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ON THE (DELTA,f)-LACUNARY STATISTICAL CONVERGENCE OF THE FUNCTIONS

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Abstract. In this paper, we introduce the concept of Δ_f-lacunary statistical convergence for a Δ-measurable real-valued function defined on time scale, where f is an unbounded modulus. Our motivation here is that this definition includes many well-known concepts which already exist in the literature. We also define strong Δ_f-lacunary Cesáro summability on a time scale and give some results related to these new concepts. Furthermore, we obtain necessary and sufficient conditions for the equivalence of Δ_f-convergence and Δ_f-lacunary statistical convergence on a time scale.

1. Introduction

The idea of statistical convergence for sequences of real and complex numbers, which was introduced by Fast [1] and Steinhaus [2] independently, is a generalization of ordinary convergence. This concept depends on density of subset of natural numbers $\mathbb{N}$. The natural (or asymptotic density) of a set $K \subset \mathbb{N}$ defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leqslant n : k \in K\}\right|,$$

if the limit exist, where $|A|$ indicates the cardinality of any set A. A sequence $x = (x_k)$ is said to be statistically convergent to L, if for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}$ has zero natural density, i.e., for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left|\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}\right| = 0,$$

and written as $st - \lim x = L$. The set of all statistical convergent sequences is denoted by S. Over the years, statistical convergence and related notions have been studied by many researchers [3–15].

The idea of a modulus function was introduced by Nakano [16]. Later, Ruckle [17], Maddox [18] and many authors used this concept to construct some sequence spaces. A function $f : [0,\infty) \to [0,\infty)$ is called modulus function, or simply modulus, if

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i) $f(x) = 0$ if and only if $x = 0$,
ii) $f(x+y) \leqslant f(x) + f(y)$ for $x \geqslant 0$, $y \geqslant 0$,
iii) f is increasing,
iv) f is continuous from right at 0.

From the above properties (ii) and (iv), it is clear that a modulus function f is continuous everywhere on $[0,\infty)$. A modulus function may be bounded or unbounded. For example, $f(x) = \frac{x}{1+x}$ is bounded, but $f(x) = x^p$, where $0 < p \leqslant 1$, is unbounded.

Aizpuru et al. [9] defined a new concept of density by using an unbounded modulus function, and also with this way, they defined f-statistical convergence for sequences as follows:

A sequence $x = (x_k)$ is said to be f-statistically convergent to L, if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f(n)} f\left(\left|\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}\right|\right) = 0,$$

where f is an unbounded modulus function, and one writes it as $st^f - \lim x_k = L$.

A time scale is any arbitrary nonempty closed subset of real numbers $\mathbb{R}$ and is denoted by $\mathbb{T}$. The time scales calculus was first introduced by Hilger [20]. This new theory allows one to unify discrete and continuous analysis as it has the differentiation and integration of independent domain used. Because of this feature, it has received much attention and its applications have been studied in many areas of science [21–24]. In addition, the first studies related to the statistical convergence and summability theory on time scales were done in [25] and [26], independently. In the following years, as a continuation and generalization of these studies, many researchers have moved well known some topics in summability theory for sequences or Lebesque measurable functions to time scale calculus [27–33]. Before giving these definitions, we shall mention some basic concepts of the time scale calculus that we will use in later sections.

The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ can be defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

for $t \in \mathbb{T}$. In this definition we put $\inf \emptyset = \sup \mathbb{T}$, where $\emptyset$ is an empty set. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ can be defined by $\mu(t) = \sigma(t) - t$. A closed interval in a time scale $\mathbb{T}$ is given by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. Open intervals or half-open intervals are defined similarly.

Let F denote the family of all left closed and right open intervals on $\mathbb{T}$ of the form $[a,b)_{\mathbb{T}}$. Let $m : F \to [0,\infty)$ be a set function on F such that $m([a,b)_{\mathbb{T}}) = b - a$. Then, the set function m is a countably additive measure on F. Now, the Caratheodory extension of the set function m associated with the family F is said to be the Lebesgue Δ-measure on $\mathbb{T}$ and this is denoted by μ_Δ. In this case, it is known that if $a \in \mathbb{T}\setminus\{\max \mathbb{T}\}$, then the single point set $\{a\}$ is $\Delta-$measurable and $\mu_\Delta(\{a\}) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leqslant b$, then $\mu_\Delta((a,b)_{\mathbb{T}}) = b - \sigma(a)$. If $a, b \in \mathbb{T}\setminus\{\max \mathbb{T}\}$ and $a \leqslant b$, then $\mu_\Delta((a,b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$ and $\mu_\Delta([a,b]_{\mathbb{T}}) = \sigma(b) - a$ (see [23]).

Now, we recall some basic concepts related to summability theory on time scales. We should note that throughout the paper, we consider that $\mathbb{T}$ is a time scale satisfying $\inf \mathbb{T} = t_0 > 0$ and $\sup \mathbb{T} = \infty$.

Definition 1.1. *[26] Let $g : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. We say that g is statistically convergent to a number L on $\mathbb{T}$, if for every $\varepsilon > 0$,*

$$\lim_{t\to\infty} \frac{\mu_\Delta\left(\{s \in [t_0, t]_\mathbb{T} : |g(s) - L| \geqslant \varepsilon\}\right)}{\mu_\Delta\left([t_0, t]_\mathbb{T}\right)} = 0,$$

which is denoted by $st_\mathbb{T} - \lim_{t\to\infty} g(t) = L$.

Let $\theta = (k_r)$ is an increasing sequence of non-negative integers with $k_0 = 0$ and $\sigma(k_r) - \sigma(k_{r-1}) \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence with respect to $\mathbb{T}$ [27].

Definition 1.2. *[27] Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. A Δ-measurable function $g : \mathbb{T} \to \mathbb{R}$ is said to be lacunary statistically convergent to a number L on $\mathbb{T}$, if for every $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{s \in (k_{r-1}, k_r]_\mathbb{T} : |g(s) - L| \geqslant \varepsilon\}\right)}{\mu_\Delta\left((k_{r-1}, k_r]_\mathbb{T}\right)} = 0,$$

which is denoted by $st_{\theta-\mathbb{T}} - \lim_{t\to\infty} g(t) = L$.

Definition 1.3. *[27] Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ and let $g : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then g is strongly lacunary Cesáro summable to L on $\mathbb{T}$, if there exists an $L \in \mathbb{R}$ such that*

$$\lim_{r\to\infty} \frac{1}{\mu_\Delta\left((k_{r-1}, k_r]_\mathbb{T}\right)} \int_{(k_{r-1}, k_r]_\mathbb{T}} |g(s) - L| \,\Delta s = 0.$$

The set of all strongly lacunary Cesáro summable functions on $\mathbb{T}$ is denoted by $N_{\theta-\mathbb{T}}$.

Definition 1.4. *[32] Let f be a modulus function and $g : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. We say that g is Δ_f-convergent to a number L on $\mathbb{T}$, if for every $\varepsilon > 0$,*

$$\lim_{t\to\infty} \frac{1}{f\left(\mu_\Delta\left([t_0, t]_\mathbb{T}\right)\right)} f\left(\mu_\Delta\left(\{s \in [t_0, t]_\mathbb{T} : |g(s) - L| \geqslant \varepsilon\}\right)\right) = 0,$$

and we write it as $\Delta_f - \lim_{t\to\infty} g(t) = L$. Also, we denote the set of all Δ_f-convergent functions on $\mathbb{T}$ by $S_\mathbb{T}^f$.

Our aim here is to introduce the concepts of Δ_f-lacunary statistical convergence and strong Δ_f-lacunary Cesáro summability on a time scale $\mathbb{T}$ with respect to a modulus function f, by continuing of [32]. We also present several results related to these new concepts.

2. Δ_f-Lacunary Statistical Convergence on Time Scale

We start this section by defining the concept of Δ_f-lacunary statistical convergence on a time scale.

Definition 2.1. *Let f be an unbounded modulus function and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. Then a Δ-measurable function $g : \mathbb{T} \to \mathbb{R}$ is Δ_f-lacunary statistically convergent to a number L on $\mathbb{T}$, if for every $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{f\left(\mu_\Delta\left((k_{r-1}, k_r]_\mathbb{T}\right)\right)} f\left(\mu_\Delta\left(\{s \in (k_{r-1}, k_r]_\mathbb{T} : |g(s) - L| \geqslant \varepsilon\}\right)\right) = 0,$$

and we write it as $st_{\theta-\mathbb{T}}^{f}-\lim\limits_{t\to\infty} g(t)=L$*. The set off all* Δ_f*-lacunary statistically convergent functions on* $\mathbb{T}$ *is denoted by* $S_{\theta-\mathbb{T}}^{f}$*.*

This definition includes some special cases.

Remark. ***i)*** *If we choose* $f(x)=x$ *in Definition 2.1, then* Δ_f*-lacunary statistical convergence is reduced to lacunary statistical convergence on a time scale introduced in [27].*

ii) *If we take* $\mathbb{T}=\mathbb{N}$*, then Definition 2.1 gives us the concept of* f*-lacunary statistical convergence which is defined in [10].*

Theorem 2.1. *Let* f *be an unbounded modulus function and let* $\theta=(k_r)$ *be a lacunary sequence on* $\mathbb{T}$*. For any a* Δ*-measurable function* $g:\mathbb{T}\to\mathbb{R}$*,* $\lim\limits_{t\to\infty} g(t)=L$ *implies* $st_{\theta-\mathbb{T}}^{f}-\lim\limits_{t\to\infty} g(t)=L$*.*

Proof. Suppose that $\lim\limits_{t\to\infty} g(t)=L$. Then, for each $\varepsilon>0$, the set $\{s\in\mathbb{T}\colon |g(s)-L|\geqslant\varepsilon\}$ is bounded. Since

$$\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\}\subseteq\{s\in\mathbb{T}:|g(s)-L|\geqslant\varepsilon\}$$

and modulus function f is increasing, therefore

$$\frac{f(\mu_\Delta(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\}))}{f(\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}}))}\leqslant\frac{f(\mu_\Delta(\{s\in\mathbb{T}:|g(s)-L|\geqslant\varepsilon\}))}{f(\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}}))}.$$

Taking limit as $r\to\infty$ on both sides, we get

$$\lim_{r\to\infty}\frac{f(\mu_\Delta(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\}))}{f(\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}}))}=0,$$

which means that $st_{\theta-\mathbb{T}}^{f}-\lim\limits_{t\to\infty} g(t)=L$. □

Theorem 2.2. *Let* f *be an unbounded modulus function and let* $\theta=(k_r)$ *be a lacunary sequence on* $\mathbb{T}$*. For any a* Δ*-measurable function* $g:\mathbb{T}\to\mathbb{R}$*,* $st_{\theta-\mathbb{T}}^{f}-\lim\limits_{t\to\infty} g(t)=L$ *implies* $st_{\theta-\mathbb{T}}-\lim\limits_{t\to\infty} g(t)=L$*.*

Proof. Suppose that $st_{\theta-\mathbb{T}}^{f}-\lim\limits_{t\to\infty} g(t)=L$. Then, using the definition of limit and also using the properties of modulus function f, for every $p\in\mathbb{N}$, for sufficiently large $t\in\mathbb{T}$, we have

$$\begin{aligned}f(\mu_\Delta(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\}))&\leqslant\frac{1}{p}f(\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}}))\\&\leqslant\frac{1}{p}pf\left(\frac{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}{p}\right)\\&=f\left(\frac{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}{p}\right)\end{aligned}$$

and since f is increasing, we get

$$\frac{\mu_\Delta(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\})}{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}\leqslant\frac{1}{p},$$

which means that $st_{\theta-\mathbb{T}} - \lim_{t\to\infty} g(t) = L$. Hence, the proof is completed. $\square$

Corollary 2.3. *Let f be an unbounded modulus function and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. For any a Δ-measurable function $g : \mathbb{T} \to \mathbb{R}$, we have*

$$\lim_{t\to\infty} g(t) = L \Rightarrow st^f_{\theta-\mathbb{T}} - \lim_{t\to\infty} g(t) = L \Rightarrow st_{\theta-\mathbb{T}} - \lim_{t\to\infty} g(t) = L.$$

3. Strong Δ_f-Lacunary Cesáro Summability and Δ_f-Lacunary Statistical Convergence on Time Scale

Now, we first introduce strong Δ_f-lacunary Cesáro summability of a Δ-measurable function defined on a time scale. We also investigate the relationship between the strong Δ_f-lacunary Cesáro summability and strong lacunary Cesáro summability on a time scale.

Definition 3.1. *Let f be a modulus function and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. Then a Δ-measurable function $g : \mathbb{T} \to \mathbb{R}$ is said to be strongly Δ_f-lacunary Cesáro summable to a number L on $\mathbb{T}$ if*

$$\lim_{r\to\infty} \frac{1}{\mu_\Delta\left((k_{r-1}, k_r]_{\mathbb{T}}\right)} \int\limits_{(k_{r-1},k_r]_{\mathbb{T}}} f\left(|g(s) - L|\right) \Delta s = 0.$$

The set off all strongly Δ_f-lacunary Cesáro summable functions on $\mathbb{T}$ is denoted by $N^f_{\theta-\mathbb{T}}$.

We now give some lemmas we will use next theorem.

Lemma 3.1. *[18] Let f be any modulus function and let $0 < \delta < 1$. Then, for each $x \geqslant \delta$, we have $f(x) \leqslant 2f(1)\delta^{-1}x$.*

Lemma 3.2. *[19] Let f be any modulus function. Then $\lim_{t\to\infty} \frac{f(t)}{t}$ exists.*

Theorem 3.3. ***i)*** *For any modulus function f, we have $N_{\theta-\mathbb{T}} \subset N^f_{\theta-\mathbb{T}}$.*

ii) *Let f be any modulus function. If $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, then we have $N^f_{\theta-\mathbb{T}} \subset N_{\theta-\mathbb{T}}$.*

Proof. It is easy to see using Lemma 3.1 and Lemma 3.2. $\square$

As a corollary we have

Corollary 3.4. *Let f be any modulus function. If $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, then we have $N^f_{\theta-\mathbb{T}} = N_{\theta-\mathbb{T}}$.*

Now, we give the relationship between the Δ_f-lacunary statistical convergence and strong Δ_f-lacunary Cesáro summability on time scale. Before doing this, we remind Jensen's inequality on time scale.

Lemma 3.5. *[22] Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $\phi : [a, b] \to (c, d)$ is rd-continuous and $F : (c, d) \to \mathbb{R}$ is convex, then*

$$F\left(\frac{\int_a^b \phi(t)\,\Delta t}{b-a}\right) \leqslant \frac{\int_a^b F(\phi(t))\,\Delta t}{b-a}.$$

Lemma 3.6. *[18] There is a modulus function f for which there exists a positive constant c such that $f(xy) \geqslant cf(x) f(y)$ for all $x \geqslant 0$, $y \geqslant 0$.*

Theorem 3.7. *Let $g : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. Then, we have*

i) *Let f be an unbounded convex modulus function for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$ exists and there exists a positive constant c such that $f(xy) \geqslant cf(x) f(y)$ for all $x \geqslant 0$, $y \geqslant 0$. If g is strongly Δ_f-lacunary Cesáro summable to L, then $st^f_{\theta-\mathbb{T}} - \lim_{t\to\infty} g(t) = L$, but not conversely.*

ii) *If $st^f_{\theta-\mathbb{T}} - \lim_{t\to\infty} g(t) = L$ and g is a bounded function, then g is strongly Δ_f-lacunary Cesáro summable to L, for any unbounded modulus function f.*

Proof. It can be proved by considering similar way with in Theorem 14 of [10] and Theorem 1, Theorem 2 of [27] and also using Lemma 3.5. □

Now, under certain restrictions on $\theta = (k_r)$ and modulus function f, we investigate necessary and sufficient conditions for the equivalence of Δ_f-convergence and Δ_f-lacunary statistical convergence on a time scale.

Theorem 3.8. *Let f be an unbounded convex modulus function for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$ exists and there exists a positive constant c such that $f(xy) \geqslant cf(x) f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. Then, we have*

$$S^f_{\mathbb{T}} \subset S^f_{\theta-\mathbb{T}} \quad \textit{if and only if} \quad \liminf_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} > 1.$$

Proof. The proof can be done easily by combining the ideas in Lemma 3.1 of [28] and Lemma 17 of [10]. Hence, we omit it. □

Theorem 3.9. *Let f be an unbounded convex modulus function for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$ exists and there exists a positive constant c such that $f(xy) \geqslant cf(x) f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ such that $\mu(t) \leqslant Mt$ for some $M \geqslant 0$ and for all $t \in \mathbb{T}$. Then, we have*

$$S^f_{\theta-\mathbb{T}} \subset S^f_{\mathbb{T}} \quad \textit{if and only if} \quad \limsup_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} < \infty.$$

Proof. The proof can be done easily in view of Lemma 3.2 of [28] and Lemma 19 of [10]. Hence, we omit it. □

We here note that all the restrictions apart from $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ on the modulus function f in Theorem 3.8 and Theorem 3.9 are needed only in the necessity part of these theorems.

Combining Theorem 3.8 and Theorem 3.9, we obtain the following result.

Corollary 3.10. *Let f be an unbounded convex modulus function for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$ exists and there exists a positive constant c such that $f(xy) \geqslant$*

$cf(x)f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ such that $\mu(t) \leqslant Mt$ for some $M \geqslant 0$ and for all $t \in \mathbb{T}$. Then, we have

$$S^f_{\theta-\mathbb{T}} = S^f_{\mathbb{T}} \quad \textit{if and only if} \quad 1 < \liminf_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} \leqslant \limsup_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} < \infty.$$

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q-QUASINORMAL OPERATORS AND ITS EXTENDED EIGENVALUES

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Abstract. In this paper, the relation between q-deformed quasinormal operators and q-quasinormal operator classes is investigated. Moreover, we prove that these are same. Also, we consider the extended eigenvalue problems for bounded q-quasinormal operators.

1. Introduction

Let q be a positive number not equal to one and A be a closed operator with dense domain on a separable Hilbert space H. If A satisfies

$$AA^* = qA^*A,$$

then A is said to be a deformed normal operator with deformation parameter q or a q-normal operator. A nonzero q-normal operator is always unbounded [17, 18]. Also, if A is a closed operator with dense domain in H and its polar decomposition $A = U|A|$ such that

$$U|A| \subset \sqrt{q}|A|U,$$

then A is called a deformed quasinormal operator with deformation parameter q or a q-quasinormal operator. Every nonzero q-quasinormal operator is unbounded [1]. The basic properties for q-deformed operators can be found in [1, 2, 3, 4, 5].

Moreover, S. Lohaj defined that the bounded operator A is a q-quasinormal operator, if the equation

$$AA^*A = qA^*AA$$

is hold [6]. He showed that if any invertible operator is q-quasinormal then $q = 1$ [6]. It is clear that a bounded deformed at quasinormal with deformation parameter q operator is q-quasinormal.

A complex number λ is said to be an extended eigenvalue of a bounded operator A if there exists an operator $X \neq 0$ such that

$$XA = \lambda AX.$$

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X is called a λ eigenoperator for A and the set of extended eigenvalues is denoted by $\sigma_{ext}(A)$ [8]. Also, the extended spectrum of bounded operators has been studied by many authors such as [8, 9, 10, 11, 12, 13, 14, 15, 16]. Biswas and Petrovic proved the result

$$\sigma_{ext}\left(A\right) \subset \left\{\lambda \in \mathbb{C} : \sigma\left(A\right) \cap \sigma\left(\lambda A\right) \neq \emptyset\right\}$$

where $\sigma(A)$ is the set of spectrum of A [9].

2. q-Quasinormal Operators and Its Extended Eigenvalues

In this paper, all operators are assumed to be linear. Let us denote by H a complex separable Hilbert space. For an operator A in H, the range and the kernel of A are denoted by $R(A)$ and $KerA$, respectively.

Lemma 2.1. *$A : H \to H$ is a q-quasinormal operator if and only if the equation $U|A|^2 = q|A|^2U$ is hold.*

Proof. Let $A : H \to H$ be a q-quasinormal operator. By the q-quasinormal definition, the equation

$$A(A^*A) = q(A^*A)A$$

is satisfied. Since its polar decomposition is $A = U|A|$, then the equation

$$U|A|^3 - q|A|^2U|A| = (U|A|^2 - q|A|^2U)|A| = 0$$

is hold. When $H = Ker|A| \oplus R(|A|)$ and $Ker(|A|) = KerU$ are hold, then

$$U|A|^2 = q|A|^2U$$

is satisfied. □

Corollary 2.2. *Let A be a closed operator with dense domain in H and $A = U|A|$ be the polar decomposition. The following statements are equivalent.*
i) A is q-quasinormal.
ii)For all $a \in \mathbb{R}$,

$$Ue^{ia|A|^2} = e^{iqa|A|^2}U, i = \sqrt{-1}$$

iii)For all $\lambda \in \mathbb{C}$ with $Im\lambda \neq 0$,

$$U(\lambda - |A|^2)^{-1} = (\lambda - q|A|^2)^{-1}U.$$

iv) For all Borel sets M,

$$E(q^{-1}M)U = UE(M),$$

where $E(.)$ is the spectral measure of $|A|$.

Every q-quasinormal operator A satisfies the relation

$$Ug(|A|^2) = g(q|A|^2)U$$

for any Borel function g.

Proof. It can be proved by using the method as in [1]. □

Corollary 2.3. *If A is a bounded q-quasinormal operator in a Hilbert space iff A is a deformed at quasinormal with deformation parameter q.*

Theorem 2.4. *Suppose that $A : H \to H$ is a q-quasinormal operator, in this case $\sigma_{ext}(A) = \mathbb{C}$ is hold.*

Proof. Let $A = U|A|$ where U is a partial isometry and $|A|$ is the square root of A^*A such that $KerU = Ker\,|A|$, be the polar decomposition of A . Since is a q-quasinormal operator, $U\,|A| = \sqrt{q}\,|A|\,U$,for $q > 1$ is true by Corollary 2.3.

Firstly, we assume that $0 \in \sigma_p\,(A)$, then there exists an element y in $H\backslash\,\{0\}$ such that $Ay = 0$ and for every $x \in H$,

$$A\,(y \otimes y)\,x = A\,(x, y)\,y = (x, y)\,Ay = 0$$

and

$$(y \otimes y)\,Ax = (y \otimes y)\,U\,|A|\,x = (U\,|A|\,x, y)\,y = \sqrt{q}\,(x, U^*\,|A|\,y)\,y = 0.$$

Then,

$$(y \otimes y)\,U|A| = U|A|\,(y \otimes y) = 0$$

is obtained. This means that $\sigma_{ext}\,(A) = \mathbb{C}$ since $0 \in \sigma_p\,(A)$.

Now, let $A : H \to H$ be a q-quasinormal operator such that $0 \notin \sigma_p\,(A)$, in this case, the equation

$$AA^*A = qA^*AA$$

is hold. Since A is a bounded operator, we have $AA^* - qA^*A \neq 0$ and

$$(AA^* - qA^*A)A = 0A(AA^* - qA^*A).$$

Consequently, the zero is an extended spectrum of A. Because of $0 \notin \sigma_p(A)$, U is an isometry. Also, from the von Neuman-Wold decomposition the equality

$$H = \overset{\infty}{\underset{n=0}{\oplus}} U^n(KerU^*)$$

is verified and subspaces $U^n\,(KerU^*)$, n is a nonnegative integer, are invariant under $|A|$ [7].

Moreover, it is defined $T_\lambda := \sum_{n=0}^{\infty} \lambda^n P_n$ such that $0 < |\lambda| \leqslant 1$, where P_n are projection operators on $U^n(KerU^*)$ for all $n \geqslant 0$. It is clear that T_λ is a bonded operator for all $0 < |\lambda| \leqslant 1$. Also, the following equations

$$T_\lambda|A| = |A|T_\lambda$$

$$T_\lambda U = \lambda U T_\lambda$$

are satisfied, so

$$U^nT_\lambda A = (U^nT_\lambda)U|A| = q^{n/2}\lambda|A|(U^nT_\lambda) = q^{n/2}\lambda AU^nT_\lambda, n \geqslant 0.$$

Since $q > 1$ and $0 < |\lambda| \leqslant 1$, $\sigma_{ext}(A) = \mathbb{C}$ is obtained. □

Example 2.1. *Let H be a separable Hilbert space. If $\{e_n\}, n \geqslant 0$ is an orthonormal basis of H, and a sequence $\{w_n\}, w_n \neq 0, n \geqslant 0$ of complex numbers such that*

$$D(S_u) = \{\sum_{n=0}^{\infty} \alpha_n e_n \in H : \sum_{n=0}^{\infty} \mid \alpha_n \mid^2 \mid w_n \mid^2 < \infty\}$$

and

$$S_u e_n = w_n e_{n+1}$$

for all $n \geqslant 0$, then S_u is called a unilateral weighted shift with weights w_n. A unilateral weighted shift S_u in H with weights w_n is q-quasinormal if and only if

$$\mid w_n \mid = \left(\frac{1}{\sqrt{q}}\right)^n \mid w_0 \mid$$

for all $n \geqslant 0$ [1]*. In particular, for* $q > 1$ *a unilateral weighted shift is a bounded q-quasinormal and* $\sigma_p(S_u) = \emptyset$ [1]*. Then, from Theorem 2.4* $\sigma_{ext}(S_u) = \mathbb{C}$*.*

Corollary 2.5. *Let* $A : H \to H$ *be a q-quasinormal operator, then for every* $n \in \mathbb{N}$*,* $q^{n/2} \in \sigma_{ext}(|A|)$ *.*

Corollary 2.6. *Let* $A : H \to H$ *be a q-quasinormal operator, if* $\lambda \in \sigma_{ext}(|A|)$*, then for every* $n \in \mathbb{N}$*,* $q^{\frac{n}{2}}\lambda \in \sigma_{ext}(|A|)$*.*

Corollary 2.7. *If A is a bounded q-quasinormal operator and* $0 \notin \sigma_p(A)$*, then* $0 \in \sigma_c(|A|)$*.*

Proof. Let $A : H \to H$ be a q-quasinormal operator, then for every $n \in \mathbb{N}$, $\sigma(|A|) \cap \sigma(q^{n/2}|A|) \neq \emptyset$. $\square$

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CONVERGENCE OF NOOR, AND ABBAS AND NAZIR ITERATION PROCEDURES FOR A CLASS OF THREE NONLINEAR QUASI CONTRACTIVE MAPS IN CONVEX METRIC SPACES

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Abstract. We define Noor iteration procedure and, Abbas and Nazir iteration procedure associated with three self maps in the setting of convex metric spaces. We prove that these iterations converge strongly to a unique common fixed point of three nonlinear quasi-contractive self maps in convex metric spaces. One of our results (Theorem 2.2) extend the result of Sastry, Babu and Srinivasa Rao [10] to three self maps. Examples are provided to illustrate our results.

1. Introduction

In 1970, Takahashi [11] introduced the concept of convexity in metric spaces as follows.

Definition 1.1. *Let (X, d) be a metric space. A map $W : X \times X \times [0, 1] \to X$ is said to be a 'convex structure' on X if*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \tag{1.1}$$

for $x, y, u \in X$ and $\lambda \in [0, 1]$.

A metric space (X, d) together with a convex structure W is called a *convex metric space* and we denote it by (X, d, W).

A nonempty subset K of X is said to be *'convex'* if $W(x, y, \lambda) \in K$ for $x, y \in K$ and $\lambda \in [0, 1]$.

Remark 1.1. *Every normed linear space $(X, ||.||)$ is a convex metric space with the convex structure W defined by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ for $x, y \in X$, and*

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$\lambda \in [0,1]$. *But, there are convex metric spaces which are not normed linear spaces* [2, 7, 11].

In 1974, Ćirić [3] introduced quasi-contraction maps in the setting of metric spaces and proved that the Picard iterative sequence converges to the fixed point in complete metric spaces.

Definition 1.2. *Let (X,d) be a metric space. A self map $T : X \to X$ is said to be a quasi-contraction map if there exists a real number $0 \leq k < 1$ such that*

$$d(Tx,Ty) \leq kM(x,y) \tag{1.2}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{1.3}$$

for $x, y \in X$.

In 1974, Ishikawa [6] introduced an iteration procedure in the setting of normed linear spaces as follows: Let K be a nonempty convex subset of a normed linear space X and let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0,1]$.
For $x_0 \in K$,

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n Tx_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n, \text{ for } n = 0,1,2,\ldots . \end{aligned} \tag{1.4}$$

In 1988, Ding [5] considered Ishikawa iteration procedure in the setting of convex metric spaces as follows: Let K be a nonempty convex subset of a convex metric space (X,d,W), and let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0,1]$.
For $x_0 \in K$,

$$\begin{aligned} y_n &= W(Tx_n, x_n, \beta_n) \\ x_{n+1} &= W(Ty_n, x_n, \alpha_n) \text{ for } n = 0,1,2,..., \end{aligned} \tag{1.5}$$

and proved that the Ishikawa iteration procedure (1.5) converges strongly to a unique fixed point of a quasi-contraction map in the setting of convex metric spaces, provided $\sum_{n=0}^{\infty} \alpha_n = \infty$.

In 1999, Ćirić [4] introduced a more general quasi-contraction map and proved the convergence of the Ishikawa iteration procedure to a unique fixed point in convex metric spaces and the result is the following.

Theorem 1.1. (Ćirić [4]) *Let K be a nonempty closed convex subset of a complete convex metric space X and let $T : K \to K$ be a self map satisfying*

$$d(Tx,Ty) \leq w(M(x,y)),$$

where $M(x,y)$ is defined by (1.3) for $x, y \in K$ and
$w : (0,\infty) \to (0,\infty)$ is a map which satisfies
(i) $0 < w(t) < t$ *for each* $t > 0$,
(ii) w *increases,*
(iii) $\lim_{t\to\infty}(t - w(t)) = \infty$, and
(iv) either $t - w(t)$ is monotonically increasing on $(0,\infty)$ (1.6)

or

$$w(t) \text{ is strictly increasing and } \lim_{n\to\infty} w^n(t) = 0 \text{ for } t > 0. \tag{1.7}$$

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For $x_0 \in K$, the Ishikawa iteration procedure $\{x_n\}_{n=0}^{\infty}$ defined by (1.5) *converges strongly to the unique fixed point of T.*

Sastry, Babu and Srinivasa Rao [9] improved Theorem 1.1 by replacing (1.6) and (1.7) with a single condition, namely $0 < w(t^+) < t$ for each $t > 0$ and proved the following theorem.

Theorem 1.2. [9] *Let K be a nonempty closed convex subset of a complete convex metric space (X, d, W) and $T : K \to K$ be a map that satisfies*

$$d(Tx, Ty) \le w(M(x, y)) \tag{1.8}$$

where $M(x, y)$ is defined in (1.3) *for $x, y \in K$ and $w : (0, \infty) \to (0, \infty)$ is a map such that*

(*i*) *w increases,*
(*ii*) $\lim_{t\to\infty}(t - w(t)) = \infty$, *and*
(*iii*) $0 < w(t^+) < t$ *for* $t > 0$.

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then for any $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the iteration procedure (1.5) *converges strongly to a unique fixed point of T.*

Remark 1.2. (*i*) *and* (*iii*) *of* Theorem 1.2 *imply that* $0 < w(t) < t$ *for each* $t > 0$.

Remark 1.3. *If $w(t) = kt$ for $t \in (0, \infty)$ and $0 \le k < 1$ then the map T of* Theorem 1.2 *reduces to a quasi-contraction map.*

Sastry, Babu, and Srinivasa Rao [10] extended Theorem 1.2 to a pair of self maps as follows.

Theorem 1.3. [10] *Let (X, d) be a complete convex metric space with convex structure W. Let S, T be self maps of X satisfying the inequality*

$$\max\{d(Sx, Sy), d(Tx, Ty), d(Sx, Ty)\} \le w(M'(x, y)) \text{ for all } x, y \in X$$

where $M'(x, y) = \max\{d(x, y), d(x, Sx), d(x, Sy), d(y, Sx), d(x, Tx), d(y, Ty), d(y, Sy), d(x, Ty), d(y, Tx), d(Sx, Tx), d(Sy, Ty)\}$ *and* $w : (0, \infty) \to (0, \infty)$ *is a map such that*
(*i*) *w is increasing on* $(0, \infty)$,
(*ii*) $\lim_{t\to\infty}(t - w(t)) = \infty$, *and*
(*iii*) $0 < w(t^+) < t$ *for each* $t > 0$.

For $x_0 \in X$, define the Ishikawa iteration procedure associated with S and T by

$$\begin{aligned} y_n &= W(Tx_n, x_n, \beta_n) \\ x_{n+1} &= W(Sy_n, x_n, \alpha_n) \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $(0, 1)$ *with $\sum \alpha_n = \infty$. Then the sequence $\{x_n\}$ converges, $\lim_{n\to\infty} x_n = z$* (say), *$z \in X$ and z is the unique common fixed point of S and T.*

In 2000, Noor [8] introduced a three step iteration procedure in the setting of Banach spaces as follows: For $x_0 \in K$,

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n \\ y_n &= (1 - \beta_n)x_n + \beta_n T z_n \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n \end{aligned} \tag{1.10}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

Noor iteration procedure (1.10) in convex metric spaces is as follows:
For $x_0 \in K$,

$$\begin{aligned} z_n &= W(Tx_n, x_n, \gamma_n) \\ y_n &= W(Tz_n, x_n, \beta_n) \\ x_{n+1} &= W(Ty_n, x_n, \alpha_n) \end{aligned} \tag{1.11}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

We call the iteration $\{x_n\}$ defined by (1.11), a *'modified Noor iteration procedure'*.

In 2014, Abbas and Nazir [1] introduced the following iteration procedure in normed linear spaces.
For $x_0 \in K$,

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n \\ y_n &= (1-\beta_n)Tx_n + \beta_n Tz_n \\ x_{n+1} &= (1-\alpha_n)Ty_n + \alpha_n Tz_n, \end{aligned} \tag{1.12}$$

for $n = 0, 1, 2, \ldots$.

Abbas and Nazir iteration procedure in the setting of convex metric spaces as follows: For $x_0 \in K$,

$$\begin{aligned} z_n &= W(Tx_n, x_n, \gamma_n) \\ y_n &= W(Tz_n, Tx_n, \beta_n) \\ x_{n+1} &= W(Tz_n, Ty_n, \alpha_n) \end{aligned} \tag{1.13}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

We call the iteration $\{x_n\}$ defined by (1.13), a *'modified Abbas and Nazir iteration procedure'*.

Inspired and motivated by the results of Ćirić [4], and Sastry, Babu and Srinivasa Rao [9, 10], we define Noor iteration procedure associated with three self maps in Section 2, and prove the convergence of this iteration procedure to the common fixed point of three self maps in convex metric spaces under certain hypotheses. In Section 3, we extend it to Abbas and Nazir iteration procedure. One of our results (Theorem 2.2) extends the result of [10] to three self maps.

2. Convergence of Noor iteration procedure

We begin this section by defining an iteration procedure in convex metric spaces as follows.

Let (X, d, W) be a convex metric space, K a nonempty convex subset of X. Let $T_1, T_2, T_3 : K \to K$ be three self maps. For $x_0 \in K$,

$$\begin{aligned} z_n &= W(T_1x_n, x_n, \gamma_n) \\ y_n &= W(T_2z_n, x_n, \beta_n) \\ x_{n+1} &= W(T_3y_n, x_n, \alpha_n) \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

We call the iteration $\{x_n\}$ defined by (2.1), a Noor iteration procedure associated with T_1, T_2 and T_3 in convex metric spaces.

Lemma 2.1. *Let (X, d, W) be a convex metric space and K be a nonempty convex subset of X. Let $T_1, T_2, T_3 : K \to K$ be three self maps satisfying the inequality*

$$\max_{i,j=1,2,3}\{d(T_ix, T_jy)\} \leq w(M_1(x,y)) \text{ for } x, y \in K \text{ with } M_1(x,y) > 0, \tag{2.2}$$

where

$$M_1(x,y) = \max_{1\le i,j\le 3, i\ne j}\{d(x,y), d(x,T_ix), d(y,T_iy), d(x,T_iy),$$
$$d(y,T_ix), d(T_ix,T_jx), d(T_iy,T_jy)\}, \tag{2.3}$$

$w:(0,\infty)\to(0,\infty)$ *is a map such that*

$$w \textit{ increases}, \tag{2.4}$$

$$\lim_{t\to\infty}(t-w(t)) = \infty, \tag{2.5}$$

and

$$0 < w(t^+) < t \textit{ for } t > 0. \tag{2.6}$$

For any $x_0 \in K$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated by Noor iteration procedure (2.1) *associated with three self maps T_1, T_2, and T_3.*

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{T_ix_n\}, \{T_iy_n\}$, and $\{T_iz_n\}$ for $i = 1,2,3$ are bounded.

Proof. For each positive integer n, we define
$A_n = \{x_k\}_{k=0}^n \cup \{y_k\}_{k=0}^n \cup \{z_k\}_{k=0}^n \cup \bigcup_{i=1}^{3}(\{T_ix_k\}_{k=0}^n \cup \{T_iy_k\}_{k=0}^n \cup \{T_iz_k\}_{k=0}^n)$ and we denote the diameter of A_n by a_n.

Let $b_n = \max_{i=1,2,3}\{\sup_{0\le k\le n} d(x_0,T_ix_k), \sup_{0\le k\le n} d(x_0,T_iy_k), \sup_{0\le k\le n} d(x_0,T_iz_k)\}$ for $n = 1,2,3\ldots$.
We now prove that $a_n = b_n$ for $n = 1,2,\ldots$.
Clearly, $b_n \le a_n$ for $n = 1,2,\ldots$.
Without loss of generality, we assume that $a_n > 0$ for $n = 1,2,\ldots$.
<u>*Case (i)*</u> : $a_n = d(T_ix_k, T_jx_l)$ for $0 \le k,l \le n$ and $i,j = 1,2,3$.
Since $a_n > 0$, we have $M_1(x_k,x_l) > 0$.
Therefore from the inequality (2.2) and Remark 1.2, we have
$a_n = d(T_ix_k,T_jx_l) \le w(M_1(x_k,x_l)) \le w(a_n) < a_n$,
a contradiction.
Therefore $a_n \ne d(T_ix_k,T_jx_l)$.
<u>*Case (ii)*</u> : By proceeding as in *Case* (*i*), it is easy to see that $a_n \ne d(T_ix_k,T_jy_l)$, $a_n \ne d(T_ix_k,T_jz_l)$, $a_n \ne d(T_iy_k,T_jy_l)$, $a_n \ne d(T_iy_k,T_jz_l)$, and $a_n \ne d(T_iz_k,T_jz_l)$ for $0 \le k,l \le n$ and $i,j = 1,2,3$.
<u>*Case (iii)*</u> : $a_n = d(x_k,T_iy_l)$ for $0 \le k,l \le n$ and $i = 1,2,3$.
If $k > 0$ then from the inequality (1.1), we have

$$\begin{aligned} a_n = d(x_k,T_iy_l) &= d(W(T_3y_{k-1},x_{k-1},\alpha_{k-1}),T_iy_l) \\ &\le \alpha_{k-1}d(T_3y_{k-1},T_iy_l) + (1-\alpha_{k-1})d(x_{k-1},T_iy_l) \\ &\le \max\{d(T_3y_{k-1},T_iy_l), d(x_{k-1},T_iy_l)\} \le a_n \text{ so that} \end{aligned}$$

$a_n = d(T_3y_{k-1},T_iy_l)$ or $a_n = d(x_{k-1},T_iy_l)$.
By *Case* (*ii*), $a_n \ne d(T_3y_{k-1},T_iy_l)$ and hence we have $a_n = d(x_{k-1},T_iy_l)$.
On continuing this process, we have $a_n = d(x_0,T_iy_l)$ so that $a_n \le b_n$.
<u>*Case (iv)*</u> : Either $a_n = d(x_k,T_ix_l)$ or $a_n = d(x_k,T_iz_l)$ for $0 \le k,l \le n$ and $i = 1,2,3$.
By proceeding as in *Case* (*iii*), it follows that $a_n \le b_n$.
<u>*Case (v)*</u> : $a_n = d(x_k,x_l)$ for $0 \le k,l \le n$.
Since $a_n > 0$, we have $k \ne l$. So, without loss of generality, we assume that $k < l$.
Therefore
$a_n = d(x_k, W(T_3y_{l-1},x_{l-1},\alpha_{l-1})) \le \alpha_{l-1}d(x_k,T_3y_{l-1}) + (1-\alpha_{l-1})d(x_k,x_{l-1})$

$\leq \max\{d(x_k, T_3y_{l-1}), d(x_k, x_{l-1})\} \leq a_n$ so that
either $a_n = d(x_k, T_3y_{l-1})$ or $a_n = d(x_k, x_{l-1})$.
If $a_n = d(x_k, x_{l-m})$ for every $1 \leq m \leq l-k$ then $a_n = 0$,
a contradiction.
Therefore $a_n = d(x_k, T_3y_{l-m})$ for some $1 \leq m \leq l-k$ and hence
$a_n \leq b_n$ follows from *Case (iii)*.

$\underline{Case\ (vi)}$: $a_n = d(x_k, y_l)$ for some $0 \leq k, l \leq n$.

$$\begin{aligned} a_n = d(x_k, W(T_2z_l, x_l, \beta_l)) &\leq \beta_l d(x_k, T_2z_l) + (1-\beta_l)d(x_k, x_l) \\ &\leq \max\{d(x_k, T_2z_l), d(x_k, x_l)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(x_k, T_2z_l)$ or $a_n = d(x_k, x_l)$.
Now by *Case (iv)* and *Case (v)*, it follows that $a_n \leq b_n$.

$\underline{Case\ (vii)}$: $a_n = d(x_k, z_l)$ for some $0 \leq k, l \leq n$.

$$\begin{aligned} a_n = d(x_k, W(T_1x_l, x_l, \gamma_l)) &\leq \gamma_l d(x_k, T_1x_l) + (1-\gamma_l)d(x_k, x_l) \\ &\leq \max\{d(x_k, T_1x_l), d(x_k, x_l)\} \leq a_n \text{ so that} \end{aligned}$$

either $a_n = d(x_k, T_1x_l)$ or $a_n = d(x_k, x_l)$.
Therefore by *Case (iv)* and *Case (v)*, we have $a_n \leq b_n$.

$\underline{Case\ (viii)}$: $a_n = d(y_k, T_ix_l)$ for $0 \leq k, l \leq n$ and $i = 1, 2, 3$.

$$\begin{aligned} a_n = d(W(T_2z_k, x_k, \beta_k), T_ix_l) &\leq \beta_k d(T_2z_k, T_ix_l) + (1-\beta_k)d(x_k, T_ix_l) \\ &\leq \max\{d(T_2z_k, T_ix_l), d(x_k, T_ix_l)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(T_2z_k, T_ix_l)$ or $a_n = d(x_k, T_ix_l)$.
Hence by *Case (ii)* and *Case (iv)*, we have $a_n \leq b_n$.

$\underline{Case\ (ix)}$: Either $a_n = d(y_k, T_iy_l)$ or $a_n = d(y_k, T_iz_l)$ for $0 \leq k, l \leq n$ and $\underline{i = 1, 2, 3.}$
By proceeding as in *Case (viii)*, it is easy to see that $a_n \leq b_n$.

$\underline{Case\ (x)}$: $a_n = d(y_k, y_l)$ for $0 \leq k, l \leq n$.

$$\begin{aligned} a_n = d(y_k, y_l) = d(y_k, W(T_2z_l, x_l, \beta_l)) &\leq \beta_l d(y_k, T_2z_l) + (1-\beta_l)d(y_k, x_l) \\ &\leq \max\{d(y_k, T_2z_l), d(y_k, x_l)\} \leq a_n \text{ so that} \end{aligned}$$

either $a_n = d(y_k, T_2z_l)$ or $a_n = d(x_l, y_k)$.
Hence $a_n \leq b_n$ follows from *Case (ix)* and *Case (vi)*.

$\underline{Case\ (xi)}$: $a_n = d(y_k, z_l)$ for $0 \leq k, l \leq n$.

$$\begin{aligned} a_n = d(y_k, W(T_1x_l, x_l, \gamma_l)) &\leq \gamma_l d(y_k, T_1x_l) + (1-\gamma_l)d(y_k, x_l) \\ &\leq \max\{d(y_k, T_1x_l), d(y_k, x_l)\} \leq a_n \text{ so that} \end{aligned}$$

either $a_n = d(y_k, T_1x_l)$ or $a_n = d(x_l, y_k)$.
By *Case (viii)* and *Case (vi)*, we have $a_n \leq b_n$.

$\underline{Case\ (xii)}$: $a_n = d(z_k, T_ix_l)$ for $0 \leq k, l \leq n$ and $i = 1, 2, 3$.

$$\begin{aligned} a_n = d(z_k, T_ix_l) = d(W(T_1x_k, x_k, \gamma_k), T_ix_l) &\leq \gamma_k d(T_1x_k, T_ix_l) + (1-\gamma_k)d(x_k, T_ix_l) \\ &\leq \max\{d(T_1x_k, T_ix_l), d(x_k, T_ix_l)\} \leq a_n \end{aligned}$$

so that either $a_n = d(T_1x_k, T_ix_l)$ or $d(x_k, T_ix_l)$.
Therefore by using *Case (i)* we have $a_n \neq d(T_1x_k, T_ix_l)$ and hence $a_n = d(x_k, T_ix_l)$.
Now by *Case (iv)*, it follows that $a_n \leq b_n$.

$\underline{Case\ (xiii)}$: Either $a_n = d(z_k, T_iy_l)$ or $a_n = d(z_k, T_iz_l)$ for $0 \leq k, l \leq n$ and $\underline{i = 1, 2, 3.}$
By proceeding as in *Case (xii)*, it is easy to see that $a_n \leq b_n$.

$\underline{Case\ (xiv)}$: $a_n = d(z_k, z_l)$ for $0 \leq k, l \leq n$.

$$\begin{aligned} a_n = d(z_k, z_l) = d(z_k, W(T_1x_l, x_l, \gamma_l)) &\leq \gamma_l d(z_k, T_1x_l) + (1-\gamma_l)d(z_k, x_l) \\ &\leq \max\{d(z_k, T_1x_l), d(z_k, x_l)\} \leq a_n \end{aligned}$$

so that $a_n = d(z_k, T_1x_l)$ or $a_n = d(z_k, x_l)$.

By *Case* (xii) and *Case* (vii), we have $a_n \leq b_n$.
Hence by considering all the above cases, we have $a_n = b_n$ for $n = 1, 2, 3...$.

We write $A = \max_{i=1,2,3}\{d(x_0, T_i x_0)\}$. Without loss of generality, we assume that $A > 0$. Now by using the inequality (2.2), we have
$d(x_0, T_i x_k) \leq d(x_0, T_i x_0) + d(T_i x_0, T_i x_k) \leq A + w(a_n)$ for $0 \leq k \leq n$ and $i = 1, 2, 3$.
Therefore $\sup_{0\leq k\leq n}\{d(x_0, T_i x_k)\} \leq A + w(a_n)$ for $i = 1, 2, 3$.
Similarly, we have $\sup_{0\leq k\leq n}\{d(x_0, T_i y_k)\} \leq A + w(a_n)$ and
$\sup_{0\leq k\leq n}\{d(x_0, T_i z_k)\} \leq A + w(a_n)$ for $i = 1, 2, 3$ so that

$$b_n \leq A + w(a_n).$$

Since $a_n = b_n$, we have

$$a_n - w(a_n) \leq A \text{ for } n = 1, 2, ... \; . \tag{2.7}$$

If the sequence $\{a_n\}$ is not bounded then $\lim_{n\to\infty} a_n = \infty$ and hence it follows from (2.5) that $\lim_{n\to\infty}(a_n - w(a_n)) = \infty$ which contradicts (2.7).

Therefore the sequence $\{a_n\}$ is bounded and hence the conclusion of the lemma follows. □

Theorem 2.2. *Let (X, d, W) be a complete convex metric space and K be a nonempty closed convex subset of X. Let $T_1, T_2, T_3 : K \to K$ be self maps satisfying the inequality*

$$\max_{i,j=1,2,3}\{d(T_i x, T_j y)\} \leq w(M_1(x, y)) \textit{ for } x, y \in K \textit{ with } M_1(x, y) > 0,$$

where $M_1(x, y)$ is defined by (2.3) and let $w : (0, \infty) \to (0, \infty)$ be a map that satisfies the relations (2.4), (2.5), and (2.6). Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by the Noor iteration procedure associated with three self maps (2.1) converges strongly to a unique common fixed point of T_1, T_2 and T_3.

Proof. Without loss of generality, we assume that $x_n \neq T_i x_n$ for any $n = 0, 1, 2, ...$ and $i = 1, 2, 3$.
For every integer $n \geq 0$, we define a set C_n by
$C_n = \{x_k\}_{k=n}^{\infty} \cup \{y_k\}_{k=n}^{\infty} \cup \{z_k\}_{k=n}^{\infty} \cup \bigcup_{i=1}^{3}(\{T_i x_k\}_{k=n}^{\infty} \cup \{T_i y_k\}_{k=n}^{\infty} \cup \{T_i z_k\}_{k=n}^{\infty})$, and we define c_n to be the diameter of C_n.
By Lemma 2.1, we have the sequence $\{c_n\}$ is bounded.
Let $d_n = \max_{i=1,2,3}\{\sup_{k\geq n} d(x_n, T_i x_k), \sup_{k\geq n} d(x_n, T_i y_k), \sup_{k\geq n} d(x_n, T_i z_k)\}$ for $n = 0, 1, 2, ...$.

Now, we prove that $c_n = d_n$ for $n = 0, 1, ...$.
Without loss of generality, we assume that $c_n > 0$.
By using the same technique discussed in Lemma 2.1, it is easy to see that $c_n \leq d_n$.
Therefore

$$c_n = d_n \textit{ for } n = 0, 1, 2, ... \; .$$

Since $\{c_n\}$ is a decreasing sequence of nonnegative real numbers, we have $\lim_{n\to\infty} c_n = c$ for some $c \geq 0$.

Now we prove that $c = 0$. On the contrary, we assume that $c > 0$.
Therefore $c_n > 0$ for $n = 0, 1, 2, \dots$.
Let n be a positive integer and $k \geq n$. For $i = 1, 2, 3$, we have

$$\begin{aligned} d(x_n, T_i x_k) &= d(W(T_3 y_{n-1}, x_{n-1}, \alpha_{n-1}), T_i x_k) \\ &\leq \alpha_{n-1} d(T_3 y_{n-1}, T_i x_k) + (1 - \alpha_{n-1}) d(x_{n-1}, T_i x_k) \\ &\leq \alpha_{n-1} w(M_1(y_{n-1}, x_k)) + (1 - \alpha_{n-1}) d(x_{n-1}, T_i x_k) \\ &\qquad\qquad (\text{since } M_1(y_{n-1}, x_k) > 0) \\ &\leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1} \text{ so that} \end{aligned}$$

$\sup_{k \geq n} d(x_n, T_i x_k) \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1}$.
Similarly, we can show that $\sup_{k \geq n} d(x_n, T_i y_k) \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1}$ and $\sup_{k \geq n} d(x_n, T_i z_k) \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1}$.
Therefore

$$d_n \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1} \text{ for } n = 1, 2, \dots .$$

Since $c_n = d_n$, we have

$$\alpha_{n-1}(c_{n-1} - w(c_{n-1})) \leq c_{n-1} - c_n \text{ for } n = 1, 2, \dots . \tag{2.8}$$

Let $s = \inf\{c_n - w(c_n) : n \geq 0\}$. If $s = 0$ then there exists a subsequence $\{c_{n(k)}\}$ of the sequence $\{c_n\}$ such that $\lim_{k \to \infty} (c_{n(k)} - w(c_{n(k)})) = 0$, i.e., $c - w(c^+) = 0$ which is absurd due to (2.6).

Hence $s > 0$ and $c_n - w(c_n) \geq s$ for $n = 0, 1, 2, \dots$.

It follows from the inequality (2.8) that $s\alpha_{n-1} \leq c_{n-1} - c_n$ for $n = 1, 2, \dots$.

Now by applying the comparison test, it follows that the series $\sum \alpha_n < \infty$, a contradiction.
Therefore $c = 0$ so that the sequence $\{x_n\}$ is Cauchy and hence by the completeness of X, there exists $x \in K$ such that $\lim_{n \to \infty} x_n = x$.
Since $c = 0$, we have $\lim_{n \to \infty} d(x_n, T_i x_n) = 0$ so that $\lim_{n \to \infty} T_i x_n = x$ for $i = 1, 2, 3$.

We now prove that x is a common fixed point of T_1, T_2 and T_3. For this purpose, we let $B = \max_{i=1,2,3} \{d(x, T_i x)\}$. Suppose that $B > 0$ so that $M_1(x_n, x) > 0$ for all n.
Now, $d(T_i x_n, T_i x) \leq \max_{i,j=1,2,3} \{d(T_i x_n, T_j x)\} \leq w(M_1(x_n, x))$ for $i = 1, 2, 3$.
On letting $n \to \infty$, we have $d(x, T_i x) \leq w(B^+)$ for $i = 1, 2, 3$ so that $B \leq w(B^+)$, a contradiction.
Therefore $B = 0$ so that x is a common fixed point of T_1, T_2 and T_3.

Clearly, the uniqueness of common fixed point of T_1, T_2, T_3 follows from Remark 1.2. □

If $T_1 = T_2 = T_3$ in Theorem 2.2 then we have the following corollary.

Corollary 2.3. *Let (X, d, W) be a complete convex metric space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a map that satisfies*

$$d(Tx, Ty) \leq w(M(x, y)) \text{ for } x, y \in K \text{ with } M(x, y) > 0,$$

where $M(x, y)$ is defined by (1.3) and $w : (0, \infty) \to (0, \infty)$ be a map that satisfy the relations (2.4), (2.5) and (2.6). Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by the 'modified Noor iteration procedure (1.11)' converges strongly to a unique fixed point of T.

The following is an easy consequence of Corollary 2.3 and Remark 1.3.

Corollary 2.4. *Let $(X, ||.||)$ be a Banach space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a quasi-contraction map, i.e., T satisfies the inequality (1.2). Let $\{\alpha_n\}_{n=o}^{\infty}$, $\{\beta_n\}_{n=o}^{\infty}$ and $\{\gamma_n\}_{n=o}^{\infty}$ be sequences in $[0,1]$ such that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. Then for any $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the Noor iteration procedure (1.10) converges strongly to a unique fixed point of T.*

The following is an example in support of Theorem 2.2.

Example 2.1. *Let $X = [0,2]$ be equipped with the usual norm on the set of all real numbers. We define $W : X \times X \times [0,1] \to X$ by $W(x,y,\lambda) = (1-\lambda)y + \lambda x$ for $x, y \in X$ so that (X,d,W) is a complete convex metric space. Let $K = [\frac{7}{12}, \frac{95}{84}]$ so that K is a closed convex subset of X and we define $T_1, T_2, T_3 : K \to K$ by*

$$T_1x = \begin{cases} \frac{1}{x} - x & \text{if } x \in [\frac{7}{12}, \frac{1}{\sqrt{2}}] \\ \frac{1}{\sqrt{2}} & \text{if } x \in (\frac{1}{\sqrt{2}}, \frac{95}{84}], \end{cases}$$

$$T_2x = \begin{cases} \frac{3}{5} + \frac{\frac{3}{5}\sqrt{2}-1}{95\sqrt{2}-84}(84x-95) & \text{if } x \in [\frac{7}{12}, \frac{1}{\sqrt{2}}] \\ \frac{1}{\sqrt{2}} & \text{if } x \in (\frac{1}{\sqrt{2}}, \frac{95}{84}], \text{ and} \end{cases}$$

$$T_3x = \begin{cases} \frac{7}{10} + \frac{\frac{7}{10}\sqrt{2}-1}{95\sqrt{2}-84}(84x-95) & \text{if } x \in [\frac{7}{12}, \frac{1}{\sqrt{2}}] \\ \frac{1}{\sqrt{2}} & \text{if } x \in (\frac{1}{\sqrt{2}}, \frac{95}{84}]. \end{cases}$$

Here, we note that $T_1x \geq T_2x \geq T_3x \geq \frac{1}{\sqrt{2}}$ for $x \in [\frac{7}{12}, \frac{95}{84}]$, $F = \bigcap\limits_{i=1}^{3} F(T_i) = \{\frac{1}{\sqrt{2}}\}$, and T_1, T_2 and T_3 are decreasing functions on $[\frac{7}{12}, \frac{95}{84}]$.

We define $w : (0,\infty) \to (0,\infty)$ by $w(t) = \frac{9t}{10}$ so that w satisfies the relations (2.4), (2.5) and (2.6). In the following, we show that the inequality (2.2) holds. For this purpose, we consider the following three cases.

Case (i) : $\frac{7}{12} \leq x < y \leq \frac{1}{\sqrt{2}}$.

In this case, $M_1(x,y) = d(x, T_1x) = \frac{1}{x} - 2x$ and

$$\begin{aligned} \max_{i,j=1,2,3}\{d(T_ix, T_jy)\} &= d(T_1x, T_3y) \\ &\leq d(T_1x, \tfrac{1}{\sqrt{2}}) = \tfrac{1}{x} - x - \tfrac{1}{\sqrt{2}} \leq \tfrac{9}{10}(\tfrac{1}{x} - 2x) = w(M_1(x,y)). \end{aligned}$$

Case (ii) : $\frac{7}{12} \leq x < \frac{1}{\sqrt{2}} \leq y \leq \frac{95}{84}$.

Here, we have $\max\limits_{i,j=1,2,3}\{d(T_ix, T_jy)\} = d(T_1x, \frac{1}{\sqrt{2}}) = \frac{1}{x} - x - \frac{1}{\sqrt{2}}$ and

$$M_1(x,y) = \begin{cases} \frac{1}{x} - 2x & \text{if } T_1x \geq y \\ y - x & \text{if } T_1x \leq y. \end{cases}$$

If $T_1x \leq y$ then
$$\begin{aligned} \max_{i,j=1,2,3}\{d(T_ix, T_jy)\} &= \tfrac{1}{x} - x - \tfrac{1}{\sqrt{2}} \leq \tfrac{9}{10}(\tfrac{1}{x} - 2x) \\ &\leq \tfrac{9}{10}(y - x) = w(M_1(x,y)). \end{aligned}$$

Similarly, we show that the inequality (2.2) is true if $T_1x \leq y$.

Case (iii) : $\frac{1}{\sqrt{2}} \leq x < y \leq \frac{95}{84}$.

In this case, $\max\limits_{i,j=1,2,3}\{d(T_ix, T_jy)\} = 0$ and hence the inequality (2.2) trivially holds.

We choose $\beta_n = \gamma_n = \frac{1}{2}$ and $\alpha_n = \frac{1}{n+2}$ for $n = 0, 1, 2, \ldots$ so that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

Let $x_0 \in [\frac{7}{12}, \frac{95}{84}]$ *be arbitrary, and let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence generated by Noor iteration procedure associated with* $T_1, T_2,$ *and* T_3*, i.e, the sequence* $\{x_n\}_{n=0}^{\infty}$ *is defined by* (2.1) *so that* $z_n = W(T_1x_n, x_n, \gamma_n) = \frac{x_n + T_1x_n}{2}$, $y_n = W(T_2z_n, x_n, \beta_n) = \frac{x_n + T_2z_n}{2}$ *and* $x_{n+1} = W(T_3y_n, x_n, \alpha_n) = \frac{n+1}{n+2}x_n + \frac{1}{n+2}T_3y_n$ *for* $n = 0, 1, 2, \dots$.

We now show that the sequence $\{x_n\}_{n=0}^{\infty}$ *converges to* $\frac{1}{\sqrt{2}}$ *which is the common fixed point of* T_1, T_2 *and* T_3.

Case (i) : $\frac{7}{12} \le x_0 < \frac{1}{\sqrt{2}}$.

By induction on n*, we show that*

$x_{n+1} - \frac{1}{\sqrt{2}} = (\frac{n+1}{n+2} + \frac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)})(x_n - \frac{1}{\sqrt{2}})$ *and* $x_n < \frac{1}{\sqrt{2}}$ *for all* $n \ge 0$.

We assume that $x_n < \frac{1}{\sqrt{2}}$ *for some* $n \ge 0$ *so that* $z_n = \frac{x_n + T_1x_n}{2} = \frac{1}{2x_n} > \frac{1}{\sqrt{2}}$, $y_n = \frac{x_n + T_2z_n}{2} = \frac{1}{2}(x_n + \frac{1}{\sqrt{2}}) < \frac{1}{\sqrt{2}}$ *and*

$$\begin{aligned} x_{n+1} &= \tfrac{n+1}{n+2}x_n + \tfrac{1}{n+2}T_3y_n = \tfrac{n+1}{n+2}x_n + \tfrac{1}{n+2}(\tfrac{7}{10} + \tfrac{\frac{7}{10}\sqrt{2}-1}{95\sqrt{2}-84}(84y_n - 95)) \\ &= \tfrac{n+1}{n+2}x_n + \tfrac{1}{n+2}(\tfrac{7}{10} + \tfrac{\frac{7}{10}\sqrt{2}-1}{95\sqrt{2}-84}(42(x_n + \tfrac{1}{\sqrt{2}}) - 95)) \\ &= (\tfrac{n+1}{n+2} + \tfrac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)})x_n + \tfrac{1}{n+2}(\tfrac{7}{10} + \tfrac{42(\frac{7}{10}\sqrt{2}-1)}{\sqrt{2}(95\sqrt{2}-84)} - \tfrac{95(\frac{7}{10}\sqrt{2}-1)}{95\sqrt{2}-84}) \\ &= (\tfrac{n+1}{n+2} + \tfrac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)})(x_n - \tfrac{1}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}} + \tfrac{1}{n+2}B_n \end{aligned}$$

where $B_n = -\frac{1}{\sqrt{2}} + \frac{42(\frac{7}{10}\sqrt{2}-1)}{\sqrt{2}(95\sqrt{2}-84)} + \frac{7}{10} + \frac{42(\frac{7}{10}\sqrt{2}-1)}{\sqrt{2}(95\sqrt{2}-84)} - \frac{95(\frac{7}{10}\sqrt{2}-1)}{(95\sqrt{2}-84)} = 0$ *so that*

$$x_{n+1} - \frac{1}{\sqrt{2}} = A_n(x_n - \frac{1}{\sqrt{2}}) \tag{2.9}$$

where $A_n = (\frac{n+1}{n+2} + \frac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)})$.

Since $0 < A_n < 1$*, we have* $x_{n+1} < \frac{1}{\sqrt{2}}$.

Thus, by induction on n*, we have* $x_n < \frac{1}{\sqrt{2}}$ *and the equation* (2.9) *is true for* $n = 0, 1, 2, \dots$.

By (2.9)*, we have*

$$|x_{n+1} - \frac{1}{\sqrt{2}}| = (\prod_{i=0}^{n} A_i)|x_0 - \frac{1}{\sqrt{2}}| \text{ for } n = 0, 1, 2, \dots . \tag{2.10}$$

Since $1 - A_n = \frac{1}{n+2} - \frac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)} > \frac{1}{n+2}$ *for* $n = 0, 1, 2, \dots$*, we have the series* $\sum\limits_{n=0}^{\infty}(1 - A_n) = \infty$ *so that* $\lim\limits_{n\to\infty} \prod\limits_{i=0}^{n} A_i = 0$ *and hence* $\lim\limits_{n\to\infty} x_n = \frac{1}{\sqrt{2}}$.

Case (ii) : $\frac{1}{\sqrt{2}} < x_0 \le \frac{95}{84}$.

In this case, we show that $x_{n+1} - \frac{1}{\sqrt{2}} = \frac{n+1}{n+2}(x_n - \frac{1}{\sqrt{2}})$ *and* $x_n > \frac{1}{\sqrt{2}}$ *for all* $n \ge 0$.

We assume that $x_n > \frac{1}{\sqrt{2}}$ *for some* $n \ge 0$ *so that*

$z_n = \frac{x_n + T_1x_n}{2} = \frac{1}{2}(x_n + \frac{1}{\sqrt{2}}) > \frac{1}{\sqrt{2}}$, $y_n = \frac{x_n + T_2y_n}{2} = \frac{1}{2}(x_n + \frac{1}{\sqrt{2}}) > \frac{1}{\sqrt{2}}$*, and*

$$\begin{aligned} x_{n+1} &= (1 - \tfrac{1}{n+2})x_n + \tfrac{1}{n+2}T_3y_n \\ &= \tfrac{n+1}{n+2}x_n + \tfrac{1}{n+2}\tfrac{1}{\sqrt{2}} = \tfrac{n+1}{n+2}(x_n - \tfrac{1}{\sqrt{2}}) + \tfrac{n+1}{n+2}\tfrac{1}{\sqrt{2}} + \tfrac{1}{(n+2)}\tfrac{1}{\sqrt{2}}. \\ &= \tfrac{n+1}{n+2}(x_n - \tfrac{1}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}} \text{ so that} \end{aligned}$$

$$x_{n+1} - \frac{1}{\sqrt{2}} = \frac{n+1}{n+2}(x_n - \frac{1}{\sqrt{2}}) \tag{2.11}$$

and hence $x_{n+1} > \frac{1}{\sqrt{2}}$.

Therefore, by induction on n, we have $x_{n+1} - \frac{1}{\sqrt{2}} = \frac{n+1}{n+2}(x_n - \frac{1}{\sqrt{2}})$ *and* $x_n > \frac{1}{\sqrt{2}}$ *for* $n = 0, 1, 2, \ldots$ *so that* $|x_{n+1} - \frac{1}{\sqrt{2}}| = \frac{1}{n+2}|x_0 - \frac{1}{\sqrt{2}}|$ *for* $n = 0, 1, 2, \ldots$ *and hence the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges to* $\frac{1}{\sqrt{2}}$.

Hence the maps T_1, T_2 *and* T_3 *satisfy all the hypotheses of* Theorem 2.2 *and for any* $x_0 \in [\frac{7}{12}, \frac{95}{84}]$, *the Noor iteration procedure associated with* T_1, T_2 *and* T_3, *converges to the unique common fixed point* $\frac{1}{\sqrt{2}}$ *of* T_1, T_2 *and* T_3.

We use MATLAB 13 software to find out the number of iterations at which the sequence $\{x_n\}_{n=0}^{\infty}$ *converges to the common fixed point* $\frac{1}{\sqrt{2}}$ *of* T_1, T_2 *and* T_3.

Table 1. $x_0 = 0.6$, $\alpha_n = \frac{1}{n+2}, \beta_n = \frac{1}{2} = \gamma_n$

No. of iterations (n)	x_n	y_n	z_n
0	*0.6*	*0.65355391*	*.8333333333*
1	*0.636001577*	*0.671554179*	*0.786161573*
50	*0.703107130*	*0.705106956*	*0.711129184*
5000	*0.707066766*	*0.707086774*	*0.707146798*
50000	*0.707102855*	*0.707104818*	*0.707110708*
100000	*0.707104829*	*0.707105805*	*0.707108733*
150000	*0.707105484*	*0.707106133*	*0.707108078*
194105	**0.707105781**	*0.707106281*	*0.707107781*

The 194105th *iteration has got the value of* $x_n = 0.707105781$ *which approximates the common fixed point* $\frac{1}{\sqrt{2}}$ *of* T_1, T_2 *and* T_3 *with an error less than* 10^{-5}.

Remark 2.1. *If we choose* $\gamma_n \equiv 0$, *and* $T_1 = T_2$ *in Theorem 2.2 then Theorem 1.3 follows as a corollary to Theorem 2.2. Hence our result (Theorem 2.2) extends Theorem 1.3 to three self maps.*

3. Convergence of Abbas and Nazir iteration

We now define Abbas and Nazir iteration procedure associated with three self maps T_1, T_2 and T_3 in convex metric spaces as follows: For any $x_0 \in K$,

$$\begin{aligned} z_n &= W(T_1x_n, x_n, \gamma_n) \\ y_n &= W(T_2z_n, T_2x_n, \beta_n) \\ x_{n+1} &= W(T_3z_n, T_3y_n, \alpha_n) \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$.

Theorem 3.1. *Let* (X, d, W) *be a complete convex metric space and* K, *a nonempty closed convex subset of* X. *Let* $T_1, T_2, T_3 : K \to K$ *be self maps of* K *that satisfy*

$$\max_{i,j=1,2,3}\{d(T_ix, T_jy)\} \leq w(M_1(x, y)) \text{ for } x, y \in K \text{ with } M_1(x, y) > 0,$$

where $M_1(x, y)$ *is defined by* (2.3) *and* $w : (0, \infty) \to (0, \infty)$ *is a map that satisfies the relations* (2.4), (2.5), *and* (2.6). *Let* $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, *and* $\{\gamma_n\}_{n=0}^{\infty}$ *be sequences in* $[0, 1]$. *Then the sequence* $\{x_n\}$ *generated by Abbas and Nazir iteration procedure associated with three self maps* (3.1) *converges strongly to a unique common fixed point of* T_1, T_2 *and* T_3.

Proof. By using the same technique discussed in Lemma 2.1 and Theorem 2.2 of Section 2, it is easy to see that the diameter c_n of the set $C_n = \{x_k\}_{k\geq n}\cup\{y_k\}_{k\geq n}\cup\{z_k\}_{k\geq n}\cup\bigcup_{i=1}^{3}(\{T_ix_k\}_{k\geq n}\cup\{T_iy_k\}_{k\geq n}\cup\{T_iz_k\}_{k\geq n})$ is equal to $d_n = \max\limits_{i=1,2,3}\{\sup\limits_{k\geq n} d(x_n, T_ix_k), \sup\limits_{k\geq n} d(x_n, T_iy_k), \sup\limits_{k\geq n} d(x_n, T_iz_k)\}$ for $n = 0, 1, 2, ...$ and $\lim\limits_{n\to\infty} c_n = c$ for some $c \geq 0$.

We now prove that $c = 0$. On the contrary, we suppose that $c > 0$ so that $c_n > 0$ for $n = 0, 1, 2, ...$.

For a positive integer n, let $k \geq n$. Then for $i = 1, 2, 3$ we have

$$d(x_n, T_ix_k) = d(W(T_3z_{n-1}, T_3y_{n-1}, \alpha_{n-1}), T_ix_k) \leq \alpha_{n-1}d(T_3z_{n-1}, T_ix_k) + (1-\alpha_{n-1})d(T_3y_{n-1}, T_ix_k) \leq w(c_{n-1}).$$

Therefore $\sup\limits_{k\geq n} d(x_n, T_ix_k) \leq w(c_{n-1})$ for $i = 1, 2, 3$ and $n = 1, 2, 3, ...$.

Similarly, $\sup\limits_{k\geq n} d(x_n, T_iy_k) \leq w(c_{n-1})$ and $\sup\limits_{k\geq n} d(x_n, T_iz_k) \leq w(c_{n-1})$ for $i = 1, 2, 3$ and $n = 1, 2, ...$ so that

$$c_n = d_n \leq w(c_{n-1}).$$

On letting $n \to \infty$, we have $c \leq w(c^+)$,
a contradiction.

Therefore $c = 0$ and hence the conclusion of the theorem follows from the lines of the proof of Theorem 2.2. □

Corollary 3.2. *Let (X, d, W) be a complete convex metric space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a map such that*

$$d(Tx, Ty) \leq w(M(x, y)) \text{ for } x, y \in K \text{ with } M(x, y) > 0,$$

where $M(x, y)$ is defined by (1.3) and $w : (0,\infty) \to (0,\infty)$ is a map that satisfies the relations (2.4), (2.5), and (2.6). Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$. Then the sequence $\{x_n\}$ generated by the modified Abbas and Nazir iteration procedure (1.13) converges strongly to a unique fixed point of T.

Corollary 3.3. *Let $(X, ||.||)$ be a Banach space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a quasi-contraction map, i.e., T satisfies the inequality (1.2). Let $\{\alpha_n\}_{n=o}^{\infty}$, $\{\beta_n\}_{n=o}^{\infty}$ and $\{\gamma_n\}_{n=o}^{\infty}$ be sequences in $[0, 1]$. Then for any $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Abbas and Nazir iteration procedure (1.12) converges strongly to a unique fixed point of T.*

The following example is in support of Theorem 3.1.

Example 3.1. *Let X, K, T_1, T_2 and T_3 be as in Example 2.1. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be arbitrary sequences in $[0, 1]$. Let $x_0 \in K$ and $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by (3.1) so that $z_n = (1-\gamma_n)x_n + \gamma_nT_1x_n$,*
$y_n = (1-\beta_n)T_2x_n + \beta_nT_2z_n$ and $x_{n+1} = (1-\alpha_n)T_3y_n + \alpha_nT_3z_n$ for $n = 0, 1, 2, ...$. Here we note that T_1, T_2 and T_3 satisfy all the hypotheses of Theorem 3.1. *Further, it is easy to see that $x_1 \geq \frac{1}{\sqrt{2}}$ and $x_n = \frac{1}{\sqrt{2}}$ for $n = 2, 3, ...$ so that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the common fixed point $\frac{1}{\sqrt{2}}$ of T_1, T_2 and T_3.*

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TRACE REGULARIZATION PROBLEM FOR HIGHER ORDER DIFFERENTIAL OPERATOR

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Abstract. We establish a regularized trace formula for higher order self-adjoint differential operator with unbounded operator coefficient.

1. Introduction and History

The first study on the regularized trace of scalar differential operators was performed by Gelfand and Levitan [10]. They studied the boundary value problem

$$y'' + q(x)\, y = \lambda y, \quad y'(0) = y'(\pi) = 0 \quad \text{with } q(x) \,\epsilon\, C^1[0,\pi]$$

and they found the formula

$$\sum_{n=0}^{\infty} (\lambda_n - \mu_n) = \frac{1}{4}\left(q(0) + q(\pi)\right) \ ,$$

under the assumption $\int_0^\pi q(x)dx = 0$. Where the μ_n are the eigenvalues of this problem. $\lambda_n = n^2$ are the eigenvalues of the same problem with $q(x) = 0$.
After that original work by Gelfand-Levitan, there was a huge interest and many scientists used the same method to obtain the regularized traces of ordinary differential operators. Later, Dikii [5] gave another proof of Gelfand-Levitan's formula from a different point of view. Afterward, Dikii [6] and Gelfand [9] made significant progress in literature by computing regularized sums of powers of eigenvalues. Later on, Levitan [17] calculated the regularized traces of Sturm Liouville Problem with a new method. This research led to Faddeev [7], who connected the trace theory with singular differential operators. Gasimov [8] made the first study combining singular operators with discrete spectrum.

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Thereafter, many scientists such as Halberg and Kramer [13], Jafaev [15], Makin [19], Yang [23] investigated the regularized traces of various scalar differential operators. The list of these works is given in Levitan and Sargsyan [18] and Sadovnichii and Podolskii [21].
Among the studies, only a few of them are focused on the regularized trace of operator-differential equation with operator coefficient. Halilova [14] obtained the regularized trace of the Sturm-Liouville equation with bounded operator coefficient. Adıgüzelov [1] found a formulation of the subtracting eigenvalues of two self-adjoint operators in $[0,\infty)$ with bounded operator coefficient. Bayramoğlu and Adıgüzelov [4] examined the regularized trace of singular second order differential operator with bounded operator coefficient. Adıgüzelov and Baksi [2], Sen, Bayramov and Oruçoğlu [22] and Adıgüzelov, Avcı and Gül [12] obtained the equalities for the regularized traces of differential operators with bounded operator coefficient. Aslanova [3] calculated the trace formula of Bessel equation with spectral parameter-dependent boundary condition.
Maksudov, Bayramoğlu and Adıgüzelov [20] investigated the regularized trace formulation of the Sturm Liouville equation with unbounded operator coefficient.
In the present paper, we compute the regularized trace formula for higher order Sturm-Liouville problem

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\left(\alpha_q-\beta_q-\frac{1}{\pi}\int_0^\pi (Q(x)\varphi_{j_q},\varphi_{j_q})dx\right) = \frac{1}{4}\left(trQ(0)-trQ(\pi)\right) .$$

2. Notation and Preliminaries

Let H be an infinite dimensional separable Hilbert space with inner product $(.,.)$ and corresponding norm $\|.\|$. Let $H_1 = L_2(0,\pi;\ H)$ be the set of all strongly measurable functions f defined on $[0,\pi]$ and taking the values in the space H. The following conditions hold for every $f \in H_1$:
1. The scalar function $(f(x),g)$ is Lebesgue measurable on $[0,\pi]$, for every $g\in H$,
2. $\int_0^\pi \|f(x)\|^2dx<\infty$.
H_1 is a normed linear space. We will denote the inner product and norm by $(.,.)_{H_1}$ and $\|.\|_{H_1}$ in H_1. If the inner product is defined as $(f_1,f_2)_{H_1} = \int_0^\pi (f_1(x), f_2(x))dx$, for any arbitrary elements f_1, f_2 of H_1, then H_1 becomes a separable Hilbert space, [16]. Let $\{\Phi_q(x)\}_1^\infty$ be an orthonormal basis of H_1.
Consider the following differential expressions

$$\ell_0(v) = (-1)^m v^{(2m)}(x) + Av(x), \qquad (m\in\ \mathbb{Z}^+)$$

$$\ell(v) = (-1)^m v^{(2m)}(x) + Av(x) + Q(x)v(x). \tag{2.1}$$

with boundary conditions

$$v^{(2i+1)}(0) = v^{(2i)}(\pi) = 0, \qquad (i=0,1,\ldots,m-1)$$

in H_1. Here, A is a densely defined operator in H. This operator takes its values in H and satisfies the conditions $A=A^*\geq I$, $A^{-1}\in\sigma_\infty(H)$, where I is the identity operator of H. $\sigma_\infty(H)$ denotes the set of all completely continuous operators from H to H.

Let $\{\gamma_i\}_{i=1}^{\infty}$ be the increasing sequence of eigenvalues of the operator A counted with respect to their multiplicities and a corresponding orthonormal sequence $\{\varphi_i\}_{i=1}^{\infty}$ of eigenvectors.

Denote by $D(L_0^{'})$ the set of the functions $v(x)$ in the space H_1 , and the following conditions are satisfied:

(v1) $v(x)$ has continuous $2m^{\ th}$ order derivative on $[0,\pi]$ with respect to the norm in the space H,

(v2) $v(x) \in D(A)$ for every $x \in [0,\pi]$, and $Av(x)$ is continuous on $[0,\pi]$ with respect to the norm in H,

(v3) $v^{(2i+1)}(0) = v^{(2i)}(\pi) = 0, \qquad (i = 0, 1, 2, \cdots, m-1)$.

Here, $D(L_0^{'})$ is dense in H_1. Define a linear operator $L_0^{'} : D(L_0^{'}) \to H_1$ as $L_0^{'}v = \ell_0(v)$.

The construction above shows that $L_0^{'}$ is symmetric. Considering the linearity of $L_0^{'}$, its eigenvalues can be calculated by mathematical induction. Therefore, the eigenvalues of $L_0^{'}$ are the form $(k+\frac{1}{2})^{2m}+\gamma_j, \qquad (k = 0, 1, 2, \cdots; \quad j = 1, 2, \cdots)$ and the orthonormal eigenvectors corresponding to these eigenvalues are the form $\sqrt{\frac{2}{\pi}}\varphi_j cos\left(k+\frac{1}{2}\right)x$. We can see that the orthonormal eigenvector sequence of the symmetric operator $L_0^{'}$ is a complete orthonormal system in H_1. Since $L_0^{'}$ is symmetric, then it is closable. Thus, we can define L_0 as $L_0 = \overline{L_0^{'}}$

Assume that the operator function $Q(x)$ in 2.1 verifies the conditions:

(*Q1*) $Q(x) : H \to H$ is a self-adjoint operator for every $x \in [0,\pi]$,

(*Q2*) $Q(x)$ is weak measurable on $[0,\pi]$, that is the scalar function $(Q(x)f, g)$ is measurable on $[0,\pi]$ for every $f, g \in H$,

(*Q3*) The function $\|Q(x)\|$ is bounded on $[0,\pi]$.

In the present paper, we establish a regularized trace formula for the operator $L = L_0 + Q$.

Now, we search some inequalities for the eigenvalues and resolvent operators of L_0 and L.

Consider the closed symmetric operator $L_0 : D(L_0) \to H_1$.

Since the eigenvector system $\{\varphi_j \cos\,(\,k+\frac{1}{2})x\}_{k=0,j=1}^{\infty\quad\infty}$ of L_0 is complete, L_0 is self-adjoint, [2]. Moreover, since the bounded operator $Q : H_1 \to H_1$ is self-adjoint, the operator, $L = L_0 + Q$ is also self-adjoint. Therefore, L_0 and L have purely-discrete spectrum, [2]. Let $\{\beta_i\}_{i=1}^{\infty}$ and $\{\alpha_i\}_{i=1}^{\infty}$ be increasing sequences of eigenvalues of L_0 and L. Denote by $\rho(L_0)$ and $\rho(L)$ the resolvent sets of L_0 and L .

We can prove the Theorem 2.1, by using [2].

Theorem 2.1. *Let the operator function* $Q(x)$ *satisfy the conditions* $(Q1)$ *to* $(Q3)$. *If* $\gamma_j \sim aj^{\ell} \qquad (0 < a, \ell < \infty)$ *as* $j \to \infty$, *then* $\alpha_n, \beta_n \sim dn^{\frac{2m\ell}{2m+\ell}}$ *as* $n \to \infty$,

where $d = \left(\frac{\ell a^{\frac{1}{\ell}}}{2b}\right)^{\frac{2m\ell}{2m+\ell}}$ *and* $b = \int_0^{\frac{\pi}{2}} (sint)^{\frac{2}{\ell}-1}(cost)^{1+\frac{1}{m}}dt.$

From Theorem 2.1, one can see that the sequence $\{\beta_n\}$ has a subsequence $\beta_{n_1} < \beta_{n_2} < ... < \beta_{n_p} < ...$ such that

$$\beta_q - \beta_{n_p} > d_0\left(q^{\frac{2m\ell}{2m+\ell}} - n_p^{\frac{2m\ell}{2m+\ell}}\right), \ (q = n_p+1, n_p+2, \cdots). \tag{2.2}$$

Here, d_0 is a positive constant.
Let $R_\alpha^0 = (L_0 - \alpha I)^{-1}$, $R_\alpha = (L - \alpha I)^{-1}$ be the resolvent operators of L_0 and L .
If $\ell > \frac{2m}{2m-1}$, then by Theorem 2.1, R_α^0 and R_α are nuclear operators for $\alpha \neq \alpha_q, \beta_q \quad (q = 1, 2, ...)$. In this case, we have the formula

$$tr\left(R_\alpha - R_\alpha^0\right) = trR_\alpha - trR_\alpha^0 = \sum_{q=1}^{\infty}\left(\frac{1}{\alpha_q - \alpha} - \frac{1}{\beta_q - \alpha}\right), \tag{2.3}$$

[11]. Let $|\alpha| = b_p = 2^{-1}(\beta_{n_p+1} + \beta_{n_p})$. This says that for the large value of p , the inequalities
$\beta_{n_p} < b_p < \beta_{n_p+1}$ and $\alpha_{n_p} < b_p < \alpha_{n_p+1}$ are satisfied. By using the last inequalities, one can prove that the series $\sum_{q=1}^{\infty} \frac{\alpha}{\alpha_q - \alpha}$ *and* $\sum_{q=1}^{\infty} \frac{\alpha}{\beta_q - \alpha}$ are uniform convergent on the circle $|\alpha| = b_p$. Hence by 2.3

$$\sum_{q=1}^{n_p}(\alpha_q - \beta_q) = -\frac{1}{2\pi i}\int_{|\alpha|=b_p} \alpha tr(R_\alpha - R_\alpha^0)d\alpha. \tag{2.4}$$

We have two lemmas by using [2]:

Lemma 2.2. *If* $\gamma_j \sim aj^\ell$ *as* $j \to \infty$ *for* $a > 0, \qquad \ell > \frac{2m}{2m-1}$ *, then*

$$\|R_\alpha^0\|_{\sigma_1(H_1)} < const.n_p^{1-\delta}, \qquad (\delta = \frac{2m\ell}{2m+\ell} - 1),$$

on the circle $|\alpha| = b_p$.

Lemma 2.3. *If the operator function* $Q(x)$ *satisfies conditions (Q1) to (Q3), and*
$\gamma_j \sim aj^\ell$ *as* $j \to \infty$ *, then for the large values of* p

$$\|R_\alpha\|_{H_1} < const.n_p^{-\delta}$$

on the circle $|\alpha| = b_p$ *, where* $a > 0, \qquad \ell > \frac{2m}{2m-1}$.

3. Main Results

In this section, we will compute regularized trace formula for the operator L.
With the well-known formula $R_\alpha = R_\alpha^0 - R_\alpha Q R_\alpha^0 \qquad (\alpha \in \rho(L_0) \cap \rho(L))$ and by 2.4, we obtain:

$$\sum_{q=1}^{n_p}(\alpha_q - \beta_q) = \sum_{j=1}^{s} E_{pj} + E_p^{(s)} \qquad . \tag{3.1}$$

Here,

$$E_{pj} = \frac{(-1)^j}{2\pi i j}\int_{|\alpha|=b_p} tr\left[(QR_\alpha^0)^j\right]d\alpha, \qquad (j = 1, 2, ..), \tag{3.2}$$

$$E_p^{(s)} = \frac{(-1)^s}{2\pi i}\int_{|\alpha|=b_p} \alpha tr\left[R_\alpha(QR_\alpha^0)^{s+1}\right]d\alpha. \tag{3.3}$$

Theorem 3.1. *If the operator function* $Q(x)$ *satisfies conditions (Q1) to (Q3) and* $\gamma_j \sim aj^{\ell}$ *as* $j \to \infty$ *then*

$$\mathop{lim}_{p\to\infty} E_{pj} \;= 0, \qquad\qquad (j = 2, 3, 4, ...),$$

where $a > 0$ *and* $\ell > \frac{2m+2\sqrt{2}m}{2\sqrt{2}m-\sqrt{2}-1}$.

Proof: Substituting p= 2 into 3.2 , we obtain the equality

$$E_{p2} = \frac{1}{2\pi i}\sum_{j=1}^{n_p}\sum_{k=n_p+1}^{\infty}\left(\int_{\alpha=b_p}\frac{d\alpha}{(\alpha-\beta_j)(\alpha-\beta_k)}\right)(Q\Phi_j,\Phi_k)_{H_1}(Q\Phi_k,\Phi_j)_{H_1}. \quad (3.4)$$

It readily follows that

$$|E_{p2}| \le \|Q\|^2_{H_1}\Lambda_p. \quad (3.5)$$

Here, $\Lambda_p = \sum_{k=n_p+1}^{\infty}(\beta_k - \beta_{n_p})^{-1}$, $\qquad (p = 1, 2, \cdots)$.
Using 3.5, we obtain

$$\lim_{p\to\infty} E_{p2} \;= 0, \qquad\qquad \left(\ell > \frac{2m}{2m-1}\right) \;. \quad (3.6)$$

Now, we wish to see that

$$\lim_{p\to\infty} E_{p3} \;= 0. \quad (3.7)$$

By 3.2, we get:

$$\begin{aligned} E_{p3} \;&=\; \sum_{j=1}^{n_p}\sum_{k=1}^{n_p}\sum_{s=n_p+1}^{\infty}[F(j,k,s)+F(s,k,j)+F(j,s,k)] \\ &+\; \sum_{j=1}^{n_p}\sum_{k=n_p+1}^{\infty}\sum_{s=n_p+1}^{\infty}[F(j,k,s)+F(s,k,j)+F(k,j,s)], \end{aligned} \quad (3.8)$$

where,

$$F(j,k,s) = g(j,k,s)(Q\Phi_j,\Phi_k)_{H_1}(Q\Phi_k,\Phi_s)_{H_1}(Q\Phi_s,\Phi_j)_{H_1},$$

$$g\,(j,k,s) = \frac{1}{6\pi i}\int_{|\alpha|=b_p}\frac{1}{(\alpha-\beta_j)\,(\alpha-\beta_k)\,(\alpha-\beta_s)}d\alpha$$

If we consider $g(j,k,s) = \overline{g(j,k,s)}$ and $Q = Q^*$, then

$$F(s,k,j) = \overline{F(j,k,s)}, \qquad F(k,j,s) = \overline{F(j,k,s)}, \qquad F(j,s,k) = \overline{F(j,k,s)}. \quad (3.9)$$

Using 3.8 and 3.9, we obtain

$$E_{p3} = I_1 + I_2 \;, \qquad\qquad , \quad (3.10)$$

$$I_1 = \sum_{j=1}^{n_p}\sum_{k=1}^{n_p}\sum_{s=n_p+1}^{\infty}[F(j,k,s) + 2\overline{F(j,k,s)}],$$

$$I_2 = \sum_{j=1}^{n_p} \sum_{k=n_p+1}^{\infty} \sum_{s=n_p+1}^{\infty} [F(j,k,s) + 2\overline{F\,(j,k,s)}].$$

$$I_1 = I_{11} + 2\overline{I_{11}}, \qquad I_2 = I_{21} + 2\overline{I_{21}} \qquad , \tag{3.11}$$

$$I_{11} = \sum_{j=1}^{n_p} \sum_{k=1}^{n_p} \sum_{s=n_p+1}^{\infty} F(j,k,s),$$

$$I_{21} = \sum_{j=1}^{n_p} \sum_{k=n_p+1}^{\infty} \sum_{s=n_p+1}^{\infty} F(j,k,s).$$

Hence we get:

$$|I_{11}| \leq \frac{1+\delta}{d_0^2 \delta} \|Q\|_{H_1}^3 n_p^{\frac{1-2\delta^2}{1+\delta}} \qquad , \tag{3.12}$$

$$|I_{21}| \leq (\frac{1+\delta}{d_0 \delta})^2 \|Q\|_{H_1}^3 n_p^{-\frac{2\delta^2}{1+\delta}} \qquad , \left(\ell > \frac{2m}{2m-1}\right). \tag{3.13}$$

By 3.10, 3.11, 3.12 and 3.13, we find

$$\lim_{p\to\infty} E_{p3} \ = 0 \qquad\qquad (\ell > \frac{2m + 2\sqrt{2}m}{2\sqrt{2}m - \sqrt{2} - 1}). \tag{3.14}$$

Evaluate the limit $\lim_{p\to\infty} E_{pj}$ $\quad (j = 4, 5, ...)$ to complete the proof:
According to 3.2

$$\begin{aligned} |E_{pj}| &\leq \frac{1}{2\pi j} \int_{|\alpha|=b_p} |tr(QR_\alpha^0)^j| d\alpha \\ &\leq \int_{|\alpha|=b_p} \|(QR_\alpha^0)^j\|_{\sigma_1(H_1)} d\alpha \\ &\leq \int_{|\alpha|=b_p} \|QR_\alpha^0\|_{\sigma_1(H_1)} \|(QR_\alpha^0)^{j-1}\|_{H_1} d\alpha \\ &\leq \|Q\|_{H_1} \int_{|\alpha|=b_p} \|R_\alpha^0\|_{\sigma_1(H_1)} \|QR_\alpha^0\|_{H_1}^{j-1} d\alpha \\ &\leq const. \int_{|\alpha|=b_p} \|R_\alpha^0\|_{\sigma_1(H_1)} \|R_\alpha^0\|_{H_1}^{j-1} d\alpha. \end{aligned} \tag{3.15}$$

Since $R_\alpha = R_\alpha^0$ for $Q(x) \equiv 0$, then according to Lemma 2.3

$$\|R_\alpha^0\|_{(H_1)} < \frac{4}{d_0} n_p^{-\delta}, \qquad\qquad \left(|\alpha| = b_p; \qquad \delta = \frac{2m\ell}{2m+\ell} - 1\right). \tag{3.16}$$

By 3.15, 3.16, and Lemma 2.2, we obtain:

$$|E_{pj}| < const. \int_{|\alpha|=b_p} n_p^{1-\delta} n_p^{-\delta(j-1)} d\alpha < const. b_p n_p^{1-\delta j}.$$

For the large values of p , since $b_p = \frac{1}{2}(\beta_{n_p+1} + \beta_{n_p}) \leq const. n_p^{1+\delta}$, we arrive at the inequality $|E_{pj}| < const. n_p^{2-\delta(j-1)}$.
If $\delta > \frac{2}{3}$ or $\ell > \frac{10m}{6m-5}$, then we have:

$$\lim_{p\to\infty} E_{pj} \;= 0 \qquad (j = 4, 5, ...). \tag{3.17}$$

On the other hand, if $\frac{2m+2\sqrt{2}m}{2\sqrt{2}m-\sqrt{2}-1} > \frac{10m}{6m-5}$, then by 3.6 and 3.14 with $\ell > \frac{2m+2\sqrt{2}m}{2\sqrt{2}m-\sqrt{2}-1}$ give:

$$\lim_{p\to\infty} E_{pj} \;= 0 \quad (j = 2, 3, ...). \tag{3.18}$$

Since the eigenvalues of L_0 are the form $(k+\frac{1}{2})^{2m}+\gamma_j, \quad (k = 0, 1, 2, ...; j = 1, 2, ...)$, we have

$$\beta_q = (k_q + \frac{1}{2})^{2m} + \gamma_{j_q}, \qquad (q = 1, 2, ...). \tag{3.19}$$

Assume that the operator function $Q(x)$ holds the additional conditions:
(Q4) $Q(x)$ has weak H derivatives of the second order on $[0, \pi]$ and the function $(Q(x)'' f, g)$ is continuous for every $f, g \in H$,
(Q5) $Q^{(i)}(x) : H \to H \qquad (i = 0, 1, 2)$ are self-adjoint nuclear operators and the functions $\|Q^{(i)}(x)\|_{\sigma_1(H)} \qquad (i = 0, 1, 2)$ are bounded and measurable on $[0, \pi]$.
Our main result is the following:

Theorem 3.2. *If the operator function $Q(x)$ satisfies the conditions (Q4) to (Q5) and $\gamma_j \sim aj^\ell$ as $j \to \infty$, then we have*

$$\mathit{lim}_{p\to\infty} \sum_{q=1}^{n_p} \left(\alpha_q - \beta_q - \frac{1}{\pi} \int_0^\pi (Q(x)\varphi_{j_q}, \varphi_{j_q}) dx \right) \;= \frac{1}{4} \left(trQ(0) - trQ(\pi) \right) \;, \tag{3.20}$$

where $a > 0, \qquad \ell > \frac{2m+2\sqrt{2}m}{2\sqrt{2}m-\sqrt{2}-1} \qquad j_1, j_2, ...$ are natural numbers satisfying the equality 3.19.
The limit on the left side is called regularized trace of L

Proof: According to the formula given by 3.2

$$E_{p1} = -\frac{1}{2\pi i} \int_{|\alpha|=b_p} tr \left(QR^0_\alpha \right) d\alpha. \tag{3.21}$$

Since QR^0_α is a nuclear operator for every $\alpha \in \rho(L_0)$ and $\{\Phi_q(x)\}_1^\infty$ is an orthonormal basis of H_1 , we have:

$$tr \left(QR^0_\alpha \right) = \sum_{q=1}^{\infty} (QR^0_\alpha \Phi_q, \Phi_q)_{H_1},$$

[11]. Replacing $tr(QR^0_\alpha)$ into the equality 3.21 and considering

$$R^0_\alpha \Phi_q = (L_0 - \alpha I)^{-1} \Phi_q = (\beta_q - \alpha)^{-1} \Phi_q \;,$$

then we obtain

$$\begin{aligned} E_{p1} &= -\frac{1}{2\pi i}\int_{|\alpha|=b_p}\left(\sum_{q=1}^{\infty}(QR_\alpha^0\Phi_q,\Phi_q)_{H_1}\right)d\alpha \\ &= -\frac{1}{2\pi i}\int_{|\alpha|=b_p}[\sum_{q=1}^{\infty}\frac{1}{\beta_q-\alpha}(Q\Phi_q,\Phi_q)_{H_1}]\,d\alpha \\ &= \sum_{q=1}^{\infty}(Q\Phi_q,\Phi_q)_{H_1}\frac{1}{2\pi i}\int_{|\alpha|=b_p}\frac{d\alpha}{\alpha-\beta_q} \end{aligned} \tag{3.22}$$

Since the orthonormal eigenvectors corresponding to the eigenvalues $(k+\frac{1}{2})^{2m}+\gamma_j$ of L_0 are $\sqrt{\frac{2}{\pi}}\varphi_j cos(k+\frac{1}{2})x \qquad (j=1,2,...)$, we have:

$$\Phi_q(x)=\sqrt{\frac{2}{\pi}}\varphi_{j_q}cos(k_q+\frac{1}{2})x \qquad (q=1,2,...). \tag{3.23}$$

According to the Cauchy's integral formula:

$$\frac{1}{2\pi i}\int_{|\alpha|=b_p}\frac{d\alpha}{\alpha-\beta_q}=\begin{cases}1 & ,q\le n_p\\ 0 & ,q>n_p\end{cases} \tag{3.24}$$

Substituting 3.23 and 3.24 in 3.22, we obtain

$$\begin{aligned} E_{p1} &= \sum_{q=1}^{n_p}(Q\Phi_q,\Phi_q)_{H_1} \\ &= \sum_{q=1}^{n_p}\int_0^\pi (Q(x)\Phi_q(x),\Phi_q(x))\,dx \\ &= \sum_{q=1}^{n_p}\int_0^\pi\left(Q(x)\sqrt{\frac{2}{\pi}}\varphi_{j_q}cos(k_q+\frac{1}{2})x,\sqrt{\frac{2}{\pi}}\varphi_{j_q}cos(k_q+\frac{1}{2})x\right)dx \\ &= \frac{2}{\pi}\sum_{q=1}^{n_p}\int_0^\pi cos^2(k_q+\frac{1}{2})x\left(Q(x)\varphi_{j_q},\varphi_{j_q}\right)dx \\ &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi\left(1+cos2(k_q+\frac{1}{2})x\right)(Q(x)\varphi_{j_q},\varphi_{j_q})dx \\ &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi\left(Q(x)\varphi_{j_q},\varphi_{j_q}\right)dx+\frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi cos(2k_q+1)x(Q(x)\varphi_{j_q},\varphi_{j_q})dx \end{aligned}$$

and substituting the last equality in 3.1 , we have

$$\begin{aligned} &\sum_{q=1}^{n_p}\left(\alpha_q-\beta_q-\frac{1}{\pi}\int_0^\pi(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right) \\ &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi cos(2k_q+1)x(Q(x)\varphi_{j_q},\varphi_{j_q})dx+\sum_{j=2}^{s}E_{pj}+E_p^{(s)} \end{aligned} \tag{3.25}$$

If the operator function $Q(x)$ holds the conditions (Q4) and (Q5), the double series

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)cos2kxdx$$

is absolutely convergent. Therefore

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})cos(2k_q+1)xdx$$

$$=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)cos(2k+1)xdx\ . \tag{3.26}$$

Now, let us arrange the expression on the right side of 3.26 as follows:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)cos(2k+1)xdx\\
=\ &\frac{1}{2\pi}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)coskxdx-(-1)^k\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)coskxdx\right)\\
=\ &\frac{1}{4}\sum_{j=1}^{\infty}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)coskxdx\right)cos(k0)\right.\\
-\ &\left.\sum_{k=0}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)coskxdx\right)cos(k\pi)\right\}
\end{aligned}\tag{3.27}$$

The difference of sums according to k on the right side of 3.27 is the difference of the values at 0 and at π of the Fourier series of the function $(Q(x)\varphi_j,\varphi_j)$ having second order derivative according to the functions $\{coskx\}_{k=0}^{\infty}$ on $[0,\pi]$. Hence by 3.26 and 3.27 we find:

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})cos2k_qxdx=\frac{1}{2}\sum_{j=1}^{\infty}((Q(0)\varphi_j,\varphi_j)+(Q(\pi)\varphi_j,\varphi_j))$$

or

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})cos2k_qxdx=\frac{1}{2}\left(trQ(0)+trQ(\pi)\right). \tag{3.28}$$

By using Lemma 2.2 and Lemma 2.3, we get:

$$\lim_{p\to\infty}E_p^{(s)}=0\qquad(s>3\delta^{-1}) \tag{3.29}$$

By 3.25, 3.28, 3.29 and Theorem 3.1, we have the main result for regularized trace as

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\left(\alpha_q-\beta_q-\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right)=\frac{1}{4}\left(trQ\,(0)-trQ\,(\pi)\right)\ .$$

The proof is completed.

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A CONSTRUCTION OF A CONGRUENCE IN A UP-ALGEBRA BY A PSEUDO-VALUATION

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Abstract. In our recently published paper, we study pseudo-valuations on UP-algebras and obtain some related results. In this article, we use a pseudo-metric induced by a pseudo-valuation to introduce a congruence relation on a UP-algebra. In addition, we construct the quotient algebra induced by this relation and prove that it is also a UP-algebra.

1. INTRODUCTION

The idea that universal algebra should be analyzed by means of pseudo-valuation was first developed by D. Busneag in 1996 [1]. This author has expanded the perception of pseudo-valuation on Hilbert's algebras [2]. Logical algebras and pseudo-valuations on them have become an object of interest for researchers in recent years. For example, Doh and Kang [3, 4] introduced in the concept of pseudo-valuation on BCK/BCI - algebras. Ghorbani in 2010 [5] determined a congruence on BCI-algebras based on pseudo-valuation and describe the obtained factorial structure generated by this congruence. Song, Roh and Jun described pseudo-valuation on BCK/BCI - algebras [15] and Song, Bordbar and Jun have described the quotient structure on such algebras generated by pseudo-valuation [16]. Jun, Lee and Song analyzed in article [8] several types of quasi-valuation maps on BCK-algebra and their interactions. Also, Mehrshad and Kouhestani were interested in pseudo-valuations on BCK-algebra [10]. Jun, Ahn and Roh. in [7] described pseudo-valuation on the BCC-algebras. Koam, Haider and Ansari described in 2019 pseudo-valuations on KU-algebras [9].

The concept of UP-algebras is introduced and analyzed by Iampan in 2017 [6] as a generalization of the concept of KU-algebras. This author has participated in the analysis of the properties of UP-algebras, also (See, for example: [11, 12, 13]).

In recently published article [14], he offered one way of determining of pseudo-valuation on PU-algebras. Apart from showing he demonstrated how to construct a pseudo-metric space by such mapping.

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In this article, using the pseudo-metric induced by a pseudo-valuation the author construct the quotient algebra. In addition, it has been shown that the algebra constructed in this way is also UP-algebra.

2. PRELIMINARIES

Here we give the definition of UP-algebra and some of its substructures necessary for further work.

Definition 2.1 ([6]). *An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP- algebra if it satisfies the following axioms:*
(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, *and*
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

In A we can define a binary relation $' \leqslant '$ by

$$(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 0).$$

Definition 2.2 ([6]). *A non-empty subset J of a UP-algebra A is called a UP-ideal of A if it satisfies the following conditions:*
(1) $0 \in J$, *and*
(2) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \wedge y \in J) \Longrightarrow x \cdot z \in J)$.

Definition 2.3 ([11]). *Let A be a UP-algebra. A subset G of A is called a* proper UP-filter *of A if it satisfies the following properties:*
(3) $\neg(0 \in G)$, *and*
(4) $(\forall x, y, z \in A)((\neg(x \cdot (y \cdot z) \in G) \wedge x \cdot z \in G) \Longrightarrow y \in G)$.

In this section, we introduce the concept of pseudo-valuations on UP-algebras, describe the basics properties of such pseudo-valuation and construct a pseudo-metric space based on this mapping.

Definition 2.4 ([14], Definition 3.1). *A real-valued function v on a UP-algebra A is called a pseudo-valuation on A if it satisfies the following two conditions:*
(5) $v(0) = 0$, *and*
(6) $(\forall x, y, z \in A)(v(x \cdot z) \leqslant v(x \cdot (y \cdot z)) + v(y))$.
A pseudo-valuation v on a UP-algebra A satisfying the following condition:
(7) $(\forall x \in A)(v(x) = 0 \Longrightarrow x = 0)$
is called a valuation on X.

Theorem 2.1 ([14], Theorem 3.16). *Let A be a UP-algebra and v be a pseudo-valuation on A. Then the mapping $d_v : A \times A \ni (x, y) \longmapsto v(x \cdot y) + v(y \cdot x) \in \mathbb{R}$ is a pseudo-metric on A.*

3. THE MAIN RESULTS

3.1. Some important properties of pseudo-metric on UP-algebras.

Proposition 3.1. *Let v be pseudo-valuation on a UP-algebra A. Then*
(8) $(\forall x, y, z \in A)(d_v(x \cdot z, y \cdot z) \leqslant d_v(x, y))$;
(9) $(\forall x, y, z \in A)(d_v(z \cdot x, z \cdot y) \leqslant d_v(x, y))$.

Proof. Let x, y, z be arbitrary elements of A. Then the following holds

$$\begin{aligned}
d_v(x \cdot z, y \cdot z) &= v((x \cdot z) \cdot (y \cdot z)) + v((y \cdot z) \cdot (x \cdot z)) \\
&\leqslant v((x \cdot z) \cdot ((y \cdot x) \cdot (y \cdot z))) + v(y \cdot x) \\
&+ v((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z))) + v(x \cdot y) \\
&= (0 + v(y \cdot x)) + (0 + v(x \cdot y)) \\
&= d_v(x, y)
\end{aligned}$$

since it is $v((x \cdot z) \cdot ((y \cdot x) \cdot (y \cdot z))) = v(0) = 0$ and $v((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z))) = v(0) = 0$.

On the other hand, relying on valid inequality (4) in the article [14], we have

$$\begin{aligned}
d_v(z \cdot x, z \cdot y) &= v((z \cdot x) \cdot (z \cdot y)) + v((z \cdot y) \cdot (z \cdot x)) \\
&\leqslant v((x \cdot y) \cdot ((z \cdot x) \cdot (z \cdot y)) + v(x \cdot y) \\
&+ v((y \cdot x) \cdot ((z \cdot y) \cdot (z \cdot x))) + v(y \cdot x) \\
&= (0 + v(x \cdot y)) + (0 + v(y \cdot x) = v(x \cdot y) + v(y \cdot x) \\
&= d_v(x, y).
\end{aligned}$$

□

3.2. **A construction of a congruence on UP-algebra.**

Definition 3.1. *Let* u *be a pseudo valuation on a* UP*-algebra* A*. Define the relation* $\theta_v \subseteq A \times A$ *by:*

$$(\forall x, y \in A)((x, y) \in \theta_v \iff d_v(x, y) = 0)$$

Theorem 3.2. *Let* v *be a pseudo-valuation on a UP-algebra* A*. Then* θ_v *is a congruence relation on* A*.*

Proof. Since θ_v induced by a pseudo-metric d_v, it is an equivalence relation on A.

To prove that θ compatible with the internal operation in A, we assume that $x, y, z \in A$ are such that $(x, y) \in \theta$. Then $d_v(x, y) = 0$. Thus

$$0 \leqslant d_v(x \cdot z, y \cdot z) \leqslant d_v(x, y) = 0 \text{ and } 0 \leqslant d_v(z \cdot x, z \cdot y) \leqslant d_v(x, y) = 0.$$

by Proposition 3.1. Hence $d_v(x \cdot z, y \cdot z) = 0$ and $d_v(z \cdot x, z \cdot y) = 0$, which means $(x \cdot z, y \cdot z) \in \theta$ and $(z \cdot x, z \cdot y) \in \theta$.

So, θ_v is a congruence relation on A. □

For the congruence relation θ_v on UP-algebra A, constructed in this way, we say that it is induced by a pseudo-valuation v.

Proposition 3.3. *Let* v *be a pseudo-valuation on a UP-algebra* A *and* θ_v *be the congruence relation induced by* v*. Then the class* $[0]_v$ *in* $A/\theta_v = \{[x]_v : x \in A\}$*, generated by the element* 0 *in* A*, is an ideal in* A*.*

Proof. Obviously the following applies: $x \in [0]_v$ if and only if $(x, 0) \in \theta_v$. Then $d_v(x, 0) = 0$. This means $0 = v(x \cdot 0) + v(0 \cdot x) = v(0) + v(x) = v(x)$. Therefore, $[0]_v$ is an ideal in A, according to Theorem 3.6 in article [14]. □

Corollary 3.4. *Let* v *be a pseudo-valuation on a UP-algebra* A *and* θ_v *be the congruence relation induced by* v*. Then the set* $\bigcup\{[x]_v : x \in A \wedge \neg(x \in [0]_v)\}$ *is a proper filter in* A*.*

Proof. The proof of this corollary follows from Theorem 3.7 in article [11]. □

3.3. The quotient $A/[0]_v$ is a UP-algebra.

Theorem 3.5. *Let v be a pseudo-valuation on a UP-algebra A, θ_v be the congruence induced by v and $[0]_v$ be the class in A/θ_v. Then the factor-set $A/[0]_v$ is a UP-algebra.*

Proof. Let v be a pseudo-valuation on a UP-algebra A and let θ_v be the congruence on A induced by v. According to the previous proposition, class $[0]_v$ is an ideal in A. We can construct a congruence relation $\sim_v$ on A using this ideal, by Theorem 3.5 in article [6], as follows

$$(\forall x, y \in A)(x \sim_v y \iff (x \cdot y \in [0]_v \wedge y \cdot x \in [0]_v)).$$

On the other hand, this pseudo-valuation v allows us to determine the ideal $J_v = \{x \in A : v(x) = 0\}$ in A, by Theorem 3.6 in article [14]. Now, we have if $x \sim_v y$, then $x \cdot y \in [0]_v$ and $y \cdot x \in [0]_v$. This means $d_v(x \cdot y, 0) = 0$ and $d_v(y \cdot x, 0) = 0$. Thus $v((x \cdot y) \cdot 0) + v(0 \cdot (x \cdot y)) = 0$ and $v((y \cdot x) \cdot 0) + v(0 \cdot (y \cdot x)) = 0$. From here, considering (UP-2), (UP-3) and (5), we have $v(x \cdot y) = 0$ and $v(y \cdot x) = 0$. Therefore, $x \cdot y \in J_v$ and $y \cdot x \in J_v$. Without much difficulty it can be checked that the reverse deduction is true, too.

On the set $A/\theta_v = \{[x]_v : x \in A\}$, we define

$$(\forall x, y \in A)([x]_v * [y] = [x \cdot y]_v).$$

According to claim (4) of Theorem 3.7 in article [6], factor-set $A/[0]_v$ is a UP-algebra. $\square$

4. CONCLUSION

In 2010, Ghorbani presented the idea of constructing a congruence on BCI-algebras in [5] by using the pseudo-valuation given on that algebra. That idea, 2018, was discussed by S.-Z. Song, H. Bordbar and Y. B. Jun. in [16]. This author introduced the concept of pseudo-valuations on UP-algebras in [14]. Looking at the texts [5, 16], in this article, as a continuation of [14], we introduced a congruence relation θ_v generated by a given pseudo-valuation v on the UP-algebra A.

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