda)$ as a subautomaton of an automaton $\mathcal{A} = (X, \Sigma, \lambda)$; so $(\sigma(\mathcal{A}) \cup \{\}, \sqsubseteq)$ is a lattice.

Theorem 29. *[3,Theorem 8.4] A finite lattice L isomorphic to the lattice $(\sigma(\mathcal{A}) \cup \{\}, \sqsubseteq)$ for some automaton $\mathcal{A}$ if and only if it is distributive.*

Lemma 30. *Assume that $(L, \leq)$ is an upper semilattice and let 0^* be an element such that $0^* \notin L$. Then $(\bar{L} = \{0^*\} \cup L, \leq)$ is a lattice.*

Proof. This follows from the Definition 14. □

Theorem 31. *An upper semilattice $(L, \leq)$ is isomorphic to the upper semilattice of subautomata of an automaton if and only if $\bar{L}$ is a distributive lattice.*

Proof. Assume that $\bar{L} = \{0^*\} \cup L$ is distributive. Then there exists automaton $\mathcal{A}$ such that $\bar{L}$ is poset isomorphic to $\sigma(\mathcal{A}) \cup \{\}$ by Theorem 29. Note that 0^* and are minimum elements in $\bar{L}$ and $\sigma(\mathcal{A}) \cup \{\}$, respectively. Thus $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$. Conversely, suppose that $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ for some automaton $\mathcal{A}$. Then $(\bar{L}, \leq) \cong (\sigma(\mathcal{A}) \cup \{\}, \sqsubseteq)$ by Lemma 15. Now the assertion follows from Theorem 29. □

Example 32. *Consider upper semilattices L and L_1 as described in Example 25 (i),(ii), respectively. $\bar{L}$ and $\bar{L_1}$ are defined in Figure 8 and Figure 9. By [2,Theorem 1.7] $\bar{L}$ is not distributive and $\bar{L_1}$ is distributive. Then $(L_1, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ for some automaton $\mathcal{A}$ and $(L, \leq) \ncong (\sigma(\mathcal{A}), \sqsubseteq)$ for every automaton $\mathcal{A}$ by Theorem 31.*

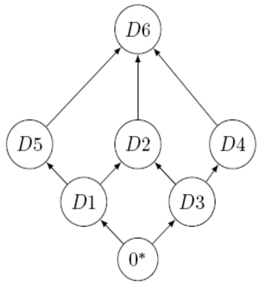


FIGURE 8. $\bar{L}$

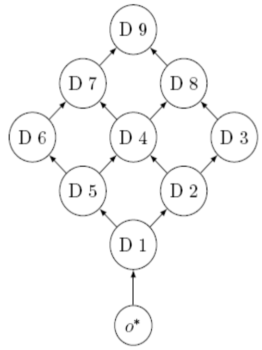


FIGURE 9. $\bar{L_1}$

Definition 33. (a) *[11, Definition 2.1]An algebra $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ is called a lattice-ordered monoid if*

(1) *$L = (L, \leq, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1.*

(2) *$(L, \bullet, 1)$ is a monoid with identity $1 \in L$ such that for all $a, b, c \in L$.*

(3) *$a \bullet 0 = 0 \bullet a = 0$.*

(4) *$a \leq b \Rightarrow \forall x \in L$, $a \bullet x \leq b \bullet x$ and $x \bullet a \leq x \bullet b$.*

(5) *$a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c)$ and $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a)$.*

(b) *[11,Definition 2.2] Let L be a lattice-ordered monoid. Then a 5-tuple $F = (X, \Sigma, \Gamma, \lambda, \theta)$ is a called crisp deterministic fuzzy automaton if*

(1) *$\mathcal{A} = (X, \Sigma, \lambda)$ is a finite automaton and Γ is a nonempty finite set.*

(2) *$\theta : X \times \Sigma^* \times \Gamma^* \longrightarrow L$ is a map called the output function such that $\theta(p, \varepsilon, \varepsilon) = 0$, $\theta(p, w, \varepsilon) = \theta(p, \varepsilon, u) = 1$ for $\varepsilon \neq w \in \Sigma^*$ and $\varepsilon \neq u \in \Gamma^*$ and $\theta(p, w_1w_2, u_1u_2) = \theta(p, w_1, u_1) \bullet \theta(\lambda(p, w_1), w_2, u_2)$, for all $p \in X$, $w_1, w_2 \in \Sigma^*$ and $u_1, u_2 \in \Gamma^*$.*

(c) *[11,Definition 2.4] A crisp deterministic fuzzy automaton $F_1 = (X_1, \Sigma, \Gamma, \lambda_1, \theta_1)$ is called a subautomaton of a crisp deterministic fuzzy automaton $F = (X, \Sigma, \Gamma, \lambda, \theta)$ if $X_1 \subseteq X$, $\lambda_1 = \lambda|_{X_1 \times \Sigma^*}$ and $\theta_1 = \theta|_{X_1 \times \Sigma^* \times \Gamma^*}$*

Remark 34. (i) *It is not to hard to see that : If $F_1 = (X_1, \Sigma, \Gamma, \lambda_1, \theta_1)$ is a subautomaton of a crisp deterministic fuzzy automaton $F = (X, \Sigma, \Gamma, \lambda, \theta)$, then $\mathcal{A}_1 = (X_1, \Sigma, \lambda_1)$ is a subautomaton of an automaton $\mathcal{A} = (X, \Sigma, \lambda)$. Moreover, if $\mathcal{A}_1 = (X_1, \Sigma, \lambda_1)$ is a subautomaton of an automaton $\mathcal{A}$, then $F_1 = (X_1, \Sigma, \Gamma, \lambda_1, \theta_1)$ where $\theta_1 = \theta|_{X_1 \times \Sigma^* \times \Gamma^*}$ is a subautomaton of a crisp deterministic fuzzy automaton $F = (X, \Sigma, \Gamma, \lambda, \theta)$. This gives the results obtained in Theorem 5, Corollary 9 , Theorem 21 is correct similarly for crisp deterministic fuzzy automata.*

(ii) *Let $\mathcal{A} = (X, \Sigma, \lambda)$ be an automaton and $L = \{0, 1\}$ be a lattice-ordered monoid where $0 \bullet 0 = 1 \bullet 0 = 0 \bullet 1 = 0$ and $1 \bullet 1 = 1$. Then $F = (X, \Sigma, \Gamma, \lambda, \theta)$ is a crisp deterministic fuzzy automaton where θ is defined as follow: if*

$w = u = \varepsilon$, then $\theta(p, w, u) = 0$. Otherwise $\theta(p, w, u) = 1$. From this, we conclude that there exists a crisp deterministic fuzzy automaton for each automaton. Which gives An upper semilattice $(L, \leq)$ is isomorphic to an upper semilattice of subautomata of a crisp deterministic fuzzy automaton if and only if $(LS(B(L)), \subseteq) \cong (L, \leq)$.

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References

[1] Birkhoff, G., Lattice theory, Amer. Math. Soc., 1973.

[2] Călugăreanu, G., Lattice Concepts of Module Theory, Springer Science+Business Media Dordrecht, 2000.

[3] Ćirić, M., Bogdanović, S., The Lattice of Subautomata of an Automaton - A Survey, *Publications de l'Institut Mathématique*, 64 (78) (1998), 165-182 .

[4] Atani, S. E., Bazari, M. Sedghi Shanbeh, Decomposable Fillters of Lattces, *Kragujevac Journal of Mathematics,* 43(1) (2019), 59-73.

[5] Halaš , R., Relative polars in ordered sets, Carleton University, *Czechoslovak Math. J.*, 50 (125) (2) 2000, 415-429.

[6] Halaš, R., Jukl, M., On Beck;s coloring of posets, *Discrete Math.,* 309 (13) (2009), 4584-4589.

[7] Ito, M., Algebraic structures of automata, *Theoretical Computer Science,* 428 (2012), 164-168.

[8] Ito, M., Algebraic structures of automata and Languages, World Scientific, Singapore, 2004.

[9] Muir, A., Warner, M. W., Lattice valued relations and automata, *Discrete App. Math.,* 7 (1984), 65-78.

[10] Saliĭ, V. N., Universal Algebra and Automata, Saratov Univ., Saratov, 1988. (in Russian).

[11] Smid, A. Maheshwari Michiel, Introduction to Theory of Computation, Ottawa, 2016.

[12] Verma, R., Tiwri, S. P., Distinguishability and Completeness of Crispdeterministic Fuzzy Automata, *Iranian Journal of Fuzzy Systems*, 5 (2017), 19-30.

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PRODUCT FACTORABLE MULTILINEAR OPERATORS DEFINED ON SEQUENCE SPACES

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Abstract. We prove a factorization theorem for multilinear operators acting in topological products of spaces of (scalar) p-summable sequences through a product. It is shown that this class of multilinear operators called *product factorable maps* coincides with the well-known class of the zero product preserving operators. Due to the factorization, we obtain compactness and summability properties by using classical functional analysis tools. Besides, we give some isomorphisms between spaces of linear and multilinear operators, and representations of some classes of multilinear maps as n-homogeneous orthogonally additive polynomials.

1. Introduction

The objective of the paper is to present a factorization theorem for multilinear operators defined on the topological product of spaces of p-summable sequences through the product of (multiple) scalar sequences. Such a factorization has been studied for multilinear operators defined on Banach algebras and vector lattices, and in the last years it has been studied for Banach spaces (see [1, 6, 12] and references therein).

Factorization through a product is closely related to a property that is called *zero product preservation,* or *orthosymmetry* in the case of vector lattices, for which orthogonality is used to generalize the notion of having product equal to 0, that is just given for the case of function lattices. This property states that a multilinear map $B : X_1 \times .... \times X_n \to Y$ is 0-valued whenever $x_i \circledast x_j = 0$ for some $x_i \in X_i$, $x_j \in X_j$ $(i, j \in \{1, 2, ..., n\})$, where $\circledast : X_1 \times .... \times X_n \to G$ is a specific map

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called *product*. For multilinear operators acting in Banach algebras, this factorization gives useful results for the weighted homomorphisms and derivations, where algebraic multiplication is considered as the specific map (see [1, 2] and references therein). For Riesz spaces, such a factorization is used to obtain interesting results regarding powers of vector lattices, in which orthogonality is involved (see [4, 6, 7] and references therein).

Recently, the author together with other mathematicians have investigated the class of multilinear operators acting in the topological product of Banach function spaces and integrable functions factoring through the pointwise product and the convolution operation, respectively (see [12–14]). Motivated by these ideas, in this paper we introduce the notion of product factorability for multilinear operators defined on topological products of spaces of (scalar) p-summable sequences, and we prove that this class coincides with the class of zero product preserving multilinear maps.

This paper is organised as follows: after some preliminaries and notations, in Section 2 we give the definitions of the specific map *product* and *product factorability* for multilinear operators with a necessary and sufficient requirement. Section 3 includes the main result of the paper, which as we said above, states that for a particular product and multilinear operators defined on the topological product of spaces of p-summable sequences, the class of product factorable maps is the same as the class of zero product preserving maps. In the sequel, some isometries between multilinear operators and linear operators are presented. Section 4 concerns compactness and summability properties based on classical functional analysis properties and theorems of product factorable maps. Section 5 is devoted to give a generalization of the main factorization theorem by using isomorphism between Banach spaces and ℓ^p spaces. In the last section, some isometries between product factorable multilinear maps and orthogonally additive n-homogeneous polynomials are given as an application, and the paper is finished with an example related to diagonal forms.

Throughout the paper, the standard notations from the Banach space theory are used. Nevertheless, before going any further let us describe some of them. The capital letters X, Y, Z will denote the Banach spaces over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We write B_X for the unit ball of a Banach space X. X^* denotes the topological dual of the Banach space X. The notations $E = Y$ and $E \cong Y$ mean E and Y are isometric and isomorphic, respectively.

Operator (linear, multilinear or polynomial) indicates *continuous operator*.
$\mathcal{L}^n(X_1 \times ... \times X_n, Y)$ denotes the Banach space of n-linear maps endowed with the norm

$$\|T\| = \sup\{\|T(x_1, ..., x_n)\| : \, x_i \in B_{X_i}, 1 \leq i \leq n\}.$$

It will be denoted by $\mathcal{L}^n(X_1 \times ... \times X_n)$, respectively, $\mathcal{L}(X, Y)$ if $Y = \mathbb{R}$, respectively, $n = 1$.

For a positive real number $p \geq 1$, ℓ^p is the Banach space of all scalar valued absolutely p-summable sequences with the norm $\|(x_i)\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ and ℓ^∞ shows the Banach space of all bounded sequences endowed with the norm $\|(x_i)\|_\infty = \sup_{i\in\mathbb{N}} |x_i|$.

$\chi_{\{1,2,...,m\}}$ will denote the sequence $\{1, \overset{m}{...} 1, 0, 0, 0....\}$ and $\chi_{\{j\}}$ shows the elements of standard basis of the space ℓ^p whose coordinates are all zero, except j^{th} that equals 1.

For brevity we will write $\times_{i=1}^{n} X_i$ for the Cartesian product space $X_1 \times .... \times X_n$ and $\times^n X$ for the n-fold Cartesian product of the Banach space X.

A linear operator $T : X \to Y$ is called *(p,q)-summing* if there exists a constant $c > 0$ such that for every choice of the elements $x_1, ..., x_m \in X$ and for all positive integers m,

$$\Big(\sum_{i=1}^{m} \big\|T(x_i)\big\|_Y^p\Big)^{1/p} \leq k \sup_{x^*\in B_{X^*}} \Big(\sum_{i=1}^{m} |\langle x_i, x^*\rangle|^q\Big)^{1/q}.$$

The space of (p,q)-summing operators from X to Y is denoted by $\Pi_{p,q}(X,Y)$ $-$ $-\Pi_p(X,Y)$, if $p = q$.

Recall that a Banach space E is said to have the *Schur property* whenever weak convergent and norm convergent sequences coincide in it. A Banach space E has the *Dunford-Pettis property* if every linear operator from E into a Banach space F maps weakly compact sets to norm compact ones.

Recall that an (linear, multilinear or polynomial) operator is called *(weakly) compact* if it maps the unit ball to a relatively (weakly) compact set.

2. Norm Preserving Products and Product Factorability

Let $X_1, X_2, ..., X_n$ and Z be Banach spaces. Consider a Banach space valued n-linear map $\circledast : X_1 \times X_2 \times ... \times X_n \to Z$ written by

$$(x_1, x_2, ..., x_n) \rightsquigarrow \circledast(x_1, x_2, ..., x_n) = x_1 \circledast x_2 \circledast ... \circledast x_n$$

for all $x_i \in X_i$ $(i = 1, 2, ..., n)$.

This particular map is called *norm preserving product (n.p. product* for short) if the inclusion $B_Y \subseteq \circledast(B_{X_1} \times B_{X_2} \times ... \times B_{X_n})$ holds and for every $x_i \in X_i$ $(i = 1, ..., n)$ and we have that

$$\| \circledast (x_1, x_2, ..., x_n)\|_Z = \inf \Big\{ \prod_{i=1}^{n} \|x_i'\|_{X_i} : x_i' \in X_i,\ i = 1, ..., n \Big\},$$

where the infimum is taken over all $\circledast(x_1, x_2, ..., x_n) = \circledast(x_1', x_2', ..., x_n')$ (see [12, Definition 2.1]) .

Example 1. *Some norm preserving products;*

- *The usual convolution operation $*$ from the product $\mathcal{L}^2(\mathbb{T})\times\mathcal{L}^2(\mathbb{T})$ of Hilbert space of integrable functions to the Wiener algebra $\mathcal{W}(\mathbb{T})$ is a norm preserving product (see [11, Remark 2.1] and references there in for the calculations),*
- *Let (Ω,Σ,μ) be a σ-finite measure space and let $\sum_{i=1}^{n}\frac{1}{p_i}=\frac{1}{r}$ and $p_i, r\geq 1$. Then the pointwise product $\odot$ defined on $\mathcal{L}^{p_1}(\mu)\times ...\times\mathcal{L}^{p_n}(\mu)$ to $\mathcal{L}^{r}(\mu)$ is a norm preserving product (see [12, Section 4]).*

A multilinear operator $B : X_1\times ...\times X_n \to Y$ is called $\circledast$-*factorable* for the n.p. product $\circledast$ if it can be factored through the product $\circledast : X_1\times ...\times X_n\to Z$ and a linear operator $T : Z\to Y$ such that $B(x_1,x_2,...,x_n) = T\circ\circledast(x_1,x_2,...,x_n) = T(x_1\circledast x_2\circledast ...\circledast x_n)$ for all $x_i\in X_i$ $(i = 1,...,n)$ (see [12, Def. 2.2]).

Thus, for a certain continuous linear operator $T : Z\to Y$, the map B admits a factorization as the form;

$$\begin{tikzcd} X_1\times X_2\times ...\times X_n \arrow[r, "B"] \arrow[d, dashed, "\circledast"] & Y. \\ Z \arrow[ur, "T"'] & \end{tikzcd}$$

The author proved in [12, Lemma 2.3.] that a necessary and sufficient condition for the $\circledast$-factorability of a multilinear operator $B : X_1\times X_2\times ...\times X_n\to Y$ is given by the existence of a constant $k > 0$ satisfying the following inequality

$$\Big\|\sum_{i=1}^{m}B(x_i^1,x_i^2,...,x_i^n)\Big\|_Y\leq k\Big\|\sum_{i=1}^{m}x_i^1\circledast x_i^2\circledast ...\circledast x_i^n\Big\|_Z \tag{1}$$

for every finite sets of vectors $\{x_i^j\}_{i=1}^m\subset X_j$ $(j = 1,2,...,n)$.

A multilinear map $B : X_1\times X_2\times ...\times X_n\to Y$ is called *zero product preserving* (or *zero $\circledast$-preserving*) if

$$B(x_1,x_2,...,x_n) = 0 \text{ if } x_k\circledast x_l = 0 \text{ for some } x_k\in X_k, x_l\in X_l$$

where k, $l\in\{1,2,...,n\}$ and $k\neq l$.

The class of zero $\circledast$-preserving multilinear operators is a Banach space endowed with the usual operator norm. The Banach space of n-linear zero $\circledast$-preserving operators defined on the $X_1\times X_2\times ...\times X_n$ to Y will be denoted by $\mathcal{L}_0^n(X_1\times X_2\times ...\times X_n, Y)$.

3. Product Factorability of Multilinear Maps acting in Sequence Spaces

Now, we will give the main theorem of the paper that states the class of zero product preserving maps defined on $\times_{i=1}^{n}\ell^{p_i}$ to the Banach space Y are equal to the class of the product factorable operators.

Remark 2. *Let* $\sum_{i=1}^{n}\frac{1}{p_i}=\frac{1}{r}$ *for* $1\le r,p_i<\infty$ $(i=1,...,n)$. *The product* $\odot:\times_{i=1}^{n}\ell^{p_i}\to\ell^r$ *defined by*

$$x_1\odot....\odot x_n=\{x^1(k)\cdot....\cdot x^n(k)\}_{k=1}^{\infty}=\{x(k)\}_{k=1}^{\infty}=x\in\ell^r$$

for all $x_i\in\ell^{p_i}\,(i=1,...,n)$ *is a norm preserving product. Indeed, consider a* $\{x(k)\}_{k=1}^{\infty}\in B_{\ell^r}$. *We can write* $x(k)=\prod_{i=1}^{n}|x(k)|^{r/p_i}sign(x(k))$ *for all* $k\in\mathbb{N}$, *where sign denotes the signum function. Since*

$$\begin{aligned}\|(|x(k)|^{r/p_i}sign(x(k)))\|_{p_i}&=\big(\sum||x(k)|^{r/p_i}sign(x(k))|^{p_i}\big)^{1/p_i}\\&=\big(\sum|x(k)|^r\big)^{1/p_i}=\|(x(k))\|_r^{r/p_i}\le 1,\end{aligned}$$

we get $\{|x(k)|^{r/p_i}sign(x(k))\}_{k=1}^{\infty}\in\ell^{p_i}$ *and* $B_{\ell^r}\subseteq\odot(B_{\ell^{p_1}}\times B_{\ell^{p_2}}\times...\times B_{\ell^{p_n}})$. *Now, let us show the equality given in the definition of the n.p. product. Take into account sequences* $x_i=\{x^i(k)\}_{k=1}^{\infty}\in\ell^{p_i}$ *for* $i=1,2,...,n$ *such that* $x_1\odot x_2\odot...\odot x_n=x$.

By the generalization of Hölder's inequlity it is easily seen that

$$\|x_1\odot x_2\odot...\odot x_n\|_r\le\|x_1\|_{p_1}\|x_2\|_{p_2}...\|x_n\|_{p_n}.$$

Now, let us show the inverse. Since for all k, *we can write*
$x(k)=\prod_{i=1}^{n}|x(k)|^{r/p_i}sign(x(k))$, *we get* $\|(|x(k)|^{r/p_i}sign(x(k)))\|_{p_i}=\|(x(k))\|_r^{r/p_i}$. *Therefore*

$$\|x\|_r=\|(x(k))\|_r=\prod_{i=1}^{n}\|(x(k))\|_r^{r/p_i}=\prod_{i=1}^{n}\|(|x(k)|^{r/p_i}sign(x(k)))\|_{p_i}.$$

Thus, we get $\|x\|_r=\inf\{\|x_1\|_{p_1}\|x_2\|_{p_2}...\|x_n\|_{p_n}\}$ *and* $\odot$ *is an n.p. product from* $\ell^{p_1}\times\ell^{p_2}\times....\times\ell^{p_n}$ *to* ℓ^r.

Theorem 3. *Let* $\sum_{i=1}^{n}\frac{1}{p_i}=\frac{1}{r}$ *for* $1\le r,p_i<\infty$ $(i=1,...,n)$. *For a multilinear operator* $B:\times_{i=1}^{n}\ell^{p_i}\to Y$ *the following statements imply each other.*

(1) *The operator* B *is zero* $\odot$*-preserving.*
(2) *The operator* B *is* $\odot$*-factorable.*
(3) *There is a constant* $k>0$ *such that for every finite sets of sequences* $\{x_1^i,....,x_m^i\}\subset\ell^{p_i}\,(i=1,2,...,n)$, *the following inequality holds;*

$$\Big\|\sum_{j=1}^{m}B(x_j^1,x_j^2,...,x_j^n)\Big\|_Y\le k\Big\|\sum_{j=1}^{m}x_j^1\odot x_j^2\odot...\odot x_j^n\Big\|_r. \tag{2}$$

Thus, B *admits the following factorization for a unique linear operator* $T:\ell^r\to Y$;

$$\begin{tikzcd}\times_{i=1}^{n}\ell^{p_i} \arrow[r, "B"] \arrow[dr, dotted, "\odot"] & Y \\ & \ell^r. \arrow[u, dotted, "T"']\end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2) Assume that B is zero $\odot$-preserving. Let us write the sequences $x_i \in \ell^{p_i}$ $(i = 1, 2, ..., n)$ in the form $x_i = \{x^i(k)\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} x^i(k)\chi_{\{k\}}$, then

$$x_1 \odot .... \odot x_n = \{x^1(k) \cdot .... \cdot x^n(k)\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} x^1(k) \cdot .... \cdot x^n(k) \cdot \chi_{\{k\}}$$

Since $\chi_{\{k\}} \odot \chi_{\{l\}} = 0$ whenever $k \neq l$, the following equality is obtained

$$\begin{aligned}
B(x_1, ..., x_n) &= B\Big(\sum_{k_1=1}^{\infty} x^1(k_1)\chi_{\{k_1\}}, ..., \sum_{k_n=1}^{\infty} x^n(k_n)\chi_{\{k_n\}}\Big)\\
&= \sum_{k_1=1}^{\infty} x^1(k_1) \cdot ... \cdot \sum_{k_n=1}^{\infty} x^n(k_1) B(\chi_{\{k_1\}}, ..., \chi_{\{k_n\}})\\
&= \sum_{k=1}^{\infty} x^1(k) \cdot ... \cdot x^n(k) B(\chi_{\{k\}}, ..., \chi_{\{k\}})\\
&= B\Big(\sum_{k=1}^{\infty} x^1(k) \cdot ... \cdot x^n(k)\chi_{\{k\}}, \chi_{\{k\}}, ..., \chi_{\{k\}}\Big)
\end{aligned}$$

by the zero product preservartion property of B.

For every natural number m, let us define the map $B_m(x_1, ..., x_n) = B(x_1 \odot \chi_{\{1,...,m\}}, ..., x_n \odot \chi_{\{1,...,m\}})$ for all $x_i \in \ell^{p_i}$ $(i = 1, 2, ..., n)$. It is easily seen that the sequence $\{B_m\}_{m=1}^{\infty}$ consists of well-defined, multilinear continuous maps. Since $x_i \odot \chi_{\{1,...,m\}} = x_i \odot \sum_{k=1}^{m} \chi_{\{k\}} = \sum_{k=1}^{m} x^i(k)\chi_{\{k\}}$, by the zero $\odot$-preservation property of B

$$\begin{aligned}
B_m(x_1, ..., x_n) &= B(x_1 \odot \chi_{\{1,...,m\}}, ..., x_n \odot \chi_{\{1,...,m\}})\\
&= B\Big(\sum_{k_1=1}^{m} x^1(k_1)\chi_{\{k_1\}}, ..., \sum_{k_n=1}^{m} x^n(k_n)\chi_{\{k_n\}}\Big)\\
&= \sum_{k_1=1}^{m} x^1(k_1) \cdot ... \cdot \sum_{k_n=1}^{m} x^n(k_n) B(\chi_{\{k_1\}}, ..., \chi_{\{k_n\}})\\
&= \sum_{k=1}^{m} x^1(k) \cdot ... \cdot x^n(k) B(\chi_{\{k\}}, ..., \chi_{\{k\}})\\
&= B\Big(\sum_{k=1}^{m} x^1(k) \cdot ... \cdot x^n(k)\chi_{\{k\}}, \sum_{k=1}^{m} \chi_{\{k\}}, ..., \sum_{k=1}^{m} \chi_{\{k\}}\Big).
\end{aligned}$$

Thus, for all m, the map B_m is written as

$$\begin{aligned}
B_m(x_1, ..., x_n) &= B\Big(\sum_{j=1}^{m} x_j^1 \cdot ... \cdot x_j^n \chi_{\{j\}}, \sum_{j=1}^{m} \chi_{\{j\}}, ..., \sum_{j=1}^{m} \chi_{\{j\}}\Big)\\
&= B(x_1 \odot ... \odot x_n \odot \chi_{\{1,...,m\}}, \chi_{\{1,...,m\}}, ..., \chi_{\{1,...,m\}})
\end{aligned}$$

for all $x_i \in \ell^{p_i}$ $(i = 1, 2, ..., n)$.

Now, for all natural number m and every $x = x_1 \odot ... \odot x_n$, define the map $T_m : \ell^r \to Y$ by $T_m(x) = T_m(x_1 \odot ... \odot x_n) = B_m(x_1, ..., x_n)$. Then, it is seen that for all m, the map T_m is well-defined, linear and continuous operator. Indeed, the linearity is seen by the linearity in the first variable of the map B_m. Let us show the continuity of the map T_m;

$$\begin{aligned}
\|T_m(x)\|_Y &= \|B_m(x_1, ..., x_n)\|_Y \\
&= \|B(x_1 \odot \chi_{\{1,...,m\}}, ..., x_n \odot \chi_{\{1,...,m\}})\|_Y \\
&\leq \|B\| \|x_1 \odot \chi_{\{1,...,m\}}\| ... \|x_n \odot \chi_{\{1,...,m\}}\| \\
&\leq \|B\| \|x_1\| ... \|x_n\|,
\end{aligned}$$

since this holds for all representations of the sequence x, it is seen that $\|T_m(x)\|_Y \leq \|B\| \|x\|_r$ by the definition of n.p. product. For all m, the operator T_m is independent of the representation of the sequence x. Indeed, let us assume $x = x_1 \odot ... \odot x_n = x'_1 \odot ... \odot x'_n$, then it is seen that

$$\begin{aligned}
T_m(x_1 \odot ... \odot x_n) &= B(x_1 \odot ... \odot x_n \odot \chi_{\{1,...,m\}}, \chi_{\{1,...,m\}}, ..., \chi_{\{1,...,m\}}) \\
&= B(x'_1 \odot ... \odot x'_n \odot \chi_{\{1,...,m\}}, \chi_{\{1,...,m\}}, ..., \chi_{\{1,...,m\}}) \\
&= T_m(x'_1 \odot ... \odot x'_n).
\end{aligned}$$

On the other hand, the set of operators $\{T_m\}_{m=1}^{\infty}$ is pointwise convergent for each $x = x_1 \odot ... \odot x_n \in \ell^r$. By the separate continuity of the multilinear map B, this is seen as follows;

$$\begin{aligned}
\lim_{m\to\infty} T_m(x_1 \odot ... \odot x_n) &= \lim_{m\to\infty} B_m(x_1, ..., x_n) \\
&= \lim_{m\to\infty} B(x_1 \odot \chi_{\{1,...,m\}}, ..., x_n \odot \chi_{\{1,...,m\}}) \\
&= B(\lim_{m\to\infty} x_1 \odot \chi_{\{1,...,m\}}, ..., \lim_{m\to\infty} x_n \odot \chi_{\{1,...,m\}}) \\
&= B(x_1, ..., x_n).
\end{aligned}$$

Thus, $\{T_m(x)\}_{m=1}^{\infty}$ converges to $B(x_1, ..., x_n)$ for all $x \in \ell^r$ such that $x = x_1 \odot ... \odot x_n$ for the elements $x_i \in \ell^{p_i}$ $(i = 1, ..., n)$. Let us define the pointwise limit $T(x) = \lim_{m\to\infty} T_m(x)$. It is clear that the limit map T is well-defined and linear. Besides it is continuous by the uniform boundedness theorem.

Summing up, the linear bounded map $T : \ell^r \to Y$ defined by $T(x_1 \odot ... \odot x_n) = B(x_1, ..., x_n)$ is the desired map.

(2) $\Rightarrow$ (3) is obtained by Lemma 2.3. given in [12].

Lastly, let us show (3) implies (1). Consider the sequences $x_i \in \ell^{p_i}$ $(i = 1, ..., n)$ such that $x_k \odot x_l = 0$ for some different $k, l \in \{1, ..., n\}$. This implies $x_1 \odot ... \odot x_n = 0$. Therefore, zero $\odot$-preservation is seen by Inequality (2) given in the statement (3). $\square$

The above theorem gives an isometry between the spaces $\mathcal{L}_0^n(\times_{i=1}^n \ell^{p_i}, Y)$ and $\mathcal{L}(\ell^r, Y)$.

Theorem 4. *The correspondence $B \longleftrightarrow T$ is an onto isometry between the Banach spaces $\mathcal{L}_0^n(\times_{i=1}^n \ell^{p_i}, Y)$ and $\mathcal{L}(\ell^r, Y)$.*
Particularly for $Y = \mathbb{R}$, we get $\mathcal{L}_0^n(\times_{i=1}^n \ell^{p_i}) = (\ell^r)^*$.

Proof. It is easily seen that the map $\mathcal{L}_0^n(\times_{i=1}^n \ell^{p_i}, Y) \to \mathcal{L}(\ell^r, Y)$ is linear. Now, let us show the isometry.

$$\begin{aligned} \|B\| &= \sup_{(x_1,\dots,x_n)\in \times_{i=1}^n B_{\ell^{p_i}}} \|B(x_1, ..., x_n)\|_Y \\ &= \sup_{(x_1,\dots,x_n)\in \times_{i=1}^n B_{\ell^{p_i}}} \|T(x_1 \odot ... \odot x_n)\|_Y \\ &\geq \sup_{x=x_1\odot\dots\odot x_n \in B_{\ell^r}} \|Tx\|_Y = \|T\|. \end{aligned}$$

For the converse inequality;

$$\|T\| = \sup_{x\in B_{\ell^r}} \|Tx\|_Y = \sup_{(x_1,\dots,x_n)\in \times_{i=1}^n B_{\ell^{p_i}}} \|B(x_1, ..., x_n)\| \leq \|B\|,$$

where $x_i = \{x^i(k)\}_{k=1}^\infty = \{|x(k)|^{r/p_i} sgn(x(k))\}_{k=1}^\infty$ for all $i = 1, ..., n$.

It is easily seen that the map $B \to T$ is onto, since an n-linear map B_T is obtained for every linear map T by defining $T(x) = B(x_1, ..., x_n)$ for all $x = x_1 \odot ... \odot x_n \in \ell^r$ for the n.p product $\odot : \times_{i=1}^n \ell^{p_i} \to \ell^r$. □

Corollary 5. *As a result of the above isometry, the following isometries are given for particular p_i values.*

- ⋆ $\mathcal{L}_0^n(\times^n \ell^p, Y) = \mathcal{L}(\ell^{p/n}, Y)$, *where $p > n$.*
- ⋆ $\mathcal{L}_0^n(\times^n \ell^n, Y) = \mathcal{L}(\ell^1, Y)$
- ⋆ $\mathcal{L}_0^n(\times^n \ell^p) = (\ell^{p/n})^* = \ell^{p/(p-n)}$
- ⋆ $\mathcal{L}_0^n(\times^n \ell^n) = (\ell^1)^* = \ell^\infty$.

4. Compactness and Summability Inquiries for $\odot$-Factorable Maps

In this section, we investigate compactness and summability for $\odot$-factorable multilinear operator that are based on the classical analysis properties and theorems like Dunford Pettis property, well-known Grothendieck's theorem or cotype related properties.

4.1. **Compactness of $\odot$-Factorable operators.** By the definition of norm preserving product, it is seen that a $\odot$-factorable multilinear map $B : \times_{i=1}^n \ell^{p_i} \to Y$ is (weakly) compact if and only if the linear operator $T : \ell^r \to Y$ appearing in the factorization is (weakly) compact. Now, we will give more specific compactness implications for $\odot$-factorable maps.

Corollary 6. *Let $\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{r}$ for $1 \leq r, p_i < \infty$ and $i = 1, ..., n$. For a $\odot$-factorable multilinear operator $B : \times_{i=1}^{n} \ell^{p_i} \to Y$, we have the following compactness results;*

1. *For $r > 1$, the map B is weakly compact.*
2. *If $r = 1$ and Y is reflexive, then the map B is compact.*
3. *For $1 \leq s < r < \infty$ and $Y = \ell^s$, the map B is compact.*

Proof. (1) This is easily seen by the weakly compactness of the factorization operator $T : \ell^r \to Y$ which is defined on the reflexive space ℓ^r.

(2) B factors through the linear map $T : \ell^r \to Y$ that is weakly compact due to reflexivity of the space Y. In addition, T –hence B– is compact by the Dunford-Pettis property of the space ℓ^1.

(3) Since the linear operator $T : \ell^r \to \ell^s$ is compact whenever $1 \leq s < r < \infty$ by the Pitt's theorem, the map B is so also (see [9, Chapter 12]). □

Corollary 7. *Let $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ for $1 \leq p_i < \infty$ $(i = 1, ..., n)$ and let $B : \times_{i=1}^{n} \ell^{p_i} \to Y$ be a $\odot$-factorable multilinear operator. For a set $A \subset \times_{i=1}^{n} \ell^{p_i}$, $B(A)$ is norm compact if $\{x_1 \odot .... \odot x_n : (x_1, ..., x_n) \in \ell^{p_1} \times .... \times \ell^{p_n}\}$ is weakly compact.*

Proof. The $\odot$-factorable multilinear operator B factors throug a linear map $T : \ell^1 \to Y$. Since $\odot(A)$ is weakly compact, $B(A) = T \circ \odot(A)$ is weakly compact. Hence it is compact by the Dunford-Pettis property of ℓ^1. □

4.2. **Summability Properties of $\odot$-Factorable Operators.** Now, let us look at the summability properties of $\odot$-factorable maps.

Theorem 8. *Let $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ for $1 \leq p_i < \infty$ $(i = 1, ..., n)$. The followings imply each other for a Hilbert-space valued multilinear map $B : \times_{i=1}^{n} \ell^{p_i} \to H$.*

i) *The map B is $\odot$-factorable,*

ii) *There is a constant $k > 0$ such that for every finite sets $\{x_1^i, ..., x_m^i\} \subset \ell^{p_i}$ $(i = 1, ..., n)$*

$$\sum_{j=1}^{m} \left\| B(x_j^1, ..., x_j^n) \right\|_H \leq k \sup_{z' \in B_{\ell^\infty}} \sum_{j=1}^{m} \left| \left\langle x_j^1 \odot ... \odot x_j^n, z' \right\rangle \right|,$$

iii) *For all $x_i \in \ell^{p_i}$ $(i = 1, ..., n)$ there is a regular Borel measure η over B_{ℓ^∞} such that*

$$\|B(x_1, ..., x_n)\|_H \leq K \int_{B_{\ell^\infty}} |\langle x_1 \odot ... \odot x_n, z' \rangle| \, d\eta(z').$$

Besides, B factors through a completely continuous linear operator due to the Dunford-Pettis property of the space ℓ^1 whenever one of the aboves holds.

Proof. i)⇒ ii) Since the map B is $\odot$-factorable, it factors through the linear map $T : \ell^1 \to H$. Since $\mathcal{L}(\ell^1, H) = \Pi_1(\ell^1, H)$ by a result of the Grothendieck's Theorem, we obtain T is a 1-summing operator and thus, B satisfies the inequality given in statement (ii).

ii) $\Rightarrow$ iii) The integral domination given in the third statement is clearly obtained by Pietsch Domination Theorem (see [9, Theorem 2.12]).

iii) $\Rightarrow$ i) If the map B has the integral domination then it is seen that $B(x_1, ..., x_n) = 0$ whenever $x_k \odot x_l = 0$ for some different $k, l \in \{1, ..., n\}$. Thus, B is zero $\odot$-preserving and it is $\odot$-factorable by the main theorem of the paper. $\square$

We obtain a weaker result by considering some cotype-related properties. It is known that cotype 2 for a Banach space implies the Orlicz property (see [8, Section 8.9]). Assume that Y has Orlicz property and let $\sum_{i=1}^n \frac{1}{p_i} = 1$ for $1 \leq p_i < \infty$ for $i = 1, ..., n$. The following domination inequality holds for an n-linear $\odot$-factorable map $B : \times_{i=1}^n \ell^{p_i} \to Y$

$$\Big(\sum_{j=1}^m \Big\| B(x_j^1, ..., x_j^n) \Big\|_Y^2 \Big)^{1/2} \leq \sup_{\varepsilon_j = \{-1,1\}} \Big\| \sum_{j=1}^m \varepsilon_j x_j^1 \odot ... \odot x_j^n \Big\|$$

for all finite sets $\{x_1^i, ..., x_m^i\} \subset \ell^{p_i}$ $(i = 1, ..., n)$.

Lastly, we will give some results for $\odot$-factorable maps that are ℓ^p-space valued. We will use Littlewood inequality that states $\mathcal{L}(\ell^1, \ell^{4/3}) = \Pi_{4/3,1}(\ell^1, \ell^{4/3})$ (see [8, Section 34.12]): if B is defined on $\times_{i=1}^n \ell^{p_i}$ to $\ell^{4/3}$, then

$$\Big(\sum_{j=1}^m \Big\| B(x_j^1, ..., x_j^n) \Big\|_{4/3}^{4/3} \Big)^{3/4} \leq k \sup_{z' \in B_{\ell^\infty}} \sum_{j=1}^m \Big| \Big\langle x_j^1 \odot ... \odot x_j^n, z' \Big\rangle \Big|$$

for all finite sets $\{x_j^i\}_{j=1}^m \subset \ell^{p_i}$ $(i = 1, ..., n)$.

5. A Generalization of the $\odot$-Factorable Operators

Let $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{r}$ for $1 \leq r, p_i < \infty$ $(i = 1, ..., n)$. Consider n Banach spaces X_i $(i = 1, ..., n)$ that are isomorphic to ℓ^{p_i} by the isomorphisms $P_i : X_i \to \ell^{p_i}$. Let us define the product $\odot_{\times_{i=1}^n P_i} : \times_{i=1}^n X_i \to \ell^r$ by

$$\odot_{\times_{i=1}^n P_i}(f_1, ..., f_n) = P_1(f_1) \odot .... \odot P_n(f_n), \quad f_i \in X_i.$$

This product can be illustrated by the following diagram;

$$\begin{tikzcd}
\times_{i=1}^n X_i \arrow[r, "\odot_{\times_{i=1}^n P_i}"] \arrow[d, dotted, "\times_{i=1}^n P_i"'] & \ell^r. \\
\times_{i=1}^n \ell^{p_i} \arrow[ur, dotted, "\odot"'] &
\end{tikzcd}$$

We will call a multilinear map $B : \times_{i=1}^n X_i \to Y$ is zero $\odot_{\times_{i=1}^n P_i}$–preserving if $B(f_1, ..., f_n) = 0$ whenever $\odot_{P_k \times P_l}(f_k, f_l) = P_k(f_k) \odot P_l(f_l) = 0$ for some $k, l \in \{1, ..., n\}$ such that $k \neq l$.

Theorem 9. *Let $\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{r}$ for $1 \leq r, p_i < \infty$ $(i = 1, ..., n)$. Consider the Banach spaces $X_i\,(i = 1, ..., n)$ that are isomorphic to ℓ^{p_i} by means of the isomorphisms $P_i : X_i \to \ell^{p_i}$. For an n-linear map $B : \times_{i=1}^{n} X_i \to Y$, the following statements are equivalent.*

1. *The operator B is zero $\odot_{\times_{i=1}^{n} P_i}-$preserving.*
2. *The map B is $\odot_{\times_{i=1}^{n} P_i}-$factorable. That is, there is a linear operator $T : \ell^r \to Y$ such that $B := T \circ \odot_{\times_{i=1}^{n} P_i}$.*
3. *There exists a $K > 0$ such that the inequality below holds for every finite sets $\{f_1^i, ..., f_m^i\} \subset X_i\,(i = 1, .., n)$;*

$$\Big\| \sum_{j=1}^{m} B(f_j^1, ..., f_j^n) \Big\|_Y \leq K \Big\| \sum_{j=1}^{m} P_1(f_j^1) \odot ... \odot P_n(f_j^n) \Big\|_r .$$

If one of the aboves is satisfied, then B admits the following factorization;

$$\begin{tikzcd}
\times_{i=1}^{n} X_i \arrow[r, "B"] \arrow[d, dotted, "\times_{i=1}^{n} P_i"'] & Y \\
\times_{i=1}^{n} \ell^{p_i} \arrow[r, dotted, "\odot"] & \ell^r . \arrow[u, dotted, "T"']
\end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2) Let us assume that B is zero $\odot_{\times_{i=1}^{n} P_i}-$preserving and define the map $\overline{B} = B \circ \times_{i=1}^{n} P_i^{-1} : \times_{i=1}^{n} \ell^{p_i} \to Y$. For the sequences $x_i \in \ell^{p_i}$ $(i \in \{1, ..., n\})$, it is seen that $\overline{B}(x_1, .., x_n) = B \circ \times_{i=1}^{n} P_i^{-1}(P_1(f_1), ..., P_n(f_n))$, where $P_i(f_i) = x_i$ for $f_i \in X_i$. Since B is zero $\odot_{\times_{i=1}^{n} P_i}-$preserving, it is obtained that $\overline{B}(x_1, .., x_n) = B(f_1, ..., f_n) = 0$ whenever $x_k \odot x_l = P_k(f_k) \odot P_l(f_l) = 0$ for some $k, l \in \{1, ..., n\}$. This shows zero $\odot$-preservation of the map $\overline{B}$ and therefore $\overline{B}$ is $\odot$-factorable by Theorem 3. So we have that there is a linear operator $T : \ell^r \to Y$ such that $\overline{B} = T \circ \odot$. By the definition of $\overline{B}$, we obtain $B = \overline{B} \circ (\times_{i=1}^{n} P_i) = T \circ \odot \circ (\times_{i=1}^{n} P_i) = T \circ \odot_{\times_{i=1}^{n} P_i}$, the desired factorization.

(2) $\Rightarrow$ (3) If the map B is $\odot_{\times_{i=1}^{n} P_i}-$factorable then the map $\overline{B} = B \circ \times_{i=1}^{n} P_i^{-1} : \times_{i=1}^{n} \ell^{p_i} \to Y$ is $\odot$-factorable. Indeed, for the $\odot_{\times_{i=1}^{n} P_i}-$factorable map B, there is a linear operator $T : \ell^r \to Y$ such that $B := T \circ \odot_{\times_{i=1}^{n} P_i}$. Thus, $\overline{B} = T \circ \odot_{\times_{i=1}^{n} P_i} \circ \times_{i=1}^{n} P_i^{-1}$. For the elements $f_i \in X_i$ that are $P_i(f_i) = x_i \in \ell^{p_i}$, we get

$$\begin{aligned}
\overline{B}(x_1, ..., x_n) &= T \circ \odot_{\times_{i=1}^{n} P_i} \circ \times_{i=1}^{n} P_i^{-1}(x_1, ..., x_n) \\
&= T \circ \odot_{\times_{i=1}^{n} P_i}(P_1^{-1}(x_1), ..., P_n^{-1}(x_n)) \\
&= T(P_1 P_1^{-1}(x_1) \odot ... \odot P_n P_n^{-1}(x_n)) \\
&= T(x_1 \odot .... \odot x_n).
\end{aligned}$$

This shows, $\overline{B}$ is $\odot$-factorable. By Lemma 2.3 given in [12] and Theorem 3, the inequality given in the statement (3) is obtained.

(3) $\Rightarrow$ (1) It is clear that B is zero $\odot_{\times_{i=1}^{n} P_i}-$preserving under the assumption of the statement (3). $\square$

6. Application: Representation As n-homogeneous Polynomial

Recall that an n-linear map $B : \times^n X \to Y$ is called symmetric if

$$B(x_1, ..., x_n) = B(x_{\sigma(1)}, ..., x_{\sigma(n)}) \quad (x_1, ..., x_n \in X)$$

for any permutation σ of the first n natural numbers. $\mathcal{L}_s^n(\times^n X, Y)$ denotes the space of symmetric multilinear operators defined on X to Y.

Remark 10. *Let $p \geq n$. It is easily seen that any $\odot$-factorable n-linear map $B : \times^n \ell^p \to Y$ is symmetric. Indeed, the map B factors through the linear map $T : \ell^{p/n} \to Y$ and thus*

$$B(x_1, ..., x_n) = T(x_1 \odot .... \odot x_n) = T(x_{\sigma(1)} \odot ... \odot x_{\sigma(n)}) = B(x_{\sigma(1)}, ..., x_{\sigma(n)})$$

for all $x_1, ..., x_n \in \ell^p$ by the commutativity of the product $\odot$.

In addition to this, a symmetry is obtained for the general version. Let X be isomorphic to the space ℓ^p by the isomorphism $P : X \to \ell^p$. Then any $\odot_{\times^n P}-$factorable n-linear map $B : \times^n X \to Y$ is symmetric.

Therefore, the following inclusions hold;

- $\mathcal{L}_0^n(\times^n \ell^p, Y) \subseteq \mathcal{L}_s^n(\times^n \ell^p, Y)$,
- $\mathcal{L}_0^n(\times^n X, Y) \subseteq \mathcal{L}_s^n(\times^n X, Y)$ *if* $X \cong \ell^p$.

We will give a counterexample to show that the symmetry does not imply zero $\odot$-preservation. Consider a bilinear map $B : \ell^p \times \ell^p \to \mathbb{R}$ defined by $B(x_1, x_2) = \sum_{k=1}^{5} x^1(k) \cdot x^2(k)$. It is seen that B is symmetric. For the sequences $x_1 = (1, 1, 0, 1, 0, -1, -1, 0, 0, ...)$ and $x_2 = (1, 1, 1, 0, 1, 1, 1, 0, 0, ...)$ in ℓ^p, it is obtained that $x_1 \odot x_2 = 0$ but $B(x_1, x_2) = 2$, thus B is not zero $\odot$-preserving.

A map $P : X \to Y$ is called n-homogeneous polynomial if it is associated with an n-linear symmetric map $B : \times^n X \to Y$ such that $P(x) = B(x, ..., x)$ for all $x \in X$. The class of n-homogeneous polynomials is a Banach space under the norm $\|P\| = \sup_{\|x\|=1} \|P(x)\|$. It will be denoted by $\mathcal{P}(^n X, Y)$. We refer the book [10] for more information about polynomials.

An n-homogeneous polynomial defined on the Banach algebra X is called orthogonally additive if $P(x + y) = P(x) + P(y)$ whenever $xy = 0$ for $x, y \in X$. Similarly we will call an n-homogeneous polynomial defined on the Banach space X orthogonally additive if $P(x+y) = P(x)+P(y)$ whenever $x \circledast y = 0$ for $x, y \in X$ and an n.p. product $\circledast$. We denote by $\mathcal{P}_0(^n X, Y)$ the space of n-homogeneous orthogonally additive polynomials from X to Y. We will write $\mathcal{P}_0(^n X)$ for $Y = \mathbb{R}$.

The Banach space of n-homogeneous orthogonally additive polynomials is closely related to the zero product preserving n-linear operators and several papers can be found in this direction in the literature (see [3, 5, 12, 15, 16] and references therein). Now we will give a generalization of the isomorphisms between orthogonally additive n-homogeneous polynomial forms and sequences given in the papers [15] and [16].

Theorem 11. *Let $1 \leq n \leq p < \infty$. There is an onto isometry between the spaces $\mathcal{L}(\ell^{p/n}, Y)$ and $\mathcal{P}_0({}^n\ell^p, Y)$. Particularly, $\mathcal{P}_0({}^n\ell^p) = (\ell^{p/n})^*$ for a scalar field range.*

Proof. Consider a linear continuous operator $T \in \mathcal{L}(\ell^{p/n}, Y)$. It is seen that T gives a $\odot$-factorable n-linear map $B_T : \times^n \ell^p \to Y$ defined by $T(x) = T(x_1 \odot ... \odot x_n) = B(x_1, ..., x_n)$ for all $x_i \in \ell^p \, (i = 1, ..., n)$ such that $x_1 \odot ... \odot x_n = x \in \ell^{p/n}$. Due to the symmetry of the $\odot$-factorable map B_T, an n-homogeneous polynomial $P_{B_T} : \ell^p \to Y$ is obtained such that it is orthogonally additive. Indeed, for all $x, y \in \ell^p$

$$\begin{aligned} P_{B_T}(x+y) &= B_T(x+y, ..., x+y) \\ &= \sum_{k=0}^{n} \binom{n}{k} B_T(x, \overset{k}{\dots}, x, y, \overset{n-k}{\dots}, y) \\ &= B_T(x, \overset{n}{\dots}, x) + B_T(y, \overset{n}{\dots}, y) \\ &= P_{B_T}(x) + P_{B_T}(y). \end{aligned}$$

whenever $x \odot y = 0$, thus P_{B_T} is orthogonally additive. Thus the linear correspondence $T \to P_{B_T}$ defines an orthogonally additive n-homogeneous polynomial P_{B_T} for every T by $T(x^n) = P_{B_T}(x)$, where $x^n = x \odot \overset{n}{\dots} \odot x$. Let us show the isometry now.

$$\|T\| = \sup_{\|x\|_{p/n} \leq 1} \|Tx\| = \sup_{\|x^{1/n}\|_p \leq 1} \|P(x^{1/n})\| = \sup_{\|y\|_p \leq 1} \|P(y)\| = \|P\|.$$

For the surjectivity, let us consider an orthogonally additive n-homogeneous polynomial $P \in \mathcal{P}_0({}^n\ell^p, Y)$. This polynomial defines a 1-homogeneous map T by $T(x) = P(x^{1/n})$ for all $x = \{x(k)\}_{k=1}^{\infty} \in \ell^{p/n}$ where $x^{1/n} = \{|x(k)|^{1/n} sign(x(k))\}_{k=1}^{\infty}$ such that $x = \{|x(k)|^{1/n} sign(x(k)) \cdot \overset{n}{\dots} \cdot |x(k)|^{1/n} sign(x(k))\}_{k=1}^{\infty} = x^{1/n} \odot \overset{n}{\dots} \odot x^{1/n} \in \ell^{p/n}$. The map T is linear. Indeed, to see this consider the sequences $x_1' = \sum_{k=1}^{m} x^1(k) \cdot \chi_{\{k\}}$ and $x_2' = \sum_{k=1}^{m} x^2(k) \cdot \chi_{\{k\}}$ defined by the sequences $x_1, x_2 \in \ell^{p/n}$.

Since $(x_1' + x_2')^{1/n} = \sum_{k=1}^{m} (x^1(k) + x^2(k))^{1/n} \cdot \chi_{\{k\}}$, by using the n-homogenity and orthogonally additivity of the polynomial P, we get that

$$\begin{aligned} T(x_1' + x_2') &= P\Big((x_1' + x_2')^{1/n}\Big) = P(\sum_{k=1}^{m} (x^1(k) + x^2(k))^{1/n} \cdot \chi_{\{k\}}) \\ &= \sum_{k=1}^{m} P((x^1(k) + x^2(k))^{1/n} \cdot \chi_{\{k\}}) = \sum_{k=1}^{m} (x^1(k) + x^2(k)) P(\chi_{\{k\}}) \\ &= P\Big(\sum_{k=1}^{m} (x^1(k))^{1/n} \cdot \chi_{\{k\}}\Big) + P\Big(\sum_{k=1}^{m} (x^2(k))^{1/n} \cdot \chi_{\{k\}}\Big) \\ &= P((x_1')^{1/n}) + P((x_2')^{1/n}) = T(x_1') + T(x_2'). \end{aligned}$$

Since $x_1 = \lim_{m\to\infty} \sum_{k=1}^{m} x^1(k) \cdot \chi_{\{k\}}$ and $x_2 = \lim_{m\to\infty} \sum_{k=1}^{m} x^2(k) \cdot \chi_{\{k\}}$, it is obtained that

$$\begin{aligned} T(x_1 + x_2) &= T(\lim_{m\to\infty} x_1' + \lim_{m\to\infty} x_2') \\ &= \lim_{m\to\infty} T(x_1' + x_2') = \lim_{m\to\infty} (T(x_1') + T(x_2')) \\ &= T(x_1 + x_2). \end{aligned}$$

Thus, every orthogonally additive n-homogeneous polynomial P defines a linear map $T \in \mathcal{L}(\ell^{p/n}, Y)$. We can illustrate this isometry by the following diagram;

$$\begin{tikzcd}
& \ell^p \arrow[dl, dotted, "\Delta_n"'] \arrow[dr, "P_{B_T}"] & \\
\times^n \ell^p \arrow[rr, "B_T"] \arrow[dr, dotted, "\odot"'] & & Y, \\
& \ell^{p/n} \arrow[ur, "T"'] &
\end{tikzcd}$$

where Δ_n is the canonical embedding called *diagonal mapping* from ℓ^p to $\times^n \ell^p$ used to define the n-homogeneous polynomials.

Particularly, every n-homogeneous polynomial form P in $\mathcal{P}_0(^n\ell^p)$ is represented by a sequence in the space $\ell^{p/(p-n)}$. $\square$

Corollary 12. $\mathcal{L}(\ell^1, Y) = \mathcal{P}_0(^n\ell^n, Y)$ *and every orthogonally additive n-homogenous polynomial $P : \ell^n \to \mathbb{R}$ is represented by a bounded scalar valued sequence.*

From Corollary 5, Theorem 11 and Corollary 12, we get the following isometries;

- $\star$ $\mathcal{L}_0^n(\times^n\ell^p, Y) = \mathcal{P}_0(^n\ell^p, Y)$, where $p \geq n$.
- $\star$ $\mathcal{L}_0^n(\times^n\ell^n, Y) = \mathcal{P}_0(^n\ell^n, Y)$
- $\star$ $\mathcal{L}_0^n(\times^n\ell^p) = \mathcal{P}_0(^n\ell^p)$
- $\star$ $\mathcal{L}_0^n(\times^n\ell^n) = \mathcal{P}_0(^n\ell^n)$.

We can give some isomorphisms for the $\odot_{\times^n P}-$factorable maps as follows;

Corollary 13. *Let $1 \leq n \leq p < \infty$ and $P : E \to \ell^p$ is an isomorphism. There is an isomorphism between the spaces $\mathcal{L}(\ell^{p/n}, Y)$ and $\mathcal{P}_0(^nE, Y)$. Particularly, $\mathcal{P}_0(^nE) = (\ell^{p/n})^*$ for a scalar field range.*

Let us finish the paper with an example.

Example 14. *Let $\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{r}$ for $1 \leq r, p_i < \infty$ for $i = 1, ..., n$. Recall that a multilinear form $B : \times_{i=1}^{n} \ell^{p_i} \to \mathbb{C}$ defined by $B(x_1, ..., x_n) = \sum_{k=1}^{\infty} \alpha_k \cdot x_k^1 \cdot .... \cdot x_k^n$ is called diagonal operator, where $\{\alpha_k\}$ is a bounded sequence. Clearly, it is seen by the definition that $B(x_1, ..., x_n) = 0$ whenever $x_k \odot x_l = 0$ for some $k, l \in \{1, 2, ..., n\}$. Therefore, it is zero product preserving and there is a linear form $T : \ell^r \to \mathbb{C}$ such that $B(x_1, ..., x_n) = T(x)$, where $x_1 \odot .... \odot x_n = x$. Besides, if we consider $p_i = ... = p_n = p$, then we obtain that the zero product preserving map $B : \times^n \ell^p \to \mathbb{C}$*

has a factorization through the linear form $T : \ell^{p/n} \to \mathbb{C}$. Since this gives the symmetry of the form $B : \times^n \ell^p \to \mathbb{C}$, we get the diagonal map B is associated with an orthogonally additive n-homogeneous diagonal polynomial form $P : \ell^p \to \mathbb{C}$.

References

[1] Alaminos, J., Brešar, M., Extremera, J., Villena, A. R., Maps preserving zero products, *Studia Math.*, 193 (2) (2009), 131-159.

[2] Alaminos, J., Brešar, M., Spenko, S., Villena A. R., Orthogonally additive polynomials and orthosymmetric maps in Banach algebras with properties $\mathbb{A}$ and $\mathbb{B}$, *Proceedings of the Edinburgh Mathematical Society*, 59 (3) (2016), 559-568.

[3] Alaminos, J., Extremera, J., Godoy, M. L. C., Villena, A. R., Orthogonally additive polynomials on convolution algebras associated with a compact group, *J Math Anal Appl.*, 472 (1) (2019),285-302.

[4] Ben Amor, F., On orthosymmetric bilinear maps, *Positivity*, 14 (2010), 123-134.

[5] Benyamini, Y., Lassalle, S., Llavona, J. G., Homogeneous orthogonally additive polynomials on Banach lattices, *Bull Lond Math Soc.*, 383 (2006), 459-469.

[6] Boulabiar, K., Buskes, G., Vector lattice powers: f-algebras and functional calculus, *Comm Algebra.*, 344 (2006), 1435-1442.

[7] Bu, Q., Buskes, G., Kusraev, A. G., Bilinear Maps on Products of Vector Lattices: A Survey, In: Boulabiar K., Buskes G., Triki A. (eds) *Positivity. Trends in Mathematics*, (2007), 97–126.

[8] Defant, A., Floret, K., Tensor norms and operator ideals, North-Holland Math. Stud. Vol. 176, North-Holland, Amsterdam, Elsevier, 1993.

[9] Diestel, J., Jarchow H., Tonge, A., Absolutely Summing Operators, Vol. 43. Cambridge University Press, 1995.

[10] Dineen, S., Complex Analysis on Infinite Dimensional Spaces, Springer-Verlag, London, 1999.

[11] Erdoğan, E., Calabuig, J. M., Sánchez Pérez, E. A., Convolution-continuous bilinear operators acting in Hilbert spaces of integrable functions, *Ann. Funct. Anal.*, 9 (2) (2018), 166-179.

[12] Erdoğan, E., Factorization of multilinear operators defined on products of function spaces, *Linear and Multilinear Algebra*, (2020), https://dx.doi.org/10.1080/03081087.2020.1715334

[13] Erdoğan, E., Gök, Ö., Convolution Factorability of Bilinear Maps and Integral Representations, *Indag. Math.*, 29 (5) (2018), 1334-1349.

[14] Erdoğan, E., Sánchez Pérez, E. A., Gök, Ö., Product factorability of integral bilinear operators on Banach function spaces, *Positivity*, 23 (3) (2019), 671-696.

[15] Ibort, A., Linares, P., Llavona, J.G., A representation theorem for orthogonally additive polynomials on Riesz spaces, *Rev Mat Complut.*, 251 (2012), 21-30.

[16] Sundaresan, K., Geometry of spaces of polynomials on Banach lattices in: Applied Geometry and Discrete Mathematics, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, *Amer. Math. Soc.*, Providence RI, vol 4 (1991), pp 571-586.

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SOME COMMENTS ON METHODOLOGY OF CUBIC RANK TRANSMUTED DISTRIBUTIONS

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Abstract. In this study, at first a new polynomial rank transmutation is proposed. Then, a new cubic rank transmutation is introduced by simplifying the set of transmutation parameters in order to improve its usefulness in statistical modeling. The purpose of this comment is to clarify some issues that exist in the methodology of obtaining the distribution by the cubic transmutation and the stage of proofing it. In this way, both the parameter space is expanded and the process of establishing the cubic transformed distribution family is given.

1. Introduction

In this study, we inspire the quadratic rank transmutation map (QRTM) proposed by [16]. The mapping is given as

$$u \to u + \lambda u\,(1-u) \tag{1}$$

where $u \in [0,1]$ and $\lambda \in [\,-1,1]$. Using this transmutation many distributions have been derived and still continue to be derived. Beside this, there are also some studies on the modifications of the QRTM. Some of the pioneering works on proposing modified QRTM can be given as follows: [1] proposed a new Weibull distribution by using exponentiated QRTM. [2] generated a new distribution family by considering exponentiated distribution as the baseline distribution. [11] studied a new distribution by taking the baseline distribution as exponentiated exponential distribution. [5] introduced transmuted exponentiated modified Weibull distribution, and [3] introduced transmuted exponentiated Lomax distribution. The last three

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studies can be seen as a special case of [2]. [9] introduced a new transmutation map by adding extra two parameters to get more flexible distribution. Then, [10] introduced a new Lindley distribution by using this new transmutation map approach. [4] introduced a kind of generalization of QRTM by considering sum of k- dimensional vector of transmutation parameters. There are two similar studies which are the generalized transmuted G family by [12] and generalized transmuted Weibull distribution by [13]. Also, by taking into account recent works, [15] introduced a new distribution named as transmuted generalized Gamma distribution. They use QRTM to generate this distribution family.
In this study, a new polynomial rank transmutation is proposed additionally to [17]. Since the parameter set is still complex, a new cubic rank transmutation is introduced in the light of the idea behind QRTM. In our study, since an extra transmutation parameter is added, the distribution has become more flexible.

2. Motivation

[17] proposed polynomial rank transmutation map to demonstrate Skew-kurtotic transmutations. *Figure 3* of [17] indicates that admissible parameter region. However this region is quite complex structure, the points on *Figure 5* of them show some special cases related to family of order statistics up to 3-sized sample. Under the leadership of this idea, we propose a new polynomial rank transmutation to get simpler structure of parameter region. Let $G(u)$ stand for the polynomial rank transmutation defined on $[0,1]$. Then, we have

$$G(u) = u + \lambda_1 u\left(1-u\right) + \lambda_2 u^2\left(1-u\right) \tag{2}$$

with $G(0)=0$ and $G(1)=1$. Note that, λ_1 and λ_2 are the transmutation parameters. Parameter region will be defined with following discussion. Since G should be non-decreasing, non-negativity of the first derivative of G with respect to u is examined. Thus, the shape of the parameter region is determined. By calling this derivative with g, we have

$$g\left(u,\lambda_1,\lambda_2\right) = -3\lambda_2 u^2 - 2u\left(\lambda_1-\lambda_2\right) + \left(1+\lambda_1\right). \tag{3}$$

Non-negativity of $g\left(u,\lambda_1,\lambda_2\right)$ at the end-points, namely the inequalities $g\left(0,\lambda_1,\lambda_2\right) = 1+\lambda_1 \geq 0$ and $g\left(1,\lambda_1,\lambda_2\right) = 1-\lambda_1-\lambda_2 \geq 0$ both requires that

$$\begin{aligned} \lambda_1 &\geq -1 \\ \lambda_1+\lambda_2 &\leq 1. \end{aligned} \tag{4}$$

From these two inequalities, it is clear that $\lambda_2 \leq 2$. When the eq. (3) is taken into account, $g\left(u,\lambda_1,\lambda_2\right)$ is a concave function for $\lambda_2 \in (0,2]$. As long as the inequality (4) is valid, $g\left(u,\lambda_1,\lambda_2\right)$ will take non-negative values. For $\lambda_2 \leq 0$, we will investigate the sufficient conditions on non-negativity of $g\left(u,\lambda_1,\lambda_2\right)$. In this case, $g\left(u,\lambda_1,\lambda_2\right)$ has a minimum point since it is a convex function. If this minimum point is within $(0,1)$, the value at that point of the function $g\left(u,\lambda_1,\lambda_2\right)$ must be positive. Accordingly, the minimum point is obtained by taking the derivative of the eq. (3) and equating them to zero as follows:

$$g'(u,\lambda_1,\lambda_2) = -6\lambda_2 u - 2(\lambda_1-\lambda_2) = 0 \Rightarrow u^* = \frac{-(\lambda_1-\lambda_2)}{3\lambda_2} \tag{5}$$

Then, the value of $g(u,\lambda_1,\lambda_2)$ at u^* must satisfy

$$g\left(\frac{-(\lambda_1-\lambda_2)}{3\lambda_2},\lambda_1,\lambda_2\right) = \frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2(3+\lambda_2)}{3\lambda_2} \geq 0 \tag{6}$$

Hence, it is necessary to say that the value of the numerator in (6) is non-positive. If this statement given by the numerator is considered as a second order polynomial of λ_1, the roots are given by

$$\lambda_{1_{1,2}} = \frac{-\lambda_2 \pm \sqrt{-3\lambda_2(\lambda_2+4)}}{2} \tag{7}$$

Here, we can say that the condition $-4 \leq \lambda_2$ must also occur in order for the roots to be real valued. Thus, under the condition $-4 \leq \lambda_2 < 0$, we have bounds for λ_1 as follows:

$$\frac{-\lambda_2 - \sqrt{-3\lambda_2(\lambda_2+4)}}{2} \leq \lambda_1 \leq \frac{-\lambda_2 + \sqrt{-3\lambda_2(\lambda_2+4)}}{2} \tag{8}$$

For these bounds, the numerator in (6) has a negative sign. This leads to the following conclusion: The range of λ_1 is as in (8) for $\lambda_2 \in [-4,0)$. However, the minimum value of the lower bound in (8) can be -1, while the maximum value of the upper bound can be 3. From this we can say that the range of λ_1 is $[-1,3]$. Thus, combining this results, the parameter region for (λ_1,λ_2) appears as shown in the Figure 1.

By considering this parameter set of (λ_1,λ_2), many well defined distributions are generated from the eq. (2) with the baseline distribution F. Now, let's get a map of the integer values of the pair (λ_2,λ_1) to see the known distributions tabulated in Table 1.

The distributions specified by the star in Table 1 are described below how they correspond to some known failure distributions.

Let $X_{r:n}$ be the rth order statistic in a sample of size n. By noting that, for $\lambda_1=-1$, $\lambda_2=-1$ generated distribution indicates the failure distribution of the lifetime of three-component parallel system, namely, this distribution indicates the distribution of the random variable $X_{3:3} = \max\{X_1,X_2,X_3\}$ where X_1, X_2 and X_3 are independent and identically distributed as F. Similarly, for $\lambda_1=2$, $\lambda_2=-1$ generated distribution indicates the distribution of the random variable $X_{1:3} = \min\{X_1,X_2,X_3\}$. For $\lambda_1=-1$, $\lambda_2=1$ generated distribution indicates the distribution of $\max\{X_1,\min\{X_2,X_3\}\}$. For $\lambda_1=0$, $\lambda_2=1$ generated distribution indicates the distribution of $\min\{X_1,\max\{X_2,X_3\}\}$. For $\lambda_1=-1$, $\lambda_2=2$ generated distribution indicates the failure distribution of the lifetime of the three-out-of- two system, namely, this distribution indicates the distribution of the random

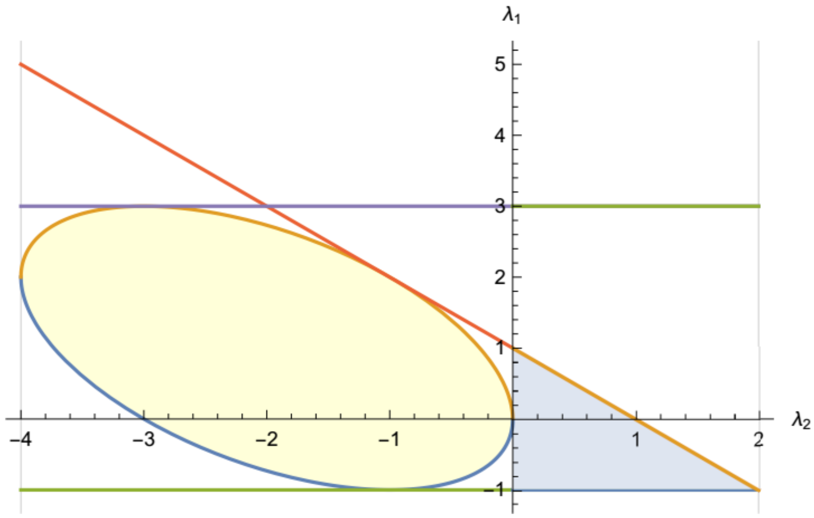

FIGURE 1. Valid parameter set (ellipsoid and triangle on the right-side)

TABLE 1. Some generated distributions according to special cases for the parameter values

λ_2	λ_1	Some Generated Distributions	λ_2	λ_1	Some Generated Distributions
-4	2	$4F^3 - 6F^2 + 3F$	-1	0	$F^3 - F^2 + F$
-3	0	$3F^3 - 3F^2 + F$	-1	1	$F^3 - 2F^2 + 2F$
-3	1	$3F^3 - 4F^2 + 2F$	-1	2	$F^3 - 3F^2 + 3F*$
-3	2	$3F^3 - 5F^2 + 3F$	0	-1	F^2*
-3	3	$3F^3 - 6F^2 + 4F$	0	0	$F*$
-2	0	$2F^3 - 2F^2 + F$	0	1	$2F - F^2*$
-2	1	$2F^3 - 3F^2 + 2F$	1	-1	$2F^2 - F^3*$
-2	2	$2F^3 - 4F^2 + 3F$	1	0	$F^2 + F - F^3*$
-1	-1	F^3*	2	-1	$3F^2 - 2F^3*$

variable $X_{2:3} = \max\{\min\{X_1, X_2\}, \min\{X_1, X_3\}, \min\{X_2, X_3\}\}$. On the other hand, for $\lambda_1 = -1$, $\lambda_2 = 0$ generated distribution indicates the failure distribution of the lifetime of the two-component parallel system, namely distribution of $X_{2:2} = \max\{X_1, X_2\}$. For $\lambda_1 = 1$, $\lambda_2 = 0$ generated distribution indicates the failure distribution of the lifetime of the two-component series system, namely distribution of $X_{1:2} = \min\{X_1, X_2\}$.
In this case, in addition to the known distributions introduced by the quadratic transmutation, more informative distribution functions occure. However, the set of

the transformation parameters of the proposed cubic transmutation is still complicated.

In order to eliminate of this complexity, by referring to the concept of reliability evaluation of coherent system by using signature (see, [6,7]), we come up with an idea inspired by both works of [16,18] as follows:

$$\Pr\left(X_{2:2} \leq t\right) = \Pr\left(\max\left\{X_1, X_2\right\} \leq t\right) = F^2\left(t\right)$$

and

$$\Pr\left(X_{1:2} \leq t\right) = \Pr\left(\min\left\{X_1, X_2\right\} \leq t\right) = 2F\left(t\right) - F^2\left(t\right)$$

where $F\left(t\right)$ indicates the failure distribution of the component lifetime, namely, $\Pr\left(X_1 \leq t\right) = F\left(t\right)$. Hence there exists a stochastic ordering relation such as $X_{1:2} \prec_{st} X \prec_{st} X_{2:2}$. In this case, these three failure distributions can be ordered as $F^2\left(t\right) \leq F\left(t\right) \leq 2F\left(t\right) - F^2\left(t\right)$. From the latter inequality, we can say that $F(t)$ is represented by a convex combination of $2F\left(t\right) - F^2\left(t\right)$ and $F^2\left(t\right)$ where the value of the combination parameter is $\frac{1}{2}$. On the other hand, it is possible to obtain many distributions besides F. Let G stand for the distribution obtained by this convex combination. Then, for $\delta \in [0,1]$, we have

$$\begin{aligned} G\left(t\right) &= \delta\left(2F\left(t\right) - F^2\left(t\right)\right) + \left(1-\delta\right)\left(F^2\left(t\right)\right) \\ &= 2\delta F\left(t\right) + \left(1-2\delta\right)F^2\left(t\right) \end{aligned} \tag{9}$$

Here, the combination parameter is reparametrized by taking $\delta = \frac{1+\lambda}{2}$ to attain quadratic rank transmutation. Now, the new parameter λ takes the values in $[\,-1,1]$. As can be seen immediately, $\lambda = 0$ corresponds to $\delta = \frac{1}{2}$. In eq. (9), substituting δ by λ, we have

$$G\left(t\right) = \left(1+\lambda\right)F\left(t\right) - \lambda F^2\left(t\right). \tag{10}$$

The above expression is the quadratic rank trasmutation proposed by [16]. Now, we concentrate on 3-component systems with similar thinking. Let X_1, X_2 and X_3 be independent random variables distributed as F. Let $X_{r:3}$ denote rth order statistic of $\left(X_1, X_2, X_3\right)$ with corresponding distribution $F_{r:3}$. Then we have

$$F_{3:3}\left(t\right) = \Pr\left(X_{3:3} \leq t\right) = \Pr\left(\max\left\{X_1, X_2, X_3\right\} \leq t\right) = F^3\left(t\right) \tag{11}$$

$$\begin{aligned} F_{2:3}\left(t\right) &= \Pr\left(X_{2:3} \leq t\right) \\ &= \Pr\left(\max\left\{\min\left\{X_1, X_2\right\}, \min\left\{X_1, X_3\right\}, \min\left\{X_2, X_3\right\}\right\} \leq t\right) \\ &= 3F^2\left(t\right) - 2F^3\left(t\right) \end{aligned} \tag{12}$$

$$\begin{aligned} F_{1:3}\left(t\right) &= \Pr\left(X_{1:3} \leq t\right) = \Pr\left(\min\left\{X_1, X_2, X_3\right\} \leq t\right) \\ &= 3F\left(t\right) - 3F^2\left(t\right) + F^3\left(t\right). \end{aligned} \tag{13}$$

According to [18], the properties $F_{3:3} \leq F_{2:3} \leq F_{1:3}$ and $F = \frac{1}{3}F_{3:3} + \frac{1}{3}F_{2:3} + \frac{1}{3}F_{1:3}$. are hold. In other words, F can be represented by a convex combination of $F_{1:3}$, $F_{2:3}$ and $F_{3:3}$. On the other hand, there is also an ordering for F such that $F_{3:3} \leq F \leq F_{1:3}$. If $F_{2:3}$ is also included in this ordering, we have for $F \geq \frac{1}{2}$, $F_{3:3} \leq F \leq F_{2:3} \leq F_{1:3}$ and for $F < \frac{1}{2}$, $F_{3:3} \leq F_{2:3} \leq F \leq F_{1:3}$. Hence, we can suggest a convex combination to cover both ordering situations. Our aim is to determine exactly where F is. In this case, we can write the following convex combination obtained by $F_{1:3}$ and $F_{2:3}$, called as G^*.

$$G^* = \delta_1 F_{1:3} + (1-\delta_1) F_{2:3} \tag{14}$$

where $\delta_1 \in [0,1]$. Now, let's write a convex combination between G^* and $F_{3:3}$. Denoting this convex combination by G, we have

$$G = \delta_2 G^* + (1-\delta_2) F_{3:3} \tag{15}$$

where $\delta_2 \in [0,1]$. Combining with the equations (14) and (15), we obtain G as

$$G = \delta_2\delta_1 F_{1:3} + \delta_2 (1-\delta_1) F_{2:3} + (1-\delta_2) F_{3:3}. \tag{16}$$

If the notation F is used for the representation of $F_{r:3}$, and rearranging with respect to polynomial degree of F, the following expression is obtained:

$$\begin{aligned} G &= \delta_2\delta_1 \left(3F - 3F^2 + F^3\right) + \delta_2 (1-\delta_1)\left(3F^2 - 2F^3\right) + (1-\delta_2) F^3 \\ &= 3\delta_1\delta_2 F + 3\delta_2 (1-2\delta_1) F^2 + (1 - 3\delta_2 + 3\delta_1\delta_2) F^3 \end{aligned} \tag{17}$$

Undoubtedly, G is a distribution function. However, reparameterization is made on the model in order to achieve the similar structure of the quadratic rank transmutation. Now, by taking $w_1 = \delta_1\delta_2$ and $w_2 = \delta_2 - \delta_1\delta_2$, eq. (17) can be rewritten as follows:

$$G = 3w_1 F + 3(w_2 - w_1) F^2 + (1-3w_2) F^3 \tag{18}$$

where $w_1, w_2 \in [0,1]$. In eq. (18), by the reparametrizating as $w_1 = \frac{1+\lambda_1}{3}$ and $w_2 = \frac{1+\lambda_2}{3}$, we have

$$G = (1+\lambda_1) F + (\lambda_2 - \lambda_1) F^2 - \lambda_2 F^3 \tag{19}$$

where $\lambda_1, \lambda_2 \in [\,-1,2]$. Since $\delta_2 = w_1 + w_2$, the parameter set is also constrained by the condition $\lambda_1 + \lambda_2 \leq 1$. Consequently, the parameter set of λ_1 and λ_2 is presented in a simpler form than the parameter region given in Figure 1. This transmutation defined in eq. (19) is called as cubic rank transmutation and transformed distribution G is named as CRT-F.

As can be seen immediately, CRT-F defines a quadratic rank transmuted distribution at $\lambda_2 = 0$, and $\lambda_1 = \lambda_2 = 0$ gives the baseline distribution F. For this reason, CRT-F can be seen as a generalized form of QRT. The parameter set of λ_1 and λ_2, which is defined as $\{(\lambda_1,\lambda_2) : \lambda_1, \lambda_2 \in [-1,2], \lambda_1 + \lambda_2 \leq 1\}$ can be figure out in Figure 2.Now, referring to the integer values of λ_1 and λ_2, we can determine the generated distribution functions by the Table 2.

Identifications given in Table 2 show that *Table 1* of [17] is included by CRT-F according to special choices of transmutation parameters.

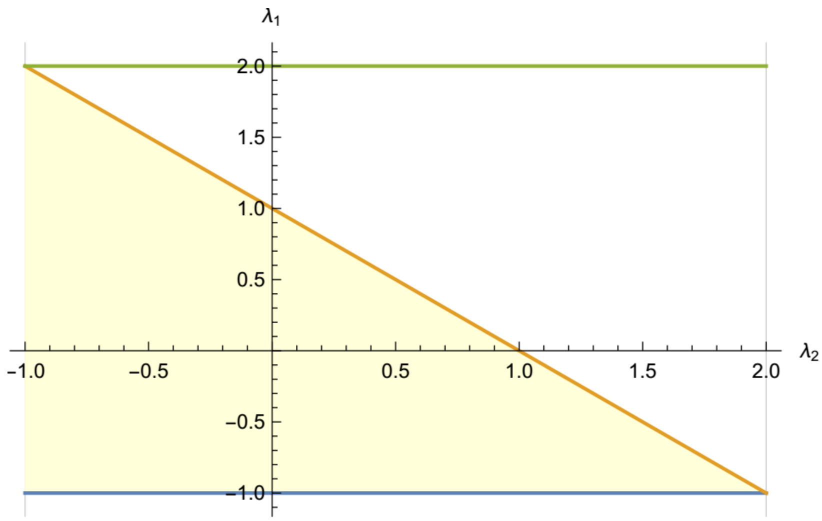

FIGURE 2. Region of the parameter set of CRT-F

TABLE 2. Identifications of CRT-F distribution for special values of transmutation parameters

λ_1	λ_2	CRT-F	Identification
-1	-1	F^3	Distribution of $T_{3:3}$
-1	0	F^2	Distribution of $T_{2:2}$
-1	1	$2F^2 - F^3$	Distribution of $\max\{X_1, \min\{X_2, X_3\}\}$
-1	2	$3F^2 - 2F^3$	Distribution of $T_{2:3}$
0	-1	$F^3 - F^2 + F$	$\frac{1}{3}F_{1:3} + \frac{2}{3}F_{3:3}$
0	0	F	Baseline Distribution
0	1	$F^2 + F - F^3$	Distribution of $\min\{X_1, \max\{X_2, X_3\}\}$
1	-1	$F^3 - 2F^2 + 2F$	$\frac{2}{3}F_{1:3} + \frac{1}{3}F_{3:3}$
1	0	$2F - F^2$	Distribution of $T_{1:2}$
2	-1	$F^3 - 3F^2 + 3F$	Distribution of $T_{1:3}$

Note that, by taking into account the parameter set of (19), the distribution family CRT-F is different as compared with the families proposed by [8, 14].

[8] proposed a cubic rank transmuted distribution family motivated by a study of [17]. The paper contained one theorem (referred to as Theorem 2.1), deriving cubic transmuted distribution. Here, We would like to point out that the result of Theorem 2.1 can be reduced to an explicit and understandable form.

Parameter space of eq. (3) of Theorem 2.1 given by [8] needs to be revised according to mixture probabilities π_i $(i = 1, 2, 3)$. Otherwise, eq. (3) does not provide the distribution function in some cases of λ_1 and λ_2. For instance, suppose that $\lambda_1 = 0$ and $\lambda_2 = -1$ in eq. (3), then we have

$$F(x) = -G^2(x) + 2G^3(x) = G^2(x)(2G(x) - 1).$$

Thus, the function $F(x)$ is positively signed when x is greater than $G^{-1}(1/2)$ which is the median of $G(x)$. Otherwise, $F(x)$ is negatively signed. More generally, according to eq. (3), we have the second-order convex polynomial as $\varphi(u) = \lambda_1 + (\lambda_2 - \lambda_1)u + (1 - \lambda_2)u^2$. After some algebraic manipulation, we say that having positively sign of $\varphi(u)$ depends on $\lambda_1 \geq \frac{(\lambda_1+\lambda_2)^2}{4}$. Furthermore, in proof of Theorem 2.1, one condition is missing which comes with $0 \leq \pi_1 + \pi_2 \leq 1$. To clarify the above claims, recall the distribution of order statistics associated with sample size of 3:

$$\begin{aligned}
\Pr(X_{1:3} \leq x) &= 1 - \Pr(X_{1:3} > x) = 1 - (1 - F(x))^3 = 3F(x) - 3F^2(x) + F^3(x), \\
\Pr(X_{2:3} \leq x) &= 3F(x)^2 - 2F^3(x), \\
\Pr(X_{3:3} \leq x) &= F^3(x).
\end{aligned}$$

Now, a new random variable T is defined by mixing the above order statistics as follows:

$$T = \begin{cases} X_{1:3}, & \text{with probability } \pi_1 \\ X_{2:3}, & \text{with probability } \pi_2 \\ X_{3:3}, & \text{with probability } \pi_3, \end{cases}$$

where $\pi_1 + \pi_2 + \pi_3 = 1$. Then the distribution of T is as follows:

$$\begin{aligned}
\Pr(T \leq t) &= \pi_1\left(3F(t) - 3F^2(t) + F^3(t)\right) + \pi_2\left(3F^2(t) - 2F^3(t)\right) + \pi_3\left(F^3(t)\right) \\
&= (3\pi_1)F(t) + 3(\pi_2 - \pi_1)F^2(t) + (\pi_1 - 2\pi_2 + \pi_3)F^3(t).
\end{aligned}$$

By letting $\pi_3 = 1 - \pi_1 - \pi_2$ then we have

$$\Pr(T \leq t) = 3\pi_1 F(t) + 3(\pi_2 - \pi_1)F^2(t) + (1 - 3\pi_2)F^3(t).$$

Since $\pi_i \in [0, 1]$, $(i = 1, 2, 3)$, appropriate parameterization for π_1 and π_2, can be taken into account as $3\pi_1 = 1 + \lambda_1$, and $3\pi_2 = 1 + \lambda_2$. New parameters are both in the interval $[-1, 2]$. Recalling the condition $0 \leq \pi_1 + \pi_2 \leq 1$, hence we have $-2 \leq \lambda_1 + \lambda_2 \leq 1$. Accordingly, latter probability is as follows:

$$\Pr(T \leq t) = (1 + \lambda_1)F(t) + (\lambda_2 - \lambda_1)F^2(t) - \lambda_2\, F^3(t),$$

with $\lambda_1, \lambda_2 \in [-1, 2]$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$. It should be noted that this is also obtained by (19).

Therefore, based on the assumptions in proof of Theorem 2.1., eq. (3) cannot be obtained. Also, comparing with the family introduced by [14], they give the similar cubic transmuted distribution family with a narrower parameter space as $\lambda_1, \lambda_2 \in [-1, 1]$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$.

3. Conclusion

In this article, we propose a new version of polynomial rank transmutation. Since the parameter set is still complex, a new cubic rank transmutation is introduced in the light of the idea behind QRTM technique. Compared to the two techniques in the literature, it is seen that the proposed technique covers them in terms of parameter space.

References

[1] Abd El Hady, N. E., Exponentiated transmuted weibull distribution a generalization of the weibull distribution, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 8* (6) (2014), 903 – 911.

[2] Alizadeh, M., Merovci, F., Hamedani, G. G., Generalized transmuted family of distributions: properties and applications, *Hacettepe Journal of Mathematics and Statistics, 46* (2017), 645–667.

[3] Ashour, S. K., Eltehiwy, M. A., Transmuted exponentiated lomax distribution, *Aust J Basic Appl Sci, 7* (7) (2013), 658–667.

[4] Das, K. K., Barman, L., On some generalized transmuted distributions, *Int. J. Sci. Eng. Res, 6* (2015), 1686–1691.

[5] Eltehiwy, M., Ashour, S., Transmuted exponentiated modified weibull distribution, *International Journal of Basic and Applied Sciences, 2* (3) (2013), 258–269.

[6] Eryilmaz, S., Computing reliability indices of repairable systems via signature, *Journal of Computational and Applied Mathematics, 260* (2014), 229 – 235, https://dx.doi.org/https://doi.org/10.1016/j.cam.2013.09.023.

[7] Franko, C., Ozkut, M., Kan, C., Reliability of coherent systems with a single cold standby component, *Journal of Computational and Applied Mathematics, 281* (2015), 230 – 238, https://dx.doi.org/https://doi.org/10.1016/j.cam.2014.12.029.

[8] Granzotto, D., Louzada, F., Balakrishnan, N., Cubic rank transmuted distributions: inferential issues and applications, *Journal of Statistical Computation and Simulation, 87* (14) (2017), 2760–2778.

[9] Mansour, M. M., Enayat, M. A., Hamed, S. M., Mohamed, M. S., A new transmuted additive weibull distribution based on a new method for adding a parameter to a family of distributions, *Int. J. Appl. Math. Sci, 8* (2015), 31–51.

[10] Mansour, M. M., Mohamed, S. M., A new generalized of transmuted lindley distribution, *Appl. Math. Sci, 9* (2015), 2729–2748.

[11] Merovci, F., Transmuted exponentiated exponential distribution, *Mathematical Sciences and Applications E-Notes, 1* (2) (2013), 112–122.

[12] Nofal, Z. M., Afify, A. Z., Yousof, H. M., Cordeiro, G. M., The generalized transmuted-g family of distributions, *Communications in Statistics-Theory and Methods, 46* (8) (2017), 4119–4136.

[13] Nofal, Z. M., El Gebaly, Y. M., The generalized transmuted weibull distribution for lifetime data, *Pakistan Journal of Statistics and Operation Research, 13* (2) (2017), 355–378.

[14] Rahman, M. M., Al-Zahrani, B., Shahbaz, M. Q., A general transmuted family of distributions, *Pakistan Journal of Statistics and Operation Research, 14* (2) (2018).

[15] Saboor, A., Khan, M. N., Cordeiro, G. M., Pascoa, M. A., Ramos, P. L., Kamal, M., Some new results for the transmuted generalized gamma distribution, *Journal of Computational and Applied Mathematics, 352* (2019), 165 – 180, https://dx.doi.org/https://doi.org/10.1016/j.cam.2018.12.002.

[16] Shaw, W. T., Buckley, I. R. C., The alchemy of probability distributions: Beyond gram-charlier & cornish-fisher expansions, and skew-normal or kurtotic-normal distributions, *Submitted, Feb, 7* (2007), 64.

[17] Shaw, W. T., Buckley, I. R. C., The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, *ArXiv e-prints* (Jan. 2009).

[18] Yilmaz, M., Potas, N., Topcu, B., Reliability properties of systems with three dependent components, In *Chaos, Complexity and Leadership 2013*, Springer, 2015, pp. 117–127.

https://communications.science.ankara.edu.tr

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INVERSE CONTINUOUS WAVELET TRANSFORM IN WEIGHTED VARIABLE EXPONENT AMALGAM SPACES

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Abstract. The wavelet transform is an useful mathematical tool. It is a mapping of a time signal to the time-scale joint representation. The wavelet transform is generated from a wavelet function by dilation and translation. This wavelet function satisfies an admissible condition so that the original signal can be reconstructed by the inverse wavelet transform. In this study, we firstly give some basic properties of the weighted variable exponent amalgam spaces. Then we investigate the convergence of the $\theta-$means of f in these spaces under some conditions. Finally, using these results the convergence of the inverse continuous wavelet transform is considered in these spaces.

1. Introduction

Recently, the variable exponent Lebesgue $L^{p(\cdot)}(\mathbb{R}^d)$ spaces and a class of nonlinear problems with variable exponential growth have been new and interesting topics. The space has several applications, such as electrorheological fluids (see [31]), elastic mechanics (see [43]) and image processing model. Moreover, the spaces $L^{p(\cdot)}(\mathbb{R}^d)$ and $L^{p}(\mathbb{R}^d)$ have many common properties, such as Banach space, reflexivity, separability, uniform convexity, Hölder inequalities and embeddings. One of the most important differences between these spaces is that the space $L^{p(\cdot)}(\mathbb{R}^d)$ is not translation invariant [27]. It is also well known that the maximal operator is bounded in $L^{p(\cdot)}(\mathbb{R}^d)$. For more comprehensive information (see [10], [12], [13] and [14]).

The amalgam of L^p and l^q on the real line is the space (L^p, l^q), which is also larger than the space L^p, consisting of functions which are locally in L^p and have l^q

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behavior at infinity. Many different forms of amalgam spaces have been studied by some authors (see [25], [33], [24], [15] and [18]). Moreover, this space play important roles in recent developments in time frequency analysis and sampling theory, which are modern branches of harmonic analysis. Signal analysis and wireless communication issues are quite popular in amalgam spaces (see [20]).

Variable exponent amalgam spaces $\left(L^{p(\cdot)}, l^q\right)$ and some basic properties, such as Banach function space, Hölder type inequalities, interpolation, bilinear multipliers and the boundedness of maximal operator, have been investigated recently. Some interesting articles have been published on this subject, but not many. So there are many open problems in this function spaces [5], [21], [26], [30], [22], [28], [3], [7], [2], [6].

The so called θ-summation method is investigated by some authors, such as [36], [32], [38], [39], [40], [34], [8]. The θ-summation is defined by

$$\sigma_T^\theta f(x) = \int\limits_{\mathbb{R}^d} f(x-t)\, T^n \theta(Tt)\, dt$$

for an integrable function θ on $\mathbb{R}$. This summability is a generalized form of the well-known summability methods, like Fejér, Riesz, Weierstrass, Abel, etc. by a suitable chosen of θ. Feichtinger and Weisz ([16], [17], [42]) showed that the θ-means $\sigma_T^\theta f$ converges to f almost everywhere and in norm as $T \to \infty$ for $f \in L^p(\mathbb{R}^d)$, (L^p, l^q). Also we characterize the points of the set of a.e. convergence as the Lebesgue points. Moreover, Uribe and Fiorenza [10], Szarvas and Weisz [34] obtained similar results for the space $L^{p(\cdot)}\left(\mathbb{R}^d\right)$.

In this study we will discuss the convergence of the inverse continuous wavelet transform in weighted variable exponent amalgam spaces. Also, we investigate the convergence of the θ-means of f almost everywhere and in norm in these spaces under which conditions. Hence we obtain more general results with respect to [34].

2. Weighted Variable Exponent Lebesgue and Amalgam spaces

In this section we give some required definitions and information about wavelet transform and weighted variable exponent amalgam spaces.

Definition 1. *Let $x \in \mathbb{R}^d$, $s \in \mathbb{R}$ and $s \neq 0$. The continuous wavelet transform is defined by*

$$W_g f(x,s) = |s|^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d} f(t)\, \overline{g\left(s^{-1}(t-x)\right)} dt = \langle f, T_x D_s g \rangle$$

for f and g, where D_s is the dilation operator, and T_x is the translation operator, i.e.,

$$D_s f(t) = |s|^{-\frac{d}{2}} f\left(\frac{t}{s}\right) \text{ and } T_x f(t) = f(t-x) \quad \left(x, t \in \mathbb{R}^d,\ 0 \neq s \in \mathbb{R}\right)$$

[11], [19]. If η is radial, non-increasing as a function on $(0,\infty)$, non-negative, bounded, $|f| \leq \eta$ and $\eta \in L^1(\mathbb{R}^d)$, then η is a radial majorant of f. If in addition $\eta(.)\ln(|.|+2) \in L^1(\mathbb{R}^d)$, then η is a radial log-majorant of f.

Definition 2. *A point $x \in \mathbb{R}^d$ is called a Lebesgue point $f \in L^1_{loc}(\mathbb{R}^d)$ if*

$$\underset{h\to 0+}{lim}\left(\frac{1}{|B(0,h)|}\int\limits_{B(0,h)}|f(x+u)-f(x)|\,du\right)=0,$$

where

$$B(a,\delta)=\left\{x\in\mathbb{R}^d:\|x-a\|<\delta\right\}.$$

Definition 3. *Let $g^*(x)=\overline{g(-x)}$ be involution operator. Then the operators $\rho_S f$ and $\rho_{S,T}f$ are defined by*

$$\rho_S f=\int\limits_S^\infty\int\limits_{\mathbb{R}^d}W_g f(x,s)\,T_x D_s\gamma\frac{dxds}{s^{d+1}}$$

and

$$\rho_{S,T} f=\int\limits_S^T\int\limits_{\mathbb{R}^d}W_g f(x,s)\,T_x D_s\gamma\frac{dxds}{s^{d+1}},$$

where $0<S<T<\infty$. Let define the operator $C'_{g,\gamma}$ with

$$C'_{g,\gamma}=-\int\limits_{\mathbb{R}^d}(g^**\gamma)(x)\ln(|x|)\,dx.$$

Then $C'_{g,\gamma}$ is finite [29], where g and γ both have radial log-majorants.

Let g and γ be radial, i.e., $\int_{\mathbb{R}^d}(g^**\gamma)(x)\,dx=0$. Assume that g and γ have a radial log-majorant. Then we get

$$\underset{S\to 0^+,T\to\infty}{lim}\rho_{S,T}f(x)=\underset{S\to 0^+}{lim}\rho_S f(x)=C'_{g,\gamma}f(x)$$

at every Lebesgue point for any $f\in L^p(\mathbb{R}^d)$ $(1\leq p<\infty)$. The convergence is proved with respect to L^p-norm for $T=\infty$, [29]. Under some similar conditions, Weisz has proved similar results [41].

Definition 4. *Let $p(.)$ be a measurable function from $\mathbb{R}^d$ into $[1,\infty)$ (called a variable exponent on $\mathbb{R}^d$) satisfying the condition $1\leq p^-\leq p(.)\leq p^+<\infty$, where*

$$p^-=\underset{x\in\mathbb{R}^d}{ess\inf}p(x),\quad p^+=\underset{x\in\mathbb{R}^d}{ess\sup}p(x).$$

The set $P(\mathbb{R}^d)$ denotes variable exponents on $\mathbb{R}^d$. Let $p(.) \in P(\mathbb{R}^d)$. The variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$ consist of all measurable functions f such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}(\frac{f}{\lambda}) \leq 1\right\},$$

where

$$\varrho_{p(.)}(f) = \int_{\mathbb{R}^d} |f(x)|^{p(x)}\, dx.$$

If $p^+ < \infty$, then $f \in L^{p(\cdot)}(\mathbb{R}^d)$ iff $\varrho_{p(.)}(f) < \infty$. The space $\left(L^{p(\cdot)}(\mathbb{R}^d), \|.\|_{p(.)}\right)$ is a Banach space. If $p(.) = p$ is a constant function, then the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$, [27]. A measurable and locally integrable function $\omega : \mathbb{R}^d \longrightarrow (0, \infty)$ is called a weight function. The weighted modular is defined by

$$\varrho_{p(.),\omega}(f) = \int_{\mathbb{R}^d} |f(x)|^{p(x)}\, \omega(x) dx.$$

The space $L^{p(\cdot)}_\omega(\mathbb{R}^d)$ is of all measurable functions such that $\|f\|_{L^{p(\cdot)}_\omega(\mathbb{R}^d)} = \left\|f\omega^{\frac{1}{p(\cdot)}}\right\|_{p(.)} < \infty$. The dual space of $L^{p(\cdot)}_\omega(\mathbb{R}^d)$ is $L^{q(\cdot)}_{\omega^*}(\mathbb{R}^d)$, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ and $\omega^* = \omega^{1-q(.)} = \omega^{-\frac{1}{p(.)-1}}$.Also, $L^{p(\cdot)}_\omega(\mathbb{R}^d)$ is a uniformly convex Banach space, thus reflexive for $1 < p^- \leq p(.) \leq p^+ < \infty$, [3], [4].

Definition 5. *The maximal operator M is defined by*

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

for $f \in L^1_{loc}(\mathbb{R})$.

Hästö and Diening [23] defined the class $A_{p(.)}$ consists of those weights ω such that

$$\|\omega\|_{A_{p(.)}} = \sup_{B\in ß} |B|^{-p_B} \|\omega\|_{L^1(B)} \left\|\frac{1}{\omega}\right\|_{L^{\frac{p'(.)}{p(.)}}(B)} < \infty,$$

where ß denotes the set of all balls in $\mathbb{R}^d$, $p_B = \left(\frac{1}{|B|}\int_B \frac{1}{p(x)} dx\right)^{-1}$ and $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$. If $p(.)$ is a constant function, then $A_{p(.)} = A_p$, where A_p is ordinary Muckenhoupt class.

If $p(.)$ satisfies the following inequality

$$|p(x) - p(y)| \leq \frac{C}{\log\left(e + \frac{1}{|x-y|}\right)}$$

for all $x, y \in \mathbb{R}^d$, then $p(.)$ provides the local log-Hölder continuity condition. Moreover, if the inequality

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}$$

holds for some $p_\infty > 1$, $C > 0$ and all $x \in \mathbb{R}^d$, then we say that $p(.)$ satisfies the local log-Hölder decay condition. We denote by $P^{\log}(\mathbb{R}^d)$ the class of variable exponents which are log-Hölder continuous, i.e. which satisfy the local log-Hölder continuity condition and local log-Hölder decay condition [4], [37].

Let $p \in P^{\log}(\mathbb{R}^d)$ and $1 < p^- \leq p(.) \leq p^+ < \infty$. Then $M : L^{p(\cdot)}_\omega(\mathbb{R}^d) \hookrightarrow L^{p(\cdot)}_\omega(\mathbb{R}^d)$ if and only if $\omega \in A_{p(.)}$ by Theorem 1.1 in [23].

The space $L^{p(.)}_{loc,\omega}(\mathbb{R}^d)$ is to be space of functions on $\mathbb{R}^d$ such that f restricted to any compact subset K of $\mathbb{R}^d$ belongs to $L^{p(.)}_w(\mathbb{R}^d)$.

In this study we take $d = 1$, and define the weighted variable exponent amalgam spaces on $\mathbb{R}$.

Definition 6. *Let $1 \leq p(.), q < \infty$ and $J_k = [k, k+1)$, $k \in \mathbb{Z}$. The weighted variable exponent amalgam spaces $\left(L^{p(\cdot)}_\omega, l^q\right)$ are defined by*

$$\left(L^{p(\cdot)}_\omega, l^q\right) = \left\{ f \in L^{p(.)}_{loc,\omega}(\mathbb{R}) : \|f\|_{\left(L^{p(\cdot)}_\omega, l^q\right)} < \infty \right\},$$

where $\|f\|_{\left(L^{p(\cdot)}_\omega, l^q\right)} = \left(\sum_{k\in\mathbb{Z}} \left\|f\chi_{J_k}\right\|^q_{L^{p(\cdot)}_\omega(\mathbb{R})}\right)^{\frac{1}{q}}$. If the weight ω is a constant function, then the space $\left(L^{p(\cdot)}_\omega, l^q\right)$ coincides with $\left(L^{p(\cdot)}, l^q\right)$ (see [7], [26]).

In 2014, Meskhi and Zaighum showed that the maximal operator is bounded in weighted variable exponent amalgam spaces under some conditions [30].

Throughout this paper, we assume that $p(.) \in P^{\log}(\mathbb{R})$, $1 < p^- \leq p(.) \leq p^+ < \infty$ and $\omega \in A_{p(.)}$.

3. $\theta-$Summability on the Weighted Variable Exponent Wiener Amalgam Spaces

Lemma 1. *Let $1 \leq p(.), q < \infty$ and $0 < c \leq \omega$. Then the inclusion $\left(L^{p(\cdot)}_\omega, l^q\right) \subset \left(L^1, l^\infty\right)$ holds.*

Proof. Take any $f \in \left(L^{p(\cdot)}_\omega, l^q\right)$. It is well known that $f \in \left(L^{p(\cdot)}_\omega, l^q\right)$ if and only if $\left\{\|f\|_{L^{p(\cdot)}_\omega[k,k+1)}\right\}_{k\in\mathbb{Z}} \in l^q$. If we use Proposition 3.5 in [3] and the definition of $\|.\|_{(L^1,l^\infty)}$, then we have $L^{p(.)}_\omega[k,k+1) \hookrightarrow L^1_\omega[k,k+1) \hookrightarrow L^1[k,k+1)$, $l^q \hookrightarrow l^\infty$ for $1 \leq p(.), q < \infty$, $0 < c \leq \omega$ and so

$$\|f\|_{(L^1,l^\infty)} = \sup_{k\in\mathbb{Z}} \|f\|_{L^1[k,k+1)} \leq C \sup_{k\in\mathbb{Z}} \|f\|_{L^{p(\cdot)}_\omega[k,k+1)}$$

$$\leq \quad C\left(\sum_{k\in\mathbb{Z}}\|f\|^{q}_{L^{p(\cdot)}_{\omega}[k,k+1)}\right)^{\frac{1}{q}} = C\,\|f\|_{\left(L^{p(\cdot)}_{\omega},l^{q}\right)} < \infty.$$

Hence we obtain that $f\in\left(L^{1},l^{\infty}\right)$ and $\left(L^{p(\cdot)}_{\omega},l^{q}\right)\subset\left(L^{1},l^{\infty}\right)$. □

Theorem 1. *Let $1\leq q\leq p^{-}\leq p\left(.\right)\leq p^{+}<\infty$ and $0<c\leq\omega$. Then the inclusion*

$$\left(L^{p(\cdot)}_{\omega},l^{q}\right)\hookrightarrow L^{q}_{\omega}\hookrightarrow L^{q},$$

hold for all $f\in\left(L^{p(\cdot)}_{\omega},l^{q}\right)$.

Proof. Let $f\in\left(L^{p(\cdot)}_{\omega},l^{q}\right)$ be given. Then we get $\left(L^{p(\cdot)}_{\omega},l^{q}\right)\hookrightarrow\left(L^{p^{-}}_{\omega},l^{q}\right)\hookrightarrow\left(L^{q}_{\omega},l^{q}\right)=L^{q}_{\omega}$ by Proposition 3.5 in [3] and [24]. Hence we have that there exists a $C>0$ such that the inequality

$$\|f\|_{L^{q}}\leq C\,\|f\|_{\left(L^{p(\cdot)}_{\omega},l^{q}\right)}$$

holds for any $f\in\left(L^{p(\cdot)}_{\omega},l^{q}\right)$. This completes the proof. □

Definition 7. *Let $\theta\in L^{1}\left(\mathbb{R}\right)$ be radial function. The θ-means of $f\in\left(L^{p(\cdot)}_{\omega},l^{q}\right)$ is defined by*

$$\sigma^{\theta}_{T}f\left(x\right):=\left(f*\theta_{T}\right)\left(x\right)=\int_{\mathbb{R}}f\left(x-t\right)\theta_{T}\left(t\right)dt,$$

where

$$\theta_{T}\left(t\right):=T^{d}\theta\left(Tt\right),\ \left(x\in\mathbb{R},\ T>0\right).$$

Theorem 2. *Let $1\leq p\left(.\right),q<\infty$ and $0<c\leq\omega$. Assume that θ has radial majorant. Then;*

i) *The limit*

$$\lim_{T\to\infty}\sigma^{\theta}_{T}f\left(x\right)=\int_{\mathbb{R}}\theta\left(y\right)dy.f(x)$$

is valid for any Lebesgue point of $f\in\left(L^{p(\cdot)}_{\omega},l^{q}\right)$.

ii) *If in addition $1\leq q\leq p^{-}\leq p\left(.\right)\leq p^{+}<\infty$, then the following limit equality*

$$\lim_{T\to 0^{+}}\sigma^{\theta}_{T}f\left(x\right)=0$$

is available for all $f\in\left(L^{p(\cdot)}_{\omega},l^{q}\right)$ and $x\in\mathbb{R}$.

Proof. **i)** Let $x \in \mathbb{R}$ be a Lebesgue point of f. Since there exists the inclusion $\left(L_{\omega}^{p(.)}, l^{q}\right) \subset \left(L^{1}, l^{\infty}\right)$ by Lemma 1, we write that

$$\underset{T\to\infty}{lim}\, \sigma_{T}^{\theta} f\left(x\right)=\int\limits_{\mathbb{R}^{n}} \theta\left(y\right) dy. f\left(x\right)$$

for $f \in \left(L_{\omega}^{p(.)}, l^{q}\right)$by Theorem 2.2 in [34].

ii) Take any $f \in \left(L_{\omega}^{p(.)}, l^{q}\right)$ and $x \in \mathbb{R}$. By Theorem 1 and Theorem 2.3 in [34], we have that $f \in L^{q}\left(\mathbb{R}\right)$ and

$$\underset{T\to 0^{+}}{lim}\, \sigma_{T}^{\theta} f\left(x\right)=0.$$

□

Proposition 1. $C_{c}\left(\mathbb{R}\right)$*, which consists of continuous functions on* $\mathbb{R}$ *whose support is compact, is dense in* $\left(L_{\omega}^{p(.)}, l^{q}\right)$ *for* $1 \leq p(.), q < \infty$ *(see Proposition 2.9 in [6]).*

Theorem 3. *For all* $f \in \left(L_{\omega}^{p(.)}, l^{q}\right)$ *the following statements are valid :*

i) $\left\|\sigma_{T}^{\theta} f\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)} \leq C\left\|f\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)}\ \left(T>0\right).$

ii) $\underset{T\to\infty}{lim}\, \sigma_{T}^{\theta} f=\int\limits_{\mathbb{R}} \theta\left(x\right) dx. f$ *in the* $\left(L_{\omega}^{p(.)}, l^{q}\right)$*-norm.*

iii) $\underset{T\to 0^{+}}{lim}\, \sigma_{T}^{\theta} f=0$ *in the* $\left(L_{\omega}^{p(.)}, l^{q}\right)$*-norm.*

Proof. **i)** It is well known that the maximal operator is bounded in $\left(L_{\omega}^{p(.)}, l^{q}\right)$ [30]. Then we have that

$$\left\|\sigma_{T}^{\theta} f\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)} \leq C\left\|f\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)}\ \left(T>0\right)$$

for all $f \in \left(L_{\omega}^{p(.)}, l^{q}\right)$ by Theorem 2.1 in [34].

ii) Also, if we follow Theorem 3.8 in [34], Theorem 2.3 in [9], Theorem 5.11 in [10], and Theorem 8 in [1], then we have that

$$\underset{T\to\infty}{lim}\, \sigma_{T}^{\theta} f=\int\limits_{\mathbb{R}} \theta\left(x\right) dx. f$$

in the $\left(L_{\omega}^{p(.)}, l^{q}\right)$-norm.

iii) Let $\epsilon>0$ be given. Using Proposition 1, it is obtained that the following inequality

$$\left\|f-g\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)}<\epsilon$$

is valid for $g \in C_c(\mathbb{R})$, whose compact support supp$g$ is K. Using i) and Proposition 2 in [7], we have that

$$\begin{aligned} \left\|\sigma_T^\theta f\right\|_{\left(L_\omega^{p(\cdot)},l^q\right)} &\leq \left\|\sigma_T^\theta (f-g)\right\|_{\left(L_\omega^{p(\cdot)},l^q\right)} + \left\|\sigma_T^\theta g\right\|_{\left(L_\omega^{p(\cdot)},l^q\right)} \\ &< C\epsilon + |S(K)|^{\frac{1}{q}} \left\|\sigma_T^\theta g\right\|_{L_\omega^{p(\cdot)}(K)}. \end{aligned}$$

Also using Theorem 3.8 in [34], we get the limit

$$\lim_{T\to 0^+} \left\|\sigma_T^\theta g\right\|_{L_\omega^{p(\cdot)}(K)} = 0.$$

So this completes the proof. □

4. Convergence of ρ_S and $\rho_{S,T}$

Theorem 4. *Assume that g,γ have radial log-majorants and $\int_{\mathbb{R}} (g^* * \gamma)(x)\,dx = 0$. If $\omega \in A_1$ and $0 < c \leq \omega$, then for all $f \in \left(L_\omega^{p(\cdot)}, l^q\right)$ the following relation holds;*

$$\rho_S f = \sigma_{\frac{1}{S}}^\theta f,\ (S>0)$$

where

$$\theta(y) = \int_1^\infty (g^* * \gamma)\left(\frac{y}{u}\right)\frac{1}{u^{n+1}}\chi_{B(0,1)}(y)\,du - \int_0^1 (g^* * \gamma)\left(\frac{y}{u}\right)\frac{1}{u^{n+1}}\chi_{\mathbb{R}^n\setminus B(0,1)}(y)\,du.$$

Proof. Let $f \in L_\omega^1 \cap \left(L_\omega^{p(\cdot)}, l^q\right)$ and $y \in \mathbb{R}$. Then we have decomposition of $\rho_S f(y)$ as

$$\begin{aligned} \rho_S f(y) &= \int_S^\infty \int_{\mathbb{R}} \frac{1}{s^3} \int_{\mathbb{R}} f(t)\, \overline{g\left(\frac{t-x}{s}\right)} \gamma\left(\frac{y-x}{s}\right) dt dx ds \\ &= \int_S^\infty \int_{|y-t|<S} \frac{1}{s^3} \int_{\mathbb{R}} f(t)\, \overline{g\left(\frac{t-x}{s}\right)} \gamma\left(\frac{y-x}{s}\right) dx dt ds \\ &\quad - \int_0^S \int_{|y-t|\geq S} \frac{1}{s^3} \int_{\mathbb{R}} f(t)\, \overline{g\left(\frac{t-x}{s}\right)} \gamma\left(\frac{y-x}{s}\right) dx dt ds \\ &\quad + \int_0^\infty \int_{|y-t|\geq S} \frac{1}{s^3} \int_{\mathbb{R}} f(t)\, \overline{g\left(\frac{t-x}{s}\right)} \gamma\left(\frac{y-x}{s}\right) dx dt ds \\ &= I - II + III \end{aligned}$$

by from [29], [34] . Also it is well known that

$$I=\left(f*\varphi_{\frac{1}{S}}\right)(y) \text{ and } II=\left(f*\psi_{\frac{1}{S}}\right)(y),$$

where

$$\varphi(t)=\int_{1}^{\infty}(g^{*}*\gamma)\left(\frac{t}{u}\right)\frac{1}{u^{n+1}}\chi_{B(0,1)}(t)\,du$$

and

$$\psi(t)=\int_{0}^{1}(g^{*}*\gamma)\left(\frac{t}{u}\right)\frac{1}{u^{n+1}}\chi_{\mathbb{R}^{n}\setminus B(0,1)}(t)\,du$$

by proof of Theorem 1.1 in [29]. On the other hand, Szarvas and Weisz proved that φ and ψ have radial majorants by Theorem 5.1 in [34] in case g and γ have radial log-majorants. Since g,γ have radial log-majorants, $f\in L_{\omega}^{1}$, $\omega\in A_{1}$ and

$$\int_{\mathbb{R}}(g^{*}*\gamma)(x)\,dx=0,$$

then we have

$$\begin{aligned} III &= \int_{0}^{\infty}\int_{|y-t|\geq S}\frac{1}{s^{3}}\int_{\mathbb{R}}f(t)\,\overline{g\left(\frac{t-x}{s}\right)}\gamma\left(\frac{y-x}{s}\right)dxdtds \\ &= \frac{1}{\omega_{0}}\int_{|y-t|\geq S}\frac{f(t)}{|y-t|}\int_{\mathbb{R}}(g^{*}*\gamma)(u)\,dudt=0 \end{aligned}$$

by Lemma 2.5 in [29]. Therefore we get

$$\begin{aligned} \rho_{S}f(y) &= \left(f*\varphi_{\frac{1}{S}}\right)(y)-\left(f*\psi_{\frac{1}{S}}\right)(y)+0 \\ &= f*\left(\varphi_{\frac{1}{S}}-\psi_{\frac{1}{S}}\right)(y)=f*\theta_{\frac{1}{S}}(y)=\sigma_{\frac{1}{S}}^{\theta}f(y), \end{aligned}$$

where

$$\begin{aligned} \theta(y) &= \varphi(y)-\psi(y) \\ &= \int_{1}^{\infty}(g^{*}*\gamma)\left(\frac{y}{u}\right)\frac{1}{u^{n+1}}\chi_{B(0,1)}(y)\,du-\int_{0}^{1}(g^{*}*\gamma)(y)\,\frac{1}{u^{n+1}}\chi_{\mathbb{R}^{n}\setminus B(0,1)}(y)\,du. \end{aligned}$$

If φ,ψ have radial majorants, then $\theta=\varphi-\psi$ have radial majorant, that is, θ is a non-negative and non-increasing function, and belongs to the space $L^{1}\cap L^{\infty}$. So it is obtained that

$$\|\theta\|_{(L^{\infty},l^{1})}=\sum_{k\in\mathbb{Z}}\left\|\theta\chi_{[k,k+1)}\right\|_{\infty}\leq\sum_{k\in\mathbb{Z}}\theta(k)<\infty$$

and $\theta\in\left(L^{\infty},l^{1}\right)$. Then using Hölder inequality and Lemma 1, we have

$$\begin{aligned} |\rho_S f(y)| &= \left|\sigma^{\theta}_{\frac{1}{S}} f(y)\right| \leq \frac{1}{S}\int\limits_{\mathbb{R}} |f(y-t)| \left|\theta\left(\frac{t}{S}\right)\right| dt \\ &\leq C\,\|f\|_{(L^1,l^\infty)}\,\|\theta\|_{(L^\infty,l^1)} \\ &\leq C\,\|f\|_{\left(L^{p(\cdot)}_\omega,l^q\right)}\,\|\theta\|_{(L^\infty,l^1)}\,. \end{aligned}$$

Hence the function ρ_S is linear and bounded from $L^1_\omega \cap \left(L^{p(\cdot)}_\omega, l^q\right)$ to $\mathbb{C}$. Also, it is well known that the inclusion $C_c \subset L^1_\omega \cap \left(L^{p(\cdot)}_\omega, l^q\right) \subset \left(L^{p(\cdot)}_\omega, l^q\right)$. Since C_c is dense in $\left(L^{p(\cdot)}_\omega, l^q\right)$ [6], then we find that $L^1_\omega \cap \left(L^{p(\cdot)}_\omega, l^q\right)$ is dense in $\left(L^{p(\cdot)}_\omega, l^q\right)$. Therefore, from the density principle, the function ρ_s is extended from $\left(L^{p(\cdot)}_\omega, l^q\right)$ to $\mathbb{C}$. This completes the proof. □

Theorem 5. *Let $f \in \left(L^{p(\cdot)}_\omega, l^q\right)$. Moreover, assume that g, γ have radial log-majorants and $\int_{\mathbb{R}} (g^* * \gamma)(x)\,dx = 0$. If $\omega \in A_1$ and $0 < c \leq \omega$, then*

i) $\lim\limits_{S\to 0^+} \rho_S f(x) = C'_{g,\gamma} f(x)$
for any Lebesgue point of the function f.
ii) If in addition $1 \leq q \leq p(.) \leq p^+ < \infty$, then
$\lim\limits_{S\to 0^+, T\to\infty} \rho_{S,T} f(x) = C'_{g,\gamma} f(x)$
for any Lebesgue point of the function f .

Proof. **i)** Since $p(.) \in P^{\log}(\mathbb{R})$ and $1 < p^- \leq p(.) \leq p^+ < \infty$, then $A_1 \subset A_{p(.)}$ [4]. By Theorem 2 and Theorem 4, we deduce that

$$\lim_{S\to 0^+} \rho_S f(x) = \lim_{S\to 0^+} \sigma^{\theta}_{\frac{1}{S}} f(x) = \int\limits_{\mathbb{R}} \theta(y)\,dy f(x)$$

for all Lebesgue points of $f \in \left(L^{p(\cdot)}_\omega, l^q\right)$. On the other hand, using Theorem 5.2 in [34] , we have that $\int_{\mathbb{R}} \theta(y)\,dy = C'_{g,\gamma}$ and

$$\lim_{S\to 0^+} \rho_S f(x) = C'_{g,\gamma} f(x)\,.$$

ii) By Theorem 5.2 in [34] we can write the equality $\rho_{S,T} f(x) = \rho_S f(x) - \rho_T f(x)$ for $x \in \mathbb{R}$. Then using (i), Theorem 2 and Theorem 4, we obtain that

$$\begin{aligned} \lim_{S\to 0^+, T\to\infty} \rho_{S,T} f(x) &= \lim_{S\to 0^+} \rho_S f(x) - \lim_{T\to\infty} \rho_T f(x) \\ &= \lim_{S\to 0^+} \sigma^{\theta}_{\frac{1}{S}} f(x) - \lim_{T\to\infty} \sigma^{\theta}_{\frac{1}{T}} f(x) \\ &= C'_{g,\gamma} f(x) - 0 = C'_{g,\gamma} f(x)\,. \end{aligned}$$

□

Corollary 1. *Assume that g, γ have radial log-majorants, $\int_{\mathbb{R}} (g^* * \gamma)(x)\, dx = 0$. If $\omega \in A_1$ and $0 < c \leq \omega$, then the following statements are valid for any $f \in \left(L_{\omega}^{p(.)}, l^q\right)$;*

i) $\underset{S\to 0^+}{lim} \rho_s f(x) = C'_{g,\gamma} f(x)$ *a.e.*

ii) *If in addition $1 \leq q \leq p(.) \leq p^+ < \infty$, then* $\underset{S\to 0^+, T\to\infty}{lim} \rho_{s,T} f(x) = C'_{g,\gamma} f(x)$ *a.e.*

Proof. Let $f \in \left(L_{\omega}^{p(.)}, l^q\right)$. Then by Lemma 1, we have $f \in \left(L^1, l^\infty\right)$. It is known that if $f \in \left(L^1, l^\infty\right)$, then real numbers almost everywhere is a Lebesgue point of f, [16], [17]. Hence by the Theorem 5, we complete the proof. □

Theorem 6. *Assume that g, γ have radial log-majorants and $\int_{\mathbb{R}} (g^* * \gamma)(x)\, dx = 0$. If $\omega \in A_1$ and $0 < c \leq \omega$, then the following results*

i) $\underset{S\to 0^+}{lim} \rho_s f = C'_{g,\gamma} f$,

ii) $\underset{S\to 0^+, T\to\infty}{lim} \rho_{s,T} f = C'_{g,\gamma} f$

are satisfied in the $\left(L_{\omega}^{p(.)}, l^q\right)$-norm for all $f \in \left(L_{\omega}^{p(.)}, l^q\right)$.

Proof. **i)** Using $\omega \in A_1 \subset A_{p(.)}$, Theorem 3 and Theorem 4, we have

$$\underset{S\to 0^+}{lim} \rho_S f = \underset{S\to 0^+}{lim} \sigma^{\theta}_{\frac{1}{S}} = \int_{\mathbb{R}} \theta(y)\, dy f$$

in the $\left(L_{\omega}^{p(.)}, l^q\right)$-norm for all $f \in \left(L_{\omega}^{p(.)}, l^q\right)$. On the other hand, since $\int_{\mathbb{R}} \theta(y)\, dy = C'_{g,\gamma}$, then we obtain that

$$\underset{S\to 0^+}{lim} \rho_S f = C'_{g,\gamma} f$$

in the $\left(L_{\omega}^{p(.)}, l^q\right)$-norm.

ii) Since $\rho_{S,T} f = \rho_S f - \rho_T f$, then we have that

$$\begin{aligned} \underset{S\to 0^+, T\to\infty}{lim} \rho_{s,T} f &= \underset{S\to 0^+}{lim} \rho_S f - \underset{T\to\infty}{lim} \rho_T f \\ &= \underset{S\to 0^+}{lim} \sigma^{\theta}_{\frac{1}{S}} f - \underset{T\to\infty}{lim} \sigma^{\theta}_{\frac{1}{T}} f = C'_{g,\gamma} f \end{aligned}$$

in the $\left(L_{\omega}^{p(.)}, l^q\right)$-norm by (i), Theorem 3 and Theorem 4. □

References

[1] Aydın, I., Unal, C., The Kolmogorov–Riesz theorem and some compactness criterions of bounded subsets in weighted variable exponent amalgam and Sobolev spaces, *Collectanea Mathematica*, (2019), https://doi.org/10.1007/s13348-019-00262-5, 1-19.

[2] Aydın, I., Unal, C., On some multipliers of vector-valued amalgam spaces, *Int. Journal of Pure and Appl. Math.*, 116 (2) (2017), 547-557.

[3] Aydın, I., On variable exponent amalgam spaces, *Analele Stiint. Univ.*, 20(3) (2012), 5-20.

[4] Aydın, I., Weighted variable Sobolev spaces and capacity, *J. Funct. Space Appl.*, Volume 2012, Article ID 132690, 17 pages, doi:10.1155/2012/132690.

[5] Aydın, I., Gürkanlı, A.T., Weighted variable exponent amalgam spaces $W(L^{p(x)}; L^q_w)$, *Glas. Mat.*, 47(67) (2012), 165-174.

[6] Aydın, I., Unal, C., Birkhoff's ergodic theorem for weighted variable exponent amalgam spaces, *Applications and Applied Mathematics: An International Journal (AAM)*, Special Issue No. 3 (2019), 1-10.

[7] Aydın, I., On vector-valued classical and variable exponent amalgam spaces, *Commun. Fac. Sci. Univ. Ank. Series A1* , 66 (2) (2017), 100-114.

[8] Butzer, P.L., Nessel, R.J., Fourier Analysis and Approximation, Academic Press, Newyork-London, Volume 1, 1971.

[9] Cruz-Uribe, D., Fiorenza, A., Approximate identities in variable Lp spaces, *Math. Nach.*, 280 (2007), 256-270.

[10] Cruz-Uribe, D., Fiorenza, A., Variable Lebesgue Spaces, Foundations and harmonic analysis, New York, NY, Birkhauser/Springer, 2013.

[11] Daubechies, I., Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.

[12] Diening, L., Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Mathematical Inequalities and Applications*, 7 (2004), 245-253.

[13] Diening, L., Harjulehto, P., Hästö, P., Ružicka, M., Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Math. 2017, Springer, Berlin, 2011.

[14] Fan, X., Zhao, D., On the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, 263 (2) (2001), 424-446.

[15] Feichtinger, H. G., Banach convolution algebras of Wiener type. In: Functions, Series, Operators, *Proc. Conf. Budapest 38, Colloq. Math. Soc., Janos Bolyai,* (1980),509–524.

[16] Feichtinger, H. G., Weisz, F., The Segal algebra $S_0(\mathbb{R}^d)$ and norm summability of Fourier series and Fourier Transforms, *Monatshefte Mth.*, 148 (2006), 333-349.

[17] Feichtinger, H. G., Weisz, F., Wiener amalgams and pointwise summability of Fourier Transforms and Fourier series, *Math. Proc. Camb. Phil. Soc.*, 140 (2006), 509-536.

[18] Fournier, J.J., Stewart, J., Amalgams of $L^p$and l^q, *Bull. Amer. Math. Soc.*, 13 (1985), 1–21.

[19] Gröchenig, K., Foundations of Time-Frequency Analysis, Birkhauser, Boston, 2001.

[20] Gröchenig, K., Heil, C., Okoudjou, K., Gabor analysis in weighted amalgam spaces, *Sampl. Theory Signal Image Process., Int. J.,* 1 (2002), 225-259.

[21] Gürkanlı, A.T., Aydın, I., On the weighted variable exponent amalgam space $W(L^{p(x)}; L^q_m)$, *Acta Math. Sci.*, 4(34B) (2014), 1–13.

[22] Gürkanlı, A.T., The amalgam spaces $W(L^{p(x)}; L^{\{p_n\}})$ and boundedness of Hardy-Littlewood maximal operators, *Current Trends in Analysis and Its Applications*, Proceedings of the 9th ISAAC Congress, Krakow (2013).

[23] Hästö, P., Diening, L., Muckenhoupt weights in variable exponent spaces, preprint, http://www.helsinki.fi/~pharjule/varsob/publications.shtml.

[24] Heil, C., An introduction to weighted Wiener amalgams. In: Wavelets and Their Applications, Allied Publishers, New Delhi, (2003), 183-216.

[25] Holland, F., Harmonic analysis on amalgams of $L^p$and l^q, *J. London Math. Soc.*, 2(10) (1975), 295–305.

[26] Kokilashvili, V., Meskhi, A., Zaighum, M.A., Weighted kernel operators in variable exponent amalgam spaces, *J. Inequal. Appl.*, (2013), DOI:10.1186/1029-242X-2013-173.

[27] Kovacik, O., Rakosnik, J., On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.*, 41(116) (1991), 592-618.

[28] Kulak, O., Gürkanlı, A.T., Bilinear multipliers of weighted Wiener amalgam spaces and variable exponent Wiener amalgam spaces, *J. Inequal. Appl.*, (2014), 476.

[29] Li, K., Sun, W., Pointwise convergence of the Calderon reproducing formula, *J. Fourier Anal. Appl.,* 18 (2012), 439-455.
[30] Meskhi, A., Zaighum, M.A., On the boundedness of maximal and potential operators in variable exponent amalgam spaces, *Journal of Mathematical Inequalities*, 8(1) (2014), 123–152.
[31] Ruzicka, M., Electrorheological fluids: modeling and mathematical theory, Springer-Verlag, Berlin, 2000.
[32] Simon, P., (C, α) summability of Walsh-Kaczmarz-Fourier Series, *J. Approx. Theory*, 127(1) (2004), 39-60.
[33] Squire, M.L.T., Amalgams of L^p and l^q, PhD, McMaster University, 1984.
[34] Szarvas, K., Weisz, F., Continuous wavelet transform in variable Lebesgue spaces, *Stud. Univ. Babeş-Bolyai Math.*, 59(4) (2014), 497-512.
[35] Szarvas, K., Variable Lebesgue spaces and continuous wavelet transforms, *Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis*, 32 (2016), 313-325.
[36] Trigub, R.M., Bellinsky, E.S., Fourier Analysis and Approximation of Function, Kluwer Academic Publishers, Dordrecht, 2004.
[37] Unal, C., Aydın, I., On some properties of the space $L_w^p(\mathbb{R}^n) \cap L_\vartheta^{q(.)}(\mathbb{R}^n)$, *Çankaya University Journal of Science and Engineering,* 13(2) (2016), 001-010.
[38] Weisz, F., Summability of multi-dimensional Fourier series and Hardy spaces, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 541, 2002.
[39] Weisz, F., Summability of multi-dimensional trigonometric Fourier series. Surv Approx Theory, 7 (2012), 1-179.
[40] Weisz, F., Convergence of the inverse continuous wavelet transform in Wiener amalgam spaces, *Journal of Analysis*, 35 (2015), 33–46.
[41] Weisz, F., Inversion formulas for the continuous wavelet transform, Acta. Math. Hungar., 138 (2013), 237-258.
[42] Weisz, F., Pointwise convergence in Pringsheim's sense of the summability of fourier transforms on Wiener amalgam spaces, *Monatsh Math.*, 175(1) (2014), 143-160.
[43] Zhikov, V.V., Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Akad. Nauk SSSR Ser. Mat.*, 29(4) (1987), 33-66.

https://communications.science.ankara.edu.tr

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ON THE GEOMETRY OF FIXED POINTS FOR SELF-MAPPINGS ON S-METRIC SPACES

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Abstract. In this paper, we focus on some geometric properties related to the set $Fix(T)$, the set of all fixed points of a mapping $T : X \to X$, on an S-metric space $(X, \mathcal{S})$. For this purpose, we present the notions of an S-Pata type x_0-mapping and an S-Pata Zamfirescu type x_0-mapping. Using these notions, we propose new solutions to the fixed circle (resp. fixed disc) problem. Also, we give some illustrative examples of our main results.

1. Introduction and Preliminaries

The notion of an S-metric space was introduced as a generalization of a metric space as follows:

Definition 1. *[20] Let X be a nonempty set and $\mathcal{S} : X^3 \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:*

(1) $\mathcal{S}(x, y, z) = 0$ *if and only if* $x = y = z$,
(2) $\mathcal{S}(x, y, z) \leq \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)$.

Then S is called an S-metric on X and the pair $(X, \mathcal{S})$ is called an S-metric space.

Many researchers have studied on S-metric spaces to obtain new fixed point results and some applications (see [7, 9, 10, 15, 21] and the references therein). Also, the relationship between a metric and an S-metric was investigated in various studies and some examples of an S-metric which is not generated by any metric were given (see [4, 5, 11] for more details).

Recently, the fixed circle problem (resp. fixed disc problem) raised by Özgür and Taş (see [12, 18] and the references therein) has been studied as an geometric

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approach to the fixed-point theory on metric spaces and some generalized metric spaces (for example, S-metric spaces) (see [8, 9, 13, 14, 23, 24]).

Now we recall the notions of a circle and a disc on an S-metric space given in [13, 20], respectively.

Let $(X, \mathcal{S})$ be an S-metric space and $T : X \to X$ be a self-mapping. A circle $C^S_{x_0,r}$ and a disc $D^S_{x_0,r}$ are defined as follows:

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) = r\}$$

and

$$D^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) \leq r\},$$

where $r \in [0, \infty)$.

If $Tx = x$ for all $x \in C^S_{x_0,r}$ (resp. $x \in D^S_{x_0,r}$), then the circle $C^S_{x_0,r}$ (resp. the disc $D^S_{x_0,r}$) is called as the fixed circle (resp. the fixed disc) of T.

A recent solution to the fixed-circle problem was given using the notion of S-Zamfirescu type x_0-mapping on S-metric spaces as follows:

Definition 2. *[16] Let $(X, \mathcal{S})$ be an S-metric space and $T : X \to X$ be a self-mapping. Then T is called an S-Zamfirescu type x_0-mapping if there exist $x_0 \in X$ and $a, b \in [0, 1)$ such that*

$$\begin{aligned} \mathcal{S}(Tx, Tx, x) &> 0 \Longrightarrow \\ \mathcal{S}(Tx, Tx, x) &\leq \max\left\{ a\mathcal{S}(x, x, x_0), \frac{b}{2}\left[\mathcal{S}(Tx_0, Tx_0, x) + \mathcal{S}(Tx, Tx, x_0)\right] \right\}, \end{aligned}$$

for all $x \in X$.

Let the number δ be defined as

$$\delta = \inf\{\mathcal{S}(Tx, Tx, x) : Tx \neq x,\ x \in X\}. \tag{1}$$

Theorem 3. *[16] Let $(X, \mathcal{S})$ be an S-metric space, $T : X \to X$ be a self-mapping and δ be the real number defined in (1). If the following conditions hold:*

(i) T is an S-Zamfirescu type x_0-mapping with $x_0 \in X$,

(ii) $\mathcal{S}(Tx, Tx, x_0) \leq \delta$ for each $x \in C^S_{x_0,\delta}$,

then $C^S_{x_0,\delta}$ is a fixed circle of T, that is, $C^S_{x_0,\delta} \subset Fix(T)$.

In this paper, we give new solutions to the fixed circle (resp. fixed disc) problem on S-metric spaces. In Section 2, we prove some fixed circle and fixed disc results using different approaches. In Section 3, we give some illustrative examples of our obtained results and deduce some important remarks. In Section 4, we summarize our study and recommend some future works.

2. Main Results

In this section, we inspire the methods given in [2, 6, 19, 26] and use the number defined in (1) to obtain new fixed circle (resp. fixed disc) results on S-metric spaces. In [19], Pata proved a fixed point theorem to generalize the well-known Banach's contraction principle on a metric space. However, Berinde showed that the main result in [19] does not hold at least in two extremal cases for the used parameter ε. The corrected version of this theorem was given with some necessary examples in [2]. In our results, we will not use the Picard iteration. Hence, our main results hold for all the parameters $\mu \in [0,1]$ and this situation will be supported by some illustrative examples given in the next section.

Let Θ denotes the class of all increasing functions $\psi : [0,1] \to [0,\infty)$ with $\psi(0) = 0$.

Definition 4. *Let $(X,\mathcal{S})$ be an S-metric space, $T : X \to X$ be a self-mapping, $\alpha \geq 0$, $\beta \geq 1$ and $\gamma \in [0,\beta]$ be any constants. Then T is called an S-Pata type x_0-mapping if there exist $x_0 \in X$ and $\psi \in \Theta$ such that*

$$\mathcal{S}(Tx,Tx,x) > 0 \Longrightarrow \mathcal{S}(Tx,Tx,x) \leq \frac{1-\mu}{2}\left\|x\right\|_s + \alpha\mu^{\beta}\psi(\mu)\left[1+\left\|x\right\|_s+\left\|Tx\right\|_s\right]^{\gamma},$$

for all $x \in X$ and each $\mu \in [0,1]$, where $\left\|x\right\|_s = \mathcal{S}(x,x,x_0)$.

Notice that $\left\|x_0\right\|_s = \mathcal{S}(x_0,x_0,x_0) = 0$. Let us consider the inequality given in the notion of S-Pata type x_0-mapping under the cases $\mu = 0$ and $\mu = 1$, respectively. For $\mu = 0$, we have

$$\mathcal{S}(Tx,Tx,x) > 0 \Longrightarrow \mathcal{S}(Tx,Tx,x) \leq \frac{1}{2}\left\|x\right\|_s = \frac{\mathcal{S}(x,x,x_0)}{2}$$

and also for $\mu = 1$, we get

$$\begin{aligned} \mathcal{S}(Tx,Tx,x) &> 0 \Longrightarrow \mathcal{S}(Tx,Tx,x) \leq \alpha\psi(1)\left[1+\left\|x\right\|_s+\left\|Tx\right\|_s\right]^{\gamma} \\ &= L\left[1+\left\|x\right\|_s+\left\|Tx\right\|_s\right]^{\gamma} \\ &= L\left[1+\mathcal{S}(x,x,x_0)+\mathcal{S}(Tx,Tx,x_0)\right]^{\gamma}, \end{aligned}$$

where $L = \alpha\psi(1) > 0$.

Theorem 5. *Let $(X,\mathcal{S})$ be an S-metric space, $T : X \to X$ be an S-Pata type x_0-mapping with $x_0 \in X$ and δ be the real number defined in (1). Then $C^S_{x_0,\delta}$ is a fixed circle of T, that is, $C^S_{x_0,\delta} \subset Fix(T)$.*

Proof. At first, we show that x_0 is a fixed point of T. On the contrary, assume that $Tx_0 \neq x_0$. Using the S-Pata type x_0-mapping property, we obtain

$$\mathcal{S}(Tx_0,Tx_0,x_0) \leq \frac{1-\mu}{2}\left\|x_0\right\|_s + \alpha\mu^{\beta}\psi(\mu)\left[1+\left\|x_0\right\|_s+\left\|Tx_0\right\|_s\right]^{\gamma}. \tag{2}$$

For $\mu = 0$, by inequality (2), we find

$$\mathcal{S}(Tx_0,Tx_0,x_0) \leq 0,$$

this is a contradiction. So, the assumption is false. This shows that $Tx_0 = x_0$ and hence $\left\|Tx_0\right\|_s = \left\|x_0\right\|_s = 0$.

Let $\delta = 0$. Then we have $C^S_{x_0,\delta} = \{x_0\}$. Clearly, $C^S_{x_0,\delta}$ is a fixed circle of T, that is, $C^S_{x_0,\delta} \subset Fix(T)$.

Let $\delta > 0$ and $x \in C^S_{x_0,\delta}$ be any point such that $Tx \neq x$. Using the S-Pata type x_0-mapping hypothesis, we obtain

$$\mathcal{S}(Tx, Tx, x) \leq \frac{1-\mu}{2} \left\|x\right\|_s + \alpha\mu^\beta\psi(\mu) \left[1 + \left\|x\right\|_s + \left\|Tx\right\|_s\right]^\gamma . \tag{3}$$

For $\mu = 0$, by inequality (3), we get

$$\mathcal{S}(Tx, Tx, x) \leq \frac{1}{2} \left\|x\right\|_s = \frac{\mathcal{S}(x, x, x_0)}{2} = \frac{\delta}{2},$$

a contradiction with the definition of δ. Hence it should be $Tx = x$. Consequently, T fixes the circle $C^S_{x_0,\delta}$ and so $C^S_{x_0,\delta} \subset Fix(T)$. □

Corollary 6. *Let $(X, \mathcal{S})$ be an S-metric space, $T : X \to X$ be an S-Pata type x_0-mapping with $x_0 \in X$ and δ be the real number defined in (1). Then T fixes whole of the disc $D^S_{x_0,\delta}$, that is, $D^S_{x_0,\delta} \subset Fix(T)$.*

Proof. By the similar arguments used in the proof of Theorem 5, the proof follows easily. □

We define another contractive condition to obtain a new fixed-circle result.

Definition 7. *Let $(X, \mathcal{S})$ be an S-metric space, $T : X \to X$ be a self-mapping, $\alpha \geq 0$, $\beta \geq 1$ and $\gamma \in [0, \beta]$ be any constants. If there exist $x_0 \in X$ and $\psi \in \Theta$ such that*

$$\begin{aligned} \mathcal{S}(Tx, Tx, x) &> 0 \Longrightarrow \mathcal{S}(Tx, Tx, x) \leq \frac{1-\mu}{2} M_S(x, x_0) \\ &\quad + \alpha\mu^\beta\psi(\mu) \left[1 + \left\|x\right\|_s + \left\|Tx\right\|_s + \left\|Tx_0\right\|_s\right]^\gamma , \end{aligned}$$

for all $x \in X$ and each $\mu \in [0, 1]$, where $\left\|x\right\|_s = \mathcal{S}(x, x, x_0)$ and

$$\begin{aligned} &M_S(x, y) \\ = \ &\max\left\{\mathcal{S}(x, x, y), \frac{\mathcal{S}(Tx, Tx, x) + \mathcal{S}(Ty, Ty, y)}{2}, \frac{\mathcal{S}(Ty, Ty, x) + \mathcal{S}(Tx, Tx, y)}{2}\right\}, \end{aligned}$$

then T is called an S-Pata Zamfirescu type x_0-mapping with respect to $\psi \in \Theta$.

In the above definition, we consider the extremal cases $\mu = 0$ and $\mu = 1$, respectively. For $\mu = 0$, we have

$$\mathcal{S}(Tx, Tx, x) > 0 \Longrightarrow \mathcal{S}(Tx, Tx, x) \leq \frac{1}{2} M_S(x, x_0)$$

and also for $\mu = 1$, we get

$$\begin{aligned} \mathcal{S}(Tx, Tx, x) &> 0 \Longrightarrow \mathcal{S}(Tx, Tx, x) \leq \alpha\psi(1) \left[1 + \left\|x\right\|_s + \left\|Tx\right\|_s + \left\|Tx_0\right\|_s\right]^\gamma \\ &= L \left[1 + \left\|x\right\|_s + \left\|Tx\right\|_s + \left\|Tx_0\right\|_s\right]^\gamma , \end{aligned}$$

where $L = \alpha\psi(1) > 0$.

Now we investigate the relationship between the notions of an S-Zamfirescu type x_0-mapping and an S-Pata Zamfirescu type x_0-mapping.

Let $\xi = \max\{a, b\}$ in Definition 2 and let us consider Bernoulli's inequality $1 + pt \leq (1+t)^p$, $p \geq 1$, $t \in [-1, \infty)$. Then we get

$$
\begin{aligned}
&\mathcal{S}(Tx,Tx,x) > 0 \Longrightarrow \\
&\mathcal{S}(Tx,Tx,x) \leq \max\left\{a\mathcal{S}(x,x,x_0), \tfrac{b}{2}\left[\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)\right]\right\} \\
&\leq \xi \max\left\{\mathcal{S}(x,x,x_0), \tfrac{\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)}{2}\right\} \\
&\leq \xi \max\left\{\mathcal{S}(x,x,x_0), \tfrac{\mathcal{S}(Tx,Tx,x)+\mathcal{S}(Tx_0,Tx_0,x_0)}{2}, \tfrac{\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)}{2}\right\} \\
&\leq \tfrac{1-\mu}{2} \max\left\{\mathcal{S}(x,x,x_0), \tfrac{\mathcal{S}(Tx,Tx,x)+\mathcal{S}(Tx_0,Tx_0,x_0)}{2}, \tfrac{\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)}{2}\right\} \\
&+ \left(\xi + \tfrac{\mu-1}{2}\right)\left[1 + \max\left\{\|x\|_s, \tfrac{\|x\|_s+\|Tx\|_s+\|Tx_0\|_s}{2}\right\}\right] \\
&\leq \tfrac{1-\mu}{2} M_S(x,x_0) + \xi\left(1 + \tfrac{\mu-1}{\xi}\right)\left[1 + \|x\|_s + \|Tx\|_s + \|Tx_0\|_s\right] \\
&\leq \tfrac{1-\mu}{2} M_S(x,x_0) + \xi\mu^{\frac{1}{\xi}}\left[1 + \|x\|_s + \|x_0\|_s + \|Tx\|_s + \|Tx_0\|_s\right] \\
&\leq \tfrac{1-\mu}{2} M_S(x,x_0) + \xi\mu\mu^{\frac{1-\xi}{\xi}}\left[1 + \|x\|_s + \|x_0\|_s + \|Tx\|_s + \|Tx_0\|_s\right].
\end{aligned}
$$

Hence we get that an S-Zamfirescu type x_0-mapping is a special case of an S-Pata Zamfirescu type x_0-mapping with $\alpha = \xi$, $\psi(x) = x^{\frac{1-\xi}{\xi}}$ and $\beta = \gamma = 1$.

Now we prove the following fixed circle theorem.

Theorem 8. *Let $(X, \mathcal{S})$ be an S-metric space, $T : X \to X$ be a self-mapping and δ be the real number defined in (1). If the following conditions hold:*

(i) T is an S-Pata Zamfirescu type x_0-mapping with respect to $\psi \in \Theta$ for $x_0 \in X$,

(ii) $\mathcal{S}(Tx,Tx,x_0) \leq \delta$ for each $x \in C^S_{x_0,\delta}$,

then $C^S_{x_0,\delta}$ is a fixed circle of T, that is, $C^S_{x_0,\delta} \subset Fix(T)$.

Proof. Using the condition (i), we can easily obtain that $Tx_0 = x_0$ and hence $\|Tx_0\|_s = \|x_0\|_s = 0$. Let $\delta = 0$. Then we have $C^S_{x_0,\delta} = \{x_0\}$. Clearly, $C^S_{x_0,\delta}$ is a fixed circle of T, that is, $C^S_{x_0,\delta} \subset Fix(T)$.

Let $\delta > 0$ and $x \in C^S_{x_0,\delta}$ be any point such that $Tx \neq x$. Using the conditions (i) and (ii), we obtain

$$
\begin{aligned}
\mathcal{S}(Tx,Tx,x) &\leq \frac{1-\mu}{2} M_S(x,x_0) + \alpha\mu^\beta\psi(\mu)\left[1 + \|x\|_s + \|Tx\|_s + \|Tx_0\|_s\right]^\gamma \\
&\leq \frac{1-\mu}{2} \max\left\{\delta, \frac{\mathcal{S}(Tx,Tx,x)}{2}\right\} \\
&\quad + \alpha\mu^\beta\psi(\mu)\left[1 + \|x\|_s + \|Tx\|_s\right]^\gamma . \qquad (4)
\end{aligned}
$$

For $\mu = 0$, using the inequality (4), we get

$$
\mathcal{S}(Tx,Tx,x) \leq \frac{1}{2} \max\left\{\delta, \frac{\mathcal{S}(Tx,Tx,x)}{2}\right\},
$$

a contradiction with the definition of δ. Consequently, it should be $Tx = x$ whence T fixes the circle $C^S_{x_0,\delta}$ and so $C^S_{x_0,\delta} \subset Fix(T)$. $\square$

Corollary 9. *Let $(X,\mathcal{S})$ be an S-metric space, $T:X\to X$ be a self-mapping and δ be the real number defined in (1). If the following conditions hold:*
(i) T is an S-Pata Zamfirescu type x_0-mapping with respect to $\psi\in\Theta$ for $x_0\in X$,
(ii) $\mathcal{S}(Tx,Tx,x_0)\le\delta$ for each $x\in D^S_{x_0,\delta}$
then T fixes whole of the disc $D^S_{x_0,\delta}$, that is, $D^S_{x_0,\delta}\subset Fix(T)$.

Proof. By the similar arguments used in the proof of Theorem 8, the proof follows easily. □

Remark 10. *If a self-mapping T satisfies the conditions of Theorem 8, then we have $\|Tx_0\|_s=\|x_0\|_s=0$. Therefore, Theorem 8 coincides with Theorem 5 in the case where $M_S(x,x_0)=\|x\|_s$ for all $x\in X$. On the other hand, if T satisfies the conditions of Theorem 5 then clearly, T satisfies the conditions of Theorem 8 since $M_S(x,x_0)\ge\|x\|_s$.*

3. Illustrative Examples and Some Remarks

In this section, we give some examples to show the validity of our results obtained in the previous section.

Example 11. *Let $X=\mathbb{R}$ be the S-metric space with the S-metric defined by*

$$\mathcal{S}(x,y,z)=|x-z|+|x+z-2y|,$$

for all $x,y,z\in\mathbb{R}$ [11]. Let us define the self-mapping $T_1:\mathbb{R}\to\mathbb{R}$ as

$$T_1x=\begin{cases} x & , \quad x\in[-2,2] \\ x+\frac{1}{2} & , \quad x\in(-\infty,-2)\cup(2,\infty)\end{cases},$$

for all $x\in\mathbb{R}$. Then T_1 is both an S-Pata type x_0-mapping and an S-Pata Zamfirescu type x_0-mapping with $x_0=0$, $\alpha=\beta=\gamma=1$ and

$$\psi(x)=\begin{cases} 0 & , \quad x=0 \\ \frac{1}{2} & , \quad x\in(0,1]\end{cases}.$$

Also we have $\delta=1$. Consequently, by Theorem 5 and Theorem 8 (resp. Corollary 6 and Corollary 9), T_1 fixes the circle $C^S_{0,1}=\left\{-\frac{1}{2},\frac{1}{2}\right\}$ $\left(\text{resp. the disc } D^S_{0,1}=\left[-\frac{1}{2},\frac{1}{2}\right]\right)$.

Example 12. *Let $X=\mathbb{R}$ be the S-metric space with the S-metric considered in Example 11. Let us define the self-mapping $T_2:\mathbb{R}\to\mathbb{R}$ as*

$$T_2x=\begin{cases} x & , \quad x\in[-4,\infty) \\ x+1 & , \quad x\in(-\infty,-4)\end{cases},$$

for all $x\in\mathbb{R}$. Then T_2 is both an S-Pata type x_0-mapping and an S-Pata Zamfirescu type x_0-mapping with $x_0=0$ (or $x_0=3$), $\alpha=\beta=\gamma=1$ and

$$\psi(x)=\begin{cases} 0 & , \quad x=0 \\ \frac{1}{2} & , \quad x\in(0,1]\end{cases}.$$

Also we obtain $\delta=2$. Consequently, T_2 fixes the circles $C^S_{0,2}$ and $C^S_{3,2}$ (resp. the discs $D^S_{0,2}$ and $D^S_{3,2}$).

Example 13. *Let $X=\mathbb{R}$ be the S-metric space with the S-metric considered in Example 11. Let us define the self-mapping $T_3:\mathbb{R}\to\mathbb{R}$ as*

$$T_3x=\begin{cases} x & , \quad x\in[-2,2] \\ 0 & , \quad x\in(-\infty,-2)\cup(2,\infty) \end{cases},$$

for all $x\in\mathbb{R}$. Then T_3 is not an S-Pata type x_0-mapping and an S-Pata Zamfirescu type x_0-mapping with $x_0=0$. But T_3 fixes the circle $C^S_{0,4}=\{-2,2\}$ and the disc $D^S_{0,4}=[-2,2]$.

Now, we give an example of a self-mapping that satisfies the conditions of Theorem 8 but does not satisfy the conditions of Theorem 5.

Example 14. *Let $X=\mathbb{R}$ be the S-metric space with the usual S-metric defined by*

$$\mathcal{S}(x,y,z)=|x-z|+|y-z|,$$

for all $x,y,z\in X$ [21]. Now, we define the self-mapping $T_4:X\to X$ by

$$T_4x=\begin{cases} \frac{5}{3}x & , \quad |x|=1 \\ x & , \quad |x|\neq 1 \end{cases}.$$

We have

$$\begin{aligned}\delta &= \inf\{\mathcal{S}(T_4x,T_4x,x):|x|=1\} \\ &= \inf\{2|T_4x-x|:|x|=1\} \\ &= \inf\left\{2\left|\frac{5}{3}x-x\right|:|x|=1\right\} \\ &= \inf\left\{\frac{4}{3}|x|:|x|=1\right\}=\frac{4}{3}.\end{aligned}$$

Then, it is easy to verify that T_4 is not an S-Pata type x_0-mapping for the point $x_0=0$ independent from the choice of the parameters α, β, γ and the function ψ. But, if we choose $\alpha=\beta=\gamma=1$ then T_4 is an S-Pata Zamfirescu type x_0-mapping for the point $x_0=0$ with the function

$$\psi(x)=\begin{cases} 0 & , \quad x=0 \\ \frac{1}{4} & , \quad x\in(0,1] \end{cases}.$$

Clearly, T_4 fixes the circle

$$\begin{aligned}C^S_{0,\frac{4}{3}} &= \left\{x:2|x-0|=\frac{4}{3}\right\} \\ &= \left\{x:|x|=\frac{2}{3}\right\} \\ &= \left\{-\frac{2}{3},\frac{2}{3}\right\}\end{aligned}$$

and the disc $D^S_{0,\frac{4}{3}}=\left\{x:|x|\leq\frac{2}{3}\right\}$.

The following remarks can be deduced from the obtained results and the given examples.

Remark 15. (*i*) *The point x_0 satisfying the conditions of an S-Pata type x_0-mapping and an S-Pata Zamfirescu type x_0-mapping is always a fixed point of the self-mapping T. Moreover, the choice of x_0 is independent from the number δ* (*see Example 11 and Example 12*). *Also the number of x_0 can be more than one* (*see Example 12*).

(*ii*) *The converse statements of Theorem 5, Corollary 6, Theorem 8 and Corollary 9 are not always true* (*see Example 13*). *That is, a self-mapping having a fixed circle* (*resp. fixed disc*) *need not to be an S-Pata type x_0-mapping or an S-Pata Zamfirescu type x_0-mapping with x_0 where the point x_0 is the center of the fixed circle* (*resp. fixed disc*).

4. Conclusion

In this paper, we have presented some new solutions to the fixed circle problem on S-metric spaces. To do this, we have inspired by the Pata and Zamfirescu type methods. We have proved two main fixed circle theorems and some related results. Also, we have given necessary illustrative examples supporting our obtained results. On the other hand, there are many generalized metric spaces in the literature (for example, see [3, 25] and the references therein). Hence, the fixed circle (resp. fixed disc) problem can be studied on these generalized metric spaces using similar approaches as a future work.

On the other hand, a related problem is the best proximity point problem since the best proximity point theorems investigate an optimal solution of the minimization problem $\{d(x, Tx) : x \in A\}$ for a mapping $T : A \to B$ where A and B are two non-empty subsets of a metric space (see [1] and the references therein). In [6], the existence of best proximity point was investigated using the Pata type proximal mappings. In [17], the notion of a best proximity circle is introduced and some proximal contractions for a non-self-mapping are determined. In this context, a related future work is the investigation of the existence of a best proximity circle via the notions of p-proximal contraction and p-proximal contractive mapping defined in [22].

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References

[1] Altun, İ., Aslantaş, M., Şahin, H., Best proximity point results for p-proximal contractions, *Acta Math. Hungar.*, (2020). https://doi.org/10.1007/s10474-020-01036-3.

[2] Berinde, V., Comments on some fixed point theorems in metric spaces, *Creat. Math. Inform.,* 27 (1) (2018), 15-20.

[3] Dosenovic, T., Radenovic, S. and Sedghi, S., Generalized metric spaces: survey, *TWMS J. Pure Appl. Math.,* 9 (1) (2018), 3-17.

[4] Gupta, A., Cyclic contraction on S-metric space, *Int. J. Anal. Appl.,* 3 (2) (2013), 119-130.

[5] Hieu, N. T., Ly, N. T., Dung, N. V., A generalization of Ciric quasi-contractions for maps on S-metric spaces, *Thai J. Math.,* 13 (2) (2015), 369-380.

[6] Jacob, G. K., Khan, M. S., Park, C., Yun, S., On generalized Pata type contractions, *Mathematics,* 6 (2018), 25.

[7] Mlaiki, N., α-ψ-contractive mapping on S-metric space, *Math. Sci. Lett.,* 4 (1) (2015), 9-12.

[8] Mlaiki, N., Çelik, U., Taş, N., Özgür, N. Y., Mukheimer, A., Wardowski type contractions and the fixed-circle problem on S-metric spaces, *J. Math.,* (2018), Article ID 9127486.

[9] Mlaiki, N., Özgür, N. Y., Taş, N., New fixed-point theorems on an S-metric space via simulation functions, *Mathematics,* 7(7) (2019), 583.

[10] Özgür, N. Y. , Taş, N., Some fixed point theorems on S-metric spaces, *Mat. Vesnik,* 69 (1) (2017), 39-52.

[11] Özgür, N. Y., Taş, N., Some new contractive mappings on S-metric spaces and their relationships with the mapping ($S25$), *Math. Sci. (Springer),* 11 (1) (2017), 7-16.

[12] Özgür, N. Y., Taş, N., Some fixed-circle theorems on metric spaces, *Bull. Malays. Math. Sci. Soc.,* 42 (4) (2019), 1433-1449.

[13] Özgür, N. Y., Taş, N., Fixed-circle problem on S-metric spaces with a geometric viewpoint, *Facta Universitatis. Series: Mathematics and Informatics,* 34 (3) (2019), 459-472.

[14] Özgür, N. Y., Taş, N., Çelik, U., New fixed-circle results on S-metric spaces, *Bull. Math. Anal. Appl.,* 9 (2) (2017), 10-23.

[15] Özgür, N. Y., Taş, N., The Picard theorem on S-metric space, *Acta Math. Sci.,* 38B (4) (2018), 1245-1258.

[16] Özgür, N. Y., Taş, N., A new solution to the Rhoades' open problem with an application, arXiv preprint. arXiv:1910.12304

[17] Özgür, N. Y., Taş, N., Pata Zamfirescu type fixed-disc results with a proximal application, arXiv preprint. arXiv:1910.12302

[18] Özgür, N. Y., Fixed-disc results via simulation functions, *Turkish J. Math.,* 43 (6) (2019), 2794-2805.

[19] Pata, V., A fixed point theorem in metric spaces, *J. Fixed Point Theory Appl.,* 10 (2011), 299-305.

[20] Sedghi, S., Shobe, N., Aliouche, A., A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik,* 64 (3) (2012), 258-266.

[21] Sedghi, S., Dung, N. V., Fixed point theorems on S-metric spaces, *Mat. Vesnik,* 66 (1) (2014), 113-124.

[22] Şahin, H., Aslantas, M., Altun, İ., Feng-Liu type approach to best proximity point results for multivalued mappings. *J. Fixed Point Theory Appl.* 22 (2020), 11.

[23] Taş, N., Suzuki-Berinde type fixed-point and fixed-circle results on S-metric spaces, *J. Linear Topol. Algebra,* 7 (3) (2018), 233-244.

[24] Taş, N., Various types of fixed-point theorems on S-metric spaces, *J. BAUN Inst. Sci. Technol.,* 20 (2) (2018), 211-223.

[25] Taş, N., Özgür, N. Y., New generalized fixed point results on S_b-metric spaces, arXiv preprint. arXiv:1703.01868

[26] Zamfirescu, T., Fixed point theorems in metric spaces, *Arch. Math.,* 23 (1972), 292-298.

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PARAMETER ESTIMATION BY TYPE-2 FUZZY LOGIC IN CASE THAT DATA SET HAS OUTLIER

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Abstract. One of the problems encountered in estimating the unknown parameters of the regression models is the presence of outliers in the data set. This situation may cause problems in providing some assumptions such as the normal distribution for the parameter estimation process and the homogeneity of the variances. The case of the presence of outlier observations in the data set, estimation methods based on fuzzy logic that can be minimized the level of impact of this data are emerged as available methods. If fuzzy logic is used in regression analysis, there are two main steps for parameter estimation. The first of these is to define the clusters that compose the data set, and the other is calculate the degree of membership to determining the contributions of the data to each model for the clusters. In this study, type-2 fuzzy clustering algorithm defined as an expansion of fuzzy c-means algorithm in the determination of membership degrees of data sets was benefited. The presence of outliers in the data set is addressed. An algorithm has been proposed to estimate the unknown belonging to parameters of the regression model using the membership degrees obtained relating to the cluster elements. The parameters were estimated using regression methods to examine the effectiveness of the algorithm that called robust methods, and the results obtained were compared.

1. Introduction

The concept of fuzzy sets was first described by Zadeh in 1965 with his work Fuzzy Sets [1]. The fuzzy c-means method was introduced by Dunn in 1973. The method was developed with the studies carried out by Bezdek [2]. Zadeh et al. published their studies on fuzzy sets and their application to decision processes in 1975 [3]. The fuzzy c-means algorithm developed by Bezdek et al., includes euclidean, diagonal and mahalanobis distance measurements and the output from

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this algorithm is controlled by validity criteria [4]. Mendel and John precisely defined type-2 fuzzy sets and used the expansion principle for type-2 fuzzy sets in 2002 [5].

They discussed the uncertainty of the m fuzzifier parameter of the fuzzy c-means algorithm, and they defined the m parameter as the interval number. The fuzzy c-means algorithm that include m fuzzifier parameter uncertainty is applied type reduction, and solutions are obtained in 2007 [6]. Dalkilic and Apaydin determined the optimal class number using the validity criterion when the independent variables had an exponential distribution and has been made parameter prediction using fuzzy neural network [7]. Juang et al. proposes a repetitive type-2 Fuzzy Neural Networks (FNN) for dynamic system processing. The self-developing network is a structure that does not require a pre-assignment task and can automatically develop its parameters according to the training data [8]. Fazel Zarandi et al. suggested a new type-2 fuzzy c-regression clustering method for the application of the steel industry in the Takagi-Sugeno (T-S) system identification stage and the model was tested on the actual data set from a steel company [9]. Enke and Mehdiyev proposed the use of a hybrid model for the estimation of short-term US interest rates. The model consists of fuzzy type-2 inferential neural network that performs input pretreatment with multiple regression analysis and fuzzy type-2 clustering based on differential evolution. The proposed model was applied to estimate US 3-Month T-bill ratios in 2013 [10]. In 2015, Kalhori and Fazel Zarandi presents a new approach to type-2 fuzzy clustering. This approach proposed to separating clusters that does not use only the distance from the centers, and a new validity index is suggested to determine the optimal number of clusters [11]. In 2016, Golsefid and Zarandi present an algorithm for clustering. In the clustering algorithm that developed according to the dicentric type-2 fuzzy clustering model, the centers of the clusters are defined by the double object. There are no type reduction or blurring steps in this algorithm [12]. In 2016, Hwak proposed a method for the design of the linear regression and this method is designed using Type-2 Fuzzy C-Means (T2FCM) clustering. This clustering approach takes into account the uncertainty associated with the fuzzification factor when estimating cluster centers. The method was also supported by experimental results [13]. Rubio et al. presented an extension of the Fuzzy Possibilistic C-Means (FPCM) algorithm. In this algorithm, Type-2 Fuzzy Logic Techniques were used to increase the efficiency of the Fuzzy Probabilistic C-Means (FPCM) method. In addition, the performance of the method was controlled by experimental data [14].

In this study, membership degrees for cluster elements are obtained by using type-2 fuzzy clustering algorithm, and an algorithm has been proposed for the regression model to include these degrees based on parameter estimates. The situation of the outlier observation in the data set was discussed, and estimation values from the parameters obtained based on type-2 fuzzy clustering were obtained.

Remainder of this paper is organized as follows. In the second part of the study, type-2 fuzzy clustering method was described. In the third section, definitions of robust regression methods to be used in comparison were given briefly. In the fourth section, an algorithm was proposed for parameter estimation based on the type-2 fuzzy membership. In the application part, Proposed algorithm for data set that has outlier, and estimates concerning models obtained using regression methods were compared.

2. Fuzzy Clustering Based Type-2 Fuzzy Logic

While equal fuzzifier index is given to each set-in type-1 fuzzy clustering, the fuzzifier index is defined as an interval in type-2 fuzzy clustering. Different fuzzifier index is given each cluster. Performance loss is prevented with description when sets have different set volumes. First, volumes of the sets obtained with fuzzy clustering algorithm are determined. Fuzzifier indexes based on obtained volumes and cluster center based on these fuzzifier indexes are calculated. The objective function values are determined based on the principle of minimizing the distance between cluster centers and cluster elements [15].

The center value and membership values are updated until the objective function reaches the smallest target value. Fuzzifier indexes that have the optimal center value and membership degrees are determined, and observations are divided into sets based on obtained membership degrees. The parameters of the linear regression model related clusters are estimated based on the membership degree that obtained from type-2 fuzzy clustering.

In the clustering that based on type-2 fuzzy logic, the objective function that calculated for interval $m = [m_1, m_2]$ defined as

$$J_{m_1}\big(U, v\big) = \sum_{i=1}^{n}\sum_{j=1}^{c} u_{ji}^{m_1} d_{ji}^2, \tag{1}$$

$$J_{m_2}\big(U, v\big) = \sum_{i=1}^{n}\sum_{j=1}^{c} u_{ji}^{m_2} d_{ji}^2.$$

The aim of the function given by the Eq. (1) is minimizing the error. In the system of Eq. (1), m_1 and m_2 are the fuzzifier index of the first and second sets, respectively. The weighted least squares function $J_{m_1}\big(U, v\big)$ is the sum of the weighted error squares of the first set and $J_{m_2}\big(U, v\big)$ is the sum of the weighed error squares of the second set, and $d_{ji}^2 = \|x_i - v_j\|^2$ is used to express the distance between data and cluster centers. Type-2 fuzzy clustering algorithm that is based on the aim of the function that given by Eq. (1) can be given by the following steps [16, 17].

Step 1: Initial values given as; c: number of sets, m_1 and m_2: fuzzifier indexes of the first and second sets, U: matrix showing the membership degrees and ε: termination criterion, are determined.

Step 2: Set centers are calculated using U matrix fuzzifier indexes m which is arbitrarily determined in first step and $m = [m_1, m_2]$ fuzzifier parameters,

$$v_{Lj} = \frac{\sum_{i=1}^{n} u_{ji}^{m_1} x_i}{\sum_{i=1}^{n} u_{ji}^{m_1}}, \ j = 1, ..., c, \tag{2}$$

$$v_{Rj} = \frac{\sum_{i=1}^{n} u_{ji}^{m_2} x_i}{\sum_{i=1}^{n} u_{ji}^{m_2}}, \ j = 1, ..., c.$$

Step 3: $\overline{u}_{ji}$ and $\underline{u}_{ji}$ are indicates the upper membership degree and the lower membership degree respectively [6]. These degrees updated with Eq. (3) and Eq. (4).

$$\overline{u}_{ji} = \begin{cases} \frac{1}{\sum_{k=1}^{c} \left(\frac{d_{ji}}{d_{ki}}\right)^{\frac{2}{m_1 - 1}}}, & if \frac{1}{\sum_{k=1}^{c} \left(\frac{d_{ji}}{d_{ki}}\right)} < \frac{1}{c} \\ \sum_{k=1}^{c} \left(\frac{d_{ji}}{d_{ki}}\right)^{\frac{2}{m_2 - 1}}, & in\ other\ situations \end{cases} \tag{3}$$

$$\underline{u}_{ji} = \begin{cases} \frac{1}{\sum_{k=1}^{c} \left(\frac{d_{ji}}{d_{ki}}\right)^{\frac{2}{m_1 - 1}}}, & if \frac{1}{\sum_{k=1}^{c} \left(\frac{d_{ji}}{d_{ki}}\right)} \geq \frac{1}{c} \\ \sum_{k=1}^{c} \left(\frac{d_{ji}}{d_{ki}}\right)^{\frac{2}{m_2 - 1}}, & in\ other\ situations \end{cases} \tag{4}$$

Step 4: When the v_{Lj} and v_{Rj} are indicated the centers that obtained by using m_1 and m_2 respectively Eq. (5) is used to type-reduction for set centers.

$$v_j = \frac{v_{Lj} + v_{Rj}}{2}, \ j = 1, ..., c \tag{5}$$

Step 5: Type-reduction process is also performed for membership degrees with Eq. (6).

$$u_{ji} = \frac{\overline{u}_{ji} + \underline{u}_{ji}}{2}, \ j = 1, ..., c; \ i = 1, ..., n \tag{6}$$

Step 6: Type-reduction process performed for objective functions of weighted sets with Eq. (7).

$$J\big(U, v\big) = \frac{J_{m_1}\big(U, v\big) + J_{m_2}\big(U, v\big)}{2} \tag{7}$$

Step 7: If $\|v_{L(t)} - v_{L(t-1)}\| < \varepsilon$ and $\|v_{R(t)} - v_{R(t-1)}\| < \varepsilon$ the iteration is ended. In other case it returns to Step 2.

3. Robust Regression Methods

In the case of an outlier in the data set, the resulting regression model moves away from observations other than the outlier by the effect of the outliers. Residues of observations other than outliers are increased. In Robust analysis, it is assumed that these deviations do not significantly affect the performance of the algorithm [18]. In the case of Robust regression analysis with outlier, parameter estimation that is less affected by the Least Square Method (LSM) is obtained [19]. In this study, estimations were obtained by using M methods from Robust methods. The M method minimizes the function of the residues rather than minimizing the sum of the squares of the residuals. Regression coefficients are obtained by the minimizing the sum.

$$\sum_{i=1}^{n} \rho[\big(y_i - \sum_{j=1}^{p} x_{ij}\widehat{\beta}_j\big)/d] \tag{8}$$

Huberâ's ρ function is defined as

$$\rho(z) = \begin{cases} \frac{z^2}{2} & |z| \leq k \\ k|z| - \frac{k^2}{2} & |z| > k \end{cases} \tag{9}$$

$$z = \frac{r_i}{d}$$

$$d = median|r_i - median(r_i)|/0.6745$$

where k is called tuning constant and k is set at 1.5. Sometimes the numerator of d is called the median of the absolute deviations (MAD) [20].

Hampel Ψ function is defined as

$$\Psi(z) = \begin{cases} |z| & 0 < |z| \leq a \\ asgn(z) & a < |z| \leq b \\ a\big(\frac{c-|z|}{c-b}\big)sgn(z) & b < |z| \leq c \\ 0 & c < |z| \end{cases} \tag{10}$$

$$sgn(z) = \begin{cases} +1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases}$$

Constant values are selected as $a = 1.7$, $b = 3.4$ and $c = 8.5$ in general [21]. Andrews (sinus estimate) Ψ function is defined as

$$\Psi(z) = \begin{cases} sin(\frac{z}{k}) & |z| \leq k\pi \\ 0 & |z| > k\pi \end{cases} \tag{11}$$

where if the scale is known, $k = 1.339$ requires a premium of 5% otherwise $k = 1.5$ or $k = 2.1$ [19].

Tukeyâ's two weighted estimate, Ψ function is defined as

$$\Psi(z) = \begin{cases} z\left(1 - \left(\frac{z}{k}\right)^2\right)^2 & |z| \leq k \\ 0 & |z| > k \end{cases} \tag{12}$$

where if the scale is known, $k = 4.685$ implies a premium of 5% otherwise $k = 5$ or $k = 6$ [19, 20].

4. Parameter Estimation based on Type-2 Fuzzy Logic When Data Set Has Outlier

The general purpose of the regression analysis is to determine the mathematical structure of the functional relationship between the dependent variable (Y) and independent variables $(X_1, \cdots, X_p)$. Determination of the mathematical structure is carried out by estimating regression coefficients (β).

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_p X_p + \varepsilon \tag{13}$$

The least square method is one of the important methods used to estimate the parameters of the linear regression model that given by Eq. (13). The important assumptions to use this method; that error terms related to the model should have normal distribution with zero-averaged and fixed variance. This assumption is expressed mathematically with $\varepsilon \sim N(0, \sigma^2)$. Estimators of the regression coefficients denoted by $\widehat{\beta}$ and determinate by,

$$\widehat{\beta} = (X'X)^{-1} X'Y. \tag{14}$$

The estimators of the dependent variable Y is shown as $\widehat{Y}$ and determined by

$$\widehat{Y} = X\widehat{\beta}. \tag{15}$$

The error for the linear regression model that expressed as the difference between observation values Y and estimation values $\widehat{Y}$ is given by,

$$\varepsilon = \left(Y - \widehat{Y}\right). \tag{16}$$

In classical regression analysis, the observations that make up the dataset belong to a single class. If the data set has different distributions in regression analysis, different methods should be used in parameter estimation, other than classical methods. These methods do not have to provide the necessary assumptions to use the classical method. If the data set has different distributions, fuzzy methods are among the methods that do not require the assumptions of classical regression. First step of fuzzy regression analysis is to determine the different clusters for the data set and the other is to obtain the degrees of membership to be used in the prediction process.

In the process of separating data with different distributions into clusters, fuzzy clustering algorithms suitable for distribution are used. With fuzzy clustering algorithms, the degree of membership is determined for each cluster. These membership degrees are used to weight the data. Parameters of the regression model are determined to have a minimum error using data that weighted by fuzzy membership degree.

Using the type-2 fuzzy clustering process, the algorithm proposed for parameter estimation of the regression model for data weighted by membership degrees given by the following steps,

Step 1: Beginning values given as; c: number of sets, m: fuzzifier indexes of the first and second sets, U: matrix showing the degrees of membership and ε: termination criterion, are determined.

Step 2: Set centers are calculated using U matrix and fuzzifier indexes m;

$$v_j = \frac{\sum_{i=1}^{n} u_{ji}^{m} x_i}{\sum_{i=1}^{n} u_{ji}^{m}},\ j = 1, ..., c. \tag{17}$$

Step 3: Objective function that depend on membership degree and set centers is calculated by,

$$J\left(U, v\right) = \sum_{i=1}^{n} \sum_{j=1}^{c} u_{ji}^{m} \|x_i - v_j\|^2. \tag{18}$$

Step 4: Membership degrees of each set are updated with,

$$u_{ji} = \frac{1}{\sum_{k=1}^{c} \left(\frac{\|x_i - v_j\|}{\|x_i - v_k\|}\right)^{\frac{2}{m-1}}}. \tag{19}$$

Step 5: Set centers and objective function are updated with Eq. (17) and Eq. (18) by use the new membership degrees.

Step 6: If the difference between the membership degree in t^{th} step and the membership degree in $(t-1)^{th}$ step is smaller than ε stops. It means that the optimal membership degrees and center are calculated.

Step 7: The membership degrees obtained from Eq. (19) are used to cluster the data set. m_1 and m_2 fuzzifier indexes are calculate and set centers based on fuzzifier indexes are calculated with Eq. (2) given in Section (2). Objective functions values that according to these centers are calculated by using Eq. (1) that given in Section (2).

Step 8: To reduce the type of fuzziness type-reduction applied to the set centers by use the Eq. (5) and, objective function value calculated by use the Eq. (7) given in Section (2).

Step 9: To clustering that use type-2 fuzzy logic, membership degrees determined by Eq. (3) and Eq. (4), and type-reduction operation is applied to the membership degree by Eq. (6).

Step 10: After the center value in t^{th} step and in $(t-1)^{th}$ step is calculated, the difference between them is determined. If the difference is less than termination criterion ε for existing sets the optimal center and membership degree achieved.

Step 11: Estimate the linear regression modelâ's parameters are realized by using membership degree as weight with obtained from type-2 fuzzy clustering [6].

Independent variable is weight by membership degree

$$X_{W_{i(Type-2)}} = u_{ij(Type-2)} x_j,\ i = 1, \ldots, c;\ j = 1, \ldots, n \tag{20}$$

and model parameters are obtained by,

$$\widehat{B}_{i(Type-2)} = \left(\left(x_j u_{ij(Type-2)}\right)' x_j\right)^{-1} \left(\left(x_j u_{ij(Type-2)}\right)' Y_j\right),\quad i = 1, \ldots, c;\ j = 1, \ldots, n_i. \tag{21}$$

The estimated values are calculated by,

$$\widehat{Y}_{i(Type-2)} = X_{W_{i(Type-2)}} \widehat{B}_{i(Type-2)},\ i = 1, \ldots, c. \tag{22}$$

Step 12: Error related to the models measured as

$$\varepsilon_i = \sum_{i=1}^{n_i} \frac{\left(Y_i - \widehat{Y}_i\right)^2}{n}. \tag{23}$$

The model that has the smallest error is used as estimated linear regression model.

5. Application

In this application, which will be discussed to determine the effectiveness of the proposed algorithm to obtain the linear regression model the dataset contains a dependent and an independent variable and set has 61 pairs of observations. Scatter chart of the data that are shown in Figure 1.

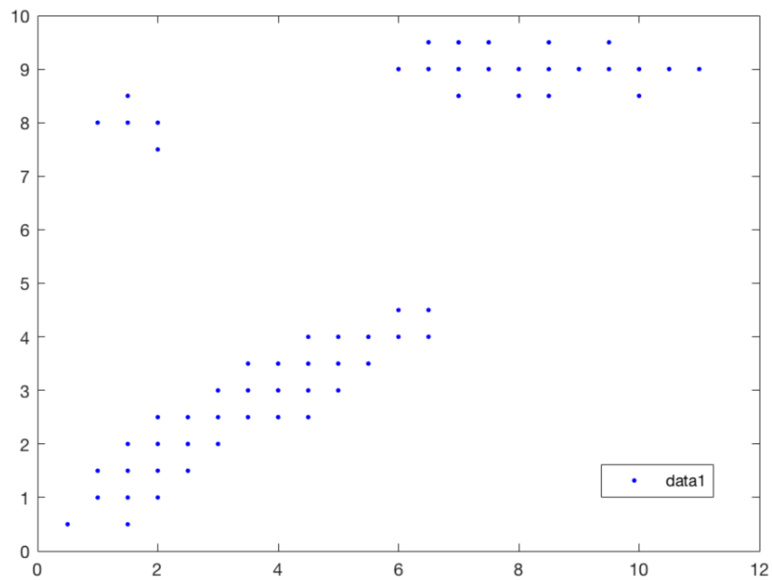

Figure 1. Scatter chart of data

As can be seen from Figure 1, there are five outliers in the dataset. Observation values, estimation values for the all related methods and the error amounts related to the estimations are given in Table 1. In Table 2, models generated by the parameters obtained using related methods and the amount of error calculated from the models.

A graph of error for the models obtained using the relevant methods are shown in Figure 2.

6. Results and Discussion

As a result, estimations that obtained with determined of fuzzy parameters in proposed algorithm for parameter estimation, and the results obtained by robust regression methods in the literature are compared. As a result of the seven methods examined, error amounts were obtained. The errors amount belonging to these

TABLE 1. Predictions values for related method and error values of related predictions

X	Y	$\widehat{Y}_{Type-1}$	ε_{Type-1}	$\widehat{Y}_{Type-2}$	ε_{Type-2}	$\widehat{Y}_{Huber}$	ε_{Huber}	$\widehat{Y}_{Hampel}$	ε_{Hampel}	$\widehat{Y}_{Tukey}$	ε_{Tukey}	$\widehat{Y}_{Andrews}$	$\varepsilon_{Andrews}$
4.0	3.5	3.1810	0.3190	3.4014	0.0986	3.9118	-0.4118	3.8906	-0.3906	3.6701	-0.1701	3.9673	-0.4673
0.5	0.5	3.4977	-2.9977	2.9990	-2.4990	0.6169	-0.1169	0.5357	-0.0357	0.1133	0.3867	0.6718	-0.1718
1.0	8.0	3.4839	4.5161	3.0431	4.9569	1.0876	6.9124	1.0149	6.9851	0.6214	7.3786	1.1426	6.8574
1.5	8.0	3.4590	4.5410	3.0894	4.9106	1.5583	6.4417	1.4942	6.5058	1.1295	6.8705	1.6134	6.3866
1.5	8.5	3.4652	5.0348	3.0884	5.4116	1.5583	6.9417	1.4942	7.0058	1.1295	7.3705	1.6134	6.8866
2.0	7.5	3.4273	4.0727	3.1368	4.3632	2.0290	5.4710	1.9735	5.5265	1.6376	5.8624	2.0842	5.4158
2.0	8.0	3.4372	4.5628	3.1352	4.8648	2.0290	5.9710	1.9735	6.0265	1.6376	6.3624	2.0842	5.9158
1.0	1.0	3.4537	-2.4537	3.0493	-2.0493	1.0876	-0.0876	1.0149	-0.0149	0.6214	0.3786	1.1426	-0.1426
1.0	1.5	3.4529	-1.9529	3.0497	-1.5497	1.0876	0.4124	1.0149	0.4851	0.6214	0.8786	1.1426	0.3574
1.5	0.5	3.4108	-2.9108	3.0993	-2.5993	1.5583	-1.0583	1.4942	-0.9942	1.1295	-0.6295	1.6134	-1.1134
1.5	1.0	3.4091	-2.4091	3.1000	-2.1000	1.5583	-0.5583	1.4942	-0.4942	1.1295	-0.1295	1.6134	-0.6134
1.5	1.5	3.4076	-1.9076	3.1008	-1.6008	1.5583	-0.0583	1.4942	0.0058	1.1295	0.3705	1.6134	-0.1134
1.5	2.0	3.4066	-1.4066	3.1015	-1.1015	1.5583	0.4417	1.4942	0.5058	1.1295	0.8705	1.6134	0.3866
2.0	1.0	3.3638	-2.3638	3.1512	-2.1512	2.0290	-1.0290	1.9735	-0.9735	1.6376	-0.6376	2.0842	-1.0842
2.0	1.5	3.3615	-1.8615	3.1527	-1.6527	2.0290	-0.5290	1.9735	-0.4735	1.6376	-0.1376	2.0842	-0.5842
2.0	2.0	3.3598	-1.3598	3.1541	-1.1541	2.0290	-0.0290	1.9735	0.0265	1.6376	0.3624	2.0842	-0.0842
2.0	2.5	3.3589	-0.8589	3.1554	-0.6554	2.0290	0.4710	1.9735	0.5265	1.6376	0.8624	2.0842	0.4158
2.5	1.5	3.3151	-1.8151	3.2052	-1.7052	2.4997	-0.9997	2.4528	-0.9528	2.1458	-0.6458	2.5550	-1.0550
2.5	2.0	3.3126	-1.3126	3.2079	-1.2079	2.4997	-0.4997	2.4528	-0.4528	2.1458	-0.1458	2.5550	-0.5550
2.5	2.5	3.3112	-0.8112	3.2107	-0.7107	2.4997	0.0003	2.4528	0.0472	2.1458	0.3542	2.5550	-0.0550
3.0	2.0	3.2659	-1.2659	3.2624	-1.2624	2.9704	-0.9704	2.9320	-0.9320	2.6539	-0.6539	3.0257	-1.0257
3.0	2.5	3.2639	-0.7639	3.2681	-0.7681	2.9704	-0.4704	2.9320	-0.4320	2.6539	-0.1539	3.0257	-0.5257
3.0	3.0	3.2637	-0.2637	3.2742	-0.2742	2.9704	0.0296	2.9320	0.0680	2.6539	0.3461	3.0257	-0.0257
3.5	2.5	3.2184	-0.7184	3.3256	-0.8256	3.4411	-0.9411	3.4113	-0.9113	3.1620	-0.6620	3.4965	-0.9965
3.5	3.0	3.2183	-0.2183	3.3421	-0.3421	3.4411	-0.4411	3.4113	-0.4113	3.1620	-0.1620	3.4965	-0.4965
3.5	3.5	3.2214	0.2786	3.3543	0.1457	3.4411	0.0589	3.4113	0.0887	3.1620	0.3380	3.4965	0.0035
4.0	2.5	3.1764	-0.6764	3.3773	-0.8773	3.9118	-1.4118	3.8906	-1.3906	3.6701	-1.1701	3.9673	-1.4673
4.0	3.0	3.1767	-0.1767	3.3933	-0.3933	3.9118	-0.9118	3.8906	-0.8906	3.6701	-0.6701	3.9673	-0.9673
4.5	2.5	3.1396	-0.6396	3.4217	-0.9217	4.3825	-1.8825	4.3699	-1.8699	4.1782	-1.6782	4.4381	-1.9381
4.5	3.0	3.1411	-0.1411	3.4287	-0.4287	4.3825	-1.3825	4.3699	-1.3699	4.1782	-1.1782	4.4381	-1.4381
4.5	3.5	3.1475	0.3525	3.4289	0.0711	4.3825	-0.8825	4.3699	-0.8699	4.1782	-0.6782	4.4381	-0.9381
4.5	4.0	3.1602	0.8398	3.4192	0.5808	4.3825	-0.3825	4.3699	-0.3699	4.1782	-0.1782	4.4381	-0.4381
5.0	3.0	3.1132	-0.1132	3.4655	-0.4655	4.8532	-1.8532	4.8491	-1.8491	4.6864	-1.6864	4.9089	-1.9089
5.0	3.5	3.1227	0.3773	3.4637	0.0363	4.8532	-1.3532	4.8491	-1.3491	4.6864	-1.1864	4.9089	-1.4089
5.0	4.0	3.1395	0.8605	3.4569	0.5431	4.8532	-0.8532	4.8491	-0.8491	4.6864	-0.6864	4.9089	-0.9089
5.5	3.5	3.1078	0.3922	3.5008	-0.0008	5.3239	-1.8239	5.3284	-1.8284	5.1945	-1.6945	5.3797	-1.8797
5.5	4.0	3.1298	0.8702	3.4947	0.5053	5.3239	-1.3239	5.3284	-1.3284	5.1945	-1.1945	5.3797	-1.3797
6.0	4.0	3.1303	0.8697	3.5328	0.4672	5.7946	-1.7946	5.8077	-1.8077	5.7026	-1.7026	5.8505	-1.8505
6.0	4.5	3.1678	1.3322	3.5245	0.9755	5.7946	-1.2946	5.8077	-1.3077	5.7026	-1.2026	5.8505	-1.3505
6.5	4.0	3.1390	0.8610	3.5713	0.4287	6.2653	-2.2653	6.2870	-2.2870	6.2107	-2.2107	6.3213	-2.3213
6.5	4.5	3.1818	1.3182	3.5630	0.9370	6.2653	-1.7653	6.2870	-1.7870	6.2107	-1.7107	6.3213	-1.8213
6.0	9.0	9.1277	-0.1277	9.1145	-0.1145	5.7946	3.2054	5.8077	3.1923	5.7026	3.2974	5.8505	3.1495
6.5	9.0	9.0993	-0.0993	9.0911	-0.0911	6.2653	2.7347	6.2870	2.7130	6.2107	2.7893	6.3213	2.6787
6.5	9.5	9.1000	0.4000	9.0918	0.4082	6.2653	3.2347	6.2870	3.2130	6.2107	3.2893	6.3213	3.1787
7.0	8.5	9.0738	-0.5738	9.0689	-0.5689	6.7360	1.7640	6.7662	1.7338	6.7189	1.7811	6.7921	1.7079
7.0	9.0	9.0727	-0.0727	9.0625	-0.0625	6.7360	2.2640	6.7662	2.2338	6.7189	2.2811	6.7921	2.2079
7.0	9.5	9.0742	0.4258	9.0668	0.4332	6.7360	2.7640	6.7662	2.7338	6.7189	2.7811	6.7921	2.7079
7.5	9.0	9.0485	-0.0485	8.9368	0.0632	7.2067	1.7933	7.2455	1.7545	7.2270	1.7730	7.2629	1.7371
7.5	9.5	9.0504	0.4496	9.0428	0.4572	7.2067	2.2933	7.2455	2.2545	7.2270	2.2730	7.2629	2.2371
8.0	8.5	9.0275	-0.5275	9.0320	-0.5320	7.6774	0.8226	7.7248	0.7752	7.7351	0.7649	7.7337	0.7663
8.0	9.0	9.0270	-0.0270	9.0249	-0.0249	7.6774	1.3226	7.7248	1.2752	7.7351	1.2649	7.7337	1.2663
8.5	8.5	9.0087	-0.5087	9.0222	-0.5222	8.1481	0.3519	8.2041	0.2959	8.2432	0.2568	8.2045	0.2955
8.5	9.0	9.0080	-0.0080	9.0191	-0.0191	8.1481	0.8519	8.2041	0.7959	8.2432	0.7568	8.2045	0.7955
8.5	9.5	9.0098	0.4902	9.0202	0.4798	8.1481	1.3519	8.2041	1.2959	8.2432	1.2568	8.2045	1.2955
9.0	9.0	8.9915	0.0085	9.0091	-0.0091	8.6188	0.3812	8.6833	0.3167	8.7513	0.2487	8.6752	0.3248
9.5	9.0	8.9771	0.0229	8.9976	0.0024	9.0895	-0.0895	9.1626	-0.1626	9.2595	-0.2595	9.1460	-0.1460
9.5	9.5	8.9780	0.5220	8.9973	0.5027	9.0895	0.4105	9.1626	0.3374	9.2595	0.2405	9.1460	0.3540
10.0	8.5	8.9662	-0.4662	8.9865	-0.4865	9.5602	-1.0602	9.6419	-1.1419	9.7676	-1.2676	9.6168	-1.1168
10.0	9.0	8.9643	0.0357	8.9851	0.0149	9.5602	-0.5602	9.6419	-0.6419	9.7676	-0.7676	9.6168	-0.6168
10.5	9.0	8.9529	0.0471	8.9721	0.0279	10.0309	-1.0309	10.1212	-1.1212	10.2757	-1.2757	10.0876	-1.0876
11.0	9.0	8.9425	0.0575	8.9587	0.0413	10.5016	-1.5016	10.6004	-1.6004	10.7838	-1.7838	10.5584	-1.5584
Error		$\varepsilon_{Type-1} = 4.2887$		$\varepsilon_{Type-2} = 4.2767$		$\varepsilon_{Huber} = 5.0208$		$\varepsilon_{Hampel} = 5.0646$		$\varepsilon_{Tukey} = 5.4003$		$\varepsilon_{Andrews} = 4.9785$	

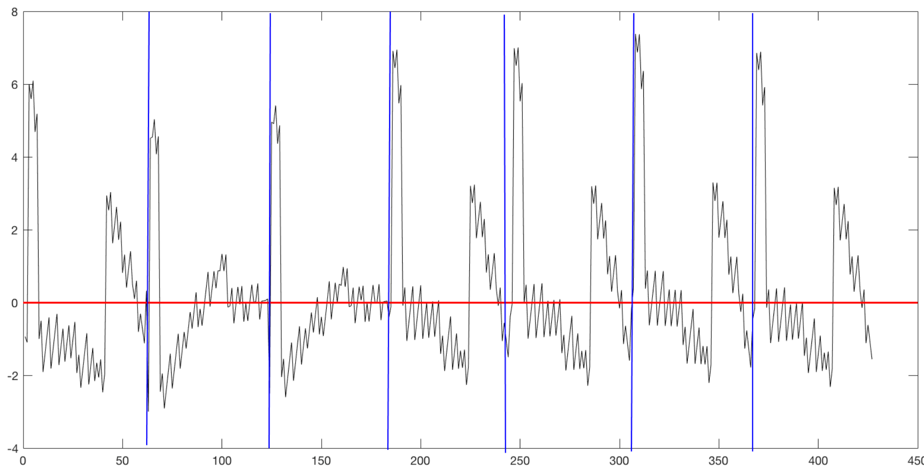

FIGURE 2. Error amount graph belonging to data.

TABLE 2. Linear regression models and errors

Methods	Models	Errors
Least Square Method	$\widehat{Y} = 1.1909 + 0.8119X$	$\varepsilon_{LSM} = 4.7141$
Type-1 Fuzzy Clustering	$\widehat{Y}_1 = 3.5395 - 0.0920X_{1j}$ $\widehat{Y}_2 = 9.3577 - 0.0414X_{2j}$	$\varepsilon_{Type-1} = 4.2887$
Type-2 Fuzzy Clustering	$\widehat{Y}_1 = 2.9495 + 0.1848X_{1j}$ $\widehat{Y}_2 = 9.3152 - 0.0601X_{2j}$	$\varepsilon_{Type-2} = 4.2767$
Huber Method	$\widehat{Y} = 0.1462 + 0.9414X$	$\varepsilon_{Huber} = 5.0208$
Hampel Method	$\widehat{Y} = 0.0564 + 0.9585X$	$\varepsilon_{Hampel} = 5.0646$
Tukey Method	$\widehat{Y} = 0.3949 + 1.0162X$	$\varepsilon_{Tukey} = 5.4003$
Andrews Method	$\widehat{Y} = 0.2010 + 0.9416X$	$\varepsilon_{Andrews} = 4.9785$

methods are obtained that LSM is 4.7141, type-1 fuzzy clustering is 4.2887, type-2 fuzzy clustering is 4.2767, Huber method is 5.0208, Hampel method is 5.0646, Tukey method is 5.4003, Andrews methods is 4.9785. As can be seen from the results, the model with the lowest error is the model obtained from type-2 fuzzy clustering. It can be said that if there are outlier observations in the data set, the method that using type-2 fuzzy clustering can be preferable as an effective method.

REFERENCES

[1] Zadeh, L. A., Fuzzy Sets, *Information and Control*, 8 (1965), 338-353.

[2] Dunn, J. C., A Fuzzy Relative of ISODATA Process and its Use in Detecting Compact Well-Separated Clusters, *Journal of Cybernetics*, 3 (1973), 32-57.

[3] Zadeh, L. A., Fu, K., Tanaka K. and Shimura, M., Fuzzy Sets and Their Applications to Cognitive and Decision Processes, 1975.

[4] Bezdek, J. C., Ehrlich, R. and Full, W., FCM: The fuzzy C-means clustering algorithm, *Computers and Geosciences*, 10 (1984), 191-203.

[5] Mendel, J. and John, R. I., Type-2 fuzzy sets made simple, *IEEE Transactions on Fuzzy Systems*, 10(2) (2002), 117-127.

[6] Hwang C. and Rhee, F., Uncertain fuzzy clustering: interval type-2 fuzzy approach to C-means, *IEEE Transactions on Fuzzy Systems*, 15(1) (2007), 107-120.

[7] Dalkilic, E. T. and Apaydin,A., A fuzzy adaptive network approach to parameter estimation in cases where independent variable come from an exponential distribution, *Journal of Computational and Applied Mathematics*, 233, (2009), 36-45.

[8] Juang, C. F., Huang, R. B. and Lin, Y. Y., A recurrent self-evolving interval type-2 fuzzy neural network for dynamic system processing, *IEEE Trans. Fuzzy Syst.*, 17 (2009), 1092-1105.

[9] Fazel Zarandi, M. H., Turksen, I. B. and Gamasaee, R., A type-2 fuzzy C-regression clustering algorithm for Takagi-Sugeno system identification and its application in the steel industry, *Inform. Sci.*, 187 (2012), 179-203.

[10] Enke, D. and Mehdiyew, N., Type-2 fuzzy clustering and type-2 fuzzy inference neural network for the prediction of short-term interest rates, *Procedia Computer Science*, 20 (2013), 115-120.

[11] Kalhori, M. R. N. and Fazel Zarandi, M. H., Interval type-2 credibilistic clustering for pattern recognition, *Pattern Recognition*, 48 (2015), 3652-3672 .

[12] Golsefid, S. M. M. and Zarandi, M. H. F., Dual-centers type-2 fuzzy clustering framework and its verification and validation indices, *Applied Soft Computing*, 47 (2016), 600-613.

[13] Hwak, K. C., A design of incremental granular model using context-based interval type-2 fuzzy C-means algorithm, *IEICE Trans. Inf. and Syst.*, E99-D (1) (2016), 309-312.

[14] Rubio, E., Castillo, O., Valdez, F., Melin, P., Gonzalez, C. and Martinez, G., An extension of the fuzzy possibilistic clustering algorithm using type-2 fuzzy logic techniques, *Adv. Fuzzy Systems*, 2017 (23) (2017).

[15] Mendel, J., Computing derivatives in interval type-2 fuzzy logic systems, *IEEE Trans. Fuzzy Syst.*, 12(1) (2004), 84-98.

[16] Rhee, F. and Hwang, C., A type-2 fuzzy C-means clustering algorithm, *in Proc. Joint Conf. IFSA/NAFIPS*, (2001), 1926-1919.

[17] Karnik, N. N. and Mendel, J. M., Type-2 fuzzy logic systems: type-reduction, *in IEEE Systems, Man, Cybernet Conference*, San Diego, CA, (1998).

[18] Rousseeuw, P.J and Leroy, A.M., Robust regression and outlier detection. John Willey and Son, 1987.

[19] Huynh, H. A., Comparison of For Approaches to Robust Regression, *Psychological Bulletin*, 92 (1982), 505-512.

[20] Huber, P.J., Robust statistics. John Willey and Son, 1981.

[21] Hampel, F.R., Ronchetti, E.M., Rousseeuw, P.J. and Stahel, W.A., Robust Statistics, John-Willey and Sons, New-York, 1986.

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AN ADAPTIVE EXTENDED KALMAN FILTERING APPROACH TO NONLINEAR DYNAMIC GENE REGULATORY NETWORKS VIA SHORT GENE EXPRESSION TIME SERIES

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Abstract. Sleep spindles, which are believed to have important role of reinforcing the sleep duration, are the characteristic wave shapes that are seen in non-REM sleep stage. Detecting and analyzing the wave forms of spindles as well as determining the areas and durations of sleep spindles are quite important to understand the sleeping process thoroughly. However, the fact that spindles have temporary regime features and lower amplitudes compared to the background EEG signals makes resolving and distinguishing between them difficult. Although there have been extensive research on the decomposition of EEG signals and about the general characteristics of the spindles, the existing studies do not decompose the components in a dynamic fashion. This study takes this argument as its starting point and comes up with a methodology to detect the spindles in the sleep EEG. In particular, this study separates EEG signals into trend and cycle components via frequency analysis, where the methodology allows for system parameters and the components to be estimated simultaneously. Since the methodology allows for the parameters to vary over time, observing the time patterns of the estimated parameters have the potential to reveal further information about the sleep process.

1. Introduction

Gene regulation is one of the most amazing processes taking place in living cells. From the sequences of hundreds of thousands of genes, cells must decide which genes to express at a particular time. As the development of the cell evolves, different conditions and functions require an efficient mechanism to turn on the required genes leaving the others behind. Cells may also activate new genes to respond to the environmental changes effectively and play specific roles. The knowledge of which

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gene triggers a particular genetic condition may help preventing the potentially harmful effects by turning that gene off. For instance, cancer may be controlled by deactivating the gene that causes it.

Gene expression is the production of functional gene materials, e.g., mRNA. The level of gene functionality may be measured using microarrays or gene chips to produce data on gene expression. Using this data reasonably may help us to have an understanding of how the genes are interacting in a living organism.

Different genes may cooperate to produce a particular reaction while a gene may repress other genes as well. The potential benefits of gene regulation may be obtained if only a complete and accurate picture of gene interactions is available. A network specifying how different genes are interconnected may go a long way in helping us to understand the gene regulation mechanism. The control and interaction of genes may be described through a gene regulatory network.

DNA microarray technology has provided an efficient way of measuring the expression levels of thousands of genes in a single experiment on a single â€œchip.â€İ It enables the monitoring of expression levels of thousands of genes simultaneously. Measuring gene expression levels in different conditions may prove useful techniques in medical diagnosis, treatment, and drug design. In order to infer useful biological information and determine the relationships between individual genes, many research efforts have currently focused on clustering.

Recently, there has been an increasing interest of research to reconstruct models for gene regulatory networks from time series data. Obviously, choosing a good model that fits gene regulatory networks is essential to make a meaningful analysis on the expression data.

Many gene expression experiments produce short time series data with only a few time points due to its high measurement costs. The time series usually represents the dynamic response of an organism to a change in conditions, e.g., application of some drug or other treatment. Therefore, it is highly desired to extract the functional information from the data on the time series of gene expressions, and the modeling of gene expression time series has become an increasingly interesting field of research.

Since it is well known that the gene expression is an inherently stochastic phenomenon, the network should be of a â€œstochasticâ€İ nature. Recently, dynamic modeling of gene regulatory networks from time series data has received more and more research interest.

The state-space model assumes that the gene expression value depends not only on the current internal state variables but also on the external inputs, which reflects the nature of a dynamic network. Unfortunately, most results reported on state-space models have been focused on linear systems, and therefore, the non-linear phenomenon of the gene networks may not be taken into account. Most of the literature available concerning the modeling of the time series of gene expressions have not explicitly dealt with these two features, and therefore, there is a need to seek alternative approaches to identify the parameters of a nonlinear stochastic gene regulatory network through real-time gene expression time series. In search of such an approach, EKF approach appears to be an appropriate candidate.

The traditional KF addresses the general problem of estimating the state of a discrete-time system governed by a linear stochastic difference equation. EKF linearizes about the current mean and covariance, and therefore may handle nonlinearities that may be associated either with the process model or with the observation model, or with the both. On the other hand, EKF is known as an effective recursive estimator of process variables, which may be suitable for identifying large number of parameters using a short time series.

In paper [1], the gene regulatory network is considered as a nonlinear dynamic stochastic model that consists of the gene measurement equation and the gene regulation equation. In order to reflect the reality, it is considered that the gene measurement from microarray was noisy; and it is assumed that the gene regulation equation was a nonlinear dynamic process which is autoregressively stochastic where the nonlinearity stems from the inherently non-linear regulatory relationship and the degree among genes. After specifying the model structure, they applied the EKF algorithm for identifying both the parameters of the model and the actual value of the levels of gene expression. Note that the EKF algorithm is an online estimation algorithm that may identify a large number of parameters (including parameters of nonlinear functions) through iterative procedure by using a small number of observations. Four sets of data regarding the real-world gene expression were processed to demonstrate the effectiveness of the EKF algorithm, and the obtained models are evaluated from the aspect of bioinformatics.

The EKF is extensively used in nonlinear state estimation problems. As long as the system characteristics are correctly known, EKF gives the best performance. However, when the system information is partially known or incorrect, EKF may diverge or give biased estimates. An extensive number of works has been published to improve the performance of EKF.

Many researchers have proposed the introduction of a forgetting factor, both into the KF and EKF, to improve the performance. However, there are two fundamental problems with this approach: the incorporation of the optimal forgetting factor into EKF and the selection of the optimal forgetting factor.

In paper [2–5], they proposed a new AEKF with a forgetting factor, and two methods are analyzed for the selection of the optimal forgetting factor. The stability properties of the proposed filter are also investigated. Results of the stability analysis show that the proposed filter is an exponential observer for nonlinear deterministic systems.

In this study, application of the developed model on the gene regulatory networks has been examined. With the aim of corroborating estimation method, it has been decided that the AEKF was proper for being used and malaria gene expression has been applied for the set of data on the time series. A results have been compared with the results of the former research [1], and it has been understood that the estimation results obtained through the developed model were more preferable.

2. GENE MODEL AND PROBLEM FORMULATION

The measured gene expression levels may be modeled as

$$y_i(k) = x_i(k) + v_i(k) \quad i = 1, 2, \ldots, n \quad k = 1, 2, \ldots, m. \tag{1}$$

where $y(k) = [y_1(k), y_2(k), \ldots, y_n(k)]^T$ is the measurement data from microarray experiments at time k with $y_i(k)$ describing the ith gene expression levels at time k, $x_i(k)$ are the actual levels of ith gene expression which stand for mRNA concentrations and/or protein concentrations at time k, $v_i(k)$ is the measurement noise, n is the number of the genes, and m is the number of the measurement time points. Here, $v(k) = [v_1(k), v_2(k), \ldots, v_n(k)]^T$ s assumed to be a zero-mean Gaussian white noise sequence with constant covariance $R > 0$, i.e., $v(k)\ N(0, R)$. The gene regulatory network containing n genes is described by the following discrete-time nonlinear stochastic dynamical system **??**:

$$x_i(k+1) = \sum_{j=1}^{n} a_{ij} x_j(k) + \sum_{j=1}^{n} b_{ij} f_{ij}\left(x_j(k), \mu_j\right) + I_{ai} + \xi_i(k) \tag{2}$$
$$i = 1, 2, \ldots, n \quad k = 1, 2, \ldots, m-1.$$

Where $A = (a_{ij})_{nn}$ is the linear regulatory relationship and the degree among genes, $B = (b_{ij})_{n\times n}$ represents the nonlinear regulatory relationship and degree among genes; $I_0 = [I_{01}, I_{02}, \ldots, I_{0n}]^T$ is the constant vector with I_{0i} standing for the external bias on the ith gene; $\xi(k) = [\xi_1(k), \xi_2(k), \ldots, \xi_n(k)]^T \sim N(0, Q_0)$; and the nonlinear function $f_j\left(x_j, \mu_j\right)$ is given by

$$f_j\left(x_j, \mu_j\right) = \frac{1}{1 + e^{-\mu_j x_j}} \tag{3}$$

with μ_j being a parameter to be identified. Setting

$$\mu(k) = [\mu_1, \mu_2, \ldots, \mu_n]^T \tag{4}$$

and

$$f(x(k), \mu) = [f_1(x_1(k), \mu_1), f_2(x_2(k), \mu_2) \ldots, f_n(x_n(k), \mu_n)]^T \tag{5}$$

we can rewrite 1 and 2 in the following vector form:

$$x(k+1) = Ax(k) + Bf(x(k), \mu) + I_0 + \xi(k) \tag{6}$$

$$y(k) = x(k) + v(k) \tag{7}$$

Letting

$$A_e = [a_{11}, a_{21}, \ldots, a_{n1}, a_{12}, a_{22}, \ldots, a_{n2}, a_{1n}, a_{2n}, \ldots, a_{nn}]^T \tag{8}$$

$$B_e = [b_{11}, b_{21}, \ldots, b_{n1}, b_{12}, b_{22}, \ldots, b_{n2}, b_{1n}, b_{2n}, \ldots, b_{m}]^T \tag{9}$$

$$\mu(k) = [\mu_1, \mu_2, \dots, \mu_n]^T \tag{10}$$

$$\theta = \left[A_e^T B_e^T \mu^T I_0^T\right]^T \tag{11}$$

all the parameters to be estimated are denoted by $\theta = \left[A_e^T B_e^T \mu^T I_0^T\right]^T$ In order to establish the gene expression model 2, it is necessary to identify the parameter vector θ. In this paper, we aim at estimating the parameters of the model2 via the AEKF method from the measurement data.

3. THE ADAPTIVE EKF APPROACH TO PARAMETER ESTIMATION

The data set is from the time series of malaria gene expression [7]. It consists of 530 genes expressed in 48 equally spaced time points. We choose the time series of expressions of the first six genes given by $z = [z_1, z_2, z_3, z_4, z_5, z_6]$

In this study, regulatory network models have been examined; and in the framework of the model 6, the AEKF proposed in [2,3] and used in the real data (Figure 7, Table 1) application studies have been conducted. Estimation results were given in Figure 2- 7 .State estimation results were given in Figure 2- 6.

In order to compare the estimated observation and the squares of actual observation values, error criterion is used and given in Figure-1. As it may be seen in Figure 1 adaptive EKF has a value of estimation more accurate than the normal EKF.

4. Conclusion

In this paper research, application of the developed model on the gene regulatory networks has been examined. With the aim of corroborating the Kalman Filter estimation method, it has been decided that the adaptive extended Kalman filter was proper for being used and malaria gene expression has been applied for the set of data on the time series. The results have been compared with the results of the former research [1] and it has been understood that the estimation results obtained through the developed model were more preferable. AEKF has a value of estimation more accurate than the normal EKF.

5. Extended Kalman Filter

The optimum linear filtering and prediction methods introduced by Kalman (1960) have been considered as one of the greatest achievements among the theories of estimation. The Kalman Filter solves the problem of estimating the instantaneous states of a linear dynamic system distorted by Gaussian white noise, using measurements that are linear functions of the system state and corrupted by additive white noise. Therefore, it is the appropriate estimation procedure for the state space systems. However, since the the simultaneous estimation of the parameters and the state problem has a nonlinear nature, the standard linear KF needs to be modified to solve such a problem. The EKF is one of the most popular estimation techniques largely investigated for state estimation of nonlinear systems. It consists

TABLE 1. Data Set.

Z=[4,314	2,271	2,789	3,788	4,162	2,208
3,2789	1,8179	2,3653	2,5943	2,9244	2,0724
1,6684	0,7923	1,4219	1,2601	0,9809	0,9977
1,7445	1,2726	1,3902	1,8115	2,1758	1,3763
1,0716	0,7282	1,068	0,9243	0,9998	0,7307
0,9868	0,5669	0,8739	0,8472	0,8891	0,4528
0,99	0,528	0,649	0,831	0,745	0,489
0,778	0,4488	0,7413	0,624	0,5897	0,5092
0,8355	0,5778	0,5219	0,9553	0,9722	0,4854
0,5796	0,3129	0,5056	0,4316	0,3823	0,3545
0,491	0,254	0,368	0,423	0,37	0,258
0,3782	0,2401	0,3691	0,2943	0,3343	0,2504
0,3446	0,2036	0,3232	0,2634	0,3019	0,2264
0,146	0,126	0,173	0,136	0,128	0,117
0,1465	0,1608	0,1002	0,128	0,1482	0,1313
0,2114	0,1577	0,1133	0,1028	0,1168	0,1554
0,2061	0,171	0,1239	0,0811	0,1129	0,1666
0,172	0,211	0,101	0,097	0,118	0,214
0,1678	0,2138	0,0642	0,0518	0,0839	0,2089
0,17	0,262	0,063	0,049	0,081	0,279
0,2155	0,3233	0,0632	0,0427	0,0948	0,2675
0,2226	0,2806	0,0655	0,0524	0,0917	0,3096
0,2101	0,3582	0,0467	0,0496	0,0995	0,3894
0,1976	0,4357	0,028	0,0469	0,1074	0,4691
0,2375	0,3711	0,0608	0,0544	0,1016	0,4062
0,2131	0,4639	0,041	0,0475	0,109	0,5582
0,253	0,641	0,044	0,075	0,128	0,592
0,1947	0,6707	0,0391	0,0707	0,1381	0,7738
0,2148	0,8082	0,085	0,1066	0,1739	0,8656
0,2349	0,9458	0,1309	0,1425	0,2098	0,9574
0,265	1,144	0,205	0,21	0,303	1,251
0,6056	1,3391	0,3874	0,5808	0,5905	1,2578
1,013	1,9144	0,9661	1,0017	0,8967	1,9266
1,4945	2,0826	1,3078	1,7174	1,6631	2,0004
1,991	2,319	1,8535	1,9343	1,7467	2,4258
2,5285	2,5555	2,493	2,2905	2,3982	2,4844
1,7578	2,9656	1,7872	2,0121	1,8186	2,8291
1,8211	2,3457	2,0033	1,9548	1,5144	2,3201
2,5851	3,3361	3,4185	4,0059	3,6226	7,6102
3,884	3,2779	4,6765	4,5845	2,8834	2,9527
3,8805	3,1208	4,7711	5,1805	3,6588	2,7262
6,0726	4,1553	6,6787	6,1378	6,9146	4,197
5,4836	2,2738	4,1907	4,4675	5,1801	2,3114
4,6334	2,0388	4,6189	4,125	4,6347	2,3628
3,2207	1,8348	2,5593	3,2643	3,9337	2,0484
1,0636	1,5575	2,3816	1,9541	2,8011	1,6607
1,561	1,9512	2,9104	2,6247	3,4341	2,0003
1,1717	1,4513	2,3003	1,9389	2,1344	1,3854]

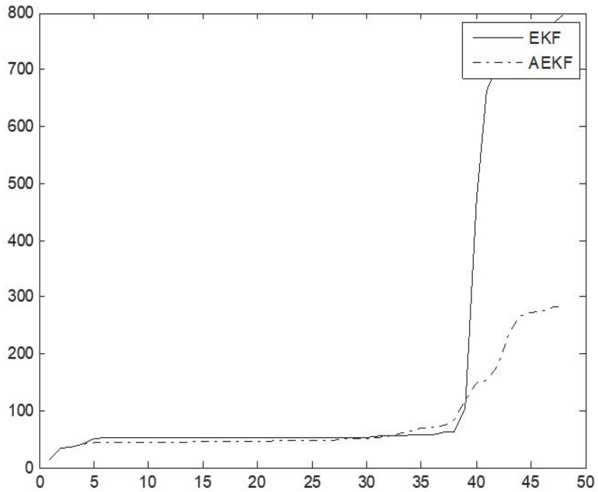

FIGURE 1. Squares Error

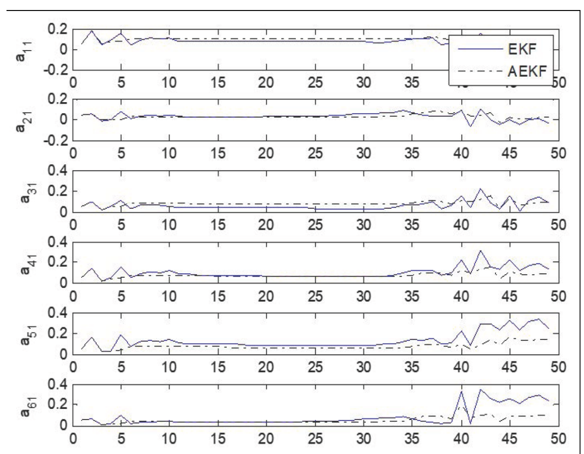

FIGURE 2. Estimation of parameters

of using the standard Kalman filter equations to the first-order approximation of the nonlinear model about the last estimate. It should also be noted that the EKF is very sensitive to its initialization and filter divergence is inevitable if the arbitrary matrices have not been chosen appropriately. [2, 8]

A non-linear state space model can be written as

$$x_t = f(x_{t-1}, t-1) + G_{t-1} w_{t-1} \tag{12}$$

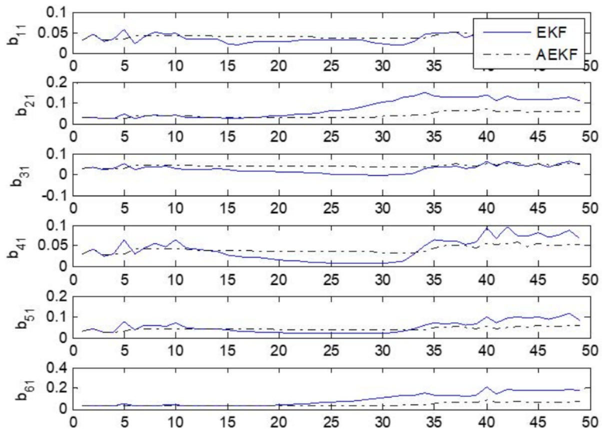

FIGURE 3. Estimation of parameters

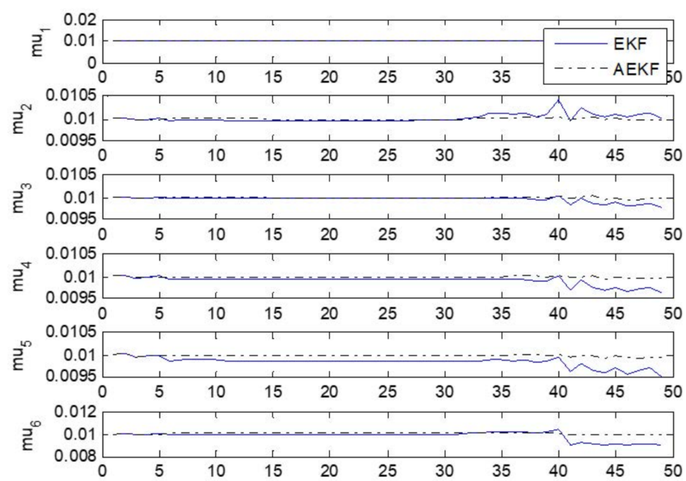

FIGURE 4. Estimation of parameters

$$y_t = h(x_t, t) + v_t \tag{13}$$

where f_t and h_t are vector-valued functions, W_t and v_t are uncorrelated zero mean white noise sequences with covariance matrix Q_t and R_t respectively. The EKF algorithm is

$$P_0 = \mathrm{Cov}(x_0) \tag{14}$$

$$\bar{x}_0 = E(x_0) \tag{15}$$

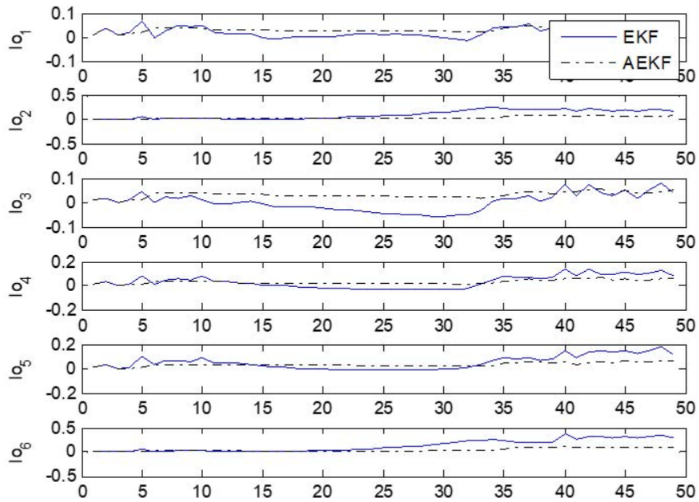

FIGURE 5. Estimation of parameters

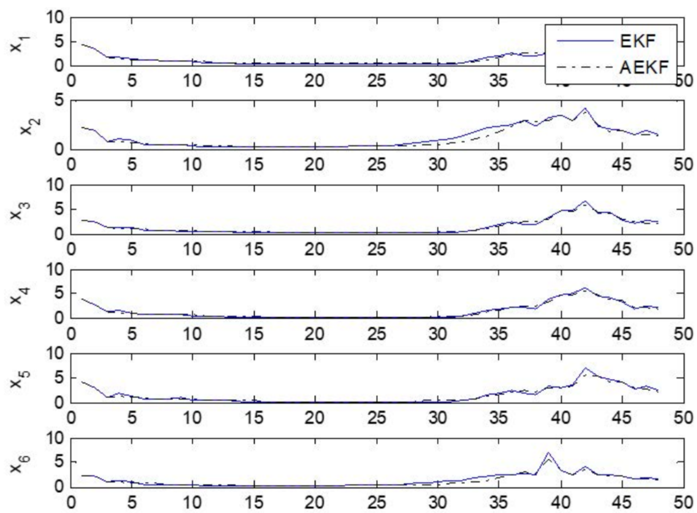

FIGURE 6. Estimation of parameters

As it is shown in [2] and [8], the updating equations are:

$$P_{t|t-1} = \alpha_t \left[\frac{\partial f_{t-1}}{\partial x_{t-1}} (\hat{x}_{t-1}) \right] P_{t-1} \left[\frac{\partial f_{t-1}}{\partial x_{t-1}} (\hat{x}_{t-1}) \right] + \alpha_t G_{t-1} Q_{t-1} G_{t-1} \tag{16}$$

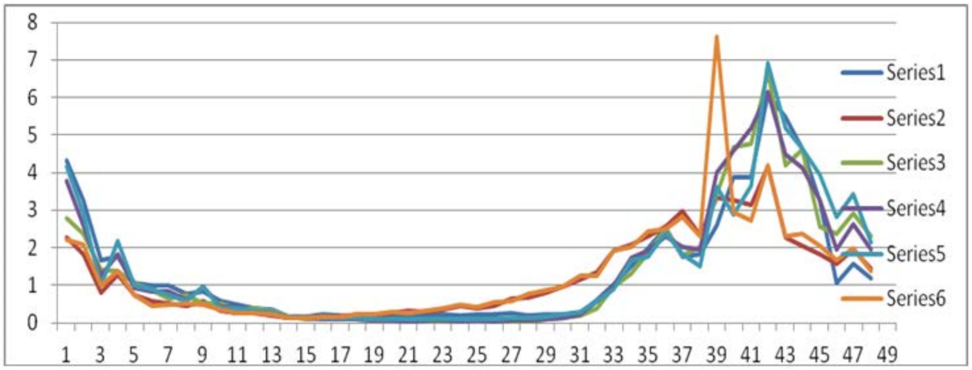

FIGURE 7. Real data

$$\hat{x}_{t|t-1} = f_{t-1}\left(\hat{x}_{t-1}\right) \tag{17}$$

$$K_k = P_{t|t-1}\left[\frac{\partial h_t}{\partial x_t}\left(\hat{x}_{t|t-1}\right)\right]\left[\left[\frac{\partial h_t}{\partial x_t}\left(\hat{x}_{t|t-1}\right)\right]P_{t|t-1}\left[\frac{\partial h_t}{\partial x_t}\left(\hat{x}_{t|t-1}\right)\right]' + R_t\right]^{-1} \tag{18}$$

$$P_t = \left[I - t_t\left[\frac{\partial h_t}{\partial x_t}\left(\hat{x}_{t|t-1}\right)\right]P_{t|t-1}\right. \tag{19}$$

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t\left[y_t - h_t\left(\hat{x}_{t|t-1}\right)\right] \tag{20}$$

$t = 1, 2, \ldots$

REFERENCES

[1] Wang, Z., Liu, X., Liu, Y., Liang, J., Vinciotti, V., An Extended Kalman Filtering Approach to Modeling Nonlinear Dynamic Gene Regulatory Networks via Short Gene Expression Time Series, Zidong Wang, Xiaohui Liu, Yurong Liu, Jinling Liang, and Veronica Vinciotti, *IEEE/ACM Transactions on Computational Biology and Bioinformatic*, 6 (3) (2009), 410-419.

[2] Özbek, L, Efe, M., An Adaptive Extended Kalman Filter with Application To Compartment Models, *Communications In Statistics-Simulation and Computation*, 33 (1) (2004), 145-158.

[3] Özbek, L., Aliev, F.A., Comments on "Adaptive Fading Kalman Filter With An Application", *Automatica*, 34 (12) (1998), 1663-1664.

[4] Jwo, D., Weng, T., An Adaptive Sensor Fusion Method with Applications in Integrated Navigation, *The Journal of Navigation*, 61 (2008), 705-721.

[5] Biçer, C., Babacan., E.K., Özbek, L., Stability of the adaptive fading extended Kalman filter with the matrix forgetting factor, *Turk. J. Elec. Eng. and Comp. Sci.*, 20 (5) (2012), 819-833.

[6] Chen, L., Aihara, K., Chaos and Asymptotical Stability in Discrete-Time Neural Networks, *Physica D: Nonlinear Phenomena*, 104 (1997), 286-325.

[7] Bozdech, Z., Llinas, M., Pulliam, B.L., Wong, E.D., Zhu, J., The Transcriptome of the Intraerythrocytic Developmental Cycle of Plasmodium Falciparum, *PLoS Biology*, 1 (1) (2003), 85-100.

[8] Özbek, L., Kalman Filtresi, Akademisyen Yayınevi, 2018.

https://communications.science.ankara.edu.tr

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HARARY ENERGY OF COMPLEMENT OF LINE GRAPHS OF REGULAR GRAPHS

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Abstract. The Harary matrix of a graph G is defined as $H(G) = [h_{ij}]$, where $h_{ij} = \frac{1}{d(v_i, v_j)}$, if $i \neq j$ and $h_{ij} = 0$, otherwise, where $d(v_i, v_j)$ is the distance between the vertices v_i and v_j in G. The H-energy of G is defined as the sum of the absolute values of the eigenvalues of Harary matrix. Two graphs are said to be H-equienergetic if they have same H-energy. In this paper we obtain the H-energy of the complement of line graphs of certain regular graphs in terms of the order and regularity of a graph and thus constructs pairs of H-equienergetic graphs of same order and having different H-eigenvalues.

1. Introduction

Let G be a simple, undirected, connected graph with n vertices and m edges. Let the vertices of G be labeled as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* of a graph G is the square matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are the *adjacency eigenvalues* of G, and they are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These form the *adjacency spectrum* of G [4]. Two graphs are said to be cospectral if they have same spectra.

The *distance* between the vertices v_i and v_j, denoted by $d(v_i, v_j)$, is the length of the shortest path joining v_i and v_j. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G. A graph G is said to be r*-regular graph* if all of its vertices have same degree equal to r.

The *Harary matrix* [9] of a graph G is a square matrix $H(G) = [h_{ij}]$ of order n, where

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$$h_{ij} = \begin{cases} \frac{1}{d(v_i,v_j)}, & \text{if} \quad i \neq j \\ 0, & \text{if} \quad i = j. \end{cases}$$

The Harary matrix was used in the study of molecules in the quantitative structure property relationship (QSPR) models [9].

The *Harary index* defined as the sum of the reciprocal of the distances between all pairs of vertices and it can be derived from the Harary matrix. It has interesting properties in structure-property correlations [11, 16].

The eigenvalues of $H(G)$ labeled as $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ are said to be the *Harary eigenvalues* or *H-eigenvalues* of G and their collection is called *Harary spectrum* or *H-spectrum* of G. Two non-isomorphic graphs are said to be *H-cospectral* if they have same H-spectra.

The *Harary energy* or *H-energy* of a graph G, denoted by $HE(G)$, is defined as [5]

$$HE(G) = \sum_{i=1}^{n} |\xi_i| \ . \tag{1}$$

The Harary energy is defined in full analogy with the *ordinary graph energy* $E(G)$, defined as [6]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ . \tag{2}$$

The ordinary graph energy has a relation with the total π-electron energy of a molecule in quantum chemistry [10]. Bounds for the Harary energy of a graph are reported in [3, 5].

Two connected graphs G_1 and G_2 are said to be *Harary equienergetic* or *H-equienergetic* if $HE(G_1) = HE(G_2)$. The H-equienergetic graphs are reported in [12, 13]. The distance energy of complements of iterated line graphs of regular graphs has been obtained in [8]. In this paper we use similar technique of [8] to obtain the H-energy of the complement of line graphs of certain regular graphs and thus construct H-equienergetic graphs having different H-spectra.

The *complement* of a graph G is a graph $\overline{G}$, with vertex set same as of G and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *line graph* of G, denoted by $L(G)$ is the graph whose vertices corresponds to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. For $k = 1, 2, \ldots$ the k-th iterated line graph of G is defined as $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$ [7].

If G is a regular graph of order n_0 and of degree r_0 then the line graph $L(G)$ is a regular graph of order $n_1 = (n_0 r_0)/2$ and of degree $r_1 = 2r_0 - 2$. Consequently the order and degree of $L^k(G)$ are [1, 2]

$$n_k = \frac{r_{k-1} n_{k-1}}{2} \tag{3}$$

and

$$r_k = 2r_{k-1} - 2, \tag{4}$$

where n_i and r_i stands for order and degree of $L^i(G)$, $i = 0, 1, \ldots$.

Therefore

$$r_k = 2^k r_0 - 2^{k+1} + 2 \tag{5}$$

and

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2) \tag{6}$$

We need following results.

Theorem 1. *[4] If G is an r-regular graph, then its maximum adjacency eigenvalue is equal to r.*

Theorem 2. *[15] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $L(G)$ are*

$$\begin{array}{rll} \lambda_i + r - 2, & i = 1, 2, \ldots, n, & \textit{and} \\ -2, & n(r-2)/2 \ \textit{ times}. & \end{array}$$

Theorem 3. *[14] Let G be an r-regular graph of order n. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, then the adjacency eigenvalues of $\overline{G}$, the complement of G, are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \ldots, n$.*

Theorem 4. *[3] Let G be an r-regular graph of order n and let $diam(G) \leq 2$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, then its H-eigenvalues are $\frac{1}{2}(n + r - 1)$ and $\frac{1}{2}(\lambda_i - 1)$, $i = 2, 3, \ldots, n$.*

Lemma 5. *[8] Let G be an r-regular graph of order n. If $r \leq \frac{n-1}{2}$, then $diam\left(\overline{L^k(G)}\right) = 2$, $k \geq 1$.*

2. Results

Theorem 6. *Let G be an r-regular graph of order n. If $r \leq \frac{n-1}{2}$, then*

$$HE\left(\overline{L(G)}\right) = r(n-2).$$

Proof. Let the adjacency eigenvalues of G be $r, \lambda_2, \ldots, \lambda_n$. From Theorem 2, the adjacency eigenvalues of $L(G)$ are

$$\left.\begin{array}{rll} 2r - 2, & \text{and} & \\ \lambda_i + r - 2, & i = 2, 3, \ldots, n, & \text{and} \\ -2, & n(r-2)/2 \ \text{ times}. & \end{array}\right\} \tag{7}$$

From Theorem 3 and Eq. (7), the adjacency eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{rll} (nr/2)-2r+1, & \text{and} & \\ -\lambda_i - r + 1, & i=2,3,\ldots,n, & \text{and} \\ 1, & n(r-2)/2 \ \text{ times.} & \end{array}\right\} \tag{8}$$

The graph $\overline{L(G)}$ is a regular graph of order $nr/2$ and of degree $(nr/2)-2r+1$. Since $r \leq \frac{n-1}{2}$, by Lemma 5, $diam\left(\overline{L(G)}\right) = 2$. Therefore by Theorem 4 and Eq. (8), the H-eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{rll} (nr-2r)/2, & \text{and} & \\ -(\lambda_i + r)/2, & i=2,3,\ldots,n, & \text{and} \\ 0, & n(r-2)/2 \ \text{ times.} & \end{array}\right\} \tag{9}$$

All adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$ [4]. Therefore $\lambda_i + r \geq 0$, $i = 1, 2, \ldots, n$. Therefore by (9),

$$\begin{aligned} HE\left(\overline{L(G)}\right) &= \frac{nr-2r}{2} + \sum_{i=2}^{n} \frac{(\lambda_i + r)}{2} + |0| \times \frac{n(r-2)}{2} \\ &= r(n-2) \qquad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r. \end{aligned}$$

□

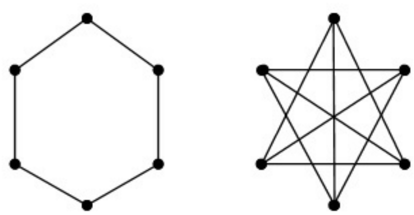

FIGURE 1. Cycle C_6 and $\overline{L(C_6)}$.

Example 7. *Consider the cycle C_6. It satisfies the conditions of Theorem 6. Complement of $L(C_6)$ is shown in the Figure 1. The H-eigenvalues of $\overline{L(C_6)}$ are 4, 0, -0.5, -0.5, -1.5, -1.5. Hence $HE(\overline{L(C_6)}) = 8$ and by Theorem 6 also, $HE(\overline{L(C_6)}) = 8$.*

Corollary 8. *Let G be a regular graph of order n_0 and of degree r_0. Let n_k and r_k be the order and degree respectively of the k-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then*

$$HE(\overline{L^k(G)}) = r_{k-1}(n_{k-1} - 2).$$

Proof. If $r_0 \leq \frac{n_0-1}{2}$, then by Eqs. (3) and (4), we have

$$r_1 = 2r_0 - 2 \leq n_0 - 3 \leq \frac{1}{2}\left(\frac{n_0 r_0}{2} - 1\right) = \frac{n_1 - 1}{2}.$$

Hence

$$r_{k-1} \leq \frac{n_{k-1} - 1}{2}.$$

Therefore by Theorem 6,

$$HE\left(\overline{L^k(G)}\right) = HE\left(\overline{L(L^{k-1}(G))}\right) = r_{k-1}(n_{k-1} - 2).$$

□

Corollary 9. *Let G be a regular graph of order n_0 and of degree r_0. Let n_k and r_k be the order and degree respectively of the k-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then*

$$HE\left(\overline{L^k(G)}\right) = \left[\frac{n_0}{2^{k-1}} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2)\right] - 2(2^{k-1} r_0 - 2^k + 2).$$

3. H-equienergetic graphs

If G_1 and G_2 are the regular graphs of same order and of same degree. Then $L(G_1)$ and $L(G_2)$ are of the same order and of same degree. Further their complements are also of same order and of same degree.

Lemma 10. *Let G_1 and G_2 be regular graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ are H-cospectral if and only if G_1 and G_2 are cospectral.*

Proof. Follows from Eqs. (7), (8) and (9). □

Lemma 11. *Let G_1 and G_2 be regular graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are H-cospectral if and only if G_1 and G_2 are cospectral.*

Theorem 12. *Let G_1 and G_2 be regular, non H-cospectral graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ form a pair of non H-cospectral, H-equienergetic graphs of equal order and of equal number of edges.*

Proof. Follows from Lemma 11 and Corollary 9. □

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References

[1] Buckley, F., Iterated line graphs, *Congr. Numer.*, 33 (1981), 390–394.
[2] Buckley, F., The size of iterated line graphs, *Graph Theory Notes New York*, 25 (1993), 33–36.
[3] Cui, Z., Liu, B., On Harary matrix, Harary index and Harary energy, *MATCH Commun. Math. Comput. Chem.*, 68 (2012), 815-823.
[4] Cvetković, D., Rowlinson, P., Simić, S., An Introduction to the Theory of Graph Spectra, Cambridge Univ. Press, Cambridge, 2010.
[5] Güngör, A. D., Çevik, A. S., On the Harary energy and Harary Estrada index of a graph, *MATCH Commun. Math. Comput. Chem.*, 64 (2010), 280-296.
[6] Gutman, I., The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz*, 103 (1978), 1-22.
[7] Harary, F., Graph Theory, Addison–Wesley, Reading, 1969.
[8] Indulal, G., D-spectrum and D-energy of complements of iterated line graphs of regular graphs, *J. Alg. Stru. Appl.*, 4 (2017), 51-56.
[9] Ivanciuc, O., Balaban, T. S., Balaban, A. T., Design of topological indices. Part 4. Reciprocal distance matrix, related local vertex invariants and topological indices, *J. Math. Chem.*, 12 (1993), 309-318.
[10] Li, X., Shi, Y., Gutman, I., Graph Energy, Springer, New York, 2012.
[11] Plavšić, D., Nikolić, S., Trinajstić, N., On the Harary index for the characterization of chemical graphs, *J. Math. Chem.*, 12 (1993), 235-250.
[12] Ramane, H. S., Manjalapur V. V., Harary equienergetic graphs, *Int. J. Math. Arch.*, 6 (2015), 81-86.
[13] Ramane, H. S., Jummannaver R. B., Harary spectra and Harary energy of line graphs of regular graphs, *Gulf J. Math.*, 4 (2016), 39-46.
[14] Sachs, H., Über selbstkomplementare Graphen, *Publ. Math. Debrecen*, 9 (1962), 270-288.
[15] Sachs, H., Über Teiler, Faktoren und charakteristische Polynome von Graphen, Teil II, *Wiss. Z. TH Ilmenau*, 13 (1967), 405-412.
[16] Xu, K., Das, K. C., Trinajstić, N., The Harary Index of a Graph, Springer, Heidelberg, 2015.

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SOME NUMERICAL CHARACTERISTICS OF DIRECT SUM OF OPERATORS

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Abstract. The connection between some numerical characteristics (numerical range, numerical radius, Crawford number and sectoriality) of direct sum of operators in the direct sum of Hilbert spaces and their coordinate operators has been investigated.

1. Introduction

The general information on numerical characteristics (as numerical range, numerical radius, Crawford number, sectoriality and etc.) can be found in [1-7]. The obtained results may be applied in perturbation theory, generalized eigenvalue problems, numerical analysis, system theory, dilation theory and etc. [1-4].

It is known that infinite direct sum of Hilbert spaces H_n, $n \geq 1$ and infinite direct sum of operators A_n in H_n, $n \geq 1$ are defined as

$$H = \overset{\infty}{\underset{n=1}{\oplus}} H_n = \left\{ u = (u_n) : u_n \in H_n,\ n \geq 1, \|u\|_H^2 = \sum_{n=1}^{\infty} \|u_n\|_{H_n}^2 < +\infty \right\}$$

and

$$A = \overset{\infty}{\underset{n=1}{\oplus}} A_n, D(A) = \{u = (u_n) \in H : u_n \in D(A_n),\ n \geq 1,\ Au = (A_n u_n) \in H\},$$

$$A : D(A) \subset H \to H$$

(see [8]).

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The general theory of linear closed operators in Hilbert spaces and its applications to physical problems has been investigated by many mathematicians (see for example [8]).

However, many physical problems of today arising in the modelling of processes of multi-particle quantum mechanics, quantum field theory and in the physics of rigid bodies support to study a theory of linear direct sum of operators in the direct sum of Hilbert spaces (see [9-11] and references there in).

Connections between numerical range and numerical radius for the direct sum of two operators in the direct sum of Hilbert spaces and coordinate operators have been investigated in [6].

In this work, the connection between some numerical characteristics of direct sum of operators in direct sum of Hilbert spaces and their coordinate operators will be investigated.

2. Numerical Range of Direct Sum Operators

Definition 1. [1] *Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ be induced norm by this inner product. In this case, the numerical range of a linear bounded operator T in $\mathcal{H}$ is the subset of the complex numbers $\mathbb{C}$ given by*

$$W(T) = \{(Tx, x) \in \mathbb{C} : x \in \mathcal{H}, \|x\| = 1\}.$$

Recall that a numerical range of an operator is convex (Toeplitz-Housdorff) and spectrum of an operator is contained in the closure of its numerical range.

The following result is true.

Theorem 2. *If for any $n \geq 1$, H_n is a Hilbert space, $A_n \in L(H_n), H = \bigoplus\limits_{n=1}^{\infty} H_n$ and $A = \bigoplus\limits_{n=1}^{\infty} A_n, A \in L(H)$, then numerical range of the operator A is in the form*

$$W(A) = co\left(\bigcup_{n=1}^{\infty} W(A_n)\right),$$

where $co\ (\Omega), \Omega \subset \mathbb{C}$ denotes the convex hull of Ω.

Proof. Indeed in this case, for any element $f \in H$ with norm

$$\|f\|_H^2 = \sum_{n=1}^{\infty} \|f_n\|_n^2 = 1, f_n \neq 0,\ n \geq 1,$$

we have

$$(Af, f)_H = \sum_{n=1}^{\infty} (A_n f_n, f_n)_n = \sum_{n=1}^{\infty} \|f_n\|_n^2 \left(A_n\left(f_n/\|f_n\|_n\right), f_n/\|f_n\|_n\right)_n = \sum_{n=1}^{\infty} \alpha_n \mu_n,$$

where

$$\alpha_n = \|f_n\|_n^2,\ \mu_n = \left(A_n\left(f_n/\left\|f_n\right\|_n\right), f_n/\left\|f_n\right\|_n\right)_n \in W(A_n), n \geq 1.$$

It is clear that $\sum\limits_{n=1}^{\infty} \alpha_n = 1$.

Now, assume that there exists a number $n_k \in \mathbb{N} \cup \{+\infty\}$ such that

$$f_{n_1} \neq 0\ ,\ f_{n_2} \neq 0\ , ..., f_{n_k} \neq 0\ ,\ k \leq \infty$$

and

$$f = (0, ..., 0, f_{n_1}, 0, ..., 0, f_{n_2}, 0, ..., 0, f_{n_k}, 0, ..., 0, ...)\,.$$

In this case,

$$\begin{aligned}
(Af, f)_H &= \sum_{j=1}^{k} \left(A_{n_j} f_{n_j}, f_{n_j}\right)_{n_j} \\
&= \sum_{j=1}^{k} \left\|f_{n_j}\right\|_{n_j}^2 \left(A_{n_j}\left(f_{n_j}/\left\|f_{n_j}\right\|_{n_j}\right), f_{n_j}/\left\|f_{n_j}\right\|_{n_j}\right)_{n_j} \\
&= \sum_{j=1}^{k} \alpha_{n_j} \mu_{n_j},
\end{aligned}$$

where

$$\alpha_{n_j} = \left\|f_{n_j}\right\|_{n_j}^2,\ \mu_{n_j} = \left(A_{n_j}\left(f_{n_j}/\left\|f_{n_j}\right\|_{n_j}\right), f_{n_j}/\left\|f_{n_j}\right\|_{n_j}\right)_{n_j} \in W\left(A_{n_j}\right).$$

It is clear that $\sum\limits_{j=1}^{k} \alpha_{n_j} = 1$. □

3. Numerical Radius of Direct Sum Operators

Definition 3. [1] *Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ be induced norm by this inner product. In this case, the numerical radius of a linear bounded operator T in $\mathcal{H}$ is a number which is given by*

$$w(T) = sup\{|\lambda| : \lambda \in W(T)\}.$$

Recall that for any vector $x \in \mathcal{H}$, it is true that

$$|(Tx, x)| \leq w(T)\|x\|^2.$$

The following result is true.

Theorem 4. *If for any $n \geq 1$, H_n is a Hilbert space, $A_n \in L(H_n)$, $H = \bigoplus\limits_{n=1}^{\infty} H_n$ and $A = \bigoplus\limits_{n=1}^{\infty} A_n, A \in L(H)$, then numerical radius of the operator A is in the following form*

$$w(A) = \sup_{n\geq 1} w(A_n).$$

Proof. For any $n \geq 1$, since $w(A_n) \leq w(A)$, then

$$\underset{n\geq 1}{sup}\ w(A_n) \leq w(A). \tag{1}$$

On the other hand, if $\mu \in W(A)$, then $\mu = \sum_{n=1}^{\infty} \alpha_n \mu_n, \sum_{n=1}^{\infty} \alpha_n = 1$ and $\mu_n \in W(A_n)$ for $n \geq 1$. Then

$$|\mu| = \left|\sum_{n=1}^{\infty} \alpha_n \mu_n\right| \leq \sum_{n=1}^{\infty} \alpha_n |\mu_n| \leq \sum_{n=1}^{\infty} \alpha_n w(A_n) \leq \underset{n\geq 1}{sup}\ w(A_n) \sum_{n=1}^{\infty} \alpha_n = \underset{n\geq 1}{sup}\ w(A_n).$$

Hence, by the definition of numerical radius we get

$$w(A) \leq \underset{n\geq 1}{sup}\ w(A_n). \tag{2}$$

Combining two inequalities (1) and (2), we reach to the desired inequality. □

4. Crawford Number of Direct Sum Operators

Definition 5. [1] *Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ be induced norm by this inner product. In this case, the Crawford number of a linear bounded operator T in $\mathcal{H}$ is given by*

$$c(T) = inf\{|\lambda| : \lambda \in W(T)\}.$$

The following results are true.

Theorem 6. *If for any $n \geq 1$, H_n is a Hilbert space, $A_n \in L(H_n), H = \bigoplus_{n=1}^{\infty} H_n$ and $A = \bigoplus_{n=1}^{\infty} A_n, A \in L(H)$, then Crawford number of the operator A is in the following form*

$$c(A) \leq \underset{n\geq 1}{inf}\ c(A_n).$$

Proof. Since $c(A) \leq c(A_n)$, for any $n \geq 1$, $c(A) \leq \underset{n\geq 1}{inf}\ c(A_n)$ satisfies. □

Theorem 7. *If for any $n \geq 1$, H_n is a Hilbert space, $A_n \in L(H_n)$, $Re(A_n) \geq 0$ (≤ 0), $H = \bigoplus_{n=1}^{\infty} H_n$ and $A = \bigoplus_{n=1}^{\infty} A_n, A \in L(H)$, then Crawford number of the operator A is in the following form*

$$c(A) = \underset{n\geq 1}{inf}\ c(A_n).$$

Proof. If $\mu \in W(A)$, then by Theorem 2, $\mu = \sum_{n=1}^{\infty} \alpha_n \mu_n, \sum_{n=1}^{\infty} \alpha_n = 1$ and $\mu_n \in W(A_n), n \geq 1$.

On the other hand, since for any $n \geq 1$, $\mu_n \geq 0$ (≤ 0), we have

$$|\mu| = \left|\sum_{n=1}^{\infty} \alpha_n \mu_n\right| = \sum_{n=1}^{\infty} \alpha_n |\mu_n| \geq \sum_{n=1}^{\infty} \alpha_n c(A_n) \geq \underset{n\geq 1}{inf}\ c(A_n) \sum_{n=1}^{\infty} \alpha_n = \underset{n\geq 1}{inf}\ c(A_n).$$

Therefore,

$$c(A) \geq \underset{n\geq 1}{inf}\ c(A_n). \tag{3}$$

In this case, equation (3) and Theorem 6 complete the proof of proposition. □

5. Sectoriality of Direct Sum Operators

Definition 8. [1] *Let T be a linear bounded operator in Hilbert space $\mathcal{H}$. If $Re(Tx, x) \geq 0$ for any $x \in \mathcal{H}$, then it is called an accretive operator in $\mathcal{H}$.*

Definition 9. [1] *Assume that $T \in L(\mathcal{H})$ is a accretive operator. If $W(T) \subset \{z \in \mathbb{C} : |argz| < \varphi\}$ for any $\varphi \in [0, \pi/2)$, then it is called a sectorial operator with vertex $\gamma = 0$ and semi-angle φ. In this case, $T \in S_\varphi(\mathcal{H})$.*

The following result is true.

Theorem 10. *If for any $n \geq 1, H_n$ is a Hilbert space, $A_n \in L(H_n), A_n \in S_{\varphi_n}(H_n), H = \bigoplus_{n=1}^{\infty} H_n$ and $A = \bigoplus_{n=1}^{\infty} A_n, A \in L(H)$, then for some $\varphi \in [0, \pi/2)$,*

$$A \in S_\varphi(H)\ thenecessaryandsufficientconditionis\ \underset{n\geq 1}{sup}\ \varphi_n < \varphi.$$

Proof. From Theorem 2, it is clear that $W(A_n) \subset W(A)$ for any $n \geq 1$.

On the other hand, if $A_n \in S_{\varphi_n}(H_n), n \geq 1$, then for any $x \in H$ with norm

$$\|x\|_H = \left(\sum_{n=1}^{\infty} \|x\|_n^2\right)^{1/2} = 1,\ n \geq 1,$$

we have

$$Re(Ax, x) = \sum_{n=1}^{\infty} Re\,(A_n x_n, x_n)_n \geq 0.$$

Then $A : H \to H$ is an accretive operator in H.

Moreover, it is clear that

$$\bigcup_{n=1}^{\infty} \{z \in W(A_n) : |argz| \leq \varphi_n\} \subset W(A).$$

If we choose $\varphi = \underset{n\geq 1}{sup}\ \varphi_n$, then

$$W(A) \subset \{z \in \mathbb{C} : |argz| \leq \varphi\}.$$

Since $\varphi \in [0, \pi/2)$, then $A \in S_\varphi(H)$.

On the contrary, if $A \in S_\varphi(H)$ for some $\varphi \in [0, \pi/2)$, then from the relation $W(A_n) \subset W(A)$ and accretivity of each coordinate operator $A_n \in L(H_n),\ n \geq 1$, we get

$$W(A_n) \subset \{z \in \mathbb{C} : |argz| \leq \varphi\}.$$

Consequently, for any $n \geq 1,\ A_n \in S_{\varphi_n}(H_n)$. This completes the validity of the assertion. □

6. Applications of Some Numerical Characteristics of Direct Sum Operators

Example 11. *Let for any $n \geq 1,\ H_n = (\mathbb{C}, |\cdot|),\ A_n = \alpha_n E,\ \alpha_n \in \mathbb{C}$ and $H = \bigoplus_{n=1}^{\infty} H_n = l_2(\mathbb{C}),\ A = \bigoplus_{n=1}^{\infty} (\alpha_n E) : H \to H$. In this case,*

$$\begin{aligned} W(A_n) &= \alpha_n, n \geq 1, \\ w(A_n) &= |\alpha_n|, n \geq 1, \\ c(A_n) &= |\alpha_n|, n \geq 1 \end{aligned}$$

and for $Re(\alpha_n) \geq 0, n \geq 1$, A_n is accretive. In addition

$$\varphi_n = arg\alpha_n, n \geq 1.$$

Therefore,

$$\begin{aligned} W(A) &= co\{\alpha_n, n \geq 1\}, \\ w(A) &= \sup_{n \geq 1} |\alpha_n|, \\ c(A) &= \inf_{n \geq 1} |\alpha_n| \end{aligned}$$

and this case when $\sup_{n \geq 1} |arg\alpha_n| < \pi/2$, angle of sectoriality of operator A is

$$\varphi = \sup_{n \geq 1} |arg\alpha_n|.$$

i.e. $A \in S_\varphi(l_2)$.

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References

[1] Gustafson, K.E., Rao, D.K.M., Numerical Range: The Field of Values of Linear Operators and Matrices, Springer, New York, 1997.
[2] Halmos, P., A Hilbert Space Problem Book (sec. ed.), Springer-Verlag, New York, 1974.
[3] Bonsall, F.F., Duncan, J., Studies in Functional Analysis-Numerical Ranges, Stud. Math., vol.21, Mathematical Association of America, 1980.
[4] Kato, T., Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1980.

[5] Shapiro, J.H., Lecture Notes, Notes on the Numerical Range, 2003.
[6] Goldberg, M., Tadmor, E., On the Numerical Radius and Its Applications, *Linear Algebra and Its Applications,* 42 (1982), 263-284.
[7] Ando, T., Li, C. K., The Numerical Range and Numerical Radius, Special Issue, *Linear and Multilinear Algebra* 37 (1994), 1-238.
[8] Dunford, N., Schwartz, J. T., Linear Operators I; II, Second ed., Interscience Publishers, New York, 1958; 1963.
[9] Ismailov, Z.I., Multipoint Normal Differential Operators for First Order, *Opusc. Math.* 29 (4) (2009), 399-414.
[10] Kochubei, A. N., Symmetric Operators and Nonclassical Spectral Problems, *Mat. Zametki* 25 (3) (1979), 425-434.
[11] Zettl, A., Sturm-Liouville Theory, First ed., Amer. Math. Soc., USA, 2005.

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ON EQUITABLE COLORING OF BOOK GRAPH FAMILIES

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Abstract. A proper vertex coloring of a graph is equitable if the sizes of color classes differ by atmost one. The notion of equitable coloring was introduced by Meyer in 1973. A proper $h-$colorable graph K is said to be equitably h-colorable if the vertex sets of K can be partioned into h independent color classes $V_1, V_2, ..., V_h$ such that the condition $||V_i| - |V_j|| \leq 1$ holds for all different pairs of i and j and the least integer h is known as equitable chromatic number of K. In this paper, we find the equitable coloring of book graph, middle, line and central graphs of book graph.

1. Introduction

The idea of equitable coloring was discovered by Meyer [4] in 1973. Hajmal and Szemeredi [3] proved that graph K with degree Δ is equitable h-colorable, if $h \geq \Delta + 1$. Later Equitable Coloring Conjecture for bipartite graphs was proved. Equitable vertex coloring of corona graphs is NP- hard.

The graphs considered here are simple. Vertex coloring is a particular case of Graph coloring. The collection of vertices receiving same color is known as color class. A proper $h-$colorable graph K is said to be *equitably $h-$colorable* if the vertex sets of K can be partitioned into h independent color classes $V_1, V_2, ..., V_h$ such that the condition $||V_i| - |V_j|| \leq 1$ holds for all different pairs of i and j [1]. And the least integer h is known as *equitable chromatic number* of K [1]. Here we found equitable coloring of book graph, middle, line and central graphs of book graph.

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2. Preliminaries

Line graph [2] of K, $L(K)$ is attained by considering the edges of K as the vertices of $L(K)$. The adjacency of any two vertices of $L(K)$ is a consequence of the corresponding adjacency of edges in K.

Middle graph [5] of K, $M(K)$ is attained by adding new vertex to all the edges of K. The adjacency of any two new vertices of $M(K)$ is a consequence of the corresponding adjacency of edges in K or adjacency of a vertex and an edge incident with it.

Central graph [6] of K, $C(K)$ is attained by the insertion of new vertex to all the edges of K and connecting any two new vertices of K which were previously non-adjacent.

The q-book graph is defined as the graph Cartesian product $S_{(q+1)} \times P_2$, where S_q is a star graph and P_2 is the path graph.

3. Results

3.1. On Equitable Coloring of Middle Graph of Book Graph.

- Order of $M(B_q)$ is $5q + 3$
- Number of incidents of $M(B_q)$ is $q^2 + 9q + 2$
- Maximum degree of $M(B_q)$ is $2(q + 1)$
- Minimum degree of $M(B_q)$ is 2

Algorithm A

Input: The value 'q' of B_q, for $q \geq 3$

Outcome: Equitably colored $V[M(B_q)]$

Procedure:

$start$

{

$V_a = \{g, h, z\}$;

$C(g) = C(h) = 1$;

$C(z) = q + 2$;

for $s = 1$ to q

{

$V_b = \{g_s, h_s\}$;

$C(g_s) = s$;

$C(h_s) = s$;

}

for $s = 1$ to q

{

$V_c = \{k_s, l_s\}$;

$C(k_s) = s + 1$;

$C(l_s) = s + 1$;

}

for $s = 1$ to q
{
$V_d = \{m_s\}$;
if s is *odd*
$C(m_s) = q + 1$;
else
$C(m_s) = q + 2$;
}
}
V $= V_a \cup V_b \cup V_c \cup V_d$
end

Theorem 3.1. *For any book graph $M(B_q)$ the equitable chromatic number,*

$$\boldsymbol{\chi_=[\mathbf{M}(\mathbf{B_q})] = \mathbf{q}+\mathbf{2}}, \forall q \geq 3$$

Proof. For $q \geq 3$, $V(B_q) = \{g, h, g_s, h_s : 1 \leq s \leq q\}$.
$V[M(B_q)] = \{g, h, z\} \cup \{g_s : 1 \leq s \leq q\} \cup \{h_s : 1 \leq s \leq q\} \cup \{k_s : 1 \leq s \leq q\} \cup \{l_s : 1 \leq s \leq q\} \cup \{m_s : 1 \leq s \leq q\}$, where z, k_s, l_s and m_s are the subdivision of the edges gh, gg_s, hh_s and g_sh_s respectively.

Let us consider $V[M(B_q)]$ and the color set C $= \{c_1, c_2, ..., c_{q+2}\}$. Assign the equitable coloring by Algorithm A. Therefore,

$$\chi_=[M(B_q)] \leq q + 2.$$

And since, there exists a maximal induced complete subgraph of order $q+2$ by the vertices z, g, k_s and therefore $\chi_=[M(B_q)] \geq q + 2$.

$c_1, c_2, ..., c_{q+2}$ are independent sets of $M(B_q)$. And $||c_i| - |c_j|| \leq 1$, for every different pair of i and j. Hence,

$$\boldsymbol{\chi_=[\mathbf{M}(\mathbf{B_q})] = \mathbf{q}+\mathbf{2}}.$$

□

3.2. **On Equitable Coloring of Central Graph of Book Graph. Features of Central Graph of Book Graph**

- Order of $C(B_q)$ is $5q + 3$
- Number of incidents of $C(B_q)$ is $2(q^2 + 3q + 1)$
- Maximum degree of $C(B_q)$ is $2q + 1$
- Minimum degree of $C(B_q)$ is 2

Algorithm B
Input: The value 'q' of B_q, for $q \geq 3$
Outcome: Equitably colored $V[C(B_q)]$
Procedure:
start

{
for $s = 1$ to q
$V_a = \{g, h, m_s\}$;
{
if $s = 1$ to 3
$C(g) = C(h) = C(m_s) = 1$;
else
$C(m_s) = s - 1$;
}
for $s = 1$ to q
{
$V_b = \{g_s, h_s\}$;
$C(g_s) = s + 1$;
$C(h_s) = s + 1$;
}
if q is *odd*
{
$V_c = \{k_s, l_s, z\}$;
for $s = 1$ to q
{
if $s = 1$ to $q - 1$
{
$C(k_s) = s + 2$;
$C(l_s) = s + 2$;
}
else
$C(z) = C(k_s) = C(l_s) = 2$;
}
else
{
for $s = 1$ to q
$V_c = \{k_s, l_s, z\}$;
$C(z) = 2$;
$C(k_s) = C(l_s) = q - s + 2$;
}
}
}
V $= V_a \cup V_b \cup V_c$
end

Theorem 3.2. *For any book graph $C(B_q)$ the equitable chromatic number,*

$$\boldsymbol{\chi_=[\mathbf{C}(\mathbf{B_q})] = \mathbf{q} + \mathbf{1}, \forall \mathbf{q} \geq \mathbf{3}}$$

Proof. For $q \geq 3$,

$$V(B_q) = \{g, h, g_s, h_s : 1 \leq s \leq q\}.$$

$$\begin{aligned} V[C(B_q)] &= \{g, h, z\} \cup \{g_s : 1 \leq s \leq q\} \cup \{h_s : 1 \leq s \leq q\} \\ \cup\{k_s &: 1 \leq s \leq q\} \cup \{l_s : 1 \leq s \leq q\} \cup \{m_s : 1 \leq s \leq q\}, \end{aligned}$$

where z, k_s, l_s and m_s are the subdivision of the edges gh, gg_s, hh_s and g_sh_s respectively.

Let us consider $V[C(B_q)]$ and the color set C $= \{c_1, c_2, ..., c_{q+1}\}$. Assign the equitable coloring by Algorithm B. Therefore,

$$\chi_=[C(B_q)] \leq q + 1$$

And $\chi[C(B_q)] = q + 1$. That is, $\chi_=[C(B_q)] \geq \chi[C(B_q)] = q + 1$. Therefore,

$$\chi_=[C(B_q)] \geq q + 1.$$

$c_1, c_2, ..., c_{q+1}$ are independent sets of $C(B_q)$. And $||c_i| - |c_j|| \leq 1$, for every different pair of i and j. Thus,

$$\chi_=[C(B_q)] = q + 1.$$

□

3.3. **On Equitable Coloring of Line Graph of Book Graph.**

- Order of $L(B_q)$ is $3q + 1$
- Number of incidents of $L(B_q)$ is $q(q + 3)$
- Maximum degree of $L(B_q)$ is $2q$
- Minimum degree of $L(B_q)$ is 2

Algorithm C

Input: The value 'q' of B_q, for $q \geq 3$

Outcome: Equitably coloring $V[L(B_q)]$

Procedure:

begin

{

for $s = 1$ to q

{

$V_a = \{g, z\} \cup \{m_s\}$;

$C(m_s) = s$;

$C(z) = C(g) = 1$;

}

for $s = 1$ to q

{

$V_b = \{k_s, l_s\}$;

$C(k_s) = s + 1$;

$C(l_s) = s + 1$;

}

}

V = $V_a \cup V_b$
end

Theorem 3.3. *For any book graph $L(B_q)$ the equitable chromatic number,*

$$\boldsymbol{\chi_=[L(B_q)] = q+1, \forall q \geq 3}$$

Proof. For $q \geq 3$,

$$V(B_q) = \{g, h, g_s, h_s : 1 \leq s \leq q\}.$$

The edge set of B_q is $\{z, k_s, l_s, m_s : 1 \leq s \leq q\}$ where z be the edge corresponding to the vertices gh, each k_s be the edge corresponding to the vertex gg_s, each edge l_s be the edge corresponding to the vertex hh_s , each edge m_s be the edge corresponding to the vertex g_sh_s. By the definition of line graph, the edge set of line graph is converted into vertices of $L(B_q)$.

$$\begin{aligned} V[L(B_q)] &= \{z\} \cup \{k_s : 1 \leq s \leq q\} \cup \{l_s : 1 \leq s \leq q\} \\ \cup\{m_s &: 1 \leq s \leq q\}. \end{aligned}$$

Let us consider the $V[L(B_q)]$ and the color set $C = \{c_1, c_2, ..., c_{q+1}\}$. Assign the equitable coloring by Algorithm C. Therefore,

$$\chi_=[L(B_q)] \leq q+1$$

And since, there exists a maximal induced complete subgraph of order $q+1$ by the vertices z, k_s and therefore

$$\chi_=[L(B_q)] \geq q+1.$$

$c_1, c_2, ..., c_{q+1}$ are independent sets of $L(B_q)$. And $||c_i| - |c_j|| \leq 1$, for every different pair of i and j. Thus,

$$\chi_=[L(B_q)] = q+1.$$

□

3.4. **On Equitable Coloring of Book Graph. Features of Book Graph.**

- Order of B_q is $2(q+1)$
- Number of incidents of B_q is $3q+1$
- Maximum degree of B_q is $q+1$
- Minimum degree of B_q is 2

Algorithm D

Input: The value 'q' of B_q, for q ≥ 3
Outcome: Equitably colored $V(B_q)$
Procedure:
start
{
for $s = 1$ to q
{
$V_a = \{g_s, h\}$;
C(h)=1;

$C(g_s) = 1;$
}
for $s = 1$ to q
{
$V_b = \{g, h_s\}$;
$C(g) = 2;$
$C(h_s) = 2;$
}
}
V $= V_a \cup V_b$
end

Theorem 3.4. *For any book graph B_q the equitable chromatic number,*

$$\boldsymbol{\chi_=(B_q) = 2}, \forall q \geq 3.$$

Proof. For $n \geq 3$,

$$V(B_q) = \{g, h\} \cup \{g_s : 1 \leq s \leq q\} \cup \{h_s : 1 \leq s \leq q\}.$$

Let us consider the $V(B_q)$ and the color set C $= \{c_1, c_2\}$. Assign the equitable coloring by Algorithm D. Therefore,

$$\chi_=(B_q) \leq 2.$$

And since, there exists a maximal induced complete subgraph of order 2 in B_q (say path P_2). Therefore,

$$\chi_=(B_q) \geq 2$$

c_1, c_2 are independent sets of B_q. And $||c_i| - |c_j|| \leq 1$, for every different pair of i and j. Hence,

$$\boldsymbol{\chi_=(B_q) = 2}.$$

□

References

[1] Furmanczyk, H., Equitable coloring of Graph products, *Opuscula Mathematica,* Vol 26. No.1, (2006).

[2] Harary, F., Graph theory, Narosa Publishing home, New Delhi, 1969.

[3] Hajnal, A., Szemeredi, E., Proof of a conjecture of Endos, in: Combinatorial theory and its applications, *Colloq. Math. Soc. Janos Bolyai,* 4 (2) (1970), 601-623.

[4] Meyer, W., Equitable coloring, *Amer. Math. Monthly*, 80, (1973).

[5] Michalak, D., On middle and total graphs with coarseness number equal 1, *Springer Verlag Graph Theory*, *Lagow Proceedings*, Berlin Heidelberg, New York, Tokyo, (1981), 139-150.

[6] Vernold Vivin, J., Harmonious coloring of total graphs, n-leaf, central graphs and circumdetic graphs, Bharathiar University, Ph.D Thesis, Coimbatore, India, 2007.

https://communications.science.ankara.edu.tr

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ON STAR COLORING OF MODULAR PRODUCT OF GRAPHS

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Abstract. A star coloring of a graph G is a proper vertex coloring in which every path on four vertices in G is not bicolored. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G. In this paper, we find the exact values of the star chromatic number of modular product of complete graph with complete graph $K_m \diamond K_n$, path with complete graph $P_m \diamond K_n$ and star graph with complete graph $K_{1,m} \diamond K_n$.

1. Introduction

All graphs in this paper are finite, simple, connected and undirected graph and we follow [2,3,7] for terminology and notation that are not defined here. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Branko Grünbaum introduced the concept of star chromatic number in 1973. A star coloring [1, 5, 6] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.

During the years star coloring of graphs has been studied extensively by several authors, for instance see [1, 4, 5].

Definition 1. *A trail is called a path if all its vertices are distinct. A closed trail whose origin and internal vertices are distinct is called a cycle.*

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Definition 2. *A graph G is complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n.*

Definition 3. *A star graph is a complete bipartite graph in which $m-1$ vertices have degree 1 and a single vertex have degree $(m-1)$. It is denoted by $K_{1,m}$.*

Definition 4. *The modular product [8] $G \diamond H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which a vertex (v, w) is adjacent to a vertex (v', w') if and only if either*

- *$v = v'$ and w is adjacent to w', or*
- *$w = w'$ and v is adjacent to v', or*
- *v is adjacent to v' and w is adjacent to w', or*
- *v is not adjacent to v' and w is not adjacent to w'.*

2. Main Results

In this section, we find the exact values of the star chromatic number of modular product of complete graph with complete graph $K_m \diamond K_n$, path with complete graph $P_m \diamond K_n$ and star graph with complete graph $K_{1,m} \diamond K_n$.

2.1. Star chromatic number of $K_m \diamond K_n$.

Theorem 1. *For any positive integers $m, n \geq 2$,*

$$\chi_s(K_m \diamond K_n) = \begin{cases} m, & \textit{when } n = 2 \\ n(m-1), & \textit{Otherwise.} \end{cases}$$

Proof. Let K_m be the complete graph on m vertices and K_n be the complete graph on n vertices. Let

$$V(K_m) = \{u_i : 1 \leq i \leq m\}$$

and

$$V(K_n) = \{v_j : 1 \leq j \leq n\}.$$

By the definition of the modular product, the vertices of $K_m \diamond K_n$ is denoted as follows:

$$V(K_m \diamond K_n) = \bigcup_{i=1}^{m} \{(u_i, v_j) : 1 \leq j \leq n\}.$$

Case(i): When $m \geq 2$ and $n = 2$

Let $\{c_1, c_2, \ldots, c_m\}$ be the set of m distinct colors. The vertices (u_i, v_j) where $1 \leq i \leq m$ and $1 \leq j \leq 2$ can be colored with color c_i. Thus $\chi_s(K_m \diamond K_n) = m$.
Suppose $\chi_s(K_m \diamond K_n) < m$, say $m-1$. Then the vertices (u_i, v_j) where $2 \leq i \leq m, 1 \leq j \leq 2$ has to be colored with one of the existing colors $\{1, 2, \ldots, m-1\}$ which results in improper coloring and also gives bicolored paths on four vertices (since the vertices (u_i, v_1), $1 \leq i \leq m$ and the vertices (u_i, v_2), $1 \leq i \leq m$ forms bipartite graphs) and so contradicts the star coloring. Hence $\chi_s(K_m \diamond K_n) = m$.

Case(ii): When $m \geq 2$ and $n > 2$
Let $\{c_1, c_2, \ldots, c_{n(m-1)}\}$ be the set of $n(m-1)$ distinct colors. For $1 \leq i \leq 2$ and $1 \leq j \leq n$, the vertices (u_i, v_j) can be colored with color c_j, and for $3 \leq i \leq m$ and $1 \leq j \leq n$, the vertices (u_i, v_j) can be colored with color $c_{(i-2)n+j}$. Thus $\chi_s(K_m \diamond K_n) = n(m-1)$ when $m \geq 2$, $n \geq 3$.
Suppose $\chi_s(K_m \diamond K_n) < n(m-1)$, say $n(m-1)-1$. Then the vertex (u_m, v_n) has to be colored with one of the existing colors $\{1, 2, \ldots, n(m-1)-1\}$ which results in improper coloring and also gives bicolored paths on four vertices (since (u_m, v_n) is adjacent to every vertices (u_i, v_j), $1 \leq i \leq m-1, 1 \leq j \leq n-1$) and this contradicts the star coloring. Hence $\chi_s(K_m \diamond K_n) = n(m-1)$. $\square$

2.2. Star chromatic number of $P_m \diamond K_n$.

Theorem 2. *For any positive integers $m, n > 1$,*

$$\chi_s(P_m \diamond K_n) = \begin{cases} 3, & \text{when } m > 4,\ n = 2 \\ n, & \text{when } m = 2, 3 \text{ and } n > 2 \\ n+1, & \text{when } m = 4,\ n \geq 2 \\ 2n, & \text{Otherwise.} \end{cases}$$

Proof. Let P_m be the path graph on m vertices and K_n be the complete graph on n vertices. Let

$$V(P_m) = \{u_i : 1 \leq i \leq m\}$$

and

$$V(K_n) = \{v_j : 1 \leq j \leq n\}.$$

By the definition of the modular product, the vertices of $P_m \diamond K_n$ is denoted as follows:

$$V(P_m \diamond K_n) = \bigcup_{i=1}^{m} \{(u_i, v_j) : 1 \leq j \leq n\}.$$

Case(i): When $m > 4$ and $n = 2$
Let $\{c_1, c_2, c_3\}$ be the set of 3 distinct colors. Then the vertices (u_i, v_j) where $1 \leq i \leq \lceil \frac{m}{2} \rceil$ and $1 \leq j \leq 2$ are colored with color c_1. For $i \equiv 2 \pmod 4), 1 \leq i \leq m$ and $1 \leq j \leq 2$, the vertices (u_i, v_j) can be colored with color c_2. Similarly, the vertices (u_i, v_j) where $i \equiv 0 \pmod 4), 1 \leq i \leq m$ and $1 \leq j \leq 2$ can be colored with color c_3. It is obvious that $\chi_s(P_m \diamond K_n) = 3$ when $m > 4$ and $n = 2$.

Case(ii): When $m = 2, 3$ and $n > 2$
Let $\{c_1, c_2, \ldots, c_n\}$ be the set of n distinct colors. The vertices (u_i, v_j) where $1 \leq j \leq n$ and $i = 1, 2, 3$ can be colored with color c_j. Thus $\chi_s(P_m \diamond K_n) = n$ when $m = 2, 3$ and $n > 2$.
Suppose $\chi_s(P_m \diamond K_n) < n$, say $n-1$. Then the vertices (u_i, v_n), $1 \leq i \leq m$ has to be colored with one of the existing colors $\{1, 2, \ldots, n-1\}$ which results in improper coloring since the vertices (u_i, v_n), $1 \leq i \leq m$ is adjacent to the vertices colored with colors $1, 2, \ldots, n-1$ and so contradicts the star coloring. Hence $\chi_s(P_m \diamond K_n) = n$.

Case(iii): When $m = 4$ and $n \geq 2$
Let $\{c_1, c_2, \ldots, c_{n+1}\}$ be the set of $n+1$ distinct colors. For $1 \leq i \leq 3$ and $1 \leq j \leq n$, the vertices (u_i, v_j) can be colored with color c_j. And the vertices (u_4, v_j), $1 \leq j \leq n$, can be given the color c_{j+1}. Thus $\chi_s(P_m \diamond K_n) = n+1$ when $m = 4$ and $n \geq 2$.
Suppose $\chi_s(P_m \diamond K_n) < n+1$, say n. Then the vertices (u_4, v_j), $1 \leq j \leq n$ has to be colored with the j^{th} color which results in bicolored paths on four vertices and so contradicts the star coloring. Hence $\chi_s(P_m \diamond K_n) = n+1$.

Case(iv): When $m > 4$ and $n \geq 3$
Let $\{c_1, c_2, \ldots, c_{2n}\}$ be the set of $2n$ distinct colors. The vertices (u_i, v_j) where $i \equiv 1, 2, 3 \pmod 4, 1 \leq i \leq m$ and $1 \leq j \leq n$ can be colored with color c_j, and the vertices (u_i, v_j) where $i \equiv 0 \pmod 4, 1 \leq i \leq m$ and $1 \leq j \leq n$ can be given the color c_{n+j}. Thus $\chi_s(P_m \diamond K_n) = 2n$ when $m > 4$ and $n \geq 3$.
Suppose $\chi_s(P_m \diamond K_n) < 2n$, say $2n-1$. Then the vertices (u_i, v_n) where $i \equiv 0 \pmod 4, 1 \leq i \leq m$ has to be colored with one of the colors $\{1, 2, \ldots, 2n-1\}$ which results in bicolored paths on four vertices and so contradicts the star coloring. Hence $\chi_s(P_m \diamond K_n) = 2n$. □

2.3. Star chromatic number of $K_{1,m} \diamond K_n$.

Theorem 3. *For any positive integers $m \geq 2$ and $n \geq 3$,*

$$\chi_s(K_{1,m} \diamond K_n) = n.$$

Proof. Let $K_{1,m}$ be the star graph on $m+1$ vertices and K_n be the complete graph on n vertices. Let

$$V(K_{1,m}) = \{u_1\} \cup \{u_i : 2 \leq i \leq m+1\}$$

and

$$V(K_n) = \{v_j : 1 \leq j \leq n\}.$$

By the definition of the modular product, the vertices of $K_{1,m} \diamond K_n$ is denoted as follows:

$$V(K_{1,m} \diamond K_n) = \bigcup_{i=1}^{m+1} \{(u_i, v_j) : 1 \leq j \leq n\}.$$

Let $\{c_1, c_2, \ldots, c_n\}$ be the set of n distinct colors. The vertices (u_i, v_j) where $1 \leq i \leq m+1$ and $1 \leq j \leq n$ can be colored with the color c_j. Thus $\chi_s(K_{1,m} \diamond K_n) = n$.
Suppose $\chi_s(K_{1,m} \diamond K_n) < n$, say $n-1$. Then the vertices (u_i, v_n), $1 \leq i \leq m+1$ has to be colored with one of the existing colors $\{1, 2, \ldots, n-1\}$ which results in improper coloring (since the vertices (u_i, v_n), $2 \leq i \leq m+1$ is adjacent to every vertices (u_1, v_j), $1 \leq j \leq n-1$ which are colored $1, 2, \ldots, n-1$ and also since the vertex (u_1, v_n) is adjacent to every vertices (u_i, v_j), $2 \leq i \leq m+1, 1 \leq j \leq n-1$ which are colored $1, 2, \ldots, n-1$ and this contradicts the star coloring. Hence $\chi_s(K_{1,m} \diamond K_n) = n$, when $m \geq 1, n \geq 3$. □

References

[1] Albertson, M.O., Chappell, G.G., Kierstead, H.A., Kündgen, A., Ramamurthi, R., Coloring with no 2-colored P_4's. *The Electronic Journal of Combinatorics* 11 (2004), R26, doi:10.37236/1779.

[2] Bondy, J.A., Murty, U.S.R. Graph theory with applications, MacMillan, London 1976.

[3] Clark, J., Holton, D. A., A first look at graph theory, World Scientific, 1991, doi:10.1142/1280.

[4] Coleman, T.F., Moré, J., Estimation of sparse Hessian matrices and graph coloring problems, *Mathematical Programming*, 28(3) (1984), 243–270, doi:10.1007/BF02612334.

[5] Fertin, G., Raspaud, A., Reed, B., On Star coloring of graphs, *Journal of Graph theory*, 47(3) (2004), 163–182, doi:10.1002/jgt.20029

[6] Grünbaum, B., Acyclic colorings of planar graphs, *Israel Journal of Mathematics*, 14 (1973), 390–408, doi:10.1007/BF02764716

[7] Harary, F., Graph theory, Narosa Publishing Home, New Delhi, 1969.

[8] Imrich, W., Klavžar, S., Product Graphs: Structure and Recognition, Wiley, New York 2000.

https://communications.science.ankara.edu.tr

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A NEW CLASS OF GENERATING FUNCTIONS OF BINARY PRODUCTS OF GAUSSIAN NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we introduce a operator in order to derive some new symmetric properties of Gaussian Fibonacci numbers. By making use of the operator defined in this paper, we give some new generating functions for Gaussian Fibonacci numbers, Gaussian Lucas numbers, Gaussian Pell numbers, Gaussian Pell Lucas numbers, Gaussian Jacobsthal numbers, Gaussian Jacobsthal polynomials, Gaussian Jacobsthal Lucas polynomials and Gaussian Pell polynomials.

1. Introduction

In the paper [9, 10], a second-order linear recurrence sequence $(U_n(a,b;p,q))_{n\geq 0}$ or briefly $(U_n)_{n\geq 0}$ is considered by the recurrence relation:

$$U_{n+2} = pU_{n+1} + qU_n,$$

with the initial conditions $U_0 = a$ and $U_1 = b$, where $a, b \in \mathbb{C}$ and $p, q \in \mathbb{Z}_+$ for $n \geq 0$. The special cases are listed below:

- For $p = q = b = 1$, $a = i$ one gets the Gaussian Fibonacci numbers;
- For $p = q = 1$, $a = 2 - i$, $b = 1 + 2i$ one has the Gaussian Lucas numbers;
- For $p = 2, q = 1$, $a = i$, $b = 1$ one has the Gaussian Pell numbers;
- For $p = 2, q = 1$, $a = 2 - 2i$, $b = 2 + 2i$ one has the Gaussian Pell Lucas numbers;
- For $p = 1, q = 2$, $a = \frac{i}{2}$, $b = 1$ it yields Gaussian Jacobsthal numbers;
- For $p = 1, q = 2$, $a = 2 - \frac{i}{2}$, $b = 1 + 2i$ one has the Gaussian Jacobsthal Lucas numbers.

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In this paper, a second-order linear recurrence polynomials $P_n(x)$ is given by the following recurrence relation:

$$P_{n+2}(x) = \alpha P_{n+1}(x) + \beta x P_n(x),$$

with $P_0(x) = p + qx$ and $P_1(x) = r + sx$, where $p, q, r, s \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{Z}_+$ for $n \geq 0$. The special cases of the polynomials $P_n(x)$ are listed as follows:

- For $\alpha = r = 1, \beta = 2$; $p = \frac{i}{2}$, $q = s = 0$ it yields the Gaussian Jacobsthal polynomials $GJ_n(x)$;
- For $\alpha = 1, \beta = 2$; $p = 2 - \frac{i}{2}$, $q = 0, r = 1$, $s = 2i$ it reduces to the Gaussian Jacobsthal Lucas polynomials $Gj_n(x)$.

In [11], Djordjevic and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers.

The Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials $GJ_n(x)$ and $Gj_n(x)$ are defined and studied by authors [15]. They give generating function, Binet formula, explicit formula, Q matrix, determinantal representations and partial derivation of these polynomials. S. Halici and S. Oz are defined in 2016 the Gaussian Pell and Gaussian Pell-Lucas numbers. They give generating functions and Binet formulas of Gaussian Pell and Gaussian Pell-Lucas numbers. The authors in [16] defined Gaussian Pell polynomials, they give the generating functions and Binet formulas for this type polynomials. On the other hand, many kinds of generalizations of Gaussian Fibonacci numbers have been presented in the literature. In particular, a generalization is the Gaussian Fibonacci numbers, the Gaussian Fibonacci numbers, say $(GF_n)_{n\in\mathbb{N}}$, is defined in [14], recurrently by

$$\begin{cases} GF_0 = i, GF_1 = 1 \\ GF_n = GF_{n-1} + GF_{n-2}, \ \forall n \geq 2 \end{cases}.$$

For $n > 1$: One can see that

$$GF_n = F_n + iF_{n-1},$$

where F_n is the n-th usual Fibonacci numbers.

The Gaussian Pell polynomials, say $(GP_n(x))_{n\in\mathbb{N}}$, is defined in [16] recurrently by

$$\begin{cases} GP_0(x) = i, GP_1(x) = 1 \\ GP_n(x) = 2xGP_{n-1}(x) + GP_{n-2}(x), \ \forall n \geq 2 \end{cases}.$$

It is note that we have an important relation between Gaussian Pell polynomials and usual Pell polynomials as follows.

$$GP_n(x) = P_{n-1}(x) + iP_{n-2}(x), \quad \forall n \geq 2.$$

In this contribution, we shall define a useful operator denoted by $\delta^k_{a_1a_2}$ for which we can formulate, extend and prove new results based on our previous ones, (see [4–6]). In order to determine new generating functions of some well-known numbers and polynomials, we combine between our indicated past techniques and these presented polishing approaches.

In section 2, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions which are used in the subsequent sections. we find generating functions of the products of Gaussian Fibonacci numbers, Gaussian Lucas numbers, Gaussian Pell numbers, Gaussian Pell Lucas numbers, Gaussian Jacobsthal numbers, Gaussian Jacobsthal polynomails, Gaussian Jacobsthal Lucas polynomails and Gaussian Pell polynomails, in section 3. In section 4 generating functions of some well-known polynomials.

2. Definitions and some Properties

In this section, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under permutation of the letters of A. Taking an extra indeterminate z, one has two fundamental series [3].

$$\lambda_z(A) = \Pi_{a\in A}(1+az),\ \sigma_z(A) = \frac{1}{\Pi_{a\in A}(1-az)},$$

the expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$:

$$\lambda_z(A) = \sum_{n=0}^{+\infty} \Lambda_n(A)z^n,\ \sigma_z(A) = \sum_{n=0}^{+\infty} S_n(A)z^n.$$

Let us now start at the following definition.

Definition 1. *(see [1]) Let A and B be any two alphabets, then we give $S_n(A-B)$ by the following form:*

$$\frac{\Pi_{b\in B}(1-bz)}{\Pi_{a\in A}(1-az)} = \sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A-B), \tag{2.1}$$

with the condition $S_n(A-B)=0$ for $n<0$.

Corollary 2. *Taking $A=0$ in* (2.1) *gives*

$$\Pi_{b\in B}(1-bz) = \sum_{n=0}^{+\infty} S_n(-B)z^n = \lambda_z(-B). \tag{2.2}$$

Further, in the case $A = 0$ or $B = 0$, we have

$$\sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A) \times \lambda_z(-B). \tag{2.3}$$

Thus,

$$S_n(A-B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B) \text{ (see [2])}. \tag{2.4}$$

Definition 3. *[12] Let n be positive integer and $A = \{a_1, a_2\}$ are set of given variables, then, the n-th symmetric function $S_n(a_1 + a_2)$ is defined by*

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$\begin{aligned} S_0(A) &= S_0(a_1 + a_2) = 1, \\ S_1(A) &= S_1(a_1 + a_2) = a_1 + a_2, \\ S_2(A) &= S_2(a_1 + a_2) = a_1^2 + a_1a_2 + a_2^2, \\ &\vdots \end{aligned}$$

Definition 4. *[7] Given an alphabet $A = \{a_1, a_2\}$, the symmetrizing operator $\delta^k_{a_1a_2}$ is defined by*

$$\delta^k_{a_1a_2} f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}. \tag{2.5}$$

Example 5. *If $f(a_1) = a_1$, the operator (2.5) gives us*

$$\delta^k_{a_1a_2} f(a_1) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2} = S_k(a_1 + a_2).$$

3. Generating functions of some well-known numbers

The following theorem is one of the key tools of the proof of our main result. It has been proved in [7] for the completeness of the paper we state its proof here.

Theorem 6. *Given two alphabets $A = \{a_1, a_2\}$ and $E = \{e_1, e_2\}$, then*

$$\sum_{n=0}^{+\infty} S_n(A)S_n(E)z^n = \frac{1 - a_1a_2e_1e_2z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^n z^n\right)}. \tag{3.1}$$

Proof. By applying the operator $\delta^1_{e_1e_2}$to the series $f(e_1) = \sum\limits_{n=0}^{+\infty} S_n(A)e_1^n z^n$, the left hand side of formula (3.1) can be written as

$$\delta^1_{e_1e_2} f(e_1) \quad = \quad \delta^1_{e_1e_2}\left(\sum_{n=0}^{+\infty} S_n(A)e_1^n z^n\right)$$

$$= \frac{e_1 \sum\limits_{n=0}^{+\infty} S_n(A) e_1^n z^n - e_2 \sum\limits_{n=0}^{+\infty} S_n(A) e_2^n z^n}{e_1 - e_2}$$
$$= \sum_{n=0}^{+\infty} S_n(A) \left(\frac{e_1^{n+1} - e_2^{n+1}}{e_1 - e_2} \right) z^n$$
$$= \sum_{n=0}^{+\infty} S_n(A) S_n(E) z^n,$$

and the right hand side of this formula can be written as

$$\delta^1_{e_1 e_2} \left(\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-A) e_1^n z^n} \right) = \frac{e_1 \sum\limits_{n=0}^{+\infty} S_n(-A) e_2^n z^n - e_2 \sum\limits_{n=0}^{+\infty} S_n(-A) e_1^n z^n}{(e_1 - e_2) \left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_1^n z^n \right) \left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_2^n z^n \right)}$$
$$= \frac{e_1 (1 - a_1 e_2 z)(1 - a_2 e_2 z) - e_2 (1 - a_1 e_1 z)(1 - a_2 e_1 z)}{(e_1 - e_2) \left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_1^n z^n \right) \left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_2^n z^n \right)}$$
$$= \frac{e_1 (1 - e_2 (a_1 + a_2) z + a_1 a_2 e_2^2 z^2) - e_2 (1 - e_1 (a_1 + a_2) z + a_1 a_2 e_1^2 z^2)}{(e_1 - e_2) \left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_1^n z^n \right) \left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_2^n z^n \right)}$$
$$= \frac{1 - a_1 a_2 e_1 e_2 z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_1^n z^n \right) \left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_2^n z^n \right)}.$$

The proof is completed. □

In this part, we now derive the new generating functions of the products of some known numbers.

For the case $A = \{a_1, -a_2\}$ and $E = \{e_1, -e_2\}$ with replacing a_2 by $(-a_2)$, e_2 by $(-e_2)$ in (3.1), we have

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{1 - a_1 a_2 e_1 e_2 z^2}{(1 - a_1 e_1 z)(1 + a_2 e_1 z)(1 + a_1 e_2 z)(1 - a_2 e_2 z)}. \tag{3.2}$$

• Based on the relationship (3.2), we obtain

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{z - a_1 a_2 e_1 e_2 z^3}{(1 - a_1 e_1 z)(1 + a_2 e_1 z)(1 + a_1 e_2 z)(1 - a_2 e_2 z)}. \tag{3.3}$$

This case consists of three related parts. **Firstly**, the substitutions

$$\begin{cases} a_1 - a_2 = 1 \\ a_1 a_2 = 2 \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 1 \\ e_1 e_2 = 2 \end{cases},$$

in (3.2) and (3.3), we obtain

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{1 - 4z^2}{1 - z - 12z^2 - 4z^3 + 16z^4}, \tag{3.4}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{z - 4z^3}{1 - z - 12z^2 - 4z^3 + 16z^4}, \tag{3.5}$$

from which we have the following theorems.

Theorem 7. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal numbers is given by*

$$\sum_{n=0}^{+\infty} GJ_n^2 z^n = \frac{-1 + 5z + 8(i+1)z^2 - 4(3-4i)z^3}{4 - 4z - 48z^2 - 16z^3 + 64z^4}.$$

Proof. We know that

$$GJ_n = (\frac{i}{2} S_n(a_1 + [-a_2]) + (1 - \frac{i}{2}) S_{n-1}(a_1 + [-a_2])), \text{(see [13])}.$$

We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} GJ_n^2 z^n \quad &= \quad \sum_{n=0}^{+\infty} (\frac{i}{2} S_n(a_1 + [-a_2]) + (1 - \frac{i}{2}) S_{n-1}(a_1 + [-a_2])) \\
&\quad \times (\frac{i}{2} S_n(e_1 + [-e_2]) + (1 - \frac{i}{2}) S_{n-1}(e_1 + [-e_2])) z^n \\
&= \quad \frac{-1}{4} \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n \\
&+ \quad (\frac{i}{2} + \frac{1}{4}) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\
&+(\quad \frac{i}{2} + \frac{1}{4}) \sum_{n=0}^{+\infty} S_n(e_1 + [-e_2]) S_{n-1}(a_1 + [-a_2]) z^n \\
&+ \quad (1 - \frac{i}{2})^2 \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\
&= \quad \frac{-1 + 4z^2}{4 - 4z - 48z^2 - 16z^3 + 64z^4} + \frac{2(2i+1)(z + 2z^2)}{4 - 4z - 48z^2 - 16z^3 + 64z^4}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{(2-i)^2(z-4z^3)}{4-4z-48z^2-16z^3+64z^4} \\
&= \quad \frac{-1+5z+8(i+1)z^2-4(3-4i)z^3}{4-4z-48z^2-16z^3+64z^4}.
\end{aligned}
$$

This completes the proof. □

Theorem 8. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} Gj_n^2 z^n = \frac{15-8i+(24i-27)z+(120i-72)z^2+(80i+84)z^3}{4-4z-48z^2-16z^3+64z^4}.$$

Proof. Since

$$Gj_n = (2-\frac{i}{2})S_n(a_1+[-a_2]) + (\frac{5i}{2}-1)S_{n-1}(a_1+[-a_2]), \text{ (see [13]).}$$

From which we have

$$
\begin{aligned}
\sum_{n=0}^{+\infty} Gj_n^2 z^n \quad &= \quad \sum_{n=0}^{+\infty}((2-\frac{i}{2})S_n(a_1+[-a_2]) + (\frac{5i}{2}-1)S_{n-1}(a_1+[-a_2])) \\
&\quad \times((2-\frac{i}{2})S_n(e_1+[-e_2]) + (\frac{5i}{2}-1)S_{n-1}(e_1+[-e_2]))z^n \\
&= \quad (2-\frac{i}{2})^2 \sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&+ \quad (2-\frac{i}{2})(\frac{5i}{2}-1)\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&+ \quad (2-\frac{i}{2})(\frac{5i}{2}-1)\sum_{n=0}^{+\infty} S_n(e_1+[-e_2])S_{n-1}(a_1+[-a_2])z^n \\
&+ \quad (\frac{5i}{2}-1)^2 \sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.
\end{aligned}
$$

Then

$$
\begin{aligned}
\sum_{n=0}^{+\infty} Gj_n^2 z^n \quad &= \quad \frac{(15-8i)(1-4z^2)}{4-4z-48z^2-16z^3+64z^4} + \frac{(44i-6)(z+2z^2)}{4-4z-48z^2-16z^3+64z^4} \\
&- \quad \frac{(20i+21)(z-4z^3)}{4-4z-48z^2-16z^3+64z^4} \\
&= \quad \frac{15-8i+(24i-27)z+(120i-72)z^2+(80i+84)z^3}{4-4z-48z^2-16z^3+64z^4}.
\end{aligned}
$$

The proof is completed. □

Secondly, the substitutions

$$\begin{cases} a_1 - a_2 = 1 \\ a_1 a_2 = 1 \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = 1 \\ e_1 e_2 = 1 \end{cases},$$

in (3.2) and (3.3) we obtain

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{1 - z^2}{1 - z - 4z^2 - z^3 + z^4}, \tag{3.6}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{z - z^3}{1 - z - 4z^2 - z^3 + z^4}. \tag{3.7}$$

We have the following theorems.

Theorem 9. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Fibonacci numbers is given by*

$$\sum_{n=0}^{+\infty} GF_n^2 z^n = \frac{-1 + 2z + (2i + 3) z^2 + 2iz^3}{1 - z - 4z^2 - z^3 + z^4}.$$

Proof. From the reference [13] we have

$$GF_n = iS_n(a_1 + [-a_2]) + (1 - i) S_{n-1}(a_1 + [-a_2]),$$

and from it

$$\begin{aligned} \sum_{n=0}^{+\infty} GF_n^2 z^n &= \sum_{n=0}^{+\infty} (iS_n(a_1 + [-a_2]) + (1 - i) S_{n-1}(a_1 + [-a_2])) \\ &\quad \times (iS_n(e_1 + [-e_2]) + (1 - i) S_{n-1}(e_1 + [-e_2]))\, z^n \\ &= -\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n \\ &+ i(1 - i) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &+ i(1 - i) \sum_{n=0}^{+\infty} S_n(e_1 + [-e_2]) S_{n-1}(a_1 + [-a_2]) z^n \\ &+ (1 - i)^2 \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n. \end{aligned}$$

Therefore

$$\sum_{n=0}^{+\infty} GF_n^2 z^n = \frac{z^2 - 1}{1 - z - 4z^2 - z^3 + z^4} + \frac{2(1 + i)(z + z^2)}{1 - z - 4z^2 - z^3 + z^4}$$

$$
\begin{aligned}
& +\frac{(1-i)^2(z-z^3)}{1-z-4z^2-z^3+z^4} \\
= \quad & \frac{-1+2z+(2i+3)z^2+2iz^3}{1-z-4z^2-z^3+z^4}.
\end{aligned}
$$

This completes the proof. □

Theorem 10. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} GL_n^2 z^n = \frac{3-4i+(8i-6)z+(18i-1)z+(6i+8)z^3}{1-z-4z^2-z^3+z^4}.$$

Proof. According to [13], we have

$$GL_n = (2-i)S_n(a_1+[-a_2]) + (3i-1)S_{n-1}(a_1+[-a_2]).$$

From which we have

$$
\begin{aligned}
\sum_{n=0}^{+\infty} GL_n^2 z^n \quad = \quad & \sum_{n=0}^{+\infty}((2-i)S_n(a_1+[-a_2]) + (3i-1)S_{n-1}(a_1+[-a_2])) \\
& \times((2-i)S_n(e_1+[-e_2]) + (3i-1)S_{n-1}(e_1+[-e_2]))z^n \\
= \quad & (2-i)^2 \sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
+ \quad & (2-i)(3i-1)\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
+ \quad & (2-i)(3i-1)\sum_{n=0}^{+\infty} S_n(e_1+[-e_2])S_{n-1}(a_1+[-a_2])z^n \\
+ \quad & (3i-1)^2 \sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.
\end{aligned}
$$

By using the relationships (3.6) and (3.7), we obtain

$$
\begin{aligned}
\sum_{n=0}^{+\infty} GL_n^2 z^n \quad = \quad & \frac{(3-4i)(1-z^2)}{1-z-4z^2-z^3+z^4} + \frac{2(7i+1)(z+z^2)}{1-z-4z^2-z^3+z^4} \\
& +\frac{(3i-1)^2(z-z^3)}{1-z-4z^2-z^3+z^4} \\
= \quad & \frac{3-4i+(8i-6)z+(18i-1)z^2+(6i+8)z^3}{1-z-4z^2-z^3+z^4}.
\end{aligned}
$$

The proof is completed. □

Thirdly, the substitutions

$$\begin{cases} a_1 - a_2 = 2 \\ a_1 a_2 = 1 \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 2 \\ e_1 e_2 = 1 \end{cases},$$

in (3.2) and (3.3) we obtain

$$\sum_{n=0}^{+\infty} S_n(e_1 + [-e_2]) S_n(e_1 + [-e_2]) z^n = \frac{1 - z^2}{1 - 4z - 10z^2 - 4z^3 + z^4}, \tag{3.8}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(e_1 + [-e_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{z - z^3}{1 - 4z - 10z^2 - 4z^3 + z^4}, \tag{3.9}$$

and we have the following theorems.

Theorem 11. *For $n \in \mathbb{N}$, the new generating functions of the product of Gaussian Pell numbers is given by*

$$\sum_{n=0}^{+\infty} GP_n^2 z^n = \frac{-1 + 5z + (9 + 4i)z^2 + (3 + 4i)z^3}{1 - 4z - 10z^2 - 4z^3 + z^4}.$$

Proof. By [13], we have $GP_n = iS_n(a_1 + [-a_2]) + (1 - 2i)S_{n-1}(a_1 + [-a_2])$.

Then, we can see that

$$\begin{aligned} \sum_{n=0}^{+\infty} GP_n^2 z^n &= \sum_{n=0}^{+\infty} (iS_n(a_1 + [-a_2]) + (1 - 2i)S_{n-1}(a_1 + [-a_2])) \\ &\quad \times (iS_n(e_1 + [-e_2]) + (1 - 2i)S_{n-1}(e_1 + [-e_2]))z^n \\ &= i^2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n \\ &+ i(1 - 2i) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &+ i(1 - 2i) \sum_{n=0}^{+\infty} S_n(e_1 + [-e_2]) S_{n-1}(a_1 + [-a_2]) z^n \\ &+ (1 - 2i)^2 \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=0}^{+\infty} GP_n^2 z^n &= \frac{-1(1 - z^2)}{1 - 4z - 10z^2 - 4z^3 + z^4} + \frac{2i(1 - 2i)(2z + 2z^2)}{1 - 4z - 10z^2 - 4z^3 + z^4} \\ &\quad + \frac{(1 - 2i)^2 (z - z^3)}{1 - 4z - 10z^2 - 4z^3 + z^4} \end{aligned}$$

$$= \frac{-1+5z+(9+4i)z^2+(3+4i)z^3}{1-4z-10z^2-4z^3+z^4}.$$

This completes the proof. □

Theorem 12. *For $n \in \mathbb{N}$, the new generating functions of the product of Gaussian Pell Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} GQ_n^2 z^n = \frac{-8i+40iz+(32+72i)z^2+(32+24i)z^3}{1-4z-10z^2-4z^3+z^4}.$$

Proof. Since

$$GQ_n = (2-2i)S_n(a_1+[-a_2]) + (6i-2)S_{n-1}(a_1+[-a_2]), \text{ (see [13]).}$$

From which we have

$$\begin{aligned}
\sum_{n=0}^{+\infty} GQ_n^2 z^n &= \sum_{n=0}^{+\infty}((2-2i)S_n(a_1+[-a_2]) + (6i-2)S_{n-1}(a_1+[-a_2])) \\
&\quad \times((2-2i)S_n(e_1+[-e_2]) + (6i-2)S_{n-1}(e_1+[-e_2]))z^n \\
&= (2-2i)^2 \sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&+ (2-2i)(6i-2)\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&+ (2-2i)(6i-2)\sum_{n=0}^{+\infty} S_n(e_1+[-e_2])S_{n-1}(a_1+[-a_2])z^n \\
&+ (6i-2)^2 \sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{n=0}^{+\infty} GQ_n^2 z^n &= \frac{(2-2i)^2(1-z^2)}{1-4z-10z^2-4z^3+z^4} + \frac{2(2-2i)(6i-2)(2z+2z^2)}{1-4z-10z^2-4z^3+z^4} \\
&\quad + \frac{(6i-2)^2(z-z^3)}{1-4z-10z^2-4z^3+z^4} \\
&= \frac{-8i+40iz+(32+72i)z^2+(32+24i)z^3}{1-4z-10z^2-4z^3+z^4}.
\end{aligned}$$

The proof is completed. □

4. Generating functions of some well-known polynomials

In this part, we now derive the new generating functions of the products of some known polynomials.

This case consists of two related parts. **Firstly**, the substitutions

$$\begin{cases} a_1 - a_2 = 1 \\ a_1 a_2 = 2x \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 1 \\ e_1 e_2 = 2y \end{cases},$$

in (3.2) and (3.3) we obtain

$$\sum_{n=0}^{+\infty} S_n\left(a_1 + [-a_2]\right) S_n\left(e_1 + [-e_2]\right) z^n = \frac{1 - 4xyz^2}{1 - z - 2\left(x + y + 4xy\right) z^2 - 4xyz^3 + 16x^2y^2z^4}, \tag{4.1}$$

$$\sum_{n=0}^{+\infty} S_{n-1}\left(a_1 + [-a_2]\right) S_{n-1}\left(e_1 + [-e_2]\right) z^n = \frac{z - 4xyz^3}{1 - z - 2\left(x + y + 4xy\right) z^2 - 4xyz^3 + 16x^2y^2z^4}, \tag{4.2}$$

from which we have the following theorems.

Theorem 13. *For $n \in \mathbb{N}$, the new generating function of product of Gaussian Jacobsthal polynomials is given by*

$$\sum_{n=0}^{+\infty} GJ_n(x) GJ_n(y) z^n = \frac{-1 + 5z + ((4i+2)(x+y) + 4xy)z^2 - 4xy(3-4i)z^3}{4 - 4z - 8\left(x + y + 4xy\right) z^2 - 16xyz^3 + 64x^2y^2z^4}.$$

Proof. From the reference [13] we have

$$GJ_n(x) = \frac{i}{2} S_n(a_1 + [-a_2]) + (1 - \frac{i}{2}) S_{n-1}(a_1 + [-a_2]),$$

and from it

$$\begin{aligned}
\sum_{n=0}^{+\infty} GJ_n(x) GJ_n(y) z^n &= \sum_{n=0}^{+\infty} (\frac{i}{2} S_n(a_1 + [-a_2]) + (1 - \frac{i}{2}) S_{n-1}(a_1 + [-a_2])) \\
&\quad \times (\frac{i}{2} S_n(e_1 + [-e_2]) + (1 - \frac{i}{2}) S_{n-1}(e_1 + [-e_2])) z^n \\
&= \frac{-1}{4} \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n \\
&+ \frac{i}{2}(1 - \frac{i}{2}) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n
\end{aligned}$$

$$+ \quad \frac{i}{2}(1-\frac{i}{2})\sum_{n=0}^{+\infty} S_n(e_1+[-e_2])S_{n-1}(a_1+[-a_2])z^n$$
$$+ \quad (1-\frac{i}{2})^2\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.$$

Therefore

$$\begin{aligned}\sum_{n=0}^{+\infty} GJ_n(x)GJ_n(y)z^n &= \frac{4xyz^2-1}{4-4z-8\left(x+y+4xy\right)z^2-16xyz^3+64x^2y^2z^4}\\ &+ \frac{(2i+1)(z+2xz^2)}{4-4z-8\left(x+y+4xy\right)z^2-16xyz^3+64x^2y^2z^4}\\ &+ \frac{(2i+1)(z+2yz^2)}{4-4z-8\left(x+y+4xy\right)z^2-16xyz^3+64x^2y^2z^4}\\ &+ \frac{(3-4i)(z-4xyz^3)}{4-4z-8\left(x+y+4xy\right)z^2-16xyz^3+64x^2y^2z^4}\\ &= \frac{-1+5z+((4i+2)(x+y)+4xy)z^2-4xy(3-4i)z^3}{4-4z-8\left(x+y+4xy\right)z^2-16xyz^3+64x^2y^2z^4}.\end{aligned}$$

This completes the proof. □

Theorem 14. *For $n\in\mathbb{N}$, the new generating function of product of Gaussian Jacobsthal Lucas polynomials is given by*

$$\begin{aligned}&\sum_{n=0}^{+\infty} Gj_n(x)Gj_n(y)z^n\\ &= \frac{15-8i \ +(8i(x+y)-16xy-11+8i)z-((14-12i)(x+y)+(44-96i)xy)z^2}{4-4z-8\left(x+y+4xy\right)z^2-16xyz^3+64x^2y^2z^4}\\ &\qquad +\frac{4xy(16xy+(4+8i)(x+y)-(3-4i))z^3}{4-4z-8\left(x+y+4xy\right)z^2-16xyz^3+64x^2y^2z^4}.\end{aligned}$$

Proof. We know that

$$Gj_n(x)=(2-\frac{i}{2})S_n(a_1+[-a_2])+(2ix+\frac{i}{2}-1)S_{n-1}(a_1+[-a_2]),\ \text{(see [13])}.$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} Gj_n(x)Gj_n(y)z^n &= \sum_{n=0}^{+\infty}((2-\frac{i}{2})S_n(a_1+[-a_2])+(2ix+\frac{i}{2}-1)S_{n-1}(a_1+[-a_2]))\\ &\quad \times((2-\frac{i}{2})S_n(e_1+[-e_2])+(2iy+\frac{i}{2}-1)S_{n-1}(e_1+[-e_2]))z^n\\ &= (2-\frac{i}{2})^2\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n\end{aligned}$$

$$
\begin{aligned}
&+ \quad (2-\frac{i}{2})(2iy+\frac{i}{2}-1)\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&+ \quad (2-\frac{i}{2})(2ix+\frac{i}{2}-1)\sum_{n=0}^{+\infty} S_n(e_1+[-e_2])S_{n-1}(a_1+[-a_2])z^n \\
&+ \quad (2ix+\frac{i}{2}-1)(2iy+\frac{i}{2}-1) \\
&\quad\quad \times\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&= \quad \frac{(2-\frac{i}{2})^2(1-4xyz^2)}{1-z-2\,(x+y+4xy)\,z^2-4xyz^3+16x^2y^2z^4} \\
&+ \quad \frac{(2-\frac{i}{2})(2iy+\frac{i}{2}-1)(z+2xz^2)}{1-z-2\,(x+y+4xy)\,z^2-4xyz^3+16x^2y^2z^4} \\
&+ \quad \frac{(2-\frac{i}{2})(2ix+\frac{i}{2}-1)(z+2yz^2)}{1-z-2\,(x+y+4xy)\,z^2-4xyz^3+16x^2y^2z^4} \\
&+ \quad \frac{(2ix+\frac{i}{2}-1)(2iy+\frac{i}{2}-1)(z-4xyz^3)}{1-z-2\,(x+y+4xy)\,z^2-4xyz^3+16x^2y^2z^4} \\
&= \quad \frac{15-8i\ +(8i(x+y)-16xy-11+8i)z-((14-12i)(x+y)}{4-4z-8\,(x+y+4xy)\,z^2-16xyz^3+64x^2y^2z^4} \\
&+ \quad \frac{(44-96i)xy)z^2+4xy(16xy+(4+8i)(x+y)-(3-4i))z^3}{4-4z-8\,(x+y+4xy)\,z^2-16xyz^3+64x^2y^2z^4}.
\end{aligned}
$$

This completes the proof. $\square$

Secondly, the substitutions

$$
\left\{\begin{array}{c} a_1-a_2=2x \\ a_1a_2=1 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{c} e_1-e_2=2y \\ e_1e_2=1 \end{array}\right. ,
$$

in (3.2) and (3.3) we give

$$
\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{1-z^2}{1-4xyz-(4(x^2+y^2)+2)z^2-4xyz^3+z^4}, \tag{4.3}
$$

$$
\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{z-z^3}{1-4xyz-(4(x^2+y^2)+2)z^2-4xyz^3+z^4}, \tag{4.4}
$$

from which we have the following theorem.

Theorem 15. *For $n \in \mathbb{N}$, the new generating functions of the product of Gaussian Pell polynomials is given by*

$$\sum_{n=0}^{+\infty} GP_n(x)GP_n(y)z^n$$
$$= \frac{-1+(4xy+1)z+(4(x^2+y^2)+2i(x+y)+1)z^2+(2i(x+y)+4xy-1)z^3}{1-4xyz-(4(x^2+y^2)+2)z^2-4xyz^3+z^4}.$$

Proof. We know that

$$GP_n(x) = iS_n(a_1+[-a_2]) + (1-2ix)S_{n-1}(a_1+[-a_2]), \text{ (see [13])}.$$

and from it we obtain

$$\begin{aligned}
\sum_{n=0}^{+\infty} GP_n(x)GP_n(y)z^n &= \sum_{n=0}^{+\infty}(iS_n(a_1+[-a_2]) + (1-2ix)S_{n-1}(a_1+[-a_2])) \\
&\times(iS_n(e_1+[-e_2]) + (1-2iy)S_{n-1}(e_1+[-e_2]))z^n \\
&= i^2\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&+i(1-2iy)\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&+i(1-2ix)\sum_{n=0}^{+\infty} S_n(e_1+[-e_2])S_{n-1}(a_1+[-a_2])z^n \\
&+(1-2ix)(1-2iy)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.
\end{aligned}$$

By using the relationships (4.3) and (4.4), we obtain

$$\begin{aligned}
\sum_{n=0}^{+\infty} GP_n(x)GP_n(y)z^n &= \frac{z^2-1}{1-4xyz-(4(x^2+y^2)+2)z^2-4xyz^3+z^4} \\
&+\frac{i(1-2iy)(2xz+2yz^2)}{1-4xyz-(4(x^2+y^2)+2)z^2-4xyz^3+z^4} \\
&+\frac{i(1-2ix)(2yz+2xz^2)}{1-4xyz-(4(x^2+y^2)+2)z^2-4xyz^3+z^4} \\
&+\frac{(1-2ix)(1-2iy)(z-z^3)}{1-4xyz-(4(x^2+y^2)+2)z^2-4xyz^3+z^4}
\end{aligned}$$
$$= \frac{-1+(4xy+1)z+(4(x^2+y^2)+2i(x+y)+1)z^2+(2i(x+y)+4xy-1)z^3}{1-4xyz-(4(x^2+y^2)+2)z^2-4xyz^3+z^4}.$$

The proof is completed. $\square$

5. Conclusion

In this paper, by making use of Eq. (3.1), we have derived some new generating functions for the products of Gaussian Fibonacci numbers, Gaussian Lucas numbers, Gaussian Pell numbers, Gaussian Pell Lucas numbers, Gaussian Jacobsthal numbers, Gaussian Jacobsthal polynomials, Gaussian Jacobsthal Lucas polynomials and Gaussian Pell polynomials. The derived theorems are based on symmetric functions and products of these numbers and polynomials.

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References

[1] Abderrezzak, A., Généralisation de la transformation d'Euler d'une série formelle, *Adv. Math.,* 103 (1994), 180–195.

[2] Abderrezzak, A., Généralisation d'identities de Carlitz, *Howard et Lehmer. Aequ. Math.,* 49 (1995), 36–46.

[3] Boussayoud, A., Boughaba, S., Kerada, M., Araci, S., Acikgoz, M., Symmetric Functions of Binary Products of k-Fibonacci and Orthogonal Polynomials, *Rev. R. Acad. Cienc. Exactas Fís. Nat, Ser. A Mat, RACSAM,* 113 (2019), 2575-2586.

[4] Boussayoud, A., On some identities and generating functions for Pell-Lucas numbers, *Online J. Anal. Comb.,* 12, (2017), 1-10.

[5] Boussayoud, A., Kerada, Boulyer, M. M., A simple and accurate method for determination of some generalized sequence of numbers, *Int. J. Pure. Appl. Math.,* 108, (2016), 503-511.

[6] Boussayoud, A., Sahali, R., The application of the operator $L_{b_1b_2}^{-k}$ in the series $\sum_{j=0}^{+\infty} a_j b_1^j z^j$, *J. Adv. Res. Appl. Math.,* 7, (2015), 68-75.

[7] Boussayoud, A., Kerada, M., Symmetric and Generating Functions, *Int. Electron. J. Pure Appl. Math.,* 7, (2014), 195-203.

[8] Boussayoud, A., L'action de l'opérateur $\delta_{e_1e_2}^k$sur la série $\sum\limits_{n=0}^{+\infty} S_n(A)e_1^n z^n$, (Doctoral dissertation), Mohamed Seddik Ben Yahia University, Jijel, Algeria, (2017.

[9] Boussayoud, A., Boughaba,S., On Some Identities and Symmetric Functions for k-Pell sequences and Chebychev polynomials, *Online J. Anal. Comb.* 14 (2019), 1-13.

[10] Boussayoud, A., Kerada, M., Araci, S., Acikgoz, M., Esi, A., Generating Functions of Binary Products of Fibonacci and Tchebyshev Polynomials of first and second kinds, *Filomat* 33 (2019), 1495-1504.

[11] Djordjevic, G. B., Srivastava, H. M., Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers, *Math. Comput. Modelling*, 42 (2005), 1049–1056.

[12] Boubellouta, Kh., Boussayoud, A., Kerada, M., Symmetric Functions for Second-Order Recurrence Sequences, *Tbil. Math. J.,* 13 (2020), 225-237.

[13] Boughaba, S., Boussayoud, A., Araci, S., Kerada, M., Construction of Generating Functions of Gaussian Fibonacci Numbers and Polynomials, *Universal Journal of Mathematics and Applications*, (Submitted).

[14] Jordan, J. H., Gaussian Fibonacci and Lucas numbers, *Fibonacci Q.,* 3 (1965), 315–318.

[15] Asci, M., Gurel, E., Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials, *Notes Number Theory Discrete Math.,* 19 (2013), 25-36.

[16] Halici, S., Oz, S., On Gaussian Pell Polynomials and Their Some Properties, *Palest. J. Math.,* 7(1) (2018), 251–256.

https://communications.science.ankara.edu.tr

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SOME FLATNESS CONDITIONS ON NORMAL METRIC CONTACT PAIRS

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Abstract. In this paper, the geometry of normal metric contact pair manifolds is studied under the flatness of conformal, concircular and quasi-conformal curvature tensors. It is proved that a conformal flat normal metric contact pair manifold is an Einstein manifold with a positive scalar curvature and has positive sectional curvature. It is also shown that a concircular flat normal metric contact pair manifold is an Einstein manifold. Finally, it is obtained that a quasi-conformally flat normal metric contact pair manifold is an Einstein manifold with a positive scalar curvature and, is a space of constant curvature.

1. Introduction

Contact transformations were defined as a geometric tool to study system of differential equations in 1872 by S.Lie [1]. Afterward the notion of contact manifolds has occurred in the manifold theory. Contact manifolds have many applications in mathematics and some applied areas such as mechanics, optics, thermodynamics, control theory and theoretical physics [2]. The Riemannian geometry of contact manifolds give us some geometric interpretation about Einstein manifolds which are arisen from the theory of relativity.

A conformal transformation is a map which converts a metric to another with preserving angle between two vector fields. Conformal curvature tensor on a Riemann manifold is a curvature tensor of the $(1,3)-$type that is invariant under conformal transformations. This tensor gives important information about the Riemann geometry of the manifold. If it vanishes then the manifold is said to be conformally flat, that's mean the manifold is flat under conformal transformations. A concircular transformation is a special conformal transformation and, preserves the geodesic circle. These type of transformations and their applications to differential geometry

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were studied by Yano [3]. In same paper Yano defined concircular curvature tensor and showed that this tensor is invariant under concircular transformations. A manifold is called concircularly flat if this tensor vanishes. Yano and Sawaski [4] introduced quasi-conformal curvature which includes both concircular and conformal curvature tensor as special cases. If this tensor vanishes on the manifold identically then the manifold is called quasi-conformally flat. Flatness conditions of conformal, concircular and quasi-conformal curvature tensors on contact manifolds has many geometric and physical applications. For example, while a conformal flat Sasakian manifold is of constant curvature [5], a normal complex contact metric manifold is not conformal, concircular and quasi-conformal flat [6].

Blair, Ludden and Yano [7] studied on complex manifolds consider the results on Calabi-Eckman manifolds $S^{2p+1} \times S^{2q+1}$. By consider two Sasakian structure on S^{2p+1} and S^{2q+1} they gave the second fundamental form on Calabi-Eckman manifold, defined Hermitian bicontact manifold and obtained an $f-$structure on bicontact manifolds. Also normality of bicontact manifolds was given in same work. Bande and Hadjar [8] studied on bicontact manifolds under the name contact pairs. Further they considered a special type of $f-$structure with complementary frames related to a contact pair and, called by contact pair structure. The normality of contact pair structures were given by same authors [9–11]. The conformal flatness of a normal metric contact pair manifold were studied by Bande, Blair and Hadjar [12]. They proved that a conformal flat normal metric contact pair manifold is locally isometric to Hopf manifold $S^{2p+1}(1) \times S^1$.

In this paper we studied on conformal, concircular and quasi-conformal curvature flatness of normal metric contact pair manifold. We prove that a conformal flat normal metric contact pair manifold is an Einstein manifold with a positive scalar curvature and, has positive sectional curvature. Also we obtain that a concircular flat normal metric contact pair manifold is an Einstein manifold. Finally we prove that a quasi-conformal flat normal metric contact pair manifold is an Einstein manifold with a positive scalar curvature and, is a space of constant curvature.

2. Preliminaries

In this section a short survey is given for contact manifolds and contact pair structures. For detail we refer to reader [8–10, 13].

2.1. **Real and Complex Contact Manifolds.** A real contact manifold is defined by a contact form η which is a volume form on a real $(2p+1)-$ dimensional differentiable manifold M. The kernel of η defines $2p-$dimensional a non-integrable distribution of TM:

$$\mathcal{D} = \{X : \eta(X) = 0,\ X \in \Gamma(TM)\}.$$

We also recall $\mathcal{D}$ contact or horizontal distribution. Let take a vector field ξ on M which is dual vector of η. Then for $(1,1)-$tensor field φ, M is called an almost contact metric manifold if following conditions are satisfied:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi\bullet, \bullet) = -g(\bullet, \varphi\bullet)$$

where I is identity map on TM and g is a Riemannian metric [13]. Also we call g by compatible metric. Similar to Kähler manifold we have a second fundamental form on an almost contact metric manifold $\Omega(\bullet, \bullet) = d\eta(\bullet, \bullet)$. Also $d\eta(\bullet, \bullet) = g(\bullet, \varphi\bullet)$ and in this case we recall g is an associated metric.

The geometry of contact manifold is studied in different classes. One of them is Sasakian manifold which has a Kähler form on Riemannian cone $M \times \mathbb{R}^+$. A Sasakian manifold has also an almost contact metric structure. The almost contact structure on a Sasakian manifold is normal i.e. $N(\varphi\bullet, \varphi\bullet) + 2d\eta(\bullet, \bullet)\xi = 0$ where $N(\varphi\bullet, \varphi\bullet)$ is the Nijenhuis tensor field of φ.

In 1959 Kobayashi [14] defined complex analogue of a real contact manifold. Therefore the concept of complex contact manifold entered to the literature. 1980s Ishihara and Konishi [15] constructed almost contact structure on a complex contact manifold and they defined compatible metric. A complex almost contact metric manifold is a complex odd $(2p+1)-$dimensional complex manifold with $(J, \varphi, \varphi \circ J, \xi, -J \circ \xi, \eta, \eta \circ J, g)$ structure such that

$$\varphi^2 = (\varphi J)^2 = -I + \eta \otimes \xi - (\eta \circ J) \otimes (J \circ \xi),$$
$$\eta(\xi) = 1 \quad , \eta(-J \circ \xi) = 0, (\eta \circ J)(-J \circ \xi) = 1, \ (\eta \circ J)(\xi) = 0,$$
$$g(\varphi\bullet, \bullet) = -g(\bullet, \varphi\bullet), \quad g((\varphi \circ J)\bullet, \bullet) = -g(\bullet, (\varphi \circ J)\bullet)$$

where g is a Hermitian metric on M, J is a natural almost complex structure. The normality of complex almost contact metric manifolds were given by Ishihara-Konishi and Korkmaz [15, 16]. Normal complex contact metric manifolds were studied by several authors [6, 16, 17].

2.2. Metric contact pair manifold.

Definition 1. *Let M be a $(2p+2q+2)$-dimensional differentiable manifold. A pair of (α_1, α_2) on M is said to be a contact pair of type (p, q) if*

- $\alpha_1 \wedge (d\alpha_1)^p \wedge \alpha_2 \wedge (d\alpha_2)^q \neq 0$
- $(d\alpha_1)^{p+1} = 0$ *and* $(d\alpha_2)^{q+1} = 0$

where p, q are positive integers [8].

For $1-$forms α_1 and α_2 we have two integrable subbundles of TM; $\mathcal{D}_1 = \{X : \alpha_1(X) = 0, X \in \Gamma(TM)\}$ and $\mathcal{D}_2 = \{X : \alpha_2(X) = 0, X \in \Gamma(TM)\}$. Then we have two characteristic foliations of M, denoted by $\mathcal{F}_1 = \mathcal{D}_1 \cap ker d\alpha_1$ and $\mathcal{F}_2 = \mathcal{D}_2 \cap ker d\alpha_2$, respectively. $\mathcal{F}_1$ and $\mathcal{F}_2$ are $(2p+1)$ and $(2q+1)-$dimensional contact manifolds with contact form induced by α_2 and α_1 respectively. For a contact pair (α_1, α_2) of type (p, q) there are associated two commuting vector fields Z_1 and Z_2, called Reeb vector fields of the pair, which are determined uniquely by the following equations:

$$\alpha_1(Z_1) = \alpha_2(Z_2) = 1, \ \alpha_1(Z_2) = \alpha_2(Z_1) = 0,$$

$$i_{Z_1} d\alpha_1 = i_{Z_1} d\alpha_2 = i_{Z_2} d\alpha_1 = i_{Z_2} d\alpha_2 = 0$$

where i_X is the contraction with the vector field X. In particular, since the Reeb vector fields commute, they determine a locally free $\mathbb{R}^2$-action, called the Reeb action.

The tangent bundle of $(M, (\alpha_1, \alpha_2))$ can be decomposable by different ways. For the two subbundle of TM which are given by

$$T\mathcal{G}_i = ker d\alpha_i \cap ker\alpha_1 \cap ker\alpha_2, \; i = 1, 2,$$

we can write

$$T\mathcal{F}_1 = T\mathcal{G}_1 \oplus \mathbb{R}Z_2 \text{ and } T\mathcal{F}_2 = T\mathcal{G}_2 \oplus \mathbb{R}Z_1$$

Therefore we get $TM = T\mathcal{G}_1 \oplus T\mathcal{G}_2 \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$. Also we can state $TM = \mathcal{H} \oplus \mathcal{V}$ for $\mathcal{H} = T\mathcal{G}_1 \oplus T\mathcal{G}_2$ and $\mathcal{V} = \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$, we call $\mathcal{H}$ is horizontal subbundle and $\mathcal{V}$ is vertical subbundle of TM.

Let X be an arbitrary vector field on M. We can write $X = X^{\mathcal{H}} + X^{\mathcal{V}}$, where $X^{\mathcal{H}}, X^{\mathcal{V}}$ horizontal and vertical component of X respectively. For $X^1 \in T\mathcal{F}_1$ and $X^2 \in T\mathcal{F}_2$ we have $X = X^1 + X^2$. Also we can write $X^1 = X^{1^h} + \alpha_2(X^1)Z_2$ and $X^2 = X^{2^h} + \alpha_1(X^2)Z_1$, where X^{1^h} and X^{2^h} are horizontal parts of X^1, X^2 respectively. Thus we have $X^{\mathcal{H}} = X^{1^h} + X^{2^h}$, $X^{\mathcal{V}} = \alpha_1(X^2)Z_1 + \alpha_2(X^1)Z_2$. From all these decompositions of X finally we get

$$X = X^{1^h} + X^{2^h} + \alpha_1(X^2)Z_1 + \alpha_2(X^1)Z_2,$$
$$\alpha_1(X^{1^h}) = \alpha_1(X^{2^h}) = 0, \quad \alpha_2(X^{1^h}) = \alpha_2(X^{2^h}) = 0.$$

Since we have two different $1-$forms by above decomposition we understand the components of $X \in \Gamma(TM)$ in which distributions.

Definition 2. *An almost contact pair structure on a $(2p+2q+2)-$dimensional manifold M is a structure $(\alpha_1, \alpha_2, \phi, Z_1, Z_2)$, where (α_1, α_2) is a contact pair, ϕ is a $(1,1)$ tensor field on M and Z_1, Z_2 are Reeb vector fields such that:*

$$\phi^2 = -I + \alpha_1 \otimes Z_1 + \alpha_2 \otimes Z_2, \quad \phi Z_1 = \phi Z_2 = 0.$$

The rank of ϕ is $(2p+2q)$ and $\alpha_i(\phi) = 0$ for $i = 1, 2$.

The endomorphism ϕ is said to be decomposable if $T\mathcal{F}_i$ is invariant under ϕ. If ϕ is decomposable then we have almost contact structure on $T\mathcal{F}_i$ for $i = 1, 2$ are induced from (α_i, Z_i, ϕ), $i \neq j$, [10]. Unless otherwise stated we assume that ϕ is decomposable.

Definition 3. *Let $(\alpha_1, \alpha_2, Z_1, Z_2, \phi)$ be an almost contact pair structure on a manifold M. A Riemannian metric g is called*

(1) *compatible if $g(\phi X_1, \phi X_2) = g(X_1, X_2) - \alpha_1(X_1)\alpha_1(X_2) - \alpha_2(X_1)\alpha_2(X_2)$ for all $X_1, X_2 \in \Gamma(TM)$.*
(2) *associated if $g(X_1, \phi X_2) = (d\alpha_1 + d\alpha_2)(X_1, X_2)$ and $g(X_1, Z_i) = \alpha_i(X_1)$, for $i = 1, 2$ and for all $X_1, X_2 \in \Gamma(TM)$ [10].*

We call $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2, g)$ by a metric almost contact pair manifold and we have the following properties on M [10]:

$$g(Z_i, X) = \alpha_i(X), \quad g(Z_i, Z_j) = \delta_{ij},$$
$$\nabla_{Z_i} Z_j = 0, \ \nabla_{Z_i}\phi = 0,$$

and for every X tangent to M, $i = 1, 2$ we have

$$\nabla_X Z_1 = -\phi_1 X, \quad \nabla_X Z_2 = -\phi_2 X$$

where $\phi = \phi_1 + \phi_2$ [10].

We have two almost complex structures on $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2, g)$ as;

$$J = \phi - \alpha_2 \otimes Z_1 + \alpha_1 \otimes Z_2, \quad T = \phi + \alpha_2 \otimes Z_1 - \alpha_1 \otimes Z_2.$$

Definition 4. *A metric contact pair manifold is said to be normal if J and T are integrable [9].*

Theorem 5. *Let $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2, g)$ be a normal metric contact pair manifold. Then we have*

$$g((\nabla_{X_1}\phi)X_2, X_3) = \sum_{i=1}^{2}(d\alpha_i(\phi X_2, X_1)\alpha_i(X_3) - d\alpha_i(\phi X_3, X_1)\alpha_i(X_2)),$$

where X_1, X_2, X_3 are arbitrary vector fields on M [10].

On a normal metric contact pair manifold we have $\nabla_X Z = -\phi X$ for $X \in \Gamma(TM)$ and $Z = Z_1 + Z_2$.

We can consider a natural question: could any metric contact pair structure be considered locally the product of two contact metric manifold? An example of metric contact pair were given in [18], which is not locally product of two contact metric manifold. So metric contact pair structure has some different properties from contact metric manifolds and their results will be useful interpretation for the geometry of contact and complex manifolds.

2.3. **Curvature properties of normal metric contact pair manifolds.** We use the following statements for the Riemann curvature;

$$R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3,$$
$$\mathcal{R}(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4)$$

for all $X_1, X_2, X_3, X_4 \in \Gamma(TM)$. Also the Ricci operator is defined by

$$QX = \sum_{i=1}^{dim(M)} R(X, E_i)E_i,$$

and the Ricci curvature and scalar curvature are given by

$$Ric(X_1, X_2) = g(QX_1, X_2),$$

$$scal = \sum_{i=1}^{dim(M)} Ric(E_i, E_i).$$

where $E_i,\ 1 \leq i \leq dim(M)$ are orthonormal basis of M. Let $Z = Z_1 + Z_2$; from Lemma 3 of [12] for X_1, X_2, X_3 on a normal metric contact pair manifold with decomposable ϕ we have

$$\begin{aligned}\mathcal{R}(X_1, Z, Z, X_3) &= d\alpha_1(\phi X_3, X_1)\alpha_1(X_2) + d\alpha_2(\phi X_3, X_1)\alpha_2(X_2) \\ &\quad -d\alpha_1(\phi X_3, X_1)\alpha_1(X_2) - d\alpha_2(\phi X_3, X_2)\alpha_2(X_1).\end{aligned}$$

Thus for $X_4, X_5 \in \Gamma(\mathcal{H})$ we get

$$R(X_4, Z, Z, X_5) = -g(X_4, X_5), \tag{1}$$

$$R(Z, X_4)X_5 = d\alpha_1(\phi X_5, X_4)Z_1 + d\alpha_2(\phi X_5, X_4)Z_2,$$

$$R(X_1, Z)Z = -\phi^2 X_1.$$

Let take an orthonormal basis of M

$$\{E_1, E_2, ..., E_p, \phi E_1, \phi E_2, ..., \phi E_p, E_{p+1}, E_{p+2}, ..., E_{p+q}, \phi E_{p+1}, \phi E_{p+2}, ..., \phi E_{p+q}, Z_1, Z_2\}$$

then for all $X_1 \in \Gamma(TM)$ we get the Ricci curvature of M as

$$Ric(X_1, Z) = \sum_{i=1}^{2p+2q} [d\alpha_1(\phi E_i, E_i)\alpha_1(X_1) + d\alpha_2(\phi E_i, E_i)\alpha_2(X_1)].$$

So, we obtain the following results:

$$Ric(X_1, Z) = 0, \quad \text{for } X_1 \in \Gamma(\mathcal{H}), \tag{2}$$

$$Ric(Z, Z) = 2p + 2q, \tag{3}$$

$$Ric(Z_1, Z_1) = 2p,\ Ric(Z_2, Z_2) = 2q,\ Ric(Z_1, Z_2) = 0.$$

Conformal $\mathcal{C}$, concircular $\mathcal{W}$ and quasi-conformal curvature tensor $\widetilde{\mathcal{C}}$ on a $(2p+2q+2)$-dimensional normal contact metric pair manifold are given by:

$$\begin{aligned}\mathcal{C}(X_1, X_2)X_3 &= R(X_1, X_2)X_3 \\ &\quad + \frac{scal}{(2p+2q+1)(2p+2q)}(g(X_2, X_3)X_1 - g(X_1, X_3)X_2) \\ &\quad + \frac{1}{2p+2q}(g(X_1, X_3)QX_2 - g(X_2, X_3)QX_1 \\ &\quad + Ric(X_1, X_3)X_2 - Ric(X_2, X_3)X_1), \\ \mathcal{W}(X_1, X_2)X_3 &= R(X_1, X_2)X_3 \\ &\quad - \frac{scal}{(2p+2q+2)(2p+2q+1)}[g(X_2, X_3)X_1 - g(X_1, X_3)X_2] \\ \widetilde{\mathcal{C}}(X_1, X_2)X_3 &= aR(X_1, X_2)X_3 + b[Ric(X_2, X_3)X_1 - Ric(X_1, X_3)X_2 \\ &\quad + g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2]\end{aligned}$$

$$-\frac{scal}{2p+2q+2}[\frac{a}{2p+2q+1}+2b][g(X_2,X_3)X_1-g(X_1,X_3)X_2]$$

where $X_1, X_2, X_3 \in \Gamma(TM)$, a and b are constants.

3. Hermitian Contact Pair Manifolds

In this section we give an almost contact pair structure on a Hermitian manifold.

Let (M, g, J) be $(2p+2q+2)-$dimensional Hermitian manifold and $(\varphi_1, \eta_1, \xi_1)$, $(\varphi_2, \eta_2, \xi_2)$ be two almost contact structures on M with following properties.

$$\begin{aligned}
&g(\varphi_i X_1, X_2) = -g(X_1, \varphi_i X_2), \quad \text{for } i = 1, 2,\\
&J\xi_1 = -\xi_2, \quad J\xi_2 = \xi_1,\\
&{\varphi_i}^2 X_1 = -X_1 + \eta_1(X_1)\xi_1 + \eta_2(X_1)\xi_2,\\
&\varphi_1(JX_1) = -J\varphi_1 X_1 = \varphi_2 X_1,\\
&\varphi_2(JX_1) = -J\varphi_2 X_1 = -\varphi_1 X_1,\\
&\varphi_2(\varphi_1 X_1) = -\varphi_1(\varphi_2 X_1) = JX_1 + \eta_1(X_1)\xi_2 - \eta_2(X_1)\xi_1
\end{aligned} \tag{4}$$

where X_1, X_2 are two arbitrary vector fields on M [19].

Let take $\phi = \varphi_1 \circ \varphi_2$. Then ϕ is a $(1,1)$ tensor field on M. By direct computation we get

$$\phi^2 X_1 = -X_1 + \eta_1(X_1)\xi_1 + \eta_2(X_1)\xi_2.$$

Thus we obtain an almost contact pair structure on M with the contact pair η_1, η_2 and we state:

Corollary 6. *Let $(M^{2p+2q+2}, J, g)$ be an almost Hermitian manifold and $(\varphi_i, \eta_i, \xi_i)_{i=0}^2$ be two almost contact structure on M with properties are given in (4). Then (η_1, η_2, ϕ) is an almost contact pair structure on M such that*

$$\begin{aligned}
\phi^2 X_1 &= -X_1 + \eta_1(X_1)\xi_1 + \eta_2(X_1)\xi_2,\\
\eta_i(\xi_j) &= \delta_{ij}, \ 1 \le i, j, \le 2,\\
\phi(\xi_i) &= 0,
\end{aligned}$$

for all $X_1 \in \Gamma(TM)$.

Also for $X_1, X_2 \in \Gamma(TM)$ we have

$$g(\phi X_1, X_2) = -g(X_1, \phi X_2)$$

and

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta_1(X_1)\eta_1(X_2) - \eta_2(X_1)\eta_2(X_2).$$

Thus we obtain compatible metric with contact pair structure.

These results show that a contact pair structure on an almost Hermitian manifold could be obtained from two almost contact structure on this manifold. Since contact pair manifolds have some significant properties, some future works could be done for Hermitian and contact structure. Also if the manifold is complex, then

$(M, \varphi_1, \varphi_2, J, \xi_1, \xi_2, \eta_1, \eta_2, g)$ is a complex almost contact metric manifold. This type of manifolds were studied by several authors [6, 14–17].

4. Flatness Conditions on Normal Contact Pair Manifolds

In this section we give some results on the flatness of conformal, concircular and quasi-conformal curvature tensors.

Theorem 7. *A conformal flat normal metric contact pair manifold is an Einstein manifold with positive scalar curvature and has positive sectional curvature.*

Proof. Let $(M, \phi, \alpha_1, \alpha_2)$ be a normal metric contact pair manifold. Suppose that M is conformal flat. Then we have

$$\begin{aligned} \mathcal{R}(X_1, X_2, X_3, X_4) &= -A[g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)] \\ &\quad -B(g(X_1, X_3)Ric(X_2, X_4) - g(X_2, X_3)Ric(X_1, X_4) \\ &\quad +Ric(X_1, X_3)g(X_2, X_4) - Ric(X_2, X_3)g(X_1, X_4)), \end{aligned} \tag{5}$$

where $A = \frac{scal}{(2p+2q+1)(2p+2q)}$ and $B = \frac{1}{2p+2q}$. Taking $X_2 = X_3 = Z$ and $X_1, X_4 \in \Gamma(\mathcal{H})$ in (5) , since $g(Z, Z) = 2$ and from (1), we obtain

$$\mathcal{R}(X_1, Z, Z, X_4) = -2Ag(X_1, X_4) - 2BRic(X_1, X_4) - (2p + 2q)g(X_1, X_4).$$

Also from (1) we get

$$2BRic(X_1, X_4) = (-2A - 2(p + q)B + 1)g(X_1, X_4)$$

and thus we obtain

$$Ric(X_1, X_4) = -\frac{-2A - 2(p + q)B + 1}{2B} g(X_1, X_4). \tag{6}$$

So, the manifold is Einstein. On the other hand by direct computation from (6) the scalar curvature is

$$scal = \frac{(2p + 2q)(2p + 2q + 1)}{4(p + q) + 1}.$$

This shows the scalar curvature is positive. Let choose $X_1 = X_4$, $X_2 = X_3$ unit and orthogonal vector fields in (5). Then the sectional curvature is obtained by

$$k(X_1, X_2) = A + 2(p + q)B - 1 = A = \frac{scal}{(2p + 2q + 1)(2p + 2q)}$$

Thus, the proof is completed. □

An Einstein manifold is also Einstein under concircular transformation. Yano proved that a concircular flat Riemann manifold is Einstein [3]. By similar way we can easily obtain following result.

Theorem 8. *A concircular flat normal metric contact pair manifold is Einstein.*

Our finally result is about quasi-conformal flatness of normal metric contact pair manifold.

Theorem 9. *A quasi-conformally flat normal metric contact pair manifold;*

(1) *is an Einstein manifold with a positive scalar curvature*
(2) *is a space of constant curvature.*

Proof. Let M be a quasi-conformally flat normal metric contact pair manifold. Then for $X_1, X_2, X_3, X_4 \in \Gamma(TM)$ we have

$$\begin{aligned} 0 &= a\mathcal{R}(X_1,X_2,X_3,X_4) + b[Ric(X_2,X_3)g(X_1,X_4) \\ &\quad -Ric(X_1,X_3)g(X_2,X_4) \\ &\quad +g(X_2,X_3)Ric(X_1,X_4) - g(X_1,X_3)g(X_2,X_4)] \\ &\quad -\frac{scal}{2p+2q+2}[\frac{a}{2p+2q+1}+2b][g(X_2,X_3)g(X_1,X_4) \\ &\quad -g(X_1,X_3)g(X_2,X_4)]. \end{aligned} \tag{7}$$

Let write $\frac{scal}{2p+2q+2}[\frac{a}{2p+2q+1}+2b] = K$ for brevity. In (7), by taking $X_1 = X_4 = E_i$ and getting sum from $i = 1$ to $i = 2p+2q+2$ we obtain

$$0 = (a + b(2p+2q))Ric(X_2,X_3) + (bscal - K(2p+2q+1))g(X_2,X_3)$$

and therefore we get

$$0 = [a + b(2p+2q)][Ric(X_2,X_3) - \frac{scal}{2p+2q+2}g(X_2,X_3)].$$

Assume that $a + b(2p+2q) \neq 0$. Then we have

$$Ric(X_2,X_3) = \frac{scal}{2p+2q+2}g(X_2,X_3). \tag{8}$$

By taking $X_2 = X_3 = Z$ in (8) and from (3), we get positive scalar curvature as

$$scal = 2(p+q)(p+q+1).$$

So, the Ricci curvature has the following form:

$$Ric(X_2,X_3) = (p+q)g(X_2,X_3). \tag{9}$$

This shows manifold is Einstein.
On the other hand consider (9) in (7) we get

$$\begin{aligned} 0 &= a[\mathcal{R}(X_1,X_2,X_3,X_4) - \frac{p+q}{2p+2q+1}[g(X_2,X_3)g(X_1,X_4) \\ &\quad -g(X_1,X_3)g(X_2,X_4)]]. \end{aligned}$$

If $a \neq 0$ we get

$$\mathcal{R}(X_1,X_2,X_3,X_4) = \frac{p+q}{2p+2q+1}[g(X_2,X_3)g(X_1,X_4) - g(X_1,X_3)g(X_2,X_4)].$$

Thus the manifold is a space of constant curvature. $\square$

References

[1] Geiges, H., A brief history of contact geometry and topology, *Expositiones Mathematicae*, 19(1) (2001), 25-53.

[2] Kholodenko, A. L., Applications of contact geometry and topology in physics, *World Scientific*, (2013)

[3] Yano, K., Concircular geometry I: concircular transformations, *Proceedings of the Imperial Academy*, 16(6) (1940), 195-200.

[4] Yano, K., Sawaski, S., Riemannian manifolds admitting a conformal transformation group, *J. Diff. Geo,.* 2 (1968), 161-184

[5] De, U. C., Shaikh, A. A., Complex manifolds and contact manifolds, *Narosa Publishing House*, (2009).

[6] Turgut Vanli, A., Unal, I., Conformal, concircular, quasi-conformal and conharmonic flatness on normal complex contact metric manifolds. *International Journal of Geometric Methods in Modern Physics*, 14(05) (2017), 1750067.

[7] Blair, D. E., Ludden, G. D., Yano, K., Geometry of complex manifolds similar to the Calabi-Eckmann manifolds, *Journal of Differential Geometry*, 9(2) (1974), 263-274.

[8] Bande, G. and Hadjar, A., Contact pairs. *Tohoku Mathematical Journal, Second Series*, 57(2) (2005), 247-260.

[9] Bande, G., Hadjar, A., On normal contact pairs, *International Journal of Mathematics*, 21(06) (2010), 737-754.

[10] Bande, G., Hadjar, A., Contact pair structures and associated metrics, *In Differential Geometry*, (2009), 266-275

[11] Bande, G., Blair, D. E., Hadjar, A., On the curvature of metric contact pairs, *Mediterranean journal of mathematics*, 10(2) (2013), 989-1009.

[12] Bande, G., Blair, D. E., Hadjar, A., Bochner and conformal flatness of normal metric contact pairs, *Annals of Global Analysis and Geometry*, 48(1) (2015), 47-56.

[13] Blair, D. E., Riemannian geometry of contact and symplectic manifolds, *Springer Science Business Media*, (2010).

[14] Kobayashi, S., Remarks on complex contact manifolds, *Proc. Amer. Math. Soc.,* 10 (1959), 164-167.

[15] Ishihara, S., Konishi, M., Complex almost contact structures in a complex contact manifold, *Kodai Math. J.,* 5 (1982), 30–37

[16] Korkmaz, B., Normality of complex contact manifolds, *Rocky Mountain J. Math.,* 30 (2000), 1343-1380

[17] Turgut Vanli, A., Blair, D. E., The Boothby-Wang Fibration of the Iwasawa Manifold as a Critical Point of the Energy, *Monatsh. Math.* 147 (2006), 75-84.

[18] Bande, G., Hadjar, A., On the characteristic foliations of metric contact pairs, *Harmonic Maps and Differential Geometry. Contemp. Math*, 542 (2011), 255-259.

[19] Beldjilali, G., Belkhelfa, M., Structures on the product of two almost Hermitian almost contact manifolds, *International Electronic Journal of Geometry*, 9(2) (2016), 80-86.

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PSEUDO-SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

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Abstract. In this paper, we introduce the notion of pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds giving characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions D_1, D_2 and $RadTM$ on pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold have been obtained. We also obtain necessary and sufficient conditions for foliations determined by above distributions to be totally geodesic.

1. Introduction

In 1990, B.Y. Chen defined slant immersions in complex geometry as a natural generalization of both holomorphic immersions and totally real immersions ([4], [5]). Further, A. Carriazo defined and studied bi-slant submanifolds of almost Hermitian and almost contact metric manifolds and further gave the notion of pseudo-slant submanifolds ([3]). The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu ([7]). Various classes of lightlike submanifolds of indefinite Kaehler manifolds are defined according to the behaviour of distributions on these submanifolds with respect to the action of (1,1) tensor field $\overline{J}$ in Kaehler structure of the ambient manifolds. Such submanifolds have been studied by Duggal and Sahin in ([8]). The geometry of slant and screen-slant lightlike submanifolds of indefinite Hermitian manifolds was studied by Sahin in ([14], [15]). The theory of slant, Cauchy-Riemann lightlike submanifolds of indefinite Kaehler manifolds has been studied in ([7], [8]).

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The objective of this paper is to introduce the notion of pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds. This new class of lightlike submanifolds of an indefinite Kaehler manifold includes slant, Cauchy-Riemann lightlike submanifolds as its sub-cases. The paper is arranged as follows. There are some basic results in section 2. In section 3, we study pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold, giving some examples. Section 4 is devoted to the study of foliations determined by distributions on pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds.

2. Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold ([7]) if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM). \tag{2.1}$$

Now consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.2}$$

$$T\overline{M}|_M = TM \oplus tr(TM), \tag{2.3}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}). \tag{2.4}$$

Following are four cases of a lightlike submanifold $\big(M, g, S(TM), S(TM^{\perp})\big)$:

Case.1 r-lightlike if $r < min\,(m, n)$,

Case.2 co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,

Case.3 isotropic if $r = m < n$, $S\,(TM) = \{0\}$,

Case.4 totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y, A_V X$ belong to $\Gamma(TM)$ and $h(X, Y), \nabla^t_X V$ belong to $\Gamma(tr(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.5) and (2.6), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l(X,W) = L(\nabla^t_X W)$, $D^s(X,N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M respectively.

Now by using (2.5), (2.7)-(2.9) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.10}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X). \tag{2.11}$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.13}$$

By using above equations, we obtain

$$\overline{g}(h^l(X,\overline{P}Y),\xi) = g(A^*_\xi X, \overline{P}Y), \tag{2.14}$$

$$\overline{g}(h^*(X,\overline{P}Y),N) = g(A_N X, \overline{P}Y), \tag{2.15}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.16}$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is metric connection, by using (2.7), we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y). \tag{2.17}$$

An indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, \overline{J})$ is a 2m-dimensional semi-Riemannian manifold $\overline{M}$ with semi-Riemannian metric $\overline{g}$ of constant index q, $0 < q < 2m$ and a (1, 1) tensor field $\overline{J}$ on $\overline{M}$ such that following conditions are satisfied:

$$\overline{J}^2 X = -X, \tag{2.18}$$

$$\overline{g}(\overline{J}X, \overline{J}Y) = \overline{g}(X,Y), \tag{2.19}$$

for all $X, Y \in \Gamma(T\overline{M})$.

An indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, \overline{J})$ is called an indefinite Kaehler manifold if $\overline{J}$ is parallel with respect to $\overline{\nabla}$, i.e.,

$$(\overline{\nabla}_X \overline{J})Y = 0, \tag{2.20}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ is Levi-Civita connection with respect to $\overline{g}$.

3. Pseudo-Slant Lightlike Submanifolds

In this section, we introduce the notion of pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds. At first, we state the following Lemmas for later use:

Lemma 3.1. *Let M be a r-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$. Suppose that $\overline{J}RadTM$ is a distribution on M such that $RadTM \cap \overline{J}RadTM = \{0\}$. Then $\overline{J}ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\overline{J}RadTM \cap \overline{J}ltr(TM) = \{0\}$.*

Lemma 3.2. *Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$. Suppose $\overline{J}RadTM$ is a distribution on M such that $RadTM \cap \overline{J}RadTM = \{0\}$. Then any complementary distribution to $\overline{J}RadTM \oplus \overline{J}ltr(TM)$ in $S(TM)$ is Riemannian.*

The proofs of Lemma 3.1 and Lemma 3.2 follow as in Lemma 3.1 and Lemma 3.2 of [15], respectively, so we omit them.

Definition 3.1. Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$ such that $q < dim(M)$. Then we say that M is a pseudo-slant lightlike submanifold of $\overline{M}$ if following conditions are satisfied:
(i) $\overline{J}RadTM$ is a distribution on M such that $RadTM \cap \overline{J}RadTM = \{0\}$,
(ii) there exists non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = (\overline{J}RadTM \oplus \overline{J}ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2$,
(iii) the distribution D_1 is anti-invariant, i.e. $\overline{J}D_1 \subset S(TM^{\perp})$,
(iv) the distribution D_2 is slant with angle $\theta(\neq \pi/2)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between $\overline{J}X$ and the vector subspace $(D_2)_x$ is a constant$(\neq \pi/2)$, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$. This constant angle θ is called slant angle of distribution D_2. A screen pseudo-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq 0$.
From the above definition, we have the following decomposition

$$TM = RadTM \oplus_{orth} (\overline{J}RadTM \oplus \overline{J}ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2. \tag{3.1}$$

In particular, we have
(i) if $D_1 = 0$, then M is a slant lightlike submanifold,
(ii) if $D_1 \neq 0$ and $\theta = 0$, then M is a CR-lightlike submanifold.
Thus above new class of lightlike submanifolds of an indefinite Kaehler manifold includes slant, Cauchy-Riemann lightlike submanifolds as its sub-cases which have been studied in ([7],[8]).
Let $(\mathbb{R}^{2m}_{2q}, \overline{g}, \overline{J})$ denote the manifold $\mathbb{R}^{2m}_{2q}$ with its usual Kaehler structure given by
$\overline{g} = \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i)$,
$\overline{J}(\sum_{i=1}^{m}(X_i \partial x_i + Y_i \partial y_i)) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i)$,
where (x^i, y^i) are the Cartesian coordinates on $\mathbb{R}^{2m}_{2q}$. Now, we construct some examples of pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold.

Example 1. Let $(\mathbb{R}_2^{12}, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.
Suppose M is a submanifold of $\mathbb{R}_2^{12}$ given by $x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = y^4 = u_4$, $x^4 = y^3 = u_5$, $x^5 = u_6 \cos u_7$, $y^5 = u_6 \sin u_7$, $x^6 = \cos u_6$, $y^6 = \sin u_6$, where u_i are real parameters and $u_6 \neq 0$.
The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$Z_1 = 2(\partial x_1 + \partial y_2)$, $\quad Z_2 = 2\partial x_2$, $\quad Z_3 = 2\partial y_1$,
$Z_4 = 2(\partial x_3 + \partial y_4)$, $\quad Z_5 = 2(\partial x_4 + \partial y_3)$,
$Z_6 = 2(\cos u_7 \partial x_5 + \sin u_7 \partial y_5 - \sin u_6 \partial x_6 + \cos u_6 \partial y_6)$,
$Z_7 = 2(-u_6 \sin u_7 \partial x_5 + u_6 \cos u_7 \partial y_5)$.

Hence $RadTM = Span\{Z_1\}$ and $S(TM) = Span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$.
Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2$ and $S(TM^\perp)$ is spanned by

$W_1 = 2(\partial x_3 - \partial y_4)$, $\quad W_2 = 2(\partial x_4 - \partial y_3)$,
$W_3 = 2(\cos u_7 \partial x_5 + \sin u_7 \partial y_5 + \sin u_6 \partial x_6 - \cos u_6 \partial y_6)$,
$W_4 = 2(u_6 \cos u_6 \partial x_6 + u_6 \sin u_6 \partial y_6)$.

It follows that $\overline{J} Z_1 = Z_2 - Z_3$, which implies that $\overline{J} RadTM$ is a distribution on M. On the other hand, we can see that $D_1 = span\{Z_4, Z_5\}$ such that $\overline{J} Z_4 = W_2$, $\overline{J} Z_5 = W_1$, which implies that D_1 is anti-invariant with respect to $\overline{J}$ and $D_2 = span\{Z_6, Z_7\}$ is a slant distribution with slant angle $\pi/4$. Hence M is a pseudo-slant 2-lightlike submanifold of $\mathbb{R}_2^{12}$.

Example 2. Let $(\mathbb{R}_2^{12}, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.
Suppose M is a submanifold of $\mathbb{R}_2^{12}$ given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 \cos\beta$, $y^3 = u_4 \sin\beta$, $x^4 = u_5 \sin\beta$, $y^4 = u_5 \cos\beta$, $x^5 = u_6 \cos\theta$, $y^5 = u_7 \cos\theta$, $x^6 = u_7 \sin\theta$, $y^6 = u_6 \sin\theta$, where u_i are real parameters.
The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$Z_1 = 2(-\partial x_1 + \partial y_2)$, $\quad Z_2 = 2\partial x_2$, $\quad Z_3 = 2\partial y_1$,
$Z_4 = 2(\cos\beta \partial x_3 + \sin\beta \partial y_3)$, $Z_5 = 2(\sin\beta \partial x_4 + \cos\beta \partial y_4)$,
$Z_6 = 2(\cos\theta \partial x_5 + \sin\theta \partial y_6)$, $Z_7 = 2(\sin\theta \partial x_6 + \cos\theta \partial y_5)$.

Hence $RadTM = Span\{Z_1\}$ and $S(TM) = Span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$.
Now $ltr(TM)$ is spanned by $N_1 = \partial x_1 + \partial y_2$ and $S(TM^\perp)$ is spanned by

$W_1 = 2(\sin\beta \partial x_3 - \cos\beta \partial y_3)$, $W_2 = 2(\cos\beta \partial x_4 - \sin\beta \partial y_4)$,
$W_3 = 2(\sin\theta \partial x_5 - \cos\theta \partial y_6)$, $W_4 = 2(\cos\theta \partial x_6 - \sin\theta \partial y_5)$.

It follows that $\overline{J} Z_1 = Z_2 + Z_3$, which implies that $\overline{J} RadTM$ is a distribution on M. On the other hand, we can see that $D_1 = span\{Z_4, Z_5\}$ such that $\overline{J} Z_4 = W_1$, $\overline{J} Z_5 = W_2$, which implies that D_1 is anti-invariant with respect to $\overline{J}$ and $D_2 = span\{Z_6, Z_7\}$ is a slant distribution with slant angle 2θ. Hence M is a pseudo-slant 2-lightlike submanifold of $\mathbb{R}_2^{12}$.
Now, for any vector field X tangent to M, we put $\overline{J} X = PX + FX$, where PX and FX are tangential and transversal parts of $\overline{J} X$ respectively. We denote the

projections on $RadTM$, $\overline{J}RadTM$, $\overline{J}ltr(TM)$, D_1 and D_2 in TM by P_1, P_2, P_3, P_4, and P_5 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$, $\overline{J}(D_1)$ and D' by Q_1, Q_2 and Q_3 respectively, where D' is non-degenerate orthogonal complementary subbundle of $\overline{J}(D_1)$ in $S(TM^{\perp})$. Then, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + P_4X + P_5X. \tag{3.2}$$

Now applying $\overline{J}$ to (3.2), we have

$$\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + \overline{J}P_3X + \overline{J}P_4X + \overline{J}P_5X, \tag{3.3}$$

which gives

$$\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + \overline{J}P_3X + \overline{J}P_4X + fP_5X + FP_5X, \tag{3.4}$$

where fP_5X (resp. FP_5X) denotes the tangential (resp. transversal) component of $\overline{J}P_5X$. Thus we get $\overline{J}P_1X \in \Gamma(\overline{J}RadTM), \overline{J}P_2X \in \Gamma(RadTM), \overline{J}P_3X \in \Gamma(ltr(TM)), \overline{J}P_4X \in \Gamma(\overline{J}D_1) \subseteq \Gamma(S(TM^{\perp})), fP_5X \in \Gamma(D_2)$ and $FP_5X \in \Gamma(D')$. Also, for any $W \in \Gamma(tr(TM))$, we have

$$W = Q_1W + Q_2W + Q_3W. \tag{3.5}$$

Applying $\overline{J}$ to (3.5), we obtain

$$\overline{J}W = \overline{J}Q_1W + \overline{J}Q_2W + \overline{J}Q_3W, \tag{3.6}$$

which gives

$$\overline{J}W = \overline{J}Q_1W + \overline{J}Q_2W + BQ_3W + CQ_3W, \tag{3.7}$$

where BQ_3W (resp. CQ_3W) denotes the tangential (resp. transversal) component of $\overline{J}Q_3W$. Thus we get $\overline{J}Q_1W \in \Gamma(\overline{J}ltr(TM)), \overline{J}Q_2W \in \Gamma(D_1)$, $BQ_3W \in \Gamma(D_2)$ and $CQ_3W \in \Gamma(D')$.

Now, by using (2.20), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on $RadTM$, $\overline{J}RadTM$, $\overline{J}ltr(TM)$, D_1, D_2, $ltr(TM)$, $\overline{J}(D_1)$ and D', we obtain

$$\begin{aligned} &P_1(\nabla_X\overline{J}P_1Y) + P_1(\nabla_X\overline{J}P_2Y) - P_1(A_{\overline{J}P_4Y}X) + P_1(\nabla_X fP_5Y) \\ &= P_1(A_{FP_5Y}X) + P_1(A_{\overline{J}P_3Y}X) + \overline{J}P_2\nabla_XY, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &P_2(\nabla_X\overline{J}P_1Y) + P_2(\nabla_X\overline{J}P_2Y) - P_2(A_{\overline{J}P_4Y}X) + P_2(\nabla_X fP_5Y) \\ &= P_2(A_{FP_5Y}X) + P_2(A_{\overline{J}P_3Y}X) + \overline{J}P_1\nabla_XY, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &P_3(\nabla_X\overline{J}P_1Y) + P_3(\nabla_X\overline{J}P_2Y) - P_3(A_{\overline{J}P_4Y}X) + P_3(\nabla_X fP_5Y) \\ &= P_3(A_{FP_5Y}X) + P_3(A_{\overline{J}P_3Y}X) + \overline{J}h^l(X,Y), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &P_4(\nabla_X\overline{J}P_1Y) + P_4(\nabla_X\overline{J}P_2Y) - P_4(A_{\overline{J}P_4Y}X) + P_4(\nabla_X fP_5Y) \\ &= P_4(A_{FP_5Y}X) + P_4(A_{\overline{J}P_3Y}X) + \overline{J}Q_2h^s(X,Y), \end{aligned} \tag{3.11}$$

$$\begin{aligned} &P_5(\nabla_X\overline{J}P_1Y) + P_5(\nabla_X\overline{J}P_2Y) - P_5(A_{\overline{J}P_4Y}X) + P_5(\nabla_X fP_5Y) \\ &= P_5(A_{FP_5Y}X) + P_5(A_{\overline{J}P_3Y}X) + fP_5\nabla_XY + BQ_3h^s(X,Y), \end{aligned} \tag{3.12}$$

$$\begin{aligned} &h^l(X,\overline{J}P_1Y)+h^l(X,\overline{J}P_2Y)+D^l(X,\overline{J}P_4Y)+h^l(X,fP_5Y)\\ &=\overline{J}P_3\nabla_XY-\nabla^l_X\overline{J}P_3Y-D^l(X,FP_5Y), \end{aligned}\tag{3.13}$$

$$\begin{aligned} &Q_2h^s(X,\overline{J}P_1Y)+Q_2h^s(X,\overline{J}P_2Y)+Q_2\nabla^s_X\overline{J}P_4Y+Q_2h^s(X,fP_5Y)\\ &=Q_2\nabla^s_XFP_5Y-Q_2D^s(X,\overline{J}P_3Y)+\overline{J}P_4\nabla_XY, \end{aligned}\tag{3.14}$$

$$\begin{aligned} &Q_3h^s(X,\overline{J}P_1Y)+Q_3h^s(X,\overline{J}P_2Y)+Q_3\nabla^s_X\overline{J}P_4Y+Q_3h^s(X,fP_5Y)\\ &=CQ_3h^s(X,Y)-Q_3\nabla^s_XFP_5Y-Q_3D^s(X,\overline{J}P_3Y)+FP_5\nabla_XY. \end{aligned}\tag{3.15}$$

Theorem 3.3. *Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$. Then M is a pseudo-slant lightlike submanifold of $\overline{M}$ if and only if*
(i) $\overline{J}RadTM$ is a distribution on M such that $RadTM\cap\overline{J}RadTM=\{0\}$,
(ii) the distribution D_1 is an anti-invariant, i.e. $\overline{J}D_1\subset S(TM^\perp)$,
(iii) there exists a constant $\lambda\in(0,1]$ such that $P^2X=-\lambda X$.
Moreover, there also exists a constant $\mu\in[0,1)$ such that $BFX=-\mu X$, for all $X\in\Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM)=(\overline{J}RadTM\oplus\overline{J}ltr(TM))\oplus_{orth}D_1\oplus_{orth}D_2$ and $\lambda=\cos^2\theta$, θ is slant angle of D_2.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then distribution D_1 is anti-invariant with respect to $\overline{J}$ and $\overline{J}RadTM$ is a distribution on M such that $RadTM\cap\overline{J}RadTM=\{0\}$.
Now for any $X\in\Gamma(D_2)$, we have $|PX|=|\overline{J}X|\cos\theta$, which implies

$$\cos\theta=\frac{|PX|}{|\overline{J}X|}.\tag{3.16}$$

In view of (3.16), we get $\cos^2\theta=\frac{|PX|^2}{|\overline{J}X|^2}=\frac{g(PX,PX)}{g(\overline{J}X,\overline{J}X)}=\frac{g(X,P^2X)}{g(X,\overline{J}^2X)}$, which gives

$$g(X,P^2X)=\cos^2\theta\, g(X,\overline{J}^2X).\tag{3.17}$$

Since M is pseudo-slant lightlike submanifold, $\cos^2\theta=\lambda(constant)\in(0,1]$ therefore from (3.17), we get $g(X,P^2X)=\lambda g(X,\overline{J}^2X)=g(X,\lambda\overline{J}^2X)$, which implies

$$g(X,(P^2-\lambda\overline{J}^2)X)=0.\tag{3.18}$$

Since X is non-null vector, we have $(P^2-\lambda\overline{J}^2)X=0$, which implies

$$P^2X=\lambda\overline{J}^2X=-\lambda X.\tag{3.19}$$

Now, for any vector field $X\in\Gamma(D_2)$, we have

$$\overline{J}X=PX+FX,\tag{3.20}$$

where PX and FX are tangential and transversal parts of $\overline{J}X$ respectively.
Applying $\overline{J}$ to (3.20) and taking tangential component, we get

$$-X=P^2X+BFX.\tag{3.21}$$

From (3.19) and (3.21), we get

$$BFX = -\sin^2\theta\, X, \quad \forall X \in \Gamma(D_2), \tag{3.22}$$

where $\sin^2\theta = 1 - \lambda = \mu(constant) \in [0, 1)$.
This proves (iii).
Conversely suppose that conditions (i), (ii) and (iii) are satisfied. From (3.21), for any $X \in \Gamma(D_2)$, we get

$$-X = P^2X - \mu X, \tag{3.23}$$

which implies

$$P^2X = -\cos^2\theta\, X, \tag{3.24}$$

where $\cos^2\theta = 1 - \mu = \lambda(constant) \in (0, 1]$.
Now $\cos\theta = \frac{g(\overline{J}X,PX)}{|\overline{J}X||PX|} = -\frac{g(X,\overline{J}PX)}{|\overline{J}X||PX|} = -\frac{g(X,P^2X)}{|\overline{J}X||PX|} = -\lambda\frac{g(X,\overline{J}^2X)}{|\overline{J}X||PX|} = \lambda\frac{g(\overline{J}X,\overline{J}X)}{|\overline{J}X||PX|}$.
From above equation, we get

$$\cos\theta = \lambda\frac{|\overline{J}X|}{|PX|}. \tag{3.25}$$

Therefore (3.16) and (3.25) give $\cos^2\theta = \lambda(constant)$.
Hence M is a pseudo-slant lightlike submanifold.

Corollary 3.1. *Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ with slant angle θ, then for any $X, Y \in \Gamma(D_2)$, we have*
(i) $g(PX, PY) = \cos^2\theta\, g(X, Y)$,
(ii) $g(FX, FY) = \sin^2\theta\, g(X, Y)$.

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.1 of [15].

Theorem 3.4. *Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $RadTM$ is integrable if and only if*
(i) $P_1(\nabla_X\overline{J}P_1Y) = P_1(\nabla_Y\overline{J}P_1X)$ *and* $P_5(\nabla_X\overline{J}P_1Y) = P_5(\nabla_Y\overline{J}P_1X)$,
(ii) $Q_2h^s(Y,\overline{J}P_1X) = Q_2h^s(X,\overline{J}P_1Y)$ *and* $h^l(Y,\overline{J}P_1X) = h^l(X,\overline{J}P_1Y)$,
(iii) $Q_3h^s(Y,\overline{J}P_1X) = Q_3h^s(X,\overline{J}P_1Y)$, *for all* $X, Y \in \Gamma(RadTM)$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(RadTM)$. From (3.8), we have $P_1(\nabla_X\overline{J}P_1Y) = \overline{J}P_2\nabla_XY$, which gives $P_1(\nabla_X\overline{J}P_1Y) - P_1(\nabla_Y\overline{J}P_1X) = \overline{J}P_2[X, Y]$. From (3.12), we get $P_5(\nabla_X\overline{J}P_1Y) = fP_5\nabla_XY + Bh^s(X, Y)$, which gives $P_5(\nabla_X\overline{J}P_1Y) - P_5(\nabla_Y\overline{J}P_1X) = fP_5[X, Y]$. In view of (3.13), we obtain $h^l(X,\overline{J}P_1Y) = \overline{J}P_3\nabla_XY$, which implies $h^l(X,\overline{J}P_1Y) - h^l(Y,\overline{J}P_1X) = \overline{J}P_3[X, Y]$. From (3.14), we have $Q_2h^s(X,\overline{J}P_1Y) = \overline{J}P_4\nabla_XY$, which gives $Q_2h^s(X,\overline{J}P_1Y) - Q_2h^s(Y,\overline{J}P_1X) = \overline{J}P_4[X, Y]$. Also from (3.15), we get $Q_3h^s(X,\overline{J}P_1Y) = Ch^s(X, Y) + FP_5\nabla_XY$, which implies $Q_3h^s(X,\overline{J}P_1Y) - Q_3h^s(Y,\overline{J}P_1X) = FP_5[X, Y]$. This concludes the theorem.

Theorem 3.5. *Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 is integrable if and only if*

(i) $P_1(A_{\overline{J}P_4Y}X) = P_1(A_{\overline{J}P_4X}Y)$ *and* $P_2(A_{\overline{J}P_4Y}X) = P_2(A_{\overline{J}P_4X}Y)$,

(ii) $D^l(Y, \overline{J}P_4X) = D^l(X, \overline{J}P_4Y)$ *and* $Q_3\nabla^s_Y\overline{J}P_4X = Q_3\nabla^s_X\overline{J}P_4Y$,

(iii) $P_5(A_{\overline{J}P_4Y}X) = P_5(A_{\overline{J}P_4X}Y)$, *for all* $X, Y \in \Gamma(D_1)$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1)$. From (3.8), we have $P_1(A_{\overline{J}P_4Y}X) + \overline{J}P_2\nabla_XY = 0$, which gives $P_1(A_{\overline{J}P_4X}Y) - P_1(A_{\overline{J}P_4Y}X) = \overline{J}P_2[X, Y]$. From (3.9), we get $P_2(A_{\overline{J}P_4Y}X) + \overline{J}P_1\nabla_XY = 0$, which gives $P_2(A_{\overline{J}P_4X}Y) - P_2(A_{\overline{J}P_4Y}X) = \overline{J}P_1[X, Y]$. In view of (3.12), we obtain $P_5(A_{\overline{J}P_4Y}X) + fP_5\nabla_XY + BQ_3h^s(X, Y) = 0$, which implies $P_5(A_{\overline{J}P_4X}Y) - P_5(A_{\overline{J}P_4Y}X) = fP_5[X, Y]$. From (3.13), we have $D^l(X, \overline{J}P_4Y) = \overline{J}P_3\nabla_XY$, which gives $D^l(X, \overline{J}P_4Y) - D^l(Y, \overline{J}P_4X) = \overline{J}P_3[X, Y]$. Also from (3.15), we obtain $Q_3\nabla^s_X\overline{J}P_4Y = CQ_3h^s(X, Y) + FP_5\nabla_XY$, which implies $Q_3\nabla^s_X\overline{J}P_4Y - Q_3\nabla^s_Y\overline{J}P_4X = FP_5[X, Y]$. Thus, we obtain the required results.

Theorem 3.6. *Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 is integrable if and only if*

(i) $P_1(\nabla_XfP_5Y - \nabla_YfP_5X) = P_1(A_{FP_5Y}X - A_{FP_5X}Y)$,

(ii) $P_2(\nabla_XfP_5Y - \nabla_YfP_5X) = P_2(A_{FP_5Y}X - A_{FP_5X}Y)$,

(iii) $h^l(X, fP_5Y) - h^l(Y, fP_5X) = D^l(Y, FP_5X) - D^l(X, FP_5Y)$,

(iv) $Q_2(\nabla^s_XFP_5Y - \nabla^s_YFP_5X) = Q_2(h^s(X, fP_5Y) - h^s(Y, fP_5X))$,

for all $X, Y \in \Gamma(D_2)$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2)$. From (3.8), we have $P_1(\nabla_XfP_5Y) - P_1(A_{FP_5Y}X) = \overline{J}P_2\nabla_XY$, which gives $P_1(\nabla_XfP_5Y - \nabla_YfP_5X) - P_1(A_{FP_5Y}X - A_{FP_5X}Y) = \overline{J}P_2[X, Y]$. From (3.9), we get $P_2(\nabla_XfP_5Y) - P_2(A_{FP_5Y}X) = \overline{J}P_1\nabla_XY$, which gives $P_2(\nabla_XfP_5Y - \nabla_YfP_5X) - P_2(A_{FP_5Y}X - A_{FP_5X}Y) = \overline{J}P_1[X, Y]$. In view of (3.13), we obtain $h^l(X, fP_5Y) + D^l(X, FP_5Y) = \overline{J}P_3\nabla_XY$, which implies $h^l(X, fP_5Y) - h^l(Y, fP_5X) + D^l(X, FP_5Y) - D^l(Y, FP_5X) = \overline{J}P_3[X, Y]$. From (3.14), we have $Q_2h^s(X, fP_5Y) - Q_2\nabla^s_XFP_5Y = \overline{J}P_4\nabla_XY$, which gives $Q_2(\nabla^s_YFP_5X - \nabla^s_XFP_5Y) + Q_2(h^s(X, fP_5Y) - Q_2h^s(Y, fP_5X)) = \overline{J}P_4[X, Y]$. This proves the theorem.

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold to be totally geodesic.

Definition 4.1. A pseudo-slant lightlike submanifold M of an indefinite Kaehler manifold $\overline{M}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is mixed geodesic pseudo-slant lightlike submanifold if $h^l(X, Y) = 0$ and $h^s(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Theorem 4.1. *Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $RadTM$ defines a totally geodesic foliation if and only if $\overline{g}(\nabla_X \overline{J}P_2Z + \nabla_X fP_5Z, \overline{J}Y) = \overline{g}(A_{\overline{J}P_3Z}X + A_{\overline{J}P_4Z}X + A_{FP_5Z}X, \overline{J}Y)$, for all $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.*

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. It is easy to see that $RadTM$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(RadTM)$, for all $X, Y \in \Gamma(RadTM)$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19), (2.20) and (3.4), for any $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X(\overline{J}P_2Z + \overline{J}P_3Z + \overline{J}P_4Z + fP_5Z + FP_5Z), \overline{J}Y)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{\overline{J}P_3Z}X + A_{FP_5Z}X + A_{\overline{J}P_4Z}X - \nabla_X \overline{J}P_2Z - \nabla_X fP_5Z, \overline{J}Y)$. This completes the proof.

Theorem 4.2. *Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(\nabla^s_X FZ, \overline{J}Y) = -\overline{g}(h^s(X, fZ), \overline{J}Y)$,

(ii) $h^s(X, \overline{J}N)$ and $D^s(X, \overline{J}W)$ have no components in $\overline{J}(D_1)$,

for all $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(\overline{J}ltr(TM))$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_1)$, for all $X, Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z, \overline{J}Y)$, which implies $\overline{g}(\nabla_X Y, Z) = \overline{g}(\nabla^s_X FZ + h^s(X, fZ), \overline{J}Y)$. In view of (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we have $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}N)$, which gives $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, h^s(X, \overline{J}N))$. Now, from (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $W \in \Gamma(\overline{J}ltr(TM))$, we get $\overline{g}(\nabla_X Y, W) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}W)$, which implies $\overline{g}(\nabla_X Y, W) = \overline{g}(\overline{J}Y, D^s(X, \overline{J}W))$. This concludes the theorem.

Theorem 4.3. *Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(A_{\overline{J}Z}X, fY) = \overline{g}(\nabla^s_X \overline{J}Z, FY)$,

(ii) $\overline{g}(fY, \nabla_X \overline{J}N) = -\overline{g}(FY, h^s(X, \overline{J}N))$,

(iii) $\overline{g}(fY, A_{\overline{J}W}X) = \overline{g}(FY, D^s(X, \overline{J}W))$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(\overline{J}ltr(TM))$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2)$, for all $X, Y \in \Gamma(D_2)$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z, \overline{J}Y)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{\overline{J}Z}X, fY) - \overline{g}(\nabla^s_X \overline{J}Z, FY)$. In view of (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we have $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(fY, \nabla_X \overline{J}N) - \overline{g}(FY, h^s(X, \overline{J}N))$. Now, from (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_2)$

and $W \in \Gamma(\overline{J}ltr(TM))$, we have $\overline{g}(\nabla_X Y, W) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}W)$, which gives $\overline{g}(\nabla_X Y, W) = \overline{g}(fY, A_{\overline{J}W}X) - \overline{g}(FY, D^s(X, \overline{J}W))$. Thus, we obtain the required results.

Theorem 4.4. *Let M be a mixed geodesic pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 defines a totally geodesic foliation if and only if $\nabla^s_X FZ$, $h^s(X, \overline{J}N)$ and $D^s(X, \overline{J}W)$ have no components in $\overline{J}(D_1)$, for all $X \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(\overline{J}ltr(TM))$.*

Proof. Let M be a mixed geodesic pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $h(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and for all $Y \in \Gamma(D_2)$. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_1)$, for all $X, Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z, \overline{J}Y)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla^s_X FZ + h^s(X, fZ), \overline{J}Y)$. In view of (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, h^s(X, \overline{J}N))$. Now, from (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $W \in \Gamma(\overline{J}ltr(TM))$, we have $\overline{g}(\nabla_X Y, W) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}W)$, which gives $\overline{g}(\nabla_X Y, W) = \overline{g}(\overline{J}Y, D^s(X, \overline{J}W))$. This proves the theorem.

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References

[1] Atceken, M., Kilic, E., Semi-Invariant Lightlike Submanidolds of a Semi-Riemannian Product Manifold, *Kodai Math. J.,* 30 (2007), 361-378.

[2] Blair, D.E., Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics, 203, Birkhauser Boston, Inc., Boston, MA, 2002.

[3] Carriazo, A., New Developments in Slant Submanifolds Theory, Narosa Publishing House, New Delhi, India, 2002.

[4] Chen, B. Y., Geometry of Slant Submanifolds, Katholieke Universiteit, Leuven, 1990.

[5] Chen, B. Y., Slant immersions, *Bull. Austral. Math. Soc.,* 41 (1990), 135- 147.

[6] Chen, B. Y., Tazawa, Y., Slant submanifolds in complex Euclidean spaces, *Tokyo J. Math.,* 14 (1991), 101-120.

[7] Duggal, K.L., Bejancu, A., Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Vol. 364 of Mathematics and its applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[8] Duggal, K.L., Sahin, B., Differential Geomety of Lightlike Submanifolds, Birkhauser Verlag AG, Basel, Boston, Berlin, 2010.

[9] Johnson, D.L., Whitt, L.B., Totally Geodesic Foliations, *J. Differential Geometry,* 15 (1980), 225-235.

[10] Kilic, E., Sahin, B., Radical Anti-Invariant Lightlike Submanifolds of Semi-Riemannian Product Manifolds, *Turkish J. Math.,* 32 (2008), 429 - 449.

[11] Lotta, A., Slant Submanifolds in Contact geometry, *Bull. Math. Soc. Roumanie,* 39 (1996), 183-198.

[12] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press New York 1983.

[13] Papaghiuc, N., Semi-slant submanifolds of a Kaehlerian manifold, *An. Stiint. Al.I.Cuza. Univ. Iasi,* 40 (1994), 55-61.

[14] Sahin, B., Screen Slant Lightlike Submanifolds, *Int. Electronic J. of Geometry,* 2 (2009), 41-54.

[15] Sahin, B., Slant lightlike submanifolds of indefinite Hermitian manifolds, *Balkan Journal of Geometry and Its Appl.,* 13(1) (2008), 107-119.

[16] Sahin, B., Gunes, R., Geodesic CR-lightlike submanifolds, *Beitrage Algebra and Geometry,* 42(2) (2001), 583-594.

[17] Shukla, S.S., Akhilesh Yadav, Pseudo-Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds, *An. Stiint. Al. I. Cuza. Univ. Iasi, TOM LXII,* 2(2) (2016), 571-583.

[18] Shukla, S.S., Akhilesh Yadav, Screen Pseudo-Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds, *Mediterian Journal of Mathematics*, 13(2) (2016), 789-802.

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ANNIHILATORS OF POWER VALUES OF b-GENERALIZED DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring with extended centroid C and maximal left ring of quotients $Q_{ml}(R)$. For a nonzero element $b \in R$, let $F : R \to R$ be a right b-generalized derivation associated with the map d of R. Suppose that $s\,(F(x))^n = 0$ for all $x \in R$ where s is a nonzero element in R and $n \geq 1$ is a fixed positive integer. Then there exist some $c \in Q_{ml}(R)$ and $\beta \in C$ such that $d(x) = ad_c(x)$, $F(x) = (c+\beta)xb$ for all $x \in R$ and either $s(c+\beta) = 0$ or $b(c+\beta) = 0$.

1. Introduction

Throughout this paper R will always denote a prime ring with center $Z(R)$, extended centroid C, left maximal ring of quotients (respectively right maximal ring of quotients) $Q_{ml}(R)$ (resp. $Q_{mr}(R)$), the Martindale ring of quotients $Q(R)$ and the symmetric Martindale quotient ring $Q_s(R)$. In case R is a prime ring, $Q(R), Q_{ml}(R), Q_{mr}(R), Q_s(R)$ are also a prime rings and C is a field. For more details the book [1] is referred.

By a *derivation* of R we mean an additive map d of R satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For $a \in R$, the map $d : R \to R$ defined by $d(x) = [a, x]$ for all $x \in R$ is called an X-*inner derivation* and denoted by $d(x) = ad_a(x)$. It is well known that any derivation d of R can be uniquely extended to $Q_{ml}(R)$ (or $Q_{mr}(R)$). Hence any extension of an X-*inner derivation* is also X- inner which can be induced by an element $q \in Q_s(R)$, i.e., $d(x) = ad_q(x)$. A *generalized derivation* g is an additive map of R satisfying $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$ where d is a derivation of R which is uniquely determined by g and is called the *associated derivation* of g.

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Let α, β be automorphisms of R. An (α, β)-*derivation* of R is an additive mapping $\delta : R \to R$ such that $\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y)$ for all $x, y \in R$. If $\alpha = Id$, the identity automorphism of R, then δ is called a β-*derivation* or simply called a *skew derivation*. An additive map δ of R is a generalized α-*derivation* if there exists an α-derivation d of R such that $\delta(xy) = \delta(x)y + \alpha(x)d(y)$ for all $x, y \in R$.

The notion of b-generalized derivations is introduced by Koşan and Lee in [10]. They define generalized b-derivations as in the following:

(1) Let $d : R \to Q_{mr}(R)$ be an additive map. An additive map $F : R \to Q_{mr}(R)$ is called a *left b-generalized derivation* with the associated map d if $F(xy) = F(x)y + bxd(y)$ for all $x, y \in R$.
(2) Let $d : R \to Q_{ml}(R)$ be an additive map. An additive map $F : R \to Q_{ml}(R)$ is called a *right b-generalized derivation* with the associated map d if $F(xy) = xF(y) + xd(y)b$ for all $x, y \in R$.

When treating an additive map of a ring R, the main goal is to give a characterization of the map or to state some structural results related to the ring itself. The works have been done related to power values of some kind of additive maps so far was initiated in [8]. Giambruno and Herstein proved that if R is a semiprime ring and d is a derivation of R such that $d(x)^n = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then $d = 0$. The generalization of this problem has been worked by a number of researchers . For instance in [2] Brešar proved that if R is a prime ring with a nonzero derivation d and a is an element of R such that $ad(x)^n = 0$ for all $x \in R$, where $n \geq 1$ is a fixed positive integer, then $a = 0$ in case $char\ R \neq (n-1)!$. In [11] Lee and Lin obtained the same result on some noncentral Lie ideal of R without the assumption on the characteristic.

In [3] J.C. Chang proved that if R is a prime ring and δ is a right generalized (α, β)-derivation of R such that $a\delta(x)^n = 0$ for all $x \in R$ and some $a \in R$ where $n \geq 1$ is a positive integer, then $\delta = 0$ or $a = 0$.

Motivated by these results we prove the following theorem.

Theorem 1. *Let R be a prime ring, F be a nonzero right b-generalized derivation of R associated with the map d of R and s, b be nonzero elements in R. If $s\,(F(x))^n = 0$ for all $x \in R$ where n is a fixed positive integer, then there exist $c \in Q_{ml}$ and $\beta \in C$ such that $d(x) = ad_c(x)$, $F(x) = (c + \beta)xb$ and either $s(c + \beta) = 0$ or $b(c + \beta) = 0$.*

2. Preliminaries

It is well known that the automorphisms, derivations, generalized derivations and α-derivations of R can be uniquely extended to $Q(R), Q_{mr}(R)$ and $Q_{ml}(R)$. For sure the results obtained for $Q_{mr}(R)$ can be adapted to $Q_{ml}(R)$ as well.

Before presenting the results we will state the following remarks:

Fact 1. Let $T = Q_{mr}(R) *_C C\{X_1, X_2, \ldots\}$ be the free product of C-algebras $Q_{mr}(R)$ and $C\{X_1, X_2, \ldots\}$ where $C\{X_1, X_2, \ldots\}$ is the free algebra in noncommutative indeterminates $\{X_1, \ldots, X_2\}$ over C. Let $f(X_i) \in T$, then f is called a *GPI*

(that is, a generalized polynomial identity) of T if $f(x_i) = 0$ for all $x_i \in T$.
Fact 2. Let R be a prime ring and I be a two-sided ideal of R. Then I, R and $Q_{mr}(R)$ satisfy the same generalized polynomial identities with coefficients in $Q_{mr}(R)$ (see [4]).
Fact 3.(Lemma 2 in [4]) Let R be a prime ring and $Q_{mr}(R)$ is the maximal right ring of quotients of R. Suppose that B is a basis of $Q_{mr}(R)$ over C. Let $\alpha_i, q_i \in B$, then $m_i = q_0 y_1 q_1 \ldots y_n q_n$ are called *monomials* and hence any element in the free product T is of the form $g = \sum_i \alpha_i m_i$. In that case, the generalized polynomial $g = \sum_i \alpha_i m_i$ is zero in T if and only if $\alpha_i = 0$ for all i. Consequently for $h_i(X_1, \ldots, X_n), k_i(X_1, \ldots, X_n) \in T$ and

$$g_1(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i h_i(X_1, \ldots, X_n),$$

$$g_2(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i k_i(X_1, \ldots, X_n)$$

if $a_1, a_2 \in Q_{mr}(R)$ are linearly independent over C and

$$a_1 g_1(X_1, \ldots, X_n) + a_2 g_2(X_1, \ldots, X_n) = 0 \in T,$$

then both $g_1(X_1, \ldots, X_n)$ and $g_2(X_1, \ldots, X_n)$ are zero in T.

In the present paper, we consider F as a right b-generalized derivation of R with associated map d of R and $b \in R$. It is proved in [10] that if R is a prime ring with $b \neq 0$, then the associated map d must be a derivation of R. For some $a, b, c \in Q_{ml}(R)$, the map $F(x) = ax + cxb$ is an example of right b-generalized derivation of R and called X-*inner right b-generalized derivation.*

3. The case of inner b-generalized derivations

Lemma 2. *Let R be a prime ring and $s, a, b, c \in R$ with $b, s \neq 0$. Suppose that $s(xa + cxb)^n = 0$ for all $x \in R$ where n is a fixed positive integer. If R does not satisfy any nontrivial generalized polynomial identity, then there exists some $\beta \in C$ such that $a = \beta b$ and either $s(c + \beta) = 0$ or $b(c + \beta) = 0$.*

Proof. If R does not satisfy any nontrivial generalized polynomial identity, then

$$s(Xa + cXb)^n = 0 \in T. \tag{1}$$

We suppose that $a \neq 0$. By (1),

$$s(Xa + cXb)^{n-1} Xa + s(Xa + cXb)^{n-1} cXb = 0 \in T.$$

First we consider the case when a and b are linearly C-independent. Then using Fact 3, we have $s(Xa + cXb)^{n-1} Xa = 0$. The monomial $s(Xa)^n$ is nontrivial in the last equation since $s \neq 0$ and $a \neq 0$, which contradicts with the assumption that R

is not a GPI-ring. Thus a and b must be linearly dependent on C. In that case, there exists some $\beta \in C$ such that $a = \beta b$. Writing this in (1), we have

$$0 = s(X\beta b + cXb)^n = s(c+\beta)(Xb(c+\beta))^n.$$

In view of Theorem 2 in [7], we obtain $s(c+\beta) = 0$ or $b(c+\beta) = 0$. □

Lemma 3. *Let R be a prime ring $s, a, b, c \in R$ with $s, b \neq 0$. Suppose that $s\left((xa+cx)b\right)^n = 0$ for all $x \in R$ where n is a fixed positive integer. Then there exists $\beta \in C$ such that $(a-\beta)b = 0$ and either $s(c+\beta) = 0$ or $b(c+\beta) = 0$.*

Proof. First suppose that R does not satisfy any nontrivial generalized polynomial identity. Then

$$s\left((Xa+cX)b\right)^n = 0 \in T. \tag{2}$$

Suppose that $s \neq 0, a \neq 0$. By (2) we have

$$s\left((Xa+cX)b\right)^{n-1} Xab + \left((Xa+cX)b\right)^{n-1} cXb = 0 \in T.$$

Now consider the case when ab and b are linearly C-independent. Using Fact 3 we obtain $s\left((Xa+cX)b\right)^{n-1} Xab = 0 \in T$. The monomial $s(Xab)^n$ is nontrivial since $s \neq 0, a \neq 0$, which contradicts with the assumption on R for not being a GPI-ring. Hence ab and b must be linearly dependent over C. In that case, there exists some $\beta \in C$ such that $(a-\beta)b = 0$. So the equation (2) becomes

$$\begin{aligned} 0 =& s\left((Xa+cX)b\right)^n \\ =& s\left((X(a-\beta) + (c+\beta)X)b\right)^n \\ =& s(c+\beta)(Xb(c+\beta))^n. \end{aligned} \tag{3}$$

In view of Theorem 2 in [7], we have $s(c+\beta) = 0$ or $b(c+\beta) = 0$.

Now suppose that R satisfies a nontrivial generalized polynomial identity, i.e., R is a GPI-ring. By Theorem 2 in [4]

$$s\left((xa+cx)b\right)^n = 0 \tag{4}$$

for all $x \in RC$. If C is an infinite field, then let F denote the algebraic closure of C and if C is a finite field, then let $F = C$. Hence the equation (4) holds for all $x \in R'$ where $R' = RC \otimes_C F$. In the light of Theorem 3.5 in [5], R' is a centrally closed prime F-algebra. By Theorem 3 in [13], R' is a primitive ring with a minimal idempotent e such that $eR'e = Fe$ and there exists a right vector space V_F such that R' acts densely on V_F.

For a given $v \in V$, the first aim is to see that $(ab)v$ and bv are linearly dependent over F. First suppose that $(ab)v$ and bv are linearly independent over F. By the density of R', there exists $x \in R'$ such that $x(ab)v = v$, $xbv = 0$. Using this in (4), we obtain

$$0 = s\left((xa+cx)b\right)^n v = sv.$$

Since the action of R' on V_F is faithful then $s = 0$, which leads a contradiction. Thus we may assume that $a(bv)$ and bv are linearly dependent on F.

Case 1. If $dim_F(bV) = 1$, then we may choose $v' \in V$ such that $bV = bv'F$ and write $abv' = bv'\gamma$ for some $\gamma \in F$. Let $v \in V$ and $bv = \alpha' bv'$ for some $\alpha' \in F$. Then

$$abv = a\alpha' bv' = \alpha' abv' = \gamma' \alpha' bv' = \gamma' bv.$$

that is abv and bv are linearly F-dependent. The equation

$$abv = \gamma' bv$$

gives $ab = \gamma' b$ since the action of R' on V_F is faithful.

Case 2. If $dim_F(bV) \geq 2$, then there exists some $\beta' \in C$ such that $abv = \beta' bv$ which means $(a - \beta')b = 0$ by a similar argument above. Using this in (4), we have

$$\begin{aligned} 0 &= s\left((x(a-\beta') + (c+\beta')x)b\right)^n \\ &= s(c+\beta')\left(xb(c+\beta')\right)^n \end{aligned}$$

for all $x \in R'$. By Theorem 2 in [7], we have $s(c+\beta') = 0$ or $b(c+\beta') = 0$. Hence in either cases there exists some $\beta' \in C$ such that $ab = \beta' b$.

Let $\{\mu_0, \mu_1, \ldots\}$ be a basis for F over C and let $\mu_0 = 1$. Writing $\beta' = \beta\mu_0 + \beta_1\mu_1 + \ldots$ for $\beta, \beta_1 \in C$, we obtain $\beta = \beta'$ and $ab = \beta b$ for $\beta \in C$ that is $(a - \beta)b = 0$. Thus it follows from the calculations in (3) that $s(c+\beta) = 0$ or $b(c+\beta) = 0$. $\square$

Proposition 4. *Let R be a prime ring and $s, a, b, c \in R$ with $s \neq 0, b \neq 0$. Suppose that $s(xa + cxb)^n = 0$ for all $x \in R$, then there exists some $\beta \in C$ such that $a = \beta b$ and either $s(c+\beta) = 0$ or $b(c+\beta) = 0$.*

Proof. It is well known that

$$s(xa + cxb)^n = 0 \tag{5}$$

holds for all $x \in Q_{ml}(R)$. First suppose that R is not a generalized polynomial identity ring. Then the desired result follows from Lemma 2.

Now suppose that R is a GPI-ring. Since $Q_{ml}(R)$ is also a prime GPI-ring and a centrally closed prime ring then it follows from Martindale's Theorem in [13] that $Q_{ml}(R)$ is a primitive ring with nonzero socle H. Let H be the socle of $Q_{ml}(R)$ and H is a regular ring, that is for any $w \in H$, $wzw = w$ for some $z \in H$. Let $Ann_r(z) = \{x \in H | zx = 0\}$ be the right annihilator of z in H and first consider the case $a, b \in H$. Let $w \in Ann_r(b)$. Substituting x by wx in (5) we have $s\left((wx)a + c(wx)b\right)^n = 0$ and hence

$$0 = s\left((wx)a + c(wx)b\right)^n w = sw(xaw)^n$$

for all $x \in Q_{ml}(R)$. Using Theorem 2 in [7], we obtain either $sw = 0$ or $aw = 0$ that is $w \in Ann_r(s)$ or $w \in Ann_r(a)$. The first situation leads a contradiction so $w \in Ann_r(a)$ and hence $Ann_r(b) \subset Ann_r(a)$.

For $a, b \in H$ there exist $u, v \in H$ such that $aua = a$ and $bvb = b$ since H is a regular ring. Let $f = ua$ and $g = vb$ be two elements in H. It is easy to see that f and g are idempotents. Since $Ann_r(b) \subset Ann_r(a)$, then $Ann_r(g) \subset Ann_r(f)$ and also $(1-g)H \subseteq (1-f)H$. Thus we have $f(1-g) = 0$. Here $a = aua = af = afg = afvb \in Hb$.

For general case let $w \in H$. Substituting x by xw in (5) we have

$$0 = s\left((xw)a + c(xw)b\right)^n$$

for all $x \in Q_{ml}(R)$. For $wa, wb \in H$ by the calculations above it is clear that $wa \in Hwb$,i.e., there exists some $t \in H$ such that $wa = twb$. Writing this in (5), we have $s\left((xt + cx)wb\right)^n = 0$. Using Lemma 3, there exists $\beta_w \in C$ such that $(t - \beta_w)wb = 0$. Since $twb = \beta_w wb$, then $wa = \beta_w wb$.

Fix an idempotent $e' \in H$ such that $e'a \neq 0$. Then

$$e'a = \beta e'b \tag{6}$$

for some $\beta \in C$. Let f be an idempotent in H. Hence for some $\beta_f \in C$, we have

$$fa = \beta_f fb. \tag{7}$$

We wish to see that if $fa \neq 0$, then $\beta_f = \beta$. Indeed there exists an idempotent $h \in H$ such that $He' + Hf = Hh$ and

$$ha = \beta_h hb. \tag{8}$$

Here it is clear that $e'h = e'$ and $fh = h$.

$$e'a = e'ha = e'\beta_h hb = \beta_h e'b.$$

Using (6) in the last equation, we obtain $\beta e'b = \beta_h e'b$ and so $\beta = \beta_h$. Analogously

$$fa = fha = f\beta_h hb = \beta_h fb.$$

In view of (7), we have $\beta_h = \beta_f$ and hence $\beta_f = \beta$. Until now we have seen that for an idempotent f in H, there exists some $\beta \in C$ such that $fa = \beta fb$ in case $fa \neq 0$.

Now let $f \in H$ be an idempotent such that $fa = 0$. Our aim is to see $fb = 0$. By Litoff's Theorem (Theorem 4.3.11 in [1]), there exists an idempotent $h \in H$ such that $e', f \in hHh$. If $ha = 0$, then $e'a = e'ha = 0$, which is a contradiction. So we may assume that $ha \neq 0$. If $ha \neq 0$, then $(h - f)a \neq 0$. Here also note that $h - f$ is an idempotent. Since $ha \neq 0$ then by (8), $ha = \beta hb$ for some $\beta \in C$. Also $(h - f)a \neq 0$ implies $(h - f)a = \beta(h - f)b$ and so $fb = 0$. Thus we obtain that if $fa = 0$, then $fb = 0$ for any idempotent $f \in H$.

In both cases, we have $f(a - \beta b) = 0$ for all $f \in H$. Since H is a regular ring then $H(a - \beta b) = 0$ implies $a = \beta b$ for some $\beta \in C$. Using this result in (5), we have $s(x\beta b + cxb)^n = s(\beta + c)\left(xb(\beta + c)\right)^n = 0$ for all $x \in Q_{ml}$ which implies $s(c + \beta) = 0$ or $b(c + \beta) = 0$. $\square$

4. The proof of the main theorem

Now we may state the proof of Theorem 1.

Proof. In view of Theorem 2.3 in [10], there exist a derivation $d : R \to Q_{ml}$ and $a \in Q_{ml}$ such that

$$F(x) = xa + d(x)b \tag{9}$$

for all $x \in R$. If d is an X-inner derivation of R, then there exists some $c \in Q_{ml}(R)$ such that $d(x) = [c, x]$ for all $x \in R$. Hence by the hypothesis, we have $s\left(x(a - cb) + cxb\right)^n = 0$ for all $x \in R$. By Proposition 4, there exists some $\beta \in C$ such that $a = (c + \beta)b$ and either $s(c + \beta) = 0$ or $b(c + \beta) = 0$.

Now consider the case when d is not an X-inner derivation of R. Using Kharchenko's well known result in [9], we have $s\left(xa + yb\right)^n = 0$ for all $x, y \in R$. In particular $s(yb)^n = 0$ for all $y \in R$. By Theorem 2 in [7], $s = 0$ or $b = 0$ which leads a contradiction. So d must be an X-inner derivation of R and $F(x) = (c + \beta)xb$ for all $x \in R$ and either $s(c + \beta) = 0$ or $b(c + \beta) = 0$. □

Example 5. *Let R be the ring of 2×2 upper triangular matrices over the field $\mathbb{F}$, i.e., $R = \{ue_{11} + we_{12} + ze_{22} | u, w, z \in \mathbb{F}\}$. For the right b-generalized derivation F of R, define $F(x) = xa + cxb$ for all $x \in R$. If we choose*

$$b = e_{11} + e_{12} + e_{22}, a = e_{12}, c = e_{11}, s = e_{22} \in R$$

then, $s(F(x))^n = 0$ for all $x \in R$ where $n \geq 1$ is a fixed positive integer but $s(c + \beta) \neq 0$ unless $\beta = 0$ in $\mathbb{F}$.

References

[1] Beidar, K. I., Martindale, W. S. III, Mikhalev, A. V., Rings with generalized identities, Marcel Dekker, Inc., New York, xiv+522 pp, 1996.

[2] Brešar, M., A note on derivations, *Math. J. Okayama Univ.*, 32 (1990), 83-88.

[3] Chang, J.C., Annihilators of power values of a right generalized (α, β)-derivation, *Bull. Inst. Math. Acad. Sin.*, 4 (1) (2009), 67-73.

[4] Chuang, C. L., GPIs having coefficients in Utumi quotient rings, *Proc. Amer. Math. Soc.*, 103 (3) (1988), 723-728.

[5] Erickson, T. S.; Martindale, W. S., 3rd; Osborn, J. M., Prime nonassociative algebras, *Pacific J. Math.*, 60 (1975), No. 1, 49-63.

[6] Faith, C., Utumi, Y., On a new proof of Litoff's theorem, *Acta Math. Acad. Sci. Hung.*, 14 (1967), 369-371.

[7] Felzenszwalb, B., On a result of Levitzki, *Canad. Math. Bull.* 21 (2) (1978), 241-242.

[8] Giambruno, A., Herstein, I.N., Derivations with nilpotent values, *Rend. Circ. Mat. Palermo*, 30 (2) (1981), 199-206.

[9] Kharchenko,V.K., Differential identities of semiprime rings, *Algebra Logika*, 18 (1979),86-119; *Algebra Logic*, (English translation), 18 (1979), 58-80.

[10] Kosan, M. T., Lee, T.K., b-generalized derivations having nilpotent values, *J. Aust. Math. Soc.*, 96 (3) (2014), 326-337.

[11] Lee, T.K., Lin, J.S., A result on derivations, *Proc. Amer. Math. Soc.*, 124 (1996), 1687-1691.

[12] Lee, T. K., Generalized derivations of left faithful rings, *Comm. Algebra*, 27(8) (1999), 4057-4073.

[13] Martindale, W. S. 3rd, Prime rings satisfying a generalized polynomial identity, *J. Algebra,* 12 (1969), 576-584.

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ROBUST STABILITY ANALYSIS FOR FUZZY STOCHASTIC HOPFIELD NEURAL NETWORKS WITH TIME-VARYING DELAYS

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Abstract. This paper investigates delay-dependent robust stability problem of fuzzy stochastic Hopfield neural networks with random time-varying delays. Moreover, in this paper, the stochastic delay is assumed to satisfy a certain probability distribution. By introducing a stochastic variable with Bernoulli distribution, the neural networks with random time delays is transformed into one with deterministic delays and stochastic parameters. Based on a Lyapunov-Krasovskii functional and stochastic analysis approach, delay-probability-distribution-dependent stability criteria have been derived in terms of linear matrix inequalities (LMIs), which can be checked easily by the LMI control toolbox. Finally two numerical examples are given to illustrate the effectiveness of the theoretical results.

1. Introduction

In recent decades, Neural Networks (NNs) especially recurrent neural networks (RNNs) and Hopfield neural networks (HNNs) have been successfully applied in various fields such as pattern recognition, optimization problems, associative memories, signal processing, etc., see [1] - [18]. One of the best important works is to study the stability of the equilibrium point of NNs. Since time delays as a source of instability and poor performance always appear in many neural networks owing to the finite speed of information processing, the stability analysis for the delayed neural network has received considerable attention [3–7].

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On the other hand, the stability analysis of stochastic systems with time delays has been investigated by many researchers since stochastic modelling plays an important role in many fields of science and engineering applications [8–16]. In a real system, time delay often exists in a random form, that is, some values of the time delay are very large. However the probability of the delay taking such large values is very small and it may lead to a more conservative result, only if the information of variation range of the time delay is considered. In addition, its probabilistic characteristic such as Bernoulli distribution and the Poisson distribution can also be obtained by statistical methods. Therefore, it is necessary and realizable to investigate the probability-distribution delay and therefore in recent years, the stability problems of NNs with probability-distribution delay have been widely investigated [17, 18].

It is well known that fuzzy logic theory has shown to be an appealing and efficient approach to dealing with the analysis and synthesis problems for complex nonlinear systems. The well-known Takagi-Sugeno (T-S) fuzzy model [19], is a popular and convenient tool to transform a complex nonlinear system to a set of linear submodels via some fuzzy models by defining a linear input/output relationship as its consequence of individual plant rule. Recently, a lot of research works have been produced on T-S fuzzy model in the existing available literature [20–22].

Based on the above discussion, we consider the problem of delay-dependent robust stability analysis for uncertain fuzzy stochastic Hopfield neural networks with time-varying delays. Some sufficient condition for delay-probability-distribution-dependent stability criteria of the addressed system have been derived in terms of linear matrix inequalities by constructing proper Lyapunov-Krasovskii functional and stochastic theory. Finally, numerical examples are provided to show the effectiveness of the theoretical results.

Notations: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n\times n$ real matrices. The superscript T denotes the transposition and the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I_n is the $n \times n$ identity matrix. $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. Moreover, let $\Big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\Big)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. the filtration contains all P-null sets and is right continuous). Denoted by $L^p_{\mathcal{F}_0}\big([-\bar{\tau}, 0]; \mathbb{R}^n\big)$ the family of all $\mathcal{F}_0$-measurable $\mathcal{C}\big([-\bar{\tau}, 0]; \mathbb{R}^n\big)$ -valued random variables $\xi = \{\xi(\theta) : -\bar{\tau} \leq \theta \leq 0\}$ such that $sup_{-\bar{\tau}\leq\theta\leq 0}\mathbb{E}|\xi(\theta)|^p < \infty$, where $\mathbb{E}\{.\}$ stands for the mathematical expectation operator with respect to the given probability measure P.

2. Problem description and preliminaries

Consider the following uncertain stochastic HNNs with time–varying delays

$$dx(t) \quad = \quad \Big[- A(t)x(t) + B(t)f(x(t)) + W(t)f(x(t - \tau(t)))\Big]dt$$

$$
\begin{aligned}
&+\Big[H_0(t)x(t)+H_1(t)x(t-\tau(t))\Big]d\omega(t),\\
x(t) \quad &= \quad \phi(t), \quad \forall t\in[-\bar{\tau},0], \qquad (1)
\end{aligned}
$$

where $x(t)\in\mathbb{R}^n$ is the neural state vector, $f(x(t))=\Big[f_1(x_1(t)),\ldots,f_n(x_n(t))\Big]^T\in\mathbb{R}^n$ is the neuron activation function with initial condition $f(0)=0$. The time-varying delay $\tau(t)$ satisfies

$$
0\le\tau(t)\le\bar{\tau},\quad \dot{\tau}(t)\le\mu, \qquad (2)
$$

where $\bar{\tau}$ and μ are constants. In (1), $A(t)=A+\Delta A(t)$, $B(t)=B+\Delta B(t)$, $W(t)=W+\Delta W(t)$, $H_0(t)=H_0+\Delta H_0(t)$ and $H_1(t)=H_1+\Delta H_1(t)$. Further $A=diag\{a_1,a_2,\ldots,a_n\}$ has positive entries $a_i>0$, B, W, H_0, H_1 are connection weight matrices with appropriate dimensions and $\Delta A(t)$, $\Delta B(t)$, $\Delta W(t)$, $\Delta H_0(t)$ and $\Delta H_1(t)$ denote the time-varying and norm-bounded uncertainties.

Assumption 2.1 The neuron activation function $f_i(x_i)$ satisfies

$$
0\le\frac{f_i(x_i)-f_i(y_i)}{x_i-y_i}\le l_i \quad \forall x_i,y_i\in\mathbb{R},\quad x_i\ne y_i,\quad i=1,...,n \qquad (3)
$$

Assumptions 2.2 Considering the information of probability distribution of the time delay $\tau(t)$, two sets and functions are defined by

$$
\Omega_1=\{t:\tau(t)\in[0,\tau_0)\} \quad and \quad \Omega_2=\{t:\tau(t)\in[\tau_0,\bar{\tau}]\}
$$

$$
\tau_1(t)=\begin{cases}\tau(t), & for \quad t\in\Omega_1\\ \bar{\tau}_1, & for \quad t\in\Omega_2,\end{cases} \qquad and \quad \tau_2(t)=\begin{cases}\tau(t), & for \quad t\in\Omega_2\\ \bar{\tau}_2, & for \quad t\in\Omega_1,\end{cases} \qquad (4)
$$

$$
\dot{\tau}_1(t)\le\mu_1<1,\quad \dot{\tau}_2(t)\le\mu_2<1, \qquad (5)
$$

where $\tau_0\in[0,\bar{\tau}]$, $\bar{\tau}_1\in[0,\tau_0)$ and $\bar{\tau}_2\in[\tau_0,\bar{\tau}]$. It is easy to know $t\in\Omega_1$ means the event $\tau(t)\in[0,\tau_0)$ occurs and $t\in\Omega_2$ means the event $\tau(t)\in[\tau_0,\bar{\tau}]$ occurs. Therefore, a stochastic variable $\alpha(t)$ can be defined as

$$
\alpha(t)=\begin{cases}1, & for \quad t\in\Omega_1\\ 0, & for \quad t\in\Omega_2.\end{cases} \qquad (6)
$$

Assumption 2.3 $\alpha(t)$ is a Bernoulli distributed sequence with

$\text{Prob}\{\alpha(t)=1\}=\mathbb{E}\{\alpha(t)\}=\alpha_0$, $\text{Prob}\{\alpha(t)=0\}=1-\mathbb{E}\{\alpha(t)\}=1-\alpha_0$,

where $0\le\alpha_0\le1$ is a constant and $\mathbb{E}\{\alpha(t)\}$ is the expectation of $\alpha(t)$.

Remark 2.4 From Assumption 2.3, it is easy to know that

$\mathbb{E}\{\alpha(t)-\alpha_0\}=0,\ \mathbb{E}\{(\alpha(t)-\alpha_0)^2\}=\alpha_0(1-\alpha_0)$.

By Assumption 2.2 and 2.3, the system (1) can be rewritten as

$$
dx(t)=\Big[-A(t)x(t)+B(t)f(x(t))+\alpha(t)W(t)f(x(t-\tau_1(t)))
$$

$$\begin{aligned}&+(1-\alpha(t))W(t)f(x(t-\tau_2(t)))\Big]dt\\&+\Big[H_0(t)x(t)+\alpha(t)H_1(t)x(t-\tau_1(t))+(1-\alpha(t))H_1(t)x(t-\tau_2(t))\Big]d\omega(t), \quad (7)\\&x(t)=\xi(t),\quad t\in[-\bar{\tau},0],\end{aligned}$$

which is equivalent to

$$\begin{aligned}dx(t) &= \Big[-A(t)x(t)+B(t)f(x(t))+\alpha_0 W(t)f(x(t-\tau_1(t)))\\&\quad +(1-\alpha_0)W(t)f(x(t-\tau_2(t)))\\&\quad +(\alpha(t)-\alpha_0)\Big(W(t)f(x(t-\tau_1(t)))-W(t)f(x(t-\tau_2(t)))\Big)\Big]dt\\&\quad +\Big[H_0(t)x(t)+\alpha_0 H_1(t)x(t-\tau_1(t))+(1-\alpha_0)H_1(t)x(t-\tau_2(t))\\&\quad +(\alpha(t)-\alpha_0)\Big(H_1(t)x(t-\tau_1(t))-H_1(t)x(t-\tau_2(t))\Big)\Big]d\omega(t), \quad (8)\\x(t) &= \xi(t),\quad t\in[-\bar{\tau},0].\end{aligned}$$

Remark 2.5 In this paper, the probability distribution of the delay taking values in some interval is assumed to be known in advance. Further, a new model of the SNNs (8) has been derived, which can be seen as an extension of the common SNNs (1). Specially, in the case of $\alpha(t)\equiv 1$, system (8) becomes system (1). Moreover, when the probability of time delay taking values is known a priori, the possible values that the delay takes may be larger than those previously obtained results based on the traditional methods, which will be illustrated via example later.
In this paper, we consider the following neural network with parameter uncertainties and stochastic perturbations which is represented by a T-S fuzzy model. The kth rule of the T-S fuzzy model is of the following form:

Plant Rule k:
IF $\theta_1(t)$ is η_1^k **and** … **and** $\theta_p(t)$ is η_p^k
THEN

$$\begin{aligned}dx(t) &= \Big[-A_k(t)x(t)+B_k(t)f(x(t))+\alpha_0 W_k(t)f(x(t-\tau_1(t)))\\&\quad +(1-\alpha_0)W_k(t)f(x(t-\tau_2(t)))\\&\quad +(\alpha(t)-\alpha_0)\Big(W_k(t)f(x(t-\tau_1(t)))-W_k(t)f(x(t-\tau_2(t)))\Big)\Big]dt\\&\quad +\Big[H_{0k}(t)x(t)+\alpha_0 H_{1k}(t)x(t-\tau_1(t))+(1-\alpha_0)H_{1k}(t)x(t-\tau_2(t))\\&\quad +(\alpha(t)-\alpha_0)\Big(H_{1k}(t)x(t-\tau_1(t))-H_{1k}(t)x(t-\tau_2(t))\Big)\Big]d\omega(t), \quad (9)\\x(t) &= \xi(t),\quad t\in[-\bar{\tau},0],\quad k=1,2,...,r,\end{aligned}$$

where η_i^k $(i = 1, 2, ..., p)$ is the fuzzy set, $\theta(t) = \Big[\theta_1(t), \theta_2(t), ..., \theta_p(t)\Big]^T$ is the premise variable vector and r is the number of **IF-THEN** rules. $\omega(t)$ is a one-dimensional Brownian motion defined on $\Big(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, P\Big)$. $\xi \in L^2_{\mathcal{F}_0}([-\bar{\tau}, 0]; \mathbb{R}^n)$ is the initial value of (9). A_k, B_k, W_k, H_{0k} and H_{1k} are constant known real matrices. $\Delta A_k(t)$, $\Delta B_k(t)$, $\Delta W_k(t)$, $\Delta H_{0k}(t)$ and $\Delta H_{1k}(t)$ denote the time-varying parameter uncertainties and we make the following assumption.

The parameter uncertainties $\Delta A_k(t)$, $\Delta B_k(t)$, $\Delta W_k(t)$, $\Delta H_{0k}(t)$ and $\Delta H_{1k}(t)$ are of the form:

$$\begin{aligned}&\Big[\Delta A_k(t) \quad \Delta B_k(t) \quad \Delta W_k(t) \quad \Delta H_{0k}(t) \quad \Delta H_{1k}(t)\Big]\\ &\quad = GF(t)\Big[E_k^A \quad E_k^B \quad E_k^W \quad E_k^{H_0} \quad E_k^{H_1}\Big],\end{aligned} \tag{10}$$

where G, E_k^A, E_k^B, E_k^W, $E_k^{H_0}$ and $E_k^{H_1}$ are known real constant matrices with appropriate dimensions, and $F(t)$ is the time-varying uncertain matrix which satisfies

$$F^T(t)F(t) \leq I. \tag{11}$$

The defuzzified output of the T-S fuzzy system (9) is represented as follows:

$$\begin{aligned}dx(t) = \sum_{k=1}^{r} \mu_k(\theta(t))\Big\{\Big[&-A_k(t)x(t) + B_k(t)f(x(t)) + \alpha_0 W_k(t)f(x(t-\tau_1(t)))\\ &+(1-\alpha_0)W_k(t)f(x(t-\tau_2(t)))\\ &+(\alpha(t)-\alpha_0)\Big(W_k(t)f(x(t-\tau_1(t))) - W_k(t)f(x(t-\tau_2(t)))\Big)\Big]dt\\ &+\Big[H_{0k}(t)x(t) + \alpha_0 H_{1k}(t)x(t-\tau_1(t)) + (1-\alpha_0)H_{1k}(t)x(t-\tau_2(t))\\ &+(\alpha(t)-\alpha_0)\Big(H_{1k}(t)x(t-\tau_1(t)) - H_{1k}(t)x(t-\tau_2(t))\Big)\Big]d\omega(t)\Big\},\end{aligned} \tag{12}$$

where

$$\mu_k(\theta(t)) = \frac{\upsilon_k(\theta(t))}{\sum\limits_{j=1}^{r}\upsilon_j(\theta(t))}, \quad \upsilon_k(\theta(t)) = \prod_{j=1}^{p}\eta_j^k(\theta_j(t))$$

in which $\eta_j^k(\theta_j(t))$ is the grade of membership of $\theta_j(t)$ in η_j^k. According to the theory of fuzzy sets, we have

$\upsilon_k(\theta(t)) \geq 0, \quad k = 1, 2, ..., r, \quad \sum\limits_{k=1}^{r}\upsilon_k(\theta(t)) > 0$ for all t. Therefore, it implies

$\mu_k(\theta(t)) \geq 0, \quad k = 1, 2, ..., r, \quad \sum\limits_{k=1}^{r}\mu_k(\theta(t)) = 1$ for all t.

Let $x(t;\xi)$ denotes the state trajectory of system (12) from the initial value $x(\theta) = \xi(\theta)$ on $-\bar{\tau} \leq \theta \leq 0$ in $L^2_{\mathcal{F}_0}([-\bar{\tau}, 0]; \mathbb{R}^n)$. It is easy to see that system (12) admits a trivial solution $x(t;0) \equiv 0$.

The following definition and lemmas are used to prove our main result.

Definition 2.6 [22] For system (9) and every $\xi \in L^2_{\mathcal{F}_0}([-\infty,0];\mathbb{R}^n)$, the trivial solution is asymptotically stable in the mean square if

$$\lim_{t\to\infty} \mathbb{E}|x(t;\xi)|^2 = 0.$$

Lemma 2.7 [23] Let D and N be real constant matrices of appropriate dimensions, matrix $F(t)$ satisfies $F^T(t)F(t) \le I$. Then (i) for any scalar $\epsilon > 0$, $DF(t)N + N^T F^T(t) D^T \le \epsilon^{-1} DD^T + \epsilon N^T N$.
(ii) For any $P > 0$, $2a^T b \le a^T P^{-1} a + b^T P b$.

Lemma 2.8 [24] For any constant matrix $M \in \mathbb{R}^{n\times n}$, $M = M^T > 0$, scalar $\eta > 0$, vector function $\omega : [0,\eta] \to \mathbb{R}^n$ such that the integrations are well defined, the following inequality holds

$$\Big(\int_0^\eta \omega(s)ds\Big)^T M \Big(\int_0^\eta \omega(s)ds\Big) \le \eta \int_0^\eta \omega^T(s) M \omega(s) ds.$$

Lemma 2.9 [25] Let M, P, Q be the given matrices such that $Q > 0$, then

$$\begin{bmatrix} P & M^T \\ M & -Q \end{bmatrix} < 0 \quad \Longleftrightarrow \quad P + M^T Q^{-1} M < 0.$$

Lemma 2.10 [26] Let U, $V(t)$, W and M be real matrices of appropriate dimension with M satisfying $M = M^T$, then

$$M + UV(t)W + W^T V^T(t) U^T < 0 \quad forall \quad V^T(t)V(t) \le I,$$

if and only if there exists a scalar $\varepsilon > 0$ such that

$$M + \varepsilon^{-1} UU^T + \varepsilon W^T W < 0.$$

Lemma 2.11 [27] Assume that $a(\cdot) \in \mathbb{R}^{n_a}$, $b(\cdot) \in \mathbb{R}^{n_b}$ and $N \in \mathbb{R}^{n_a\times n_b}$ are defined on the interval Ω. Then for any matrices $X \in \mathbb{R}^{n_a\times n_a}$, $Y \in \mathbb{R}^{n_a\times n_b}$ and $Z \in \mathbb{R}^{n_b\times n_b}$, the following holds

$$-2\int_\Omega a^T(\alpha) N b(\alpha) d\alpha \le \int_\Omega \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & Y-N \\ & Z \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha, \text{ where } \begin{bmatrix} X & Y \\ & Z \end{bmatrix} \ge 0.$$

3. Main Results

In this section, we consider a general stochastic system $dx(t) = f(x(t),t)dt + g(x(t),t)d\omega(t)$ on $t \ge t_0$ with initial value $x(t_0) = x_0 \in \mathbb{R}^n$, where $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n\times m}$. Let $\mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+;\mathbb{R}^+)$ denotes the family of all nonnegative functions $V(x(t),t)$ on $\mathbb{R}^n \times \mathbb{R}^+$ which are continuously twice differentiable in x and once differentiable in t. Let $V \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+;\mathbb{R}^+)$, an operator $\mathcal{L}V$ is defined from $\mathbb{R}^n \times \mathbb{R}^+$ to $\mathbb{R}$ by

$$\mathcal{L}V(x(t),t) \quad = \quad V_t(x(t),t) + V_x(x(t),t) f(x(t),t)$$

$$+\frac{1}{2}trace\Big[g^T(x(t),t)V_{xx}(x(t),t)g(x(t),t)\Big],$$

where

$$\begin{aligned} V_t(x(t),t) &= \frac{\partial V(x(t),t)}{\partial t},\ V_x(x(t),t) = \Big(\frac{\partial V(x(t),t)}{\partial x_1},...,\frac{\partial V(x(t),t)}{\partial x_n}\Big), \\ V_{xx}(x(t),t) &= \Big(\frac{\partial^2 V(x(t),t)}{\partial x_i \partial x_j}\Big)_{n\times n}. \end{aligned}$$

Then, by Ito's formula, one can have

$$\mathbb{E}V(x(t),t) = \mathbb{E}V(x_0,t_0) + \mathbb{E}\int_{t_0}^{t} \mathcal{L}V(x(s),s)ds. \tag{13}$$

Now, we define the following variables:

$$\bar{A} + \Delta\bar{A}(t) = \sum_{k=1}^{r} \mu_k(\theta(t))\big(A_k + \Delta A_k(t)\big),\ \bar{B} + \Delta\bar{B}(t) = \sum_{k=1}^{r} \mu_k(\theta(t))\big(B_k + \Delta B_k(t)\big),$$

$$\bar{W} + \Delta\bar{W}(t) = \sum_{k=1}^{r} \mu_k(\theta(t))\big(W_k + \Delta W_k(t)\big),$$

$$\bar{H}_0 + \Delta\bar{H}_0(t) = \sum_{k=1}^{r} \mu_k(\theta(t))\big(H_{0k} + \Delta H_{0k}(t)\big),$$

$$\bar{H}_1 + \Delta\bar{H}_1(t) = \sum_{k=1}^{r} \mu_k(\theta(t))\big(H_{1k} + \Delta H_{1k}(t)\big),$$

by using the above notations and parameter uncertainties are not taken into account, then system (12) can be rewritten as

$$\begin{aligned} dx(t) &= \Big[-\bar{A}x(t) + \bar{B}f(x(t)) + \alpha_0\bar{W}f(x(t-\tau_1(t))) + (1-\alpha_0)\bar{W}f(x(t-\tau_2(t))) \\ &\quad +(\alpha(t)-\alpha_0)\Big(\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big)\Big]dt \\ &\quad +\Big[\bar{H}_0x(t) + \alpha_0\bar{H}_1x(t-\tau_1(t)) + (1-\alpha_0)\bar{H}_1x(t-\tau_2(t)) \\ &\quad +(\alpha(t)-\alpha_0)\Big(\bar{H}_1x(t-\tau_1(t)) - \bar{H}_1x(t-\tau_2(t))\Big)\Big]d\omega(t). \end{aligned} \tag{14}$$

Now, we discuss the stability criteria for stochastic neural network (14) without uncertainties as follows

Theorem 3.1 For given scalars $\bar{\tau}_1$, $\bar{\tau}_2$, μ_1, μ_2, and $0 < \alpha_0 < 1$ satisfying $\alpha_0\mu_1 < 1$, the SNNs (14) is globally asymptotically stable in the mean square, if there exist symmetric positive definite matrices $P > 0$, $Q_i > 0$ $(i = 1, 2, 3, 4, 5, 6)$,

$R_1 > 0$, $R_2 > 0$, $Z_1 > 0$, and $Z_2 > 0$, for any matrices X, Y and positive diagonal matrices $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ such that the following LMIs

$$\begin{bmatrix} X & Y \\ & Z_1 \end{bmatrix} \geq 0 \quad and \quad \begin{bmatrix} X & Y \\ & Z_2 \end{bmatrix} \geq 0 \tag{15}$$

$$\Upsilon_k = \begin{bmatrix} \Omega_k & \bar{\tau}_1\eta_1 Z_1 & \sigma\bar{\tau}_1\eta_2 Z_1 & \bar{\tau}_2\eta_1 Z_2 & \sigma\bar{\tau}_2\eta_2 Z_2 & \vartheta_1\bar{P} & \sigma\vartheta_2\bar{P} & Y & Y \\ & -\bar{\tau}_1 Z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & * & -\bar{\tau}_1 Z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & * & * & -\bar{\tau}_2 Z_2 & 0 & 0 & 0 & 0 & 0 \\ & * & * & * & -\bar{\tau}_2 Z_2 & 0 & 0 & 0 & 0 \\ & * & * & * & * & -\bar{P} & 0 & 0 & 0 \\ & * & * & * & * & * & -\bar{P} & 0 & 0 \\ & * & * & * & * & * & * & -R_1 & 0 \\ & * & * & * & * & * & * & * & -R_2 \end{bmatrix} < 0 \tag{16}$$

hold for $k = 1, 2, ..., r$, where $\Omega_k = (\Omega^k_{i,j})_{12\times 12}$ with

$$\Omega^k_{1,1} = -PA_k - A_kP + \alpha_0\bar{\tau}_1 X + (1-\alpha_0)\bar{\tau}_2 X + 2Y + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6,$$

$$\Omega^k_{1,2} = 0,\ \Omega^k_{1,3} = -\alpha_0 Y,\ \Omega^k_{1,4} = 0,\ \Omega^k_{1,5} = -(1-\alpha_0)Y,\ \Omega^k_{1,6} = \Omega^k_{1,7} = 0,$$

$$\Omega^k_{1,8} = PB_k + LK_1,\ \Omega^k_{1,9} = \alpha_0 PW_k,\ \Omega^k_{1,10} = (1-\alpha_0)PW_k,\ \Omega^k_{1,11} = \Omega^k_{1,12} = 0,$$
$$\Omega^k_{2,2} = -(1-\alpha_0\mu_1)Q_1,\ \Omega^k_{2,3} = 0,$$

$$\Omega^k_{2,4} = \Omega^k_{2,5} = \Omega^k_{2,6} = \Omega^k_{2,7} = \Omega^k_{2,8} = \Omega^k_{2,9} = \Omega^k_{2,10} = \Omega^k_{2,11} = \Omega^k_{2,12} = 0,$$

$$\Omega^k_{3,3} = -(1-\mu_1)Q_5,\ \Omega^k_{3,4} = \Omega^k_{3,5} = \Omega^k_{3,6} = \Omega^k_{3,7} = \Omega^k_{3,8} = 0,$$
$$\Omega^k_{3,9} = LK_2,\ \Omega^k_{3,10} = \Omega^k_{3,11} = \Omega^k_{3,12} = 0,\ \Omega^k_{4,4} = -Q_2,$$

$$\Omega^k_{4,5} = \Omega^k_{4,6} = \Omega^k_{4,7} = \Omega^k_{4,8} = \Omega^k_{4,9} = \Omega^k_{4,10} = \Omega^k_{4,11} = \Omega^k_{4,12} = 0,$$

$$\Omega^k_{5,5} = -(1-\mu_2)Q_6,\ \Omega^k_{5,6} = 0,\ \Omega^k_{5,7} = \Omega^k_{5,8} = \Omega^k_{5,9} = 0,\ \Omega^k_{5,10} = LK_3,$$
$$\Omega^k_{5,11} = \Omega^k_{5,12} = 0,\ \Omega^k_{6,6} = -Q_3,\ \Omega^k_{6,7} = \Omega^k_{6,8} = \Omega^k_{6,9} = 0,$$

$$\Omega^k_{6,10} = \Omega^k_{6,11} = \Omega^k_{6,12} = 0,\ \Omega^k_{7,7} = -(1-\alpha_0\mu_2)Q_4,$$

$$\Omega^k_{7,8} = \Omega^k_{7,9} = \Omega^k_{7,10} = \Omega^k_{7,11} = \Omega^k_{7,12} = 0,\ \Omega^k_{8,8} = -2K_1,$$
$$\Omega^k_{8,9} = \Omega^k_{8,10} = \Omega^k_{8,11} = \Omega^k_{8,12} = 0,\ \Omega^k_{9,9} = -2K_2,$$

$$\Omega^k_{9,10} = \Omega^k_{9,11} = \Omega^k_{9,12} = 0,\ \Omega^k_{10,10} = -2K_3,\ \Omega^k_{10,11} = \Omega^k_{10,12} = 0,$$

$$\Omega^k_{11,11} = -\Big(\frac{1-\alpha_0}{\bar{\tau}_1}\Big)Z_1,\ \Omega^k_{11,12} = 0,$$

$$\Omega^k_{12,12} = -\Big(\frac{\alpha_0}{\bar{\tau}_2}\Big)Z_2,$$

$$\eta_1 = \big[-A_k \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad B_k \quad \alpha_0 W_k \quad (1-\alpha_0)W_k \quad 0 \quad 0\big]^T,$$

$$\eta_2 = \big[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad W_k \quad -W_k \quad 0 \quad 0\big]^T,$$

$$\vartheta_1 = \big[H_{0k} \quad 0 \quad \alpha_0 H_{1k} \quad 0 \quad (1-\alpha_0)H_{1k} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\big]^T,$$

$$\vartheta_2 = \big[0 \quad 0 \quad H_{1k} \quad 0 \quad -H_{1k} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\big]^T,$$

$$\sigma = \sqrt{\alpha_0(1-\alpha_0)}, \quad \bar{P} = P + \bar{\tau}_1 R_1 + \bar{\tau}_2 R_2.$$

Proof: Denoting,

$$\begin{aligned} y(t) &= -\bar{A}x(t) + \bar{B}f(x(t)) + \alpha_0\bar{W}f(x(t-\tau_1(t))) + (1-\alpha_0)\bar{W}f(x(t-\tau_2(t))) \\ &\quad +(\alpha(t)-\alpha_0)\Big(\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big), \qquad (17) \\ g(t) &= \bar{H}_0x(t) + \alpha_0\bar{H}_1x(t-\tau_1(t)) + (1-\alpha_0)\bar{H}_1x(t-\tau_2(t)) \\ &\quad +(\alpha(t)-\alpha_0)\Big(\bar{H}_1x(t-\tau_1(t)) - \bar{H}_1x(t-\tau_2(t))\Big). \qquad (18) \end{aligned}$$

The system (14) can be written as

$$dx(t) = y(t)dt + g(t)d\omega(t). \qquad (19)$$

Integrating (19) from $t-\tau_1(t)$ to t, and from $t-\tau_2(t)$ to t, we get the following equalities

$$\begin{aligned} x(t-\tau_1(t)) &= x(t) - \int_{t-\tau_1(t)}^{t} y(s)ds - \int_{t-\tau_1(t)}^{t} g(s)d\omega(s) \qquad (20) \\ x(t-\tau_2(t)) &= x(t) - \int_{t-\tau_2(t)}^{t} y(s)ds - \int_{t-\tau_2(t)}^{t} g(s)d\omega(s), \qquad (21) \end{aligned}$$

we can rewrite (14) as

$$\begin{aligned} dx(t) &= \Big[-\bar{A}x(t) + \bar{B}f(x(t)) + \alpha_0\bar{W}G(x(t-\tau_1(t)))x(t-\tau_1(t)) \\ &\quad +(1-\alpha_0)\bar{W}G(x(t-\tau_2(t)))x(t-\tau_2(t)) \\ &\quad +\big(\alpha(t)-\alpha_0\big)\Big(\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big)\Big]dt \end{aligned}$$

$$+\Big[\bar{H}_0x(t)+\alpha_0\bar{H}_1x(t-\tau_1(t))+(1-\alpha_0)\bar{H}_1x(t-\tau_2(t))$$
$$+\big(\alpha(t)-\alpha_0\big)\Big(\bar{H}_1x(t-\tau_1(t))$$
$$-\bar{H}_1x(t-\tau_2(t))\Big)\Big]d\omega(t), \tag{22}$$

where $G(x(t)) = diag\Big(h_1(x_1(t)),h_2(x_2(t)),...,h_n(x_n(t))\Big)$ and $0 \leq h_j(x_j(t)) = f_j(x_j(t))/(x_j(t)) \leq l_j$. Moreover, by substituting (20) and (21) into (22), we obtain

$$dx(t)=\Big[-\bar{A}x(t)+\bar{B}f(x(t))+\alpha_0\bar{W}G(x(t-\tau_1(t)))x(t)$$
$$-\alpha_0\bar{W}G(x(t-\tau_1(t)))\int\limits_{t-\tau_1(t)}^{t}y(s)ds-\alpha_0\bar{W}G(x(t-\tau_1(t)))\int\limits_{t-\tau_1(t)}^{t}g(s)d\omega(s)$$
$$+(1-\alpha_0)\bar{W}G(x(t-\tau_2(t)))x(t)-(1-\alpha_0)\bar{W}G(x(t-\tau_2(t)))\int\limits_{t-\tau_2(t)}^{t}y(s)ds$$
$$-(1-\alpha_0)\bar{W}G(x(t-\tau_2(t)))\int\limits_{t-\tau_2(t)}^{t}g(s)d\omega(s)$$
$$+\big(\alpha(t)-\alpha_0\big)\Big(\bar{W}f(x(t-\tau_1(t)))-\bar{W}f(x(t-\tau_2(t)))\Big)\Big]dt$$
$$+\Big[\bar{H}_0x(t)+\alpha_0\bar{H}_1x(t-\tau_1(t))+(1-\alpha_0)\bar{H}_1x(t-\tau_2(t))+$$
$$\big(\alpha(t)-\alpha_0\big)\Big(\bar{H}_1x(t-\tau_1(t))-\bar{H}_1x(t-\tau_2(t))\Big)\Big]d\omega(t). \tag{23}$$

Choose a Lyapunov-Krasovskii functional candidate as follows

$$V(x(t),t)=x^T(t)Px(t)+\int\limits_{t-\alpha_0\tau_1(t)}^{t}x^T(s)Q_1x(s)ds+\int\limits_{t-\tau_0}^{t}x^T(s)Q_2x(s)ds$$
$$+\int\limits_{t-\bar{\tau}}^{t}x^T(s)Q_3x(s)ds+\int\limits_{t-\alpha_0\tau_2(t)}^{t}x^T(s)Q_4x(s)ds+\int\limits_{t-\tau_1(t)}^{t}x^T(s)Q_5x(s)ds$$
$$+\int\limits_{t-\tau_2(t)}^{t}x^T(s)Q_6x(s)ds+\int\limits_{-\bar{\tau}_1}^{0}\int\limits_{t+\beta}^{t}y^T(\alpha)Z_1y(\alpha)d\alpha d\beta+\int\limits_{-\bar{\tau}_2}^{0}\int\limits_{t+\beta}^{t}y^T(\alpha)Z_2y(\alpha)d\alpha d\beta$$

$$+\int_{-\bar{\tau}_1}^{0}\int_{t+\beta}^{t} g^T(\alpha)R_1 g(\alpha)d\alpha d\beta + \int_{-\bar{\tau}_2}^{0}\int_{t+\beta}^{t} g^T(\alpha)R_2 g(\alpha)d\alpha d\beta. \tag{24}$$

By Ito's formula, we can calculate $\mathcal{L}V(x(t),t)$ along with (24), then we have

$$dV(x(t),t) = \mathcal{L}V(x(t),t) + 2x^T(t)Pg(t)d\omega(t), \tag{25}$$

where

$$\begin{aligned}
\mathcal{L}V(x(t),t) \le\ & 2x^T(t)P\Big[-\bar{A}x(t) + \bar{B}f(x(t)) + \alpha_0\bar{W}G(x(t-\tau_1(t)))x(t) \\
& -\alpha_0\bar{W}G(x(t-\tau_1(t)))\int_{t-\tau_1(t)}^{t} y(s)ds - \alpha_0\bar{W}G(x(t-\tau_1(t)))\int_{t-\tau_1(t)}^{t} g(s)d\omega(s) \\
& +(1-\alpha_0)\bar{W}G(x(t-\tau_2(t)))x(t) - (1-\alpha_0)\bar{W}G(x(t-\tau_2(t)))\int_{t-\tau_2(t)}^{t} y(s)ds \\
& -(1-\alpha_0)\bar{W}G(x(t-\tau_2(t)))\int_{t-\tau_2(t)}^{t} g(s)d\omega(s) \\
& +\big(\alpha(t)-\alpha_0\big)\Big(\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big)\Big] \\
& +g^T(t)Pg(t) + x^T(t)Q_1x(t) - (1-\alpha_0\mu_1)x^T(t-\alpha_0\tau_1(t))Q_1x(t-\alpha_0\tau_1(t)) \\
& +x^T(t)Q_2x(t) - x^T(t-\tau_0)Q_2x(t-\tau_0) + x^T(t)Q_3x(t) - x^T(t-\bar{\tau})Q_3x(t-\bar{\tau}) \\
& +x^T(t)Q_4x(t) - (1-\alpha_0\mu_2)x^T(t-\alpha_0\tau_2(t))Q_4x(t-\alpha_0\tau_2(t)) \\
& +x^T(t)Q_5x(t) - (1-\mu_1)x^T(t-\tau_1(t))Q_5x(t-\tau_1(t)) + x^T(t)Q_6x(t) \\
& -(1-\mu_2)x^T(t-\tau_2(t))Q_6x(t-\tau_2(t)) \\
& +\bar{\tau}_1y^T(t)Z_1y(t) - \int_{t-\bar{\tau}_1}^{t} y^T(s)Z_1y(s)ds \\
& +\bar{\tau}_2y^T(t)Z_2y(t) - \int_{t-\bar{\tau}_2}^{t} y^T(s)Z_2y(s)ds + \bar{\tau}_1g^T(t)R_1g(t)
\end{aligned}$$

$$-\int\limits_{t-\bar{\tau}_1}^{t} g^T(s)R_1g(s)ds + \bar{\tau}_2 g^T(t)R_2g(t) - \int\limits_{t-\bar{\tau}_2}^{t} g^T(s)R_2g(s)ds. \qquad (26)$$

Define $a(\cdot)$, $b(\cdot)$ and N in Lemma 2.11 as $a(\alpha) = x(t)$, $b(\alpha) = y(s)$, $N = P\bar{W}G(x(t-\tau_1(t)))$ and using (20), then

$$-2\alpha_0 x^T(t)P\bar{W}G(x(t-\tau_1(t))) \int\limits_{t-\tau_1(t)}^{t} y(s)ds$$

$$\leq \alpha_0 \int\limits_{t-\tau_1(t)}^{t} \begin{bmatrix} x(t) \\ y(s) \end{bmatrix}^T \begin{bmatrix} X & Y - P\bar{W}G(x(t-\tau_1(t))) \\ & Z_1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(s) \end{bmatrix} ds$$

$$\leq \alpha_0 \Big\{ \bar{\tau}_1 x^T(t)Xx(t) + 2x^T(t)[Y - P\bar{W}G(x(t-\tau_1(t)))] \int\limits_{t-\tau_1(t)}^{t} y(s)ds$$

$$+ \int\limits_{t-\tau_1(t)}^{t} y^T(s)Z_1y(s)ds \Big\}$$

$$\leq \alpha_0\bar{\tau}_1 x^T(t)Xx(t) + 2\alpha_0 x^T(t)Yx(t) - 2\alpha_0 x^T(t)Yx(t-\tau_1(t))$$

$$-2\alpha_0 x^T(t)Y \int\limits_{t-\tau_1(t)}^{t} g(s)d\omega(s) - 2\alpha_0 x^T(t)P\bar{W}G(x(t-\tau_1(t)))x(t)$$

$$2 + \alpha_0 x^T(t)P\bar{W}f(x(t-\tau_1(t))) + 2\alpha_0 x^T(t)P\bar{W}G(x(t-\tau_1(t))) \int\limits_{t-\tau_1(t)}^{t} g(s)d\omega(s)$$

$$+\alpha_0 \int\limits_{t-\tau_1(t)}^{t} y^T(s)Z_1y(s)ds. \qquad (27)$$

Define $a(.)$, $b(.)$ and N in Lemma 2.11 as $a(\alpha) = x(t)$, $b(\alpha) = y(s)$, $N = P\bar{W}G(x(t-\tau_2(t)))$ and using (21), then

$$-2(1-\alpha_0)x^T(t)P\bar{W}G(x(t-\tau_2(t))) \int\limits_{t-\tau_2(t)}^{t} y(s)ds$$

$$\leq \quad (1-\alpha_0)\int\limits_{t-\tau_2(t)}^{t} \begin{bmatrix} x(t) \\ y(s) \end{bmatrix}^T \begin{bmatrix} X & Y - P\bar{W}G(x(t-\tau_2(t))) \\ & Z_2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(s) \end{bmatrix} ds$$

$$\leq \quad (1-\alpha_0)\bar{\tau}_2 x^T(t)Xx(t) + 2(1-\alpha_0)x^T(t)Yx(t) - 2(1-\alpha_0)x^T(t)Yx(t-\tau_2(t))$$

$$-2(1-\alpha_0)x^T(t)Y\int\limits_{t-\tau_2(t)}^{t} g(s)d\omega(s) - 2(1-\alpha_0)x^T(t)P\bar{W}G(x(t-\tau_2(t)))x(t)$$

$$+2(1-\alpha_0)x^T(t)P\bar{W}f(x(t-\tau_2(t))) + 2(1-\alpha_0)x^T(t)P\bar{W}G(x(t-\tau_2(t)))$$

$$\times\int\limits_{t-\tau_2(t)}^{t} g(s)d\omega(s) + (1-\alpha_0)\int\limits_{t-\tau_2(t)}^{t} y^T(s)Z_2y(s)ds. \tag{28}$$

Here

$$\begin{aligned} f(x(t-\tau_1(t))) &= G(x(t-\tau_1(t)))x(t-\tau_1(t)), \quad \text{and} \\ f(x(t-\tau_2(t))) &= G(x(t-\tau_2(t)))x(t-\tau_2(t)) \end{aligned}$$

are used. Using (27) and (28) in (26), we have

$$\mathcal{L}V(x(t),t) \leq -2x^T(t)P\bar{A}x(t) + 2x^T(t)P\bar{B}f(x(t)) + \alpha_0\bar{\tau}_1x^T(t)Xx(t)$$

$$-2\alpha_0x^T(t)Yx(t-\tau_1(t)) - 2\alpha_0x^T(t)Y\int\limits_{t-\tau_1(t)}^{t} g(s)d\omega(s)$$

$$+2\alpha_0x^T(t)P\bar{W}f(x(t-\tau_1(t))) - (1-\alpha_0)\int\limits_{t-\tau_1(t)}^{t} y^T(s)Z_1y(s)ds$$

$$+(1-\alpha_0)\bar{\tau}_2x^T(t)Xx(t) + 2x^T(t)Yx(t) - 2(1-\alpha_0)x^T(t)Yx(t-\tau_2(t))$$

$$-2(1-\alpha_0)x^T(t)Y\int\limits_{t-\tau_2(t)}^{t} g(s)d\omega(s) + 2(1-\alpha_0)x^T(t)P\bar{W}f(x(t-\tau_2(t)))$$

$$-\alpha_0\int\limits_{t-\tau_2(t)}^{t} y^T(s)Z_2y(s)ds + 2\big(\alpha(t)-\alpha_0\big)$$

$$\times x^T(t)P\Big(\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big)$$

$$
\begin{aligned}
&+g^T(t)\Big(P+\bar{\tau}_1R_1+\bar{\tau}_2R_2\Big)g(t)+x^T(t)Q_1x(t)\\
&-(1-\alpha_0\mu_1)x^T(t-\alpha_0\tau_1(t))Q_1x(t-\alpha_0\tau_1(t))\\
&+x^T(t)Q_2x(t)-x^T(t-\tau_0)Q_2x(t-\tau_0)+x^T(t)Q_3x(t)-x^T(t-\bar{\tau})Q_3x(t-\bar{\tau})\\
&+x^T(t)Q_4x(t)-(1-\alpha_0\mu_2)x^T(t-\alpha_0\tau_2(t))Q_4x(t-\alpha_0\tau_2(t))+x^T(t)Q_5x(t)\\
&-(1-\mu_1)x^T(t-\tau_1(t))Q_5x(t-\tau_1(t))\\
&+x^T(t)Q_6x(t)-(1-\mu_2)x^T(t-\tau_2(t))Q_6x(t-\tau_2(t))\\
&+y^T(t)\Big(\bar{\tau}_1Z_1+\bar{\tau}_2Z_2\Big)y(t)\\
&-\int\limits_{t-\tau_1(t)}^{t}g^T(s)R_1g(s)ds-\int\limits_{t-\tau_2(t)}^{t}g^T(s)R_2g(s)ds. \qquad (29)
\end{aligned}
$$

For symmetric positive definite matrices R_1, and R_2, it follows from Lemma 2.7 that

$$
\begin{aligned}
\frac{1}{\alpha_0}\Big\{-2x^T(t)Y\int\limits_{t-\tau_1(t)}^{t}g(s)d\omega(s)\Big\} &\le \frac{1}{\alpha_0}\Big\{x^T(t)YR_1^{-1}Y^Tx(t)\\
&+\Big(\int\limits_{t-\tau_1(t)}^{t}g(s)d\omega(s)\Big)^TR_1\Big(\int\limits_{t-\tau_1(t)}^{t}g(s)d\omega(s)\Big)\Big\}, \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{1-\alpha_0}\Big\{-2x^T(t)Y\int\limits_{t-\tau_2(t)}^{t}g(s)d\omega(s)\Big\} &\le \frac{1}{1-\alpha_0}\Big\{x^T(t)YR_2^{-1}Y^Tx(t)\\
&+\Big(\int\limits_{t-\tau_2(t)}^{t}g(s)d\omega(s)\Big)^TR_2\Big(\int\limits_{t-\tau_2(t)}^{t}g(s)d\omega(s)\Big)\Big\}. \qquad (31)
\end{aligned}
$$

It is clear from (3) that

$$
\begin{aligned}
f_j(x_j(t))\Big[f_j(x_j(t))-l_jx_j(t)\Big] &\le 0, \qquad (32)\\
f_j(x_j(t-\tau_i(t)))\Big[f_j(x_j(t-\tau_i(t)))-l_jx_j(t-\tau_i(t))\Big] &\le 0, \quad j=1,2,..,n. \qquad (33)
\end{aligned}
$$

From inequalities (32) and (33), for any positive diagonal matrices $K_i = diag(k_{i1}, k_{i2}, ..., k_{in})$, $i = 1, 2, 3$, the following inequalities hold

$$\begin{aligned} 0 \quad \leq \quad & -2\sum_{j=1}^{n} k_{1j} f_j(x_j(t))\Big[f_j(x_j(t)) - l_j x_j(t)\Big] - 2\sum_{i=1}^{2}\sum_{j=1}^{n} k_{(i+1)j} f_j(x_j(t-\tau_i(t))) \\ & \times\Big[f_j(x_j(t-\tau_i(t))) - l_j x_j(t-\tau_i(t))\Big] \\ = \quad & 2x^T(t)LK_1 f(x(t)) - 2f^T(x(t))K_1 f(x(t)) \\ & +2x^T(t-\tau_1(t))LK_2 f(x(t-\tau_1(t))) - 2f^T(x(t-\tau_1(t))) \\ & K_2 f(x(t-\tau_1(t))) + 2x^T(t-\tau_2(t))LK_3 f(x(t-\tau_2(t))) \\ & -2f^T(x(t-\tau_2(t)))K_3 f(x(t-\tau_2(t))). \end{aligned} \tag{34}$$

where $L = diag\big(l_1, l_2, ..., l_n\big)$. By Remark 2.4, it is easy to know

$$\mathbb{E}\Big\{2\big(\alpha(t)-\alpha_0\big)x^T(t)P\Big(\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big)\Big\} = 0 \tag{35}$$

$$\begin{aligned} \mathbb{E}\Big\{y^T(t)Z_1 y(t)\Big\} = \mathbb{E}\Big\{\Big[&-\bar{A}x(t) + \bar{B}f(x(t)) + \alpha_0\bar{W}f(x(t-\tau_1(t))) \\ &+(1-\alpha_0)\bar{W}f(x(t-\tau_2(t)))\Big]^T \\ \times Z_1\Big[-\bar{A}x(t) &+ \bar{B}f(x(t)) + \alpha_0\bar{W}f(x(t-\tau_1(t))) + (1-\alpha_0)\bar{W}f(x(t-\tau_2(t)))\Big] \\ +2(\alpha(t)-\alpha_0)\Big[&-\bar{A}x(t) + \bar{B}f(x(t)) + \alpha_0\bar{W}f(x(t-\tau_1(t))) + (1-\alpha_0)\bar{W} \\ \times f(x(t-\tau_2(t)))\Big]&Z_1\Big[\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big] + \big(\alpha(t)-\alpha_0\big)^2 \\ &\times\Big[\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big]^T Z_1 \\ &\times\Big[\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big]\Big\} \\ = \Big[-\bar{A}x(t) &+ \bar{B}f(x(t)) + \alpha_0\bar{W}f(x(t-\tau_1(t))) + (1-\alpha_0)\bar{W}f(x(t-\tau_2(t)))\Big]^T Z_1 \\ \times\Big[-\bar{A}x(t) &+ \bar{B}f(x(t)) + \alpha_0\bar{W}f(x(t-\tau_1(t))) + (1-\alpha_0)\bar{W}f(x(t-\tau_2(t)))\Big] \\ &+\alpha_0(1-\alpha_0)\Big[\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big]^T Z_1 \\ &\times\Big[\bar{W}f(x(t-\tau_1(t))) - \bar{W}f(x(t-\tau_2(t)))\Big] \end{aligned}$$

$$\mathbb{E}\Big\{y^T(t)Z_1y(t)\Big\} = \zeta^T(t)\Big\{\eta_1 Z_1\eta_1^T + \alpha_0(1-\alpha_0)\eta_2 Z_1\eta_2^T\Big\}\zeta(t). \tag{36}$$

Similarly we have,

$$\mathbb{E}\Big\{y^T(t)Z_2y(t)\Big\} = \zeta^T(t)\Big\{\eta_1 Z_2\eta_1^T + \alpha_0(1-\alpha_0)\eta_2 Z_2\eta_2^T\Big\}\zeta(t), \tag{37}$$

and

$$\mathbb{E}\Big\{g^T(t)(P+\bar{\tau}_1R_1+\bar{\tau}_2R_2)g(t)\Big\} = \zeta^T(t)\Big\{\vartheta_1\bar{P}\vartheta_1^T + \alpha_0(1-\alpha_0)\vartheta_2\bar{P}\vartheta_2^T\Big\}\zeta(t). \tag{38}$$

$$\mathbb{E}\Big\{\Big(\int\limits_{t-\tau_1(t)}^{t} g(s)d\omega(s)\Big)^T R_1\Big(\int\limits_{t-\tau_1(t)}^{t} g(s)d\omega(s)\Big)\Big\} = \mathbb{E}\Big\{\int\limits_{t-\tau_1(t)}^{t} g^T(s)R_1g(s)ds\Big\} \tag{39}$$

$$\mathbb{E}\Big\{\Big(\int\limits_{t-\tau_2(t)}^{t} g(s)d\omega(s)\Big)^T R_2\Big(\int\limits_{t-\tau_2(t)}^{t} g(s)d\omega(s)\Big)\Big\} = \mathbb{E}\Big\{\int\limits_{t-\tau_2(t)}^{t} g^T(s)R_2g(s)ds\Big\}. \tag{40}$$

Using Lemma 2.8 in (29) and substituting (29)-(31), (34) into (25) and taking mathematical expectation on both sides of (25) then using (35)-(40), we can get

$$\begin{aligned}\mathbb{E}dV(x(t),t) &= \mathbb{E}\big\{\mathcal{L}V(x(t),t)\big\}\\ &\le \zeta^T(t)\Big\{\bar{\Omega} + \bar{\tau}_1\eta_1Z_1\eta_1^T + \bar{\tau}_1\alpha_0(1-\alpha_0)\eta_2Z_1\eta_2^T + \bar{\tau}_2\eta_1Z_2\eta_1^T\\ &\quad + \bar{\tau}_2\alpha_0(1-\alpha_0)\eta_2Z_2\eta_2^T + \vartheta_1\bar{P}\vartheta_1^T + \alpha_0(1-\alpha_0)\vartheta_2\bar{P}\vartheta_2^T\\ &\quad + YR_1^{-1}Y^T + YR_2^{-1}Y^T\Big\}\zeta(t),\end{aligned} \tag{41}$$

where

$$\begin{aligned}\zeta(t) = \Big[&x^T(t) \;\; x^T(t-\alpha_0\tau_1(t)) \;\; x^T(t-\tau_1(t)) \;\; x^T(t-\tau_0) \;\; x^T(t-\tau_2(t))\\ &x^T(t-\bar{\tau}) \;\; x^T(t-\alpha_0\tau_2(t)) \; f^T(x(t)) \;\; f^T(x(t-\tau_1(t))) \;\; f^T(x(t-\tau_2(t)))\\ &\Big(\int\limits_{t-\tau_1(t)}^{t} y(s)ds\Big)^T \;\; \Big(\int\limits_{t-\tau_2(t)}^{t} y(s)ds\Big)^T\Big]^T.\end{aligned}$$

Let us define,

$$\begin{aligned}\bar{\Upsilon} = \bar{\Omega} &+ \bar{\tau}_1\eta_1Z_1\eta_1^T + \bar{\tau}_1\alpha_0(1-\alpha_0)\eta_2Z_1\eta_2^T + \bar{\tau}_2\eta_1Z_2\eta_1^T\\ &+ \bar{\tau}_2\alpha_0(1-\alpha_0)\eta_2Z_2\eta_2^T + \vartheta_1\bar{P}\vartheta_1^T + \alpha_0(1-\alpha_0)\vartheta_2\bar{P}\vartheta_2^T\\ &+ YR_1^{-1}Y^T + YR_2^{-1}Y^T < 0.\end{aligned}$$

Considering $\mu_k(\theta(t)) \geq 0$ $(k = 1, 2, ..., r)$ and $\Upsilon_k < 0$ $(k = 1, 2, ..., r)$ in Theorem 3.1, we have $\sum_{k=1}^{r} \mu_k(\theta(t))\Upsilon_k < 0$. Noting that $\sum_{k=1}^{r} \mu_k(\theta(t)) = 1$. Where

$$\Upsilon_k = \Omega_k + \bar{\tau}_1\eta_1 Z_1\eta_1^T + \bar{\tau}_1\alpha_0(1-\alpha_0)\eta_2 Z_1\eta_2^T + \bar{\tau}_2\eta_1 Z_2\eta_1^T$$
$$+\bar{\tau}_2\alpha_0(1-\alpha_0)\eta_2 Z_2\eta_2^T + \vartheta_1\bar{P}\vartheta_1^T$$
$$+\alpha_0(1-\alpha_0)\vartheta_2\bar{P}\vartheta_2^T + YR_1^{-1}Y^T + YR_2^{-1}Y^T < 0,$$

Ω_k, η_1, η_2, ϑ_1, and ϑ_2 are defined as in Theorem 3.1. By Schur complement, we know that $\Upsilon_k < 0$ is equivalent to (16). Let $\lambda = \min\{\lambda_{\min}(-\Upsilon_k)\}$, then by the generalized Ito's formula [13], we have

$$\mathbb{E}V(x(t),t) - \mathbb{E}V(x(0),0) = \mathbb{E}\int_0^t \mathcal{L}V(x(s),s)ds \leq -\lambda\mathbb{E}\int_0^t \|x(s)\|^2 ds.$$

Moreover,

$$\mathbb{E}\int_0^t \|x(s)\|^2 ds \leq \frac{1}{\lambda}\mathbb{E}V(x(0),0), \quad t \geq 0.$$

which indicates that system (14) is globally asymptotically stable in the mean square. This completes the proof.

In the following part, we extend the above result to uncertain fuzzy stochastic Hopfield neural network (UFSHNN) (12) and obtain the stability criteria as the following theorem by means of the feasibility of LMIs.

Theorem 3.2 For given scalars $\bar{\tau}_1$, $\bar{\tau}_2$, μ_1, μ_2, and $0 < \alpha_0 < 1$ satisfying $\alpha_0\mu_1 < 1$, the UFSHNN (12) is globally robustly asymptotically stable in the mean square, if there exist symmetric positive definite matrices $P > 0$, $Q_i > 0$ $(i = 1, 2, 3, 4, 5, 6)$, $R_1 > 0$, $R_2 > 0$, $Z_1 > 0$, and $Z_2 > 0$, for any matrices X and Y, positive diagonal matrices $K_1 > 0$, $K_2 > 0$, $K_3 > 0$ and positive scalars $\epsilon_j > 0$ $(j = 1, ..., 7)$, such that the following LMIs

$$\begin{bmatrix} X & Y \\ & Z_1 \end{bmatrix} \geq 0 \quad and \quad \begin{bmatrix} X & Y \\ & Z_2 \end{bmatrix} \geq 0 \tag{42}$$

$$\begin{bmatrix} \Upsilon_k & \Gamma_1 & \bar{\tau}_1\Gamma_2 & \bar{\tau}_1\Gamma_3 & \bar{\tau}_2\Gamma_4 & \bar{\tau}_2\Gamma_5 & \Gamma_6 & \Gamma_7 \\ & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 \\ & * & -\bar{\tau}_1\epsilon_2 & 0 & 0 & 0 & 0 & 0 \\ & * & * & -\bar{\tau}_1\epsilon_3 & 0 & 0 & 0 & 0 \\ & * & * & * & -\bar{\tau}_2\epsilon_4 & 0 & 0 & 0 \\ & * & * & * & * & -\bar{\tau}_2\epsilon_5 & 0 & 0 \\ & * & * & * & * & * & -\epsilon_6 I & 0 \\ & * & * & * & * & * & * & -\epsilon_7 I \\ & & & & & & & \end{bmatrix} < 0 \tag{43}$$

hold for $k = 1, 2, ..., r$, where Υ_k and σ are defined as in Theorem 3.1, with

$$\Omega^k_{1,1} = -PA_k - A_kP + \alpha_0\bar{\tau}_1 X + (1-\alpha_0)\bar{\tau}_2 X + 2Y + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6$$

$$+\big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^A_k)^T(E^A_k) + \epsilon_6(E^{H_0}_k)^T(E^{H_0}_k), \Omega^k_{1,2} = 0,$$

$$\Omega^k_{1,3} = -\alpha_0 Y + \alpha_0\epsilon_6(E^{H_0}_k)^T(E^{H_1}_k),\ \Omega^k_{1,4} = 0,$$

$$\Omega^k_{1,5} = -(1-\alpha_0)Y + (1-\alpha_0)\epsilon_6(E^{H_0}_k)^T(E^{H_1}_k),\ \Omega^k_{1,6} = 0,$$

$$\Omega^k_{1,7} = 0,\ \Omega^k_{1,8} = PB_k + LK_1 - \big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^A_k)^T(E^B_k),$$

$$\Omega^k_{1,9} = \alpha_0 PW_k - \alpha_0\big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^A_k)^T(E^W_k),$$

$$\Omega^k_{1,10} = (1-\alpha_0)PW_k - (1-\alpha_0)\big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^A_k)^T(E^W_k),\ \Omega^k_{1,11} = 0,$$

$$\Omega^k_{1,12} = 0,\ \Omega^k_{2,2} = -\big(1 - \alpha_0\mu_1\big)Q_1,$$

$$\Omega^k_{2,3} = \Omega^k_{2,4} = \Omega^k_{2,5} = \Omega^k_{2,6} = \Omega^k_{2,7} = \Omega^k_{2,8} = \Omega^k_{2,9} = \Omega^k_{2,10} = 0,$$

$$\Omega^k_{2,11} = \Omega^k_{2,12} = 0,\ \Omega^k_{3,3} = -(1-\mu_1)Q_5 + \alpha_0^2\epsilon_6(E^{H_1}_k)^T(E^{H_1}_k) + \sigma^2\epsilon_7(E^{H_1}_k)^T(E^{H_1}_k),$$

$$\Omega^k_{3,4} = 0,\ \Omega^k_{3,5} = \sigma^2\epsilon_6(E^{H_1}_k)^T(E^{H_1}_k) + \sigma^2\epsilon_7(E^{H_1}_k)^T(E^{H_1}_k),$$

$$\Omega^k_{3,6} = \Omega^k_{3,7} = \Omega^k_{3,8} = 0,\ \Omega^k_{3,9} = LK_2,\ \Omega^k_{3,10} = 0,\ \Omega^k_{3,11} = \Omega^k_{3,12} = 0,$$

$$\Omega^k_{4,4} = -Q_2,\ \Omega^k_{4,5} = \Omega^k_{4,6} = \Omega^k_{4,7} = \Omega^k_{4,8} = \Omega^k_{4,9} = \Omega^k_{4,10} = \Omega^k_{4,11} = \Omega^k_{4,12} = 0,$$

$$\Omega^k_{5,5} = -(1-\mu_2)Q_6 + (1-\alpha_0)^2\epsilon_6(E^{H_1}_k)^T(E^{H_1}_k) + \sigma^2\epsilon_7(E^{H_1}_k)^T(E^{H_1}_k),$$

$$\Omega^k_{5,6} = \Omega^k_{5,7} = \Omega^k_{5,8} = \Omega^k_{5,9} = 0, \Omega^k_{5,10} = LK_3,\ \Omega^k_{5,11} = \Omega^k_{5,12} = 0,$$

$$\Omega^k_{6,6} = -Q_3,\ \Omega^k_{6,7} = \Omega^k_{6,8} = \Omega^k_{6,9} = \Omega^k_{6,10} = \Omega^k_{6,11} = \Omega^k_{6,12} = 0,$$

$$\Omega^k_{7,7} = -\big(1-\alpha_0\mu_2\big)Q_4,\ \Omega^k_{7,8} = \Omega^k_{7,9} = \Omega^k_{7,10} = \Omega^k_{7,11} = \Omega^k_{7,12} = 0,$$

$$\Omega^k_{8,8} = -2K_1 + \big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^B_k)^T(E^B_k),$$

$$\Omega^k_{8,9} = \alpha_0\big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^B_k)^T(E^W_k),$$

$$\Omega^k_{8,10} = (1-\alpha_0)\big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^B_k)^T(E^W_k),\ \Omega^k_{8,11} = \Omega^k_{8,12} = 0,$$

$$\Omega^k_{9,9} = -2K_2 + \alpha_0^2\big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^W_k)^T(E^W_k) + \sigma^2\big(\bar{\tau}_1\epsilon_3 + \bar{\tau}_2\epsilon_5\big)(E^W_k)^T(E^W_k),$$

$$\Omega^k_{9,10} = \sigma^2\big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^W_k)^T(E^W_k) - \sigma^2\big(\bar{\tau}_1\epsilon_3 + \bar{\tau}_2\epsilon_5\big) \times (E^W_k)^T(E^W_k),$$

$$\Omega^k_{9,11} = \Omega^k_{9,12} = 0,\ \Omega^k_{10,10} = -2K_3 + (1-\alpha_0)^2\big(\epsilon_1 + \bar{\tau}_1\epsilon_2 + \bar{\tau}_2\epsilon_4\big)(E^W_k)^T(E^W_k)$$

$$+\sigma^2\big(\bar{\tau}_1\epsilon_3 + \bar{\tau}_2\epsilon_5\big)(E^W_k)^T(E^W_k),\ \Omega^k_{10,11} = \Omega^k_{10,12} = 0,$$

$$\Omega^k_{11,11} = -\Big(\frac{1-\alpha_0}{\bar{\tau}_1}\Big)Z_1,\ \Omega^k_{11,12} = 0,\ \Omega^k_{12,12} = -\Big(\frac{\alpha_0}{\bar{\tau}_2}\Big)Z_2.$$

Proof Replace A_k, B_k, W_k, H_{0k} and H_{1k} in LMI (16) with $A_k + GF(t)E^A_K$, $B_k+GF(t)E^B_K$, $W_k+GF(t)E^W_K$, $H_{0k}+GF(t)E^{H_0}_K$ and $H_{1k}+GF(t)E^{H_1}_K$ respectively, we find that (16) for UFSHNN (12) is equivalent to the following condition

$$\begin{gathered}\Upsilon_k + \Gamma_1 F(t)\Pi^T_{1k} + \Pi_{1k}F^T(t)\Gamma^T_1 + \bar{\tau}_1\Gamma_2 F(t)\Pi^T_{1k} + \bar{\tau}_1\Pi_{1k}F^T(t)\Gamma^T_2 \\ +\bar{\tau}_1\Gamma_3 F(t)\Pi^T_{2k} + \bar{\tau}_1\Pi_{2k}F^T(t)\Gamma^T_3 + \bar{\tau}_2\Gamma_4 F(t)\Pi^T_{1k} \\ +\bar{\tau}_2\Pi_{1k}F^T(t)\Gamma^T_4 + \bar{\tau}_2\Gamma_5 F(t)\Pi^T_{2k} + \bar{\tau}_2\Pi_{2k}F^T(t)\Gamma^T_5 + \\ \Gamma_6 F(t)\Pi^T_{3k} + \Pi_{3k}F^T(t)\Gamma^T_6 + \Gamma_7 F(t)\Pi^T_{4k} + \Pi_{4k}F^T(t)\Gamma^T_7 < 0, \end{gathered} \tag{44}$$

where,

$$\Gamma_1 = \Big[G^TP\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\Big]^T,$$

$$\Gamma_2 = \Big[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ G^TZ_1\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\Big]^T,$$

$$\Gamma_3 = \Big[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ G^TZ_1\ \ 0\ 0\ 0\ 0\ 0\ 0\Big]^T,$$

$$\Gamma_4 = \Big[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ G^TZ_2\ \ 0\ 0\ 0\ 0\ 0\Big]^T,$$

$$\Gamma_5 = \Big[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ G^TZ_2\ \ 0\ 0\ 0\ 0\Big]^T,$$

$$\Gamma_6 = \Big[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ G^T\bar{P}\ \ 0\ 0\ 0\Big]^T,$$

$$\Gamma_7 = \Big[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ G^T\bar{P}\ \ 0\ 0\Big]^T,$$

$$\Pi_{1k} = \Big[-E^A_k\ \ 0\ 0\ 0\ 0\ 0\ 0\ \ E^B_k\ \ \alpha_0 E^W_k\ \ (1-\alpha_0)E^W_k\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\Big]^T,$$

$$\Pi_{2k} = \Big[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ \sigma E^W_k\ \ -\sigma E^W_k\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\Big]^T,$$

$$\Pi_{3k} = \Big[-E^{H_0}_k\ \ 0\ \ \alpha_0 E^{H_1}_k\ \ 0\ \ (1-\alpha_0)E^{H_1}_k\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\Big]^T,$$

$$\Pi_{4k} = \Big[0\ \ 0\ \ \sigma E^{H_1}_k\ \ 0\ \ \sigma E^{H_1}_k\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\Big]^T.$$

From Lemma 2.10, (44) holds for all $F^T(t)F(t) \le I$ if and only if there exist scalars $\varepsilon_j > 0$ $(j = 1, 2, ..., 7)$, such that

$$\begin{gathered}\Upsilon_k + \varepsilon_1^{-1}\Gamma_1\Gamma_1^T + \varepsilon_1\Pi_{1k}\Pi^T_{1k} + \bar{\tau}_1\varepsilon_2^{-1}\Gamma_2\Gamma_2^T + \bar{\tau}_1\varepsilon_2\Pi_{1k}\Pi^T_{1k} + \bar{\tau}_1\varepsilon_1^{-1}\Gamma_3\Gamma_3^T \\ +\bar{\tau}_1\varepsilon_3\Pi_{2k}\Pi^T_{2k} + \bar{\tau}_2\varepsilon_4^{-1}\Gamma_4\Gamma_4^T + \bar{\tau}_2\varepsilon_4\Pi_{1k}\Pi^T_{1k} + \bar{\tau}_2\varepsilon_5^{-1}\Gamma_5\Gamma_5^T + \bar{\tau}_2\varepsilon_5\Pi_{2k}\Pi^T_{2k}\end{gathered}$$

$$+\varepsilon_6^{-1}\Gamma_6\Gamma_6^T + \varepsilon_6\Pi_{3k}\Pi_{3k}^T + \varepsilon_7^{-1}\Gamma_7\Gamma_7^T + \varepsilon_7\Pi_{4k}\Pi_{4k}^T < 0. \tag{45}$$

By Schur complement, the Eq. (45) is equivalent to the LMI (43). Then, by Theorem 3.1, the system (12) is globally robustly asymptotically stable in the mean square. This completes the proof.

Remark 3.3 In [22], the authors dealt with the problem of delay-dependent robust stability for uncertain stochastic fuzzy Hopfield neural networks with time-varying delays. However, the probability distribution delay was not taken into account in this model. In our paper, we study delay-dependent robust stability analysis for uncertain fuzzy stochastic Hopfield neural networks with random time-varying delays. Thus, the results in this paper are lead to an improvement over the existing ones [22].

Remark 3.4 In the case of $k = 1$, the system (12) is reduced to same as in [18] and the stability criteria for the corresponding reduced system can be obtained by using Theorem 3.1. Moreover, the traditional assumption such as boundedness, monotonicity or differentiability on the neuron activation functions [22] have been removed in this paper.

4. Numerical Examples

In this section, we will give two examples showing the effectiveness of established theoretical results.

Example 1 Consider the SNNs (14) without uncertain parameters defined as

$$A_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.6 & -0.4 \\ 0.5 & 0.4 \end{bmatrix}, \; W_1 = \begin{bmatrix} 0.3 & 0.4 \\ 0.2 & -0.5 \end{bmatrix},$$

$$H_{01} = \begin{bmatrix} 0.7 & 0.5 \\ -0.8 & 0.5 \end{bmatrix}, \; H_{11} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \; A_2 = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.5 & -0.6 \\ 0.6 & 0.5 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & -0.4 \end{bmatrix},$$

$$H_{02} = \begin{bmatrix} 0.5 & -0.5 \\ 0.7 & 0.5 \end{bmatrix}, \; H_{12} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \; L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The activation function $f(x(t)) = tanh(x(t))$, the time-varying delays are chosen as $\bar{\tau}_1 = 0.4$ and $\bar{\tau}_2 = 1$. The derivative of time-varying delays $\dot{\tau}_1(t) \leq \mu_1 = 0.9$, $\dot{\tau}_2(t) \leq \mu_2 = 0.9$, $\alpha_0 = 0.2$, and using the Matlab LMI toolbox to solve the LMI in Theorem 3.1, we obtained the following matrices

$$P = \begin{bmatrix} 327.4076 & 24.6582 \\ 24.6582 & 186.7410 \end{bmatrix}, \; Q_1 = \begin{bmatrix} 29.1065 & 2.8428 \\ 2.8428 & 78.8062 \end{bmatrix},$$

$$\begin{aligned}
Q_2 &= \begin{bmatrix} 28.3027 & 2.5709 \\ 2.5709 & 73.2516 \end{bmatrix}, \; Q_3 = \begin{bmatrix} 28.3027 & 2.5709 \\ 2.5709 & 73.2516 \end{bmatrix}, \\
Q_4 &= \begin{bmatrix} 29.1065 & 2.8428 \\ 2.8428 & 78.8062 \end{bmatrix}, \; Q_5 = \begin{bmatrix} 124.1846 & 7.8651 \\ 7.8651 & 200.4389 \end{bmatrix}, \\
Q_6 &= \begin{bmatrix} 361.9906 & 25.6513 \\ 25.6513 & 445.1012 \end{bmatrix}, \; Z_1 = \begin{bmatrix} 53.0635 & 4.5276 \\ 4.5276 & 19.6996 \end{bmatrix}, \\
Z_2 &= \begin{bmatrix} 34.4196 & 3.2734 \\ 3.2734 & 10.2579 \end{bmatrix}, \; R_1 = \begin{bmatrix} 48.6955 & 15.8265 \\ 15.8265 & 45.5044 \end{bmatrix}, \\
R_2 &= \begin{bmatrix} 27.2106 & 10.5027 \\ 10.5027 & 24.3706 \end{bmatrix}, \; X = 10^5 \times \begin{bmatrix} 0.0003 & 2.3024 \\ -2.3023 & 0.0007 \end{bmatrix}, \\
Y &= \begin{bmatrix} 9.9647 & 0.3887 \\ 6.4928 & -2.2659 \end{bmatrix}, \; K_1 = \begin{bmatrix} 139.1778 & 0 \\ 0 & 139.1778 \end{bmatrix}, \\
K_2 &= \begin{bmatrix} 13.2926 & 0 \\ 0 & 13.2926 \end{bmatrix}, \; K_3 = \begin{bmatrix} 33.5343 & 0 \\ 0 & 33.5343 \end{bmatrix}.
\end{aligned}$$

Therefore, it follows from Theorem 3.1, that the SNNs without uncertain parameters (14) is globally asymptotically stable in the mean square. The response of the state dynamics for the SNNs without uncertain parameters (14) which converges to zero asymptotically in the mean square are shown in Figures 1 and 2.

Example 2 Consider the SNNs (12) with uncertain parameters defined as

$$A_1 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.5 & -0.7 \\ 0.3 & 0.6 \end{bmatrix}, \; W_1 = \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & -0.5 \end{bmatrix},$$

$$H_{01} = \begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 0.6 \end{bmatrix}, \; H_{11} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -0.4 & -0.6 \\ 0.6 & 0.4 \end{bmatrix}, \; W_2 = \begin{bmatrix} 0.2 & 0.3 \\ 0.2 & -0.4 \end{bmatrix}, \; H_{02} = \begin{bmatrix} 0.7 & 0.5 \\ 0.6 & 0.5 \end{bmatrix},$$

$$H_{12} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \; L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; G = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$E_1^A = E_2^A = E_1^B = E_2^B = E_1^W = E_2^W = E_1^{H_0} = E_2^{H_0} = E_1^{H_1} = E_2^{H_1} = 0.2I.$$

The activation function $f(x(t)) = tanh(x(t))$, the time-varying delays are chosen as $\bar{\tau}_1 = 0.2$, $\bar{\tau}_2 = 1.2$, The derivative of time-varying delays $\dot{\tau}_1(t) \leq \mu_1 = 0.8$, $\dot{\tau}_2(t) \leq \mu_2 = 0.8$, $\alpha_0 = 0.1$ and using the Matlab LMI toolbox to solve the LMI in Theorem 3.2, we obtained the following matrices

$$P = \begin{bmatrix} 518.3570 & 41.1464 \\ 41.1464 & 291.9800 \end{bmatrix}, \; Q_1 = \begin{bmatrix} 19.7588 & -25.7369 \\ -25.7369 & 50.5741 \end{bmatrix},$$

$$\begin{aligned}
Q_2 &= \begin{bmatrix} 19.4515 & -25.2105 \\ -25.2105 & 49.6365 \end{bmatrix},\ Q_3 = \begin{bmatrix} 19.4515 & -25.2105 \\ -25.2105 & 49.6365 \end{bmatrix}, \\
Q_4 &= \begin{bmatrix} 13.4952 & -17.7217 \\ -17.7217 & 34.7562 \end{bmatrix},\ Q_5 = \begin{bmatrix} 124.1216 & -9.4733 \\ -9.4733 & 95.1319 \end{bmatrix}, \\
Q_6 &= \begin{bmatrix} 970.4180 & 52.1610 \\ 52.1610 & 539.2814 \end{bmatrix},\ Z_1 = \begin{bmatrix} 58.4896 & -27.6329 \\ -27.6329 & 38.2542 \end{bmatrix}, \\
Z_2 &= \begin{bmatrix} 21.3870 & -12.1165 \\ -12.1165 & 12.1744 \end{bmatrix},\ R_1 = \begin{bmatrix} 70.7114 & -67.4364 \\ -67.4364 & 83.8331 \end{bmatrix}, \\
R_2 &= \begin{bmatrix} 34.0450 & -34.7188 \\ -34.7188 & 40.1826 \end{bmatrix},\ X = \begin{bmatrix} 19.1175 & -23.9969 \\ -23.9969 & 47.1209 \end{bmatrix}, \\
Y &= \begin{bmatrix} 6.2330 & -2.7589 \\ -2.7589 & 3.6739 \end{bmatrix},\ K_1 = \begin{bmatrix} 345.0867 & 0 \\ 0 & 345.0867 \end{bmatrix}, \\
K_2 &= \begin{bmatrix} 13.8953 & 0 \\ 0 & 13.8953 \end{bmatrix},\ K_3 = \begin{bmatrix} 111.8889 & 0 \\ 0 & 111.8889 \end{bmatrix}, \\
\epsilon_1 &= 248.1167, \quad \epsilon_2 = 167.8423, \quad \epsilon_3 = 492.0111, \quad \epsilon_4 = 34.7926, \\
\epsilon_5 &= 150.2244, \quad \epsilon_6 = 279.6797, \quad \epsilon_7 = 97.3928.
\end{aligned}$$

Therefore, it follows from Theorem 3.2, that the UFSHNN (12) is globally robustly asymptotically stable in the mean square. The response of the state dynamics for the UFSHNN (12) which converges to zero asymptotically in the mean square are shown in Figures 3 and 4.

5. Conclusion

The delay-dependent robust stability analysis for fuzzy stochastic Hopfield neural networks with random time-varying delays has been investigated. By using the combination of Lyapunov stability theory and stochastic analysis approach, some delay-dependent criteria have been derived to guarantee that the global robust asymptotic stability of the system in the mean square. This criteria can be checked easily by the LMI control toolbox in Matlab. Finally, numerical examples have been provided to illustrate the advantages and usefulness of the proposed results.

References

[1] Haykin, S., Neural Networks: A Comprehensive Foundation, Prentice Hall, NJ, 1998.
[2] Arik, S., Stability analysis of delayed neural networks, *IEEE Transactions on Circuits and Systems,* I 47 (1997), 1089-1092.
[3] Liu, Y. R., Wang, Z. D., Liu, X., Global exponential stability of generalized recurrent neural networks with discrete and distributed delays, *Neural Networks,* 19 (2006), 667-675.
[4] Zhang, Q., Wei, X., Xu, J., Delay-dependent global stability condition for delayed Hopfield neural networks, *Nonlinear Analysis: Real World Applications,* 8 (2007), 997-1002.

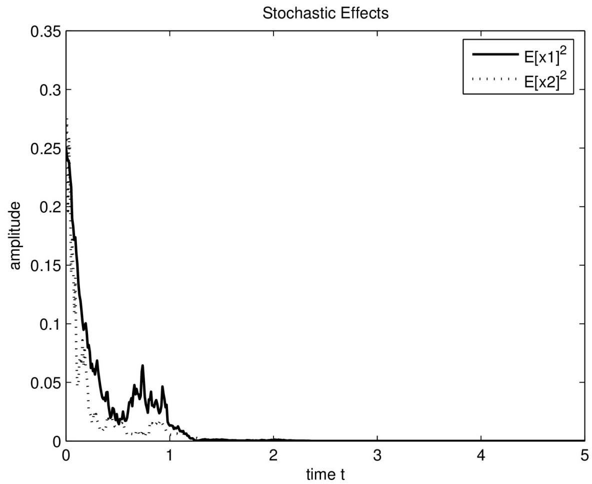

FIGURE 1. The state trajectories are converging to zero for $k = 1$ for Example 1

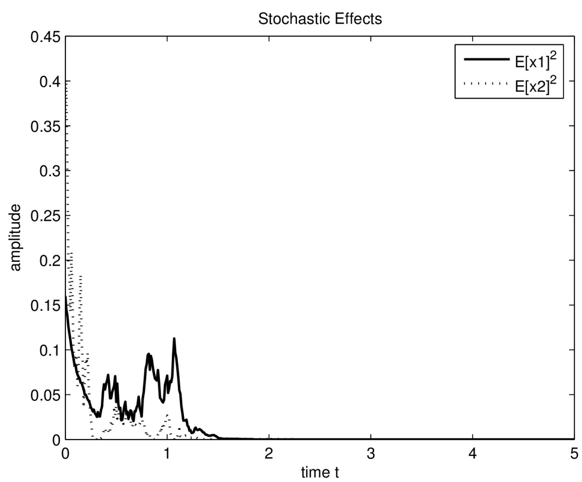

FIGURE 2. The state trajectories are converging to zero for $k = 2$ for Example 1

[5] Wu, H., Feng, W., Liang, X., New stability criteria for uncertain neural networks with interval time-varying delays, *Cognitive Neurodynamics,* 2 (2008), 363-370.
[6] Li, X., Chen, Z., Stability properties for Hopfield neural networks with delays and impulsive perturbations, *Nonlinear Analysis: Real World Applications,* 10 (2009), 3253-3265.

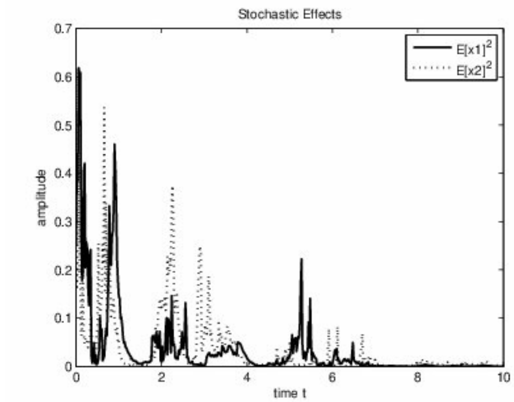

FIGURE 3. The state trajectories are converging to zero for $k = 1$ for Example 2

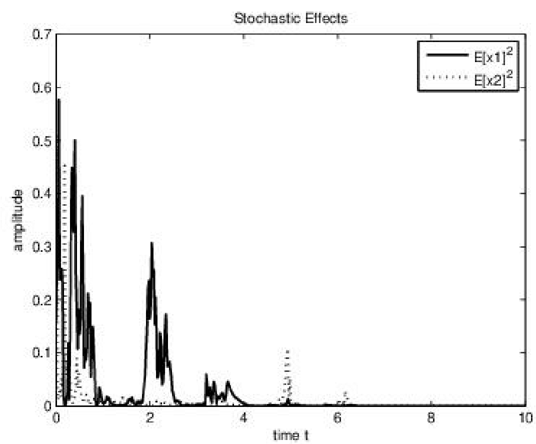

FIGURE 4. The state trajectories are converging to zero for $k = 2$ for Example 2

[7] Shao, H., Novel delay-dependent stability results for neural networks with time-varying delays, *Circuits Systems and Signal Processings*, 29 (2010), 637-647.

[8] Chen, W.H., Lu, X., Mean square exponential stability of uncertain stochastic delayed neural networks, *Physics Letters A,* 372 (2008), 1061-1069.

[9] Feng, W., Yang, S. X., Wu, H., On robust stability of uncertain stochastic neural networks with distributed and interval time-varying delays, *Chaos, Solitons and Fractals,* 42 (2009), 2095-2104.

[10] Ma, L., Da, F., Mean-square exponential stability of stochastic Hopfield neural networks with time-varying discrete and distributed delays, *Physics Letters A,* 373 (2009), 2154-2161.

[11] Lou, X., Cui, B., Delay-dependent stochastic stability of delayed Hopfield neural networks with Markovian jump parameters, *Journal of Mathematical Analysis and Applications,* 328 (2007), 316-326.

[12] Rakkiyappan, R., Balasubramaniam, P., Delay-dependent asymptotic stability for stochastic delayed recurrent neural networks with time varying delays, *Applied Mathematics and Computation,* 198 (2008), 526-533.

[13] Zhou, Q., Wan, L., Exponential stability of stochastic delayed Hopfield neural networks, *Applied Mathematics and Computation,* 199 (2008), 84-89.

[14] Sakthivel, R., Samidurai, R., Anthoni, S.M., Asymptotic stability of stochastic delayed recurrent neural networks with impulsive effects, *Journal of Optimization Thoery and Applications,* 147 (2010), 583-596.

[15] Sakthivel, R., Samidurai, R., Anthoni, S.M., New exponential stability criteria for stochastic BAM neural networks with impulses, *Physica Scripta,* 82 (2010), 045802.

[16] Sakthivel, R., Raja, R., Anthoni, S.M., Asymptotic stability of delayed stochastic genetic regulatory networks with impulses, *Physica Scripta,* 82 (2010), 055009.

[17] Zhang, Y. J., Yue, D., Tian, E. G., Robust delay-distribution-dependent stability of discrete-time stochastic neural networks with time-varying delay, *Neurocomputing,* 72 (2009), 1265-1273.

[18] Fu, J., Zhang, H. G., Ma, T., Delay-probablity-distribution-dependent robust stability analysis for stochastic neural networks with time-varying delays, *Progress Natural Science,* 19 (2009), 1333-1340.

[19] Takagi, T., Sugeno, M., Fuzzy identification of systems and its applications to modeling and control, *IEEE Transactions on System, Man, and Cybernetics,* 15 (1985), 116-132.

[20] Huang, H., Ho, D.W.C., Lam, J., Stochastic stability analysis of fuzzy Hopfield neural networks with time-varying delays, *IEEE Transactions on Circuits and Systems,* II 52 (2005) 251-255.

[21] Li, H., Chen, B., Lin, C., Zhou, Q., Mean square exponential stability of stochastic fuzzy Hopfield neural networks with discrete and distributed time-varying delays, *Neurocomputing,* 72 (2009) 2017-2023.

[22] Sheng, L., Gao, M., Yang, H., Delay-dependent robust stability for uncertain stochastic fuzzy Hopfield neural networks with time-varying delays, *Fuzzy Sets and Systems,* 160 (2009), 3503-3517.

[23] Rakkiyappan, R., Balasubramaniam, P., Delay-probability-distribution-dependent stability of uncertain stochastic genetic regulatory networks with mixed time-varying delays: An LMI approach, *Nonlinear Analysis: Hybrid Systems,* 4 (2010), 600-607.

[24] Gu, K., Integral inequality in the stability problem of time-delay Systems, Proceeding 39th IEEE CDC, Sydney, Philadelphia, 1994.

[25] Boyd, S., Ghaoui, L. E., Feron, E., Balakrishnan, V., Linear Matrix Inequalities in Systems and Control Theory, SIAM books, Philadelphia, 1994.

[26] Xie, L., Output feedback H_∞ control of systems with parameter uncertainty, *International Journal of Control,* 63(1996)741-750.

[27] Moon, Y.S., Park, P., Kwon, W.H., Lee, Y.S., Delay-dependent robust stabilization of uncertain state-delayed systems, *International Journal of Control,* 74 (2001), 1447-1455.

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POINTWISE BI-SLANT SUBMERSIONS FROM COSYMPLECTIC MANIFOLDS

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Abstract. We introduce pointwise bi-slant submersions from cosymplectic manifolds onto Riemannian manifolds as a generalization of anti-invariant, semi-invariant, semi-slant, hemi-slant, pointwise semi-slant, pointwise hemi-slant and pointwise slant Riemannian submersions. We give an example for pointwise bi-slant submersions and investigate integrability and totally geodesicness of the distributions which are mentioned in the definition of pointwise bi-slant submersions admitting vertical Reeb vector field. Also we obtain necessary and sufficient conditions for such submersions to be totally geodesic maps.

1. Introduction

The geometry of slant submanifolds was initiated by B.Y. Chen [9]. Later many geometers obtained some interesting results on this subject. As an extension of slant submanifolds, pointwise slant submanifolds were considered by F. Etayo [11] under the name of quasi-slant submanifolds. He showed that a complete totally geodesic quasi-slant submanifold of Kaehlerian manifold is a slant submanifold.

As a generalization of contact CR-manifolds, slant and semi-slant submanifolds, the geometry of bi-slant submanifolds in contact metric manifolds was studied by Carriazo [8]. A bi-slant submanifold of Kaehlerian manifold was defined by Uddin and et al. (see [27]). They investigated warped product bi-slant submanifold. Furthermore, Alqahtani and the other authors studied warped product bi-slant submanifolds of cosymplectic manifolds [4].

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The theory of submersions especially the theory of Riemannian submersions is one of the important research fields in Riemannian geometry. Riemannian submersions between Riemannian manifolds were introduced by O'Neill [18] and Gray [13]. Watson investigated the Riemannian submersions between almost Hermitian manifolds, (see [28]). Several types of Riemannian submersions have been studying in different kinds of structures, (see [1–3, 5, 10, 14–16, 19–25]).

In purpose of the present article is to investigate pointwise bi-slant submersions from cosymplectic manifolds onto Riemannian manifolds. In section 2, we review some basic properties about cosymplectic manifolds and Riemannian submersions. In section 3 we define pointwise bi-slant submersions from cosymplectic manifolds and study the geometry of leaves of distributions. Also, we obtain necessary and sufficient conditions for such submersions to be totally geodesic maps.

2. Preliminaries

In the section, we remember the basic concepts about cosymplectic manifolds and Riemannian submersions for later use.

2.1. **Cosymplectic manifolds.** Let M be $(2n+1)$-dimensional smooth manifold with an endomorphism ϕ, a vector field ξ and a 1-form η which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.$$

Then M is said to be an almost contact manifold. There always exist a compatible metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi)$$

for $X, Y \in \Gamma(TM)$. The condition for normality in terms of ϕ, ξ and η on M is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. The fundamental 2-form Φ of M is defined as $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be cosymplectic if it is normal and both $d\Phi = 0$ and $d\eta = 0$. Then considering the covariant derivative of ϕ, the structure equation of a cosymplectic manifold is characterized by the relation

$$(\nabla_X \phi) Y = 0 \ \text{ and } \ \nabla_X \xi = 0$$

for any $X, Y \in \Gamma(TM)$ [7, 17].

2.2. **Riemannian submersions.** A smooth map $\pi : M \to N$ between Riemannian manifolds M and N with dimension m and n, respectively, is called a Riemannian submersion if π_* is onto and satisfies [12]

i) π has maximal rank,

ii) π_* preserves the lengths of vectors normal to fibers.

For each $q \in N$, $\pi^{-1}(q)$ is a submanifold of M with dimension $m - n$. The submanifold $\pi^{-1}(q)$ are called fibers and a vector field X on M is called vertical (resp. horizontal) if it is always tangent (resp. orthogonal). If X is horizontal and π-related to a vector field X_* on N then X is called basic. The projection morphisms on the distributions $\ker \pi_*$ and $(\ker \pi_*)^\perp$ are denoted by $\mathcal{V}$ and $\mathcal{H}$, respectively.

The type of $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on M are given by

$$\mathcal{T}(X,Y) = \mathcal{T}_X Y = \mathcal{H}\nabla_{\mathcal{V}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{V}X}\mathcal{H}Y \tag{1}$$

$$\mathcal{A}(X,Y) = \mathcal{A}_X Y = \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y + \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y \tag{2}$$

for $X,Y \in \Gamma(TM)$ where ∇ denotes the Levi-Civita connection of (M,g). On the other hand for $U,V \in \Gamma(\ker \pi_*)$ and $X,Y \in \Gamma\left((\ker \pi_*)^\perp\right)$ the tensor fields satisfy the following equations

$$\mathcal{T}_U V = \mathcal{T}_V U \tag{3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y]. \tag{4}$$

Note that a Riemannian submersion $\pi : M \longrightarrow N$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. Considering the equations (1) and (2), one can write

$$\nabla_U V = \mathcal{T}_U V + \bar{\nabla}_U V \tag{5}$$

$$\nabla_U X = \mathcal{H}\nabla_U X + \mathcal{T}_U X \tag{6}$$

$$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U \tag{7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{8}$$

for $X,Y \in \Gamma\left((\ker \pi_*)^\perp\right)$ and $U,V \in \Gamma(\ker \pi_*)$, where $\bar{\nabla}_U V = \mathcal{V}\nabla_U V$. Moreover, if X is basic then $\mathcal{H}\nabla_U X = \mathcal{A}_X U$.

Lemma 1. *([18]) Let $\pi : M \longrightarrow N$ be a Riemannian submersion between Riemannian manifolds and suppose that X and Y are basic vector fields of M π-related to X_* and Y_* on N. Then*

i) *$\mathcal{H}[X,Y]$ is a basic vector field i.e. $\pi_*(\mathcal{H}[X,Y]) = [X_*, Y_*] \circ \pi$,*
ii) *$[U,X]$ is vertical for any vector field U of $(\ker \pi_*)$,*
iii) *$\mathcal{H}\nabla_X Y$ is the basic vector field i.e. $\pi_*(\mathcal{H}\nabla_X Y) = \overline{\nabla}_{X_*} Y_*$,*

where ∇ and $\overline{\nabla}$ are the Levi-Civita connection on M and N, respectively.

Let (M,g) and (N,g') be Riemannian manifolds and $\Psi : M \longrightarrow N$ is a smooth mapping between them. The second fundamental form of Ψ is given by

$$\nabla\Psi_*(X,Y) = \nabla^\Psi_X \Psi_*(Y) - \Psi_*(\nabla_X Y) \tag{9}$$

for $X,Y \in \Gamma(TM)$, where ∇^Ψ is the pullback connection. The smooth map Ψ is said to be harmonic if $trace\nabla\Psi_* = 0$ and ψ is called a totally geodesic map if $(\nabla\Psi_*)(X,Y) = 0$ for $X,Y \in \Gamma(TM)$ [6].

Remark 2. *Throughout this article, we consider that the characteristic vector field ξ is a vertical vector field.*

3. Pointwise Bi-Slant Submersions

In the present section of the paper we define pointwise bi-slant submersions from cosymplectic manifolds and obtain necessary and sufficient conditions for integrability and total geodesicness of the distributions.

Definition 3. *Let (M,ϕ,ξ,η,g) be a cosymplectic manifold and (N,g') a Riemannian manifold. A Riemannian submersion $\pi : M \longrightarrow N$ is called a pointwise bi-slant submersion if*

i) *for nonzero any $U \in \Gamma(D_1)_p$ and $p \in M$, the angle θ_1 between ϕU and the space $(D_1)_p$ is independent of the choice of the nonzero vector $U \in \Gamma(D_1)$,*

ii) *for nonzero any $V \in \Gamma(D_2)_q$ and $q \in M$, the angle θ_2 between ϕV and the space $(D_2)_q$ are independent of the choice of the nonzero vector $V \in \Gamma(D_2)$*

such that $\ker\pi_* = D_1 \oplus D_2 \oplus \xi$. Then the angle $\theta_{i,}$ is called the slant function of the pointwise bi-slant submersion. π is called proper if its slant functions satisfy $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

We can give the following example using cosymplectic structure (ϕ,ξ,η,g) as in Example 2.1 of [26].

Example 4. *Define $\pi : \mathbb{R}^9 \to \mathbb{R}^4$ as follows:*

$$\pi\left(x_1,...,x_8,z\right) = \left(x_1,(\cos\alpha)x_2 + (sin\alpha)x_4,(-\cos\beta)x_5 + (\sin\beta)x_7, x_6\right),$$

where $(x_1,...,x_8,z)$ are natural coordinates of $\mathbb{R}^9$. Then we obtain

$$D_1 = \{V_1 = \frac{\partial}{\partial x_3}, V_2 = \sin\beta\frac{\partial}{\partial x_5} + \cos\beta\frac{\partial}{\partial x_7}\} \text{ and}$$
$$D_2 = \{V_3 = \frac{\partial}{\partial x_8}, V_4 = \sin\alpha\frac{\partial}{\partial x_2} - \cos\alpha\frac{\partial}{\partial x_4}\}.$$

Thus π is a pointwise bi-slant submersion with slant functions $\theta_1 = \beta$ and $\theta_2 = \alpha$.

Suppose that π is a pointwise bi-slant submersion from a cosymplectic manifold (M,ϕ,ξ,η,g) onto a Riemannian manifold (N,g'). For $U \in \Gamma\left(\ker\pi_*\right)$, we have

$$U = PU + QU + \eta(U)\xi \tag{10}$$

where $PU \in \Gamma\left(D_1\right)$ and $QU \in \Gamma\left(D_2\right)$.
Also, for $U \in \Gamma\left(\ker\pi_*\right)$, we write

$$\phi U = \psi U + \omega U \tag{11}$$

where $\psi U \in \Gamma\left(\ker\pi_*\right)$ and $\omega U \in \Gamma\left(\ker\pi_*\right)^\perp$.
For $X \in \Gamma\left(\ker\pi_*\right)^\perp$, we have

$$\phi X = \mathcal{B}X + \mathcal{C}X \tag{12}$$

where $\mathcal{B}X \in \Gamma\left(\ker\pi_*\right)$ and $\mathcal{C}X \in \Gamma\left(\ker\pi_*\right)^\perp$.
The horizontal distribution $\left(\ker\pi_*\right)^\perp$ is decomposed as

$$\left(\ker\pi_*\right)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mu$$

where μ is the complementary distribution to $\omega D_1 \oplus \omega D_2$ in $(\ker \pi_*)^{\perp}$.
By using (3.2) and (3.3) we obtain

$$\psi D_1 = D_1, \quad \psi D_2 = D_2, \quad \mathcal{B}\omega D_1 = D_1, \quad \mathcal{B}\omega D_2 = D_2.$$

Considering Definition 3 we can give the following result.

Theorem 5. *Suppose that π is a Riemannian submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g'). Then π is a pointwise bi-slant submersion if and only if there exist bi-slant function θ_i defined on D_i such that*

$$\psi^2 = -\cos^2 \theta_i \left(I - \eta \otimes \xi\right), \ i = 1, 2.$$

Proof. The proof is similar to the proof of Theorem 2 of [10], so we omit it. □

Theorem 6. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g') with bi-slant functions θ_1, θ_2. Then*

i) the distribution D_1 is integrable if and only if

$$g\left(\mathcal{T}_U \omega\psi V - \mathcal{T}_V \omega\psi U, W\right) = g\left(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \psi W\right) + g\left(\mathcal{H}\nabla_U \omega V - \mathcal{H}\nabla_V \omega U, \omega W\right)$$

ii) the distribution D_2 is integrable if and only if

$$g\left(\mathcal{T}_W \omega\psi Z - \mathcal{T}_Z \omega\psi W, U\right) = g\left(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W, \psi U\right) + g\left(\mathcal{H}\nabla_W \omega Z - \mathcal{H}\nabla_Z \omega W, \omega U\right)$$

where $U, V \in \Gamma(D_1)$, $W, Z \in \Gamma(D_2)$.

Proof. From $U, V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$ we have

$$\begin{aligned} g\left([U, V], W\right) =& g\left(\nabla_U \phi V, \phi W\right) - g\left(\nabla_V \phi U, \phi W\right) \\ =& g\left(\nabla_U \psi V, \phi W\right) - g\left(\nabla_U \omega V, \phi W\right) + g\left(\nabla_V \psi U, \phi W\right) \\ & - g\left(\nabla_V \omega U, \phi W\right). \end{aligned}$$

Considering Theorem 5 we arrive

$$\begin{aligned} g\left([U, V], W\right) =& \cos^2 \theta_1 g\left(\nabla_U V, W\right) - g\left(\nabla_U \omega\psi V, \phi W\right) - \cos^2 \theta_1 g\left(\nabla_V U, W\right) \\ & + g\left(\nabla_V \omega\psi U, \phi W\right) + g\left(\nabla_U \omega V, \phi W\right) - g\left(\nabla_V \omega U, \phi W\right). \end{aligned}$$

Thus we get

$$\begin{aligned} \sin^2 \theta_1 g\left([U, V], W\right) =& - g\left(\nabla_U \omega\psi V, W\right) + g\left(\nabla_V \omega\psi U, W\right) + g\left(\nabla_U \omega V, \phi W\right) \\ & - g\left(\nabla_V \omega U, \phi W\right). \end{aligned}$$

By using the equation (6) we obtain

$$\begin{aligned} \sin^2 \theta_1 g\left([U, V], W\right) =& - g\left(\mathcal{T}_U \omega\psi V, W\right) + g\left(\mathcal{T}_V \omega\psi U, W\right) + g\left(\mathcal{T}_U \omega V, \psi W\right) \\ & + g\left(\mathcal{H}\nabla_U \omega V, \omega W\right) - g\left(\mathcal{T}_V \omega U, \psi W\right) - g\left(\mathcal{H}\nabla_V \omega U, \omega W\right). \end{aligned}$$

This completes the proof. □

Theorem 7. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M,ϕ,ξ,η,g) onto a Riemannian manifold (N,g') with bi-slant functions θ_1,θ_2. Then the distribution D_1 defines a totally geodesic foliation if and only if*

$$\begin{aligned}\sin^2\theta_1 g\left([U,X],V\right) = &\sin 2\theta_1 X\left[\theta_1\right] g\left(\phi U,\phi V\right) - g\left(\mathcal{A}_X\omega\psi U,V\right)\\ &+ g\left(\mathcal{A}_X\omega U,\psi V\right) + g\left(\mathcal{H}\nabla_X\omega U,\omega V\right)\end{aligned}$$

and

$$g\left(\mathcal{H}\nabla_U\omega V,\omega W\right) = g\left(\mathcal{T}_U\omega\psi V,W\right) - g\left(\mathcal{T}_U\omega V,\psi W\right)$$

where $U,V\in\Gamma(D_1)$, $W\in\Gamma(D_2)$ and $X\in\Gamma\left((\ker\pi_*)^\perp\right)$.

Proof. For any $U,V\in D_1$ and $X\in\Gamma\left((\ker\pi_*)^\perp\right)$ we write

$$\begin{aligned}g\left(\nabla_U V,X\right) &= -\,g\left([U,X],V\right) - g\left(\nabla_X U,V\right)\\ &= -\,g\left([U,X],V\right) + g\left(\nabla_X\phi\psi U,V\right) - g\left(\nabla_X\omega U,\phi V\right).\end{aligned}$$

From Theorem 5, the above equation is obtained as follows

$$\begin{aligned}g\left(\nabla_U V,X\right) = &-\,g\left([U,X],V\right) + \sin 2\theta_1 X[\theta_1] g\left(\phi U,\phi V\right) - \cos^2\theta_1 g\left(\nabla_X U,V\right)\\ &+ g\left(\nabla_X\omega\psi U,V\right) - g\left(\nabla_X\omega U,\phi V\right)\end{aligned}$$

Using the equation (8) we have

$$\begin{aligned}\sin^2\theta_1 g\left(\nabla_U V,X\right) = &-\,sin^2\theta_1 g\left([U,X],V\right) + \sin 2\theta_1 X[\theta_1] g\left(\phi U,\phi V\right)\\ &+ g\left(\mathcal{A}_X\omega\psi U,V\right) - g\left(\mathcal{A}_X\omega U,\psi V\right) - g\left(\mathcal{H}\nabla_X\omega U,\omega V\right)\end{aligned}$$

Similarly for $W\in\Gamma(D_2)$ we have

$$g\left(\nabla_U V,W\right) = -g\left(\nabla_U\psi^2 V,W\right) - g\left(\nabla_U\omega\psi V,W\right) + g\left(\nabla_U\omega V,\phi W\right).$$

Thus we write

$$\begin{aligned}\sin^2\theta_1 g\left(\nabla_U V,W\right) = &-\,g\left(\mathcal{T}\omega\psi V,W\right) + g\left(\mathcal{T}_U\omega V,\psi W\right)\\ &+ g\left(\mathcal{H}\nabla_U\omega V,\omega W\right).\end{aligned}$$

This completes the proof. □

Theorem 8. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M,ϕ,ξ,η,g) onto a Riemannian manifold (N,g') with bi-slant functions θ_1,θ_2. Then the distribution D_2 defines a totally geodesic foliation if and only if*

$$\begin{aligned}\sin^2\theta_2 g\left([W,X],Z\right) = &\sin 2\theta_2 X\left[\theta_2\right] g\left(\phi W,\phi Z\right) - g\left(\mathcal{A}_X\omega\psi W,Z\right)\\ &+ g\left(\mathcal{A}_X\omega W,\psi Z\right) + g\left(\mathcal{H}\nabla_X\omega W,\omega Z\right)\end{aligned}$$

and

$$g\left(\mathcal{H}\nabla_W\omega Z,\omega U\right) = g\left(\mathcal{T}_W\omega\psi Z,U\right) - g\left(\mathcal{T}_W\omega Z,\psi U\right)$$

where $U\in D_1$, $W,Z\in D_2$ and $X\in\Gamma\left((\ker\pi_*)^\perp\right)$.

Proof. The proof of this theorem is similar to the proof of Theorem 5. □

Theorem 9. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g') with bi-slant functions θ_1, θ_2. Then the distribution $(\ker \pi_*)^{\perp}$ defines a totally geodesic foliation if and only if*

$$\sin^2 \theta_1 g(\nabla_X Y, U) = \left(\cos^2 \theta_2 - \cos^2 \theta_1\right) g(A_X Y, QU) - g(\mathcal{H}\nabla_X Y, \omega\phi U) + g(\omega\mathcal{A}_X Y, \omega U) + g(C\mathcal{H}\nabla_X Y, \omega U).$$

where $X, Y \in \Gamma(\ker \pi_*)^{\perp}$ and $U \in \Gamma(\ker \pi_*)$.

Proof. For $X, Y \in \Gamma(\ker \pi_*)^{\perp}$ and $U \in \Gamma(\ker \pi_*)$ we write

$$g(\nabla_X Y, U) = g(\phi\nabla_X Y, \psi PU) + g(\phi\nabla_X Y, \psi QU) + g(\phi\nabla_X Y, \omega U)$$

From Theorem 5 we have

$$g(\nabla_X Y, U) = -g\left(\nabla_X Y, \psi^2 PU\right) - g\left(\nabla_X Y, \psi^2 QU\right) - g(\nabla_X Y, \omega\psi U) + g(\phi\nabla_X Y, \omega U)$$

By using the equation (8) we arrive

$$\sin^2 \theta_1 g(\nabla_X Y, U) = \left(\cos^2 \theta_2 - \cos^2 \theta_1\right) g(\mathcal{A}_X Y, QU) - g(\mathcal{H}\nabla_X Y, \omega\psi U) + g(\omega\mathcal{A}_X Y, \omega U) + g(C\mathcal{H}\nabla_X Y, \omega U).$$

Thus we have the desired equation. □

Theorem 10. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g') with bi-slant functions θ_1, θ_2. Then the distribution $(\ker \pi_*)$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned}\sin^2 \theta_1 g([U, X], V) = {} & \left(\cos^2 \theta_1 - \cos^2 \theta_2\right) g(\phi\nabla_X QU, \phi V) + \sin 2\theta_1 X[\theta_1]\, g(\phi U, \phi V) \\ & - (\sin 2\theta_1 X[\theta_1] - \sin 2\theta_2 X[\theta_2])\, g(\phi QU, QV) + g(\mathcal{A}_X \omega\psi U, V) \\ & - g(\mathcal{A}_X \omega U, \psi V) - g(\mathcal{H}\nabla_X \omega U, \omega V) - \sin^2 \theta_1 \eta(\nabla_X U)\eta(V)\end{aligned}$$

where $X \in \Gamma(\ker \pi_*)^{\perp}$ and $U, V \in \Gamma(\ker \pi_*)$.

Proof. Given $X \in \Gamma(\ker \pi_*)^{\perp}$ and $U, V \in \Gamma(\ker \pi_*)$. Then we derive

$$\begin{aligned}g(\nabla_U V, X) &= -g([U, X], V) - g(\nabla_X U, V) \\ &= -g([U, X], V) - g(\nabla_X \phi U, \phi V) - \eta(\nabla_X U)\eta(V)\end{aligned}$$

By using the equations (10) and (11), we have

$$g(\nabla_U V, X) = -g([U, X], V) - g(\nabla_X \psi PU, \phi V) - g(\nabla_X \psi QU, \phi V) - g(\nabla_X \omega U, \phi V) - \eta(\nabla_X U)\eta(V).$$

Thus, we obtain

$$g(\nabla_U V, X) = -g([U, X], V) + g\left(\nabla_X \psi^2 PU, V\right) + g\left(\nabla_X \psi^2 QU, V\right)$$

$$+ g(\nabla_X \omega\psi U, V) - g(\nabla_X \omega U, \phi V) - \eta(\nabla_X U)\eta(V)$$

Using Theorem 5 we arrive

$$\begin{aligned} g(\nabla_U V, X) = &- g([U, X], V) + \sin 2\theta_1 X[\theta_1] g(PU, V) \\ &- \sin 2\theta_1 X[\theta_1]\eta(PU)\eta(V) + \sin 2\theta_2 X[\theta_2] g(QU, V) \\ &- \sin 2\theta_2 X[\theta_2]\eta(QU)\eta(V) - \cos^2\theta_1 g(\nabla_X PU, V) \\ &+ \cos^2\theta_1 \eta(\nabla_X PU)\eta(V) - \cos^2\theta_2 g(\phi\nabla_X QU, \phi V) \\ &+ g(\nabla_X \omega\psi U, V) - g(\nabla_X \omega U, \phi V) - \eta(\nabla_X U)\eta(V) \end{aligned}$$

From the equation (8) we obtain

$$\begin{aligned} \sin^2\theta_1 g(\nabla_U V, X) = &- \sin^2\theta_1 g([U, X], V) + \sin 2\theta_1 X[\theta_1] g(\phi U, \phi V) \\ &+ (\sin 2\theta_2 X[\theta_2] - \sin 2\theta_1 X[\theta_1]) g(\phi QU, \phi V) \\ &+ \left(\cos^2\theta_1 - \cos^2\theta_2\right) g(\phi\nabla_X QU, \phi V) + g(\mathcal{A}_X \omega\psi U, V) \\ &- g(\mathcal{A}_X \omega U, \psi V) - g(\mathcal{H}\nabla_X \omega U, \omega V) - \sin^2\theta_1 \eta(\nabla_X U)\eta(V) \end{aligned}$$

Using above equation the proof is completed. □

Theorem 11. *Suppose that π be a pointwise bi-slant submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g') with bi-slant functions θ_1, θ_2. Then π is totally geodesic if and only if*

$$-\cos^2\theta_1 \mathcal{T}_U PV - \cos^2\theta_2 \mathcal{T}_U QV + \mathcal{H}\nabla_U \omega\psi V + C\mathcal{H}\nabla_U \omega V + \omega\mathcal{T}_U \omega V = 0$$

and

$$-\cos^2\theta_1 \mathcal{A}_X PU - \cos^2\theta_2 \mathcal{A}_X QU + \mathcal{H}\nabla_X \omega\psi U + C\mathcal{H}\nabla_X \omega U + \omega\mathcal{A}_X \omega U = 0$$

where $X \in \Gamma(\ker\pi_*)^\perp$ and $U, V \in (\ker\pi_*)$.

Proof. Since π is a Riemannian submersion for $X, Y \in \Gamma(\ker\pi_*)^\perp$ we have

$$(\nabla\pi_*)(X, Y) = 0.$$

Thus for $U, V \in \Gamma(\ker\pi_*)$ it is enough to show that $(\nabla\pi_*)(U, V) = 0$ and $(\nabla\pi_*)(X, U) = 0$. Then we can write

$$(\nabla\pi_*)(U, V) = -\pi_*(\nabla_U V).$$

Thus from the equation (9), we obtain

$$\begin{aligned} (\nabla\pi_*)(U, V) = &- \pi_*(\nabla_U V) = \pi_*(\phi\nabla_U \psi V + \phi\nabla_U \omega V) \\ =&\pi_*\left(\nabla_U \psi^2 PV + \nabla_U \psi^2 QV + +\nabla_U \omega\psi V + \phi\nabla_U \omega V\right). \end{aligned}$$

Considering Theorem 5 we find

$$(\nabla\pi_*)(U, V) = \pi_*\left(-\cos^2\theta_1 \nabla_U PV - \cos^2\theta_2 \nabla_U QV + \nabla_U \omega\psi V + \phi\nabla_U \omega V\right).$$

Therefore we obtain the first equation of Theorem 11.
On the other hand, we can write

$$(\nabla \pi_*)(X,U) = -\pi_*(\nabla_X U).$$

Using the equation (7) and (8), we arrive

$$(\nabla \pi_*)(X,U) = \pi_*\left(-\cos^2\theta_1 \mathcal{A}_X PU - \cos^2\theta_2 \mathcal{A}_X QU + \mathcal{H}\nabla_X \omega\psi U + C\mathcal{H}\nabla_X \omega U\right).$$

This concludes the proof. □

References

[1] Akyol, M.A.,Gündüzalp, Y., Hemi-slant submersions from almost product Riemannian manifolds, *Gulf Journal of Mathematics,* 4(3) (2016), 15-27.

[2] Akyol, M.A., Sarı, R., On semi-slant $\xi^{\perp}$-Riemannian submersions, *Mediterr. J. Math.,* 14 (2017), 234(20 pp).

[3] Akyol, M.A.Sahin, B., Conformal slant submersions, *Hacet. J. Math. Stat.,* 48 (2019), 28-44.

[4] Alqahtani, L.S., Stankovic, M.S., Uddin, S., Warped Product Bi-slant Submanifolds of Cosymplectic Manifolds, *Filomat,* 31(16) (2017), 5065-5071.

[5] Aykurt Sepet, S., Ergüt,M., Pointwise slant submersions from cosymplectic manifolds, *Turk. J. Math.*, 40 (2016), 582-593.

[6] Baird, P., Wood, J.C., Harmonic morphisms between Riemannian manifolds, London Mathematical Society Monographs, Oxford University Press, Oxford, 2003.

[7] Blair, D.E., Contact manifolds in Riemannian geometry, Lectures Notes in Mathematics 509, Springer-Verlag, Berlin, 146p., 1976.

[8] Carriazo, A., Bi-slant Immersions, *Proceeding of the ICRAMS*, (2000), 88-97.

[9] Chen, B.Y., Slant immersions, *Bull. Austral. Math. Soc.,* 41(1990), 135-147.

[10] Erken, İ.K., Murathan, C., On Slant Riemannian Submersions for Cosymplectic Manifolds, *Bull. Korean Math. Soc.,* (2014), 51, 1749-1771.

[11] Etayo, F., On quasi-slant submanifolds of an almost Hermitian manifold, *Publ. Math. Debrecen*, 51 (1998), 217-223.

[12] Falcitelli, M., Ianus, S., Pastore, A.M., Riemannian Submersions and Related Topics, World Scientific, River Edge, NJ, 2004.

[13] Gray, A., Pseudo-Riemannian almost product manifolds and submersions, *J. Math. Mech.,* 16 (1967), 715-737.

[14] Gündüzalp, Y., Semi-slant submersions from almost product Riemannian manifolds, *Demonstratio Mathematica*, 49(3) (2016), 345-356.

[15] Gündüzalp, Y., Akyol, M.A., Conformal slant submersions from cosymplectic manifolds, *Turk. J. Math.*, 42 (2018), 2672-2689.

[16] Lee, J.W., Şahin, B., Pointwise slant submersions, *Bull. Korean Math. Soc.,* 51(4) (2014), 1115-1126.

[17] Ludden, G.D., Submanifolds of cosymplectic manifolds, *J. Differential Geom.,* 4 (1970), 237-244.

[18] O'Neill, B., The fundamental equations of a submersion, *Mich. Math. J.*, 13 (1966), 458-469.

[19] Sayar, C., Özdemir, F., Taştan, H.M., Pointwise semi-slant submersions whose total manifolds are locally product Riemannian manifolds, *International Journal of Maps in Mathematics*, 1 (2018), 91-115.

[20] Sayar, C., Taştan, H.M., Özdemir, F., Tripathi, M.M., Generic submersions from Kaehlerian Manifolds, *Bull. Malays. Math. Sci. Soc.*, https://doi.org/10.1007/s40840-018-00716-2.

[21] Şahin, B., Anti-invariant Riemannian submersions from almost Hermitian manifolds, *Central European J. Math.*, 8(3) (2010), 437-447.

[22] Şahin, B., Semi-invariant Riemannian submersions from almost Hermitian manifolds, *Can. Math. Bull.,* 56 (2011), 173-183.

[23] Şahin, B., Slant submersions from almost Hermitian manifolds, *Bull. Math. Soc. Sci. Math. Roumanie*, 54(102) (2011), 93-105.

[24] Taştan, H.M., Şahin, B., Yanan, Ş., Hemi-slant submersions, *Mediterr. J. Math.,* 13 (2016), 2171-2184.

[25] Tastan, H.M., Lagrangian submersions from normal almost contact manifolds, *Filomat,* 31(12) (2017), 3885-3895.

[26] Tastan, H.M., Gerdan Aydın, S., Clairaut anti-invariant submersions from cosymplectic manifolds, *Honam Mathematical J.*, 41(4) (2019), 707-724.

[27] Uddin, S., Chen, B.Y., Al-Solamy, F.R., Warped Product Bi-slant Immersions in Kaehler Manifolds , *Mediterr. J. Math,* 14(95) (2017).

[28] Watson, B., Almost Hermitian submersions, *J. Differential Geometry,* 11(1) (1976), 147-165.

https://communications.science.ankara.edu.tr

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ON A JANOWSKI FORMULA BASED ON A GENERALIZED DIFFERENTIAL OPERATOR

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Abstract. The central purpose of the current paper is to consider a set of beneficial possessions including inequalities for a generalized subclass of Janowski functions (analytic functions) which are formulated here by revenues of a generalized Sàlàgean's differential operator. Numerous recognized consequences of the outcomes are also indicated. We present some results involving the subordination and superordination inequalities. Moreover, growth inequalities are indicated in the sequel. Real and special cases are suggested containing the differential operator.

1. Introduction

The differential operators regularly characterize physical capacities, the derivatives signify their proportions of modification, and the operator expresses a relationship between the two. Because such relatives are exceptionally common, differential operators play a flat role in many categories involving physics, economics, engineering and biology. In this direction, Ibrahim and Darus introduced the following mixed operator [1]: let Λ be the class of normalized function formulated by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U = \{z : |z| < 1\}. \tag{1}$$

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then, we have

$$\begin{aligned}
&D_\kappa^0 f(z) = f(z)\\
&D_\kappa^1 f(z) = zf'(z) + \frac{\kappa}{2}\left(f(z) - f(-z) - 2z\right), \quad \kappa \in \mathbb{R}\\
&\vdots\\
&D_\kappa^m f(z) = D_\kappa(D_\kappa^{m-1} f(z))\\
&\qquad = z + \sum_{n=2}^{\infty}[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m\, a_n z^n.
\end{aligned} \tag{2}$$

Obviously, if we let $\kappa = 0$, we get the Sàlàgean's differential operator [2]. We title D_κ^m the Sàlàgean-difference operator. In addition, D_κ^m is a modified Dunkl operator of complex variables [3] and for recent work [4]. Dunkl operator characterizes a major generalization of partial derivatives and attains the commutative law in $\mathbb{R}^n$. In geometry, it acquires the reflexive relation, which is plotting the space into itself as a set of fixed points.

The Hadamard product or convolution of two power series is denoted by $(*)$ achieving

$$\begin{aligned}
f(z) * h(z) &= \Big(z + \sum_{n=2}^{\infty} a_n z^n\Big) * \Big(z + \sum_{n=2}^{\infty} \eta_n z^n\Big)\\
&= z + \sum_{n=2}^{\infty} a_n \eta_n z^n.
\end{aligned} \tag{3}$$

Thus, we have

$$D_\kappa^m f(z) = \mathfrak{D}(z) * f(z),$$

where

$$\begin{aligned}
\mathfrak{D}(z) : &= z + \sum_{n=2}^{\infty}[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m\, z^n\\
&:= z + \sum_{n=2}^{\infty} \mathfrak{N}^m\, z^n
\end{aligned}$$

Recall that $f \prec g$ then there exists a Schwarz function $\omega \in U$ such that $\omega(0) = 0, |\omega(z)| < 1$, $z \in U$ satisfying $f(z) = g(\omega(z))$ for all $z \in U$ (see [5]). The inequality $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Furthermore, let $\mathbb{J}(A, B)$ denote the family of all functions φ that are analytic in the open unit disk U with $\varphi(0) = 1$ and achieve

$$\varphi(z) \prec \frac{1 + Az}{1 + Bz}, \; -1 \le B < A \le 1.$$

Note that the function $\dfrac{1+Az}{1+Bz}$ is univalent in the open unit disk U.

Recently, Arif et all. [6] introduced anew class of analytic functions along with the concepts of Janowski functions as follows:

Definition 1.1. *If $f \in \Lambda$, then $f \in \mathbb{J}^b(A,B,j)$ if and only if*

$$1+\frac{1}{b}\Big(\frac{2D_0^{j+1}f(z)}{D_0^j f(z)-D_0^j f(-z)}\Big) \prec \frac{1+Az}{1+Bz},$$

$$\Big(z\in U,\ -1\le B<A\le 1,\ j=1,2,...,\ b\in\mathbb{C}\setminus\{0\}\Big),$$

where $D_0^{j+1}f(z)$ is the Sàlàgean's differential operator.

In our study, we shall extend the above class as follows:
If $f \in \Lambda$, then $f \in \mathbb{J}^b_\kappa(A,B,j)$ if and only if

$$1+\frac{1}{b}\Big(\frac{2D_\kappa^{j+1}f(z)}{D_\kappa^j f(z)-D_\kappa^j f(-z)}\Big) \prec \frac{1+Az}{1+Bz},$$

$$\Big(z\in U,\ -1\le B<A\le 1,\ j=1,2,...,\ b\in\mathbb{C}\setminus\{0\},\ \kappa\ge 0\Big),$$

1.1. **Special cases.**

- $\kappa = 0 \Longrightarrow$ [6];
- $\kappa = 0, B = 0 \Longrightarrow$ [7];
- $\kappa = 0, A = 1, B = -1, b = 2 \Longrightarrow$ [8].

Lemma 1.2. *If $P \in \mathbb{J}(A,B)$ then its coefficients satisfy*

$$|p_n| \le (A-B), \quad \forall n \ge 1,$$

where

$$P(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ..., \quad z \in U.$$

2. Results

Our results based on two expressions involving the differential operator D_κ^m.

2.1. **Special class of the expression $D_\kappa^{j+1}/D_\kappa^j$.** We have our first result as follows:

Theorem 2.1. *If $f \in \mathbb{J}^b_\kappa(A,B,j)$ then the odd function*

$$\mathfrak{O}(z) = \frac{1}{2}[f(z) - f(-z)], \quad z \in U$$

achieves the following inequality

$$1+\frac{1}{b}\Big(\frac{D_\kappa^{j+1}\mathfrak{O}(z)}{D_\kappa^j\mathfrak{O}(z)}-1\Big) \prec \frac{1+Az}{1+Bz},$$

$$\Big(z \in U,\ -1 \le B < A \le 1,\ j = 1, 2, ...,\ b \in \mathbb{C} \setminus \{0\},\ \kappa \ge 0\Big),$$

Proof. Since $f \in \mathbb{J}_\kappa^b(A, B, j)$ then there is a function $P \in \mathbb{J}(A, B)$ such that

$$b(P(z) - 1) = \Big(\frac{2D_\kappa^{j+1} f(z)}{D_\kappa^j f(z) - D_\kappa^j f(-z)}\Big)$$

and

$$b(P(-z) - 1) = \Big(\frac{-2D_\kappa^{j+1} f(-z)}{D_\kappa^j f(z) - D_\kappa^j f(-z)}\Big).$$

This implies that

$$1 + \frac{1}{b}\Big(\frac{D_\kappa^{j+1} \mathfrak{O}(z)}{D_\kappa^j \mathfrak{O}(z)} - 1\Big) = \frac{P(z) + P(-z)}{2}.$$

Also, since

$$P(z) \prec \frac{1 + Az}{1 + Bz}$$

where $\dfrac{1 + Az}{1 + Bz}$ is univalent then by the definition of the subordination, we obtain

$$1 + \frac{1}{b}\Big(\frac{D_\kappa^{j+1} \mathfrak{O}(z)}{D_\kappa^j \mathfrak{O}(z)} - 1\Big) \prec \frac{1 + Az}{1 + Bz}.$$

□

Note that

- When $\kappa = 0$, we get the following result, which can be found in [6]

$$1 + \frac{1}{b}\Big(\frac{D_0^{j+1} \mathfrak{O}(z)}{D_0^j \mathfrak{O}(z)} - 1\Big) \prec \frac{1 + Az}{1 + Bz}.$$

- When $\kappa = 0, b = 1$, we have a result given in [10]

$$\Big(\frac{D_0^{j+1} f(z)}{D_0^j f(z)}\Big) \prec \frac{1 + Az}{1 + Bz}.$$

- When $\kappa = 0, b = 1, A = 1 - 2\alpha, -1$ we attain a result given in [10]

$$\Big(\frac{D_0^{j+1} f(z)}{D_0^j f(z)}\Big) \prec \frac{1 + (1 - 2\alpha)z}{1 - z}.$$

Corollary 2.2. *If $f \in \mathbb{J}_\kappa^b(A, B, j)$ then the odd function*

$$\mathfrak{O}(z) = \frac{1}{2}[f(z) - f(-z)], \quad z \in U$$

achieves

$$\Re\Big(\frac{z\mathfrak{O}(z)'}{\mathfrak{O}(z)}\Big) \ge \frac{1 - r^2}{1 + r^2}, \quad |z| = r < 1.$$

Proof. In view of Theorem 2.1, the function $\mathfrak{O}(z)$ is starlike in the open unit disk. The subordination concept implies that

$$\frac{z\mathfrak{O}(z)'}{\mathfrak{O}(z)} \prec \frac{1-z^2}{1+z^2}$$

that is, there exists a Schwarz function $\wp \in U, |\wp(z)| \leq |z| < 1, \wp(0) = 0$ such that

$$\Phi(z) := \frac{z\mathfrak{O}(z)'}{\mathfrak{O}(z)} \prec \frac{1-\wp(z)^2}{1+\wp(z)^2}$$

which yields that there is $\xi, |\xi| = r < 1$ such that

$$\wp^2(\xi) = \frac{1-\Phi(\xi)}{1+\Phi(\xi)}, \quad \xi \in U.$$

A calculation gives that

$$\Big|\frac{1-\Phi(\xi)}{1+\Phi(\xi)}\Big| = |\wp(\xi)|^2 \leq |\xi|^2.$$

Hence, we have the following conclusion

$$\Big|\Phi(\xi) - \frac{1+|\xi|^4}{1-|\xi|^4}\Big|^2 \leq \frac{4|\xi|^4}{(1-|\xi|^4)^2}$$

or

$$\Big|\Phi(z) - \frac{1+|\xi|^4}{1-|\xi|^4}\Big| \leq \frac{2|\xi|^2}{(1-|\xi|^4)}.$$

This implies that

$$\Re(\Phi(z)) \geq \frac{1-r^2}{1+r^2}, \quad |\xi| = r < 1,$$

which completes the proof. □

Theorem 2.3. *If $f \in \mathbb{J}^b_\kappa(A,B,j)$ then*

$$\Re\Big(D^{j+1}_\kappa f(z)\Big) \geq \Re\Big(\frac{z}{2}(1+b(P(z)-1))e^{\psi(z)}\Big),$$

where $P(z) \in \mathbb{J}(A,B)$ and

$$\psi(z) := \frac{b}{2}\int_0^z \frac{(P(\tau)+P(-\tau)-2)}{\tau}d\tau.$$

Proof. Let $f \in \mathbb{J}^b_\kappa(A,B,j)$. It has been shown in [6], Theorem 5 that

$$D^{j+1}_0 f(z) = \frac{z}{2}(1+b(P(z)-1))e^{\psi(z)}.$$

But

$$\Re\Big(D^{j+1}_\kappa f(z)\Big) \geq \Re\Big(D^{j+1}_0 f(z)\Big), \quad \kappa \geq 0,$$

it follows that

$$\Re\Big(D^{j+1}_\kappa f(z)\Big) \geq \Re\Big(\frac{z}{2}(1+b(P(z)-1))e^{\psi(z)}\Big).$$

This completes the proof. □

Theorem 2.4. *If* $f \in \mathbb{J}^b_\kappa(A,B,j)$ *then*

$$\frac{(1-|b|Ar)r+(1-|b|Br^2)}{(1+r^2)(1-Br)} \le |D^{j+1}_\kappa f(z)| \le \frac{\big((1+|b|Ar)r+(1-|b|Br^2)\big)}{(1-r^2)(1+Br)}.$$

Proof. Let $f \in \mathbb{J}^b_\kappa(A,B,j)$. This implies the following equality

$$|D^{j+1}_\kappa f(z)| = |\eta(z)|\Big|1+\frac{b(A-B)\omega(z)}{1+B\omega(z)}\Big|, \quad |\omega(z)|<|z|=r<1, \tag{4}$$

where

$$\eta(z) = \frac{1}{2}[D^j_\kappa f(z) - D^j_\kappa f(-z)].$$

The function η is univalent in the open unit disk U thus in view of the Growth Theorem, we have

$$\frac{r}{(1+r)^2} \le |\eta(z)| \le \frac{r}{(1-r)^2}, \quad |z|=r<1.$$

Moreover, by taking $|\omega(z)|<|z|=r$, a computation gives the inequality

$$\begin{aligned}\Big|1+\frac{b(A-B)\omega(z)}{1+B\omega(z)}\Big| &= \Big|\frac{(1+B\omega(z))+b(A-B)\omega(z)}{1+B\omega(z)}\Big| \\ &\le \frac{(1-|b|Ar)+(1-|b|)Br}{1-Br)} \\ &\le \frac{(1+|b|Ar)+(1-|b|)Br}{1+Br)}.\end{aligned}$$

By employing the last two inequalities in (4), we have the desire result. □

Theorem 2.5. *For* $f \in \Lambda$, *define the functional*

$$\Psi(z) = \frac{D^j_\kappa f(z)}{z}, \quad z \in U\setminus\{0\}$$

such that $\Re(\Psi(z))>0$. *Then*

$$\Re\Big(\frac{z\Psi(z)'}{\Psi(z)}\Big) \le \frac{2r}{1-r^2}, \quad |z|=r<1.$$

Proof. It is clear that $\Psi(0)=1$. According to the condition of the theorem, there exists a Schwarz function $\wp \in U, |\wp(z)| \le |z| < 1, \wp(0)=0$ such that

$$\Psi(z) = \frac{1+\wp(z)}{1-\wp(z)}.$$

This gives the equality

$$\frac{z\Psi(z)'}{\Psi(z)} = \frac{2z\wp'(z)}{1-\wp(z)^2}.$$

Hence, by utilizing the Schwarz-Pick Theorem, we attain

$$|\frac{z\Psi(z)'}{\Psi(z)}| \leq \frac{2|z|\,|\wp'(z)|}{1-|\wp(z)|^2} \leq \frac{2r}{1-|\wp(z)|^2}.\frac{1-|\wp(z)|^2}{1-r^2} = \frac{2r}{1-r^2}.$$

□

2.2. **Special class of the expression** $z(D^m_\kappa f(z))'^m_\kappa f(z)$**.** A function $f \in \Lambda$ is said to be in the class $\mathbb{S}^m_\kappa(h)$ if and only if the expression $z(D^m_\kappa f(z))'^m_\kappa f(z)$ takes all values in the conic domain $\Omega := h(U)$, where $h(z)$ is convex univalent then, we can describe the class as follows:

$$\Big(\frac{z(D^m_\kappa f(z))'}{D^m_\kappa f(z)}\Big) \prec h(z), \quad z \in U. \tag{5}$$

Next result shows the upper bound of the operator $D^m_\kappa f(z)$, when $f \in \mathbb{S}^m_\kappa(h)$ and the upper and lower bound of the expression $D^m_\kappa f(z)/z$.

Theorem 2.6. *Let $f \in \mathbb{S}^m_\kappa(h)$, where $h(z)$ is convex univalent function in U. Then*

$$D^m_\kappa f(z) \prec z\exp\Big(\int_0^z \frac{h(\omega(\xi))-1}{\xi}d\xi\Big),$$

where $\omega(z)$ is analytic in $\mathbb{U}$, with $\omega(0)=0$ and $|\omega(z)|<1$. Furthermore, for $|z|=\eta$, $D^m_\kappa f(z)$ achieves the inequality

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta))-1}{\eta}\Big)d\eta \leq \Big|\frac{D^m_\kappa f(z)}{z}\Big| \leq \exp\Big(\int_0^1 \frac{h(\omega(\eta))-1}{\eta}\Big)d\eta.$$

Proof. Since $f \in \mathbb{S}^m_\kappa(h)$, we have

$$\Big(\frac{z(D^m_\kappa f(z))'}{D^m_\kappa f(z)}\Big) \prec h(z), \quad z \in \mathbb{U},$$

which means that there exists a Schwarz function with $\omega(0)=0$ and $|\omega(z)|<1$ such that

$$\Big(\frac{z(D^m_\kappa f(z))'}{D^m_\kappa f(z)}\Big) = h(\omega(z)), \quad z \in \mathbb{U},$$

which implies that

$$\Big(\frac{(D^m_\kappa f(z))'}{D^m_\kappa f(z)}\Big) - \frac{1}{z} = \frac{h(\omega(z))-1}{z}.$$

Integrating both sides, we have

$$\log D^m_\kappa f(z) - \log z = \int_0^z \frac{h(\omega(\xi))-1}{\xi}d\xi.$$

Consequently, this yields

$$\log\frac{D^m_\kappa f(z)}{z} = \int_0^z \frac{h(\omega(\xi))-1}{\xi}d\xi. \tag{6}$$

By using the definition of subordination, we get

$$D_\kappa^m f(z) \prec z \exp\Big(\int_0^z \frac{h(\omega(\xi))-1}{\xi} d\xi\Big).$$

In addition, we note that the function $h(z)$ maps the disk $0 < |z| < \eta < 1$ onto a region which is convex and symmetric with respect to the real axis, that is

$$h(-\eta|z|) \le \Re(h(\omega(\eta z))) \le h(\eta|z|), \quad \eta \in (0,1),$$

which yields the following inequalities:

$$h(-\eta) \le h(-\eta|z|), \quad h(\eta|z|) \le h(\eta)$$

and

$$\int_0^1 \frac{h(\omega(-\eta|z|))-1}{\eta} d\eta \le \Re\Big(\int_0^1 \frac{h(\omega(\eta))-1}{\eta} d\eta\Big) \le \int_0^1 \frac{h(\omega(\eta|z|))-1}{\eta} d\eta.$$

By using the above relations and Eq. (6), we conclude that

$$\int_0^1 \frac{h(\omega(-\eta|z|))-1}{\eta} d\eta \le \log\Big|\frac{D_\kappa^m f(z)}{z}\Big| \le \int_0^1 \frac{h(\omega(\eta|z|))-1}{\eta} d\eta.$$

This equivalence to the inequality

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta|z|))-1}{\eta} d\eta\Big) \le \Big|\frac{D_\kappa^m f(z)}{z}\Big| \le \exp\Big(\int_0^1 \frac{h(\omega(\eta|z|))-1}{\eta} d\eta\Big).$$

Thus, we obtain

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta))-1}{\eta}\Big) d\eta \le \Big|\frac{D_\kappa^m f(z)}{z}\Big| \le \exp\Big(\int_0^1 \frac{h(\omega(\eta))-1}{\eta}\Big) d\eta.$$

This completes the proof. $\square$

3. Conclusion

From above, we conclude that the generalized differential operator is used to generate a set of new classes of analytic functions in terms of the Janowski formula. Different cases are recognized for recent efforts. One can develop the above work using another classes of univalent functions such as harmonic, multivalent and meromorphic.

References

[1] Ibrahim, R.W., Darus, M., Subordination inequalities of a new Salagean-difference operator, *International Journal of Mathematics and Computer Science,* 14 (3) (2019), 573–582.

[2] Sàlàgean, G.S., Subclasses of univalent functions, Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), *Lecture Notes in Math.,* vol. 1013, Springer, Berlin, (1983), 362–372.

[3] Dunkl, C.F., Differential-difference operators associated with reflections groups, *Trans. Am. Math. Soc.,* 311 (1989), 167–183.

[4] Ibrahim, R. W., New classes of analytic functions determined by a modified differential-difference operator in a complex domain, *Karbala International Journal of Modern Science,* 3 (1) (2017), 53–58.

[5] Miller, S.S., Mocanu, P. T., Differential subordinations: theory and applications, CRC Press, 2000.

[6] Arif, M. et al., A New Class of Analytic Functions Associated with Sàlàgean Operator, *Journal of Function Spaces,* 2019 (2019).

[7] Sakaguchi, K., On a certain univalent mapping, J*ournal of the Mathematical Society of Japan,* 11 (1959), 72–75.

[8] Das, R. N., Singh, P., On subclasses of schlicht mapping, *Indian Journal of Pure and Applied Mathematics,* 8 (1977), 864–872.

[9] Kuroki, K., Owa, S., Some subordination criteria concerning the Salagean operator, *Journal of Inequalities in Pure and Applied Mathematics,* 10 (2) (2009), 1–11.

[10] Noor, K.I. et al., Some applications of certain integral operators involving functions, *Tamkang Journal of Mathematics,* 49 (2018), 25–34.

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ON DOUBLY WARPED PRODUCTS

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Abstract. We give a new characterization for doubly warped products by using the geometry of their canonical foliations intersecting perpendicularly. We also give a necessary and sufficient condition for a doubly warped product to be a warped or a direct product. As a result, we prove the non-existence of Einstein proper doubly warped product pseudo-Riemannian manifold of dimension grater or equal than 4.

The characters in abstract should be between 500 to 5000

1. Introduction

The notion of warped product of pseudo-Riemannian manifolds was defined by Bishop and O' Neill in [2] in order to construct a large class of complete manifolds of negative curvature. In fact, this notion appeared in the literature before [2] under the name of semi-reducible spaces [10]. Also, this notion is a natural and fruitful generalization of the notion of direct (or Riemannian) product. One of the reasons why warped products have been studied actively is that they play very important roles in physics as well as in differential geometry, especially in the theory of relativity. In fact, the standard space-time models such as Robertson-Walker, Schwarschild, static and Kruskal are warped products. Moreover, the simplest models of neighborhoods of stars and black holes are warped products [12].

In this paper, we first prove a existence theorem for doubly warped products. Secondly, we give a necessary and sufficient condition, called the *mixed Ricci-flatness* for a doubly warped product to be a warped or a direct product. In order to achieved this, we use a result of [1] or [14] concerning Ricci tensor of a doubly warped product. Then by using this result, we prove the non-existence of Einstein

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doubly warped product pseudo-Riemannian manifold of dimension ≥ 4 in proper case.

2. Preliminaries

Let M_1 and M_2 be any pseudo-Riemannian manifolds endowed with pseudo-Riemannian metric tensors g_1 and g_2 respectively, and let f_1 and f_2 are positive smooth functions defined on $M_1 \times M_2$. Also π_1 and π_2 are canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. Then the *doubly twisted product* [13] ${}_{f_2}M_1 \times_{f_1} M_2$ of (M_1, g_1) and (M_2, g_2) is the product manifold $M = M_1 \times M_2$ equipped with metric $g = f_2^2 g_1 \oplus f_1^2 g_2$ given by

$$g = f_2^2 \pi_1^*(g_1) + f_1^2 \pi_2^*(g_2)$$

where $\pi_i^*(g_i)$ is the pullback of g_i via π_i for $i = 1, 2$. Each function f_i is called a *twisting function* of the doubly twisted product $({}_{f_2}M_1 \times_{f_1} M_2, g)$. In this case, if either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we obtain a *twisted product* [4].

If the twisting functions f_1 and f_2 only depend on the points of M_1 and M_2 respectively, then $({}_{f_2}M_1 \times_{f_1} M_2, g)$ is called a *doubly warped product pseudo-Riemannian manifold* [6]. The functions f_1 and f_2 are called *warping functions* of doubly warped product. In which case, if either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we obtain a *warped product* [2]. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we get a *direct (or Riemannian) product* [5]. If neither f_1 nor f_2 is constant, then we say that $({}_{f_2}M_1 \times_{f_1} M_2, g)$ is *proper* doubly warped product pseudo-Riemannian manifold.

Let $({}_{f_2}M_1 \times_{f_1} M_2, g)$ be a doubly warped product manifold with the Levi-Civita connection ∇ and ∇^i denote the Levi-Civita connection of M_i for $i \in \{1, 2\}$. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathfrak{L}(M_i)$ and use the same notation for a vector field and for its lift. On the other hand, each π_i is a (positive) homothety, so it preserves the Levi-Civita connection. Thus, there is no confusion using the same notation for a connection on M_i and for its pullback via π_i. Then, the covariant derivative formulas for a doubly warped product manifold [6] are given as:

$$\nabla_X Y = \nabla^1_X Y - g(X, Y)\nabla(\ln(f_2 \circ \pi_2)) \ , \tag{1}$$

$$\nabla_X V = \nabla_V X = V(\ln(f_2 \circ \pi_2))X + X(\ln(f_1 \circ \pi_1))V \ , \tag{2}$$

$$\nabla_U V = \nabla^2_U V - g(U, V)\nabla(\ln(f_1 \circ \pi_1)) \tag{3}$$

for $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$. Moreover, $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally umbilical submanifolds with closed mean curvature vector fields in $({}_{f_2}M_1 \times_{f_1} M_2, g)$ [11], where $p_1 \in M_1$ and $p_2 \in M_2$.

Remark 1. *From now on, we will use the same symbols for warping functions and their pullbacks.*

Next, we recall that some facts for later use.

Let M a pseudo-Riemannian manifold with metric tensor g. The *Ricci tensor* of M is a symmetric $(0,2)$ type tensor defined by

$$Ric(X,Y)=\sum_{i=1}^{m} g(E_i,E_i)g(R(E_i,X)Y,E_i)\ , \tag{4}$$

where X and Y are smooth vector fields on M, $\{E_1,...,E_m\}$ is an orthonormal frame field on the set of all smooth vector fields on M and R is *Riemann curvature tensor* of M defined by

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z\ , \tag{5}$$

here ∇ is the Levi-Civita connection with respect to the metric g. For more details, see [5].

For the Ricci tensor of a doubly warped product ${}_{f_2}M_1\times_{f_1}M_2$ with $dim(M_1)=m_1>1$ and $dim(M_2)=m_2>1$, we have the following result from Theorem 2.5.2 of [14] or the equation (2.19) of [1],

$$Ric(X,V)=(m_1+m_2-2)\left(\frac{Xf_1}{f_1}\right)\left(\frac{Vf_2}{f_2}\right)\ , \tag{6}$$

where $X\in\mathcal{L}(M_1)$ and $V\in\mathcal{L}(M_2)$.

3. Main Results

We need the following two facts to prove the first main theorem.

Lemma 2. *(Proposition 3-a [13]) Let $M=M_1\times M_2$ and call $(\mathcal{D}_1,\mathcal{D}_2)$ the canonical foliations. Suppose that g is a pseudo-Riemann metric such that $\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal. Then (M,g) is a doubly twisted product ${}_{f_2}M_1\times_{f_2}M_2$ if and only if $\mathcal{D}_1$ and $\mathcal{D}_2$ are totally umbilic.*

Lemma 3. *(Lemma 2.3 [9]) Let ${}_{f_2}M_1\times_{f_2}M_2$ be a doubly twisted product. It is a doubly warped product if and only if the mean curvature vector fields of canonical foliations are closed.*

We are ready to prove the main theorem.

Theorem 4. *Let (M,g) be a pseudo-Riemannian manifold and $\mathcal{D}_1$ and $\mathcal{D}_2$ be canonical foliations on M. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect perpendicularly everywhere. Then g is the metric of doubly warped product ${}_{f_2}M_1\times_{f_1}M_2$ if and only if there exists a smooth function μ_1 (resp. μ_2) on M_1 (resp. M_2) such that for any $Z\in\mathcal{L}(M_1)$ and $W\in\mathcal{L}(M_2)$, we have*

$$\mathcal{L}_W g=2W[\mu_2]g \quad on \quad M_1 \tag{7}$$

and

$$\mathcal{L}_Z g = 2Z[\mu_1]g \quad on \quad M_2 \ , \tag{8}$$

where $\mathcal{L}_W$ is the Lie derivative with respect to W and M_1 (resp. M_2) is the integral manifold of $\mathcal{D}_1$ (resp. $\mathcal{D}_2$).

Proof. Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a doubly warped product with the metric $g = f_2^2 g_1 \oplus f_1^2 g_2$. Then using the Lie derivative formula, for any $X, Y, Z \in \mathfrak{L}(M_1)$ and $U, V, W \in \mathfrak{L}(M_2)$, we have

$$(\mathcal{L}_W g)(X, Y) = -2g(h_1(X, Y), W) \tag{9}$$

and

$$(\mathcal{L}_Z g)(U, V) = -2g(h_2(U, V), Z) \ , \tag{10}$$

where h_1 (resp. h_2) denotes the second fundamental form of M_1 (resp. M_2), (e.g. see [3, p. 195]). By using (1) and (3), we obtain

$$(\mathcal{L}_W g)(X, Y) = -2g(-g(X, Y)\nabla(\ln f_2), W) \tag{11}$$

and

$$(\mathcal{L}_Z g)(U, V) = -2g(-g(U, V)\nabla(\ln f_1), Z) \tag{12}$$

from (9) and (10), respectively. By direct calculation, we get

$$(\mathcal{L}_W g)(X, Y) = 2W[\ln f_2]g(X, Y) \tag{13}$$

and

$$(\mathcal{L}_Z g)(U, V) = 2Z[\ln f_1]g(U, V) \tag{14}$$

from (11) and (12), respectively. Thus, we find the assertion (7) for $\mu_2 = \ln f_2$ from (13) and the assertion (8) for $\mu_1 = \ln f_1$ from (14).

Conversely, suppose that the conditions (7) and (8) hold. Then for any $X, Y, Z \in \mathfrak{L}(M_1)$ and $U, V, W \in \mathfrak{L}(M_2)$, using (7)$\sim$(10), we have

$$-2g(h_1(X, Y), W) = 2W[\mu_2]g(X, Y) \tag{15}$$

and

$$-2g(h_2(U, V), Z) = 2Z[\mu_1]g(U, V). \tag{16}$$

After some calculation, we obtain

$$g(h_1(X, Y), W) = g(-g(X, Y)\nabla\mu_2, W) \tag{17}$$

and

$$g(h_2(U, V), Z) = g(-g(U, V)\nabla\mu_1, Z) \tag{18}$$

from (15) and (16), respectively. We get

$$h_1(X, Y) = -g(X, Y)\nabla\mu_2 \tag{19}$$

and

$$h_2(U, V) = -g(U, V)\nabla\mu_1 \tag{20}$$

from (17) and (18), respectively. The equation (19) (resp. (20)) tells us the canonical foliation $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) is totally umbilical with the mean curvature vector field $-\nabla\mu_2$ (resp. $-\nabla\mu_1$). Moreover, the mean curvature vector field $-\nabla\mu_1$ (resp. $-\nabla\mu_2$) is closed, since its dual 1-form $-d\mu_1$ (resp. $-d\mu_2$) is closed. Thus by Lemmas 2 and 3, g is the metric of a doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$. □

Before going to give the second main result, let recall the definition of mixed Ricci-flatness.

Let $M = {}_{f_2}M_1 \times_{f_1} M_2$ be a doubly warped product pseudo-Riemannian manifold with metric tensor $g = f_2^2 g_1 \oplus f_1^2 g_2$. Then we say that (M, g) is *mixed Ricci-flat*, if we have $Ric(X, V) = 0$ for every $X \in \mathcal{L}(M_1)$ and $V \in \mathcal{L}(M_2)$ [7].

Theorem 5. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a doubly warped product of (M_1, g_1) and (M_2, g_2) with warping functions f_1 and f_2 and $dim(M_1) = m_1 > 1$ and $dim(M_2) = m_2 > 1$. Then ${}_{f_2}M_1 \times_{f_1} M_2$ is mixed Ricci-flat if and only if*

(1) *either ${}_{f_2}M_1 \times_{f_1} M_2$ can be expressed as a warped product ${}_{f_2}M_1 \times M_2$ of $(M_1, \tilde{g_1})$ and (M_2, g_2) with warping function f_2, where $\tilde{g_2} = k_1^2 g_2$ for some positive constant k_1, or*

(2) *either ${}_{f_2}M_1 \times_{f_1} M_2$ can be expressed as a warped product $M_1 \times_{f_1} M_2$ of (M_1, g_1) and $(M_2, \hat{g_2})$ with warping function f_1, where $\hat{g_1} = k_2^2 g_1$ for some positive constant k_2, or*

(3) *${}_{f_2}M_1 \times_{f_1} M_2$ is a direct product $M_1 \times M_2$ of $(M_1, \bar{g_1})$ and $(M_2, \bar{g_2})$, where $\bar{g_1} = c_2^2 g_1$ and $\bar{g_2} = c_1^2 g_2$ for some positive constants c_1 and c_2.*

Proof. If ${}_{f_2}M_1 \times_{f_1} M_2$ is mixed Ricci-flat, then we have $Ric(X, V) = 0$ for all $X \in \mathcal{L}(M_1)$ and $V \in \mathcal{L}(M_2)$. Thus, by the hypothesis and (6), we obtain

$$(Xf_1)(Vf_2) = 0 \ , \tag{21}$$

for all $X \in \mathcal{L}(M_1)$ and $V \in \mathcal{L}(M_2)$. There are three different cases.

Case 1. *$Xf_1 = 0$ and $Vf_2 \neq 0$.*

Hence, we find $f_1 = k_1$ for some positive constant k_1. Thus, we can write $g = f_2^2 g_1 \oplus \tilde{g_2}$, where $\tilde{g_2} = k_1^2 g_2$, that is ${}_{f_2}M_1 \times_{f_1} M_2$ can be expressed as a warped product ${}_{f_2}M_1 \times M_2$ with warping function f_2, where the metric tensor of M_2 is $\tilde{g_2}$ given above. This is **(1)**.

Case 2. *$Xf_1 \neq 0$ and $Vf_2 = 0$.*

Similarly, ${}_{f_2}M_1 \times_{f_1} M_2$ can be expressed as a warped product $M_1 \times_{f_1} M_2$ with warping function f_1, where the metric tensor of this warped product $M_1 \times_{f_1} M_2$ M_2 is $g = \hat{g_1} \oplus f_1^2 g_2$ such that $\hat{g_1} = k_2^2 g_1$ for some positive constant k_2, so we get **(2)**.

Case 3. $Xf_1 = Vf_2 = 0$.

Then, it follows immediately that $f_1 = c_1$ and $f_2 = c_2$, where c_1 and c_2 are positive constants. Thus, it is easy to see that ${}_{f_2}M_1 \times_{f_1} M_2$ is a direct product $M_1 \times M_2$ of $(M_1, \bar{g_1})$ and $(M_2, \bar{g_2})$, here $\bar{g_1} = c_2^2 g_1$ and $\bar{g_2} = c_1^2 g_2$. Which is **(3)**. The converse is obvious from the equation (6). $\square$

A pseudo-Riemannian manifold (M, g) is called an *Einstein manifold* if its Ricci tensor proportional to its metric, i.e., $Ric = \lambda g$ for some constant λ [5].Since, the Einstein conditions leads to mixed Ricci-flatness, by our main result Theorem 5, we have following result.

Corollary 6. *There exist no Einstein proper doubly warped product pseudo-Riemannian manifold of dimension greater or equal than 4.*

Remark 7. *This result was also obtained without dimension restriction in [1] by different manner, see Proposition 3.1 of [1].*

Remark 8. *In [8], the author asserts that the existence of Einstein doubly warped product pseudo-Riemannian manifolds, see Remark 3.3 of [8]. But Corollary 6 contradicts that result.*

Remark 9. *The mixed Ricci-flatness condition was also used for a twisted product to be a warped product by M. Fernández López et al [7].*

Remark 10. *As can be easily seen from the Preliminaries section, there exist no inclusion relation between the classes of proper twisted products and the classes of proper doubly warped products.*

Remark 11. *Some space-time models such as Robertson-Walker and Kruskal have the mixed Ricci-flatness property. Thus, in view of Theorem 5, these space-times cannot be further generalized to the proper doubly warped products.*

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References

[1] Agaoka, Y., Kim, I-B., Yeom, D.J., On doubly warped product manifolds, *Mem. Fac. Integrated Arts and Sci.*, Hiroshimo Univ., ser. IV., 24 (1998), 1-10.
[2] Bishop, R.L., O'Neill, B., Manifolds of negative curvature, *Trans. Amer. Mat. Soc.*, 145(1) (1969), 1-49.
[3] Blumenthal, R.A., Hebda, J.J., An analogue of the holonomy bundle for a foliated manifold, *Tôhoku Math. J.*, 40(2) (1988), 189-197.
[4] Chen, B.Y., Geometry of submanifolds and its applicaitons, Science University of Tokyo, Tokyo, 1981.
[5] Chen, B.Y., Differential geometry of warped product manifolds and submanifolds, World Scientific, 2017.

[6] Ehrlich, P. E., Metric deformations of Ricci and sectional curvature on compact Riemannian manifolds, Ph.D. dissertation, SUNY, Stony Brook, N.Y., 1974.

[7] Fernández López, M., García Río, E., Küpeli, D.N., Ünal, B., A curvature condition for a twisted product to be a warped product, *Manuscripta Math.*, 106 (2001), 213-217.

[8] Gupta, P., On compact Einstein doubly warped product manifolds, *Tamkang J. Math.*, 49(4) (2018), 267-275. doi:10.5556/j.tkjm.49.2018.2605.

[9] Gutierrez, M., Olea, B., Semi-Riemannian manifolds with a doubly warped structures, *Revista Mathematica Iberoamericana*, 28(1) (2012), 1-24.

[10] Kruchkovich, G.I., On semi-reducible Riemannian spaces (in Russian), *Dokl. Akad. Nauk SSSR*, 115 (1957), 862-865.

[11] Olea, B., Doubly warped product structures on semi-Riemannian manifolds, Ph.D. thesis, Universty of Malaga, 2009.

[12] O'Neill, B., Semi-Riemannian geometry with applications to relativity, Academic Press, San Diego, 1983.

[13] Ponge, R., Reckziegel, H., Twisted products in pseudo- Riemannian geometry, *Geom. Dedicata*, 48 (1993), 15-25.

[14] Ünal, B., Doubly warped products, Ph.D. thesis, Universty of Missouri-Columbia, 2000.

[15] Yano, K., Kon, M., Structures on Manifolds, World Scientific, Singapore, 1984.

https://communications.science.ankara.edu.tr

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EQUITABLE EDGE COLORING ON TENSOR PRODUCT OF GRAPHS

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Abstract. A graph G is edge colored if different colors are assigned to its edges or lines, in the order of neighboring edges are allotted with least diverse k-colors. If each of k-colors can be partitioned into color sets and differs by utmost one, then it is equitable. The minimum of k-colors required is known as equitably edge chromatic number and symbolized by $\chi'_{=}(G)$. Further the impression of equitable edge coloring was first initiated by Hilton and de Werra in 1994. In this paper, we ascertain the equitable edge chromatic number of $P_m \otimes P_n$, $P_m \otimes C_n$ and $K_{1,m} \otimes K_{1,n}$.

1. Introduction

In the midst of various coloring concepts of graphs, the motive of equitability in edge coloring on tensor product of graphs is an inventive approach. Graphs considered in this paper are of simple finite sets V and E. Each element of V is called its vertices and the elements of E are called its edges, which are the unordered pair of vertices. Therefore $G(V, E)$ is a graph. We use the standard notation of graph theory [1, 2]. The minimum number of colors needed to color edges of a graph G is utmost its maximum degree. Since all edges incident to the same vertex must be alloted with distinct colors. Noticeably $\chi'(G) \geq \Delta(G)$. In 1964, Vizing [3] conjectured that for every simple graph $\chi'(G) \leq \Delta(G) + 1$. In 1973, Meyer [4] presented the concept of equitable vertex coloring and its equitable

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chromatic number, which opened the way for introducing equitability in the fields of edge and total coloring.

The concept of equitable edge coloring was defined by Hilton and de Werra [5] and the tensor product of graph was defined by P.M.Weichsel [6]. We have merged both these conception and resolute the equitable edge chromatic number of $P_m \otimes P_n$, $P_m \otimes C_n$ and $K_{1,m} \otimes K_{1,n}$. The combined component of each of these graphs enlarges as a new structured graph and has wider applications in the areas of networks, scheduling and assignment domains.

2. Preliminaries

Definition 2.1. *An edge coloring of a graph G is a function $f : E(G) \longrightarrow C$, where C is a set of distinct colors. For any positive integer k, a k-edge coloring is an edge coloring that uses exactly k different colors. A proper edge coloring of a graph is an edge coloring such that no two adjacent edges are assigned the same color. Thus a proper edge coloring f of G is a function $f : E(G) \longrightarrow C$ such that $f(e) \neq f(e')$ whenever edges e and e' are adjacent in G.*

Definition 2.2. *The chromatic index of a graph G, denoted $\chi'(G)$, is the minimum number of different colors required for a proper edge coloring of G. The graph G is k-edge-chromatic if $\chi'(G) = k$.*

Definition 2.3. *For k-proper edge coloring f of graph G, if $||E_i| - |E_j|| \leq 1$, $i, j = 0, 1, 2, \ldots, k-1$, where $E_i(G)$ is the set of edges of color i in G, then f is called a k-equitable edge coloring of graph G, and*

$$\chi'_{=}(G) = \min\{k : \text{there exists a } k\text{-equitable edge-coloring of } G\}$$

is called the equitable edge chromatic number of graph G.

Definition 2.4. *[6] The tensor product of G and H is the graph, denoted as $G \otimes H$, whose vertex set is $V(G) \otimes V(H) = V(G \otimes H)$, and for each vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$. Thus*

$$\begin{aligned} V(G \otimes H) &= \{(g, h) / g \in V(G) \text{ and } h \in V(H)\} \\ E(G \otimes H) &= \{(g, h)(g', h') / gg' \in E(G) \text{ and } hh' \in E(H)\}. \end{aligned}$$

Lemma 2.5. *[1] For any simple graph $G(V, E)$, $\chi'_{=} \geq \Delta(G)$.*

Lemma 2.6. *[1] For any simple graph G and H, $\chi'_{=}(G) = \chi'(G)$, and if $H \subseteq G$, then $\chi'(H) \leq \chi'(G)$, where $\chi'_{=}(G)$ is the proper edge chromatic number of G. So Lemma 2.7 and Lemma 2.8 are obtained.*

Lemma 2.7. *For any simple graph G and H, if H is a subgraph of G, then $\chi'_{=}(H) \leq \chi'_{=}(G)$*

Lemma 2.8. *For any complete graph K_p with order p,*

$$\chi'_{=}(K_p) = \begin{cases} p, & p \equiv 1 \pmod 2, \\ p - 1, & p \equiv 0 \pmod 2, \end{cases}$$

Lemma 2.9. *[7] Let $G(V,E)$ be a connected graph. If there are two adjacent vertices with maximum degree, then $\chi'_{as}(G) \geq \Delta(G) + 1$.*

3. Main Results

Theorem 3.1. *For $m \leq n$, $\chi'_{=}(P_m \otimes P_n) = 4$.*

Proof. Let $V(P_m) = \{u_i : 1 \leq i \leq m\}$ and $V(P_n) = \{v_j : 1 \leq j \leq n\}$. By the definiton of tensor product, $V(P_m \otimes P_n) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$$E(P_m \otimes P_n) = \bigcup_{i=1}^{m-1} \{u_i v_j, u_{i+1} v_{j+1} : 1 \leq j \leq n-1\} \cup \bigcup_{i=1}^{m-1} \{u_i v_j, u_{i+1} v_{j-1} : 2 \leq j \leq n\}.$$

Let $e_{(i)(j),(k)(l)}$ be the edge of $P_m \otimes P_n$ connecting the vertices $u_i v_j$ and $u_k v_l$ of $P_m \otimes P_n$.

Therefore $e_{(i)(j),(k)(l)} \in E(P_m \otimes P_n)$ if and only if $|k-i| = |l-i| = 1$. Since $P_m \otimes P_n$ is isomorphic to $P_n \otimes P_m$. Without loss of generality, we assume $m \leq n$ for all cases of m and n. Now let us partition $E(P_m \otimes P_n)$ for the following cases.

Case (i) Both m and n are odd

$$\begin{aligned}
E_1 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j-1),(2i-1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j),(2i+1)(2j+1)} \right) \right\} \\
E_2 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i-1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i+1)(2j),(2i)(2j+1)} \right) \right\} \\
E_3 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i+1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i-1)(2j),(2i)(2j+1)} \right) \right\} \\
E_4 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j-1),(2i+1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j),(2i-1)(2j+1)} \right) \right\}
\end{aligned}$$

In this partition $|E_1| = |E_2| = |E_3| = |E_4| = 2\left(\frac{m-1}{2}\right)\left(\frac{n-1}{2}\right)$ and satisfies $||E_i| - |E_j|| \leq 1$ for $i \neq j$.

Case (ii) When m is odd and n is even

$$\begin{aligned}
E_1 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i)(2j-1),(2i-1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i)(2j),(2i+1)(2j+1)} \right) \right\} \\
E_2 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i-1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i+1)(2j),(2i)(2j+1)} \right) \right\}
\end{aligned}$$

$$E_3 = \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i+1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i-1)(2j),(2i)(2j+1)} \right) \right\}$$

$$E_4 = \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i)(2j-1),(2i+1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i)(2j),(2i-1)(2j+1)} \right) \right\}$$

In this partition $|E_1| = |E_2| = |E_3| = |E_4| = \left(\frac{m-1}{2}\right)\left(\frac{n}{2}\right) + \left(\frac{m-1}{2}\right)\left(\frac{n-2}{2}\right)$ which infers $||E_i| - |E_j|| \leq 1$ for $i \neq j$.

Case (iii) When m is even and n is odd

$$E_1 = \left\{ \bigcup_{i=1}^{\frac{m}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j-1),(2i-1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-2}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j),(2i+1)(2j+1)} \right) \right\}$$

$$E_2 = \left\{ \bigcup_{i=1}^{\frac{m}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i-1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-2}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i+1)(2j),(2i)(2j+1)} \right) \right\}$$

$$E_3 = \left\{ \bigcup_{i=1}^{\frac{m-2}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i+1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i-1)(2j),(2i)(2j+1)} \right) \right\}$$

$$E_4 = \left\{ \bigcup_{i=1}^{\frac{m-2}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j-1),(2i+1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j),(2i-1)(2j+1)} \right) \right\}$$

In this partition $|E_1| = |E_2| = |E_3| = |E_4| = \left(\frac{m-2}{2}\right)\left(\frac{n-1}{2}\right) + \left(\frac{m}{2}\right)\left(\frac{n-1}{2}\right)$ which facts that $||E_i| - |E_j|| \leq 1$ for $i \neq j$.

Case (iv) Both m and n are even

$$E_1 = \left\{ \bigcup_{i=1}^{\frac{m}{2}} \left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i)(2j-1),(2i-1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-2}{2}} \left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i)(2j),(2i+1)(2j+1)} \right) \right\}$$

$$E_2 = \left\{ \bigcup_{i=1}^{\frac{m}{2}} \left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i-1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-2}{2}} \left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i+1)(2j),(2i)(2j+1)} \right) \right\}$$

$$E_3 = \left\{ \bigcup_{i=1}^{\frac{m-2}{2}} \left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i+1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m}{2}} \left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i-1)(2j),(2i)(2j+1)} \right) \right\}$$

$$E_4 = \left\{ \bigcup_{i=1}^{\frac{m-2}{2}} \left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i)(2j-1),(2i+1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m}{2}} \left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i)(2j),(2i-1)(2j+1)} \right) \right\}$$

In this partition $|E_1| = |E_2| = \binom{m}{2}\binom{n}{2} + \binom{m-2}{2}\binom{n-2}{2}$ and $|E_3| = |E_4| = \binom{m-2}{2}\binom{n}{2} + \binom{m}{2}\binom{n-2}{2}$ which signifies $||E_i| - |E_j|| \le 1$ for $i \neq j$.

In all the cases by observing the suffixes of the edges of E_i and E_j $(i \neq j)$, it is inferred that there is no common edges in E_i and E_j $(i \neq j)$. i.e, $E_i \cap E_j = \phi$ for $i \neq j$. Clearly E_i's are pair wise mutually disjoint, also $\bigcup\limits_{i=1}^{4} E_i = E\left(P_m \otimes P_n\right)$. Here $P_m \otimes P_n$ is equitably edge colorable with 4 colors. Hence $\chi'_{=}\left(P_m \otimes P_n\right) \le 4$. Since $\Delta = 4$, we have $\chi'_{=}\left(P_m \otimes P_n\right) \ge \chi'\left(P_m \otimes P_n\right) \ge \Delta = 4$. This implies $\chi'_{=}\left(P_m \otimes P_n\right) \ge 4$. Therefore $\chi'_{=}\left(P_m \otimes P_n\right) = 4$. □

Theorem 3.2. *For $m \le n$, $\chi'_{=}\left(P_m \otimes C_n\right) = 4$.*

Proof. Let $V\left(P_m\right) = \{u_i : 1 \le i \le m\}$ and $V\left(C_n\right) = \{v_j : 1 \le j \le n\}$. By the definiton of tensor product, $V\left(P_m \otimes C_n\right) = \{u_i v_j : 1 \le i \le m, 1 \le j \le n\}$ and

$$E\left(P_m \otimes C_n\right) = \bigcup_{i=1}^{m-1} \{u_i v_j, u_{i+1} v_{j+1} : 1 \le j \le n-1\} \cup \bigcup_{i=1}^{m-1} \{u_i v_j, u_{i+1} v_{j-1} : 2 \le j \le n\}$$

$\cup \bigcup\limits_{i=1}^{m-1} \{u_i v_1, u_{i+1} v_n\} \cup \bigcup\limits_{i=2}^{m} \{u_i v_1, u_{i-1} v_n\}$. Let $e_{(i)(j),(k)(l)}$ be the edge of $P_m \otimes C_n$ connecting the vertices $u_i v_j$ and $u_k v_l$ of $P_m \otimes C_n$.

Therefore $e_{(i)(j),(k)(l)} \in E\left(P_m \otimes C_n\right)$ if and only if $|k - i| = |l - i| = 1$. Since $P_m \otimes C_n$ is isomorphic to $C_n \otimes P_m$. Without loss of generality, we assume $m \le n$ for all cases of m and n. Now let us partition $E\left(P_m \otimes C_n\right)$ for the following cases.
Case (i) Both m and n are odd

$$\begin{aligned}
E_1 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j-1),(2i-1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j),(2i+1)(2j+1)} \right) \right\} \\
&\quad \bigcup \left\{ e_{(2i+1)(1),(2i)(n):1 \le i \le \frac{m-1}{2}} \right\} \\
E_2 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i-1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i+1)(2j),(2i)(2j+1)} \right) \right\} \\
&\quad \bigcup \left\{ e_{(2i)(1),(2i+1)(n):1 \le i \le \frac{m-1}{2}} \right\} \\
E_3 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i+1)(2j-1),(2i)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i-1)(2j),(2i)(2j+1)} \right) \right\} \\
&\quad \bigcup \left\{ e_{(2i)(1),(2i-1)(n):1 \le i \le \frac{m-1}{2}} \right\} \\
E_4 &= \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j-1),(2i+1)(2j)} \right) \right\} \bigcup \left\{ \bigcup_{i=1}^{\frac{m-1}{2}} \left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j),(2i-1)(2j+1)} \right) \right\}
\end{aligned}$$

$$\bigcup\left\{e_{(2i-1)(1),(2i)(n):1\leq i\leq \frac{m-1}{2}}\right\}$$

In this partition $|E_1| = |E_2| = |E_3| = |E_4| = 2\left(\frac{m-1}{2}\right)\left(\frac{n-1}{2}\right) + \left(\frac{m-1}{2}\right)$.
and deduce that $||E_i| - |E_j|| \leq 1$ for $i \neq j$.
Case (ii) When m is odd and n is even

$$\begin{aligned}
E_1 &= \left\{\bigcup_{i=1}^{\frac{m-1}{2}}\left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i)(2j-1),(2i-1)(2j)}\right)\right\}\bigcup\left\{\bigcup_{i=1}^{\frac{m-1}{2}}\left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i)(2j),(2i+1)(2j+1)}\right)\right\}\\
&\quad\bigcup\left\{e_{(2i+1)(1),(2i)(n):1\leq i\leq \frac{m-1}{2}}\right\}
\end{aligned}$$

$$\begin{aligned}
E_2 &= \left\{\bigcup_{i=1}^{\frac{m-1}{2}}\left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i-1)(2j-1),(2i)(2j)}\right)\right\}\bigcup\left\{\bigcup_{i=1}^{\frac{m-1}{2}}\left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i+1)(2j),(2i)(2j+1)}\right)\right\}\\
&\quad\bigcup\left\{e_{(2i)(1),(2i+1)(n):1\leq i\leq \frac{m-1}{2}}\right\}\\
E_3 &= \left\{\bigcup_{i=1}^{\frac{m-1}{2}}\left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i+1)(2j-1),(2i)(2j)}\right)\right\}\bigcup\left\{\bigcup_{i=1}^{\frac{m-1}{2}}\left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i-1)(2j),(2i)(2j+1)}\right)\right\}\\
&\quad\bigcup\left\{e_{(2i)(1),(2i-1)(n):1\leq i\leq \frac{m-1}{2}}\right\}\\
E_4 &= \left\{\bigcup_{i=1}^{\frac{m-1}{2}}\left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i)(2j-1),(2i+1)(2j)}\right)\right\}\bigcup\left\{\bigcup_{i=1}^{\frac{m-1}{2}}\left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i)(2j),(2i-1)(2j+1)}\right)\right\}\\
&\quad\bigcup\left\{e_{(2i-1)(1),(2i)(n):1\leq i\leq \frac{m-1}{2}}\right\}
\end{aligned}$$

In this partition $|E_1| = |E_2| = |E_3| = |E_4| = \left(\frac{m-1}{2}\right)\left(\frac{n}{2}\right)+\left(\frac{m-1}{2}\right)\left(\frac{n-2}{2}\right)+\left(\frac{m-1}{2}\right)$.
and assures that$||E_i| - |E_j|| \leq 1$ for $i \neq j$.
Case (iii) When m is even and n is odd

$$\begin{aligned}
E_1 &= \left\{\bigcup_{i=1}^{\frac{m}{2}}\left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j-1),(2i-1)(2j)}\right)\right\}\bigcup\left\{\bigcup_{i=1}^{\frac{m-2}{2}}\left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j),(2i+1)(2j+1)}\right)\right\}\\
&\quad\bigcup\left\{e_{(2i+1)(1),(2i)(n):1\leq i\leq \frac{m-2}{2}}\right\}\\
E_2 &= \left\{\bigcup_{i=1}^{\frac{m}{2}}\left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i-1)(2j-1),(2i)(2j)}\right)\right\}\bigcup\left\{\bigcup_{i=1}^{\frac{m-2}{2}}\left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i+1)(2j),(2i)(2j+1)}\right)\right\}\\
&\quad\bigcup\left\{e_{(2i)(1),(2i+1)(n):1\leq i\leq \frac{m-2}{2}}\right\}
\end{aligned}$$

$$
\begin{aligned}
E_3 &= \left\{\bigcup_{i=1}^{\frac{m-2}{2}}\left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i+1)(2j-1),(2i)(2j)}\right)\right\} \bigcup \left\{\bigcup_{i=1}^{\frac{m}{2}}\left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i-1)(2j),(2i)(2j+1)}\right)\right\} \\
&\quad \bigcup \left\{e_{(2i)(1),(2i-1)(n):1\le i\le \frac{m}{2}}\right\} \\
E_4 &= \left\{\bigcup_{i=1}^{\frac{m-2}{2}}\left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j-1),(2i+1)(2j)}\right)\right\} \bigcup \left\{\bigcup_{i=1}^{\frac{m}{2}}\left(\bigcup_{j=1}^{\frac{n-1}{2}} e_{(2i)(2j),(2i-1)(2j+1)}\right)\right\} \\
&\quad \bigcup \left\{e_{(2i-1)(1),(2i)(n):1\le i\le \frac{m}{2}}\right\}
\end{aligned}
$$

In this partition $|E_1| = |E_2| = \left(\frac{m}{2}\right)\left(\frac{n-1}{2}\right) + \left(\frac{m-2}{2}\right)\left(\frac{n-1}{2}\right) + \left(\frac{m-2}{2}\right)$,
$|E_3| = |E_4| = \left(\frac{m-2}{2}\right)\left(\frac{n-1}{2}\right) + \left(\frac{m}{2}\right)\left(\frac{n-1}{2}\right) + \left(\frac{m}{2}\right)$ and verifies that $||E_i| - |E_j|| \le 1$ for $i \ne j$.

Case (iv) Both m and n are even

$$
\begin{aligned}
E_1 &= \left\{\bigcup_{i=1}^{\frac{m}{2}}\left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i)(2j-1),(2i-1)(2j)}\right)\right\} \bigcup \left\{\bigcup_{i=1}^{\frac{m-2}{2}}\left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i)(2j),(2i+1)(2j+1)}\right)\right\} \\
&\quad \bigcup \left\{e_{(2i+1)(1),(2i)(n):1\le i\le \frac{m-2}{2}}\right\} \\
E_2 &= \left\{\bigcup_{i=1}^{\frac{m}{2}}\left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i-1)(2j-1),(2i)(2j)}\right)\right\} \bigcup \left\{\bigcup_{i=1}^{\frac{m-2}{2}}\left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i+1)(2j),(2i)(2j+1)}\right)\right\} \\
&\quad \bigcup \left\{e_{(2i)(1),(2i+1)(n):1\le i\le \frac{m-2}{2}}\right\} \\
E_3 &= \left\{\bigcup_{i=1}^{\frac{m-2}{2}}\left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i+1)(2j-1),(2i)(2j)}\right)\right\} \bigcup \left\{\bigcup_{i=1}^{\frac{m}{2}}\left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i-1)(2j),(2i)(2j+1)}\right)\right\} \\
&\quad \bigcup \left\{e_{(2i)(1),(2i-1)(n):1\le i\le \frac{m}{2}}\right\} \\
E_4 &= \left\{\bigcup_{i=1}^{\frac{m-2}{2}}\left(\bigcup_{j=1}^{\frac{n}{2}} e_{(2i)(2j-1),(2i+1)(2j)}\right)\right\} \bigcup \left\{\bigcup_{i=1}^{\frac{m}{2}}\left(\bigcup_{j=1}^{\frac{n-2}{2}} e_{(2i)(2j),(2i-1)(2j+1)}\right)\right\} \\
&\quad \bigcup \left\{e_{(2i-1)(1),(2i)(n):1\le i\le \frac{m}{2}}\right\}
\end{aligned}
$$

In this partition $|E_1| = |E_2| = \left(\frac{m}{2}\right)\left(\frac{n}{2}\right) + \left(\frac{m-2}{2}\right)\left(\frac{n-2}{2}\right) + \left(\frac{m-2}{2}\right)$ and
$|E_3| = |E_4| = \left(\frac{m-2}{2}\right)\left(\frac{n}{2}\right) + \left(\frac{m}{2}\right)\left(\frac{n-2}{2}\right) + \left(\frac{m}{2}\right)$ which is evident that $||E_i| - |E_j|| \le 1$ for $i \ne j$.

In all the cases by observing the suffixes of the edges of E_i and E_j $(i \ne j)$, it is inferred that there is no common edges in E_i and E_j $(i \ne j)$ and implies

$E_i \cap E_j = \phi$ for $i \neq j$. Clearly $E_i's$ are pair wise mutually disjoint, also $\bigcup_{i=1}^{4} E_i = E(P_m \otimes C_n)$. Here $P_m \otimes C_n$ is equitably edge colorable with 4 colors. Hence $\chi'_{=}(P_m \otimes C_n) \leq 4$. Since $\Delta = 4$, $\chi'_{=}(P_m \otimes C_n) \geq \chi'(P_m \otimes C_n) \geq \Delta = 4$. This implies $\chi'_{=}(P_m \otimes C_n) \geq 4$. Therefore $\chi'_{=}(P_m \otimes C_n) = 4$. □

Theorem 3.3. *For any positive integer m and n, $\chi'_{=}(K_{1,m} \otimes K_{1,n}) = mn$.*

Proof. Let $V(K_{1,m}) = \{u_0\} \bigcup \{u_i : 1 \leq i \leq m\}$ and $V(K_{1,n}) = \{v_0\} \{v_j : 1 \leq j \leq n\}$

By the definition of tensor product,

$V(K_{1,m} \otimes K_{1,n}) = \{u_i v_j : 0 \leq i \leq m, 0 \leq j \leq n\}$ and

$E(K_{1,m} \otimes K_{1,n}) = \{\bigcup_{i=1}^{m} (u_0 v_0, u_i v_j) : 1 \leq j \leq n\} \bigcup \{\bigcup_{i=1}^{m} \{(u_i v_0, u_0 v_j) : 1 \leq j \leq n\}$

Let $e_{(i)(j),(k)(l)}$ be the edge of $K_{1,m} \otimes K_{1,n}$ connecting the vertices $u_i v_j$ and $u_k v_l$ of the tensor product of star graphs. Since $K_{1,m} \otimes K_{1,n}$ is isomorphic to $K_{1,n} \otimes K_{1,m}$. Without loss of generality, we assume $m \leq n$ for all cases of m and n.

Now we partition the edge set of $E(K_{1,m} \otimes K_{1,n})$ as

$$E_k = \left\{ e_{(0)(0),(\lceil \frac{k}{n} \rceil)((k-1 \bmod n)+1)} \right\} \bigcup \left\{ e_{(\lceil \frac{k}{n} \rceil)(0),(0)((k-1 \bmod n)+1)} \right\}, 1 \leq k \leq mn$$

Clearly each of edge classes $E_1, E_2, \ldots E_k$ are independent sets of $E(K_{1,m} \otimes K_{1,n})$, such that $|E_1| = |E_k| = 2$. It satisfies the inequality $||E_i| - |E_j|| \leq 1$ for every pair (i,j). This implies

$$\chi'_{=}(K_{1,m} \otimes K_{1,n}) \leq mn. \text{ But } \chi'_{=}(K_{1,m} \otimes K_{1,n}) \geq \chi'(K_{1,m} \otimes K_{1,n}) \geq mn.$$

Therefore $\chi'_{=}(K_{1,m} \otimes K_{1,n}) = mn$. □

4. Conclusion

The equitable edge coloring of tensor product of graphs is a new inventive approach and this field of research is wide open. The equitable edge coloration of Mycielskian of some graphs are obtained by Vivik and Girija [8]. The proofs provided in this paper establishes an optimal solution for the equitable edge coloring of tensor product of two different paths, paths with cycles and two star graphs. It would be further interesting to discern the bounds of equitable edge coloring of tensor and other product of graphs.

References

[1] Bondy, J. A., Murty, U. S. R., Graph Theory with Applications, New York; The Macmillan Press Ltd., 1976.

[2] Harary, F., Graph Theory, Narosa Publishing home, 1969.

[3] Vizing, V.G., On an estimate of the chromatic class of a p-graph, *Metody Diskret. Analiz,* 5 (1964), 25–30.

[4] Meyer, W., Equitable Coloring, *Amer. Math. Monthly,* 80 (1973), 920–922.

[5] Hilton, A.J.W., de Werra, D., A sufficient condition for equitable edge-colorings of simple graphs, *Discrete Mathematics*, 128 (1994), 179-201.

[6] Weichsel, P.M., The Kronecker product of graphs, *Proc. Amer. Math. Soc.,* 8 (1962), 47-52.

[7] Li, J.W., Zhang, Z.F., Chen, X.E., Sun, Y.R., A note on adjacent strong edge coloring of $K(n,m)$, *Acta Mathematicae Application Sinica*, 22(2) (2006), 273-276.

[8] Veninstine Vivik.J and Girija.G, Equitable edge chromatic number of mycielskian of graphs, *Far East Journal of Mathematics*,101(9) (2017), 1887-1895.

https://communications.science.ankara.edu.tr

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ANALYSIS OF FRACTIONAL DIFFERENTIAL SYSTEMS INVOLVING RIEMANN LIOUVILLE FRACTIONAL DERIVATIVE

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Abstract. This paper is devoted to studying the multiple positive solutions for a system of nonlinear fractional boundary value problems. Our analysis is based upon the Avery Peterson fixed point theorem. In addition, we include an example for the demonstration of our main result.

1. Introduction

Researchers have focused a great deal of attention on the fractional boundary value problems due to the rapid progress in the theory and applications of fractional calculus. Aside from various fields of mathematics, boundary value problems for fractional differential equations have many applications in the area of chemistry, physics, biology, aerodynamics, control theory, economics, viscoelasticity, electrical circuits, and so forth. Driven by the numerous applications, there are many works related to the existence of positive solutions for the nonlinear fractional boundary value problems. For an overview of these type of study, we mention Podlubny [12], Jiqiang Jiang, Hongchuan Wang [21], Kilbas, Srivastava, and Trujillo [9], Bai and Sun [1], Goodrich [3], Cabrera, Harjani and Sadarangani [15], He, Zhang, Liu, Yonghong Wu and Cui, [16], Wang, Liang and Wang [17],Kamal Shah,Salman Zeb,Rahmat Ali Khan [25]. Goodrich [4] studied the following fractional boundary value problem subject to the given boundary conditions

$$D^{\alpha}u(t)+f(t,u)=0, \quad 0<t<1, \quad n-1<\alpha\le n,$$
$$u^{(i)}(0)=0,\ 0\le i\le n-2, \quad D^{\delta}u(1)=0, \quad 1\le\delta\le n-2,$$

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where D^{α} is the standard Riemann-Liouville fractional derivative of order α and $f \in \mathcal{C}([0,1]\times[0,\infty))$, $n > 3$. The existence of positive solutions was analyzed by means of the Krasnoselskii's fixed point theorem on cones.

In [20], C.F.Li et al. considered the following boundary value problem of fractional derivative equations

$$\begin{aligned} D^{\alpha}u(t) + f(t,u) &= 0, \quad 0 < t < 1, \\ u(0) &= 0, \\ D^{\beta}u(1) &= aD^{\beta}u(\eta), \end{aligned}$$

where D^{α} is the standard Riemann-Liouville fractional derivative of order α, $1 < \alpha \le 2$, $0 < \beta \le 1$, $0 \le a \le 1$, $\eta \in (0,1)$ and $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous. Here, the argument relies on some fixed theorems on cones.

At the same time, boundary value problems for integer order differential systems are widely studied, despite fractional differential systems have emerged as a significant field of investigation quite recently. Thus intensive study of the existence theory of fractional systems has been carried out by means of methods of nonlinear analysis such as fixed point theory, lower and upper solutions, monotone iterative methods, see [11, 13, 14, 6, 7, 8, 5, 10, 22, 23, 24] and the references therein.

In this paper, we discuss the multiple positive solutions for the following systems of nonlinear fractional differential equations :

$$D^{q_1}u(t) + f_1(t,u(t),v(t)) = 0, \quad t \in (0,1), \tag{1}$$

$$D^{q_2}v(t) + f_2(t,u(t),v(t)) = 0, \quad t \in (0,1), \tag{2}$$

$$u(0) = u'(0) = 0, D^{p_1}u(1) = \mu D^{p_1}u(\eta) + g_1(\int_0^1 u(s)dA_1(s), \int_0^1 v(s)dA_1(s)), \tag{3}$$

$$v(0) = v'(0) = 0, D^{p_2}v(1) = \mu D^{p_2}v(\eta) + g_2(\int_0^1 u(s)dA_2(s), \int_0^1 v(s)dA_2(s)), \tag{4}$$

in which D is the Riemann-Liouville fractional derivative, $2 < q_i \le 3$ and $0 < p_i \le 1$, $0 < q_i - p_i - 1$ for $i = 1,2$, $0 < \eta < 1$, $\mu \in (0,\infty)$, $\mu\eta^{q_i-p_i-1} < 1$, $\int_0^1 u(s)dA_i(s)$ and $\int_0^1 v(s)dA_i(s)$ are the Riemann- Stieltjes integrals with positive measures, A_1 and A_2 are functions of bounded variation, $f_i \in \mathcal{C}([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$, $g_i \in \mathcal{C}([0,\infty]\times[0,\infty),[0,\infty))$ for $i = 1,2$.

Motivated by the above papers, our goal is to obtain the existence of multiple positive solutions for the fractional differential system (1)-(4). Here, we employ Riemann-Stieltjes integral boundary conditions. As they include multi-point and integral conditions as special cases, the system (1)-(4) is more general than the problems mentioned in some literature. Applying the Avery Peterson fixed point theorem, multiple positive solutions are established. An example is also presented to illustrate our main result.

In order to present our main result, we will make use of the following concepts and the Avery Peterson fixed point theorem.

Let φ and θ be nonnegative continuous convex functionals on the cone P, ϕ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then, for positive numbers a,b,c,d we define the following sets:

$$\begin{aligned}
P(\varphi, d) &= \{x \in P : \varphi(x) < d\}, \\
P(\varphi, \phi, b, d) &= \{x \in P : b \le \phi(x), \varphi(x) \le d\}, \\
P(\varphi, \theta, \phi, b, c, d) &= \{x \in P : b \le \phi(x), \theta(x) \le c, \varphi(x) \le d\}, \\
R(\varphi, \psi, a, d) &= \{x \in P : a \le \psi(x), \varphi(x) \le d\}.
\end{aligned}$$

Theorem 1. [18] *Let P be a cone in a real Banach space E. and φ, θ, ϕ, ψ be defined as above, furthermore ψ holds $\psi(kx) \le k\psi(x)$ for $0 \le k \le 1$ such that, for some positive numbers $\overline{M}$ and d,*

$$\phi(x) \le \psi(x) \text{ and } \|x\| \le \overline{M}\varphi(x)$$

for all $x \in \overline{P(\varphi, d)}$. Assume $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$ is completely continuous and there exist positive numbers a,b,c with $a < b$, such that

$(S_1) : \{x \in P(\varphi, \theta, \phi, b, c, d) : \phi(x) > b\} \neq \emptyset$ and $\phi(Tx) > b$ for $x \in P(\varphi, \theta, \phi, b, c, d)$,

$(S_2) : \phi(Tx) > b$ for $x \in P(\varphi, \phi, b, d)$ with $\theta(Tx) > c$,

$(S_3) : 0 \notin R(\varphi, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\varphi, \psi, a, d)$ with $\psi(x) = a$.

Then, T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, d)}$, such that

$\varphi(x_i) \le d$, for $i = 1, 2, 3$; $b < \phi(x_1)$, $a < \psi(x_2)$, with $\phi(x_2) < b$ and $\psi(x_3) < a$.

2. Existence Results

During the last decade, many definitions on the fractional calculus have been carried out. In our paper, our work is based upon the Riemann Liouville fractional operator defined by

$$D^{\nu} g(t) = \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{dt}\right)^{n} \int_0^t (t-s)^{n-\nu-1} g(s) ds,$$

where $g : (0, \infty) \to \mathcal{R}$ is a function, n is the smallest integer greater than or equal to ν whenever the right hand side is defined. In particular, for $\nu = n$, $D^{\nu} g(t) = D^n g(t)$.

In order to derive the main result of the system (1)-(4), we present the following lemma:

Lemma 2. *If $h, y \in \mathcal{C}[0, 1]$, then the fractional differential equation*

$$D^{q_1} u(t) + h(t) = 0, \quad t \in (0, 1), \tag{5}$$

$$D^{q_2} v(t) + y(t) = 0, \quad t \in (0, 1), \tag{6}$$

with the boundary conditions (3) *and* (4) *has the solution*

$$u(t) = \int_0^1 H_1(t, s) h(s) ds + \frac{t^{q_1 - 1}\Gamma(q_1 - p_1)}{\Gamma(q_1)\Delta_1} g_1\left(\int_0^1 u(s) dA_1(s), \int_0^1 v(s) dA_1(s)\right),$$

$$v(t) = \int_0^1 H_2(t,s)y(s)ds + \frac{t^{q_2-1}\Gamma(q_2-p_2)}{\Gamma(q_2)\Delta_2} g_2\Big(\int_0^1 u(s)dA_2(s), \int_0^1 v(s)dA_2(s)\Big),$$

where

$$H_i(t,s) = G_i(t,s) + \frac{t^{q_i-1}\mu}{\Gamma(q_i)\Delta_i}\overline{G_i}(\eta,s), \tag{7}$$

$$G_i(t,s) = \frac{1}{\Gamma(q_i)}\begin{cases} t^{q_i-1}(1-s)^{q_i-p_i-1} - (t-s)^{q_i-1}, & 0 \le s \le t \le 1, \\ t^{q_i-1}(1-s)^{q_i-p_i-1}, & 0 \le t \le s \le 1, \end{cases} \tag{8}$$

$$\overline{G}_i(\eta,s) = \begin{cases} \eta^{q_i-p_i-1}(1-s)^{q_i-p_i-1} - (\eta-s)^{q_i-p_i-1}, & 0 \le s \le \eta \le 1, \\ \eta^{q_i-p_i-1}(1-s)^{q_i-p_i-1}, & 0 \le \eta \le s \le 1, \end{cases} \tag{9}$$

and $\Delta_i = 1 - \mu\eta^{q_i-p_i-1}$, $(i \in \{1,2\})$.

Proof. The equations (5) and (6) can be translated into the following equations:

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(q_1)}\int_0^t (t-s)^{q_1-1}h(s)ds + c_1 t^{q_1-1} + c_2 t^{q_1-2} + c_3 t^{q_1-3}, \\ v(t) &= -\frac{1}{\Gamma(q_2)}\int_0^t (t-s)^{q_2-1}y(s)ds + d_1 t^{q_2-1} + d_2 t^{q_2-2} + d_3 t^{q_2-3}. \end{aligned}$$

Taking into account of (3)-(4) and $D^\sigma[t^{q-1}] = \frac{\Gamma(q)}{\Gamma(q-\sigma)}t^{q-\sigma-1}$ $(\sigma, q > 0)$, we obtain $c_2 = c_3 = 0$, $d_2 = d_3 = 0$ and

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(q_1)(1-\mu\eta^{q_1-p_1-1})}\int_0^1 (1-s)^{q_1-p_1-1}h(s)ds \\ &\quad - \frac{\mu}{\Gamma(q_1)(1-\mu\eta^{q_1-p_1-1})}\int_0^\eta (\eta-s)^{q_1-p_1-1}h(s)ds \\ &\quad + \frac{\Gamma(q_1-p_1)}{\Gamma(q_1)(1-\mu\eta^{q_1-p_1-1})} g_1\Big(\int_0^1 u(s)dA_1(s)), \int_0^1 v(s)dA_1(s)\Big), \end{aligned}$$

$$\begin{aligned} d_1 &= \frac{1}{\Gamma(q_2)(1-\mu\eta^{q_2-p_2-1})}\int_0^1 (1-s)^{q_2-p_2-1}y(s)ds \\ &\quad - \frac{\mu}{\Gamma(q_2)(1-\mu\eta^{q_2-p_2-1})}\int_0^\eta (\eta-s)^{q_2-p_2-1}y(s)ds \\ &\quad + \frac{\Gamma(q_2-p_2)}{\Gamma(q_2)(1-\mu\eta^{q_2-p_2-1})} g_2\Big(\int_0^1 u(s)dA_2(s)), \int_0^1 v(s)dA_2(s)\Big). \end{aligned}$$

So, the solution is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(q_1)}\int_0^t (t-s)^{q_1-1}h(s)ds \\ &\quad + \frac{t^{q_1-1}}{\Gamma(q_1)(1-\mu\eta^{q_1-p_1-1})}\int_0^1 (1-s)^{q_1-p_1-1}h(s)ds \end{aligned}$$

$$
\begin{aligned}
& -\frac{t^{q_1-1}\mu}{\Gamma(q_1)\Delta_1}\int_0^\eta(\eta-s)^{q_1-p_1-1}h(s)ds \\
& +\frac{t^{q_1-1}\Gamma(q_1-p_1)}{\Gamma(q_1)\Delta_1}g_1(\int_0^1 u(s)dA_1(s)),\int_0^1 v(s)dA_1(s)) \\
= \; & \int_0^1 H_1(t,s)h(s)ds+\frac{t^{q_1-1}\Gamma(q_1-p_1)}{\Gamma(q_1)\Delta_1}g_1(\int_0^1 u(s)dA_1(s)),\int_0^1 v(s)dA_1(s)),
\end{aligned}
$$

$$
\begin{aligned}
v(t) \; = \; & -\frac{1}{\Gamma(q_2)}\int_0^t(t-s)^{q_2-1}y(s)ds \\
& +\frac{t^{q_2-1}}{\Gamma(q_2)(1-\mu\eta^{q_2-p_2-1})}\int_0^1(1-s)^{q_2-p_2-1}y(s)ds \\
& -\frac{t^{q_2-1}\mu}{\Gamma(q_2)\Delta_2}\int_0^\eta(\eta-s)^{q_2-p_2-1}y(s)ds \\
& +\frac{t^{q_2-1}\Gamma(q_2-p_2)}{\Gamma(q_2)\Delta_2}g_2(\int_0^1 u(s)dA_2(s)),\int_0^1 v(s)dA_2(s)) \\
= \; & \int_0^1 H_2(t,s)y(s)ds+\frac{t^{q_2-1}\Gamma(q_2-p_2)}{\Gamma(q_2)\Delta_2}g_2(\int_0^1 u(s)dA_2(s)),\int_0^1 v(s)dA_2(s)).
\end{aligned}
$$

□

Lemma 3. *(See [2]) The function $G_i(t,s)$, $i\in\{1,2\}$ holds the following properties :*

(i) $G_i(t,s)\geq 0$ for any $t,s\in[0,1]$,

(ii) $p_i t^{q_i-1}L_i(s)\leq G_i(t,s)\leq L_i(s)$ for any $t,s\in[0,1]$,
where

$$L_i(s)=\frac{s(1-s)^{q_i-p_i-1}}{\Gamma(q_i)}. \tag{10}$$

One can easily obtain the following lemma.

Lemma 4. *The function $H_i(t,s)$, $i\in\{1,2\}$ holds the following properties :*

(i) $H_i(t,s)\geq 0$ for any $t,s\in[0,1]$,

(ii) $p_i t^{q_i-1}K_i(s)\leq H_i(t,s)\leq K_i(s)$ for any $t,s\in[0,1]$,

where $K_i(s)=\dfrac{s(1-s)^{q_i-p_i-1}}{\Gamma(q_i)}+\dfrac{\mu\overline{G}_i(\eta,s)}{\Gamma(q_i)\Delta_i}$.

Let us introduce the Banach space $\mathcal{B} = \mathcal{C}[0,1] \times \mathcal{C}[0,1]$ with the norm $\|(u,v)\| = \|u\| + \|v\|$ for $(u,v) \in \mathcal{B}$ and $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define a cone

$$P = \left\{ (u,v) \in \ \mathcal{B} : u(t) \geq 0, v(t) \geq 0, t \in [0,1], \min_{t\in[\eta,1]} (u(t)+v(t)) \geq p\|(u,v)\| \right\}$$

where $p = min\left\{p_1\eta^{q_1-1}, p_2\eta^{q_2-1}\right\}$and operators $T_i : P \to \mathcal{B}$, $i \in \{1,2\}$given by

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 H_1(t,s) f_1(s,u(s),v(s))ds \\ &\quad + \frac{t^{q_1-1}\Gamma(q_1-p_1)}{\Gamma(q_1)\Delta_1} g_1(\int_0^1 u(s)dA_1(s), \int_0^1 v(s)dA_1(s)), \\ T_2(u,v)(t) &= \int_0^1 H_2(t,s) f_2(s,u(s),v(s))ds \\ &\quad + \frac{t^{q_2-1}\Gamma(q_2-p_2)}{\Gamma(q_2)\Delta_2} g_2(\int_0^1 u(s)dA_2(s), \int_0^1 v(s)dA_2(s)). \end{aligned}$$

Let us set

$$\begin{aligned} N_i &= 4\int_0^1 K_i(s)ds, \\ m_i &= 2p\int_\eta^1 K_i(s)ds, \\ \overline{L_i} &= \frac{4\Gamma(q_i-p_i)\int_0^1 dA_i(s)}{\Gamma(q_i)\Delta_i}. \end{aligned}$$

To prove that the system $(1)-(4)$ has three positive solutions, the following three functionals are defined by

$$\phi(u,v) = \min_{t\in[\eta,1]} (u(t)+v(t)), \quad \psi(u,v) = \theta(u,v) = \varphi(u,v) = \|u\| + \|v\|.$$

The main theorem of this paper is stated as follows :

Theorem 5. *Assume that there exist constants* $0 < a < b < \frac{b}{p} < c < d$ *such that* $b \leq \frac{m_i d}{N_i}$ *and* f_i, g_i *hold the following conditions:*

(C_1) $f_i(t,u,v) \leq \dfrac{d}{N_i}$ *for* $t \in [0,1]$, $(u+v) \in [0,d]$,

(C_2) $f_i(t,u,v) > \dfrac{b}{m_i}$ *for* $t \in [\eta,1]$, $(u+v) \in [b,c]$,

(C_3) $f_i(t,u,v) \leq \dfrac{a}{N_i}$ *for* $t \in [0,1]$, $(u+v) \in [0,a]$,

(C_4) $g_i(u,v) \leq \dfrac{u+v}{\overline{L}_i}$ *for* $(u+v) \in [0, d\int_0^1 dA_i(s)]$.

Then the system $(1)-(4)$ *has at least three positive solutions* (u_i, v_i) $(i = 1, 2, 3)$ *such that* $\|(u_i, v_i)\| \leq d$, $i = 1, 2, 3$*;* $b \leq \phi(u_1, v_1)$, $a < \|\psi(u_2, v_2)\|$ *with* $\phi(u_2, v_2) < b$ *and* $\|(u_3, v_3)\| < a$.

Proof. Define the completely continuous operator $T : P \to \mathcal{B}$ by

$$T(u, v)(t) = (T_1(u, v)(t), T_2(u, v)(t)).$$

As easily seen, the fixed point of the operator T is the solution of the system $(1) - (4)$. First, we check that $T : P \to P$. Lemma 4 and the nonnegativity of f_i and g_i imply that $T_1(u, v)(t) \geq 0$, $T_2(u, v)(t) \geq 0$ for $t \in [0, 1]$. Besides, for $(u, v) \in P$

$$\begin{aligned}
\|T_1(u,v)\| &\leq \int_0^1 K_1(s) f_1(s, u(s), v(s)) ds \\
&\quad + \frac{\Gamma(q_1 - p_1)}{\Gamma(q_1)\Delta_1} g_1(\int_0^1 u(s) dA_1(s), \int_0^1 v(s) dA_1(s)), \\
\|T_2(u,v)\| &\leq \int_0^1 K_2(s) f_2(s, u(s), v(s)) ds \\
&\quad + \frac{\Gamma(q_2 - p_2)}{\Gamma(q_2)\Delta_2} g_2(\int_0^1 u(s) dA_2(s), \int_0^1 v(s) dA_2(s))
\end{aligned}$$

and

$$\begin{aligned}
\min_{t\in[\eta,1]} T_1(u,v)(t) &\geq p_1 \eta^{q_1-1} \int_0^1 K_1(s) f_1(s, u(s), v(s)) ds \\
&\quad + \frac{\eta^{q_1-1}\Gamma(q_1 - p_1)}{\Gamma(q_1)\Delta_1} g_1(\int_0^1 u(s) dA_1(s), \int_0^1 v(s) dA_1(s)) \\
&\geq p_1 \eta^{q_1-1} \|T_1(u,v)\|.
\end{aligned}$$

In a similar manner, we obtain $\min\limits_{t\in[\eta,1]} T_2(u,v)(t) \geq p_2 \eta^{q_2-1} \|T_2(u,v)\|$. Thus,

$$\begin{aligned}
\min_{t\in[\eta,1]} \{T_1(u,v)(t) + T_2(u,v)(t)\} &\geq p_1 \eta^{q_1-1} \|T_1(u,v)\| + p_2 \eta^{q_2-1} \|T_2(u,v)\| \\
&\geq p[\|T_1(u,v)\| + \|T_2(u,v)\|] \\
&= p\|T(u,v)\|,
\end{aligned}$$

so $T : P \to P$. Furthermore by employing standard methods, T is a completely continuous operator.

Now, all the conditions of Theorem 1 will be shown to be verified. First, we indicate that $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$. If $(u, v) \in \overline{P(\varphi, d)}$, then $\varphi(u, v) \leq d$, $\|u\| + \|v\| \leq d$. In view of C_4, we can get

$$g_i(\int_0^1 u(s) dA_i(s), \int_0^1 v(s) dA_i(s)) \quad \leq \quad \frac{\int_0^1 (u(s) + v(s)) dA_i(s)}{\overline{L}_i}$$

$$\leq \quad \frac{d\int_0^1 dA_i(s)}{\overline{L}_i}.$$

Hence, (C_1) yields that

$$\begin{aligned}
\max_{t\in[0,1]} T_1(u,v)(t) &= \max_{t\in[0,1]}\Big|\int_0^1 H_1(t,s)f_1(s,u(s),v(s))ds \\
&\quad + \frac{t^{q_1-1}\Gamma(q_1-p_1)g_1(\int_0^1 u(s)dA_1(s),\int_0^1 v(s)dA_1(s))}{\Gamma(q_1)\Delta_1}\Big| \\
&\leq \frac{d}{N_1}\int_0^1 K_1(s)ds + \frac{\Gamma(q_1-p_1)d}{\Gamma(q_1)\Delta_1\overline{L}_1}\int_0^1 dA_1(s) \\
&\leq \frac{d}{2}.
\end{aligned}$$

In the same way, one has $\max_{t\in[0,1]} T_2(u,v)(t) \leq \frac{d}{2}$. So, we have $T : \overline{P}(\varphi,d) \to \overline{P}(\varphi,d)$. Next, we indicate that (S_1) of Theorem 1 is fulfilled. Take $(\frac{b}{2p},\frac{b}{2p})$. Then, one may verify that $(\frac{b}{2p},\frac{b}{2p}) \in P(\varphi,\theta,\phi,b,c,d)$ and $\phi(u,v) > b$. Hence, $\{(u,v) \in P(\varphi,\theta,\phi,b,c,d) : \phi(u,v) > b\} \neq \emptyset$. Choose $(u,v) \in P(\varphi,\theta,\phi,b,c,d)$, then this means $(u(t)+v(t)) \in [b,c]$ for any $t \in [\eta,1]$. By C_2 we get

$$\begin{aligned}
\phi(T(u,v)) &= \min_{t\in[\eta,1]}(T_1(u,v)(t)+T_2(u,v)(t)) \\
&\geq p\int_\eta^1 K_1(s)f_1(s,u(s),v(s))ds + p\int_\eta^1 K_2(s)f_2(s,u(s),v(s))ds \\
&> p\frac{b}{m_1}\int_\eta^1 K_1(s)ds + p\frac{b}{m_2}\int_\eta^1 K_2(s)ds \\
&> b.
\end{aligned}$$

Thus (S_1) of Theorem 1 holds.

Finally, we need to show that the last condition of Theorem 1 is fulfilled. In fact, if $(u,v) \in P(\varphi,\phi,b,d)$ with $\theta(T(u,v)) > c$, then

$$\begin{aligned}
\min_{t\in[\eta,1]}(T_1(u,v)(t)+T_2(u,v)(t)) &\geq p\|T(u,v)\| \\
&> pc > b,
\end{aligned}$$

so, (S_2) holds.

Since $a > 0$, 0 is not member of $R(\varphi,\psi,a,d)$ with $\psi(u,v) = a$. Let $(u,v) \in R(\varphi,\psi,a,d)$ and $\psi(u,v) = a$, then using (C3), we get

$$\begin{aligned}
\psi(T(u,v)) &= \|T(u,v)\| \\
&\leq \int_0^1 K_1(s)f_1(s,u(s),v(s))ds + \int_0^1 K_2(s)f_2(s,u(s),v(s))ds
\end{aligned}$$

$$
\begin{aligned}
&\quad +\frac{\Gamma(q_1-p_1)g_1(\int_0^1 u(s)dA_1(s),\int_0^1 v(s)dA_1(s))}{\Gamma(q_1)\Delta_1}\\
&\quad +\frac{\Gamma(q_2-p_2)g_2(\int_0^1 u(s)dA_2(s),\int_0^1 v(s)dA_2(s))}{\Gamma(q_2)\Delta_2}\\
\leq &\quad \frac{a}{N_1}\int_0^1 K_1(s)ds+\frac{a}{N_2}\int_0^1 K_2(s)ds\\
&\quad +\frac{\Gamma(q_1-p_1)a}{\Gamma(q_1)\Delta_1\overline{L}_1}\int_0^1 dA_1(s)+\frac{\Gamma(q_2-p_2)a}{\Gamma(q_2)\Delta_2\overline{L}_2}\int_0^1 dA_2(s)\\
= &\quad a.
\end{aligned}
$$

Because all the condition of Theorem 1 fulfilled, the assertion of Theorem 5 is satisfied. The proof is complete. □

Example 6. *Consider*

$$
\begin{cases}
D^{5/2}u(t)+f_1(t,u(t),v(t))=0, \quad t\in(0,1),\\
D^{5/2}v(t)+f_2(t,u(t),v(t))=0, \quad t\in(0,1),\\
u(0)=u'(0)=v(0)=v'(0)=0,\\
D^{1/2}u(1)=1/2D^{1/2}u(1/2)+g_1(\int_0^1 u(s)dA_1(s),\int_0^1 v(s)dA_1(s)),\\
D^{1/2}v(1)=1/2D^{1/2}v(1/2)+g_2(\int_0^1 u(s)dA_2(s),\int_0^1 v(s)dA_2(s)),
\end{cases} \tag{11}
$$

in which $q_1=q_2=\frac{5}{2}$, $p_1=p_2=\frac{1}{2}$, $\mu=\frac{1}{2}$, $A_1(s)=A_2(s)=s^2$, $\eta=\frac{1}{2}$,

$$
f_1(t,u,v)=\begin{cases}
\frac{t}{7}+\frac{4(u+v)}{5}, & (u+v)\in[0,10],\\
\frac{t}{7}+\frac{642(u+v)-6340}{10}, & (u+v)\in[10,20],\\
\frac{t}{7}+\frac{5(u+v)+37600}{58}, & (u+v)\in[20,600],\\
\frac{t}{7}+700, & (u+v)\in[600,\infty),
\end{cases}
$$

$$
f_2(t,u,v)=\begin{cases}
\frac{t}{10}+\frac{4(u+v)}{5}, & (u+v)\in[0,10],\\
\frac{t}{10}+\frac{642(u+v)-6340}{10}, & (u+v)\in[10,20],\\
\frac{t}{10}+\frac{5(u+v)+37600}{58}, & (u+v)\in[20,600],\\
\frac{t}{10}+700, & (u+v)\in[600,\infty).
\end{cases}
$$

And

$$
g_i(u,v)=\begin{cases}
\frac{9\sqrt{\pi}}{64}ln(u+v+1), & (u+v)\in[0,600],\\
\frac{9\sqrt{\pi}}{64}ln(601), & (u+v)\in[600,\infty).
\end{cases}
$$

It is easily seen that $\Delta_1=\Delta_2=\frac{3}{4}$. *We obtain,* $N_1=N_2=\frac{4}{3\sqrt{\pi}}$, *then* $p=(\frac{1}{2})^{\frac{5}{2}}$, $m_1=m_2=\frac{1}{2^{\frac{5}{2}}3\sqrt{\pi}}$. *And* $\overline{L_1}=\overline{L_2}=\frac{64}{9\sqrt{\pi}}$. *Choosing,*

$f_1(t,u,v)\leq\frac{d}{N_1}\approx 1413,7$, *for* $t\in[0,1]$, $(u+v)\in[0,600]$,

$$f_1(t,u,v) \geq \frac{b}{m_1} \approx 601,59, \textit{ for } t \in [\tfrac{1}{2},1],\ (u+v) \in [20,200],$$

$$f_1(t,u,v) \leq \tfrac{a}{N_1} \approx 13,29 \textit{ for } t \in [0,1],\ (u+v) \in [0,10],$$

$$g_i(u,v) \leq \tfrac{u+v}{L_i} \textit{ for } (u+v) \in [0,600].$$

We conclude that all the assumptions of Theorem 5 are verified, thus the problem (11) has at least three positive solutions.

References

[1] Bai, Z., Sun,W., Existence and multiplicity of positive solutions for singular fractional boundary value problems, *Comput. Math. Appl.,* 63 (2012), 1369-1381.

[2] He, J., Jia, M., Liu, X., Chen, H., Existence of positive solutions for a high order fractional differential equation integral boundary value problem with changing sign nonlinearity, *Advances in Difference Equations,* (2018) 2018:49.

[3] Goodrich, C.S., On a fractional boundary value problem with fractional boundary conditions, *Appl. Math. Lett.,* 25 (2012), 1101-1105.

[4] Goodrich, C.S., Existence of a positive solution to a class of fractional differential equations, *Appl. Math. Lett.* (2010) 23:1050-1055

[5] Ahmad, B., Nieto J.J., Alsaedi A., Aqlan, M.H., A Coupled System of Caputo-Type Sequential Fractional Differential Equations with Coupled (Periodic/Anti-periodic Type) Boundary Conditions, *Mediterr. J. Math.* (2017) 14:227.

[6] Henderson, J., Luca, R., Positive solutions for a system of nonlocal fractional boundary value problems, *Fractional Calculus and Applied Analysis*, 16(4) (2013), 985-1008.

[7] Henderson, J., Luca, R., Systems of Riemann-Liouville fractional equations with multi-point boundary conditions, *Applied Mathematics and Computation,* 309 (2017), 303-323.

[8] Henderson, J. and Luca, R. : Positive solutions for a system of semipositone coupled fractional boundary value problems, *Boundary Value Problems* (2016), 2016:61

[9] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., Theory and applications of fractional differential equations, in: North-Holland Mathematics Studies, vol.204, Elsevier Science B.V, Amsterdam, 2006.

[10] Liu, Y., New existence results for positive solutions of boundary value problems for coupled systems of multi-term fractional differential equations, *Hacettepe Journal of Mathematics and Statistics,* 45(2) (2016), 391-416.

[11] Xie, S., Xie, Y., Positive solutions of higher order nonlinear fractional differential systems with nonlocal boundary conditions, *Journal of Applied Analysis and Computation*, 6(4), (2016), 1211-227.

[12] Podlubny, I., Fractional Differential Equations, Academic Press, San Diego, 1999.

[13] Su, X., Boundary value problem for a coupled system of nonlinear fractional differential equations, *Applied Mathematics Letters,* 22 (2009), 64-69.

[14] Wang, Y., Positive solutions for a system of fractional integral boundary value problem, *Boundary Value Problems,* (2013), 2013:256.

[15] Cabrera, I., Harjani, J., Sadarangani, K., Existence and uniqueness of solutions for a boundary value problem of fractional type with nonlocal integral boundary conditions in Holder spaces, *Mediterr. J. Math.,* (2018), 15-98.

[16] He, J., Zhang, X., Liu, L., Wu, Y., Cui, Y., Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions, *Boundary Value Problems,* (2018), 2018:189.

[17] Wang, Y., Liang, S., Wang, Q., Existence results for fractional differential equations with integral and multi-point boundary conditions, *Boundary Value Problems,* (2018), 2018:4.

[18] Avery, R.I., Peterson, A.C., Three positive fixed points of nonlinear operators on ordered Banach spaces, *Comput. Math. Appl.* 42 (2001), 313-322.

[19] Shah, K., Khan, R. A., Multiple positive solutions to a coupled systems of nonlinear fractional differential equations, *Springer Plus,* (2016), 5:1116.

[20] Li, C.F., Luo, X.N., Zuhou, Y., Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, *Computers and Mathematics with Applications* 59 (2010), 1363-1375.

[21] Jiang, J., Wang, H., Existence and uniqueness of solutions for a fractioanl differential equation with multi-point boundary value problems, *Journal of Applied Analysis and Computation,* 9(6) (2019), 2156-2168.

[22] Ali, A., Samet, B., Shah, K., Khan, R.A., Existence and stability of solution to a toppled systems of differential equations of non-integer order, *Boundary Value Problems,* (2017), 2017:16.

[23] Shah, K., Nonlocal boundary value problems for nonlinear toppled system of fractional differential equations, *Hacettepe Journal of Mathematics and Statistics,* 49 (1) (2020).

[24] Shah, K., Khan, R.A., Iterative scheme for a coupled system of fractional-order differential equations with three-point boundary conditions, *Mathematical Methods in the Applied Sciences,* 41(3) (2018), 1047-1053.

[25] Shah, K., Zeb, S., Khan, R.A., Multiplicity results of multi-point boundary value problem of nonlinear fractional differential equations, *Appl. Math. Inf. Sci.,* 12(3) (2018), 1-8.

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ON CERTAIN MULTIDIMENSIONAL NONLINEAR INTEGRALS

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Abstract. The aim of the paper is to obtain generalized convergence results for nonlinear multidimensional integrals of the form:

$$L_\eta(\omega;x)=\frac{\eta^n}{\Omega_{n-1}}\int\limits_D K(\eta\left|t-x\right|,\omega(t))dt.$$

We will prove some theorems concerning pointwise convergence of the family $L_\eta(\omega;x)$ as $\eta\to\infty$ at a fixed point $x\in D$ which represents any generalized Lebesgue point of the function $\omega\in L_1(D)$, where D is an open bounded subset of R^n. Moreover, we will consider the case $D=R^n$.

1. Introduction

The studies so far showed that Musielak [14] was the first researcher who investigated the approximation characteristics of convolution type nonlinear integral operators of the form:

$$T_w(f;s)=\int\limits_a^b K_w(x-s,f(x))dx, \tag{1.1}$$

where $s\in(a,b)\subset(-\infty,\infty)$, $w\in\mathcal{I}$ and $\mathcal{I}$ is a non-empty index set. His research was an intriguing contribution to literature related to this kind of nonlinear integral operators. Later, Swiderski and Wachnicki [19] studied the pointwise convergence of the operators of type (1.1). Extensive knowledge concerning this theory can be found in the monograph by Bardaro *et al.* [7]. Later on, multidimensional counterparts of the operators of type (1.1) were studied by Angeloni and Vinti [6] in some function spaces. Then, Jackson-type generalization of the operators defined

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in [6] were considered by Yilmaz [22]. Some studies on nonlinear operators in different settings can be found in [5, 9, 13, 21]. Also, results and applications in wide range concerning linear operators can be found in [1, 4, 8, 12, 17, 20]. Some weighted approximation results concerning well-known Gauss-Weierstrass and Picard integral operators can be found in the recent articles [23] and [24], respectively. In [16], a class of summation-integral-type operators covering many well-known ones was considered.

In the year 2016, Almali and Gadjiev [3] considered the following certain nonlinear integrals:

$$L_\eta(\omega; x) = \frac{\eta^n}{\Omega_{n-1}} \int\limits_D K(\eta\,|t-x|\,, \omega(t))dt, \tag{1.2}$$

where $D = R^n,\ t, x \in R^n,\ |t-x| = \sqrt{\sum\limits_{k=1}^{n} (t_k - x_k)^2}$ and Ω_{n-1} is the surface area of unit sphere $S^{n-1} = \{x \in R^n : |x| = 1\}$ in R^n. Here, R^n denotes usual $n-$dimensional Euclidean space. Also, a real number η is considered as a positive parameter. They obtained pointwise convergence result for Lebesgue points of integrable functions. In the same article, exponential nonlinear integrals were also introduced. Some related works can be given as [2, 11].

The aim of the current manuscript is to obtain convergence results for the operators of type (1.2) in two different settings via assigning two different definitions to a symbol D. We will prove pointwise convergence of the family $L_\eta(\omega; x)$ as $\eta \to \infty$ at a fixed point $x \in D$ which represents any generalized Lebesgue point of function $\omega \in L_1(D)$, where D is an open bounded subset of R^n, and $\omega \in L_1(R^n)$, separately. The space $L_1(D)$ consists of the measurable functions satisfying $\int\limits_D |\omega(t)|\, dt < \infty$. The norm formula in this space is given as follows: $\|\omega\|_{L_1(D)} = \int\limits_D |\omega(t)|\, dt$. The definition of the space $L_1(R^n)$ is analogous. Our results generalize and improve Theorem 2.2 in [3] in two different directions in view of the usages of generalized domain of integration and generalized characteristic point, respectively. Now, we consider the kernel function of the operators of type (1.2). Since $\eta\,|.| \in R_0^+$, for simplicity, we may denote $\eta\,|.|$ by $\eta\nu$. Therefore, $K(\eta\,|.|\,, \omega(t)) =: K(\eta\nu, \omega)$, where $\nu \in R_0^+$ and $\omega : R^n \to R$.

The conditions on the kernel function to be given below are revised versions of the conditions used by Almali and Gadjiev [3].

We assume that real-valued kernel function $K(\eta\nu, \omega)$, where $\eta\nu \in R_0^+$ and $\omega : R^n \to R$, satisfies the following conditions:

a: For every $\nu \in R_0^+$ and $\eta \in R^+$, $K(\eta\nu, 0) = 0$ and $K(\eta\nu, \omega)$ is analytic at $\omega = 0$ with radius of analyticity $\mathfrak{R} = \infty$ for all values of its first variable, that is, its Maclaurin series converges for all $\omega \in R$ and for all values of its first variable.

b: $\left.\frac{\partial^m K(\eta\nu,\omega)}{\partial\omega^m}\right|_{\omega=0}$ is a non-negative and non-increasing function with respect to ν on R_0^+ for any $m = 1, \ldots$ and for all values of $\eta \in R^+$.

c: The first partial derivative $\left.\frac{\partial K(\eta\nu,\omega)}{\partial\omega}\right|_{\omega=0}$ is a majorant function for all remaining derivatives, that is, $\left.\frac{\partial^m K(\eta\nu,\omega)}{\partial\omega^m}\right|_{\omega=0} \leq \left.\frac{\partial K(\eta\nu,\omega)}{\partial\omega}\right|_{\omega=0}$, where $m = 1, \ldots,$ for all values of $\nu \in R_0^+$ and $\eta \in R^+$.

d: $\frac{\eta^n}{\Omega_{n-1}} \int\limits_{R^n} \left.\frac{\partial^m K(\eta|t|,\omega)}{\partial\omega^m}\right|_{\omega=0} dt = A_m < \infty$, where A_m with $m = 1, \ldots$ are certain positive constants which are independent of η and

$$\lim_{\eta\to\infty} \frac{\eta^n}{\Omega_{n-1}} \int\limits_{\zeta<|t|<\infty} \left.\frac{\partial^m K(\eta\,|t|\,,\omega)}{\partial\omega^m}\right|_{\omega=0} dt = 0$$

for all $\zeta > 0$ and $m = 1, \ldots.$

Definition 1. *A point $x \in R^n$ at which the following relation holds:*

$$\lim_{r\to0^+} \frac{1}{r^{n(\alpha+1)}} \int\limits_{0<|t|\leq r} |\omega(t+x) - \omega(x)|\, dt = 0,$$

where $0 \leq \alpha < 1$, is called a generalized Lebesgue point of function $\omega \in L_1(R^n)$ (or for any function ω which is integrable on sufficiently large domain).

Definition of one-dimensional version of this point can be found in some recent papers, such as [13] and [5]. Definition of $d-$point analogue of this point in one-dimensional case was also considered by Gadjiev [12].

2. Main Theorems

Theorem 1. *Suppose that $K(.,\omega)$ satisfies conditions (a)-(d). If $x \in R^n$ is a generalized Lebesgue point of function $\omega \in L_1(R^n)$ and ω is a bounded function on R^n, that is, there exists a number $M > 0$ which depends on only ω such that $|\omega| \leq M$, then for the operators $L_\eta(\omega; x)$ which are defined in (1.2), we have*

$$\lim_{\eta\to\infty} L_\eta(\omega; x) = \sum_{m=1}^{\infty} \frac{A_m}{m!} [\omega(x)]^m$$

provided that the function

$$\eta^n \int\limits_{0<r<\infty} \left\{r^{n(\alpha+1)}\right\}'_r \left.\frac{\partial K(\eta r,\omega)}{\partial\omega}\right|_{\omega=0} dr,$$

where $r = |t|$, is bounded as $\eta \to \infty$.

Proof. By definition of generalized Lebesgue point, for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\int\limits_{0<|t|\leq r} |\omega(t+x)-\omega(x)|\,dt < \varepsilon r^{n(\alpha+1)}$$

holds provided that $r \leq \delta$ and $0 \leq \alpha < 1$.

Denoting the surface of unit sphere $\{t' \in R^n : |t'| = 1\}$ in R^n by S^{n-1}, we define

$$\int\limits_{S^{n-1}} |\omega(rt'+x)-\omega(x)|\,dt' =: u\,(r)\,,$$

where dt' is the surface element on S^{n-1} (see p.14 in [18]). For further details about polar coordinates transformation, we refer the reader to [10]. Therefore, we define the auxiliary function as

$$f(r) := \int\limits_0^r u\,(\rho)\,\rho^{n-1} d\rho \tag{2.1}$$

for which there holds:

$$f(r) \leq \varepsilon r^{n(\alpha+1)} \tag{2.2}$$

provided that $r \leq \delta$ and $0 \leq \alpha < 1$.

Following [3], we write the Maclaurin expansion of the function $K(.,\omega)$ with respect to ω as follows:

$$\begin{aligned} K(\eta\,|t-x|\,,\omega(t)) &= \sum_{m=0}^{\infty} \frac{1}{m!} K_{\omega}^{(m)}(\eta\,|t-x|\,,0)\,[\omega(t)]^m \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} K_{\omega}^{(m)}(\eta\,|t-x|\,,0)\,[\omega(t)]^m\,, \end{aligned}$$

where $K_{\omega}^{(m)}(\eta\,|t-x|\,,0) := \left.\frac{\partial^m K(\eta|t-x|,\omega)}{\partial\omega^m}\right|_{\omega=0}$ and for every $\nu \in R_0^+$ and $\eta \in R^+$ with $K(\eta\nu, 0) = 0$. Since the conditions of Lebesgue dominated converge theorem (see, for example, [15]) are fulfilled, we can change the order of summation and integration. Since R^n is a locally compact abelian group, using change of variables and binomial representation of $[\omega(t+x)]^m$, we have

$$\begin{aligned} L_\eta(\omega;x) &= \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_{R^n} \frac{1}{m!} K_{\omega}^{(m)}(\eta\,|t|\,,0) \sum_{k=0}^{m-1} \left(\begin{array}{c} m \\ k \end{array} \right) [\omega(t+x)-\omega(x)]^{m-k}\,[\omega(x)]^k\,dt \\ &\quad + \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_{R^n} \frac{1}{m!} K_{\omega}^{(m)}(\eta\,|t|\,,0)\,[\omega(x)]^m\,dt. \end{aligned}$$

Let

$$I=\sum_{k=0}^{m-1}\binom{m}{k}\left[\omega(t+x)-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}.$$

Now, without loss of generality, we consider the case ω is not identically zero on R^n. Since ω is bounded by a certain positive number M such that $|\omega(z)|\leq M$ for all $z\in R^n$, there holds

$$\begin{aligned}
|I| &= \left|\sum_{k=0}^{m-1}\binom{m}{k}\left[\omega(t+x)-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}\right| \\
&\leq \sum_{k=0}^{m-1}\binom{m}{k}\left|\omega(t+x)-\omega(x)\right|^{m-k-1}\left|\omega(t+x)-\omega(x)\right|\left|\omega(x)\right|^{k} \\
&\leq \left|\omega(t+x)-\omega(x)\right|\sum_{k=0}^{m-1}\binom{m}{k}(2M)^{m-k-1}(M)^{k} \\
&\leq \left|\omega(t+x)-\omega(x)\right|\frac{1}{2M}\sum_{k=0}^{m-1}\binom{m}{k}(2M)^{m-k}(M)^{k} \\
&\quad+\left|\omega(t+x)-\omega(x)\right|\frac{1}{2M}\binom{m}{m}(2M)^{m-m}(M)^{m} \\
&= \left|\omega(t+x)-\omega(x)\right|\frac{1}{2M}\sum_{k=0}^{m}\binom{m}{k}(2M)^{m-k}(M)^{k} \\
&= \left|\omega(t+x)-\omega(x)\right|\frac{1}{2M}(3M)^{m}.
\end{aligned}$$

Therefore, using condition (c), we can write

$$\begin{aligned}
&\left|L_{\eta}(\omega;x)-\sum_{m=1}^{\infty}\frac{1}{m!}A_m\left[\omega(x)\right]^{m}\right| \\
&\leq \frac{\eta^n}{\Omega_{n-1}}\sum_{m=1}^{\infty}\int\limits_{R^n}\frac{1}{m!}K_{\omega}^{(m)}(\eta\left|t\right|,0)\left|\omega(t+x)-\omega(x)\right|\frac{(3M)^{m}}{2M}dt \\
&\leq \frac{\eta^n}{\Omega_{n-1}}\frac{1}{2M}\sum_{m=1}^{\infty}\frac{(3M)^{m}}{m!}\int\limits_{R^n}K_{\omega}^{(1)}(\eta\left|t\right|,0)\left|\omega(t+x)-\omega(x)\right|dt.
\end{aligned}$$

Fixing $\delta>0$, we have the following inequality:

$$\left|L_{\eta}(\omega;x)-\sum_{m=1}^{\infty}\frac{1}{m!}A_m\left[\omega(x)\right]^{m}\right|$$

$$\begin{aligned}
&\leq \frac{\eta^n}{\Omega_{n-1}} \frac{\left(e^{3M}-1\right)}{2M} \left\{ \int\limits_{|t|>\delta} + \int\limits_{|t|\leq\delta} \right\} K_\omega^{(1)}(\eta\,|t|\,,0)\,|\omega(t+x)-\omega(x)|\,dt \\
&= : \frac{1}{\Omega_{n-1}} \frac{\left(e^{3M}-1\right)}{2M} \left\{\eta^n I'_\eta + \eta^n I''_\eta\right\}.
\end{aligned}$$

Let us show that $\eta^n I'_\eta \to 0$ as $\eta \to \infty$. The following deductions are the natural consequences of conditions satisfied by our kernel function. Since

$$0 \leq \Omega_{n-1} K_\omega^{(1)}(\eta r,0)\frac{1}{n}r^n\left(1-\frac{1}{2^n}\right) \leq \int\limits_{\frac{r}{2}\leq|t|\leq r} K_\omega^{(1)}(\eta\,|t|\,,0)dt,$$

by (d) and well-known squeeze theorem, we see that $r^n K_\omega^{(1)}(\eta r,0) \to 0$ as $r \to \infty$ and $r \to 0$. In particular, this observation leads to $K_\omega^{(1)}(\eta r,0) \to 0$ as $r \to \infty$ and $r \to 0$. This type analysis is also performed in [3,18]. For $\eta^n I'_\eta$, we obtain

$$\begin{aligned}
\eta^n I'_\eta &\leq \eta^n K_\omega^{(1)}(\eta\delta,0)\int\limits_{\delta}^{\infty}\int\limits_{S^{n-1}} |\omega(rt'+x)|\,r^{n-1}dt'dr \\
&\quad + \eta^n\,|\omega(x)| \int\limits_{\delta<|t|<\infty} K_\omega^{(1)}(\eta\,|t|\,,0)dt \\
&\leq \eta^n K_\omega^{(1)}(\eta\delta,0)\,\|\omega\|_{L_1(R^n)} + \eta^n\,|\omega(x)| \int\limits_{\delta<|t|<\infty} K_\omega^{(1)}(\eta\,|t|\,,0)dt.
\end{aligned}$$

The terms on the right-hand side tend to zero as $\eta \to \infty$ by overall hypotheses discussed previously. Hence, $\lim\limits_{\eta\to\infty} \eta^n I'_\eta = 0$.

Now, we consider $\eta^n I''_\eta$. By relation (2.1), we can write

$$\begin{aligned}
\eta^n I''_\eta &= \eta^n \int\limits_0^\delta \int\limits_{S^{n-1}} |\omega(rt'+x)-\omega(x)|\,K_\omega^{(1)}(\eta r,0)r^{n-1}dt'dr \\
&= \eta^n \int\limits_0^\delta u\,(r)\,K_\omega^{(1)}(\eta r,0)r^{n-1}dr \\
&= \eta^n \int\limits_0^\delta K_\omega^{(1)}(\eta r,0)df(r).
\end{aligned}$$

Using integration by parts for Stieltjes integrals and relation (2.2), we get the following inequality:

$$\eta^n I''_\eta \leq \varepsilon \eta^n \int\limits_0^\infty \left\{ r^{n(\alpha+1)} \right\}'_r K_\omega^{(1)}(\eta r, 0) dr.$$

Since $\varepsilon > 0$ is arbitrarily small and the following expression:

$$\eta^n \int\limits_0^\infty \left\{ r^{n(\alpha+1)} \right\}'_r K_\omega^{(1)}(\eta r, 0) dr$$

remains bounded as $\eta \to \infty$, we have

$$\lim_{\eta \to \infty} \eta^n I''_\eta = 0.$$

Combining all results gives

$$\lim_{\eta \to \infty} L_\eta(\omega; x) = \sum_{m=1}^{\infty} \frac{1}{m!} A_m \left[\omega(x)\right]^m.$$

Thus, the proof is completed. □

In the second theorem, we give a local approximation result for nonlinear multidimensional integrals of the form:

$$T_\eta(\omega; x) = \frac{\eta^n}{\Omega_{n-1}} \int\limits_D K(\eta \left|t - x\right|, \omega(t)) dt, \tag{2.3}$$

where $x \in D$ and D is any bounded open subset of R^n. We replaced R^n by D compared to operators of type (1.2).

Theorem 2. *Suppose that $K(.,\omega)$ satisfies conditions (a)-(d). If $x \in D$ is a generalized Lebesgue point of function $\omega \in L_1(D)$ with $\omega : R^n \to R$ and ω is a bounded function on D, that is, there exists a number $P > 0$ which depends on only ω such that $|\omega| \leq P$, then for the operators $T_\eta(\omega; x)$ which are defined in (2.3), we have*

$$\lim_{\eta \to \infty} T_\eta(\omega; x) = \sum_{m=1}^{\infty} \frac{A_m}{m!} \left[\omega(x)\right]^m$$

provided that the function

$$\eta^n \int\limits_{0 < r \leq \delta} \left\{ r^{n(\alpha+1)} \right\}'_r \left. \frac{\partial K(\eta r, \omega)}{\partial \omega} \right|_{\omega = 0} dr,$$

where $r = |t|$ and $\delta > 0$ is a number chosen to ensure the existence of the integral, is bounded as $\eta \to \infty$.

Proof. We follow mainly the proof steps of previous theorem with some additional considerations.

By definition of generalized Lebesgue point, for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\int\limits_{0<|t|\leq r} |\omega(t+x) - \omega(x)|\, dt < \varepsilon r^{n(\alpha+1)}$$

holds provided that $r \leq \delta$ and $0 \leq \alpha < 1$.

Denoting the surface of unit sphere $\{t' \in R^n : |t'| = 1\}$ in R^n by S^{n-1}, we define

$$\int\limits_{S^{n-1}} |\omega(rt' + x) - \omega(x)|\, dt' =: \widetilde{u}\,(r)\,,$$

where dt' is the surface element on S^{n-1} (see p. 14 in [18]). Therefore, we define the new function as

$$\widetilde{f}(r) := \int\limits_0^r \widetilde{u}\,(\rho)\, \rho^{n-1} d\rho$$

for which there holds:

$$\widetilde{f}\,(r) \leq \varepsilon r^{n(\alpha+1)}$$

provided that $r \leq \delta$ and $0 \leq \alpha < 1$.

Now, we define the auxiliary function g by

$$g(t) := \begin{cases} \omega(t), & t \in D, \\ 0, & t \in R^n \backslash D. \end{cases} \tag{2.4}$$

We recall the Maclaurin series of $K(.,\omega)$ at $\omega = 0$ expressed as

$$K(\eta\,|t-x|\,, \omega(t)) = \sum_{m=1}^{\infty} \frac{1}{m!} K_\omega^{(m)}(\eta\,|t-x|\,, 0)\,[\omega(t)]^m\,,$$

where $K_\omega^{(m)}(\eta\,|t-x|\,,0) := \left.\frac{\partial^m K(\eta|t-x|,\omega)}{\partial\omega^m}\right|_{\omega=0}$ and for every $\nu \in R_0^+$ and $\eta \in R^+$ with $K(\eta\nu, 0) = 0$. In view of this, we infer that

$$\begin{aligned} & T_\eta(\omega; x) \\ = \quad & \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_D \frac{1}{m!} K_\omega^{(m)}(\eta\,|t-x|\,,0) \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} [\omega(t) - \omega(x)]^{m-k}\, [\omega(x)]^k\, dt \\ & + \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_{R^n} \frac{1}{m!} K_\omega^{(m)}(\eta\,|t-x|\,,0)\,[\omega(x)]^m\, dt \\ & - \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_{R^n} \frac{1}{m!} K_\omega^{(m)}(\eta\,|t-x|\,,0)\,[\omega(x)]^m\, dt. \end{aligned}$$

Let

$$I=\sum_{k=0}^{m-1}\binom{m}{k}\left[\omega(t)-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}.$$

Without loss of generality, we consider the case ω is not identically zero on D. Since ω is bounded by a certain positive number P such that $|\omega(z)|\leq P$ for all $z\in D$, there holds

$$\begin{aligned}
|I| &= \left|\sum_{k=0}^{m-1}\binom{m}{k}\left[\omega(t)-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}\right| \\
&\leq \sum_{k=0}^{m-1}\binom{m}{k}\left|\omega(t)-\omega(x)\right|^{m-k-1}\left|\omega(t)-\omega(x)\right|\left|\omega(x)\right|^{k} \\
&\leq \left|\omega(t)-\omega(x)\right|\frac{1}{2P}\left(3P\right)^{m}.
\end{aligned}$$

Therefore, in view of (2.4) and using condition (c), we obtain the following inequality:

$$\begin{aligned}
&\left|T_{\eta}(\omega;x)-\sum_{m=1}^{\infty}\frac{1}{m!}A_{m}\left[\omega(x)\right]^{m}\right| \\
\leq\;& \frac{\eta^{n}}{\Omega_{n-1}}\sum_{m=1}^{\infty}\int\limits_{D}\frac{1}{m!}K_{\omega}^{(m)}(\eta\left|t-x\right|,0)\left|\omega(t)-\omega(x)\right|\frac{(3P)^{m}}{2P}dt \\
&+\left|\frac{\eta^{n}}{\Omega_{n-1}}\sum_{m=1}^{\infty}\int\limits_{R^{n}\setminus D}\frac{1}{m!}K_{\omega}^{(m)}(\eta\left|t-x\right|,0)\sum_{k=0}^{m-1}\binom{m}{k}\left[-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}dt\right| \\
\leq\;& \frac{\eta^{n}}{\Omega_{n-1}}\frac{1}{2P}\sum_{m=1}^{\infty}\frac{(3P)^{m}}{m!}\int\limits_{D}K_{\omega}^{(1)}(\eta\left|t-x\right|,0)\left|\omega(t)-\omega(x)\right|dt \\
&+\left|\frac{\eta^{n}}{\Omega_{n-1}}\sum_{m=1}^{\infty}\int\limits_{R^{n}\setminus D}\frac{1}{m!}K_{\omega}^{(1)}(\eta\left|t-x\right|,0)\sum_{k=0}^{m-1}\binom{m}{k}\left[-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}dt\right| \\
=\;& :I_{1}+I_{2}.
\end{aligned}$$

Fixing $\delta>0$, we define $B_{\delta}:=\{t,x\in D:|t-x|\leq\delta\}\subset D$. Therefore, we have the following inequality:

$$I_{1}\leq\frac{\eta^{n}}{\Omega_{n-1}}\frac{\left(e^{3P}-1\right)}{2P}\int\limits_{t\in D\setminus B_{\delta}}K_{\omega}^{(1)}(\eta\left|t-x\right|,0)\left|\omega(t)-\omega(x)\right|dt$$

$$\begin{aligned}
&+\frac{\eta^n}{\Omega_{n-1}}\frac{\left(e^{3P}-1\right)}{2P}\int\limits_{|t|\leq\delta}K_\omega^{(1)}(\eta\,|t|\,,0)\,|\omega(t+x)-\omega(x)|\,dt\\
=\quad&:\frac{1}{\Omega_{n-1}}\frac{\left(e^{3P}-1\right)}{2P}\left\{\eta^n I'_\eta+\eta^n I''_\eta\right\}.
\end{aligned}$$

Let us show that $\eta^n I'_\eta \to 0$ as $\eta \to \infty$. For $\eta^n I'_\eta$, we obtain

$$\eta^n I'_\eta \leq \eta^n K_\omega^{(1)}(\eta\delta,0)\,\|\omega\|_{L_1(D)}+\eta^n\,|\omega(x)|\int\limits_{\delta<|t|<\infty}K_\omega^{(1)}(\eta\,|t|\,,0)dt.$$

The terms on the right-hand side tend to zero as $\eta\to\infty$ by (d). Hence, $\lim\limits_{\eta\to\infty}\eta^n I'_\eta = 0$.

It is easy to see that I_2 tends to zero as $\eta\to\infty$. The remaining part is analogous to proof of the preceding theorem. Hence

$$\lim_{\eta\to\infty}T_\eta(\omega;x)=\sum_{m=1}^{\infty}\frac{1}{m!}A_m\left[\omega(x)\right]^m.$$

Thus, the proof is completed. □

Example 1. *In [3], the authors considered the following kernel function satisfying the hypotheses:*

$$K\left(\eta v,\omega\right)=\frac{1}{\sqrt{2\pi}}\left[\exp\left(e^{-(\eta v)^2}\omega\right)-1\right].$$

Inspiring from the kernel given above and also Picard kernel, we consider the following kernel function without scaling:

$$K\left(\eta v,\omega\right)=\exp\left(e^{-\eta v}\omega\right)-1,$$

where for $\eta v\in R_0^+$, $K\left(\eta v,0\right)=0$ and $K_\omega^{(m)}(\eta v,0)=e^{-m\eta v}$ with $m=1,\ldots$. Clearly, this function is non-negative and non-increasing with respect to ν on R_0^+ for any $m=1,\ldots$ and for all values of $\eta\in R^+$, and the first partial derivative majorizes the remaining derivatives. Lastly, in view of the well-known identity related to gamma function

$$\int\limits_0^\infty e^{-m\lambda}\lambda^{n-1}d\lambda=\frac{(n-1)!}{m^n},$$

where $\lambda=\eta v$, we see that the condition (d) easily holds there.

3. Final Comments

Some theorems which are analogous to Theorem (3.3) and Theorem (3.5) in the article by Almali and Gadjiev [3] can be stated and proved using similar arguments. Also, more general theorems with respect to other characteristic points, such as $\mu-$generalized Lebesgue point, can also be proved.

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References

[1] Aliev, I. A., Gadjiev, A. D., Aral, A., On approximation properties of a family of linear operators at critical value of parameter, *J. Approx. Theory,* 138(2) (2006), 242–253.

[2] Almali, S. E., On pointwise convergence of the family of Urysohn-type integral operators, *Math. Methods Appl. Sci.,* 42(16) (2019), 5346–5353.

[3] Almali, S. E., Gadjiev, A. D., On approximation properties of certain multidimensional nonlinear integrals, *J. Nonlinear Sci. Appl.,* 9(5) (2016), 3090-3097.

[4] Altomare, F., Campiti, M., Korovkin-type Approximation Theory and Its Applications, Appendix A by Michael Pannenberg and Appendix B by Ferdinand Beckhoff, Berlin: De Gruyter Studies in Mathematics 17., Walter de Gruyter & Company, 1994.

[5] Altin, H. E., On pointwise approximation properties of certain nonlinear Bernstein operators, *Tbilisi Math. J.,* 12(2) (2019), 47–58.

[6] Angeloni, L., Vinti, G., Convergence in variation and rate of approximation for nonlinear integral operators of convolution type, *Results Math.,* 49(1-2) (2006), 1–23. (Erratum: Results Math. 57 (2010), no. 3-4, 387–391)

[7] Bardaro, C., Musielak, J., Vinti, G., Nonlinear Integral Operators and Applications, De Gruyter Series in Nonlinear Analysis and Applications, 9., Berlin: Walter de Gruyter & Company, 2003.

[8] Butzer, P. L., Nessel, R. J., Fourier analysis and approximation Vol. 1: One-dimensional Theory, Pure and Applied Mathematics, Vol. 40., New York-London: Academic Press, 1971.

[9] Costarelli, D., Vinti, G., Approximation by nonlinear multivariate sampling Kantorovich type operators and applications to image processing, *Numer. Funct. Anal. Optim.,* 34(8) (2013), 819–844.

[10] Folland, G. B., Real Analysis: Modern Techniques and Their Applications, Pure and Applied Mathematics, New York: Wiley, 1999.

[11] Gadjiev, A. D., On nearness to zero of a family of nonlinear integral operators of Hammerstein, (Russian-Azerbaijani summary) *Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk,* 2 (1966), 32–34.

[12] Gadjiev, A. D., The order of convergence of singular integrals which depend on two parameters, (Russian), In: Special problems of functional analysis and their applications to the theory of differential equations and the theory of functions (Russian), (1968), 40–44.

[13] Karsli, H., On approximation properties of non-convolution type nonlinear integral operators, *Anal. Theory Appl.,* 26(2) (2010), 140–152.

[14] Musielak, J., On some approximation problems in modular spaces, *Constructive function theory '81 (Varna, 1981), Publ. House Bulgar. Acad. Sci., Sofia* (1983), 455–461.

[15] Rudin, W., Real and Complex Analysis, Third edition., New York: McGraw-Hill Book Company, 1987.

[16] Serenbay, S. K., Yilmaz, M. M., Ibikli, E., The convergence of a family of integral operators (in Lp space) with a positive kernel, *J. Comput. Appl. Math.,* 235(16) (2011), 4567–4575.

[17] Stein, E. M., Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, No. 30. Princeton-New Jersey: Princeton University Press, 1970.

[18] Stein, E. M., Weiss, G., Introduction to Fourier analysis on Euclidean Spaces, Princeton Mathematical Series, No. 32. Princeton-New Jersey: Princeton University Press, 1971.

[19] Świderski, T., Wachnicki, E., Nonlinear singular integrals depending on two parameters, *Comment. Math. (Prace Mat.)*, 40 (2000), 181–189.

[20] Taberski, R., Singular integrals depending on two parameters, *Prace Mat.,* 7 (1962), 173-179.

[21] Uysal, G., Mishra, V. N., Serenbay, S. K., Some weighted approximation properties of nonlinear double integral operators, *Korean J. Math.,* 26(3) (2018), 483–501.

[22] Yilmaz, B., Jackson type generalization of nonlinear integral operators, *J. Comput. Anal. Appl.,* 17(1) (2014), 77–83.

[23] Yilmaz, B., On some approximation properties of the Gauss-Weierstrass operators, *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.,* 68(2) (2019), 2154–2160.

[24] Yilmaz, B., Ari, D. A., A note on the modified Picard integral operators, *Math. Methods Appl. Sci.,* (2020), 1-6. https://doi.org/10.1002/mma.6360

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ON FRONTIER AND EXTERIOR IN INTUTIONISTIC SUPRA $\alpha-$ CLOSED SET

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Abstract. The main aim of the study of this paper is to work with the properties of frontier and exterior in intuitionistic supra topological spaces. Considering this we have introduced intuitionistic supra α-frontier and intuitionistic α-exterior in intuitionistic supra topological space. We have also deliberated the properties of intuitionistic suppra α-frontier and intuitionistic supra α-exterior in intuitionistic supra topological space. The comparative study has been done with the use of intuitionistic supra α-open set between Intuitionistic supra frontier, Intuitionistic supra exterior and intuitionistic supra α-frontier, intuitionistic α-exterior in intuitionistic supra topological space.

1. Introduction

In 1970, Levine[4] introduced the concept of generalized closed sets in topological spaces. Njastad.O[12] and Maki.H et al[6] introduced α-closed sets and gα-closed sets in topological spaces. In 1965 ,O.Njastad[12] introduced α-open sets. The concept of intuitionistic set and intuitionistic topological spaces was introduced by Coker[1][2]. Supra topology was introduced by A.S.Mashhour et.al[6] Intuitionistic supra α-open set was introduced by the Author[8] on intuitionistic supra topological spaces and discussed the properties of Intuitionistic supra α-open sets in supra topological spaces.

The purpose of this paper is to study the properties of α-frontier and α-exterior in intuitionistic supra topological spaces. Also to study the comparison between Intuitionistic supra frontier, Intuitionistic supra exterior and intuitionistic supra α-frontier, α-exterior in intuitionistic supra topological space.

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2. Preliminaries

Definition 2.1 [1] Let X be a non-empty set,an intuitionistic set (IS in short) A is an object having the form $A = \langle X, A_1, A_2\rangle$, where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \phi$. The set A_1 is called the set members of A, while A_2 is called the set of non-members of A.

Definition 2.2 [1] Let X be a non-empty set, $A = \langle X, A_1, A_2\rangle$ and $B = \langle X, B_1, B_2\rangle$ be IS's on X and let $\{A_i : i \in J\}$ be an arbitrary family of IS's in X, where $A_i = \left\langle X, A_i^{(1)}, A_i^{(2)}\right\rangle$. Then

- (i) $A \subseteq B$ iff $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$.
- (ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
- (iii) $\bar{A} = \langle X, A_2, A_1\rangle$.
- (iv) $A \cup B{=}\langle X, A_1 \cup B_1, A_2 \cap B_2\rangle$.
- (v) $A \cap B{=}\langle X, A_1 \cap B_1, A_2 \cup B_2\rangle$.
- (vi) $\bigcup A_i{=}\langle X, \bigcup A_i^1, \bigcap A_i^2\rangle$.
- (vii) $\bigcap A_i{=}\langle X, \bigcap A_i^1, \bigcup A_i^2\rangle$.
- $(viii)$ A-$B{=}A \cap \bar{B}$.
- (ix) $[]\, A{=}\langle X, A_1, (A_1)^c\rangle$.
- (x) $\langle\rangle\, A{=}\langle X, (A_2)^c, A_2\rangle$.
- (xi) $\underset{\sim}{X}{=}\langle X, X, \phi\rangle$.
- (xii) $\underset{\sim}{\phi}{=}\langle X, \phi, X\rangle$.

Definition 2.3 [6] An intuitionistic topology on a non-empty set X is a family τ of IS's in X satifying the following axioms:

- (i) $\underset{\sim}{X}, \underset{\sim}{\phi} \in \tau$.
- (ii) $A_1 \cap A_2 \in \tau$ for any $A_1, A_2 \in \tau$.
- (iii) $\cup A_i \in \tau$ for any arbitrary family $\{A_i : i \in J\} \subseteq \tau$.

The pair (X, τ) is called an intuitionistic topological space (ITS in short) and IS in τ is known as an intuitionitic open set (IOS in short) in X, the complement of IOS is called an intuitionistic closed set (ICS in short).

Definition 2.4 [6] The supra closure of a set A is denoted by $cl^{\mu}(A)$, and is defined as,
supra $cl(A) = \bigcap\{B : B \quad is \ \ supra \ \ closed \ \ and \ \ A \subseteq B\}$.
The supra interier of a set A is denoted by $int^{\mu}(A)$, and is defined as
supra $int(A) = \bigcup\{B : B \quad is \ \ supra \ \ open \ \ and \ \ A \supseteq B\}$.

Definition 2.5 [1] An Intuitionistic supra topology on a non-empty set X is a family τ of IS's in X satisfying the following axioms:

- (i) $\underset{\sim}{X}, \underset{\sim}{\phi} \in \tau$.
- (ii) $\cup A_i \in \tau$ for any arbitrary family $\{A_i : i \in J\} \subseteq \tau$.

The pair(X,τ) is called intuitionistic supra topological space (ISTS in short) and IS in τ is known as an intuitionistic supra open set (ISOS in short) in X, the complement of ISOS is called intuitionistic supra closed set(ISCS in short).

Definition 2.6 [1] Let (X,τ) be an ISTS and let $A = \langle X, A_1, A_2\rangle$ be an IS in X, then the supra closure and supra interior of A are defined by:
$cl^{\mu}(A) = \bigcap\{K : K \ is \ an \ ISCS \ in \ X \ and \ A \subseteq K\}$.
$int^{\mu}(A) = \bigcup\{K : K \ is \ an \ ISOS \ in \ X \ and \ A \supseteq K\}$.

Definition 2.7 [8] Let (X,τ) be an ISTS and let $A = \langle X, A_1, A_2\rangle$ be an IS in X , then the supra α closure and supra α interior of A are defined by:
$I\alpha cl^{\mu}(A) = \bigcap\{k : k \ is \ an \ IS\alpha CS \ in \ X \ and \ A \subseteq k\}$
$I\alpha int^{\mu}(A) = \bigcup\{k : k \ is \ an \ IS\alpha OS \ in \ X \ and \ A \supseteq k\}$

Definition 2.8 [1] Let (X,τ) be an intuitionistic supra topological space. An intuitionistic set A is called intuitionistic supra α-closed set (ISαCS in short) if $cl^{\mu}(int^{\mu}(cl^{\mu}(A))) \subseteq$ U, whenever A $\subseteq$U, U is intuitionistic supra α-open set (ICαOS).
The complement of intuitionistic supra α-closed set is intuitionistic supra α-open set (ISαOS in short).

3. Intuitionistic supra Frontier

Definition 3.1 Let X be an ISTS and for a subset A of a ISTS X, $IFr^{\mu}(A) = Icl^{\mu}(A) - Iint^{\mu}(A)$ is said to be Intuitionistic supra Frontier of A.

Theorem 3.2 Let X be an ISTS then and for any a subset A of IS in ISTS X , the following statements hold:

(i) $IFr^{\mu}(A) = Icl^{\mu}(A) \cap Icl^{\mu}(X - A)$.
(ii) $IFr^{\mu}(A) = IFr^{\mu}(X - A)$.
(iii) $IFr^{\mu}(IFr^{\mu}(A)) \subseteq IFr^{\mu}(A)$.
(iv) $Icl^{\mu}(A) = Iint^{\mu}(A) \cup IFr^{\mu}(A)$.
(v) $Iint^{\mu}(A) \cap IFr^{\mu}(A)=\phi$.
(vi) $IFr^{\mu}(\underset{\sim}{X}) = \phi, IFr^{\mu}(\widetilde{\phi}) = \underset{\sim}{X}$.
(vii) $IFr^{\mu}(Icl^{\mu}(A)) \subseteq IFr^{\mu}(A)$.

Proof. Let A be a IS in ISTS X.

(i) $IFr^{\mu}(A) = Icl^{\mu}(A) - Iint^{\mu}(A)=Icl^{\mu}(A) \cap Icl^{\mu}(X - A)$.
(ii) $IFr^{\mu}(A) = Icl^{\mu}(A) - Iint^{\mu}(A)=(X - Iint^{\mu}(A)) - (X - Icl^{\mu}(A))=Icl^{\mu}(X - A) - Iint^{\mu}(X - A)=IFr^{\mu}(X - A)$.

(iii) $IFr^{\mu}(IFr^{\mu}(A))$=$Icl^{\mu}(IFr^{\mu}(A)) \cap Icl^{\mu}(X - IFr^{\mu}(A)) \subseteq Icl^{\mu}(IFr^{\mu}(A)) = IFr^{\mu}(X - A)$. Hence $IFr^{\mu}(IFr^{\mu}(A)) \subseteq IFr^{\mu}(A)$.

(iv) $Iint^{\mu}(A) \cup IFr^{\mu}(A)$=$Iint^{\mu}(A) \cup (Icl^{\mu}(A) - Iint^{\mu}(A))$=$(Iint^{\mu}(A) \cup Icl^{\mu}(A)) - (Iint^{\mu}(A) \cup Iint^{\mu}(A))$=$(Iint^{\mu}(A) \cup Icl^{\mu}(A)) - Iint^{\mu}(A)$=$Icl^{\mu}(A)$.

(v) $Iint^{\mu}(A) \cap IFr^{\mu}(A)$=$Iint^{\mu}(A) \cap (Icl^{\mu}(A) - Iint^{\mu}(A))$=$\underset{\sim}{\phi}$.

(vi) $IFr^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}$,$IFr^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.

(vii) $IFr^{\mu}(Icl^{\mu}(A)) = Icl^{\mu}(Icl^{\mu}(A)) - Iint^{\mu}(Icl^{\mu}(A)) \subseteq Icl^{\mu}(A) - Iint^{\mu}(A) = IFr^{\mu}(A)$. Hence $IFr^{\mu}(Icl^{\mu}(A)) \subseteq IFr^{\mu}(A)$.

□

The proof of the above theorem is shown in the following example:

Example 3.3 Let $X = \{a, b, c\}$. $\tau = \left\{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\right\}$, where $A_1 = \langle X, \{a\}, \{b, c\}\rangle$, $A_2 = \langle X, \{b\}, \{c\}\rangle$ and $A_3 = \langle X, \{a, b\}, \{c\}\rangle$.
Let $A = \langle X, \{a\}, \{c\}\rangle$. $X - A = \langle X, \{c\}, \{a\}\rangle$. $Iint^{\mu}(A) = \langle X, \{a\}, \{b, c\}\rangle$. $Icl^{\mu}(A) = \underset{\sim}{X}$. $IFr^{\mu}(A) = \langle X, \{b, c\}, \{a\}\rangle$.
$Iint^{\mu}(X - A) = \underset{\sim}{\phi}$. $Icl^{\mu}(X - A) = \langle X, \{b, c\}, \{a\}\rangle$.$IFr^{\mu}(X - A) = \langle X, \{b, c\}, \{a\}\rangle$.

(i) $IFr^{\mu}(A) = Icl^{\mu}(A) - Iint^{\mu}(A) = \langle X, \{b, c\}, \{a\}\rangle$ and $Icl^{\mu}(A) \cap Icl^{\mu}(X - A) = \langle X, \{b, c\}, \{a\}\rangle$.

(ii) $IFr^{\mu}(A) = \langle X, \{b, c\}, \{a\}\rangle$ and $IFr^{\mu}(X - A) = \langle X, \{b, c\}, \{a\}\rangle$.

(iii) $IFr^{\mu}(A) = \langle X, \{b, c\}, \{a\}\rangle$. $IFr^{\mu}(IFr^{\mu}(A)) = \langle X, \{b, c\}, \{a\}\rangle$.Hence $IFr^{\mu}(IFr^{\mu}(A)) \subseteq IFr^{\mu}(A)$.

(iv) $Iint^{\mu}(A) \cup IFr^{\mu}(A) = \underset{\sim}{X}.cl^{\mu}(A) = \underset{\sim}{X}$.Hence $Icl^{\mu}(A) = Iint^{\mu}(A) \cup IFr^{\mu}(A)$.

(v) $Iint^{\mu}(A) \cap IFr^{\mu}(A) = \underset{\sim}{\phi}$.

(vi) $IFr^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}$,$IFr^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.

(vii) $IFr^{\mu}(Icl^{\mu}(A)) = \langle X, \{b, c\}, \{a\}\rangle$.$IFr^{\mu}(A) = \langle X, \{b, c\}, \{a\}\rangle$.Hence $IFr^{\mu}(Icl^{\mu}(A)) \subseteq IFr^{\mu}(A)$.

Definition 3.4 Let X be an ISTS and for a subset A of a ISTS, $I\alpha Fr^{\mu}(A)$=$I\alpha cl^{\mu}(A) - I\alpha int^{\mu}(A)$ said to be Intuitionistic supra α-Frontier of A.

Theorem 3.5 For a subset A of ISTS, I$\alpha Fr^{\mu}(A) \subseteq IFr^{\mu}(A)$.

proof Let x $\in I\alpha Fr^{\mu}(A)$ then x $\in I\alpha cl^{\mu}(A) - I\alpha int^{\mu}(A)$, implies x $\in Icl^{\mu}(A) - Iint^{\mu}(A)$, since every intuitionistic supra closed set is intuitionistic supra α-closed set. Hence x $\in IFr^{\mu}(A)$. Therefore $I\alpha Fr^{\mu}(A) \subseteq IFr^{\mu}(A)$.
Converse of the above theorem need not be true. It is shown in the following example.

Example 3.6 Let $X = \{a, b, c\}$. $\tau = \left\{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\right\}$, where$A_1 = \langle X, \{a\}, \{b, c\}\rangle$. $A_2 = \langle X, \{b\}, \{c\}\rangle$ and $A_3 = \langle X, \{a, b\}, \{c\}\rangle$.

Let $A = \langle X, \{\phi\}, \{a\}\rangle$, $IFr^{\mu}(A) = \langle X, \{b, c\}, \{a\}\rangle$, and $I\alpha Fr^{\mu}(A) = \langle X, \phi\ \{a\}\rangle$. Here $I\alpha Fr^{\mu}(A) \subseteq IFr^{\mu}(A)$ is true but converse is not true.

Theorem 3.7 Let X be an ISTS then and for any a subset A of IS in ISTS X, the following statements holds:

(i) $I\alpha Fr^{\mu}(A) = I\alpha cl^{\mu}(A) \cap I\alpha cl^{\mu}(X - A)$.
(ii) $I\alpha Fr^{\mu}(A) = I\alpha Fr^{\mu}(X - A)$.
(iii) $I\alpha Fr^{\mu}(I\alpha Fr^{\mu}(A)) \subseteq I\alpha Fr^{\mu}(A)$.
(iv) $I\alpha cl^{\mu}(A) = I\alpha int^{\mu}(A) \cup I\alpha Fr^{\mu}(A)$.
(v) $I\alpha int^{\mu}(A) \cap I\alpha Fr^{\mu}(A) = \underset{\sim}{\phi}$.
(vi) $I\alpha Fr^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}$, $I\alpha Fr^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.
(vii) $I\alpha Fr^{\mu}(I\alpha cl^{\mu}(A)) \subseteq I\alpha Fr^{\mu}(A)$.

Proof

(i) $I\alpha Fr^{\mu}(A) = I\alpha cl^{\mu}(A) - I\alpha int^{\mu}(A) = I\alpha cl^{\mu}(A) \cap I\alpha cl^{\mu}(X - A)$.
(ii) $I\alpha Fr^{\mu}(A) = I\alpha cl^{\mu}(A) - I\alpha int^{\mu}(A) = (X - I\alpha int^{\mu}(A)) - (X - I\alpha cl^{\mu}(A)) = I\alpha cl^{\mu}(X - A) - I\alpha int^{\mu}(X - A) = I\alpha Fr^{\mu}(X - A)$.
(iii) $I\alpha Fr^{\mu}(I\alpha Fr^{\mu}(A)) = I\alpha cl^{\mu}(I\alpha Fr^{\mu}(A)) \cap I\alpha cl^{\mu}(X - I\alpha Fr^{\mu}(A)) \subseteq I\alpha cl^{\mu} I\alpha Fr^{\mu}(A)) = I\alpha Fr^{\mu}(X - A)$. Hence $I\alpha Fr^{\mu}(I\alpha Fr^{\mu}(A)) \subseteq I\alpha Fr^{\mu}(A)$.
(iv) $I\alpha int^{\mu}(A) \cup I\alpha Fr^{\mu}(A) = I\alpha int^{\mu}(A) \cup (I\alpha cl^{\mu}(A) - I\alpha int^{\mu}(A)) = (I\alpha int^{\mu}(A) \cup I\alpha cl^{\mu}(A)) - (I\alpha int^{\mu}(A) \cup I\alpha int^{\mu}(A)) = (I\alpha int^{\mu}(A) \cup I\alpha cl^{\mu}(A)) - I\alpha int^{\mu}(A) = I\alpha cl^{\mu}(A)$.
(v) $I\alpha int^{\mu}(A) \cap I\alpha Fr^{\mu}(A) = I\alpha int^{\mu}(A) \cap (I\alpha cl^{\mu}(A) - I\alpha int^{\mu}(A)) = \underset{\sim}{\phi}$.
(vi) $I\alpha Fr^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}, I\alpha Fr^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.
(vii) $I\alpha Fr^{\mu}(I\alpha cl^{\mu}(A)) = I\alpha cl^{\mu}(I\alpha cl^{\mu}(A)) - I\alpha int^{\mu}(I\alpha cl^{\mu}(A)) \subseteq I\alpha cl^{\mu}(A) - I\alpha int^{\mu}(A) = I\alpha Fr^{\mu}(A)$. Hence $I\alpha Fr^{\mu}(I\alpha cl^{\mu}(A)) \subseteq I\alpha Fr^{\mu}(A)$.

Example 3.8 Let $X = \{a, b, c\}$. $\tau = \left\{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\right\}$, where $A_1 = \langle X, \{a\}, \{b, c\}\rangle$, $A_2 = \langle X, \{c\}, \{a, b\}\rangle$, and $A_3 = \langle X, \{a, c\}, \{b\}\rangle$.
Let $A = \langle X, \{b\}, \{a, c\}\rangle$. $X - A = \langle X, \{a, c\}, \{b\}\rangle$. $I\alpha int^{\mu}(A) = \underset{\sim}{\phi}$, $I\alpha cl^{\mu}(A) = \langle X, \{b\}, \{a, c\}\rangle$. $I\alpha Fr^{\mu}(A) = \langle X, \{b\}, \{a, c\}\rangle$.
$I\alpha int^{\mu}(X - A) = \langle X, \{a, c\}, \{b\}\rangle$. $I\alpha cl^{\mu}(X - A) = \langle X, \{a, c\}, \{b\}\rangle$. $I\alpha Fr^{\mu}(X - A) = \langle X, \{a, c\}, \{b\}\rangle$.

(i) $I\alpha Fr^{\mu}(A) = I\alpha cl^{\mu}(A) - I\alpha int^{\mu}(A) = \langle X, \{b\}, \{a, c\}\rangle$, , and $I\alpha cl^{\mu}(A) \cap I\alpha cl^{\mu}(X - A) = \langle X, \{b\}, \{a, c\}\rangle$.
(ii) $I\alpha Fr^{\mu}(A) = \langle X, \{b\}, \{a, c\}\rangle$ and $I\alpha Fr^{\mu}(X - A) = \langle X, \{b\}, \{a, c\}\rangle$.
(iii) $I\alpha Fr^{\mu}(A) = \langle X, \{b\}, \{a, c\}\rangle$. $I\alpha Fr^{\mu}(I\alpha Fr^{\mu}(A)) = \langle X, \{b\}, \{a, c\}\rangle$. Hence $I\alpha Fr^{\mu}(I\alpha Fr^{\mu}(A)) \subseteq I\alpha Fr^{\mu}(A)$.
(iv) $I\alpha int^{\mu}(A) \cup I\alpha Fr^{\mu}(A) = \langle X, \{b\}, \{a, c\}\rangle . I\alpha cl^{\mu}(A) = \langle X, \{b\}, \{a, c\}\rangle$. Hence $Icl^{\mu}(A) = I\alpha int^{\mu}(A) \cup I\alpha Fr^{\mu}(A)$.

(v) $I\alpha int^{\mu}(A) \cap I\alpha Fr^{\mu}(A) = \underset{\sim}{\phi}$.
(vi) $I\alpha Fr^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}, I\alpha Fr^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.
(vii) $I\alpha Fr^{\mu}(I\alpha cl^{\mu}(A)) = \langle X, \{b\}, \{a,c\}\rangle$. Hence $I\alpha Fr^{\mu}(I\alpha cl^{\mu}(A)) \subseteq I\alpha Fr^{\mu}(A)$.

4. Intuitionistic supra Exterior

Definition 4.1 Let X be an ISTS and for a subset A of a ISTS X, $IExt^{\mu}(A) = Iint^{\mu}(\underset{\sim}{X} - A)$ s said to be Intuitionistic supra Exterior of A.

Theorem 4.2 Let X be an ISTS then and for any a subset A of IS in ISTS X, the following statements hold:

(i) $IExt^{\mu}(A) = \underset{\sim}{X} - Icl^{\mu}(A)$.
(ii) $IExt^{\mu}(IExt^{\mu}(A)) = Iint^{\mu}(Icl^{\mu}(A)) \supseteq Iint^{\mu}(A)$.
(iii) $A \subseteq B = IExt^{\mu}(B) \subseteq IExt^{\mu}(A)$.
(iv) $IExt^{\mu}(A \cup B) = IExt^{\mu}(A) \cap IExt^{\mu}(B)$.
(v) $IExt^{\mu}(A \cap B) = IExt^{\mu}(A) \cup IExt^{\mu}(B)$.
(vi) $IExt^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}, IExt^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.
(vii) $IExt^{\mu}(A) = IExt^{\mu}(\underset{\sim}{X} - IExt^{\mu}(A))$.

Proof

(i) $IExt^{\mu}(A) = Iint^{\mu}(\underset{\sim}{X} - A) = \underset{\sim}{X} - Icl^{\mu}(A)$.
(ii) $IExt^{\mu}(IExt^{\mu}(A)) = Iint^{\mu}(\underset{\sim}{X} - (IExt^{\mu}(A)) = Iint^{\mu}(Icl^{\mu}(A)) \supseteq Iint^{\mu}(A)$.
(iii) $A \subseteq B \Longrightarrow Iint^{\mu}(A) \subseteq Iint^{\mu}(B)$.
$IExt^{\mu}(B) = Iint^{\mu}(\underset{\sim}{X} - B) \subseteq Iint^{\mu}(\underset{\sim}{X} - A) = IExt^{\mu}(A) \Longrightarrow IExt^{\mu}(B) \subseteq IExt^{\mu}(A)$.
(iv) $IExt^{\mu}(A \cup B) = Iint^{\mu}(\underset{\sim}{X} - (A \cup B)) = Iint^{\mu}((\underset{\sim}{X} - A) \cap (\underset{\sim}{X} - B)) \subseteq Iint^{\mu}(\underset{\sim}{X} - A) \cap Iint^{\mu}(\underset{\sim}{X} - B) = IExt^{\mu}(A) \cap IExt^{\mu}(B)$.
Hence $IExt^{\mu}(A \cup B) = IExt^{\mu}(A) \cap IExt^{\mu}(B)$.
(v) $IExt^{\mu}(A \cap B) = Iint^{\mu}(\underset{\sim}{X} - (A \cap B)) = Iint^{\mu}((\underset{\sim}{X} - A) \cup (\underset{\sim}{X} - B)) \supseteq Iint^{\mu}(\underset{\sim}{X} - A) \cup Iint^{\mu}(\underset{\sim}{X} - B) = IExt^{\mu}(A) \cup IExt^{\mu}(B)$.
Hence $IExt^{\mu}(A \cap B) = IExt^{\mu}(A) \cup IExt^{\mu}(B)$.
(vi) $IExt^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}, IExt^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.
(vii) $IExt^{\mu}(\underset{\sim}{X} - IExt^{\mu}(A)) = IExt^{\mu}(\underset{\sim}{X} - Iint^{\mu}(\underset{\sim}{X} - A)) = Iint^{\mu}(\underset{\sim}{X} - A) = IExt^{\mu}(A)$.

The proof of the above theorem is shown in the following example:

Example 4.3 Let $X = \{a, b, c\}$. $\tau = \left\{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\right\}$, where $A_1 = \langle X, \{a\}, \{b,c\}\rangle$, $A_2 = \langle X, \{b\}, \{c\}\rangle$, and $A_3 = \langle X, \{a,b\}, \{c\}\rangle$.

Let $A = \langle X, \{a,b\}, \{c\}\rangle$. $B = \langle X, \{a,b\}, \phi\rangle$. $X - A = \langle X, \{c\}, \{a,b\}\rangle$. $Iint^{\mu}(A) = \langle X, \{a,b\}, \{c\}\rangle$. $Icl^{\mu}(A) = \underset{\sim}{X}$.
$Iint^{\mu}(X - A) = \langle X, \{c\}, \{a,b\}\rangle$. $Icl^{\mu}(X - A) = \underset{\sim}{\phi}$. $Iext^{\mu}(A) = \underset{\sim}{\phi}$, $Iext^{\mu}(B) = \underset{\sim}{\phi}$

(i) $\underset{\sim}{X} - Icl^{\mu}(A) = \underset{\sim}{\phi}$. Hence $IExt^{\mu}(A) = \underset{\sim}{X} - Icl^{\mu}(A)$.
(ii) $IExt^{\mu}(IExt^{\mu}(A)) = \underset{\sim}{X}$ Hence $IExt^{\mu}(IExt^{\mu}(A)) \supseteq Iint^{\mu}(A)$.
(iii) $A = \langle X, \{a,b\}, \{c\}\rangle \subseteq B = \langle X, \{a,b\}, \phi\rangle$ implies $IExt^{\mu}(A){=}\underset{\sim}{\phi} and IExt^{\mu}(B){=}\underset{\sim}{\phi}$ implies $IExt^{\mu}(B) \subseteq IExt^{\mu}(A)$.
(iv) $IExt^{\mu}(A \cup B) = \underset{\sim}{\phi}, IExt^{\mu}(A) = \underset{\sim}{\phi} \cup IExt^{\mu}(B) = \underset{\sim}{\phi}$.
(v) $IExt^{\mu}(A \cap B) = \underset{\sim}{\phi}, IExt^{\mu}(A) = \underset{\sim}{\phi} \cap IExt^{\mu}(B) = \underset{\sim}{\phi}$.
(vi) $IExt^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}, IExt^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.
(vii) $IExt^{\mu}(\underset{\sim}{X} - IExt^{\mu}(A)) = \underset{\sim}{\phi}$.

Definition 4.4 Let X be an ISTS and for a subset A of a ISTS X, $I\alpha Ext^{\mu}(A) = I\alpha int^{\mu}(\underset{\sim}{X} - A)$ said to be Intuitionistic supra α-Exterior of A.

Theorem 4.5 Let X be an ISTS then and for any a subset A of IS in ISTS X, the following statements hold:

(i) $I\alpha Ext^{\mu}(A) = \underset{\sim}{X} - I\alpha cl^{\mu}(A)$.
(ii) $I\alpha Ext^{\mu}(I\alpha Ext^{\mu}(A)) = I\alpha int^{\mu}(I\alpha cl^{\mu}(A)) \supseteq I\alpha int^{\mu}(A)$.
(iii) $A \subseteq B = I\alpha Ext^{\mu}(B) \subseteq I\alpha Ext^{\mu}(A)$.
(iv) $I\alpha Ext^{\mu}(A \cup B) = I\alpha Ext^{\mu}(A) \cap I\alpha Ext^{\mu}(B)$.
(v) $I\alpha Ext^{\mu}(A \cap B) = I\alpha Ext^{\mu}(A) \cup I\alpha Ext^{\mu}(B)$.
(vi) $I\alpha Ext^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}, I\alpha Ext^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.
(vii) $I\alpha Ext^{\mu}(A) = I\alpha Ext^{\mu}(\underset{\sim}{X} - I\alpha Ext^{\mu}(A))$.

Proof

(i) $I\alpha Ext^{\mu}(A) = I\alpha int^{\mu}(\underset{\sim}{X} - A) = \underset{\sim}{X} - I\alpha cl^{\mu}(A)$.
(ii) $I\alpha Ext^{\mu}(I\alpha Ext^{\mu}(A)) = I\alpha int^{\mu}(\underset{\sim}{X} - (I\alpha Ext^{\mu}(A)) = I\alpha int^{\mu}(I\alpha cl^{\mu}(A)) \supseteq I\alpha int^{\mu}(A)$.
(iii) $A \subseteq$ B implies $I\alpha int^{\mu}(A) \subseteq I\alpha int^{\mu}(B)$.
$I\alpha Ext^{\mu}(B) = I\alpha int^{\mu}(\underset{\sim}{X} - B) \subseteq I\alpha int^{\mu}(\underset{\sim}{X} - A) = I\alpha Ext^{\mu}(A)$ implies $I\alpha Ext^{\mu}(B) \subseteq I\alpha Ext^{\mu}(A)$.
(iv) $I\alpha Ext^{\mu}(A \cup B) = I\alpha int^{\mu}(\underset{\sim}{X} - (A \cup B)) = I\alpha int^{\mu}((\underset{\sim}{X} - A) \cap (\underset{\sim}{X} - B)) \subseteq I\alpha int^{\mu}(\underset{\sim}{X} - A) \cap I\alpha int^{\mu}(\underset{\sim}{X} - B) = I\alpha Ext^{\mu}(A) \cap I\alpha Ext^{\mu}(B)$.
Hence $I\alpha Ext^{\mu}(A \cup B) = I\alpha Ext^{\mu}(A) \cap I\alpha Ext^{\mu}(B)$.

(v) $I\alpha Ext^{\mu}(A\cap B) = I\alpha int^{\mu}(\underset{\sim}{X} - (A\cap B)) = I\alpha int^{\mu}((\underset{\sim}{X} - A)\cup(\underset{\sim}{X} - B)) \supseteq I\alpha int^{\mu}(\underset{\sim}{X} - A)\cup I\alpha int^{\mu}(\underset{\sim}{X} - B) = I\alpha Ext^{\mu}(A)\cup I\alpha Ext^{\mu}(B)$.
Hence $I\alpha Ext^{\mu}(A\cap B) = I\alpha Ext^{\mu}(A)\cup I\alpha Ext^{\mu}(B)$.

(vi) $I\alpha Ext^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}, I\alpha Ext^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.

(vii) $I\alpha Ext^{\mu}(\underset{\sim}{X} - I\alpha Ext^{\mu}(A)) = I\alpha Ext^{\mu}(\underset{\sim}{X} - I\alpha int^{\mu}(\underset{\sim}{X} - A)) = I\alpha int^{\mu}(\underset{\sim}{X} - A) = I\alpha Ext^{\mu}(A)$.

The proof of the above theoremis shown in the following example:

Example 4.6 Let $X = \{a, b, c\}$. $\tau = \left\{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\right\}$, where $A_1 = \langle X, \{b\}, \{c\}\rangle$, $A_2 = \langle X, \{a\}, \{c\}\rangle$, and $A_3 = \langle X, \{a, b\}, \{c\}\rangle$.
Let $A = \langle X, \{a, c\}, \{\phi\}\rangle$. $B = \langle X, X, \phi\rangle$. $X - A = \langle X, , \{\phi\}, \{a, c\}\rangle$. $I\alpha int^{\mu}(A) = \langle X, , \{a, c\}, \{b, c\}\rangle$, $I\alpha cl^{\mu}(A) = \underset{\sim}{X}$. $\alpha Fr^{\mu}(A) = \langle X, \{b, c\}, \{\phi\}\rangle$.
$I\alpha int^{\mu}(X-A) = \langle X, \{\phi\}, \{a, c\}\rangle$. $I\alpha cl^{\mu}(X-A) = \underset{\sim}{\phi}$. $I\alpha Fr^{\mu}(X-A) = \langle X, \{\phi\}, \{a, c\}\rangle$.
$I\alpha ext^{\mu}(A) = \underset{\sim}{\phi}$, $I\alpha Ext^{\mu}(B) = \underset{\sim}{\phi}$.

(i) $\underset{\sim}{X} - I\alpha cl^{\mu}(A) = \underset{\sim}{\phi}$. Hence $I\alpha Ext^{\mu}(A) = \underset{\sim}{X} - I\alpha cl^{\mu}(A)$.

(ii) $I\alpha Ext^{\mu}(I\alpha Ext^{\mu}(A)) = \underset{\sim}{X}$. Hence $I\alpha Ext^{\mu}(I\alpha Ext^{\mu}(A)) \supseteq I\alpha int^{\mu}(A)$.

(iii) $A = \langle X, \{a, c\}, \{\phi\}\rangle \subseteq B = \langle X, X, \phi\rangle\, implies I\alpha Ext^{\mu}(A) = \underset{\sim}{\phi}$ and $I\alpha Ext^{\mu}(B) = \underset{\sim}{\phi}$ implies $I\alpha Ext^{\mu}(B) \subseteq I\alpha Ext^{\mu}(A)$.

(iv) $I\alpha Ext^{\mu}(A\cup B) = \underset{\sim}{\phi}$, $I\alpha Ext^{\mu}(A) = \underset{\sim}{\phi}\cup I\alpha Ext^{\mu}(B) = \underset{\sim}{\phi}$.

(v) $I\alpha Ext^{\mu}(A\cap B) = \underset{\sim}{\phi}$, $I\alpha Ext^{\mu}(A) = \underset{\sim}{\phi}\cap I\alpha Ext^{\mu}(B) = \underset{\sim}{\phi}$.

(vi) $I\alpha Ext^{\mu}(\underset{\sim}{X}) = \underset{\sim}{\phi}, I\alpha Ext^{\mu}(\underset{\sim}{\phi}) = \underset{\sim}{X}$.

(vii) $I\alpha Ext^{\mu}(\underset{\sim}{X} - I\alpha Ext^{\mu}(A)) = \underset{\sim}{\phi}$.

References

[1] Coker, D., A note on intutionistic sets and intutionistic points, *Turk.J.Math.,* 20-3 (1996), 343-351.

[2] Coker, D., An introduction to intutionistic fuzzy topological spaces, *Fuzzy sets and systems,* (1997), 81-89.

[3] Coker, D., An introduction to Intutionistic topological spaces, *Busefal,* 81 (2000), 51-56.

[4] Levin, N., semi open sets and semi continunity in topological spaces, *Amer.Math.,*12 (1991), 5-13.

[5] Devi, R., Sampathkumar, S., Caldas, M., On supra α-open sets ans Sα-continuous functions, *General Mathematics,* 16(2) (2008),77-84.

[6] Mashhour, A.S., Allam, A.A., Mahmoud, F.S., Khedr, F.H., On Supra topological spaces, *Indian J.Pure and Appl. Math.,* 14(4) (1983),502-510.

[7] Zadeh, L.A., Fuzzy sets, *Information and control*, 8 (1965), 338-353.

[8] Vidyarani, L., Vigneshwaran, M., On some intutionistic supra closed sets on intutionistic supra topological spaces, *Bulletin of Mathematics and Statistics Research*, 3(3) (2015), 1-9.

[9] Ilango, G., Albinaa, T.A., Properties of a α-interior and α-closure in intuitionistic topological spaces, *IOSR Journal of Mathematics,* 12 (2015), 91-95.
[10] Andrijevic, D., Some properties of the topology of α-sets, *Mat.Vesnik,* 36 (1984) , 1-10.
[11] Girija, S., Selvanayaki, S., Ilango, G., Frontier and semi frontier in intuitionistic topological spaces, *EAI Endorsed Transactions on Energy Web and Information Technologies,* 5 (2018) , 1-5.
[12] Njastad, O., On some classes of nearly open sets, *Pacfic J.Math.,* 15 (1965), 961-970.
[13] Asmaa, Y.J., Raouf, G., On generalization closed set and generalized coninunity on intuitionistic topological spaces, *J.of Al-anbar University of Pure Science* 3 (2009) , 1-11.

https://communications.science.ankara.edu.tr

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RECOGNITION OF COMPLEX POLYNOMIAL BÉZIER CURVES UNDER SIMILARITY TRANSFORMATIONS

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Abstract. In this paper, similarity groups in the complex plane $\mathbb{C}$, polynomial curves and complex Bézier curves in $\mathbb{C}$ are introduced. Global similarity invariants of polynomial curves and complex Bézier curves in $\mathbb{C}$ are given in terms of complex functions. The problem of similarity of two polynomial curves in $\mathbb{C}$ are solved. Moreover, in case two polynomial curve (complex Bézier curve) are similar for the similarity group, a general form of all similarity transformations, carrying one curve into the other curve, are obtained.

1. Introduction

The invariance is a very important tool in areas data registration, object recognition, computer aided design applications. In computer aided applications, the iterative closest point(ICP) algorithm is an accurate and efficient method for rigid registration problem and curve matching. The aim of registration or object recognition is to find the corresponding relationship between two point sets(or two curves) and compute the transformation which aligns two point sets(or two curves)(see [1–4]) Generally, Euclidean invariant features are used in above mentioned methods and a representation of polynomial curve or Bézier curve in the complex plane $\mathbb{C}$ are a useful method to investigate of their global invariants. (see [5, 7–10, 16]) In [16], taking customary rational Bézier curves in complex plane, complex rational Bézier curves are investigated. For Bézier curves, rational curves and implicit algebraic curves, detecting whether two plane curves are similar by an orientation preserving similarity transformation is important. (see [11–19]).

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This paper presents the similarity conditions of two point sets and the similarity conditions of two polynomial paths(two complex Bézier curve) in the complex plane $\mathbb{C}$.

The polynomial curve $Z(u), W(u), u \in [0,1]$ in defined in terms of monomial complex control points $p_j, q_j \in \mathbb{C}$ as

$Z(u) = \sum_{j=0}^{m} p_j u^j$ and $W(u) = \sum_{j=0}^{m} p_j u^j$, resp.

The complex Bézier curves $Z(u), W(u), u \in [0,1]$ in defined in terms of degree m Bernstein polynomials $B_j^m(u)$ and complex control points $z_j, w_j \in \mathbb{C}$ as

$Z(u) = \sum_{j=0}^{m} z_j B_j^m(u)$ and $W(u) = \sum_{j=0}^{m} w_j B_j^m(u)$.

Let $GM(\mathbb{C}^*)$ be the group of all similarities of $\mathbb{C}$, $GM^+(\mathbb{C}^*)$ be the group of all orientation-preserving similarities of $\mathbb{C}$. The group of all linear similarities of $\mathbb{C}$ is denoted by $M(\mathbb{C}^*)$. The group of all orientation-preserving linear similarities of $\mathbb{C}$ is denoted by $M^+(\mathbb{C}^*)$.

The problem of similarity of two polynomial curves(or two complex Bézier curves) $Z(u), W(u)$ for the groups $GM(\mathbb{C}^*)$ and $GM^+(\mathbb{C}^*)$ is reduced to the problem of similarity of two polynomial curves(or two complex Bézier curves) $Z(u), W(u)$ for the groups $M(\mathbb{C}^*)$ and $M^+(\mathbb{C}^*)$, resp. Moreover, since a complex Bézier curve can be define in terms of complex control points, these problems of similarity of two complex Bézier curves is reduced to the problem of similarity of sets of complex control points for these groups. Similarly, same problem can given for polynomial curves. Otherwise, the problem of similarity of sets of complex control points for the above mentioned groups can be applied to the point set rigid registration problem.

For the groups of Euclidean motions $M(n)$ and Euclidean rigid motions $M^+(n)$ in the n-dimensional Euclidean space, the problems of equivalence two Bézier curves of degree m and its global invariants are investigated in [15]. In [9], similar problem in this paper is solved for the groups $M(2)$ and $M^+(2)$. For orientation-preserving similarity group $Sim^+(n)$ in similarity geometry, *local* differential invariants, existence and rigidity theorems for a regular curve are obtained in [20]. For only similarity group $Sim(2)$ and linear similarity group $LSim(2)$, the problems of equivalence two Bézier curves of degree m are investigated in [18]. For orthogonal group $O(2)$, special orthogonal group $O^+(2)$, linear similarity group $LSim(2)$ and orientation linear similarity group $LSim^+(2)$, the conditions of the global G-equivalence of two regular paths are given in [10, 21].

So the paper contains solutions of problems of global similarity of complex Bézier curves and polynomial curves for the above mentioned groups without using differential invariants of a complex Bézier curve and a polynomial curve. In order to make this paper more self contained from a mathematical points of view, the structure of the present paper is the following. In Sect.2, relations between complex plane and two-dimensional Euclidean space and definitions of similarity groups in terms of complex numbers are introduced. In Sect.3, global invariants of a polynomial curve and a complex Bézier curve are given. For above mentioned similarity groups, the problem of similarity of two complex Bézier curves are given. In Sect.4, conditions of similarity for two m-uples complex number sets and a general form of all similarity transformations, carrying one set into the other set, are obtained. In Sect.5, conditions of similarity for two complex Bézier curves and a general form of all similarity transformations, carrying one curve into the other curve, are obtained.

2. Similarity groups in the complex plane

Let $\mathbb{C}$ be the field of complex numbers. The product of two complex numbers z_1 and z_2 has the form

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \tag{2.1}$$

Consider the complex number $z = a + ib$ in the matrix form $z = \begin{pmatrix} a \\ b \end{pmatrix}$.

Then, the equality (2.1) has the following form

$$z_1 z_2 = \begin{pmatrix} a_1 a_2 - b_1 b_2 \\ a_1 b_2 + a_2 b_1 \end{pmatrix} = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}. \tag{2.2}$$

Here we denote by L_z the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for all $z = a + ib \in \mathbb{C}$. Then $L_z : \mathbb{C} \to \mathbb{C}$ is a mapping and the equality (2.2) has the form, $\forall z_1, z_2 \in \mathbb{C}$,

$$z_1 z_2 = L_{z_1} z_2. \tag{2.3}$$

The field $\mathbb{C}$ can be used to represents $\mathbb{R}^2$ with the inner product $< z_1, z_2 >= a_1 a_2 + b_1 b_2, \forall z_1 = a_1 + ib_1, z_2 = a_2 + ib_2 \in \mathbb{C}$. Here, the quadratic form on $\mathbb{R}^2$ is $< z_1, z_1 >= |z_1|^2, \forall z_1 \in \mathbb{C}$. The conjugate of z_1, denoted by $\overline{z_1}$, is defined as $\overline{z_1} = a_1 - ib_1$. Clearly, from definition we have $z_1 + \overline{z_1} = 2a_1$, $z_1 \overline{z_1} = |z_1|^2$, $|z_1| = |\overline{z_1}|$ and $< \overline{z_1}, \overline{z_2} >=< z_1, z_2 >$. For $|z_1| \neq 0$, the inverse of z_1 is defined as $\frac{1}{z_1} = \frac{\overline{z_1}}{|z_1|^2}$. Moreover, let $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we have $\overline{z_1} = \Lambda z_1$.

For $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$, the determinant of matrix $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ will be denoted by $[z_1\ z_2]$.

Then we put $Re(\overline{z_1} z_2) =< z_1, z_2 >$ and $Im(\overline{z_1} z_2) = [z_1\ z_2]$.

For $z_1, z_2 \in \mathbb{C}$, in the case $z_1 \overline{z_1} \neq 0$, the element $\frac{z_2}{z_1}$ exists and the following equality hold:

$$L_{\frac{z_2}{z_1}} = \begin{pmatrix} Re(\frac{z_2}{z_1}) & -Im(\frac{z_2}{z_1}) \\ Im(\frac{z_2}{z_1}) & Re(\frac{z_2}{z_1}) \end{pmatrix}. \tag{2.4}$$

Put $\mathbb{C}^* = \{z \in \mathbb{C} | z \neq 0\}$, $S(\mathbb{C}^*) = \{z \in \mathbb{C} | z\overline{z} = 1\}$, $M^+(\mathbb{C}^*) = \{L_z | z \in \mathbb{C}^*\}$ and $MS(\mathbb{C}^*) = \{L_z | z \in S(\mathbb{C}^*)\}$.

It is easy to see that $\mathbb{C}^*$ is a group and $S(\mathbb{C}^*)$ is a subgroup of $\mathbb{C}^*$.

We denote the set $M^-(\mathbb{C}^*) = \left\{ L_z\Lambda | \Lambda = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, L_z \in M^+(\mathbb{C}^*) \right\}$.

Let $M^+(\mathbb{C}^*)$ and $M^-(\mathbb{C}^*)$ be sets generated by all orientation-preserving and orientation-reversing linear similarities of $\mathbb{R}^2$, resp. Clearly, $M^+(\mathbb{C}^*) \cap M^-(\mathbb{C}^*) = \varnothing$. The set $M(\mathbb{C}^*)$ of all linear similarities of $\mathbb{R}^2$ can be written in the form $M(\mathbb{C}^*) = M^+(\mathbb{C}^*) \cup M^-(\mathbb{C}^*)$.

The following theorem is known from [23, p.229].

Theorem 1. (i) $GM^+(\mathbb{C}^*) = \{F : \mathbb{C} \to \mathbb{C} | F(v) = L_z v + b, L_z \in M^+(\mathbb{C}^*), \forall v \in \mathbb{C}, b \in \mathbb{C}\}$.

(ii) $GM^-(\mathbb{C}^*) = \{F : \mathbb{C} \to \mathbb{C} | F(v) = (L_z\Lambda)v + b, L_z \in M^+(\mathbb{C}^*), \forall v \in \mathbb{C}, b \in \mathbb{C}\}$.

(iii) $GM(\mathbb{C}^*) = GM^+(\mathbb{C}^*) \cup GM^+(\mathbb{C}^*)$.

Remark 1. *For the essential notations of the group of all similarity transformations and the group of all orientation-preserving similarity transformations, see some references [10, 18, 20].*

3. On invariant functions of an complex Bézier curve and the theorem on reduction

Let G be a group $GM^+(\mathbb{C}^*)$ or $GM(\mathbb{C}^*)$.

Definition 1. *A function $f(z_0, z_1, \ldots, z_m)$ of complex numbers $z_0, z_1, \ldots, z_m$ in $\mathbb{C}$ will be called G-invariant if $f(Fz_0, Fz_1, \ldots, Fz_m) = f(z_0, z_1, \ldots, z_m)$ for all $F \in G$.*

Example 1. *Let z_0, z_1 be two complex number and $z_0 \neq 0$. The function $f(z_0, z_1) = Re(\frac{z_1}{z_0})$ is $M(\mathbb{C}^*)$-invariant. Really, let $L_z \in M(\mathbb{C}^*)$. Then by the equality (2.3), we have $L_z w = zw, \forall z, w \in \mathbb{C}$. We consider $L_z \frac{z_1}{z_0}$. Then, we obtain $L_z \frac{z_1}{z_0} = \frac{zz_1}{zz_0} = \frac{z_1}{z_0}$. Hence, we obtain that $Re(L_z \frac{z_1}{z_0}) = Re(\frac{z_1}{z_0})$. So, $Re(\frac{z_1}{z_0})$ is $M(\mathbb{C}^*)$-invariant.*

Similarly, the function $f(z_0, z_1) = Im(\frac{z_1}{z_0})$ is $M^+(\mathbb{C}^*)$-invariant.

Example 2. *Let z_0, z_1, z_2 be three complex number and $z_0 \neq z_1$. The function $f(z_0, z_1, z_2) = Re(\frac{z_2 - z_0}{z_1 - z_0})$ is $GM(\mathbb{C}^*)$-invariant. Really, let $F \in GM^+(\mathbb{C}^*)$. Then by Theorem 1, we have $F(v) = L_z v + w, \forall z \in \mathbb{C}^*$ and $v, w \in \mathbb{C}$. We consider $\frac{F(z_2 - z_0)}{F(z_1 - z_0)}$. Using above the equality, we have $\frac{F(z_2 - z_0)}{F(z_1 - z_0)} = \frac{L_z(z_2 - z_0)}{L_z(z_1 - z_0)}) = \frac{z_2 - z_0}{z_1 - z_0}$. By above example, we obtain $Re(\frac{z_2 - z_0}{z_1 - z_0})$ is $GM(\mathbb{C}^*)$-invariant. Similarly, the function $f(z_0, z_1, z_2) = Im(\frac{z_2 - z_0}{z_1 - z_0})$ is $GM^+(\mathbb{C}^*)$-invariant.*

A Bézier curve in $\mathbb{C}$ is a parametric curve(or U-path, where $U = [0, 1]$) whose complex points $Z(u)$ are defined by $Z(u) = \sum_{j=0}^{m} z_j B_j^m(u)$, where $z_j \in \mathbb{C}$ and $B_j^m(u)$ is the Bernstein basis polynomials.

A polynomial curve in $\mathbb{C}$ is a parametric curve (or U-path, where $U = [0, 1]$) whose complex points $Z(u)$ are defined by $Z(u) = \sum_{j=0}^{m} p_j(u)$, where $p_j \in \mathbb{C}$ is monomial complex control points(for more details, see [7, 8, 16, 22])

By lemma in [22, p.166], all polynomial curves can be represented in Bézier curve form.

Definition 2. *A G-invariant function $f(z_0, z_1, \ldots, z_m)$ of control complex points $z_0, z_1, \ldots, z_m$ of a Bézier curve $Z(u) = \sum_{j=0}^{m} z_j B_j^m(u)$ will be called a control G-invariant of $Z(u)$. A G-invariant function $f(p_0, p_1, \ldots, p_m)$ of monomial control complex points $p_0, p_1, \ldots, p_m$ of a polynomial curve $Z(u) = \sum_{j=0}^{m} p_j u^j$ will be called a monomial G-invariant of $Z(u)$.*

Now we define similarity of two Bézier curves of degree m and similarity of two m-uples of complex points in $\mathbb{C}$.

Definition 3. *Bézier curves $Z(u)$ and $W(u)$ in $\mathbb{C}$ will be called G -similar if there exists $F \in G$ such that $W(u) = FZ(u)$ for all $u \in [0, 1]$.*

Definition 4. *m-uples $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ of complex numbers in $\mathbb{C}$ are called G-similar if there is $F \in G$ such that $w_j = Fz_j$ for all $j = 1, 2, \ldots, m$.*

Since Bézier curves can be introduced by control points, the following two theorems means that the problem of G-similarity of Bézier curves reduce to the problem of G-similarity of two m-uples complex numbers.

Remark 2. *Throughout paper, we consider the curves in forms $Z(u) = \sum_{j=0}^{m} z_j B_j^m(u) = \sum_{j=0}^{m} p_j u^j$ and $W(u) = \sum_{j=0}^{m} w_j B_j^m(u) = \sum_{j=0}^{m} q_j u^j$ in $\mathbb{C}$ of degree m, where $m \geq 1$. Moreover, $Z'(u)$ and $W'(u)$ are their first derivatives.*

Theorem 2. *Let $Z(u)$ and $W(u)$ be Bézier curves. Then the following statements are equivalent:*

(*i*) *$Z(u)$ and $W(u)$ are $GM^+(\mathbb{C}^*)$-similar.*
(*ii*) *$Z'(u)$ and $W'(u)$ are $M^+(\mathbb{C}^*)$-similar.*
(*iii*) *m-uples $\{z_0, z_1, \ldots, z_m\}$ and $\{w_0, w_1, \ldots, w_m\}$ are $GM^+(\mathbb{C}^*)$-similar.*
(*iv*) *m-uples $\{z_1 - z_0, z_2 - z_0, \ldots, z_m - z_0\}$ and $\{w_1 - w_0, w_2 - w_0, \ldots, w_m - w_0\}$ are $M^+(\mathbb{C}^*)$-similar.*
(*v*) *$\{p_1, p_2, \ldots, p_m\}$ and $\{q_1, q_2, \ldots, q_m\}$ are $M^+(\mathbb{C}^*)$-similar.*

Proof. Proof is similar to proof of Theorem 2 in [15] and Theorem 4.1 in [9]. □

Theorem 3. *Let $Z(u)$ and $W(u)$ be Bézier curves. Then the following statements are equivalent:*

(*i*) *$Z(u)$ and $W(u)$ are $GM(\mathbb{C}^*)$-similar.*
(*ii*) *$Z'(u)$ and $W'(u)$ are $M(\mathbb{C}^*)$-similar.*
(*iii*) *m-uples $\{z_0, z_1, \ldots, z_m\}$ and $\{w_0, w_1, \ldots, w_m\}$ are $GM(\mathbb{C}^*)$-similar.*
(*iv*) *m-uples $\{z_1 - z_0, z_2 - z_0, \ldots, z_m - z_0\}$ and $\{w_1 - w_0, w_2 - w_0, \ldots, w_m - w_0\}$ are $M(\mathbb{C}^*)$-similar.*
(*v*) *$\{p_1, p_2, \ldots, p_m\}$ and $\{q_1, q_2, \ldots, q_m\}$ are $M(\mathbb{C}^*)$-similar.*

Proof. Proof is similar to proof of Theorem 1 in [15] and Theorem 4.1 in [9]. □

Remark 3. (1)

(i) *Let* $\{z_1, z_2, \ldots, z_m\}$ *and* $\{w_1, w_2, \ldots, w_m\}$ *in* $\mathbb{C}$ *be two m-uples such that* $z_k \neq 0$ *and* $w_k = 0$*. Then* $\{z_1, z_2, \ldots, z_m\}$ *and* $\{w_1, w_2, \ldots, w_m\}$ *are not* G*-similar. In the case* $z_k = w_k = 0$*, we obtain the problem of* G*- similarity of these* $m-1$*-uples* $\{z_1,\ z_2,\ \ldots, z_{k-1}, z_{k+1}, \ldots, z_m\}$ *and* $\{w_1,\ w_2,\ \ldots,\ w_{k-1}, w_{k+1},\ \ldots,\ w_m\}$*, Therefore, we put* $z_k \neq 0$ *and* $w_k \neq 0$ *for* $k \in \{1, 2, \ldots, m\}$*.*

(ii) *Let* z_1 *and* w_1 *in* $\mathbb{C}$ *be two complex number such that* $z_1 \neq 0$ *and* $w_1 \neq 0$*. Then there always is an element* L_z *in* G *such that* $w_1 = L_z z_1$*. Therefore, we put* $m > 1$ *for m-uples* $\{z_1, z_2, \ldots, z_m\}$ *and* $\{w_1, w_2, \ldots, w_m\}$ *in* $\mathbb{C}$*.*

4. Conditions of similarity for two m-uple complex number sets

Theorem 4. *Let* $\{z_1, z_2, \ldots, z_m\}$ *and* $\{w_1, w_2, \ldots, w_m\}$ *be two m-uples in* $\mathbb{C}$ *such that* $z_k \neq 0$ *and* $w_k \neq 0$*, where* $k \in \{1, 2, \ldots, m\}$*. Then these m-uples are* $M^+(\mathbb{C}^*)$*-similar if and only if*

$$\begin{cases} Re(\dfrac{z_i}{z_k}) = Re(\dfrac{w_i}{w_k}) \\ Im(\dfrac{z_i}{z_k}) = Im(\dfrac{w_i}{w_k}) \end{cases} \tag{4.1}$$

for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$*.*
Furthermore, there is the unique $L_z \in M^+(\mathbb{C}^*)$ *such that* $w_i = L_z z_i$ *for all* $i = 1, 2, \ldots, m$*, where the matrix* L_z *can be written as*

$$L_z = \begin{pmatrix} Re(\frac{w_k}{z_k}) & -Im(\frac{w_k}{z_k}) \\ Im(\frac{w_k}{z_k}) & Re(\frac{w_k}{z_k}) \end{pmatrix}. \tag{4.2}$$

Proof. $\Rightarrow$: Assume that $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ in $\mathbb{C}$ are $M^+(\mathbb{C}^*)$)-similar. Since the functions $Re(\frac{z_i}{z_k})$ and $Im(\frac{z_i}{z_k})$ are $M^+(\mathbb{C}^*)$)-invariant, we obtain that the equalities (4.1) hold.
$\Leftarrow$: Assume that the equalities (4.1) hold. By the equality (4.1), we have

$$\frac{z_i}{z_k} = \frac{w_i}{w_k} \tag{4.3}$$

for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$. Consider the element $z = \frac{w_k}{z_k} \in \mathbb{C}$. By the equality (4.3), we have $w_i = w_k \frac{w_i}{w_k} = w_k \frac{z_i}{z_k} = \frac{w_k}{z_k} z_i$ for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$. So, by the equality (2.3),we have $w_i = L_z z_i$ for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$. Clearly $L_z \in M^+(\mathbb{C}^*)$. For uniqueness, assume that $L_v \in M^+(\mathbb{C}^*)$ exists such that $w_i = L_v z_i$ for all $i = 1, 2, \ldots, m$. Then, by this equality and the equality (2.3), we have $v \in M^+(\mathbb{C}^*)$ such that $w_i = v z_i$ for all $i = 1, 2, \ldots, m$. Since $z_k \neq 0$, the equality $w_k = v z_k$ implies that $v = \frac{w_k}{z_k} = z$. Hence the uniqueness of L_z is proved. Moreover, using the equality (2.3), the element $z = \frac{w_k}{z_k}$ can be written as the matrix L_z, where L_z has the form (4.2). □

Denote by $rank\,\{z_1, z_2, \ldots, z_m\}$ the rank of the m-uple $\{z_1, z_2, \ldots, z_m\}$ in $\mathbb{C}$. It is easy to see that $rank\,\{z_1, z_2, \ldots, z_m\}$ is $M(\mathbb{C}^*)$-invariant.

Theorem 5. *Let $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ be two m-uples in $\mathbb{C}$ such that $z_k \neq 0$, $w_k \neq 0$ for $k \in \{1, 2, \ldots, m\}$ and $rank\,\{z_1, z_2, \ldots, z_m\} = rank\,\{w_1, w_2, \ldots, w_m\} = 1$. Then these m-uples are $M(\mathbb{C}^*)$-similar if and only if*

$$Re(\frac{z_i}{z_k}) = Re(\frac{w_i}{w_k}) \tag{4.4}$$

for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$.
Furthermore, there is the unique $L_z \in M(\mathbb{C}^*)$ such that $w_i = L_z z_i$ for all $i = 1, 2, \ldots, m$, where the matrix L_z can be written as the form (4.2).

Proof. $\Rightarrow$: The proof is similar to the proof of Theorem 4.
$\Leftarrow$: Assume that the equality (4.4) holds.
Since $rank\,\{z_1, z_2, \ldots, z_m\} = rank\,\{w_1, w_2, \ldots, w_m\} = 1$, we have $Im(\frac{z_i}{z_k}) = Im(\frac{w_i}{w_k}) = 0$ for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$. Hence the equalities (4.1) in Theorem 4 hold. Using Theorem 4, we have $w_i = L_z z_i$ for all $i = 1, 2, \ldots, m$ and the matrix L_z has the form (4.2). □

Let m-uple $\{z_1, z_2, \ldots, z_m\}$ in $\mathbb{C}$. In the case $rank\,\{z_1, z_2, \ldots, z_m\} = 2$, denote by
$ind\,\{z_1, z_2, \ldots, z_m\}$ the smallest of p, $1 \leq p \leq m$, such that $z_p \neq \lambda z_k$ for all $\lambda \in \mathbb{R}$ and $z_k \neq 0$.

Theorem 6. *Let $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ be two m-uples in $\mathbb{C}$ such that $z_k \neq 0$, $w_k \neq 0$, $rank\,\{z_1, z_2, \ldots, z_m\} = rank\,\{w_1, w_2, \ldots, w_m\} = 2$ and $ind\,\{z_1, z_2, \ldots, z_m\} = ind\,\{w_1, w_2, \ldots, w_m\} = l$ for $k, l \in \{1, 2, \ldots, m\}$ and $k \neq l$. Then these m-uples are $M(\mathbb{C}^*)$-similar if and only if*

$$\begin{cases} Re(\frac{z_i}{z_k}) = Re(\frac{w_i}{w_k}) \\ \left[Im(\frac{z_l}{z_k})\right]^2 = \left[Im(\frac{w_l}{w_k})\right]^2 \\ \frac{Im(\frac{z_i}{z_k})}{Im(\frac{z_l}{z_k})} = \frac{Im(\frac{w_i}{w_k})}{Im(\frac{w_l}{w_k})} \end{cases} \tag{4.5}$$

for all $i = 1, 2, \ldots, m$, $i \neq k$.
Furthermore, there is the unique $L_z \in M(\mathbb{C}^*)$ such that $w_i = L_z z_i$ for all $i = 1, 2, \ldots, m$. Then there exist two statements:

(*i*) *In the case $Im(\frac{z_l}{z_k}) = Im(\frac{w_l}{w_k})$, the element $L_z \in M^+(\mathbb{C}^*)$ and it can be represented by (4.2).*

(*ii*) *In the case $Im(\frac{z_l}{z_k}) = -Im(\frac{w_l}{w_k})$, the element $L_z\Lambda \in M^-(\mathbb{C}^*)$ and it can be written as*

$$L_z\Lambda = \begin{pmatrix} Re(\frac{w_k}{\overline{z}_k}) & -Im(\frac{w_k}{\overline{z}_k}) \\ Im(\frac{w_k}{\overline{z}_k}) & Re(\frac{w_k}{\overline{z}_k}) \end{pmatrix}. \tag{4.6}$$

Proof. $\Rightarrow$: Let m-uples $\{z_1, z_2, \dots, z_m\}$ and $\{w_1, w_2, \dots, w_m\}$ are $M(\mathbb{C}^*)$-similar. Since the functions $Re(\frac{z_i}{z_k})$, $\left[Im(\frac{z_l}{z_k})\right]^2$ and $\frac{Im(\frac{z_i}{z_k})}{Im(\frac{z_l}{z_k})}$ are $M(\mathbb{C}^*)$-invariant, we obtain that the equalities (4.6).

$\Leftarrow$: Assume that the equality (4.6) holds. Using the conditions $z_k \neq 0$, $w_k \neq 0$ and $ind\{z_1, z_2, \dots, z_m\} = ind\{w_1, w_2, \dots, w_m\} = l$ for $k, l \in \{1, 2, \dots, m\}, k \neq l$ and the equality $\left[Im(\frac{z_l}{z_k})\right]^2 = \left[Im(\frac{w_l}{w_k})\right]^2$, we have the equality $\left[Im(\frac{z_l}{z_k})\right] = \left[Im(\frac{w_l}{w_k})\right]$ or $\left[Im(\frac{z_l}{z_k})\right] = -\left[Im(\frac{w_l}{w_k})\right]$. Moreover, since $rank\{z_1, z_2, \dots, z_m\} = rank\{w_1, w_2, \dots, w_m\} = 2$ and
$ind\{z_1, z_2, \dots, z_m\} = ind\{w_1, w_2, \dots, w_m\} = l$, we have $Im(\frac{z_l}{z_k}) \neq 0$.

(i) Assume that $Im(\frac{z_l}{z_k}) = Im(\frac{w_l}{w_k})$. Then, using this equality and the equality $\frac{Im(\frac{z_i}{z_k})}{Im(\frac{z_l}{z_k})} = \frac{Im(\frac{w_i}{w_k})}{Im(\frac{w_l}{w_k})}$, we have $Im(\frac{z_i}{z_k}) = Im(\frac{w_i}{w_k})$ for all $i \neq k$. So the equalities (4.1) hold. Then by Theorem 4, there is the unique $L_z \in M^+(\mathbb{C}^*)$ such that $w_i = L_z z_i$ for all $i = 1, 2, \dots, m$. The element L_z has the form (4.2).

(ii) Assume that $Im(\frac{z_l}{z_k}) = -Im(\frac{w_l}{w_k})$. Then, using this equality and the equality $\frac{Im(\frac{z_i}{z_k})}{Im(\frac{z_l}{z_k})} = \frac{Im(\frac{w_i}{w_k})}{Im(\frac{w_l}{w_k})}$, we have $Im(\frac{z_i}{z_k}) = -Im(\frac{w_i}{w_k})$ for all $i \neq k$. Hence, we obtain $Im(\frac{z_i}{z_k}) = -Im(\frac{\overline{z_i}}{\overline{z_k}})$. Then by this equality and the equality $Re(\frac{z_i}{z_k}) = Re(\frac{\overline{z_i}}{\overline{z_k}})$, we have $Re(\frac{w_i}{w_k}) = Re(\frac{\overline{z_i}}{\overline{z_k}})$ and $Im(\frac{w_i}{w_k}) = Im(\frac{\overline{z_i}}{\overline{z_k}})$. In this case, by Theorem 4, there is the unique $L_z \in M^+(\mathbb{C}^*)$ such that $w_i = L_z \overline{z_i} = L_z(\Lambda z_i) = (L_z \Lambda) z_i$ for all $i = 1, 2, \dots, m$. Then the element $L_z \Lambda$ has the form (4.6).

$\square$

5. Conditions of similarity for two complex Bézier curves and its applications

Using Theorem 2 and Theorem 4, the following corollary obtain.

Corollary 1. *Let $Z(u) = \sum_{j=0}^{m} p_j u^j$ and $W(u) = \sum_{j=0}^{m} q_j u^j$ be two polynomial curves in $\mathbb{C}$ of degree $m > 1$. Then $Z(u)$ and $W(u)$ are $GM^+(C^*)$-similar if and only if*

$$\begin{cases} Re(\dfrac{p_i}{p_m}) = Re(\dfrac{q_i}{q_m}) \\ Im(\dfrac{p_i}{p_m}) = Im(\dfrac{q_i}{q_m}) \end{cases} \tag{5.1}$$

for all $i = 1, 2, \dots, m-1$.
Furthermore, there is the unique $F \in GM^+(\mathbb{C}^*)$ such that $q_i = F p_i = L_z p_i + b$ for all $i = 0, 1, 2, \dots, m$, where the matrix $L_z \in M^+(\mathbb{C}^*)$ and the constant $b \in \mathbb{C}$ can

be written as

$$L_z = \begin{pmatrix} Re(\frac{q_m}{p_m}) & -Im(\frac{q_m}{p_m}) \\ Im(\frac{q_m}{p_m}) & Re(\frac{q_m}{p_m}) \end{pmatrix} \tag{5.2}$$

and

$$b = q_0 - L_z p_0. \tag{5.3}$$

Example 3. *Consider two polynomial curves $Z(u) = (2 + 2u, 3 - 8u + 11u^2)$ and $W(u) = (-10 + 44u - 55u^2, 20 - 6u + 22u^2)$ with complex monomial control points $p_0 = 2 + 3i, p_1 = 2 - 8i, p_2 = 11i$ and $q_0 = -10 + 20i, q_1 = 44 - 6i, q_2 = -55 + 22i$ in $\mathbb{C}$, resp. It is easy to see that the equalities in (5.1) hold for the curves $Z(u)$ and $W(u)$. Then by Theorem 1, $Z(u)$ and $W(u)$ are $GM^+(C^*)$-similar and $L_z = 2+5i$ and $b = 1 + 4i$.*

Using Theorem 3 and Theorem 6, the following corollary obtain.

Corollary 2. *Let $Z(u) = \sum_{j=0}^{m} p_j u^j$ and $W(u) = \sum_{j=0}^{m} q_j u^j$ be two polynomial curves in $\mathbb{C}$ of degree $m > 1$. Let Then $Z(u)$ and $W(u)$ are $GM(C^*)$-similar if and only if*

$$\begin{cases} Re(\dfrac{p_i}{p_m}) = Re(\dfrac{q_i}{q_m}) \\ \left[Im(\dfrac{p_l}{p_m})\right]^2 = \left[Im(\dfrac{q_l}{q_m})\right]^2 \\ \dfrac{Im(\frac{p_j}{p_m})}{Im(\frac{p_l}{p_m})} = \dfrac{Im(\frac{q_j}{q_m})}{Im(\frac{q_l}{q_m})} \end{cases} \tag{5.4}$$

for all $i = 1, 2, \ldots, m - 1$ and for all $j = 1, 2, \ldots, l - 1, l + 1, \ldots, m - 1$, where $ind\{p_1, p_2, \ldots, p_m\} = ind\{q_1, q_2, \ldots, q_m\} = l$ for $l \in \{1, 2, \ldots, m - 1\}$.
Furthermore, there is the unique $F \in GM(\mathbb{C}^*)$ such that $q_i = Fp_i$ for all $i = 0, 1, 2, \ldots, m$. There are the following two cases:

(*i*) *In the case $Im(\frac{p_l}{p_m}) = Im(\frac{q_l}{q_m})$, F has the form $Fp_i = L_z p_i + b_1$ for all $i = 0, 1, 2, \ldots, m$, where the element $L_z \in M^+(\mathbb{C}^*)$ and the constant $b_1 \in \mathbb{C}$ can be written as (5.2) and (5.3), resp.*

(*ii*) *In the case $Im(\frac{p_l}{p_m}) = -Im(\frac{q_l}{q_m})$, F has the form $Fp_i = L_z \Lambda p_i + b_2$ for all $i = 0, 1, 2, \ldots, m$, where the element $L_z \in M^+(\mathbb{C}^*)$ and and the constant $b_2 \in \mathbb{C}$ can be written as*

$$L_z = \begin{pmatrix} Re(\frac{q_m}{\overline{p_m}}) & -Im(\frac{q_m}{\overline{p_m}}) \\ Im(\frac{q_m}{\overline{p_m}}) & Re(\frac{q_m}{\overline{p_m}}) \end{pmatrix} \tag{5.5}$$

and

$$b_2 = q_0 - L_z \Lambda p_0. \tag{5.6}$$

Example 4. *Consider two polynomial curves $Z(u) = (2 + 2u, 3 - 8u + 11u^2)$ and $W(u) = (20 - 36u + 55u^2, 8 + 26u - 22u^2)$ with complex monomial control points*

$p_0 = 2+3i, p_1 = 2-8i, p_2 = 11i$ *and* $q_0 = 20+8i, q_1 = -36+26i, q_2 = 55-22i$ *in* $\mathbb{C}$, *resp. It is easy to see that the equalities in* (5.4) *hold for the curves* $Z(u)$ *and* $W(u)$. *Then by Theorem* 2, $Z(u)$ *and* $W(u)$ *are* $GM(C^*)$*-similar and* $L_z = 2+5i$ *and* $b = 1+4i$. *But* $Z(u)$ *and* $W(u)$ *are not* $GM^+(C^*)$*-similar.*

Using Theorem 2 and Theorem 4, the following corollary obtain.

Corollary 3. *Let* $Z(u) = \sum_{j=0}^{m} z_j B_j^m$ *and* $W(u) = \sum_{j=0}^{m} w_j B_j^m$ *be two Bézier curves in* $\mathbb{C}$ *of degree* $m > 1$ *such that* $z_m - z_0 \neq 0$ *and* $w_m - w_0 \neq 0$. *Then* $Z(u)$ *and* $W(u)$ *are* $GM^+(C^*)$*-similar if and only if*

$$\begin{cases} Re(\dfrac{z_i - z_0}{z_m - z_0}) = Re(\dfrac{w_i - w_0}{w_m - w_0}) \\ Im(\dfrac{z_i - z_0}{z_m - z_0}) = Im(\dfrac{w_i - w_0}{w_m - w_0}) \end{cases} \tag{5.7}$$

for all $i = 1, 2, \ldots, m-1$.
Furthermore, there is the unique $F \in GM^+(\mathbb{C}^*)$ *such that* $w_i = Fz_i = L_z z_i + b$ *for all* $i = 0, 1, 2, \ldots, m$, *where the matrix* $L_z \in M^+(\mathbb{C}^*)$ *and the constant* $b \in \mathbb{C}$ *can be written as*

$$L_z = \begin{pmatrix} Re(\frac{w_m - w_0}{z_m - z_0}) & -Im(\frac{w_m - w_0}{z_m - z_0}) \\ Im(\frac{w_m - w_0}{z_m - z_0}) & Re(\frac{w_m - w_0}{z_m - z_0}) \end{pmatrix} \tag{5.8}$$

and

$$b = w_0 - L_z z_0. \tag{5.9}$$

Example 5. *Consider two complex Bézier curves* $Z(u) = \sum_{j=0}^{2} z_j B_j^m$ *and* $W(u) = \sum_{j=0}^{2} w_j B_j^m$ *with complex control points* $z_0 = 2+3i, z_1 = 3-i, z_2 = 4+6i$ *and* $w_0 = -10+20i, w_1 = 12+17i, w_2 = -21+36i$ *in* $\mathbb{C}$, *resp. It is easy to see that the equalities in* (5.7) *hold for the curves* $Z(u)$ *and* $W(u)$. *Then by Theorem* 3, $Z(u)$ *and* $W(u)$ *are* $GM^+(C^*)$*-similar and* $L_z = 2+5i$ *and* $b = 1+4i$.

Using Theorem 3 and Theorem 6, the following corollary obtain.

Corollary 4. *Let* $Z(u) = \sum_{j=0}^{m} z_j B_j^m$ *and* $W(u) = \sum_{j=0}^{m} w_j B_j^m$ *be two Bézier curves in* $\mathbb{C}$ *of degree* $m > 1$ *such that* $z_m - z_0 \neq 0$ *and* $w_m - w_0 \neq 0$. *Let Then* $Z(u)$ *and* $W(u)$ *are* $GM(C^*)$*-similar if and only if*

$$\begin{cases} Re(\dfrac{z_i - z_0}{z_m - z_0}) = Re(\dfrac{w_i - w_0}{w_m - w_0}) \\ \left[Im(\dfrac{z_l - z_0}{z_m - z_0})\right]^2 = \left[Im(\dfrac{w_l - w_0}{w_m - w_0})\right]^2 \\ \dfrac{Im(\frac{z_j - z_0}{z_m - z_0})}{Im(\frac{z_l - z_0}{z_m - z_0})} = \dfrac{Im(\frac{w_j - w_0}{w_m - w_0})}{Im(\frac{w_l - w_0}{w_m - w_0})} \end{cases} \tag{5.10}$$

for all $i = 1, 2, \ldots, m-1$ *and for all* $j = 1, 2, \ldots, l-1, l+1, \ldots, m-1$, *where* $ind\{z_1 - z_0, z_2 - z_0, \ldots, z_m - z_0\} = ind\{w_1 - w_0, w_2 - w_0, \ldots, w_m - w_0\} = l$ *for*

$l \in \{1, 2, \ldots, m-1\}$.

Furthermore, there is the unique $F \in GM(\mathbb{C}^*)$ *such that* $w_i = Fz_i$ *for all* $i = 0, 1, 2, \ldots, m$. *There are the following two cases:*

(i) *In the case* $Im(\frac{z_l - z_0}{z_m - z_0}) = Im(\frac{w_l - w_0}{w_m - w_0})$, F *has the form* $Fz_i = L_z z_i + b_1$ *for all* $i = 0, 1, 2, \ldots, m$, *where the element* $L_z \in M^+(\mathbb{C}^*)$ *and the constant* $b_1 \in \mathbb{C}$ *can be written as* (5.8) *and* (5.9), *resp.*

(ii) *In the case* $Im(\frac{z_l - z_0}{z_m - z_0}) = -Im(\frac{w_l - w_0}{w_m - w_0})$, F *has the form* $Fz_i = L_z \Lambda z_i + b_2$ *for all* $i = 0, 1, 2, \ldots, m$, *where the element* $L_z \in M^+(\mathbb{C}^*)$ *and and the constant* $b_2 \in \mathbb{C}$ *can be written as*

$$L_z = \begin{pmatrix} Re(\frac{w_m - w_0}{z_m - z_0}) & -Im(\frac{w_m - w_0}{z_m - z_0}) \\ Im(\frac{w_m - w_0}{z_m - z_0}) & Re(\frac{w_m - w_0}{z_m - z_0}) \end{pmatrix} \tag{5.11}$$

and

$$b_2 = w_0 - L_z \Lambda z_0. \tag{5.12}$$

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References

[1] Zhang, C. , Du, S., Liu, J.,Li, Y., Xue,J., Liu, Y., Robust iterative closest point algorithm with bounded rotation angle for 2D registration, *Neurocomputing*, 195(2016),172-180.

[2] Besl, P. J., McKay, N. D., A method for registration of 3-D shapes, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 14(2) (1992), 239-256.

[3] Du, S., Zheng, N., Ying, S., Liu, J., Affine iterative closest point algorithm for point set registration, *Pattern Recognition Letters*, 31(2010),791-799.

[4] Chen, H., Zhang, X., Du, S., Wu, Z., Zheng, N., A Correntropy-based affine iterative closest point algorithm for robust point set registration, *IEEE/CAA J. Autom. Sin.*, 6 (4) (2019), 981-991.

[5] Weiss, I., Geometric invariants and object recognition, *J. Math. Imaging Vision*, 10 (3) (1993), 201-231 .

[6] Sanchez-Reyes, J., Complex rational Bézier curves, *Comput. Aided Geom. Design*, 26(8) (2009), 865-876.

[7] Tsianos, K. I., Goldman, R., Bézier and B-spline curves with knots in the complex plane, *Fractals*, 19(1) (2011), 67-86.

[8] Ait-Haddou, R., Herzog, W., Nomura, T., Complex Bézier curves and the geometry of polygons, *Comput. Aided Geom. Design*, 27(7) (2010), 525-537.

[9] Ören, İ., On the control invariants of planar Bézier curves for the groups M(2) and SM(2), *Turk. J. Math. Comput. Sci.*,10 (2018), 74-81.

[10] Khadjiev, D., Ören, İ., Pekşen,Ö., Global invariants of paths and curves for the group of all linear similarities in the two-dimensional Euclidean space, *Int. J. Geom. Methods Mod. Phys.*, 15(6)(2018), 1-28.

[11] Alcazar, J. G., Hermoso, C., Muntingh, G., Detecting similarity of rational plane curves, *J. Comput. Appl. Math.*, 269 (2014), 1-13.

[12] Alcazar, J. G., Hermoso, C„ Muntingh, G., Similarity detection of rational space curves, *J. Symbolic Comput.*, 85 (2018), 4-24.

[13] Alcazar, J. G., Gema, M. Diaz-Toca, Hermoso, C., On the problem of detecting when two implicit plane algebraic curves are similar, *Internat. J. Algebra Comput*, 29 (5) (2019),775-793.
[14] Hauer, M., Jüttler, B., Detecting affine equivalences of planar rational curves, EuroCG 2016, Lugano, Switzerland, March 30-April 1, 2016.
[15] Ören, İ., Equivalence conditions of two Bézier curves in the Euclidean geometry, *Iran J Sci Technol Trans Sci*, 42(3) (2016),1563-1577.
[16] Reyes, J. S., Detecting symmetries in polynomial Bézier curves,*J. Comput. Appl. Math.*, 288 (2015), 274-283.
[17] Mozo-Fernandez, J., Munuera, C., Recognition of polynomial plane curves under affine transformations, *AAECC*, 13 (2002), 121-136.
[18] Gürsoy, O., İncesu, ., LS(2)-Equivalence conditions of control points and application to planar Bézier curves, *New Trends in Mathematical Science*, 3 (2017), 70-84.
[19] Bez, H.E., Generalized invariant-geometry conditions for the rational Bézier paths, *Int J Comput Math*, 87 (2010), 793-811 .
[20] Encheva, R. P., Georgiev, G. H., Similar Frenet curves, *Result. Math*, 55 (2009), 359-372.
[21] Khadjiev, D., Ören, İ., Global invariants of paths and curves for the group of orthogonal transformations in the two-dimensional Euclidean space, *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.*, 27(2) (2019), 37-65.
[22] Marsh, D., Applied Geometry For Computer Graphics and CAD, Springer-Verlag, London,1999.
[23] Berger, M., Geometry I, Springer-Verlag, Berlin, Heidelberg, 1987.

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AN APPROACH TO PRE-SEPARATION AXIOMS IN NEUTROSOPHIC SOFT TOPOLOGICAL SPACES

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Abstract. In this study, we introduce the concept of neutrosophic soft pre-open (neutrosophic soft pre-closed) sets and pre-separation axioms in neutrosophic soft topological spaces. In particular, the relationship between these separation axioms are investigated. Also, we give a new definition for neutrosophic soft topological subspace and define neutrosophic soft pre irresolute soft and neutrosophic pre irresolute open soft functions.

1. Introduction

In 2005, Smarandache introduced the concept of a neutrosophic set [20] as a generalization of classical sets, fuzzy set theory [20] (see also [10]) , intuitionistic fuzzy set theory [4] (see also [14]) etc. By using this theory of neutrosophic set, some scientists made researches in many areas of mathematics [7, 18]. Many inherent difficulties exist in classical methods for the inadequacy of the theories of parametrization tools. So, classical methods are insufficient in dealing with several practical problems in some other disciplines such as economics, engineering, environment, social science, medical science, etc. In 1999, Molodtsov [16] pointed out the inherent difficulties of these theories. A different approach was initiated by Molodtsov for modeling uncertainties. This approach was applied in some other directions such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration and so on. The theory of soft topological spaces was introduced by Shabir and Naz [19] for the first time in 2011. Soft topological spaces were defined over an initial universe with a fixed set of parameters and showed that a soft topological space gives a parameterized family of topological

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spaces. In [1, 2, 5, 6, 9, 11, 13], some scientists made researches and did theoretical studies in soft topological spaces. In 2013, Maji [15] defined the concept of neutrosophic soft sets for the first time. Then, Deli and Broumi [12] modified this concept. In 2017, Bera presented neutrosophic soft topological spaces in [8].

In this study, our pupose is to adapt the concepts of neutrosophic pre open soft set, neutrosophic pre closed soft set to neutrosophic soft topological spaces. Then, we define neutrosophic soft pre interior point, neutrosophic soft pre cluster point, neutrosophic soft pre interior operator and neutrosophic soft pre closure operator. By using these definitions and concepts, the concept of pre-separation axioms of neutrosophic soft topological spaces is introduced. Furthermore, we analyze properties of neutrosophic soft pre T_i -spaces $(i = 0, 1, 2, 3, 4)$ and focus on some relations between them. Characterization theorems of them are also proved. We hope that, the findings in this document will help scientists to enhance and promote the further studies on neutrosophic soft topology to carry out a general framework for their applications in practical life.

2. Preliminaries

In this section, we present the basic definitions and theorems related to neutrosophic soft set theory.

Definition 1. *[20] A neutrosophic set A on the universe set X is defined as:*
$A = \{\langle x, T_A(x), I_A(x), F_A x\rangle : x \in X\}$,
where
T,I, $F : X \to]^-0, 1^+[$ *and* $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

Definition 2. *[16] Let X be an initial universe, E be a set of all parameters, and $P(X)$ denote the power set of X. A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$. In other words, the soft set is a parameterized family of subsets of the set X. For $e \in E$, $F(e)$ may be considered as the set of e-elements of the soft set (F, E), or as the set of e-approximate elements of the soft set, i.e.*
$(F, E) = \{(e, F(e)) : e \in E,\ F : E \to P(X)\}$.

After the neutrosophic soft set was defined by Maji [15], this concept was modified by Deli and Broumi [12] as given below:

Definition 3. *[12] Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denote the set of all neutrosophic sets of X. Then a neutrosophic soft set $\left(\widetilde{F}, E\right)$ over X is a set defined by a set valued function $\widetilde{F}$ representing a mapping $\widetilde{F} : E \to P(X)$, where $\widetilde{F}$ is called the approximate function of the neutrosophic soft set $\left(\widetilde{F}, E\right)$. In other words, the neutrosophic soft set is a parametrized family of some elements of the set $P(X)$ and therefore it can be written as a set of ordered pairs:*

$\left(\widetilde{F},E\right)=\left\{\left(e,\ \left\langle x,\ T_{\widetilde{F}(e)}\left(x\right),I_{\widetilde{F}(e)}\left(x\right),F_{\widetilde{F}(e)}\left(x\right)\right\rangle :x\in X\right):e\in E\right\}$

where $T_{\widetilde{F}(e)}(x)$, $I_{\widetilde{F}(e)}(x)$, $F_{\widetilde{F}(e)}(x)\in[0,1]$ *are respectively called the truth-membership,*
indeterminacy-membership and falsity-membership function of $\widetilde{F}(e)$. *Since the supremum of each* T, I, F *is* 1, *the inequality*

$0\leq T_A(x)+I_A(x)+F_A(x)\leq 3$

is obvious.

Definition 4. *[8] Let* $\left(\widetilde{F},E\right)$ *be a neutrosophic soft set over the universe set* X. *The complement of* $\left(\widetilde{F},E\right)$ *is denoted by* $\left(\widetilde{F},E\right)^c$ *and is defined by:*

$\left(\widetilde{F},E\right)^c=\left\{\left(e,\ \left\langle x,\ F_{\widetilde{F}(e)}\left(x\right),\ 1-I_{\widetilde{F}(e)}\left(x\right),\ T_{\widetilde{F}(e)}\left(x\right)\right\rangle :x\in X\right):e\in E\right\}.$

It is obvious that $\left[\left(\widetilde{F},E\right)^c\right]^c=\left(\widetilde{F},E\right)$.

Definition 5. *[15] Let* $\left(\widetilde{F},E\right)$ *and* $\left(\widetilde{G},E\right)$ *be two neutrosophic soft sets over the universe set* X. $\left(\widetilde{F},E\right)$ *is said to be a neutrosophic soft subset of* $\left(\widetilde{G},E\right)$ *if*

$T_{\widetilde{F}(e)}(x)\leq T_{\widetilde{G}(e)}(x)$, $I_{\widetilde{F}(e)}(x)\leq I(x)$, $F_{\widetilde{F}(e)}(x)\geq F_{\widetilde{G}(e)}(x)$, $\forall e\in E$, $\forall x\in X$.

It is denoted by $\left(\widetilde{F},E\right)\subseteq\left(\widetilde{G},E\right)$. $\left(\widetilde{F},E\right)$ *is said to be neutrosophic soft equal to* $\left(\widetilde{G},E\right)$ *if* $\left(\widetilde{F},E\right)\subseteq\left(\widetilde{G},E\right)$ *and* $\left(\widetilde{G},E\right)\subseteq\left(\widetilde{F},E\right)$. *It is denoted by* $\left(\widetilde{F},E\right)=\left(\widetilde{G},E\right)$.

Definition 6. *[3] Let* $\left(\widetilde{F}_1,E\right)$ *and* $\left(\widetilde{F}_2,E\right)$ *be two neutrosophic soft sets over the universe set* X. *Then their union is denoted by* $\left(\widetilde{F}_1,E\right)\cup\left(\widetilde{F}_2,E\right)=\left(\widetilde{F}_3,E\right)$ *and is defined by:*

$\left(\widetilde{F}_3,E\right)=\left\{\left(e,\ \left\langle x,\ T_{\widetilde{F}_3(e)}\left(x\right),I\left(x\right),F_{\widetilde{F}_3(e)}\left(x\right)\right\rangle :x\in X\right):e\in E\right\},$

where

$T_{\widetilde{F}_3(e)}(x)=max\left\{T_{\widetilde{F}_1(e)}(x),\ T_{\widetilde{F}_2(e)}(x)\right\},$

$I_{\widetilde{F}_3(e)}(x)=max\left\{I_{\widetilde{F}_1(e)}(x),\ I_{\widetilde{F}_2(e)}(x)\right\},$

$F_{\widetilde{F}_3(e)}(x)=min\left\{F_{\widetilde{F}_1(e)}(x),\ F_{\widetilde{F}_2(e)}(x)\right\}.$

Definition 7. *[3] Let* $\left(\widetilde{F}_1,E\right)$ *and* $\left(\widetilde{F}_2,E\right)$ *be two neutrosophic soft sets over the universe set* X. *Then their* intersection *is denoted by* $\left(\widetilde{F}_1,E\right)\cap\left(\widetilde{F}_2,E\right)=\left(\widetilde{F}_3,E\right)$ *and is defined by:*

$\left(\widetilde{F}_3,E\right)=\left\{\left(e,\ \left\langle x,\ T_{\widetilde{F}_3(e)}\left(x\right),I\left(x\right),F_{\widetilde{F}_3(e)}\left(x\right)\right\rangle :x\in X\right):e\in E\right\},$

where

$$T_{\widetilde{F}_3(e)}(x) = min\left\{T_{\widetilde{F}_1(e)}(x),\, T_{\widetilde{F}_2(e)}(x)\right\},$$

$$I_{\widetilde{F}_3(e)}(x) = min\left\{I_{\widetilde{F}_1(e)}(x),\, I_{\widetilde{F}_2(e)}(x)\right\},$$

$$F_{\widetilde{F}_3(e)}(x) = max\left\{F_{\widetilde{F}_1(e)}(x),\, F_{\widetilde{F}_2(e)}(x)\right\}.$$

Definition 8. *[3] A neutrosophic soft set $\left(\widetilde{F}, E\right)$ over the universe set X is said to be a null neutrosophic soft set if $T_{\widetilde{F}(e)}(x) = 0$, $I_{\widetilde{F}(e)}(x) = 0$, $F_{\widetilde{F}(e)}(x) = 1$; $\forall e \in E$, $\forall x \in X$. It is denoted by $0_{(X,E)}$.*

Definition 9. *[3] A neutrosophic soft set $\left(\widetilde{F}, E\right)$ over the universe set X is said to be an absolute neutrosophic soft set if $T_{\widetilde{F}(e)}(x) = 1$, $I_{\widetilde{F}(e)}(x) = 1$, $F_{\widetilde{F}(e)}(x) = 0$; $\forall e \in E$, $\forall x \in X$ It is denoted by $1_{(X,E)}$.*

Clearly $0^c_{(X,E)} = 1_{(X,E)}$ and $1^c_{(X,E)} = 0_{(X,E)}$.

Definition 10. *[3] Let $NSS(X,E)$ be the family of all neutrosophic soft sets over the universe set X and $\tau \subset NSS(X,E)$. Then τ is said to be a neutrosophic soft topology on X if:*

1. $0_{(X,E)}$ and $1_{(X,E)}$ belong to τ,

2. the union of any number of neutrosophic soft sets in τ belongs to τ,

3. the intersection of a finite number of neutrosophic soft sets in τ belongs to τ.

Then (X, τ, E) is said to be a neutrosophic soft topological space over X. Each member of τ is said tobe a neutrosophic soft open set [3].

Definition 11. *[3] Let (X, τ, E) be a neutrosophic soft topological space over X and $\left(\widetilde{F}, E\right)$ be a neutrosophic soft set over X. Then $\left(\widetilde{F}, E\right)$ is said to be a neutrosophic soft closed set iff its complement is a neutrosophic soft open set.*

Definition 12. *[3] Let $NSS(X,E)$ be the family of all neutrosophic soft sets over the universe set X. Then neutrosophic soft set $x^e_{(\alpha,\beta,\gamma)}$ is called* a neutrosophic soft point *for every $x \in X$, $0 < \alpha$, β, $\gamma \leq 1$, $e \in E$ and is defined as follows:*

$$x^e_{(\alpha,\beta,\gamma)}(e')(y) = \begin{cases} (\alpha,\beta,\gamma), & \text{if } e' = e \text{ and } y = x \\ (0,0,1), & \text{if } e' \neq e \text{ or } y \neq x \end{cases}$$

It is clear that every neutrosophic soft set is the union of its neutrosophic soft points.

Definition 13. *[3] Let $\left(\widetilde{F}, E\right)$ be a neutrosophic soft set over the universe set X. We say that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F}, E\right)$ read as belonging to the neutrosophic soft set $\left(\widetilde{F}, E\right)$ whenever*
$\alpha \leq T_{\widetilde{F}(e)}(x)$, $\beta \leq I_{\widetilde{F}(e)}(x)$ and $\gamma \geq F_{\widetilde{F}(e)}(x)$.

Definition 14. *[3] Let $x^{e}_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ be two neutrosophic soft points. For the neutrosophic soft points $x^{e}_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ over a common universe X, we say that the neutrosophic soft points are distinct points if $x^{e}_{(\alpha,\beta,\gamma)} \cap y^{e'}_{(\alpha',\beta',\gamma')} = 0_{(X,E)}$. It is clear that $x^{e}_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ are distinct neutrosophic soft points if and only if $x \neq y$ or $e \neq e'$.*

Definition 15. *[7] Let $\left(\widetilde{F}, E_1\right)$,$\left(\widetilde{G}, E_2\right)$ be two neutrosophic sets over the universal set X. Then their cartesian product is another neutrosophic set $\left(\widetilde{K}, E_3\right) = \left(\widetilde{F}, E_1\right) \times \left(\widetilde{G}, E_2\right)$, where $E_3 = E_1 \times E_2$ and $\widetilde{K}\left(e_1, e_2\right) = \widetilde{F}\left(e_1\right) \times \widetilde{G}\left(e_2\right)$. The truth, indeterminacy and falsity membership of $\left(\widetilde{K}, E_3\right)$ are given by $\forall e_1 \in E_1$, $\forall e_2 \in E_2$, $\forall x \in X$,*

$$T_{\widetilde{K}(e_1,e_2)}(x) = min\left\{T_{\widetilde{F}(e_1)}(x)\,,\, T_{\widetilde{G}(e_2)}(x)\right\},$$

$$I_{\widetilde{K}(e_1,e_2)}(x) = min\left\{I_{\widetilde{F}(e_1)}(x)\,,\, I_{\widetilde{G}(e_2)}(x)\right\},$$

$$F_{\widetilde{K}(e_1,e_2)}(x) = max\left\{F_{\widetilde{F}(e_1)}(x)\,,\, F_{\widetilde{G}(e_2)}(x)\right\}.$$

This definition can be extended for more than two neutrosophic soft sets.

Definition 16. *[7] A neutrosophic soft relation $\widetilde{R}$ between two neutrosophic soft sets $\left(\widetilde{F}, E_1\right)$ and $\left(\widetilde{G}, E_2\right)$ over the common universe X is the neutrosophic soft subset of $\left(\widetilde{F}, E_1\right) \times \left(\widetilde{G}, E_2\right)$. Clearly, it is another neutrosophic soft set $\left(\widetilde{R}, E_3\right)$ where $E_3 \subset E_1 \times E_2$ and*
$\widetilde{R}\left(e_1, e_2\right) = \widetilde{F}\left(e_1\right) \times \widetilde{G}\left(e_2\right)$ for $\left(e_1, e_2\right) \in E_3$.

Definition 17. *[7] Let $\left(\widetilde{F}, E_1\right)$, $\left(\widetilde{G}, E_2\right)$ be two neutrosophic sets over the universal set X and f be a neutrosophic soft relation defined on $\left(\widetilde{F}, E_1\right) \times \left(\widetilde{G}, E_2\right)$. Then f is called neutrosophic soft function if f associates each element of $\left(\widetilde{F}, E_1\right)$ with the unique element of $\left(\widetilde{G}, E_2\right)$. We write $f : \left(\widetilde{F}, E_1\right) \to \left(\widetilde{G}, E_2\right)$ as a neutrosophic soft function or a mapping. For $x^{e}_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F}, E_1\right)$ and $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G}, E_2\right)$ when $x^{e}_{(\alpha,\beta,\gamma)} \times y^{e'}_{(\alpha',\beta',\gamma')} \in f$, we denote it by $f\left(x^{e}_{(\alpha,\beta,\gamma)}\right) = y^{e'}_{(\alpha',\beta',\gamma')}$. Here $\left(\widetilde{F}, E_1\right)$ and $\left(\widetilde{G}, E_2\right)$ are called domain and codomain respectively and $y^{e'}_{(\alpha',\beta',\gamma')}$ is the image of $x^{e}_{(\alpha,\beta,\gamma)}$ under f.*

Definition 18. *[8] Let (X,τ,E) be a neutrosophic soft topological space and $\left(\widetilde{F},E\right) \in NSS\,(X,E)$ be arbitrary. Then the interior of $\left(\widetilde{F},E\right)$ is denoted by $\left(\widetilde{F},E\right)^{\circ}$ and is defined as:*

$$\left(\widetilde{F},E\right)^{\circ} = \bigcup\left\{\left(\widetilde{G},E\right) : \left(\widetilde{G},E\right) \subset \left(\widetilde{F},E\right),\ \left(\widetilde{G},E\right) \in \tau\right\}$$

i.e., it is the union of all open neutrosophic soft subsets of $\left(\widetilde{F},E\right)$.

Definition 19. *[8] Let (X,τ,E) be a neutrosophic soft topological space and $\left(\widetilde{F},E\right) \in NSS\,(X,E)$ be arbitrary. Then the closure of $\left(\widetilde{F},E\right)$ is denoted by $\overline{\left(\widetilde{F},E\right)}$ and is defined as:*

$$\overline{\left(\widetilde{F},E\right)} = \bigcap\left\{\left(\widetilde{G},E\right) : \left(\widetilde{G},E\right) \subset \left(\widetilde{F},E\right),\ \left(\widetilde{G},E\right)^{c} \in \tau\right\}$$

i.e., it is the intersection of all closed neutrosophic soft super sets of $\left(\widetilde{F},E\right)$.

3. Some Properties

Definition 20. *A subset $\left(\widetilde{F},E\right)$ of a neutrosophic soft topological space (X,τ,E) is said to be neutrosophic pre open soft, if $\left(\widetilde{F},E\right) \subset \left[\overline{\left(\widetilde{F},E\right)}\right]^{\circ}$. The family of all neutrosophic pre open soft sets of (X,τ,E) is denoted by $NSPO\,(X)$. The family of all neutrosophic pre open soft sets of (X,τ,E) containing a neutrosophic soft point $x^{e}_{(\alpha,\beta,\gamma)}$ is denoted by $NSPO\left(X, x^{e}_{(\alpha,\beta,\gamma)}\right)$.*

Definition 21. *A neutrosophic soft point $x^{e}_{(\alpha,\beta,\gamma)}$ of a neutrosophic soft topological space*
(X,τ,E) is said to be neutrosophic soft pre interior point of a neutrosophic soft set $\left(\widetilde{F},E\right)$, if there exists $\left(\widetilde{G},E\right) \in NSPO\left(X, x^{e}_{(\alpha,\beta,\gamma)}\right)$ such that $x^{e}_{(\alpha,\beta,\gamma)} \nsubseteq \left(\widetilde{G},E\right)^{c}$ and $\left(\widetilde{G},E\right) \subset \left(\widetilde{F},E\right)$.

Definition 22. *The set of all neutrosophic soft pre interior points of $\left(\widetilde{F},E\right)$ is said to be neutrosophic soft pre interior of $\left(\widetilde{F},E\right)$ and denoted by $NSPint\left(\widetilde{F},E\right)$.*

Definition 23. *The complement of a neutrosophic pre open soft set is called neutrosophic pre closed soft. The intersection of all neutrosophic pre closed soft sets containing a neutrosophic soft set $\left(\widetilde{F},E\right)$ is called neutrosophic pre closure of $\left(\widetilde{F},E\right)$ and is denoted by $NSPcl\left(\widetilde{F},E\right)$.*

Definition 24. *A neutrosophic soft point $x^{e}_{(\alpha,\beta,\gamma)}$ of a neutrosophic soft topological space*

(X,τ,E) is said to be neutrosophic soft pre cluster point of a neutrosophic soft set $\left(\widetilde{F},E\right)$, if $\left(\widetilde{G},E\right) \nsubseteq \left(\widetilde{F},E\right)^c$ for any $\left(\widetilde{G},E\right) \in NSPO\left(X, x^e_{(\alpha,\beta,\gamma)}\right)$.

Definition 25. *A neutrosophic soft topological space (X,τ,E) is said to be a neutrosophic soft pre T_0-space if for every pair of distinct neutrosophic soft points $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ there exist neutrosophic pre-open soft sets $\left(\widetilde{F},E\right)$, $\left(\widetilde{G},E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F},E\right)$,*
$y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{F},E\right)^c$ or $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{G},E\right)^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G},E\right)$.

Definition 26. *Let (X,τ,E) be a neutrosophic soft topological space and $Y \subseteq X$. Let $\left(\widetilde{H},E\right)$ be a neutrosophic soft set over Y such that*

$$T_{\widetilde{H}(e)}(x) = \begin{cases} 1, & \text{if } x \in Y \\ 0, & \text{if } x \notin Y \end{cases}$$

$$I_{\widetilde{H}(e)}(x) = \begin{cases} 1, & \text{if } x \in Y \\ 0, & \text{if } x \notin Y \end{cases}$$

$$F_{\widetilde{H}(e)}(x) = \begin{cases} 1, & \text{if } x \in Y \\ 0, & \text{if } x \notin Y \end{cases}$$

for any $e \in E$.

Let $\tau_Y = \left\{\left(\widetilde{H},E\right) \cap \left(\widetilde{F},E\right) : \left(\widetilde{F},E\right) \in \tau\right\}$, then (Y,τ_Y,E) is called neutrosophic soft subspace of (X,τ,E). If $\left(\widetilde{H},E\right) \in \tau$ (resp. $\left(\widetilde{H},E\right)^c \in \tau$), then (Y,τ_Y,E) is called neutrosophic open (resp.closed) soft subspace of (X,τ,E).

Theorem 27. *A neutrosophic soft subspace (Y,τ_Y,E) of a neutrosophic soft pre T_0-space*
(X,τ,E) is neutrosophic soft pre T_0.

Proof. Let $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ be two distinct neutrosophic soft points in (Y,τ_Y,E). Then, these neutrosophic soft points are also in (X,τ,E). Hence, there exist neutrosophic pre-open soft sets $\left(\widetilde{F},E\right)$, $\left(\widetilde{G},E\right)$ in τ such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F},E\right)$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{F},E\right)^c$ or $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{G},E\right)^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G},E\right)$. Let $\left(\widetilde{H},E\right)$ be a neutrosophic soft set over Y as described in Definition 26. Thus, $\left(\widetilde{H},E\right) \cap \left(\widetilde{F},E\right)$ and $\left(\widetilde{H},E\right) \cap \left(\widetilde{G},E\right)$ are neutrosophic pre-open soft sets in (Y,τ_Y,E) such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{H},E\right) \cap \left(\widetilde{F},E\right)$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left[\left(\widetilde{H},E\right) \cap \left(\widetilde{F},E\right)\right]^c$ or

$x^e_{(\alpha,\beta,\gamma)} \in \left[\left(\widetilde{H},E\right) \cap \left(\widetilde{G},E\right)\right]^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{H},E\right) \cap \left(\widetilde{G},E\right)$. Therefore, (Y,τ_Y,E) is neutrosophic soft pre T_0. □

Definition 28. *A neutrosophic soft topological space (X,τ,E) is said to be a neutrosophic soft pre T_1-space if for every pair of distinct neutrosophic soft points $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ there exists neutrosophic pre-open soft sets $\left(\widetilde{F},E\right)$ and $\left(\widetilde{G},E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F},E\right)$,*
$y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{F},E\right)^c$ and $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{G},E\right)^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G},E\right)$.

Theorem 29. *A neutrosophic soft subspace (Y,τ_Y,E) of a neutrosophic soft pre T_1-space*
(X,τ,E) is neutrosophic soft pre T_1.

Proof. It is similar to the proof of Theorem 27. □

Theorem 30. *Every neutrosophic soft point with the truth-membership value 1, the*
indeterminacy-membership value 1 and falsity-membership value 0, is neutrosophic pre-closed soft in a neutrosophic soft topological space (X,τ,E) if and only if (X,τ,E) is neutrosophic soft pre T_1.

Proof. $(\Rightarrow)$ Suppose that $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ be two distinct neutrosophic soft points of (X,τ,E). Then, $x^e_{(\alpha,\beta,\gamma)} \subset x^e_{(1,1,0)}$ and $y^{e'}_{(\alpha',\beta',\gamma')} \subset y^{e'}_{(1,1,0)}$. By hypothesis, $y^{e'}_{(1,1,0)}$ and $y^{e'}_{(1,1,0)}$ are neutrosophic pre-closed soft sets. Then, $\left[x^e_{(1,1,0)}\right]^c$ and $\left[y^{e'}_{(1,1,0)}\right]^c$ are neutrosophic pre-open soft sets such that $x^e_{(\alpha,\beta,\gamma)} \in \left[y^{e'}_{(1,1,0)}\right]^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left[\left[y^{e'}_{(1,1,0)}\right]^c\right]^c$ and $x^e_{(\alpha,\beta,\gamma)} \in \left[\left[x^e_{(1,1,0)}\right]^c\right]^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left[x^e_{(1,1,0)}\right]^c$. Therefore, (X,τ,E) is neutrosophic soft pre T_1.

$(\Leftarrow)$ Suppose that (X,τ,E) is neutrosophic soft pre T_1. Let $x^e_{(1,1,0)}$ be a neutrosophic soft point with the truth-membership value 1, the indeterminacy-membership value 1 and falsity- membership value 0. Take any neutrosophic soft point $y^{e'}_{(\alpha',\beta',\gamma')} \in \left[x^e_{(1,1,0)}\right]^c$. It is easily seen that $x^e_{(1,1,0)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ are distinct. From our assumption, there exist neutrosophic pre-open soft sets $\left(\widetilde{F},E\right)$ and $\left(\widetilde{G},E\right)$ such that
$x^e_{(1,1,0)} \in \left(\widetilde{F},E\right)$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{F},E\right)^c$ and
$x^e_{(1,1,0)} \in \left(\widetilde{G},E\right)^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G},E\right)$. Then, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G},E\right) \subset \left[x^e_{(1,1,0)}\right]^c$. This means that $\left[x^e_{(1,1,0)}\right]^c$ is neutrosophic pre-open soft. Therefore, $x^e_{(1,1,0)}$ is neutrosophic pre-closed soft. □

Definition 31. *A neutrosophic soft topological space* (X, τ, E) *is said to be a neutrosophic soft pre* T_2*-space if for every pair of distinct neutrosophic soft points* $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ *there exists neutrosophic pre-open soft sets* $\left(\widetilde{F}, E\right)$ *and* $\left(\widetilde{G}, E\right)$ *such that* $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F}, E\right)$,
$y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{F}, E\right)^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G}, E\right)$, $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{G}, E\right)^c$ *and* $\left(\widetilde{F}, E\right) \subset \left(\widetilde{G}, E\right)^c$.

For a neutrosophic soft topological space (X, τ, E) we have the following diagram:

$$\begin{array}{c} \text{neutrosophic soft pre } T_2\text{−space} \\ \downarrow \\ \text{neutrosophic soft pre } T_1\text{−space} \\ \downarrow \\ \text{neutrosophic soft pre } T_0\text{−space} \end{array}$$

Converse statements may not be true as shown in the examples below;

Example 32. *Let* $X = \{x, y\}$ *be a universe,* $E = \{a.b\}$ *be a parameteric set and* $\left(\widetilde{F}_a, E\right)$ *be a neutrosophic soft set defined as* $\widetilde{F}_a(a) = \{\langle x, a, a, 1-a\rangle, \langle y, a, a, 1-a\rangle\}$ *and*
$\widetilde{F}_a(b) = \{\langle x, 0, 0, 1\rangle, \langle y, a, a, 1-a\rangle\}$ *for any* $\alpha \in (0, 1]$. *Then, the family*
$\tau = \{0_{(X,E)},\ 1_{(X,E)}\} \cup \left\{\left(\widetilde{F}_a, E\right) : a \in (0, 1]\right\}$
is a neutrosophic soft topology over X. *So,* (X, τ, E) *is a neutrosophic soft topological space.* (X, τ, E) *is a neutrosophic soft pre* T_0 *-space but not a neutrosophic soft pre* T_1 *–space. Because,* $x^b_{(0.9,\ 0.6,\ 0.2)}$ *and* $y^a_{(0.8,\ 0.7,\ 0.4)}$ *are distinct neutrosophic soft points in* (X, τ, E) *and there doesn't exist any neutrosophic pre-open soft set that contains* $x^b_{(0.9,\ 0.6,\ 0.2)}$ *but doesn't contain* $y^a_{(0.8,\ 0.7,\ 0.4)}$.

Example 33. *Let* $X = \{x, y\}$ *be a universe,* $E = \{a.b\}$ *be a parameteric set and* $\left(\widetilde{F}, E\right)$ *be a neutrosophic soft set defined as* $\widetilde{F}(a) = \{\langle x, 0, 0, 1\rangle,\ \langle y, 0, 0, 1\rangle\}$ *and* $\widetilde{F}(b) = \{\langle x, 0, 0, 1\rangle,\ \langle y, 0, 0, 0.9\rangle\}$. *Then, the family* $\tau = \left\{0_{(X,E)},\ 1_{(X,E)}, \left(\widetilde{F}, E\right)\right\}$ *is a neutrosophic soft topology over* X. *So,* (X, τ, E) *is a neutrosophic soft topological space.* (X, τ, E) *is a neutrosophic soft pre* T_1 *–space. But, it is not a neutrosophic soft pre* T_2 *–space for the existence of distinct neutrosophic soft points* $x^a_{(0.5,\ 0.5,\ 0.1)}$ *and* $y^b_{(0.4,\ 0.4,\ 0.6)}$.

Theorem 34. *Let* (X, τ, E) *be a neutrosophic soft topological space.* (X, τ, E) *is neutrosophic soft pre* T_2*-space if and only if for any pair of distinct neutrosophic* *soft points* $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$, *there exists a neutrosophic pre-open soft set* $\left(\widetilde{F},E\right)$ *such that*

$$x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F},E\right),\ y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{F},E\right)^c \text{ and } y^{e'}_{(\alpha',\beta',\gamma')} \in \left[NSPcl\left(\widetilde{F},E\right)\right]^c.$$

Proof. ($\Rightarrow$) Let $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ be two distinct neutrosophic soft points in (X,τ,E) . Since (X,τ,E) is a neutrosophic soft pre T_2-space, there exist two neutrosophic pre-open soft sets $\left(\widetilde{F},E\right)$ and $\left(\widetilde{G},E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F},E\right)$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G},E\right)$ and $\left(\widetilde{F},E\right) \subset \left(\widetilde{G},E\right)^c$. So, it is implied that $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{F},E\right)^c$. Since $\left(\widetilde{G},E\right)^c$ is a neutrosophic pre-closed soft set, $NSPcl\left(\widetilde{F},E\right) \subset \left(\widetilde{G},E\right)^c$. This means that, $\left(\widetilde{G},E\right) \subset \left[NSPcl\left(\widetilde{F},E\right)\right]^c$. So,
$y^{e'}_{(\alpha',\beta',\gamma')} \in \left[NSPcl\left(\widetilde{F},E\right)\right]^c$.

($\Leftarrow$)Take any pair of distinct neutrosophic soft points $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ in (X,τ,E). From our assumption, there exists a neutrosophic pre-open soft set $\left(\widetilde{F},E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F},E\right)$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{F},E\right)^c$ and $y^{e'}_{(\alpha',\beta',\gamma')} \in \left[NSPcl\left(\widetilde{F},E\right)\right]^c$. Since $\left[NSPcl\left(\widetilde{F},E\right)\right]^c$ is a neutrosophic pre-open soft set and $\left(\widetilde{F},E\right) \subset \left[\left[NSPcl\left(\widetilde{F},E\right)\right]^c\right]^c$, (X,τ,E) is neutrosophic soft pre T_2-space. □

Theorem 35. *A neutrosophic soft subspace* (Y,τ_Y,E) *of neutrosophic soft pre* T_2-*space*
(X,τ,E) *is neutrosophic soft pre* T_2.

Proof. Let (X,τ,E) be a neutrosophic soft pre T_2-space, $Y \subseteq X$ and (Y,τ_Y,E) be a neutrosophic soft subspace. Take any distinct neutrosophic soft points $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ in (Y,τ_Y,E).

So, these neutrosophic soft points are also contained in (X,τ,E). Hence, there exist neutrosophic pre-open soft sets $\left(\widetilde{F},E\right)$ and $\left(\widetilde{G},E\right)$ in τ such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F},E\right)$,
$y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G},E\right)$ and $\left(\widetilde{F},E\right) \subset \left(\widetilde{G},E\right)^c$. Let $\left(\widetilde{H},E\right)$ be a neutrosophic soft set over Y as described in Definition 26. Then, $\left(\widetilde{H},E\right) \cap \left(\widetilde{F},E\right)$ and $\left(\widetilde{H},E\right) \cap \left(\widetilde{G},E\right)$ are neutrosophic pre-open soft sets in (Y,τ_Y,E) such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{H},E\right) \cap \left(\widetilde{F},E\right)$, $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{H},E\right) \cap \left(\widetilde{G},E\right)$ and $\left(\widetilde{H},E\right) \cap \left(\widetilde{F},E\right) \subset \left[\left(\widetilde{H},E\right) \cap \left(\widetilde{G},E\right)\right]^c$. This means that (Y,τ_Y,E) is neutrosophic soft pre T_2. □

Definition 36. *Let (X,τ,E) be a neutrosophic soft topological space, $\left(\widetilde{G},E\right)$ be a neutrosophic pre-closed soft set and $x^e_{(\alpha,\beta,\gamma)}$ be a neutrosophic soft point such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F},E\right)^c$. If there exist neutrosophic pre-open soft sets $\left(\widetilde{G},E\right)$ and $\left(\widetilde{K},E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{G},E\right)$, $\left(\widetilde{F},E\right) \subseteq \left(\widetilde{K},E\right)$ and $\left(\widetilde{K},E\right) \subset \left(\widetilde{G},E\right)^c$, then (X,τ,E) is said to be a neutrosophic soft pre regular space.*

Definition 37. *A neutrosophic soft topological space (X,τ,E) is said to be a strong neutrosophic soft pre T_1-space if every neutrosophic soft point is a neutrosophic pre-closed soft set in (X,τ,E).*

Definition 38. *A neutrosophic soft pre regular space (X,τ,E) is said to be a neutrosophic soft pre T_3-space if it is also a strong neutrosophic soft pre T_1-space.*

Theorem 39. *Every neutrosophic soft pre T_3-space is a neutrosophic soft pre T_2-space.*

Proof. Let $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ be two distinct neutrosophic soft points of a neutrosophic soft pre T_3-space (X,τ,E). Then, $y^{e'}_{(\alpha',\beta',\gamma')}$ is neutrosophic pre-closed soft set and
$x^e_{(\alpha,\beta,\gamma)} \in \left[y^{e'}_{(\alpha',\beta',\gamma')}\right]^c$. From the neutrosophic soft pre-regularity, there exist disjoint neutrosophic pre-open soft sets $\left(\widetilde{G},E\right)$ and $\left(\widetilde{K},E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} \in$ $\left(\widetilde{G},E\right)$ and
$y^{e'}_{(\alpha',\beta',\gamma')} \subset \left(\widetilde{K},E\right)$. Thus, $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{G},E\right)$ and $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{K},E\right)$. Therefore, (X,τ,E) is neutrosophic soft pre T_2-space □

Theorem 40. *A neutrosophic soft subspace (Y,τ_Y,E) of a neutrosophic soft pre T_3-space (X,τ,E) is neutrosophic soft pre T_3.*

Proof. Let (X,τ,E) be a neutrosophic soft pre T_3-space, $Y \subseteq X$ and (Y,τ_Y,E) be a neutrosophic soft subspace. Let $x^e_{(\alpha,\beta,\gamma)}$ be any neutrosophic soft point in (Y,τ_Y,E). It is obvious that $x^e_{(\alpha,\beta,\gamma)}$ is also a neutrosophic soft point in (X,τ,E). Since (X,τ,E) is a strong neutrosophic soft pre T_1-space, $x^e_{(\alpha,\beta,\gamma)}$ is a neutrosophic pre-closed soft set in (X,τ,E). Consider the neutrosphic soft set $\left(\widetilde{H},E\right)$ over Y defined in Definition 26. It is easily seen that $\left(\widetilde{H},E\right) \cap x^e_{(\alpha,\beta,\gamma)}$ is neutrosophic pre-closed soft in (Y,τ_Y,E). This means that (Y,τ_Y,E) is a strong neutrosophic soft pre-T_1-space. Now, we must show that (Y,τ_Y,E) is also a neutrosophic soft pre-regular space. Let $\left(\widetilde{G},E\right)$ be a neutrosophic pre-closed soft set in (Y,τ_Y,E) and $x^e_{(\alpha,\beta,\gamma)}$ be a neutrosophic soft point in (Y,τ_Y,E) such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{G},E\right)^c$. Then, $\left(\widetilde{G},E\right) = \left(\widetilde{H},E\right) \cap \left(\widetilde{F},E\right)$ for some neutrosophic pre-closed soft set $\left(\widetilde{F},E\right)$ in (X,τ,E). Hence, $x^e_{(\alpha,\beta,\gamma)} \in \left[\left(\widetilde{H},E\right)\cap\left(\widetilde{F},E\right)\right]^c$. So, $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{H},E\right)^c \cup \left(\widetilde{F},E\right)^c$. Because of the description of the neutrosophic soft set $\left(\widetilde{H},E\right)$ in Definition 5.2, it is clear that $x^e_{(\alpha,\beta,\gamma)} \notin \left(\widetilde{H},E\right)^c$. This means that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F},E\right)^c$. From the neutrosophic soft pre-regularity of (X,τ,E), there exists neutrosophic pre-open soft sets $\left(\widetilde{K},E\right)$ and $\left(\widetilde{L},E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{K},E\right)$, $\left(\widetilde{F},E\right) \subseteq \left(\widetilde{L},E\right)$ and $\left(\widetilde{K},E\right) \subseteq \left(\widetilde{L},E\right)^c$. This implies that $\left(\widetilde{H},E\right)\cap\left(\widetilde{K},E\right)$ and $\left(\widetilde{H},E\right)\cap\left(\widetilde{L},E\right)$ are neutrosophic pre-open soft sets in (Y,τ_Y,E) such that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{H},E\right)\cap\left(\widetilde{K},E\right)$, $\left(\widetilde{F},E\right) \subseteq \left(\widetilde{H},E\right)\cap\left(\widetilde{L},E\right)$ and $\left(\widetilde{H},E\right)\cap\left(\widetilde{K},E\right) \subset \left(\widetilde{H},E\right)\cap\left(\widetilde{L},E\right)^c$. Therefore, (Y,τ_Y,E) is neutrosophic soft pre T_3. □

Definition 41. *Let (X,τ,E) be a neutrosophic soft topological space, $\left(\widetilde{F}_1,E\right)$ and $\left(\widetilde{F}_2,E\right)$ be neutrosophic pre-closed soft sets such that $\left(\widetilde{F}_1,E\right) \subset \left(\widetilde{F}_2,E\right)^c$. If there exist neutrosophic pre-open soft sets $\left(\widetilde{G},E\right)$ and $\left(\widetilde{K},E\right)$ such that $\left(\widetilde{F}_1,E\right) \subseteq \left(\widetilde{G},E\right)$, $\left(\widetilde{F}_2,E\right) \subseteq \left(\widetilde{K},E\right)$ and $\left(\widetilde{G},E\right) \subset \left(\widetilde{K},E\right)^c$, then (X,τ,E) is said to be a neutrosophic soft pre normal space.*

Definition 42. *A neutrosophic soft pre normal space (X,τ,E) is said to be a neutrosophic soft pre T_4-space, if it is also a strong neutrosophic soft pre T_1-space.*

Theorem 43. *Let (X,τ,E) be a fuzzy soft topological space. Then, the following statements are equivalent:*

(1) *(X,τ,E) is a neutrosophic soft pre normal space.*

(2) *For every neutrosophic pre closed soft set $\left(\widetilde{K},E\right)$ and neutrosophic preopen soft set $\left(\widetilde{L},E\right)$ such that $\left(\widetilde{K},E\right) \subseteq \left(\widetilde{L},E\right)$, there exists a neutrosophic pre open soft set $\left(\widetilde{F},E\right)$ such that $\left(\widetilde{K},E\right) \subseteq \left(\widetilde{F},E\right)$, $NSPcl\left(\widetilde{F},E\right) \subseteq \left(\widetilde{L},E\right)$.*

Proof. (1) ⇒ (2) Let $\left(\widetilde{K},E\right)$ be a pre closed soft set and $\left(\widetilde{L},E\right)$ be a fuzzy pre open soft set such that $\left(\widetilde{K},E\right) \subseteq \left(\widetilde{L},E\right)$. Then, $\left(\widetilde{K},E\right)$, $\left(\widetilde{L},E\right)^c$ are neutrosophic pre closed soft sets such that $\left(\widetilde{L},E\right)^c \subseteq \left(\widetilde{K},E\right)^c$. It follows from (1), there exist neutrosophic pre open soft sets $\left(\widetilde{F},E\right)$ and $\left(\widetilde{G},E\right)$ such that $\left(\widetilde{K},E\right) \subseteq \left(\widetilde{F},E\right)$, $\left(\widetilde{L},E\right)^c \subseteq \left(\widetilde{G},E\right)$ and $\left(\widetilde{F},E\right) \subseteq \left(\widetilde{G},E\right)^c$. Since $\left(\widetilde{G},E\right)^c$ is neutrosophic pre

closed soft, $NSPcl\left(\widetilde{F},E\right) \subseteq \left(\widetilde{G},E\right)^c$. So,
$NSPcl\left(\widetilde{F},E\right) \subseteq \left(\widetilde{L},E\right)$. Therefore, the neutrosophic pre open soft set $\left(\widetilde{F},E\right)$ satisfies the conditions.

(2) $\Rightarrow$ (1) Let $\left(\widetilde{F}_1,E\right)$ and $\left(\widetilde{F}_2,E\right)$ be neutrosophic pre-closed soft sets such that
$\left(\widetilde{F}_1,E\right) \subset \left(\widetilde{F}_2,E\right)^c$, where $\left(\widetilde{F}_2,E\right)^c$ is neutrosophic pre open soft. From our hypothesis, there exists a neutrosophic pre open soft set $\left(\widetilde{F},E\right)$ such that $\left(\widetilde{F}_1,E\right) \subset$ $\left(\widetilde{F},E\right)$ and
$NSPcl\left(\widetilde{F},E\right) \subseteq \left(\widetilde{F}_2,E\right)^c$. So, $\left(\widetilde{F}_2,E\right) \subseteq \left[NSPcl\left(\widetilde{F},E\right)\right]^c$, $\left(\widetilde{F}_1,E\right) \subset \left(\widetilde{F},E\right)$ and
$\left[NSPcl\left(\widetilde{F},E\right)\right]^c \subseteq \left(\widetilde{F},E\right)^c$, where $\left[NSPcl\left(\widetilde{F},E\right)\right]^c$ and $\left(\widetilde{F},E\right)$ are are neutrosophic pre open soft sets. Thus, (X,τ,E) is neutrosophic soft pre normal space. □

Theorem 44. *A neutrosophic pre closed neutrosophic soft subspace (Y,τ_Y,E) of a neutrosophic soft pre normal space (X,τ,E) is neutrosophic soft pre normal.*

Proof. Let $\left(\widetilde{F},E\right)$ and $\left(\widetilde{G},E\right)$ be neutrosophic pre closed soft sets in (Y,τ_Y,E) such that $\left(\widetilde{F},E\right) \subset \left(\widetilde{G},E\right)^c$. Consider the neutrosphic soft set $\left(\widetilde{H},E\right)$ over Y defined in Definition 26. Then, $\left(\widetilde{H},E\right)$ is neutrosophic pre closed soft in (X,τ,E), $\left(\widetilde{F},E\right) = \left(\widetilde{H},E\right) \cap \left(\widetilde{K},E\right)$ and $\left(\widetilde{G},E\right) = \left(\widetilde{H},E\right) \cap \left(\widetilde{L},E\right)$ for some neutrosophic pre closed soft sets $\left(\widetilde{K},E\right)$ and $\left(\widetilde{L},E\right)$ in (X,τ,E). Hence, $\left(\widetilde{H},E\right) \cap \left(\widetilde{K},E\right)$ and $\left(\widetilde{H},E\right) \cap \left(\widetilde{L},E\right)$ are neutrosophic pre closed soft sets in (X,τ,E) and $\left(\widetilde{H},E\right) \cap$ $\left(\widetilde{K},E\right) \subseteq \left[\left(\widetilde{H},E\right) \cap \left(\widetilde{L},E\right)\right]^c$. Since (X,τ,E) is neutrosophic soft pre normal, there exist neutrosophic pre open sets $\left(\widetilde{M},E\right)$ and $\left(\widetilde{N},E\right)$ such that
$\left(\widetilde{H},E\right) \cap \left(\widetilde{K},E\right) \subseteq \left(\widetilde{M},E\right)$, $\left(\widetilde{H},E\right) \cap \left(\widetilde{L},E\right) \subseteq \left(\widetilde{N},E\right)$ and $\left(\widetilde{M},E\right) \subseteq \left(\widetilde{N},E\right)^c$. So,
$\left(\widetilde{H},E\right) \cap \left(\widetilde{M},E\right)$ and $\left(\widetilde{H},E\right) \cap \left(\widetilde{N},E\right)$ are neutrosophic pre open sets in (Y,τ_Y,E) such that $\left(\widetilde{F},E\right) \subset \left(\widetilde{H},E\right) \cap \left(\widetilde{M},E\right)$, $\left(\widetilde{G},E\right) \subset \left(\widetilde{H},E\right) \cap \left(\widetilde{N},E\right)$ and
$\left(\widetilde{H},E\right) \cap \left(\widetilde{M},E\right) \subset \left[\left(\widetilde{H},E\right) \cap \left(\widetilde{N},E\right)\right]^c$. Therefore, (Y,τ_Y,E) is neutrosophic soft pre normal. □

Definition 45. *Let (X,τ_1,E), (Y,τ_2,K) be neutrosophic soft topological spaces and*

$f:(X,\tau_1,E)\to(Y,\tau_2,K)$ be a neutrosophic soft function. The function f is said to be neutrosophic pre irresolute soft, if $f^{-1}\left(\left(\widetilde{G},E\right)\right)\in\tau_1$ for any $\left(\widetilde{G},E\right)\in\tau_2$.

Definition 46. *Let (X,τ_1,E), (Y,τ_2,K) be neutrosophic soft topological spaces and*
$f:(X,\tau_1,E)\to(Y,\tau_2,K)$ be a neutrosophic soft function. The function f is said to be neutrosophic pre irresolute open soft, if $f\left(\left(\widetilde{F},E\right)\right)\in\tau_2$ for any $\left(\widetilde{F},E\right)\in\tau_1$.

Theorem 47. *Let (X,τ_1,E) and (Y,τ_2,K) be neutrosophic soft topological spaces and*
$f:(X,\tau_1,E)\to(Y,\tau_2,K)$ be a neutrosophic soft function which is bijective, neutrosophic pre irresolute soft and neutrosophic pre irresolute open soft. If (X,τ_1,E) is a neutrosophic soft pre normal space, then (Y,τ_2,K) is also a neutrosophic soft pre normal space.

Proof. Let $\left(\widetilde{F},E\right)$ and $\left(\widetilde{G},E\right)$ be neutrosophic pre closed soft sets in (Y,τ_2,K) such that
$\left(\widetilde{F},E\right)\subset\left(\widetilde{G},E\right)^c$. Since f is neutrosophic pre irresolute soft, then $f^{-1}\left(\left(\widetilde{F},E\right)\right)$ and
$f^{-1}\left(\left(\widetilde{G},E\right)\right)$ are neutrosophic pre closed soft sets in (X,τ_1,E) such that
$f^{-1}\left(\left(\widetilde{F},E\right)\right)\subset\left[f^{-1}\left(\left(\widetilde{G},E\right)\right)\right]^c$. Since (X,τ_1,E) is a neutrosophic soft pre normal space, there exist neutrosophic pre open soft sets $\left(\widetilde{K},E\right)$ and $\left(\widetilde{L},E\right)$ such that
$f^{-1}\left(\left(\widetilde{F},E\right)\right)\subset\left(\widetilde{K},E\right)$, $f^{-1}\left(\left(\widetilde{G},E\right)\right)\subset\left(\widetilde{L},E\right)$ and $\left(\widetilde{K},E\right)\subset\left(\widetilde{L},E\right)^c$. It follows that
$\left(\widetilde{F},E\right)=f\left[f^{-1}\left(\left(\widetilde{F},E\right)\right)\right]\subset f\left(\left(\widetilde{K},E\right)\right)$, $\left(\widetilde{G},E\right)=f\left[f^{-1}\left(\left(\widetilde{G},E\right)\right)\right]\subset f\left(\left(\widetilde{L},E\right)\right)$ and $f\left(\left(\widetilde{K},E\right)\right)\subset f\left(\left(\widetilde{L},E\right)^c\right)=\left[f\left(\left(\widetilde{L},E\right)\right)\right]^c$. From the fact that f is neutrosophic pre irresolute open soft, $f\left(\left(\widetilde{K},E\right)\right)$ and $f\left(\left(\widetilde{L},E\right)\right)$ are neutrosophic pre open soft sets such that $\left(\widetilde{F},E\right)\subset f\left(\left(\widetilde{K},E\right)\right)$, $\left(\widetilde{G},E\right)\subset f\left(\left(\widetilde{L},E\right)\right)$ and $f\left(\left(\widetilde{K},E\right)\right)\subset\left[f\left(\left(\widetilde{L},E\right)\right)\right]^c$. This means that (Y,τ_2,K) is a neutrosophic soft pre normal space. □

4. Conclusion

The notions of neutrosophic pre open soft sets, neutrosophic pre closed soft sets, neutrosophic pre soft interior, neutrosophic pre soft closure, neutrosophic soft pre-interior point, neutrosophic soft pre-cluster point and neutrosophic soft pre separation axioms are introduced, and some properties of the notions are studied. Also, several interesting properties have been established. Additionally, a new approach is made to the concept of neutrosophic soft topological subspace. Since topological structures on neutrosophic soft sets have been introduced by many scientists, we generalize the pre topological properties to the neutrosophic soft sets which may be useful in some other disciplines. For the existence of compact connections between soft sets and information systems [17, 21] the results obtained from the studies on neutrosophic soft topological space can be used to develop these connections. We hope that many researchers will benefit from the findings in this document to further their studies on neutrosophic soft topology to carry out a general framework for their applications in practical life.

References

[1] Al-shami, T.M., Kocinac, Lj.D.R., The equivalence between the enriched and extended soft topologies, *Appl. Comput. Math.*, 18 (2) (2019), 149-162.
[2] Aras, C.G., Sonmez, A., Çakalli, H., An approach to soft functions, *J. Math. Anal.,* 8 2 (2017), 129-138.
[3] Aras, C.G., Ozturk, T.Y., Bayramov, S., Separation axioms on neutrosophic soft topological spaces, *Turk. J. Math.*, 43 (2019), 498-510.
[4] Atanassov, K., Intuitionistic fuzzy sets, *Fuzzy Sets Syst.,* 20 (1986), 87-96.
[5] Bayramov, S., Gunduz, C., On intuitionistic fuzzy soft topological spaces, *TWMS J. Pure Appl. Math.,* 5 (2014), 66-79.
[6] Bayramov, S., Gunduz, C., A new approach to separability and compactness in soft topological spaces, *TWMS J Pure Appl. Math.*, 9 (2018), 82-93.
[7] Bera, T., Mahapatra, N.K., On neutrosophic soft function, *Annals of Fuzzy Mathematics and Informatics,* 12 (1) (July 2016), 101-119.
[8] Bera, T., Mahapatra, N.K., Introduction to neutrosophic soft topological space, *Opsearch*, 54 (2017), 841-867.
[9] Cagman, N., Karatas, S., Enginoglu, S., Soft topology, *Comput Math. Appl.,* 62 (2011), 351-358.
[10] Çakalli, H., Das, P., Fuzzy compactness via summability, *Appl. Math. Lett.,* 22 (11) (2009), 1665-1669.
[11] Coskun, A.E., Aras, C.G., Cakalli, H., Sonmez, A., Soft matrices on soft multisets in an optimal decision process, *AIP Conference Proceedings,* 1759, 1, 020099 (2016); doi: 10.1063/1.4959713.
[12] Deli, I., Broumi, S., Neutrosophic soft relations and some Properties, *Ann. Fuzzy Math. Inform,* 9 (2015), 169-182.
[13] Gunduz, C., Bayramov, S., On the Tietze extension theorem in soft topological spaces, P*roceedings of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan*, 43 (2017), 105-115.
[14] Hussain, S., On some properties of intuitionistic fuzzy soft boundary, *Commun. Fac. Sci. Univ. Ank. Series A1*, 69 (2) (2020), 39-50.
[15] Maji, P.K., Neutrosophic soft set, *Ann. Fuzzy Math. Inform,* 5 (2013), 157-168.
[16] Molodtsov, D., Soft set theory-first results, *Comput Math. Appl.,* 37 (1999), 19-31.
[17] Pei, D., Miao, D., From soft sets to information systems, in: X. Hu, Q. Liu, A. Skowron, T. Y. Lin, R. R. Yager, B. Zhang (Eds.), *Proceedings of Granular Computing, in: IEEE* 2 (2005), 617-621.
[18] Salma, A.A., Alblowi, S.A., Neutrosophic set and neutrosophic topological spaces, *IOSR J. Math.*, 3 (2012), 31-35.
[19] Shabir, M., Naz, M., On soft topological spaces, *Comput Math. Appl.,* 61 (2011), 1786-1799.

[20] Smarandache, F., Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, *Int. J. Pure Appl. Math.,* 24 (2005), 287-297.

[21] Xiao, Z., Chen, L., Zhong, B., Ye, S., Recognition for soft information based on the theory of soft sets, in: J. Chen (Ed.), *Proceedings of ICSSSM-05*, 2 (2005), 1104-1106.

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BEST PROXIMITY PROBLEMS FOR NEW TYPES OF $\mathcal{Z}$-PROXIMAL CONTRACTIONS WITH AN APPLICATION

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Abstract. In this study, we establish existence and uniqueness theorems of best proximity points for new types of $\mathcal{Z}$-proximal contractions defined on a complete metric space. The presented results improve and generalize some recent results in the literature. Several examples are constructed to demonstrate the generality of our results. As applications of the obtained results, we discuss sufficient conditions to ensure the existence of a unique solution for a variational inequality problem.

1. Introduction

Khojasteh et al. [14] presented the notion of $\mathcal{Z}$-contraction involving a new class of mappings namely simulation functions and proved new fixed point theorems by using different methods than others in literature.

Definition 1.1 ([14]). *A simulation function is a mapping $\zeta : [0,\infty)\times[0,\infty)\to\mathbb{R}$ satisfying the following conditions:*

(ζ_1) $\zeta(0,0)=0$,
(ζ_2) $\zeta(a,b)<b-a$ *for all* $a,b>0$,
(ζ_3) *If $(a_n),(b_n)$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} a_n=\lim_{n\to\infty} b_n>0$, then*

$$\limsup_{n\to\infty}\ \zeta(a_n,b_n)<0. \tag{1.1}$$

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Theorem 1.2 ([14]). *Let (M,d) be a complete metric space and $\mathcal{T}: M \to M$ be a $\mathcal{Z}$-contraction with respect to ζ satisfying the conditions (ζ_1)-(ζ_3) in Definition 1.1, that is,*

$$\zeta(d(\mathcal{T}u,\mathcal{T}v),d(u,v)) \geq 0, \qquad \text{for all } u,v \in M.$$

Then $\mathcal{T}$ has a unique fixed point and, for every initial point $u_0 \in M$, the Picard sequence $\{\mathcal{T}^n u_0\}$ converges to this fixed point.

Afterwards, Argoubi et al. [3] partly modified Definition 1.1, by removing the condition (ζ_1), because of the fact that the condition (ζ_1) was not used in the proof of Theorem 1.2. On the other hand, Roldan-Lopez-de-Hierro et al. [17] extended the family of all simulation functions by replacing the condition (ζ_3) in Definition 1.1 with the following proviso.

(ζ_4) If $(a_n),(b_n)$ are sequences in $(0,\infty)$ such that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n > 0$ and $a_n < b_n$ for all $n \in \mathbb{N}$, then the inequality (1.1) is satisfied.

In this study, we will consider simulation functions satisfying the conditions (ζ_2) and (ζ_4). For the sake of openness, we identify the following families of function.

$$\begin{aligned}
\mathcal{Z}_1 &= \{\zeta :\ \zeta \text{ satisfies conditions } (\zeta_1),(\zeta_2) \text{ and } (\zeta_3)\},\\
\mathcal{Z}_2 &= \{\zeta :\ \zeta \text{ satisfies conditions } (\zeta_2) \text{ and } (\zeta_3)\},\\
\mathcal{Z}_3 &= \{\zeta :\ \zeta \text{ satisfies conditions } (\zeta_1),(\zeta_2) \text{ and } (\zeta_4)\},\\
\mathcal{Z}_4 &= \{\zeta :\ \zeta \text{ satisfies conditions } (\zeta_2) \text{ and } (\zeta_4)\}.
\end{aligned}$$

Remark 1.3. *It is obvious that $\mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \mathcal{Z}_4$ and also $\mathcal{Z}_3 \subset \mathcal{Z}_4$.*

Example 1.4. *Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a function defined by*

$$\zeta(t,s) = \begin{cases} 1 & \text{if } (s,t) = (0,0),\\ 2(s-t) & \text{if } s < t,\\ \lambda s - t & \text{otherwise}, \end{cases}$$

where $\lambda \in (0,1)$. Then it is easy to see that $\zeta \in \mathcal{Z}_4$, but $\zeta \notin \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$.

The main concern of the paper is to establish existence and uniqueness theorems of best proximity points for new types of $\mathcal{Z}$-proximal contractions in complete metric spaces. The obtained results extend and complement some known results from the literature. Several examples are constructed to demonstrate the new concepts and the generality of our results. Also, sufficient conditions to guarantee the existence of a unique solution to the problem of variational inequality are discussed.

2. Preliminaries

A best proximity point generates to a fixed point if the mapping under consideration is a self-mapping. For more details on this research subject, we refer the reader to [1, 2, 4–7, 9–13, 16, 18–22].

Let P and Q two nonempty subsets of a metric space (M,d). We will use the following notations:

$$\begin{aligned} d(P,Q) &:= \inf\{d(p,q) : p \in P, q \in Q\}; \\ P_0 &:= \{p \in P : d(p,q) = d(P,Q) \text{ for some } q \in Q\}; \\ Q_0 &:= \{q \in Q : d(p,q) = d(P,Q) \text{ for some } p \in P\}. \end{aligned}$$

Throughout this study, the set of all best proximity points of a non-self-mapping $\mathcal{T} : P \to Q$ will be denoted by

$$B_{est}(\mathcal{T}) = \{u \in P : d(u, \mathcal{T}u) = d(P,Q)\}.$$

Jleli and Samet [12] introduced the concepts of α-ψ-proximal contractive and α-proximal admissible mappings and established best proximity point theorems for such mappings defined on complete metric spaces. Subsequently, Hussain et al. [9] modified the aforesaid notions and substantiated certain best proximity point theorems.

Definition 2.1 ([12]). *Let $\mathcal{T} : P \to Q$ and $\alpha : P \times P \to [0,\infty)$ be given mappings. Then $\mathcal{T}$ is said to be α-proximal admissible, if*

$$\left.\begin{array}{l} \alpha(u_1,u_2) \geq 1 \\ d(p_1, \mathcal{T}u_1) = d(P,Q) \\ d(p_2, \mathcal{T}u_2) = d(P,Q) \end{array}\right\} \Longrightarrow \alpha(p_1,p_2) \geq 1,$$

for all $u_1, u_2, p_1, p_2 \in P$.

Definition 2.2 ([9]). *Let $\mathcal{T} : P \to Q$ and $\alpha, \eta : P \times P \to [0,\infty)$ be given mappings. Then $\mathcal{T}$ is said to be (α,η)-proximal admissible, if*

$$\left.\begin{array}{l} \alpha(u_1,u_2) \geq \eta(u_1,u_2) \\ d(p_1, \mathcal{T}u_1) = d(P,Q) \\ d(p_2, \mathcal{T}u_2) = d(P,Q) \end{array}\right\} \Longrightarrow \alpha(p_1,p_2) \geq \eta(p_1,p_2),$$

for all $u_1, u_2, p_1, p_2 \in P$.

Note that if we take $\eta(u,v) = 1$ for all $u, v \in P$, then the previous definition reduces to Definition 2.1.

Very recently, Tchier et al. [22] introduced the concept of $\mathcal{Z}$-proximal contractions as follows.

Definition 2.3 ([22]). *Let P and Q be two nonempty subsets of a metric space (M,d). A non-self-mapping $\mathcal{T} : P \to Q$ is said to be a $\mathcal{Z}$-proximal contraction, if there exists a simulation function $\zeta \in \mathcal{Z}_2$ such that*

$$\left.\begin{array}{l} d(p, \mathcal{T}u) = d(P,Q) \\ d(q, \mathcal{T}v) = d(P,Q) \end{array}\right\} \Longrightarrow \zeta(d(p,q), d(u,v)) \geq 0, \tag{2.1}$$

for all $p, q, u, v \in P$.

Let us introduce the following notions which will be used in our main results.

Definition 2.4. *Let $\mathcal{T} : P \to Q$ and $\alpha, \eta : P \times P \to [0, \infty)$ be given mappings. Then $\mathcal{T}$ is said to be triangular (α, η)-proximal admissible, if*

(1) *$\mathcal{T}$ is (α, η)-proximal admissible;*
(2) *$\alpha(u, v) \geq \eta(u, v)$ and $\alpha(v, z) \geq \eta(v, z)$ implies that $\alpha(u, z) \geq \eta(u, z)$, for all $u, v, z \in P$.*

Definition 2.5. *Let P and Q be two nonempty subsets of a metric space (M, d), $\zeta \in \mathcal{Z}_4$ and $\alpha, \eta : P \times P \to [0, \infty)$ be mappings. A non-self mapping $\mathcal{T} : P \to Q$ is said to be (α, η)-$\mathcal{Z}$-proximal contraction, if*

$$\left.\begin{array}{l} \alpha(u, v) \geq \eta(u, v) \\ d(p, \mathcal{T}u) = d(P, Q) \\ d(q, \mathcal{T}v) = d(P, Q) \end{array}\right\} \Longrightarrow \zeta(d(p, q), d(u, v)) \geq 0, \tag{2.2}$$

for all $p, q, u, v \in P$.

We provide the following examples illustrating Definition 2.5 where Definition 2.3 is not applicable.

Example 2.6. *Let $M = \mathbb{R}$ be endowed with the usual metric d, $P = [0, \frac{1}{2}] \cup \{1, 10\}$ and $Q = [0, \frac{1}{6}] \cup \{1, 10\}$. Define a mapping $\mathcal{T} : P \to Q$ by*

$$\mathcal{T}u = \begin{cases} 10, & \text{if } u = 1, \\ 1, & \text{if } u = 10, \\ \frac{u}{6}, & \text{if } u \in \left[0, \frac{1}{2}\right]. \end{cases}$$

It is obvious that $d(P, Q) = 0$ and $P_0 = Q_0 = Q$. Now, define $\alpha, \eta : P \times P \to [0, \infty)$ by

$$\alpha(u, v) = \begin{cases} 4, & \text{if } u, v \in \left[0, \frac{1}{2}\right], \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \eta(u, v) = 2.$$

Then $\mathcal{T}$ is (α, η)-$\mathcal{Z}$-proximal contraction, but not a $\mathcal{Z}$-proximal contraction where $\zeta(t, s) = \frac{1}{2}s - t$ for all $t, s \in [0, \infty)$. Indeed, let us consider

$$\begin{array}{l} \alpha(u, v) \geq \eta(u, v), \\ d(p, \mathcal{T}u) = d(q, \mathcal{T}v) = d(P, Q). \end{array} \tag{2.3}$$

Taking into account (2.3), we get that $u, v \in \left[0, \frac{1}{2}\right]$, and so $p = \mathcal{T}u = \frac{u}{6}$ and $q = \mathcal{T}v = \frac{v}{6}$. Then

$$\begin{aligned} \zeta\left(d(p, q), d(u, v)\right) &= \frac{1}{2}d(u, v) - d(p, q) \\ &= \frac{1}{2}\left|u - v\right| - \frac{1}{6}\left|u - v\right| \geq 0. \end{aligned}$$

It means that $\mathcal{T}$ is (α, η)-$\mathcal{Z}$-proximal contraction. On the other hand, let

$$\begin{array}{l} d(0, \mathcal{T}0) = d(P, Q) = 0, \\ d(10, \mathcal{T}1) = d(P, Q) = 0. \end{array}$$

Then

$$\begin{aligned}\zeta\left(d(0,10),d(0,1)\right)&=\frac{1}{2}d(0,1)-d(0,10)\\&=\frac{1}{2}-10\ngeq 0,\end{aligned}$$

and hence $\mathcal{T}$ is not a $\mathcal{Z}$-proximal contraction.

Example 2.7. *Let $M=\{(0,1),(1,0),(-1,0),(0,-1)\}$ be endowed with the Euclidian metric d. Consider $P=\{(0,1),(1,0)\}$ and $Q=\{(0,-1),(-1,0)\}$. We have $d(P,Q)=\sqrt{2}$. Let $\mathcal{T}:P\to Q$ be given as $\mathcal{T}(u,v)=(-v,-u)$. Choose $\zeta(t,s)=ks-t$ for $s,t\geq 0$, with $k\in(0,1)$. Take $\alpha,\eta:P\times P\to[0,\infty)$ as*

$$\alpha(u,v)=\begin{cases}1, & \text{if } u=v,\\ 0, & \text{otherwise,}\end{cases}\qquad \text{and}\quad \eta(u,v)=\begin{cases}\frac{1}{4}, & \text{if } u=v,\\ 3, & \text{otherwise.}\end{cases}$$

Let $u,v,p,q\in P$ such that

$$\alpha(u,v)\geq\eta(u,v)\quad\text{and}\quad d(p,\mathcal{T}u)=d(q,\mathcal{T}v)=d(P,Q)=\sqrt{2}.$$

We should have $u=v=p=q=(0,1)$ or $u=v=p=q=(1,0)$. Then, $\zeta(d(p,q),d(u,v))=\zeta(0,0)=0$, that is, $\mathcal{T}$ is (α,η)-$\mathcal{Z}$-proximal contraction.

On the other hand, by taking $u=p=(0,1)$ and $q=v=(1,0)$, we have

$$d(p,\mathcal{T}u)=d(q,\mathcal{T}v)=d(P,Q),$$

but $\zeta(d(p,q),d(u,v))=\zeta(\sqrt{2},\sqrt{2})=(k-1)\sqrt{2}<0$, that is, $\mathcal{T}$ is not a $\mathcal{Z}$-proximal contraction.

3. Main Results

The first result of this study is the following.

Theorem 3.1. *Let (P,Q) be a pair of nonempty subsets of a complete metric space (M,d) such that P_0 is nonempty, $\mathcal{T}:P\to Q$ and $\alpha,\eta:P\times P\to[0,\infty)$ be given mappings. Suppose the following conditions are satisfied:*

- *(i) P_0 is closed and $\mathcal{T}(P_0)\subseteq Q_0$;*
- *(ii) $\mathcal{T}$ is triangular (α,η)-proximal admissible;*
- *(iii) there exist $u_0,u_1\in P_0$ such that $d(u_1,\mathcal{T}u_0)=d(P,Q)$ and $\alpha(u_0,u_1)\geq\eta(u_0,u_1)$;*
- *(iv) $\mathcal{T}$ is a continuous (α,η)-$\mathcal{Z}$-proximal contraction.*

Then $\mathcal{T}$ has a best proximity point in P. If $\alpha(u,v)\geq\eta(u,v)$ for all $u,v\in B_{est}(\mathcal{T})$, then $\mathcal{T}$ has a unique best proximity point $u^*\in P$. Moreover, for each $u\in M$, we have $\lim_{n\to\infty}\mathcal{T}^n u=u^*$.

Proof. By virtue of the assertion (iii), there exist $u_0,u_1\in P_0$ such that

$$d(u_1,\mathcal{T}u_0)=d(P,Q)\quad\text{and}\quad\alpha(u_0,u_1)\geq\eta(u_0,u_1).$$

Since $\mathcal{T}(P_0) \subseteq Q_0$, there exists $u_2 \in P_0$ such that

$$d(u_2, \mathcal{T}u_1) = d(P, Q).$$

Thus, we get

$$\begin{aligned}&\alpha(u_0, u_1) \geq \eta(u_0, u_1),\\ &d(u_1, \mathcal{T}u_0) = d(u_2, \mathcal{T}u_1) = d(P, Q).\end{aligned}$$

Since $\mathcal{T}$ is an (α, η)-proximal admissible, we conclude that $\alpha(u_1, u_2) \geq \eta(u_1, u_2)$. Now, we have

$$d(u_2, \mathcal{T}u_1) = d(P, Q) \quad \text{and} \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2).$$

Again, since $\mathcal{T}(P_0) \subseteq Q_0$, there exists $u_3 \in P_0$ such that

$$d(u_3, \mathcal{T}u_2) = d(P, Q),$$

and thus

$$\begin{aligned}&\alpha(u_1, u_2) \geq \eta(u_1, u_2),\\ &d(u_2, \mathcal{T}u_1) = d(u_3, \mathcal{T}u_2) = d(P, Q).\end{aligned}$$

Since $\mathcal{T}$ is (α, η)-proximal admissible, this implies that $\alpha(u_2, u_3) \geq \eta(u_2, u_3)$. Thereby, we have

$$d(u_3, \mathcal{T}u_2) = d(P, Q) \quad \text{and} \quad \alpha(u_2, u_3) \geq \eta(u_2, u_3).$$

By repeating this process, a sequence $\{u_n\}$ in P_0 can be constituted by the following way:

$$d(u_{n+1}, \mathcal{T}u_n) = d(P, Q) \quad \text{and} \quad \alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1}), \tag{3.1}$$

for all $n \in \mathbb{N} \cup \{0\}$. If there exists n_0 such that $u_{n_0} = u_{n_0+1}$, then

$$d(u_{n_0}, \mathcal{T}u_{n_0}) = d(u_{n_0+1}, \mathcal{T}u_{n_0}) = d(P, Q).$$

This means that u_{n_0} is a best proximity point of $\mathcal{T}$ and the proof is finalized. Due to this reason, we suppose that $u_n \neq u_{n+1}$ for all n. Using (3.1), for all $n \in \mathbb{N}$, we get

$$\begin{aligned}&\alpha(u_{n-1}, u_n) \geq \eta(u_{n-1}, u_n),\\ &d(u_n, \mathcal{T}u_{n-1}) = d(u_{n+1}, \mathcal{T}u_n) = d(P, Q).\end{aligned}$$

Since $\mathcal{T}$ is an (α, η)-$\mathcal{Z}$-proximal contraction, for all $n \in \mathbb{N}$, we obtain

$$0 \leq \zeta(d(u_n, u_{n+1}), d(u_{n-1}, u_n)) < d(u_{n-1}, u_n) - d(u_n, u_{n+1}). \tag{3.2}$$

It follows from the above inequality that

$$0 < d(u_n, u_{n+1}) < d(u_{n-1}, u_n), \qquad \text{for all } n \in \mathbb{N}.$$

Therefore the sequence $\{d(u_n, u_{n+1})\}$ is decreasing and so there exists $r \geq 0$ such that $\lim_{n\to\infty} d(u_n, u_{n+1}) = r$. Now, our purpose is to show that $r = 0$. On the contrary, assume that $r > 0$. Set the sequences $\{a_n\}$ and $\{b_n\}$ as $a_n = d(u_n, u_{n+1})$ and $b_n = d(u_{n-1}, u_n)$. Then since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r$ and $a_n < b_n$ for all n, by the axiom (ζ_4), we deduce

$$0 \leq \limsup_{n\to\infty} \zeta(d(u_n, u_{n+1}), d(u_{n-1}, u_n)) < 0,$$

which is a contraction. That's why $r = 0$, that is,

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{3.3}$$

Let us prove now that $\{u_n\}$ is a Cauchy sequence in P_0. Suppose, to the contrary, that $\{u_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two subsequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$ and

$$d(u_{m_k}, u_{n_k}) \geq \varepsilon \quad \text{and} \quad d(u_{m_k}, u_{n_k-1}) < \varepsilon. \tag{3.4}$$

Using the triangular inequality and (3.4), we have

$$\begin{aligned} \varepsilon \leq d(u_{m_k}, u_{n_k}) &\leq d(u_{m_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{n_k}) \\ &< \varepsilon + d(u_{n_k-1}, u_{n_k}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.3), we obtain

$$\lim_{k\to\infty} d(u_{m_k}, u_{n_k}) = \varepsilon. \tag{3.5}$$

Again, using the triangular inequality,

$$|d(u_{m_k+1}, u_{n_k+1}) - d(u_{m_k}, u_{n_k})| \leq d(u_{m_k+1}, u_{m_k}) + d(u_{n_k}, u_{n_k+1}),$$

which yields that

$$\lim_{k\to\infty} d(u_{m_k+1}, u_{n_k+1}) = \varepsilon. \tag{3.6}$$

Since $\mathcal{T}$ is triangular (α, η)-proximal admissible, by using (3.1), we infer that

$$\alpha(u_m, u_n) \geq \eta(u_m, u_n), \text{ for all } n, m \in \mathbb{N} \cup \{0\} \text{ with } m < n. \tag{3.7}$$

By combining (3.1) and (3.7), for all $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} &\alpha(u_{m_k}, u_{n_k}) \geq \eta(u_{m_k}, u_{n_k}), \\ &d(u_{m_k+1}, \mathcal{T}u_{m_k}) = d(u_{n_k+1}, \mathcal{T}u_{n_k}) = d(P, Q). \end{aligned}$$

Since $\mathcal{T}$ is an (α, η)-$\mathcal{Z}$-proximal contraction, the last equation gives us that, for all $k \in \mathbb{N} \cup \{0\}$

$$0 \leq \zeta(d(u_{m_k+1}, u_{n_k+1}), d(u_{m_k}, u_{n_k})) < d(u_{m_k}, u_{n_k}) - d(u_{m_k+1}, u_{n_k+1}). \tag{3.8}$$

Choose the sequences $\{a_k = d(u_{m_k+1}, u_{n_k+1})\}$ and $\{b_k = d(u_{m_k}, u_{n_k})\}$. Then, from (3.5), (3.6) and (3.8), we conclude that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \varepsilon$ and $a_k < b_k$ for all k. Taking lim sup of (3.8) and considering (ζ_4), we get

$$0 \leq \limsup_{n\to\infty} \zeta((d(u_{m_k+1}, u_{n_k+1}), d(u_{m_k}, u_{n_k})) < 0,$$

which is a contradiction. Accordingly, $\{u_n\}$ is a Cauchy sequence in P_0. Since P_0 is a closed subset of the complete metric space (M, d), there exists $u \in P_0$ such that

$$\lim_{n\to\infty} d(u_n, u) = 0.$$

In view of the fact that $\mathcal{T}$ is continuous, we deduce that

$$\lim_{n\to\infty} d(\mathcal{T}u_n, \mathcal{T}u) = 0.$$

Thus, using the last two equations and (3.1), we have

$$d(P,Q) = \lim_{n\to\infty} d(u_{n+1}, \mathcal{T}u_n) = d(u, \mathcal{T}u),$$

which means that $u \in P_0 \subseteq P$ is a best proximity point of $\mathcal{T}$. As the final step, we shall show that the set $B_{est}(\mathcal{T})$ is a singleton. Assume that v is another best proximity point of $\mathcal{T}$. Then, by hypothesis, we have $\alpha(u,v) \geq \eta(u,v)$, and thus

$$\begin{aligned}
&\alpha(u,v) \geq \eta(u,v),\\
&d(u,\mathcal{T}u) = d(v,\mathcal{T}v) = d(P,Q).
\end{aligned}$$

Then, by the argument (iv), we infer that

$$0 \leq \zeta\left(d(u,v), d(u,v)\right) < d(u,v) - d(u,v) = 0,$$

which is a contradiction. Thus, the best proximity point of $\mathcal{T}$ is unique. □

The following example illustrates Theorem 3.1.

Example 3.2. *Let $M = [0,\infty)\times[0,\infty)$ be endowed with the metric $d((u_1,u_2),(v_1,v_2)) = |u_1 - v_1| + |u_2 - v_2|$. Take $P = \{1\} \times [0,\infty)$ and $Q = \{0\} \times [0,\infty)$. We mention that $d(P,Q) = 1$, $P_0 = P$ and $Q_0 = Q$. Consider the mapping $\mathcal{T} : P \to Q$ as*

$$\mathcal{T}(1,u) = \begin{cases} (0, \frac{u^2+1}{4}) & \text{if } 0 \leq u \leq 1\\ (0, u - \frac{1}{2}) & \text{if } u > 1.\end{cases}$$

Note that $\mathcal{T}$ is continuous at $u_0 = 1$ and $\mathcal{T}(P_0) \subseteq Q_0$. Consider $\zeta(a,b) = kb - a$ with $k \in (\frac{1}{2}, 1)$, for all $a, b \geq 0$. Define $\alpha, \eta : P \times P \to [0,\infty)$ as follows

$$\begin{cases}\alpha((1,u),(1,v)) = 1 & \text{if } \quad u,v \in [0,1]\\ \alpha((1,u),(1,v)) = 0 & \text{if not,}\end{cases} \quad \text{and} \quad \begin{cases}\eta((1,u),(1,v)) = \frac{1}{3} & \text{if } \quad u,v \in [0,1]\\ \eta((1,u),(1,v)) = 2 & \text{if not.}\end{cases}$$

Let $(1,u), (1,v), (1,p)$ and $(1,q)$ in P such that

$$\begin{cases}\alpha((1,u),(1,v)) \geq \eta((1,u),(1,v))\\ d((1,p),\mathcal{T}(1,u)) = d(P,Q) = 1,\\ d((1,q),\mathcal{T}(1,v)) = d(P,Q) = 1.\end{cases}$$

Then, necessarily, $(u,v) \in [0,1] \times [0,1]$. Also, $p = \frac{1+u^2}{4}$ and $q = \frac{1+v^2}{4}$. Here, we have that $\alpha((1,p),(1,q)) \geq \eta((1,p),(1,q))$, that is, $\mathcal{T}$ is (α,η)-proximal admissible.

Moreover,

$$\begin{aligned}\zeta(d((1,p),(1,q)),d((1,u),(1,v))) &= \zeta(d((1,\frac{1+u^2}{4}),(1,\frac{1+v^2}{4})),d((1,u),(1,v)))\\ &= \zeta(|\frac{u^2}{4}-\frac{v^2}{4}|,|u-v|)\\ &= k|u-v|-|\frac{u^2}{4}-\frac{v^2}{4}|\\ &= k|u-v|-\frac{1}{4}(u+v)|u-v|\\ &\geq (k-\frac{1}{2})|u-v|\geq 0.\end{aligned}$$

Then $\mathcal{T}$ is an (α,η)-$\mathcal{Z}$-proximal contraction. Also, for $u_0=(1,1)$ and $u_1=(1,\frac{1}{2})$, we have

$$d(u_1,\mathcal{T}u_0)=d((1,\frac{1}{2}),(0,\frac{1}{2}))=1=d(P,Q)\quad \text{and}\quad \alpha(u_0,u_1)\geq\eta(u_0,u_1),$$

that is, condition (iii) holds. Moreover, it is obvious that $\mathcal{T}$ is triangular (α,η)-proximal admissible. All hypotheses of Theorem 3.1 are verified, so $\mathcal{T}$ admits a best proximity point, which is $u=(1,2-\sqrt{3})$.

In the subsequent result, we replace the continuity assertion in the previous theorem with the following condition in P:

- (C) If a sequence $\{u_n\}$ in P converges to $u\in P$ such that $\alpha(u_n,u_{n+1})\geq\eta(u_n,u_{n+1})$, then $\alpha(u_n,u)\geq\eta(u_n,u)$ for all $n\in\mathbb{N}$.

Theorem 3.3. *Let (P,Q) be a pair of nonempty subsets of a complete metric space (M,d) such that P_0 is nonempty, $\mathcal{T}:P\to Q$ and $\alpha,\eta:P\times P\to[0,\infty)$ be given mappings. Suppose the following conditions are satisfied:*

- (i) *P_0 is closed and $\mathcal{T}(P_0)\subseteq Q_0$;*
- (ii) *$\mathcal{T}$ is triangular (α,η)-proximal admissible;*
- (iii) *there exist $u_0,u_1\in P_0$ such that $d(u_1,\mathcal{T}u_0)=d(P,Q)$ and $\alpha(u_0,u_1)\geq\eta(u_0,u_1)$;*
- (iv) *(C) holds and $\mathcal{T}$ is an (α,η)-$\mathcal{Z}$-proximal contraction.*

Then $\mathcal{T}$ has a best proximity point in P. If $\alpha(u,v)\geq\eta(u,v)$ for all $u,v\in B_{est}(\mathcal{T})$, then $\mathcal{T}$ has a unique best proximity point $u^*\in P$. Moreover, for each $u\in M$, we have $\lim_{n\to\infty}\mathcal{T}^n u=u^*$.

Proof. By pursuing on the lines of the proof of Theorem 3.1, there exists a Cauchy sequence $\{u_n\}\subset P_0$ satisfying the expression (3.1) and $u_n\to p$. In view of (i), P_0 is closed and so $p\in P_0$. Also, since $\mathcal{T}(P_0)\subseteq Q_0$, there exists $z\in P_0$ such that

$$d(z,\mathcal{T}p)=d(P,Q). \tag{3.9}$$

On the other hand, by (C), we get

$$\alpha(u_n,p)\geq\eta(u_n,p),\quad \text{for all } n\in\mathbb{N}.$$

Thus, from (3.1), we have

$$\begin{aligned}&\alpha(u_n, p) \geq \eta(u_n, p),\\ &d(u_{n+1}, \mathcal{T}u_n) = d(z, \mathcal{T}p) = d(P, Q).\end{aligned}$$

Therefore, from the assertion (iv), we conclude

$$0 \leq \zeta(d(u_{n+1}, z), d(u_n, p)) < d(u_n, p) - d(u_{n+1}, z), \tag{3.10}$$

and so

$$\lim_{n\to\infty} d(u_{n+1}, z) \leq 0.$$

By the uniqueness of limit, we obtain $z = p$. Consequently, from (3.9), we have $d(p, \mathcal{T}p) = d(P, Q)$. Uniqueness of the best proximity point follows from the proof of Theorem 3.1. □

Example 3.4. *Let* $X = \mathbb{R}^2$ *be endowed with the Euclidian metric,* $P = \{(0, u) : u \geq 0\}$ *and* $Q = \{(1, u) : u \geq 0\}$*. Note that* $d(P, Q) = 1$*,* $P_0 = P$ *and* $Q_0 = Q$*. Define* $\mathcal{T} : P \to Q$ *and* $\alpha : P \times P \to [0, \infty)$ *by*

$$\mathcal{T}(0, u) = \begin{cases} (1, \frac{u}{9}), & \textit{if } 0 \leq u \leq 1,\\ (1, \frac{1}{2}), & \textit{if } u > 1,\end{cases}$$

and

$$\alpha((0, u), (0, v)) = \begin{cases} 2\eta((0, u), (0, v)), & \textit{if } u, v \in [0, 1] \textit{ or } u = v,\\ 0 = \eta((0, u), (0, v)), & \textit{otherwise.}\end{cases}$$

Choose $\zeta(a, b) = \frac{2}{3}b - a$ *for all* $a, b \in [0, \infty)$*. Let* $u, v, p, q \geq 0$ *such that*

$$\begin{cases} \alpha((0, u), (0, v)) \geq \eta((0, u), (0, v))\\ d((0, p), \mathcal{T}(0, u)) = d(P, Q) = 1\\ d((0, q), \mathcal{T}(0, v)) = d(P, Q) = 1.\end{cases}$$

Then $u, v \in [0, 1]$ *or* $u = v$*. We distinguish the following cases.*
Case 1: $u, v \in [0, 1]$*. Here,* $\mathcal{T}(0, u) = (1, \frac{u}{9})$ *and* $\mathcal{T}(0, v) = (1, \frac{v}{9})$*. Also,*

$$\sqrt{1 + (p - \frac{u}{9})^2} = \sqrt{1 + (q - \frac{v}{9})^2} = 1,$$

that is, $p = \frac{u}{9}$ *and* $q = \frac{v}{9}$*. So,* $\alpha((0, p), (0, q)) \geq \eta((0, p), (0, q))$*. Moreover,*

$$\begin{aligned}\zeta(d((0, p), (0, q)), d((0, u), (0, v))) &= \frac{2}{3}d((0, u), (0, v)) - d((0, \frac{u}{9}), (0, \frac{v}{9}))\\ &= \frac{2}{3}|u - v| - \frac{|u - v|}{9} \geq 0.\end{aligned}$$

Case 2: $u = v > 1$*. Here,* $\mathcal{T}(0, u) = (1, \frac{1}{2})$ *and* $\mathcal{T}(0, v) = (1, \frac{1}{2})$*. Similarly, we get that* $p = q = \frac{1}{2}$*. So,* $\alpha((0, p), (0, q)) \geq \eta((0, p), (0, q))$*. Also,* $\zeta(d((0, p), (0, q)), d((0, u), (0, v))) \geq 0$*.*
Case 3: $u, v > 1$ *with* $u \neq v$*. Then, the proof is similar to that in Case 2.*

In each case, we get that $\mathcal{T}$ is (α, η)-proximal admissible. It is also easy to see that $\mathcal{T}$ is triangular (α, η)-proximal admissible. Also, $\mathcal{T}$ is (α, η)-$\mathcal{Z}$-proximal contraction. Moreover, if $\{u_n = (0, p_n)\}$ is a sequence in P such that $\alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1})$ for all n and $u_n = (0, p_n) \to u = (0, p)$ as $n \to \infty$, then $p_n \to p$. We have $p_n, p_{n+1} \in [0,1]$ or $p_n = p_{n+1}$. We get that $p \in [0,1]$ or $p_n = p$. This implies that $\alpha(u_n, u) \geq \eta(u_n, u)$ for all n.

Also, there exists $(u_0, u_1) = ((0,1), (0, \frac{1}{9})) \in P_0 \times P_0$ such that

$$d(u_1, \mathcal{T}u_0) = 1 = d(P, Q) \quad \textit{and} \quad \alpha(u_0, u_1) \geq \eta(u_0, u_1).$$

Consequently, all conditions of Theorem 3.3 are satisfied. Therefore, $\mathcal{T}$ has a unique best proximity point in P which is $(0,0)$.

Corollary 3.5. *Let (P, Q) be a pair of nonempty subsets of a complete metric space (M, d). Suppose that $\mathcal{T} : P \to Q$ is a $\mathcal{Z}$-proximal contraction and P_0 is nonempty closed subset of M with $\mathcal{T}(P_0) \subseteq Q_0$. Then $\mathcal{T}$ has a unique best proximity point $u^* \in P$. Moreover, for each $u \in M$, we have $\lim_{n\to\infty} \mathcal{T}^n u = u^*$.*

Proof. The proof follows from Theorem 3.1 (Theorem 3.3), if we take $\alpha(u, v) = \eta(u, v) = 1$. □

Remark 3.6. *Theorem 3.1 (Theorem 3.3) extend and improve various best proximity point and fixed point results in complete metric spaces. Furthermore, some best proximity point and fixed point results in metric spaces endowed with a graph or a binary relation can be derived from our results under some suitable α-admissible mappings.*

4. A Variational Inequality Problem

Let C be a nonempty, closed and convex subset of a real Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. A variational inequality problem can be stated as follows:

$$\text{Find } u \in C \text{ such that } \langle Su, v - u \rangle \geq 0 \text{ for all } v \in C, \tag{4.1}$$

where $S : H \to H$ is a given operator. This problem has been a classical subject in economics, operations research and mathematical physics, particularly in the calculus of variations associated with the minimization of infinite-dimensional functionals; see, for instance, [15] and the references therein. It is closely related to many problems of nonlinear analysis, such as optimization, complementarity and equilibrium problems and finding fixed points; see, for instance, [8, 15, 23]. To solve problem (4.1), we define the metric projection operator $P_C : H \to C$. Here, we recall that for each $u \in H$, there exists a unique nearest point $P_C u \in C$ satisfying the inequality

$$\|u - P_C u\| \leq \|u - v\|, \quad \text{for all } v \in C.$$

The following lemmas correlate the solvability of a variational inequality problem to the solvability of a special fixed point problem.

Lemma 4.1. *Let $z \in H$. Then $u \in C$ satisfies the inequality $\langle u - z, y - u\rangle \geq 0$, for all $y \in C$ if and only if $u = P_C z$.*

Lemma 4.2. *Let $S : H \to H$. Then $u \in C$ is a solution of $\langle Su, v - u\rangle \geq 0$, for all $v \in C$, if and only if $u = P_C(u - \lambda Su)$, with $\lambda > 0$.*

Theorem 4.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $S : H \to H$ is such that $P_C(I - \lambda S) : C \to C$ is a $\mathcal{Z}$-proximal contraction. Then there exists a unique element $u^* \in C$ such that $\langle Su^*, v - u^*\rangle \geq 0$ for all $v \in C$. Moreover, for any arbitrary element $u_0 \in C$, the sequence $\{u_n\}$ defined by $u_{n+1} = P_C(u_n - \lambda Su_n)$ where $\lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$, converges to u^*.*

Proof. We consider the operator $F : C \to C$ defined by $Fx = P_C(x - \lambda Sx)$ for all $x \in C$. By Lemma 4.2, $u \in C$ is a solution of $\langle Su, v - u\rangle \geq 0$ for all $v \in C$, if and only if $u = Fu$. Now, F satisfies all the hypotheses of Corollary 3.5 with $P = Q = C$. It now follows from Corollary 3.5 that the fixed point problem $u = Fu$ admits a unique solution $u^* \in C$. □

References

[1] Al-Thagafi, M. A., Shahzad, N., Best proximity pairs and equilibrium pairs for Kakutani multimaps, *Nonlinear Anal.*, 70 (2009), 1209-1216.

[2] Altun, I., Aslantas, M., Sahin, H., Best proximity point results for p-proximal contractions, *Acta Math. Hungar.*, (2020), https://doi.org/10.1007/s10474-020-01036-3.

[3] Argoubi, H., Samet, B., Vetro, C., Nonlinear contractions involving simulation functions in a metric space with a partial order, *J. Nonlinear Sci. Appl.*, 8 (6) (2015), 1082-1094.

[4] Aydi, H., Felhi, A., On best proximity points for various α-proximal contractions on metric-like spaces, *J. Nonlinear Sci. Appl.*, 9 (2016), 5202–5218.

[5] Aydi, H., Felhi, A., Best proximity points for cyclic Kannan-Chatterjea-Ćirić type contractions on metric-like spaces, *J. Nonlinear Sci. Appl.*, 9 (2016), 2458–2466.

[6] Caballero, J., Harjani, J., Sadarangani, K., A best proximity point theorem for Geraghty-contractions, *Fixed Point Theory Appl.*, 2012:231 (2012).

[7] Eldred, A. A., Veeramani, P., Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, 323 (2) (2006), 1001-1006.

[8] Fang, S. C., Petersen, E. L., Generalized variational inequalities, *J. Optim. Theory Appl.*, 38 (1982), 363-383.

[9] Hussain, N., Kutbi, M. A., Salimi, P., Best proximity point results for modified α-ψ-proximal rational contractions, *Abstr. Appl. Anal.*, 2013, Article ID 927457 (2013).

[10] Işık, H., Aydi, H., Mlaiki, N., Radenović, S., Best proximity point results for Geraghty type Z-proximal contractions with an application, *Axioms*, 8 (3) (2019), 81.

[11] Işık, H., Sezen, M. S., Vetro, C., φ-Best proximity point theorems and applications to variational inequality problems, *J. Fixed Point Theory Appl.*, 19 (4) (2017), 3177-3189.

[12] Jleli, M., Samet, B.: Best proximity points for α-ψ-proximal contractive type mappings and application, *Bull. Sci. Math.*, 137 (2013), 977-995.

[13] Karapınar, E., Khojasteh, F., An approach to best proximity points results via simulation functions, *J. Fixed Point Theory Appl.*, 19 (2017), 1983-1995.

[14] Khojasteh, F., Shukla, S., Radenović, S., A new approach to the study of fixed point theorems via simulation functions, *Filomat*, 29 (6) (2015), 1189-1194.

[15] Kinderlehrer, D., Stampacchia, G., An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.

[16] Kumam, P., Aydi, H., Karapınar, E., Sintunavarat, W., Best proximity points and extension of Mizoguchi-Takahashi's fixed point theorems, *Fixed Point Theory Appl.*, 2013:242 (2013).

[17] Roldan-Lopez-de-Hierro, A. F., Karapınar, E., Roldan-Lopez-de-Hierro, C., Martinez-Moreno, J., Coincidence point theorems on metric spaces via simulation functions, *J. Comput. Appl. Math.*, 275 (2015), 345-355.

[18] Sadiq Basha, S., Veeramani, P., Best proximity pair theorems for multifunctions with open fibres, *J. Approx. Theory*, 103 (2000), 119-129.

[19] Sahin, H., Aslantas, M., Altun, I., Feng-Liu type approach to best proximity point results, for multivalued mappings, *J. Fixed Point Theory Appl.*, 22 (2020), 11.

[20] Samet, B., Best proximity point results in partially ordered metric spaces via simulation functions, *Fixed Point Theory Appl.*, 2015:232 (2015).

[21] Sankar Raj, V., A best proximity point theorem for weakly contractive non-self-mappings, *Nonlinear Anal.*, 74 (14) (2011), 4804-4808.

[22] Tchier, F., Vetro, C., Vetro, F., Best approximation and variational inequality problems involving a simulation function, *Fixed Point Theory Appl.*, 2016:26 (2016).

[23] Todd, M. J., The Computations of Fixed Points and Applications, Springer, Berlin, 1976.

https://communications.science.ankara.edu.tr

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ON PROXIMITY SPACES AND TOPOLOGICAL HYPER NEARRINGS

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Abstract. In 1934 the concept of algebraic hyperstructures was first introduced by a French mathematician, Marty. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the result of this composition is a set. In this paper, we prove some results in topological hyper nearring. Then we present a proximity relation on an arbitrary hyper nearring and show that every hyper nearring with a topology that is induced by this proximity is a topological hyper nearring. In the following, we prove that every topological hyper nearring can be a proximity space.

1. Introduction

In 1934, the concept of hypergroups was first introduced by a French mathematician, Marty [22]. In the following, it was studied and extended by many researchers, namely, Corsini [3], Corsini and Leoreanu [4], Davvaz [6–8], Frenni [12], Koskas [20], Mittas [23], Vougiouklis, and others. The topological hyper nearring notion is defined and studied by Borhani and Davvaz in [2].

In the 1950's, Efremovič [10, 11], a Russian mathematician, gave the definition of proximity space, which he called infinitesimal space in a series of his papers. He axiomatically characterized the proximity relation A is near B for subsets A and B of any set X. The set X, together with this relation, was called an infinitesimal (proximity) space. Defining the closure of a subset A of X to be the collection of all points of X near A, Efremovič [10,11] showed that a topology can be introduced in a proximity space.

In this paper, we study some remarks on topological hyper nearring, then we

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define a proximity relation on hyper nearring and, we will prove that every hyper nearring with a topology that is induced by this proximity is a topological hyper nearring. In the following, we show that every topological hyper nearring is a proximity space.

2. Preliminaries

In this section, we recall some basic classical definitions of topology from [21] and definitions related to hyperstructures that are used in what follows.

Definition 1. [6] A *hyper nearring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a quasi canonical hypergroup, i.e., in $(R, +)$ the following conditions hold:

(i) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$;
(ii) There is $0 \in R$ such that $x + 0 = 0 + x = x$, for all $x \in R$;
(iii) For any $x \in R$ there exists one and only one $x' \in R$ such that $0 \in x + x'$ (we shall write $-x$ for x' and we call it the opposite of x);
(iv) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

If A and B are two non-empty subsets of R and $x \in R$, then we define:

$$A + B = \bigcup_{\substack{a \in A \\ b \in B}} a + b,\ x + A = \{x\} + A and A + x = A + \{x\}.$$

(2) $(R, \cdot)$ is a semigroup respect to the multiplication, having 0 as a left absorbing element, i.e., $x \cdot 0 = 0$ for all $x \in R$. But, in general, $0 \cdot x \neq 0$ for some $x \in R$.

(3) The multiplication is left distributive with respect to the hyperoperation $+$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Note that for all $x, y \in R$, we have $-(-x) = x$, $0 = -0$, $-(x + y) = -y - x$ and $x(-y) = -xy$. Let R and S be two hyper nearrings. The map $f : R \to S$ is called a homomorphism if for all $x, y \in R$, the following conditions hold: $f(x+y) = f(x) + f(y)$, $f(x \cdot y) = f(x) \cdot f(y)$ and $f(0) = 0$. It is easy to see that if f is a homomorphism, then $f(-x) = -f(x)$, for all $x \in R$. A nonempty subset H of a hyper nearring R is called a *subhyper nearring* if $(H, +)$ is a subhypergroup of $(R, +)$, i.e., (1) $a, b \in H$ implies $a + b \subseteq H$; (2) $a \in H$ implies $-a \in H$; and (3) $(H, \cdot)$ is a subsemigroup of $(R, \cdot)$. A subhypergroup A of the hypergroup $(R, +)$ is called *normal* if for all $x \in R$, we have $x + A - x \subseteq A$. Let H be a normal hyper R-subgroup of hyper nearrring R. In [14], Heidari et al. defined the relation

$$x \sim y (\mathrm{mod}\, H) \text{ if and only if } (x - y) \cap H \neq \emptyset, \text{ for all } x, y \in H.$$

This relation is a regular equivalence relation on R. Let $\rho(x)$ be the equivalence class of the element $x \in H$ and denote the quotient set by R/H. Define the hyperoperation $\oplus$ and multiplication $\odot$ on R/H by

$$\begin{aligned}\rho(a) \oplus \rho(b) &= \{\rho(c) \ : \ c \in \rho(a) + \rho(b)\}, \\ \rho(a) \odot \rho(b) &= \rho(a \cdot b),\end{aligned}$$

for all $a, b \in R$. Let $(R, +, \cdot)$ be a hyper nearring and τ a topology on R. Then, we consider a topology τ^* on $\mathcal{P}^*(R)$ which is generated by $\mathcal{B} = \{S_V \ : \ V \in \tau\}$, where $S_V = \{U \in \mathcal{P}^*(R) \ : \ U \subseteq V, U \in \tau\}$, $V \in \tau$. In the following we consider the product topology on $R \times R$ and the topology τ^* on $\mathcal{P}^*(R)$ [2].

Definition 2. [2] Let $(R, +, \cdot)$ be a hyper nearring and (R, τ) be a topological space. Then, the system $(R, +, \cdot, \tau)$ is called a *topological hyper nearring* if

(1) the mapping $(x, y) \mapsto x + y$, from $R \times R$ to $\mathcal{P}^*(R)$,
(2) the mapping $x \mapsto -x$, from R to R,
(3) the mapping $(x, y) \mapsto x.y$, from $R \times R$ to R,

are continuous.

Example 1. [2] The hyper nearring $R = (\{0, a, b, c\}, +, \cdot)$ defined as follows:

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0,a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{0,a,c\}$	$\{b,c\}$
c	$\{c\}$	$\{c\}$	$\{b,c\}$	$\{0,a,b\}$

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Let $\tau = \{\varnothing, R, \{0, a\}\}$. Then$(R, +, \cdot, \tau)$ is a topological hyper nearrring.

Lemma 1. *[2] Let $(R, +, \cdot, \tau)$ be a topological hyper nearring. If U is an open set and a complete part of R, then for every $c \in R$, $c + U$ and $U + c$ are open sets.*

Definition 3. [24] A binary relation δ on $P(X)$ is called a *proximity* on X if and only if δ satisfies the following conditions:

(P1) $A\delta B$ implies $B\delta A$,
(P2) $A\delta B$ implies $A \neq \emptyset$,
(P3) $A \cap B \neq \emptyset$ implies $A\delta B$,
(P4) $A\delta(B \cup C)$ if and only if $A\delta B$ or $A\delta C$,
(P5) $A \not\delta B$ implies there exists $E \subseteq X$ such that $A \not\delta E$ and $B \not\delta E^c$.

The pair (X, δ) is called a *proximity space*. If the sets $A, B \subseteq X$ are δ-related, then we write $A\delta B$, otherwise we write $A \not\delta B$.

Example 2. Let $A, B \subseteq X$ and $A\delta B$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$. Then δ is a *proximity* on X.

The following theorem shows a proximity relation δ on X induces a topology on X.

Theorem 1. *[24] If a subset A of a proximity space (X, δ) is defined to be closed if and only if $x\delta A$ implies $x \in A$, then the collection of complements of all closed sets so defined yields a topology $\tau = \tau(\delta)$ on X.*

3. Some results on topological hyper nearrings

In this section, we present some results and properties in topological hyper nearring.

Lemma 2. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring. Then, $0 \in \bigcup_{R \neq U \in \tau} U$.*

Proof. If $0 \notin \bigcup_{R \neq U \in \tau} U$, then for every $R \neq U \in \tau$, $0 \notin U$. Let $U \in \tau$, $U \neq \emptyset$ and $0 \neq x \in U$. By the continuity of the mapping $+$, there exist neighborhoods $V_1, V_2 \in \tau$ of x and 0, respectively, such that $V_1 + V_2 \subseteq U$. Hence, we conclude that $V_2 = R$ and $V_1 + R \subseteq U$. Hence, we have $0 \in x + (-x) \subseteq V_1 + R \subseteq U$ and it is a contradiction. Therefore, we have $0 \in \bigcup_{R \neq U \in \tau} U$. □

Lemma 3. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part of R. Let $\mathcal{U}$ be the system of all neighborhoods of 0, then for any subset A of R,*

$$\overline{A} = \bigcap_{U \in \mathcal{U}} (A + U).$$

Proof. Suppose that $x \in \overline{A}$ and $U \in \mathcal{U}$. $x - U$ is an open neighborhood of x, hence we have $x - U \cap A \neq \emptyset$. Thus there exists $a \in A$ such that $a \in x - U$. So, $x \in a + U \subseteq A + U$, for all $U \in \mathcal{U}$. Therefore, $\overline{A} \subseteq \bigcap_{U \in \mathcal{U}} (A + U)$. Now, let $x \in A + U$, for every $U \in \mathcal{U}$ and let V be a neighborhood of x. $x - V$ is a neighborhood of 0, hence $x \in A + (x - V)$. So, there exist $a \in A$ and $t \in x - V$ such that $x \in a + t$. Thus $a \in x - t \subseteq x + V - x = V$. Then $A \cap V \neq \emptyset$ and this proves that $x \in \overline{A}$ and $\bigcap_{U \in \mathcal{U}} (A + U) \subseteq \overline{A}$. Therefore, $\overline{A} = \bigcap_{U \in \mathcal{U}} (A + U)$. □

Corollary 1. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part of R and let $\mathcal{U}$ be the system of all neighborhoods of 0. Then,*

(i) $\overline{\{0\}} = \bigcap_{U \in \mathcal{U}} U$;

(ii) *For every open set V and every closed set F such that $V \cap \overline{\{0\}} \neq \emptyset$ and $F \cap \overline{\{0\}} \neq \emptyset$, we have $\overline{\{0\}} \subseteq V$ and $\overline{\{0\}} \subseteq F$;*

(iii) *$\{0\}$ is dense in R if and only if R has trivial topology $\{\emptyset, R\}$.*

Proof. (i) It follows immediately from of Lemma 3.

(ii) Let V be open, $V \cap \overline{\{0\}} \neq \emptyset$ and $t \in V \cap \overline{\{0\}}$. V is a neighborhood of t and $t \in \overline{\{0\}}$, thus V is a neighborhood of 0 and by (i), $\overline{\{0\}} \subseteq V$. Now, suppose that

F is a closed subset and $F \cap \overline{\{0\}} \neq \emptyset$. Then, $\overline{\{0\}} \not\subseteq F^c$. F^c is open thus we have $\overline{\{0\}} \cap F^c = \emptyset$. Consequently, we get $\overline{\{0\}} \subseteq F$.

(iii) Let $\{0\}$ is dense in R and U be nonempty and open in R. Then, $R = \overline{\{0\}}$ and by (ii) $\overline{\{0\}} \subseteq U$. Therefore, $R = U$. □

Lemma 4. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part of R. Then $\{0\}$ is open if and only if τ is discrete.*

Proof. It is straightforward. □

Theorem 2. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part and H be a normal subhyper group of it. Then R/H is discrete if and only if H is open.*

Proof. Suppose that R/H is discrete and π is the natural mapping $x \mapsto \pi(x) = H + x$ of R onto R/H. Then, the identity, $\pi(0)$ of R/H is an isolated point. So, $\pi^{-1}(\pi(0)) = H$ is open of R. Now, if H is open, since π is open, it follows that $\pi(H)$ is open. Hence the identity $\pi(H)$ of R/H is an isolated point. Therefore, we conclude that R/H is discrete. □

Theorem 3. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part. Then, the following conditions are equivalent:*

(1) *R is a T_0- space;*
(2) *$\{0\}$ is closed.*

Proof. (1⇒2) Let R be a T_0- space and let $x \in \overline{\{0\}}$. We prove that $x = 0$. If $x \neq 0$, then by (1) there exists an open neighborhood U containing only 0 or x, but since $x \in \overline{\{0\}}$, hence U is a neighborhood of 0, such that $x \notin U$. So, $x \in -U + x$. By Lemma 1, $-U + x$ is an open neighborhood of x, such that $0 \notin -U + x$ (Because if $0 \in -U + x$, then there exists $u \in U$ such that $0 \in -u + x$. So, $x = u + 0 \in U$), this is a contradiction. Thus, $x = 0$ and it follows that 0 is closed.

(2⇒1) Let $\{0\}$ be closed and $x, y \in R, x \neq y$. We show that there exist an open neighborhood U containing only x or y. If $y = 0$, since $\{0\}$ is closed and $x \neq 0$, then x is an interior point of $R \setminus \{0\}$. Hence, there exists a neighborhood U of x such that $0 \notin U$. Now, if $x \neq 0, y \neq 0$ and $x \neq y$, then $0 \notin x - y$. Consequently, by the previous part, for every $t \in x - y$ there exists a neighborhood U_t of t, such that $0 \notin U_t$. We consider $U = \bigcup\limits_{t \in x-y} U_t$. Then $x - y \subseteq U$ and $0 \notin U$.Thus $U + y$ is a neighborhood of x such that $y \notin U + y$ (since $0 \notin U$). Therefore, R is a T_0-space. □

Let (X, τ) be a topological space. If f is a arbitrary mapping from X onto Y, then consider the family $\tau_f = \{U \ : \ U \subseteq Y, f^{-1}(U) \in \tau\}$. Obviously τ_f is a topology on Y.

Theorem 4. *[25] Let $f : (X, \tau) \to (Y, \tau')$ be a continuous function. Then $\tau' \leq \tau_f$.*

Lemma 5. *Let $f : R \to R'$ be a homomorphism of hyper nearrings. Then for every subset $A \subseteq R$, $f^{-1}(f(A)) = kerf + A$.*

Proof. Let $A \subseteq R$ and $t \in f^{-1}(f(A))$. Then $f(t) \in f(A)$ and it follows that there exists $a \in A$ such that $f(t) = f(a)$. Thus $0 \in f(t) - f(a) = f(t - a)$. Hence there exists $x \in t - a$ such that $f(x) = 0$. Then $x \in kerf$. Thus $t \in x + a \subseteq kerf + A$ and this shows that $f^{-1}(f(A)) \subseteq kerf + A$. It is obvious that $kerf + A \subseteq f^{-1}(f(A))$. Therefore, $f^{-1}(f(A)) = kerf + A$. □

Theorem 5. *Let $(R, +, \cdot, \tau)$ and $(R', +', \cdot', \tau')$ be two topological hyper nearring such that every open subset of them is a complete part and f from R onto R' be a homomorphism. Then (R', τ_f) is a topological hyper nearring.*

Proof. We should show that $+'$, $\cdot'$ and inverse operation are continuous on (R', τ'). Suppose that $x', y' \in R'$ and $x' +' y' \subseteq U' \in \tau_f$. Since f is onto, then there exist $x, y \in R'$ such that $f(x) = x'$ and $f(y) = y'$. Hence $f(x + y) = f(x) +' f(y) = x' +' y' \subseteq U'$. So, $x + y \subseteq f^{-1}(U') \in \tau$(since $U' \in \tau_f$). Since $+$ is continuous, then there exist neighborhoods $U_x \in \tau$ and $U_y \in \tau$ of elements x and y, respectively, such that $U_x + U_y \subseteq f^{-1}(U')$. By Lemmas 1 and 5, $f^{-1}(f(U_x)) = kerf +' U_x \in \tau$ and $f^{-1}(f(U_y)) \in \tau$. Hence $f(U_x) \in \tau_f$ and $f(U_y) \in \tau_f$. Therefore, we obtain

$$f(U_x) +' f(U_y) = f(U_x + U_y) \subseteq f(f^{-1}(U')) = U'.$$

This completes the proof. □

Theorem 6. *Let f from (R, τ) onto (R', τ') be a homomorphism of topological hyper nearrings. Then $f : (R, \tau) \to (R', \tau_f)$ is continuous and open.*

Proof. If $U \in \tau_f$, by the definition of τ_f, $f^{-1}(U) \in \tau$. Thus, f is continuous.
Now, let U be an open subset in R. Then by Theorem 5 $f^{-1}(f(U)) = kerf + U$ is open in (R, τ). Thus by the definition of τ_f, $f(U) \in \tau_f$. This means $f(U)$ is open in R'. Therefore, f is open. □

Let R be a topological hyper nearring, H be normal hyper R-subgroup of R and π be natural mapping of R onto R/H by $x \mapsto \pi(x) = H + x$. Then, by Theorem 3.30 [2] $(R/H, \tau_\pi)$ is a topological hyper nearring. It is called the quotient space of topological hyper nearring R that we showed τ_π by $\overline{\tau}$ in [2].

Theorem 7. *Let R be a T_0-topological hyper nearring such that every open subset of it is a complete part of R and H be a discrete subhypergroup of R. Then H is closed.*

Proof. Let $x \in \overline{H}$. Since H is a discrete subhypergroup of R, then $0 \in H$ and there exists an open neighborhood V of 0 such that $V \cap H = \{0\}$. By Lemma 1, $x - V$ is an open neighborhood of x. Therefore, $x - V \cap H \neq \emptyset$ (because $x \in \overline{H}$). Hence there exists $h \in H$ such that $h \in x - V$ and $h \in x - v$, for some $v \in V$. Thus $v \in -h + x \subseteq V \cap \overline{H} \subseteq \overline{V \cap H}$ (let $t \in V \cap \overline{H}$ and U_t is a neighborhood of t. $U_t \cap V$ is an open neighborhood of t and since $t \in \overline{H}$, then $(U_t \cap V) \cap H \neq \emptyset$

and $U_t \cap (V \cap H) \neq \emptyset$. It follows that $t \in \overline{V \cap H}$ and $V \cap \overline{H} \subseteq \overline{V \cap H}$). Thus $v \in \overline{V \cap H} = \overline{\{0\}} = \{0\}$(by Theorem 3) and it follows that $x = h \in H$ and H is closed. □

Theorem 8. *Let R be a topological hyper nearring and H a dense subhypergroup of R. If V is a neighborhood of 0 in H, then $\overline{V}$ is a neighborhood of 0 of R.*

Proof. Since V is a neighborhood of 0 in H, it follows that there exists an open neighborhood U of 0 in R such that $U \cap H \subseteq V$. Hence, we obtain $U = U \cap G = U \cap \overline{H} \subseteq \overline{U \cap H} \subseteq \overline{V}$. Therefore, 0 is an interior point $\overline{V}$ and $\overline{V}$ is open in R. □

4. Topological hyper nearring derived from a proximity space

In this section, we define a proximity relation on an arbitrary hyper nearring and prove that every hyper nearring with topology whose is induced by this proximity relation is a topological hyper nearring. Also, we show that every topological hyper nearring is a proximity space.

Theorem 9. *Let $(R, +, \cdot)$ be a hyper nearring, N be a normal subhypergroup of R and $A, B \subseteq R$. We define $A\delta B$ if and only if there exist $a \in A$ and $b \in B$ such that $-b + a \subseteq N$, then (R, δ) is a proximity space.*

Proof. (P_1) Suppose that $A\delta B$. Then, there exist $a \in A$ and $b \in B$ such that $-b + a \subseteq N$. So, we get $-a + b \subseteq -N = N$. Therefore, $B\delta A$.

(P_2) It is obvious.

(P_3) Let there exists $x \in A \cap B \neq \emptyset$. Then $-x + x \subseteq -x + N + x \subseteq N$. So, we conclude that $A\delta B$.

(P_4) It is straightforward.

(P_5) Let $A \not\delta B$ and $E := B + N$. If $A\delta E = B + N$, then there exist $a \in A$ and $b \in B$ such that $-(b + N) + a \subseteq N$. Therefore, $-N - b + a \subseteq N$ and this implies that $-b + a \subseteq N + N \subseteq N$. Thus, $A\delta B$ and it is a contradiction. Hence $A \not\delta E$. Also, $B \not\delta E^c$. If $B\delta E^c$, then there exist $b \in B$ and $x \in (B + N)^c$ such that $-x + b \subseteq N$. Therefore, $x \in b + N \subseteq B + N$ and it is a contradiction. □

Theorem 10. *In the proximity space (R, δ) that $(R, +, \cdot)$ is a hyper nearring and δ is defined relation in Theorem 9, the set $\beta = \{x + N \ : \ x \in R\}$ is a base for the topology $\tau = \tau(\delta)$.*

Proof. Let U be an open subset of R and let $y \in U$. We should show that $y + N \subseteq U$. Let $t \notin U$, then $t \in U^c$ and $t\delta U^c$ (since U^c is closed). $-y + t \subseteq -y + y + N \subseteq -y + N + y \subseteq N$. Hence $t\delta y$ and by $(P4)$, $y\delta U^c$. Thus $y \in U^c$ and it is a contradiction. This implies that β is a base for the topology $\tau(\delta)$. □

Lemma 6. *The normal subhypergroup N of R is a clopen set in the topology $\tau(\delta)$ is defined in Theorem 10.*

Proof. By Theorem 10, N is open. Now, let $x\delta N$, for $x \in R$. Then there exists $n \in N$ such that $-n + x \subseteq N$. Therefore $x \in n - n + x \subseteq n + N = N$. Thus N is a closed subset in R. $\square$

Theorem 11. *Let $(R, +, \cdot)$ be a hyper nearring, the normal subhypergroup N be a complete part of R and the relation δ is defined in Theorem 9. Then the system $(R, +, \cdot, \tau(\delta))$ is a topological hyper nearring.*

Proof. We should show that $+$, $\cdot$ and inverse operation are continuous. Suppose that U is an open subset of R such that $x + y \subseteq U$, for $x, y \in R$. Then by Theorem 10, there exists $t \in R$ such that $x + y \subseteq t + N \subseteq U$. Therefore, $x + N$ and $y + N$ are neighborhoods of x and y such that $(x + N) + (y + N) = x + y + N \subseteq t + N + N = t + N \subseteq U$. Thus $+$ is continuous on R. Now, Suppose that U is an open neighborhood of $-x$. By Theorem 10, there exists $t \in R$ such that $-x \in t + N \subseteq U$. Therefore, $x \in -N - t = -t + N$. Hence $-t + N$ is a neighborhoods of x and $-(-t + N) = -N + t = N + t = t + N \subseteq U$. This proves that inverse operation is continuous. Now, we show that $\cdot$ is continuous. Suppose that U is an open subset of R such that $x \cdot y \in U$, for $x, y \in R$. Then there exist $t \in R$ such that $x \cdot y \in t + N \subseteq U$(by Theorem10). $x + N$ and $y + N$ are neighborhoods of x and y such that $(x + N) \cdot (y + N) \subseteq x \cdot y + N$ (N is a complete part of R, then $x \cdot y + N$ is a complete part of R. Hence $(x + N) \cdot (y + N) \subseteq x \cdot y + N$). So, $(x + N) \cdot (y + N) \subseteq x \cdot y + N \subseteq t + N + N = t + N \subseteq U$. Thus $\cdot$ is continuous on R. $\square$

Example 3. Let $R = \{0, a, b\}$ be a set with a hyperoperation $+$ and a binary operation $\cdot$ as follows:

$+$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

$\cdot$	0	a	b
0	0	a	b
a	0	a	b
b	0	a	b

Then, $(R, +, \cdot)$ is a hyper nearring. We consider a normal subhyperring $N = \{0, a\}$ of R and define:

$A\delta B$ if and only if there exist $a \in A$ and $b \in B$ such that $-b + a \subseteq N$.

Therefore, $\tau(\delta) = \{\varnothing, \{0, a, b\}, \{0, a\}, \{b\}\}$. Simply, we can show that $(R, +, \cdot, \tau(\delta))$ is a topological hyper nearring.

The following theorem, show that every topological hyper nearring is a proximity space.

Theorem 12. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part of R. Then there exists a proximity relation δ such that (R, δ) is a proximity space.*

Proof. Let $\mathcal{U}$ be the system of symmetric neighborhoods at 0, for every $A, B \subseteq R$ and $V \in \mathcal{U}$. We define

$A\delta B$ if and only if $A \cap B + V \neq \emptyset$.

Now, we show that δ is a proximity relation.

(P_1) Suppose that $A\delta B$. Then, there exist $a \in A$ and $b \in B$ such that $a \in b + V$. Hence $b \in a - V = a + V \subseteq A + V$. Therefore, $B\delta A$.

(P_2) It is obvious.

(P_3) Let $A \cap B \neq \emptyset$. Then, there exists $x \in A \cap B$. Therefore, $x \in A \cap B + V \neq \emptyset$. Thus $A\delta B$.

(P_4) It is straightforward..

(P_5) Let $A \not\delta B$ and $E := B + V$. If $A\delta B + V$, then $A \cap (B + V) + V \neq \emptyset$. Therefore $A \cap B + V \neq \emptyset$ (since V is a complete part of R, then $V + V \subseteq V$) and this proves that $A\delta B$, that it is a contradiction. Hence $A \not\delta E$. Also, if $B\delta E^c$, it follows that $B \cap (B + V)^c + V \neq \emptyset$. Hence there exist $b \in B$, $x \in (B + V)^c$ and $v \in V$ such that $b \in x + v$. Thus $x \in b - v \subseteq B + V$ and it is a contradiction. Therefore, $B \not\delta E^c$. □

5. Conclusion

In this paper we expressed the relationship between two important subjects: algebraic hyperstructures and topology. We studied several characteristics of topological hyper nearrings and in the following, we related them to proximity spaces.

References

[1] Ameri, R., Topological (transposition) hypergroups, *Italian Journal of Pure and Applied Mathematics*, (13) (2003), 181-186.

[2] Borhani-Nejad, S., Davvaz, B., Topological hyper nearrings, submitted.

[3] Corsini, P., Prolegomena of Hypergroup Theory, Second ed., Aviani Editore, Tricesimo, Italy, 1993.

[4] Corsini, P., Leoreanu, V., Applications of Hypergroup Theory, Kluwer Academic Publishers, 2003.

[5] Dasic, V., Hypernearrings, *Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990)*, World Scientific, (1991), 75-85.

[6] Davvaz, B., Hypernearrings and weak hypernearrings, *11th Algebra Seminar of Iranian Math. Soc. Isfahan University of Technology*, Isfahan, October 27-29, (1999), 68-78.

[7] Davvaz, B., Polygroup Theory and Related Systems, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[8] Davvaz, B., Semihypergroup Theory, Elsevier, 2016.

[9] Davvaz, B., Leoreanu-Fotea, V., Hyperring Theory and Applications, International Academic Press, 115, Palm Harber, USA, 2007.

[10] Efremovč, V.A., The geometry of proximity, *Mat. Sb.*, 31 (1952), 189-200.

[11] Efremovič, V.A., Infinitesimal spaces, *Doklady Akademii Nauk SSSR (N.S.)*, (in Russian), 76 (1951), 341-343.

[12] Freni, D., A note on the core of a hypergroup and the transitive closure β^* of β, *Riv. Mat. Pura Appl.*, 8 (1991), 153-156.

[13] Gontineac, V.M., On hypernear-rings and H-hypergroups, *Fifth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1993)*, Hadronic Press, Inc., USA, (1994), 171-179.

[14] Heidari, D., Davvaz, B., Modarres, S.M.A., Topological hypergroups in the sense of Marty, *Comm. algebra*, 42 (2014), 4712-4721.

[15] Heidari, D., Davvaz, B., Modarres, S.M.S., Topological polygroups, *Bull. Malays. Math. Sci. Soc.*, 3942 (2016), 707-721.

[16] Hošková-Mayerová, S., Topological hypergroupoids, *Comput. Math. Appl.*, 64(9) (2012), 2845-2849.

[17] Jancic-Rašovic S., Cristea, I., Division hypernear-rings, *An. Stiint. Univ. "Ovidius" Constanta Ser. Mat.*, 26(3) (2018), 109-126.

[18] Jancic-Rašovic S., Cristea, I., Hypernear-rings with a defect of distributivity, *Filomat*, 32(4) (2018), 1133-1149.

[19] Kim, K.H., Davvaz, B., Roh, E.H., On Hyper R-subgroups of hypernear-rings, *Scientiae Mathematicae Japonicae*, Online (2007), 649-656.

[20] Koskas, M., Groupoides, demi-hypergroupes et hypergroupes, *J. Math. Pure Appl.*, 49(9) (1970), 155-192.

[21] Munkres, J.R., Topology, Prentice Hall, Upper Saddle River, NJ, 2000.

[22] Marty, F., Sur une generalization de la notion de group, *8th congress Math. Scandinaves, Stockholm*, (1934), 45-49.

[23] Mittas, J., Hypergroups canoniques, *Math. Balkanica*, 2 (1972), 165-179.

[24] Naimpally S.A., Warrack, B.D., Proximity Spaces, *Cambridge Tract*, 1970.

[25] Ursul, M., Topological rings satisfying compactness conditions, Springer Science Business Media (2002), vol 549.

https://communications.science.ankara.edu.tr

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ON HERMITE-HADAMARD TYPE INEQUALITIES FOR INTERVAL-VALUED MULTIPLICATIVE INTEGRALS

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Abstract. In this work, we define multiplicative integrals for interval-valued functions. We establish some new Hermite-Hadamard type inequalities in the setting of interval-valued multiplicative calculus and give some examples to illustrate our main results. We also discuss special cases of our main results which are the extension of already established results.

1. Introduction

The Hermite–Hadamard inequality discovered by C. Hermite and J. Hadamard, (see [14], [32, pp. 137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that, if $F : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$F\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b F(x)dx \leq \frac{F(a)+F(b)}{2}. \tag{1.1}$$

Both inequalities in (1.1) hold in the reversed direction if F is concave. We note that Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite–Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied, see [1, 2, 7, 8, 11, 15–18, 26, 31, 36–40] and reference therein.

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On the other hand, interval analysis is a particular case of set–valued analysis which is the study of sets in the spirit of mathematical analysis and general topology. It was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real–world phenomena. An old example of interval enclosure is Archimede's method which is related to compute of the circumference of a circle. In 1966, the first book related to interval analysis was given by Moore who is known as the first user of intervals in computational mathematics, see [27]. After his book, several scientists started to investigate theory and application of interval arithmetic. Nowadays, because of its applications, interval analysis is a useful tool in various area which are interested intensely in uncertain data. You can see applications in computer graphics, experimental and computational physics, error analysis, robotics and many others. What's more, several important inequalities (Hermite–Hadamard, Ostrowski, etc.) have been studied for the interval–valued functions in recent years. In [9, 10], Chalco–Cano et al. obtained Ostrowski type inequalities for interval–valued functions by using Hukuhara derivative for interval–valued functions. In [33], Román-Flores et al. established Minkowski and Beckenbach's inequalities for interval–valued functions. For the others, please see [12, 13, 19, 33, 34]. However, inequalities were studied for more general set–valued maps. For example, in [35], Sadowska gave the Hermite–Hadamard inequality. In [21, 41], authors established Hermite-Hadamard type inequalities for co-ordinated convex interval-valued functions. For the other studies, you can see [3, 25, 29].

The main purpose of this paper is to define *integral/multiplicative integral for interval–valued functions and to obtain Hermite–Hadamard inequality via these integrals.
The overall structure of the study takes the form of six sections including introduction. The remainder of this work is organized as follows: we first recall the interval calculus by giving the several definitions and properties in Section 2. In section 3, we define multiplicative integral for interval-valued functions and give some basic properties of this newly define integral. In Section 4, we define logarithmically interval-valued h-convex functions and discuss special cases and properties of this class of functions. In section 5, we obtain Hermite-Hadamard inequalities and related inequalities for our new class of convex functions by utilizing our newly define integral. At the end, in section 6, we give concluding remarks about our work.

2. Interval Calculus

A real valued interval X is bounded, closed subset of $\mathbb{R}$ defined by

$$X = \left[\underline{X}, \overline{X}\right] = \left\{t \in \mathbb{R} : \underline{X} \ \leq t \leq \overline{X}\right\}$$

where $\underline{X}$, $\overline{X} \in \mathbb{R}$ and $\underline{X} \leq \overline{X}$. The numbers $\underline{X}$ and $\overline{X}$ are called the left and the right endpoints of interval X, respectively. When $\overline{X} = \underline{X} = a$, the interval X is said to be degenerate and we use the form $X = a = [a, a]$. Also, we call X positive if $\underline{X} > 0$ or negative if $\overline{X} < 0$. The set of all closed intervals of $\mathbb{R}$, the sets of all closed positive intervals of $\mathbb{R}$ and closed negative intervals of $\mathbb{R}$ is denoted by $\mathbb{R}_{\mathcal{I}}$, $\mathbb{R}_{\mathcal{I}}^{+}$ and $\mathbb{R}_{\mathcal{I}}^{-}$ respectively. The Hausdorff-Pompeiu distance between the intervals X and Y is defined by

$$d(X,Y) = d\left(\left[\underline{X}, \overline{X}\right], \left[\underline{Y}, \overline{Y}\right]\right) = \max\left\{\left|\underline{X} - \underline{Y}\right|, \left|\overline{X} - \overline{Y}\right|\right\}.$$

It is known that $(\mathbb{R}_{\mathcal{I}}, d)$ is a complete metric space [4].
Now, we give the definitions of basic interval arithmetic operations for the intervals X and Y as follows:

$$\begin{aligned} X + Y &= \left[\underline{X} + \underline{Y}, \overline{X} + \overline{Y}\right], \\ X - Y &= \left[\underline{X} - \overline{Y}, \overline{X} - \underline{Y}\right], \\ X.Y &= [\min S, \max S] \text{ where } S = \left\{\underline{X}\,\underline{Y}, \underline{X}\,\overline{Y}, \overline{X}\underline{Y}, \overline{X}\,\overline{Y}\right\}, \\ X/Y &= [\min T, \max T] \text{ where } T = \left\{\underline{X}/\underline{Y}, \underline{X}/\overline{Y}, \overline{X}/\underline{Y}, \overline{X}/\overline{Y}\right\} \text{ and } 0 \notin Y. \end{aligned}$$

Scalar multiplication of the interval X is defined by

$$\lambda X = \lambda\left[\underline{X}, \overline{X}\right] = \begin{cases} \left[\lambda\underline{X}, \lambda\overline{X}\right], & \lambda > 0 \\ \{0\}, & \lambda = 0 \\ \left[\lambda\overline{X}, \lambda\underline{X}\right], & \lambda < 0. \end{cases}$$

where $\lambda \in \mathbb{R}$.
The opposite of the interval X is

$$-X := (-1)X = [-\overline{X}, -\underline{X}]$$

for $\lambda = -1$.
The subtraction is given by

$$X - Y = X + (-Y) = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}].$$

Use of monotonic functions

$$F(X) = [F(\underline{X}), F(\overline{X})].$$

For example, $F(x) = e^{x}$, $x \in \mathbb{R}$ and $F(x) = \ln x$, $x > 0$ then we have

$$\begin{aligned} \exp(X) &= \left[\exp(\underline{X}), \exp(\overline{X})\right] \\ \ln(X) &= \left[\ln(\underline{X}), \ln(\overline{X})\right]. \end{aligned}$$

In general, $-X$ is not additive inverse for X i.e $X - X \neq 0$.

The definitions of operations lead to a number of algebraic properties which allows $\mathbb{R}_{\mathcal{I}}$ to be quasilinear space (see, [24]). They can be listed as follows (see, [4, 22–24, 27]):
(1) (Associativity of addition) $(X+Y)+Z=X+(Y+Z)$ for all $X,Y,Z\in\mathbb{R}_{\mathcal{I}}$,
(2) (Additive element) $X+0=0+X=X$ for all $X\in\mathbb{R}_{\mathcal{I}}$,
(3) (Commutativity of addition) $X+Y=Y+X$ for all $X,Y\in\mathbb{R}_{\mathcal{I}}$,
(4) (Cancelation law) $X+Z=Y+Z\Longrightarrow X=Y$ for all $X,Y,Z\in\mathbb{R}_{\mathcal{I}}$,
(5) (Associativity of multiplication) $(X.Y).Z=X.(Y.Z)$ for all $X,Y,Z\in\mathbb{R}_{\mathcal{I}}$,
(6) (Commutativity of multiplication) $X.Y=Y.X$ for all $X,Y\in\mathbb{R}_{\mathcal{I}}$,
(7) (Unit element) $X.1=1.X$ for all $X\in\mathbb{R}_{\mathcal{I}}$,
(8) (Associate law) $\lambda(\mu X)=(\lambda\mu)X$ for all $X\in\mathbb{R}_{\mathcal{I}}$ and all $\lambda,\mu\in\mathbb{R}$,
(9) (First distributive law) $\lambda(X+Y)=\lambda X+\lambda Y$ for all $X,Y\in\mathbb{R}_{\mathcal{I}}$ and all $\lambda\in\mathbb{R}$,
(10) (Second distributive law) $(\lambda+\mu)X=\lambda X+\mu X$ for all $X\in\mathbb{R}_{\mathcal{I}}$ and all $\lambda,\mu\in\mathbb{R}$.
Besides these properties, the distributive law is not always valid for intervals. For example, $X=[1,2], Y=[2,3]$, $Z=[-2,-1]$

$$X.(Y+Z)=[0,4]$$

whereas

$$X.Y+X.Z=[-2,5].$$

But, this law hold in certain cases. If $YZ>0$, then

$$X.(Y+Z)=X.Y+X.Z.$$

What's more, one of the set property is the inclusion "$\subseteq$" that is given by

$$X\subseteq Y\Longleftrightarrow \underline{Y}\leq\underline{X} \text{ and } \overline{X}\leq\overline{Y}.$$

Considering together with arithmetic operations and inclusion, one has the following property which is called inclusion isotony of interval operations:
Let $\odot$ be the addition, multiplication, subtraction or division. If X,Y,Z and T are intervals such that

$$X\subseteq Y \quad \text{and } Z\subseteq T,$$

then the following relation is valid

$$X\odot Z\subseteq Y\odot T.$$

3. *Integral of Interval-Valued Functions

In this section, we define *integral or multiplicative integral for the interval-valued functions and give properties of this new integral. Throughout in this section, we shall use $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$ is positive interval-valued function, IR is the notation for the interval-valued integrals and I^* means the multiplicative integral.
First, we recall that the concept of *integral is denoted by $\int_a^b (F(x))^{dx}$ which introduced by Bashirov et al. in [5]. In multiplicative integrals we replace the sum by product and the product by raising to power of a function F on $[a,b]$.
We give the following relation between Riemann integral and *integral:

Proposition 1. *If a positive function F is Riemann integrable on $[a,b]$, then F is *integrable on $[a,b]$ and*

$$\int_a^b (F(x))^{dx} = e^{\int_a^b \ln(F(x))dx}.$$

*For further details of *integral reader can read [5].*

Now we recall the concept of Interval-valued integral given by R. E. Moore in [28]. Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$. The interval-valued Riemann integral of function F is defined by

$$\int_a^b F(x)\,dx = \int_a^b \left[\underline{F}(x), \overline{F}(x)\right] dx.$$

Let's define interval-valued *integral or multiplicative integral (I^*R):
A function F is said to be an interval-valued function of t on $[a,b]$ if it assigns a nonempty interval to each $t \in [a,b]$

$$F(t) = \left[\underline{F}(t), \overline{F}(t)\right].$$

A partition of $[a,b]$ is any finite ordered subset $\mathcal{P}$ having the form

$$\mathcal{P} : a = t_0 < t_1 < ... < t_n = b.$$

The mesh of a partition $\mathcal{P}$ is defined by

$$mesh(\mathcal{P}) = \max\{t_i - t_{i-1} : i = 1, 2, ..., n\}.$$

We denote by $\mathcal{P}([a,b])$ the set of all partition of $[a,b]$. Let $\mathcal{P}(\delta,[a,b])$ be the set of all $\mathcal{P}_1 \in \mathcal{P}([a,b])$ such that $mesh(\mathcal{P}) < \delta$. Choose an arbitrary point ξ_i in interval $[t_{i-1}, t_i]$, $i = 1, 2, ..., n$ and we define the product

$$P(F, \mathcal{P}_1, \delta) = \prod_{i=1}^{n} F(\xi_i)^{[t_i - t_{i-1}]}$$

where $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ is a positive function. We call $P(F, \mathcal{P}_1, \delta)$ a Riemann product of F corresponding to $\mathcal{P}_1 \in \mathcal{P}(\delta,[a,b])$.

Definition 1. A positive function $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ is said to be integrable in multiplicative sense or *integrable (I^*Rintegrable) on $[a,b]$ if there exists $A \in \mathbb{R}_I$ such that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d\left(P(F, \mathcal{P}_1, \delta), A\right) < \varepsilon$$

for every Riemann product P of F corresponding to each $\mathcal{P}_1 \in \mathcal{P}(\delta,[a,b])$ and independent of choice of $\xi_i \in [t_{i-1}, t_i]$ for $1 \le i \le n$. In this case, A is called the I^*R-integral of F on $[a,b]$ and is denoted by

$$A = (I^*R)\int_a^b (F(t))^{dt}.$$

The collection of all functions that are I^*R integrable on $[a,b]$ will be denote by $\mathcal{I}^*\mathcal{R}_{([a,b])}$.

The following theorem gives relation between I^*R-integral and multiplicative integral (I^*-integral):

Theorem 1. *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be a positive interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right] \in \mathcal{I}^*\mathcal{R}_{([a,b])}$ if and only if $\underline{F}(t)$, $\overline{F}(t) \in \mathcal{I}^*_{([a,b])}$ and*

$$(I^*R)\int_a^b (F(t))^{dt} = \left[(I^*)\int_a^b \left(\underline{F}(t)\right)^{dt}, (I^*)\int_a^b \left(\overline{F}(t)\right)^{dt}\right]$$

*where $\mathcal{I}^*_{([a,b])}$ denotes the all *integrable functions.*

It is seen easily that if $F(t) \subseteq \mathcal{G}(t)$ for all $t \in [a,b]$, then $(I^*R)\int_a^b (F(t))^{dt} \subseteq (I^*R)\int_a^b (\mathcal{G}(t))^{dt}$.

It is very easy to notice that if positive function F is interval-valued integrable ($IR-$integrable), then F is I^*R integrable and

$$(I^*R)\int_a^b (F(t))^{dt} = e^{\int_a^b (\ln\circ F)(t)dt}.$$

As we know that $\ln\circ F$ is (IR) integrable on $[a,b]$ and continuity of the exponential we have

$$P(F,\mathcal{P}_1,\delta) = e^{\sum\limits_{i=1}^{n}(t_i-t_{i-1})(\ln\circ F)(\xi_i)}$$

imply the above statement and conversely, we have

$$\int_a^b F(t)\,dt = \ln\int_a^b \left(e^{F(t)}\right)^{dt}.$$

Example 1. A positive interval-valued function $F : [1,2] \to \mathbb{R}_I$ be defined by

$$F(t) = \left[t^2, e^{t^2}\right],$$

then F is interval-valued *integrable on $[1,2]$ and $\int_1^2 \left(\left[t^2, e^{t^2}\right]\right)^{dt} = e^{\int_1^2 [2\ln t, t^2]dt} = [2.1651, 10.3119]$.

Now we give some properties of *integral for interval valued functions. We consider F and $\mathcal{G}$ are positive interval-valued functions then the following equalities hold:

(1) $\int_a^b (F(t)^p)^{dt} = \left(\int_a^b (F(t))^{dt}\right)^p$

(2) $\int_a^b (F(t)\mathcal{G}(t))^{dt} = \int_a^b (F(t))^{dt} \cdot \int_a^b (\mathcal{G}(t))^{dt}$

(3) $\int_a^b \left(\frac{F(t)}{\mathcal{G}(t)}\right)^{dt} = \frac{\int_a^b (F(t))^{dt}}{\int_a^b (\mathcal{G}(t))^{dt}}$

(4) $\int_a^b (F(t))^{dt} = \int_a^c (F(t))^{dt} . \int_c^b (F(t))^{dt}$, where $a \leq c \leq b$.

Proof. Now we give the proofs of above properties.

(1) $\Longrightarrow$

$$\begin{aligned} \int_a^b (F(t)^p)^{dt} &= e^{\int_a^b \ln(F(t)^p)dt} \\ &= \left(e^{\int_a^b \ln(F(t))dt}\right)^p \\ &= \left(\int_a^b (F(t))^{dt}\right)^p \end{aligned}$$

(2) $\Longrightarrow$

$$\begin{aligned} \int_a^b (F(t)\,\mathcal{G}(t))^{dt} &= e^{\int_a^b \ln(F(t)\mathcal{G}(t))dt} \\ &= e^{\int_a^b \ln(F(t))dt+\int_a^b \ln(\mathcal{G}(t))dt} \\ &= \int_a^b (F(t))^{dt} . \int_a^b (\mathcal{G}(t))^{dt} \end{aligned}$$

(3) $\Longrightarrow$

$$\begin{aligned} \int \left(\frac{F(t)}{\mathcal{G}(t)}\right)^{dt} &= e^{\int_a^b \ln\left(\frac{F(t)}{\mathcal{G}(t)}\right)dt} \\ &= e^{\int_a^b \ln(F(t))dt-\int_a^b \ln(\mathcal{G}(t))dt} \\ & \frac{\int_a^b (F(t))^{dt}}{\int_a^b (\mathcal{G}(t))^{dt}} \end{aligned}$$

(4) $\Longrightarrow$

$$\begin{aligned} \int_a^b (F(t))^{dt} &= e^{\int_a^b \ln(F(t))dt} \\ &= e^{\int_a^c \ln(F(t))dt+\int_c^d \ln(F(t))dt} \\ &= \int_a^c (F(t))^{dt} . \int_c^b (F(t))^{dt} . \end{aligned}$$

□

4. Logarithmically Interval-valued Convex Functions

In [42], Zhao et al. introduced a kind of interval-valued convex function as follows:

Definition 2. Let $h : [c, d] \to \mathbb{R}$ be a non-negative function, $(0, 1) \subseteq [c, d]$ and $h \neq 0$. We say that $F : [a, b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is an interval-valued h-convex function if for all $x, y \in [a, b]$ and $t \in (0, 1)$, we have

$$h(t)F(x) + h(1-t)F(y) \subseteq F(tx + (1-t)y). \tag{4.1}$$

$SX(h, [a, b], \mathbb{R}_{\mathcal{I}}^{+})$ will show the set of all interval-valued h-convex functions.

The usual notion of convex interval-valued function corresponds to relation (4.1) with $h(t) = t$ [35]. Also, if $h(t) = t^s$ in (4.1), then Definition 2 gives the other interval-valued convex function defined by Breckner [6].

Definition 3. [30]Let $h : [c, d] \to \mathbb{R}$ be a non-negative function, $(0, 1) \subseteq [c, d]$ and $h \neq 0$. A function $F : [a, b] \to (0, \infty)$ is called log-$h-$convex function, if

$$F\left(tx + (1-t)\, y\right) \leq \left[F\left(x\right)\right]^{h(t)} \left[F\left(y\right)\right]^{h(1-t)},$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. For brevity, we can say that $F \in SXL[h, [a, b], \mathbb{R}_{I}^{+}]$ instead of logarithmically interval-valued-h-convex function.

In [20], Guo et al. gave the concept of interval-valued log-h-convex functions as follows:

Definition 4. Let $h : [c, d] \to \mathbb{R}$ be a non-negative function, $(0, 1) \subseteq [c, d]$ and $h \neq 0$. A function $F : [a, b] \to \mathbb{R}_{I}^{+}$ is said to be logarithmically interval-valued-h-convex function if for all $x, y \in [a, b]$ and $t \in [0, 1]$, we have

$$\left[F\left(x\right)\right]^{h(t)} \left[F\left(y\right)\right]^{h(1-t)} \subseteq F(tx + (1-t)y).$$

Remark 1. If we set $h(t) = t$ in Definition 4, then we have new definition of interval-valued convex function which is called log-interval-valued convex function.

Definition 5. A function $F : [a, b] \to \mathbb{R}_{I}^{+}$ is said to be log-interval-valued convex function if for all $x, y \in [a, b]$ and $t \in [0, 1]$, we have

$$\left[F\left(x\right)\right]^{t} \left[F\left(y\right)\right]^{1-t} \subseteq F(tx + (1-t)y).$$

Remark 2. If we use $h\left(t\right) = t^s$ in Definition 4, then we have a new definition of interval-valued convex function which is called s-logarithmically interval-valued convex function.

Definition 6. A function $F : [a, b] \to \mathbb{R}_{I}^{+}$ is said to be s-log-interval-valued convex function if for all $x, y \in [a, b]$, $s \in [0, 1]$ and $t \in [0, 1]$, we have

$$\left[F\left(x\right)\right]^{t^s} \left[F\left(y\right)\right]^{(1-t)^s} \subseteq F(tx + (1-t)y).$$

Remark 3. If we use $h\left(t\right) = 1$ in Definition 4, then we have a new definition of interval-valued convex function which is called logarithmically interval–valued P-convex function.

Definition 7. A function $F:[a,b]\to\mathbb{R}_I^+$ is said to be log-interval-valued P-convex function if for all $x,y\in[a,b]$, and $t\in[0,1]$, we have

$$[F(x)][F(y)]\subseteq F(tx+(1-t)y).$$

Remark 4. If we put $h(t)=\frac{1}{t}$ in Definition 4, then we have a new definition of interval-valued convex function which is called logarithmically interval-valued Q-convex function.

Definition 8. A function $F:[a,b]\to\mathbb{R}_I^+$ is said to be log-interval-valued Q-convex function if for all $x,y\in[a,b]$, and $t\in[0,1]$, we have

$$[F(x)]^{\frac{1}{t}}[F(y)]^{\frac{1}{1-t}}\subseteq F(tx+(1-t)y).$$

Remark 5. If we put $h(t)=\frac{1}{t^s}$ in Definition 4, then we have a new definition of interval-valued convex function which is called s-logarithmically interval-valued Q-convex function.

Definition 9. A function $F:[a,b]\to\mathbb{R}_I^+$ is said to be s-log-interval-valued Q-convex function if for all $x,y\in[a,b]$, $s\in[0,1]$ and $t\in[0,1]$, we have

$$[F(x)]^{\frac{1}{t^s}}[F(y)]^{\frac{1}{(1-t)^s}}\subseteq F(tx+(1-t)y).$$

Remark 6. If we put $h(t)=h\left(\frac{1}{2}\right)$ in Definition 4, then we have a new definition of interval-valued convex function which is called logarithmically interval-valued jensen type convex function.

Definition 10. A function $F:[a,b]\to\mathbb{R}_I^+$ is said to be log-interval-valued jensen type convex function if for all $x,y\in[a,b]$ and $t\in[0,1]$, we have

$$[F(x)]^{h\left(\frac{1}{2}\right)}[F(y)]^{h\left(\frac{1}{2}\right)}\subseteq F(tx+(1-t)y).$$

Proposition 2. *If $F,\mathcal{G}\in SXL[h,[a,b],\mathbb{R}_I^+]$, then $F\mathcal{G}\in SXL[h,[a,b],\mathbb{R}_I^+]$.*

5. Hermite-Hadamard inequalities

Theorem 2. *Let $F:[a,b]\to\mathbb{R}_\mathcal{I}^+$, $h:[0,1]\to\mathbb{R}^+$ and $h\left(\frac{1}{2}\right)\neq 0$. If $F\in SXL\left(h,\ [a,b],\ \mathbb{R}_\mathcal{I}^+\right)$ and $F\in\mathcal{I}^*\mathcal{R}_{([a,b])}$, then following double inequality holds:*

$$\left[F\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}}\supseteq\left(\int_a^b F(x)^{dx}\right)^{\frac{1}{b-a}}\supseteq[F(a)F(b)]^{\int_0^1 h(t)dt}. \tag{5.1}$$

Proof. Since $F\in SXL\left(h,\ [a,b],\ \mathbb{R}_\mathcal{I}^+\right)$, then we have

$$F\left(\frac{x+y}{2}\right)\supseteq[F(x)]^{h\left(\frac{1}{2}\right)}[F(y)]^{h\left(\frac{1}{2}\right)}.$$

By setting $x=ta+(1-t)b$ and $y=tb+(1-t)a$, we get

$$\ln F\left(\frac{a+b}{2}\right)\supseteq h\left(\frac{1}{2}\right)[\ln F(ta+(1-t)b)+\ln F(tb+(1-t)a)]. \tag{5.2}$$

Integrating inequality (5.2) with respect to t over $[0,1]$, we have

$$\begin{aligned}\ln F\left(\frac{a+b}{2}\right) &\supseteq h\left(\frac{1}{2}\right)\left[\int_0^1 \ln F\left(ta+(1-t)\,b\right)dt\right.\\ &\left.+\int_0^1 \ln F\left(tb+(1-t)\,a\right)dt\right]\end{aligned}$$

and by changing the variable of integration, we have

$$\ln F\left(\frac{a+b}{2}\right) \supseteq \frac{2h\left(\frac{1}{2}\right)}{b-a}\int_a^b \ln F\left(x\right)dx$$

which implies that

$$\ln\left[F\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}} \supseteq \frac{1}{b-a}\int_a^b \ln F\left(x\right)dx.$$

Hence

$$\begin{aligned}\left[F\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}} &\supseteq \left(e^{\int_a^b \ln F(x)dx}\right)^{\frac{1}{b-a}}\\ &= \left(\int_a^b \left(F\left(x\right)\right)^{dx}\right)^{\frac{1}{b-a}}\end{aligned}$$

which is the first inequality in (5.1).

Now we have to prove second inequality in (5.1), for this first we note that $F \in SXL\left(h,\ [a,b],\ \mathbb{R}_{\mathcal{I}}^{+}\right)$, so we have

$$\ln F\left(ta+(1-t)\,b\right) \supseteq h\left(t\right)\ln F\left(a\right)+h\left(1-t\right)\ln F\left(b\right). \tag{5.3}$$

Integrating inequality (5.3) with respect to t over $[0,1]$ and change the variable of integration we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b \ln F\left(x\right)dx &\supseteq \left[\ln F\left(a\right)+\ln F\left(b\right)\right]\int_0^1 h\left(t\right)dt\\ &= \ln\left(F\left(a\right)F\left(b\right)\right)^{\int_0^1 h(t)dt}\end{aligned}$$

which implies that

$$\left(e^{\int_a^b \ln F(x)dx}\right)^{\frac{1}{b-a}} \supseteq \left(F\left(a\right)F\left(b\right)\right)^{\int_0^1 h(t)dt}.$$

Thus

$$\left(\int_a^b \left(F\left(x\right)\right)^{dx}\right)^{\frac{1}{b-a}} \supseteq \left(F\left(a\right)F\left(b\right)\right)^{\int_0^1 h(t)dt}$$

which is the second inequality in (5.1).

The proof of the theorem is completed. □

Remark 7. If $\underline{F} = \overline{F}$, then our Theorem 2 reduces to [30, Theorem 3.1].

Corollary 1. *Under the assumptions of Theorem 2, if we put $h(t) = t$, then we have following new Hermite-Hadamard inequality for log-interval-valued convex functions*

$$F\left(\frac{a+b}{2}\right) \supseteq \left(\int_a^b (F(x))^{dx}\right)^{\frac{1}{b-a}} \supseteq G(F(a), F(b)) \tag{5.4}$$

where $G(F(a), F(b))$ referes to the geometric mean of $F(a)$ and $F(b)$.

Example 2. A function $F : [x, y] \to \mathbb{R}_I^+$ with $0 < x < y$ is defined as

$$F(x) = \left[\underline{F}(x), \overline{F}(x)\right] = \left[\frac{1}{x}, e^x - 1\right],$$

then $F \in SXL\left([a, b],\ \mathbb{R}_{\mathcal{I}}^+\right)$ and we have

$$\left(\int_a^b (F(x))^{dx}\right)^{\frac{1}{b-a}} = e^{\frac{1}{b-a}\left[\ln\left(\frac{b^b}{a^a}\right)^{-1}+1, \frac{b^2-a^2}{2}\right]}.$$

On the other hand

$$F\left(\frac{a+b}{2}\right) = \left[\frac{2}{a+b}, e^{\frac{a+b}{2}} - 1\right],$$

and

$$\begin{aligned} G(F(a), F(b)) &= \sqrt{F(a)F(b)} \\ &= \left[\sqrt{\frac{1}{ab}}, \sqrt{(e^a - 1)(e^b - 1)}\right]. \end{aligned}$$

Thus for all $a, b \in [x, y]$ inequalities (5.4) satisfies and we have

$$\left[\frac{2}{a+b}, e^{\frac{a+b}{2}} - 1\right] \supseteq e^{\frac{1}{b-a}\left[\ln\left(\frac{b^b}{a^a}\right)^{-1}+1, \frac{b^2-a^2}{2}\right]} \supseteq \left[\sqrt{\frac{1}{ab}}, \sqrt{(e^a - 1)(e^b - 1)}\right]$$

for verification we suppose $a = 1$ and $b = 2$, then we have

$$[0.6666, 3.4816] \supseteq [0.6796, 3.4260] \supseteq [0.7071, 3.3131].$$

Remark 8. If $\underline{F} = \overline{F}$ in Corollary 1, then we have inequality for log-convex function which can be found in [14, p. 197, (5.3)].

Corollary 2. *Under the assumptions of Theorem 2, if we let $h(t) = t^s$, then we have a new inequality of s-logarithmically interval-valued convex functions*

$$\left[F\left(\frac{a+b}{2}\right)\right]^{2^{s-1}} \supseteq \left(\int_a^b (F(x))^{dx}\right)^{\frac{1}{b-a}} \supseteq [F(a)F(b)]^{\frac{1}{s+1}}.$$

Remark 9. If $\underline{F} = \overline{F}$ in Corollary 2, then Corollary 2 reduces to [30, Corollary 3.3].

Corollary 3. *Under the conditions of Theorem 2, if we use $h(t) = 1$, then we have a new inequality for the logarithmically interval-valued P-convex functions*

$$\sqrt{F\left(\frac{a+b}{2}\right)} \supseteq \left(\int_a^b (F(x))^{dx}\right)^{\frac{1}{b-a}} \supseteq F(a)\,F(b)\,.$$

Remark 10. If $\underline{F} = \overline{F}$ in Corollary 3, then Corollary 3 reduces to [30, Corollary 3.4].

Corollary 4. *Under the assumptions of Theorem 2, if we set $h(t) = t^{-1}$ in first inequality, then we have following inequality for the logarithmically interval-valued Q-convex functions*

$$\left[F\left(\frac{a+b}{2}\right)\right]^{\frac{1}{4}} \supseteq \left(\int_a^b (F(x))^{dx}\right)^{\frac{1}{b-a}}.$$

Remark 11. If $\underline{F} = \overline{F}$ in Corollary 4, Then Corollary 4 reduces to [30, Corollary 3.5].

Corollary 5. *Under the conditions of Theorem 2, if suppose $h(t) = t^{-s}$ in first inequality, then we have new inequality for s-logarithmically interval-valued Q-convex functions*

$$\left[F\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2^{s+1}}} \supseteq \left(\int_a^b (F(x))^{dx}\right)^{\frac{1}{b-a}}.$$

Remark 12. If $\underline{F} = \overline{F}$ in Corollary 5, then we have following new inequality for the s-logarithmically Q-convex functions

$$\left[F\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2^{s+1}}} \leq \left(\int_a^b (F(x))^{dx}\right)^{\frac{1}{b-a}}.$$

Theorem 3. *Let $F, \mathcal{G} : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$, $h : [0,1] \to \mathbb{R}^{+}$ and $h\left(\frac{1}{2}\right) \neq 0$. If $F, \mathcal{G} \in SXL\left(h,\ [a,b],\ \mathbb{R}_{\mathcal{I}}^{+}\right)$ and $F, \mathcal{G} \in \mathcal{I}^{*}\mathcal{R}_{([a,b])}$, then following double inequality holds:*

$$\begin{aligned}\left[F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}} \quad &\supseteq \quad \left(\int_a^b F(x)^{dx} \cdot \int_a^b \mathcal{G}(x)^{dx}\right)^{\frac{1}{b-a}} \\ &\supseteq \quad \left[F(a)\,F(b)\mathcal{G}(a)\,\mathcal{G}(b)\right]^{\int_0^1 h(t)dt}. \qquad (5.5)\end{aligned}$$

Proof. Since $F, \mathcal{G} \in SXL\left(h,\ [a,b],\ \mathbb{R}_{\mathcal{I}}^{+}\right)$, so F and $\mathcal{G}$ can be written as

$$F\left(\frac{x+y}{2}\right) \supseteq [F(x)]^{h\left(\frac{1}{2}\right)}\,[F(y)]^{h\left(\frac{1}{2}\right)} \qquad (5.6)$$

and

$$\mathcal{G}\left(\frac{x+y}{2}\right) \supseteq [\mathcal{G}(x)]^{h\left(\frac{1}{2}\right)}\,[\mathcal{G}(y)]^{h\left(\frac{1}{2}\right)}. \qquad (5.7)$$

By setting $x = ta + (1-t)\,b$ and $y = tb + (1-t)\,a$ in (5.6) and (5.7), we have

$$F\left(\frac{a+b}{2}\right) \supseteq \left[F\left(ta + (1-t)\,b\right)\right]^{h\left(\frac{1}{2}\right)} \left[F\left(tb + (1-t)\,a\right)\right]^{h\left(\frac{1}{2}\right)} \tag{5.8}$$

and

$$\mathcal{G}\left(\frac{a+b}{2}\right) \supseteq \left[\mathcal{G}\left(ta + (1-t)\,b\right)\right]^{h\left(\frac{1}{2}\right)} \left[\mathcal{G}\left(tb + (1-t)\,a\right)\right]^{h\left(\frac{1}{2}\right)}. \tag{5.9}$$

From (5.8) and (5.9), we have

$$\begin{aligned} & \ln F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right) \\ \supseteq \quad & h\left(\frac{1}{2}\right)\left[\ln F\left(ta + (1-t)\,b\right) + \ln F\left(tb + (1-t)\,a\right)\right. \\ & \left. + \ln \mathcal{G}\left(ta + (1-t)\,b\right) + \ln \mathcal{G}\left(tb + (1-t)\,a\right)\right] \end{aligned} \tag{5.10}$$

and integrating (5.10) with respect to t over $[0,1]$, we have

$$\begin{aligned} & \ln F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right) \\ \supseteq \quad & h\left(\frac{1}{2}\right)\left[\int_0^1 \ln F\left(ta + (1-t)\,b\right) dt + \int_0^1 \ln F\left(tb + (1-t)\,a\right) dt\right. \\ & \left. + \int_0^1 \ln \mathcal{G}\left(ta + (1-t)\,b\right) dt + \int_0^1 \ln \mathcal{G}\left(tb + (1-t)\,a\right) dt\right]. \end{aligned} \tag{5.11}$$

By changing the variable in last inequality, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} \ln F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}\left[\int_a^b \ln F\left(x\right) dx + \int_a^b \ln \mathcal{G}\left(x\right) dx\right]$$

which implies that

$$\left[F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}} \supseteq \left(e^{\int_a^b \ln F(x)dx + \int_a^b \ln \mathcal{G}(x)dx}\right)^{\frac{1}{b-a}}.$$

Hence

$$\left[F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}} \supseteq \left(\int_a^b F\left(x\right)^{dx} \cdot \int_a^b \mathcal{G}\left(x\right)^{dx}\right)^{\frac{1}{b-a}}$$

which is the first inequality in (5.5).
To prove the second inequality in (5.5), first we note that $F, \mathcal{G} \in SXL\left(h,\ [a,b],\ \mathbb{R}_{\mathcal{I}}^{+}\right)$, so we have

$$F\left(ta + (1-t)\,b\right) \supseteq \left[F\left(a\right)\right]^{h(t)} \left[F\left(b\right)\right]^{h(1-t)} \tag{5.12}$$

and

$$\mathcal{G}\left(ta + (1-t)\,b\right) \supseteq \left[\mathcal{G}\left(a\right)\right]^{h(t)} \left[\mathcal{G}\left(b\right)\right]^{h(1-t)}. \tag{5.13}$$

From (5.12) and (5.13), we have

$$\begin{aligned} & \ln F(ta+(1-t)b)\,\mathcal{G}(ta+(1-t)b) \\ \supseteq\ & h(t)[\ln F(a)+\ln \mathcal{G}(a)]+h(1-t)[\ln F(b)+\ln \mathcal{G}(b)] \end{aligned} \tag{5.14}$$

and integrating inequality (5.14) with respect to t over $[0,1]$, we get

$$\begin{aligned} & \int_0^1 \ln F(ta+(1-t)b)\,\mathcal{G}(ta+(1-t)b) \\ \supseteq\ & [\ln F(a)+\ln \mathcal{G}(a)]\int_0^1 h(t)\,dt \\ & +[\ln F(b)+\ln \mathcal{G}(b)]\int_0^1 h(1-t)\,dt. \end{aligned}$$

By changing the variable of integration, we have

$$\frac{1}{b-a}\int_0^1 \ln F(x)\,\mathcal{G}(x)\,dx \supseteq \ln[F(a)F(b)\mathcal{G}(a)\mathcal{G}(b)]^{\int_0^1 h(t)dt}$$

which implies that

$$\left(e^{\int_a^b \ln F(x)dx+\int_a^b \ln \mathcal{G}(x)dx}\right)^{\frac{1}{b-a}} \supseteq [F(a)F(b)\mathcal{G}(a)\mathcal{G}(b)]^{\int_0^1 h(t)dt}.$$

Thus

$$\left(\int_a^b F(x)^{dx}\cdot\int_a^b \mathcal{G}(x)^{dx}\right)^{\frac{1}{b-a}} \supseteq [F(a)F(b)\mathcal{G}(a)\mathcal{G}(b)]^{\int_0^1 h(t)dt}$$

which is the second inequality in (5.5).
The proof of theorem is completed. □

Corollary 6. *If we use $h(t)=t$ in Theorem 3, then following new inequalities for logarithmically interval-valued convex functions hold:*

$$\begin{aligned} F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right) &\supseteq \left(\int_a^b F(x)^{dx}\cdot\int_a^b \mathcal{G}(x)^{dx}\right)^{\frac{1}{b-a}} \\ &\supseteq G(F(a),F(b)).G(\mathcal{G}(a),\mathcal{G}(b)) \end{aligned}$$

where $G(.,.)$ referes to the geometric mean.

Remark 13. If $\underline{F}=\overline{F}$ in Corollary 6, then Corollary 6 reduces to [3, Theorem 7].

Corollary 7. *Under the assumptions of Theorem 3, if we let $h(t)=t^s$, then following inequalities for the s-logarithmically interval-valued convex functions hold:*

$$\left[F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right)\right]^{2^{s-1}} \supseteq \left(\int_a^b F(x)^{dx}\cdot\int_a^b \mathcal{G}(x)^{dx}\right)^{\frac{1}{b-a}}$$

$$\supseteq \quad [F(a)F(b)\mathcal{G}(a)\mathcal{G}(b)]^{\frac{1}{s+1}}.$$

Corollary 8. *Under the conditions of Theorem 3, if we choose $h(t)=1$, then we have following inequalities for the logarithmically P-convex functions*

$$\begin{aligned} F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right) &\supseteq \left(\int_a^b F(x)^{dx}.\int_a^b \mathcal{G}(x)^{dx}\right)^{\frac{2}{b-a}} \\ &\supseteq [F(a)F(b)\mathcal{G}(a)\mathcal{G}(b)]^2. \end{aligned}$$

Corollary 9. *If we put $h(t)=t^{-1}$ in first inequality of Theorem 3, then we have following inequalities for the logarithmically interval-valued Q-convex functions*

$$\left[F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right)\right]^{\frac{1}{4}} \supseteq \left(\int_a^b F(x)^{dx}.\int_a^b \mathcal{G}(x)^{dx}\right)^{\frac{1}{b-a}}.$$

Corollary 10. *Under the assumptions of Theorem 3, if we set $h(t)=t^{-s}$ in first inequality, then following inequalities for s-logarithmically interval-valued Q-convex functions hold*

$$\left[F\left(\frac{a+b}{2}\right)\mathcal{G}\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2^{s+1}}} \supseteq \left(\int_a^b F(x)^{dx}.\int_a^b \mathcal{G}(x)^{dx}\right)^{\frac{1}{b-a}}.$$

Theorem 4. *Let $F:[a,b]\to\mathbb{R}_{\mathcal{I}}^{+}$, $h:[0,1]\to\mathbb{R}^{+}$ and $h\left(\frac{1}{2}\right)\neq 0$. If $F\in SXL\left(h,\ [a,b],\ \mathbb{R}_{\mathcal{I}}^{+}\right)$ and $F\in\mathcal{I}^*\mathcal{R}_{([a,b])}$, then following inequalities hold:*

$$\begin{aligned} \left(F\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2h^2\left(\frac{1}{2}\right)}} &\supseteq \Delta_1^{\frac{1}{4h\left(\frac{1}{2}\right)}} \\ &\supseteq \left(\int_a^b F(x)^{dx}\right)^{\frac{1}{b-a}} \\ &\supseteq \Delta_2^{\frac{1}{2}\int_0^1 h(t)dt} \\ &\supseteq (F(a)F(b))^{\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(t)dt}, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= F\left(\frac{3a+b}{4}\right)F\left(\frac{a+3b}{4}\right) \\ \Delta_2 &= F(a)F(b)F^2\left(\frac{a+b}{2}\right). \end{aligned}$$

Proof. Since $F\in SXL\left(h,\ [a,b],\ \mathbb{R}_{\mathcal{I}}^{+}\right)$ then for $t\in[a,\frac{a+b}{2}]$, we have

$$\ln F\left(\frac{3a+b}{4}\right)$$

$$\begin{aligned}
&= \ln F\left(\frac{ta+(1-t)\,\frac{a+b}{2}}{2}+\frac{(1-t)\,a+t\frac{a+b}{2}}{2}\right) \\
&\supseteq \ln F\left(\frac{ta+(1-t)\,\frac{a+b}{2}}{2}\right)^{h\left(\frac{1}{2}\right)} F\left(\frac{(1-t)\,a+t\frac{a+b}{2}}{2}\right)^{h\left(\frac{1}{2}\right)} \\
&= h\left(\frac{1}{2}\right)\left[\ln F\left(\frac{ta+(1-t)\,\frac{a+b}{2}}{2}\right)+\ln F\left(\frac{(1-t)\,a+t\frac{a+b}{2}}{2}\right)\right]. \quad (5.15)
\end{aligned}$$

Integrating inequality (5.15) with respect to t over $[0,1]$, we have

$$\begin{aligned}
\ln F\left(\frac{3a+b}{2}\right) \supseteq h\left(\frac{1}{2}\right)&\left[\int_0^1 \ln F\left(\frac{ta+(1-t)\,\frac{a+b}{2}}{2}\right)dt\right. \\
&\left.+\int_0^1 \ln F\left(\frac{(1-t)\,a+t\frac{a+b}{2}}{2}\right)dt\right]
\end{aligned}$$

and by changing the variable of integration, we have

$$\ln\left(F\left(\frac{3a+b}{4}\right)\right)^{\frac{1}{4h\left(\frac{1}{2}\right)}} \supseteq \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \ln F(x)\,dx$$

which implies that

$$\left(F\left(\frac{3a+b}{4}\right)\right)^{\frac{1}{4h\left(\frac{1}{2}\right)}} \supseteq \left(\int_a^{\frac{a+b}{2}} F(x)^{dx}\right)^{\frac{1}{b-a}}. \quad (5.16)$$

Similarly, we have

$$\left(F\left(\frac{a+3b}{4}\right)\right)^{\frac{1}{4h\left(\frac{1}{2}\right)}} \supseteq \left(\int_{\frac{a+b}{2}}^b F(x)^{dx}\right)^{\frac{1}{b-a}}. \quad (5.17)$$

Multiplying (5.16) and (5.17), we have

$$\Delta_1^{\frac{1}{4h\left(\frac{1}{2}\right)}} = \left(F\left(\frac{3a+b}{4}\right)F\left(\frac{a+3b}{4}\right)\right)^{\frac{1}{4h\left(\frac{1}{2}\right)}} \supseteq \left(\int_a^b F(x)^{dx}\right)^{\frac{1}{b-a}}.$$

Now from Theorem 2, one has

$$\begin{aligned}
\left(F\left(\frac{a+b}{2}\right)\right)^{\frac{1}{4h^2\left(\frac{1}{2}\right)}} &= \left[F\left(\frac{1}{2}\frac{3a+b}{4}+\frac{1}{2}\frac{a+3b}{4}\right)\right]^{\frac{1}{4h^2\left(\frac{1}{2}\right)}} \\
&\supseteq \left[F\left(\frac{3a+b}{4}\right)^{h\left(\frac{1}{2}\right)} F\left(\frac{a+3b}{4}\right)^{h\left(\frac{1}{2}\right)}\right]^{\frac{1}{4h^2\left(\frac{1}{2}\right)}} \\
&\supseteq \Delta_1^{\frac{1}{4h\left(\frac{1}{2}\right)}}
\end{aligned}$$

$$
\begin{aligned}
&\supseteq \left(\int_a^b F(x)^{dx}\right)^{\frac{1}{b-a}} \\
&\supseteq \left(F(a)F(b)F^2\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2}\int_0^1 h(t)dt} \\
&= \Delta_2^{\frac{1}{2}\int_0^1 h(t)dt} \\
&\supseteq \left([F(a)F(b)]^{\frac{1}{2}}\,[F(a)F(b)]^{h\left(\frac{1}{2}\right)}\right)^{\int_0^1 h(t)dt} \\
&= (F(a)F(b))^{\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(t)dt}
\end{aligned}
$$

which completes the proof. □

Corollary 11. *Under the assumptions of Theorem (4), if we set $h(t) = t$ in Theorem 4, then following inequalities for the logarithmically interval-valued convex functions hold*

$$
\begin{aligned}
\left(F\left(\frac{a+b}{2}\right)\right)^2 &\supseteq \sqrt{\Delta_1} \supseteq \left(\int_a^b F(x)^{dx}\right)^{\frac{1}{b-a}} \\
&\supseteq \Delta_2^{\frac{1}{4}} \supseteq G(F(a), F(b))
\end{aligned}
\tag{5.18}
$$

where $G(.,.)$ referes to the geometric mean.

Example 3. A function $F : [x, y] \to \mathbb{R}_I^+$ with $0 < x < y$ is defined as

$$F(x) = \left[\underline{F}(x), \overline{F}(x)\right] = \left[\frac{1}{x}, e^x - 1\right],$$

then $F \in SXL\left([a, b],\ \mathbb{R}_{\mathcal{I}}^+\right)$ and we have

$$\left(\int_a^b (F(x))^{dx}\right)^{\frac{1}{b-a}} = e^{\frac{1}{b-a}\left[\ln\left(\frac{b^b}{a^a}\right)^{-1}+1,\frac{b^2-a^2}{2}\right]}.$$

On the other hand

$$
\begin{aligned}
F\left(\frac{a+b}{2}\right) &= \left[\frac{2}{a+b}, e^{\frac{a+b}{2}} - 1\right] \\
\Delta_1 &= \left[\frac{16}{(3a+b)(a+3b)}, \left(e^{\frac{3a+b}{4}} - 1\right)\left(e^{\frac{a+3b}{4}} - 1\right)\right] \\
\Delta_2 &= \left[\frac{1}{ab}, (e^a - 1)\left(e^b - 1\right)\right]\left[\frac{2}{a+b}, e^{\frac{a+b}{2}} - 1\right]^2
\end{aligned}
$$

and

$$G(F(a), F(b)) = \sqrt{F(a)F(b)}$$

$$= \left[\sqrt{\frac{1}{ab}}, \sqrt{(e^a-1)(e^b-1)}\right].$$

Thus for all $a, b \in [x, y]$ inequalities (5.18) satisfies and we have

$$\begin{aligned}
\left[\frac{2}{a+b}, e^{\frac{a+b}{2}}-1\right]^2 &\supseteq \sqrt{\left[\frac{16}{(3a+b)(a+3b)}, \left(e^{\frac{3a+b}{4}}-1\right)\left(e^{\frac{a+3b}{4}}-1\right)\right]} \\
&\supseteq e^{\frac{1}{b-a}\left[\ln\left(\frac{b^b}{a^a}\right)^{-1}+1, \frac{b^2-a^2}{2}\right]} \\
&\supseteq \left(\left[\frac{1}{ab}, (e^a-1)(e^b-1)\right]\left[\frac{2}{a+b}, e^{\frac{a+b}{2}}-1\right]^2\right)^{\frac{1}{4}} \\
&\supseteq \left[\sqrt{\frac{1}{ab}}, \sqrt{(e^a-1)(e^b-1)}\right]
\end{aligned}$$

for verification we suppose $a = 1$ and $b = 2$, then we have

$$\begin{aligned}
[0.4444, 12.1215] &\supseteq [0.6761, 3.4409] \\
&\supseteq [0.6796, 3.4260] \\
&\supseteq [0.6865, 3.3963] \\
&\supseteq [0.7071, 3.3131].
\end{aligned}$$

Corollary 12. *Under the assumptions of Theorem 4, if we let $h(t) = t^s$, then following inequalities for the s-logarithmically interval-valued convex functions hold:*

$$\begin{aligned}
\left(F\left(\frac{a+b}{2}\right)\right)^{2^{2s-1}} &\supseteq \Delta_1^{2^{s-2}} \supseteq \left(\int_a^b F(x)^{dx}\right)^{\frac{1}{b-a}} \\
&\supseteq \Delta_2^{\frac{1}{2(s+1)}} \supseteq (F(a)F(b))^{\frac{1}{s+1}\left[\frac{1}{2}+\frac{1}{2^s}\right]}.
\end{aligned}$$

Corollary 13. *Under the conditions of Theorem 4, if we choose $h(t) = 1$, then we have following inequalities for the logarithmically P-convex functions*

$$\begin{aligned}
\sqrt{F\left(\frac{a+b}{2}\right)} &\supseteq \Delta_1^{\frac{1}{4}} \supseteq \left(\int_a^b F(x)^{dx}\right)^{\frac{1}{b-a}} \\
&\supseteq \sqrt{\Delta_2} \supseteq (F(a)F(b))^{\frac{3}{2}}.
\end{aligned}$$

Corollary 14. *If we put $h(t) = t^{-1}$ in first two inequalities of Theorem 4, then we have following inequalities for the logarithmically interval-valued Q-convex functions*

$$\left(F\left(\frac{a+b}{2}\right)\right)^{\frac{1}{8}} \supseteq \Delta_1^{\frac{1}{8}} \supseteq \left(\int_a^b F(x)^{dx}\right)^{\frac{1}{b-a}}.$$

Corollary 15. *Under the assumptions of Theorem 4, if we set $h(t) = t^{-s}$ in first inequality, then following inequalities for s-logarithmically interval-valued Q-convex functions hold*

$$\left(F\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2^{2s+1}}} \supseteq \Delta_1^{\frac{1}{2^{2s+1}}} \supseteq \left(\int_a^b F(x)^{dx}\right)^{\frac{1}{b-a}}.$$

6. Conclusion

In this paper, authors define multiplicative integral for the interval-valued functions and derived some new Hermite-Hadamard and related inequalities for logarithmically interval-valued h-convex functions by utilizing our newly define integral. Authors also gave some more new results in the special cases of our main results. Interested readers can obtain more results by using the notions used in this paper. The results in this paper can be a new contribution in the field of Hermite-Hadamard integral inequalities.

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References

[1] Ali, M. A., Abbas, M., Budak, H., Kashuri, A., Some new Hermite–Hadamard integral inequalities in multiplicative calculus, *TWMS Journal of Applied and Engineering Mathematics* (2020, In Press).

[2] Ali, M. A., Abbas, M., Zafar, A. A., On some Hermite-Hadamard integral inequalities in multiplicative calculus, *Journal of Inequalities and Special Functions, 10* (1) (2019), 111–122.

[3] Ali, M. A., Abbas, M., Zhang, Z., Sial, I. B., Arif, R., On integral inequalities for product and quotient of two multiplicatively convex functions, *Asian Research Journal of Mathematics* (2019), 1–11.

[4] Aubin, J.-P., Cellina, A., Differential inclusions: set-valued maps and viability theory, Springer Science & Business Media, Springer-Verlage Berlin Heidelberg New York Tokyo, 2012.

[5] Bashirov, A. E., Kurpınar, E. M., Özyapıcı, A., Multiplicative calculus and its applications, *Journal of Mathematical Analysis and Applications, 337* (1) (2008), 36–48.

[6] Breckner, W. W., Continuity of generalized convex and generalized concave set-valued functions, *Rev. Anal. Numér. Théor. Approx., 22* (1) (1993), 39–51.

[7] Budak, H., Ali, M. A., Tarhanaci, M., Some new quantum Hermite–Hadamard-like inequalities for coordinated convex functions, *Journal of Optimization Theory and Applications, 186* (3) (2020), 899–910.

[8] Budak, H., Erden, S., Ali, M. A., Simpson and Newton type inequalities for convex functions via newly defined quantum integrals, *Mathematical Methods in the Applied Sciences* (2020).

[9] Chalco-Cano, Y., Flores-Franulic, A., Román-Flores, H., Ostrowski type inequalities for interval-valued functions using generalized hukuhara derivative, *Computational & Applied Mathematics, 31* (3) (2012).

[10] Chalco-Cano, Y., Lodwick, W. A., Condori-Equice, W., Ostrowski type inequalities and applications in numerical integration for interval-valued functions, *Soft Computing, 19* (11) (2015), 3293–3300.

[11] Chen, F., A note on Hermite-Hadamard inequalities for products of convex functions., *Journal of Applied Mathematics* (2013).
[12] Costa, T., Jensen's inequality type integral for fuzzy-interval-valued functions, *Fuzzy Sets and Systems, 327* (2017), 31–47.
[13] Costa, T., Román-Flores, H., Some integral inequalities for fuzzy-interval-valued functions, *Information Sciences, 420* (2017), 110–125.
[14] Dragomir, S., Pearce, C., Selected topics on Hermite-Hadamard inequalities and applications, rgmia monographs, victoria university, 2000, *ONLINE: http://rgmia. vu. edu. au/monographs* (2004).
[15] Dragomir, S., Pecaric, J., Persson, L.-E., Some inequalities of Hadamard type, *Soochow J. Math, 21* (3) (1995), 335–341.
[16] Dragomir, S. S., Two mappings in connection to Hadamard's inequalities, *Journal of Mathematical Analysis and Applications, 167* (1) (1992), 49–56.
[17] Dragomir, S. S., Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces, *Proyecciones (Antofagasta), 34* (4) (2015), 323–341.
[18] Ertugral, F., Sarikaya, M. Z., Simpson type integral inequalities for generalized fractional integral, *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, 113* (4) (2019), 3115–3124.
[19] Flores-Franulič, A., Chalco-Cano, Y., Román-Flores, H., An ostrowski type inequality for interval-valued functions, In *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)* (2013), IEEE, pp. 1459–1462.
[20] Guo, Y., Ye, G., Zhao, D., Liu, W., Some integral inequalities for log-h-convex interval-valued functions, *IEEE Access, 7* (2019), 86739–86745.
[21] Kara, H., Ali, M. A., Budak, H., Hermite–Hadamard-type inequalities for interval-valued coordinated convex functions involving generalized fractional integrals, *Mathematical Methods in the Applied Sciences* (2020).
[22] Lupulescu, V., Fractional calculus for interval-valued functions, *Fuzzy Sets and Systems, 265* (2015), 63–85.
[23] Markov, S., Calculus for interval functions of a real variable, *Computing, 22* (4) (1979), 325–337.
[24] Markov, S., On the algebraic properties of convex bodies and some applications, *Journal of convex analysis, 7* (1) (2000), 129–166.
[25] Mitroi, F.-C., Nikodem, K., Wasowicz, S., Hermite–Hadamard inequalities for convex set-valued functions, *Demonstratio Mathematica, 46* (4) (2013), 655–662.
[26] Mohammed, P., Some new Hermite-Hadamard type inequalities for mt-convex functions on differentiable coordinates, *Journal of King Saud University-Science, 30* (2) (2018), 258–262.
[27] Moore, R. E., Interval analysis, Prentice-Hall, Englewood Cliffs, 1966.
[28] Moore, R. E., Kearfott, R. B., Cloud, M. J., Introduction to interval analysis, Siam, Philadelphia, P. A., 2009.
[29] Nikodem, K., Sanchez, J. L., Sanchez, L., Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps, *Mathematica Aeterna, 4* (8) (2014), 979–987.
[30] Noor, M. A., Qi, F., Awan, M. U., Some Hermite-Hadamard type inequalities for log-h-convex functions, *Analysis, 33* (4) (2013), 367–375.
[31] Pachpatte, B., On some inequalities for convex functions, *RGMIA Res. Rep. Coll, 6* (1) (2003), 1–9.
[32] Peajcariaac, J. E., Tong, Y. L., Convex functions, partial orderings, and statistical applications, Academic Press, Bostan SanDiego New York London Sydney Tokyo Toronto, 1992.
[33] Román-Flores, H., Chalco-Cano, Y., Lodwick, W., Some integral inequalities for interval-valued functions, *Computational and Applied Mathematics, 37* (2) (2018), 1306–1318.

[34] Román-Flores, H., Chalco-Cano, Y., Silva, G. N., A note on gronwall type inequality for interval-valued functions, In *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)* (2013), IEEE, pp. 1455–1458.

[35] Sadowska, E., Hadamard inequality and a refinement of Jensen inequality for set valued functions, *Results in Mathematics, 32* (3-4) (1997), 332–337.

[36] Sarikaya, M. Z., Yildirim, H., On generalization of the riesz potential, *Indian Jour. of Math. and Mathematical Sci, 3* (2) (2007), 231–235.

[37] Tseng, K.-L., Hwang, S.-R., New Hermite-Hadamard-type inequalities and their applications, *Filomat, 30* (14) (2016), 3667–3680.

[38] Vivas-Cortez, M., Aamir Ali, M., Kashuri, A., Bashir Sial, I., Zhang, Z., Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus, *Symmetry, 12* (9) (2020), 1476.

[39] Wang, J., Li, X., Zhu, C., et al., Refinements of Hermite-Hadamard type inequalities involving fractional integrals, *Bulletin of the Belgian Mathematical Society-Simon Stevin, 20* (4) (2013), 655–666.

[40] Zhao, D., Ali, M. A., Kashuri, A., Budak, H., Generalized fractional integral inequalities of Hermite–Hadamard type for harmonically convex functions, *Advances in Difference Equations, 2020* (1) (2020), 1–14.

[41] Zhao, D., Ali, M. A., Murtaza, G., Zhang, Z., On the Hermite–Hadamard inequalities for interval-valued coordinated convex functions, *Advances in Difference Equations, 2020, 570 (2020).*

[42] Zhao, D., An, T., Ye, G., Liu, W., New Jensen and Hermite–Hadamard type inequalities for h-convex interval-valued functions, *Journal of Inequalities and Applications, 2018* (1) (2018), 302.

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A SOLUTION OF A VISCOSITY CESÀRO MEAN ALGORITHM

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Abstract. Based on the viscosity approximation method, we introduce a new cesàro mean approximation method for finding a common solution of split generalized equilibrium problem in real Hilbert spaces. Under certain conditions control on parameters, we prove a strong convergence theorem for the sequences generated by the proposed iterative scheme. Some numerical examples are presented to illustrate the convergence results. Our results can be viewed as a generalization and improvement of various existing results in the current literature.

1. **Introduction**

Let $\mathbb{R}$ denote the set of all real number, H_1 and H_2 be real Hilbert spaces and C and Q be nonempty closed convex subset of H_1 and H_2, respectively. A mapping $T : C \to C$ said to be a k-strictly pseudocontractive if there exists a constant $0 \leq k < 1$ such that

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \qquad \forall x, y \in C.$$

When $k = 1$, T is said to be pseudocontractive if

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \qquad \forall x, y \in C.$$

If $k = 0$, T is called nonexpansive on C.

The fixed point problem (FPP) for a nonexpansive mapping T is: Find $x \in C$ such that $x \in Fix(T)$, where $Fix(T)$ is the fixed point set of the nonexpansive mapping T.

The class of k-strictly pseudocontractive falls into the one between classes of nonexpansive mapping and pseudocontractive mapping.

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A set-valued $M : H \to 2^H$ is called monotone if for all $x, y \in H, u \in M(x)$ and $v \in M(y)$ such that $\langle x - y, u - v \rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if the $Graph(M)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, u) \in H \times H, \langle x - y, u - v \rangle \geq 0$, for every $(y, v) \in Graph(M)$ implies that $u \in M(x)$.

Let $E : H \to H$ be a single-valued nonlinear mapping, and let $M : H \to 2^H$ be a set-valued mapping. We consider the following variational inclusion problem (VIP), which is: Find $x \in H$ such that

$$\theta \in E(x) + M(x),$$

where θ is the zero vector in H. The solution set of (VIP) is denoted by $I(E, M)$.

Let the set-valued mapping $M : H \to 2^H$ be a maximal monotone. We define the resolvent operator $J_{M,\lambda}$ associate with M and λ as follows:

$$J_{M,\lambda}(x) = (I + \lambda M)^{-1}(x), \qquad x \in H$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}(x)$ is single-valued, nonexpansive and 1-inverse strongly monotone [2, 22].

In 1994 Blum and Oettli [1] introduced and studied the following equilibrium problem (EP): Find $x \in C$ such that $F(x, y) \geq 0$, $\forall y \in C$, where $F : C \times C \to \mathbb{R}$ is a bifunction.

Kumam et al. [11] considered an iterative algorithm in a Hilbert space:

$$\begin{aligned} t_n &= T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\ u_n &= T_{q_n}^{(F_2,\varphi_2)}(t_n - q_n B t_n), \\ v_n &= J_{M_1,\lambda_1}(u_n - \lambda_1 E_1 u_n), \\ w_n &= J_{M_2,\lambda_2}(v_n - \lambda_2 E_2 v_n), \end{aligned}$$

$$y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) \tfrac{1}{t_n} \int_0^{t_n} S(s) W_n w_n ds,$$

$$C_{n+1,i} = \{z \in C_{n,i} : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\},$$

$$C_{n+1} = \textstyle\bigcap_{i=1}^{\infty} C_{n+1,i},$$

$$x_{n+1} = P_{C_{n+1}} x_0.$$

Moudafi [15] introduced the following split equilibrium problem (SEP):
Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bimappings and let $A : H_1 \to H_2$ be a bounded linear operator, then the SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \ \forall x \in C$$

and such that

$$y^* = Ax^* \in Q \ solves \ F_2(y^*, y) \geq 0, \ \forall y \in Q$$

The solution set of (SEP) is denoted by $\Omega = \{p \in EP(F_1) : Ap \in EP(F_2)\}$. (SEP) includes the split variational inequality problem, split zero problem, and split feasibility problem (see, for instance, [3–6, 14, 15]).

Recently, Kazmi and Rizvi [10] introduced a split generalized equilibrium problem $(SGEP)$: Find $x^* \in C$ such that

$$F_1(x^*, x) + \psi_1(x^*, x) \geq 0, \ \forall x \in C$$

and such that

$$y^* = Ax^* \in Q \ solves \ F_2(y^*, y) + \psi_2(y^*, y) \geq 0, \ \forall y \in Q$$

where $F_1, \psi_1 : C \times C \to \mathbb{R}$ and $F_2, \psi_2 : C \times C \to \mathbb{R}$ be nonlinear bi functions and $A : H_1 \to H_2$ is bounded linear operator. The solution set of $(SGEP)$ is denoted by $\Gamma = \{p \in GEP(F_1, \psi_1) : Ap \in GEP(F_2, \psi_2)\}$. They considered the following iterative method:

$$\begin{aligned} &u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,\psi_2)} - I)Ax_n); \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n B)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds. \end{aligned}$$

In 2015 Wang [19] introduced and studied the following iterative method to prove a strong convergence theorem for $F(T)$ and VIP in real Hilbert space:

$$\begin{aligned} y_n \quad &= \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} \quad &= \beta_n x_n + (1 - \beta_n)TJ_{r_n}(y_n - r_n Ay_n), \qquad \forall n \geq 1, \end{aligned}$$

where u is fixed element and $J_{r_n} = (1 + r_n B)^{-1}$.

In 2017 Zhang and Gui [21] introduced an iterative algorithm in a Hilbert space as follows:

$$\begin{aligned} u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{s_n}^{F_2} - I)Ax_n) \\ x_{n+1} &= \alpha_n f(x_n) + \tfrac{(1-\alpha_n)}{l}\textstyle\sum_{i=0}^{l} T_i^n u_n, \end{aligned}$$

where $T_i : C \to C$ is an asymptotically nonexpansive mapping for $i = 0, 1, \ldots, n$.

Motivated by the works of Kumam et al. [11], Kazmi and Rizvi [10], Zhang and Gui [21], Wang [19] and by the ongoing research in direction, we introduce and study an iterative method for approximating a common solution of $SGEP, VIP$ and FPP for a nonexpansive semigroup in real Hilbert spaces.

2. **Preliminaries**

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. For each point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C y\rangle \leq 0. \tag{2.1}$$

Further, it is well known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \le (\frac{1}{2}) \|(T(x) - x) - (T(y) - y)\|^2, \quad (2.2)$$

and therefore, we get, for all $(x, y) \in H \times Fix(T)$,

$$\langle (x - T(x)), (y - T(y)) \rangle \le (\frac{1}{2}) \|(T(x) - x)\|^2, \quad (2.3)$$

see, e.g. [9].It is also known that H satisfies Opial's condition [16], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad (2.4)$$

holds for every $y \in H$ with $y \neq x$.

Definition 2.1. *A mapping $T : H \to H$ is said to be firmly nonexpansive, if*

$$\langle Tx - Ty, x - y \rangle \ge \|Tx - Ty\|^2, \ \forall x, y \in H.$$

Lemma 2.2. *[7] The following inequality holds in real space H:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle, \ \forall x, y \in H.$$

Definition 2.3. *A mapping $T : C \to H$ is said to be monotone, if*

$$\langle Tx - Ty, x - y \rangle \ge 0, \quad \forall x, y \in C.$$

T is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Tx - Ty, x - y \rangle \ge \alpha \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Lemma 2.4. *[2] Let $M : H \to 2^H$ be a maximal monotone mapping, and let $E : H \to H$ be a monotone mapping, then the mapping $M + E : H \to 2^H$ is a maximal monotone mapping.*

Lemma 2.5. *[22] Let $x \in H$ be a solution of variational inclusion if and only if $x = J_{M,\lambda}(x - \lambda Ex), \forall \lambda > 0$, that is*

$$I(E, M) = Fix(J_{M,\lambda}(I - \lambda E)), \qquad \forall \lambda > 0.$$

Lemma 2.6. *[13] Assume that B is a strong positive linear bounded self adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|B\|^{-1}$. Then $\|I - \rho B\| \le 1 - \rho\bar{\gamma}$.*

Lemma 2.7. *[17] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.8. *[20] Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that* $a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 0$ *where* α_n *is a sequence in* $(0,1)$ *and* δ_n *is a sequence in* $\mathbb{R}$ *such that (i)* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$; *(ii)* $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or (iii)* $\Sigma_{n=1}^{\infty}\delta_n < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Assumption 2.9. *[12] Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying the following assumption:*

(1) $F(x,x) \geq 0,\ \forall x \in C$,
(2) F *is monotone, i.e.,* $F(x,y) + F(y,x) \leq 0,\ \forall x \in C$,
(3) F *is upper hemicontinuous, i.e., for each* $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$,
(4) *For each* $x \in C$ *fixed, the function* $x \to F(x,y)$ *is convex and lower semicontinuous;*

let $\psi : C \times C \to \mathbb{R}$ *such that*

(1) $\psi(x,x) \geq 0,\ \forall x \in C$,
(2) *For each* $y \in C$ *fixed, the function* $x \to \psi(x,y)$ *is upper semicontinuous,*
(3) *For each* $x \in C$ *fixed, the function* $y \to \psi(x,y)$ *is convex and lower semicontinuous;*

Lemma 2.10. *[10] Assume that* $F_1, \psi_1 : C \times C \to \mathbb{R}$ *satisfy Assumption 2.9. Let* $r > 0$ *and* $x \in H_1$. *Then, there exists* $z \in C$ *such that*

$$F_1(z,y) + \psi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.11. *[4] Assume that the bifunctions* $F_1, \psi_1 : C \times C \to \mathbb{R}$ *satisfy Assumption 2.9 and* ψ_1 *is monotone. For* $r > 0$ *and for all* $x \in H_1$, *define a mapping* $T_r^{(F_1,\psi_1)} : H_1 \to C$ *as follows:*

$$T_r^{(F_1,\psi_1)}x = \{z \in C : F_1(z,y) + \psi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0\}, \quad \forall y \in C.$$

Then the followings hold:

$(i) T_r^{(F_1,\psi_1)}\ is\ single-valued.$

$(ii) T_r^{(F_1,\psi_1)}\ is\ firmly\ nonexpansive, i.e.,$

$$\|T_r^{(F_1,\psi_1)}(x) - T_r^{(F_1,\psi_1)}(y)\|^2 \leq \langle T_r^{(F_1,\psi_1)}(x) - T_r^{(F_1,\psi_1)}(y), x - y\rangle, \quad x, y \in H_1.$$

$(iii) Fix(T_r^{(F_1,\psi_1)}) = GEP(F_1, \psi_1).$

$(iv) GEP(F_1, \psi_1)$ *is compact and convex.*

Further, assume that $F_2, \psi_2 : Q \times Q \to \mathbb{R}$ *satisfy Assumption 2.9. For* $s > 0$ *and for all* $w \in H_2$, *define a mapping* $T_s^{(F_2,\psi_2)} : H_2 \to Q$ *as follows:*

$$T_s^{(F_2,\psi_2)}w = \{d \in Q : F_2(d,e) + \psi_2(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0\}, \quad \forall e \in Q.$$

Then, we easily observe that $T_a^{(F_2,\psi_2)}$ *satisfies in Lemma 2.11 and* $GEP(F_1,\psi_1)$ *is compact and convex.*

Lemma 2.12. *[8] Let* $F_1 : C \times C \to \mathbb{R}$ *be a bifunction satisfying Assumption 2.9 and let* $T_r^{F_1}$ *be defined as in Lemma 2.11, for* $r > 0$*. Let* $x, y \in H_1$ *and* $r_1, r_2 > 0$*. Then,*

$$\|T_{r_2}^{F_1} y - T_{r_1}^{F_1} x\| \leq \|x - y\| + |\frac{r_2 - r_1}{r_2}| \|T_{r_2}^{F_1} y - y\|.$$

Lemma 2.13. *[18] Let* $F_1 : C \times C \to \mathbb{R}$ *be a bifunction satisfying Assumption 2.9 and let* $T_r^{F_1}$ *be defined as in Lemma 2.11, for* $r > 0$*. Let* $x \in H_1$ *and* $r_1, r_2 > 0$*. Then,*

$$\|T_{r_2}^{F_1} x - T_{r_1}^{F_1} x\|^2 \leq \frac{r_2 - r_1}{r_2} \langle T_{r_2}^{F_1}(x) - T_{r_1}^{F_1}(x), T_{r_2}^{F_1}(x) - x \rangle.$$

Notation. Let $\{x_n\}$ be a sequence in H, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) denotes strong (respectively, weak) convergence of the sequence $\{x_n\}$ to a point $x \in H$.

3. Viscosity Iterative Algorithm

In this section, we prove a strong convergence theorem based on the explicit iterative for fixed point of nonexpansive semigroup. We firstly present the following unified algorithm.

Let H_1 and H_2 be two real Hilbert spaces; Let $C \subseteq H_1$, $Q \subseteq H_2$ be nonempty, closed and convex subsets; Let $F_1, \psi_1 : C \times C \to \mathbb{R}$ and $F_2, \psi_2 : Q \times Q \to \mathbb{R}$ are nonlinear mappings satisfying Assumption 2.9 and F_2 is upper semicontinuous in first argument. Let $\{V_i : C \to C\}$ be a uniformly k-strict pseudocontractions and $T^i : C \to C$ be a nonexpansive mapping on C for $i = 0, 1, 2, \ldots, n$ defined by $T^i x = tx + (1-t)V_i$ for each $x \in C$, $t \in [k, 1)$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0, 1)$, $A : H_1 \to H_2$ be a bounded linear operator, B be a strongly positive bounded linear self adjoint operators on H_1 with constant $\bar{\gamma}_1 > 0$, such that $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$, E be a $\bar{\gamma}_2$- inverse strongly monotone mapping on H_1 such that $\bar{\gamma}_2 > 0, \lambda \in (0, 2\bar{\gamma}_2)$ and $M : H_1 \to 2^{H_1}$ be a maximal monotone mapping. Suppose that $\Theta = \bigcap_{i=0}^n Fix(T^i) \cap \Gamma \cap I(E, M) \neq \emptyset$.

Algorithm 3.1. *For given* $x_0 \in C$ *arbitrary, let the sequence* $\{x_n\}$ *be generated by the manner:*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \delta A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n) \\ w_n = J_{M,\lambda}(u_n - \lambda E u_n) \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)\frac{1}{n+1}\sum_{i=0}^n T^i w_n + \gamma_n e_n, \end{cases} \tag{3.1}$$

where $\{e_n\}$ *is a bounded error sequence in* H_1*,* $\delta \in (0, \frac{1}{L^2})$*,* L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A*,* $\{\alpha_n\}$*,* $\{\beta_n\}$ *are the sequence in*

$(0,1)$ *and* $\{r_n\} \subset [r,\infty)$ *with* $r>0$, $\{s_n\} \subset [s,\infty)$ *with* $s>0$ *satisfying conditions:*
$(C1)\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0,\ \Sigma_{n=1}^{\infty}\alpha_n = \infty;$

$(C2)\lim_{n\to\infty}\frac{\gamma_n}{\alpha_n} = 0$

$(C3)\lim_{n\to\infty}|r_{n+1}-r_n| = 0,\ \liminf_{n\to\infty} r_n > 0,\ \lim_{n\to\infty}|s_{n+1}-s_n| = 0.$

Lemma 3.2. *Let $p \in \Theta$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded.*

Proof. By Lemma 2.11 (ii), using the similar argument in Remark 3.1 [21], for $\delta \in (0, \frac{1}{2L^2})$, $I + \delta A^*(T_{s_n}^{(F_2,\psi_2)} - I)A$ is a nonexpansive mapping and $A^*(T_{s_n}^{(F_2,\psi_2)} - I)A$ is a $\frac{1}{2L^2}$-inverse strongly monotone mapping. Take $p \in \Theta$.
And similar to Theorem 3.1 [21], we have

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \delta(\delta - \tfrac{1}{L^2})\|A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n\|^2. \tag{3.2}$$

Since $\delta \in (0, \frac{1}{2L^2})$, we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2. \tag{3.3}$$

Now, we show that $I - \lambda E$ is a nonexpansive mapping. Indeed for $x, y \in C$ and $\lambda \in (0, 2\bar{\gamma})$, we have

$$\begin{aligned}\|(I-\lambda E)x - (I-\lambda E)y\|^2 &= \|x - y - \lambda(Ex - Ey)\|^2\\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ex-Ey\rangle + \lambda^2\|Ex-Ey\|^2\\ &\le \|x-y\|^2 - 2\lambda\bar{\gamma}_2\|Ex-Ey\|^2 + \lambda^2\|Ex-Ey\|^2\\ &\le \|x-y\|^2 + \lambda(\lambda - 2\bar{\gamma}_2)\|Ex-Ey\|^2\\ &\le \|x-y\|^2,\end{aligned} \tag{3.4}$$

then $I - \lambda E$ is a nonexpansive mapping.
Since $J_{M,\lambda}(u_n - \lambda E u_n)$ is a nonexpansive mapping, we have

$$\begin{aligned}\|w_n - p\|^2 &= \|J_{M,\lambda}(u_n - \lambda E u_n) - J_{M,\lambda}(p - \lambda E p)\|^2\\ &\le \|(u_n - \lambda E u_n) - (p - \lambda E p)\|^2\\ &\le \|u_n - p\|^2,\end{aligned} \tag{3.5}$$

then

$$\|w_n - p\| \le \|u_n - p\|. \tag{3.6}$$

Then

$$\|w_n - p\| \leq \|x_n - p\|. \tag{3.7}$$

From Theorem 1 [10], we obtain $(1-\beta_n)I-\alpha_n B$ is positive and $\|(1-\beta_n)I-\alpha_n B\| \leq 1-\beta_n-\alpha_n\bar{\gamma}_1$, for any $x, y \in C$.
Now, on setting $t^n := \frac{1}{n+1}\sum_{i=0}^{n} T^i$, we can easily observe that the mapping t^n is nonexpansive. Since $p \in \Theta$, we have

$$t^n p = \frac{1}{n+1}\sum_{i=0}^{n} T^i p = \frac{1}{n+1}\sum_{i=0}^{n} p = p.$$

Since $\{e_n\}$ is bounded, using condition $(C2)$, we obtain that $\{\frac{\gamma_n\|e_n\|}{\alpha_n}\}$ is bounded. Then, there exists a nonnegative real number K such that

$$\|\gamma f(p) - Bp\| + \frac{\gamma_n\|e_n\|}{\alpha_n} \leq K, \qquad \forall n \geq 0, \tag{3.8}$$

therefore

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)t^n w_n + \gamma_n e_n - p\| \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\| + \beta_n\|x_n - p\| \\
&\quad + \|((1-\beta_n)I - \alpha_n B)\|\|t^n w_n - t^n p\| + \gamma_n\|e_n\| \\
&\leq \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Bp\|) + \beta_n\|x_n - p\| \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\|w_n - p\| + \gamma_n\|e_n\| \\
&\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| + \beta_n\|x_n - p\| \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma}_1)\|x_n - p\| + \gamma_n\|e_n\| \\
&\leq (1-(\bar{\gamma}_1 - \gamma\alpha)\alpha_n)\|x_n - p\| + \alpha_n K \\
&\leq \max\{\|x_n - p\|, \tfrac{K}{\bar{\gamma}_1 - \gamma\alpha}\} \\
&\;\vdots \\
&\leq \max\{\|x_0 - p\|, \tfrac{K}{\bar{\gamma}_1 - \gamma\alpha}\}.
\end{aligned} \tag{3.9}$$

Hence $\{x_n\}$ is bounded. $\square$

We deduce that $\{u_n\}$, $\{w_n\}$, $\{t^n\}$ and $\{f(x_n)\}$ are bounded.

Lemma 3.3. *The following properties are satisfied for the Algorithm 3.1*

$P1.$ $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$

$P2.$ $\lim_{n\to\infty}\|x_n-t^n w_n\|=0.$

$P3.$ $\lim_{n\to\infty}\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2=0, \quad \lim_{n\to\infty}\|Eu_n-Ep\|=0.$

$P4.$ $\lim_{n\to\infty}\|u_n-x_n\|=0, \quad \lim_{n\to\infty}\|w_n-u_n\|=0, \quad \lim_{n\to\infty}\|t^n w_n-w_n\|=0.$

Proof. P1: Similar to Theorem 3.1 [21], we obtain

$$\|u_{n+1}-u_n\| \le \|x_{n+1}-x_n\| + \delta\|A\|(\frac{|s_{n+1}-s_n|}{s_{n+1}}\eta_n)^{\frac{1}{2}} + \frac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} \quad (3.10)$$

where

$\sigma_{n+1}=\sup_{n\in\mathbb{N}}\|T_{r_{n+1}}^{(F_1,\psi_1)}(x_{n+1}+\delta A^*(T_{s_{n+1}}^{(F_2,\psi_2)}-I)Ax_{n+1})-(x_{n+1}+\delta A^*(T_{s_{n+1}}^{(F_2,\psi_2)}-I)Ax_{n+1})\|,$

$\eta_n=\sup_{n\in\mathbb{N}}\langle T_{s_{n+1}}^{(F_2,\psi_2)}Ax_n-T_{s_n}^{(F_2,\psi_2)}Ax_n, T_{s_{n+1}}^{(F_2,\psi_2)}Ax_n-Ax_n\rangle.$

Since $J_{M,\lambda}(u_n-\lambda Eu_n)$ is a nonexpansive mapping, we have

$$\begin{aligned}\|w_{n+1}-w_n\| &= \|J_{M,\lambda}(u_{n+1}-\lambda Eu_{n+1})-J_{M,\lambda}(u_n-\lambda Eu_n)\|\\ &\le \|(u_{n+1}-\lambda Eu_{n+1})-(u_n-\lambda Eu_n)\|\\ &\le \|u_{n+1}-u_n\|.\end{aligned} \quad (3.11)$$

Next we easily estimate that

$$\|t^{n+1}w_{n+1}-t^n w_n\| \le \|w_{n+1}-w_n\| + \frac{2}{n+2}\|w_n-p\| + \frac{2}{n+2}\|p\|$$

By (3.10) and (3.11) we can write

$$\begin{aligned}\|t^{n+1}w_{n+1}-t^n w_n\| &\le \|x_{n+1}-x_n\| + \delta\|A\|(\tfrac{|s_{n+1}-s_n|}{s_{n+1}}\eta_n)^{\frac{1}{2}}\\ &\quad + \tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} + \tfrac{2}{n+2}(\|x_n-p\|+\|p\|),\end{aligned} \quad (3.12)$$

Setting $x_{n+1} = \beta_n x_n + (1-\beta_n) y_n$, then we have

$$\begin{aligned} y_{n+1} - y_n &= \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - Bt^{n+1}w_{n+1} + \tfrac{\gamma_{n+1}e_{n+1}}{\alpha_{n+1}}) \\ &\quad + t^{n+1}w_{n+1} - t^n w_n + \tfrac{\alpha_n}{1-\beta_n}(Bt_n - \gamma f(x_n) - \tfrac{\gamma_n e_n}{\alpha_n}). \end{aligned}$$

Using (3.12), we have

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1}) - Bt^{n+1}w_{n+1}\| + \tfrac{\gamma_{n+1}\|e_{n+1}\|}{\alpha_{n+1}}\|) \\ &\quad + \|t^{n+1}w_{n+1} - t^n w_n\| + \tfrac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n) - Bt^n w_n\| + \tfrac{\gamma_n\|e_n\|}{\alpha_n}\|) \\ &\leq \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1}) - Bt^{n+1}w_{n+1}\| + \tfrac{\gamma_{n+1}\|e_{n+1}\|}{\alpha_{n+1}}) + \|x_{n+1} - x_n\| \\ &\quad + \delta\|A\|(\tfrac{|s_{n+1}-s_n|}{s_{n+1}}\eta_n)^{\frac{1}{2}} + \tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} + \tfrac{2}{n+2}(\|x_n - p\| + \|p\|) \\ &\quad + \tfrac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n) - Bt^n w_n\| + \tfrac{\gamma_n\|e_n\|}{\alpha_n}\|) \end{aligned}$$

which implies that

$$\begin{aligned} &\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\ &\leq \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1}) - Bt^{n+1}w_{n+1}\| + \tfrac{\gamma_{n+1}\|e_{n+1}\|}{\alpha_{n+1}}) + \delta\|A\|(\tfrac{|s_{n+1}-s_n|}{s_{n+1}}\eta_n)^{\frac{1}{2}} \\ &\quad + \tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} + \tfrac{2}{n+2}(\|x_n - p\| + \|p\|) + \tfrac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n) - Bt^n w_n\| + \tfrac{\gamma_n\|e_n\|}{\alpha_n}\|). \end{aligned}$$

Hence, it follows by conditions $(C1) - (C3)$ that

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.13}$$

From Lemma 2.7 and (3.13), we get $\lim_{n\to\infty}\|y_{n+1} - x_n\| = 0$, and

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|y_{n+1} - x_n\| = 0. \tag{3.14}$$

Then $\lim_{n\to\infty}\|t^{n+1}w_{n+1} - t^n w_n\| = 0$.

P2: We can write

$$\begin{aligned} \|x_n - t^n w_n\| &\leq \|x_{n+1} - x_n\| \\ &\quad + \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)t^n w_n + \gamma_n e_n - t^n w_n\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(x_n) - Bt^n w_n\| + \beta_n\|x_n - t^n w_n\| + \gamma_n\|e_n\|. \end{aligned}$$

Then

$$(1-\beta_n)\|x_n - t^n w_n\| \le \|x_{n+1} - x_n\| + \alpha_n \|\gamma f(x_n) - Bt^n w_n\| + \gamma_n \|e_n\|.$$

Therefore we have

$$\|x_n - t^n w_n\| \quad \le \tfrac{1}{1-\beta_n}\|x_{n+1} - x_n\| + \tfrac{\alpha_n}{1-\beta_n}(\|(\gamma f(x_n) - Bt^n w_n\| + \tfrac{\gamma_n \|e_n\|}{\alpha_n}).$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ we obtain

$$\lim_{n\to\infty} \|x_n - t^n w_n\| = 0. \tag{3.15}$$

P3: Since $\{x_n\}$ is bounded, we may assume a nonnegative real number N such that $\|x_n - p\| \le N$. From (3.5) and (3.2), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)t^n w_n + \gamma_n e_n - p\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Bp) + \beta_n(x_n - t^n w_n) + (1-\alpha_n B)(t^n w_n - p) + \gamma_n e_n\|^2 \\
&\le \|(1-\alpha_n B)(t^n w_n - p) + \beta_n(x_n - t^n w_n)\|^2 + 2\langle \alpha_n(\gamma f(x_n) - Bp) + \gamma_n e_n, x_{n+1} - p\rangle \\
&\le (\|(1-\alpha_n B)(t^n w_n - p)\| + \beta_n \|x_n - t^n w_n\|)^2 + 2\alpha_n \langle \gamma f(x_n) - Bp, x_{n+1} - p\rangle \\
&\quad +2\langle \gamma_n e_n, x_{n+1} - p\rangle \\
&\le ((1-\alpha_n \bar{\gamma}_1)\|w_n - p\| + \beta_n \|x_n - t^n w_n\|)^2 + 2\alpha_n \langle \gamma f(x_n) - Bp, x_{n+1} - p\rangle \\
&\quad +2\gamma_n \|e_n\| N \\
&= (1-\alpha_n \bar{\gamma}_1)^2 \|w_n - p\|^2 + \beta_n^2 \|x_n - t^n w_n\|^2 + 2(1-\alpha_n \bar{\gamma}_1)\beta_n \|w_n - p\|\|x_n - t^n w_n\| \\
&\quad +2\alpha_n \langle \gamma f(x_n) - Bp, x_{n+1} - p\rangle + 2\gamma_n \|e_n\| N
\end{aligned} \tag{3.16}$$

$$
\begin{aligned}
&\le (1-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\| \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N \\
&= (1-\alpha_n\bar{\gamma}_1)^2(\|x_n-p\|^2+\delta(\delta-\tfrac{1}{L^2})\|A^*(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2)+(\beta_n)^2\|x_n-t^nw_n\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N \\
&\le \|x_n-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+(1-\alpha_n\bar{\gamma}_1)^2\delta(\delta-\tfrac{1}{L^2})\|A^*(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2 \\
&\quad +\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\| \\
&\quad +2\alpha_n(\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(1-\alpha_n\bar{\gamma}_1)^2\delta(\tfrac{1}{L^2}-\delta)\|A^*(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2 \\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N) \\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\gamma\|f(x_n)\|+\|Bp\|+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N).
\end{aligned}
$$

Because of $\delta(\frac{1}{L^2}-\delta)>0$, $\|x_n-x_{n+1}\|\to 0$ and $\|x_n-t^nw_n\|\to 0$ as $n\to\infty$ and $(C1)$ we obtain $\lim_{n\to\infty}\|A^*(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2=0$

which implies that

$$
\lim_{n\to\infty}\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2=0. \tag{3.17}
$$

It follows from (3.16)

$$\begin{aligned}\|x_{n+1}-p\|^2 &= (1-\alpha_n\bar{\gamma}_1)^2\|w_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2\\ &\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\ &\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\ &\le (1-\alpha_n\bar{\gamma}_1)^2(\|u_n-p\|^2+\lambda(\lambda-2\bar{\gamma}_2)\|Eu_n-Ep\|^2)\\ &\quad +\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\ &\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N.\end{aligned}$$

Therefore

$$\begin{aligned}&(1-\alpha_n\bar{\gamma}_1)^2\lambda(2\bar{\gamma}_2-\lambda)\|Eu_n-Ep\|^2\\ &\le (1-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2\\ &\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2\\ &\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\ &\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2\\ &\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\gamma\|f(x_n)\|+\|Bp\|+\tfrac{\gamma_n\|e_n\|}{\alpha_n})N.\end{aligned}$$

Because of $\lambda(2\bar{\gamma}_2-\lambda)>0, \|x_n-x_{n+1}\|\to 0$ and $\|x_n-t^nw_n\|\to 0$ as $n\to\infty$ and $(C1)$ we obtain

$$\lim_{n\to\infty}\|Eu_n-Ep\|=0. \tag{3.18}$$

P4: Since $p\in\Theta$, we can obtain

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|u_n-x_n\|\|A^*(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|,$$

see [21]. It follows from (3.16) that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le (1-\alpha_n\bar{\gamma}_1)^2\|w_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\
&\le (1-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\
&\le (1-\alpha_n\bar{\gamma}_1)^2(\|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|)\\
&\quad +\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\
&\le \|x_n-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2-(1-\alpha_n\bar{\gamma}_1)^2\|u_n-x_n\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)^2\delta\|A(u_n-x_n)\|\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|+\beta_n^2\|x_n-t^nw_n\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2(\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&(1-\alpha_n\bar{\gamma}_1)^2\|u_n-x_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)^2\delta\|A(u_n-x_n)\|\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|+\beta_n^2\|x_n-t^nw_n\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N)\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)^2\delta\|A(u_n-x_n)\|\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|+\beta_n^2\|x_n-t^nw_n\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\gamma\|f(x_n)\|+\|Bp\|+\tfrac{\gamma_n\|e_n\|}{\alpha_n})N.
\end{aligned}
$$

Since $\alpha_n\to 0$, $\beta_n\to 0$, $\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|\to 0$ and $\|x_n-t^nw_n\|\to 0$ as $n\to\infty$ and from $(C1)$, we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.19}$$

Since $p \in \Theta$ and $J_{M,\lambda}$ is 1-inverse strongly monotone [2,22], we can obtain

$$\begin{aligned}
\|w_n - p\|^2 &= \|J_{M,\lambda}(u_n - \lambda E u_n) - J_{M,\lambda}(p - \lambda E p)\|^2 \\
&\le \langle (u_n - \lambda E u_n) - (p - \lambda E p), w_n - p\rangle \\
&= \tfrac{1}{2}(\|(u_n - \lambda E u_n) - (p - \lambda E p)\|^2 + \|w_n - p\|^2 \\
&\quad - \|(u_n - \lambda E u_n) - (p - \lambda E p) - (w_n - p)\|^2) \\
&\le \tfrac{1}{2}(\|u_n - p\|^2 + \|w_n - p\|^2 - \|(u_n - \lambda E u_n) - (p - \lambda E p) - (w_n - p)\|^2) \\
&\le \tfrac{1}{2}(\|x_n - p\|^2 + \|w_n - p\|^2 - \|w_n - u_n\|^2 + 2\lambda\langle u_n - w_n, E u_n - E p\rangle \\
&\quad - \lambda^2\|E u_n - E p\|^2),
\end{aligned}$$

and hence,

$$\|w_n - p\|^2 \le \|x_n - p\|^2 - \|w_n - u_n\|^2 + 2\lambda\|u_n - w_n\|\|E u_n - E p\|. \tag{3.20}$$

It follows from (3.16) and (3.20) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le (1 - \alpha_n\bar{\gamma}_1)^2\|w_n - p\|^2 + \beta_n^2\|x_n - t^n w_n\|^2 + 2(1 - \alpha_n\bar{\gamma}_1)\beta_n\|w_n - p\|\|x_n - t^n w_n\| \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Bp, x_{n+1} - p\rangle + 2\gamma_n\|e_n\|N \\
&\le (1 - \alpha_n\bar{\gamma}_1)^2(\|u_n - p\|^2 - \|w_n - u_n\|^2 + 2\lambda\|u_n - w_n\|\|E u_n - E p\|) \\
&\quad + \beta_n^2\|x_n - t^n w_n\|^2 + 2(1 - \alpha_n\bar{\gamma}_1)\beta_n\|w_n - p\|\|x_n - t^n w_n\| \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Bp, x_{n+1} - p\rangle + 2\gamma_n\|e_n\|N,
\end{aligned}$$

therefore we have

$$
\begin{aligned}
&(1-\alpha_n\bar{\gamma}_1)^2\|w_n-u_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+(1-\alpha_n\bar{\gamma}_1)^2 2\lambda\|u_n-w_n\|\|Eu_n-Ep\|\\
&\quad+\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\
&\quad+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad+(1-\alpha_n\bar{\gamma}_1)^2 2\lambda\|u_n-w_n\|\|Eu_n-Ep\|+\beta_n^2\|x_n-t^nw_n\|^2\\
&\quad+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\gamma\|f(x_n)\|+\|Bp\|+\tfrac{\gamma_n\|e_n\|}{\alpha_n})N.
\end{aligned}
$$

Since $\|x_n-x_{n+1}\|\to 0, \|x_n-t^nw_n\|\to 0$ and $\|Eu_n-Ep\|\to 0$ and from $(C1)$, we obtain

$$\lim_{n\to\infty}\|w_n-u_n\|=0. \tag{3.21}$$

Using (3.15), (3.19) and (3.21), we obtain

$$\|t^nw_n-w_n\|\le\|t^nw_n-x_n\|+\|x_n-u_n\|+\|u_n-w_n\|\to 0, \text{ as } n\to\infty$$

which implies

$$\lim_{n\to\infty}\|t^nw_n-w_n\|=0.$$

□

4. Main Result

Theorem 4.1. *The Algorithm defined by (3.1) convergence strongly to $z\in\bigcap_{i=1}^n Fix(T^i)\cap\Gamma\cap I(E,M)$, which is a unique solution of the variational inequality $\langle(\gamma f-B)z,y-z\rangle\le 0,\quad\forall y\in\Theta$.*

Proof. Let $s=P_\Theta$. We get

$$
\begin{aligned}
\|s(I-B+\gamma f)(x)-s(I-B+\gamma f)(y)\| &\le \|(I-B+\gamma f)(x)-(I-B+\gamma f)(y)\|\\
&\le \|I-B\|\|x-y\|+\gamma\|f(x)-f(y)\|\\
&\le (1-\bar{\gamma_1})\|x-y\|+\gamma\alpha\|x-y\|\\
&= (1-(\bar{\gamma}_1-\gamma\alpha))\|x-y\|.
\end{aligned}
$$

Then $s(I-B+\gamma f)$ is a contraction mapping from H_1 into itself. Therefore by Banach contraction principle, there exists $z\in H_1$ such that $z=s(I-B+\gamma f)z=P_\Theta(I-B+\gamma f)z$.

We show that $\limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle \leq 0$ where $z = P_\Theta(I - B + \gamma f)$. To show this inequality, we choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - B)z, w_n - z\rangle = \lim_{n\to\infty}\langle(\gamma f - B)z, w_{n_i} - z\rangle. \tag{4.1}$$

Since $\{w_{n_i}\}$ is bounded, there exists a subsequence $\{w_{n_{i_j}}\}$ of $\{w_{n_i}\}$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $w_{n_i} \rightharpoonup w$. From $\|t^n w_n - w_n\| \to 0$, we obtain $t^n w_{n_i} \rightharpoonup w$.

Now, we prove that $w \in \bigcap_{i=0}^{n} Fix(T^i) \cap \Gamma \cap I(E, M)$. Let us first show that $w \in Fix(t^n) = \frac{1}{n+1}\sum_{i=0}^{n} Fix(T^i)$. Assume that $w \notin \frac{1}{n+1}\sum_{i=0}^{n} Fix(T^i)$. Since $w_{n_i} \rightharpoonup w$ and $t^n w \neq w$, from Opial's conditions (2.4) and Lemma 3.3 (P4), we have

$$\begin{aligned}
\liminf_{n\to\infty}\|w_{n_i} - w\| &< \liminf_{n\to\infty}\|w_{n_i} - t^n w\| \\
&\leq \liminf_{n\to\infty}(\|w_{n_i} - t^n w_{n_i}\| + \|t^n w_{n_i} - t^n w\|) \\
&\leq \liminf_{n\to\infty}\|w_{n_i} - w\|,
\end{aligned}$$

which is a contradiction. Thus, we obtain $w \in Fix(t^n)$. We show that $w \in \Gamma$. Since $u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \delta A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n)$, where $d_n = x_n + \delta A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n$, we have

$$F_1(u_n, y) + \psi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - d_n\rangle \geq 0, \qquad \forall y \in C.$$

It follows from the monotonicity of F_1 that

$$\psi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - d_n\rangle \geq F_1(u_n, y), \qquad \forall y \in C$$

which implies that

$$\psi_1(u_n, y) + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_n} + \delta A^*(\frac{(T_{s_{n_i}}^{(F_2,\psi_2)} - I)Ax_{n_i}}{r_n})\rangle \geq F_1(y, u_{n_i}), \qquad \forall y \in C.$$

Because of $\|u_n - x_n\| \to 0$, we get $u_{n_i} \rightharpoonup w$ and $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$.

Since $\lim_{n\to\infty}\|A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n\| = 0$ then $A^*(\frac{(T_{s_{n_i}}^{(F_2,\psi_2)} - I)Ax_{n_i}}{r_n}) \to 0$.

Therefore

$$\psi_1(u_{n_i}, y) \geq F_1(y, u_{n_i}), \qquad h_1(w, y) \geq F_1(y, w).$$

Let $y_t = ty + (1-t)w$ for all $t \in (0,1]$. Since $y \in C$ and $w \in C$, we get $y_t \in C$. It follows from Assumption 2.9 that

$$\begin{aligned}0 = F_1(y_t, y_t) + \psi_1(y_t, y_t) \quad &\leq tF_1(y_t, y) + (1-t)F_1(y_t, w)\\ &\quad +t\psi_1(y_t, y) + (1-t)\psi_1(y_t, w)\\ &= t(F_1(y_t, y) + \psi_1(y_t, y))\\ &\quad +(1-t)(F_1(y_t, w) + \psi_1(y_t, w))\\ &\leq F_1(y_t, y) + \psi_1(y_t, y),\end{aligned}$$

so $0 \leq F_1(y_t, y) + \psi_1(y_t, y)$.

Letting $t \to 0$, we obtain $0 \leq F_1(w, y) + \psi_1(w, y)$. This implies that $w \in GEP(F_1, \psi_1)$. Now we show that $Aw \in GEP(F_2, \psi_2)$. Since $\|u_n - x_n\| \to 0$, $u_n \rightharpoonup w$ as $n \to \infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ and since A is bounded linear operator so that $Ax_{n_j} \rightharpoonup Aw$.

Because of $\|(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n\| \to 0$, we have $T_{s_n}^{(F_2,\psi_2)} Ax_{n_j} \rightharpoonup Aw$. Therefore from Lemma 2.11, we have

$$\begin{aligned}&F_2(T_{s_{n_j}}^{(F_2,\psi_2)} Ax_{n_j}, v) + \psi_2(T_{s_{n_j}}^{(F_2,\psi_2)} Ax_{n_j}, v)\\ &\qquad + \frac{1}{s_{n_j}} \langle v - T_{s_{n_j}}^{(F_2,\psi_2)} Ax_{n_j}, T_{s_{n_j}}^{(F_2,\psi_2)} Ax_{n_j} - Aw\rangle \geq 0, \quad \forall v \in Q.\end{aligned}$$

Since F_2 is upper semicontinuous in first argument, taking lim sup to above inequality as $j \to \infty$, we obtain

$$F_2(Aw, v) + \psi_2(Aw, v) \geq 0, \qquad \forall v \in Q,$$

which means that $Aw \in GEP(F_2, \psi_2)$ and hence $w \in \Gamma$.

Now we show that $w \in I(E, M)$. It follows from Lemma 2.4 that $M + E$ is a maximal monotone. Let $(y, g) \in G(M + E)$, that is $g - Ey \in M(y)$.

Since $w_{n_i} = J_{M,\lambda}(u_{n_i} - \lambda E u_{n_i})$, we have $u_{n_i} - \lambda E u_{n_i} \in (I + \lambda M)(w_{n_i})$, then $\frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda E u_{n_i}) \in M(w_{n_i})$.

Since $M + E$ is a maximal monotone, we have

$$\langle y - w_{n_i}, g - Ey - \frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda E u_{n_i})\rangle \geq 0,$$

and so

$$\begin{aligned}\langle y - w_{n_i}, g\rangle \quad &\geq \langle y - w_{n_i}, Ey + \tfrac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda E u_{n_i})\rangle\\ &= \langle y - w_{n_i}, Ey - Ew_{n_i} + Ew_{n_i} - Eu_{n_i} + \tfrac{1}{\lambda}(u_{n_i} - w_{n_i})\rangle\\ &\geq 0 + \langle y - w_{n_i}, Ew_{n_i} - Eu_{n_i}\rangle + \langle y - w_{n_i}, \tfrac{1}{\lambda}(u_{n_i} - w_{n_i})\rangle.\end{aligned}$$

Since E is a $\bar{\gamma}_2$-inverse strongly monotone, we can easily observe that $\|Ew_n -$

$Eu_n\| \to 0$.
It follows from (3.21), $\|Ew_n - Eu_n\| \to 0$ and $w_{n_i} \rightharpoonup w$ that

$$\lim_{n\to\infty} \langle y - w_{n_i}, g\rangle = \langle y - w, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $M + E$ that $0 \in (M + E)(w)$, that is $w \in I(E, M)$.
We claim that $\limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle \leq 0$, where $z = P_\Theta(I - B + \gamma f)$. Now from (2.1), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle &= \limsup_{n\to\infty}\langle(\gamma f - B)z, t^n w_n - z\rangle \\ &\leq \limsup_{i\to\infty}\langle(\gamma f - B)z, t^n w_{n_i} - z\rangle \\ &= \langle(\gamma f - B)z, w - z\rangle \leq 0.\end{aligned} \tag{4.2}$$

Next, we show that $x_n \to z$. It follows from (3.3) that

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &= \alpha_n\langle\gamma f(x_n) - Bz, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\ &\quad + \langle((1-\beta_n)I - \alpha_n B)(t^n w_n - z) + \gamma_n e_n, x_{n+1} - z\rangle \\ &\leq \alpha_n(\gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle + \langle\gamma f(z) - Bz, x_{n+1} - z\rangle) + \beta_n\|x_n - z\|\|x_{n+1} - z\| \\ &\quad + \|(1-\beta_n)I - \alpha_n B\|\|t^n w_n - z\|\|x_{n+1} - z\| + \gamma_n\|e_n\|N \\ &\leq \alpha_n\alpha\gamma\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle\gamma f(z) - Bz, x_{n+1} - z\rangle + \beta_n\|x_n - z\|\|x_{n+1} - z\| \\ &\quad + (1 - \beta_n - \alpha_n\bar{\gamma}_1)\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|e_n\|N \\ &\leq \tfrac{1-\alpha_n(\bar{\gamma}_1 - \alpha\gamma)}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle\gamma f(z) - Bz, x_{n+1} - z\rangle + \gamma_n\|e_n\|N \\ &\leq \tfrac{1-\alpha_n(\bar{\gamma}_1 - \alpha\gamma)}{2}\|x_n - z\|^2 + \tfrac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n\langle\gamma f(z) - Bz, x_{n+1} - z\rangle + \gamma_n\|e_n\|N.\end{aligned}$$

This implies that

$$\begin{aligned}2\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n(\bar{\gamma}_1 - \alpha\gamma))\|x_n - z\|^2 + \|x_{n+1} - z\|^2 \\ &\quad + 2\alpha_n(\langle\gamma f(z) - Bz, x_{n+1} - z\rangle + \tfrac{\gamma_n\|e_n\|}{\alpha_n}N).\end{aligned}$$

Then we have

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n(\bar{\gamma}_1 - \alpha\gamma))\|x_n - z\|^2 + 2\alpha_n M_n, \tag{4.3}$$

where $k_n = \alpha_n(\bar{\gamma}_1 - \alpha\gamma)$ and $M_n = \langle \gamma f(z) - Bz, x_{n+1} - z\rangle + \frac{\gamma_n \|e_n\|}{\alpha_n} N$.
Since $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, it is easy to see that $\lim_{n\to\infty} k_n = 0$, $\Sigma_{n=0}^{\infty} k_n = \infty$ and $\limsup_{n\to\infty} M_n \leq 0$. Hence, from (4.2) and (4.3) and Lemma 2.8, we deduce that $x_n \to z$, where $z = P_{\Theta}(I - B + \gamma f)z$. □

5. Numerical Examples

In this section, we give some examples and numerical results for supporting our main theorem.

Example 5.1. *Let $H_1 = H_2 = R$, $C = [0, 2]$ and $Q = [-4, -2]$; let $F_1, \psi_1 : C \times C \to R$ and $F_2, \psi_2 : Q \times Q \to R$ be defined by $F_1(x, y) = x(y - x), \psi_1(x, y) = 2x(y-x)$, $\forall x, y \in C$ and $F_2(u, v) = -2u(u-v), \psi_2(u, v) = 3u(v-u)$, $\forall u, v \in Q$, and let for each $x \in R$, we define $f(x) = \frac{1}{6}x$, $A(x) = -2x$, $B(x) = \frac{1}{2}x$, $E(x) = 2x - 6$, and*

$$Mx = \begin{cases} \{x\}, & x > 2 \\ \{2\}, & x \leq 2 \end{cases}$$

and let, for each $x \in C$, $V_i x = -2\alpha_i x$, where $\alpha_i = i + 1$, $i = 0, 1, \cdots, 5$ and $e_n = \sin n$. Then there exist unique sequences $\{w_n\}$, $\{x_n\} \subset R$, $\{u_n\} \subset C$, and $\{z_n\} \subset Q$ generated by the iterative schemes

$$\begin{gathered} z_n = T_{s_n}^{(F_2,\psi_2)}(Ax_n); \qquad u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \tfrac{1}{32}A^*(z_n - Ax_n)); \\ w_n = (I + M)^{-1}(u_n - Eu_n); \end{gathered} \tag{5.1}$$

$$x_{n+1} = (\frac{1}{3n} + \frac{1}{2(n+1)^2})x_n + ((1 - \frac{1}{2(n+1)^2})I - \frac{1}{n}B)\frac{1}{n+1}\sum_{i=0}^{n} T^i w_n + \gamma_n e_n \tag{5.2}$$

where $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{2(n+1)^2}$, $\gamma_n = \frac{1}{n^3}$, $r_n = 1 + \frac{2}{n}$ and $s_n = \frac{n}{2n+1}$.
It is easy to prove that the bifunctions F_1, ψ_1 and F_2, ψ_2 satisfy the Assumption 2.9 and F_2 is upper semicontinuous, A is a bounded linear operator on R with adjoint operator A^* and $\|A\| = \|A^*\| = 1$. Hence $\delta \in (0, 1)$, so we can choose $\delta = \frac{1}{32}$. Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and B is a strongly positive bounded linear operator with constant $\bar{\gamma}_1 = \frac{1}{4}$ on R. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. And E is a inverse strongly monotone mapping on R with $\bar{\gamma}_2 \in (0, 1]$, then $\lambda \in (0, 2)$. We can choose $\lambda = 1$. Furthermore, it is easy to observe that $2 \in I(E, M)$, $2 \in EP(F_1, \psi_1)$, $-4 \in EP(F_2, \psi_2)$. Hence $\Theta = \{2\} \neq \emptyset$. After simplification, schemes (5.5) and (5.6) reduce to

$$\begin{gathered} z_n = -\tfrac{16n + (4n+2)x_n}{6n+1}; \\ u_n = \tfrac{592n^2 + 1248n + 192 + (88n^2 + 16n)x_n}{32(2n+3)(6n+1)}; \\ w_n = -u_n + 6 \end{gathered} \tag{5.3}$$

$$x_{n+1} = (\frac{1}{3n} + \frac{1}{2(n+1)^2})x_n + \frac{1}{6}(1 - \frac{1}{2(n+1)^2} - \frac{1}{2n})(24t - 20)w_n + \frac{1}{n^3}\sin n, \quad (5.4)$$

where $t \in [\frac{7}{9}, 1)$. *Following the proof of Theorem 4.1, we obtain that* $\{z_n\}$ *converges strongly to* $\{-4\} \in GEP(F_2, \psi_2)$ *and* $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ *converges strongly to* $w = \{2\} \in \bigcap_{i=0}^{3} Fix(T^i) \cap \Omega \cap I(E, M) \neq \emptyset$ *as* $n \to \infty$. *Figure* 1 *indicates the behavior of* x_n *for algorithm (5.4).*

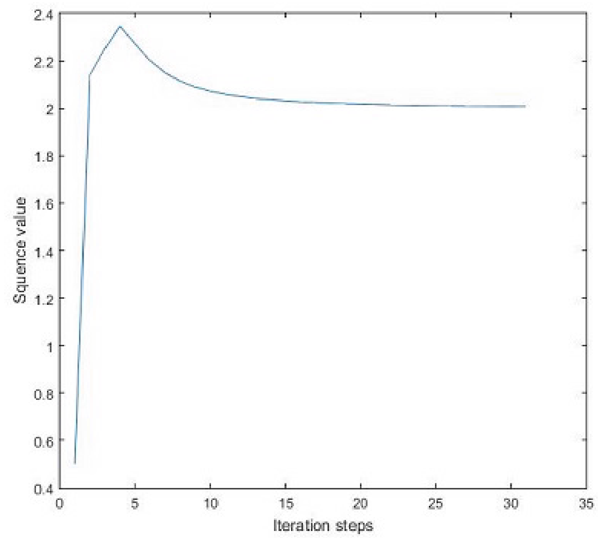

FIGURE 1. The graph of $\{x_n\}$ with initial value $x_1 = 0.5$.

Example 5.2. *Let* $H_1 = H_2 = R$, $C = [0, 4]$ *and* $Q = [0, 2]$; *let* $F_1 : C \times C \to R$ *and* $F_2 : Q \times Q \to R$ *be defined by* $F_1(x, y) = x(y - x)$, $\forall x, y \in C$ *and* $F_2(u, v) = -2u(u - v)$, $\forall u, v \in Q$, *and let for each* $x \in R$, *we define* $f(x) = \frac{1}{8}x$, $A(x) = -x$, $B(x) = x$, $E(x) = 2x$, *and*

$$Mx = \begin{cases} \{2x\}, & x > 0 \\ \{0\}, & x \leq 0 \end{cases}$$

and let, for each $x \in C$, $V_i x = -\alpha_i x$, *where* $\alpha_i = \frac{2}{i+1}$, $i = 0, 1, \cdots, 5$ *and* $e_n = \cos n$. *Then there exist unique sequences* $\{w_n\}$, $\{x_n\} \subset R$, $\{u_n\} \subset C$, *and* $\{z_n\} \subset Q$ *generated by the iterative schemes*

$$\begin{aligned} z_n = T_{s_n}^{F_2}(Ax_n); \qquad u_n = T_{r_n}^{F_1}(x_n + \tfrac{1}{4}A^*(z_n - Ax_n)); \\ w_n = (I + \tfrac{3}{2}M)^{-1}(u_n - \tfrac{3}{2}Eu_n); \end{aligned} \quad (5.5)$$

$$x_{n+1} = (\frac{1}{4\sqrt{n}} + \frac{1}{n+1})x_n + ((1 - \frac{1}{n+1})I - \frac{1}{\sqrt{n}}B)\frac{1}{n+1}\sum_{i=0}^{n} T^i w_n + \gamma_n e_n \quad (5.6)$$

where $\alpha_n = \frac{1}{\sqrt{n}},\ \beta_n = \frac{1}{n+1},\ \gamma_n = \frac{1}{n^2},\ r_n = 1 + \frac{1}{n}$ *and* $s_n = \frac{2n}{3n-1}$.
It is easy to prove that the bifunctions F_1 *and* F_2 *satisfy the Assumption 2.9 and* F_2 *is upper semicontinuous,* A *is a bounded linear operator on* R *with adjoint operator* A^* *and* $\|A\| = \|A^*\| = 1$. *Hence* $\delta \in (0,1)$, *so we can choose* $\delta = \frac{1}{4}$. *Further,* f *is contraction mapping with constant* $\alpha = \frac{1}{7}$ *and* B *is a strongly positive bounded linear operator with constant* $\bar{\gamma}_1 = 1$ *on* R. *Therefore, we can choose* $\gamma = 2$ *which satisfies* $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. *And* E *is a inverse strongly monotone mapping on* R *with* $\bar{\gamma}_2 \in (0,1]$, *then* $\lambda \in (0,2)$. *We can choose* $\lambda = \frac{3}{2}$. *Furthermore, it is easy to observe that* $0 \in I(E,M)$, $0 \in EP(F_1)$, $0 \in EP(F_2)$. *Hence* $\Theta = \{0\} \neq \emptyset$. *After simplification, schemes (5.5) and (5.6) reduce to*

$$z_n = \frac{(3n-1)x_n}{7n-1}; \qquad u_n = \frac{(18n-2)x_n}{4(7n-1)}; \qquad w_n = -\frac{1}{8}u_n \quad (5.7)$$

$$x_{n+1} = (\frac{1}{4\sqrt{n}} + \frac{1}{n+1})x_n + \frac{1}{1080}(1 - \frac{1}{n+1} - \frac{1}{\sqrt{n}})(227t - 67)w_n + \frac{1}{n^2}\cos n, \quad (5.8)$$

where $t \in [\frac{1}{3}, 1)$. *Following the proof of Theorem 4.1, we obtain that* $\{z_n\}$ *converges strongly to* $\{0\} \in EP(F_2)$ *and* $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ *converges strongly to* $w = \{0\} \in \bigcap_{i=0}^{5} Fix(T^i) \cap \Omega \cap I(E,M) \neq \emptyset$ *as* $n \to \infty$. *Figure* 2 *indicates the behavior of* x_n *for algorithm (5.8).*

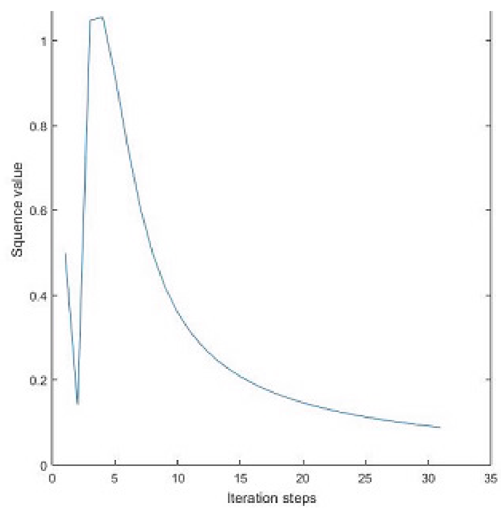

FIGURE 2. The graph of $\{x_n\}$ with initial value $x_1 = 0.45$.

References

[1] Blum, E., Oettli, W., From optimization and variational inequalities to equilibrium problems, *Math. Stud.,* 63 (1994), 123-145.

[2] Brezis, H., Operateurs Maximaux Monotones et Semi-Groups de contractions dans les Espaces de Hilbert , North-Holland Mathematics Studies, no. 5. Notas de Mathematica no. 50, North-Holland, Amsterdam, The Netherlands, 1973.

[3] Byrne, C., Censor, Y., Gibali, A., Reich, S., The split common null point problem, *J. Nonlinear Convex Anal.,* 13(4) (2012), 759-775.

[4] Censor, Y., Elfving, T., Amultiprojection algorithm using Bregman projections in product space, *Numer. Algorithm,* 8(1994)221-239.

[5] Censor, Y., Bortfeld, T., Martin, B., Trofimov, A., A unified approach for inversion problems in intensity modulated radiation theory, *Phys. Med. Biol.,* 51 (2006), 2353-2365.

[6] Censor, Y., Gibali, A., Reich, S., Algorithms for the split variational inequality problem, *Numer. Algorithm,* 59(2) (2012), 301-323.

[7] Chang, S. S., Lee, J., Chan, H. W., An new method for solving equilibrium problem, fixed point problem and variational inequality problem with application to optimization, *Nonlinear Analysis*, 70 (2009), 3307-3319.

[8] Cianciaruso, F., Marino, G., Muglia, L., Yao, Y., A hybrid projection algorithm for finding solution of mixed equilibrium problem and variational inequality problem, *Fixed Point Theory Appl.,* 2010 (2010), 383740.

[9] Crombez, G., A hicrarchical presentation of operators with fixed points on Hilbert spaces, *Numer. Funct. Anal. Optim.,* 27 (2006), 259-277.

[10] Kazmi, K. R., Rizvi, S. H., Iterative approximation of a common solution of a split generalized equilibrium problem and a fixed point problem for nonexpansive semigroup, *Math. Sci.,* 7 (2013), Art. 1.

[11] Kumam, P., Hamphries, U., Katchang, P., Common solution of generalized mixed equilibrium problems, variational inclusions, and common fixed points for nonexpansive semigroups and strictly pseudocontractive mappings, *Journal of Applied Mathematics,* 2011(2011).

[12] Mahdioui, H., Chadli, O., On a system of generalized mixed equilibrium problem involving variational-like inequalities in Banach spaces: existence and algorithmic aspects, *Advances in Operations Research*, 2012 (2012), 843486.

[13] Marino, G., Xu, H.K, A general iterative method for nonexpansive mappings in Hilbert spaces, *Math. Appl.,* 318 (2006), 43.-52.

[14] Moudafi, A. The split common fixed poin problem for demicontractive mappings, *Invers Probl.,* 26 (2010), 055007.

[15] Moudafi, A., Split monotone variational inclusions, *J. Optim. Theory Appl.,* 150 (2011), 275-283.

[16] Opial, Z., Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Am. Math. Soc.,* 73(4) (1967), 595-597.

[17] Suzuki, T., Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.,* 305 (2005), 227-239.

[18] Takahashi, S., Takahashi, W., Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, *Nonlinear Anal.,* 69 (2008), 1025-1033.

[19] Wang, S., On fixed point and variational inclusion problems, *Faculty of Sciences and Mathematics, University of Nis, Serbia*, 6(2015), 1409-1417.

[20] Xu, H. K., Viscosity approximation method for nonexpansive semigroups, *J. Math. Anal. Appl.,* 298 (2004), 279-291.

[21] Zhang, Y., Gui, Y., Strong convergence theorem for split equilibrium problem and fixed point problem in Hilbert spaces, *Int. Math.,* 12(9) (2017), 413-427.

[22] Zhang, S.-S., Lee, J. H. W., Chan, C. K., Algorithms of common solutions to quasi variational inclusion and fixed point problems, *Applied Mathematics and Mechanics*, 29(5) (2008), 571-581.

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A COMMON FIXED POINT THEOREM FOR MULTI-VALUED θ_δ CONTRACTIONS VIA SUBSEQUENTIAL CONTINUITY

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Abstract. The main objective of this paper is to present a common fixed point theorem for two pairs of single and set valued mappings via subsequential continuity and δ- compatibility. To illustrate the validity of our results, an example is provided and we give also an application for a system of integral inclusions of Volterra type.

1. Introduction

Fixed point theory is one of the important tools in the study of several problems in non linear analysis, physics, economics,.... Starting from Banach principle, some results and generalizations were given in this way. A common fixed point theorem generally involves conditions on commutativity, continuity and contractive condition of the given mappings, with completeness, or closedness of the underlying space or subspaces, along with conditions on suitable containment amongst the ranges of involved mappings. Sessa [20] has weakened the notion of commuting mappings to weakly commuting, later Jungck [14] introduced the concept of compatibility for a pair of self maps, which was extended to hybrid pair of mappings by Kaneko and Sessa [16]. Afterwards Jungck et al. [15] have furnished an extension to compatible mappings notion, called weak compatibility in the setting of single-valued and

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multi-valued mappings. Recently, Bouhadjera and Godet Tobie [7] introduced subsequential continuity which is weaker than the reciprocal continuity introduced by Pant [19]. In fact every non-vacuously pair of reciprocally continuous maps is naturally subsequentially continuous. However subsequentially continuous mappings are neither sequentially continuous nor reciprocally continuous. Quite recently, Beloul et al. [5] extended the notion of subsequential continuity to the context of set value maps in order to establish a common fixed point by using Hausdorff distance, while there is a function called δ-distance which defined by Fisher [10], although δ-distance is not a metric like the Hausdorff distance, but shares most of the properties of a metric, some results on δ-distance can be found in [1,4,6]. Common fixed point theorem commonly require commutativity, continuity, completeness together with a suitable condition on containment of ranges of involved maps beside an appropriate contraction condition. Thus, research in this field is aimed at weakening one or more of these conditions.
In this paper we will utilize a θ-contraction introduced by Jleli and Samet [12] and δ-distance to establish a strict coincidence and a strict common fixed point of a δ-compatible and subsequentially hybrid pair of mappings, without continuity or reciprocal continuity, weak reciprocal continuity, completeness and containment of ranges.

2. Preliminaries

Let (X, d) be a metric space, $B(X)$ is the set of all non-empty bounded subsets of X. For all $A, B \in B(X)$ we define the two functions:$D, \delta : B(X) \times B(X) \to \mathbb{R}_+$ such that

$$D(A, B) = \inf\{d(a, b); a \in A, b \in B\},$$

$$\delta(A, B) = \sup\{d(a, b); a \in A, b \in B\}.$$

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$ and $D(A, B) = D(a, B)$, also if $B = \{b\}$ is a singleton we write

$$\delta(A, B) = D(A, B) = d(a, b).$$

Clearly that δ satisfies the following properties:

$$\delta(A, B) = \delta(B, A) \geq 0,$$

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B),$$

$$\delta(A, A) = \operatorname{diam} A,$$

$$\delta(A, B) = 0 \text{ implies } A = B = \{a\},$$

for all $A, B, C \in B(X)$.

Notice that for all $a \in A$ and $b \in B$ we have

$$D(A, B) \leq d(a, b) \leq \delta(A, B),$$

where $A, B \in B(X)$.

Definition 1. *[20] Two mappings $S : X \to B(X)$ and $f : X \to X$ are to be weakly commuting on X if $fSx \in B(X)$ and for all $x \in X$:*

$$\delta(Sfx, fSx) \leq \max\{\delta(fx, Sx), diam(fSx)\}.$$

Definition 2. *[17] A hybrid pair of mappings (f, S) of a metric space (X, d) is δ-compatible if*

$$\lim_{n\to\infty} \delta(Sfx_n, fSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $fSx_n \in B(X), \lim_{n\to\infty} Sx_n = \{z\}$, and $\lim_{n\to\infty} fx_n = z$, for some $z \in X$.

Definition 3. *[19] The pair of self mappings (f, g) on a metric space(X, d) is said to be reciprocally continuous if*

$$\lim_{n\to\infty} fgx_n = ft$$

and

$$\lim_{n\to\infty} gfx_n = gt,$$

where $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X.

Later, Singh and Mishra [21] generalized the concept of reciprocal continuity to the setting of single and set-valued maps as follows.

Definition 4. *[21] Two maps $f : X \to X$ and $S : X \to B(X)$ are reciprocally continuous on X (resp. at $t \in X$) if and only if $fSx \in B(X)$ for each $x \in X$ (resp. $fSt \in B(X)$) and*

$$\lim_{n\to\infty} fSx_n = fM, \ \lim_{n\to} Sfx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = M \in B(X), \lim_{n\to\infty} fx_n = t \in M$

Definition 5. *[7] Two self-mappings f and g on a metric space (X, d) are said to be subcompatible if there exists a sequence $\{x_n\}$ such that:*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \ \text{ and } \ \lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

for some $t \in X$.

Definition 6. *[7] The pair (f, g) of self mappings is said to be subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} fgx_n = fz, \lim_{n\to\infty} gfx_n = gz$.*

Definition 7. *[5] Let $f : X \to X$ and $S : X \to CB(X)$ two single and set-valued mappings respectively, the hybrid pair (f, S) is to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} Sx_n = M \in CB(X) \quad \text{and} \quad \lim_{n\to\infty} fx_n = z \in M,$$

for some $z \in X$ and $\lim_{n\to\infty} fSx_n = fM, \lim_{n\to\infty} Sfx_n = Sz$.

Notice that continuity or reciprocal continuity implies subsequential continuity, but the converse may be not.

Example 8. *Let $X = [0,1]$ and d the euclidian metric, we define f, S by*

$$fx = \begin{cases} 1-x, & 0 \le x \le \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} < x \le 1 \end{cases} \qquad Sx = \begin{cases} [0,x], & 0 \le x \le \frac{1}{2} \\ [x-\frac{1}{2}, x], & \frac{1}{2} < x \le 1 \end{cases}$$

We consider a sequence $\{x_n\}$ such that for each $n \ge 1$ we have: $x_n = \frac{1}{2} - \frac{1}{n+1}$, clearly that $\lim_{n\to\infty} fx_n = \frac{1}{2} \in [0, \frac{1}{2}]$ and $\lim_{n\to\infty} Sx_n = [0, \frac{1}{2}] \in B(X)$, also we have:

$$\lim_{n\to\infty} fSx_n = \lim_{n\to\infty} [\frac{1}{2} + \frac{1}{n}, 1] = [\frac{1}{2}, 1] = f([0, \frac{1}{2}]),$$

and

$$\lim_{n\to\infty} Sfx_n = \lim_{n\to\infty} [\frac{1}{n}, \frac{1}{2} + \frac{1}{n}] = [0, \frac{1}{2}] = S(\frac{1}{2}),$$

then (f, S) is subsequentially continuous.
On the other hand, consider a sequence $\{y_n\}$ which defined for all $n \ge 1$ by: $y_n = \frac{1}{2} + \frac{1}{n+1}$, we have

$$\lim_{n\to\infty} fx_n = \frac{1}{2} \in [0,1], \quad \text{and} \quad \lim_{n\to\infty} Sx_n = [0,1] \in B(X),$$

however

$$\lim_{n\to\infty} fSx_n = \lim_{n\to\infty} f([\frac{1}{n}, 1 + \frac{1}{n}]) \ne f([0,1]),$$

then f and S are never reciprocally continuous.

Let Θ be the set of all functions $\theta : (0, +\infty) \to (1, +\infty)$ be a function satisfying:

(θ_1) : θ is non decreasing,
(θ_2) : for each sequence $\{t_n\}$ in $(0, +\infty)$, $\lim_{n\to\infty} t_n = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,
(θ_3) : there exists $r \in (0,1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Example 9. *For all $i \in \{1,2,3\}$, the following functions are elements of Θ.*
1) $\theta_1(t) = e^t$. 2) $\theta_2(t) = e^{te^t}$. 3) $\theta_3(t) = e^{\sqrt{t}}$. 4) $\theta_4(t) = e^{\sqrt{t}e^t}$.

Definition 10. *[12] Let (X, d) be a metric space and $T : X \to X$ be a mapping. For $\theta \in \Theta$, we say T is θ-contraction, if there exists $k \in [0,1]$ such that for $x, y \in X$, $d(Tx, Ty) > 0$ implies $\theta(d(Tx, Ty)) \le [\theta(d(x,y))]^k$.*

Theorem 11. *[12] Let (X, d) be a complete metric space and let $T : X \to X$ be an θ-contraction. Then T has a unique fixed point in X.*

3. Main results

In this section, we introduce a multivalued θ_δ-contraction and prove a common fixed point theorem for hybrid pair mappings with δ-distance.

Definition 12. *Let (X,d) be a metric space and $T : X \to B(X)$ be a mapping. For $\theta \in \Theta$, we say T is θ_δ-contraction, if there exists $k \in [0,1]$ such that for $x, y \in X$, $\delta(Tx,Ty) > 0$ implies $\theta(\delta(Tx,Ty)) \leq [\theta(d(x,y))]^k$.*

Definition 13. *Let f be a self mapping on a metric space (X,d) and let $T : X \to B(X)$ be a multivalued mapping. Then T is called generalized multivalued $(f, \theta_\delta$-contraction if for all $x, y \in X$ there exists $k \in [0,1]$ such that,*

$$\delta(Tx,Ty) > 0 \quad \textit{implies} \;\; \theta(\delta(Tx,Ty)) \leq [\theta(R(x,y))]^k,$$

where $\theta \in \Theta$

$$R(x,y) = \max\{d(fx,fy), D(fx,Tx), D(fy,Ty), \frac{1}{2}[D(fx,Ty) + D(fy,Tx)]\}.$$

Now we extend the last definition for two pairs of hybrid pair, in order to establish a common fixed point for set valued and single valued mapping in metric space, without continuity and completeness of space, we use only subsequential continuity with δ-compatibility.

Theorem 14. *Let $f, g : X \to X$ be single valued mappings and $S, T : X \to B(X)$ be multi-valued mappings of metric space (X,d). If the two pairs (f,S) and (g,T) are subsequentially continuous and δ-compatible. Then the pair (f,S) as well as (g,T) has a strict coincidence point. Moreover, f, g, S and T have a common strict fixed point provided that there exists $k \in (0,1)$ such that for all x, y in X we have:*

$$\delta(Sx,Ty) > 0 \;\; \textit{implies} \;\; \theta(\delta(Tx,Ty)) \leq [\theta(R(x,y))]^k, \tag{1}$$

where $\theta \in \Theta$. and

$$R(x,y) = \max\{d(fx,fy), D(fx,Tx), D(gy,Ty), \frac{1}{2}[D(fx,Ty) + D(fy,Tx)]\}.$$

Proof. Since (f,S) is subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = M \in B(X), \quad \lim_{n\to\infty} fx_n = z \in M.$$
$$\lim_{n\to\infty} fSx_n = fM, \quad \lim_{n\to\infty} Sfx_n = Sz,$$

Also, the pair (f,S) is δ-compatible implies that

$$\lim_{n\to\infty} \delta(fSx_n, Sfx_n) = \delta(fM, Sz) = 0,$$

which gives that $fM = Sz = \{fz\}$, and so z is a coincidence point of f and S. Similarly, for the pair (g,T) there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ty_n = N \in B(X) \quad \text{and} \quad \lim_{n\to\infty} gy_n = t \in N$$

and

$$\lim_{n\to\infty} gTy_n = gN, \quad \lim_{n\to\infty} Tgy_n = Tt.$$

The pair (g, T) is δ-compatible, implies that

$$\lim_{n\to\infty} \delta(gTy_n, Tgy_n) = \delta(gN, Tt) = 0.$$

Then $gN = Tt$ and Tt is a singleton, i.e, $Tt = \{gt\}$ and t is a strict coincidence point of g and T.
Now, we claim $fz = gt$, if not so $\delta(Sz, Tt) > 0$, otherwise

$$d(fz, gt) \leq \delta(Sz, Tt) = 0,$$

which is a contradiction. Then by using (1), we get:

$$\theta(\delta(Sz, Tt)) \leq [\theta(R(z, t))]^k.$$

Since $Sz = \{fz\}$ and $Tt = \{gt\}$, we get

$$D(fz, Sz) = D(gt, Tt) = 0,$$

$$D(fz, Tt) = d(fz, gt)$$

and $D(gt, Sz) = d(fz, gt)$. Hence

$$R(z, t) = \max\{d(fz, gt), D(fz, Sz), D(gt, Tt), \frac{1}{2}(D(fz, Tt) + D(gt, Sz))\}.$$

Subsisting in (1) we get

$$\theta(d(fz, gt)) = \theta(\delta(Sz, Tt)) \leq [\theta(d(fz, gt))]^k < \theta(d(fz, gt)),$$

which is a contradicts that $\theta(t) > 1$ for all $t \geq 0$. Then $fz = gt$.
Now we claim $z = fz$, if not by taking $x = x_n$ and $y = t$ in (1), $\delta(Sx_n, Tt) > 0$, otherwise letting $n \to \infty$, we get

$$d(z, fz) = d(z, gt) \leq \delta(M, Tt) = 0,$$

which contradicts that $z \neq fz$, and so we have

$$\theta(\delta(Sx_n, Tt)) \leq [\theta(R(x_n, t))]^k.$$

Letting $n \to \infty$, we get $M(x_n, t) \to d(z, fz)$ and so we have:

$$\theta(d(z, fz)) \leq \theta(\delta(M, Tt) \leq [\theta(d(z, fz))]^k < \theta(d(z, fz)),$$

which is a contradiction. Hence z is a fixed point for f and S.
We will show $z = t$, if not by taking $x = x_n$ and $y = y_n$ in (1), $\delta(Sx_n, Ty_n) > 0$, if not letting $n \to \infty$, we get:

$$d(z, t) \leq \delta(M, N) = 0,$$

which is a contradiction, so we have:

$$\theta[\delta(Sx_n, Ty_n)] \leq \theta[R(x_n, y_n)]^k$$

Taking $n \to \infty$, we get:

$$\theta(d(z,t)) < \theta(\delta(M,N) \leq [\theta(d(z,t))]^k < \theta(d(z,t)),$$

which is a contradiction. Hence $z = t$ and consequently z is a common fixed point for f, g, S and T. For the uniqueness, let w be another fixed point, by and using (1), $\delta(Sz, Tw) > 0$, if not $d(z,t) \leq \delta(Sz, Tt) = 0$, which is a contradiction. Then we have:

$$\theta(d(z,w)) < \theta(\delta(Sz,Tw)) \leq [\theta(d(z,w))]^k < \theta(d(z,w)),$$

which is a contradiction. Then z is unique. □

If $f = g$ and $S = T$ we obtain the following corollary:

Corollary 15. *Let $f : X \to X$ be a single valued mapping and $S : X \to B(X)$ be a multi-valued mapping of metric space (X,d). Suppose that the pair (f,S) is subsequentially continuous, as well is δ-compatible and there exist $\theta \in \Theta$ and $k \in [0,1)$ such that for all x, y in X we have:*

$$\delta(Sx,Sy) > 0 \ \textit{ implies } \ \theta(\delta(Sx,Sy)) \leq [\theta(M(x,y))]^k,$$

where

$$R(x,y) = \max\{d(fx,fy), D(fx,Sx), D(fy,Sy), \frac{1}{2}[D(fx,Sy) + D(fy,Sx)]\}.$$

Then f and S have a strict common fixed point.

If S and T are single valued maps, we get the following corollary:

Corollary 16. *Let (X,d) a metric space and let $f, g, S, T : X \to X$ be self mappings, if the pair (f,S) is subsequentially continuous and compatible as well as (g,T). Then f and S have a coincidence point as well as g and T. Moreover, f, g, S and T have a common fixed point provided that there exists $k \in [0,1)$ and $\theta \in \Theta$ such that for all x, y in X we have:*

$$d(Sx,Ty) > 0 : \ \textit{implies}: \ \theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^k,$$

where

$$R(x,y) = \max\{d(fx,gy), d(fx,Sx), d(gy,Ty), \frac{1}{2}[d(fx,Ty) + d(gy,Sx)]\}.$$

Example 17. *Let $X = [0,2]$, $d(x,y) = |x-y|$ and f, g, S and T defined by*

$$fx = gx = \begin{cases} \frac{x+1}{2}, & 0 \leq x \leq 1 \\ 0, & 1 < x \leq 2 \end{cases} \qquad Fx = Tx = \begin{cases} \{1\}, & 0 \leq x \leq 1 \\ [\frac{5}{4}, 2], & 1 < x \leq 2 \end{cases}$$

Consider a sequence $\{x_n\}$ for all $n \geq 1$ such that $x_n = 1 - \frac{1}{n}$, it is clear that

$$\lim_{n\to\infty} fx_n = 1 \in \{1\}$$

and

$$\lim_{n\to\infty} Tx_n = \{1\},$$

which implies that the pair (f,T) is subsequentially continuous. On other hand, we have

$$\lim_{n\to\infty}\delta(fTx_n,Tfx_n)=\delta(\{1\},\{1\})=0.$$

so (f,S) is δ-compatible.
For the inequality (1), by taking $\theta(t)=e^t$ and $k=\frac{9}{10}$, we discuss the following cases:

(1) *For $x,y\in[0,1]$, we have $\delta(Tx,Ty)=0$.*
(2) *For $x\in[0,1]$ and $y\in(1,2]$, we have:*

$$\delta(Tx,Ty)=1\leq\frac{3}{2}\leq\frac{9}{10}D(fy,Ty),$$

which implies

$$e^{\delta(Tx,Ty)}\leq(e^{D(fy,Ty)})^{\frac{9}{10}}\leq(e^{R(x,y)})^{\frac{9}{10}}.$$

(3) *For $x\in(1,2]$ and $y\in[0,1]$, we have*

$$\delta(Tx,Ty)=1\leq\frac{3}{2}\leq\frac{9}{10}D(fy,Ty),$$

this yields

$$e^{\delta(Tx,Ty)}\leq(e^{D(fx,Tx)})^{\frac{9}{10}}\leq(e^{R(x,y)})^{\frac{9}{10}}.$$

(4) *For $x,y\in(1,2]$ we have*

$$\delta(Tx,Ty)=1\leq\frac{3}{2}\leq\frac{9}{10}D(fx,Tx),$$

then

$$e^{\delta(Tx,Ty)}\leq(e^{D(fx,Tx)})^{\frac{9}{10}}\leq(e^{R(x,y)})^{\frac{9}{10}}.$$

hence f and T satisfy (1), therefore 1 is the unique common strict fixed point of f and S.

4. Application to integral inclusions

In this section, we apply the obtained results to assert the existence of solution for a system of integral inclusions.
Consider the following integral inclusions system's.

$$x_i(t)\in g(t)+\int_0^t K_i(t,s,x_i(s))ds, i=1,2 \tag{2}$$

where g is a continuous function on $[0,1]$, i,e., $f\in C([0,1],\mathbb{R})$ and $K_i:[0,1]\times[0,1]\times\mathbb{R}\to CB(\mathbb{R})$ are a set valued functions.
Clearly $X=C([0,1])$ with convergence uniform metric's $d_\infty(x,y)=\sup_{x\in X}|x(t)-y(t)|$ is a complete metric space. Define two set valued mappings:

$$Sx_1(t)=\{z\in X, z(t)\in f(t)+\int_0^t K_1(t,s,x_1(s))ds\},$$

$$Tx_2(t) = \{z \in X, z(t) \in f(t) + \int_0^t K_2(t,s,x_2(s))ds\}.$$

Assume that;

A_1 : The function $K_i : (t,s) \mapsto K_i(t,s,x_i(s))$ are continuous on $[0,1] \times (0,1]$ for all $x \in C((0,1])$;

A_2 : For all $x_i \in X$ and $k_i \in K_i$ $(i = 1,2)$, there exists a function $\varphi : [0,1] \times [0,1] \to [0,+\infty)$ such that

$$|k_1(t,s,x_1(s)) - k_2(t,s,x_2(s))| \leq \varphi(t,s)|x_1 - x_2|;$$

A_3 : There exists $\tau > 0$ such that

$$\sup_{t\in[0,1]} \int_0^t \varphi(t,s)ds \leq e^{-\tau};$$

A_4 : There exist two sequences $\{x_n\}, \{y_n\}$ and two elements x, y in X such that

$$\lim_{n\to\infty} Sx_n = M \in B(X),$$

$$\lim_{n\to\infty} x_n = x \in M$$

and

$$\lim_{n\to\infty} Ty_n = N \in B(X),$$
$$\lim_{n\to\infty} y_n = y \in N.$$

Theorem 18. *Under assumptions* $(A_1)-(A_4)$ *the system of integral inclusions (2) has a solution in* $C((0,1]) \times C([0,1])$.

Proof. The system (2) has a solution if and only if S and T have a common fixed point.
Denote I_X the identity operator on X.
From condition (4), the two pairs (I_X, S) and (I_X, T) are subsequentially continuous as well as δ-compatible.

For the contractive condition (1), let $x_1, x_2 \in C([0,1])$ and $z_1 \in Sx_1$, then there exists $k_1 \in K_1$ such that

$$z_1(t) = f(t) + \int_0^t k_1(t,s,x_1(s))ds.$$

Let $z_2 \in f(t) + \int_0^t K_2(t,s,x_2(t))ds$, i.e.,

$$z_2(t) = f(t) + \int_0^1 k_2(t,s,x_2(s))ds,$$

for some $k_2 \in K_2$, so we have

$$|z_1 - z_2| \leq \int_0^t |k_1(t,s,x_1(s)) - k_2(t,s,x_2(s))|ds$$

$$\leq \int_0^t |x_1 - x_2|\varphi(t,s)ds.$$

Since $K_i, i = 1,2$ are bounded, so we have

$$\sup_{z_i \in X} |z_1 - z_2| \leq \|x_1 - x_2\|_\infty \int_0^t \varphi(t,s)ds,$$

which implies that

$$\delta(Sx_1, Tx_2) \leq e^{-\tau} d(x_1, x_2)$$

$$\leq e^{-\tau} \max\{d(x_1,x_2), d(x_1,Sx_1), d(x_2,Tx_2), \frac{1}{2}(d(x_1,Tx_2) + d(x_2,Sx_1))\}.$$

Since θ is non decreasing we get

$$e^{\sqrt{\delta(Sx_1,Tx_2)}} \leq \left(e^{\frac{\sqrt{L(x_1,x_2)}}{2}}\right)^{e^{-\tau}},$$

where $L(x_1,x_2) = \max\{d(x_1,x_2), d(x_1,Sx_1), d(x_2,Tx_2), \frac{1}{2}(d(x_1,Tx_2)+d(x_2,Sx_1))\}$. Hence all hypotheses of Theorem 14 are satisfied, with $\theta(t) = e^{\sqrt{t}}$, $k = e^{-\tau}$ and $f = g = I_X$, therefore the system (2) has a solution. □

5. Conclusion

We have established common fixed point theorems for two hybrid pairs θ-contraction using δ-distance without exploiting the notion of continuity or reciprocal continuity, weak reciprocal continuity. Since θ-contraction is a proper generalization of ordinary contraction, our results generalize, extend and improve the results of Jleli and Samet [12] and others existing in literature, without using completeness of space or subspace, containment requirement of range space.

References

[1] Acar, O., A Fixed Point Theorem for multivalued almost F_δ-contraction, *Results Math.,* 72(3) (2017), 1545-1553.

[2] Beloul, S., Common fixed point theorems for weakly subsequentially continuous generalized contractions with applications, *Appl. Maths. E-Notes,* 15 (2015), 173-186.

[3] Beloul, S., Chauhan, S., Gregus type fixed points for weakly subsequentially continuous mappings satisfying strict contractive condition of integral type, *Le Mathematiche,* 71(2) (2017), 3-15.

[4] Beloul, S., Common Fixed Point Theorem for strongly tangential and weakly Compatible mappings satisfying implicit relations, *Thai. J. Math.,* 15(2) (2017), 349-358.

[5] Beloul, S., Tomar, A., A coincidence and common fixed point theorem for subsequentially continuous hybrid pairs of maps satisfying an implicit relation, *Math. Moravica,* 21(2) (2017), 15-25.

[6] Beloul, S., Kaddouri, H., Fixed Point Theorems For Subsequentially Multi-Valued F_δ-Contractions In Metric Spaces, *Facta Univ Nis Ser. Math. Inform.,* 35(2) (2020), 379-392.

[7] Bouhadjera, H., Godet Thobie, C., Common fixed point theorems for pairs of subcompatible maps, .(2009), arXiv:0906.3159v1 [math.FA].

[8] Ćirić, L. B., Generalization of Banach's Contraction Principle, *Proc. Amer. Math. Soc.,* 45 (1974), 267-273.

[9] Durmaz, G., Some theorems for a new type of multivalued contractive maps on metric space, *Turkish J. Math.,* 41, (2017), 1092-1100.

[10] Fisher, B., Fixed points for set-valued mappings on metric spaces, *Bull. Malays. Math. Sci. Soc.,* 2(4) (1981), 95-99.

[11] Isik. H, Ionescu, C., New type of multivalued contractions with related results and applications, *U.P.B. Sci. Bull., Series A,* 80(2) (2018), 13-22.

[12] Jleli. M, Samet, B., A new generalization of the Banach contraction principle, *J. Inequal. Appl.,* 2014:38, (2014), 8 pp.

[13] Jungck, G., Commuting mappings and fixed points, *Amer. Math. Monthly*, 83(4) (1976), 261-263.

[14] Jungck, G., Compatible mappings and common fixed points, I*nt.J.Math.and Math. Sci.,* 9(4) (1986), 771-779.

[15] Jungck, G., Rhoades, B.E.,Fixed point theorems for occasionally weakly compatible mappings, *Fixed Point Theory,* 9 (2008), 383-384.

[16] Kaneko, H., Sessa, S., Fixed point theorems for compatible multi-valued and single-valued mappings, *Internat. J. Math. Math. Sci.,* 12(2) (1989), 257-262.

[17] Liu, C., Li-Shan, Common fixed points of a pair of single valued mappings and a pair of set valued mappings, *Qufu Shifan Daxue Xuebao Ziran Kexue Ban,* 18 (1992), 6-10.

[18] Nadler, S. B., Multi-valued contraction mappings, *Pacific J. Math.,* 30 (1969), 475-488.

[19] Pant, R. P., A common fixed point theorem under a new condition, *Indian J. Pure Appl. Math.* 30(2) (1999), 147-152.

[20] Sessa, S., On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math. Beograd,* 32(46) (1982), 149-153.

[21] Singh, S. L., Mishra, S. N., Coincidence and fixed points of reciprocally continuous and compatible hybrid maps, *Internat. J. Math. Math. Sci.,* 10 (2002), 627-635.

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A-STATISTICALLY LOCALIZED SEQUENCES IN n-NORMED SPACES

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Abstract. In 1974, Krivonosov defined the concept of localized sequence that is defined as a generalization of Cauchy sequence in metric spaces. In this present work, the A-statistically localized sequences in n-normed spaces are defined and some main properties of A-statistically localized sequences are given. Also, it is shown that a sequence is A-statistically Cauchy iff its A-statistical barrier is equal to zero. Moreover, we define the uniformly A-statistically localized sequences on n-normed spaces and investigate its relationship with A-statistically Cauchy sequences.

1. Introduction and Background

In 1922, Banach defined normed linear spaces as a set of axioms. Since then, mathematicians keep on trying to find a proper generalization of this concept. The first notable attempt was by Vulich [41]. He introduced K-normed space in 1937. In another process of generalization, Siegfried Gähler [5] introduced 2-metric in 1963. As a continuation of his research, Gähler [6] proposed a mathematical structure, called 2-normed space, as a generalization of normed linear spaces. A.H. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with Gähler and Gupta [8] also provide valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by Siddiqi [40]. As a further extension, he introduced n-metric and n-norm in his subsequent works Gähler [7] and regarded normed linear spaces as 1-normed spaces. However, many researchers disagree to consider 2-norm and n-norm as generalization of norm. In spite of this disagreement, several researchers have

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worked on this topic for decades Gürdal and Pehlivan [10,11], Gürdal and Açık [12], Gürdal and Şahiner [13], Gürdal et al. [14], Mohiuddine et al. [23], Mursaleen [24], Savaş and Sezer [37], Savaş and Gürdal [31–33], Savaş et al. [34] and Yegül and Dündar [45,46]. They have found out many interesting properties of this space and lots of fixed point theorems are established.

This paper was inspired by Krivonosov [18], where the concept of a localized sequence was introduced, which can be treated as a generalization of a Cauchy sequence in metric spaces. We will often quote some results from Krivonosov [18], that can be easily transferred to the concept of A-statistically localized sequence and the A-statistical localor of a sequence in n-normed space. Let X is a metric space with a metric $d(\cdot,\cdot)$ and (x_n) is a sequence of points in X. It is an interesting fact that if $F : X \to X$ is a mapping with the condition $d(Fx, Fy) \leq d(x, y)$ for all $x, y \in X$, then for every $x \in X$ the sequence $(F^n x)$ is localized at every fixed point of the mapping F. This means that fixed points of the mapping F is contained in the localor of the sequence $(F^n x)$. Motivating the above facts and the fact that the localor of a sequence can be extended by changing the usual limit to the statistical limit (see Fridy [4]) of a sequence. Recently, the authors in [25] have extended the concepts and results, which were given in [18], by changing the usual limit to the statistical limit in metric spaces. This definition has been extended to statistical localized and ideal localized in metric space Nabiev et al. [25,26] and in 2-normed spaces Yamancı et al. [43,44], and they obtained interested results about this concept.

This paper consists of three sections with the new results in sections 2-3. In Section 2 the concept of the A-statistically localized sequence and the A-statistical localor of a sequence in n-normed space is introduced and fundamental properties of A-statistically localized sequences are studied. In Section 3, we prove that a sequence is A-statistically Cauchy sequence if and only if its A-statistical barrier is equal to zero. Moreover, we define the uniformly A-statistically localized sequences on n-normed spaces and investigate its relationship with A-statistically Cauchy sequences and prove that in n-normed linear spaces each A-statistically bounded sequence has everywhere A-statistically localized subsequence.

Throughout this paper, let A be a nonnegative regular matrix and $\mathbb{N}$ will denote the set of all positive integers. Let X and Y be two sequence spaces and $A = (a_{n_k})$ be an infinite matrix. If for each $x \in X$ the series $A_n(x) = \sum_{k=1}^{\infty} a_{n_k} x_k$ converges for each n and the sequence $Ax = \{A_n(x)\} \in Y$, we say that A maps X into Y. By (X, Y) we denote the set of all matrices which maps X into Y. In addition if the limit is preserved, then we denote the class of such matrices by $(X, Y)_{\text{reg}}$. A matrix A is called regular if $A \in (c, c)$ and $\lim_{k\to\infty} A_k(x) = \lim_{k\to\infty} x_k$ for all $x = \{x_k\}_{k\in N} \in c$ when c, as usual, stands for the set of all convergent sequences. It is well known that the necessary and sufficient condition for A to be regular are

i) $\|A\| = \sup_n \sum\limits_k |a_{n_k}| < \infty$;
ii) $\lim a_{n_k} = 0$, for each k;
iii) $\lim_n \sum\limits_k a_{n_k} = 1$.

The idea of A-statistical convergence was introduced by Kolk [17] using a nonnegative regular matrix A. For a nonnegative regular matrix $A = (a_{n_k})$, a set $K \subset \mathbb{N}$ will be said to have A-density if $\delta_A(K) = \lim_{n\to\infty} \sum\limits_{k\in K} a_{n_k}$ exists. The real number sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be A-statistically convergent to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in N : |x_k - L| \geqslant \varepsilon\}$ has A-density zero. Note that the idea of A-statistical convergence is an extension of the idea of statistical convergence introduced by Fast [3] using the idea of asymptotic density and later studied by Fridy [4], Connor [1], Salat [29], Gürdal and Yamancı [15], Mohiuddine and Alamri [20], Yamancı and Gürdal [42] and Savaş [30] (also, see [16,19,21,22,35,36,38]). Let $K = \{k(j) : k(1) < k(2) < k(3) < ...\} \subset \mathbb{N}$ and $\{x\}_K = \left\{x_{k(j)}\right\}$ be a subsequence of x. If the set K has A-density zero (i.e. $\delta_A(K) = 0$) the subsequence $\{x\}_K$ of the sequence x is called an A-thin subsequence. If the set K does not have A-density zero, the subsequence $\{x\}_K$ is called an A-nonthin subsequence of x. The statement $\delta_A(K) \neq 0$ means that either $\delta_A(K) > 0$ or $\delta_A(K)$ is not defined (i.e. K does not have A-density).

In [2], Connor and Kline extended the concept of a statistical limit (cluster) point of a number sequence x to a A-statistical limit (cluster) point replacing the matrix C_1 by a nonnegative regular matrix A. Recall that the number λ is a A-statistical limit point of the number sequence x provided that there is a subset $K = \{k(j)\}_{j=1}^{\infty}$ of positive integers with $\delta_A(K) \neq 0$ and $x_{k(j)} \to \lambda$ is $j \to \infty$ (see [2]). The number γ is a A-statistical cluster point of the number sequence $x = (x_k)$ provided that for every $\varepsilon > 0$, $\delta_A(K_\varepsilon) \neq 0$ where $K_\varepsilon := \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$ (see [2]).

Now we recall the n-normed space which was introduced in [9] and some definitions on n-normed space (see [39]).

Definition 1. *Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $\|.,...,.\|$ on X^n satisfying the following four properties*

(i) $\|x_1, x_2, ..., x_n\| = 0$ *if and only if $x_1, x_2, ..., x_n$ are linearly dependent;*
(ii) $\|x_1, x_2, ..., x_n\|$ *is invariant under permutation;*
(iii) $\|x_1, x_2, ..., x_{n-1}, \alpha x_n\| = |\alpha| \ \|x_1, x_2, ..., x_{n-1}, x_n\|$, *for any $\alpha \in \mathbb{R}$;*
(iv) $\|x_1, x_2, ..., x_{n-1}, y + z\| \leq \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$,
is called an n-norm on X and the pair $(X, \|.,...,.\|)$ is called an n-normed space.

It is well-known fact from the following corollary that n-normed spaces are actually normed spaces (see also [7]).

Corollary 1. *([9]) Every n-normed space is an $(n-r)$-normed space for all $r = 1, ..., n-1$. In particular, every n-normed space is a normed space.*

Example 1. *A standard example of an n-normed space is $X = \mathbb{R}^n$ equipped with the n-norm is*

$\|x_1, x_2, ..., x_{n-1}, x_n\| :=$ *the volume of the n-dimensional parallelepiped spanned by $x_1, x_2, ..., x_{n-1}, x_n$ in X.*

Observe that in any n-normed space $(X, \|., ..., .\|)$ we have

$$\|x_1, x_2, ..., x_{n-1}, x_n\| \geq 0$$

and

$$\|x_1, x_2, ..., x_{n-1}, x_n\| = \|x_1, x_2, ..., x_{n-1}, x_n + \alpha_1 x_1 + ... + \alpha_{n-1} x_{n-1}\|$$

for all $x_1, x_2, ..., x_n \in X$ and $\alpha_1, ..., \alpha_{n-1} \in \mathbb{R}$.

Let X be a real inner product space of dimension $d \geq n$. Equip X with the standard n-norm

$$\|x_1, x_2, ..., x_n\|_S := \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{1/2},$$

where $\langle ., . \rangle$ denotes the inner product on X. If $X = \mathbb{R}^n$, then this n-norm is the same as the n-norm in Example 1.

Definition 2. *A sequence $\{x_k\}$ in an n-normed space $(X, \|., ..., .\|)$ is said to convergent to an $l \in X$ if*

$$\lim_{k \to \infty} \|x_k - l, z_1, z_2, ..., z_{n-1}\| = 0$$

for every $z_1, z_2, ..., z_{n-1} \in X$.

Definition 3. *A sequence $\{x_k\}$ in an n-normed space $(X, \|., ..., .\|)$ is called a Cauchy sequence if*

$$\lim_{k,l \to \infty} \|x_k - x_l, z_1, z_2, ..., z_{n-1}\| = 0$$

for every $z_1, z_2, ..., z_{n-1} \in X$.

Let $a, x_1, ..., x_{n-1} \in X$ and for each $\varepsilon > 0$ define the ε-neighborhood of the points $a, x_1, ..., x_{n-1}$ as the set

$$U_\varepsilon (a, x_1, ..., x_{n-1}) = \{z : \|a - z, x_1 - z, ..., x_{n-1} - z\| < \varepsilon\}.$$

As it is known (see [28]) that the family of all sets

$$W_\Sigma = \bigcap_{i=1}^{n} U_{\varepsilon_i} \left(a, x_{1i}, ..., x_{(n-1)i}\right)$$

with arbitrary pairs $\Sigma = \left\{ \left(x_{11}, ..., x_{(n-1)1}, \varepsilon_1 \right), ..., \left(x_{1n}, ..., x_{(n-1)n}, \varepsilon_n \right) \right\}$ forms a complete system of neighborhoods of the point $a \in X$. Note that a set M in a linear n-normed space $(X, \|., ..., .\|)$ is said to be bounded if $\beta(M) < \infty$, where

$$\beta(M) = \sup \{ \|a - z, x_1 - z, ..., x_{n-1} - z\| : a, x_1, ..., x_{n-1}, z \in M \}.$$

We also suppose that for any $\varepsilon > 0$ there exists a neighborhood U of 0 such that $\|x_1^*, x_2^*, ..., x_n^*\| < \varepsilon$ for all points $x_1^*, x_2^*, ..., x_n^*$ in U.

2. Definitions and notations

In this section, we introduce some basic definitions and notations in n-normed space $(X, \|., ..., .\|)$.

Definition 4. (a) *A sequence (x_n) in n-normed space $(X, \|., ..., .\|)$ is said to be A-statistically localized in the subset $K \subset X$ if the sequence $\|x_n - x, z_1, z_2, ..., z_{n-1}\|$ A-statistically converges for all $x, z_1, z_2, ..., z_{n-1} \in K$.*

(b) *the maximal set on which a sequence is A-statistically localized is said to be a A-statistical localor of the sequence. We denote by $\mathrm{loc}^{\mathrm{st}_A}(x_n)$ the A-statistically localor of the sequence (x_n).*

(c) *A sequence (x_n) in n-normed space $(X, \|., ..., .\|)$ is said to be A-statistically localized everywhere if the A-statistical localor of (x_n) coincides with X.*

(d) *A sequence (x_n) in n-normed space $(X, \|., ..., .\|)$ is called A-statistically localized in itself if the A-statistically localor contains x_n for almost all n, that is,*

$$\delta_A\left(\left\{n : x_n \notin \mathrm{loc}^{\mathrm{st}_A}(x_n)\right\}\right) = 0 \ \textit{or} \ \delta_A\left(\left\{n : x_n \in \mathrm{loc}^{\mathrm{st}_A}(x_n)\right\}\right) = 1.$$

(e) *A sequence (x_n) is said to be A-statistically localized if the $\mathrm{loc}^{\mathrm{st}_A}(x_n)$ is not empty.*

Definition 5. *Let (x_n) be a sequence in an n-normed space $(X, \|., ..., .\|)$. Then the sequence (x_n) is said to be A-statistical convergent to L if for each $\varepsilon > 0$ and any nonzero $z_1, z_2, ..., z_{n-1}$ in X,*

$$\delta_A(\{k \in \mathbb{N} : \|x_n - L, z_1, z_2, ..., z_{n-1}\| \geq \varepsilon\}) = 0.$$

In this case we write $x_n \overset{st_A}{\to} L$ or

$$st_A - \lim_{n \to \infty} \|x_n - L, z\| = 0.$$

Definition 6. *A sequence (x_n) in a linear n-normed space $(X, \|., ..., .\|)$ is said to be a A-statistically Cauchy sequence in X if for every $\varepsilon > 0$ and any nonzero $z_1, z_2, ..., z_{n-1} \in X$ there exists a number $N = N(\varepsilon, z_1, z_2, ..., z_{n-1})$ such that*

$$\delta_A(\{k \in \mathbb{N} : \|x_k - x_m, z_1, z_2, ..., z_{n-1}\| \geq \varepsilon\}) = 0$$

for all $m \geq N$.

We can see from the above definitions that every A-statistically Cauchy sequence in n-normed space $(X, \|., ..., .\|)$ is A-statistically localized everywhere in $(X, \|., ..., .\|)$. Actually, due to

$$|\|x_n - L, z_1, z_2, ..., z_{n-1}\| - \|x - x_m, z_1, z_2, ..., z_{n-1}\|| \leqslant \|x_n - x_m, z_1, z_2, ..., z_{n-1}\|,$$

we get

$$\begin{aligned}&\{n \in \mathbb{N} : \|x_n - x_m, z_1, z_2, ..., z_{n-1}\| \geqslant \varepsilon\}\\ &\supset \{n \in \mathbb{N} : |\|x_n - L, z_1, z_2, ..., z_{n-1}\| - \|x_m - L, z_1, z_2, ..., z_{n-1}\|| \geqslant \varepsilon\}.\end{aligned}$$

Hence, the number sequence $\|x_n - L, z_1, z_2, ..., z_{n-1}\|$ is an A-statistically Cauchy sequence, then $\|x_n - L, z_1, z_2, ..., z_{n-1}\|$ is A-statistically convergent for every $L \in X$ and every nonzero $z \in X$. So, $\|x_n - L, z_1, z_2, ..., z_{n-1}\|$ in n-normed space $(X, \|., ..., .\|)$ is A-statistically localized everywhere.

Lemma 1. *A sequence (x_n) in linear n-normed space $(X, \|., ..., .\|)$ is an A-statistically Cauchy sequence if and only if there exists a subsequence $K = (k_n)$ of $\mathbb{N}$ with $\delta_A(K) = 1$ such that*

$$\lim_{n,m\to\infty} \|x_{k_n} - x_{k_m}, z_1, z_2, ..., z_{n-1}\| = 0$$

for all $z_1, z_2, ..., z_{n-1}$ in X.

Proof. Let (x_n) be an A-statistically Cauchy sequence in $(X, \|., ..., .\|)$. By definition, we can construct a decreasing sequence

$$(K_j) \subset \mathbb{N} \ (K_{j+1} \subset K_j, \ j = 1, 2, ...)$$

such that $\delta_A(K_j) = 1$ and $\|x_{k_1} - x_{k_2}, z_1, z_2, ..., z_{n-1}\| \leq \frac{1}{j}$ for all $z_1, z_2, ..., z_{n-1} \in X$, $k_1, k_2 \in K_j$, $j \in \mathbb{N}$. Further, let $v_1 \in K_1$. Then we can find $v_2 \in K_2$ with $v_2 > v_1$ such that $\frac{|K_2(n)|}{n} > \frac{1}{2}$ for each $n > v_2$. Inductively, we can construct a subsequence $(v_j) \in \mathbb{N}$ such that $v_j \in K_j$ for each $j \in \mathbb{N}$ and

$$\frac{|K_j(n)|}{n} > \frac{j-1}{j}$$

for each $n \geq v_j$. Then, as in [27], it is easy to prove that $\delta_A(K) = 1$ if

$$K = \{k \in \mathbb{N} : 1 \leq k < v_1\} \cup \left[\bigcup_{j\in\mathbb{N}} \{k : v_j \leq k < v_{j+1}\} \bigcap K_j\right].$$

Now, for $\varepsilon > 0$ choose $j \in \mathbb{N}$ such that $j > \frac{1}{\varepsilon}$. If $m, n \in K$ and $m, n > v_j$ we can find $r, s \geq j$ such that $v_r \leq m < v_{r+1}$, $v_s \leq n < v_{s+1}$. Hence, $m \in K_r$ and $n \in K_s$. For definite, suppose that $r \leq s$. Then $K_s \subset K_r$ which implies $m, n \in K_r$. Therefore, for every $z \in X$ we have

$$\|x_m - x_n, z_1, z_2, ..., z_{n-1}\| \leq \frac{1}{r} \leq \frac{1}{j} < \varepsilon.$$

Then we have

$$\lim_{\substack{n,m\to\infty \\ m,n\in K}} \|x_m - x_n, z_1, z_2, ..., z_{n-1}\| = 0.$$

Let us prove the converse. Suppose that $K = (k_n) \subset \mathbb{N}$ is a subsequence of subsets $\mathbb{N}$ such that $\delta_A(K) = 1$ and $\lim\limits_{n,m\to\infty} \|x_{k_n} - x_{k_m}, z_1, z_2, ..., z_{n-1}\| = 0$ for all z in X. Then, for any $\varepsilon > 0$ there exists $p_0 = p_0(\varepsilon, z) \in \mathbb{N}$ such that $\|x_{k_n} - x_{k_m}, z_1, z_2, ..., z_{n-1}\| < \varepsilon$ for all $n, m \geq p_0$. This yields

$$\left\{k \in \mathbb{N} : \left\|x_k - x_{k_{p_0}}, z_1, z_2, ..., z_{n-1}\right\| \geq \varepsilon\right\} \subset \mathbb{N}\backslash \{k_{p_0+1}, k_{p_0+2}, ...\}.$$

Hence

$$\delta_A \left\{k \in \mathbb{N} : \left\|x_k - x_{k_{p_0}}, z_1, z_2, ..., z_{n-1}\right\| \geq \varepsilon\right\} \leq \delta_A \left(\mathbb{N}\backslash \{k_{p_0+1}, k_{p_0+2}, ...\}\right) = 0.$$

So, (x_k) is an A-statistically Cauchy sequence in X. □

Lemma 2. *A sequence (x_k) in $(X, \|., ..., .\|)$ is a A-statistically Cauchy sequence if and only if for every neighborhood U of the origin there is an integer $N(U)$ such that $n, m \geq N(U)$ implies that $x_{k_n} - x_{k_m} \in U$, where $K = (k_n) \subset \mathbb{N}$ and $\delta_A(K) = 1$.*

Proof. Let $z \in X$ and $\varepsilon > 0$. Suppose that there is $K = (k_n) \subset \mathbb{N}$ such that $x_{k_n} - x_{k_m} \in U_\varepsilon(0, z_1, z_2, ..., z_{n-1})$ for $n, m \geq N(U)$, where $U_\varepsilon(0, z_1, z_2, ..., z_{n-1})$ is a neighborhood of zero. This implies $\|x_{k_n} - x_{k_m}, z_1, z_2, ..., z_{n-1}\| < \varepsilon$ for every $n, m \geq N(U)$. Then $\lim\limits_{n,m\to\infty} \|x_{k_n} - x_{k_m}, z_1, z_2, ..., z_{n-1}\| = 0$, i.e., (x_k) is an A-statistically Cauchy sequence in X.

Conversely, assume that $\lim\limits_{n,m\to\infty} \|x_{k_n} - x_{k_m}, z_1, z_2, ..., z_{n-1}\| = 0$, where $K = (k_n) \subset \mathbb{N}$ and $\delta_A(K) = 1$. Let $W_\Sigma(0)$ be an arbitrary neighborhood of 0 with $\Sigma = \left\{\left(b_{11}, ..., b_{(n-1)1}, \alpha_1\right), ..., \left(b_{1r}, ..., b_{(n-1)r}, \alpha_r\right)\right\}$. By hypothesis, we have

$$\lim_{n,m\to\infty} \left\|x_{k_n} - x_{k_m}, b_{1j}, b_{2j}, ..., b_{(n-1)j}\right\| = 0 \text{ for } j = 1, ..., r.$$

Thus for each α_j there exists an integer N_j such that

$$\left\|x_{k_n} - x_{k_m}, b_{1j}, b_{2j}, ..., b_{(n-1)j}\right\| < \alpha_j$$

for $n, m \geq N_j$. Let $N = \max\{N_1, ..., N_r\}$. Then

$$\begin{aligned} &\left\|x_{k_n} - x_{k_m} - b_{1j}, ..., x_{k_n} - x_{k_m} - b_{(n-1)j}, x_{k_n} - x_{k_m}\right\| \\ &= \left\|x_{k_n} - x_{k_m}, b_{1j}, b_{2j}, ..., b_{(n-1)j}\right\| < \alpha_j \end{aligned}$$

for $n, m \geq N$ implies that $x_{k_n} - x_{k_m} \in W_\Sigma(0)$ for $n, m \geq N$ and thus it follows that (x_k) is an A-statistically Cauchy sequence in X. □

3. Main Results

Proposition 1. *Let (x_n) be an A-statistically localized sequence in linear n-normed space $(X, \|., ..., .\|)$. Then (x_n) is A-statistically bounded in X.*

Proof. Let (x_n) be an A-statistically localized sequence. So, the number sequence $(\|x_n - L, z_1, z_2, ..., z_{n-1}\|)$ A-statistically converges for some $L \in X$ and every $z \in X$. Then the number sequence $(\|x_n - L, z_1, z_2, ..., z_{n-1}\|)$ is A-statistically bounded, i.e., there is $S > 0$ such that

$$\delta_A \left(\{n \in \mathbb{N} : \|x_n - L, z_1, z_2, ..., z_{n-1}\| \geq S\}\right) = 0.$$

This implies that almost all elements of (x_k) are located in the neighborhood $U_S(0, z_1, z_2, ..., z_{n-1})$ of the origin. Then, sequence (x_k) is A-statistically bounded in X. □

Proposition 2. *Let $M = \mathrm{loc}^{\mathrm{st}_A}(x_n)$ and the point $y \in X$ be such that there exists $x \in M$ for any $\varepsilon > 0$ and every nonzero $z_1, z_2, ..., z_{n-1} \in M$ satisfying*

$$\delta_A \left(\{n \in \mathbb{N} : |\|x - x_n, z_1, z_2, ..., z_{n-1}\| - \|y - x_n, z_1, z_2, ..., z_{n-1}\|| \geqslant \varepsilon\}\right) = 0. \quad (1)$$

Then $y \in M$.

Proof. To show that the sequence $\beta_n = \|x_n - y, z_1, z_2, ..., z_{n-1}\|$ satisfies the A-statistically Cauchy criteria is enough. Let $\varepsilon > 0$ and $x \in M = \mathrm{loc}^{\mathrm{st}_A}(x_n)$ is a point that has the property (1). Because the sequence $\|x_n - x, z_1, z_2, ..., z_{n-1}\|$ satisfying the property (1) is A-statistically Cauchy sequence, then there exists a subsequence $K = (k_n)$ of $\mathbb{N}$ with $\delta_A(K) = 1$ such that

$$|\|x - x_{k_n}, z_1, z_2, ..., z_{n-1}\| - \|y - x_{k_n}, z_1, z_2, ..., z_{n-1}\|| \to 0$$

and

$$|\|x_{k_n} - x, z_1, z_2, ..., z_{n-1}\| - \|x_{k_m} - x, z_1, z_2, ..., z_{n-1}\|| \to 0$$

as $m, n \to \infty$. Clearly, there exists $n_0 \in \mathbb{N}$ for any $\varepsilon > 0$ and every nonzero $z_1, z_2, ..., z_{n-1} \in M$ such that for all $n \geq n_0$, $m \geq m_0$, we get

$$|\|x - x_{k_n}, z_1, z_2, ..., z_{n-1}\| - \|y - x_{k_n}, z_1, z_2, ..., z_{n-1}\|| < \frac{\varepsilon}{3} \quad (2)$$

$$|\|x - x_{k_n}, z_1, z_2, ..., z_{n-1}\| - \|x - x_{k_m}, z_1, z_2, ..., z_{n-1}\|| < \frac{\varepsilon}{3}. \quad (3)$$

From (2), (3) and (4)

$$\begin{aligned}
&|\|y - x_{k_n}, z_1, z_2, ..., z_{n-1}\| - \|y - x_{k_m}, z_1, z_2, ..., z_{n-1}\||\\
&\leq |\|y - x_{k_n}, z_1, z_2, ..., z_{n-1}\| - \|x - x_{k_n}, z_1, z_2, ..., z_{n-1}\||\\
&+ |\|x - x_{k_n}, z_1, z_2, ..., z_{n-1}\| - \|x - x_{k_m}, z_1, z_2, ..., z_{n-1}\||\\
&+ |\|x - x_{k_m}, z_1, z_2, ..., z_{n-1}\| - \|y - x_{k_n}, z_1, z_2, ..., z_{n-1}\|| \quad (4)
\end{aligned}$$

we have that

$$|\|y - x_{k_n}, z_1, z_2, ..., z_{n-1}\| - \|y - x_{k_m}, z_1, z_2, ..., z_{n-1}\|| < \varepsilon \quad (5)$$

for all $n \geq n_0$, $m \geq n_0$, i.e.,

$$\left|\|y - x_{k_n}, z_1, z_2, ..., z_{n-1}\| - \|y - x_{k_m}, z_1, z_2, ..., z_{n-1}\|\right| \to 0 \text{ as } m, n \to \infty$$

for the subset $K = (k_n) \subset N$ with $\delta_A(K) = 1$. This means that the sequence $\|y - x_n, z_1, z_2, ..., z_{n-1}\|$ is an A-statistically Cauchy sequence, which finishes the proof. □

Definition 7. *A point a in a n-normed space $(X, \|.,...,.\|)$ is called a limit point of a set M in X if for an arbitrary $\Sigma = \left\{\left(x_{11}, ..., x_{(n-1)1}, \varepsilon_1\right), ..., \left(x_{1n}, ..., x_{(n-1)n}, \varepsilon_n\right)\right\}$ there is a point $a_\Sigma \in M$, $a_\Sigma \neq a$ such that $a_\Sigma \in W_\Sigma(a)$.*

Moreover, a subset $Y \subset X$ is called a closed subset of X if Y contains every its limit point. If Y^0 is the set of all points of a subset $Y \subset X$, then the set $\overline{Y} = Y \cup Y^0$ is called the closure of the set Y.

Proposition 3. *A-statistically localor of any sequence is a closed subset of the n-normed space $(X, \|.,...,.\|)$.*

Proof. Let $y \in \overline{\operatorname{loc}^{\operatorname{st}_A}}(x_n)$. Then, for arbitrary

$$\Sigma = \left\{\left(x_{11}, ..., x_{(n-1)1}, \varepsilon_1\right), ..., \left(x_{1n}, ..., x_{(n-1)n}, \varepsilon_n\right)\right\}$$

there is a point $x \in \operatorname{loc}^{\operatorname{st}_A}(x_n)$ such that $x \neq y$ and $x \in W_\Sigma(y)$. Hence

$$\delta_A\left(\left\{n \in \mathbb{N} : \left|\|x - x_n, z_1, z_2, ..., z_{n-1}\| - \|y - x_n, z_1, z_2, ..., z_{n-1}\|\right| \geqslant \varepsilon\right\}\right) = 0$$

for any $\varepsilon > 0$ and every $z_1, z_2, ..., z_{n-1} \in \operatorname{loc}^{\operatorname{st}_A}(x_n)$, because we get

$$\begin{aligned}&\left|\|x - x_n, z_1, z_2, ..., z_{n-1}\| - \|y - x_n, z_1, z_2, ..., z_{n-1}\|\right| \\ &\leq \|y - x_n, z_1, z_2, ..., z_{n-1}\| < \varepsilon\end{aligned}$$

for almost all n. As a result, the hypothesis of Proposition 2 is satisfied. So, $y \in \operatorname{loc}^{\operatorname{st}_A}(x_n)$, that is, $\operatorname{loc}^{\operatorname{st}_A}(x_n)$ is closed. □

Recall that the point y is an A-statistical limit point of the sequence (x_n) in n-normed space $(X, \|.,...,.\|)$ if there is a set $K = \{k_1 < k_2 < ...\} \subset \mathbb{N}$ such that $\delta_A(K) \neq 0$ and $\lim_{n\to\infty} \|x_{k_n} - y, z_1, z_2, ..., z_{n-1}\| = 0$. A point ξ is called an A-statistical cluster point if

$$\delta_A\left(\left\{n \in \mathbb{N} : \|x_n - \xi, z_1, z_2, ..., z_{n-1}\| < \varepsilon\right\}\right) \neq 0$$

for each $\varepsilon > 0$ and every $z_1, z_2, ..., z_{n-1} \in X$.

We can give the following results because of the inequality

$$\left|\|x_n - y, z_1, z_2, ..., z_{n-1}\| - \|x - y, z_1, z_2, ..., z_{n-1}\|\right| \leq \|x_n - x, z_1, z_2, ..., z_{n-1}\|.$$

Proposition 4. *Let $y \in X$ be an A-statistical limit point (an A-statistical cluster point) of a sequence (x_n) in n-normed space $(X, \|.,...,.\|)$. Then the number $\|y - x, z_1, z_2, ..., z_{n-1}\|$ is an A-statistical limit point (an A-statistical cluster point) of the sequence $\{\|x_n - x, z_1, z_2, ..., z_{n-1}\|\}$ for each $x \in X$ and every nonzero $z_1, z_2, ..., z_{n-1} \in X$.*

Proposition 5. *All A-statistical limit points (A-statistical cluster points) of the A-statistically localized sequence (x_n) in n-normed space $(X, \|., ..., .\|)$ have the same distance from each point x of the A-statistical localor* $\mathrm{loc}^{\mathrm{st}_A}(x_n)$.

Proof. Actually, if y_1, y_2 are two A-statistical limit points (A-statistical cluster points) of the sequence (x_n) in n-normed space $(X, \|., ..., .\|)$, then the numbers $\|y_1 - x, z_1, z_2, ..., z_{n-1}\|$ and $\|y_2 - x, z_1, z_2, ..., z_{n-1}\|$ are A-statistical limit points of the A-statistically convergent sequence $\|x - x_n, z_1, z_2, ..., z_{n-1}\|$. As a result, $\|y_1 - x, z_1, z_2, ..., z_{n-1}\| = \|y_2 - x, z_1, z_2, ..., z_{n-1}\|$. □

Proposition 6. $\mathrm{loc}^{\mathrm{st}_A}(x_n)$ *only contains one A-statistical limit (cluster) point of the sequence (x_n) in n-normed space $(X, \|., ..., .\|)$. In particular, everywhere localized sequence only has one A-statistical limit (cluster) point.*

Proof. Let $x, y \in \mathrm{loc}^{\mathrm{st}_A}(x_n)$ be two A-statistical limit or cluster points of the sequence (x_n) in n-normed space $(X, \|., ..., .\|)$. Then, we have that

$$\|x - x, z_1, z_2, ..., z_{n-1}\| = \|x - y, z_1, z_2, ..., z_{n-1}\|$$

from the Proposition 5. But $\|x - x, z_1, z_2, ..., z_{n-1}\| = 0$. This means $\|x - y, z_1, z_2, ..., z_{n-1}\| = 0$ for $x \neq y$. This is a contradiction. □

Proposition 7. *Let $y \in \mathrm{loc}^{\mathrm{st}_A}(x_n)$ be an A-statistical limit point of the sequence (x_n). Then $x_n \overset{st_A}{\to} y$.*

Proof. The sequence $\{\|x_n - y, z_1, z_2, ..., z_{n-1}\|\}$ A-statistically converges and some subsequence of this sequence converges to zero, i.e., $x_n \overset{st_A}{\to} y$. □

Definition 8. *Let (x_n) be the A-statistically localized sequence with the A-statistically localor $M = \mathrm{loc}^{\mathrm{st}_A}(x_n)$. The number*

$$\mu = \inf_{x \in M} \left(st_A\text{-} \lim_{n \to \infty} \|x - x_n, z_1, z_2, ..., z_{n-1}\| \right)$$

is said to be the A-statistical barrier of (x_n).

Theorem 1. *Let (x_n) be the A-statistically localized sequence in n-normed space $(X, \|., ..., .\|)$. Then (x_n) is A-statistically Cauchy sequence if and only if its A-statistical barrier is equal to zero.*

Proof. Let (x_n) be an A-statistically Cauchy sequence in n-normed space $(X, \|., ..., .\|)$. Then, there exists the set $K = \{k_1 < k_2 < ... < k_n < ...\} \subset \mathbb{N}$ such that $\delta_A(K) = 1$ and $\lim_{n,m \to \infty} \|x_{k_n} - x_{k_m}, z_1, z_2, ..., z_{n-1}\| = 0$. Hence, there exists $n_0 \in \mathbb{N}$ for each $\varepsilon > 0$ and every nonzero $z_1, z_2, ..., z_{n-1} \in X$ such that

$$\left\| x_{k_n} - x_{k_{n_0}}, z_1, z_2, ..., z_{n-1} \right\| < \varepsilon$$

for all $n \geq n_0$. Because an A-statistically Cauchy sequence is A-statistically localized everywhere, we get $st_A\text{-}\lim_{n \to \infty} \left\| x_n - x_{k_{n_0}}, z_1, z_2, ..., z_{n-1} \right\| \leq \varepsilon$, that is, $\mu \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\mu = 0$.

In contrast, if $\mu = 0$ then there is $x \in M = \mathrm{loc}^{\mathrm{st}_A}(x_n)$ for each $\varepsilon > 0$ such that $\|x, z_1, z_2, ..., z_{n-1}\| = st_A\text{-}\lim_{n\to\infty} \|x - x_n, z_1, z_2, ..., z_{n-1}\| < \frac{\varepsilon}{2}$ for every nonzero $z_1, z_2, ..., z_{n-1} \in M$. At this stage,

$$\delta_A \left(\left[n \in \mathbb{N} : |\|x, z_1, z_2, ..., z_{n-1}\| - \|x - x_n, z_1, z_2, ..., z_{n-1}\|| \right.\right.$$
$$\left.\left. \geq \frac{\varepsilon}{2} - \|x, z_1, z_2, ..., z_{n-1}\| \right] \right) = 0.$$

So,

$$\delta_A \left(\left\{ n \in \mathbb{N} : \|x - x_n, z_1, z_2, ..., z_{n-1}\| \geq \frac{\varepsilon}{2} \right\} \right) = 0,$$

that is, $st_A\text{-}\lim_{n\to\infty} \|x - x_n, z_1, z_2, ..., z_{n-1}\| = 0$. Therefore, (x_n) is an A-statistically Cauchy sequence. □

Theorem 2. *Let (x_n) be A-statistically localized in itself and let (x_n) contain a A-nonthin Cauchy subsequence. Then (x_n) is an A-statistically Cauchy sequence in itself.*

Proof. Let (x_n') be a A-nonthin Cauchy subsequence of (x_n). Without loss of generality we can suppose that all elements of (x_n') are in $\mathrm{loc}^{\mathrm{st}_A}(x_n)$. Because (x_n') is a Cauchy sequence by Theorem 1,

$$\inf_{x_n'} \lim_{m\to\infty} \|x_m' - x_n', z_1, z_2, ..., z_{n-1}\| = 0.$$

In other hand, because (x_n) is A-statistically localized in itself, then

$$st_A\text{-}\lim_{m\to\infty} \|x_m - x_n', z_1, z_2, ..., z_{n-1}\| = st_A - \lim_{m\to\infty} \|x_m' - x_n', z_1, z_2, ..., z_{n-1}\| = 0.$$

This means

$$\mu = \inf_{x\in M} \left(st_A\text{-}\lim_{m\to\infty} \|x_m - x, z_1, z_2, ..., z_{n-1}\| \right) = 0,$$

that is, (x_n) is an A-statistically Cauchy sequence in itself. □

Let $x \in X$ and $\delta > 0$. Recall that the sequence (x_n) in n-normed space $(X, \|., ..., .\|)$ is said to be A-statistically bounded if there is a subset $K = \{k_1 < k_2 < ... < k_n \subset ...\}$ of $\mathbb{N}$ such that $\delta_A(K) = 1$ and $(x_{k_n}) \subset U_\delta(0, z_1, z_2, ..., z_{n-1})$, where $U_\delta(0, z_1, z_2, ..., z_{n-1})$ is some neighborhood of the origin. It is obvious that (x_{k_n}) is a bounded sequence in X and it has a localized in itself subsequence. As a result, the following statement is correct:

Theorem 3. *Each A-statistically bounded sequence in n-normed space $(X, \|., ..., .\|)$ has an A-statistically localized in itself subsequence.*

Definition 9. *An infinite subset $L \subset (X, \|., .\|)$ is called thick relatively to a non-empty subset $Y \subset X$ if for each $\varepsilon > 0$ there is the a point $y \in Y$ such that the neighborhood $U_\varepsilon(0, z_1, z_2, ..., z_{n-1})$ has infinitely many points of L. In particular, if the set L is thick relatively to its subset $Y \subset L$ then L is said to be thick in itself.*

Theorem 4. *The following statements are equivalent to each other in n-normed space $(X, \|.,...,.\|)$:*

(i) Each bounded subset of X is totally bounded.

(ii) Each bounded infinite set of X is thick in itself.

(iii) Each A-statistically localized in itself sequence in X is an A-statistically Cauchy sequence.

Proof. It is obvious that (i) implies (ii). Now, we prove that (ii) implies (iii). Let $(x_n) \subset X$ be an A-statistically localized in itself. Then (x_n) is A-statistically bounded sequence in X. Then here is an infinite set L of points of (x_n) such that L is a bounded subset of X. By the supposition, the set L is thick in itself. So, we can choose $x_k \in L$ for every $\varepsilon > 0$ such that the neighborhood $U_\varepsilon(0, z_1, z_2, ..., z_{n-1})$ contains infinitely many points of X, say $x'_1, ..., x'_n, ...$ The sequence $(\|x'_n - x_k, z_1, z_2, ..., z_{n-1}\|)$ A-statistically converges and

$$st_A - \lim_{n\to\infty} \|x'_n - x_K, z_1, z_2, ..., z_{n-1}\| \leq \varepsilon$$

for the sequence (x'_n). Therefore, the A-statistically barrier of (x_n) is equal to zero. Then (x_n) is a Cauchy sequence.

Suppose that (iii) is satisfied, but (i) is not. Then, there is a subset $L \subset X$ such that L is not totally bounded. This means that there exists $\varepsilon > 0$ and a sequence $(x_n) \subset L$ such that $\|x_n - x_m, z_1, z_2, ..., z_{n-1}\| > \varepsilon$ for any $n \neq m$ and every nonzero $z_1, z_2, ..., z_{n-1} \in L$.

Because (x_n) is A-statistically bounded by Theorem 3, it has an A-statistically localized in itself sequence (x'_n). Due to $\|x'_n - x'_m, z_1, z_2, ..., z_{n-1}\| > \varepsilon$ for any $n \neq m$, the subsequence is not an A-statistically Cauchy sequence. This contradicts (iii). Therefore, (iii) implies (ii), which finish the proof. □

From Theorem 2 and 3, we get the property (iii) is equivalent to

(iv) each A-statistically bounded sequence has an A-statistically Cauchy subsequence.

Definition 10. *A sequence (x_n) in n-normed space $(X, \|.,...,.\|)$ is said to be uniformly A-statistically localized on the subset L of X if the sequence $\{\|x - x_n, z_1, z_2, ..., z_{n-1}\|\}$ uniformly A-statistically converges for all $x \in L$ and every nonzero $z_1, z_2, \ldots, z_{n-1}$ in L.*

Proposition 8. *Let (x_n) be uniformly A-statistically localized on the set $L \subset X$ and $w \in Y$ is such that for every $\varepsilon > 0$ and every nonzero $z_1, z_2, ..., z_{n-1}$ in L, there is $y \in L$ satisfying the property*

$$\delta_A\left(\{n \in \mathbb{N} : |\|w - x_n, z_1, z_2, ..., z_{n-1}\| - \|y - x_n, z_1, z_2, ..., z_{n-1}\|| \geqslant \varepsilon\}\right) = 0.$$

Then $w \in \mathrm{loc}^{\mathrm{st}_A}(x_n)$ and (x_n) is uniformly A-statistically localized on a set that contains such points w.

References

[1] Connor, B. J., The statistical and strong p-Cesaro convergence of sequences, *Analysis (Munich)*, 8 (1988), 47-63.

[2] Connor, J. , Kline, J., On statistical limit points and the consistency of statistical convergence, *J. Math. Anal. Appl.*, 197 (1996), 392-399.

[3] Fast, H., Sur la convergence statistique, *Colloq. Math.,* 2 (1951), 241-244.

[4] Fridy, J. A., On statistical convergence, *Analysis (Munich),* 5 (1985), 301-313.

[5] Gähler, S., 2-metrische Räume und ihre topologische Struktur, *Math. Nachr.,* 26 (1963), 115-148.

[6] Gähler, S., Lineare 2-normierte räume, *Math. Nachr.,* 28(1-2) (1964), 1-43.

[7] Gähler, S., Untersuchungen über verallgemeinerte m-metrische räume. I., *Math. Nachr.,* 40(1-3) (1969), 165-189

[8] Gähler, S., Siddiqi, A. H., Gupta, S.C., Contributions to non-archimedean functional analysis, *Math. Nachr.,* 69 (1963), 162-171.

[9] Gunawan, H., Mashadi, On n-normed Spaces, *Int. J. Math. Sci.,* 27(10) (2001), 631-639.

[10] Gürdal, M., Pehlivan, S., The statistical convergence in 2-Banach spaces, *Thai. J. Math.,* 2(1) (2004), 107-113.

[11] Gürdal, M., Pehlivan, S., The statistical convergence in 2-normed spaces, *Southeast Asian Bull. Math.,* 33(2) (2009), 257-264.

[12] Gürdal, M., Açık, I., On $\mathcal{I}$-cauchy sequences in 2-normed spaces, *Math. Inequal. Appl.*, 11(2) (2008), 349-354.

[13] Gürdal, M., Şahiner, A., Statistical approximation with a sequence of 2-Banach spaces, *Math. Comput. Modelling,* 55(3-4) (2012), 471-479.

[14] Gürdal, M., Şahiner, A., Açık, I., Approximation theory in 2-Banach spaces, *Nonlinear Anal.,* 71(5-6) (2009), 1654-1661.

[15] Gürdal, M., Yamancı, U., Statistical convergence and some questions of operator theory, *Dynam. Syst. Appl.*, 24 (2015), 305-312.

[16] Kadak, U., Mohiuddine, S. A., Generalized statistically almost convergence based on the difference operator which includes the (p, q)-gamma function and related approximation theorems, *Results Math.,* (2018), 73:9.

[17] Kolk, E., The statistical convergence in Banach spaces, *Acta et Commentationes Universitatis Tartuensis,* 928 (1991), 41-52.

[18] Krivonosov, L. N., Localized sequences in metric spaces, *Izv. Vyssh. Uchebn. Zaved. Mat.,* 4 (1974), 45-54; *Soviet Math. (Iz. VUZ),* 18(4), 37-44.

[19] Mohiuddine, S. A., Statistical weighted A-summability with application to Korovkin's type approximation theorem, *J. Inequal. Appl.,* (2016), 2016:101.

[20] Mohiuddine, S. A., Alamri, B. A. S., Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems, *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM,* 113 (2019), 1955-1973.

[21] Mohiuddine, S. A., Asiri, A., Hazarika, B., Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems, *Int. J. Gen. Syst.,* 48(5) (2019), 492-506.

[22] Mohiuddine, S. A., Hazarika, B., Alghamdi, M. A., Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems, *Filomat,* 33(14) (2019), 4549-4560.

[23] Mohiuddine, S. A., Şevli, H., Cancan, M., Statistical convergence in fuzzy 2-normed space, *J. Comput. Anal. Appl.,* 12(4) (2010), 787-798.

[24] Mursaleen, M., On statistical convergence in random 2-normed spaces, *Acta Sci. Math. (Szeged),* 76 (2010), 101-109.

[25] Nabiev, A. A., Savaş, E., Gürdal, M., Statistically localized sequences in metric spaces. *J. App. Anal. Comp.,* 9(2) (2019), 739-746.
[26] Nabiev, A. A., Savaş, E., Gürdal, M., $\mathcal{I}$-localized sequences in metric spaces, *Facta Univ. Ser. Math. Inform.,* 35(2) (2020), 459-469.
[27] Rath, D., Tripathy, B.C., On statistically convergent and statistically Cauchy sequences, *Indian J. Pure Appl. Math.,* 25(4) (1994), 381-386.
[28] Raymond, W. F., Cho, Ye. J., Geometry of linear 2-normed spaces, Huntington, N.Y. Nova Science Publishers, 2001.
[29] Šalát, T., On statistically convergent sequences of real numbers, *Mathematica Slovaca,* 30(2) (1980), 139-150.
[30] Savaş, E., A-statistical convergence of order α via φ-function, *Appl. Anal. Discrete Math.,* 13 (2019), 918-926.
[31] Savaş E. and Gürdal M., Certain summability methods in intuitionistic fuzzy normed spaces, *J. Intell. Fuzzy Syst.,* 27(4) (2014), 1621-1629.
[32] Savaş E., Gürdal M., Generalized statistically convergent sequences of functions in fuzzy 2-normed spaces, *J. Intell. Fuzzy Syst.,* 27(4) (2014), 2067-2075.
[33] Savaş, E., Gürdal, M., Ideal convergent function sequences in random 2-normed spaces, *Filomat,* 30(3) (2016), 557-567.
[34] Savaş, E., Yamancı, U., Gürdal, M., $\mathcal{I}$-lacunary statistical convergence of weighted g via modulus functions in 2-normed spaces, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.,* 68(2) (2019), 2324-2332.
[35] Savaş, R., λ-double statistical convergence of functions, *Filomat,* 33(2) (2019), 519-524.
[36] Savaş, R., Öztürk, M., On generalized ideal asymptotically statistical equivalent of order α for functions, *Ukr. Math. J.,* 70 (2019), 1901-1912.
[37] Savaş, R.„ Sezer, S. A., Tauberian theorems for sequences in 2-normed spaces, *Results Math.,* 72 (2017), 1919-1931.
[38] Savaş, R. , Patterson, R. F., $\mathcal{I}_2$-lacunary strongly summability for multidimensional measurable functions, *Pub. I. Math-Beograd,* 107(121) (2020), 93-107.
[39] Şahiner, A., Gürdal, M., Yiğit, T., Ideal convergence characterization of the completion of linear n-normed spaces, *Comput. Math. Appl.,* 61(3) (2011), 683-689.
[40] Siddiqi, A. H., 2-normed spaces, *Aligarh Bull. Math.,* (1980), 53-70.
[41] Vulich, B., On a generalized notion of convergence in a Banach space, *Ann. Math.,* 38(1) (1937), 156-174.
[42] Yamancı, U., Gürdal, M., Statistical convergence and operators on Fock space, *New York J. Math.*, 22 (2016), 199-207.
[43] Yamancı, U., Nabiev, A. A., Gürdal, M., Statistically localized sequences in 2-normed spaces, *Honam Math. J.*, 42(1) (2020), 161-173.
[44] Yamancı, U., Savaş, E., Gürdal, M., I-localized sequences in two normed spaces, *Malays. J. Math. Sci.,* 14(3) (2020), 491-503.
[45] Yegül, S., Dündar, E., On statistical convergence of sequences of functions in 2-normed spaces, *J. Classical Anal.*, 10(1) (2017), 49-57.
[46] Yegül, S., Dündar, E., Statistical convergence of double sequences of functions and some properties in 2-normed spaces, *Facta Univ. Ser. Math. Inform.,* 33(5) (2018), 705-719.

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ON THE CHINESE CHECKERS SPHERICAL INVERSIONS IN THREE DIMENSIONAL CHINESE CHECKERS SPACE

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Abstract. In this paper, we study an inversion with respect to a Chinese checkers sphere in the three dimensional Chinese Checkers space, and prove several properties of this inversion. We also study cross ratio, harmonic conjugates and the inverse images of lines, planes and Chinese Checkers spheres in three dimensional Chinese Checkers space.

1. Introduction

In the game of Chinese checkers, checkers are allowed to move in the vertical (north and south), horizontal (east and west), and diagonal (northeast, northwest, southeast and southwest) directions. In [7], Krause asked how to develop a distance function that measures and made a suggestion for the idea of Chinese Checkers geometry and in G. Chen, introduced it by defining the metric in the coordinate plane [22]. The inversion was introduced by Perga and then studied and applied by Steiner about 1820s [2]. During the following decades, many physicists and mathematicians independently rediscovered inversions, proving the properties that were most useful for their particular applications (for some references see [1], [4], [10]). Many kinds of generalizations of inversion transform have been presented in literature. The inversions with respect to the central conics in real Euclidean plane was introduced in [3]. Then the inversions with respect to ellipse was studied detailed in [13]. In three-dimensional space a generalization of the spherical inversion is given in [16]. Also, the inversions with respect to the taxicab distance, alpha-distance [17], [21], [19] or in general a p-distance [20]. The circle inversion have been generalized in three-dimensional space by using a sphere as the circle of inversion [16].

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H. Minkowski was one of the developers in "non-Euclidean" geometry and found taxicab geometry [9]. The taxicab geometry has been studied and improved by some mathematicians (for some references see [5], [6], [8], [14], [18]).

An inversion in a sphere is a transformation of the space that flips the sphere inside-out. That is, points outside the sphere get mapped to points inside the sphere, and points inside the sphere get mapped outside the sphere. In the present article, we define a notion of inversion valid in three dimensional Chinese Checkers space. In particular, we define an inversion with respect to a Chinese Checkers sphere and prove several properties of this new transformation. Also we introduce inverse points, cross ratio, harmonic conjugates and the inverse images of lines, planes and Chinese Checkers spheres in three dimensional Chinese Checkers space.

2. Chinese Checkers Spherical Inversions

In this section, we introduce the inversion in a Chinese Checkers sphere.

The Chinese Checkers space $\mathbb{R}^3_C$ is almost the same as the Euclidean space $\mathbb{R}^3$. The points and lines are the same, and the angles are measured the same way, but the distance function is different. In $\mathbb{R}^3$ the Chinese Checkers metric is defined using the distance function

$$d_C(A,B) = d_L(A,B) + \left(\sqrt{2}-1\right) d_S(A,B)$$

where

$$d_L(A,B) = \max\left\{|x_1-x_2|, |y_1-y_2|, |z_1-z_2|\right\}$$

and

$$d_S(A,B) = \min\left\{|x_1-x_2| + |y_1-y_2|, |x_1-x_2| + |z_1-z_2|, |y_1-y_2| + |z_1-z_2|\right\}$$

where $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$ in $\mathbb{R}^3$. The unit ball in $\mathbb{R}^3_C$ is the set of points (x, y, z) in space which satisfy the equation

$$\max\left\{|x|, |y|, |z|\right\} + \left(\sqrt{2}-1\right) \min\left\{|x| + |y|, |x| + |z|, |y| + |z|\right\} = 1$$

We can define the notion of inversion in $\mathbb{R}^3_C$ as an analogue of inversion in $\mathbb{R}^3$.

Definition 1. *Let $\mathcal{S}$ be a Chinese Checkers sphere centered at a point O with radius r in $\mathbb{R}^3_C$. The inversion in the Chinese Checkers sphere $\mathcal{S}$ or the Chinese Checkers spherical inversion respect to $\mathcal{S}$ is the function such that*

$$I_{(O,r)} : \mathbb{R}^3_C - \{O\} \to \mathbb{R}^3_C - \{O\}$$

defined by $I_{(O,r)}(P) = P'$, for $P \neq O$ where P' is on the $\overrightarrow{OP}$ and

$$d_C(O,P) \cdot d_C(O,P') = r^2.$$

The point P' is said to be the Chinese Checkers spherical inverse of P in $\mathcal{S}$, $\mathcal{S}$ is called the sphere of inversion and O is called the center of inversion.

The Chinese Checkers spherical inversions with respect to the sphere, like reflections, are involutions. The fixed points of $I_{(O,r)}$ are the points on the Chinese Checkers sphere $\mathcal{S}$ centered at O with radius r.

Some basic properties about spherical inversion are given in the following items. Note that it is possible to extend every property of the Chinese Checkers circle inversion to Chinese Checkers spherical inversion.

Theorem 2. *Let $\mathcal{S}$ be an Chinese Checkers sphere with the center O in the Chinese Checkers spherical inversion $I_{(O,r)}$. If the point P is in the exterior of $\mathcal{S}$ then the point P', the inverse of P, is interior to $\mathcal{S}$, and conversely.*

Proof. Let the point P be in the exterior of $\mathcal{S}$, then $d_C(O,P) > r$. If $P' = I_{(O,r)}(P)$; then $d_C(O,P) \cdot d_C(O,P') = r^2$. Hence $r^2 = d_C(O,P) \cdot d_C(O,P') > r \cdot d_C(O,P')$ and $d_C(O,P') < r$. $\square$

The inversion $I_{(O,r)}$ is undefined at the point O. However, we can add to the Chinese Checkers space a single point at infinite O_∞, which is the inverse of the center O of Chinese Checkers inversion sphere $\mathcal{S}$. So, the inversion $I_{(O,r)}$ is one-to-one map of extended Chinese Checkers sphere.

Theorem 3. *Let $\mathcal{S}$ be a Chinese Checkers sphere with the center $O = (0,0,0)$ and the radius r in $\mathbb{R}^3_C$. If $P = (x,y,z)$ and $P' = (x',y',z')$ are inverse points with respect to the Chinese Checkers spherical inversion $I_{(O,r)}$, then*

$$P' = \frac{r^2}{\left(d_C(O,P)\right)^2} P$$

Proof. The equation of $\mathcal{S}$ is $d_C(O,P) = d_L(O,P) + \left(\sqrt{2}-1\right) d_S(O,P)$. Suppose that $P = (x,y,z)$ and $P' = (x',y',z')$ are inverse points with respect to the Chinese Checkers spherical inversion $I_{(O,r)}$. Since the points O, P and P' are collinear and the rays $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are same direction,

$$\overrightarrow{OP'} = k\overrightarrow{OP},\ k \in \mathbb{R}^+$$

$$(x',y',z') = (kx,ky,kz).$$

From $d_C(O,P) \cdot d_C(O,P') = r^2$, $k = \dfrac{r^2}{\left(d_C(O,P)\right)^2}$. Replacing the value of k in $(x',y',z') = (kx,ky,kz)$, the equations of x', y' and z' are obtained. $\square$

Corollary 4. *Let $\mathcal{S}$ be a Chinese Checkers sphere with the center $O = (a,b,c)$ and the radius r in $\mathbb{R}^3_C$. If $P = (x,y,z)$ and $P' = (x',y',z')$ are inverse points with respect to the Chinese Checkers spherical inversion $I_{(O,r)}$, then*

$$P' - O = \frac{r^2}{\left(d_C(O,P)\right)^2} (P - O)$$

Proof. Since the translation preserve distances in the Chinese Checkers space [15], [12] by translating in $\mathbb{R}^3_C$ $(0,0,0)$ to (a,b,c) one can easily get the value of x', y' and z'. □

Theorem 5. *Let P, Q and O be any three collinear different points in $\mathbb{R}^3_C$. If the Chinese Checkers spherical inversion $I_{(O,r)}$ transform P and Q into P' and Q' respectively, then*

$$d_C(P', Q') = \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}.$$

Proof. Assume first that O, P, Q are collinear. From the Definition 1 we conclude that $d_C(O,P)d_C(O,P') = r^2 = d_C(O,Q)d_C(O,Q')$. Since the ratios of the Euclidean and Chinese Checkers distances along a line are same,

$$\begin{aligned} d_C(P',Q') &= |d_C(O,P') - d_C(O,Q')| \\ &= \left| \frac{r^2}{d_C(O,P)} - \frac{r^2}{d_C(O,Q)} \right| \\ &= \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)} \end{aligned}$$

is obtained. □

When O, P, Q are not collinear, the theorem is not valid in Chinese Checkers space, generally $\mathbb{R}^3_C$. For example, for $O = (0,0,0)$, $P = (1,0,0)$, $Q = (2,2,2)$ and $r = 2\sqrt{2}$, the inversion $I_{(O,r)}$ transform P and Q into $P' = (8,0,0)$ and $Q' = \left(\frac{4(9+4\sqrt{2})}{45}, \frac{4(9+4\sqrt{2})}{45}, \frac{4(9+4\sqrt{2})}{45} \right)$. It follows that $d_C(O,P) = 1$, $d_C(O,Q) = 4\sqrt{2} - 2$ and $d_C(P,Q) = 3\sqrt{2} - 1$, $d_C(P',Q') = \frac{4(9+4\sqrt{2})}{45} + 8\left(\sqrt{2} - 1\right)$.

Theorem 6. *Let P, Q and O be any three non-collinear different points in $\mathbb{R}^3_C$ and $I_{(O,r)}$ be the inversion such that transform P and Q into P' and Q' respectively. If the direction of PQ line is any element of D sets $i = 1,2,3$ such that*

$$D_1 = \{u_i \mid i \in 1, 2, ..., 18\}$$

for $u_1 = (1,0,0)$, $u_2 = (-1,0,0)$, $u_3 = (0,1,0)$, $u_4 = (0,-1,0)$, $u_5 = (0,0,1)$, $u_6 = (0,0,-1)$, $u_7 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $u_8 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$, $u_9 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $u_{10} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$, $u_{11} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $u_{12} = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$,
$u_{13} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $u_{14} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $u_{15} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$,
$u_{16} = \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $u_{17} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $u_{18} = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$,

$$D_2 = \{v_i \mid i \in 1, 2, ..., 8\}$$

for $v_1 = \left(\frac{1}{2\sqrt{2}-1}, \frac{1}{2\sqrt{2}-1}, \frac{1}{2\sqrt{2}-1}\right)$, $v_2 = \left(-\frac{1}{2\sqrt{2}-1}, \frac{1}{2\sqrt{2}-1}, \frac{1}{2\sqrt{2}-1}\right)$,

$$v_3 = \left(\tfrac{1}{2\sqrt{2}-1}, -\tfrac{1}{2\sqrt{2}-1}, \tfrac{1}{2\sqrt{2}-1}\right),\ v_4 = \left(\tfrac{1}{2\sqrt{2}-1}, \tfrac{1}{2\sqrt{2}-1}, -\tfrac{1}{2\sqrt{2}-1}\right),$$
$$v_5 = \left(\tfrac{1}{2\sqrt{2}-1}, -\tfrac{1}{2\sqrt{2}-1}, -\tfrac{1}{2\sqrt{2}-1}\right),\ v_6 = \left(-\tfrac{1}{2\sqrt{2}-1}, \tfrac{1}{2\sqrt{2}-1}, -\tfrac{1}{2\sqrt{2}-1}\right),$$
$$v_7 = \left(-\tfrac{1}{2\sqrt{2}-1}, -\tfrac{1}{2\sqrt{2}-1}, \tfrac{1}{2\sqrt{2}-1}\right),\ v_8 = \left(-\tfrac{1}{2\sqrt{2}-1}, -\tfrac{1}{2\sqrt{2}-1}, -\tfrac{1}{2\sqrt{2}-1}\right),$$

$$D_3 = \{t_i \mid i \in 1, 2, ..., 48\}$$

$$t_1 = \left(1, \sqrt{2}+1, 0\right),\ t_2 = \left(1, -\left(\sqrt{2}+1\right), 0\right),\ t_3 = \left(-1, \sqrt{2}+1, 0\right),$$
$$t_4 = \left(-1, -\left(\sqrt{2}+1\right), 0\right),\ t_5 = \left(1, 0, \sqrt{2}+1\right),\ t_6 = \left(1, 0, -\left(\sqrt{2}+1\right)\right),$$
$$t_7 = \left(-1, 0, \sqrt{2}+1\right),\ t_8 = \left(-1, 0, -\left(\sqrt{2}+1\right)\right),\ t_9 = \left(0, \sqrt{2}+1, 1\right),$$
$$t_{10} = \left(0, -\left(\sqrt{2}+1\right), 1\right),\ t_{11} = \left(0, \sqrt{2}+1, -1\right),\ t_{12} = \left(0, -\left(\sqrt{2}+1\right), -1\right),$$
$$t_{13} = \left(0, 1, \sqrt{2}+1\right),\ t_{14} = \left(0, 1, -\left(\sqrt{2}+1\right)\right),\ t_{15} = \left(0, -1, \sqrt{2}+1\right),$$
$$t_{16} = \left(0, -1, -\left(\sqrt{2}+1\right)\right), t_{17} = \left(\sqrt{2}+1, 0, 1\right),\ t_{18} = \left(-\left(\sqrt{2}+1\right), 0, 1\right),$$
$$t_{19} = \left(\sqrt{2}+1, 0, -1\right),\ t_{20} = \left(-\left(\sqrt{2}+1\right), 0, -1\right),\ t_{21} = \left(\sqrt{2}+1, 1, 0\right),$$
$$t_{22} = \left(-\left(\sqrt{2}+1\right), 1, 0\right),\ t_{23} = \left(\sqrt{2}+1, -1, 0\right),\ t_{24} = \left(-\left(\sqrt{2}+1\right), -1, 0\right),$$
$$t_{25} = \left(1, \sqrt{2}-1, 0\right),\ t_{26} = \left(1, -\left(\sqrt{2}-1\right), 0\right),\ t_{27} = \left(-1, \sqrt{2}-1, 0\right),$$
$$t_{28} = \left(-1, -\left(\sqrt{2}-1\right), 0\right),\ t_{29} = \left(1, 0, \sqrt{2}-1\right),\ t_{30} = \left(1, 0, -\left(\sqrt{2}-1\right)\right),$$
$$t_{31} = \left(-1, 0, \sqrt{2}-1\right),\ t_{32} = \left(-1, 0, -\left(\sqrt{2}-1\right)\right),\ t_{33} = \left(0, \sqrt{2}-1, 1\right),$$
$$t_{34} = \left(0, -\left(\sqrt{2}-1\right), 1\right),\ t_{35} = \left(0, \sqrt{2}-1, -1\right),\ t_{36} = \left(0, -\left(\sqrt{2}-1\right), -1\right),$$
$$t_{37} = \left(0, 1, \sqrt{2}-1\right),\ t_{38} = \left(0, 1, -\left(\sqrt{2}-1\right)\right),\ t_{39} = \left(0, -1, \sqrt{2}-1\right),$$
$$t_{40} = \left(0, -1, -\left(\sqrt{2}-1\right)\right),\ t_{41} = \left(\sqrt{2}-1, 0, 1\right),\ t_{42} = \left(-\left(\sqrt{2}-1\right), 0, 1\right),$$
$$t_{43} = \left(\sqrt{2}-1, 0, -1\right),\ t_{44} = \left(-\left(\sqrt{2}-1\right), 0, -1\right),\ t_{45} = \left(\sqrt{2}-1, 1, 0\right),$$
$$t_{46} = \left(-\left(\sqrt{2}-1\right), 1, 0\right),\ t_{47} = \left(\sqrt{2}-1, -1, 0\right),\ t_{48} = \left(-\left(\sqrt{2}-1\right), -1, 0\right)$$

$$d_C(P', Q') = \frac{r^2 d_C(P, Q)}{d_C(O, P) d_C(O, Q)}$$

is valid.

Proof. Since translations preserve the Chinese Checkers distances, it is enough to consider the center O of the inversion sphere as the origin. Let P, $Q \in D_1$ in $\mathbb{R}^3_C$. If $P = (p, 0, 0)$ and $Q = (0, q, 0)$, the images of P and Q respect to $I_{(O,r)}$ are $P' = (\frac{r^2}{p}, 0, 0)$ and $Q' = (0, \frac{r^2}{q}, 0)$. It follows that

$$d_C(P', Q') = d_L(P', Q') + \left(\sqrt{2}-1\right) d_S(P', Q')$$

$$d_L(P', Q') = \max\left\{\left|\frac{r^2}{p}\right|, \left|\frac{r^2}{q}\right|, 0\right\}$$

$$d_S(P', Q') = \min\left\{\left|\frac{r^2}{p} + \frac{r^2}{q}\right|, \left|\frac{r^2}{p}\right|, \left|\frac{r^2}{q}\right|\right\}$$

and then on D_1

$$d_C(P',Q') = \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}.$$

If P, $Q \in D_2$ in $\mathbb{R}^3_C$. then the images of $P = \left(\frac{p}{2\sqrt{2}-1}, \frac{p}{2\sqrt{2}-1}, \frac{p}{2\sqrt{2}-1}\right)$ and $Q = \left(-\frac{q}{2\sqrt{2}-1}, \frac{q}{2\sqrt{2}-1}, \frac{q}{2\sqrt{2}-1}\right)$ respect to $I_{(O,r)}$ are $P' = \left(\frac{r^2}{p\left(2\sqrt{2}-1\right)}, \frac{r^2}{p\left(2\sqrt{2}-1\right)}, \frac{r^2}{p\left(2\sqrt{2}-1\right)}\right)$ and $Q' = \left(-\frac{r^2}{q\left(2\sqrt{2}-1\right)}, \frac{r^2}{q\left(2\sqrt{2}-1\right)}, \frac{r^2}{q\left(2\sqrt{2}-1\right)}\right)$. It follows that

$$d_C(P',Q') = d_L(P',Q') + \left(\sqrt{2}-1\right) d_S(P',Q')$$

$$d_L(P',Q') = \max\left\{\left|\frac{r^2}{p\left(2\sqrt{2}-1\right)} + \frac{r^2}{q\left(2\sqrt{2}-1\right)}\right|, \left|\frac{r^2}{p\left(2\sqrt{2}-1\right)} - \frac{r^2}{q\left(2\sqrt{2}-1\right)}\right|\right\}$$

$$d_S(P',Q') = \min\left\{\begin{array}{c} \left|\frac{r^2}{p\left(2\sqrt{2}-1\right)} + \frac{r^2}{q\left(2\sqrt{2}-1\right)}\right| + \left|\frac{r^2}{p\left(2\sqrt{2}-1\right)} - \frac{r^2}{q\left(2\sqrt{2}-1\right)}\right|, \\ 2\left|\frac{r^2}{p\left(2\sqrt{2}-1\right)} - \frac{r^2}{q\left(2\sqrt{2}-1\right)}\right| \end{array}\right\}$$

and then on D_2

$$d_C(P',Q') = \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}.$$

If P, $Q \in D_3$ in $\mathbb{R}^3_C$. then the images of $P = \left(p, p\left(\sqrt{2}+1\right), 0\right)$ and $Q = \left(q, -q\left(\sqrt{2}+1\right), 0\right)$ respect to $I_{(O,r)}$ are $P' = \left(\frac{r^2}{8p}, \frac{r^2\left(\sqrt{2}+1\right)}{8p}, 0\right)$ and $Q' = \left(\frac{r^2}{8q}, -\frac{r^2\left(\sqrt{2}+1\right)}{8q}, 0\right)$. So, we get

$$d_C(P',Q') = d_L(P',Q') + \left(\sqrt{2}-1\right) d_S(P',Q')$$

$$d_L(P',Q') = \max\left\{\left|\frac{r^2}{8p} - \frac{r^2}{8q}\right|, \left|\frac{r^2\left(\sqrt{2}+1\right)}{8p} + \frac{r^2\left(\sqrt{2}+1\right)}{8q}\right|, 0\right\}$$

$$d_S(P',Q') = \min\left\{\begin{array}{c} \left|\frac{r^2}{8p} - \frac{r^2}{8q}\right| + \left|\frac{r^2\left(\sqrt{2}+1\right)}{8p} + \frac{r^2\left(\sqrt{2}+1\right)}{8q}\right|, \left|\frac{r^2}{8p} - \frac{r^2}{8q}\right|, \\ \left|\frac{r^2\left(\sqrt{2}+1\right)}{8p} + \frac{r^2\left(\sqrt{2}+1\right)}{8q}\right| \end{array}\right\}$$

and then on D_3

$$d_C(P',Q') = \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}.$$

$\square$

3. Cross Ratio and Harmonic Conjugates

The Chinese Checkers directed distance from the point A to the point B along a line l in $\mathbb{R}_C^2$ is denoted by $d_C[AB]$. If the ray with initial point A containing B has the positive direction of orientation, $d_C[AB] = d_C(A,B)$ and if the ray has the opposite direction, $d_C[AB] = -d_C(A,B)$ [11]. The Chinese Checkers cross ratio is preserved by the inversion in the Chinese Checkers circle as in the taxicab plane in [17].

Now, we show the properties related to the Chinese Checkers cross ratio and harmonic conjugates in $\mathbb{R}_C^3$.

Definition 7. *Let A, B, C and D be four distinct points on an oriented line in $\mathbb{R}_C^3$. We define the their Chinese Checkers cross ratio $(AB,CD)_C$ in $\mathbb{R}_C^3$ by*

$$(AB,CD)_C = \frac{d_C[AC]}{d_C[AD]}\frac{d_C[BD]}{d_C[BC]}.$$

It is known that the cross ratio is positive if both C and D are between A and B or if neither C nor D is between A and B, whereas the cross ratio is negative if the pairs $\{A,B\}$ and $\{C,D\}$ separate each other. Also, the cross ratio is an invariant under inversion in a sphere whose center is not any of the four points A, B, C and D in the taxicab plane, [11]. Similarly, this property is valid in Chinese Checkers sphere.

Theorem 8. *The inversion in a Chinese Checkers sphere in $\mathbb{R}_C^3$ preserves the Chinese Checkers cross ratio.*

Proof. Let A, B, C and D be four collinear points on an oriented line l with the center of the inversion $I_{(O,r)}$ in $\mathbb{R}_C^3$. Let $I_{(O,r)}$ transform A, B, C and D into A', B', C' and D', respectively. The Chinese Checkers spherical inversion reverses the Chinese Checkers directed distance from the point A to the point B along a line l in $\mathbb{R}_C^3$ to the Chinese Checkers directed distance from the point B' to the point A' and preserves the separation or non separation of the pair A, B and C, D. Hence it is suffices to show that $|(A'B',C'D')_C| = |(AB,CD)_C|$. This follows Theorem 2

$$\begin{aligned}
\frac{d_C\left(A',C'\right)}{d_C\left(A',D'\right)}\frac{d_C\left(B',D'\right)}{d_C\left(B',C'\right)} &= \frac{\dfrac{r^2 d_C(A,C)}{d_C(O,A)\,d_C(O,C)}\;\dfrac{r^2 d_C(B,D)}{d_C(O,B)\,d_C(O,D)}}{\dfrac{r^2 d_C(A,D)}{d_C(O,A)\,d_C(O,D)}\;\dfrac{r^2 d_C(B,C)}{d_C(O,B)\,d_C(O,C)}} \\
&= \frac{d_C(A,C)}{d_C(A,D)}\frac{d_C(B,D)}{d_C(B,C)}.
\end{aligned}$$

□

Definition 9. *Let A and B be two points on a line l in $\mathbb{R}_C^3$, any pair C and D on the line l for which $\frac{d_C[AC]}{d_C[CB]} = \frac{d_C[AD]}{d_C[DB]}$ is said to divide A and B harmonically. The points*

C and D are called Chinese Checkers harmonic conjugates with respect to A and B, and the Chinese Checkers harmonic set of points is denoted by $H(AB,CD)_C$.

It is clear that two distinct points C and D are Chinese Checkers harmonic conjugates with respect to A and B if and only if $(AB,CD)_C = -1$.

Theorem 10. *Let $\mathcal{S}$ be a Chinese Checkers sphere with the center O, and the line segment $[AB]$ a diameter of $\mathcal{S}$ in $\mathbb{R}^3_C$. Let P and P' be distinct points of the ray OA, which divide the segment $[AB]$ internally and externally. Then P and P' are Chinese Checkers harmonic conjugates with respect to A and B if and only if P and P' are inverse points with respect the Chinese Checkers spherical inversion $I_{(O,r)}$.*

Proof. Suppose that P and P' are Chinese Checkers harmonic conjugates with respect to A and B in $\mathbb{R}^3_C$. Then

$$(AB,PP')_C = -1$$
$$\frac{d_C\,[AP]}{d_C\,[AP']}\frac{d_C\,[BP']}{d_C\,[BP]} = -1.$$

Since P divides the line segment $[AB]$ internally and P is on the ray OB, $d_C(P,B) = r - d_C(O,P)$ and $d_C(A,P) = r + d_C(O,P)$. Since P' divides the line segment $[AB]$ externally and P' is on the ray OB, $d_C(A,P') = d_C(O,P') + r$ and $d_C(B,P') = d_C(O,P') - r$.

Hence

$$\frac{r + d_C(O,P)}{d_C(O,P') + r}\cdot\frac{d_C(O,P') - r}{d_C(O,P) - r} = -1$$
$$(r + d_C(O,P))\,(d_C(O,P') - r) = (d_C(O,P') + r)\cdot(r - d_C(O,P))\,.$$

Simplifying the last equality, $d_C(O,P)d_C(O,P') = r^2$ is obtained. Therefore P and P' are Chinese Checkers inverse points with respect to the Chinese Checkers spherical inversion $I_{(O,r)}$.

Conversely, if P and P' are Chinese Checkers inverse points with respect to the Chinese Checkers spherical inversion $I_{(O,r)}$, the proof is similar. □

4. Chinese Checkers Spherical Inversions of Lines, Planes and Chinese Checkers Spheres

It is well known that inversions with respect to circle transform lines and circles into lines and/ or circles in Euclidean plane and Hyperbolic plane.

The following features are well known for inversion in Euclidean plane:

i) Lines passing through the inversion center are invariant.

ii) Lines that do not pass through the center of inversion transform circles passing through the center of inversion.

iii) Circles passing through the center of inversion transform lines does not pass through the center of the inversion.

iv) Circles not passing through the center of inversion transform circles does not pass through the center of the inversion.

v) Circles with center of inversion transform circles with center of inversion.

In this section, we study the Chinese Checkers spherical inversion of lines, planes and Chinese Checkers spheres. The Chinese Checkers spherical inversion $I_{(O,r)}$ maps the lines, planes passing through O onto themselves.

The Chinese Checkers spherical inversion $I_{(O,r)}$ maps Chinese Checkers spheres with centered O onto Chinese Checkers spheres. But the Chinese Checkers spherical inversion of a sphere not passing through the centre of inversion is another Chinese Checkers sphere that does not contain the centre of inversion.

Theorem 11. *Consider the inversion $I_{(O,r)}$ in a Chinese Checkers sphere $\mathcal{S}$ with the centre O. Every line and plane containing O is invariant under the inversion.*

Proof. It is clear that the straight lines containing O onto themselves.

Let $\mathcal{S}$ be a Chinese Checkers sphere of inversion and $P=(x,y,z)$ with equation $d_L(O,P)+\left(\sqrt{2}-1\right)d_S(O,P)=r$ and the plane $Mx+Ny+Tz=0$. Applying $I_{(O,r)}$ to this plane gives

$$M\frac{r^2x'}{\left(d_C(O,P)\right)^2}+N\frac{r^2y'}{\left(d_C(O,P)\right)^2}+T\frac{r^2z'}{\left(d_C(O,P)\right)^2}=0.$$

So, $Mx'+Ny'+Tz'=0$ is obtained. □

The inverse of a plane not containing O is not a Chinese Checkers sphere containing O.

Theorem 12. *The inverse of a Chinese Checkers sphere with the centre O with respect to the Chinese Checkers spherical inversion $I_{(O,r)}$ is a Chinese Checkers sphere containing O.*

Proof. Since the translation preserve distance in $\mathbb{R}^3_C$, we can take a Chinese Checkers sphere $\mathcal{S}$ of inversion and $P=(x,y,z)$ with equation
$d_L(O,P)+\left(\sqrt{2}-1\right)d_S(O,P)=r$ and $\mathcal{S}$ the Chinese Checkers sphere
$d_L(O,P)+\left(\sqrt{2}-1\right)d_S(O,P)=k$, $k\in\mathbb{R}^+$. Applying $I_{(O,r)}$ to $\mathcal{S}$ gives

$$d_L(O,P')+\left(\sqrt{2}-1\right)d_S(O,P')=\frac{r^2}{k}.$$

Note that this is a Chinese Checkers sphere with the centre O. □

Theorem 13. *The inversion $I_{(O,r)}$ in a Chinese Checkers sphere $\mathcal{S}$ with centre O. Every edges, vertices and faces of Chinese Checkers sphere is invariant under the inversion.*

Proof. The points of Chinese Checkers sphere are mapped by $I_{(O,r)}$ back onto Chinese Checkers sphere from the Definition 1. Hence every edges, vertices and faces of Chinese Checkers sphere is invariant under $I_{(O,r)}$. □

References

[1] Blair, D., Inversion Theory and Conformal Mapping, *Student Mathematical Library,* American Mathematical Society, 9, 2000.

[2] Blåsjö Jakob, V., Steiner's Systematische Entwickelung: The Culmination of Classical Geometry, *The Mathematical Intelligencer,* 31(1) (2009), 21-29.

[3] Childress, N., Inversion with respect to the central conics, Mathematics Magazine, 38(3) (1965).

[4] Deza, E., Deza, M., Dictionary of Distances, Elsevier Science, 2006.

[5] Gelisgen, Ö., Kaya, R., The Taxicab Space Group, *Acta Mathematica Hungarica*, 122 (1) (2008), 187-200.

[6] Kaya, R., Akça, Z., Günaltılı, İ., Özcan, M., General Equation For taxicab Conics and Their Classification, *Mitt. Math. Ges. Hamburg,* 19 (2000), 135-148.

[7] Krause, E. F., Taxicab Geometry, Addison-Weley, Menlo Park, California,1975.

[8] Menger, K., You Will Like Geometry, Guildbook of Illinois Institute of Technology Geometry Exhibit, Museum of Science & Industry,Chicago, Illinois, 1952.

[9] Minkowski, H., Gasammelte Abhandlungen, Chelsea Publishing Co., New York,1967.

[10] Nickel, J.A., A Budget of Inversion, *Math. Comput. Modelling,* 21(6) (1995), 87-93.

[11] Özcan, M., Kaya, R., On the Ratio of Directed Lengths in the Taxicab Plane and Related Properties, *Missouri Journal of Mathematical Sciences,* 14(2) (2002).

[12] Gelisgen, Ö., Kaya, R., Özcan, M., Distance Formulae in the Chinese Checker Space, *Int.J. Pure Appl. Math.,* 26(1) (2006), 35-44.

[13] Ramirez, J.L., An Introduction to Inversion in an Ellipse, arXiv:1309.6378v1, Sept. 2013.

[14] Schattschneider, D. J., The Taxicab Group, *American Mathematical Monthly,* 91 (1984), 423-428.

[15] Kaya, R., Gelisgen, Ö., Bayar, A., Ekmekçi, S., Group of Isometries of CC-Plane, *Missouri J. Math. Sci.,* 18 (2006), 221-233.

[16] Ramirez, J.L., Rubiano, G. N., A generalization of the spherical inversion, *International Journal of Mathematical Education in Science and Technology*, 48(1) (2016), 132-149.

[17] Bayar,A., Ekmekçi, S., On circular inversions in taxicab plane. *J. Adv. Res. Pure Math.,* 6(4) (2014) , 33–39.

[18] Akca, Z., Kaya, R., On the Distance Formulae In three Dimensional Taxicab Space, *Hadronic Journal,* 27 (2006), 521-532.

[19] Gelisgen, Ö., Ermiş, T., Some properties of inversions in alpha plane, *Forum Geometricorum,* 19 (2019), 1-9.

[20] Ramirez, J.L., Rubiano, G. N., Jurcic-Zlobec, B., Generating fractal patterns by using $p-$circle, *Fractals*, 23(4) (2015), 1-13.

[21] Pekzorlu, A., Bayar, A., Taxicab Spherical Inversions in Taxicab Space, *Journal Of Mahani, Math. Research Center*, 9(1-2) (2020), 45-54.

[22] Chen, G., Lines and Circles in Taxicab Geometry Master Thesis, Department of Mathematic and Computer Science, Central Missouri State Uni, 1992.

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A NOTE ON QUASI BI-SLANT SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

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Abstract. The aim of the present paper is to define and study the notion of quasi bi-slant submanifolds of almost contact metric manifolds. We mainly concerned with quasi bi-slant submanifolds of cosymplectic manifolds as a generalization of slant, semi-slant, hemi-slant, bi-slant and quasi hemi-slant submanifolds. First, we give non-trivial examples in order to demostrate the method presented in this paper is effective and investigate the geometry of distributions. Moreover, We study these types of submanifolds with parallel canonical structures.

1. Introduction

Study of submanifolds theory has shown an increasing development in image processing, computer design, economic modeling as well as in mathematical physics and in mechanics. In this manner, B-Y. Chen [6] initiated the notion of slant submanifold as a generalization of both holomorphic (invariant) and totally real submanifold (anti-invariant) of an almost Hermitian manifold. Inspried by B-Y. Chen's paper, many geometers have studied this notion in the different kind of structures: (see [7], [22], [23]). Many consequent results on slant submanifolds are collected in his book [5]. After this notion, as a generalization of semi-slant submanifold which was defined by N. Papaghiuc [19] (see also [8]). A. Carriazo [3] and [4] introduced the notion of bi-slant submanifold under the name anti-slant submanifold. However, B. Şahin called these submanifolds hemi-slant submanifolds in [21]. (See also [9] and [10], [20], [24]).

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Furthermore, the submanifolds of a cosymplectic manifold have been studied by many geometers: See [11], [12], [13], [14], [15], [16], [18]. Taking into account of the above studies, we are motivated to fill a gap in the literature by giving the notion of quasi bi-slant submanifolds in which the tangent bundle consist of one invariant and two slant distributions and the Reeb vector field. In this paper, as a generalization of slant, semi-slant, hemi-slant, bi-slant and quasi hemi-slant submanifolds, we introduce quasi bi-slant submanifolds and investigate the geometry of distributions in detail.

The paper is organized as follows: In section 2, we recall basic formulas and definitions for a cosymplectic manifold and their submanifolds. In section 3, we introduce the notion of quasi bi-slant submanifolds, giving a non-tirivial example and obtain some basic results for the next sections. In section 4, we give some necessary and sufficient conditions for the geometry of distributions. Finally, we study these types of submanifolds with parallel canonical structures.

2. Preliminaries

In this section, we give the definition of cosymplectic manifold and some background on submanifolds theory.

A $(2m+1)$-dimensional C^∞-manifold M said to have an almost contact structure if there exist on M a tensor field φ of type (1,1), a vector field ξ and 1-form η satisfying:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta o\varphi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold M satisfying the following conditions

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.2}$$

where X, Y are vector fields on M.

An almost contact structure (φ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $M \times \mathbb{R}$ is given by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

where f is a C^∞-function on $M \times \mathbb{R}$ has no torsion i.e., J is integrable. The condition for normality in terms of φ, ξ and η is $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ on M, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ. Finally, the fundamental two-form Φ is defined $\Phi(X, Y) = g(X, \varphi Y)$.

An almost contact metric structure (φ, ξ, η, g) is said to be cosymplectic, if it is normal and both Φ and η are closed ([1], [2], [16]), and the structure equation of a cosymplectic manifold is given by

$$(\nabla_X \varphi)Y = 0 \tag{2.3}$$

for any X, Y tangent to M, where ∇ denotes the Riemannian connection of the metric g on M. Moreover, for cosymplectic manifold

$$\nabla_X \xi = 0. \tag{2.4}$$

Example. ([17]) $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_i, y_i, z)(i = 1, ..., n)$ and its usual contact form

$$\eta = dz \quad \text{and} \quad \xi = \frac{\partial}{\partial z},$$

here ξ is the characteristic vector field and its Riemannian metric g and tensor field φ are given by

$$g = \sum_{i=1}^{n}((dx_i)^2 + (dy_i)^2) + (dz)^2, \qquad \varphi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1, ..., n.$$

This gives a cosymplectic manifold on $\mathbb{R}^{2n+1}$. The vector fields $e_i = \frac{\partial}{\partial y_i}$, $e_{n+i} = \frac{\partial}{\partial x_i}$, ξ form a φ-basis for the cosymplectic structure. On the other hand, it can be shown that $\mathbb{R}^{2n+1}(\varphi, \xi, \eta, g)$ is a cosymplectic manifold.

Let M be a Riemannian manifold isometrically immersed in $\bar{M}$ and induced Riemannian metric on M is denoted by the same symbol g throughout this paper. Let $\mathcal{A}$ and h denote the shape operator and second fundamental form, respectively, of immersion of M into $\bar{M}$. The Gauss and Weingarten formulas of M into $\bar{M}$ are given by [6]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

and

$$\bar{\nabla}_X V = -\mathcal{A}_V X + \nabla^{\perp}_X V, \tag{2.6}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ is the induced connection on M and $\nabla^{\perp}$ represents the connection on the normal bundle $T^{\perp}M$ of M and A_V is the shape operator of M with respect to normal vector $V \in \Gamma(T^{\perp}M)$. Moreover, $\mathcal{A}_V$ and h are related by

$$g(h(X, Y), V) = g(\mathcal{A}_V X, Y) \tag{2.7}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

If $h(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$, then M is said to be totally geodesic.

3. Quasi bi-slant submanifolds of cosmyplectic manifolds

In this section, we define the concept of quasi bi-slant submanifolds of cosymplectic manifolds, giving a non-trivial example and obtain some related results for later use.

Definition 3.1. A submanifold M of cosymplectic manifold $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ is called quasi bi-slant if there exists four orthogonal distributions $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ and ξ of M, at the point $p \in M$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi >$
(ii) The distribution $\mathcal{D}$ is invariant, i.e. $\varphi\mathcal{D} = \mathcal{D}$.
(iii) $\varphi\mathcal{D}_1 \perp \mathcal{D}_2$ and $\varphi\mathcal{D}_2 \perp \mathcal{D}_1$;
(iv) The distributions $\mathcal{D}_1, \mathcal{D}_2$ are slant with slant angle θ_1, θ_2, respectively.

Taking the dimension of distributions $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are m_1, m_2 and m_3, respectively. One can easily see the following cases:

- If $m_1 \neq 0$ and $m_2 = m_3 = 0$, then M is a invariant submanifold.
- If $m_1 = m_2 = 0$ and $\theta_2 = \frac{\pi}{2}$ then M is an anti-invariant submanifold.
- If $m_1 = 0, m_2 \neq m_3 \neq 0$, $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$ then M is a semi-invariant submanifold.
- If $m_1 = m_2 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$ then M is a slant submanifold.
- If $m_1 = 0, m_2 \neq m_3 \neq 0$, $\theta_1 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$ then M is a semi-slant submanifold.
- If $m_1 = 0, m_2 \neq m_3 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $0 < \theta_2 < \frac{\pi}{2}$ then M is a hemi-slant submanifold.
- If $m_1 = 0, m_2 \neq m_3 \neq 0$, and θ_1 and θ_2 are different from either 0 and $\frac{\pi}{2}$, then M is a bi-slant submanifold.

If $m_1 \neq m_2 \neq m_3 \neq 0$ and $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$, then M is called a ***proper quasi bi-slant submanifold***.

Remark 3.2. In this paper, we assume that M is *proper* quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$.

Now, we present an example of proper quasi bi-slant submanifold in $\mathbb{R}^{11}$.

Example. We will use the canonical contact structure φ defined by

$$\varphi(x_1, y_1, ..., x_n, y_n, z) = (y_1, -x_1, ..., y_n, -x_n, 0).$$

Thus we have $\varphi(\partial x_i) = \partial y_i$, $\varphi(\partial y_j) = -\partial x_j$ and $\varphi(\partial z) = 0$, $\quad 1 \leq i, j \leq 5$ where $\partial x_i = \frac{\partial}{\partial x_i}$. For any pair of real numbers θ_1, θ_2 satisfying $0 < \theta_1, \theta_2 < \frac{\pi}{2}$, let us consider submanifold M_{θ_1,θ_2} of $\mathbb{R}^{11}$ defined by

$\pi_{\theta_1,\theta_2}(u, s, w, k, t, r, z) = (u, s\cos\theta_1, 0, s\sin\theta_1, \omega, k\cos\theta_2, 0, k\sin\theta_2, t, r, z)$. If we take

$$e_1 = \partial x_1, \quad e_2 = \cos\theta_1 \partial y_1 + \sin\theta_1 \partial y_2,$$
$$e_3 = \partial x_3, \quad e_4 = \cos\theta_2 \partial y_3 + \sin\theta_2 \partial y_4,$$
$$e_5 = \partial x_5, \quad e_6 = \partial y_5, \quad e_7 = \xi = \partial z$$

then the restriction of $e_1, ..., e_7$ to M forms an orthonormal frame of the tangent bundle TM. Obviously, we get

$$\varphi e_1 = \partial y_1, \ \varphi e_2 = -\cos\theta_1 \partial x_1 - \sin\theta_1 \partial x_2, \ \varphi e_3 = \partial y_3$$
$$\varphi e_4 = -\cos\theta_2 \partial x_3 - \sin\theta_2 \partial x_4, \ \varphi e_5 = \partial y_5, \ \varphi e_6 = -\partial x_5.$$

Let us put $\mathcal{D}_1 = Span\{e_1, e_2\}$, $\mathcal{D}_2 = Span\{e_3, e_4\}$, and $\mathcal{D} = Span\{e_5, e_6\}$. Then obviously $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}$, satisfy the definition of quasi bi-slant submanifold M_{θ_1,θ_2}

defined by π_{θ_1,θ_2} is a proper quasi bi-slant submanifold of $\mathbb{R}^{11}$ with θ_1, θ_2 as its bi-slant angles.

Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then, for any $X \in \Gamma(TM)$, we have

$$X = \mathcal{P}X + \mathcal{Q}X + \mathcal{R}X + \eta(X)\xi \tag{3.1}$$

where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ denote the projections on the distributions $\mathcal{D}, \mathcal{D}_1$ and $\mathcal{D}_2$, recpectively.

$$\varphi X = \mathcal{T}X + \mathcal{F}X, \tag{3.2}$$

where $\mathcal{T}X$ and $\mathcal{F}X$ are tangential and normal components on M. Making now use of (3.1) and (3.2), we get immediately

$$\varphi X = \mathcal{TP}X + \mathcal{TQ}X + \mathcal{FQ}X + \mathcal{TR}X + \mathcal{FR}X, \tag{3.3}$$

here since $\varphi\mathcal{D} = \mathcal{D}$, we have $\mathcal{FP}X = 0$. Thus we get

$$\varphi(TM) = \mathcal{D} \oplus \mathcal{T}\mathcal{D}_1 \oplus \mathcal{T}\mathcal{D}_2 \tag{3.4}$$

and

$$T^{\perp}M = \mathcal{F}\mathcal{D}_1 \oplus \mathcal{F}\mathcal{D}_2 \oplus \mu \tag{3.5}$$

where μ is the orthogonal complement of $\mathcal{F}\mathcal{D}_1 \oplus \mathcal{F}\mathcal{D}_2$ in $T^{\perp}M$ and it is invariant with recpect to φ. Also, for any $Z \in T^{\perp}M$, we have

$$\varphi Z = \mathcal{B}Z + \mathcal{C}Z, \tag{3.6}$$

where $\mathcal{B}Z \in \Gamma(TM)$ and $\mathcal{C}Z \in \Gamma(T^{\perp}M)$.

Taking into account of the condition (iii) in Definition (3.1), (3.2) and (3.6), we obtain the followings:

Lemma 3.3. *Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then, we have*

$$(\mathbf{a})\ T\mathcal{D}_1 \subset \mathcal{D}_1, \quad (\mathbf{b})\ T\mathcal{D}_2 \subset \mathcal{D}_2, \quad (\mathbf{c})\ \mathcal{BF}\mathcal{D}_1 = \mathcal{D}_1, \quad (\mathbf{d})\ \mathcal{BF}\mathcal{D}_2 = \mathcal{D}_2.$$

With the help of (3.2) and (3.6), we obtain the following Lemma.

Lemma 3.4. *Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then, we have*

$$\begin{aligned}
(\mathbf{a})\ \mathcal{T}^2U_1 &= -\cos^2\theta_1 U_1, \quad &(\mathbf{b})\ \mathcal{T}^2U_2 &= -\cos^2\theta_2 U_2,\\
(\mathbf{c})\ \mathcal{BF}U_1 &= -\sin^2\theta_1 U_1, \quad &(\mathbf{d})\ \mathcal{BF}U_2 &= -\sin^2\theta_1 U_2,\\
(\mathbf{e})\ \mathcal{T}^2U_1 + \mathcal{BF}U_1 &= -U_1, \quad &(\mathbf{f})\ \mathcal{T}^2U_2 + \mathcal{BF}U_2 &= -U_2,\\
(\mathbf{g})\ \mathcal{FT}U_1 + \mathcal{CF}U_1 &= 0, \quad &(\mathbf{h})\ \mathcal{FT}U_2 + \mathcal{CF}U_2 &= 0,
\end{aligned}$$

for any $U_1 \in \mathcal{D}_1$ and $U_2 \in \mathcal{D}_2$.

By using (2.3), Definition (3.1), (3.2) and (3.6), we obtain the following Lemma.

Lemma 3.5. *Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then, we have*

(i) $\mathcal{T}_i^2 U_i = -\cos^2\theta_i U_i$,
(ii) $g(\mathcal{T}_i U_i, \mathcal{T}_i V_i) = (\cos^2\theta_i) g(U_i, V_i)$,
(iii) $g(\mathcal{F}_i U_i, \mathcal{F}_i V_i) = (\sin^2\theta_i) g(U_i, V_i)$

for any $i = 1, 2$, $U_1, V_1 \in \Gamma(\mathcal{D}_1)$ *and* $U_2, V_2 \in \Gamma(\mathcal{D}_2)$.

We need the following lemma for later use.

Lemma 3.6. *Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$, then for any $Z_1, Z_2 \in \Gamma(TM)$, we have the following*

$$\nabla_{Z_1}\mathcal{T}Z_2 - \mathcal{T}\nabla_{Z_1}Z_2 = \mathcal{A}_{\mathcal{F}Z_2}Z_1 + \mathcal{B}h(Z_1, Z_2) \tag{3.7}$$

and

$$\nabla^{\perp}_{Z_1}\mathcal{F}Z_2 - \mathcal{F}\nabla_{Z_1}Z_2 = \mathcal{C}h(Z_1, Z_2) - h(Z_1, \mathcal{T}Z_2). \tag{3.8}$$

Proof. Since $\bar{M}$ is a cosmyplectic manifold, we have that

$$(\bar{\nabla}_{Z_1}\varphi)Z_2 = 0$$

which implies that

$$\bar{\nabla}_{Z_1}\varphi Z_2 - \varphi\bar{\nabla}_{Z_1}Z_2 = 0.$$

By using (2.5) and (3.2), we get

$$\bar{\nabla}_{Z_1}\mathcal{T}Z_2 + \bar{\nabla}_{Z_1}\mathcal{F}Z_2 - \varphi(\nabla_{Z_1}Z_2 + h(Z_1, Z_2)) = 0.$$

Taking into account of (2.5), (2.6), (3.2) and (3.6), we obtain

$$\begin{aligned}&\nabla_{Z_1}\mathcal{T}Z_2 + h(Z_1, \mathcal{T}Z_2) - \mathcal{A}_{\mathcal{F}Z_2}Z_1 + \nabla^{\perp}_{Z_1}\mathcal{F}Z_2\\ &- \mathcal{T}\nabla_{Z_1}Z_2 - \mathcal{F}\nabla_{Z_1}Z_2 - \mathcal{B}hZ_1, Z_2 - \mathcal{C}h(Z_1, Z_2) = 0.\end{aligned}$$

Comparing the tangential and normal components, we have the required results. $\square$

In a similar way, we have:

Lemma 3.7. *Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$, then we have the following*

$$\nabla_{Z_1}\mathcal{B}W_1 - \mathcal{B}\nabla^{\perp}_{Z_1}W_1 = \mathcal{A}_{\mathcal{C}W_1}Z_1 - \mathcal{T}\mathcal{A}_{W_1}Z_1 \tag{3.9}$$

and

$$\nabla^{\perp}_{Z_1}\mathcal{C}W_1 - \mathcal{C}\nabla^{\perp}_{Z_1}W_1 = -\mathcal{F}\mathcal{A}_{W_1}Z_1 - h(Z_1, \mathcal{B}W_1) \tag{3.10}$$

for any $Z_1 \in \Gamma(TM)$ *and* $W_1 \in \Gamma(T^{\perp}M)$.

4. Integrability and totally geodesic foliations

In this section we give some necessary and sufficient conditions for the integrability of the distributions.

First, we have the following theorem:

Theorem 4.1. *Let M be a quasi bi-slant submanifold of $\bar{M}$. The invariant distribution $\mathcal{D}$ is integrable if and only if*

$$g(\mathcal{T}(\nabla_X \mathcal{T}Y - \nabla_Y \mathcal{T}X), Z) = g(h(X, \mathcal{T}Y) - h(Y, \mathcal{T}X), \varphi QZ + \varphi RZ)$$

for any $X, Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$.

Proof. The distribution $\mathcal{D}$ is integrable on M if and only if

$$g([X,Y],\xi) = 0 \quad \text{and} \quad g([X,Y],Z) = 0$$

for any $X, Y \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$ and $\xi \in \Gamma(TM)$. Since M is a cosymplectic manifold, we immediately have $g([X,Y],\xi) = 0$. Thus $\mathcal{D}$ is integrable if and only if $g([X,Y],Z) = 0$. Now, for any $X, Y \in \mathcal{D}$ and $Z = \mathcal{Q}Z + \mathcal{R}Z \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$, by using (2.2), (2.5), we obtain

$$g([X,Y],Z) = g(\varphi \bar{\nabla}_X Y, \varphi Z) - \eta(\bar{\nabla}_X Y)\eta(Z) - g(\varphi \bar{\nabla}_Y X, \varphi Z) + \eta(\bar{\nabla}_Y X)\eta(Z).$$

Now, using (2.4), (3.2) and $\mathcal{F}Y = 0$ for any $Y \in \Gamma(\mathcal{D})$, we have

$$\begin{aligned} g([X,Y],Z) &= g(\bar{\nabla}_X \varphi Y, \varphi Z) - g(\bar{\nabla}_Y \varphi X, \varphi Z) \\ &= g(\bar{\nabla}_X \mathcal{T}Y, \varphi Z) - g(\bar{\nabla}_Y \mathcal{T}X, \varphi Z). \end{aligned}$$

Taking into account of (2.5) and (3.3) in the above equation, we get

$$\begin{aligned} g([X,Y],Z) = &-g(\varphi \nabla_X \mathcal{T}Y, Z) + g(h(X, \mathcal{T}Y), \varphi Z) \\ &+ g(\varphi \nabla_Y \mathcal{T}X, Z) - g(h(Y, \mathcal{T}X), \varphi Z). \end{aligned}$$

Now again taking into account the equation (3.2), we obtain

$$\begin{aligned} g([X,Y],Z) = &\, g(\mathcal{T}(\nabla_Y \mathcal{T}X - \nabla_X \mathcal{T}Y), Z) \\ &+ g(h(X, \mathcal{T}Y) - h(Y, \mathcal{T}X), \varphi QZ + \varphi RZ) \end{aligned}$$

which completes the proof. □

For the slant distribution $\mathcal{D}_1$, we have:

Theorem 4.2. *Let M be a quasi bi-slant submanifold of $\bar{M}$. The slant distribution $\mathcal{D}_1$ is integrable if and only if*

$$\begin{aligned} g(\nabla^{\perp}_{U_1} \mathcal{F}V_1 + \nabla^{\perp}_{V_1} \mathcal{F}U_1, \mathcal{F}\mathcal{R}Z) = &\, g(\mathcal{A}_{\mathcal{F}\mathcal{T}V_1} U_1 - \mathcal{A}_{\mathcal{F}\mathcal{T}U_1} V_1, Z) \\ &+ g(\mathcal{A}_{\mathcal{F}V_1} U_1 + \mathcal{A}_{\mathcal{F}U_1} V_1, \mathcal{T}Z) \end{aligned}$$

for any $U_1, V_1 \in \Gamma(\mathcal{D}_1)$, $Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}_2)$.

Proof. The distribution $\mathcal{D}_1$ is integrable on M if and only if

$$g([U_1, V_1], \xi) = 0 \quad \text{and} \quad g([U_2, V_2], Z) = 0$$

for any $U_1, V_1 \in \Gamma(\mathcal{D}_1)$, $Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}_2)$ and $\xi \in \Gamma(TM)$. The first case is trivial. Thus $\mathcal{D}_1$ is integrable if and only if $g([U_1, V_1], Z) = 0$. Now, for any $U_1, V_1 \in \mathcal{D}_1$ and $Z = \mathcal{P}Z + \mathcal{R}Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}_2)$, by using (2.2), (2.5), we obtain

$$\begin{aligned} g([U_1, V_1], Z) = &-g(\bar{\nabla}_{U_1}\varphi\mathcal{T}V_1, Z) - g(\bar{\nabla}_{U_1}\mathcal{F}V_1, \varphi Z) \\ &+ g(\bar{\nabla}_{V_1}\varphi\mathcal{T}U_1, Z) - g(\bar{\nabla}_{V_1}\mathcal{F}U_1, \varphi Z). \end{aligned}$$

Taking into account the equation lemma (3.5) (i) in the above equation, we get

$$\begin{aligned} g([U_1, V_1], Z) = &\cos^2\theta_1 g([U_1, V_1], Z) - g(\bar{\nabla}_{U_1}\mathcal{F}\mathcal{T}V_1 - \bar{\nabla}_{V_1}\mathcal{F}\mathcal{T}U_1, Z) \\ &- g(\bar{\nabla}_{U_1}\mathcal{F}V_1 + \bar{\nabla}_{V_1}\mathcal{F}U_1, \varphi\mathcal{P}Z + \varphi\mathcal{R}Z). \end{aligned}$$

Now, using (2.6) and (3.3), we obtain

$$\begin{aligned} g([U_1, V_1], Z) = &\cos^2\theta_1 g([U_1, V_1], Z) + g(\mathcal{A}_{\mathcal{F}\mathcal{T}V_1}U_1 - \mathcal{A}_{\mathcal{F}\mathcal{T}U_1}V_1, Z) \\ &+ g(\mathcal{A}_{\mathcal{F}V_1}U_1 + \mathcal{A}_{\mathcal{F}U_1}V_1, \mathcal{T}Z) \\ &- g(\nabla^{\perp}_{U_1}\mathcal{F}V_1 + \nabla^{\perp}_{V_1}\mathcal{F}U_1, \mathcal{F}\mathcal{R}Z) \end{aligned}$$

or

$$\begin{aligned} \sin^2\theta_1 g([U_1, V_1], Z) = &g(\mathcal{A}_{\mathcal{F}\mathcal{T}V_1}U_1 - \mathcal{A}_{\mathcal{F}\mathcal{T}U_1}V_1, Z) \\ &+ g(\mathcal{A}_{\mathcal{F}V_1}U_1 + \mathcal{A}_{\mathcal{F}U_1}V_1, \mathcal{T}Z) \\ &- g(\nabla^{\perp}_{U_1}\mathcal{F}V_1 + \nabla^{\perp}_{V_1}\mathcal{F}U_1, \mathcal{F}\mathcal{R}Z) \end{aligned}$$

which gives the assertion. □

In a similar way, we obtain the following case for the slant distribution $\mathcal{D}_2$.

Theorem 4.3. *Let M be a quasi bi-slant submanifold of $\bar{M}$. The slant distribution $\mathcal{D}_2$ is integrable if and only if*

$$\mathcal{T}(\nabla_{U_2}\mathcal{T}V_2 - \mathcal{A}_{\mathcal{F}V_2}U_2) \in \Gamma(\mathcal{D}_2),$$
$$\mathcal{B}(h(U_2, TV_2) + \nabla^{\perp}_{U_2}\mathcal{F}V_2) \in \Gamma(T^{\perp}M)$$

and

$$g(\mathcal{A}_{\mathcal{F}Z}V_2 - \nabla_{V_2}\mathcal{T}Z, \mathcal{T}U_2) = g(h(V_2, \mathcal{T}Z) + \nabla^{\perp}_{V_2}\mathcal{F}Z, \mathcal{F}U_2)$$

for any $U_2, V_2 \in \Gamma(\mathcal{D}_2)$, $Z = \mathcal{P}Z + \mathcal{Q}Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}_1)$ and $W \in \Gamma(T^{\perp}M)$.

Theorem 4.4. *Let M be a quasi bi-slant submanifold of $\bar{M}$. The invariant distribution $\mathcal{D}$ defines totally geodesic foliation on M if and only if*

$$g(\nabla_X\mathcal{T}Y, \mathcal{T}Z) = -g(h(X, \mathcal{T}Y), \mathcal{F}Z)$$

and

$$\mathcal{F}\nabla_X\mathcal{T}Y_1 + \mathcal{C}h(X, \mathcal{T}Y) \in \Gamma(TM)$$

for any $X, Y \in \Gamma(\mathcal{D})$, $Z = \mathcal{Q}Z + \mathcal{R}Z \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$ and $W \in \Gamma(T^{\perp}M)$.

Proof. The distribution $\mathcal{D}$ defines a totaly geodesic foliation on M if and only if $g(\bar{\nabla}_X Y, \xi) = 0$, $g(\bar{\nabla}_X Y, Z) = 0$ and $g(\bar{\nabla}_X Y, W) = 0$ for any $X, Y \in \Gamma(\mathcal{D})$, $Z = \mathcal{Q}Z + \mathcal{R}Z \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$ and $W \in \Gamma(T^{\perp}M)$. Then by using (2.2) and (2.4), we obtain

$$g(\bar{\nabla}_X Y, \xi) = Xg(Y, \xi) - g(Y, \bar{\nabla}_X \xi) = -g(Y, \bar{\nabla}_X \xi) = 0. \tag{4.1}$$

On the other hand, using (2.2), we find

$$g(\bar{\nabla}_X Y, Z) = g(\bar{\nabla}_X \varphi Y, \varphi Z) = g(\bar{\nabla}_X \mathcal{T}Y, \varphi Z)$$

here we have used $\mathcal{F}Y = 0$ for any $Y \in \Gamma(\mathcal{D})$. Now, by using (3.3) and (2.5), we have

$$\begin{aligned} g(\bar{\nabla}_X Y, Z) &= g(\nabla_X \mathcal{T}Y + h(X, \mathcal{T}Y), \varphi\mathcal{Q}Z + \varphi\mathcal{R}Z) \\ &= g(\nabla_X \mathcal{T}Y + h(X, \mathcal{T}Y), \mathcal{T}\mathcal{Q}Z + \mathcal{F}\mathcal{Q}Z + \mathcal{T}\mathcal{R}Z + \mathcal{F}\mathcal{R}Z)) \\ &= g(\nabla_X \mathcal{T}Y, \mathcal{T}\mathcal{Q}Z + \mathcal{T}\mathcal{R}Z) + g(h(X, \mathcal{T}Y), \mathcal{F}\mathcal{Q}Z + \mathcal{F}\mathcal{R}Z)) \\ &= g(\nabla_X \mathcal{T}Y, \mathcal{T}Z) + g(h(X, \mathcal{T}Y), \mathcal{F}Z) \end{aligned} \tag{4.2}$$

for any $X, Y \in \Gamma(\mathcal{D})$ and $Z = \mathcal{Q}Z + \mathcal{R}Z \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$. Now, for any $X, Y \in \Gamma(\mathcal{D})$ and $W \in \Gamma(T^{\perp}M)$, we have

$$\begin{aligned} g(\bar{\nabla}_X Y, W) &= -g(\varphi\bar{\nabla}_{X_1} \varphi Y_1, W) = -g(\varphi(\nabla_X \mathcal{T}Y + h(X, \mathcal{T}Y)), W)) \\ &= -g(\mathcal{T}\nabla_X \mathcal{T}Y + \mathcal{F}\nabla_X \mathcal{T}Y + \mathcal{B}h(X, \mathcal{T}Y) + \mathcal{C}h(X, \mathcal{T}Y), W)) \\ &= -g(\mathcal{F}\nabla_X \mathcal{T}Y + \mathcal{C}h(X, \mathcal{T}Y), W). \end{aligned} \tag{4.3}$$

Thus proof follows (4.1), (4.2) and (4.3). □

Theorem 4.5. *Let M be a quasi bi-slant submanifold of $\bar{M}$. The slant distribution $\mathcal{D}_1$ defines totally geodesic foliation on M if and only if*

$$\begin{aligned} &g(\mathcal{A}_{\mathcal{F}\mathcal{T}V_1} U_1, Z) - g(\mathcal{A}_{\mathcal{F}V_1} U_1, \mathcal{T}\mathcal{P}Z) \\ &= g(\mathcal{A}_{\mathcal{F}V_1} U_1, \mathcal{T}\mathcal{R}Z) - g(\nabla^{\perp}_{U_1} \mathcal{F}V_1, \mathcal{F}\mathcal{R}Z) \end{aligned} \tag{4.4}$$

and

$$\mathcal{F}\mathcal{A}_{\mathcal{F}V_1} U_1 - \nabla^{\perp}_{U_1} \mathcal{F}\mathcal{T}V_1 - \mathcal{C}\nabla^{\perp}_{U_1} \mathcal{F}V_1 \in \Gamma(TM)$$

for any $X, Y \in \Gamma(\mathcal{D})$, $Z = \mathcal{Q}Z + \mathcal{R}Z \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$ and $W \in \Gamma(T^{\perp}M)$.

Proof. The distribution $\mathcal{D}_1$ defines a totaly geodesic foliation on M if and only if $g(\bar{\nabla}_{U_1} V_1, \xi) = 0$, $g(\bar{\nabla}_{U_1} V_1, Z) = 0$ and $g(\bar{\nabla}_{U_1} V_1, W) = 0$, for any $U_1, V_1 \in \Gamma(\mathcal{D}_1)$, $Z = \mathcal{P}Z + \mathcal{R}Z \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$ and $W \in \Gamma(T^{\perp}M)$. Since M is a cosymplectic manifold, we immediately have $g(\bar{\nabla}_{U_1} V_1, \xi) = 0$. Now, for any $U_1, V_1 \in \Gamma(\mathcal{D}_1)$, and $Z = \mathcal{P}Z + \mathcal{R}Z \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$, by using (2.2) and (2.4), we obtain

$$g(\bar{\nabla}_{U_1} V_1, Z) = -g(\bar{\nabla}_{U_1} \varphi\mathcal{T}V_1, Z) + g(\bar{\nabla}_{U_1} \mathcal{F}V_1, \varphi\mathcal{P}Z + \varphi\mathcal{R}Z).$$

Now, by using lemma (3.5) (i), we get

$$g(\bar{\nabla}_{U_1} V_1, Z) = cos^2\theta_1 g(\bar{\nabla}_{U_1} V_1, Z) - g(-\mathcal{A}_{\mathcal{F}\mathcal{T}V_1} U_1 + \nabla^{\perp}_{U_1} \mathcal{F}\mathcal{T}V_1, Z)$$

$$+ g(-\mathcal{A}_{\mathcal{F}V_1}U_1 + \nabla^{\perp}_{U_1}\mathcal{F}V_1, \mathcal{T}\mathcal{P}Z)$$
$$+ g(-\mathcal{A}_{\mathcal{F}V_1}U_1 + \nabla^{\perp}_{U_1}\mathcal{F}V_1, \mathcal{T}\mathcal{R}Z + \mathcal{F}\mathcal{R}Z)$$

or

$$\begin{aligned} sin^2\theta_1 g(\bar{\nabla}_{U_1}V_1, Z) &= g(\mathcal{A}_{\mathcal{F}\mathcal{T}V_1}U_1, Z) - g(\mathcal{A}_{\mathcal{F}V_1}U_1, \mathcal{T}\mathcal{P}Z) \\ &\quad - g(\mathcal{A}_{\mathcal{F}V_1}U_1, \mathcal{T}\mathcal{R}Z) + g(\nabla^{\perp}_{U_1}\mathcal{F}V_1, \mathcal{F}\mathcal{R}Z). \end{aligned} \tag{4.5}$$

Now, for any $U_1, V_1 \in \Gamma(\mathcal{D})$ and $W \in \Gamma(TM)^{\perp}$, we have

$$\begin{aligned} g(\bar{\nabla}_{U_1}V_1, W) &= -g(\bar{\nabla}_{U_1}\varphi\mathcal{T}V_1, W) - g(\varphi(\bar{\nabla}_{U_1}\mathcal{F}V_1), W) \\ &\quad - g(\varphi(-\mathcal{A}_{\mathcal{F}V_1}U_1 + \nabla^{\perp}_{U_1}\mathcal{F}V_1), W) \\ &= \cos^2\theta_1 g(\bar{\nabla}_{U_1}V_1, W) - g(-\mathcal{A}_{\mathcal{F}\mathcal{T}V_1}U_1 + \nabla^{\perp}_{U_1}\mathcal{F}\mathcal{T}V_1, W) \\ &\quad - g(-\mathcal{T}\mathcal{A}_{\mathcal{F}V_1}U_1 - \mathcal{F}\mathcal{A}_{\mathcal{F}V_1}U_1 + \mathcal{B}\nabla^{\perp}_{U_1}\mathcal{F}V_1 + \mathcal{C}\nabla^{\perp}_{U_1}\mathcal{F}V_1, W) \end{aligned}$$

or

$$\begin{aligned} \sin^2\theta_1 g(\bar{\nabla}_{U_1}V_1, W) &= -g(\nabla^{\perp}_{U_1}\mathcal{F}\mathcal{T}V_1, W) + g(\mathcal{F}\mathcal{A}_{\mathcal{F}V_1}U_1 - \mathcal{C}\nabla^{\perp}_{U_1}\mathcal{F}V_1, W) \\ &= g(\mathcal{F}\mathcal{A}_{\mathcal{F}V_1}U_1 - \nabla^{\perp}_{U_1}\mathcal{F}\mathcal{T}V_1 - \mathcal{C}\nabla^{\perp}_{U_1}\mathcal{F}V_1, W). \end{aligned} \tag{4.6}$$

Thus proof follows (4.5) and (4.6). □

Theorem 4.6. *Let M be a quasi bi-slant submanifold of $\bar{M}$. The slant distribution $\mathcal{D}_2$ defines totally geodesic foliation on M if and only if*

$$\mathcal{T}(\nabla_{U_2}\mathcal{T}V_2 - \mathcal{A}_{\mathcal{F}V_2}U_2) \in \Gamma(\mathcal{D}_2),$$

$$\mathcal{B}(h(U_2, TV_2) + \nabla^{\perp}_{U_2}\mathcal{F}V_2) \in \Gamma(T^{\perp}M)$$

and

$$g(\nabla^{\perp}_{U_2}\mathcal{F}\mathcal{T}V_2 - \mathcal{F}\mathcal{A}_{V_2}U_2, W) = g(\nabla^{\perp}_{U_2}\mathcal{F}V_2, \mathcal{C}W)$$

for any $U_2, V_2 \in \Gamma(\mathcal{D}_2)$, $Z = \mathcal{P}Z + \mathcal{Q}Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}_1)$ and $W \in \Gamma(T^{\perp}M)$.

From theorem (4.4), (4.5) and (4.6), we have the following decomposition theorem:

Theorem 4.7. *Let M be a proper quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then M is a local product Riemannian manifold of the form $M_{\mathcal{D}} \times M_{\mathcal{D}_1} \times M_{\mathcal{D}_2}$, where $M_{\mathcal{D}}$, $M_{\mathcal{D}_1}$ and $M_{\mathcal{D}_2}$ are leaves of $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$, recpectively, if and only if the conditions* (**??**), (**??**), (4.4), (**??**), (**??**), (**??**) *and* (**??**) *hold.*

5. Quasi bi-slant submanifolds with parallel canonical structures

In this section, we obtain some results for the quasi bi-slant submanifolds with parallel canonical structure. Let M be a proper quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then we define

$$(\bar{\nabla}_{Z_1}\mathcal{T})Z_2 = \nabla_{Z_1}\mathcal{T}Z_2 - \mathcal{T}\nabla_{Z_1}Z_2 \tag{5.1}$$

$$(\bar{\nabla}_{Z_1}\mathcal{F})Z_2 = \nabla^{\perp}_{Z_1}\mathcal{F}Z_2 - \mathcal{F}\nabla_{Z_1}Z_2 \tag{5.2}$$

$$(\bar{\nabla}_{Z_1}\mathcal{B})W_1 = \nabla_{Z_1}\mathcal{B}W_1 - \mathcal{B}\nabla^{\perp}_{Z_1}W_1 \tag{5.3}$$

$$(\bar{\nabla}_{Z_1}\mathcal{C})W_1 = \nabla^{\perp}_{Z_1}\mathcal{C}W_1 - \mathcal{C}\nabla^{\perp}_{Z_1}W_1 \tag{5.4}$$

where $Z_1, Z_2 \in \Gamma(TM)$ and $W_1 \in \Gamma(T^{\perp}M)$.
Then, the endomorphism $\mathcal{T}$ (*resp.* $\mathcal{F}$) and the endomorphism $\mathcal{B}$ (*resp.* $\mathcal{C}$) are parallel if $\bar{\nabla}\mathcal{T} \equiv 0$ (*resp.* $\bar{\nabla}\mathcal{F} \equiv 0$) and $\bar{\nabla}\mathcal{B} \equiv 0$ (*resp.* $\bar{\nabla}\mathcal{C} \equiv 0$), respectively.
Taking into account of (3.7), (3.8), (3.9), (3.10) and (5.1)-(5.4), we have the following lemma.

Lemma 5.1. *Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then for any $Z_1, Z_2 \in \Gamma(TM)$ and $W_1 \in \Gamma(T^{\perp}M)$ we obtain*

$$(\bar{\nabla}_{Z_1}\mathcal{T})Z_2 = \mathcal{A}_{\mathcal{F}Z_2}Z_1 + \mathcal{B}h(Z_1, Z_2) \tag{5.5}$$

$$(\bar{\nabla}_{Z_1}\mathcal{F})Z_2 = \mathcal{C}h(Z_1, Z_2) - h(Z_1, \mathcal{T}Z_2) \tag{5.6}$$

$$(\bar{\nabla}_{Z_1}\mathcal{B})W_1 = \mathcal{A}_{\mathcal{C}W_1}Z_1 + \mathcal{T}A_{W_1}Z_1 \tag{5.7}$$

$$(\bar{\nabla}_{Z_1}\mathcal{C})W_1 = -\mathcal{F}\mathcal{A}_{W_1}Z_1 - h(Z_1, \mathcal{B}W_1). \tag{5.8}$$

First, we have the following theorem:

Theorem 5.2. *Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then, $\mathcal{T}$ is parallel if and only if the invariant distribution $\mathcal{D}$ is totally geodesic.*

Proof. For any $X, Y \in \Gamma(\mathcal{D})$, from (5.5), we have

$$(\bar{\nabla}_X\mathcal{T})Y = \mathcal{B}h(X, Y) \tag{5.9}$$

here we have used $\mathcal{A}_{\mathcal{F}Y}X = 0$ since $\mathcal{F}Y = 0$ for any $Y \in \Gamma(\mathcal{D})$. Thus, our assertion comes from (5.9). □

Theorem 5.3. *Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then if $\mathcal{F}$ is parallel if and only if*

$$g(\mathcal{A}_{\mathcal{C}V}Z_2, Z_1) = -g(\mathcal{A}_V Z_1, \mathcal{T}Z_2) \tag{5.10}$$

for any $Z_1, Z_2 \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. Assume that F is parallel. Now, from (5.6), we have

$$(\bar{\nabla}_{Z_1}\mathcal{F})Z_2 = \mathcal{C}h(Z_1, Z_2) - h(Z_1, \mathcal{T}Z_2). \tag{5.11}$$

Now, taking inner product with $V \in \Gamma(T^{\perp}M)$ in the above equation and using (2.5), we obtain

$$\begin{aligned} g((\bar{\nabla}_{Z_1}\mathcal{F})Z_2, V) &= g(\mathcal{C}h(Z_1, Z_2) - h(Z_1, \mathcal{T}Z_2), V) \\ &= g(\mathcal{C}h(Z_1, Z_2), V) - g(h(Z_1, \mathcal{T}Z_2), V) \\ &= -g(h(Z_1, Z_2), \varphi V) - g(\bar{\nabla}_{Z_1}\mathcal{T}Z_2, V) \\ &= -g(\mathcal{A}_{\mathcal{C}V}Z_2, Z_1) + g(\mathcal{T}Z_2, \bar{\nabla}_{Z_1}V) \\ &= -g(\mathcal{A}_{\mathcal{C}V}Z_2, Z_1) + g(\mathcal{T}Z_2, -\mathcal{A}_V Z) \end{aligned}$$

which gives the assertion. □

Theorem 5.4. *Let M be a quasi bi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then $\mathcal{F}$ is parallel if and only if $\mathcal{B}$ is parallel.*

Proof. By using (2.5), (5.6) and (5.7), we get

$$\begin{aligned} g((\bar{\nabla}_{Z_1}\mathcal{F})Z_2, W_1) &= g(\mathcal{C}h(Z_1, Z_2), W_1) - g(h(Z_1, \mathcal{T}Z_2), W_1) \\ &= -g(h(Z_1, Z_2), \mathcal{C}W_1) - g(\mathcal{A}_{W_1}Z_1, \mathcal{T}Z_2) \\ &= -g(\mathcal{A}_{\mathcal{C}W_1}Z_1, Z_2) + g(\mathcal{T}\mathcal{A}_{W_1}Z_1, Z_2) \\ &= -g(\mathcal{A}_{\mathcal{C}W_1}Z_1 - \mathcal{T}\mathcal{A}_{W_1}Z_1, Z_2) \\ &= -g((\bar{\nabla}_{Z_1}\mathcal{B})W_1, Z_2) \end{aligned}$$

for any $Z_1, Z_2 \in \Gamma(TM)$ and $W_1 \in \Gamma(T^{\perp}M)$. This proves our assertion. □

Finally, we mention another non-trivial example of quasi bi-slant submanifold of a cosymplectic manifold.

Example. Let M be a submanifold of $\mathbb{R}^{11}$ defined by

$$x(u, v, t, r, s, k, z) = (u, v, t, \frac{1}{\sqrt{2}}r, \frac{1}{\sqrt{2}}r, 0, s, k\cos\alpha, k\sin\alpha, 0, z).$$

We can easily to see that the tangent bundle of M is spanned by the tangent vectors

$$e_1 = \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial y_1}, e_3 = \frac{\partial}{\partial x_2}, e_4 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3},$$

$$e_5 = \frac{\partial}{\partial x_4}, e_6 = \cos\alpha\frac{\partial}{\partial y_4} + \sin\alpha\frac{\partial}{\partial x_5}, e_7 = \frac{\partial}{\partial z} = \xi.$$

We define the almost contact structurev φ of $\mathbb{R}^{11}$, by

$$\varphi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad \varphi(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}, \quad \varphi(\frac{\partial}{\partial z}) = 0, \quad 1 \leq i, j \leq 5.$$

For any vector field $Z = \lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu \frac{\partial}{\partial z} \in \Gamma(T\mathbb{R}^{11})$, then we have

$$g(Z, Z) = \lambda_i^2 + \mu_j^2 + \nu^2, \quad g(\varphi Z, \varphi Z) = \lambda_i^2 + \mu_j^2$$

and

$$\varphi^2 Z = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_j} = -Z$$

for any $i, j = 1, ..., 5$. It follows that $g(\varphi Z, \varphi Z) = g(Z, Z) - \eta^2(Z)$. Thus (φ, ξ, η, g) is an is an almost contact metric structure on $\mathbb{R}^{11}$. Thus we have

$$\varphi e_1 = \frac{\partial}{\partial y_1},\ \varphi e_2 = \frac{\partial}{\partial x_1},\ \varphi e_3 = \frac{\partial}{\partial y_2},\ \varphi e_4 = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial y_3},$$
$$\varphi e_5 = \frac{\partial}{\partial y_4},\ \varphi e_6 = -\cos\alpha \frac{\partial}{\partial x_4} + \sin\alpha \frac{\partial}{\partial y_5},\ \varphi e_7 = 0.$$

By direct calculations, we obtain the distribution $\mathcal{D} = span\{e_1, e_2\}$ is an invariant distribution, the distribution $\mathcal{D}_1 = span\{e_3, e_4\}$ is a slant distribution with slant angle $\theta_1 = \frac{\pi}{4}$ and the distribution $\mathcal{D}_2 = span\{e_5, e_6\}$ is also a slant distribution with slant angle $\theta_2 = \alpha,\ 0 < \alpha < \frac{\pi}{2}$. Thus M is a $7-$dimensional proper quasi bi-slant submanifold of $\mathbb{R}^{11}$ with its usual almost contact metric structure.

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References

[1] Blair, D. E., Contact manifolds in Riemannian geometry, Lecture Notes in Mathematic Springer-Verlag, New York, Vol. 509, 1976.

[2] Blair, D. E., The theory of quasi-Sasakian structure, *J. Differential Geom.* 1, (3-4) (1967), 331-345.

[3] Carriazo, A., New developments in slant submanifolds theory, Narasa Publishing Hause New Delhi, India, 2002.

[4] Carriazo, A., Bi-slant immersions, *Proceeding of the ICRAMS 2000,* Kharagpur, (2000), 88-97.

[5] Chen, B. Y., Geometry of slant submanifolds, Katholieke Universiteit Leuven, Leuven, Belgium, View at Zentralblatt Math., 1990.

[6] Chen, B. Y., Slant immersions, *Bull. Austral. Math. Soc.,* 41 (1990), 135-147.

[7] Cabrerizo, J. L., Carriazo, A., Fernandez, L. M., Fernandez, M., Slant submanifolds in Sasakian manifolds, *Glasgow Math. J.,* 42 (2000), 125-138.

[8] Cabrerizo, J. L., Carriazo, A., Fernandez, L. M., Fernandez, M., Semi-slant submanifolds of a Sasakian manifold, *Geom. Dedic.,* 78(2) (1999), 183-199.

[9] Dirik, S., Atçeken, M., On the geometry of pseudo-slant submanifolds of a cosymplectic manifold, *International Electronic Journal of Geometry,* 9(1) (2016), 45-56.

[10] Dirik, S., Atceken, M., Contact pseudo-slant submanifolds of a cosymplectic manifold, *New trends in Mathematical Sciences*, 6(4) (2018), 154-164.

[11] Gupta, R. S., Haider, S. K., Sharfuddin, A. Slant submanifolds in cosymplectic manifolds, *Colloquium Mathematicum,* 105 (2016), 207-219.

[12] Khan, M. A., Totally umbilical hemi slant submanifolds of Cosymplectic manifolds, *Mathematica Aeterna,* 3(8) (2013), 645-653.

[13] Kim, U. K., On anti-invariant submanifolds of cosymplectic manifolds, *Bulletin of the Korean Mathematical Society*, 21(1) (1984), 35-37.

[14] Lone, M. A., Lone, M. S., Shahid, M. H., Hemi-slant submanifolds of cosymplectic manifolds, *Cogent Mathematics,* 3(1) (2016),1204143.

[15] Lotta, A., Slant submanifolds in contact geometry, *Bulletin Mathematical Society Roumanie,* 39 (1996), 183-198.

[16] Ludden, G. D., Submanifolds of cosymplectic manifolds, *Journal of Differential Geometry,* 4 (1970), 237-244.

[17] Olzsak, Z., On almost cosymplectic manifolds, *Kodai Math J.,* 4 (1981), 239-250.

[18] Uddin, S., Ozel, C., Khan, V. A., A classication of a totally umbilical slant submanifold of cosymplectic manifolds, *Hindawi puplishing corporation abstract applied analysis,* article ID 716967, 8 pages (2012).

[19] Papaghuic, N., Semi-slant submanifolds of a Kaehlarian manifold, *An. St. Univ. Al. I. Cuza. Univ. Iasi,* 40 (2009), 55-61.

[20] Prasad, R., Verma, S. K., Kumar, S., Quasi hemi-slant submanifolds of sasakian manifolds, J*ournal of Mathematical and Computational Science,* 10(2) (2020), 418-435.

[21] Şahin, B., Warped product submanifolds of a Kaehler manifold with a slant factor, *Annales Polonici Mathematici,* 95 (2009), 107-226.

[22] Şahin, B., Slant submanifolds of an almost product Riemannian manifold, *Journal of the Korean Mathematical Society,* 43(4) (2006), 717-732.

[23] Şahin, B., Keleş, S. Slant submanifolds of Kaehler product manifolds, *Turkish Journal of Mathematics,* 31 (2007), 65-77.

[24] Taştan, H. M., Özdemir, F,. The geometry of hemi-slant submanifolds of a locally product Riemannian manifold, *Turkish Journal of Mathematics,* 39 (2015), 268-284.

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APPROXIMATION BY BÉZIER VARIANT OF JAKIMOVSKI-LEVIATAN-PĂLTĂNEA OPERATORS INVOLVING SHEFFER POLYNOMIALS

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Abstract. In the present paper, the Bézier variant of Jakimovski-Leviatan-Păltănea operators involving Sheffer polynomials is introduced and the degree of approximation by these operators is investigated with the aid of Ditzian-Totik modulus of smoothness, Lipschitz type space and for functions with derivatives of bounded variations.

Introduction

Approximation theory is a crucial branch of Mathematical analysis. The fundamental property of approximation theory is to approximate a function f by another functions which have better properties than f. In 1950, Szasz [14] introduced a generalization of Bernstein polynomials on the infinite interval $[0,\infty)$ and established the convergence properties of these operators. Subsequently, Jakimovski-Leviatan [8] generalised the Szász operators as

$$P_n(f;x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right), \tag{0.1}$$

by means of Appell polynomials which are generated by:

$$g(u)e^{ux}=\sum_{k=0}^{\infty}p_k(x)u^k, \tag{0.2}$$

where $g(u)=\sum_{k=0}^{\infty}a_ku^k, a_0\neq 0$ is an analytic function, on the disk $|u|<r$ $(r>1)$, under the assumption $p_k(x)\geq 0$, for $x\in[0,\infty)$.

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In 2008, Păltănea [11] defined a generalisation of the Phillips operators [12] based on a parameter $\rho > 0$, as

$$G_n^\rho(f;x) = \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^\infty \Phi_{n,k}^\rho(t) f(t)dt + e^{-nx} f(0), \quad x \in [0,\infty), \tag{0.3}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$ and $\Phi_{n,k}^\rho(t) = \frac{n\rho^{k\rho}}{\Gamma(k\rho)} e^{-n\rho t}(nt)^{k\rho-1}$, which includes Szász operators for $\rho \to \infty$ and Phillips operators for $\rho = 1$. For $f \in C[0,\infty)$, Verma and Gupta [15] defined the Jakimovski-Leviatan-Păltănea operator as follows:

$$P_{n,\rho}^*(f;x) = \sum_{k=1}^{\infty} L_{n,k}(x) \int_0^\infty Q_{n,k}^\rho(t) f(t)dt + L_{n,0}(x) f(0), \quad \rho > 0, \tag{0.4}$$

where $L_{n,k}(x) = \frac{e^{-nx}}{g(1)} p_k(nx)$ and $Q_{n,k}^\rho(t) = \frac{n\rho}{\Gamma(k\rho)} e^{-n\rho t}(n\rho t)^{k\rho-1}$ and established an asymptotic formula and rate of convergence for these operators. Goyal and Agrawal [4] defined the Bézier variant of these operators (0.4) and established the degree of approximation using Ditzian-Totik modulus of smoothness, Lipschitz type space and for functions having a derivative of bounded variation.

Let $C(z) = \sum_0^\infty c_k z^k, (c_0 \neq 0)$ and $D(z) = \sum_{k=1}^\infty d_k z^k, (d_1 \neq 0)$ be analytic functions on the disc $|z| < r,\ r > 1$ where c_k and d_k are real. The Sheffer type polynomials $\{p_k(x)\}$ are given by the generating functions of the form

$$C(z)e^{tD(z)} = \sum_{k=0}^{\infty} p_k(t) z^k, \quad |z| < r. \tag{0.5}$$

Under the following assumptions:

(i) for $t \in [0,\infty)$, $p_k(t) \geq 0$, $k = 0, 1, 2, \cdots$

(ii) $C(1) \neq 0$ and $D'(1) = 1$,

Ismail [6] defined another generalisation of the Szász operators and the Jakimovski-Leviatan operators [8] using the Sheffer polynomials as

$$T_n(f;x) = \frac{e^{-nxD(1)}}{C(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{0.6}$$

and estabilished some approximation properties of these operators. For the special case $D(t) = t$ and $C(t) = 1$, we find $p_k(x) = \frac{x^k}{k!}$, therefore (0.6) reduces to Szasz operators and for the case $D(t) = t$, the operators $T_n(f;x)$ yield the operators $P_n(f;x)$ defined in (0.1). Inspired by the work of Verma and Gupta [15], Mursaleen et al. [9] defined the Jakimovski-Leviatan-Păltănea operators by means of Sheffer polynomials, and integral modification of the operators given by (0.6), as

$$M_{n,\rho}(f;x) = \sum_{k=1}^{\infty} L_{n,k}(x) \int_0^\infty Q_{n,k}^\rho(t) f(t)dt + L_{n,0}(x) f(0), \quad \rho > 0, \tag{0.7}$$

where $L_{n,k}(x)=\frac{e^{-nxD(1)}}{C(1)}p_k(nx)$ and $Q^{\rho}_{n,k}(t)=\frac{n\rho}{\Gamma(k\rho)}e^{-n\rho t}(n\rho t)^{k\rho-1}$ and established some convergence properties of these operators with the help of the Korovkin-type theorem, rate of convergence by using Ditzian-Totik modulus of smoothness and approximation properties for the functions having derivatives of bounded variation.

Since the Bézier curves have important applications in computer aided graphics and applied mathematics, Zeng and Piriou [16] initiated the study of a Bézier variant of Bernstein operators. Zeng [17] introduced the Szasz-Bézier operators and discussed the rate of convergence of these operators for the functions of bounded variations. Subsequently several researchers defined the Bézier variants of some other sequences of positive linear operators and studied their approximation properties (see, e.g., [1, 2, 4, 5, 7, 13]).

Motivated by the above work, we introduce the Bézier variant of the operators defined in (0.7). Let $\lambda>0$ and $C_\lambda[0,\infty):=\{f\in C[0,\infty): f(t)=O(e^{\lambda t})\text{ as } t\to\infty\}$. For $\beta\geq 1$ and $f\in C_\lambda[0,\infty)$, the Bézier variant of (0.7) is defined as

$$M^{\beta}_{n,\rho}(f;x)=\sum_{k=1}^{\infty}N^{(\beta)}_{n,k}(x)\int_0^{\infty}Q^{\rho}_{n,k}(t)f(t)dt+N^{(\beta)}_{n,0}(x)f(0),\quad \rho>0, \tag{0.8}$$

where $N^{(\beta)}_{n,k}(x)=\left[J_{n,k}(x)\right]^{\beta}-\left[J_{n,k+1}(x)\right]^{\beta}$, $\beta\geq 1$; $L_{n,k}(x)=\frac{e^{-nxD(1)}}{C(1)}p_k(nx)$ and $J_{n,k}(x)=\sum_{j=k}^{\infty}L_{n,j}(x)$ with the following properties:

(1) $J_{n,k}(x)-J_{n,k+1}(x)=L_{n,k}(x)$, $\quad k=0,1,2,\cdots$,
(2) $J_{n,0}(x)>J_{n,1}(x)>J_{n,2}(x)>\cdots J_{n,n}(x),\quad x\in[0,\infty)$.

In particular,

(i) if $\beta=1$, the operators $M^{\beta}_{n,\rho}(f;x)$ include the operators given by (0.7),
(ii) if $\beta=1$ and $D(t)=t$, the operators $M^{\beta}_{n,\rho}(f;x)$ reproduce the operators defined in [15],
(iii) if $C(t)=1$, $D(t)=t$, $\rho=1$ and $\beta=1$, the operators $M^{\beta}_{n,\rho}(f;x)$ reduce to the well known Phillips operators [12].

The organization of the paper as follows: In Section 1, the Bézier variant of Jakimovski-Leviatan-Păltănea operators involving Sheffer polynomials has been introduced. In Section 2, some auxiliary results such as moments, central moments and lemmas have been presented. In Section 3, the rate of convergence by using Ditzian-Totik modulus of smoothness and Lipschitz type space have been discussed. In Section 4, the approximation result for the functions having derivatives of bounded variation has been discussed.

1. Auxiliary Results

Lemma 1.1. *The r^{th} order moments $M_{n,\rho}(t^r;x)$, for $r=0,1,2,$ are given by the following identities:*

(i) $M_{n,\rho}(1;x)=1$;
(ii) $M_{n,\rho}(t;x)=x+\frac{C'(1)}{nC(1)}$;

(iii) $M_{n,\rho}(t^2;x) = x^2 + \frac{x}{n}\left(1 + \frac{1}{\rho} + \frac{2C'(1)}{C(1)} + D''(1)\right) + \frac{1}{n^2\rho}\left(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\right)$.

As a consequence of the above lemma, we obtain

Lemma 1.2. *The central moments $M_{n,\rho}((t-x)^r;x)$, $r = 1,2$, are given by the following equalities:*

(i) $M_{n,\rho}(t-x;x) = \frac{C'(1)}{nC(1)}$;

(ii) $M_{n,\rho}((t-x)^2;x) = \frac{x}{n}\left(1 + \frac{1}{\rho} + D''(1)\right) + \frac{1}{n^2\rho}\left(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\right)$.

In what follows, we denote $M_{n,\rho}((t-x)^2;x) = \xi_{n,\rho}(x)$.

Remark 1.3. *For sufficiently large n and $\mu > 2$, one has*

$$M_{n,\rho}((t-x)^2;x) \leq \frac{\mu x}{n}\left(1 + \frac{1}{\rho} + D''(1)\right). \tag{1.1}$$

Let $C_B[0,\infty)$ be the family of all continuous and bounded functions defined on $[0,\infty)$.

Lemma 1.4. *For every $f \in C_B[0,\infty)$, we have*

$$\|M_{n,\rho}(f;x)\| \leq \|f\|. \tag{1.2}$$

Proof. The proof of this lemma is readily follow with the help of Lemma 1.1(i). Hence, the details are omitted. □

Lemma 1.5. *For $\lambda > 0$, let $f \in C_\lambda[0,\infty)$. Then*

$$|M^{\beta}_{n,\rho}(f;x)| \leq \beta M_{n,\rho}(|f|;x). \tag{1.3}$$

Proof. For $0 \leq u, v \leq 1$ and $\beta \geq 1$, the following inequality holds

$$|u^\beta - v^\beta| \leq \beta|u - v|. \tag{1.4}$$

Since, $N^{(\beta)}_{n,k}(x) = \left[J_{n,k}(x)\right]^\beta - \left[J_{n,k+1}(x)\right]^\beta$, for all $\beta \geq 1$ and

$$J_{n,k}(x) = \sum_{j=k}^{\infty} L_{n,j}(x) \leq \sum_{j=0}^{\infty} L_{n,j}(x) = 1,$$

in view of the inequality (1.4), we have

$$\left|N^{(\beta)}_{n,k}(x)\right| = \left|\left[J_{n,k}(x)\right]^\beta - \left[J_{n,k+1}(x)\right]^\beta\right| \leq \beta|J_{n,k}(x) - J_{n,k+1}(x)| = \beta L_{n,k}(x). \tag{1.5}$$

Further,

$$\left|M^{\beta}_{n,\rho}(f;x)\right| \leq \sum_{k=1}^{\infty}\left|N^{(\beta)}_{n,k}(x)\right|\int_0^\infty Q^{\rho}_{n,k}(t)\left|f(t)\right|dt + \left|N^{(\beta)}_{n,0}(x)\right||f(0)|, \quad \rho > 0. \tag{1.6}$$

From (1.5) and (1.6), we get the desired result. □

2. Main Results

For $t \geq 0$, $x > 0$, and $0 < \alpha \leq 1$, the Lipschitz type space [10] is defined as:

$$Lip_K^*(\alpha) := \left\{ f \in C[0,\infty) : |f(x) - f(t)| \leq K \frac{|x-t|^\alpha}{(x+t)^{\frac{\alpha}{2}}} \right\},$$

where K is some positive constant.

In the next theorem, we investigate the rate of convergence of the operators $M_{n,\rho}^{\beta}(\cdot;x)$ for the function $f \in Lip_K^*(\alpha)$.

Theorem 2.1. *Let $f \in Lip_K^*(\alpha)$. Then for each $x > 0$, we have*

$$|M_{n,\rho}^{\beta}(f;x) - f(x)| \leq \frac{\beta K}{x^{\frac{\alpha}{2}}} \big(\xi_{n,\rho}(x)\big)^{\frac{\alpha}{2}}.$$

Proof. In view of Lemma 1.5 and the fact that, $M_{n,\rho}^{\beta}(1;x) = 1$, we have

$$\begin{aligned} |M_{n,\rho}^{\beta}(f;x) - f(x)| &\leq |M_{n,\rho}^{\beta}(f(t) - f(x);x)| \\ &\leq \beta M_{n,\rho}(|f(t) - f(x)|;x) \\ &\leq \beta K M_{n,\rho}\left(\frac{|x-t|^\alpha}{(x+t)^{\frac{\alpha}{2}}};x\right) \\ &\leq \frac{\beta K}{x^{\frac{\alpha}{2}}} M_{n,\rho}\Big(|x-t|^\alpha;x\Big). \end{aligned} \tag{2.1}$$

Now, applying Hölder's inequality by setting $p = 2/\alpha$ and $q = 2/(2-\alpha)$ and using Lemma 1.1

$$\begin{aligned} M_{n,\rho}(|x-t|^\alpha;x) &\leq \Big(M_{n,\rho}((x-t)^2;x)\Big)^{\frac{\alpha}{2}} \Big(M_{n,\rho}(1^{\frac{2}{2-\alpha}};x)\Big)^{\frac{2-\alpha}{2}} \\ &\leq \Big(M_{n,\rho}((x-t)^2;x)\Big)^{\frac{\alpha}{2}} = \big(\xi_{n,\rho}(x)\big)^{\frac{\alpha}{2}}. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we get the required result. □

Let us recall the definitions of the Peetre's K-functional and the Ditzian-Totik first order modulus of smoothness. Let $\phi(x) = \sqrt{x}$ and $f \in C_B[0,\infty)$.

Definition 2.1. *[3] The Ditzian-Totik first order modulus of smoothness $\omega_\phi(f;\delta)$, $\delta > 0$, is defined by*

$$\omega_\phi(f;\delta) := \sup_{0<h\leq\delta} \left| f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right)\right|, \quad \forall\, x \pm \frac{h\phi(x)}{2} \in [0,\infty).$$

Definition 2.2. *[3] The Peetre's K-functional is defined by*

$$K_\phi(f;\delta) := \inf\{\|f-g\| + \delta\|\phi g'\| + \delta^2\|g'\|,\ \delta > 0\}, \quad \forall\, g \in W_\phi,$$

where $W_\phi := \{g : g \in AC_{loc}, \|\phi g'\| < \infty, \|g'\| < \infty\}$ and $g \in AC_{loc}$ means that g is a locally absolutely continuous function in $[0,\infty)$.

From [3], it is known that $\omega_\phi(f;\delta) \sim K_\phi(f;\delta)$, i.e. there exists a constant $\gamma > 0$, such that

$$\gamma^{-1}\omega_\phi(f;\delta) \leq K_\phi(f;\delta) \leq \gamma\omega_\phi(f;\delta). \tag{2.3}$$

In the next theorem, Ditzian-Totik first order modulus of smoothness is used to establish a direct approximation theorem.

Theorem 2.2. *Let $f \in C_B[0,\infty)$ and $\phi(x) = \sqrt{x}$, then for every $x \in [0,\infty)$ we have*

$$|M_{n,\rho}^\beta(f;x) - f(x)| \leq C\omega_\phi\bigg(f; \frac{1}{\sqrt{n}}\bigg),$$

where C is a constant and independent on f and n.

Proof. Let $x \in [0,\infty)$ be arbitrary but fixed. For $g \in W_\phi$, we have the following representation

$$g(t) = g(x) + \int_x^t g'(u)du.$$

Applying the operator $M_{n,\rho}^\beta(f;x)$ on both sides of the above equation, we obtain

$$\begin{aligned} M_{n,\rho}^\beta(g;x) - g(x) &= M_{n,\rho}^\beta\bigg(\int_x^t g'(u)du;x\bigg), \\ |M_{n,\rho}^\beta(g;x) - g(x)| &= \bigg|M_{n,\rho}^\beta\bigg(\int_x^t g'(u)du;x\bigg)\bigg| \leq M_{n,\rho}^\beta\bigg(\bigg|\int_x^t g'(u)du\bigg|;x\bigg) \end{aligned} \tag{2.4}$$

In view of Lemma 1.5, we have

$$M_{n,\rho}^\beta((t-x)^2;x) = |M_{n,\rho}^\beta((t-x)^2;x)| \leq \beta M_{n,\rho}((t-x)^2;x).$$

Hence, using Lemma 1.2, we get

$$M_{n,\rho}^\beta((t-x)^2;x) \leq \beta\left\{\frac{x}{n}\left(1 + \frac{1}{\rho} + D''(1)\right) + \frac{1}{n^2\rho}\left(\frac{(1+\rho)C'(1) + \rho C''(1)}{C(1)}\right)\right\}. \tag{2.5}$$

To estimate the right hand side of (2.4), we split our domain $[0,\infty)$ into two parts $A = [0, 1/n]$ and $B = (1/n, \infty]$.

Case-I:

If $x \in [0, 1/n]$, then from (2.5), for sufficiently large n, we have $M_{n,\rho}^\beta((t-x)^2;x) \sim \frac{\beta}{n^2\rho}\bigg(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\bigg)$, i.e. there exists some $k_1 > 0$, such that

$$M_{n,\rho}^\beta((t-x)^2;x) \leq \frac{k_1\beta}{n^2\rho}\bigg(\frac{(1+\rho)C'(1) + \rho C''(1)}{C(1)}\bigg).$$

Hence, applying Cauchy-Schwarz inequality in equation (2.4), we have

$$|M_{n,\rho}^\beta(g;x) - g(x)| \quad \leq \quad \|g'\|M_{n,\rho}^\beta\bigg(|t-x|;x\bigg)$$

$$
\begin{aligned}
&\leq \|g'\|\Big(M^{\beta}_{n,\rho}((t-x)^2;x)\Big)^{1/2}\\
&\leq \|g'\|\Big\{\frac{k_1\beta}{n^2\rho}\Big(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\Big)\Big\}^{1/2}\\
&= \frac{\Delta_1}{n}\|g'\|, \qquad (2.6)
\end{aligned}
$$

where $\Delta_1=\Big\{\frac{k_1\beta}{\rho}\Big(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\Big)\Big\}^{1/2}$.

Case-II: If $x\in(1/n,\infty]$, then from (2.5), we obtain $M^{\beta}_{n,\rho}((t-x)^2;x)\sim\frac{\beta x}{n}\Big(1+\frac{1}{\rho}+D''(1)\Big)$. Hence, there exists some constant $k_2>0$, such that

$$M^{\beta}_{n,\rho}((t-x)^2;x)\leq\frac{k_2\beta x}{n}\Big(1+\frac{1}{\rho}+D''(1)\Big).$$

Since

$$\Big|\int_x^t g'(u)du\Big|\leq\|\phi g'\|\Big|\int_x^t\frac{1}{\phi(u)}du\Big|,$$

and for any $x,t\in(0,\infty)$,

$$\Big|\int_x^t\frac{1}{\phi(u)}du\Big|=\Big|\int_x^t\frac{1}{\sqrt{u}}du\Big|=2|(\sqrt{t}-\sqrt{x})|=2\frac{|t-x|}{\sqrt{t}+\sqrt{x}}\leq2\frac{|t-x|}{\phi(x)},$$

we have

$$\Big|\int_x^t g'(u)du\Big|\leq2\|\phi g'\|\frac{|t-x|}{\phi(x)}. \qquad (2.7)$$

Now, combining equations (2.4) and (2.7) and using Cauchy-Schwarz inequality, for any $x\in(1/n,\infty)$, we have

$$
\begin{aligned}
|M^{\beta}_{n,\rho}(g;x)-g(x)| &\leq 2\|\phi g'\|\phi^{-1}(x)M^{\beta}_{n,\rho}\Big(|t-x|;x\Big)\\
&\leq 2\|\phi g'\|\phi^{-1}(x)\Big(M^{\beta}_{n,\rho}((t-x)^2;x)\Big)^{1/2}\\
&\leq 2\|\phi g'\|\phi^{-1}(x)\Big(\frac{k_2\beta x}{n}\Big(1+\frac{1}{\rho}+D''(1)\Big)\Big)^{1/2}\\
&= \Delta_2\frac{\|\phi g'\|}{\sqrt{n}}, \qquad (2.8)
\end{aligned}
$$

where $\Delta_2 = \left(k_2\beta\left(1+\frac{1}{\rho}+D''(1)\right)\right)^{1/2}$.

Again, combining equations (2.4), (2.6) and (2.8), for $x \in [0,\infty)$ we have

$$\begin{aligned}
|M_{n,\rho}^{\beta}(g;x)-g(x)| &\leq \Delta_2\frac{\|\phi g'\|}{\sqrt{n}}+\frac{\Delta_1}{n}\|g'\| \\
&\leq \Delta\left(\frac{\|\phi g'\|}{\sqrt{n}}+\frac{1}{n}\|g'\|\right), \quad \text{where} \quad \Delta=\max(\Delta_1,\Delta_2).
\end{aligned}$$

Hence, using Lemma 1.4 and above equation, we get

$$\begin{aligned}
|M_{n,\rho}^{\beta}(f;x)-f(x)| &\leq |M_{n,\rho}^{\beta}(g;x)-g(x)|+|f(x)-g(x)|+|M_{n,\rho}^{\beta}(f-g;x)| \\
&\leq 2\|f-g\|+\Delta\left(\frac{\|\phi g'\|}{\sqrt{n}}+\frac{1}{n}\|g'\|\right) \\
&\leq \Delta'\left(\|f-g\|+\frac{\|\phi g'\|}{\sqrt{n}}+\frac{1}{n}\|g'\|\right), \quad \Delta'=\max(2,\Delta).
\end{aligned}$$

Finally, taking the infimum on the right side of the above equation over all $g \in W_\phi$,

$$|M_{n,\rho}^{\beta}(f;x)-f(x)| \leq \Delta' K_\phi\left(f;\frac{1}{\sqrt{n}}\right),$$

and using the relation (2.3), we get

$$|M_{n,\rho}^{\beta}(f;x)-f(x)| \leq \Delta'\gamma\ \omega_\phi\left(f;\frac{1}{\sqrt{n}}\right).$$

Now taking $C=\Delta'\gamma$, the proof of the theorem is completed. □

3. Functions with Derivatives of Bounded Variation

Let $DBV_2[0,\infty)$, be the class of all functions f defined on $[0,\infty)$ with $|f(t)| \leq C(1+t^2)$, $C>0$ and having a derivative f' equivalent to a function of bounded variation on every finite subinterval of $[0,\infty)$. Then we observe that for all functions $f \in DBV_2[0,\infty)$, there holds the following representation

$$f(x)=\int_0^x g(t)dt+f(0),$$

where g is a function of bounded variation on every finite subinterval of $(0,\infty)$.
In view of the Dirac-delta function, the alternate form of the operator $M_{n,\rho}^{\beta}(f;x)$ can be written as

$$M_{n,\rho}^{\beta}(f;x)=\int_0^\infty F_{n,\rho}^{\beta}(x,t)f(t)dt, \quad \rho>0, \tag{3.1}$$

where $F_{n,\rho}^{\beta}(x,t)=\sum_{k=1}^{\infty}N_{n,k}^{(\beta)}(x)Q_{n,k}^{\rho}(t)+N_{n,0}^{(\beta)}(x)\delta(t)$ and $\delta(t)$ is a Dirac-delta function.

To establish the rate of convergence of the operators given by (3.1) for $f \in DBV_2[0,\infty)$, the following lemma is needed:

Lemma 3.1. *Let $x \in (0,\infty)$ and $\mu > 2$. Then for sufficiently large n, we have*

(i) $\Phi^{\beta}_{n,\rho}(x,x_1) = \int_0^{x_1} F^{\beta}_{n,\rho}(x,t)dt \le \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\frac{1}{(x-x_1)^2}, \quad 0 \le x_1 < x,$

(ii) $1-\Phi^{\beta}_{n,\rho}(x,x_2) = \int_{x_2}^{\infty} F^{\beta}_{n,\rho}(x,t)dt \le \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\frac{1}{(x-x_2)^2}, \quad x < x_2 < \infty.$

Proof. (i) Using (3.1) and Remark 1.3, we have

$$\begin{aligned}
\Phi^{\beta}_{n,\rho}(x,x_1) &= \int_0^{x_1} F^{\beta}_{n,\rho}(x,t)dt \le \int_0^{x_1}\left(\frac{x-t}{x-x_1}\right)^2 F^{\beta}_{n,\rho}(x,t)dt\\
&= (x-x_1)^{-2}M^{\beta}_{n,\rho}((t-x)^2;x) \le \beta(x-x_1)^{-2}M_{n,\rho}((t-x)^2;x)\\
&\le \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\frac{1}{(x-x_1)^2}.
\end{aligned}$$

In the same way, assertion (ii) can be easily proved. □

Theorem 3.2. *Let $f \in DBV_2[0,\infty)$ and $\mu > 2$. Then for each $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}
|M^{\beta}_{n,\rho}(f;x)-f(x)| &\le \frac{\beta^{1/2}}{\beta+1}|f'(x+)+\beta f'(x-)|\sqrt{\frac{\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)}\\
&+ \frac{\beta^{3/2}}{\beta+1}|f'(x+)-f'(x-)|\sqrt{\frac{\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)}\\
&+ \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=0}^{[\sqrt{n}]} V_{x-\frac{x}{k}}^{x+\frac{x}{k}}(f'_x)+\frac{x}{\sqrt{n}}V_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}(f'_x)\\
&+ \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)\{|f(2x)-f(x)-xf'(x+)|\}\\
&+\left\{\frac{|f(x)|}{x^2}+C\left(4+\frac{1}{x^2}\right)\right\}\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\\
&+|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)},
\end{aligned}$$

where $V_c^d(f'_x)$ denotes the total variation of f'_x on $[c,d]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t)-f'(x-), & 0 \le t < x,\\ 0, & x = t,\\ f'(t)-f'(x+), & x < t < \infty. \end{cases} \tag{3.2}$$

Proof. In view of the fact that $M^{\beta}_{n,\rho}(1;x) = 1$, and the alternate form (3.1) of the operators given by (0.8), for every $x \in (0,\infty)$ we have

$$M^{\beta}_{n,\rho}(f(t);x)-f(x) = M^{\beta}_{n,\rho}(f(t);x)-M^{\beta}_{n,\rho}(f(x);x) = M^{\beta}_{n,\rho}(f(t)-f(x);x)$$

$$\begin{aligned} &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)(f(t)-f(x))dt \\ &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\bigg(\int_x^t f'(\nu)d\nu\bigg)dt. \end{aligned} \tag{3.3}$$

For any $f \in DBV_2[0,\infty)$, and using (3.2), we can write

$$\begin{aligned} f'(\nu) &= \delta_x(\nu)\left[f'(\nu)-\frac{f'(x+)+f'(x-)}{2}\right]+\left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right] \\ &+ \frac{f'(x+)-f'(x-)}{2}\left[sgn(\nu-x)+\frac{\beta-1}{\beta+1}\right]+f'_x(\nu), \end{aligned} \tag{3.4}$$

where

$$\delta_x(\nu)=\begin{cases} 1, & x=\nu \\ 0, & x\neq\nu. \end{cases}$$

Combining equations (3.3) and (3.4), we get

$$\begin{aligned} M_{n,\rho}^{\beta}(f(t);x)-f(x) &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\bigg(\int_x^t \bigg\{\delta_x(\nu)\left[f'(\nu)-\frac{f'(x+)+f'(x-)}{2}\right] \\ &\quad +\left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right] \\ &+ \frac{f'(x+)-f'(x-)}{2}\left[sgn(\nu-x)+\frac{\beta-1}{\beta+1}\right] \\ &\quad +f'_x(\nu)\bigg\}d\nu\bigg)dt \\ &= \Psi_1+\Psi_2+\Psi_3+\Psi_4, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \Psi_1 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \delta_x(\nu)\left[f'(\nu)-\frac{f'(x+)+f'(x-)}{2}\right]d\nu dt \\ \Psi_2 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right]d\nu dt \\ \Psi_3 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \frac{f'(x+)-f'(x-)}{2}\left[sgn(\nu-x)+\frac{\beta-1}{\beta+1}\right]d\nu dt \\ \Psi_4 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t f'_x(\nu)d\nu dt. \end{aligned}$$

We can easily see from the definition of $\delta_x(t)$ that

$$\Psi_1=\int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \delta_x(\nu)\left[f'(\nu)-\frac{f'(x+)+f'(x-)}{2}\right]d\nu dt=0 \tag{3.6}$$

and

$$\begin{aligned}
\Psi_2 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right] d\nu dt \\
&= \left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right]\int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t d\nu dt \\
&= \left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right]\int_0^\infty F_{n,\rho}^{\beta}(x,t)(t-x)dt \\
&= \left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right] M_{n,\rho}^{\beta}(t-x;x). \qquad (3.7)
\end{aligned}$$

Now, we evaluate Ψ_3,

$$\begin{aligned}
\Psi_3 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \frac{f'(x+)-f'(x-)}{2}\left[sgn(\nu-x)+\frac{\beta-1}{\beta+1}\right] d\nu dt \\
&= \left(\frac{\beta-1}{\beta+1}\right)\frac{f'(x+)-f'(x-)}{2} M_{n,\rho}^{\beta}((t-x),x) \\
&+ \frac{f'(x+)-f'(x-)}{2}\int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t sgn(\nu-x)d\nu dt \\
&= \left(\frac{\beta-1}{\beta+1}\right)\frac{f'(x+)-f'(x-)}{2} M_{n,\rho}^{\beta}((t-x),x) \\
&- \frac{f'(x+)-f'(x-)}{2}\int_0^x F_{n,\rho}^{\beta}(x,t)(t-x)dt \\
&\quad + \frac{f'(x+)-f'(x-)}{2}\int_x^\infty F_{n,\rho}^{\beta}(x,t)(t-x)dt \\
&= \left(\frac{\beta-1}{\beta+1}\right)\frac{f'(x+)-f'(x-)}{2} M_{n,\rho}^{\beta}((t-x),x) \\
&\quad + \frac{f'(x+)-f'(x-)}{2}\int_0^\infty F_{n,\rho}^{\beta}(x,t)|x-t|dt \\
&= \left(\frac{\beta-1}{\beta+1}\right)\frac{f'(x+)-f'(x-)}{2} M_{n,\rho}^{\beta}((t-x),x) \\
&\quad + \frac{f'(x+)-f'(x-)}{2} M_{n,\rho}^{\beta}(|t-x|,x). \qquad (3.8)
\end{aligned}$$

Combining equations (3.5)-(3.8), we have

$$\begin{aligned}
|M_{n,\rho}^{\beta}(f(t);x)-f(x)| &\leq \left|\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right| |M_{n,\rho}^{\beta}(t-x;x)| \\
&\quad + \frac{\beta-1}{\beta+1}\left|\frac{f'(x+)-f'(x-)}{2}\right| |M_{n,\rho}^{\beta}((t-x),x)|
\end{aligned}$$

$$+ \quad \left|\frac{f'(x+)-f'(x-)}{2}\right| M_{n,\rho}^{\beta}(|t-x|,x)+|\Psi_4|. \qquad (3.9)$$

Now, applying Lemma 1.5 and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|M_{n,\rho}^{\beta}(f(t);x)-f(x)| &\leq \frac{1}{\beta+1}|f'(x+)+\beta f'(x-)|(\beta M_{n,\rho}((t-x)^2;x))^{1/2} \\
&+ \frac{\beta}{\beta+1}|f'(x+)-f'(x-)|(\beta M_{n,\rho}((t-x)^2;x))^{1/2}+|\Psi_4| \\
&\leq \frac{\beta^{1/2}}{\beta+1}|f'(x+)+\beta f'(x-)|\sqrt{\frac{\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)} \\
&+\frac{\beta^{3/2}}{\beta+1}|f'(x+) \\
&-f'(x-)|\sqrt{\frac{\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)}+|\Psi_4|. \qquad (3.10)
\end{aligned}$$

We now estimate $|\Psi_4|$. We may write

$$\Psi_4=\int_0^{\infty} F_{n,\rho}^{\beta}(x,t)\int_x^t f_x'(\nu)d\nu dt=\Psi_5+\Psi_6,$$

where

$$\Psi_5=\int_0^{x} F_{n,\rho}^{\beta}(x,t)\int_x^t f_x'(\nu)d\nu dt$$

$$\Psi_6=\int_x^{\infty} F_{n,\rho}^{\beta}(x,t)\int_x^t f_x'(\nu)d\nu dt.$$

Since $\int_c^d d_t\Phi_{n,\rho}^{\beta}(x,t)\leq 1$, for each $[c,d]\subset[0,\infty)$ and $f_x'(x)=0$, using Lemma 3.1 and integration by parts with $x_1=x-\frac{x}{\sqrt{n}}$, we have

$$\begin{aligned}
|\Psi_5| &= \left|\int_0^{x} F_{n,\rho}^{\beta}(x,t)\int_x^t f_x'(\nu)d\nu dt\right| \\
&= \left|\int_0^{x}\left(\int_x^t f_x'(\nu)d\nu\right)d_t\Phi_{n,\rho}^{\beta}(x,t)\right| = \left|\int_0^{x} f_x'(t)\Phi_{n,\rho}^{\beta}(x,t)dt\right| \\
&\leq \int_0^{x_1}|f_x'(t)-f_x'(x)||\Phi_{n,\rho}^{\beta}(x,t)|dt+\int_{x_1}^{x}|f_x'(t)-f_x'(x)||\Phi_{n,\rho}^{\beta}(x,t)|dt \\
&\leq \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_0^{x_1}V_t^x(f_x')\frac{1}{(x-t)^2}dt+\int_{x_1}^{x}V_t^x(f_x')dt \\
&\leq \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_0^{x_1}V_t^x(f_x')\frac{1}{(x-t)^2}dt+\frac{x}{\sqrt{n}}V_{x-\frac{x}{\sqrt{n}}}^{x}(f_x'). \qquad (3.11)
\end{aligned}$$

Substituting $t = x - \frac{x}{u}$, we obtain

$$\begin{aligned}
&\frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_0^{x-\frac{x}{\sqrt{n}}} V_t^x(f_x')\frac{1}{(x-t)^2}dt \\
&= \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_1^{\sqrt{n}} V_{x-\frac{x}{u}}^x(f_x')du \\
&\leq \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1} V_{x-\frac{x}{k}}^x(f_x')du \\
&\leq \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]} V_{x-\frac{x}{k}}^x(f_x'). \qquad (3.12)
\end{aligned}$$

Combining (3.11) and (3.12), we have

$$\Psi_5 \leq \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]} V_{x-\frac{x}{k}}^x(f_x') + \frac{x}{\sqrt{n}}V_{x-\frac{x}{\sqrt{n}}}^x(f_x'). \qquad (3.13)$$

From Lemma 3.1(ii), $F_{n,\rho}^{\beta}(x,t) = -d_t(1-\Phi_{n,\rho}^{\beta}(x,t)),\ t > x$, hence we may write

$$\begin{aligned}
|\Psi_6| &\leq \left|\int_x^{2x}\left(\int_x^t f_x'(\nu)d\nu\right)d_t(1-\Phi_{n,\rho}^{\beta}(x,t))\right| + \left|\int_{2x}^{\infty}\left(\int_x^t f_x'(\nu)d\nu\right)d_tF_{n,\rho}^{\beta}(x,t)\right| \\
&= \Psi_7+\Psi_8, \text{say.}
\end{aligned}$$

First estimate Ψ_7,

$$\begin{aligned}
\Psi_7 &= \left|\int_x^{2x}\left(\int_x^t f_x'(\nu)d\nu\right)d_t(1-\Phi_{n,\rho}^{\beta}(x,t))\right| \\
&\leq \left|\int_x^{2x} f_x'(\nu)d\nu\right||(1-\Phi_{n,\rho}^{\beta}(x,2x))| + \left|\int_x^{2x} f_x'(t)(1-\Phi_{n,\rho}^{\beta}(x,t))dt\right| \\
&\leq \left|\int_x^{2x}(f'(\nu)-f'(x+))d\nu\right||(1-\Phi_{n,\rho}^{\beta}(x,2x))| + \left|\int_x^{2x} f_x'(t)(1-\Phi_{n,\rho}^{\beta}(x,t))dt\right| \\
&\leq \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)|f(2x)-f(x)-xf'(x+)| \\
&\quad + \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_{x+x/\sqrt{n}}^{2x}\frac{V_x^t(f_x')}{(x-t)^2}dt + \int_x^{x+x/\sqrt{n}} V_x^t(f_x')dt.
\end{aligned}$$

Now, substituting $t = x + \frac{x}{u}$, we have

$$\begin{aligned}
&\leq \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)|f(2x)-f(x)-xf'(x+)| \\
&\quad + \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_1^{\sqrt{n}} V_x^{x+\frac{x}{u}}(f_x')du + \int_x^{x+x/\sqrt{n}} V_x^t(f_x')dt
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)|f(2x)-f(x)-xf'(x+)| \\
&\quad +\frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]} V_x^{x+\frac{x}{k}}(f'_x)+\frac{x}{\sqrt{n}}V_x^{x+\frac{x}{\sqrt{n}}}(f'_x). \qquad (3.14)
\end{aligned}$$

Using Cauchy-Schwarz inequality

$$\begin{aligned}
\Psi_8 &= \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(\nu)d\nu\right)F_{n,\rho}^{\beta}(t,x)dt\right| \\
&= \left|\int_{2x}^{\infty}\left(\int_x^t (f'(\nu)-f'(x+))d\nu\right)F_{n,\rho}^{\beta}(x,t)dt\right| \\
&\leq \left|\int_{2x}^{\infty}(f(t)-f(x))F_{n,\rho}^{\beta}(t,x)dt\right|+\int_{2x}^{\infty}|t-x||f'(x+)|F_{n,\rho}^{\beta}(t,x)dt \\
&\leq \left|\int_{2x}^{\infty}f(t)F_{n,\rho}^{\beta}(t,x)dt\right|+|f(x)|\left|\int_{2x}^{\infty}F_{n,\rho}^{\beta}(t,x)dt\right| \\
&\quad +|f'(x+)|\left(\int_{2x}^{\infty}(t-x)^2F_{n,\rho}^{\beta}(t,x)dt\right)^{1/2} \\
&\leq C\left|\int_{2x}^{\infty}(1+t^2)F_{n,\rho}^{\beta}(t,x)dt\right|+|f(x)|\left|\int_{2x}^{\infty}F_{n,\rho}^{\beta}(t,x)dt\right| \\
&\quad +|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)}
\end{aligned}$$

Since $t\geq 2x$, we have $t\leq 2(t-x)$, hence

$$\begin{aligned}
\Psi_8 &\leq C\left(4+\frac{1}{x^2}\right)\left(\int_{2x}^{\infty}(t-x)^2F_{n,\rho}^{\beta}(t,x)dt\right)+\frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)|f(x)| \\
&\quad +|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)} \\
&= \left\{\frac{|f(x)|}{x^2}+C\left(4+\frac{1}{x^2}\right)\right\}\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right) \\
&\quad +|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)} \qquad (3.15)
\end{aligned}$$

From (3.14) and (3.15), we obtain

$$\begin{aligned}
|\Psi_6| &\leq \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)\{|f(2x)-f(x)-xf'(x+)|\} \\
&\quad +\frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]} V_x^{x+\frac{x}{k}}(f'_x)+\frac{x}{\sqrt{n}}V_x^{x+\frac{x}{\sqrt{n}}}(f'_x)
\end{aligned}$$

$$+ \quad \left\{\frac{|f(x)|}{x^2} + C\left(4 + \frac{1}{x^2}\right)\right\}\sqrt{\frac{\mu x\beta}{n}\left(1 + \frac{1}{\rho} + D''(1)\right)}$$
$$+|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1 + \frac{1}{\rho} + D''(1)\right)}. \tag{3.16}$$

Combining the estimates (3.10), (3.13) and (3.16), we obtain the desired result. □

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References

[1] Abel, U., Gupta,V., An estimate of the rate of convergence of a Bezier variant of the Baskakov-Kantorovich operators for bounded variation functions. *Demonstr. Math.*, 36(1):123–136, 2003.

[2] Agrawal, P. N., Ispir, N., Kajla, A., Approximation properties of Bezier-summation-integral type operators based on Polya– Bernstein functions, *Appl. Math. Comput.*, 259 (2015), 533–539.

[3] Ditzian, Z., Totik, V., Moduli of smoothness, Springer Ser. Comput. Math., *Berlin, itd: Springer Verlag*, 9, 1987.

[4] Goyal, M., Agrawal, P.N., Bézier variant of the Jakimovski– Leviatan– Păltănea operators based on Appell polynomials, *Ann. Univ. Ferrara*, 63(2) (2017), 289–302.

[5] Guo, S. S., Liu, G. F., Song, Z. J., Approximation by Bernstein- Durrmeyer- Bezier operators in Lp spaces. *Acta Math. Sci. Ser. A Chin. Ed*, 30(6):1424–1434, 2010.

[6] Ismail, M., On a generalization of Szász operators, *Mathematica (Cluj)*, 39 (1974), 259–267.

[7] Ispir, N., Yuksel, I., On the Bezier variant of Srivastava- Gupta operators, *Appl. Math. E-Notes*, 5 (2005), 129–137.

[8] Jakimovski, A., Leviatan, D., Generalized Szász operators for the approximation in the infinite interval, *Mathematica (Cluj)*, 11(34) (1969) , 97–103.

[9] Mursaleen, M., AL-Abeid, A., Ansari, K. J., Approximation by Jakimovski- Leviatan-Păltănea operators involving Sheffer polynomials, *Rev. R. Acad. Cienc. Exactas FÂťÄśs. Nat. Ser. A Mat. RACSAM*, 113(2) (2019), 1251–1265.

[10] Özarslan, M. A., Local approximation behavior of modified SMK operators, *Miskolc Math. Notes*, 11(1) (2010) , 87–99.

[11] Păltănea, R., Modified Szász- Mirakjan operators of integral form, *Carpathian J. Math.*, (2008), 378–385.

[12] Phillips, R. S., An inversion formula for Laplace transforms and semi-groups of linear operators, *Ann. Math.*, (1954) , 325-356.

[13] Srivastava, H. M., Gupta, V., Rate of convergence for the Bézier variant of the Bleimann-Butzer- Hahn operators, *Appl. Math. Lett.*, 18(8) (2005), 849–857.

[14] Szasz, O., Generalization of S. Bernstein's polynomials to the infinite interval, *J. Res. Nat. Bur. Standards*, 45 (1950) , 239-245.

[15] Verma, D., Gupta,V., Approximation for Jakimovski- Leviatan- Păltănea operators, *Ann. Univ. Ferrara*, 61(2) (2015) , 367-380.

[16] Zeng, X., Piriou, A., On the rate of convergence of two Bernstein- Bézier type operators for bounded variation functions, *J. Approx. Theory*, 95(3) (1998), 369–387.

[17] Zeng, X. M., On the rate of convergence of the generalized Szasz type operators for functions of bounded variation, *J. Math. Anal. Appl.*, 226(2) (1998), 309–325.

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