FUNDAMENTAL JOURNAL OF MATHEMATICS AND APPLICATIONS

VOLUME III ISSUE I

http://dergipark.gov.tr/fujma

ISSN 2645-8845

VOLUME III ISSUE I
ISSN 2645-8845

June 2020
http://dergipark.gov.tr/fujma

FUNDAMENTAL JOURNAL OF MATHEMATICS AND APPLICATIONS

Contents

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 1-7
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.623500

Numerical Oscillation Analysis for Gompertz Equation with One Delay

Qian Yang[1*] and Qi Wang[1]

[1]*School of Applied Mathematics, Guangdong University of Technology, Guangzhou, China*
[*]*Corresponding author*

Article Info

Keywords: *Asymptotic behavior, Gompertz equation, Oscillation, θ-method*
2010 AMS: *65L10, 65L30*
Received: *23 September 2019*
Accepted: *20 March 2020*
Available online: *10 June 2020*

Abstract

This paper concerns with the oscillation of numerical solutions of a kind of nonlinear delay differential equation proposed by Benjamin Gompertz, this equation usually be used to describe the population dynamics and tumour growth. We obtained some conditions under which the numerical solutions are oscillatory. The non-oscillatory behaviors of numerical solutions are also analyzed. Numerical examples are given to test our theoretical results.

1. Introduction

In recent years, the studies on oscillation of the solutions of delay differential equations (DDEs) are developing rapidly (see [1, 2]). This research has been applied to many fields including biology, physics, ecology and so on. Nonetheless there are few papers have been published on the oscillation of numerical solutions of DDEs (see [3]-[6]). So we will consider numerical oscillation for Gompertz equation with one delay in this paper. In the past few years, Gompertz equation has been generally used to describe the population dynamics and tumour growth (see [7, 8]). In 1825, Benjamin Gompertz proposed the classical Gompertz model[9]

$$\dot{V}(t) = -rV(t)\ln\frac{V(t)}{K}, \quad V(0) = V_0 > 0.$$

In 1932, Winsor analyzed some analytical properties of a modified Gompertz model and pointed that it can be used to describe empirically the deceleration of tumour growth[10]. In 2000, Ferrante et al. considered a stochastic version of the Gompertz model to describe vivo tumor growth [11]. While to study the investigated phenomena better, some researchers prefer to incorporate various equations with the time delays in different ways. In [12], four kinds of models were derived by introducing the discrete delays into the classical Gompertz model. One of them, which occurs in the following form will be discussed in the rest paper

$$\dot{V}(t) = -rV(t)\ln\frac{V(t-\tau)}{K}, \quad t \geq 0, \tag{1.1}$$

with $r, K \in (0,\infty)$, where r is the growth rate, V is the number of individuals or cells and K is the plateau number of individuals or cells. The time delay figures maturation period of the individuals in the context of population growth. While it may figure the time lag during the course of tumor growth (or degradation) owing to the time which is required for the cells to identify and accommodate to changes in the environment. The existence, uniqueness and asymptotic properties of the solutions of (1.1)

Email addresses and ORCID numbers: **y727445149@163.com, https://orcid.org/0000-0002-9316-7285 (Q. Yang), bmwzwq@126.com, https://orcid.org/0000-0003-3578-2551 (Q. Wang)**

were studied in [12]. In [13], the occurrence of period orbits owing to the Hopf bifurcation was analyzed. Meanwhile, the existence of periodic solutions was confirmed and some results for the asymptotically stability of the periodic solutions were given. Later, for the Gompetrz model with one delay, the stability and Hopf bifurcation were studied in [14]. However, to the best of our knowledge, until now very few results dealing with the oscillation of solutions of (1.1) were found. Therefore, from the viewpoint of analytically and numerically, our objective in this paper is to acquire some sufficient conditions for oscillation of all positive solutions of (1.1) about the equilibrium. We also prove that every non-oscillatory solution will tend to the equilibrium when the time approaches to infinity.

In the rest paper, we only study the solutions of (1.1) with initial condition of the form

$$V(t) = \phi(t), \quad -\tau \leq t \leq 0,$$

where $\phi \in C([-\tau,0],[0,\infty))$ and $\phi(0) > 0$. By the method of steps one can prove that (1.1) has positive solutions for all $t \geq 0$. From [15], we know the difference equation

$$a_{n+1} - a_n + \sum_{j=-k}^{l} q_j a_{n+j} = 0 \tag{1.2}$$

is oscillatory if and only if the characteristic equation of (1.2) has no positive roots. So we introduce a useful theorem.

Theorem 1.1. *[15] Consider the difference equation*

$$a_{n+1} - a_n + p a_{n-k} = 0, \tag{1.3}$$

where $p \in \mathbb{R}$, $k \in \mathbb{Z}$. Then every solution of (1.3) oscillates if and only if one of the following conditions holds:

1. *$k = -1$ and $p \leq -1$;*
2. *$k = 0$ and $p \geq 1$;*
3. *$k \in \{\dots,-3,-2\} \cup \{1,2,\dots\}$ and $p\frac{(k+1)^{k+1}}{k^k} > 1$.*

2. The oscillation of solutions

In this section, we will illustrate some sufficient conditions for oscillation of (1.1) about the equilibrium K analytically and numerically.

Theorem 2.1. *Every positive solution of (1.1) oscillates about K if*

$$r\tau > \frac{1}{e}. \tag{2.1}$$

Proof. Set $V(t) = Ke^{y(t)}$, then $V(t)$ oscillates about K if and only if $y(t)$ oscillates about zero. So from (1.1) we find that

$$\dot{y}(t) = -ry(t-\tau). \tag{2.2}$$

Then by Theorem 2.2.3 in [15], we know that every solution of (1.1) oscillates if and only if (2.1) holds. □

Next, we transfer to discuss the numerical case. Applying the linear θ-method to (2.2), one has

$$y_{n+1} = y_n - h\theta r y_{n+1-m} - h(1-\theta) r y_{n-m}, \tag{2.3}$$

where $0 \leq \theta \leq 1$, $h = \tau/m$ is stepsize and m is a positive integer. y_{n+1} and y_{n+1-m} are approximations to $y(t)$ and $y(t-\tau)$ at t_{n+1}, respectively. Let $y_n = \ln(V_n/K)$, then (2.3) reads

$$\ln\frac{V_{n+1}}{K} = \ln\frac{V_n}{K} - h\theta r \ln\frac{V_{n+1-m}}{K} - h(1-\theta) r \ln\frac{V_{n-m}}{K} = \ln\left[\frac{V_n}{K}\left(\frac{K}{V_{n+1-m}}\right)^{h\theta r}\left(\frac{K}{V_{n-m}}\right)^{h(1-\theta)r}\right],$$

that is

$$V_{n+1} = V_n K^{hr} \frac{1}{V_{n+1-m}^{h\theta r}} \frac{1}{V_{n-m}^{h(1-\theta)r}}. \tag{2.4}$$

It is obvious that V_n is oscillatory about K if and only if y_n is oscillatory. In the following we seek the conditions under which (2.4) is oscillatory.

Lemma 2.2. *The characteristic equation of (2.3) is given by*

$$\lambda = R(-hr\lambda^{-m}), \tag{2.5}$$

where $R(x) = (1+(1-\theta)x)/(1-\theta x)$ is the stability function of the linear θ-method.

The proof of this Lemma can be given directly and we omit it.

Lemma 2.3. *Under the condition (2.1), (2.5) has no positive roots for $0 \le \theta \le 1/2$.*

Proof. Let $P(\lambda) = \lambda - R(-hr\lambda^{-m})$. From [16], we have

$$R(-hr\lambda^{-m}) \le \exp(-hr\lambda^{-m}), \quad \lambda > 0, \quad 0 \le \theta \le 1/2.$$

Further, we will prove $Q(\lambda) = \lambda - \exp(-hr\lambda^{-m}) > 0$ for $\lambda > 0$. Assume there is a $\lambda_0 > 0$ such that $Q(\lambda_0) \le 0$, then $\lambda_0 \le \exp(-hr\lambda_0^{-m})$, and $\lambda_0^m \le \exp(-r\tau\lambda_0^{-m})$. Thus

$$r\tau e \le r\tau\lambda_0^{-m}\exp(1-r\tau\lambda_0^{-m}).$$

So we have

- If $1-r\tau\lambda_0^{-m} = 0$, then $r\tau e \le 1$, which contradicts to (2.1).
- If $1-r\tau\lambda_0^{-m} \ne 0$, since $e^x < 1/(1-x)$ for $x < 1$ and $x \ne 0$, we get $r\tau e \le 1$, which also contradicts to (2.1).

Therefore, for $\lambda > 0$, $P(\lambda) = \lambda - R(-hr\lambda^{-m}) \ge \lambda - \exp(-hr\lambda^{-m}) = Q(\lambda) > 0$, which suggests that (2.5) has no positive roots. □

Next we consider the case $1/2 < \theta \le 1$ under the assumption $m > 1$.

Lemma 2.4. *Under the conditions (2.1) and $1/2 < \theta \le 1$, (2.5) has no positive roots for $h < h_0$, where*

$$h_0 = \begin{cases} \infty, & r\tau \ge 1, \\ \tau(1+\ln r\tau), & r\tau < 1. \end{cases} \tag{2.6}$$

Proof. It can be noted that $R(-hr\lambda^{-m})$ is an increasing function for θ when $\lambda > 0$, then

$$R(-hr\lambda^{-m}) = \frac{1-h(1-\theta)r\lambda^{-m}}{1+h\theta r\lambda^{-m}} \le \frac{1}{1+hr\lambda^{-m}}.$$

Next, we will illustrate that $\lambda - 1/(1+hr\lambda^{-m})$ is positive under some conditions. Actually

$$\lambda - \frac{1}{1+hr\lambda^{-m}} = \frac{\lambda^{-m+1}}{1+hr\lambda^{-m}}S(\lambda),$$

we need to prove $S(\lambda) = \lambda^m - \lambda^{m-1} + hr > 0$ for each $\lambda > 0$. Obviously, $S(\lambda)$ is the characteristic polynomial of the difference equation

$$w_{n+1} = w_n - hrw_{n-m+1}.$$

According to Theorem 1.1, $S(\lambda)$ has no positive roots if and only if

$$hr\frac{m^m}{(m-1)^{m-1}} > 1,$$

equivalently

$$\ln r\tau + (m-1)\ln\left(1+\frac{1}{m-1}\right) > 0. \tag{2.7}$$

If $r\tau \ge 1$, then (2.7) holds. If $r\tau < 1$ and $h < \tau(1+\ln r\tau)$, from the fact "$\ln(1+x) > x/(1+x)$ holds for $x > -1$ and $x \ne 0$" we have

$$\ln r\tau + (m-1)\ln\left(1+\frac{1}{m-1}\right) > \ln r\tau + \frac{m-1}{m} > 0.$$

Thus we find

$$P(\lambda) = \lambda - R(-hr\lambda^{-m}) \ge \lambda - 1/(1+hr\lambda^{-m}) > 0,$$

which implies that (2.5) has no positive roots. □

In view of Lemmas 2.3, 2.4 and Theorem 2.1, we get the following theorem.

Theorem 2.5. *Under the condition* (2.1), (2.4) *is oscillatory for*

$$h < \begin{cases} \infty, & for \quad 0 \le \theta \le 1/2, \\ h_0, & for \quad 1/2 < \theta \le 1, \end{cases}$$

where h_0 *is defined in* (2.6).

3. Non-oscillatory solutions

In this section, we study the asymptotic behavior of non-oscillatory solutions of (1.1) and (2.4).

Theorem 3.1. *Let* $V(t)$ *be a positive solution of* (1.1)*, which does not oscillate about* K*, then* $\lim_{t\to\infty} V(t) = K$.

Proof. Since $V(t) = Ke^{y(t)}$ we only need to prove that $\lim_{t\to\infty} y(t) = 0$. Assume that $y(t) \ge 0$ for sufficiently large t (the case $y(t) < 0$ is similar and will be omitted). Then from (2.2) we have $\dot{y}(t) \le 0$. So $y(t)$ is decreasing and

$$\lim_{t\to\infty} y(t) = Y \in [0,\infty), \tag{3.1}$$

we prove $Y = 0$ by contradiction. Assume $Y > 0$ and (2.2) produces

$$\lim_{t\to\infty} \dot{y}(t) = -r \lim_{t\to\infty} y(t-\tau) = -rY < 0.$$

Then $\lim_{t\to\infty} y(t) = -\infty$, which is a contradiction to (3.1). □

In the following, we will prove that the numerical solution V_n can inherit this property.

Theorem 3.2. *Let* y_n *be a solution of* (2.3)*, which does not oscillate, then* $\lim_{t\to\infty} y_n = 0$.

Proof. Assume that $y_n > 0$ for n sufficiently large (the case $y_n < 0$ is similar and will be omitted). From (2.3) we know

$$y_{n+1} - y_n = -(h\theta r y_{n+1-m} + h(1-\theta) r y_{n-m}) < 0, \tag{3.2}$$

then y_n is decreasing. So there exists a constant Z such that

$$\lim_{n\to\infty} y_n = Z \in [0,\infty). \tag{3.3}$$

We argue $Z = 0$ by contradiction. Suppose $Z > 0$, then there is $N \in \mathbb{N}$ and $\varepsilon > 0$ such that $0 < Z-\varepsilon < y_n < Z+\varepsilon$ for $n-m > N$, hence $y_{n-m} > Z-\varepsilon$ and $y_{n-m+1} > Z-\varepsilon$. So (3.2) gives

$$y_{n+1} - y_n < -(h\theta r Z + h(1-\theta) r Z),$$

which indicates that $y_{n+1} - y_n < A$, where $A = -(h\theta r Z + h(1-\theta) r Z) < 0$. Thus $y_n \to -\infty$ as $n \to \infty$, which contradicts to (3.3). □

Theorem 3.3. *Let* V_n *be a positive solution of* (2.4)*, which does not oscillate about* K *, then* $\lim_{n\to\infty} V_n = K$.

4. Numerical examples

In this section we give two numerical examples to verify the previous results.

Firstly, in order to test Theorems 2.1 and 2.5, we consider the following equation

$$\dot{V}(t) = -\frac{1}{15} V(t) \ln \frac{V(t-13)}{2}, \quad t \ge 0 \tag{4.1}$$

with the initial condition

$$V(t) = 7, \quad -13 \le t \le 0.$$

In (4.1), we have $1/e < r\tau = 13/15 < 1$, which implies that the solutions of (4.1) are oscillatory according to Theorem 2.1. In Figure 4.1, we draw the figures of the analytic solutions and the numerical solutions with $\theta = 0.1 \le 1/2$ and $h = \tau/m =$

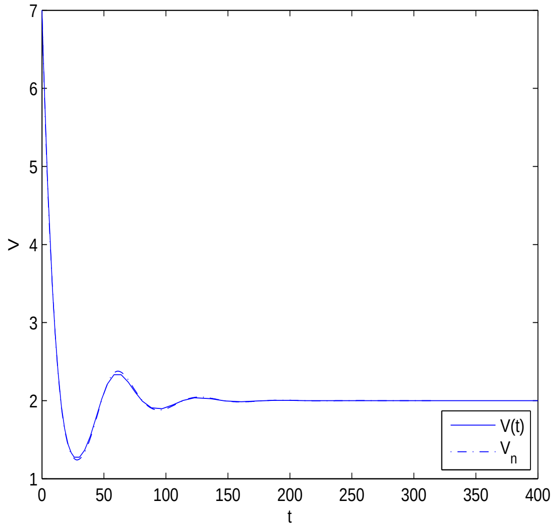

Figure 4.1: The analytic solutions and the numerical solutions of (4.1) with $h = 0.52$, $\theta = 0.1$.

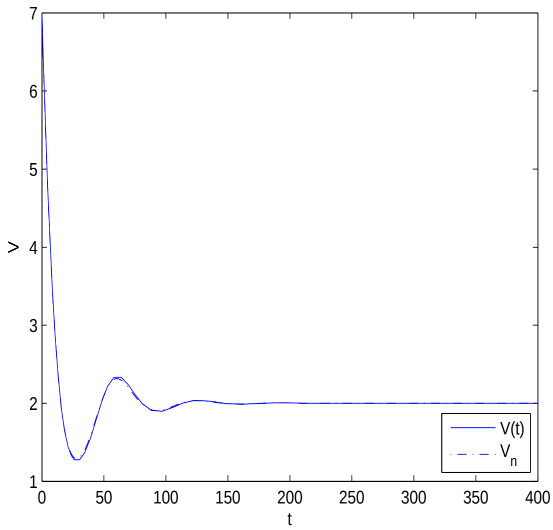

Figure 4.2: The analytic solutions and the numerical solutions of (4.1) with $h = 0.65$, $\theta = 0.9$.

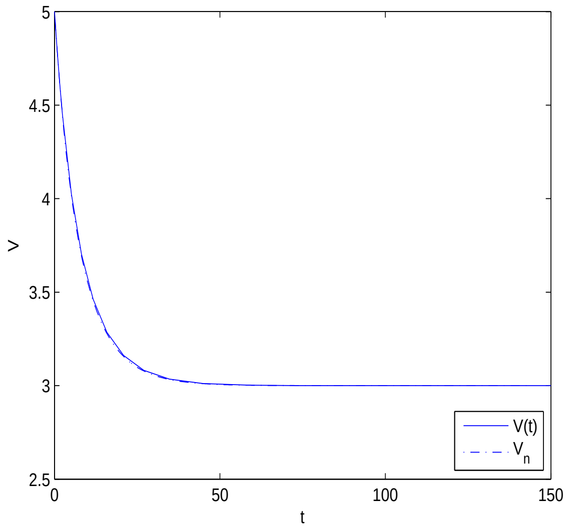

Figure 4.3: The analytic solutions and the numerical solutions of (4.2) with $h = 0.4$ and $\theta = 0.2$.

$13/25 = 0.52 < +\infty$. On the other hand, we set $1/2 < \theta = 0.9 \leq 1$ and $m = 20$ in Figure 4.2. Then $h_0 = \tau(1 + \ln r\tau) \approx 8.1140$ and $h = \tau/m = 13/20 = 0.65 < h_0$. Therefore, according to Theorem 2.5, the numerical solutions of (4.1) are also oscillatory for these two cases, which are all the same with Figures 4.1 and 4.2.

Next, we illustrate the validity of Theorems 3.1 and 3.2 in the second example. Consider the equation

$$\dot{V}(t) = -\frac{1}{10}V(t)\ln\frac{V(t-\frac{4}{5})}{3}, \quad t \geq 0 \tag{4.2}$$

with the initial condition

$$V(t) = 5, \quad -\frac{4}{5} \leq t \leq 0.$$

In (4.2), we have $r\tau = 0.08 < 1/e$, which does not satisfy Theorem 2.1. So the analytic solutions and the numerical solutions of (4.2) are non-oscillatory. In Figure 4.3, we draw the figures of the analytic solutions and the numerical solutions of (4.2). From this figure, we can see that $V(t) \to K = 3$ as $t \to \infty$ and $V_n \to K = 3$ as $n \to \infty$. That is, the linear $\theta-$method preserves the asymptotic behavior of non-oscillatory solutions, which coincides with Theorems 3.1 and 3.2.

5. Conclusion

In this paper, numerical oscillation and asymptotic behavior for Gompertz equation with one delay are studied. Some sufficient conditions are proposed. Numerical examples are provided to illustrate the validity of our results. In the future, we will consider the multidimensional and stochastic case.

Acknowledgement

This work is supported by Natural Science Foundation of Guangdong Province (No. 2017A030313031).

References

[1] J. Džurina, I. Jadlovská, *Oscillation theorems for fourth-order delay differential equations with a negative middle term*, Math. Meth. Appl. Sci., **40** (2017), 7830-7842.
[2] K. M. Chudinov, *On exact sufficient oscillation conditions for solutions of linear differential and difference equations of the first order with after effect*, Russian Math., **62** (2018), 79-84.
[3] J. F. Gao, M. F. Song, M. Z. Liu, *Oscillation analysis of numerical solutions for nonlinear delay differential equations of population dynamics*, Math. Model. Anal., **16** (2011), 365-375.
[4] J. F. Gao, M. F. Song, *Oscillation analysis of numerical solutions for nonlinear delay differential equations of hematopoiesis with unimodal production rate*, Appl. Math. Comput., **264** (2015), 72-84.
[5] Q. Wang, *Oscillation analysis of θ-methods for the Nicholson's blowflies model*, Math. Meth. Appl. Sci., **39** (2016), 941-948.
[6] Y. Z. Wang, J. F. Gao, *Oscillation analysis of numerical solutions for delay differential equations with real coefficients*, J. Comput. Appl. Math., **337** (2018), 73-86.
[7] M. Bodnar, U. Foryś, *Three types of simple DDE's describing tumor growth*, J. Biol. Syst., **15** (2007), 453-471.
[8] L. E. B. Cabrales, A. R. Aguilera, R. P. Jiménez, et al., *Mathematical modeling of tumor growth in mice following low-level direct electric current*, Math. Comput. Simulat., **78** (2008), 112-120.
[9] B. Gompertz, *On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies*, Philos. T. R. Soc. B, **115** (1825), 513-583.
[10] C. P. Winsor, *The Gompertz Curve as a Growth Curve*, Proc. Natl Acad. Sci., **18** (1932), 1-8.

[11] L. Ferrante, S. Bompadre, L. P. Leone, *Parameter estimation in a Gompertzian stochastic model for tumor growth*, Biometrics, **56** (2000), 1076-1081.
[12] M. J. Piotrowska, U. Foryś, *The nature of Hopf bifurcation for the Gompertz model with delays*, Math. Comput. Model., **54** (2011), 2183-2198.
[13] M. J. Piotrowska, U. Foryś, *Analysis of the Hopf bifurcation for the family of angiogenesis models*. J. Math. Anal. Appl., **382** (2011), 180-203.
[14] M. Bodnar, M. J. Piotrowska, U. Foryś, *Gompertz model with delays and treatment: Mathematical analysis*, Math. Biosci. Eng., **10** (2013), 551-563.
[15] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations: with Applications*, Oxford University Press, 1991.
[16] M. H. Song, Z. W. Yang, M. Z. Liu, *Stability of θ-methods for advanced differential equations with piecewise continuous arguments*, Comput. Math. Appl., **49** (2005), 1295-1301.

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 8-16
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.708816

Null (Lightlike) f-Rectifying Curves in the Three Dimensional Minkowski Space $\mathbb{E}_1^3$

Zafar Iqbal[1*] and Joydeep Sengupta[2]

[1]*Department of Mathematics, Kaliyaganj College, Kaliyaganj, Uttar Dinajpur, West Bengal, INDIA*
[2]*Department of Mathematics and Statistics, Aliah University, Newtown, Kolkata, West Bengal, INDIA*
[*]*Corresponding author*

Article Info

Keywords: *Curvature, Minkowski 3-space, Null (lightlike) curve, Rectifying curve, Serret-Frenet formulae, Torsion*
2010 AMS: *53A04, 53B30, 53C40, 53C50*
Received: *24 March 2020*
Accepted: *15 May 2020*
Available online: *10 June 2020*

Abstract

A rectifying curve γ in the Euclidean 3-space $\mathbb{E}^3$ is defined as a space curve whose position vector always lies in its rectifying plane (i.e., the plane spanned by the unit tangent vector field T_γ and the unit binormal vector field B_γ of the curve γ), and an f-rectifying curve γ in the Euclidean 3-space $\mathbb{E}^3$ is defined as a space curve whose f-position vector γ_f, defined by $\gamma_f(s) = \int f(s)d\gamma$, always lies in its rectifying plane, where f is a nowhere vanishing real-valued integrable function in arc-length parameter s of the curve γ. In this paper, we introduce the notion of f-rectifying curves which are null (lightlike) in the Minkowski 3-space $\mathbb{E}_1^3$. Our main aim is to characterize and classify such null (lightlike) f-rectifying curves having spacelike or timelike rectifying plane in the Minkowski 3-Space $\mathbb{E}_1^3$.

1. Introduction

Let $\mathbb{E}^3$ denote the Euclidean 3-space. Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit-speed curve parametrized by arc-length function s with at least four continuous derivatives. Needless to mention, I denotes a non-trivial interval in $\mathbb{R}$, i.e., a connected set in $\mathbb{R}$ containing at least two points. For the curve γ in $\mathbb{E}^3$, we consider the Frenet apparatus $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma, \tau_\gamma\}$, where T_γ is the unit tangent vector field, N_γ is the unit principal normal vector field, $B_\gamma = T_\gamma \times N_\gamma$ is the unit binormal vector field of the curve γ, and $\kappa_\gamma : I \longrightarrow \mathbb{R}$ is a differentiable function with $\kappa_\gamma > 0$, known as the curvature of γ, and $\tau_\gamma : I \longrightarrow \mathbb{R}$ is a differentiable function, called the torsion of γ. Then the Serret-Frenet formulae for the curve γ are given by ([1]-[4])

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_\gamma \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\gamma & 0 \\ -\kappa_\gamma & 0 & \tau_\gamma \\ 0 & -\tau_\gamma & 0 \end{pmatrix} \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}.$$

The planes spanned by $\{T_\gamma, N_\gamma\}$, $\{N_\gamma, B_\gamma\}$ and $\{T_\gamma, B_\gamma\}$ are called the osculating plane, the normal plane and the rectifying plane of the curve γ, respectively ([2, 5]).

In the Euclidean 3-space $\mathbb{E}^3$, the notion of a rectifying curve was introduced by B.Y. Chen in [5] as a tortuous curve whose position vector always lies in the rectifying plane of the curve. That is, for a rectifying curve $\gamma : I \longrightarrow \mathbb{E}^3$, the position vector of γ can be expressed as

$$\gamma(s) = \lambda(s)T_\gamma(s) + \mu(s)B_\gamma(s), \;\; s \in I,$$

Email addresses and ORCID numbers: **zafariqbal_math@yahoo.com, https://orcid.org/0000-0003-4405-1160 (Z. Iqbal), joydeep1972@yahoo.com, https://orcid.org/0000-0002-1609-0798 (J. Sengupta)**

for two differentiable functions $\lambda, \mu : I \longrightarrow \mathbb{R}$ in arc-length parameter s of γ.

Several characterizations and classification of the rectifying curves in $\mathbb{E}^3$ were studied in [5]-[8]. Meanwhile the notion of rectifying curves were extended to several sort of Riemannian and pseudo-Riemannian spaces. As for example, many characterizations and classification of rectifying curves in the Minkowski 3-space $\mathbb{E}^3_1$ were studied in [9]-[11].

In this paper, we study null f-rectifying curves in the Minkowski 3-space $\mathbb{E}^3_1$. We organize this paper with three sections. In the first section, we give some basic preliminaries and then introduce the notion of f-rectifying curves which are null (or lightlike) in $\mathbb{E}^3_1$. Thereafter the second section is devoted to investigate some characterizations of null f-rectifying curves in $\mathbb{E}^3_1$. In the concluding section, we classify null f-rectifying curves in terms of their f-position vectors in $\mathbb{E}^3_1$.

2. Preliminaries

The Minkowski 3-space $\mathbb{E}^3_1$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with the standard flat metric g (called the Lorentzian inner product) defined by

$$g(v,w) = v_1w_1 + v_2w_2 - v_3w_3$$

for all tangent vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ to $\mathbb{E}^3_1$ (see [12, 13]). A tangent vector v to $\mathbb{E}^3_1$ is called a

spacelike vector		if and only if	$g(v,v) > 0$	or	$v = 0$,
lightlike vector	(null vector)	if and only if	$g(v,v) = 0$	and	$v \neq 0$,
timelike vector		if and only if	$g(v,v) < 0$	([12, 13]).	

As usual, the norm of a tangent vector v to $\mathbb{E}^3_1$ is denoted and defined by $\|v\| = \sqrt{|g(v,v)|}$. It is trivial to mention that a tangent vector v to $\mathbb{E}^3_1$ is called a unit vector if and only if $\|v\| = 1$, i.e., if and only if $|g(v,v)| = 1$, i.e., if and only if $g(v,v) = \pm 1$. Two tangent vectors v and w to $\mathbb{E}^3_1$ are said to be orthogonal if and only if $g(v,w) = 0$. For any two tangent vectors v and w to $\mathbb{E}^3_1$, the vectorial product of v and w is defined by

$$v \times w = \begin{vmatrix} e_1 & e_2 & -e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - v_3w_2)e_1 + (v_3w_1 - v_1w_3)e_2 + (v_2w_1 - v_1w_2)e_3,$$

where $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ for each $i \in \{1,2,3\}$, $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$ such that $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$ ([12, 13]).

Let $\gamma : I \longrightarrow \mathbb{E}^3_1$ be a curve in $\mathbb{E}^3_1$ and γ' stands for its velocity vector field. The curve γ is said to be a spacelike curve, a lightlike curve (null curve) or a timelike curve in $\mathbb{E}^3_1$ if and only if its velocity vector $\gamma'(t)$ is a spacelike vector, a lightlike vector (null vector) or a timelike vector, respectively, for each $t \in I$. To elaborate, the curve γ in $\mathbb{E}^3_1$ is a

spacelike curve		if and only if	$g(\gamma'(t), \gamma'(t)) > 0$	or	$\gamma'(t) = 0$,
lightlike curve	(null curve)	if and only if	$g(\gamma'(t), \gamma'(t)) = 0$	and	$\gamma'(t) \neq 0$,
timelike curve		if and only if	$g(\gamma'(t), \gamma'(t)) < 0$		

for all $t \in I$ (see [12, 13]). Thus, the curve γ is said to be a non-null curve in $\mathbb{E}^3_1$ if and only if it is either a spacelike curve or a timelike curve in $\mathbb{E}^3_1$, i.e., if and only if $g(\gamma'(t), \gamma'(t)) \neq 0$ for all $t \in I$. If γ is a non-null (spacelike or timelike) curve in $\mathbb{E}^3_1$ and we change the parameter t by the function $s = s(t)$ given by $s(t) = \int_0^t \|\gamma'(u)\|\, du$ such that $\|\gamma'(s)\| = \sqrt{|g(\gamma'(s), \gamma'(s))|} = 1$, i.e., $g(\gamma'(s), \gamma'(s)) = \pm 1$ for all $s \in I$, then the non-null curve γ is said to be parametrized by arc-length function s or a unit-speed non-null curve in $\mathbb{E}^3_1$. Again, if γ is a null (lightlike) curve in $\mathbb{E}^3_1$ and we change the parameter t by the function $s = s(t)$ given by $s(t) = \int_0^t \sqrt{\|\gamma''(u)\|}\, du$ such that $g(\gamma''(s), \gamma''(s)) = 1$ for all $s \in I$, then the null curve γ is said to be parametrized by pseudo arc-length function s or a unit-speed null curve in $\mathbb{E}^3_1$.

Let $\gamma : I \longrightarrow \mathbb{E}^3_1$ be a unit-speed null or non-null curve in the Minkowski 3-space $\mathbb{E}^3_1$ parametrized by arc-length function or pseudo arc-length function s with Frenet apparatus $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma, \tau_\gamma\}$, where $\{T_\gamma, N_\gamma, B_\gamma = T_\gamma \times N_\gamma\}$ is the dynamic Frenet frame along the curve γ in $\mathbb{E}^3_1$ and κ_γ, τ_γ are two differentiable functions in the parameter s called, respectively, the curvature and the torsion of the curve γ in $\mathbb{E}^3_1$. Then to write the Serret-Frenet formulae for the curve γ the following mutually distinct cases come up for consideration:

Case I: Let γ be a spacelike curve with a spacelike principal normal N_γ in $\mathbb{E}^3_1$. Then the Serret-Frenet formulae for the curve γ are given by

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_\gamma \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\gamma & 0 \\ -\kappa_\gamma & 0 & \tau_\gamma \\ 0 & \tau_\gamma & 0 \end{pmatrix} \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix},$$

where $g\left(T_\gamma(s),T_\gamma(s)\right)=1$, $g\left(N_\gamma(s),N_\gamma(s)\right)=1$, $g\left(B_\gamma(s),B_\gamma(s)\right)=-1$, $g\left(T_\gamma(s),N_\gamma(s)\right)=0$, $g\left(T_\gamma(s),B_\gamma(s)\right)=0$, $g\left(N_\gamma(s),B_\gamma(s)\right)=0$ for all $s\in I$.

Case II: Let γ be a spacelike curve with a timelike principal normal N_γ in $\mathbb{E}^3_1$. Then the Serret-Frenet formulae for the curve γ are given by

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_\gamma \end{pmatrix}=\begin{pmatrix} 0 & \kappa_\gamma & 0 \\ \kappa_\gamma & 0 & \tau_\gamma \\ 0 & \tau_\gamma & 0 \end{pmatrix}\begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix},$$

where $g\left(T_\gamma(s),T_\gamma(s)\right)=1$, $g\left(N_\gamma(s),N_\gamma(s)\right)=-1$, $g\left(B_\gamma(s),B_\gamma(s)\right)=1$, $g\left(T_\gamma(s),N_\gamma(s)\right)=0$, $g\left(T_\gamma(s),B_\gamma(s)\right)=0$, $g\left(N_\gamma(s),B_\gamma(s)\right)=0$ for all $s\in I$.

Case III: Let γ be a spacelike curve with a null principal normal N_γ in $\mathbb{E}^3_1$. Then the Serret-Frenet formulae for the curve γ are given by

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_\gamma \end{pmatrix}=\begin{pmatrix} 0 & \kappa_\gamma & 0 \\ 0 & \tau_\gamma & 0 \\ -\kappa_\gamma & 0 & -\tau_\gamma \end{pmatrix}\begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix},$$

where $g\left(T_\gamma(s),T_\gamma(s)\right)=1$, $g\left(N_\gamma(s),N_\gamma(s)\right)=0$, $g\left(B_\gamma(s),B_\gamma(s)\right)=0$, $g\left(T_\gamma(s),N_\gamma(s)\right)=0$, $g\left(T_\gamma(s),B_\gamma(s)\right)=0$, $g\left(N_\gamma(s),B_\gamma(s)\right)=1$ for all $s\in I$. In this case, κ_γ can take only two values: $\kappa_\gamma=0$ if γ is a straight line and $\kappa_\gamma=1$ in the remaining cases.

Case IV: Let γ be a timelike curve in $\mathbb{E}^3_1$. Then the Serret-Frenet formulae for the curve γ are given by

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_\gamma \end{pmatrix}=\begin{pmatrix} 0 & \kappa_\gamma & 0 \\ \kappa_\gamma & 0 & \tau_\gamma \\ 0 & -\tau_\gamma & 0 \end{pmatrix}\begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix},$$

where $g\left(T_\gamma(s),T_\gamma(s)\right)=-1$, $g\left(N_\gamma(s),N_\gamma(s)\right)=1$, $g\left(B_\gamma(s),B_\gamma(s)\right)=1$, $g\left(T_\gamma(s),N_\gamma(s)\right)=0$, $g\left(T_\gamma(s),B_\gamma(s)\right)=0$, $g\left(N_\gamma(s),B_\gamma(s)\right)=0$ for all $s\in I$.

Case V: Let γ be a null (lightlike) curve in $\mathbb{E}^3_1$. Then the Serret-Frenet formulae for the curve γ are given by

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_\gamma \end{pmatrix}=\begin{pmatrix} 0 & \kappa_\gamma & 0 \\ \tau_\gamma & 0 & -\kappa_\gamma \\ 0 & -\tau_\gamma & 0 \end{pmatrix}\begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}, \tag{2.1}$$

where $g\left(T_\gamma(s),T_\gamma(s)\right)=0$, $g\left(N_\gamma(s),N_\gamma(s)\right)=1$, $g\left(B_\gamma(s),B_\gamma(s)\right)=0$, $g\left(T_\gamma(s),N_\gamma(s)\right)=0$, $g\left(T_\gamma(s),B_\gamma(s)\right)=1$, $g\left(N_\gamma(s),B_\gamma(s)\right)=0$ for all $s\in I$. In this case, κ_γ can take only two values: $\kappa_\gamma=0$ if γ is a straight null line and $\kappa_\gamma=1$ in the remaining cases.

The two-dimensional pseudo-Riemannian sphere of unit radius and centred at the origin in $\mathbb{E}^3_1$ is denoted and defined by

$$\mathbb{S}^2_1(1):=\left\{v\in\mathbb{E}^3_1 : g(v,v)=1\right\},$$

and the two-dimensional pseudo-hyperbolic space of unit radius and centred at the origin in $\mathbb{E}^3_1$ is denoted and defined by

$$\mathbb{H}^2_0(1):=\left\{v\in\mathbb{E}^3_1 : g(v,v)=-1\right\}.$$

For more elaborations of the above discussion please see [9]-[13].

An arbitrary plane π in $\mathbb{E}^3_1$ is spacelike, timelike or lightlike if the induced Lorentzian metric $g|_\pi$ is respectively positive definite, non-degenerate of index 1, or degenerate. A unit-speed null curve $\gamma : I \longrightarrow \mathbb{E}^3_1$ parametrized by pseudo arc-length function s is called a rectifying curve in $\mathbb{E}^3_1$ if its position vector always lies in its rectifying plane in $\mathbb{E}^3_1$, i.e., if its position vector γ in $\mathbb{E}^3_1$ can be expressed as

$$\gamma(s)=\lambda(s)T_\gamma(s)+\mu(s)B_\gamma(s),\ \ s\in I,$$

for some differentiable functions $\lambda,\mu : I\longrightarrow\mathbb{R}$ in pseudo arc-length parameter s of γ. Now, for some non-zero integrable function $f : I\longrightarrow\mathbb{R}$ in pseudo arc-length function s, the f-position vector of the curve γ in $\mathbb{E}^3_1$ is denoted by γ_f and is defined by

$$\gamma_f(s):=\int f(s)\,d\gamma$$

for all $s\in I$. Keeping in mind this notion of position vector of a curve in $\mathbb{E}^3_1$, we define a null f-rectifying curve in $\mathbb{E}^3_1$ as follows:

Definition 2.1. *(Null f-Rectifying Curve) Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit-speed null curve in $\mathbb{E}_1^3$ parametrized by pseudo arc-length function s with Frenet apparatus $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma, \tau_\gamma\}$, and let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in pseudo arc-length parameter s. The curve γ is called an f-rectifying curve in $\mathbb{E}_1^3$ if its f-position vector $\gamma_f = \int f\, d\gamma$ always lies in its rectifying plane in $\mathbb{E}_1^3$, i.e., if its f-position vector $\gamma_f = \int f\, d\gamma$ in $\mathbb{E}_1^3$ can be expressed as*

$$\gamma_f(s) = \int f(s)\, d\gamma = \lambda(s) T_\gamma(s) + \mu(s) B_\gamma(s), \ \ s \in I,$$

for two differentiable functions $\lambda, \mu : I \longrightarrow \mathbb{R}$ in pseudo arc-length parameter s.

In the next section, we shall see that if the function f vanishes on I, then the ratio $\dfrac{\tau_\gamma}{\kappa_\gamma}$ for the curve γ in $\mathbb{E}_1^3$ is constant, and hence it becomes a helix in $\mathbb{E}_1^3$. This is why we have taken here the function f as nowhere vanishing integrable function on I. And if the function f is a non-zero constant on I, then the ratio $\dfrac{\tau_\gamma}{\kappa_\gamma}$ for the curve γ in $\mathbb{E}_1^3$ is a non-constant linear function in pseudo arc-length parameter s, and hence it reduces to a rectifying curve in $\mathbb{E}_1^3$.

3. Characterizations of null f-rectifying curves in the Minkowski 3-space $\mathbb{E}_1^3$

First, we mention (and then prove) a theorem in which we characterize unit-speed null (lightlike) f-rectifying curves in the Minkowski 3-space $\mathbb{E}_1^3$ in terms of the norm functions, tangential components and binormal components of their f-position vectors.

Theorem 3.1. *Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit-speed null f-rectifying curve in $\mathbb{E}_1^3$ parametrized by pseudo arc-length function s with the curvature function $\kappa_\gamma \equiv 1$ and the torsion function τ_γ, and let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in pseudo arc-length parameter s with primitive function F. Then the following statements hold:*

1. *The norm function $\rho = \|\gamma_f\|$ is given by*
$$\rho(s) = \sqrt{|2cF(s)|}$$
for all $s \in I$, where c is a non-zero constant.
2. *The tangential component $g(\gamma_f, T_\gamma)$ of the f-position vector γ_f of the curve γ is a non-zero constant.*
3. *The torsion function τ_γ is non-zero, and the binormal component $g(\gamma_f, B_\gamma)$ of the f-position vector γ_f of the curve γ is given by*
$$g\left(\gamma_f(s), B_\gamma(s)\right) = F(s) = \int f(s)\, ds$$
for all $s \in I$.

Conversely, if $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in pseudo arc-length function s with primitive function F, and if $\gamma : I \longrightarrow \mathbb{E}_1^3$ is a unit-speed null curve in $\mathbb{E}_1^3$ and with the curvature function $\kappa_\gamma \equiv 1$ and the torsion function τ_γ, and any one of the statements 1, 2 or 3 holds, then γ is an f-rectifying curve in $\mathbb{E}_1^3$.

Proof. Let us first assume that $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit-speed null f-rectifying curve in $\mathbb{E}_1^3$ parametrized by pseudo arc-length function s with the curvature function $\kappa_\gamma \equiv 1$ and the torsion function τ_γ, where $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in pseudo arc-length parameter s with primitive function F. Then the f-position vector γ_f of the curve γ can be expressed as

$$\gamma_f(s) = \int f(s)\, d\gamma = \lambda(s) T_\gamma(s) + \mu(s) B_\gamma(s), \ \ s \in I, \tag{3.1}$$

for two derivable functions $\lambda, \mu : I \longrightarrow \mathbb{R}$ in pseudo arc-length parameter s. Differentiating both the sides of the equation (3.1) with respect to s and then applying the Serret-Frenet formulae (2.1), we obtain

$$f(s)\, T_\gamma(s) = \lambda'(s) T_\gamma(s) + \big(\lambda(s) - \mu(s)\tau_\gamma(s)\big) N_\gamma(s) + \mu'(s) B_\gamma(s) \tag{3.2}$$

for all $s \in I$. Equating the coefficients of like-terms from both the sides of equation (3.2), we find

$$\lambda'(s) = f(s), \ \ \lambda(s) - \mu(s)\tau_\gamma(s) = 0, \ \ \mu'(s) = 0$$

which implies

$$\begin{cases} \lambda(s) = \displaystyle\int f(s)\, ds = F(s), \\ \tau_\gamma(s) = \dfrac{\lambda(s)}{\mu(s)}, \\ \mu(s) = \text{a non-zero constant} = c \text{ (suppose)} \end{cases} \tag{3.3}$$

for all $s \in I$. We have the following:

1. Using the equation (3.1) and the relations (3.3), the norm function $\rho = \|\gamma_f\|$ is given by
$$\rho^2(s) = \left\|\gamma_f(s)\right\|^2 = \left|g\left(\gamma_f(s), \gamma_f(s)\right)\right| = |2cF(s)|$$
for all $s \in I$. That is,
$$\rho(s) = \sqrt{|2cF(s)|}$$
for all $s \in I$, where c is a non-zero constant.
2. Using the equation (3.1) and the relations (3.3), the tangential component $g(\gamma_f, T_\gamma)$ of the f-position vector γ_f of γ is given by
$$g\left(\gamma_f(s), T_\gamma(s)\right) = \mu(s) = c$$
for all $s \in I$. Hence, the tangential component $g(\gamma_f, T_\gamma)$ of the f-position vector γ_f of the curve γ is a non-zero constant.
3. From the relations (3.3) it is evident that $\tau_\gamma(s) \neq 0$ for all $s \in I$. Using the equation (3.1) and the relations (3.3), the binormal component $g(\gamma_f, B_\gamma)$ of the f-position vector γ_f of γ is given by
$$g\left(\gamma_f(s), B_\gamma(s)\right) = \lambda(s) = F(s)$$
for all $s \in I$.

Conversely, we assume that $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in pseudo arc-length function s with primitive function F, and we also assume that $\gamma : I \longrightarrow \mathbb{E}^3_1$ is a unit-speed null (lightlike) curve in $\mathbb{E}^3_1$ and with the curvature function $\kappa_\gamma \equiv 1$ and the torsion function τ_γ, and the statement 1 or 2 holds. For the statement 1, we have

$$g\left(\gamma_f(s), \gamma_f(s)\right) = 2cF(s) \tag{3.4}$$

for all $s \in I$, where c is a non-zero constant. Differentiating both the sides of the equation (3.4), and using the relations $\gamma_f'(s) = f(s)T_\gamma(s)$ and $F'(s) = f(s)$ for all $s \in I$, we obtain

$$g\left(\gamma_f(s), T(s)\right) = c \tag{3.5}$$

for all $s \in I$. This is nothing but the statement 2. So, in either case, we find the equation (3.5). Now, differentiating both the sides of the equation (3.5) with respect to s, and applying the relations $\gamma_f'(s) = f(s)T_\gamma(s)$, $T_\gamma'(s) = \kappa_\gamma(s)N_\gamma(s)$, $\kappa_\gamma(s) = 1$ and $g\left(T_\gamma(s), T_\gamma(s)\right) = 0$ for all $s \in I$, we obtain

$$\begin{aligned} f(s)\, g\left(T_\gamma(s), T_\gamma(s)\right) + \kappa_\gamma(s)\, g\left(\gamma_f(s), N_\gamma(s)\right) &= 0 \\ \Longrightarrow \qquad g\left(\gamma_f(s), N_\gamma(s)\right) &= 0 \end{aligned}$$

for all $s \in I$. This asserts us that γ is an f-rectifying curve in $\mathbb{E}^3_1$.

Finally, we assume that the statement 3 holds. Then for all $s \in I$, we have

$$g\left(\gamma_f(s), B_\gamma(s)\right) = F(s). \tag{3.6}$$

Differentiating both the sides of the equation (3.6) with respect to s, and in virtue of the relations $\gamma_f'(s) = f(s)T_\gamma(s)$, $B_\gamma'(s) = -\tau_\gamma(s)N_\gamma(s)$, $\tau_\gamma(s) \neq 0$, $g\left(T_\gamma(s), B_\gamma(s)\right) = 1$ and $F'(s) = f(s)$ for all $s \in I$, we obtain

$$\begin{aligned} f(s)\, g\left(T_\gamma(s), B_\gamma(s)\right) - \tau_\gamma(s) g\left(\gamma_f(s), N_\gamma(s)\right) &= f(s) \\ \Longrightarrow \qquad g\left(\gamma_f(s), N_\gamma(s)\right) &= 0 \end{aligned}$$

for all $s \in I$. This asserts us that γ is an f-rectifying curve in $\mathbb{E}^3_1$. □

In the next theorem, we characterize a unit-speed null f-rectifying curve in the Minkowski 3-space $\mathbb{E}_1^3$ by virtue of the ratio $\dfrac{\tau_\gamma}{\kappa_\gamma}$ of the curvature function κ_γ and the torsion function τ_γ.

Theorem 3.2. *Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit-speed null curve in $\mathbb{E}_1^3$ parametrized by pseudo arc-length function s with the curvature function $\kappa_\gamma \equiv 1$ and the torsion function τ_γ. Also, let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in pseudo arc-length parameter s with primitive function F. Then, up to isometries of $\mathbb{E}_1^3$, the curve γ is congruent to an f-rectifying curve in $\mathbb{E}_1^3$ if and only if the ratio $\dfrac{\tau_\gamma}{\kappa_\gamma}$ satisfies*

$$\frac{\tau_\gamma(s)}{\kappa_\gamma(s)} = \frac{1}{c}\, F(s)$$

for all $s \in I$, where c is a non-zero constant.

Proof. Let us first assume that $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit-speed null f-rectifying curve in $\mathbb{E}_1^3$ parametrized by pseudo arc-length function s with the curvature function $\kappa_\gamma \equiv 1$ and the torsion function τ_γ, and $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in pseudo arc-length parameter s with primitive function F. Then from the second one of the relations (3.3), we have

$$\frac{\tau_\gamma(s)}{\kappa_\gamma(s)} = \frac{\lambda(s)}{\mu(s)} = \frac{1}{c}\, F(s)$$

for all $s \in I$, where c is a non-zero constant.

Conversely, we assume that $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit-speed null curve in $\mathbb{E}_1^3$ parametrized s with the curvature function $\kappa_\gamma \equiv 1$ and the torsion function τ_γ, where $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in pseudo arc-length parameter s with primitive function F such that the ratio $\dfrac{\tau_\gamma}{\kappa_\gamma}$ is given by

$$\frac{\tau_\gamma(s)}{\kappa_\gamma(s)} = \frac{1}{c}\, F(s)$$

for all $s \in I$, where c is a non-zero constant. Then by applying the Serret-Frenet formulae (2.1), we obtain

$$\frac{d}{ds}\big(\gamma_f(s) - F(s)\,T_\gamma(s) - c\,B_\gamma(s)\big) = 0$$

for all $s \in I$. This proves that, up to isometries of $\mathbb{E}_1^3$, γ is an f- rectifying curve in $\mathbb{E}_1^3$. □

Remark 3.3. *Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit-speed null curve in $\mathbb{E}_1^3$ parametrized by pseudo arc-length function s with curvature function $\kappa_\gamma \equiv 1$ and the torsion function τ_γ. If the function f vanishes identically on I, then its primitive function F is a constant on I. Hence, by the previous theorem, the ratio $\dfrac{\tau_\gamma}{\kappa_\gamma}$ for the curve γ in $\mathbb{E}_1^3$ is given by*

$$\frac{\tau_\gamma(s)}{\kappa_\gamma(s)} = \frac{1}{c}\, F(s) = a \text{ constant}$$

for all $s \in I$. Consequently, the curve γ reduces to becomes a helix in $\mathbb{E}_1^3$ ([1]).

Again, if the function f is a non-zero constant on I, then its primitive function F is given by

$$F(s) = c_1 s + c_2$$

for all $s \in I$, where c_1 and c_2 are constants. Hence, by the previous theorem, the ratio $\dfrac{\tau_\gamma}{\kappa_\gamma}$ for the curve γ in $\mathbb{E}_1^3$ is given by

$$\frac{\tau_\gamma(s)}{\kappa_\gamma(s)} = \frac{1}{c}\, F(s) = \frac{1}{c}\,(c_1 s + c_2) = a\,s + b$$

for all $s \in I$, where $a = \dfrac{c_1}{c} (\neq 0)$ and $b = \dfrac{c_2}{c}$ are constants. Thus, the ratio $\dfrac{\tau_\gamma}{\kappa_\gamma}$ is a non-constant linear function in pseudo arc-length parameter s. Consequently, the curve γ reduces to a rectifying curve in $\mathbb{E}_1^3$ ([11]).

4. Classification of null f-rectifying curves in the Minkowski 3-space $\mathbb{E}_1^3$

In this section, we determine explicitly all unit-speed null f-rectifying curves in the Minkowski 3-space $\mathbb{E}_1^3$ in terms of their f-position vectors. The main theorem reads as follows:

Theorem 4.1. *Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit-speed null curve in $\mathbb{E}_1^3$ parametrized by pseudo arc-length function s and $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in s with primitive function F. Then γ is an f-rectifying curve in $\mathbb{E}_1^3$ having a spacelike (or timelike) f-position vector γ_f if and only if, up to a parametrization, its f-position vector γ_f is given by*

$$\gamma_f(t) = \sqrt{2cF(0)}\, e^t\, y(t)$$

for all possible t, where c is a positive constant, $F(0) > 0$ and $y = y(t)$ is a unit-speed timelike (respectively spacelike) curve in the pseudo-sphere $\mathbb{S}_1^2(1)$ (respectively the pseudo-hyperbolic space $\mathbb{H}_0^2(1)$).

Proof. First, we assume that γ is a unit-speed null f-rectifying curve in $\mathbb{E}_1^3$ having a spacelike f-position vector γ_f, where $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in s with primitive function F. Then we have

$$g\left(\gamma_f(s), \gamma_f(s)\right) > 0, \quad g\left(T_\gamma(s), T_\gamma(s)\right) = 0$$

for all $s \in I$, and from the proof of the Theorem 3.1, we obtain

$$\rho^2(s) = \left\|\gamma_f(s)\right\|^2 = \left|g\left(\gamma_f(s), \gamma_f(s)\right)\right| = 2cF(s), \tag{4.1}$$

for all $s \in I$, where we may choose c as an arbitrary positive constant. Now, we define a curve $y = y(s)$ by

$$y(s) := \frac{\gamma_f(s)}{\rho(s)} \tag{4.2}$$

for all $s \in I$. Then we have

$$g(y(s), y(s)) = \frac{g\left(\gamma_f(s), \gamma_f(s)\right)}{\rho^2(s)} = 1, \tag{4.3}$$

for all $s \in I$. Therefore, $y = y(s)$ is a curve in the pseudo-sphere $\mathbb{S}_1^2(1)$. Differentiating both the sides of the equation (4.3) with respect to s, we obtain

$$g\left(y(s), y'(s)\right) = 0 \tag{4.4}$$

for all $s \in I$. Now, from the equations (4.1) and (4.2), we find

$$\gamma_f(s) = y(s)\sqrt{2cF(s)} \tag{4.5}$$

for all $s \in I$. Differentiating both the sides of the equation (4.5) with respect to s, we get

$$f(s)\, T_\gamma(s) = y'(s)\sqrt{2cF(s)} + \frac{c\, f(s)\, y(s)}{\sqrt{2cF(s)}}, \tag{4.6}$$

for all $s \in I$. From the equations (4.3), (4.4) and (4.6), we obtain

$$g\left(y'(s), y'(s)\right) = -\frac{f^2(s)}{4F^2(s)} \tag{4.7}$$

for all $s \in I$. This indicates that y is a timelike curve. From the equation (4.7), we find

$$\left\|y'(s)\right\| = \sqrt{\left|g\left(y'(s), y'(s)\right)\right|} = \frac{f(s)}{2F(s)}$$

for all $s \in I$. Let t be arc-length parameter of the curve y in $\mathbb{S}_1^2(1)$ given by

$$t = \int_0^s \left\|y'(u)\right\| du.$$

Then we obtain

$$\begin{aligned} t &= \int_0^s \frac{f(u)}{2F(u)}\, du \\ \Longrightarrow \quad t &= \frac{1}{2}\ln F(s) - \frac{1}{2}\ln F(0) \\ \Longrightarrow \quad F(s) &= F(0)\, e^{2t}. \end{aligned} \tag{4.8}$$

It is obvious that $F(0) > 0$. Substituting the result (4.8) in (4.5), we obtain the f-position vector of γ as follows:

$$\gamma_f(t) = y(t)\sqrt{2\,c\,F(0)\,e^{2t}} = \sqrt{2\,c\,F(0)}\,e^t\,y(t)$$

for all possible t, where c is a positive constant, $F(0) > 0$ and $y = y(t)$ is a unit-speed timelike curve in the pseudo-sphere $\mathbb{S}_1^2(1)$.

Conversely, we assume that γ is a unit-speed null curve in $\mathbb{E}_1^3$ such that for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ in s with primitive function F the f-position vector γ_f of γ is given by

$$\gamma_f(t) := \sqrt{2\,c\,F(0)}\,e^t\,y(t) \tag{4.9}$$

for all possible t, where c is a positive constant, $F(0) > 0$ and $y = y(t)$ is a unit-speed timelike curve in the pseudo-sphere $\mathbb{S}_1^2(1)$. Since $y = y(t)$ is a unit-speed timelike curve in the pseudo-sphere $\mathbb{S}_1^2(1)$, we have $g(y'(t), y'(t)) = -1$, $g(y(t), y(t)) = 1$ and consequently $g(y(t), y'(t)) = 0$ for all t. Therefore, from the equation (4.9), we have

$$g\left(\gamma_f(t), \gamma_f(t)\right) = 2\,c\,F(0)\,e^{2t} \tag{4.10}$$

for all t. Now, we may reparametrize the curve γ by

$$t = \frac{1}{2}\left(\ln F(s) - \ln F(0)\right),$$

where s stands for arc-length parameter of γ. Then from (4.10), we have

$$g\left(\gamma_f(s), \gamma_f(s)\right) = 2\,c\,F(s)$$

for all $s \in I$. Therefore, the norm function $\rho = \|\gamma_f\|$ is given by

$$\rho^2(s) = \left\|\gamma_f(s)\right\|^2 = \left|g\left(\gamma_f(s), \gamma_f(s)\right)\right| = |2\,c\,F(s)|$$

for all $s \in I$, that is,

$$\rho(s) = \sqrt{|2\,c\,F(s)|}$$

for all $s \in I$, where c is a positive constant. Therefore, by applying Theorem 3.1, we conclude the nature of γ as an f-rectifying curve in $\mathbb{E}_1^3$.

The proof is analogous when γ is considered as a unit-speed null f-rectifying curve in $\mathbb{E}_1^3$ having a timelike f-position vector γ_f. □

5. Conclusion

In this paper, we introduced the notion of null (lightlike) f-rectifying curves in the Minkowski 3-Space $\mathbb{E}_1^3$ for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ in pseudo arc-length parameter s with primitive function F. Then we characterized such curves in $\mathbb{E}_1^3$. In Theorem 3.1, we have shown that for a unit-speed f-rectifying curve γ in $\mathbb{E}_1^3$, the norm function of its f-position vector γ_f is expressed in terms of the primitive function F, the tangential component of its f-position vector γ_f is a non-zero constant and the binormal component of its f-position vector γ_f is nothing but the primitive function F. Thereafter, in Theorem 3.2, it is shown that for a unit-speed f-rectifying curve γ in $\mathbb{E}_1^3$, the ratio $\frac{\tau_\gamma}{\kappa_\gamma}$ of the curvature function κ_γ and the torsion function τ_γ is a non-zero constant multiple of the primitive function F. Finally, in Theorem 4.1, we classified all such unit-speed null f-rectifying curves having spacelike or timelike f-position vectors in $\mathbb{E}_1^3$.

Acknowledgement

We would like to express our heartiest gratitude to Prof. Subenoy Chakraborty, Professor, Department of Mathematics, Jadavpur University, INDIA, for guiding us with his pearls of wisdom during the preparation of this article. We are also grateful to the anonymous honourable referees for their valuable time dedicated to this article and for their thorough comments which helped to improve this article.

References

[1] A. Pressley, *Elementary Differential Geometry*, 2nd ed., Springer, 2010.
[2] M. P. do Carmo, *Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition*, Courier Dover Publications, 2016.
[3] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. 2, 3rd ed., Publish or Perish, Houston, Texas, 1999.
[4] R. S. Millman, G. D. Parker, *Elements of Differential Geometry*, Prentice-Hall, Inc., New Jersey, 1977.

[5] B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?*, Amer. Math. Monthly, **110** (2003), 147–152.
[6] B. Y. Chen, *Rectifying curves and geodesics on a cone in the Euclidean 3-space*, Tamkang J. Math., **48** (2017), 209-214.
[7] B. Y. Chen, F. Dillen, *Rectifying curves as centrodes and extremal curves*, Bull. Inst. Math. Acad. Sinica, **33** (2005), 77-90.
[8] S. Deshmukh, B. Y. Chen, S. Alshamari, *On rectifying curves in Euclidean 3-space*, Turk. J. Math., **42** (2018), 609-620.
[9] K. Ilarslan, E. Nešović, *Timelike and null normal curves in Minkowski space* $\mathbb{E}_1^3$, Indian J. Pure Appl. Math., **35**(7) (2004), 881-888.
[10] K. Ilarslan, E. Nešović, *On rectifying curves as centrodes and extremal curves in the Minkowski 3-Space*, Novi Sad J. Math., **37** (2007), 53-64.
[11] K. Ilarslan, E. Nešović, T. M. Petrović, *Some characterization of rectifying curves in the Minkowski 3-Space*, Novi Sad J. Math., **33** (2003), 23-32.
[12] R. Lopez, *Differential Geometry of curves and surfaces in Lorentz-Minkowski space*, Int. Electron. J. Geom., **7(1)** (2014), 44–107.
[13] B. O'Neill, *Semi–Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 17-24
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.643374

Gauss Map and Local Approach of Isoparametric Surfaces in Lorentz and Euclidean Space

Emre Öztürk[1]

[1] *Turkish Court of Accounts, Ankara, Turkey*

Article Info

Keywords: *Gauss map, Isoparametric surface, Linear operator*
2010 AMS: *14J25, 32A25*
Received: *5 November 2019*
Accepted: *23 April 2020*
Available online: *10 June 2020*

Abstract

In this study, we determine the isoparametric surfaces and we give the Gauss map of these surfaces by semi symmetric matrix, in Lorentz space. Also we define any chord property and we show that the surfaces which have the chord property corresponds to isoparametric surfaces. Moreover, we consider the chord property locally and we give some examples in the Euclidean space.

1. Introduction and preliminaries

Isoparametric surfaces, surfaces with constant principal curvatures, are studied in [1]-[3] in terms of the chord property and helical points of the surface in the Euclidean space. In [4], the unit disk characterized by the following:

Lemma 1.1. *The only bounded, smooth and simply-connected plane region whose Szegö kernel coincides with the Cauchy kernel is the disc.*

Kerzman and Stein [4] used complex analysis technics related with the chord of the curve and they proved the Lemma above. Then, Boas [5] extended this idea to $n-$ dimensional Euclidean space. Boas gave the following theorem, by the help of Bochner-Martinelli kernel:

Theorem 1.2. *Ball is the only bounded C^1 domain in $\mathbb{R}^m$ such that given any two points of the boundary, the chord joining them meets the normals at the two endpoints with equal angles.*

Thus, in ([5], Proof of Theorem 2, pp. 277-278), the chord property idea of [4] extended to the hyperspheres. Moreover, Boas [2], extended his study [5], to all isoparametric surfaces in the Euclidean space. He gave such a local characterization theorem for hyperspheres and spherical cylinders and proved that these surfaces satisfy

$$\left\langle x-y, \overrightarrow{\nabla f}(x)\right\rangle = \left\langle y-x, \overrightarrow{\nabla f}(y)\right\rangle \tag{1.1}$$

where x,y are points on surface and $\overrightarrow{\nabla f}$ is the unit normal (gradient) vector field. Wegner [6], gave the short proof of ([2], Local characterization theorem, p.120). In [1], in the light of [2, 5], the equation (1.1) considered on a hypersurface such that a unit normal vector field G is naturally defined on the surface. Such G is called the Gauss map of surface. For any hypersphere, the chord joining any two points on it meets the sphere at the same angle at the two points, that is, the sphere satisfies

$$\langle y-x, G(x)+G(y)\rangle = 0 \tag{1.2}$$

Email addres and ORCID number: **emreozturk1471@gmail.com https://orcid.org/0000-0001-6638-3233 (E. Öztürk)**

In [1], the following question considered:

What are the hypersurfaces of Euclidean space that satisfy the (1.2)?

They used algebraic approaches and stated that Gauss map of surfaces which satisfy (1.2) is written as $G(x) = Ax + b$ where A is constant symmetric matrix, b is column vector. In [7], some special curves are defined and relations between these curves and isoparametric surfaces are given in Lorentz-Minkowski space. In this study, we are looking for answers of the followings:

What are the hypersurfaces of Lorentz space that satisfy the (1.2)?

and

What are the hypersurfaces of Euclidean space that satisfy the (1.2) locally?

In Lorentz space, vectors have different causal characters such as if $\langle u,u\rangle > 0$ or $u = 0$, $\langle u,u\rangle < 0$ and $\langle u,u\rangle = 0$ $(u \neq 0)$ then u is called by spacelike, timelike and lightlike (or null) vector respectively. The number of timelike vectors of the orthonormal basis of the vector space is called the index of space and usually denoted by ν. Through the [8], we give the followings:

Definition 1.3. *Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be a two vector distinct from zero in $n-$ dimensional real vector space $\mathbb{R}^n$. Following inner product,*

$$\langle X,Y\rangle = -x_1y_1 + \sum_{i=2}^{n} x_iy_i$$

is called by Lorentzian inner product of X and Y, and $\langle , \rangle$ is called metric tensor of vector space. $(\mathbb{R}^n, \langle , \rangle)$ is called Lorentz space and denoted by $\mathbb{L}^n$ or $\mathbb{R}^n_1$. If $\langle u,v\rangle = 0$ implies that u=0 for all v where $u,v \in T_P\mathbb{R}^n$, then $\langle , \rangle$ is called canonical non-degenerated inner product with arbitrary index.

Norm of the vector $u \in \mathbb{R}^n_1$ is given by $\|u\| = \sqrt{|\langle u,u\rangle|}$. Let the index of $n-$dimensional non-degenerated inner product space of V be $1 \leq \nu \leq n$ and its orthonormal base be $\{e_1, e_2, ..., e_n\}$. Then $\varepsilon_1 = \varepsilon_2 = ... = \varepsilon_\nu = -1$ and $\varepsilon_{\nu+1} = \varepsilon_{\nu+2} = ... = \varepsilon_n = 1$, where $\varepsilon_i = \langle e_i, e_i\rangle$. Therefore, the diagonal matrix $(\delta_{ij}\varepsilon_j)$ is called by the sign matrix of V such that

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \quad 1 \leq i,j \leq n$$

is Kronecker delta.

Definition 1.4. *[8] Let $n \geq 2$ and $0 \leq \nu \leq n$,*

(1) The pseudosphere of radius $r > 0$ in $\mathbb{R}^{n+1}_\nu$ is the hyperquadric

$$\mathbb{S}^n_\nu(r) = \left\{P \in \mathbb{R}^{n+1}_\nu \mid \langle P,P\rangle = r^2\right\}$$

with dimension n and index ν.

(2) The pseudohyperbolic space of radius $r > 0$ in $\mathbb{R}^{n+1}_{\nu+1}$ is the hyperquadric

$$\mathbb{H}^n_\nu(r) = \left\{P \in \mathbb{R}^{n+1}_{\nu+1} \mid \langle P,P\rangle = -r^2\right\}$$

with dimension n and index ν.

Definition 1.5. *Let M be a hypersurface in the Minkowski space and $\overrightarrow{n}$ be a unit normal vector field of M. If $\langle \overrightarrow{n}, \overrightarrow{n}\rangle < 0$, $(\langle \overrightarrow{n}, \overrightarrow{n}\rangle > 0)$ then M is said to be spacelike (timelike) surface.*

Lemma 1.6. *[8] Let S be a shape operator (Weingarten map) and v be a tangent vector on M. $S(v) = -\nabla_v \overrightarrow{n}$ and for all $P \in M$, linear operator of S is self-adjoint on T_PM. Here ∇ is Levi-Civita connection on M in $\mathbb{R}^n_1$space.*

2. Linear operators and isoparametric surfaces

An integral operator in complex space $\mathbb{C}^n$ is given by

$$\zeta(f,x) = \int_a^b f(x)K(t,x)dt$$

such that $K(t,x)$ is continuous according to parameter t and it is called the kernel of the operator ζ. Suppose that $\mathscr{B}$ is a bounded smooth domain in the complex plane. The Cauchy kernel represents holomorphic functions f in $\mathscr{B}$ in terms of the boundary values on γ. Here γ is the boundary of the domain $\mathscr{B}$. Cauchy integral operator on any $\mathscr{B}$ domain whose bounded by the curve γ in complex plane is given by

$$\zeta(z,w)=\frac{1}{2\pi i}\int\limits_{z\in\gamma}\frac{f(z)}{z-w}\dot{\gamma}(z)d\sigma(z)$$

where $w\in\mathbb{C}$ is on γ. Here $dz=\dot{\gamma}(z)d\sigma$, $z=\gamma(s)$ unit speed curve and $d\sigma$ is Lebesgue measure (arc lenght). Hence, Cauchy kernel of the ζ operator is $\frac{1}{2\pi i}\frac{1}{z-w}\dot{\gamma}(z)$. Similarly the $\mathscr{S}(z,w)$ Szegö integral operator is given by

$$\zeta(z,w)=\frac{1}{2\pi i}\int\limits_{z\in\gamma}f(z)\mathscr{S}(z,w)d\sigma(z)$$

where the kernel is considered the orthogonal projection of $\mathbb{S}:\mathscr{L}^2(\gamma,d\sigma)\to\mathscr{H}^2(\gamma)$. Here $\mathscr{H}^2(\gamma)$ is closed subspace of $\mathscr{L}^2(\gamma,d\sigma)$ of boundary values of holomorphic functions in $\mathscr{B}$. It is easy to see that $\mathbb{S}\zeta=\zeta$ holds identically and the curve γ satisfy

$$\langle\gamma(t)-\gamma(s),T(t)-T(s)\rangle=0 \tag{2.1}$$

where T is the unit tangent normal vector field of the curve. It follows from the definitions above and (2.1) that γ is non-null hyperbolic curve in Lorentz plane.
Now we extend the chord idea to the surfaces in the high dimensions and we give some characterizations about these surfaces by the help of Gauss map itself, in terms of [1] and [7], in Lorentz space. Throughout this chapter, the metric tensor will be considered as a Lorentzian unless otherwise mentioned. Let us give the following definition first.

Definition 2.1. *Let M be a non-null hypersurface and G is Gauss map of M. If*

$$\langle Q-P,G(P)+G(Q)\rangle=0$$

for all $P,Q\in M$ then, M is called by $\mathbf{G}-$hypersurface.

Theorem 2.2. *Let M be a $\mathbf{G}-$hypersurface. Gauss map of this surface is given by*

$$G(x)=Ax+b$$

where A is the semi-symmetric matrix and $b\in E_1^n$ column vector.

Proof. Let the hypersurface M fully lies in space and consider the points $y_0,y_1,...,y_n$ on M such that $\{y_{j-}y_0 \mid 1\le j\le n\}$ spans $\mathbb{R}_1^n$. Similar to [1], we find

$$A^T=\varepsilon\left(B_jA_j^{-1}\right)\varepsilon \tag{2.2}$$

where $\varepsilon=diag(-1,1,...,1)$ is the sign matrix, A_j and B_j are the $n\times n$ matrices that accepts the $y_{j-}y_0$ and $G(y_j)-G(y_0)$ as $j-$column respectively. Also $b=\sum\limits_{k=1}^{n}b_k\alpha_k$ such that b_k is given by

$$\begin{pmatrix}\langle\alpha_1,\alpha_1\rangle & \langle\alpha_1,\alpha_2\rangle & \dots & \langle\alpha_1,\alpha_n\rangle\\ \langle\alpha_2,\alpha_1\rangle & \ddots & & \langle\alpha_2,\alpha_n\rangle\\ \vdots & & \ddots & \vdots\\ \langle\alpha_n,\alpha_1\rangle & \langle\alpha_n,\alpha_2\rangle & \dots & \langle\alpha_n,\alpha_n\rangle\end{pmatrix}\begin{pmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{pmatrix}=\begin{pmatrix}c_1\\ c_2\\ \vdots\\ c_n\end{pmatrix}$$

where $\alpha_j=y_j-y_0$ and $c_j=\langle G(y_0),y_0\rangle-\langle G(y_j),y_j\rangle$. Hence we write

$$c_j=\langle\alpha_j,\alpha_1\rangle b_1+\langle\alpha_j,\alpha_2\rangle b_2+\cdots+\langle\alpha_j,\alpha_n\rangle b_n. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$G(x)=Ax+b.$$

We note that A is constant matrix (see [1]). Now we prove that A is semi-symmetric matrix. Let $X,Y \in T_xM$ be a tangent vector. Due to Lemma 1.6

$$\nabla_X G(X) = AX = -S(X) \tag{2.4}$$

and

$$\langle S(X),Y\rangle = \langle X,S(Y)\rangle. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\begin{aligned}\langle -AX,Y\rangle = \langle X,-AY\rangle \quad &\Leftrightarrow \quad \langle AX,Y\rangle = \langle X,AY\rangle \\ &\Leftrightarrow \quad \left\langle X,(\varepsilon A^T\varepsilon)Y\right\rangle = \langle X,AY\rangle \\ &\Leftrightarrow \quad A = \varepsilon A^T\varepsilon\end{aligned}$$

which is intended. □

Theorem 2.3. *Let us assume that hypersurface M have diagonalized shape operator. For non-null hypersurface M, the following statements are equivalent:*

i) *M is the **G**-hypersurface.*

ii) *M is an isopametric surface.*

iii) *M is the open part of non-null hyperplane, pseudosphere, pseudohyperbolic space, pseudospherical cylinder or pseudohyperbolic cylinder.*

Proof. Let M be a $\mathbf{G}-$hypersurface and $\{E_1,E_2,...,E_{n-1}\}$ orthonormal frame on surface such that $E_i, 1\leq i\leq n-1$ are characteristic vectors corresponding to characteristic values μ_i of the shape operator. Hence, $S(E_i)=\mu_iE_i$ for all i. Due to Theorem 2.2, $G(x)=Ax+b$ where A is semi-symmetric matrix. It follows from (2.4) that

$$AE_j(x) = -S(E_j(x)) = -\mu_j(x)E_j(x)$$

and

$$(A+\mu_jI)\,E_j(x) = 0.$$

In order to the existence of non-zero characteristic vectors

$$\det(A+\mu_jI) = 0. \tag{2.6}$$

From equation (2.6), it is obvious that μ_i is constant. Therefore, M is an isoparametric surface.
Let M be an isoparametric surface. Let us define

$$f(x) = \langle Ax+b,Ax+b\rangle \tag{2.7}$$

where $f:\mathbb{R}^n_1\to\mathbb{R}$ and $M\subset f^{-1}(\pm1)$. It follows from (2.7) that

$$f(x) = \left\langle x,A^2x\right\rangle + 2\langle x,Ab\rangle + \langle b,b\rangle.$$

By straightforward calculations we get $\overrightarrow{\nabla f}(x)=2A(Ax+b)$. Since the gradient of f and Gauss map G is linear dependent

$$A(Ax+b) = \lambda(x)(Ax+b),\ x\in M \tag{2.8}$$

for some real valued $\lambda(x)$ functions. It follows from (2.8) that $(A-\lambda(x)I)(Ax+b)=0$ and $\det(A-\lambda(x)I)=0$. Obviously $\lambda(x)$ is constant. Let us consider $V=\{Ax+b \mid x\in M\}$ as a characteristic space that corresponding to $\lambda(x)=\lambda$ and $Sp\{V\}$ is normal space at $x\in M$. Let us determine the surface M depends on the norm of V.
a) Let $\|V\|=1$. In this case $G(x)=b$ is constant and M is the open part of non-null hyperplane.
b) If $\|V\|=n$ then $V=Im(A)$. For some $r>0$ and $\lambda=\pm\frac{1}{r}$ we have $A\mid_V=\pm\frac{1}{r}I$ such that $G(x)=\pm\frac{1}{r}x+b$. Depends on causal character of $Sp\{V\}$, we get pseudosphere of $\mathbb{S}^{n-1}_1$ or pseudohyperbolic space of $\mathbb{H}^{n-1}_1$.

c) Let $\|V\| = p, 2 \le p \le n-1$. Dimension of $V^{\perp}$ orthogonal complement is $n-p$ and $V^{\perp} \subseteq T_xM$. In the neighborhood of $x_0 \in M$, we choose $\{E_1(x), E_2(x), ..., E_{n-1}(x)\}$ orthonormal frame such that $E_1, E_2, ..., E_{n-p}$ are constant in $V^{\perp}$., we write

$$V = Sp\left\{E_{n-p+1}(x), E_{n-p+2}(x), ..., E_{n-1}(x), G(x)\right\}$$

The tangent subspace spanned by $\left\{E_{n-p+1}(x), E_{n-p+2}(x), ..., E_{n-1}(x)\right\}$ is integrable and integral submanifold of M_1 through the point x_0 is given as $M_1 = M \cap (x_0 + V)$. Hence $\mathbb{R}_1^{n-p} = V^{\perp}$ and $M = M_1 \times \mathbb{R}_1^{n-p}$ where M_1 is hypersurface in $\mathbb{R}_1^p$ and Gauss map of M_1 satisfy the $G_1(x) = G(x)$. Besides, $A_1 = A\mid_V$ satisfy

$$G_1(x) = A_1 x + b. \tag{2.9}$$

It follows from (2.8) and (2.9) that $A_1 = \pm\frac{1}{r}I$. So M_1 is pseudosphere of $\mathbb{S}_1^{p-1}(r)$ or pseudohyperbolic space of $\mathbb{H}_1^{p-1}(r)$. Hence M is the open part of $\mathbb{S}_1^{p-1}(r) \times \mathbb{R}^{n-p}$ or $\mathbb{H}_1^{p-1}(r) \times \mathbb{R}^{n-p}$, respectively.
Let us consider isometric immersion $f_1 : M \to \mathbb{R}_1^n$ with respect to $[x_1, x_2, ..., x_n]$ rectangular coordinate system. Let us give

$$f_1(x_1, x_2, ..., x_n) = a_1x_1 + a_2x_2 + ... + a_nx_n = c$$

where $a_i, 1 \le i \le n$ are constant coefficients, $x = (x_1, x_2, ..., x_n) \in M$ and c is real number. The Gauss map of this immersion is given by

$$G(P) := \frac{\overrightarrow{\nabla f_1}}{\left\|\overrightarrow{\nabla f_1}\right\|}\mid_P = \frac{1}{m}(-a_1, a_2, ..., a_n)\mid_P$$

where $P \in M$ and $m = \sqrt{\left|-a_1^2 + a_2^2 + ... + a_n^2\right|}$. It can be easily seen that

$$\langle Q - P, G(P) + G(Q)\rangle = 0$$

where $P, Q \in f_1$. By Definition 1.4, pseudosphere with center x_0 and radius r is given by

$$\mathbb{S}_1^{n-1}(r) = \left\{x \in \mathbb{R}_1^n : \langle x - x_0, x - x_0\rangle = r^2\right\}.$$

Without loss of generality, we can consider $x_0 = (0, 0, ..., 0)$ and $x = (x_1, x_2, ..., x_n)$. In this case,

$$f_2(x_1, x_2, ..., x_n) = -x_1^2 + x_2^2 + ... + x_n^2 = r^2$$

where $f_2 : \mathbb{S}_1^{n-1} \to \mathbb{R}_1^n$ is an isometric immersion. The Gauss map of this immersion is given by

$$G(P) := \frac{\overrightarrow{\nabla f_2}}{\left\|\overrightarrow{\nabla f_2}\right\|}\mid_P = \frac{1}{r}(x_1, x_2, ..., x_n)\mid_P.$$

By straightforward calculations we get

$$\langle Q - P, G(P) + G(Q)\rangle = 0$$

where $P, Q \in f_2$. Therefore, $\mathbb{S}_1^{n-1}(r)$ is $\mathbf{G}-$hypersurface (similarly $\mathbb{H}_1^{n-1}(r)$ is $\mathbf{G}-$hypersurface). Moreover, we consider

$$-x_1^2 + x_2^2 + ... + x_p^2 = r^2,\ x_{p+1} = u_{p+1}, x_{p+2} = u_{p+2}, ..., x_n = u_n$$

where u_i are real variables such that $p+1 \le i \le n$. By the help of natural Lorentz projection $\pi_L : \mathbb{R}_1^p \times \mathbb{R}_1^{n-p} \to \mathbb{R}_1^p \hookrightarrow \mathbb{R}_1^n$ onto $\mathbb{S}_1^{p-1}$ we get

$$G(P) = \frac{1}{r}(x_1, x_2, ..., x_p, 0, ..., 0)\mid_P.$$

Hence,

$$\langle Q - P, G(P) + G(Q)\rangle = 0$$

where $P, Q \in \mathbb{S}_1^{p-1}(r) \times \mathbb{R}^{n-p}$. Proof is similar for $\mathbb{H}_1^{p-1}(r) \times \mathbb{R}^{n-p}$. □

3. Local isoparametric surfaces in Euclidean space

Chord property of isoparametric surfaces are examined locally and globally in [2, 5]. Our point of view to localization is totally different from previous studies. Let us give the following definition and results.

Definition 3.1. *Let M be a hypersurface in Euclidean space and $\langle , \rangle$ be the metric tensor of the space. If some points such $P, Q \in M$ satisfy*

$$\langle Q-P, G(P)+G(Q)\rangle = 0$$

then, M is called as local isoparametric surface.

Theorem 3.2. *The helicoid surface given by*

$$\Phi(s,t) = (0,0,bs) + t(\cos s, \sin s, 0), \ b \neq 0$$

is local isoparametric surface if and only if

$$p_1 = q_1 \quad or \quad p_2 = -q_2$$

where $\varphi(p_1,p_2) = P$, $\varphi(q_1,q_2) = Q$, $\varphi : U \subseteq \mathbb{R}^2 \to \Phi$ and $(p_1,p_2), (q_1,q_2) \in U$.

Proof. Let us consider the surface $\varphi(U) = \Phi$ and $\varphi : U \to \Phi$ differentiable map where $U \subseteq \mathbb{R}^2$. Let $P, Q \in \Phi$ two points on surface such that $\varphi(p_1,p_2) = P$ and $\varphi(q_1,q_2) = Q$. By straightforward calculations we get

$$\Phi_s(s,t) = (-t\sin s, t\cos s, b)$$

and

$$\Phi_t(s,t) = (\cos s, \sin s, 0)$$

Unit normal vector field Z of helicoid is given by

$$Z \circ \varphi = \frac{\Phi_s(s,t) \times \Phi_t(s,t)}{\|\Phi_s(s,t) \times \Phi_t(s,t)\|} = \left(-\frac{b}{\sqrt{b^2+t^2}}\sin s, \frac{b}{\sqrt{b^2+t^2}}\cos s, -\frac{1}{\sqrt{b^2+t^2}}t\right).$$

Therefore,

$$G(P)+G(Q) = (\alpha_1, \alpha_2, \alpha_3) \tag{3.1}$$

where

$$\begin{aligned}
\alpha_1 &= -\frac{b}{\sqrt{b^2+p_2^2}}\sin p_1 - \frac{b}{\sqrt{b^2+q_2^2}}\sin q_1 \\
\alpha_2 &= \frac{b}{\sqrt{b^2+p_2^2}}\cos p_1 + \frac{b}{\sqrt{b^2+q_2^2}}\cos q_1 \\
\alpha_3 &= -\frac{b}{\sqrt{b^2+p_2^2}}p_2 - \frac{1}{\sqrt{b^2+q_2^2}}q_2.
\end{aligned}$$

Besides,

$$Q-P = (q_2\cos q_1 - p_2\cos p_1, q_2\sin q_1 - p_2\sin p_1, bq_1 - bp_1). \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\langle Q-P, G(P)+G(Q)\rangle = b\left(\frac{p_2}{\sqrt{b^2+p_2^2}} + \frac{q_2}{\sqrt{b^2+q_2^2}}\right)(\sin(q_1-p_1) + (p_1-q_1)).$$

Obviously $\langle Q-P, G(P)+G(Q)\rangle = 0$ if and only if $p_1 = q_1$ or $p_2 = -q_2$. By Definition 3.1, Φ is local isoparametric surface. □

Theorem 3.3. *The hyperbolic paraboloid surface given by*

$$\Phi(u,v)=(u,v,\frac{v^2}{b^2}-\frac{u^2}{a^2}),\ a,b\in\mathbb{R}\setminus\{0\}$$

is local isoparametric surface if and only if

$$\frac{a}{b}=\left|\frac{p_1-q_1}{p_2-q_2}\right|$$

where $\varphi(p_1,p_2)=P$, $\varphi(q_1,q_2)=Q$,$\varphi:U\subseteq\mathbb{R}^2\to\Phi$ *and* (p_1,p_2) , $(q_1,q_2)\in U$.

Proof. By straightforward calculations we get

$$Z\circ\varphi=\left(-\frac{2u}{a^2\sqrt{1+\frac{4u^2}{a^4}+\frac{4v^2}{b^4}}},\frac{2v}{b^2\sqrt{1+\frac{4u^2}{a^4}+\frac{4v^2}{b^4}}},-\frac{1}{\sqrt{1+\frac{4u^2}{a^4}+\frac{4v^2}{b^4}}}\right)$$

and

$$\langle Q-P,G(P)+G(Q)\rangle=\frac{\left(-b^2(p_1-q_1)^2+a^2(p_2-q_2)^2\right)\left(\sqrt{1+\frac{4p_1^2}{a^4}+\frac{4p_2^2}{b^4}}-\sqrt{1+\frac{4q_1^2}{a^4}+\frac{4q_2^2}{b^4}}\right)}{a^2b^2\sqrt{1+\frac{4p_1^2}{a^4}+\frac{4p_2^2}{b^4}}\sqrt{1+\frac{4q_1^2}{a^4}+\frac{4q_2^2}{b^4}}}.$$

Hence $\langle Q-P,G(P)+G(Q)\rangle=0$ if and only if

$$\frac{a}{b}=\left|\frac{p_1-q_1}{p_2-q_2}\right|$$

which is intended. □

Let us give the following surface, in the light of [9].

Theorem 3.4. *The Viviani ruled surface given by*

$$\Phi(u,v)=(\frac{5}{2}+\frac{5}{2}\cos u,\frac{5}{2}\sin v,5\sin\frac{u}{2})+4v(1+\cos u,\sin u,2\sin\frac{u}{2})$$

is local isoparametric surface if and only if

$$p_1=q_1+4k\pi,\ k\in\mathbb{Z}$$

where $\varphi(p_1,p_2)=P$, $\varphi(q_1,q_2)=Q$,$\varphi:U\subseteq\mathbb{R}^2\to\Phi$ *and* (p_1,p_2) , $(q_1,q_2)\in U$.

Proof. Similar to previous operations we get

$$Z\circ\varphi=\left(\frac{\sqrt{2}\sin^3\frac{u}{2}}{\sqrt{3+\cos u}},\frac{-5\cos\frac{u}{2}+\cos\frac{3u}{2}}{2\sqrt{2}\sqrt{3+\cos u}},\frac{\sqrt{2}\cos^2\frac{u}{2}}{\sqrt{3+\cos u}}\right)$$

and

$$\langle Q-P,G(P)+G(Q)\rangle=\alpha\sin^2(\frac{p_1-q_1}{4})$$

where α is non-zero constant. Hence $\langle Q-P,G(P)+G(Q)\rangle=0$ if and only if

$$p_1=q_1+4k\pi,\ k\in\mathbb{Z}$$

and this completes the proof. □

4. Conclusion

In this study, we showed that the Gauss map of isoparametric surfaces is written by $G(x) = Ax + b$ where A is semi-symmetric matrix and b is column vector, in Lorentz space. Moreover, in the Euclidean space; we gave the definition of local isoparametric surface, and we examined the some of them such as helicoid, hyperbolic paraboloid and Viviani ruled surface, by different point of view from the previous studies.

References

[1] D. S. Kim, Y. H. Kim, *New characterizations of spheres, cylinders and $W-$Curves*, Linear Algebra Appl., **432** (2010), 3002-3006.
[2] H. P. Boas, *Spheres and cylinders: A local geometric characterization*, Illinois J. Math., **28**(1) (1984), 120-124.
[3] Y. H. Kim, K. E. Lee, *Surfaces of Euclidean 4-space whose geodesics are W-Curves*, Nihonkai Math. J., **4** (1993), 221-232.
[4] N. Kerzman, E. M. Stein, *The Cauchy kernel, the Szegö kernel, and the Riemann mapping function*, Math. Ann., **236** (1978), 85-93.
[5] H. P. Boas, *A geometric characterization of the ball and the Bochner-Martinelli kernel*, Math. Ann., **248** (1980), 275-278.
[6] B. Wegner, *A differential geometric proof of the local geometric characterization of spheres and cylinders by Boas*, Math. Balkanica, **2** (1988), 294-295.
[7] E. Öztürk, Y. Yaylı, *W-curves in Lorentz-Minkowski space*, Math. Sci. Appl. E-Notes, **5**(2) (2017), 76-88.
[8] B. O'Neill, *Semi-Riemann geometry with applications to relativity*, Academic Press. Inc., 1983.
[9] C. Ekici, E. Özüsağlam, M. Çimdiker, E. Öztürk, *On the curvatures of Viviani ruled surface*, Ciencia e Tecnica Vitivinicola, **29**(7) (2014), 449-475.

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 25-28
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.725481

Topological Properties of The Digital Line

Paulraj Gnanachandra[1*], Saeid Jafari[2] and Lellis Thivagar[3]

[1]*Centre for Research and Post Graduate Studies in Mathematics, Ayya Nadar Janaki Ammal College (Autonomous) , Sivakasi, India*
[2]*College of Vestsjaelland South, Herrestraede 11, Slagelse, Denmark*
[3]*School of Mathematics,Madurai Kamaraj University, Madurai, Tamil Nadu, India*
[*]*Corresponding author*

Article Info

Keywords: *Digital interval, Digital topology, Khalimsky line*
2010 AMS: *57M05, 68R99, 68U10*
Received: *22 April 2020*
Accepted: *31 May 2020*
Available online: *10 June 2020*

Abstract

The main purpose of Digital topology is the study of topological properties of discrete objects which are obtained digitizing continuous objects. Digital topology plays a very important role in computer vision, image processing and computer graphics. The ultimate aim of this article is to analyze the behavior of various general topological concepts in the Khalimsky topology. In this article, we provide some results and examples of topology on $\mathbb{Z}$, the set of all integers. Also, we explain the concepts of digital line and digital intervals with illustrative counterexamples.

1. Introduction

Digital topology is a term that has arisen in the study the digital images. Topological properties of images on a Cathode ray tube are essential in studying graphics, digital processing, pattern analysis and artificial intelligence. There are two fundamental approaches to the digital images. They are graph theoretic and topological approaches. The first approach was initiated by A. Rosenfeld [1]-[4] and the topological approach was originated by Kong, Kopperman, Meyer and Khalimsky et. al.[5] in the 1990s. For finite spaces, these two approaches are equivalent. The study commences with the Jordan Curve Theorem and elucidates that a simple closed curve separates the real plane $\mathbb{R}\times\mathbb{R}$ into exactly two connected components. Khalimsky et.al [5] utilized a connected topology on a finite ordered set in the context of computer graphics. One such a topology on $\mathbb{Z}$, (the set of all integers) is the topology generated by the triples $\{2m-1,2m,2m+1\}$ as a subbase. This topology was introduced by Khalimsky and so it is called the Khalimsky topology.

2. Preliminaries

Let $\mathscr{D}$ stand for the set of all triples $\{2m-1,2m,2m+1\}$ where $m\in\mathbb{Z}$. Then $\mathscr{D}$ is a subbase for some topology on $\mathbb{Z}$, symbolized by k. The set of all integers $\mathbb{Z}$ with this topology k that is $(\mathbb{Z},\mathsf{k})$ is called the digital line. Throughout, $\mathbb{O}$ and $\mathbb{E}$ denote the set of all odd and even integers respectively and $\check{D}$ denotes the set of all dense subsets of $\mathbb{Z}$. The closure and interior of a set A of a topological space (X,τ) is denoted by $cl_\tau(\mathsf{A})$ and $int_\tau(\mathsf{A})$. Similarly the interior and closure of $(\mathbb{Z},\mathsf{k})$ is denoted by $cl_{\mathsf{k}}(\mathsf{A})$ and $int_{\mathsf{k}}(\mathsf{A})$. Beyond doubt, a base for $(\mathbb{Z},\mathsf{k})$ is $\mathscr{D}\cup\mathscr{B}$ where $\mathscr{B}=\{\{m\}:m\in\mathbb{O}\}$. This follows from the fact that : Let $\mathscr{C}=\{2m-1,2m,2m+1\}$ and $\mathscr{D}=\{2n-1,2n,2n+1\},m,n\in\mathbb{Z}$. Then

Email addresses and ORCID numbers: **pgchandra07@gmail.com, https://orcid.org/ 0000-0001-6089-6441 (P. Gnanachandra), jafaripersia@gmail.com, https://orcid.org/ 0000 - 0001- 5191-9330 (S. Jafari),mlthivagar@yahoo.co.in, https://orcid.org/ 0000-0001-5997-5185 (L. Thivagar)**

$$\mathscr{C} \cap \mathscr{D} = \left\{ \begin{array}{cc} \mathscr{C} & \text{if } n = m \\ (2m+1) & \text{if } m = n-1 \\ (2m-1) & \text{if } m = n+1 \\ \emptyset & \text{elsewhere} \end{array} \right\}$$

Also, Let $\mathscr{D} \subseteq \mathbb{Z}$. $\mathscr{D}$ is open (resp. closed) $\Leftrightarrow$ for every $d \in \mathscr{D}$, (d is odd)(resp. d is even) or (d is even with $d-1, d+1 \in \mathscr{D}$) (resp. d is odd with $d-1, d+1 \in \mathscr{D}$). Let $\mathscr{N}(n)$ (resp. $\mathscr{N}[n]$) denote the smallest neighbourhood (resp. closed neighbourhood) of n in $(\mathbb{Z}, \mathsf{k})$. Now $\mathscr{N}(n) = \left\{ \begin{array}{cc} \{n\} & \text{if } n \in \mathbb{O} \\ \{n-1, n, n+1\} & \text{if } n \in \mathbb{E} \end{array} \right\}$ and $\mathscr{N}[n] = \left\{ \begin{array}{cc} \{n\} & \text{if } n \in \mathbb{E} \\ \{n-1, n, n+1\} & \text{if } n \in \mathbb{O} \end{array} \right\}$. Also pay attention to $cl_{\mathsf{k}}(\{2n+1\}) = \{2n, 2n+1, 2n+2\}$, $int_{\mathsf{k}}(cl_{\mathsf{k}}(\{2n+1\})) = \{2n+1\}$, $cl_{\mathsf{k}}(int_{\mathsf{k}}(\{2n+1\})) = \{2n, 2n+1, 2n+2\}$, $int_{\mathsf{k}}(cl_{\mathsf{k}}(int_{\mathsf{k}}(\{2n+1\}))) = \{2n+1\}$, $cl_{\mathsf{k}}(int_{\mathsf{k}}(cl_{\mathsf{k}}(int_{\mathsf{k}}(\{2n+1\})))) = \{2n, 2n+1, 2n+2\}$.

Definition 2.1. *Let (X, τ) be a topological space and $\mathsf{S} \subseteq \mathsf{X}$. S is semi-open [6] if $\mathsf{S} \subseteq cl_\tau(int_\tau(\mathsf{S}))$, semi-closed if $int_\tau(cl_\tau(\mathsf{S})) \subseteq \mathsf{S}$, p-set [7] if $cl_\tau(int_\tau(\mathsf{S})) \subseteq int_\tau(cl_\tau(\mathsf{S}))$, q-set if $int_\tau(cl_\tau(\mathsf{S})) \subseteq cl_\tau(int_\tau(\mathsf{S}))$, G_δ-set if it equals the countable intersection of open sets of X, F_σ-set if it equals the countable union of closed sets of X, pointwise dense [8] if $\mathsf{Y}_{x \in \mathscr{B}} cl_\tau(\{x\}) : \{x\}$ is open $= \mathsf{X}$ and g-closed [9] if $cl_\tau(\mathsf{S}) \subseteq \mathsf{U}$ whenever $S \subseteq \mathsf{U}$ and U is open in X.*

Definition 2.2. *A topological space (X, τ) is $\mathsf{T}_{\frac{1}{2}}$ [9] if every g-closed set is closed, semi-T_0 [10] if for any two distinct points x and y of X, there exists a semi-open set S such that ($x \in \mathsf{S}$ and $y \notin \mathsf{S}$) or ($y \in \mathsf{S}$ and $x \notin \mathsf{S}$), semi-T_1 [10] if for $x \neq y \in \mathsf{X}$, there exist semi-open sets S_1 and S_2 such that $x \in \mathsf{S}_1$ but $y \notin \mathsf{S}_1$ and $y \in \mathsf{S}_2$ but $x \notin \mathsf{S}_2$, semi-R_0 [10] if for each semi-open set $\mathsf{S}, x \in \mathsf{S}$ implies $scl(\{x\}) \subseteq \mathsf{S}$, where $scl(\{x\})$ is the set of all semi-closed sets that containing $\{x\}$, semi-R_1 [11] if for $x, y \in \mathsf{X}$ such that $scl(\{x\}) \neq scl(\{y\})$, there are disjoint semi-open sets U and V such that $scl(\{x\}) \subseteq \mathsf{U}$ and $scl(\{y\}) \subseteq \mathsf{V}$, Urysohn [12] if whenever $x \neq y$ in X, there are neighbourhoods S of x and T of y with $cl(\mathsf{S}) \cap cl(\mathsf{T}) \neq \emptyset$, door [13] if every subset is either open or closed, extremally disconnected [12] if the closure of every open set is open, Alexandroff [14] if every intersection of open sets is open locally finite [8] if each point lies in a finite open set and in a finite closed set.*

3. Properties

In this section, we investigate the properties of the topological space $(\mathbb{Z}, \mathsf{k})$ and discuss the subspaces of the digital intervals of the digital line.

Proposition 3.1.

(i) $cl_{\mathsf{k}}(\mathbb{O}) = \mathbb{Z}$.
(ii) *$\mathbb{O}$ is pointwise dense in $\mathbb{Z}$.*
(iii) *Every dense subset of $\mathbb{Z}$ is open.*
(iv) *$(\mathbb{Z}, \mathsf{k})$ is second countable, Lindelof and separable.*
(v) *$(\mathbb{Z}, \mathsf{k})$ is $\mathsf{T}_{\frac{1}{2}}, \mathsf{T}_0$ and neither T_1 nor R_0.*
(vi) *$(\mathbb{Z}, \mathsf{k})$ is semi-T_0, semi-T_1 and semi-R_0.*
(vii) *$(\mathbb{Z}, \mathsf{k})$ is locally finite, connected, Alexandroff and neither door nor extremally disconnected.*
(viii) *Every F_σ set is closed and G_δ set is open in $(\mathbb{Z}, \mathsf{k})$*

Proof.

(i) Let $n \in \mathbb{Z}$ and $\mathscr{N}$ be neighbourhood of n in $(\mathbb{Z}, \mathsf{k})$. Since $\mathscr{N}$ contains an odd integer, $cl_{\mathsf{k}}(\mathbb{O}) = \mathbb{Z}$.
(ii) Clearly $\mathbb{O}$ is open in $\mathbb{Z}$, and for each $n \in \mathbb{O}$, $cl_{\mathsf{k}}(n) = \{n-1, n, n+1\}$ and hence $\mathbb{Z} = \mathsf{Y}_{n \in \mathbb{O}} cl_{\mathsf{k}}(n)$.
(iii) Let $\mathscr{D} \in \check{D}$. Then $\mathscr{D} = \mathsf{A} \cup \mathsf{B}$ where $\mathsf{A} \in \mathbb{O}$ and $\mathsf{B} \in \mathbb{E}$. Take $\mathsf{A} = \{n\}$, and n is even implies $\mathscr{D} = \mathsf{A} \cup \{n-1, n, n+1\}$ and hence $\mathscr{D}$ is open.
(iv) Follows from the fact that $\mathbb{Z}$ is countable.
(v) Since any neighbourhood of $2n$ contains $2n-1$, $(\mathbb{Z}, \mathsf{k})$ is not T_1. We can easily verify that $(\mathbb{Z}, \mathsf{k})$ is T_0. Since $cl_{\mathsf{k}}(2n-1) = \{2n, 2n-1, 2n+1\}$ and $\{2n-1\}$ is open implies that $(\mathbb{Z}, \mathsf{k})$ is not R_0. $\mathsf{T}_{\frac{1}{2}}$ follows from every singleton is either open or closed in $\mathbb{Z}$.
(vi) $(\mathbb{Z}, \mathsf{k})$ is semi-T_0, semi-T_1 and semi-R_0 follows from the fact that every singleton is semi-closed.
(vii) Let $n \in \mathbb{Z}$. If $n \in \mathbb{O}$, then $\{n\}$ is a finite open set such that $n \in \{n\}$ and $\{n-1, n, n+1\}$ is a finite closed set and $n \in \{n-1, n, n+1\}$. This implies $(\mathbb{Z}, \mathsf{k})$ is locally finite. A locally finite space is Alexandroff implies $(\mathbb{Z}, \mathsf{k})$ is an Alexandroff space. $\{2n, 2n-1\}$ is neither open nor closed implies $(\mathbb{Z}, \mathsf{k})$ is not door. Also $\{2n+1\}$ is open and $cl_{\mathsf{k}}(\{2n+1\}) = \{2n+2n+1, 2n+2\}$ is not open. Therefore $(\mathbb{Z}, \mathsf{k})$ is not extremally disconnected. Let A be a non-empty clopen subset of $(\mathbb{Z}, \mathsf{k})$. Fix $n \in \mathsf{A}$. If $n \in \mathbb{O}$ and A is closed, $n-1, n+1 \in \mathsf{A}$. Thus $\{n-1, n, n+1\} \subseteq \mathsf{A}$. Since $n-1$ and $n+1$ are even and A is open, $\{n-2, n-1, n, n+1, n+2\} \subseteq \mathsf{A}$. Continuing we get $\mathbb{Z} = \mathsf{A}$. If $n \in \mathbb{E}$ and A is open, $n-1, n+1 \in \mathsf{A}$. Thus $\{n-1, n, n+1\} \subseteq \mathsf{A}$. Since $n-1$ and $n+1$ are odd and A is closed, $\{n-2, n-1, n, n+1, n+2\} \subseteq \mathsf{A}$. Continuing, we get τ equals A. That is $(\mathbb{Z}, \mathsf{k})$ is connected.

(viii) Let $\mathsf{A} \in (\mathbb{Z}, \mathsf{k})$ where $\mathsf{A} = \cup \mathsf{A}_n$ each A_n is closed if x is in A, then x is in some A_n. If x is even, it is evident. If x is odd, then $\{x-1, x, x+1\} \subseteq \mathsf{A}_n \subseteq \mathsf{A}$. That is A is closed.

(ix) Let $\mathsf{A} \in (\mathbb{Z}, \mathsf{k})$ where $\mathsf{A} = \cap \mathsf{A}_n$ each A_n is open. If x is in A, then x is in every A_n. If x is even, then the open set $\{x-1, x, x+1\} \subseteq \mathsf{A}_n \subseteq \mathsf{A}$. That is A is open.

□

Levine's Property $\mathscr{Q}$

Levine defined that a set S has the property $\mathscr{Q}$ if the interior and the closure operators commute on S and characterized the sets having the property $\mathscr{Q}$. That is a set A in a topological space (X, τ) has the property $\mathscr{Q}$ [15] if $int_\tau(cl_\tau(\mathsf{S})) = cl_\tau(int_\tau(\mathsf{S}))$.

Example 3.2.

(i) *Let X be an infinite set with co-finite topology. Then every non-empty open subset is infinite and hence every finite subset of X has the property $\mathscr{Q}$.*

(ii) *Let X be an uncountable set with co-countable topology. Then every non-empty open subset is uncountable and hence every countable subset of X has the property $\mathscr{Q}$.*

(iii) *Let X be an non-empty set with $x \in \mathsf{X}$. Assign X with x-inclusion topology, then X does not have the property $\mathscr{Q}$.*

(iv) *In (X, τ), $\mathsf{X} = \mathsf{W}$, the set of all whole numbers and $\tau = \{\emptyset, \{0\}, \mathsf{X}\}$. Then every subset of X has the property $\mathscr{Q}$.*

4. Digital subspaces

From now on we consider subspaces of $(\mathbb{Z}, \mathsf{k})$, and investigate the behaviour of cardinalities of some kind of subspace topologies. We will now prove that the cardinalities of topologies on the intervals $\{1\}, \{1,2\}, \{1,2,3\}, ..., \{1,2,3,..,n\}, ...$ form a subsequence of the well known Fibonacci sequence. Also observe that

(i) If $\mathsf{S} = \{1\}$, $\tau_1 = \{\emptyset, \mathsf{S}\}$, then $|\tau_1| = 2$.

(ii) If $\mathsf{S} = \{1,2\}$, $\tau_2 = \{\emptyset, \{1\}, \mathsf{S}\}$, then $|\tau_2| = 3$.

(iii) If $\mathsf{S} = \{1,2,3\}$, subbase = $\{\{1\}, \{1,2,3\}, \{3\}\}$, Base = $\{\{1\}, \{3\}, \mathsf{S}\}$ and $\tau_3 = \{\emptyset, \{1\}, \{3\}, \{1,3\}, \mathsf{S}\}$ then $|\tau_2| = 5$.

(iv) Let $\mathsf{S} = \{1,2,3,4\}$, subbase = $\{\{1\}, \{1,2,3\}, \{3,4\}\}$, Base = $\{1\}, \{3\}, \{1,2,3\}, \{3,4\}, \mathsf{S}\}$ and $\tau_4 = \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \mathsf{S}\}$ then $|\tau_4| = 8$.

(v) Let $\mathsf{S} = \{1,2,3,4,5\}$ with $\mathsf{S}_\mathsf{k} = \tau_5$, the subspace topology generated by $\{\{1\}, \{5\}, \{1,2,3\}, \{3,4,5\}\}$. Here $\tau_5 = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}, \{1,2,3\}, \{1,3,5\}, \{3,4,5\}, \{1,2,3,5\}, \{1,3,4,5\}, \mathsf{S}\}$. Then (S, τ_5) is a subspace of $(\mathbb{Z}, \mathsf{k})$. Also from this we observe that $q(\tau_5)$ and $p(\tau_5)$ are discrete topology on S. Also, $\mathscr{Q}(\tau_5) = \{\emptyset, \{4\}, \{2,4\}, \{1,3,5\}, \{1,2,3,5\}, \{1,3,4,5\}, \mathsf{S}\}$. Then $|\tau_5| = 13$.

(vi) Let $\mathsf{S} = \{1,2,3,4,5,6\}$, then subbase = $\{\{1\}, \{1,2,3\}, \{3,4,5\}, \{5,6\}\}$ and Base = $\{\{1\}, \{3\}, \{1,2,3\}, \{3,4,5\}, \{5,6\}\}$ and $\tau_6 = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}, \{5,6\}, \{1,2,3\}, \{1,3,5\}, \{3,4,5\}, \{1,5,6\}, \{3,5,6\}, \{1,2,3,5\}, \{1,3,4,5\}, \{1,3,5,6\}, \{3,4,5,6\}, \{1,2,3,5,6\}, \{1,2,3,4,5\}, \{1,3,4,5,6\}, \mathsf{S}\}$. Then $|\tau_6| = 21$.

(vii) Let $\mathsf{S} = \{1,2,3,4,5,6,7\}$, then subbase = $\{\{1\}, \{7\}, \{1,2,3\}, \{3,4,5\}, \{5,6,7\}\}$ and Base = $\{\{1\}, \{3\}, \{5\}, \{7\}, \{1,2,3\}, \{3,4,5\}, \{5,6,7\}\}$ and $\tau_7 = \{\emptyset, \{1\}, \{3\}, \{5\}, \{7\}, \{1,3\}, \{1,5\}, \{1,7\}, \{3,5\}, \{3,7\}, \{5,6\}, \{5,7\}, \{1,2,3\}, \{3,4,5\}, \{5,6,7\}, \{1,3,5\}, \{1,3,7\}, \{1,5,7\}, \{3,5,7\}, \{1,2,3,5\}, \{1,2,3,7\}, \{1,3,4,5\}, \{1,3,5,7\}, \{1,5,6,7\}, \{3,4,5,7\}, \{3,5,6,7\}, \{1,2,3,4,5\}, \{3,4,5,6,7\}, \{1,2,3,5,7\}, \{1,3,5,6,7\}, \{1,3,4,5,7\}, \{1,2,3,4,5,7\}, \{1,2,3,5,6,7\}, \{1,3,4,5,6,7\}, \mathsf{S}\}$. Then $|\tau_7| = 34$.

Lemma 4.1. *Let $\mathsf{S} = \{1,2,3,4,5\}$. Then*

(i) *$q(\tau_5)$ is the discrete topology on S,*

(ii) *$p(\tau_5)$ is the discrete topology on S,*

(iii) *$\mathscr{Q}(\tau_5)$ is a topology on S other than discrete topology and the indiscrete topology on S, where $q(\tau_5)$, $p(\tau_5)$ and $\mathscr{Q}(\tau_5)$ respectively denote the collection of all q-sets, p-sets and collection of all subsets of S having the property Q in (S, τ_5).*

Lemma 4.2. *If $\tau_{m-1}, \tau_m, \tau_{m+1}$ are the topologies on the digital intervals $\mathbb{Z} \cap [1, m-1]$, $\mathbb{Z} \cap [1,m]$, $\mathbb{Z} \cap [1, m+1]$ respectively inherited from the Khalimsky topology k on $\mathbb{Z}$, then $|\tau_{m-1}| = |\tau_{m+1}| = |\tau_m|$.*

Proof.

Case (a) Since $[1, m-1] \subseteq [1, m+1]$ and $m-1 \in \mathbb{O}$. Then $\tau_{m+1} \subseteq \tau_{m+2}$. Now $\{m+1\}$ is the basic open set in $[1, m+1]$. $\tau = \{\{m+1\} \cup \mathsf{A} : \mathsf{A} \in \tau_m\}$. Then $\tau_{m-1} \cup \tau = \tau_{m+1}$ and $\tau_{m-1} \cap \tau = \emptyset$.

Case (b) Since $[1,m] \subseteq [1, m+1]$ and $m \in \mathbb{O}$. Then $\tau_m \subseteq \tau_{m+1}$. Now $\{m, m+1\}$ is the base open set in $[1, m+1]$. $\tau = \{\{m, m+1\} \cup \mathsf{A} : \mathsf{A} \in \tau_{m+1}\}$. Then $\tau_m \cup \tau = \tau_{m+1}$ and $\tau_m \cap \tau = \emptyset$.

In both cases $|\tau_{m+1}| = |\tau_{m-1}| = |\tau_m|$. □

5. Conclusion

In this work, we provide some results and examples of the topology on $\mathbb{Z}$, the set of all integers. Also we explain the concepts of digital line and digital intervals. We prove that the cardinalities of topologies on the digital intervals form a sub sequence of the Fibonacci sequence.

Acknowledgement

We thank the unavowed referees for their valuable commands and suggestions. Also the first author express her gramercy to Professor **Mahmut Akyigit** for his keen interest about this article.

References

[1] A. Rosenfeld, *Digital topology*, Amer. Math. Monthly, **86** (1979), 621 - 630.
[2] A. Rosenfeld, *Digital straight line segments*, IEEE Trans. Computers, **23** (1994), 1264 - 1269.
[3] A. Rosenfeld, *Digital Picture Analysis*, Springer, Berlin, 1976.
[4] A. Rosenfeld, A. C. Kak, *Digital Picture Processing*, Academic Press, New York, 1976.
[5] E. Khalimsky, R. Kopperman, P. R. Meyer, *Computer graphics and connected topologies on finite ordered sets*, Top. Appl., **36** (1990), 1 - 17.
[6] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70**(1) (1963), 36 - 41.
[7] P. Thangavelu, K. C. Rao, *p-sets in topological spaces*, Bull. Pure Appl. Math., **21E**(2) (2002), 341 - 355.
[8] T. Y. Kong, R. Kopperman, *A topological approach to digital topology*, Amer. Math. Monthly, **98** (10), (1991), 901 - 907.
[9] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, **19**(2) (1970), 89 - 96.
[10] S. N. Maheswari, R. Prasad, *Some new separation axioms*, Ann. Soc. Sci. Bruxelle Ser., **89** III (1975), 395 - 402.
[11] C. H. Dorsett, *Semi-T_2, Semi-R_1 and semi-R_0 topological spaces*, Ann. Soc. Sci. Bruxelle Ser., **(I,92)** (1978), 143 - 150.
[12] S. Willard, *General Topology*, Addison Wesley, 1970.
[13] J. L. Kelly, *General Topology*, Princeton, NJ.D.Van Nastrand, 1955.
[14] P. Alexandroff, *Diskrete raume*, Mat. Sbornik, **2** (1937), 507 - 519.
[15] N. Levine, *On the community of the closure and interior operators in topological spaces*, Amer. Math. Monthly, **68** (1961), 474 - 477.

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 29-32
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.716406

Contact Hamiltonian Description of Systems with Exponentially Decreasing Force and Friction that is Quadratic in Velocity

Furkan Semih Dündar[1]

[1] *Department of Mechanical Engineering, Faculty of Technology, Amasya University, Amasya, Turkey*

Article Info

Keywords: *Contact geometry, Contact Hamiltonian, Friction*
2010 AMS: *53D10, 70G45*
Received: *8 April 2020*
Accepted: *21 May 2020*
Available online: *10 June 2020*

Abstract

We have given a simple contact Hamiltonian description of a system with exponentially vanishing (or zero) potential under a friction term that is quadratic in velocity. We have given two applications: to cavity solitons and to a free body under air friction.

1. Introduction

Hamiltonian mechanics is done in phase space which is a symplectic manifold. Symplectic mainfolds are even dimensional mainfolds equipped with a symplectic 2-form ω. On the other hand, contact geometry is carried on odd dimensional manifolds equipped with a contact 1-form η. η is a contact form if $\eta \wedge d\eta \wedge \cdots \wedge d\eta \neq 0$ (in a $2n+1$ dimensional contact manifold, the term $d\eta$ is wedge-multiplied n times). For more information on contact manifolds, the reader is referred to [1] and to [2] for the relation between contact geometry and the Huygens' Principle.

Recently, Hamiltonian mechanics has been generalized to work in contact manifolds with the addition of extra parameter S [3]. Ref. [4] studied variational aspects of contact Hamiltonian mechanics and ref. [5] applied contact geometric methods to a theory of gravity called shape dynamics [6–8] (see ref. [9] for a review of shape dynamics). Also in ref. [3] time-dependent contact Hamiltonians are introduced. In our study, we will use time-independent contact Hamiltonian mechanics in 1D. So the variables we have are q, p, S. In this variables the equations of motion derived from the contact Hamiltonian (H) read as [3]:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial S}, \\ \dot{S} &= p\frac{\partial H}{\partial p} - H. \end{aligned}$$

The organization of the paper is as follows: in Section 2 we introduce the contact Hamiltonian we use, in Section 3 we give a solution of the equations of motion derived from the contact system, in Section 4 we give two applications where our description can be used and finally in Section 5 we conclude the paper.

***Email address and ORCID number*:** furkan.dundar@amasya.edu.tr, https://orcid.org/0000-0001-5184-5749 (F. S. Dündar)

2. Contact Hamiltonian description

It is known in the literature (see ref. [3]) that the contact Hamiltonian $H = p^2/2m + V(q) + \gamma S$ describes a system with frictional force linear in velocity for an arbitrary potential term $V(q)$. In this study we make a minor change in the last term and use the following contact Hamiltonian:

$$H = \frac{p^2}{2m} + V(q) + \gamma p S.$$

The equations of motion that follow are as:

$$\dot{q} = \frac{p}{m} + \gamma S, \tag{2.1}$$

$$\dot{p} = -V'(q) - \gamma p^2, \tag{2.2}$$

$$\dot{S} = \frac{p^2}{2m} - V(q). \tag{2.3}$$

When we take the time derivative of Equation (2.1) we obtain:

$$\begin{aligned} \ddot{q} &= \frac{\dot{p}}{m} + \gamma \dot{S}, \\ &= -\frac{1}{m}(V'(q) + \gamma p^2) + \gamma\left(\frac{p^2}{2m} - V(q)\right), \\ m\ddot{q} &= -V'(q) - \frac{\gamma}{2}p^2 - m\gamma V(q). \end{aligned} \tag{2.4}$$

Using Equation (2.1) we can write $p = m\dot{q} - m\gamma S$. As an ansatz let us assume $S = \alpha \dot{q}$ for some α. Then we get $p = m(1-\gamma\alpha)\dot{q}$. If we put this form of p into Equation (2.4) we obtain:

$$m\ddot{q} + \frac{m^2\gamma}{2}(1-\gamma\alpha)^2\dot{q}^2 = -V'(q) - m\gamma V(q). \tag{2.5}$$

On the other hand, when we use the ansatz $S = \alpha\dot{q}$ in Equation (2.3) we obtain:

$$\alpha\ddot{q} = \dot{S} = \frac{m}{2}(1-\gamma\alpha)^2\dot{q}^2 - V(q). \tag{2.6}$$

In order to be consistent, Equation (2.5) and Equation (2.6) must give the same answer. So we must have the following:

$$\frac{m^2}{2\alpha}(1-\gamma\alpha)^2\dot{q}^2 - \frac{m}{\alpha}V(q) = -\frac{m^2\gamma}{2}(1-\gamma\alpha)^2\dot{q}^2 - V'(q) - m\gamma V(q).$$

Equating the terms in front of $\dot{q}^2$ on both sides gives us $\alpha = -1/\gamma$. There appears a condition on the potential $V(q)$:

$$V'(q) = -2m\gamma V(q). \tag{2.7}$$

So with the ansatz we put, arbitrary potentials are not allowed. The solution of Equation (2.7) is elementary:

$$V(q) = A\exp(-2m\gamma q),$$

for some constant A. Now we have determined α in $S = \alpha\dot{q}$ as $\alpha = -1/\gamma$. As a consistency check let us put this in Equation (2.1) and obtain $p = 2m\dot{q}$. On the other hand we have $\dot{p} = -V'(q) - \gamma p^2$ from Equation (2.2). This yields:

$$2m\ddot{q} = -\frac{\partial}{\partial q}(A\exp(-2m\gamma q)) - \gamma p^2,$$

and

$$m\ddot{q} + 2\gamma m^2\dot{q}^2 = Am\gamma\exp(-2m\gamma q).$$

Comparing this Equation with Equation (2.6) we see that there is no inconsistency. So the consistent equation of motion is as follows:

$$m\ddot{q} + \gamma_n\dot{q}^2 = -\frac{\partial}{\partial q}A_n e^{-\gamma_n q/m}, \tag{2.8}$$

where γ_n (new γ) is given through $\gamma_n = 2m^2\gamma$ and A_n (new A) is $A_n = A/2$.

3. Solution of the equation of motion

In this Section, we will solve the equation of motion of the system given by Equation (2.8). Let us define $Q = \gamma_n q/m$. So we have:

$$\ddot{Q} + \dot{Q}^2 = \frac{A_n\gamma_n^2}{m^3}e^{-Q}.$$

We now define F via $Q = \log F$. Then we get:

$$\frac{\ddot{F}}{F} - \frac{\dot{F}^2}{F^2} + \frac{\dot{F}^2}{F^2} = \frac{A_n\gamma_n^2}{m^3}\frac{1}{F}.$$

The second and third terms cancel out with each other and we obtain $\ddot{F} = A_n\gamma_n^2/m^3$. Making changes of variables in the reverse order, one obtains:

$$q(t) = \frac{m}{\gamma_n}\log\left(\frac{A_n\gamma_n^2}{2m^3}t^2 + c_1 t + c_2\right),$$

where c_1 and c_2 are two constants of integration.

4. Possible applications

The equation of motion (see Equation (2.8)) derived from the contact Hamiltonian, $H = p^2/2m + V(q) + (\gamma_n/2m^2)pS$, is:

$$m\ddot{q} + \gamma_n\dot{q}^2 = -\frac{\partial}{\partial q}A_n e^{-\gamma_n q/m},$$

where γ_n, A_n are some constant parameters. In this Section, we give two possible applications of our choice of contact Hamiltonian. The first one is cavity solitons with friction quadratic in velocity, and the other one is air friction with quadratic dependency on velocity.

4.1. Cavity solitons

Recently Ref. [10] put forward that cavity solitons (for a review see Ref. [11]) may be modeled with an effective potential of the form $-K^2e^{-r/R}$ with the strength K and the range R. Our potential term is $V(q) = A_n e^{-\gamma_n q/m}$. So if we choose $A = -K^2$ and $\gamma_n = m/R$ we can model cavity solitons. But our model has a quadratic friction term: $\gamma_n\dot{q}^2 = (m/R)\dot{q}^2$. It may be possible that our contact Hamiltonian can model cavity solitons with an exponentially decreasing force and a friction term that is quadratic in velocity.

4.2. Air friction

It is well known in literature that for large bodies, air friction can be modelled with frictional force that has quadratic dependency on the velocity. Therefore our contact Hamiltonian may also model a free particle under air friction if $A_n = 0$ or with a driving force equal to $(A_n\gamma_n/m)e^{-\gamma_n q/m}$.

5. Conclusion

Recently contact Hamiltonian mechanics has gained some interest [3–5]. In this paper we used a simple contact Hamiltonian to account for quadratic dependence on velocity. As we mentioned in Section 4 this description may be useful for modelling cavity solitons with a quadratic friction term or air friction for free particles. We note that our work is only an initial step towards giving a contact Hamiltonian description of a system with an arbitrary potential under a friction that is quadratic in velocity.

Acknowledgements

We would like to thank Metin Arık and Bayram Tekin for useful discussions. We also would like to Gülhan Ayar for bringing contact geometry to our attention.

References

[1] H. Geiges, *A brief history of contact geometry and topology*, Expo. Math., **19**(1) (2001), 25–53.
[2] H. Geiges, *Christiaan huygens and contact geometry*, (2005) arXiv:math/0501255.
[3] A. Bravetti, H. Cruz, D. Tapias, *Contact Hamiltonian mechanics*, Ann. Phys.-New York, **376** (2017), 17–39.
[4] Q. Liu, P. J. Torres, C. Wang, *Contact Hamiltonian dynamics: Variational principles, invariants, completeness and periodic behavior*, Ann. Phys.-New York, **395** (2018), 26–44.
[5] D. Sloan, *Dynamical similarity.* Phys. Rev. D, **97**(12) (2018), 123541.
[6] E. Anderson, J. Barbour, B. Foster, N. Ó Murchadha, *Scale-invariant gravity: Geometrodynamics*. Classical Quant. Grav., **20** (2003), 1571–1604.
[7] E. Anderson, J. Barbour, B. Z. Foster, B. Kelleher, N. Ó. Murchadha, *The physical gravitational degrees of freedom*, Classical Quant. Grav., **22** (2005), 1795–1802.
[8] J. Barbour, N. O Murchadha, *Classical and Quantum Gravity on Conformal Superspace*, (1999), arXiv:gr-qc/9911071.
[9] F. Mercati, A Shape Dynamics Tutorial, (2014), arXiv:1409.0105.
[10] S. R. Anbardan, C. Rimoldi, R. Kheradmand, G. Tissoni, F. Prati, *Exponentially decaying interaction potential of cavity solitons*, Phys. Rev. E, **97**(3) (2018), 032208.
[11] L. A. Lugiato, F. Prati, M. Brambilla, L. Columbo, S. Barland, G. Tissoni, K. M. Aghdami, R. Kheradmand, H. Tajalli, H. Vahed, *Cavity solitons*, In Without Bounds: A Scientific Canvas of Nonlinearity and Complex Dynamics, Springer, 2013, 395–404.

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 33-44
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.645321

An Optimization Method for Semilinear Parabolic Relaxed Constrained Optimal Control Problems

Basil Kokkinis[1]

[1]*Department of Mathematics, School of Applied Mathematical and Physical Sciences, National Technical University, Athens, Greece*

Article Info

Keywords: *Conditional descent method, Discretization, Optimal control, Relaxed controls, Semilinear parabolic equations, State constraints*
2010 AMS: *49M05, 49M25, 65N30*
Received: *11 November 2019*
Accepted: *7 April 2020*
Available online: *30 June 2020*

Abstract

This paper addresses optimal control problems governed by semilinear parabolic partial differential equations, subject to control constraints and state constraints of integral type. Since such problems may not have classical solutions, a relaxed optimal control problem is considered. The relaxed control problem is discretized by using a finite element method and the behavior in the limit of discrete optimality, admissibility and extremality properties is studied. A conditional descent method with penalties applied to the discrete problems is proposed. It is shown that the accumulation points of sequences produced by this method are admissible and extremal for the discrete problem. Finally, numerical examples are given.

1. Introduction

In the absence of any convexity assumptions, optimal control problems, in general, have no classical solutions. To study them, they usually need to be reformulated to their corresponding relaxed form. Warga [1], Roubíček [2] and Fattorini [3] have extensively studied the concept of relaxation on optimal control problems. Relaxation had been introduced initially to prove the existence of optimal controls and then to derive necessary optimality conditions. Additionally, relaxed controls are used as a tool to develop optimization methods (Warga [4], Chryssoverghi et al. [5]) and discrete approximation schemes (Chryssoverghi et al. [6], Roubíček [7], Azhmyakov et al. [8]). Relaxed controls have been applied to optimal control problems for systems defined by PDEs in [3], [2] as well as in many papers, among them [6], [9]-[13]. In particular, Arada and Raymond in [9] prove existence and a Pontryagin's minimum principle for relaxed solutions of state-constrained relaxed optimal control problems governed by semilinear elliptic equations under a stability condition. The approximation of similar problems was studied by the same authors in [10] and by Casas in [11]. Chryssoverghi and Bacopoulos in [6] present approximation results for relaxed semilinear parabolic optimal control problems. In [12] relaxed controls have been used to develop iterative optimization methods applied directly on a relaxed problem. Finally, Luan in [13], using relaxed controls obtains some results on the nonexistence and existence of multisolution semilinear elliptic optimal control problems.

In this paper, an optimal control problem with distributed control is considered for systems defined by a semilinear parabolic PDE, in the presence of constraints on the control and the state. The parabolic equation has two separate semilinear terms in order to allow more general assumptions, monotonicity for the term on the left-hand side and Lipschitz continuity for the term on the right-hand side. The state constraints depend both on the state and its gradient and are of integral type. The cost functional depends also on the state gradient. Convexity assumptions are not imposed, so this problem may have no classical solutions. To deal with this, the problem is reformulated in its relaxed form using relaxed controls. The state equation in relaxed

Email address and ORCID number: **bkok@math.ntua.gr, https://orcid.org/0000-0002-1632-0266, (B. Kokkinis)**

form is then discretized in space using a Galerkin finite element method (semi-discretization). The spatial discretization is done with continuous piecewise linear functions. The controls are approximated by piecewise constant relaxed ones. Necessary conditions for optimality are stated for the discrete relaxed problem. Then it is shown that sequences of optimal (resp. extremal) relaxed controls for the discrete problem have subsequences which converge to optimal (resp. extremal) controls for the continuous relaxed problem. Next, an algorithm based on a penalized conditional descent method is proposed, applied to the discrete problems, which generates Gamkrelidze controls. It is shown that accumulation points of sequences constructed by the algorithm satisfy the necessary conditions for optimality for the discrete problem and such accumulation points always exist. For implementation reasons relaxed controls have to be approximated by classical ones. So, using standard techniques, the Gamkrelidze controls computed by the above method can be approximated by piecewise constant classical ones, see [5]. Thus the above method using relaxed controls has all the theoretical advantages of them and gives us at last, through the above-mentioned approximation, classical controls. Finally, two numerical examples are presented.

The novelty points of this paper are: (i) the study of such nonconvex optimal control problems with relaxation, (ii) the discretization of such problems, and (iii) the construction of methods applied to the discrete problem with relaxed controls. In order to solve these problems numerically one must necessarily disretize them and then apply some optimization method to the resulting discrete problem. Since the structures of the continuous and the discrete problems are basically different it is necessary to know if discrete optimality (or extremality) carries over in the limit to continuous optimality (resp. extremality). This paper actually extends the results of [14] by semi-discretizing the problem and studying the behavior in the limit and then by applying an optimization method to this discretized problem.

The paper is organized as follows. In section 2 the relaxed controls are introduced, and the classical and the relaxed optimal control problems are formulated. The existence of optimal relaxed controls is also proved. In section 3 the relaxed problem is discretized and in section 4 the behavior in the limit of discrete relaxed optimality and extremality is studied. A penalized conditional descent method is presented in section 5. Two numerical examples are given in section 6.

2. The continuous optimal control problems

Let Ω be a bounded domain in $\mathbb{R}^d$, with boundary Γ, and set $Q := \Omega\times(0,T)$, $\Sigma := \Gamma\times(0,T)$ with given final time $T > 0$. Consider the following semilinear parabolic initial boundary value problem

$$y_t + A(t)y + \sum_{i=1}^{d} a_{0i}(x,t)\partial y/\partial x_i + b(x,t,y(x,t),u(x,t)) = f(x,t,y(x,t),u(x,t)) \text{ in } Q, \tag{2.1}$$

$$y(x,t) = 0 \text{ on } \Sigma \tag{2.2}$$

$$y(x,0) = y^0(x) \text{ in } \Omega. \tag{2.3}$$

Here $A(t)$ is the elliptic differential operator

$$A(t)y := -\sum_{j,i=1}^{d} (\partial/\partial x_i)[a_{ij}(x,t)\partial y/\partial x_j].$$

Throughout the paper, we shall use the notation $(\cdot,\cdot)$, $(\cdot,\cdot)_1$, $(\cdot,\cdot)_Q$ for the inner product and $\|\cdot\|$, $\|\cdot\|_1$, $\|\cdot\|_Q$ for the norm of the spaces $L^2(\Omega)$, $V := H^1_0(\Omega)$, $L^2(Q)$ respectively. We define on $V\times V$ the bilinear form associated with $A(t)$

$$a(t,y,v) := \sum_{j,i=1}^{d} \int_\Omega a_{ij}(x,t)\frac{\partial y}{\partial x_i}\frac{\partial v}{\partial x_j}dx. \tag{2.4}$$

Also, q_1 and q_2 are given nonnegative integers.

The set of classical controls is defined by

$$\mathbb{U} := \{u : Q \to U \,|\, u \text{ measurable}\} \subset L^\infty(Q),$$

$U \subset \mathbb{R}$ is compact, not necessarily convex and the functionals by

$$J_m(u) := \int_Q g_m(x,t,y,\nabla y,u)dxdt, \quad m = 0,...,q_2.$$

The continuous classical optimal control problem is

$$\text{minimize} J_0(u)$$

subject to the state equation (2.1 - 2.3), the control constraints $u \in \mathbb{U}$ and the state constraints

$$\begin{aligned} J_m(u) &= 0, \quad m = 1,...,q_1, \\ J_m(u) &\leq 0, \quad m = q_1+1,...,q_2. \end{aligned}$$

The above problem in order to have a solution has to be endowed with undesirable convexity assumptions (for example, Cesari property), which are usually not realistic for nonlinear systems. But when we formulate the problem in its relaxed form using relaxed controls then the new problem has a solution in a larger space under weaker assumptions.

Let $C(U)$ be the set of continuous functions on U and $M(U)$ (resp. $M_1(U)$) the set of Radon (resp. probability) measures on U. We endow $M(U) = C(U)^*$ with the weak* topology. We define the set of relaxed controls ([1], [2])

$$R := \{r : \bar{Q} \to M_1(U) \,|\, r \text{ weakly measurable} \} \subset L_w^\infty(Q; M(U)) \equiv L^1(Q; C(U))^*.$$

The topology of R is the weak* topology induced by $L^1(Q;C(U))^*$. R is convex, and with the above topology metrizable and compact. We identify each element $u \in \mathbb{U}$ with the relaxed control $r(\cdot) = \delta_{u(\cdot)}$, where $\delta_{u(\cdot)}$ denotes the Dirac measure concentrated at $u(\cdot)$ and thus we can regard $\mathbb{U}$ as a subset of R. Furthermore $\mathbb{U}$ is dense in R. For simplicity reasons, for $h \in L^1(Q;C(U))$ and $r \in R$,we write

$$h(x,t,r(x,t)) := \int_U h(x,t,u)r(x,t)(du). \tag{2.5}$$

It follows (see [1]) that $h(x,t,r(x,t))$ is linear in r. Let (r_k) be a sequence of relaxed controls and $r \in R$. Then, (r_k) is said to converge to r if and only if

$$\lim_{k\to\infty} \int_Q h(x,t,r_k(x,t))dxdt = \int_Q h(x,t,r(x,t))dxdt,$$

for all $h \in L^1(Q;C(U))$.

For the case of noncompact U, Fattorini in [3], gives a new definition of relaxed controls based on finitely additive measures on U.

The weak relaxed form of the state equation (2.1), using the notation (2.5), is given by

$$< y_t, v > + a(t,y,v) + \sum_{i=1}^{d} (a_{0i}(t)\partial y/\partial x_i, v) + (b(t,y,r),v) = (f(t,y,r),v), \tag{2.6}$$

for every $v \in V$, a.e. in $(0,T)$, $y(t) \in V$, a.e. in $(0,T)$

$$y(0) = y^0, \tag{2.7}$$

where $< \cdot,\cdot >$ denotes the dual pairing between V and its dual space $V^* = H^{-1}(\Omega)$ and $a(t,y,v)$ is the bilinear form given in (2.4) .

The continuous relaxed optimal control problem (ROCP) is

$$\text{minimize } J_0(r)$$

subject to the relaxed state equation (2.6), (2.7), the control constraints $r \in R$ and the state constraints

$$\begin{aligned} J_m(r) &= 0, \quad m = 1,\dots,q_1, \\ J_m(r) &\leq 0, \quad m = q_1+1,\dots,q_2, \end{aligned}$$

where

$$J_m(r) := \int_Q \int_U g_m(x,t,y,\nabla y,u)r(du)dxdt, \quad m = 0,\dots,q_2.$$

We introduce the following assumptions.

(H1) $\Omega \subset \mathbb{R}^d, d \leq 3$, is a bounded domain with C^1-boundary Γ.

(H2) The coefficient functions a_{ij} of $A(t)$ belong to $L^\infty(Q)$ and

$$\sum_{j=1}^{d}\sum_{i=1}^{d} a_{ij}(x,t)\xi_i\xi_j \geq \gamma_0 \sum_{i=1}^{d} \xi_i^2, \quad \forall \xi_i, \xi_j \in \mathbb{R},\ (x,t) \in Q, \text{with } \gamma_0 > 0,$$

from which easily follow the inequalities

$$|a(t,y,v)| \leq \alpha_1 \|y\|_1 \|v\|_1, \quad a(t,v,v) \geq \alpha_2 \|v\|_1^2, \quad \forall y,v \in V, \quad t \in (0,T),$$

for some $\alpha_1 \geq 0,\ \alpha_2 > 0$.

(H3) $a_0 = (a_{01},...,a_{0d})^T \in L^\infty(Q)^d$. The functions b and $f : Q\times\mathbb{R}\times U \to \mathbb{R}$ are measurable w.r.t. $(x,t)\in Q$ for any fixed y,u, continuous for fixed $(x,t)\in Q$ and satisfy the conditions

$$
\begin{aligned}
&|b(x,t,y,u)| \le \varphi(x,t)+\beta\,|y|^2, \quad (x,t,y,u)\in Q\times\mathbb{R}\times U,\\
&|f(x,t,y,u)| \le \psi(x,t)+\gamma|y|, \quad (x,t,y,u)\in Q\times\mathbb{R}\times U,\\
&|f(x,t,y_1,u)-f(x,t,y_2,u)| \le L|y_1-y_2|, \quad (x,t,y_1,y_2,u)\in Q\times\mathbb{R}^2\times U,
\end{aligned}
$$

where $\varphi,\psi\in L^2(Q)$, $\beta,\gamma,L\ge 0$.

The function b is monotone increasing with respect to y for almost every $(x,t)\in Q$. Assuming that $b(\cdot,\cdot,y,\cdot)=0$, (if not, we subtract this term from both sides of (2.1)) it follows that $b(x,t,y,u)y\ge 0$.

(H4) The functions $g_m : Q\times\mathbb{R}^{d+1}\times U\to\mathbb{R}$ are measurable for fixed $(y,\bar{y},u)\in\mathbb{R}^{d+1}\times U$, continuous for fixed $(x,t)\in Q$ and satisfy

$$
|g_m(x,t,y,\bar{y},u)| \le \zeta_m(x,t)+\delta_m y^2+\bar{\delta}_m\,|\bar{y}|^2, \quad (x,t,y,\bar{y},u)\in Q\times\mathbb{R}^{d+1}\times U,
$$

with $\zeta_m\in L^1(Q)$, $\delta_m\ge 0$, $\bar{\delta}_m\ge 0$.

(H5) The functions $b_y, f_y : Q\times\mathbb{R}\times U\to\mathbb{R}$ are measurable on Q for fixed $(y,u)\in\mathbb{R}\times U$ and continuous on $\mathbb{R}\times U$ for fixed $(x,t)\in Q$ and satisfy

$$
\begin{aligned}
&\left|b_y(x,t,y,u)\right| \le \xi(x,t)+\eta\,|y|, \quad (x,t,y,u)\in Q\times\mathbb{R}\times U,\\
&\left|f_y(x,t,y,u)\right| \le L_1, \quad (x,t,y,u)\in Q\times\mathbb{R}\times U,
\end{aligned}
$$

with $\xi\in L^2(Q)$, $\eta\ge 0$, $L_1\ge 0$.

(H6) The functions $g_{my}, g_{m\bar{y}} : Q\times\mathbb{R}^{d+1}\times U\to\mathbb{R}$ are measurable on Q for fixed $(y,\bar{y},u)\in\mathbb{R}^{d+1}\times U$ and continuous on $\mathbb{R}^{d+1}\times U$ for fixed $(x,t)\in Q$ and satisfy

$$
\begin{aligned}
&\left|g_{my}(x,t,y,\bar{y},u)\right| \le \zeta_{m1}(x,t)+\delta_{m1}\,|y|+\bar{\delta}_{m1}\,|\bar{y}|^2, \quad (x,t,y,\bar{y},u)\in Q\times\mathbb{R}^{d+1}\times U,\\
&\left|g_{m\bar{y}}(x,t,y,\bar{y},u)\right| \le \zeta_{m2}(x,t)+\delta_{m2}y^2+\bar{\delta}_{m2}\,|\bar{y}|, \quad (x,t,y,\bar{y},u)\in Q\times\mathbb{R}^{d+1}\times U,
\end{aligned}
$$

with $\zeta_{m1},\zeta_{m2}\in L^2(Q)$, $\delta_{m1},\bar{\delta}_{m1},\delta_{m2},\bar{\delta}_{m2}\ge 0$.

Using assumptions (H1-H3) and the fact that V is compactly embedded in $L^4(\Omega)$, we can see that equation (2.6) is well defined.

Theorem 2.1. *Under Assumptions (H1-H3), for every $r\in R$ and $y^0\in L^2(\Omega)$ (or $y^0\in V$), there exist a unique $y := y_r$ such that $y\in L^2((0,T),V)$, $y_t\in L^2((0,T),V^*)$ satisfying (2.6), (2.7). In addition, y is essentially equal to a function in $C([0,T],L^2(\Omega))$, and thus the initial condition (2.7) is well defined.*

Proof. The proof is based on compactness arguments (see [15]). □

Next lemma describes the continuity of the state and the functionals w.r.t. the corresponding relaxed control. This result is the basic tool to prove the existence of optimal relaxed controls.

Lemma 2.2. *Under Assumptions (H1-H3), the mapping $r\mapsto y_r$, from R to $L^2(Q)$ and $L^2((0,T),V)$, is continuous. Under Assumptions (H1-H4), the functionals $r\mapsto J_m(r)$, $m=0,...,q_2$, from R to $\mathbb{R}$, are continuous.*

Proof. Let $r_k\to r$ in R and set $y_k := y_{r_k}$. Taking $y=v=y_k$ in (2.6), using Assumptions (H2-H3) and the basic inequality $2ab\le\frac{1}{\varepsilon}a^2+\varepsilon b^2$, $\varepsilon>0$, we have

$$
\frac{1}{2}\frac{d}{dt}\|y_k\|^2+a_2\|y_k\|_1^2 \le \frac{1}{2}\|\psi(t)\|^2+\left(\frac{1}{2}+\gamma+\frac{c}{2\varepsilon}\right)\|y_k\|^2+c\frac{\varepsilon}{2}\|y_k\|_1^2, \text{ where } c := \|a_0\|_{L^\infty}.
$$

Integrating w.r.t. t on $[0,t]$ for $t\le T$ and selecting appropriate $\varepsilon>0$ to hide the term $\int_0^t \|y_k(s)\|_1^2\,ds$ to the left-hand side we obtain

$$
\frac{1}{2}\|y_k\|^2+\left(a_2-c\frac{\varepsilon}{2}\right)\int_0^t\|y_k(s)\|_1^2\,ds \le \frac{1}{2}\|y^0\|^2+\frac{1}{2}\|\psi\|_{L^2(Q)}^2+\left(\frac{1}{2}+\gamma+\frac{c}{2\varepsilon}\right)\int_0^t\|y_k(s)\|^2\,ds. \tag{2.8}
$$

Using Gronwall's inequality we deduce from (2.8) that y_k is bounded in $L^2(Q)$. Then again from (2.8) we obtain that y_k is bounded in $L^2((0,T),V)$. One can also check using (2.6) that y_k' is bounded in $L^2((0,T),V^*)$. Thus, there exist a subsequence still denoted by (y_k) such that $y_k \longrightarrow y$ in $L^2((0,T),V)$ weakly and $y_k' \longrightarrow y'$ in $L^2((0,T),V^*)$ weakly. Since V is compactly embedded in $L^2(\Omega)$ by Theorem 2.1 chap. III in [16] follows that $y_k \longrightarrow y$ in $L^2(Q)$ strongly. It follows easily that $y = y_r$ and that the convergence holds for the original sequence. The strong convergence $y_k \longrightarrow y$ in $L^2((0,T),V)$ can be proved as in Lemma 4.2 here. Finally, from Proposition 2.1 in [6] we derive that the functionals $r \mapsto J_m(r)$, $m = 0,\dots,q_2$ are continuous. □

Theorem 2.3. *Under Assumptions (H1-H4) and supposing the existence of a feasible control the ROCP has a solution.*

Proof. It follows from Lemma 2.2 and the compactness of R. □

Necessary conditions for optimality for the ROCP are given in Chryssoverghi et al. [14].

3. The semi-discrete optimal control problems

(H7) a, a_0 are independent of t (for simplicity), b, b_y, f, f_y are continuous on $\bar{Q}\times\mathbb{R}\times U$, $g_m, g_{my}, g_{m\bar{y}}$ are continuous on $\bar{Q}\times\mathbb{R}^{d+1}\times U$ and $y^0 \in V$.

For each integer $n \geq 0$, let Ω^n be a subdomain of Ω with polyhedral boundary Γ^n such that $\mathrm{dist}(\Gamma^n,\Gamma) = o(h^n)$, $\{E_i^n\}_{i=1}^{M^n}$ be an admissible regular quasi-uniform triangulation of $\bar{\Omega}^n$ into closed d-simplices (finite elements), with $h^n = \max_i[\mathrm{diam}(E_i^n)] \to 0$ as $n \to \infty$. Associated with the above triangulation we define

$$V^n := \left\{ y^n \in V | y^n \in C(\bar{\Omega}), \text{affine on each } E_i^n,\ y^n = 0 \text{ in } \Omega - \Omega^n \right\}$$

with $\dim V^n = N^n$ and v_i^n, $i = 1,\dots,N^n$ be a basis of V^n,

$$R^n := \{r^n \in R \,|\, r^n = \{r_i^n, i = 1,\dots M^n\}, r_i^n \text{ is equal to a constant measure w.r.t. } x \text{ in } M_1(U) \text{ on the interior of } (E_i^n \times (0,T)), i = 1,\dots M^n\}$$

the set of (semi)discrete relaxed controls and $\mathbb{U}^n := R^n \cap U$ the set of (semi)discrete classical controls. Clearly, we have $\mathbb{U}^n \subset R^n$.

For a given $r^n \in R^n$, the corresponding (semi)discrete state y^n is given by the (semi)discrete state equation (system of ODE's w.r.t. c^n)

$$(y^{n'}, v_i^n) + a(y^n, v_i^n) + (a_0^T(t)\nabla y^n, v_i^n) + (b(t, y^n, r^n), v_i^n) = (f(t, y^n, r^n), v_i^n),\ i = 1,\dots,N^n, \forall t \in (0,T), \tag{3.1}$$

$$(y^n(0) - y^0, v_i^n)_1 = 0,\ i = 1,\dots,N^n, \tag{3.2}$$

where $y^n(t) = \sum_{i=1}^{N^n} c_i^n(t)\, v_i^n$. Note that $y^n(0)$ is the orthogonal projection of y^0 onto V.

Theorem 3.1. *Under Assumptions (H2-H3) and (H7), for every n and $r^n \in R^n$ the equation (3.1), (3.2) admits a unique solution y^n. In addition, the solutions are uniformly (w.r.t. r^n) bounded and equicontinuous.*

The discrete functionals are defined by

$$J_m^n(r^n) := \int_Q g_m(x,t,y^n,\nabla y^n, r^n)dxdt, \quad m = 0,\dots,q_2.$$

We consider the following two discrete problems:

$$\text{minimize } J_0^n(r^n)$$

subject to (3.1), (3.2), the control constraints $r^n \in R^n$ and the state constraints

$$\text{Case (a)} \quad \begin{array}{l} |J_m^n(r^n)| \leq \varepsilon_m^n, \quad m = 1,\dots,q_1 \\ J_m^n(r^n) \leq \varepsilon_m^n, \quad \varepsilon_m^n \geq 0, \quad m = q_1+1,\dots,q_2, \end{array} \tag{3.3}$$

and

$$\text{Case (b)} \quad \begin{array}{l} J_m^n(r^n) = \varepsilon_m^n, \quad m = 1,\dots,q_1, \\ J_m^n(r^n) \leq \varepsilon_m^n, \quad \varepsilon_m^n \geq 0, \quad m = q_1+1,\dots,q_2, \end{array} \tag{3.4}$$

where ε_m^n are non-negative given numbers, introduced for feasibility reasons.

The first of the above discrete problems with state constraints (3.3) is denoted by $DROCP_a$ and the second one with state constraints (3.4) by $DROCP_b$.

Theorem 3.2. *Under Assumptions (H2-H4) and (H7), the mappings $r^n \mapsto y^n$ and $r^n \mapsto J_m^n(r^n)$, defined on R^n, are continuous. If any of the discrete problems is feasible, then it has a solution.*

Proof. The continuity of the operator $r^n \mapsto y^n$ is proved by Theorem 3.1 and using Ascoli's theorem to pass in the limit in (3.1), (3.2). The continuity of $r^n \to J_m^n(r^n)$ follows from the continuity of g_m. Since the set R^n is compact with the relative weak* topology of $M(U)^{M^n}$ it follows that the discrete problems $DROCP_a$, $DROCP_b$ defined above have a solution. □

To compute the directional derivative of the functional J^n, where for simplicity reasons the index m is omitted, we introduce the linear adjoint state equation

$$-(z^{n'},v)+a(v,z^n)+(a_0^T\nabla v,z^n)+(z^n b_y(t,y^n,r^n),v)=(z^n f_y(t,y^n,r^n)+g_y(t,y^n,\nabla y^n,r^n),v)+(g_{\bar{y}}(t,y^n,\nabla y^n,r^n),\nabla v),\forall v\in V^n, \quad (3.5)$$

$$z^n(T)=0, \quad (3.6)$$

which has a unique solution $z^n = z_{r^n}$ with $y^n = y_{r^n}$.

We define, for each function g, the Hamiltonian H

$$H(x,t,y,\bar{y},z,u) := z[f(x,t,y,u)-b(x,t,y,u)]+g(x,t,y,\bar{y},u).$$

The following lemma and theorem can be proved by using the techniques of [1], [6]. See also [17], where necessary optimality conditions on signomial constrained optimal control problems are proved.

Lemma 3.3. *Under Assumptions (H2-H7), the directional derivative of the functional J^n is given by*

$$DJ^n(r^n,r'^n-r^n)=\int_Q H(x,t,y^n,\nabla y^n,z^n,r'^n-r^n)dxdt,\ r^n,r'^n\in R^n,$$

where z^n is given by (3.5), (3.6). Moreover, the mappings $r^n \mapsto z^n$ and $(r^n,r'^n)\mapsto DJ^n(r^n,r'^n-r^n)$ are continuous.

Proof. For simplicity of notation we drop the index n. For $r,r'\in R$, $0<\varepsilon\le 1$, set $r_\varepsilon = r+\varepsilon(r'-r), y:=y_r, y_\varepsilon := y_{r_\varepsilon}$, $\delta_\varepsilon y := y_\varepsilon - y$. Now, by our assumptions, for fixed $r\in R$, the functional

$$\Phi(y,\bar{y},r):=\int_Q g(x,t,y,\bar{y},r)dxdt,$$

is Fréchet differentiable uniformly in r, i.e.

$$\Phi(y+\delta y,\bar{y}+\delta\bar{y},r)-\Phi(y,\bar{y},r)=\int_Q [g_y(x,t,y,\bar{y},r)\delta y+g_{\bar{y}}(x,t,y,\bar{y},r)\delta\bar{y}]\,dxdt+\theta(\delta y,\delta\bar{y})(\|\delta y\|_\infty+\|\delta\bar{y}\|),$$

where $\theta(\delta y,\delta\bar{y})\to 0$ as $\|\delta y\|_\infty+\|\delta\bar{y}\|\to 0$, with θ independent of the control $r\in R$. This can be shown under our assumptions by using the Mean Value Theorem, Hölder's inequality and Proposition 2.1 in [6] for a fixed control. By Lemma 2.2 in [6], we have

$$\begin{aligned} J(r_\varepsilon)-J(r) &= \int_Q [g(y_\varepsilon,\nabla y_\varepsilon,r_\varepsilon)-g(y,\nabla y,r_\varepsilon)+g(y,\nabla y,r_\varepsilon)-g(y,\nabla y,r)]\,dxdt \\ &= \int_Q g_y(y,\nabla y,r)\delta_\varepsilon y dxdt+\int_Q g_{\bar{y}}(y,\nabla y,r)\nabla\delta_\varepsilon y dxdt+\varepsilon\int_Q g(y,\nabla y,r'-r)\,dxdt+o(\varepsilon). \end{aligned} \quad (3.7)$$

Since $\delta_\varepsilon y(0)=z(T)=0$, by similar arguments, the state equation (3.1) yields

$$-\int_0^T (z',\delta_\varepsilon y)dt+\int_0^T a(\delta_\varepsilon y,z)\,dt+\int_0^T (a_0^T\nabla\delta_\varepsilon y,z)dt=\int_Q (f_y(y,r)-b_y(y,r))\,\delta_\varepsilon yzdxdt+\varepsilon\int_Q (f(y,r'-r)-b(y,r'-r))\,zdxdt+o(\varepsilon). \quad (3.8)$$

On the other hand, the adjoint equation (3.5) yields

$$\begin{aligned} -\int_0^T (z',\delta_\varepsilon y)dt &+\int_0^T a(\delta_\varepsilon y,z)dt+\int_0^T (a_0^T\nabla\delta_\varepsilon y,z)dt \\ &= \int_Q (f_y(y,r)-b_y(y,r))\,\delta_\varepsilon yzdxdt+\int_Q g_y(y,\nabla y,r)\delta_\varepsilon ydxdt+\int_Q g_{\bar{y}}(y,\nabla y,r)\nabla\delta_\varepsilon ydxdt. \end{aligned} \quad (3.9)$$

Gathering (3.7), (3.8) and (3.9), we obtain

$$DJ(r,r'-r)=\int_Q \left[z\left(f(x,t,y,r'-r)-b(x,t,y,r'-r)\right)+g(x,t,y,\nabla y,r'-r)\right]dxdt.$$

□

Theorem 3.4. *(i) Under Assumptions (H2-H7), if $r^n \in R^n$ is a solution of the DROCP$_b$, then it is extremal, i.e. there exist multipliers $\lambda_m^n \in \mathbb{R}$, $m = 0,...,q_2$, with $\lambda_0^n \geq 0$, $\lambda_m^n \geq 0$, $m = q_1+1,...,q_2$, $\sum\limits_{m=0}^{q_2} |\lambda_m^n| = 1$, such that*

$$\sum_{m=0}^{q_2} \lambda_m^n DJ_m^n(r^n, r'^n - r^n) = \int_Q H(x,t,y^n,\nabla y^n, z^n, r'^n - r^n)dxdt \geq 0, \forall r'^n \in R^n, \tag{3.10}$$

$$\lambda_m^n[J_m^n(r^n) - \varepsilon_m^n] = 0, \quad m = q_1+1,...,q_2, \tag{3.11}$$

where H and z^n are defined with $g := \sum\limits_{m=0}^{q_2} \lambda_m^n g_m$. Condition (3.10) is equivalent to the strong discrete block pointwise minimum principle

$$\int_{E_i^n} H(x,t,y^n,\nabla y^n, z^n, r^n)dx = \min_{u\in U} \int_{E_i^n} H(x,t,y^n,\nabla y^n, z^n, u)dx, \; i = 1,...,M^n, \; a.e. \, in \, (0,T). \tag{3.12}$$

(ii) With Assumptions (H2-H7) and assuming that $J_0^n, J_{q_1+1}^n, ..., J_{q_2}^n$ are convex and $J_1^n, ..., J_{q_1}^n$ are affine, if $r^n \in R^n$ is admissible and extremal for the DROCP$_b$, with $\lambda_0^n > 0$, then r^n is optimal for this problem.

Proof. (i) The global condition (3.10) and the conditions (3.11) follow from the general multiplier theorem V.2.3 in [1]. The equivalence of the conditions (3.10) and (3.12) is standard (see [1]), since the closed set U has a dense denumerable subset.

(ii) The assumptions imply that the functional $J^n(r^n) := \sum\limits_{m=0}^{q} \lambda_m^n J_m^n(r^n)$ is convex. The condition (3.10) is then satisfied if and only if r^n minimizes J^n on R^n. Supposing now that r^n does not minimize J_0^n and using the constraints and the conditions (3.11), easily follows that r^n does not minimize J^n, which is a contradiction. □

4. Behavior in the limit

Here we study the limiting behavior of the discrete problems as $n \to \infty$. Next proposition gives us a control approximation result. It is proved in [6] for totally (i.e. in space and time) discrete controls, from which it follows for semidiscrete ones.

Proposition 4.1. *For every $r \in R$, there exist $(u^n) \in \mathbb{U}^n$ such that $u^n \to r$ in R.*

Lemma 4.2 (Consistency of states and functionals)**.** *Under Assumptions (H2-H3) and (H7), if $r^n \to r$ in R, then the corresponding discrete states y^n converge to y_r in $L^2((0,T),V)$ strongly and*

$$\lim_{n\to\infty} J_m^n(r^n) = J_m(r), \quad m = 0,...,q_2. \tag{4.1}$$

Proof. Multiplying (3.1) by c_i^n and summing over i we obtain for every $t \in (0,T)$

$$\langle y^{n'}, y^n\rangle + a(y^n,y^n) + (a_0^T(t)\nabla y^n, y^n) + (b(t,y^n,r^n), y^n) = (f(t,y^n,r^n), y^n). \tag{4.2}$$

Integrating (4.2) on $[0,t]$, $t \leq T$ and working similarly to the proof of Lemma 2.2, we deduce that

$$\frac{1}{2}\|y^n(t)\|^2 + c_1 \int_0^t \|y^n(s)\|_1^2 ds \leq \frac{1}{2}\|y^n(0)\|^2 + \frac{1}{2}\int_0^t \|\psi(s)\|^2 ds + c_2 \int_0^t \|y^n(s)\|^2 ds, \; \forall t \in (0,T), \tag{4.3}$$

for some appropriate constants c_1, c_2.

Using Gronwall's inequality and the fact that $y^n(0)$ is bounded (since clearly $y^n(0) \to y^n$ in V strongly, due to the projection) we deduce from (4.3) that y^n is bounded in $L^2(Q)$. Then from (4.3) we obtain that y^n is bounded in $L^2((0,T),V)$. One can also prove using (3.1) that $y^{n'}$ is bounded in $L^2((0,T),V^*)$. Since the injection of V in $L^2(\Omega)$ is compact, by the compactness Theorem 2.2 chap. III in [16], there exists a subsequence still denoted by y^n : $y^n \to y$ in $L^2(Q)$ strongly.

Let $v \in C_0^1(\bar{\Omega})$ an arbitrary given function and $(v^n) \in V^n$ a sequence of functions interpolating the function v at the vertices inside Ω^n and vanishing on Γ^n. The sequence converges to v in V strongly. Then, the integral form of the discrete equation (3.1) is written

$$(y^n(T), v^n) - (y_0^n, v^n) + \int_0^T a(y^n, v^n)dt + \int_0^T (a_0^T \nabla y^n, v^n)dt = \int_0^T (f(t,y^n,r^n) - b(t,y^n,r^n), v^n)dt. \tag{4.4}$$

We obtain from (4.4) using the above convergences and Proposition 2.1 in [6] that, $\forall v \in C_0^1(\bar{\Omega})$

$$
\begin{aligned}
&(y^n(T),v) = (y^n(T),\, v-v^n) + (y^n(T),\, v^n) = (y^n(T),\, v-v^n) + (y_0^n, v^n)\\
&+\int_0^T (f(t,y^n,r^n) - b(t,y^n,r^n), v^n)dt\, - \int_0^T a(y^n,v^n)dt - \int_0^T (a_0^T \nabla y^n, v^n)dt\\
&\longrightarrow (y^0,v) + \int_0^T (f(y,r) - b(y,r),v)dt - \int_0^T a(y,v)dt - \int_0^T (a_0^T \nabla y, v)dt\, = (y(T),v).
\end{aligned}
$$

Since $C_0^1(\bar{\Omega})$ is dense in $L^2(\Omega)$ it follows that $(y^n(T),v) \to (y(T),v)\, \forall v \in L^2(\Omega)$, i.e. $y^n(T) \to y(T)$ in $L^2(\Omega)$ weakly. By the above convergences, we get from (4.4)

$$(y(T),v) - (y^0,v) + \int_0^T a(y,v)dt + \int_0^T (a_0^T \nabla y, v)dt = \int_0^T (f(y,r) - b(y,r),v)dt,$$

hence $y = y_r$.

Next, we prove that $y^n \to y$ in $L^2((0,T),V)$ strongly. We have

$$
\begin{aligned}
\alpha_2 \|y^n - y\|^2_{L^2((0,T),V)} &\le \textstyle\int_0^T a(y^n-y, y^n-y)dt + \frac{1}{2}\|y^n(T)-y(T)\|^2 = \frac{1}{2}\left\|y_0^n\right\|^2 - \frac{1}{2}(y^n(T),y(T)) - \frac{1}{2}(y(T),y^n(T)-y(T))\\
&+ \textstyle\int_0^T (f(y^n,r^n) - b(y^n,r^n),y^n)dt - \int_0^T (a_0^T \nabla y^n, y^n)dt - \int_0^T a(y^n,y)dt - \int_0^T a(y,y^n-y)dt,
\end{aligned}
$$

and as $n \to \infty$ the right-hand side of the above inequality convergence to zero.

Finally, the convergences (4.1) follow from Proposition 2.1 in [6]. □

In what follows the feasibility of the ROCP is assumed. Next theorem addresses the limit behavior of optimal discrete relaxed controls for the $DROCP_a$.

Theorem 4.3. *Under Assumptions (H2-H4) and (H7) and the additional assumption that the sequences (ε_m^n) converge to zero as $n \to \infty$ and satisfy*

$$|J_m^n(\tilde{r}^n)| \le \varepsilon_m^n, \quad m = 1,\dots,q_1, \quad J_m^n(\tilde{r}^n) \le \varepsilon_m^n, \quad \varepsilon_m^n \ge 0, \quad m = q_1+1,\dots,q_2,$$

for every n, where $(\tilde{r}^n) \in R^n$ is a sequence which converges in R to some $\tilde{r} \in R$ optimal for the ROCP. Then, for each n, we consider (r^n) which is optimal for the $DROCP_a$. The above sequence (r^n) has accumulation points in R that are optimal for the ROCP.

Proof. From theorem's assumptions the feasibility of the $DROCP_a$, for every n, follows. Let a subsequence of (r^n), still denoted by (r^n), such that $r^n \to r,\ r \in R$. Since r^n is admissible as optimal and $\tilde{r}^n$ is admissible for the $DROCP_a$, it follows

$$J_0^n(r^n) \le J_0^n(\tilde{r}^n), |J_m^n(r^n)| \le \varepsilon_m^n, m = 1,\dots,q_1, J_m^n(r^n) \le \varepsilon_m^n, m = q_1+1,\dots,q_2.$$

Taking limits as $n \to \infty$, with the help of Lemma 4.2, we conclude that r is optimal for the ROCP. □

Lemma 4.4. *Under Assumptions (H2-H7), if $r^n \to r$ in R, then $z^n \to z_r$ in $L^2((0,T),V)$ strongly, where z^n the corresponding discrete adjoint states. If $r^n \to r$ and $r'^n \to r'$, then*

$$\lim_{n\to\infty} DJ_m^n(r^n, r'^n - r^n) = DJ_m(r, r'-r), \quad m = 0,\dots,q_2.$$

Proof. It follows easily from Lemma 4.2 and the same arguments as those in the proof of that Lemma. □

Next, we consider the $DROCP_b$. We can choose (ε_m^n), $m = 1,\dots,q_2$, such that $\varepsilon_m^n \to 0$, $n \to \infty$ and the $DROCP_b$ is feasible for every n (see [6]).

Theorem 4.5. *Under Assumptions (H2-H7), for each n, let r^n be admissible and extremal for the $DROCP_b$. Then the sequence (r^n) has accumulation points that are admissible and extremal for the ROCP.*

Proof. Since R is compact, consider a subsequence (r^n) such that $r^n \to r$ in R. From Theorem 3.4, there exist multipliers $\lambda_m^n \in \mathbb{R}$, $m = 0,\dots,q_2$ with $\sum_{m=0}^{q_2} |\lambda_m^n| = 1$, thus there exist subsequences (λ_m^n), $m = 0,\dots,q_2$, such that $\lambda_m^n \to \lambda_m$, $m = 0,\dots,q_2$. Let any $r' \in R$ and (r'^n) be a sequence such that $r'^n \to r'$ (Proposition 4.1). Using the above convergences, Lemmas 4.2, 4.4 and Proposition 2.1 in [6] and passing to the limit in (3.10), (3.11) we have

$$\int_Q H(x,t,y,\nabla y, z, r'(x,t) - r(x,t))dxdt \ge 0, \quad \forall r' \in R,$$

$$\lambda_m J_m(r) = \lim_{n\to\infty} \lambda_m^n [J_m^n(r^n) - \varepsilon_m^n] = 0, \quad m = q_1+1,\ldots,q_2.$$

Also,

$$J_m(r) = \lim_{n\to\infty} [J_m^n(r^n) - \varepsilon_m^n] = 0, \quad m = 1,\ldots,q_1,$$
$$J_m(r) = \lim_{n\to\infty} [J_m^n(r^n) - \varepsilon_m^n] \leq 0, \quad m = q_1+1,\ldots,q_2.$$

Therefore, r is admissible and extremal for the ROCP (see [14]). □

5. Discrete penalized conditional descent method

We choose a fixed discretization and for notational simplicity we shall drop the index n in the data. Let (M_m^l), $m = 1,\ldots,q_2$, be increasing sequences with $(M_m^l) > 0$ and $M_m^l \to \infty$ as $l \to \infty$. Define the discrete functionals with penalties

$$J^l(r) := J_0(r) + 0.5\{\sum_{m=1}^{q_1} M_m^l [J_m(r)]^2 + \sum_{m=q_1+1}^{q_2} M_m^l [\max(0, J_m(r))]^2\}.$$

Let $\rho, \sigma \in (0,1)$, and let (β^l), (ζ_k) be positive sequences, with (β^l) decreasing and converging to zero, and $\zeta_k \leq 1$. A penalized conditional descent method with Armijo line step search applied on the $DROCP_b$ is presented in the following algorithm.

Algorithm

Step 1. $k = 0$, $l = 1$. Choose an initial discrete control $r_0^1 \in R$.

Step 2. Compute the state and the adjoint associated with r_k^l. Find $\bar{r}_k^l \in R$ such that

$$\bar{r}_k^l = \arg\min \left\{ DJ^l(r_k^l, r' - r_k^l),\ r' \in R \right\}$$

and set $d_k := DJ^l(r_k^l, \bar{r}_k^l - r_k^l)$.

Step 3. If $|d_k| > \beta^l$, then go to Step 4, else $r^l = r_k^l$, $\bar{r}^l = \bar{r}_k^l$, $d^l = d_k$, $r_k^{l+1} = r_k^l$, $l = l+1$ and return to Step 2.

Step 4. Find the smallest nonnegative integer s, denoted $\bar{s}$:

$$J^l(r_k^l + \sigma^s \zeta_k (\bar{r}_k^l - r_k^l)) - J^l(r_k^l) \leq \sigma^s \zeta_k \rho d_k.$$

Set $\alpha_k = \sigma^{\bar{s}} \zeta_k$.

Step 5. Choose an equivalent $r_{k+1}^l \in R$ such that

$$J^l(r_{k+1}^l) = J^l(r_k^l + \alpha_k(\bar{r}_k^l - r_k^l)),$$

set $k = k+1$, and return to Step 2.

We now define the sequences of multipliers

$$\lambda_m^l := M_m^l J_m(r^l),\ m = 1,\ldots,q_1,\ \lambda_m^l := M_m^l \max(0, J_m(r^l)),\ m = q_1+1,\ldots,q_2, \tag{5.1}$$

where r^l are defined in Step 3 of the Algorithm.

In the following theorem we study the convergence properties of the above algorithm.

Theorem 5.1. *Consider the sequence (r^l) constructed in Step 3 of the Algorithm. If the sequences $(\lambda_m^l), m = 1,\ldots,q_2$ remain bounded, then any accumulation point of (r^l) satisfies the optimality conditions (3.10), (3.11) for the discrete problem.*

Proof. We can prove that $l \to \infty$ in the Algorithm as in Theorem 5.1 in [12].

If $r \in R$ is an accumulation point of the sequence (r^l), there exist a subsequence of it, still denoted by (r^l), converging to $r \in R$ as $l \to \infty$. If the sequences (λ_m^l), $m = 1,\ldots,q_2$ defined in (5.1) are bounded, then they have subsequences, again denoted by (λ_m^l), such that $\lambda_m^l \to \lambda_m$. Using Lemma 4.2, we obtain

$$0 = \lim_{l\to\infty} \frac{\lambda_m^l}{M_m^l} = \lim_{l\to\infty} J_m(r^l) = J_m(r), \quad m = 1,\ldots,q_1,$$

$$0=\lim_{l\to\infty}\frac{\lambda_m^l}{M_m^l}=\lim_{l\to\infty}[\max(0,J_m(r^l))]=\max(0,J_m(r)),\quad m=q_1+1,\ldots,q_2,$$

thus r is admissible. Next, for every $r'\in R$, Steps 2, 3 of the Algorithm give

$$DJ^l(r^l,r'-r^l)=\lambda_0^l DJ_0(r^l,r'-r^l)+\sum_{m=1}^{q_1}\lambda_m^l DJ_m(r^l,r'-r^l)+\sum_{m=q_1+1}^{q_2}\lambda_m^l DJ_m(r^l,r'-r^l)\geq d^l, \tag{5.2}$$

with $\lambda_0^l:=1$. From Step 3 of the Algorithm we have $\left|d^l\right|\leq\beta^l\to 0$. We use the above convergences and Lemma 3.3 to pass to the limit in (5.2), as $l\to\infty$ and obtain

$$\lambda_0 DJ_0(r,r'-r)+\sum_{m=1}^{q_1}\lambda_m DJ_m(r,r'-r)+\sum_{m=q_1+1}^{q_2}\lambda_m DJ_m(r,r'-r)\geq 0. \tag{5.3}$$

Obviously, $\lambda_0=1$ and the construction of λ_m^l implies that in the limit $\lambda_m\geq 0$, $m=q_1+1,\ldots,q_2$. Dividing (5.3) by $\sum\limits_{m=0}^{q_2}|\lambda_m|\geq 1$ we can suppose that $\sum\limits_{m=0}^{q_2}|\lambda_m|=1$. Also, if $J_m(r)<0$, for some $m\in[q_1+1,\,q_2]$, then for l sufficiently large, we have $J_m^l(r^l)<0$ and $\lambda_m^l=0$, hence $\lambda_m=0$, i.e. the conditions (3.11) hold. Therefore, r is also extremal. □

Under the additional assumptions of Theorem 3.4 the Algorithm computes optimal controls.

Finally, we can show, see [5], that the constructed control r_k^l in Step 5 of the Algorithm can be chosen to be of Gamkrelidze type and these controls can be approximated by classical controls. So, the relaxed controls can be implemented.

6. Numerical examples

In this section, two examples are presented. The first one without state constraints and the second one with an equality state constraint. The Algorithm applied on both problems (in the first one without penalties) with $\rho=\sigma=0.5$ and initial control $r:=(r_0+r_1)/2$, where $r_0(x,t):=\delta_0,\ r_1(x,t):=\delta_1$ (Dirac measures).

Example 6.1. *Let $Q:=(0,1)\times(0,1)$ and $U:=\{0,1\}$. Consider the following optimal control problem*

$$\text{minimize } J_0(u):=\int_Q\{0.5\,[(y-\bar{y})^2+|\nabla y-\nabla\bar{y}|^2]-u^2+u\}dxdt$$

subject to

$$y_t-y_{xx}+0.5y\,|y|+(1+u-\bar{u})y=0.5\,\bar{y}\,|\bar{y}|+\bar{y}+x(1-x)\,(-1+t)+2-2t+t^2+\sin y-\sin\bar{y}+3(u-\bar{u})\text{ in }Q,$$
$$y(0,t)=y(1,t)=0,$$
$$y(x,0)=x(1-x)\text{ in }(0,1),$$

and the control constraints $u\in U$, where

$$\bar{u}(x,t):=\begin{cases}1, & \text{if } 0\leq t\leq 0.5,\\ 1-2(t-0.5)(-0.4x+0.7), & \text{if } 0.5<t\leq 1,\end{cases}$$

$$\bar{y}(x,t):=x(1-x)\left(1-t+0.5t^2\right).$$

It is easy to verify that

$$r(x,t)\{1\}=\bar{u}(x,t),\quad r(x,t)\{0\}=1-r(x,t)\{1\},\quad (x,t)\in Q,$$

is the unique optimal relaxed control distributed between the points 0 and 1 with optimal state $\bar{y}$ and optimal cost 0.

These are the results when the Algorithm was applied for 90 iterations.

$$J_0^n(r_k^n)=3.5376\cdot 10^{-5},\quad d_k=-1.2321\cdot 10^{-4},$$

where d_k was defined in Step 2 of the Algorithm. Figure 6.1 shows the last control probability function $p_1(x,t):=r_k^n(x,t)\{1\}$. The state y for the final iteration is shown in Figure 6.2.

Example 6.2. *Consider the above problem under the equality state constraint*

$$J_1(u):=\int_Q ydxdt=0.$$

These are the results when the Algorithm was applied for 210 iterations.

$$J_0^n(r_k^{nl})=8.3807\cdot 10^{-2},\quad J_1^n(r_k^{nl})=3.8048\cdot 10^{-5},\quad d_k=-3.8013\cdot 10^{-3}.$$

Here, $p_1(x,t):=r_k^{nl}(x,t)\{1\}$ is shown in Figure 6.3 and the state y for the final iteration in Figure 6.4.

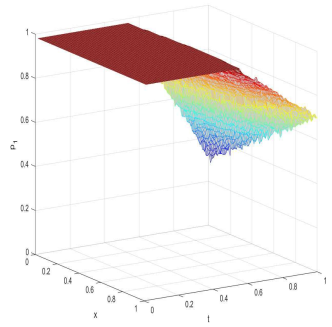

Figure 6.1: Example 6.1: Last relaxed control probability p_1

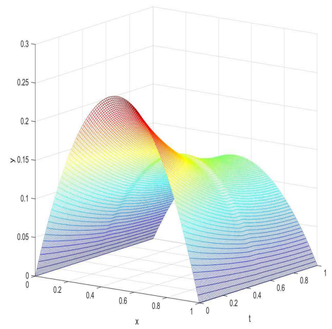

Figure 6.2: Example 6.1: State y for the final iteration

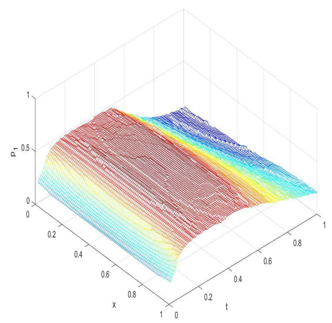

Figure 6.3: Example 6.2: Last relaxed control probability p_1

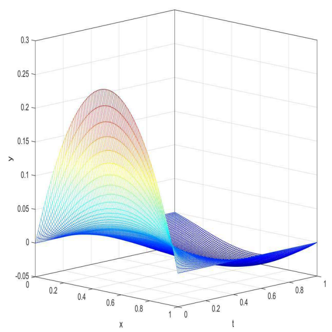

Figure 6.4: Example 6.2: State y for the final iteration

7. Conclusion

In the absence of any convexity assumptions, relaxed controls are an important tool to prove existence of optimal controls. Thus, the corresponding relaxed optimal control problem is introduced, which is then discretized and the behavior in the limit of sequences of optimal and admissible extremal controls was studied. Finally, a penalized conditional descent method using relaxed controls, applied to the discrete relaxed problem, is proposed. This method constructs discrete Gamkrelidze controls which, for implementation reasons, can be approximated by piecewise classical ones.

References

[1] J. Warga, *Optimal Control of Differential and Functional Equations*, Academic Press, New York, 1972.
[2] T. Roubíček, *Relaxation in Optimization Theory and Variational Calculus*, Walter de Gruyter, Berlin, 1997.
[3] H. O. Fattorini, *Infinite Dimensional Optimization Theory and Optimal Control*, Cambridge Univ. Press, Cambridge, 1999.
[4] J. Warga, *Steepest descent with relaxed controls*, SIAM J. Control Optim., **15** (1977), 674-682.
[5] I. Chryssoverghi, A. Bacopoulos, B. Kokkinis, J. Coletsos, *Mixed Frank-Wolfe penalty method with applications to nonconvex optimal control problems*, J. Optimiz. Theory App., **94** (1997) 311-334.
[6] I. Chryssoverghi, A. Bacopoulos, *Approximation of relaxed nonlinear parabolic optimal control problems*, J. Optimiz. Theory App., **77** (1993) 31-50.
[7] T. Roubíček, *A convergent computational method for constrained optimal relaxed control problems*, J. Optimiz. Theory App., **69** (1991) 589-603.
[8] V. Azhmyakov, W. Schmidt, *Approximations of relaxed optimal control problems*, J. Optimiz. Theory App., **130** (2006) 61-77.
[9] N. Arada, J. P. Raymond, *State-constrained relaxed problems for semilinear elliptic equations*, J. Math. Anal. Appl., **223** (1998) 248-271.
[10] N. Arada, J. P. Raymond, *Approximation of optimal control problems with state constraints*, Numer. Funct. Anal. and Optimiz., **21** (2000) 601-621.
[11] E. Casas, *The relaxation theory applied to optimal control problems of semilinear elliptic equations*, J. Dolezal, J. Fidler (editors), *System Modelling and Optimization*, Chapman & Hall, London, 1996, pp. 187-194.
[12] I. Chryssoverghi, J. Coletsos, B. Kokkinis, *Discrete relaxed method for semilinear parabolic optimal control problems*, Control Cybernet., **28** (1999) 157-176.
[13] S. Luan, *Nonexistence and existence of an optimal control problem governed by a class of semilinear elliptic equations*, J. Optimiz. Theory App., **158** (2013) 1-10.
[14] I. Chryssoverghi, J. Coletsos, B. Kokkinis, *Classical and relaxed optimization methods for nonlinear parabolic optimal control problems* I. Lirkov, S. Margenov and J. Waśniewski (editors), *Large-scale scientific computing*, Springer-Verlag, Berlin, 2010, pp. 247-255.
[15] J. L. Lions, *Quelques Méthodes de Résolution des Probèmes aux Limites Non Linéaires*, Dunod Gauthier-Villars, Paris, 1969.
[16] R. Temam, *Navier-Stokes Equations*, North-Holland, New York, 1977.
[17] S. Treanta, *On signomial constrained optimal control problems*, Commun. Adv. Math. Sci., **2** (2019), 55-59.

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 45-51
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.721287

Certain Geometric Properties and Matrix Transformations on a Newly Introduced Banach Space

Merve İlkhan[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey*

Article Info

Keywords: *Banach-Saks type p, Fixed point property, Matrix transformations*
2010 AMS: *46B45, 46B20, 46A45*
Received: *16 April 2020*
Accepted: *31 May 2020*
Available online: *10 June 2020*

Abstract

The main purpose of this study is to characterize some matrix classes from classical sequence spaces into a newly introduced space and find the norm of some special matrix operators. Also, we give certain geometric properties of this space.

1. Introduction

The matrix transformations in sequence spaces have been studied by many authors over years. Since the most general linear operators from a sequence space to another one can be given by an infinite matrix, the theory of matrix transformations has been of great importance in the study of sequence spaces. For the relevant literature consult to [1]-[6].

In the recent times, the interest in investigating geometric properties of sequence spaces with topological properties have increased. Over years several papers on the geometric properties of various spaces have appeared. For instance, Mursaleen et al. [7] examined the geometric properties of Euler sequence space. More information about the relevant literature can be found in [8]-[14].

The main purpose of this work is to characterize some matrix classes on a newly introduced sequence space and find the norm of certain bounded linear matrix operators. Also, we prove that the resulting space is of type p Banach-Saks and it has the weak fixed point property. Finally, we investigate the strictly convexity and uniformly convexity of this space.

2. Preliminaries and notations

A sequence space is a linear subspace of the space of all real valued sequences ω. ℓ_∞, c, c_0 and ℓ_p $(1 \leq p < \infty)$ are the sequence spaces of all bounded, convergent, null sequences and absolutely p-summable sequences, respectively.

Given any sequence spaces X and Y and an infinite matrix $T = (t_{ij})$, T is called a matrix mapping from X into Y if for every sequence $x = (x_j) \in X$, $Tx = (T_i(x))$ with

$$T_i(x) = \sum_{j=1}^{\infty} t_{ij} x_j$$

is in Y and the series is convergent for each $i \in \mathbb{N} = \{1, 2, \ldots\}$. Then, Tx is called the T-transform of x.

Email address and ORCID number: **merveilkhan@duzce.edu.tr, https://orcid.org/0000-0002-0831-1474 (M. İlkhan)**

The set

$$X_T = \{x = (x_j) \in \omega : Tx \in X\}$$

is called the matrix domain of T in the space X and it is also a sequence space.

Let $\varphi : \mathbb{N} \to \mathbb{N}$ be the Euler function defined as

$$\varphi(i) = \sum_{j=1,(j,i)=1}^{i} 1,$$

where (j,i) is the greatest common divisor of j and i. That is, $\varphi(i)$ gives the number of positive integers less than i which are coprime with j.

The Euler function φ satisfies the following properties:

1. $i = \sum_{j|i} \varphi(j)$ holds for every $i \in \mathbb{N}$.
2. $\varphi(i) = i \prod_{p|i}(1 - \frac{1}{p})$, where p is the prime divisor of i.
3. $\varphi(ij) = \varphi(i)\varphi(j)$ holds for $(i,j) = 1$.

Let $i = p_1^{\alpha_1} p_2^{\alpha_2} ... p_l^{\alpha_l}$. The Möbius function $\mu : \mathbb{N} \to \{-1,0,1\}$ is defined as

$$\mu(i) = (-1)^l \text{ if } \alpha_1 = \alpha_2 = ... = \alpha_l = 1$$

$$\mu(i) = 0 \text{ if } \alpha_k \neq 1 \text{ for at least one } k \in \{1,2,...,l\},$$

where $p_1, p_2, ..., p_l$ are non-equivalent prime numbers and $p_1^{\alpha_1} p_2^{\alpha_2} ... p_l^{\alpha_l}$ is the prime factorization of $i > 1$. Also,

$$\mu(1) = 1$$

and for $i \neq 1$

$$\sum_{p|i} \mu(p) = 0$$

holds.

Φ-summability was introduced by Schoenberg [15] in order to study the Riemann integrability of a generalized Dirichlet function in $[0,1]$. It is said that a sequence $x = (x_j)$ is φ-convergent to l if

$$\lim_{i\to\infty} \frac{1}{i} \sum_{j|i} \varphi(j) x_j = l.$$

Let $\Phi = (\phi_{ij})$ be the matrix defined as

$$\phi_{ij} = \begin{cases} \frac{\varphi(j)}{i} & , \quad \text{if } j \mid i, \\ 0 & , \quad \text{if } j \nmid i. \end{cases}$$

The regularity of this special matrix is also observed by Schoenberg [15]. This means that the matrix Φ maps c into c and the limit is preserved.

In [16], by using this matrix, the sequence spaces

$$\ell_p(\Phi) = \left\{ x = (x_i) \in \omega : \sum_i \left| \frac{1}{i} \sum_{j|i} \varphi(j) x_j \right|^p < \infty \right\} \quad (1 \leq p < \infty)$$

and

$$\ell_\infty(\Phi) = \left\{ x = (x_i) \in \omega : \sup_i \left| \frac{1}{i} \sum_{j|i} \varphi(j) x_j \right| < \infty \right\}$$

are introduced and proved that these spaces are Banach spaces with the norms

$$\|x\|_{\ell_p(\Phi)} = \left(\sum_i \left| \frac{1}{i} \sum_{j|i} \varphi(j) x_j \right|^p \right)^{1/p} \quad (1 \leq p < \infty)$$

and

$$\|x\|_{\ell_\infty(\Phi)} = \sup_i \left| \frac{1}{i} \sum_{j|i} \varphi(j) x_j \right|,$$

respectively.

Unless otherwise stated, $\tilde{x} = (\tilde{x}_i)$ will be the Φ-transform of a sequence $x = (x_i)$, that is,

$$\tilde{x}_i = \Phi_i(x) = \frac{1}{i} \sum_{j|i} \varphi(j) x_j \tag{2.1}$$

for all $i \in \mathbb{N}$.

3. Some matrix transformations and norms of matrix operators

In this part of the study, we firstly give the characterization of matrix classes $(X, \ell_p(\Phi))$, where $X \in \{\ell_\infty, c, c_0, \ell_1\}$ and $1 \le p \le \infty$. For this aim, we give the following results, where $\mathscr{F}$ denotes the collection of all finite subsets of $\mathbb{N}$. q is conjugate of p; that is $p^{-1} + q^{-1} = 1$ with $1 < p, q < \infty$.

Lemma 3.1. *[17] Let $1 \le p < \infty$.*

(a) $T = (t_{ij}) \in (\ell_\infty, \ell_p) = (c, \ell_p) = (c_0, \ell_p)$ if and only if

$$\sup_{K \in \mathscr{F}} \sum_i \left| \sum_{j \in K} t_{ij} \right|^p < \infty.$$

(b) $T = (t_{ij}) \in (\ell_1, \ell_p)$ if and only if

$$\sup_j \sum_i |t_{ij}|^p < \infty.$$

(c) $T = (t_{ij}) \in (\ell_\infty, \ell_\infty) = (c, \ell_\infty) = (c_0, \ell_\infty)$ if and only if

$$\sup_i \sum_j |t_{ij}| < \infty.$$

(d) $T = (t_{ij}) \in (\ell_1, \ell_\infty)$ if and only if

$$\sup_{i,j} |t_{ij}| < \infty.$$

Theorem 3.2. *Let $1 \le p < \infty$.*

(a) $T = (t_{ij}) \in (\ell_\infty, \ell_p(\Phi)) = (c, \ell_p(\Phi)) = (c_0, \ell_p(\Phi))$ if and only if

$$\sup_{K \in \mathscr{F}} \sum_i \left| \sum_{j \in K} \sum_{l|i} \frac{\varphi(l)}{i} t_{lj} \right|^p < \infty.$$

(b) $T = (t_{ij}) \in (\ell_1, \ell_p(\Phi))$ if and only if

$$\sup_j \sum_i \left| \sum_{l|i} \frac{\varphi(l)}{i} t_{lj} \right|^p < \infty.$$

(c) $T = (t_{ij}) \in (\ell_\infty, \ell_\infty(\Phi)) = (c, \ell_\infty(\Phi)) = (c_0, \ell_\infty(\Phi))$ if and only if

$$\sup_i \sum_j \left| \sum_{l|i} \frac{\varphi(l)}{i} t_{lj} \right| < \infty.$$

(d) $T = (t_{ij}) \in (\ell_1, \ell_\infty(\Phi))$ if and only if

$$\sup_{i,j} \left| \sum_{l|i} \frac{\varphi(l)}{i} t_{lj} \right| < \infty.$$

Proof. Given any infinite matrix $T=(t_{ij})\in(\ell_\infty,\ell_p(\Phi))$, define a new matrix $\hat{T}=(\hat{t}_{ij})$ by

$$\hat{t}_{ij}=\sum_{l|i}\frac{\varphi(l)}{i}t_{lj}$$

for all $i,j\in\mathbb{N}$. Then, for any $x=(x_j)\in\ell_\infty$, the equality

$$\sum_j \hat{t}_{ij}x_j=\sum_{l|i}\frac{\varphi(l)}{i}\sum_j t_{lj}x_j$$

means that $\hat{T}_i(x)=\Phi_i(Tx)$ for all $i\in\mathbb{N}$. This implies that $Tx\in\ell_p(\Phi)$ for $x=(x_j)\in\ell_\infty$ if and only if $\hat{T}x\in\ell_p$ for $x=(x_j)\in\ell_\infty$. Hence, we conclude from Lemma 3.1 (a) that

$$\sup_{K\in\mathscr{F}}\sum_i\left|\sum_{j\in K}\sum_{l|i}\frac{\varphi(l)}{i}t_{lj}\right|^p<\infty.$$

The other results follow with the same technique by using Lemma 3.1 (b), (c) and (d). □

Now, we investigate the norm of the bounded linear matrix operators from $\ell_p(\Phi)$ into $\ell_1(\Phi)$ and $\ell_\infty(\Phi)$ for $1\le p\le\infty$. Firstly, we have a lemma which is essential for our investigation.

Lemma 3.3. *Given any infinite matrix $T=(t_{ij})$, the following statements hold:*

(a) The norm of $T\in B(\ell_p,\ell_\infty)$ is defined by

$$\|T\|_{(\ell_1,\ell_\infty)}=\sup_{i,j}|t_{ij}|$$

and

$$\|T\|_{(\ell_p,\ell_\infty)}=\sup_i\sum_j|t_{ij}|^q \qquad (1<p\le\infty).$$

(b) The norm of $T\in B(\ell_p,\ell_1)$ is defined by

$$\|T\|_{(\ell_1,\ell_1)}=\sup_j\sum_i|t_{ij}|$$

and

$$\|T\|_{(\ell_p,\ell_1)}=\sup_{K\in\mathscr{F}}\sum_j\left|\sum_{i\in K}t_{ij}\right|^q \qquad (1<p\le\infty).$$

Theorem 3.4. *Let $T=(t_{ij})$ be an infinite matrix.*

(a) If $T\in B(\ell_1(\Phi),\ell_\infty(\Phi))$, then

$$A_1^\infty=\sup_{i,j}\left|\sum_{j|l}\frac{\mu(\frac{l}{j})}{\varphi(l)}j\sum_{k|i}\frac{\varphi(k)}{i}t_{kl}\right|$$

is finite. In this case, $\|T\|_{(\ell_1(\Phi),\ell_\infty(\Phi))}=A_1^\infty$.

(b) Let $1<p\le\infty$. If $T\in B(\ell_p(\Phi),\ell_\infty(\Phi))$, then

$$A_p^\infty=\sup_i\sum_j\left|\sum_{j|l}\frac{\mu(\frac{l}{j})}{\varphi(l)}j\sum_{k|i}\frac{\varphi(k)}{i}t_{kl}\right|^q$$

is finite. In this case, $\|T\|_{(\ell_p(\Phi),\ell_\infty(\Phi))}=A_p^\infty$.

(c) If $T\in B(\ell_1(\Phi),\ell_1(\Phi))$, then

$$A_1^1=\sup_j\sum_i\left|\sum_{j|l}\frac{\mu(\frac{l}{j})}{\varphi(l)}j\sum_{k|i}\frac{\varphi(k)}{i}t_{kl}\right|$$

is finite. In this case, $\|T\|_{(\ell_1(\Phi),\ell_1(\Phi))}=A_1^1$.

(d) Let $1<p\le\infty$. If $T\in B(\ell_p(\Phi),\ell_1(\Phi))$, then

$$A_p^1=\sup_{K\in\mathscr{F}}\sum_j\left|\sum_{i\in K}\sum_{j|l}\frac{\mu(\frac{l}{j})}{\varphi(l)}j\sum_{k|i}\frac{\varphi(k)}{i}t_{kl}\right|^q$$

is finite. In this case, $\|T\|_{(\ell_p(\Phi),\ell_1(\Phi))}=A_p^1$.

Proof. Let $\tilde{T} = \Phi T \Phi^{-1}$. From Theorem 3 in [16], it is known that the spaces $\ell_p(\Phi)$ and ℓ_p are linearly isomorphic, where $1 \leq p \leq \infty$. Hence, we deduce from the following diagram

$$\begin{tikzcd}
\ell_p(\Phi) \arrow[r, "T"] & X(\Phi) \arrow[d, "\Phi"] \\
\ell_p \arrow[u, "\Phi^{-1}"] \arrow[r, "\tilde{T}=\Phi T\Phi^{-1}"'] & X
\end{tikzcd}$$

that $\|T\|_{(\ell_p(\phi),X(\Phi))} = \|\tilde{T}\|_{(\ell_p,X)}$, where $X \in \{\ell_\infty, \ell_1\}$ and $1 \leq p \leq \infty$. Thus, the desired results follows from Lemma 3.3. $\square$

4. Certain geometric properties of $\ell_p(\Phi)$

In this part of the study, some geometric properties of the space $\ell_p(\Phi)$ for $1 < p < \infty$ is given. $\mathscr{B}_X$ denotes the unit ball in a normed space $(X, \|.\|)$.

It is said that a Banach space X satisfies the *Banach-Saks property* if every sequence (u_n) in $X \cap \ell_\infty$ has a subsequence (t_n) such that the sequence $(a_k(t))$ is convergent, where

$$a_k(t) = \frac{1}{k+1}(t_0 + t_1 + \ldots + t_k); \ (k \in \mathbb{N}).$$

It is said that a Banach space X satisfies the *weak Banach-Saks property* if there exists a subsequence (t_n) of a given weakly null sequence (u_n) in X such that the sequence $(a_k(t))$ is strongly convergent to zero.

It is said that a Banach space satisfies the property *Banach-Saks type p* if every weakly null sequence (u_k) has a subsequence (u_{k_j}) such that for some $C > 0$,

$$\left\| \sum_{j=1}^{n} u_{k_j} \right\| < Cn^{1/p}$$

for all $n \in \mathbb{N}$. Note that $n^{1/\infty} = 1$ for all $n \in \mathbb{N}$ ([18]).

Theorem 4.1. *The space $\ell_p(\Phi)$ is of type p Banach-Saks for $1 < p < \infty$.*

Proof. Let (δ_n) be a sequence such that $\delta_n > 0$ for all $n \in \mathbb{N}$ and $\sum_n \delta_n \leq 1/2$. Choose a weakly null sequence (u_n) in $\mathscr{B}_{\ell_p(\Phi)}$. Put $t_1 = u_{n_1} = u_1$. There exists $m_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=m_1+1}^{\infty} t_1^i \varepsilon^i \right\|_{\ell_p(\Phi)} < \delta_1.$$

Since (u_n) is weakly null sequence implies $u_n \to 0$ coordinatewise, there is an $n_2 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{m_1} u_n^i \varepsilon^i \right\|_{\ell_p(\Phi)} < \delta_1,$$

for all $n \geq n_2$. Put $t_2 = u_{n_2}$. Then, there exists an $m_2 > m_1$ such that

$$\left\| \sum_{i=m_2+1}^{\infty} t_2^i \varepsilon^i \right\|_{\ell_p(\Phi)} < \delta_2.$$

Again using the fact that $u_n \to 0$ coordinatewise, there exists an $n_3 > n_2$ such that

$$\left\| \sum_{i=1}^{m_2} u_n^i \varepsilon^i \right\|_{\ell_p(\Phi)} < \delta_2,$$

for all $n \geq n_3$.

By continuing this process, we obtain two sequences (m_i) with $m_1 < m_2 < \ldots < m_i < \ldots$ and (n_i) with $n_1 < n_2 < \ldots < n_i < \ldots$ such that

$$\left\| \sum_{i=1}^{m_j} u_n^i \varepsilon^i \right\|_{\ell_p(\Phi)} < \delta_j,$$

for all $n \geq n_{j+1}$ and

$$\left\| \sum_{i=m_j+1}^{\infty} t_j^i \varepsilon^i \right\|_{\ell_p(\Phi)} < \delta_j,$$

where $t_j = u_{n_j}$. It follows that

$$\begin{aligned} \left\| \sum_{j=1}^{n} t_j \right\|_{\ell_p(\Phi)} &= \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{m_{j-1}} t_j^i \varepsilon^i + \sum_{i=m_{j-1}+1}^{m_j} t_j^i \varepsilon^i + \sum_{i=m_j+1}^{\infty} t_j^i \varepsilon^i \right) \right\|_{\ell_p(\Phi)} \\ &\leq \left\| \sum_{j=1}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} t_j^i \varepsilon^i \right) \right\|_{\ell_p(\Phi)} + 2 \sum_{j=0}^{n} \delta_j. \end{aligned}$$

Also, given any $u \in \mathscr{B}_{\ell_p(\Phi)}$, we have $\|u\|_{\ell_p(\Phi)}^p = \sum_{i=1}^{\infty} \left| \frac{1}{i} \sum_{k|i} \varphi(k) u_k \right|^p < 1$. Therefore, we have that

$$\begin{aligned} \left\| \sum_{j=1}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} t_j^i \varepsilon^i \right) \right\|_{\ell_p(\Phi)}^p &= \sum_{j=1}^{n} \sum_{i=m_{j-1}+1}^{m_j} \left| \frac{1}{i} \sum_{k|i} \varphi(k) t_j^k \right|^p \\ &\leq \sum_{j=1}^{n} \sum_{i=1}^{\infty} \left| \frac{1}{i} \sum_{k|i} \varphi(k) t_j^k \right|^p \leq n. \end{aligned}$$

Hence, we obtain

$$\left\| \sum_{j=1}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} t_j^i \varepsilon^i \right) \right\|_{\ell_p(\Phi)} \leq n^{1/p}.$$

Since $n^{1/p} \geq 1$ holds for all $n \in \mathbb{N}$ and $1 < p < \infty$, we have

$$\left\| \sum_{j=1}^{n} t_j \right\|_{\ell_p(\Phi)} \leq n^{1/p} + 1 \leq 2n^{1/p}.$$

Hence, we conclude that $\ell_p(\Phi)$ is of type p Banach-Saks for $1 < p < \infty$. □

García-Falset [19] introduce the following coefficient:

$$R(X) = \sup \left\{ \liminf_{n \to \infty} \|u_n - L\| : (u_n) \text{ is a sequence in } \mathscr{B}_X,\ u_n \xrightarrow{w} 0,\ L \in \mathscr{B}_X \right\}.$$

Here $u_n \xrightarrow{w} 0$ means that (u_n) is weakly convergent to zero. A Banach space X with $R(X) < 2$ has the *weak fixed point property* ([20]).

Remark 4.2. *$R(\ell_p(\Phi)) = R(\ell_p) = 2^{1/p}$ since $\ell_p(\Phi)$ is linearly isomorphic to ℓ_p.*

Hence, we have the following result.

Theorem 4.3. *The space $\ell_p(\Phi)$ has the weak fixed point property for $1 < p < \infty$.*

Let $\mathscr{S}_X = \{u \in X : \|u\| = 1\}$. The *Gurarii's modulus of convexity* is

$$\beta_X(\delta) = \inf \left\{ 1 - \inf_{0 \leq \lambda \leq 1} \|\lambda u + (1-\lambda) v\| : u, v \in \mathscr{S}_X, \|u - v\| = \delta \right\},$$

where $0 \leq \delta \leq 2$ ([21]).

Theorem 4.4. *The inequality $\beta_{\ell_p(\Phi)}(\delta) \leq 1 - [1 - (\frac{\delta}{2})^p]^{1/p}$ holds, where $0 \leq \delta \leq 2$.*

Proof. Let $0 \leq \delta \leq 2$. Consider the sequences

$$\tilde{u} = \left(\left(1 - \left(\frac{\delta}{2} \right)^p \right)^{1/p}, \frac{\delta}{2}, 0, 0, 0, \ldots \right)$$

and

$$\tilde{v} = \left(\left(1 - \left(\frac{\delta}{2}\right)^p\right)^{1/p}, -\frac{\delta}{2}, 0, 0, 0, \ldots \right).$$

Set $u = \Phi^{-1}\tilde{u}$ and $v = \Phi^{-1}\tilde{v}$. By using the relation (2.1), we obtain that

$$\|u\|^p_{\ell_p(\Phi)} = \|\Phi u\|^p_{\ell_p} = \|\tilde{u}\|^p_{\ell_p} = \left| \left(1 - \left(\frac{\delta}{2}\right)^p\right)^{1/p} \right|^p + \left|\frac{\delta}{2}\right|^p = 1$$

and

$$\|v\|^p_{\ell_p(\Phi)} = \|\Phi v\|^p_{\ell_p} = \|\tilde{v}\|^p_{\ell_p} = \left| \left(1 - \left(\frac{\delta}{2}\right)^p\right)^{1/p} \right|^p + \left|-\frac{\delta}{2}\right|^p = 1.$$

Also, we have

$$\|u - v\|^p_{\ell_p(\Phi)} = \|\tilde{u} - \tilde{v}\|^p_{\ell_p} = (|\delta|^p)^{1/p} = \delta.$$

Hence, we conclude that

$$\begin{aligned}
\beta_{\ell_p(\Phi)}(\delta) &\leq 1 - \inf_{0\leq\lambda\leq 1} \|\lambda u + (1-\lambda)v\|_{\ell_p(\Phi)} \\
&\leq 1 - \inf_{0\leq\lambda\leq 1} \|\lambda \tilde{u} + (1-\lambda)\tilde{v}\|_{\ell_p} \\
&\leq 1 - \inf_{0\leq\lambda\leq 1} \left[\left| \lambda\left(1 - \left(\frac{\delta}{2}\right)^p\right)^{1/p} + (1-\lambda)\left(1 - \left(\frac{\delta}{2}\right)^p\right)^{1/p} \right|^p + \left|\lambda\frac{\delta}{2} - (1-\lambda)\frac{\delta}{2}\right|^p \right]^{1/p} \\
&\leq 1 - \inf_{0\leq\lambda\leq 1} \left[1 - \left(\frac{\delta}{2}\right)^p + |2\lambda - 1|^p \frac{\delta}{2}^p \right]^{1/p} \\
&\leq 1 - \left[1 - \left(\frac{\delta}{2}\right)^p\right]^{1/p}.
\end{aligned}$$

□

Corollary 4.5. *If $\beta_{\ell_p(\Phi)}(\delta) = 1$, then $\ell_p(\Phi)$ is strictly convex.*

Corollary 4.6. *If $0 < \beta_{\ell_p(\Phi)}(\delta) \leq 1$, then $\ell_p(\Phi)$ is uniformly convex.*

References

[1] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, İstanbul, 2012.
[2] M. Et, *On some difference sequence spaces*, Turkish J. Math., **17** (1993), 18-24.
[3] E. E. Kara, M. Başarır, *On compact operators and some Euler $B^{(m)}$ difference sequence spaces*, J. Math. Anal. Appl., **379**(2) (2011), 499-511.
[4] M. Kirişçi, F. Başar, *Some new sequence spaces derived by the domain of generalized difference matrix*, Comput. Math. Appl., **60** (2010), 1299-1309.
[5] M. Mursaleen, A. K. Noman, *On the spaces of λ-convergent and bounded sequences*, Thai J. Math., **8**(2) (2012), 311–329.
[6] P. Zengin Alp, M. İlkhan, *On the difference sequence space $\ell_p(\hat{T}^q)$*, Math. Sci. Appl. E-Notes, **7**(2) (2019), 161-173.
[7] M. Mursaleen, F. Başar, B. Altay, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ II*, Nonlinear Anal., **65**(3) (2006), 707–717.
[8] S. Demiriz, C. Çakan, *Some topological and geometrical properties of a new difference sequence space*, Abstr. Appl. Anal., **2011** (2011), Article ID 213878, 14 pages.
[9] M. Et, M. Karakaş, V. Karakaya, *Some geometric properties of a new difference sequence space defined by de la Vallee-Poussin mean*, Appl. Math. Comput., **234** (2014), 237-244.
[10] E. E. Kara, *Some topological and geometrical properties of new Banach sequence spaces*, J. Inequal. Appl., **2013** (2013), 38, 15 pages.
[11] V. Karakaya, *Some geometric properties of sequence spaces involving lacunary sequence*, J. Inequal. Appl., **2007** (2007), Article ID 81028, 8 pages.
[12] E. Savaş, V. Karakaya, N. Şimşek, *Some $\ell(p)$-type new sequence spaces and their geometric properties*, Abstr. Appl. Anal., **2009** (2009), Article ID 696971, 12 pages.
[13] S. Suantai, *On the H-property of some Banach sequence spaces*, Arch. Math., **39**(4) (2003), 309–316.
[14] N. Şimşek, V. Karakaya, *On some geometrical properties of generalized modular spaces of Cesaro type defined by weighted means*, J. Inequal. Appl., **2009** (2009), Article ID 932734, 13 pages.
[15] I. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.
[16] M. İlkhan, E. E. Kara, *A new Banach space defined by Euler totient matrix operator*, Oper. Matrices, **13**(2) (2019), 527-544.
[17] M. Stieglitz, H. Tietz, *Matrix transformationen von folgenraumen eine ergebnisübersicht*, Math. Z., **154** (1977), 1-16.
[18] H. Knaust, *Orlicz sequence spaces of Banach-Saks type*, Arch. Math., **59**(6) (1992), 562-565.
[19] J. García-Falset, *Stability and fixed points for nonexpansive mappings*, Houston J. Math., **20**(3) (1994), 495-506.
[20] J. García-Falset, *The fixed point property in Banach spaces with the NUS-property*, J. Math. Anal. Appl., **215**(2) (1997), 532-542.
[21] L. Sànchez, A. Ullàn, *Some properties of Gurarii's modulus of convexity*, Arch. Math., **71** (1998), 399-406.

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 52-56
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.725913

High-Order Coefficients of Second-Order ODEs in Relation to Pre-Factors for Complex Parameter

Fatih Say[1]

[1]*Department of Mathematics, Faculty of Arts and Sciences, Ordu University, Ordu, Turkey*

Article Info

Keywords: *Asymptotic expansions, Approximation, Asymptotic beyond all orders, Perturbation, Series solutions*
2010 AMS: *34E20, 34M35, 34M40, 34M60*
Received: *23 April 2020*
Accepted: *3 June 2020*
Available online: *10 June 2020*

Abstract

In this study, we asymptotically reconsider the relations between the pre-factors of a general inhomogeneous second-order ordinary differential equation and the high-order coefficients of its asymptotic power series for complex values of the asymptotic parameter ε_1. The study provides a general formula for its generic high-order coefficients with the associated pre-factors for complex ε_1 based on the use of a well-known factorial divided by a power approach.

1. Introduction

Many of the essential properties of the ordinary differential equations (ODE) can be investigated by using asymptotic expansion methods such as perturbation methods of Poincaré [1, 2], method of matched asymptotic expansion [3], WKB approximation method [4, 5] and SCEM method [6]. The generic feature of the singular differential equations is that the high-order coefficients of the singular perturbation expansions always behave in the characteristic factorial divided by a power (factorial/power) form, and they factorially diverge for a wide range of singular perturbation problems. It is principally first discussed in detail by Dingle [7] and Berry [8]. In the companion paper [9], we already considered the link between the pre-factor functions of a particular type of second-order inhomogeneous ODEs and the associated high-order coefficients of the asymptotic expansion. Motivated by the previous study, in this paper, we will reapply the same idea permitted us to obtain the formulae in [9] to the asymptotic solution of the general differential equation in the case of small parameter ε_1 is complex-valued. We will address what difference it will make in the derived formulae. Once it is done, one could use them while addressing the asymptotic properties of the differential equations such as superasymptotics, hyperasymptotics and Stokes rays [8], [10]-[12] since the exponential asymptotics is usually discussed in the complex plane. For instance, Stokes rays are the local properties of the differential equations and across which the exponentially growing terms occur along the complex plane. For this reason alone, it is nice to interpret the findings of [9] in terms of the complex values of a small parameter. Moreover, the neglected highest derivative of the singular differential equations at leading order becomes important as it varies rapidly. Therefore, the asymptotic behavior of the differential equations (and integrals) has been comprehensively studied in detail in the last few decades and, as a consequence, the subject of exponential asymptotics is introduced, see for example [13]-[17] and references therein. For this reason, studying the asymptotic behavior of such equations, especially for the ones whose exact solutions cannot be derived via conventional asymptotic techniques, are always of great interest.

In this paper, albeit briefly, we reconsider whether the formulas in [9] can be further extended for complex values of the

Email address and ORCID number: **fatihsay@odu.edu.tr, https://orcid.org/0000-0002-4500-2830 (F. Say)**

small parameter while addressing the general representations of the general inhomogeneous second-order ODE. We again take into account the factorial/power ansatz of high-order coefficients [7, 15, 18] to capture the formulas in the derivation of the asymptotic expansions for the particular case of this paper. We present that the links between the pre-factors and the coefficients of the asymptotic expansion of the ODE work for the complex values of ε_1. The outline of the article is as follows: First, we introduce the illustrative singular ordinary differential equation and re-define the small parameter of the ODE in terms of its complex values in Section 2. We next expand the equation in the traditional asymptotic expansion method where we derive the leading order solution along with the recurrence relationship of the successive terms of the expansion. To be able to address and interpret the general form of the high-order coefficients in terms of the pre-factor functions of the ODE for the complex parameter, we employ the common and powerful factorial/power formula whereby we determine their relationships in the limit $n \to \infty$ in Section 3. We finish the study with the concluding remarks in Section 4.

2. The asymptotic expansion of ODE

In order to be able to capture the relationship between the high-order terms of the expansion and the pre-factor functions, we will address the asymptotic expansion of the following singular inhomogeneous ODE of [9], that is,

$$\varepsilon_1 \frac{d^2 w(z)}{dz^2} + \varepsilon_1 f(z) \frac{dw(z)}{dz} + g(z) w(z) = t(z), \tag{2.1}$$

in which $\varepsilon_1 \in \mathbb{C}$ is the small perturbation parameter and pre-factor functions $f(z)$, $g(z)$ and $t(z)$ are not constant. Before starting to study this section, let us first discuss the form of the asymptotic expansions occurring in exponential asymptotics. Divergent solutions of the differential equations including this particular case mostly appear in the following nature in exponential asymptotics [19]

$$\begin{aligned} w(z) = \Bigg(& w_0(z) + \varepsilon_1 w_1(z) + \varepsilon_1^2 w_2(z) + \cdots + \varepsilon_1^n \frac{\Gamma(2n+\beta)}{\chi_1(z)^{2n+\beta}} \left(\sum_{k=0}^{\infty} \frac{A_k(z)}{(2n+\beta)^k} \right) \Bigg) \\ & + \left(\sum_{l=0}^{m-1} \varepsilon_1^l B_l(z) \right) \exp\left(-\frac{\chi_1(z)}{\varepsilon_1} \right) + \left(\sum_{t=0}^{s-1} \varepsilon_1^t C_t(z) \right) \exp\left(-\frac{\chi_2(z)}{\varepsilon_1} \right) + \mathrm{R}(\varepsilon_1, z), \end{aligned} \tag{2.2}$$

in which $\chi_{i\geq 1}(z)$s are subject to every single singularity of the early ordered terms of each level, and $\mathrm{R}(\varepsilon_1, z)$ is the resultant remainder of the expansion with respect to the order of the first neglected term. As it suffices for this particular case, only $\chi_1(z)$, $A_k(z)$ and β will be addressed in (3.1) of the following section. Functions $\chi_{i\geq 2}(z)$, $B_l(z)$ and $C_t(z)$ can be addressed, in a similar way, when needed. It is indeed one of the main ideas lying behind the exponential asymptotics, see [19]. This will particularly be discussed in the succeeding section. The reason that such expansions diverge is in fact the singular point(s) of their early terms; most particularly, it is $w_0(z)$ in this general case. Moreover, the Stokes rays usually sprout from the singular point(s) of the early terms. The exponentially small terms which occur in the form of $\exp(-\chi_{i\geq 1}(z)/\varepsilon_1)$ appear and disappear across the active Stokes rays, and this can be observed when analytically continued in the Argand diagram; particularly, this jump occurs smoothly via error function. Based on the sectors occurred by the Stokes rays in the diagram, associated sub-dominant exponential terms come into play. Thence, the subject of exponential asymptotics deals with this divergence and its relation with the exponentially small terms hidden behind algebraic order terms [20]. Furthermore, the magnitude of the powers of the exponentially small terms of (2.2) shows at which point the asymptotic expansions change their behavior from decreasing to diverging to infinity; for more details, see [19].

The equation (2.1) currently contains no complex parameter besides ε_1. Since we are only interested in finding the asymptotic solutions for ε_1 in terms of pre-factor functions $f(z)$, $g(z)$ and $t(z)$, we need to first introduce complex ε_1 in a useful way. Unlike to [21] where the independent variable is changed by multiplying complex factor, we re-scale the small perturbation parameter in this case. In particular, to address whether the links between the factors and the expansion coefficients derived in [9] work for the complex values of ε_1 in the asymptotic procedure or not, we principally re-scale ε_1, without loss of generality, by

$$\varepsilon_1 = e^{i\theta} \varepsilon, \tag{2.3}$$

where $0 < \varepsilon \ll 1$. To express herein that we will focus on the general form of the first summation of (2.2) in terms of pre-factor functions in our derivation since we are only concerned with the limits $n \to \infty$ and $\varepsilon \to 0$ for the singular ODEs in the form of this paper. Upon substitution of this re-scaled values of ε_1 into the original differential equation (2.1), we may find the following singular differential equation depending on θ

$$e^{i\theta} \varepsilon \frac{d^2 w(z)}{dz^2} + e^{i\theta} \varepsilon f(z) \frac{dw(z)}{dz} + g(z) w(z) = t(z). \tag{2.4}$$

The solution of ODEs by conventional asymptotic methods usually proceeds in a similar way, see [1, 2] for details. Therefore, we will first assume that a regular asymptotic expansion of the solution of equation (2.4) exists. We then substitute it into the equation and equate the factors of like powers of the small parameter on both sides. In particular, let us proceed with the usual approach that its asymptotic power series solution in powers of ε is

$$w(z) \sim \sum_{n=0}^{\infty} \varepsilon^n w_n(z), \tag{2.5}$$

which is valid in the limit as $\varepsilon \to 0$. Because this series expansion of $w(z)$ must satisfy the differential equation (2.4), we employ the summation in the equation. After rearranging into a hierarchy of powers of ε, we find

$$\sum_{n=1}^{\infty} \varepsilon^n \left[w''_{n-1}(z) + f(z) w'_{n-1}(z) + e^{-i\theta} g(z) w_n(z) \right] + e^{-i\theta} g(z) w_0(z) = e^{-i\theta} t(z), \tag{2.6}$$

where the prime $'$ indicates the differentiation of the functions respecting to z. Once the factors of like powers of ε are equated for both sides of the asymptotic equality (2.6), the leading order solution $w_0(z)$ at $\mathrm{O}(1)$ and the differential recurrence relation of $w_{n-1}(z)$ and $w_n(z)$ of the expansion in (2.6) at $\mathrm{O}(\varepsilon^n)$ are derived, respectively, as follows

$$w_0(z) = \frac{t(z)}{g(z)}, \tag{2.7}$$

$$w''_{n-1}(z) + f(z) w'_{n-1}(z) + e^{-i\theta} g(z) w_n(z) = 0, \tag{2.8}$$

for $n \geq 1$. An observant reader may notice that the low-ordered term $w_0(z)$ is not affected with the complex values of the small parameter and it is the same as the corresponding one of [9]; in fact, this reinforces the consistency between the two pieces of the works. When the leading order term and then the associated succeeding terms of the expansion are employed repeatedly in the above sequence (2.8), one can derive the high-order terms as n increases in practice by earlier terms. However, one must make sure that singularity or singularities of the low-ordered term(s) must be secured in the high-order terms of the expansion. Calculation of the exact expansion coefficients at each order by this relation, unlike for the low-ordered terms, could be challenging at times. Therefore, to describe the $n \to \infty$ behavior of the high-order terms as well as the size of the approximation by seeking an asymptotic expansion of the solution in terms of the pre-factors, we may employ the factorial/power formula as it generates the form of the expansion coefficients, without loss of generality. It is worth to point out that as one may notice these approximated solutions will clearly be not exact when ε is small but nonzero, they only define their asymptotic equality for sufficiently large n and small ε. Moreover, the presence of the singularity or singularities of (2.7) forces the asymptotic expansion to diverge in the standard factorial divided by a power nature in the limits $n \to \infty$ and $\varepsilon \to 0$ as the calculation of the general terms requires the differentiation of the preceding terms at each order.

3. Asymptotic formula of the high-order terms

As discussed earlier, finding the exact solution of such equations in the form of (2.1) could be extremely difficult sometimes in the asymptotic procedure. However, as before, our motivation is to study the general asymptotic form of the coefficients for sufficiently large values of n in terms of the pre-factor functions of the particular ODE. These coefficients are governed by the nearest singularity of the expansions. For this reason, we will approximate the higher-order coefficients of the expansion using the powerful factorial/power method as they are naturally divergent in this nature in many cases in the limit $\varepsilon \to 0$. We consider the high-order terms $w_n(z)$ of a function $w(z)$, which is asymptotic to a factorially divergent power series [7, 15], diverge in the following form

$$w_n(z) = \frac{\Gamma(2n+\beta)}{\chi_1(z)^{2n+\beta}} \left(\sum_{k=0}^{\infty} \frac{A_k(z)}{(2n+\beta)^k} \right), \quad \text{as} \quad n \to \infty, \tag{3.1}$$

where $\chi_1(z) = 0$ at the singular point(s) of the leading order term (2.7), β is a constant and Γ is the gamma function, or factorial function, as described in [22]; it enables to extend the domain of the factorial to complex arguments for negative values of the non-integers, see [23, pg. 149]. We remark that the ansatz given in (3.1) is the only leading order approximation of the expansion (2.5) or (2.2) in most general form. There must be separate factorial/power ansatz for each singular points exist. It indeed extracts the high-order term behavior of the expansion wherein derivation of the behavior of $\chi_1(z)$ plays a pivotal role in the asymptotic procedure. To fully determine all components of the high-order terms in (3.1), we substitute the ansatz (3.1) into the relation (2.8). After performing some computations, we find at the leading order for sufficiently large values of n that

$$\chi_1'(z)^2 + e^{-i\theta} g(z) = 0,$$

through which we find that

$$\chi_1'(z) = \pm\sqrt{-e^{-i\theta} g(z)}. \tag{3.2}$$

After having the integration of both sides in (3.2), we subsequently derive the denominator $\chi_1(z)$ as a function of the pre-factor $g(z)$ such that

$$\begin{aligned}\chi_1(z) &= \pm\int\sqrt{-e^{-i\theta}g(z)}dz + c_{\chi_1}\\ &= \pm\int\sqrt{\exp(i(\pi-\theta))g(z)}dz + c_{\chi_1},\end{aligned} \tag{3.3}$$

in which c_{χ_1} is an integration constant. $\chi_1(z)$ requires to satisfy the singularity or singularities of the early term $w_0(z)$, which precisely causes the general terms to diverge, by which the integration constant can be derived. The denominator of the high-order coefficients is expressed as a multiplication of the pre-factor $g(z)$ and $e^{i\theta}$ as a result of the choice of re-scaled ε_1 in (2.3). Next, we will focus on deriving the general form of the leading $A_0(z)$ term in relation to the pre-factors $f(z)$ and $g(z)$ for sufficiently large values of n since it contributes to the expansion before the subsequent $A_{n\geq 1}(z)$ functions in the limit $n\to\infty$. To be able to do this, we carry on the next order of balancing when the summation index n is sufficiently large. Similarly to previous order of balancing, after doing the required calculations and simplifications, we attain the differential equation of $A_0(z)$ as

$$\chi_1''(z)A_0(z) + 2\chi_1'(z)A_0'(z) + f(z)\chi_1'(z)A_0(z) = 0. \tag{3.4}$$

Although it looks the same as its corresponding one in [9], $A_0(z)$ of (3.1) will be a complex function as well in this case as the denominator function $\chi_1(z)$ depends upon θ given by the relation in (3.3). Particularly for this expression (3.4), after doing the simple separation and then doing the direct integration with respect to z, unknown $A_0(z)$ may be evaluated in the following form

$$A_0(z) = c_0\frac{\exp\left(-\int\frac{f(z)dz}{2}\right)}{\sqrt{\chi_1'(z)}}. \tag{3.5}$$

Back substitution of the relation obtained in (3.2) into (3.5) completely derives $A_0(z)$ as

$$A_0(z) = c_0\frac{\exp\left(-\int\frac{f(z)dz}{2}\right)}{[\pm\exp(i(\pi-\theta))g(z)]^{1/4}}.$$

Note that all of the integration constants obtained so far can be absorbed into a single constant c_0, without loss of generality. In this conjecture, substituting all the relations derived by now for $\chi_1(z)$ and $A_0(z)$ in (3.3) and (3.5) into the factorial/power form in (3.1), we may generate the most general form of the high-order coefficients as following

$$w_n(z) \sim c_0\frac{\Gamma(2n+\beta)}{\chi_1(z)^{2n+\beta}}\left(\frac{\exp\left(-\int\frac{f(z)dz}{2}\right)}{[\pm\exp(i(\pi-\theta))g(z)]^{1/4}}\right), \quad \text{as} \quad n\to\infty. \tag{3.6}$$

Substituting this coefficient (3.6) back into the summation of the singularly perturbed ODE completes the derivation of the general asymptotic representation of the singular ODE in terms of $f(z)$, $g(z)$ and $t(z)$ of (2.4) in powers of ε, wherein $t(z)$ and its zeros are crucial while deriving the low-ordered terms of the expansion, so does the high-order terms. Finally, to establish the most general form of the solutions by the complex values of ε_1, we should use the equation (2.3) and leave ε alone. Once doing this and substituting it into the summation, we establish the leading order approximation of $w(z)$ as a function of pre-factor functions in powers of ε_1 and $\exp(-i\theta)$ such that

$$w(z) \sim c_0\sum_{n=0}^{\infty}\left(\varepsilon_1 e^{-i\theta}\right)^n\frac{\Gamma(2n+\beta)}{\chi_1(z)^{2n+\beta}}\left(\frac{\exp\left(-\int\frac{f(z)dz}{2}\right)}{[\pm\exp(i(\pi-\theta))g(z)]^{1/4}}\right), \quad \text{as} \quad n\to\infty,$$

which is the leading high-order behavior of the asymptotic solution $w(z)$ of equation (2.1) derived based on using the factorial/power representation (3.1) with the limit $n\to\infty$. The choice of rescaling ε_1 permits us to expand (2.1) as a power series of its complex values. Again, the region of its validity depends on its singularity structure which may be addressed via exponential asymptotics and it preserves all the features of the differential equation. Because the exact solutions of such type of equations are rare in physics applications, one can implement this for the suitable choices of the pre-factor functions and can find the limiting behavior of $w(z)$ when needed. However, if it is not sufficient, this means that the perturbation parameter is not small enough. Furthermore, as this expansion is naturally divergent due to increasing powers of the low-ordered terms, by taking the ratio of the adjacent terms of the expansion, a general form of the optimal truncation point as well as the relation of the resultant remainder, which is exponentially small, and divergent series can be directly and easily formulated and interpreted by these specified formulas.

4. Concluding remarks

This work has taken into consideration how to straightforwardly address the general form of the tail of the expansions for complex ε_1 by focusing on the pre-factor functions of the certain ODEs in the form of this paper along with their effects in the asymptotic expansions. The obtained links in relation to pre-factors can be implemented for the complex values of ε_1, whence they are extendable to complex region. Moreover, as one may notice that being ε_1 complex turns the pre-factors and the right-hand side of the inhomogeneous singular equations into complex factors. Therefore, the formulas we have attained are applicable for the study of the singular ODEs having the complex pre-factors as well.

Competing interest: The author declares that no competing interests exist.

Acknowledgement

I would like to thank the referees for their helpful comments on the manuscript.

References

[1] H. Poincaré, *Sur les intégrales irrégulières*, Acta Math., **8** (1) (1886), 295-344.
[2] E. J. Hinch, *Perturbation Methods*, Cambridge University Press, Cambridge, 1991.
[3] M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964.
[4] C. M. Bender, S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*, Reprint of the 1978 original, Springer-Verlag, New York, 1999.
[5] V. F. Zaitsev, A. D. Polyanin, *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd edition, Chapman & Hall/CRC Press Company, Boca Raton, FL, 2003.
[6] J. Cousteix, J. Mauss, *Successive Complementary Expansion Method*, Asymptotic Analysis and Boundary Layers, Springer, 2007, pp. 59-98.
[7] R. B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation*, Academic Press, London-New York, 1973.
[8] M. V. Berry, *Asymptotics, superasymptotics, hyperasymptotics*, H. Segur, S. Tanveer, H. Levine (editors), Asymptotics Beyond All Orders, NATO Adv. Sci. Inst. Ser. B Phys., vol. 284, Springer Science & Business Media, Boston, 1991, 1-14.
[9] F. Say, *Late-order terms of second order ODEs in terms of pre-factors*, Hacet. J. Math. Stat., (in press).
[10] M. V. Berry, *Uniform asymptotic smoothing of Stokes's discontinuities*, P. Roy. Soc. Lond. A Mat., **422** (1862) (1989), 7-21.
[11] M. V. Berry, *Stokes's phenomenon; smoothing a Victorian discontinuity*, Inst. Hautes Études Sci. Publ. Math., **68** (1) (1988), 211-221.
[12] B. L. J. Braaksma, G. K. Immink, M. Van der Put, J. Top (editors), *Differential equations and the Stokes phenomenon*, Proceedings of the workshop held at the University of Groningen, Groningen, May 28–30, 2001, World Scientific Publishing Co., Inc., River Edge, NJ, 2002, doi:10.1142/5107.
[13] P. J. Langman, *When is a Stokes line not a Stokes line?*, Ph.D. Thesis, University of Southampton, 2005.
[14] M. D. Kruskal, H. Segur, *Asymptotics beyond all orders in a model of crystal growth*, Stud. Appl. Math., **85** (2) (1991), 129-181.
[15] J. P. Boyd, *The devil's invention: Asymptotic, superasymptotic and hyperasymptotic series*, Acta Appl. Math., **56** (1) (1999), 1-98.
[16] F. W. J. Olver, *Asymptotic expansions of the coefficients in asymptotic series solutions of linear differential equations*, Methods Appl. Anal., **1** (1) (1994), 1-13.
[17] R. E. Meyer, *A simple explanation of the Stokes phenomenon*, SIAM Rev., **31** (3) (1989), 435-445.
[18] J. P. Boyd, *A hyperasymptotic perturbative method for computing the radiation coefficient for weakly nonlocal solitary waves*, J. Comput. Phys., **120** (1) (1995), 15-32.
[19] F. Say, *Exponential asymptotics: multi-level asymptotics of model problems*, Ph.D. Thesis, University of Nottingham, 2016.
[20] R. E. Meyer, *Exponential asymptotics*, SIAM Rev., **22** (2) (1980), 213-224.
[21] P. Hsieh, Y. Sibuya, *On the asymptotic integration of second order linear ordinary differential equations with polynomial coefficients*, J. Math. Anal. Appl., **16** (1) (1966), 84-103.
[22] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.
[23] E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, Oxford, 1939.

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 57-60
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.692916

Average Values of Triangles

Péter Kórus[1]

[1] *Department of Mathematics, Juhász Gyula Faculty of Education, University of Szeged, Hattyas utca 10, H-6725 Szeged, Hungary*

Article Info

Keywords: *Area, Circumradius, Inradius, Perimeter, Triangle*
2010 AMS: *51M04*
Received: *22 February 2020*
Accepted: *20 April 2020*
Available online: *10 June 2020*

Abstract

The average area and perimeter of triangles defined by three random points on the circumference of a unit circle are already known. In this paper, we calculate the average value of the inradius along with its variance. We also consider the average area, inradius and circumradius of triangles of unit perimeter.

1. Triangles of unit circumradius

How to choose a triangle randomly on the circumference of a unit circle? An answer is selecting three points A,B,C on the circumference uniformly. Many questions arise regarding these triangles ABC. What is the average area or average perimeter or average inradius of the triangles? The average area of these triangles is known to be $\frac{3}{2\pi}=0.47746\ldots$, see [2], while the average perimeter is $\frac{12}{\pi}=3.81971\ldots$, see [1]. Our goal is to calculate the average value of the inradius of the triangles. We use the same characterization of the triangle as in [2]. Suppose without loss of generality that the center of the circumcircle of the triangle is $O(0,0)$, one vertex is $A(1,0)$ and the other two vertices are determined by directed angles $\angle AOB=\theta_1\in[0,\pi]$ and $\angle AOC=\theta_2\in[0,2\pi)$. Then the inradius of the triangle is equal to

$$r(\theta_1,\theta_2)=r(\triangle ABC)=\frac{A(\triangle ABC)}{\frac{P(\triangle ABC)}{2}}=\frac{2\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left|\sin\frac{\theta_2-\theta_1}{2}\right|}{\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\left|\sin\frac{\theta_2-\theta_1}{2}\right|}.$$

We calculate the average value of the inradius. First note that

$$\begin{aligned}\bar{r} &= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{2\pi}\frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left|\sin\frac{\theta_2-\theta_1}{2}\right|}{\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\left|\sin\frac{\theta_2-\theta_1}{2}\right|}\,d\theta_2\,d\theta_1\\ &= \frac{1}{\pi^2}\int_0^{\pi}\int_{\theta_1}^{2\pi}\frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_2-\theta_1}{2}}{\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\sin\frac{\theta_2-\theta_1}{2}}\,d\theta_2\,d\theta_1+\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\theta_1}\frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1-\theta_2}{2}}{\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\sin\frac{\theta_1-\theta_2}{2}}\,d\theta_2\,d\theta_1\\ &= \frac{1}{\pi^2}\left[\int_0^{\pi}I_1\,d\theta_1+\int_0^{\pi}I_2\,d\theta_1\right].\end{aligned}$$

Email address and ORCID number: **korpet@jgypk.szte.hu, https://orcid.org/0000-0001-8540-6293 (P. Kórus)**

Simplify the integrand in I_1 as follows.

$$\begin{aligned}
\frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_2-\theta_1}{2}}{\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\sin\frac{\theta_2-\theta_1}{2}} &= \frac{2\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_2-\theta_1}{4}\cos\frac{\theta_2-\theta_1}{4}}{2\sin\frac{\theta_2+\theta_1}{4}\cos\frac{\theta_2-\theta_1}{4}+2\sin\frac{\theta_2-\theta_1}{4}\cos\frac{\theta_2-\theta_1}{4}} \\
&= \frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_2-\theta_1}{4}}{\sin\frac{\theta_2+\theta_1}{4}+\sin\frac{\theta_2-\theta_1}{4}} \\
&= \frac{4\sin\frac{\theta_1}{4}\cos\frac{\theta_1}{4}\sin\frac{\theta_2}{4}\cos\frac{\theta_2}{4}\sin\frac{\theta_2-\theta_1}{4}}{2\sin\frac{\theta_2}{4}\cos\frac{\theta_1}{4}} \\
&= 2\sin\frac{\theta_1}{4}\cos\frac{\theta_2}{4}\sin\frac{\theta_2-\theta_1}{4}. \qquad (1.1)
\end{aligned}$$

Analogously, for the integrand in I_2,

$$\frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1-\theta_2}{2}}{\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\sin\frac{\theta_1-\theta_2}{2}} = 2\cos\frac{\theta_1}{4}\sin\frac{\theta_2}{4}\sin\frac{\theta_1-\theta_2}{4}.$$

Using (1.1), we calculate I_1 similarly as in [2]:

$$\begin{aligned}
I_1 &= 2\sin\frac{\theta_1}{4}\int_{\theta_1}^{2\pi}\cos\frac{\theta_2}{4}\sin\frac{\theta_2-\theta_1}{4}\,d\theta_2 \\
&= 2\sin\frac{\theta_1}{4}\int_{\theta_1}^{2\pi}\cos\frac{\theta_2}{4}\left[\sin\frac{\theta_2}{4}\cos\frac{\theta_1}{4}-\sin\frac{\theta_1}{4}\cos\frac{\theta_2}{4}\right]d\theta_2 \\
&= 2\sin\frac{\theta_1}{4}\cos\frac{\theta_1}{4}\int_{\theta_1}^{2\pi}\cos\frac{\theta_2}{4}\sin\frac{\theta_2}{4}\,d\theta_2-2\sin^2\frac{\theta_1}{4}\int_{\theta_1}^{2\pi}\cos^2\frac{\theta_2}{4}\,d\theta_2 \\
&= \sin\frac{\theta_1}{2}\left[-\cos\frac{\theta_2}{2}\right]_{\theta_1}^{2\pi}-2\sin^2\frac{\theta_1}{4}\left[\sin\frac{\theta_2}{2}+\frac{\theta_2}{2}\right]_{\theta_1}^{2\pi} \\
&= \sin\frac{\theta_1}{2}\left[1+\cos\frac{\theta_1}{2}+2\sin^2\frac{\theta_1}{4}\right]-2\pi\sin^2\frac{\theta_1}{4}+\theta_1\sin^2\frac{\theta_1}{4} \\
&= 2\sin\frac{\theta_1}{2}-2\pi\sin^2\frac{\theta_1}{4}+\theta_1\sin^2\frac{\theta_1}{4},
\end{aligned}$$

while

$$I_2 = 2\cos\frac{\theta_1}{4}\int_0^{\theta_1}\sin\frac{\theta_2}{4}\sin\frac{\theta_1-\theta_2}{4}\,d\theta_2 = 2\sin\frac{\theta_1}{2}-\theta_1\cos^2\frac{\theta_1}{4}.$$

Therefore, integration by parts gives us

$$\begin{aligned}
\int_0^{\pi} I_1\,d\theta_1 &= \int_0^{\pi} 2\sin\frac{\theta_1}{2}-\pi\left(1-\cos\frac{\theta_1}{2}\right)+\frac{\theta_1}{2}\left(1-\cos\frac{\theta_1}{2}\right)d\theta_1 \\
&= \left[-4\cos\frac{\theta_1}{2}-\pi\theta_1+2\pi\sin\frac{\theta_1}{2}+\frac{\theta_1^2}{4}\right]_0^{\pi}-\left[\theta_1\sin\frac{\theta_1}{2}\right]_0^{\pi}+\int_0^{\pi}\sin\frac{\theta_1}{2}\,d\theta_1 \\
&= -\pi^2+2\pi+\frac{\pi^2}{4}+4-\pi+2 = 6+\pi-\frac{3}{4}\pi^2
\end{aligned}$$

and analogously,

$$\begin{aligned}
\int_0^{\pi} I_2\,d\theta_1 &= \int_0^{\pi} 2\sin\frac{\theta_1}{2}-\frac{\theta_1}{2}\left(1+\cos\frac{\theta_1}{2}\right)d\theta_1 \\
&= \left[-4\cos\frac{\theta_1}{2}-\frac{\theta_1^2}{4}\right]_0^{\pi}-\left[\theta_1\sin\frac{\theta_1}{2}\right]_0^{\pi}+\int_0^{\pi}\sin\frac{\theta_1}{2}\,d\theta_1 \\
&= 6-\pi-\frac{1}{4}\pi^2,
\end{aligned}$$

and we obtain the average value of the inradius as follows.

Formula 1.

$$\bar{r} = \frac{1}{\pi^2}\left[\int_0^{\pi} I_1\,d\theta_1+\int_0^{\pi} I_2\,d\theta_1\right] = \frac{12}{\pi^2}-1 = 0.21585\ldots.$$

1.1. Variance of the inradius

Calculate the second moment of the inradius.

$$\begin{aligned}
\overline{r^2} &= \frac{2}{\pi^2}\int_0^{\pi}\int_{\theta_1}^{2\pi}\frac{\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\sin^2\frac{\theta_2-\theta_1}{2}}{\left(\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\sin\frac{\theta_2-\theta_1}{2}\right)^2}d\theta_2\,d\theta_1+\frac{2}{\pi^2}\int_0^{\pi}\int_0^{\theta_1}\frac{\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\sin^2\frac{\theta_1-\theta_2}{2}}{\left(\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\sin\frac{\theta_1-\theta_2}{2}\right)^2}d\theta_2\,d\theta_1 \\
&= \frac{2}{\pi^2}\left[\int_0^{\pi} J_1\,d\theta_1+\int_0^{\pi} J_2\,d\theta_1\right].
\end{aligned}$$

Simplification for the integrand in J_1 is

$$\frac{\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\sin^2\frac{\theta_2-\theta_1}{2}}{\left(\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\sin\frac{\theta_2-\theta_1}{2}\right)^2}=4\sin^2\frac{\theta_1}{4}\cos^2\frac{\theta_2}{4}\sin^2\frac{\theta_2-\theta_1}{4}$$

and for the integrand in J_2 is

$$\frac{\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\sin^2\frac{\theta_1-\theta_2}{2}}{\left(\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}+\sin\frac{\theta_1-\theta_2}{2}\right)^2}=4\cos^2\frac{\theta_1}{4}\sin^2\frac{\theta_2}{4}\sin^2\frac{\theta_1-\theta_2}{4}.$$

Calculate J_1 as follows:

$$\begin{aligned} J_1 &= \tfrac{1}{2}\left(1-\cos\tfrac{\theta_1}{2}\right)\int_{\theta_1}^{2\pi}\left(1+\cos\tfrac{\theta_2}{2}\right)\left(1-\cos\tfrac{\theta_2-\theta_1}{2}\right)d\theta_2 \\ &= \tfrac{1}{2}\left(1-\cos\tfrac{\theta_1}{2}\right)\left[\theta_2+2\sin\tfrac{\theta_2}{2}-2\sin\tfrac{\theta_2-\theta_1}{2}\right]_{\theta_1}^{2\pi}-\tfrac{1}{2}\left(1-\cos\tfrac{\theta_1}{2}\right)\int_{\theta_1}^{2\pi}\tfrac{1}{2}\left(\cos\tfrac{\theta_1}{2}+\cos\left(\theta_2-\tfrac{\theta_1}{2}\right)\right)d\theta_2 \\ &= \tfrac{1}{2}\left(1-\cos\tfrac{\theta_1}{2}\right)\left(2\pi-\theta_1-3\sin\tfrac{\theta_1}{2}-\pi\cos\tfrac{\theta_1}{2}+\tfrac{\theta_1}{2}\cos\tfrac{\theta_1}{2}\right). \end{aligned}$$

Similarly,

$$J_2=\tfrac{1}{2}\left(1+\cos\tfrac{\theta_1}{2}\right)\int_0^{\theta_1}\left(1-\cos\tfrac{\theta_2}{2}\right)\left(1-\cos\tfrac{\theta_2-\theta_1}{2}\right)d\theta_2=\tfrac{1}{2}\left(1+\cos\tfrac{\theta_1}{2}\right)\left(\theta_1-3\sin\tfrac{\theta_1}{2}+\tfrac{\theta_1}{2}\cos\tfrac{\theta_1}{2}\right).$$

Calculation shows that

$$\int_0^{\pi}J_1\,d\theta_1=\int_0^{\pi}\tfrac{1}{2}\left(1-\cos\tfrac{\theta_1}{2}\right)\left(2\pi-\theta_1-3\sin\tfrac{\theta_1}{2}-\pi\cos\tfrac{\theta_1}{2}+\tfrac{\theta_1}{2}\cos\tfrac{\theta_1}{2}\right)d\theta_1=-\tfrac{17}{4}-\tfrac{3}{2}\pi+\tfrac{15}{16}\pi^2.$$

and

$$\int_0^{\pi}J_2\,d\theta_1=\int_0^{\pi}\tfrac{1}{2}\left(1+\cos\tfrac{\theta_1}{2}\right)\left(\theta_1-3\sin\tfrac{\theta_1}{2}+\tfrac{\theta_1}{2}\cos\tfrac{\theta_1}{2}\right)d\theta_1=-\tfrac{31}{4}+\tfrac{3}{2}\pi+\tfrac{5}{16}\pi^2.$$

Hence

$$\overline{r^2}=\frac{2}{\pi^2}\left[\int_0^{\pi}J_1\,d\theta_1+\int_0^{\pi}J_2\,d\theta_1\right]=\frac{5}{2}-\frac{24}{\pi^2}=0.06829\ldots,$$

and we get the variance of the inradius.

Formula 2.

$$\mathrm{var}(r)=\overline{r^2}-\overline{r}^2=\frac{3}{2}-\frac{144}{\pi^4}=0.02169\ldots.$$

2. Triangles of unit perimeter

Consider those triangles, whose perimeter is equal to 1. Our goal is to calculate the average area, average inradius and average circumradius of these triangles. The first question is, how to choose a random triangle ABC of sides a,b,c with $P(\triangle ABC)=a+b+c=1$? Note that necessary and sufficient conditions for a,b,c to generate such a triangle are $a,b,c>0$, $a+b+c=1$ and triangle inequalities $a+b>c$, $a+c>b$, $b+c>a$. These necessary and sufficient conditions become

$$a+b+c=1 \ \text{ and } \ 0<a,b,c<\frac{1}{2}. \tag{2.1}$$

Our method of random choosing is the following. Choose number a uniformly from $\left(0,\frac{1}{2}\right)$, then choose number b uniformly from $\left(\frac{1}{2}-a,\frac{1}{2}\right)$ and then fix $c=1-a-b$. This method ensures that (2.1) holds. Then the average value of area

$$A(\triangle ABC)=\frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}=\frac{1}{4}\sqrt{(1-2a)(1-2b)(2a+2b-1)}$$

given by Heron's formula is

$$\overline{A}=8\int_0^{\frac{1}{2}}\frac{\sqrt{1-2a}}{4}\left[\int_{\frac{1}{2}-a}^{\frac{1}{2}}\sqrt{(1-2b)(2a+2b-1)}\,db\right]da.$$

Calculation [3] gives us

$$\int\sqrt{(1-2b)(2a+2b-1)}\,db=\frac{(a+2b-1)\sqrt{(1-2b)(2a+2b-1)}}{4}-\frac{a^2}{2}\arctan\frac{\sqrt{1-2b}}{\sqrt{2a+2b-1}}+C,$$

whence

$$\int_{\frac{1}{2}-a}^{\frac{1}{2}}\sqrt{(1-2b)(2a+2b-1)}\,db=\frac{\pi}{4}a^2.$$

Formula 3.

$$\overline{A}=\frac{\pi}{2}\int_0^{\frac{1}{2}}\sqrt{1-2a}\,a^2da=\frac{\pi}{105}=0.02991\ldots.$$

One can easily obtain

$$\overline{A^2}=8\int_0^{\frac{1}{2}}\frac{1-2a}{16}\left[\int_{\frac{1}{2}-a}^{\frac{1}{2}}(1-2b)(2a+2b-1)\,db\right]da=\frac{1}{960},$$

and then the variance of the area.

Formula 4.

$$\operatorname{var}(A)=\overline{A^2}-\overline{A}^2=\frac{1}{960}-\frac{\pi^2}{11025}=0.000146\ldots.$$

Since the inradius of triangle ABC is $r(\triangle ABC)=\frac{A(\triangle ABC)}{\frac{P(\triangle ABC)}{2}}=2A(\triangle ABC)$, we can easily obtain the average value and the variance of the inradius.

Formula 5.

$$\begin{aligned}\overline{r}&=\frac{2\pi}{105}=0.05983\ldots.\\ \operatorname{var}(r)&=\overline{r^2}-\overline{r}^2=\frac{4}{960}-\frac{4\pi^2}{11025}=0.000585\ldots.\end{aligned}$$

The circumradius of triangle ABC is equal to $R(\triangle ABC)=\frac{abc}{4A(\triangle ABC)}$. Therefore, the average value of the circumradius is

$$\overline{R}=8\int_0^{\frac{1}{2}}\frac{a}{\sqrt{1-2a}}\left[\int_{\frac{1}{2}-a}^{\frac{1}{2}}\frac{b(1-a-b)}{\sqrt{(1-2b)(2a+2b-1)}}db\right]da.$$

After some calculation [3] we have

$$\int\frac{b(1-a-b)}{\sqrt{(1-2b)(2a+2b-1)}}db=\frac{a^2-4a+2}{16}\arctan\frac{a+2b-1}{\sqrt{(1-2b)(2a+2b-1)}}+\frac{(a+2b-1)\sqrt{(1-2b)(2a+2b-1)}}{16}+C.$$

From this, we obtain

$$\int_{\frac{1}{2}-a}^{\frac{1}{2}}\frac{b(1-a-b)}{\sqrt{(1-2b)(2a+2b-1)}}db=\frac{\pi}{16}(a^2-4a+2).$$

Finally, we get the average value of the circumradius.

Formula 6.

$$\overline{R}=\frac{\pi}{2}\int_0^{\frac{1}{2}}\frac{a(a^2-4a+2)}{\sqrt{1-2a}}da=\frac{2\pi}{21}=0.29919\ldots.$$

Note that since

$$\overline{R^2}=8\int_0^{\frac{1}{2}}\frac{a^2}{1-2a}\left[\int_{\frac{1}{2}-a}^{\frac{1}{2}}\frac{b^2(1-a-b)^2}{(1-2b)(2a+2b-1)}db\right]da$$

does not converge, the circumradius has infinite variance.

3. Conclusion

Our main aim in this study was to examine the average area, perimeter, inradius, circumradius of triangles. One approach for considering certain triangles is to fix their circumradiuses to 1. Then the average area and perimeter of triangles were already known to be $\frac{3}{2\pi}$ and $\frac{12}{\pi}$. In the paper, we calculated the average inradius and the variance being $\frac{12}{\pi^2}-1$ and $\frac{3}{2}-\frac{144}{\pi^4}$. Another approach is to consider triangles of unit perimeter. We calculated the average area, inradius and circumradius of such triangles being $\frac{\pi}{105}$, $\frac{2\pi}{105}$ and $\frac{2\pi}{21}$ along with the accompanying variances. A possible extension of these results may be the calculation of the average values of polygons with more than three sides.

References

[1] A. Madras, S. KC, *Randomly generated triangles whose vertices are vertices of regular polygons*, Rose-Hulman Undergrad. Math. J., **7**(2) (2006), Article 12.
[2] E. W. Weisstein, *Circle triangle picking*. From MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com/CircleTrianglePicking.html
[3] Wolfram Alpha LLC, 2018, *Wolfram|Alpha*. https://www.wolframalpha.com

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 61-69
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.691602

Pseudoblocks of Finite Dimensional Algebras

Afaf S. Alharthi[1*] and Ahmed A. Khammash[1]

[1] *Department of Mathematical Sciences, Umm Al-Qura University, Makkah, Saudi Arabia*
[*] *Corresponding author*

Article Info

Keywords: *Endomorphism algebra, Extensions, Hecke algebras, Modules, Pseudoblocks, Representations, Tensor algebras*
2010 AMS: *20C30*
Received: *21 February 2020*
Accepted: *27 May 2020*
Available online: *10 June 2020*

Abstract

The notion of pseudoblocks is borrowed from [1] and introduced to finite dimensional algebras. We determine the pseudoblocks for several known algebras such as the triangular algebra and the cyclic group algebra. Also, we determine the pseudoblocks for the group algebra of the special linear group $SL(2,p)$ in the natural characteristic being the only finite group of Lie type of finite representation type.

1. Introduction

In [1], the concept of pseudoblocks of the endomorphism algebra of a module over an algebra was introduced and shown to have a control on the (Brauer) block distribution of the simple modules for the endomorphism algebra in the light of the Brauer-Fitting correspondence. In this paper, we borrow the concept of pseudoblock from [1] to introduce it to finite dimensional (not only endomorphism) algebras. We investigate the pseudoblocks for several known algebras such as the triangular algebra and the cyclic group algebra. Towards the end, we investigate the pseudoblock distribution for the group algebra of the special linear group $SL(2,p)$ in the natural characteristic being the only finite group of Lie type of finite representation type.

2. The pseudoblocks

The Brauer-Fitting correspondence relates the isomorphism classes of indecomposable direct summands of a module to the projective indecomposable modules for its endomorphism algebra. This correspondence is shown in [1] to be incompatible with the (Brauer) block distribution of modules in both sides. Instead, the concept of the pseudoblock of an endomorphism algebra of a module over an algebra was introduced to ensure such compatibility. Here, we borrow this notion and introduce it for any finite dimensional algebra. Let A be a finite dimensional algebra over an algebraically closed field F, $modA$ denotes the category of finitely generated A-modules, and we write $IndA$ for the class of indecomposable A-modules. We also write $(X,Y)_A$ for the A-homomorphism space $Hom_A(X,Y)$ between two modules $X,Y \in modA$. The **pseudoblock linkage relation** $\underset{PSA}{\approx}$ is an equivalence relation defined on $IndA$ in terms of the homomorphism space.

Definition 2.1. *If $X,Y \in IndA$, then $X \underset{PSA}{\approx} Y$ iff there is a sequence of modules $X = X_1, X_2, ..., X_t = Y$ in $IndA$ such that for all $i \in \{1,2,...,t\}$ either $(X_i, X_{i+1})_A \neq 0$ or $(X_{i+1}, X_i)_A \neq 0$.*

***Email addresses and ORCID numbers*: s43880619@st.uqu.edu.sa, afaf.s@tu.edu.sa, afaf.saad.1438@hotmail.com, 0000-0001-5313-5919 (A. S. Alharthi), aakhammash@uqu.edu.sa, prof.khammash@gmail.com, 0000-0001-9404-1732 (A. A. Khammash).**

*Clearly, $\underset{PSA}{\approx}$ is an equivalence relation on IndA. We call the equivalence classes $IndA/\underset{PSA}{\approx}$ are called **pseudoblocks** of the algebra A.*

3. Connection with the Brauer blocks

The following shows that the pseudoblock linkage principle $\underset{PSA}{\approx}$ is stronger than the Brauer linkage principle $\underset{A}{\approx}$ relating indecomposable modules which belong to the same block.

Lemma 3.1. *If $X,Y \in IndA$ and $X \underset{PSA}{\approx} Y$, then $X \underset{A}{\approx} Y$.*

Proof. If $X \underset{PSA}{\approx} Y$, then there is a sequence of modules $X = X_1, X_2, \ldots, X_t = Y$ in $IndA$ such that for all $i \in \{1,2,\ldots,t\}$ either $(X_i, X_{i+1})_A \neq 0$ or $(X_{i+1}, X_i)_A \neq 0$. But this implies (see [2], p.93) that for all $i \in \{1,2,\ldots,t\}$ either $X_i \underset{A}{\approx} X_{i+1}$ or $X_{i+1} \underset{A}{\approx} X_i$, and so $X \underset{A}{\approx} Y$. □

Remark 3.2. *The converse of lemma 3.1 does not hold. If we take $A = FSL(2,4)$ and $CharF = 2$, then A has four simple modules namely $1, 2_1, 2_2, 4$ (the latter being the Steinberg module) distributed into two Brauer blocks $\underbrace{1,2_1,2_2}_{B_1}, \underbrace{4}_{B_2}$. The two indecomposable modules 1, $\begin{matrix} 2_1 \\ 1 \\ 2_2 \end{matrix} \in IndA$ belong to the same (Brauer) block, but they lie in a two different pseudoblocks of A. To see this,*

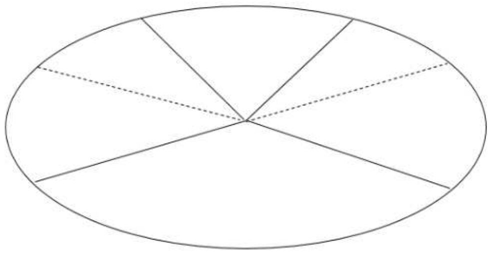

Figure 3.1: Some blocks in *IndA* split into union of pseudoblocks

It follows that, in principle, some (Brauer) blocks of A split into a union of pseudoblocks, and so we have $|IndA/\underset{A}{\approx}| \leqslant |IndA/\underset{PSA}{\approx}|$.

Motivation 3.3. *If we take $Y \in modA$ (not necessary indecomposable) and write $Inds(Y)$ for the isomorphism class of indecomposable A-summands of Y, then applying the linkage relation $\underset{PSA}{\approx}$ on $Inds(Y)$, it was shown in [1] that the (Brauer) block distribution of the simple modules of the endomorphism algebra $E(Y) = End_A(Y)$ is controlled by the pseudoblocks distribution of $Inds(Y)$; that is if $Y_i, Y_j \in Inds(Y)$ and $S_i, S_j \in Irr(E(Y))$ are the corresponding simple $E(Y)$-modules under the Brauer-Fitting correspondence, then $S_i \underset{E(Y)}{\approx} S_j \Leftrightarrow Y_i \underset{PSA}{\approx} Y_j$.*

A Useful Criterion 3.4. *The pseudoblock equivalence relation $\underset{PSA}{\approx}$ is defined in terms of the homomorphism space $(X,Y)_A$. If $X,Y \in IndA$, then $(X,Y)_A \neq 0$ if and only if $\exists K \leq_A X : X/K \cong$ submodule of Y. For, if $0 \neq f \in (X,Y)_A$, then $K = kerf \lneqq X$ and $X/K \cong Imf \leqslant_A Y$. Conversely, if $\exists K \leqslant_A X : X/K \cong T \leqslant_A Y$, then composing the map $X/K \cong T \longrightarrow Y$ with the natural map $X \to X/K$ we get a nonzero map $\theta : X \to Y$. Therefore, we have the figure 3.2*

Lemma 3.5. *$(X,Y)_A \neq 0$ if and only if $\exists K \leqslant_A X$: $X/K \cong$ a submodule of Y.*

4. Connection with tensor algebras

Suppose that A_1, A_2 are two finite dimensional F-algebras. If $X_i \in Ind(A_i); i = 1,2$, then it is known (by considering endomorphism algebras) that $X_1 \otimes X_2 \in Ind(A_1 \otimes A_2)$. The following theorem shows that the concept of pseudo-blocks is compatible with tensor operation of modules.

Theorem 4.1. *[3]. If $X_i, X_i' \in Ind(A_i); i = 1,2$, then $X_1 \otimes X_2 \underset{PS(A_1 \otimes A_2)}{\approx} X_1' \otimes X_2'$ if and only if $X_1 \underset{PSA_1}{\approx} X_1' \quad \wedge \quad X_2 \underset{PSA_2}{\approx} X_2'$.*

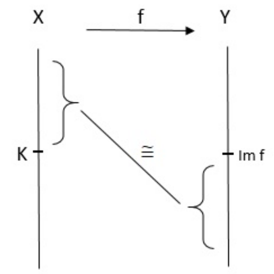

Figure 3.2

Proof. Since $X_1 \underset{PSA_1}{\approx} X_1'$, there is a sequence $X_1 = U_1, U_2, \ldots, U_t = X_1'$ in $IndA_1$ such that for all $j \in \{1,2,\ldots,t\}$ either $(U_j, U_{j+1})_{A_1} \neq 0$ or $(U_{j+1}, U_j)_{A_1} \neq 0$. Similarly, since $X_2 \underset{PSA_2}{\approx} X_2'$, there is a sequence $X_2 = V_1, V_2, \ldots, V_t = X_2'$ in $IndA_2$ such that for all $j \in \{1,2,\ldots,t\}$ either $(V_j, V_{j+1})_{A_2} \neq 0$ or $(V_{j+1}, V_j)_{A_2} \neq 0$ if and only if we have a sequence (with refining sequences if necessary) $X_1 \otimes X_2 = U_1 \otimes V_1, U_2 \otimes V_2, \ldots, U_t \otimes V_t = X_1' \otimes X_2'$ such that for all $j \in \{1,2,\ldots,t\}$ either

$$(U_j \otimes V_j, U_{j+1} \otimes V_{j+1})_{A_1 \otimes A_2} \neq 0 \quad \text{or} \quad (U_{j+1} \otimes V_{j+1}, U_j \otimes V_j)_{A_1 \otimes A_2} \neq 0$$

(by taking the tensor homomorphisms). Therefore, $X_1 \otimes X_2 \underset{PS(A_1 \otimes A_2)}{\approx} X_1' \otimes X_2'$.

□

5. The pseudoblocks of certain finite dimensional algebras

Here, we determine the pseudoblocks for some finite dimensional algebras. It turns out that the two concepts; blocks and pseudo-blocks, coincide for all.

5.1. Semisimple algebras

It is clear that the two notions; blocks and pseudoblocks, coincide for any finite dimensional semisimple algebra A; that is $IndA/\underset{PSA}{\approx} = IndA/\underset{A}{\approx}$. □

5.2. The symmetric group algebra FS_3

Let $A = FS_3$.

1. If $CharF \nmid |S_3|$, then $A = FS_3$ is semisimple, and so $IndA/\underset{PSA}{\approx} = IndA/\underset{A}{\approx}$ as shown above.
2. If $CharF = 2$, then A has two simple module $1,2$ and $IndA$ (consists of three indecomposable modules) has the following block distribution: $\underbrace{1, \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}}_{B_1}, \underbrace{2}_{B_2}$ which clearly coincides with the pseudoblock distribution.
3. If $CharF = 3$, then A has two simple modules both of dimension 1; S_0 (the trivial module) and S_1 (the sign module), and $IndA$ consists of six indecomposable modules all lie in one Brauer block and are connected by the following sequence of A-maps

$$S_1 \to \begin{smallmatrix} S_0 \\ S_1 \end{smallmatrix} \to \begin{smallmatrix} S_1 \\ S_0 \\ S_1 \end{smallmatrix} \to \begin{smallmatrix} S_1 \\ S_0 \end{smallmatrix} \to \begin{smallmatrix} S_0 \\ S_1 \\ S_0 \end{smallmatrix} \to \begin{smallmatrix} S_0 \\ S_1 \end{smallmatrix} \to S_0.$$

Hence, $A = FS_3$ has a single pseudoblock in this case. Therefore, we have the following

Theorem 5.1. *For $A = FS_3$ and in all characteristic of F, we have $IndA/\underset{PSA}{\approx} = IndA/\underset{A}{\approx}$.* □

5.3. The triangular algebra

Now take

$$A=\{(a_{ij})\in M_n(F)|a_{ij}=0;\forall i>j\}=\left\{a=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ & a_{22} & \dots & a_{2n}\\ & & \ddots & \vdots\\ & & & a_{nn}\end{pmatrix};a_{ij}\in F\right\};$$

the algebra of $n\times n$ upper triangular matrices (which is isomorphic to the algebra of lower triangular matrices). Then, A is isomorphic to the path algebra of an equi-oriented quiver of type A_n. By Gabriel's theorem (see [4, Chapter11]), this quiver has $n(n+1)/2$ indecomposable modules corresponding to the positive roots of Lie algebra of type A_n. In fact, A acts on the space of column vectors $U=F^n$ by matrix multiplication and

$$N=\left\{\begin{pmatrix} 0 & a_{12} & \dots & \dots & a_{1n}\\ & 0 & a_{23} & \dots & a_{2n}\\ & & \ddots & \ddots & \vdots\\ & & & 0 & a_{n-1n}\\ & & & & 0\end{pmatrix}\right\}=J(A);$$

the Jacobson radical of A, and consequently A has n simple (1-dimensional) representations $\psi_v:A\longrightarrow F\quad (a\longmapsto a_{vv});v=1,2,\dots,n$ (ψ_v is an algebra map $\psi_v=\psi_\mu\Leftrightarrow v=\mu$). We also have

$NU=\left\{\begin{pmatrix} v_1\\ v_2\\ \vdots\\ v_{n-1}\\ 0\end{pmatrix}:v_i\in F\right\}$, and $N^iU=\left\{\begin{pmatrix} v_1\\ \vdots\\ v_{n-i}\\ 0\\ 0\end{pmatrix}:v_i\in F\right\}$, so $U\supset NU\supset N^2U\supset\dots\supset N^{n-1}U\supset 0$ is a composition series with $dimN^{i-1}U/N^iU=1;\forall i=1,2,\dots,n$ and $N^{i-1}U/N^iU\cong\psi_{n-i+1}$. Therefore, as A-module, $U=F^n$ has the following (unique) composition series

$$\begin{array}{c} U\supset NU\supset N^2U\supset\dots\supset N^{n-1}U\supset 0\\ \psi_n\quad \psi_{n-1}\quad \psi_{n-2}\dots\psi_2\quad \psi_1.\end{array}$$

It follows that the quotient module $U_{i,\alpha}=N^{n-i}U/N^{n-i+\alpha}U$ is a uniserial (hence indecomposable) with the following (unique) composition series

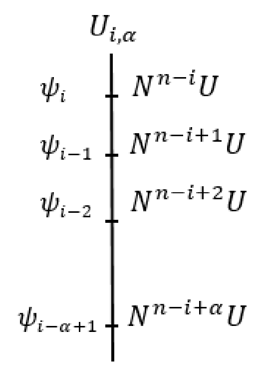

Figure 5.1

and hence $U_{i,\alpha}=N^{n-i}U/N^{n-i+\alpha}U$; $(i=1,2,\dots,n\ and\ \alpha=1,2,\dots,i)$ give a complete set of indecomposable A-modules. Not that $U_{i,\alpha}\cong U_{j,\beta}\Leftrightarrow i=j\wedge\alpha=\beta$ and $U_{i,1}=\psi_i$. The modules $U_{1,1},U_{2,2},\dots,U_{n,n}$ give a complete set of projective indecomposable A-modules. In fact, it is clear that $U_{v,v}=L_v=\left\{\begin{pmatrix} 0 & 0 & a_{1v} & \dots & 0\\ 0 & 0 & a_{2v} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & a_{vv} & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}:a_{iv}\in F,i\leq v;v=1,2,\dots,n\right\}\lhd A$. Note that the composition factors of $U_{v,v}=N^{n-v}U/N^nU=N^{n-1}U$ are as follows:

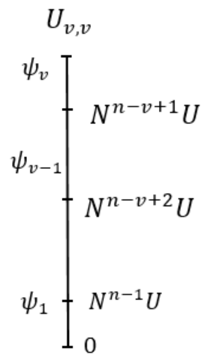

Figure 5.2

The triangular algebra A is not semisimple ($J(A) = N \neq 0$), hence it has a nontrivial block theory. In fact, $A = \sum_{1\leq v\leq n}^{\oplus} U_{v,v}$ (projective indecomposable A-modules PIM decomposition) is known to be connected; i.e. it has a single non-zero central idempotent, namely I_n, and so it has a single block. On the other hand, from the structure of the objects $U_{i,\alpha} = N^{n-i}U/N^{n-i+\alpha}U$ ($i = 1,2,\ldots,n$ *and* $\alpha = 1,2,\ldots,i$) of $Ind(A)$, the objects of the class $Ind(A)$ can be connected by a series of A-maps as follows:

$$
\begin{array}{ccccccccccc}
 & & & & & & & & & & U_{n,1} \\
 & & & & & & & & & & \uparrow \\
 & & & & & & & & & & \vdots \\
 & & & & & & & & & & \uparrow \\
 & & & & & & U_{4,1} & \to & \ldots & \to & U_{n,n-3} \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & U_{3,1} & \to & U_{4,2} & \to & \ldots & \to & U_{n,n-2} \\
 & & & & \uparrow & & \uparrow & & & & \uparrow \\
 & & U_{2,1} & \to & U_{3,2} & \to & U_{4,3} & \to & \ldots & \to & U_{n,n-1} \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
U_{1,1} & \to & U_{2,2} & \to & U_{3,3} & \to & U_{4,4} & \to & \ldots & \to & U_{n,n}
\end{array}
$$

n=2

n=3

n=4

Therefore, A has a single pseudo-block, and so we have:

Theorem 5.2. *For the triangular algebra A over a field F, we have* $IndA/\underset{PSA}{\approx} = IndA/\underset{A}{\approx}$. $\square$

5.4. The group algebra of cylic groups

We now consider the group algebra of cyclic group $A = FC_n; n = p^a e; p \nmid e$ over a field of characteristic p. It is known (see [2], p.34) that $A = FC_n$ has e simple (all are 1-dimensional) modules $\{S_\lambda | \lambda$ is an e-th root of $1\}$, where $S_\lambda = F$ on which C_n acts by multiplication with λ. It is also known that $A = FC_n$ has a total of $n = p^a e$ indecomposable modules. For each integer $1 \leq m \leq p^a$, there is a uniserial module $L_{\lambda,m}$ of dimension m with all composition factors are isomorphic to S_λ (note that $L_{\lambda,1} = S_\lambda$). The set $\{L_{\lambda,m} | \lambda, m\}$ gives a complete set of $n = p^a e$ indecomposable FC_n-modules. Clearly, **PIM**$= \{L_{\lambda,p^a} | \lambda\}$ ($L_{\lambda,p^a} = P(S_\lambda)$ is the projective cover of S_λ), and $FC_n = \sum_\lambda^{\oplus} L_{\lambda,p^a}$. The group algebra FC_n has e blocks $\{B_\lambda | \lambda\}$, where $B_\lambda = \{L_{\lambda,m} | 1 \leq m \leq p^a\}$. It is clear from the structure of $L_{\lambda,m}$ that FC_n has e pseudo-blocks.

Theorem 5.3. *For the group algebra* FC_n *over a field F,* $IndFC_n/\underset{PSFC_n}{\approx} = IndFC_n/\underset{FC_n}{\approx}$. $\square$

5.5. p-group algebra in characteristic p

The group algebra FG of a finite p-group over a field F of characteristic p is known to be indecomposable and has a single simple module, namely the trivial module $1 = F_G$, and hence has a single block. All indecomposable FG-modules are uniserial with all of its composition factors are isomorphic to F_G. Hence, $IndFG$ forms a single pseudo-block of FG.

Theorem 5.4. *For the group algebra FG of a finite p-group over a field F of characteristic p,* $IndFG/\underset{PSFG}{\approx} = IndFG/\underset{FG}{\approx}$. $\square$

6. The special linear group $SL(2,p)$

We now consider the group algebra $A = FSL(2,p)$ in characteristic odd prime number p. It is known that $SL(2,p)$ is the only finite group of Lie type which is of finite representation type in the natural characteristic (see [5, Chapter1]). It is known that $SL(2,p)$ has p (p-regular) conjugacy classes and (hence) p isomorphism classes of simple $FSL(2,p)$-modules of dimensions $1,2,3,\dots,p$ distributed in three blocks B_1,B_2,B_3 (see [6], p.469). We refer to each simple module by its dimension; hence 1 is the natural representation of $SL(2,p)$ and p is the Steinberg representation. There are p^2-p+1 indecomposable $FSL(2,p)$-modules of which $2p-1$ of them are either simple or projective (The Steinberg representation is both simple and projective). The number of remaining indecomposable (non-simple non-projective) $FSL(2,p)$-modules is $(p-1)(p-2)$. Denote by $P_i; 1\le i\le p$, the projective cover of the simple $FSL(2,p)$-module i. The following theorem describes the structure of the projective indecomposable modules.

Theorem 6.1. *[2]. The projective indecomposable $FSL(2,p)$-modules have the following structures:*

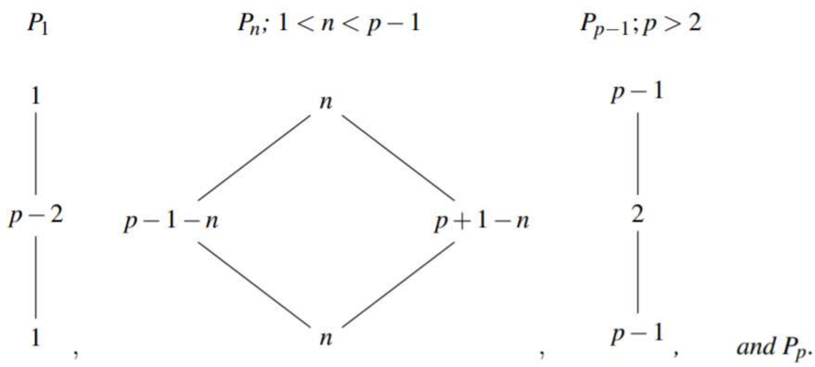

Figure 6.1

The structures of the other indecomposable (non-simple, non-projective) $FSL(2,p)$-modules are explained in the following theorem

Theorem 6.2. *[7]. Every (non-simple,non-projective) indecomposable $FSL(2,p)$-module M has two socle layers. The socle of M consists of the modules $i,i+2,\dots,j (i\leqslant j)$, and the top consists of the modules $p-j+\varepsilon, p-j+\varepsilon+2,\dots,p-i+\delta$, where $\varepsilon,\delta=\pm 1$.*

The following theorem shows that, the compatiblity between the pseudoblock of $FSL(2,p)$ and block theory.

Theorem 6.3. *For the group algebra $A = FG; G = SL(2,p)$ over a field F of characteristic prime number p,*

$$IndA/\underset{PSA}{\approx} \quad = \quad IndA/\underset{A}{\approx}.$$

Proof. **<u>First:</u>** The block B_3 (which contains the Steinberg module $p\cong P_p$) is clearly pseudoblock.

<u>Second:</u> Since B_1 contains all odd-dimensional simple A-modules except p, let P_m, P_i be projective indecomposable A-modules, let m,i be simple A-modules; for all $m,i\in\{1,3,\dots,p-2\}$, and let $M_{\grave{i}}$ be non-simple, non-projective, indecomposable A-modules; $\grave{i}=\{1,2,\dots,r\}$; in which P_m, P_i, m, i and $M_{\grave{i}}$ in B_1 for all $m,i,\grave{i}$. Let $P_i = i/p-1-i, p+1-i/i$, $P_m = m/p-1-m, p+1-m/m$, $M_1 = i/p-1-i, p+1-i$, $M_2 = p+1-i/i$, $M_3 = p+1-m, p-1-m/m$, $M_4 = p+1-m, p-1-m/i,m$, $M_5 = m/p-1-m, p+1-m$, $M_6 = p+1-i, p-1-i/i$.

Then, we have six cases as follows:

1. Let i,m be any two simple A-modules. Hence,
$$i\to M_2\to m.$$
Then, all odd-dimensional simple A-modules are connected either ways by a sequence of A-module homomorphisms.
2. Let i,m be simple A-modules, and let P_i,P_m be projective indecomposable A-modules. Hence,
$$P_i\to M_1\to i, \quad m\to M_3\to P_m.$$

Then, all odd-dimensional simple A-modules and all projective indecomposable A-modules are connected either ways by a sequence of A-module homomorphisms.

3. Let $M_{i^\backprime}; i^\backprime = \{1,2,3,5\}$ be any non-simple, non-projective, indecomposable A-modules, and let i, m be any two simple A-modules. Hence,
$$M_1 \to i, \quad M_2 \to m, \quad M_3 \to P_m \to M_5 \to m.$$
Then, all odd-dimensional simple A-modules and all non-simple, non-projective, indecomposable A-modules $M_{i^\backprime}; i^\backprime = \{1,2,\ldots,r\}$ are connected either ways by a sequence of A-module homomorphisms.
4. Let P_i, P_m be any two projective indecomposable A-modules. Hence,
$$P_i \to M_1 \to i \to M_2 \to p+1-i \to M_3 \to P_m.$$
Then, all projective indecomposable A-modules $P_m, \forall m = \{1,3,\ldots,p-2\}$ are connected either ways by a sequence of A-module homomorphisms.
5. Let P_i, P_m be any two projective indecomposable A-modules, and let M_1, M_3, M_5, M_6 be non-simple, non-projective, indecomposable A-modules. Hence,
$$P_i \to M_1, \quad P_m \to M_5.$$
Also,
$$M_6 \to P_i, \quad M_3 \to P_m.$$
Then, all projective indecomposable A-modules $P_m, \forall m = \{1,3,\ldots,p-2\}$ and all non-simple, non-projective, indecomposable A-modules $M_{i^\backprime}; i^\backprime = \{1,2,\ldots,r\}$ are connected either ways by a sequence of A-module homomorphisms.
6. Let $M_1, M_2, M_3, M_4, M_5, M_6$ be any non-simple, non-projective, indecomposable A-modules. Hence,
$$M_6 \to P_i \to M_1,$$
$$M_1 \to i \to M_2,$$
$$M_3 \to P_m \to M_5,$$
and
$$M_4 \to M_3.$$
Then, all non-simple, non-projective, indecomposable A-modules are connected either ways by a sequence of A-module homomorphisms.

The previous six cases are enough without loss of generality. So, all indecomposable A-modules in B_1 are connected either ways by a sequence of A-module homomorphisms as follows:
$$P_i \to M_{i^\backprime} \to i \to \ldots \leftarrow M^\backprime_{i^\backprime} \leftarrow m \leftarrow M^{\backprime\backprime}_{i^\backprime} \leftarrow P_m;$$
for all $i, m \in \{1,3,5,\ldots,p-2\}$ and $i^\backprime = \{1,2,\ldots,r\}$.

Thus, the block B_1 does not split into union of pseudoblocks. So, B_1 is one pseudoblock.

Third: Similarly, since the block B_2 contains all even-dimensional simple A-modules.
Let P_e, P_j be projective indecomposable A-modules, let e, j be simple A-modules; for all $j, e \in \{2,4,\ldots,p-1\}$, and let $N_{j^\backprime}$ be non-simple, non-projective, indecomposable A-modules; $j^\backprime = \{1,2,\ldots,r\}$; in which P_e, P_j, e, j, and $N_{j^\backprime}$ in B_2 for all $e, j, j^\backprime$. Let $P_j = j/p-1-j, p+1-j/j$, $\quad P_e = e/p-1-e, p+1-e/e$, $\quad N_1 = j/p-1-j, p+1-j$, $\quad N_2 = p+1-j/j$, $N_3 = p-1-e, p+1-e/e$, $\quad N_4 = p-1-e, p+1-e/e, j$, $\quad N_5 = e/p-1-e, p+1-e$, $\quad N_6 = p-1-j, p+1-j/j$.

Then, we have six cases as follows:

1. Let j, e be any two simple A-modules. Hence,
$$j \to N_2 \to e.$$
Then, all even-dimensional simple A-modules are connected either ways by a sequence of A-module homomorphisms.
2. Let j, e be simple A-modules, and let P_j, P_e be projective indecomposable A-modules. Hence,
$$P_j \to N_1 \to j, \quad e \to N_3 \to P_e.$$
Then, all even-dimensional simple A-modules and all projective indecomposable A-modules are connected either ways by a sequence of A-module homomorphisms.

3. Let $N_{j^\backslash}; j^\backslash = \{1,2,3,5\}$ be any non-simple, non-projective, indecomposable A-modules, and let j,e be any two simple A-modules. Hence,
$$N_1 \to j, \quad N_2 \to e, \quad N_3 \to P_e \to N_5 \to e.$$
Then, all even-dimensional simple A-modules and all non-simple, non-projective, indecomposable A-modules $N_{j^\backslash}; j^\backslash = \{1,2,\dots,r\}$ are connected either ways by a sequence of A-module homomorphisms.
4. Let P_j, P_e be any two projective indecomposable A-modules. Hence,
$$P_j \to N_1 \to j \to N_2 \to p+1-j \to N_3 \to P_e.$$
Then, all projective indecomposable A-modules $P_e, \forall e = \{2,4,\dots,p-1\}$ are connected either ways by a sequence of A-module homomorphisms.
5. Let P_j, P_e be any two projective indecomposable A-modules, and let N_1, N_3, N_5, N_6 be non-simple, non-projective, indecomposable A-modules. Hence,
$$P_j \to N_1, \quad P_e \to N_5.$$
Also,
$$N_6 \to P_j, \quad N_3 \to P_e.$$
Then, all projective indecomposable A-modules $P_e; \forall e = \{2,4,\dots,p-1\}$ and all non-simple, non-projective, indecomposable A-modules $N_{j^\backslash}; j^\backslash = \{1,2,\dots,r\}$ are connected either ways by a sequence of A-module homomorphisms.
6. Let N_1,N_2,N_3,N_4,N_5,N_6 be any non-simple, non-projective, indecomposable A-modules. Hence,
$$N_6 \to P_j \to N_1,$$
$$N_1 \to j \to N_2,$$
$$N_3 \to P_e \to N_5,$$
and
$$N_4 \to N_3.$$
Then, all non-simple, non-projective, indecomposable A-modules are connected either ways by a sequence of A-module homomorphisms.

The previous six cases are enough without loss of generality. So, all indecomposable A-modules in B_2 are connected either ways by a sequence of A-module homomorphisms as follows:
$$P_j \to N_{j^\backslash} \to j \to \dots \leftarrow N^\backslash_{j^\backslash} \leftarrow e \leftarrow N^{\backslash\backslash}_{j^\backslash} \leftarrow P_e;$$
for all $j,e \in \{2,4,\dots,p-1\}$ and $j^\backslash = \{1,2,\dots,r\}$.

Thus, the block B_2 does not split into union of pseudoblocks. So, B_2 is one pseudoblock.

Thus, for group algebra $FSL(2,p)$ in characteristic odd prime p the two notions blocks and pseudoblocks coincide.

Example 6.4. *If $p=2$, then the representations of $SL(2,2) \cong S_3$ in characteristic 2; hence the two notions blocks and pseudoblocks coincide as stated in section 5.*

If $p=7$, then the following are the indecomposable $FSL(2,7)$-modules:

- *The simple $FSL(2,7)$-modules are:* $\underbrace{1,3,5}_{B_1}, \underbrace{2,4,6}_{B_2}, \underbrace{7}_{B_3}$.
- *The projective indecomposable $FSL(2,7)$-modules are:*
$$1/5/1, \quad 3/3,5/3, \quad 5/1,3/5, \quad 4/2,4/4, \quad 2/4,6/2, \quad 6/2/6, \quad 7.$$
- *The (non-projective non-simple) indecomposable $FSL(2,7)$-modules are:*
$5/1$, $1/5$, $3/5$, $5/3$, $3/3$, $3,5/3$, $3/3,5$ $1,3/5$, $5/1,3$, $3,5/1,3,5$, $1,3,5/3,5$, $1,3,5/1,3,5$, $3,5/1,3$, $3,5/3,5$, $1,3/3,5$. *(in B_1)*

$2/6$, $6/2$, $4/2$, $2/4$, $4/4$, $4,6/2$, $2/4,6$, $2,4/4$, $4/2,4$, $2,4/2,4,6$, $2,4,6/2,4$, $2,4,6/2,4,6$, $2,4/2,4$, $2,4/4,6$, $4,6/2,4$. *(in B_2)*

The total number of indecomposable modules is $43 = 7^2 - 7 + 1$, *where* $Ext_{FSL(2,7)}(i, j)$ *are* 1*-dimension for all indecomposable* $FSL(2,7)$*-modules as stated in ([5], p.117).*

The indecomposable $FSL(2,7)$*-modules in* B_1 *forms a single pseudoblock via the following sequence of homomorphisms:*
$3/3 \to 1,3,5/1,3,5 \to 1,3,5/3,5 \to 1,3/3,5 \to 1,3/5 \to 5/1,3/5 \to 5/1,3 \to 3,5/1,3 \to 3,5/1,3,5 \to 3,5/3,5 \to 3,5/3 \to 3/3,5/3 \to 3/3,5 \to 3/5 \to 3 \to 5/3 \to 5 \to 1/5 \to 1 \to 5/1 \to 1/5/1.$

The indecomposable $FSL(2,7)$*-modules in* B_2 *forms a single pseudoblock via the following sequence of homomorphisms:*
$4/4 \to 2,4,6/2,4,6 \to 2,4,6/2,4 \to 4,6/2,4 \to 4,6/2 \to 2/4,6/2 \to 2/4,6 \to 2,4/4,6 \to 2,4/2,4,6 \to 2,4/2,4 \to 2,4/4 \to 4/2,4/4 \to 4/2,4 \to 4/2 \to 4 \to 2/4 \to 2 \to 6/2 \to 6 \to 2/6 \to 6/2/6.$

$\square$

Acknowledgement

The main results of this paper are taken from a dissertation written by the first author as a part of a master degree fulfillments from Umm Al-Qura University. The first author thanks her supervisor Prof. Ahmed A. Khammash for his help and guidance and Taif University for their generous grant.

References

[1] A. Khammash, *The pseudoblocks of endomorphism algebras*, Int. Math. Forum, **4**(48) (2009), 2363-2368.
[2] J. Alperin, *Local Representation Theory: Modular Representations as an Introduction to the Local Representation Theory of Finite Groups*, Cambridge University Press, 1986.
[3] A. Khammash, *Brauer-fitting correspondence on tensor algebra,* Int. J. Algebra, **8**(19) (2014), 895-901.
[4] K. Erdmann, T. Holm, *Algebras and Representation Theory*, Springer, 2018.
[5] J. Humphreys, *Modular Representations of Finite Groups of Lie Type*, Cambridge University Press, 2006.
[6] L. Dornhoff, *Group Representation Theory: Modular Representation Theory*, M. Dekker, 1972.
[7] D. Craven, *Maximal* psl_2 *subgroups of exceptional groups of lie type*, (2019), arXiv:1610.07469.

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 70-85
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.670266

Maximal and Willmore Null Hypersurfaces in Generalized Robertson-Walker Spacetimes

Cyriaque Atindogbe[1*] **and Hippolyte Hounnon**[1]

[1]*Department of Mathematics, Faculty of Science and Techniques, University of Abomey-Calavi, Benin*
[*]*Corresponding author*

Article Info

Keywords: *Generalized Robertson-Walker spacetimes, Maximal null hypersurface, Willmore null hypersurface*
2010 AMS: *53C42, 53C50, 53C21*
Received: *4 January 2020*
Accepted: *25 April 2020*
Available online: *10 June 2020*

Abstract

We establish after some technical results a characterization of maximal null hypersurfaces in terms of a constant mean curvature screen foliation (in the slices) induced by the Chen's vector field. Thereafter, bounds are provided for both the squared norm of the (screen) shape operator for non totally geodesic maximal null hypersurfaces and the scalar curvature of the fiber. In terms of the scalar curvature of the fiber and the warping function, we establish necessary and sufficient conditions for Null Convergence Condition (NCC) to be satisfied in which case we prove that there are no non totally geodesic maximal null hypersurfaces. A generic example consisting of graphs of functions defined on the fiber is given to support our results. Finally, we provide lower bounds for the extrinsic scalar curvature and give a characterization result for Willmore null hypersurfaces in generalized Robertson-Walker spacetimes.

1. Introduction

The study of maximal spacelike hypersurfaces in Lorentzian manifolds is an important topic as evidenced by the considerable amount of papers devoted to this purpose ([1]-[13], and references therein). The big amount of interest to these objects is due to to the fact that they play a key role in the dynamic aspects of general relativity and are solutions of existence and uniqueness Calabi-Berstein type problems [2, 6]. The reason for the terminology maximal (in contrast to the minimality in Riemannian setting) is that the vanishing of the mean curvature is equivalent to the fact that the hypersurface realizes a local maximum of the area functional for compactly supported normal variation. Willmore hypersurfaces are generalization of the maximal ones. They are critical point of the total squared mean curvature functional whose study was proposed by Willmore in 1965. Most of the works done since then are on (nondegenerate) $2-$dimensional surfaces in (semi-)Riemannian setting [14]-[17].

Null hypersurfaces are genuine objects in Lorentzian geometry in the sense that they have not Riemannian counterpart. They are very interesting in general relativity and black hole horizons are one of the most remarkable examples, and recent works show that there is an increasing interest on null hypersurfaces both from a physical and geometrical point of view [18]-[24].

In this paper we are interested in maximal and Willmore null hypersurfaces in generalized Robertson-Walker spacetimes. Our main aim is to give existence and characterization results both for maximal and Willmore null hypersurfaces and bring out some of their geometric properties.

The paper is organized as follows. In Section 2 we revise some facts about null hypersurfaces in Lorentzian manifolds with special attention paid to their connection with Chen's concircular vector field and symmetries of generalized Robertson-Walker

Email addresses and ORCID numbers: **atincyr@gmail.com, https://orcid.org/ 0000-0001-8346-4027 (C. Atindogbe), hiph14@yahoo.fr, https://orcid.org/ 0000-0002-4068-6225 (H. Hounnon)**

spacetimes. Section 3, after some technical results (Proposition 3.1), presents a characterization of maximal null hypersurfaces in terms of a constant mean curvature screen foliation induced by Chen's vector field in the slices (Theorem 3.2). Also, necessary and sufficient conditions to obey the Null Convergence Condition (NCC) are provided in terms of the scalar curvature of the fiber and the warping function. In this case, we prove that there are no non totally geodesic maximal null hypersurfaces. Upper and lower bounds are provided for both the squared norm of the (screen) shape operator of non totally geodesic maximal null hypersurfaces and the scalar curvature of the fiber (Theorem 3.4). In Section 4 we support above results by a generic example consisting of graphs of functions defined on the fiber and establish the maximality condition in terms of the Laplacian of the involved functions (Theorem 4.1). Section 5 is concerned with providing a lower bound for the extrinsic scalar curvature for maximal null hypersurfaces (Theorem 5.2). Finally we give in Section 6 a characterization of Willmore null hypersurfaces in generalized Robertson-Walker spacetimes (Theorem 6.3).

2. Preliminaries

2.1. Some symmetries of generalized Robertson-Walker spacetimes

A Generalized Robertson-Walker spacetime (GRW in short) is the warped product $\overline{M} = -I \times_f F$, where I (the base) is an open interval of the real line $\mathbb{R}$, (F, g_F) the fiber is a Riemannian manifold of dimension $n-1$ and $f > 0$ is a smooth warping function (or scale factor) defined on I. It is then endowed with the Lorentzian metric

$$\overline{g} = -dt^2 + f^2(t) g_F$$

where t stands for the natural (global) parameter on $\mathbb{R}$. In particular, when the Riemannian fiber F has constant sectional curvature, then $-I \times_f F$ is classically called a Robertson-Walker (RW) spacetime, and it is a spatially homogeneous spacetime. Throughout, π_I (resp. π_F) will denote projection on the base space I (resp. on the fiber F).

Observe that the existence of a globally defined timelike coordinate vector field ∂_t makes a GRW time-orientable. The vector field

$$\zeta = f\partial_t$$

is timelike, closed and conformal. If $\overline{\nabla}$ denotes the Levi-Civita connection of $\overline{M}$, it holds for vector fields V tangent to $\overline{M}$,

$$\overline{\nabla}_V \zeta = f'(t)V.$$

The above definition of GRW spacetimes highlights the existence of a spacelike hypersurface foliation with leaves the slices $\{t\} \times F$ (spatial universes), $(t \in I)$.

A nice characterization theorem by Chen [25, Theorem 1] states that a Lorentzian manifold of dimension $n \geq 3$ is a GRW spacetime if and only if it admits a timelike concircular vector field. Following Fialkow [26], a concircular vector field is a vector field v which satisfies

$$\overline{\nabla}_X v = \mu X$$

for vector fields X tangent to $\overline{M}$, where $\overline{\nabla}$ denotes the Levi-Civita connection of $\overline{M}$ and μ is a smooth function on $\overline{M}$. A vector field v as above is called Chen's vector field. It is an eigenvector of the Ricci tensor of $\overline{M}$ with eigenvalue we denote by σ. The Weyl tensor of $\overline{M}$ is

$$W = \overline{R} + \frac{\overline{s}}{2n(n-1)} \overline{g} \owedge \overline{g} + \frac{1}{n-2}\left(\overline{Ric} - \frac{\overline{s}}{n}\overline{g}\right) \owedge \overline{g}$$

where we use the following definition for the Ricci tensor : $\overline{Ric}(X,Y) =$ trace $\left(Z \longmapsto \overline{R}(Z,X)Y\right)$, being $\overline{s}$ the scalar curvature of $\overline{M}$. It can be shown ([11]) that the components of the Ricci curvature are given by

$$\overline{R}_{jk} = \frac{n-2}{\langle v,v\rangle} W(v,\partial_j,v,\partial_k) + \frac{\overline{s}-\sigma}{n-1}\left(\overline{g}_{jk} - \frac{\langle v,\partial_j\rangle\langle v,\partial_k\rangle}{\langle v,v\rangle}\right) + \sigma\frac{\langle v,\partial_j\rangle\langle v,\partial_k\rangle}{\langle v,v\rangle}. \tag{2.1}$$

Using warping coordinate frames, it holds

$$\begin{aligned}
\overline{Ric}(\partial_t,\partial_t) &= -(n-1)\frac{f''}{f} \\
\overline{Ric}(\partial_t,X) &= 0 \\
\overline{Ric}(X,Y) &= Ric_{g_F}(X,Y) + \left[ff'' + (n-2)f'^2\right] g_F(X,Y)
\end{aligned} \tag{2.2}$$

where X and Y are tangent to the fiber F.

2.2. Geometry of null hypersurfaces

Let M be a null hypersurface in a Lorentzian manifold $(\overline{M}^n, \overline{g})$, i.e a hypersurface for which the induced metric tensor $g = \overline{g}_{|M}$ is degenerate on it. A *screen distribution* on M^{n-1} $(n \geq 3)$, is a complementary bundle of $TM^{\perp}$ in TM. It is then a rank $n-2$ non-degenerate distribution over M. In fact, there are infinitely many possibilities of choices for such a distribution. Each of them is canonically isomorphic to the factor vector bundle $TM/TM^{\perp}$. From [20], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\overline{g}(N,\xi) = 1, \quad \overline{g}(N,N) = \overline{g}(N,W) = 0, \qquad \forall\, W \in \mathscr{S}(N)|_{\mathscr{U}})$$

where $\mathscr{S}(N)$ denotes the fixed screen distribution.

Then $T\overline{M}$ admits the splitting:

$$T\overline{M}|_M = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus \mathscr{S}(N).$$

We call $tr(TM)$ a *(null) transverse vector bundle* along M. Now, we need to clarify the (general) concept of rigging for null hypersurfaces (see [23] for details).

Definition 2.1. *Let M be a null hypersurface in a Lorentzian manifold. A rigging for M is a vector field ζ defined on some open set containing M such that $\zeta_p \notin T_pM$ for each $p \in M$.*

Given a rigging ζ in a neighborhood of M in $(\overline{M}, \overline{g})$ let α denote the 1-form $\overline{g}$-metrically equivalent to ζ, i.e. $\alpha = \overline{g}(\zeta, \,.)$. Take $\omega = i^{\star}\alpha$, being $i : M \hookrightarrow \overline{M}$ the canonical inclusion. Next, consider the tensors

$$\breve{g} = \overline{g} + \alpha \otimes \alpha \quad \text{and} \quad \widetilde{g} = i^{\star}\breve{g}.$$

It is easy to show that $\widetilde{g}$ defines a Riemannian metric on the (whole) hypersurface M. The *rigged vector field* of ζ is the $\widetilde{g}$-metrically equivalent vector field to the 1-form ω and it is denoted by ξ. In fact the rigged vector field ξ is the unique lightlike vector field in M such that $\overline{g}(\zeta,\xi) = 1$. Moreover, ξ is $\widetilde{g}$-unitary. To a rigging ζ for M is associated the screen distribution $\mathscr{S}(\zeta)$ given by $\mathscr{S}(\zeta) = TM \cap \zeta^{\perp}$. It is the $\widetilde{g}$-orthogonal subspace to ξ and the corresponding null transverse vector field on M is

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta,\zeta)\xi.$$

A null hypersurface M equipped with a rigging ζ is said to be normalized and is denoted (M,ζ) (the latter is called a normalization of the null hypersurface). A normalization (M,ζ) is said to be closed (resp. conformal) if the rigging ζ is closed i.e the 1-form α is closed (resp. ζ is a conformal vector field, i.e there exists a function ρ on the domain of ζ such that $L_{\zeta}\overline{g} = 2\rho\overline{g}$). We say that ζ is a *null rigging* for M if the restriction of ζ to the null hypersurface M is a null vector field.

Let ζ be a rigging for a null hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$. The screen distribution $\mathscr{S}(\zeta) = \ker\omega$ is integrable whenever ω is closed, in particular if the rigging is closed. Throughout, the ambient Lorentzian metric $\overline{g}$ will also be denoted $\langle\ ,\ \rangle$.

On a normalized null hypersurface (M,ζ), the Gauss and Weingarten formulas are given by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + B(X,Y)N, \\ \overline{\nabla}_X N &= -A_N X + \tau(X)N, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \nabla_X PY &= \overset{\star}{\nabla}_X PY + C(X,PY)\xi, \\ \nabla_X \xi &= -\overset{\star}{A}_{\xi} X - \tau(X)\xi, \end{aligned} \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M}, \overline{g})$, ∇ denotes the connection on M induced from $\overline{\nabla}$ through the projection along the null transverse vector field N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(\zeta)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma(\mathscr{S}(\zeta))$ with respect to the decomposition (2.4). Now the $(0,2)$ tensors B and C are the second fundamental forms on TM and $\mathscr{S}(\zeta)$ respectively, A_N and $\overset{\star}{A}_{\xi}$ are the shape operators on TM with respect to the rigging ζ and the rigged vector field ξ respectively and τ a 1-form on TM defined by

$$\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi).$$

For the second fundamental forms B and C the following hold

$$B(X,Y) = g(\overset{\star}{A}_\xi X, Y), \quad C(X, PY) = g(A_N X, Y) \;\; \forall X, Y \in \Gamma(TM),$$

and

$$B(X,\xi) = 0, \quad \overset{\star}{A}_\xi \xi = 0 \quad \text{and} \quad C(X,Y) - C(Y,X) = \langle [X,Y], N\rangle,$$

and the last equality shows that the screen structure $\mathscr{S}(\zeta)$ is integrable if and only if C is symmetric on it. In this case, $\overset{\star}{\nabla}$ is the Levi-Civita connection of the screen foliation from $(\overline{M}, \overline{g})$ and Equations (2.3) and (2.4) show that its second fundamental form is

$$\mathbb{I}^\zeta(X,Y) = C(X,Y)\xi + B(X,Y)N, \quad X, Y \in \mathscr{S}(\zeta).$$

Let denote by $\overline{R}$ and R the Riemannian curvature tensors of $\overline{\nabla}$ and ∇, respectively. Then the following are the Gauss-Codazzi equations [20, p. 93].

$$\begin{aligned}
\left\langle \overline{R}(X,Y)Z, \xi \right\rangle &= (\nabla_X B^N)(Y,Z) - (\nabla_Y B^N)(X,Z) \\
&\quad + \tau^N(X) B^N(Y,Z) - \tau^N(Y) B^N(X,Z), \\
\left\langle \overline{R}(X,Y)Z, PW \right\rangle &= \left\langle R(X,Y)Z, PW \right\rangle + B^N(X,Z) C^N(Y,PW) \\
&\quad - B^N(Y,Z) C^N(X,PW), \\
\left\langle \overline{R}(X,Y)\xi, N \right\rangle &= \left\langle R(X,Y)\xi, N \right\rangle = C^N(Y, \overset{\star}{A}_\xi X) - C^N(X, \overset{\star}{A}_\xi Y) \\
&\quad - 2d\tau^N(X,Y), \quad \forall X, Y, Z, W \in \Gamma(TM|_{\mathscr{U}}). \\
\left\langle \overline{R}(X,Y)PZ, N \right\rangle &= \left\langle (\nabla_X A_N)Y, PZ \right\rangle - \left\langle (\nabla_Y A_N)X, PZ \right\rangle \\
&\quad + \tau^N(Y) \left\langle A_N X, PZ \right\rangle - \tau^N(X) \left\langle A_N Y, PZ \right\rangle.
\end{aligned}$$

A null hypersurface M is said to be *totally umbilic* (resp. *totally geodesic*) if there exists a smooth function ρ on M such that at each $p \in M$ and for all $u, v \in T_p M$, $B(p)(u,v) = \rho(p)\overline{g}(u,v)$ (resp. B vanishes identically on M). These are intrinsic notions on any null hypersurface in the sense that they are independent of the normalization. Remark that M is *totally umbilic* (resp. *totally geodesic*) if and only if $\overset{\star}{A}_\xi = \rho P$ (resp. $\overset{\star}{A}_\xi = 0$). The trace of $\overset{\star}{A}_\xi$ is the null (non normalized) mean curvature of M, explicitly given by

$$H_p = \sum_{i=2}^{n-1} \overline{g}(\overset{\star}{A}_\xi(e_i), e_i) = \sum_{i=2}^{n-1} B(e_i, e_i),$$

being $(e_2, \ldots, e_{n-1})$ an orthonormal basis of $\mathscr{S}(\zeta)$ at p. Let $\widetilde{\nabla}$ denote the Levi-Civita connection on the rigged Riemannian structure $(M, \widetilde{g})$. It holds [23],

$$(L_\xi \widetilde{g})(X,Y) = -2B(X,Y), \quad X, Y \in \mathscr{S}(\zeta).$$

In particular,

$$H = -\widetilde{\operatorname{div}}\, \xi. \tag{2.5}$$

Observe that if M is orientable compact without boundary it follows from (2.5) that $\displaystyle\int_M H d\widetilde{g} = 0$.

Now, we recall from [18], the following generalized Raychaudhury equation,

$$\overline{Ric}(\xi) = \xi(H) + \tau(\xi)H - \|\overset{\star}{A}_\xi\|^2. \tag{2.6}$$

3. Maximal null hypersurfaces in GRW spacetimes

Due to the causal character $(0, n-2)$ of a null hypersurface in any $n-$dimensional $(n \geq 3)$ Lorentzian manifold, the normalization problem in a GRW spacetime has the outstanding feature that the Chen's closed timelike concircular vector field $\zeta = f\partial_t$ can act as rigging vector field for each of them. Let ξ denote the corresponding rigged vector field. The associated null transverse vector field is

$$N = f\partial_t + \frac{1}{2} f^2(t)\xi.$$

By the Weingarten formula, it holds

$$-A_N X+\tau(X)N=f'(t)X+\frac{1}{2}\Big(X\cdot f^2\Big)\xi+\frac{1}{2}f^2\Big(-\overset{\star}{A}_\xi X-\tau(X)\xi\Big)\quad \text{for } X\in\mathfrak{X}(M). \tag{3.1}$$

Hence, for $X\in\mathscr{S}(\zeta)$,

$$\Big(-A_N X+\frac{1}{2}f^2\overset{\star}{A}_\xi X-f'(t)X\Big)+\Big(\frac{1}{2}f^2\tau(X)-(X\cdot f)f\Big)\xi+\tau(X)N=0.$$

Then,

$$\tau(X)=0,\qquad X\cdot f=0\quad\text{and}\quad A_N X=\frac{1}{2}f^2\overset{\star}{A}_\xi X-f'(t)X$$

for all $X\in\mathscr{S}(\zeta)$.

Now take $X=\xi$ in (3.1) and get

$$A_N\xi=0,\quad \tau(\xi)=0\quad\text{and}\quad \xi\cdot f=-\frac{f'}{f}. \tag{3.2}$$

Then, we can state:

Proposition 3.1. *Let M be a null hypersurface in a GRW spacetime normalized with the Chen's vector field $\zeta=f\partial_t$. Then,*

(i) the 1−form τ vanishes identically on M and the rigged vector field ξ is geodesic.
(ii) $\xi\cdot f=-\frac{f'}{f}$ and for all vector field X tangent to the screen structure $\mathscr{S}(\zeta)$, it holds $X\cdot f=0$.
(iii) For all $X\in\mathfrak{X}(M)$,

$$A_N X=\frac{1}{2}f^2\overset{\star}{A}_\xi X^{\mathscr{S}}-f'(t)X^{\mathscr{S}},$$

where $X^{\mathscr{S}}=PX$ with P the projection morphism of $\Gamma(TM)$ onto $\Gamma(\mathscr{S}(\zeta))$ with respect to the decomposition (2.4).

Since ζ is closed, the screen structure $\mathscr{S}(\zeta)$ induces a foliation on M. For $p=(t,x)\in M$ let $\mathscr{F}_p$ denote the leaf of $\mathscr{S}(\zeta)$ through p. Every $X\in\mathfrak{X}(M)$ splits as follows

$$X=-\frac{1}{f}\alpha(X)\partial_t+X^F$$

where X^F is the lift of the projection of X onto the fiber F and $\alpha=\overline{g}(\zeta,\cdot)=-f\,dt$ is the 1−form metrically equivalent to ζ respect to $\overline{g}$. It follows that $X\in\mathfrak{X}(M)$ is tangent to the screen structure if and only if $X=X^F\in\mathfrak{X}(F)$. In other words, each leaf $\mathscr{F}_p$ is a hypersurface in the slice $\{\pi_I(p)\}\times F$. Furthermore, for $X\in\mathfrak{X}(M)$, it holds $X=\alpha(X)\xi+X^{\mathscr{S}}$ which gives

$$X^F=X^{\mathscr{S}}+\alpha(X)\xi^F,$$

where $\xi=-\frac{1}{f}\partial_t+\xi^F$. In particular $g_F(X^{\mathscr{S}},\xi^F)=0$. Then, since $g_F(\xi^F,\xi^F)=\frac{1}{f^4}$ it follows that at each point $p=(t,x)\in M$, the vector $f(t)\xi^F$ is a unit normal in $\{t\}\times F$ to the leaf $\mathscr{F}_p$ of $\mathscr{S}(\zeta)$ through p. Let $A^F_{f\xi^F}$ and $H^{\mathscr{F}}$ denote respectively the shape operator and the mean curvature of $\mathscr{F}_p$ as a hypersurface of the slice $\{t\}\times F$ which inherits the metric $f^2(t)g_F$.

Theorem 3.2. *Let $I\times_f F$ be a GRW spacetime. A null hypersurface M is maximal if and only if the screen foliation induced by the Chen's vector field $\zeta=f\partial_t$ has constant mean curvature $-\frac{f'(t)}{f(t)}$ in each slice $\{t\}\times F$.*

Proof. Given $X\in\mathfrak{X}(M)$, we have

$$\begin{aligned}
-\overset{\star}{A}_\xi X &= \overline{\nabla}_X\xi=\overline{\nabla}_X\Big(-\frac{1}{f}\partial_t+\xi^F\Big)=\frac{X\cdot f}{f^2}\partial_t-\frac{1}{f}\overline{\nabla}_X\partial_t+\overline{\nabla}_X\xi^F\\
&= \frac{X\cdot f}{f^2}\partial_t-\frac{1}{f^2}\Big[\overline{\nabla}_X f\partial_t-(X\cdot f)\partial_t\Big]+\overline{\nabla}_X\xi^F\\
&= 2\frac{X\cdot f}{f^2}\partial_t-\frac{f'}{f^2}X-\frac{1}{f}\alpha(X)\overline{\nabla}_{\partial_t}\xi^F+\overline{\nabla}_{X^F}\xi^F\\
&= 2\frac{X\cdot f}{f^2}\partial_t-\frac{f'}{f^2}X-\frac{f'}{f}\alpha(X)\xi^F+\overline{\nabla}_{X^F}\xi^F
\end{aligned} \tag{3.3}$$

As $\overset{\star}{A}_\xi\xi=0$, we just need to compute (3.3) for $X\in\mathscr{S}(\zeta)$ i.e $X=X^F=X^{\mathscr{S}}$ for which $X\cdot f=0$ as seen from item (ii) in Proposition 3.1 and $\alpha(X)=0$. Let ∇^F denote the Levi-Civita connection of (F,g_F). We have for all $X\in\mathscr{S}(\zeta)$,

$$\overset{\star}{A}_\xi X = \frac{f'}{f^2}X-\overline{\nabla}_X\xi^F=\frac{f'}{f^2}X-\frac{1}{f}\overline{\nabla}_X f\xi^F$$

$$= \frac{f'}{f^2}X - \frac{1}{f}\Big[\nabla^F_X f\xi^F + \langle X, f\xi^F\rangle \frac{f'}{f}\partial_t\Big],$$

thus for all $X \in \mathscr{S}(\zeta)$,

$$\overset{\star}{A}_\xi X = \frac{f'}{f^2}X + \frac{1}{f}A^F_{f\xi^F}X. \tag{3.4}$$

Let $(\partial_t, f\xi^F, e_2, \ldots, e_{n-1})$ be a frame field on $\overline{M}$ along M such that $(e_2, \ldots, e_{n-1})$ represents an orthonormal basis for $\mathscr{S}(\zeta)$. The null mean curvature H_p at $p = (t, x)$ is given by

$$\begin{aligned} H_p &= \sum_{i=2}^{n-1}\langle \overset{\star}{A}_\xi e_i, e_i\rangle = \sum_{i=2}^{n-1}\left[\frac{f'}{f^2}\langle e_i, e_i\rangle + \frac{1}{f}\Big\langle A^F_{f\xi^F}e_i, e_i\Big\rangle\right] \\ &= (n-2)\frac{f'}{f^2} + \frac{n-2}{f}H^{\mathscr{F}} \end{aligned}$$

that is

$$H_p = \frac{n-2}{f^2}\Big[f'(t) + f(t)H^{\mathscr{F}}\Big] \tag{3.5}$$

and the claim follows from (3.5). □

In the way of above proof, we have established (combining (3.4) and $\overset{\star}{A}_\xi \xi = 0$) the following:

Proposition 3.3. *Let $I \times_f F$ be a GRW spacetime. A null hypersurface M is totally umbilic if and only if the screen foliation induced by the Chen's vector field $\zeta = f\partial_t$ is totally umbilic in the slices.*

Theorem 3.4. *Let $I \times_f F$ be a GRW spacetime and suppose M is a maximal null hypersurface normalized by the Chen's vector field $\zeta = f\partial_t$.*

(i) The squared norm of the screen shape operator has the following upper bound

$$\| \overset{\star}{A}_\xi \|^2 \leq \frac{n-2}{f^2}W(\zeta,\xi,\zeta,\xi) + \frac{1}{(n-1)f^4}\Big[(n-1)(n-2)ff'' - s_F\Big] \tag{3.6}$$

with equality if and only if the scale factor f is constant, in particular the slices are minimal.

(ii) If the warping function f has a convex logarithm i.e $(\ln f)'' \geq 0$ (resp. a concave logarithm, i.e $(\ln f)'' \leq 0$) then

$$\| \overset{\star}{A}_\xi \|^2 \geq \frac{n-2}{f^2}W(\zeta,\xi,\zeta,\xi) - \frac{1}{(n-1)f^4}s_F$$

$$\Big(resp.\ \ \| \overset{\star}{A}_\xi \|^2 \leq \frac{n-2}{f^2}W(\zeta,\xi,\zeta,\xi) - \frac{1}{(n-1)f^4}s_F\Big)$$

with equality if and only if the warping function is given by

$$f(t) = k\exp(\lambda t), \quad k \in \mathbb{R}^\star_+, \quad \lambda \in \mathbb{R}.$$

(iii) If the Weyl tensor satisfies $i_{\partial_t}W = 0$, then the Null Convergence Condition (NCC) holds if and only if the scalar curvature s_F of the fiber F has the following lower bound

$$s_F \geq (n-1)(n-2)f^2(\ln f)'', \tag{3.7}$$

in which case the GRW spacetime admits no non totally geodesic maximal null hypersurface. Otherwise, on the set of non geodesic points it holds

$$\| \overset{\star}{A}_\xi \|^2 \leq \frac{1}{(n-1)f^4}\Big[(n-1)(n-2)ff'' - s_F\Big] \tag{3.8}$$

with

$$s_F < (n-1)(n-2)ff''.$$

Proof. From (2.1) and (2.2) it is easy to see that the eigenvalue σ associated to the Chen's vector ζ and the scalar curvature $\overline{s}$ are given by

$$\sigma = (n-1)\frac{f''}{f}$$

and

$$\overline{s} = 2(n-1)\frac{f''}{f} + (n-1)(n-2)\frac{f'^2}{f^2} + \frac{s_F}{f^2}.$$

It follows from (2.1) that for all null vector U,

$$\overline{Ric}(U) = -(n-2)W(\partial_t, U, \partial_t, U) - \frac{\langle \partial_t, U\rangle^2}{n-1}\Big[(n-1)(n-2)(\ln f)'' - \frac{s_F}{f^2}\Big]. \tag{3.9}$$

In particular,

$$\overline{Ric}(\xi) = -\frac{(n-2)}{f^2}W(\zeta,\xi,\zeta,\xi) - \frac{1}{(n-1)f^4}\Big[(n-1)(n-2)f^2(\ln f)'' - s_F\Big]. \tag{3.10}$$

It follows from (3.10) and (2.6) that for a maximal M,

$$\|\overset{\star}{A}_\xi\|^2 = -\overline{Ric}(\xi) \quad = \quad \frac{n-2}{f^2}W(\zeta,\xi,\zeta,\xi) + \frac{1}{(n-1)f^4}\Big[(n-1)(n-2)f^2(lnf)'' - s_F\Big]$$

which gives (3.6) with equality if and only if $f' = 0$ in particular the slices (with mean curvature $\frac{f'(t)}{f(t)}$) are minimal and (i) is proved. For (ii), since $n \geq 3$, the hypothesis implies $(n-1)(n-2)f^2(lnf)'' \geq 0$ (resp. $(n-1)(n-2)f^2(lnf)'' \leq 0$). The equality case for both estimations is obtained for $(\ln f)'' = 0$ that is $f(t) = k\exp(\lambda t), \quad k \in \mathbb{R}^\star_+, \quad \lambda \in \mathbb{R}$.

Suppose the Weyl tensor satisfies $i_{\partial_t}W = 0$. Then, from (3.9) the null convergence condition holds if and only if the scalar curvature s_F of the fiber satisfies (3.7). In this case, using (2.6) for a maximal M leads to $-\|\overset{\star}{A}_\xi\|^2 = \overline{Ric}(\xi) \geq 0$ i.e $\overset{\star}{A}_\xi \ = 0$ and M is totally geodesic. Otherwise (i.e if the null convergence condition failed), for the set of non geodesic points,

$$s_F < (n-1)(n-2)f^2(\ln f)'' = (n-1)(n-2)(ff'' - f'^2) \leq (n-1)(n-2)ff''$$

and (3.8) follows from (3.6) in which the first term vanishes. □

4. A generic example

Let $I \times_f F$ be a GRW spacetime and consider $\psi : F \longrightarrow I$ a differentiable function with graph

$$M = \{(\psi(x), x), \quad x \in F\}.$$

This is a null hypersurface if and only if

$$\|\mathrm{grad}^F \psi\|_F = f \circ \psi, \tag{4.1}$$

that is ψ is a generalized eikonal function. In this case, a rigging for it is

$$\zeta = (f \circ \psi)\partial_t$$

with associated rigged vector field given by

$$\xi = -\frac{1}{f \circ \psi}\partial_t - \frac{1}{(f\circ\psi)^3}\mathrm{grad}^F \psi. \tag{4.2}$$

In particular $\xi^F = -\dfrac{1}{(f\circ\psi)^3}\mathrm{grad}^F\psi$. The screen structure is given by the level sets of ψ: for each $p = (t_0, x_0) \in M$, $\mathscr{F}_p = \{t_0\} \times \psi^{-1}(t_0)$. Let us compute the second fundamental form and screen shape operator for $\{t_0\} \times_f \psi^{-1}(t_0)$ relative to $\{t_0\} \times_f F$. Let

$$u_\psi = -\frac{1}{(f\circ\psi)^2}\mathrm{grad}^F\psi.$$

As per (4.1), u_ψ is a unit normal vector to $\mathscr{F}_{(t,x)}$ in $\{t\}\times_f F$ and we have

$$\xi = -\frac{1}{f\circ\psi}\partial_t + \frac{1}{(f\circ\psi)}u_\psi = \frac{1}{(f\circ\psi)}\big[-\partial_t + u_\psi\big].$$

Then, for any $X \in \mathfrak{X}(M)$,

$$\begin{aligned} -\overset{\star}{A}_\xi X &= \overline{\nabla}_X \xi = \overline{\nabla}_{-\langle X,\partial_t\rangle\partial_t + X^F}\Big[\frac{1}{f\circ\psi}(-\partial_t + u_\psi)\Big] \\ &= -\langle X,\partial_t\rangle \frac{1}{f\circ\psi}\frac{f'\circ\psi}{f\circ\psi}u_\psi - \frac{1}{f\circ\psi}\frac{f'\circ\psi}{f\circ\psi}X^F + \frac{X^F\cdot(f\circ\psi)}{(f\circ\psi)^2}\partial_t \\ &\quad + \frac{1}{f\circ\psi}\nabla^F_{X^F}u_\psi + (f'\circ\psi)g_F(X^F,u_\psi)\partial_t - \frac{X^F\cdot(f\circ\psi)}{(f\circ\psi)^2}u_\psi. \end{aligned}$$

But the left hand side belongs to the screen structure, which is orthogonal to $\zeta = f\partial_t$, so it holds

$$\frac{X^F\cdot(f\circ\psi)}{(f\circ\psi)^2} = -(f'\circ\psi)g_F(X^F,u_\psi) = -\frac{f'\circ\psi}{(f\circ\psi)^2}\langle\partial_t,X\rangle, \tag{4.3}$$

that is

$$X^F\cdot(f\circ\psi) = -(f'\circ\psi)\langle\partial_t,X\rangle,$$

where in (4.3) we use $\langle\partial_t,X\rangle = \langle X^F,u_\psi\rangle = (f\circ\psi)^2 g_F(X^F,u_\psi)$ due to $\langle X,\xi\rangle = 0$.

Then,

$$\begin{aligned} \overset{\star}{A}_\xi X &= \langle X,\partial_t\rangle\frac{1}{f\circ\psi}\frac{f'\circ\psi}{f\circ\psi}u_\psi + \frac{1}{f\circ\psi}\frac{f'\circ\psi}{f\circ\psi}X^F - \frac{1}{f\circ\psi}\nabla^F_{X^F}u_\psi + \frac{X^F\cdot(f\circ\psi)}{(f\circ\psi)^2}u_\psi \\ &\overset{(4.3)}{=} \frac{f'\circ\psi}{(f\circ\psi)^2}X^F - \frac{1}{f\circ\psi}\nabla^F_{X^F}u_\psi. \end{aligned}$$

Replacing u_ψ leads to

$$\overset{\star}{A}_\xi X = \frac{1}{(f\circ\psi)^3}\nabla^F_{X^F}\mathrm{grad}^F\psi + \frac{f'\circ\psi}{(f\circ\psi)^2}X^F + 2\frac{f'\circ\psi}{(f\circ\psi)^4}\langle X,\partial_t\rangle\mathrm{grad}^F\psi.$$

In particular, for any $X \in \mathscr{S}(\zeta)$,

$$\overset{\star}{A}_\xi X = \frac{1}{(f\circ\psi)^3}\nabla^F_X\mathrm{grad}^F\psi + \frac{f'\circ\psi}{(f\circ\psi)^2}X.$$

Hence, for $X,Y \in \mathfrak{X}(M)$,

$$\begin{aligned} B(X,Y) &= \frac{1}{f\circ\psi}Hess^F_\psi(X^F,Y^F) + (f'\circ\psi)g_F(X^F,Y^F) \\ &\quad +2\langle X,\partial_t\rangle\frac{f'\circ\psi}{(f\circ\psi)^2}d\psi(Y^F). \end{aligned}$$

Then the restriction $B_{|\mathscr{S}(\zeta)\times\mathscr{S}(\zeta)}$ of the second fundamental form reads for all $X,Y \in \mathscr{S}(\zeta)$,

$$B(X,Y) = \frac{1}{f\circ\psi}Hess^F_\psi(X,Y) + (f'\circ\psi)g_F(X,Y).$$

Consider a quasi $\overline{g}-$orthonormal frame field $(\partial_t, f\xi^F, e_2,\dots,e_{n-1})$ on $\overline{M}$ with $(e_i)_{2\le i\le n-1}$ tangent to $\mathscr{F}$. Then, $(f^2\xi^F, fe_2,\dots,fe_{n-1})$ is an orthonormal frame field for g_F. Therefore, the null mean curvature reads

$$\begin{aligned} H &= \sum_{i=2}^{n-1} B(e_i,e_i) \\ &= \frac{1}{f\circ\psi}\sum_{i=2}^{n-1} g_F\Big(\nabla^F_{e_i}\mathrm{grad}^F\psi, e_i\Big) + (f'\circ\psi)\sum_{i=2}^{n-1} g_F(e_i,e_i) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(f\circ\psi)^3}\sum_{i=2}^{n-1} g_F\Big(\nabla^F_{fe_i}\mathrm{grad}^F\psi, fe_i\Big) + \frac{f'\circ\psi}{(f\circ\psi)^2}\sum_{i=2}^{n-1} g_F(fe_i, fe_i) \\
&= \frac{1}{(f\circ\psi)^3}\Bigg[\sum_{i=2}^{n-1} g_F\Big(\nabla^F_{fe_i}\mathrm{grad}^F\psi, fe_i\Big) + g_F\Big(\nabla^F_{f^2\xi^F}\mathrm{grad}^F\psi, f^2\xi^F\Big)\Bigg] \\
&\quad -\frac{1}{(f\circ\psi)^3} g_F\Big(\nabla^F_{f^2\xi^F}\mathrm{grad}^F\psi, f^2\xi^F\Big) + (n-2)\frac{f'\circ\psi}{(f\circ\psi)^2} \\
&= \frac{1}{(f\circ\psi)^3}\Delta^F\psi + (n-2)\frac{f'\circ\psi}{(f\circ\psi)^2} \\
&\quad -\frac{1}{(f\circ\psi)^3} g_F\Big(\nabla^F_{f^2\xi^F}\mathrm{grad}^F\psi, f^2\xi^F\Big).
\end{aligned}
$$

Let us compute the term $g_F\Big(\nabla^F_{(f\circ\psi)^2\xi^F}\mathrm{grad}^F\psi, (f\circ\psi)^2\xi^F\Big)$. Note that from (4.2) $(f\circ\psi)^2\xi^F = -\frac{1}{f\circ\psi}\mathrm{grad}^F\psi$. So,

$$
\begin{aligned}
g_F\Big(\nabla^F_{(f\circ\psi)^2\xi^F}\mathrm{grad}^F\psi, (f\circ\psi)^2\xi^F\Big) &= -(f\circ\psi) g_F\Big(\nabla^F_{\xi^F}\mathrm{grad}^F\psi, \mathrm{grad}^F\psi\Big) \\
&= -\frac{(f\circ\psi)}{2}\xi^F\cdot\Big(\|\mathrm{grad}^F\psi\|_F^2\Big) \\
&= -(f\circ\psi)^2\xi^F\cdot(f\circ\psi) \\
&\overset{(4.3)}{=} (f\circ\psi)^2\frac{f'\circ\psi}{f\circ\psi} = (f'\circ\psi)(f\circ\psi).
\end{aligned}
$$

It follows that

$$
H = \frac{1}{(f\circ\psi)^3}\Big[\Delta^F\psi + (n-3)(f'\circ\psi)(f\circ\psi)\Big]. \tag{4.4}
$$

From (4.4) we can state the following:

Theorem 4.1. *Let $I\times_f F$ be a GRW spacetime and consider $\psi: F\longrightarrow I$ a differentiable function with graph*

$$
M = \{(\psi(x), x), \quad x\in F\} \quad \text{such that} \quad \|grad^F\psi\|_F = f\circ\psi.
$$

(i) M is a null hypersurface which is maximal if and only if

$$
\Delta^F\psi = -(n-3)(f'\circ\psi)(f\circ\psi)
$$

(ii) If M is maximal then $\Delta^F\psi$ is constant on each leaf $\mathscr{F}_p$, $p\in M$.

(iii) The (Riemannian) mean curvature $H^{\mathscr{F}}$ of a leaf $\mathscr{F}_p$ as a hypersurface in the slice $\{\pi_I(p)\}\times F$ is given by

$$
H^{\mathscr{F}} = \frac{1}{(n-2)(f\circ\psi)^2}\Big[\Delta^F\psi - (f'\circ\psi)(f\circ\psi)\Big]. \tag{4.5}
$$

It follows that leaves of the foliation $\mathscr{F}$ has constant mean curvature if and only if $\Delta^F\psi$ is constant leafwise. In particular they are minimal if and only if $\Delta^F\psi_{|\mathscr{F}} = (f'\circ\psi)(f\circ\psi)$.

Proof. The first claim (i) follows from (4.1) and (4.4). For (ii) just recall that ψ is constant on each leaf of the foliation $\mathscr{F}$. Now, for (iii), use (3.5) and (4.4) to derive (4.5). □

5. The extrinsic scalar curvature estimates

In [19] we pointed out that there are no natural ways to induce a *scalar curvature* analogue on null hypersurfaces (or more generally a null submanifolds) as usual. The drawback in considering this concept on null hypersurfaces is twofold: since the induced connection is not a Levi-Civita connection (unless M be totally geodesic) the $(0,2)$ induced Ricci tensor is not symmetric in general. Also, as the induced metric is degenerate, its inverse does not exist and it is not possible to proceed in the usual way by contracting the Ricci tensor to get a scalar quantity. A first attempt in this way was made in [27] which consists to restrict the concept to a very limited class of null hypersurfaces: those admitting a "canonical screen distribution" that induces a canonical transversal vector bundle and a symmetric induced Ricci tensor. Although the above two conditions are interesting to compensate lacking due to the above quoted difficulties, to admit symmetric induced Ricci tensor in lightlike setting is very restrictive. Also, the problem in contracting with respect to the noninvertible induced metric is still unsolved for the general setting. However, when the null hypersurface is provided with a normalization (a rigging), thanks to the nondegenerate associated rigged structure, we can drop above restrictions and construct an analog of this scalar quantity (see [19]) called extrinsic scalar curvature, referring to the use of an extra structure ζ, the rigging.

Our purpose in this section is to determine this scalar quantity for null hypersurfaces in GRW spacetimes, normalized by the Chen's vector field $\zeta = f\partial_t$ and give some bounds for it in case the null hypersurface is maximal.

Let Ric denote the induced Ricci tensor on the normalized (M,ζ). We define the symmetrized $(0,2)-$Ricci tensor Ric^{sym} by

$$Ric^{sym}(X,Y) = \frac{1}{2}\Big[Ric(X,Y) + Ric(Y,X)\Big].$$

By direct computation ([19]), we see that

$$\begin{aligned} Ric^{sym}(X,Y) &= \overline{Ric}(X,Y) + B(X,Y)tr(A_N) \\ &\quad -\frac{1}{2}\Big[\Big\langle \overline{R}(\xi,X)Y,N\Big\rangle + \Big\langle \overline{R}(\xi,Y)X,N\Big\rangle + \langle A_N X, \overset{\star}{A}_\xi Y\rangle + \langle A_N Y, \overset{\star}{A}_\xi X\rangle\Big]. \end{aligned} \tag{5.1}$$

Now, the extrinsic scalar curvature s_ζ on (M,ζ) is the $\widetilde{g}-$trace of Ric^{sym}, where $\widetilde{g}$ is the associated Riemannian rigged structure, i.e

$$s_\zeta = \widetilde{g}^{\alpha\beta} Ric^{sym}_{\alpha\beta} \tag{5.2}$$

being $(e_1 = \xi, e_2, \ldots, e_{n-1})$ a $\widetilde{g}-$ orthonormal frame field on M with $(e_i)_{1\le i\le n-1}$ tangent to the screen structure. We compute the components $Ric^{sym}_{\alpha\beta}$ using (5.1), (3.2) and symmetries in (2.2).

$$\begin{aligned} Ric^{sym}_{00} = Ric^{sym}(\xi,\xi) &= \overline{Ric}(\xi,\xi) = \overline{Ric}\Big(-\frac{1}{f}\partial_t + \xi^F, -\frac{1}{f}\partial_t + \xi^F\Big) \\ &= -(n-1)\frac{f''}{f^3} + \frac{ff'' + (n-2)f'^2}{f^4} + Ric^F\Big(\xi^F,\xi^F\Big) \\ &= \frac{2-n}{f^2}(\ln f)'' + Ric^F\Big(\xi^F,\xi^F\Big). \end{aligned}$$

$$\begin{aligned} Ric^{sym}_{0i} = Ric^{sym}(\xi,e_i) &= \overline{Ric}(\xi,e_i) - \frac{1}{2}\Big\langle \overline{R}(\xi,e_i)\xi,N\Big\rangle \\ &= Ric^F(\xi^F,e_i) \end{aligned}$$

$$\begin{aligned} Ric^{sym}_{ij} = Ric^{sym}(e_i,e_j) &= \overline{Ric}(e_i,e_j) + B(e_i,e_j)tr(A_N) \\ &\quad -\frac{1}{2}\Big(\Big\langle f\partial_t + \frac{1}{2}f^2\xi, \overline{R}(\xi,e_j)e_i\Big\rangle \\ &\quad +\Big\langle f\partial_t + \frac{1}{2}f^2\xi, \overline{R}(\xi,e_i)e_j\Big\rangle\Big) \\ &\quad +\Big\langle A_N e_i, \overset{\star}{A}_\xi e_j\Big\rangle + +\Big\langle A_N e_j, \overset{\star}{A}_\xi e_i\Big\rangle \end{aligned}$$

Then, using the following relations of curvature tensor [28],

$$\begin{aligned} \overline{R}(\partial_t,\partial_t)\partial_t &= \overline{R}(\partial_t,\partial_t)X^F = \overline{R}(X^F,Y^F)\partial_t = 0 \\ \overline{R}(X^F,\partial_t)\partial_t &= -\frac{f''}{f}X^F, \qquad \overline{R}(\partial_t,X^F)Y^F = ff''g_F(X^F,Y^F)\partial_t \\ \overline{R}(X^F,Y^F)Z^F &= R^F((X^F,Y^F)Z^F) + f'^2\Big[g_F(X^F,Z^F)Y^F - g_F(Y^F,Z^F)X^F\Big] \end{aligned}$$

we have

$$\begin{aligned} Ric^{sym}(e_i,e_j) &= \overline{Ric}(e_i,e_j) + B(e_i,e_j)tr(A_N) \\ &\quad -\frac{1}{2}\Big(2ff''g_F(e_i,e_j) + \frac{1}{2}\Big\langle R^F(\xi^F,e_j)e_i,\xi^F\Big\rangle \\ &\quad +\frac{1}{2}\Big\langle R^F(\xi^F,e_i)e_j,\xi^F\Big\rangle - 2\frac{f'^2}{f^2}g_F(e_i,e_j) \\ &\quad +\Big\langle \overset{\star}{A}_\xi A_N e_i, e_j\Big\rangle + \Big\langle \overset{\star}{A}_\xi A_N e_j, e_i\Big\rangle\Big) \end{aligned}$$

Now, relation (iii) in Proposition 3.1 implies

$$\left\langle \overset{\star}{A}_{\xi} A_N e_j, e_i \right\rangle = \frac{1}{2} f^2 \left\langle \overset{\star}{A}_{\xi}{}^2 e_i, e_j \right\rangle - f' B(e_i, e_j)$$

and

$$tr(A_N) = \frac{1}{2} f^2 H - (n-2) f'.$$

Then,

$$\begin{aligned} Ric^{sym}(e_i,e_j) &= Ric^F(e_i,e_j) + \Big[(n-2) + \frac{1}{f^2}\Big] f'^2 g_F(e_i,e_j) \\ &\quad + \Big[\frac{1}{2} f^2 H + (3-n) f'\Big] B(e_i,e_j) \\ &\quad - \frac{1}{2} f^2 \Big[\Big\langle R^F(e_i,\xi^F)\xi^F, e_j \Big\rangle + \Big\langle \overset{\star}{A}_{\xi} e_i, \overset{\star}{A}_{\xi} e_j \Big\rangle\Big] \end{aligned}$$

Also, the components $\widetilde{g}^{ab}$ in the same frame field are the following:

$$\widetilde{g}^{00} = 1, \qquad \widetilde{g}^{0i} = \widetilde{g}^{i0} = 0, \qquad \widetilde{g}^{ij} = \frac{1}{f^2} g_F^{ij}.$$

Finally, after substitution in (5.2) and reducing we get the following expression for the extrinsic scalar curvature s_ζ on the normalized (M,ζ).

Proposition 5.1. *Let $I \times_f F$ be a GRW spacetime and suppose M is a null hypersurface normalized by the Chen's vector field $\zeta = f\partial_t$. Then the extrinsic scalar curvature on (M,ζ) is given by*

$$\begin{aligned} s_\zeta &= \frac{1}{f^4} s_F + \left[\frac{1}{2} f^2 H + (3-n) f'\right] H \\ &\quad - \frac{1}{2}\Big[Ric^F(\xi^F,\xi^F) + f^2 \| \overset{\star}{A}_{\xi} \|^2\Big] \\ &\quad + \frac{2-n}{f^2} (\ln f)'' + \frac{n-2}{f^2}\Big[(n-2) f + 1\Big] (\ln f)'. \end{aligned} \tag{5.3}$$

Theorem 5.2. *Let $I \times_f F$ be a $n-$dimensional GRW spacetime $(n \geq 3)$ and M a maximal null hypersurface normalized with the Chen's vector field $\zeta = f\partial_t$.*

(i) If f is a convex decreasing warping function then s_ζ has the following upper bound

$$s_\zeta \leq \frac{1}{f^4} s_F - \frac{1}{2} Ric^F(\xi^F,\xi^F).$$

(ii) If $i_\zeta W = 0$ (a quasi-Einstein space), then on the set of non totally geodesic points of a maximal null hypersurface, it holds

$$\begin{aligned} s_\zeta &\geq \frac{1}{f^4}\left[1 + \frac{f^2}{2(n-1)}\right] s_F + -\frac{1}{2} Ric^F(\xi^F,\xi^F) \\ &\quad - \frac{n-2}{f^4}\left[\frac{1}{2} f^3 f'' + f f'' - f'^2 - \Big((n-2) f + 1\Big) f f'\right]. \end{aligned} \tag{5.4}$$

Proof. Item (i) is immediate using (5.3). Indeed, we have $H = 0$ (maximality), $-\frac{1}{2} f^2 \| \overset{\star}{A}_{\xi} \|^2 < 0$ and as $n \geq 3$ the hypothesis on the warping function f leads to $\frac{2-n}{f^2}(\ln f)'' + \frac{n-2}{f^2}\Big[(n-2)f + 1\Big](\ln f)' < 0$. For (ii), the hypothesis implies from (3.8) that

$$-\frac{1}{2} f^2 \| \overset{\star}{A}_{\xi} \|^2 \geq \frac{-1}{2(n-1) f^2}\Big[(n-1)(n-2) f f'' - s_F\Big].$$

Then relation (5.4) follows from (5.3). $\square$

6. Willmore null hypersurfaces

6.1. The Willmore functional

For a normalized null hypersurface (M,ζ) with compactly supported (non normalized) mean curvature H (in particular if M is compact), the Willmore action is given by

$$\mathcal{W}(M,\zeta)=\int_M H^2\,dM$$

where dM is the volume element induced on M.

An obvious fact is that

$$H^2\leq (n-2)\|\overset{\star}{A}_\xi\|^2 \tag{6.1}$$

with equality if and only if M is totally umbilic.

Also,

$$\widetilde{\mathrm{div}}\left((\widetilde{\mathrm{div}}\xi)\xi\right)=\widetilde{g}\left(\widetilde{\nabla}\widetilde{\mathrm{div}}\xi,\xi\right)+\left(\widetilde{\mathrm{div}}\xi\right)^2$$

i.e

$$-\widetilde{\mathrm{div}}\left(H\xi\right)=-\xi(H)+H^2. \tag{6.2}$$

In [18] we established (2.6), that is $\overline{Ric}(\xi)=\xi(H)+\tau(\xi)H-\|\overset{\star}{A}_\xi\|^2$. Assume that there is a constant λ such that $(\tau(\xi)+\lambda)H\leq 0$. Then

$$H^2\geq \overline{Ric}(\xi)+\lambda H-\widetilde{\mathrm{div}}\left(H\xi\right). \tag{6.3}$$

Combining (6.1) and (6.3), we get

$$\overline{Ric}(\xi)+\lambda H-\widetilde{\mathrm{div}}\left(H\xi\right)\leq H^2\leq (n-2)\|\overset{\star}{A}_\xi\|^2.$$

The equality case in the upper bound (resp. in the lower bound) is attained if and only if M is totally umbilic (resp. M is totally geodesic). Recall from (2.5) that $H=-\widetilde{\mathrm{div}}\xi$. Hence if M is compact without boundary and orientable, by the divergence theorem it holds

$$\int_M \overline{Ric}(\xi)\,dM\leq \mathcal{W}(M,\zeta)\leq (n-2)\int_M \|\overset{\star}{A}_\xi\|^2\,dM.$$

The equality case in the upper bound is equivalent to $\int_M\left(H^2-(n-2)\|\overset{\star}{A}_\xi\|^2\right)dM=0$ and by use of (6.1) this means that $H^2=(n-2)\|\overset{\star}{A}_\xi\|^2$ and M is totally umbilic. For the lower bound, the equality case reads

$$\int_M\left(\widetilde{\mathrm{div}}(H\xi)+\tau(\xi)H-\|\overset{\star}{A}_\xi\|^2\right)dM=\int_M(\tau(\xi)H-\|\overset{\star}{A}_\xi\|^2)dM=0=\int_M -\lambda H dM.$$

Hence,

$$\int_M\left((\tau(\xi)+\lambda)H-\|\overset{\star}{A}_\xi\|^2\right)dM=0.$$

As $(\tau(\xi)+\lambda)H\leq 0$ it follows that $(\tau(\xi)+\lambda)H-\|\overset{\star}{A}_\xi\|^2=0$ and $\|\overset{\star}{A}_\xi\|^2=0$ which means that M is totally geodesic. Thus, we can state.

Theorem 6.1. *Let (M,ζ) be a normalized orientable compact (without boundary) null hypersurface. Suppose there existe a constant λ such that $(\tau(\xi)+\lambda)H\leq 0$. Then, the Willmore action has the following bounds:*

$$\int_M \overline{Ric}(\xi)\,dM\leq \mathcal{W}(M,\zeta)\leq (n-2)\int_M \|\overset{\star}{A}_\xi\|^2\,dM.$$

The equality case in the upper bound (resp. in the lower bound) is attained if and only if M is totally umbilic (resp. M is totally geodesic).

Remark 6.2. *Suppose M is a normalized orientable compact (without boundary) null hypersurface. If H is constant then it vanishes identically. Indeed, from equality in (6.2), it holds by use of the divergence theorem,*

$$0=\int_M \xi(H)\,dM=\int_M H^2\,dM \quad \text{which shows that} \quad H=0.$$

6.2. The Euler equation in GRW spacetimes

In a recent work [29], we pointed out the fact that in Lorentzian manifolds with a closed timelike vector field, there is no compact simply connected null hypersurfaces. But GRW spacetimes do admit such vector fields, (the Chen's ones). So, considering the Willmore problem for null hypersurfaces in GRW spacetimes, we restrict to the family of normalized orientable null hypersurfaces (M,ζ) for which the mean curvature H has compactly supported variations and for critical points of the Willmore action (the Willmore null hypersurfaces) we apply standard techniques of the calculus of variations.

Let us consider on $\overline{M}$ the frame field $(\partial_t,\xi,\partial_{u^2},\ldots,\partial_{u^{n-1}})$ with $(\partial_{u^i})_{2\leq i\leq n-1}$ tangent to the screen structure $\mathscr{S}(\zeta)$, in which

$$\overline{g}_{\alpha\beta}\simeq\begin{pmatrix} -1 & \frac{1}{f} & 0 & \ldots & 0\\ \frac{1}{f} & 0 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & f^2(t)g_{F_{ij}} & \\ 0 & 0 & & & \end{pmatrix}\quad\text{and}\quad \widetilde{g}_{ab}\simeq\widetilde{G}=\begin{pmatrix} 1 & 0 & \ldots & 0\\ 0 & & & \\ \vdots & & f^2(t)g_{F_{ij}} & \\ 0 & & & \end{pmatrix}.$$

Hence

$$\det\overline{g}_{\alpha\beta}=-\frac{1}{f^2}\det\widetilde{g}_{ab}.$$

Let v be a null coordinate with $\xi=\partial_v$. We have

$$d\overline{g}=\sqrt{-\det\overline{g}_{\alpha\beta}}\,dt\wedge dv\wedge du^2\cdots\wedge du^{n-1}$$

and

$$\begin{aligned} dM=i_\zeta d\overline{g} &= f(t)\sqrt{-\det\overline{g}_{\alpha\beta}}\,dv\wedge du^2\cdots\wedge du^{n-1}\\ &= f(t)\sqrt{\frac{1}{f^2}\det\widetilde{g}_{ab}}\,dv\wedge du^2\cdots\wedge du^{n-1}\\ &= \sqrt{\det\widetilde{g}_{ab}}\,dv\wedge du^2\cdots\wedge du^{n-1}\\ &= d\widetilde{g}.\end{aligned}$$

Hence

$$dM=d\widetilde{g}.$$

It follows that

$$\mathcal{W}(M,\zeta)=\int_M H^2\,dM=\int_M(\widetilde{\operatorname{div}}\,\xi)^2 d\widetilde{g}.$$

Let $p=(t,x)\in I\times_f F$ be a generic point on the null hypersurface. We have $x=x(u^1,\ldots,u^{n-1})$ where $(u^1,\ldots,u^{n-1})$ denotes coordinates on the fiber F. Then $p=p(t,u^1,\ldots,u^{n-1})$. Now consider a variation of the null hypersurface in the normal direction ξ given by

$$\overline{p}(t,u^1,\ldots,u^{n-1},s)=\exp_{(t,u^1,\ldots,u^{n-1})}\left(s\phi(t,u^1,\ldots,u^{n-1})\xi\right)\tag{6.4}$$

where ϕ is a smooth real-valued function and s is real number in a neighborhood of 0. We denote by δ the operator

$$\delta=\frac{\partial}{\partial s}_{|_{s=0}}.$$

Willmore null hypersurfaces are those for which

$$\delta\int_M H^2\,dM=0.$$

The following ranges of indices are in use

$$\begin{aligned}\alpha,\ \beta,\ \gamma,\ldots,&=0,1,\ \ldots,\ n-1\\ a,\ b,\ c\ldots,&=1,\ \ldots,\ n-1\\ i,\ j,\ k\ldots,&=2,\ \ldots,\ n-1.\end{aligned}$$

Without lost of generality, we may assume $\partial_i:=\frac{\partial}{\partial u^i}\in\mathscr{S}(\zeta)$. Then at each p, $\operatorname{span}\{\partial_t|_p\ ,\ \partial u^1|_p\}=T_pM^\perp\oplus\mathbb{R}\zeta$.

From (6.4), we have

$$\delta p = \phi \xi.$$

Then,

$$\begin{aligned}\delta \partial_{i|_p} &= (\partial_i \phi)\xi + \phi \overline{\nabla}_{\partial_i}\xi \\ &= (\partial_i \phi)\xi - \phi \overset{\star}{A}_{\xi}\partial_i.\end{aligned}$$

$$\begin{aligned}\delta \xi_{|_p} &= (\xi\cdot\phi)\xi + \phi \overline{\nabla}_{\xi}\xi \\ &= (\xi\cdot\phi)\xi \quad \text{as } \xi \text{ is geodesic.}\end{aligned}$$

$$\begin{aligned}\delta \partial_{t|_p} &= (\partial_t \phi)\xi + \phi \overline{\nabla}_{\partial_t}\xi \\ &= (\partial_t \phi)\xi + \phi\left(\frac{f'(t)}{f^2(t)}\partial_t + \frac{f'(t)}{f(t)}\xi^F\right). \\ &= \left(-\frac{\partial_t\phi}{f(t)} + \phi\frac{f'(t)}{f^2(t)}\right)\partial_t + \left((\partial_t\phi) + \phi\frac{f'(t)}{f(t)}\right)\xi^F\end{aligned}$$

and

$$\delta \partial_{u^1|_p} = (\partial_{u^1}\phi)\xi + \phi\overline{\nabla}_{\partial_{u^1}}\xi$$

Now, we compute $\delta\widetilde{g}_{ab}$.

$$\begin{aligned}\overline{g}_{ij}(s) &= \overline{g}\Big(\partial_i + s(\partial_i\phi)\xi - s\phi\overset{\star}{A}_{\xi}\partial_i, \partial_j + s(\partial_j\phi)\xi - s\phi\overset{\star}{A}_{\xi}\partial_j\Big) \\ &= \widetilde{g}_{ij} + s\Big[-\phi\widetilde{g}(\partial_i,\overset{\star}{A}_{\xi}\partial_j) - \phi\widetilde{g}(\overset{\star}{A}_{\xi}\partial_i,\partial_j)\Big] + s^2\phi^2\widetilde{g}(\overset{\star}{A}_{\xi}\partial_i,\overset{\star}{A}_{\xi}\partial_j).\end{aligned}$$

Hence

$$\delta\widetilde{g}_{ij} = -2\phi B(\partial_i,\partial_j).$$

Also,

$$\delta\widetilde{g}_{0i} = \delta\widetilde{g}_{i0} = \partial_i\phi \qquad \text{and} \qquad \delta\widetilde{g}_{00} = 2(\xi\cdot\phi).$$

Using the relation $\sum_k \widetilde{g}^{ik}\widetilde{g}_{kj} = \delta^i_j$,

$$\sum_k \delta\widetilde{g}^{ik}\widetilde{g}_{kj} + \sum_k \widetilde{g}^{ik}\delta\widetilde{g}_{kj} = 0.$$

Then,

$$\begin{aligned}\sum_k \delta\widetilde{g}^{ik}\widetilde{g}_{kj} &\overset{(6.5)}{=} -\sum_k \widetilde{g}^{ik}\Big(-2\phi B(\partial_k,\partial_j)\Big) \\ &= 2\phi\sum_l \widetilde{g}^{il}B(\partial_l,\partial_j).\end{aligned}$$

It follows that

$$\delta\widetilde{g}^{ik} = 2\phi\sum_{lj}\widetilde{g}^{kj}\widetilde{g}^{il}B(\partial_l,\partial_j).$$

Next, put $B_{ij} := B(\partial_i,\partial_j)$. We compute δB_{ij}. First, we have

$$\Big\langle \overline{\nabla}_{\partial_i}\partial_j, \delta\xi\Big\rangle = \Big\langle \nabla_{\partial_i}\partial_j + B_{ij}N, (\xi\cdot\phi)\xi\Big\rangle = (\xi\cdot\phi)B_{ij}.$$

Also,

$$\begin{aligned}\overline{\nabla}_{\partial_i(s)}\partial_j(s) &= \overline{\nabla}_{\partial_i+s(\partial_i\phi)\xi-s\phi\overset{\star}{A}_\xi\partial_i}\Big(\partial_j+s(\partial_j\phi)\xi-s\phi\overset{\star}{A}_\xi\partial_j\Big)\\ &= \overline{\nabla}_{\partial_i}\partial_j+s\Big[\overline{\nabla}_i(\partial_j\phi)\xi-\overline{\nabla}_{\partial_i}\Big(\phi\overset{\star}{A}_\xi\partial_j\Big)\Big]+\text{second order term in } s\end{aligned}$$

which gives

$$\begin{aligned}\delta\overline{\nabla}_{\partial_i}\partial_j &= \overline{\nabla}_i(\partial_j\phi)\xi-\overline{\nabla}_{\partial_i}\Big(\phi\overset{\star}{A}_\xi\partial_j\Big)\\ &= (\partial_i\partial_j\phi)\xi+(\partial_j\phi)(-\overset{\star}{A}_\xi\partial_i)-(\partial_i\phi)\overset{\star}{A}_\xi\partial_j-\phi\Big(\nabla_{\partial_i}\overset{\star}{A}_\xi\partial_j\Big)\\ &\quad -\phi B(\partial_i\overset{\star}{A}_\xi\partial_j)N+(\partial\phi)\nabla_\xi\partial_j-\phi\nabla_{\overset{\star}{A}_\xi\partial_i}\partial_j-\phi B(\overset{\star}{A}_\xi\partial_i,\partial_j).\end{aligned}$$

It follows that

$$\Big\langle\delta\overline{\nabla}_{\partial_i}\partial_j,\xi\Big\rangle=-2\phi B(\partial_i\overset{\star}{A}_\xi\partial_j)=-2\phi\widetilde{g}\Big(\overset{\star}{A}_\xi\partial_i,\overset{\star}{A}_\xi\partial_j\Big).$$

From $B_{ij}=\langle\overline{\nabla}_{\partial_i}\partial_j,\xi\rangle$ it follows,

$$\begin{aligned}\delta B_{ij} &= \Big\langle\delta\overline{\nabla}_{\partial_i}\partial_j,\xi\Big\rangle+\Big\langle\overline{\nabla}_{\partial_i}\partial_j,\delta\xi\Big\rangle\\ &= -2\phi\Big\langle\overset{\star}{A}_\xi\partial_i,\overset{\star}{A}_\xi\partial_j\Big\rangle+(\xi\cdot\phi)B_{ij}.\end{aligned}$$

But

$$H=\sum_{ij}\widetilde{g}^{ij}\widetilde{g}\Big(\overset{\star}{A}_\xi\partial_i,\partial_j\Big)=\sum_{ij}\widetilde{g}^{ij}B_{ij}.$$

Then,

$$\begin{aligned}\delta H &= \sum_{ij}\delta\widetilde{g}^{ij}B_{ij}+\sum_{ij}\widetilde{g}^{ij}\delta B_{ij}\\ &= \sum_{ij}\Big(2\phi\sum_{lm}\widetilde{g}^{il}\widetilde{g}^{jm}B_{lm}\Big)B_{ij}+\sum_{ij}\widetilde{g}^{ij}\Big[-2\phi\widetilde{g}(\overset{\star}{A}_\xi\partial_i,\overset{\star}{A}_\xi\partial_j)+(\xi\cdot\phi)B_{ij}.\Big]\\ &= 2\phi\|\overset{\star}{A}_\xi\|^2-2\phi\|\overset{\star}{A}_\xi\|^2+(\xi\cdot\phi)H\end{aligned}$$

since $\overset{\star}{A}_\xi\partial_i=\widetilde{g}^{lm}B_{im}\partial_l$ and the first term at the right hand side is $2\phi\|\overset{\star}{A}_\xi\|^2$. Thus,

$$\delta H=(\xi\cdot\phi)H.$$

Now, let $\Omega=\sqrt{\det\widetilde{g}_{ab}}=\sqrt{\det\widetilde{g}_{ij}}$. Then

$$2\Omega\frac{\partial\Omega}{\partial s}=\Omega^2\operatorname{trace}\Big(\widetilde{G}^{-1}\cdot\frac{\partial\widetilde{G}}{\partial s}\Big),$$

i.e

$$\begin{aligned}\delta\Omega &= \frac{1}{2}\Omega\operatorname{trace}\Big(\widetilde{g}^{ik}\delta\widetilde{g}_{kj}\Big)\overset{(6.5)}{=}\frac{1}{2}\Omega\operatorname{trace}\Big(\widetilde{g}^{ik}\Big(-2\phi B_{kj}\Big)\Big)\\ &= -\phi\Omega\operatorname{trace}\widetilde{g}^{ik}B_{kj}=-\phi\Omega\widetilde{g}^{ij}B_{ij}=-\phi\Omega H.\end{aligned}$$

Finally,

$$\begin{aligned}\delta\int_M H^2 d\widetilde{g} &= \int_M 2H\delta H d\widetilde{g}+\int_M H^2\delta\widetilde{g}\\ &= \int_M 2H(\xi\cdot\phi)Hd\widetilde{g}+\int_M H^2(-\phi H)d\widetilde{g},\end{aligned}$$

i.e

$$\delta\int_M H^2 d\widetilde{g}=\int_M H^2\Big[2(\xi\cdot\phi)-H\phi\Big]d\widetilde{g}.$$

Thus the condition that the integral is stationary for all smooth function ϕ is $H=0$. Indeed, take $\phi=H$. Then

$$0 = \int_M H^2\Big[2(\xi\cdot H)-H^2\Big]d\widetilde{g}=\int_M\Big[2H^2(\xi\cdot H)-H^4\Big]d\widetilde{g}$$

$$\begin{aligned} &= \int_M \left[\frac{2}{3}(\xi \cdot H^3) - H^4\right] d\widetilde{g} = \int_M \left[-\frac{2}{3}H^3 \widetilde{\mathrm{div}}\xi - H^4\right] d\widetilde{g} \\ &= \int_M \left[\frac{2}{3}H^4 - H^4\right] d\widetilde{g} = -\frac{1}{3}\int_M H^4 d\widetilde{g}. \end{aligned}$$

Then we can state the following:

Theorem 6.3. *In a generalized Robertson-Walker spacetime, the only Willmore normalized null hypersurfaces (M,ζ) where ζ is the closed conformal timelike concircular vector field $f\partial_t$ are the maximal ones.*

References

[1] J. Aledo, R. Rubio, J. Salamanca, *Complete spacelike hypersurfaces in generalized Robertson–Walker and the null convergence condition: Calabi–Bernstein problem*, RACSAM Rev. R. Acad. Cienc. Exactas F´ıs. Nat. Ser. A Mat., doi: 10.1007/s13398-016-0277-3 (2016).
[2] L. J. Alias, P. Mira, *On the Calabi-Bernstein theorem for maximal hypersurfaces in the Lorentz-Minkowski space*, Publ. R. Soc. Mat. Esp., **4** (2003), 23–55.
[3] L. J. Alias, A. Romero, M. Sánchez, *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson–Walker spacetimes*, Gen. Relativity Gravitation, **27** (1995), 71–84.
[4] R. Bartnik, *Existence of maximal surfaces in asymptotically flat spacetimes*, Comm. Math. Phys., **94** (1984), 155–175.
[5] D. Brill, F. Flaherty, *Isolated maximal surfaces in spacetime*, Comm. Math. Phys., **50** (1976), 157–165.
[6] S. Y. Cheng, S. T. Yau, *Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces*, Ann. of Math. (2), **104**(3) (1976), 407–419.
[7] Y. Choquet-Bruhat, A. Fischer, J. Marsden, *Isolated Gravitating Systems in General Relativity*, Italian, 1979, 396–456.
[8] Y. Choquet-Bruhat, *Quelques propriétés des sous-variétés maximales d'une variété lorentzienne*, C. R. Math. Acad. Sci. Paris, Serie A, **281** (1975), 577–580.
[9] H. F. de Lima, U. L. Parente, *On the geometry of maximal spacelike hypersurfaces immersed in a generalized Robertson–Walker spacetime*, Ann. Mat. Pura Appl. **192** (2013), 649–663.
[10] C. A. Mantica, L. G. Molinardi, *Generalized Robertson-Walker spacetimes - A survey*, Int. J. Geom. Methods Mod. Phys., **14** (2017), 1730001 (27 pages) doi: 10.1142/S021988781730001X.
[11] C. A. Mantica, L. G. Molinardi, *On the Weyl and Ricci tensors of deneralized Robertson-Walker spacetimes*, J. Math. Phys., **57**, 102502 (2016), doi: 10.1063/1.4965714.
[12] J. E. Marsden, F. J. Tipler, *Maximal hypersurfaces and foliations of constant mean curvature in general relativity*, Phys. Rep., **66**(3) (1980), 109-139.
[13] A. Romero, R. Rubio, J. J. Salamanca, *Uniqueness of complete maximal hypersurfaces in spatially parabolic generalized Robertson-Walker spacetimes*, Classical Quantum Gravity, **30** (2013), 115007.
[14] M. Barros, *Free elasticae and Willmore tori in warped product spaces*, Glasg. Math. J., **40** (1998), 265-272.
[15] M. Barros, *A geometric algorithm to construct new solitons in the O(3) nonlinear sigma model*, Phys. Lett. B, **553**(3-4) (2003), 325-331.
[16] M. Barros, O. J. Garay, A. D. Singer, *Elasticae with constant slant in the complex projective plane and new examples of Willmore tori in five spheres*, Tohoku Math. J. (2), **51**(2) (1999), 177–192.
[17] J. L. Weiner, *On the problem of Chen, Willmore, et al.*, Indiana Univ. Math.l J., **27** (1978), 19–35.
[18] C. Atindogbe, H. Fotsing Tetsing, *Newton transformations on null hypersurfaces*, Commun. Math., **23** (2015), 57–83.
[19] C. Atindogbe, *Scalar curvature on lightlike hypersurfaces*, Appl. Sci., **11** (2009), 9-18.
[20] K. L. Duggal, A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic, 364, Dordrecht, 1996.
[21] K. L. Duggal B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Birkhaüser Verlag AG, Basel, 2010.
[22] G. J. Galloway, *Maximum principles for null hypersurfaces and null splitting theorems*, Annes de l'institut Henri Poincaré, **1**, (2000), 543–567.
[23] M. Gutiérrez, B. Olea, *Induced Riemannian structures on null hypersurfaces*, Math. Nachr., (2015), 1–18, doi: https://doi.org/10.1002/mana.201400355.
[24] M. Gutiérrez, B. Olea, *Totally umbilic null hypersurfaces in generalized Robertson-Walker spaces* Differ. Geom. Appl., **42** (2015), 15–30.
[25] B. Y. Chen, *A simple characterization of generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation, **46**:1833 (2014), doi: 10.1007/s10714-014-1833-9.
[26] A. Fialkow, *Conformals geodesics*, Trans. Amer. Math. Soc., **45**(3) (1939), 443–473.
[27] K. L. Duggal, *On scalar curvature in lightlike geometry*, J. Geom. Phys., **57** (2007), 473–481.
[28] H. K. El-Sayied, S. Shenawy, N. Syed, *On symmetries of generalized Robertson-Walker spacetimes and applications*, J. Dyn. Syst. Geom. Theor., **15**(1) (2017), 51–69.
[29] C. Atindogbe M. Gutierrez, R. Hounnonkpè, *New properties on normalized null hypersurfaces*, Mediterranean J. Math. Appl., **15**:166 (2018), doi: 10.1007/s00009-018-1210-0

Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 86-93
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.718298

On Dual $k-$ Pell Bicomplex Numbers and Some Identities Including Them

Serpil Halıcı[1*] and Şule Çürük[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Pamukkale University, Denizli, Turkey*
[*]*Corresponding author : shalici@pau.edu.tr*

Article Info

Keywords: *Bicomplex numbers, Dual numbers, Pell numbers*
2010 AMS: *11Y55, 11B39, 11B37*
Received: *11 April 2020*
Accepted: *9 June 2020*
Available online: *10 June 2020*

Abstract

In the paper, we have considered the real and dual bicomplex numbers separately. Firstly, we examine the dual numbers and investigate the characteristic properties of them. Then, we give the definition, feature and related concepts about bicomplex numbers. And we define the number of dual $k-$ Pell bicomplex numbers that are not found for the first time in the literature and we examine the norm and conjugate properties of these numbers. We give equations about conjugates and give also some important characteristic of these newly defined numbers, and we write the recursive correlations of these numbers. Using these relations we give some important identities such as Vajda's, Honsberger's and d'Ocagne identities.

1. Introduction

The sequence $k-$ Pell is defined as follows[1]:

$$\left\{P_{k,n}\right\}_{n\geq 0}=\left\{0,1,2,4+k,8+4k,16+12k+k^2,\ \dots,2P_{k,n-1}+kP_{k,n-2},\ \dots\right\}.$$

The elements of this set are satisfied the following relation:

$$P_{k,n}=2P_{k,n-1}+kP_{k,n-2},\ \ k\in Z^{+},\ \ n\geq 2$$

and the initial values are

$$P_{k,0}=0,P_{k,1}=1.$$

In [2], Binet-like formula related to these numbers is given as

$$P_{k,n}=\frac{(1+\sqrt{1+k})^n-(1-\sqrt{1+k})^n}{2\sqrt{1+k}}.$$

The characteristic equation that gives these numbers and the roots of this equation are as follows:

$$x^2-2x-k=0,\ \ \alpha+\beta=2,\alpha\beta=-k,\alpha-\beta=2\sqrt{1+k}.$$

Email addresses and ORCID numbers: **shalici@pau.edu.tr, https://orcid.org/0000-0002-8071-0437 (S. Halici), sule9220@gmail.com, https://orcid.org/0000-0002-4514-6156 (S. Cürük)**

For more details on Pell sequence can be seen the references [3]. Bicomplex numbers is a $4-$ dimensional real vector space and it is defined as follows.

$$BC = \{b = b_1 + b_2 i + b_3 j + b_4 ij : b_1, b_2, b_3, b_4 \in R\}.$$

Hence, any bicomplex number b consists of a real and three imaginary units. It should be noted that the multiplication of bicomplex numbers is similar to multiplication of real quaternions. That is,

$$i^2 = j^2 = -1,\ ij^2 = 1,\ ij = ji = k.$$

It is noted that there are some differences between these two sets of numbers. According to this, we can list these as bicomplex numbers are commutative and they have zero divisors and non-trivial idempotent elements. On the other hand the real quaternions are non-commutative and don't have zero divisors and non-trivial idempotent elements. Also, the commutative property is satisfied for elements of the set BC.
In this work, we first investigate some properties of bicomplex numbers by examining the conjugates and norms. Then, we have introduced a new set of bicomplex numbers with coefficient from Pell number sequence, and gave some fundamental properties of this new set. Also, we gave some generalized identities such as Catalan's identity, d'Ocagne's identity, Honsberger formula, that the elements of this set provided. Working the mathematical structure of quantum mechanics on the bicomplex number field, there are many studies in this topic(see, [4]-[10]). $n-th$, $k-$ Pell bicomplex number $BP_{k,n}$ is as follows:

$$BP_{k,n} = P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + ijP_{k,n+3}.$$

That is the $k-$ Pell bicomplex number sequence is

$$\{BP_{k,n}\}_{n\geq 0} = BP_{k,n} : BP_{k,n} = 2BP_{k,n-1} + kBP_{k,n-2},\ k \in Z^+,\ n \geq 2.$$

Here $BP_{k,0} = i + 2j + (4+k)ij$ and $BP_{k,1} = 1 + 2i + (4+k)j + (8+4k)ij$.
Follows from that we have

$$BP_{k,2} = 2BP_{k,1} + kBP_{k,0} = 2 + (4+k)i + (8+4k)j + (16+12k+k^2)ij.$$

So, we can write

$$BP_{k,n+1} = 2BP_BP_{k,n} + kBP_BP_{k,n-1}$$

which is a useful equation.

2. Dual $k-$ Pell bicomplex numbers

As known that the dual numbers are binary members or a member of the 2 parameter families of the complex numbers system, called generalized complex numbers. Then, any dual number can be written as $z = x + \varepsilon y$, where $(x,y) \in R^2$ and ε is a nilpotent number, also $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. Then, the dual numbers set is

$$D = R[\varepsilon] = \{z = x + \varepsilon y : (x,y) \in R^2,\ \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Now, for the numbers $k \in Z^+$, we define dual $k-$ Pell number as follows:

$$\widehat{P_{k,n}} = P_{k,n} + \varepsilon P_{k,n+1}.$$

Hence, we can define any element of the dual bicomplex sequence $\{\widehat{BP_{k,n}}\}_{n\geq 0}$ as

$$\widehat{BP_{k,n}} = BP_{k,n} + \varepsilon BP_{k,n+1}.$$

Here $BP_{k,n}$ is the $n-th$, $k-$ Pell bicomplex number.

Theorem 2.1. *The elements of dual bicomplex sequence $\{\widehat{BP_{k,n}}\}_{n\geq 0}$ are satisfied the following relation:*

$$\widehat{BP_{k,n}} = 2\widehat{BP_{k,n-1}} + k\widehat{BP_{k,n-2}},\ n \geq 1.$$

Where the initial values $\widehat{BP_{k,0}}$ and $\widehat{BP_{k,1}}$ are follows.

$$\widehat{BP_{k,0}} = BP_{k,0} + \varepsilon BP_{k,1} \text{ and } \widehat{BP_{k,1}} = BP_{k,1} + \varepsilon BP_{k,2},$$

respectively.

Proof. For $n = 2$, we have

$$\widehat{BP_{k,2}} = 2\widehat{BP_{k,1}} + k\widehat{BP_{k,0}}.$$

From the following the fact we get

$$\widehat{BP_{k,1}} = BP_{k,1} + \varepsilon BP_{k,2} \ \textit{and} \ k\widehat{BP_{k,0}} = kBP_{k,0} + k\varepsilon BP_{k,1}.$$

So, we write

$$2\widehat{BP_{k,1}} + k\widehat{BP_{k,0}} = (2BP_{k,1} + kBP_{k,0}) + \varepsilon(2BP_{k,2} + kBP_{k,1}).$$

Follows from that, we have this:

$$2(BP_{k,1} + \varepsilon BP_{k,2}) + k(BP_{k,0} + \varepsilon BP_{k,1}) = 2\widehat{BP_{k,1}} + k\widehat{BP_{k,0}} = \widehat{BP_{k,2}}.$$

In here the initial values are

$$\widehat{BP_{k,0}} = (i + 2j + (4+k)ij) + \varepsilon(1 + 2i + (4+k)j + (8+4k)ij)$$

and

$$\widehat{BP_{k,1}} = 1 + 2i + (4+k)j + (8+4k)ij + \varepsilon(2 + (4+k)i + (8+4k)j + (16+12k+k^2)ij).$$

Furthermore, we can also write the number $\widehat{BP_{k,n}}$ differently as follows:

$$\widehat{BP_{k,n}} = (P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + ijP_{k,n+3}) + \varepsilon(P_{k,n+1} + iP_{k,n+2} + jP_{k,n+3} + ijP_{k,n+4}).$$

Then, we get

$$\widehat{BP_{k,n}} = \widehat{P_{k,n}} + i\widehat{P_{k,n+1}} + j\widehat{P_{k,n+2}} + ij\widehat{P_{k,n+3}}$$

where $\widehat{P_{k,n}}$ is the $n-th$ dual $k-$ Pell number. $\square$

Since, usually the absolute values and arguments of bicomplex numbers are defined for each conjugation it is important to consider the conjugates of these numbers. Since there are four different units in this set, it means that four separate conjugates will be defined. According to this, for the bicomplex number $\widehat{BP_{k,n}}$, we can define four different conjugates as follows:

$$\widehat{BP_{k,n}} = \widehat{P_{k,n}} - i\widehat{P_{k,n+1}} - j\widehat{P_{k,n+2}} - ij\widehat{P_{k,n+3}},$$

$$\widehat{BP_{k,n}}^{i} = \widehat{P_{k,n}} - i\widehat{P_{k,n+1}} + j\widehat{P_{k,n+2}} - ij\widehat{P_{k,n+3}},$$

$$\widehat{BP_{k,n}}^{j} = \widehat{P_{k,n}} + i\widehat{P_{k,n+1}} - j\widehat{P_{k,n+2}} - ij\widehat{P_{k,n+3}},$$

$$\widehat{BP_{k,n}}^{ij} = \widehat{P_{k,n}} - i\widehat{P_{k,n+1}} - j\widehat{P_{k,n+2}} + ij\widehat{P_{k,n+3}}.$$

Using this definition, we can give equations provided by conjugates. So, the following theorem is about them.

Theorem 2.2. *For the numbers $\widehat{BP_{k,n}}$ the following equalities are satisfied:*

$$\widehat{BP_{k,n}} + \bar{\widehat{BP_{k,n}}} = 2\widehat{P_{k,n}}.$$

$$\widehat{BP_{k,n}} + \widehat{BP_{k,n}}^{i} = 2(\widehat{P_{k,n}} + j\widehat{P_{k,n+2}}).$$

$$\widehat{BP_{k,n}} + \widehat{BP_{k,n}}^{j} = 2(\widehat{P_{k,n}} + i\widehat{P_{k,n+1}}).$$

$$\widehat{BP_{k,n}} + \widehat{BP_{k,n}}^{ij} = 2(\widehat{P_{k,n}} + ij\widehat{P_{k,n+3}}).$$

$$\widehat{BP_{k,n}}^{i} + \widehat{BP_{k,n}}^{j} = 2(\widehat{P_{k,n}} - ij\widehat{P_{k,n+3}}).$$

$$\widehat{BP_{k,n}}^{i} + \widehat{BP_{k,n}}^{ij} = 2(\widehat{P_{k,n}} - i\widehat{P_{k,n+1}}).$$

$$\widehat{BP_{k,n}}^{j} + \widehat{BP_{k,n}}^{ij} = 2(\widehat{P_{k,n}} - j\widehat{P_{k,n+2}}).$$

$$\widehat{BP_{k,n}} + \widehat{BP_{k,n}}^{i} + \widehat{BP_{k,n}}^{j} + \widehat{BP_{k,n}}^{ij} = 4\widehat{P_{k,n}}.$$

$$\widehat{BP_{k,n}} - \widehat{BP_{k,n}}^{i} = -2j\widehat{P_{k,n+2}}.$$

$$\bar{\widehat{BP_{k,n}}} - \widehat{BP_{k,n}}^{ij} = -2k\widehat{P_{k,n+3}}.$$

$$\widehat{BP_{k,n}}^{i} - \widehat{BP_{k,n}}^{j} = -2(i\widehat{P_{k,n+1}} - j\widehat{P_{k,n+2}}).$$

$$\widehat{BP_{k,n}}^{i} - \widehat{BP_{k,n}}^{ij} = 2j(\widehat{P_{k,n+2}} - i\widehat{P_{k,n+3}}).$$

Proof. From the definition and properties of dual numbers the proofs can be easily seen. □

Now, using the definitions norm and conjugate we also give the following theorem.

Theorem 2.3. *For the numbers $\widehat{BP_{k,n}}$ the following equalities are satisfied:*

$$\textbf{i)}\ Nr(\widehat{BP_{k,n}}) = \widehat{P_{k,n}}^2 + \widehat{P_{k,n+1}}^2 + \widehat{P_{k,n+2}}^2 - \widehat{P_{k,n+3}}^2 + 2(i\widehat{P_{k,n+2}}\widehat{P_{k,n+3}} + j\widehat{P_{k,n+1}}\widehat{P_{k,n+3}} - ij\widehat{P_{k,n+1}}\widehat{P_{k,n+2}}). \tag{2.1}$$

$$\textbf{ii)}\ Nr(\widehat{BP_{k,n}})^i = \widehat{P_{k,n}}^2 + \widehat{P_{k,n+1}}^2 - \widehat{P_{k,n+2}}^2 - \widehat{P_{k,n+3}}^2 + 2j(\widehat{P_{k,n}}\widehat{P_{k,n+2}} - \widehat{P_{k,n+1}}\widehat{P_{k,n+3}}).$$

$$\textbf{iii)}\ Nr(\widehat{BP_{k,n}})^j = \widehat{P_{k,n}}^2 - \widehat{P_{k,n+1}}^2 + \widehat{P_{k,n+2}}^2 - \widehat{P_{k,n+3}}^2 + 2i(\widehat{P_{k,n}}\widehat{P_{k,n+1}} - \widehat{P_{k,n+2}}\widehat{P_{k,n+3}}).$$

$$\textbf{vi)}\ Nr(\widehat{BP_{k,n}})^{ij} = \widehat{P_{k,n}}^2 + \widehat{P_{k,n+1}}^2 + \widehat{P_{k,n+2}}^2 - \widehat{P_{k,n+3}}^2 + 2ij(\widehat{P_{k,n}}\widehat{P_{k,n+3}} - \widehat{P_{k,n+1}}\widehat{P_{k,n+2}}).$$

Proof. As per the definition of norm, we write

$$\textbf{i)}\ Nr(\widehat{BP_{k,n}}) = (\widehat{BP_{k,n}})(\overline{\widehat{BP_{k,n}}}) = (\widehat{P_{k,n}} + i\widehat{P_{k,n+1}} + j\widehat{P_{k,n+2}} + ij\widehat{P_{k,n+3}})(\widehat{P_{k,n}} - i\widehat{P_{k,n+1}} - j\widehat{P_{k,n+2}} - ij\widehat{P_{k,n+3}}).$$

After some calculations, we get

$$Nr(\widehat{BP_{k,n}}) = \widehat{P_{k,n}}^2 + \widehat{P_{k,n+1}}^2 + \widehat{P_{k,n+2}}^2 - \widehat{P_{k,n+3}}^2 + 2(i\widehat{P_{k,n+2}}\widehat{P_{k,n+3}} + j\widehat{P_{k,n+1}}\widehat{P_{k,n+3}} - ij\widehat{P_{k,n+1}}\widehat{P_{k,n+2}}).$$

In other cases, proof can be made in a similar way. □

Note here that the dual $k-$ Pell bicomplex numbers with the negative indices can be given.

Corollary 2.4. *Negative dual $k-$ Pell bicomplex numbers $\widehat{BP_{k,-n}}$ are given as*

$$(-1)^{n-1}\left\{P_{k,n} - iP_{k,n-1} + jP_{k,n-2} - ijP_{k,n-3} + \varepsilon(-P_{k,n-1} + iP_{k,n-2} - jP_{k,n-3} + ijP_{k,n-4})\right\}.$$

Proof. From the equalities $P_{-n} = (-1)^{n-1}P_n$ and $P_0 = 0$, we get

$$P_{k,-n} = (-1)^{n-1}P_{k,n}$$

and writing negative of its instead of n in the equation

$$BP_{k,n} = P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + ijP_{k,n+3},$$

$$BP_{k,-n} = (-1)^{n-1}P_{k,n} + i(-1)^{n}P_{k,n-1} + j(-1)^{n+1}P_{k,n-2} + ij(-1)^{n+2}P_{k,n-3}$$

can be written. It follows from that

$$BP_{k,-n} = (-1)^{n-1}(P_{k,n} - iP_{k,n-1} + jP_{k,n-2} - ijP_{k,n-3}).$$

On the other hand for dual of these numbers, by the aid of the equality

$$\widehat{BP_{k,n}} = BP_{k,n} + \varepsilon BP_{k,n+1}$$

we have

$$\widehat{BP_{k,-n}} = BP_{k,-n} + \varepsilon BP_{k,-n+1}.$$

Hence, the term $\widehat{BP_{k,-n}}$ is as follows:

$$(-1)^{n-1}(P_{k,n} - iP_{k,n-1} + jP_{k,n-2} - ijP_{k,n-3}) + \varepsilon(-1)^{n}(P_{k,n-1} - iP_{k,n-2} + jP_{k,n-3} - ijP_{k,n-4}).$$

That is we have

$$\widehat{BP_{k,-n}} = (-1)^{n-1}\left\{P_{k,n} - iP_{k,n-1} + jP_{k,n-2} - ijP_{k,n-3} + \varepsilon(-P_{k,n-1} + iP_{k,n-2} - jP_{k,n-3} + ijP_{k,n-4})\right\}$$

which is desired result. □

Generating functions and their properties are a powerful tool for solving recurrences and combinatorial problems. Generally, a generating function is a series of formal power containing the information of the inputs of a given sequence in its coefficients. There are various generating functions according to usage and application areas.

In the following theorem, the generating function will be given for the dual $k-$ Pell bicomplex sequence.

Theorem 2.5. *The function that generates the elements of the sequence* $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$ *is*

$$G(t)=\frac{\widehat{BP_{k,0}}+\left(\widehat{BP_{k,1}}-2\widehat{BP_{k,0}}\right)t}{1-2t-kt^2}.$$

Here $\widehat{BP_{k,0}}$ *and* $\widehat{BP_{k,1}}$ *are*

$$i+2j+(4+k)ij+(1+2i+(4+k)j+(8+4k)ij)$$

and

$$1+2i+(4+k)j+(8+4k)ij+\varepsilon(2+(4+k)i+(8+4k)j+(16+12k+k^2)ij).$$

Proof. The generating function of $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$ is as follows:

$$\begin{aligned}
&g_{\widehat{BP_{k,n}}t==G(t)=\sum\limits_{n=0}^{\infty}\widehat{BP_{k,n}}t^n},\\
&G(t)=\widehat{BP_{k,0}}+\widehat{BP_{k,1}}t+\widehat{BP_{k,2}}t^2+\ldots+\widehat{BP_{k,n}}t^n+\ldots\ ,\\
&-2tG(t)=-2(\widehat{BP_{k,0}}t+\widehat{BP_{k,1}}t^2+\widehat{BP_{k,2}}t^3+\ldots+\widehat{BP_{k,n}}t^n+\widehat{BP_{k,n+1}}t^{n+1}\ldots),\\
&-kt^2G(t)=-k(\widehat{BP_{k,0}}t^2+\widehat{BP_{k,1}}t^3+\widehat{BP_{k,2}}t^4+\ldots+\widehat{BP_{k,n}}t^{n+1}+\widehat{BP_{k,n}}t^{n+2}\ldots).
\end{aligned}$$

Using above equations, we write the following formula:

$$(1-2t-kt^2)G(t)=\widehat{BP_{k,0}}+(\widehat{BP_{k,1}}-2\widehat{BP_{k,0}})t.$$

Then, it follows that

$$G(t)=\frac{\widehat{BP_{k,0}}+\left(\widehat{BP_{k,1}}-2\widehat{BP_{k,0}}\right)t}{1-2t-kt^2}$$

that the desired generating function. Here $\widehat{BP_{k,0}}$ and $\widehat{BP_{k,1}}$ are

$$\widehat{BP_{k,0}}=i+2j+(4+k)ij+\varepsilon(1+2i+(4+k)j+(8+4k)ij)$$

and

$$\widehat{BP_{k,1}}=1+2i+(4+k)j+(8+4k)ij+\varepsilon(2+(4+k)i+(8+4k)j+(16+12k+k^2)ij)$$

respectively. □

Theorem 2.6. *Elements of the sequence* $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$ *satisfy in the following formula:*

$$\widehat{BP_{k,n}}=\frac{\underline{\alpha}\alpha^n-\underline{\beta}\beta^n}{\alpha-\beta}$$

where

$$\underline{\alpha}=1+i\alpha+j\alpha^2+ij\alpha^3+\varepsilon(\alpha+i\alpha^2+j\alpha^3+ij\alpha^4)$$

and

$$\underline{\beta}=-\left\{(1+i\beta+j\beta^2+ij\beta^3)+\varepsilon(\beta+i\beta^2+j\beta^3+ij\beta^4)\right\}.$$

Proof. The general solution of the characteristic equation of the sequence $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$ is

$$\widehat{BP_{k,n}}=A\alpha^n+B\beta^n.$$

The initial conditions $\widehat{BP_{k,0}}$ and $\widehat{BP_{k,1}}$ yields the following equations

$$\widehat{BP_{k,0}} = A + B$$

and

$$\widehat{BP_{k,1}} = A\alpha + B\beta$$

respectively. Solving these equations, we get

$$A = \frac{\widehat{BP_{k,1}} - \beta\widehat{BP_{k,0}}}{\alpha - \beta},\ B = \frac{\alpha\widehat{BP_{k,0}} - \widehat{BP_{k,1}}}{\alpha - \beta}.$$

So, we have the following formula for the sequence $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$:

$$\widehat{BP_{k,n}} = \frac{1}{\alpha - \beta}\left\{(\widehat{BP_{k,1}} - \beta\widehat{BP_{k,0}})\alpha^n + (\alpha\widehat{BP_{k,0}} - \widehat{BP_{k,1}})\beta^n\right\} = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta}.$$

□

We also note that the remarkable fact the last formula can be rewritten as follows:

$$\widehat{BP_{k,n}} = \widehat{BP_{k,1}}P_{k,n} + k\widehat{BP_{k,0}}P_{k,n-1}$$

or

$$\widehat{BP_{k,n}} = BP_{k,1}\widehat{P_{k,n}} + kBP_{k,0}\widehat{P_{k,n-1}}.$$

Here, the values $P_{k,n}$, $P_{k,n-1}$, $BP_{k,1}$ and $BP_{k,0}$ are known.

The relation given in the theorem above theorem is known as the Binet formula. Many identities related to all Fibonacci-like integer sequences are obtained with the help of this formula.

Theorem 2.7. *The Cassini's identity for the sequence* $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$ *is follows:*

$$\widehat{BP_{k,n-1}}\widehat{BP_{k,n+1}} - \widehat{BP_{k,n}}^2 = (-1)^n\underline{\alpha\beta}k^{n-1} \tag{2.2}$$

where $\underline{\alpha\beta}$ *is equal to this:*

$$(1+k)\left\{k^2 - 1 - 2i(1-k) - 2j(k+2) + 8ij\right\} - 2\varepsilon\left\{(1+k) - k^2(1+k) + 2i(1-k^2) + 2j(k^2+3k+2) + 4ij(2+k)\right\}.$$

Proof. If we use the Binet formula for proof, then we get

$$\widehat{BP_{k,n-1}}\widehat{BP_{k,n+1}} - \widehat{BP_{k,n}}^2 = \frac{1}{4(1+k)}\left\{(\underline{\alpha}\alpha^{n-1} - \underline{\beta}\beta^{n-1})(\underline{\alpha}\alpha^{n+1} - \underline{\beta}\beta^{n+1}) - (\underline{\alpha}\alpha^n - \underline{\beta}\beta^n)^2\right\}.$$

$$\widehat{BP_{k,n-1}}\widehat{BP_{k,n+1}} - \widehat{BP_{k,n}}^2 = \frac{1}{4(1+k)}(\underline{\alpha}^2\alpha^{2n} - \underline{\alpha\beta}\alpha^{n-1}\beta^{n+1} - \underline{\alpha\beta}\beta^{n-1}\alpha^{n+1} + \underline{\beta}^2\beta^{2n} - \underline{\alpha}^2\alpha^{2n} + 2\underline{\alpha\beta}\alpha^n\beta^n - \underline{\beta}^2\beta^{2n}).$$

If the required simplifications are made, then

$$\widehat{BP_{k,n-1}}\widehat{BP_{k,n+1}} - \widehat{BP_{k,n}}^2 = (-1)^n\underline{\alpha\beta}k^{n-1}$$

is obtained. Thus, the proof is completed. □

Theorem 2.8. *The Catalan's identity for the sequence* $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$ *is*

$$\widehat{BP_{k,n+m}}\widehat{BP_{k,n-m}} - \widehat{BP_{k,n}}^2 = \frac{(-1)^{n-m+1}k^{n-m}}{4(1-k)}\underline{\alpha\beta}\left\{\alpha^{2m} + \beta^{2m} - 2(-k)^m\right\}.$$

Proof. From the Binet formula, we get the following equation.

$$\widehat{BP_{k,n+m}}\widehat{BP_{k,n-m}}-\widehat{BP_{k,n}}^2=\frac{1}{4(1+k)}\left\{(\underline{\alpha}\alpha^{n+m}-\underline{\beta}\beta^{n+m})(\underline{\alpha}\alpha^{n-m}-\underline{\beta}\beta^{n-m})-(\underline{\alpha}\alpha^{n}-\underline{\beta}\beta^{n})^2\right\}.$$

If necessary corrections are made,

$$\widehat{BP_{k,n+m}}\widehat{BP_{k,n-m}}-\widehat{BP_{k,n}}^2=\frac{(-1)^{n-m+1}k^{n-m}}{4(1-k)}\underline{\alpha\beta}\left\{\alpha^{2m}+\beta^{2m}-2(-k)^m\right\}.$$

is obtained. Thus, the proof is completed.
Specially, in the Catalan identity, if we take $m=1$ then we get the Cassini's identity. □

Theorem 2.9. *The Honsberger's identity for the sequence* $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$ *is*

$$\widehat{BP_{k,m-1}}\widehat{BP_{k,n}}+\widehat{BP_{k,m}}\widehat{BP_{k,n+1}}=\frac{1}{4(1+k)}\left\{\underline{\alpha}^2\alpha^{n+m-1}(1+\alpha^2)+\underline{\beta}^2\beta^{n+m-1}(1+\beta^2)-\underline{\alpha\beta}(-k)^{m-1}\alpha^{n-m+1}+\beta^{n-m+1})(1-k)\right\}.$$

Here,

$$\underline{\alpha}^2=(1-\alpha^2)(1-\alpha^4)+2i\alpha(1-\alpha^4)+2j\alpha^2(1-\alpha^2)+4ij\alpha^3+2\alpha\varepsilon\left\{(1-\alpha^2)(1-\alpha^4)+2i\alpha(1-\alpha^4)+2j\alpha^2(1-\alpha^2)+4ij\alpha^3\right\}.$$

$$\underline{\beta}^2=(1-\beta^2)(1-\beta^4)+2i\beta(1-\beta^4)+2j\beta^2(1-\beta^2)+4ij\beta^3+2\beta\varepsilon\left\{(1-\beta^2)(1-\beta^4)+2i\beta(1-\beta^4)+2j\beta^2(1-\beta^2)+4ij\beta^3\right\}.$$

Proof. Let us use the Binet formula. Then

$$\widehat{BP_{k,m-1}}\widehat{BP_{k,n}}+\widehat{BP_{k,m}}\widehat{BP_{k,n+1}}=\frac{1}{4(1+k)}\left\{(\underline{\alpha}\alpha^{m-1}-\underline{\beta}\beta^{m-1})(\underline{\alpha}\alpha^{n}-\underline{\beta}\beta^{n})+(\underline{\alpha}\alpha^{m}-\underline{\beta}\beta^{m})(\underline{\alpha}\alpha^{n+1}-\underline{\beta}\beta^{n+1})\right\}.$$

$$=\frac{1}{4(1+k)}(\underline{\alpha}^2\alpha^{n+m-1}-\underline{\alpha\beta}\alpha^{m-1}\beta^{n}-\underline{\alpha\beta}\alpha^{n}\beta^{m-1}+\underline{\beta}^2\beta^{n+m-1}+\underline{\alpha}^2\alpha^{n+m+1}-\underline{\alpha\beta}\alpha^{m}\beta^{n+1}-\underline{\alpha\beta}\beta^{m}\alpha^{n+1}+\underline{\beta}^2\beta^{n+m+1})$$

can be written. When the necessary actions are performed, we get the following equation.

$$=\frac{1}{4(1+k)}\left\{(\underline{\alpha}^2\alpha^{n+m-1}(1+\alpha^2)+\underline{\beta}^2\beta^{n+m-1}(1+\beta^2)-\underline{\alpha\beta}(-k)^{m-1}(\alpha^{n-m+1}+\beta^{n-m+1})(1-k)\right\}.$$

Thus, the proof is completed. □

Theorem 2.10. *The d'Ocagne identity for the sequence* $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$ *is*

$$\widehat{BP_{k,m}}\widehat{BP_{k,n+1}}-\widehat{BP_{k,n}}\widehat{BP_{k,m+1}}=\frac{-\underline{\alpha}\ \underline{\beta}}{4(1+k)}\left\{(-k)^m(\beta^{n-m+1}+\alpha^{n-m+1})-(-k)^n(\beta^{m-n+1}+\alpha^{m-n+1})\right\}.$$

Proof. Binet formula can be used to prove the proof. So,

$$\widehat{BP_{k,m}}\widehat{BP_{k,n+1}}-\widehat{BP_{k,n}}\widehat{BP_{k,m+1}}$$

is equal to this:

$$\frac{1}{4(1+k)}\left\{(\underline{\alpha}\alpha^{m}-\underline{\beta}\beta^{m})(\underline{\alpha}\alpha^{n+1}-\underline{\beta}\beta^{n+1})-(\underline{\alpha}\alpha^{n}-\underline{\beta}\beta^{n})(\underline{\alpha}\alpha^{m+1}-\underline{\beta}\beta^{m+1})\right\}.$$

When the necessary arrangements are performed, we get

$$\widehat{BP_{k,m}}\widehat{BP_{k,n+1}}-\widehat{BP_{k,n}}\widehat{BP_{k,m+1}}=\frac{-\underline{\alpha\beta}}{4(1+k)}\left\{\alpha^m\beta^{n+1}+\beta^m\alpha^{n+1}-\alpha^n\beta^{m+1}-\beta^n\alpha^{m+1}\right\}.$$

Thus, we get the desired result. That is,

$$\widehat{BP_{k,m}}\widehat{BP_{k,n+1}}-\widehat{BP_{k,n}}\widehat{BP_{k,m+1}}=\frac{-\underline{\alpha\beta}}{4(1+k)}\left\{(-k)^m(\beta^{n-m+1}+\alpha^{n-m+1})-(-k)^n(\beta^{m-n+1}+\alpha^{m-n+1})\right\}.$$

□

Theorem 2.11. *For the positive integers n, i, j, Vajda's identity related with the sequence $\left\{\widehat{BP_{k,n}}\right\}_{n\geq 0}$ is follows:*

$$\widehat{BP_{k,n+i}}\widehat{BP_{k,n+j}}-\widehat{BP_{k,n}}\widehat{BP_{k,n+i+j}}=\frac{(-1)^{n+1}k^{n}}{4(1+k)}\left\{\underline{\alpha}\underline{\beta}(\beta^{j}-\alpha^{j})(\alpha^{i}-\beta^{i})\right\}.$$

Proof. Let us use the Binet formula for the proof. The desired this value, that is

$$\widehat{BP_{k,n+i}}\widehat{BP_{k,n+j}}-\widehat{BP_{k,n}}\widehat{BP_{k,n+i+j}}$$

is follows:

$$\frac{1}{4(1+k)}\left\{(\alpha^{n+i}-\beta^{n+i})(\alpha^{n+j}-\beta^{n+j})-(\alpha^{n}-\beta^{n})(\alpha^{n+i+j}-\beta^{n+i+j})\right\}.$$

When the necessary algebraic operations are performed, we get

$$\frac{-\underline{\alpha}\underline{\beta}}{4(1+k)}\left\{\alpha^{n+i}\beta^{n+j}+\beta^{n+i}\alpha^{n+j}-\alpha^{n}\beta^{n+i+j}-\beta^{n}\alpha^{n+i+j}\right\}.$$

From here, we get

$$\widehat{BP_{k,n+i}}\widehat{BP_{k,n+j}}-\widehat{BP_{k,n}}\widehat{BP_{k,n+i+j}}=\frac{(-1)^{n+1}k^{n}}{4(1+k)}\left\{\underline{\alpha}\underline{\beta}(\beta^{j}-\alpha^{j})(\alpha^{i}-\beta^{i})\right\}.$$

which is desired. □

3. Conclusion

In this study, we examine the dual numbers and give them the characteristics of these numbers. Also, we give the definition and related concepts about bicomplex numbers. Moreover, we define the number of dual $k-$ Pell bicomplex numbers which are not found for the first time in the literature and we examined the norm and conjugate properties of these numbers. We have given some important characteristics of these newly defined numbers, and we have obtain the recursive relations of these numbers. Using obtained relations one can investigate the other important identities.

Acknowledgement

We would like to thank the referees who reviewed this work and submitted their positive reviews.

References

[1] P. Catarino, *On some identities and generating functions for k− Pell numbers*, Int. J. of Math. Anal., **7**(38) (2013), 1877-1884.
[2] P. Catarino, *Bicomplex k− Pell quaternions*, Comput. Methods Funct. Theory, **19**(1) (2019), 65-76.
[3] S. Halici, On Some Pell Polynomials. Acta Uni. Apul., **29**(2012), 105-112.
[4] D. Alpay, M. E. Luna-Elizarrarás, M. Shapiro, D. C. Struppa, *Basics of Functional Analysis with Bicomplex Scalars, and Bicomplex Schur analysis*. Springer Sci. and Business Media,(2014).
[5] F. Babadag, *Fibonacci, Lucas numbers with dual bicomplex numbers*, J. Math. Sci., **10**(1-2) (2018), 161-172.
[6] A. T. Benjamin, S. S. Plott, J. A. Sellers, *Tiling proofs of recent sum identities involving Pell numbers*, Ann. Comb., **12**(3) (2008), 271-278.
[7] M. A. Gungor, A. Cihan, *On dual-hyperbolic numbers with generalized Fibonacci and Lucas numbers components*, Fundam. J. Math. Appl., **2**(2) (2019), 162-172, doi: 10.33401/fujma.617415.
[8] S. Halici, *On Bicomplex Fibonacci Numbers and Their Generalization*, In Models and Theories in Social Systems (pp. 509-524), Springer, Cham, (2019).
[9] S. Ö. Karakuş, F. K. Aksoyak, *Generalized bicomplex numbers and Lie groups*, Adv. Appl. Clifford Alg., **25**(4) (2015), 943-963.
[10] M. E. Luna-Elizarraras, M. Shapiro, D. C. Struppa, A. Vajiac, *Bicomplex numbers and their elementary functions*, Cubo(Temuco), **14**(2) (2012), 61-80.