UJMA

Universal Journal of Mathematics and Applications

VOLUME III
ISSUE II

ISSN 2619-9653

http://dergipark.gov.tr/ujma

VOLUME III ISSUE II
ISSN 2619-9653

June 2020
http://dergipark.gov.tr/ujma

UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

Editors in Chief

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Editorial Board of Universal Journal of Mathematics and Applications

Contents

Universal Journal of Mathematics and Applications, 3 (2) (2020) 44-52
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.644837

Multi-Parametric Families of Solutions of Order N to the Boussinesq and KP Equations and the Degenerate Rational Case

Pierre Gaillard[1]

[1]*Institut de mathématiques de Bourgogne, Université de Bourgogne, 9 avenue Alain Savary, BP 47870, 21078 Dijon Cedex, France*

Article Info

Keywords: *Boussinesq equation, Kadomtsev Petviashvili equation, Rational solutions*
2010 AMS: *33Q55, 37K10*
Received: *10 November 2019*
Accepted: *26 March 2020*
Available online: *22 June 2020*

Abstract

From elementary exponential functions which depend on several parameters, we construct multi-parametric solutions to the Boussinesq equation. When we perform a passage to the limit when one of these parameters goes to 0, we get rational solutions as a quotient of a polynomial of degree $N(N+1)-2$ in x and t, by a polynomial of degree $N(N+1)$ in x and t for each positive integer N depending on $3N$ real parameters. We restrict ourself to give the explicit expressions of these rational solutions for $N=1$ until $N=3$ to shortened the paper.
We easily deduce the corresponding explicit rational solutions to the Kadomtsev Petviashvili equation for the same orders from 1 to 3.

1. Introduction

The Boussinesq equation in the following normalization is considered

$$u_{tt} - u_{xx} + (u^2)_{xx} + \frac{1}{3}u_{xxxx} = 0. \tag{1.1}$$

The subscripts x and t denote as usual partial derivatives.
This equation (1.1) is an equation solvable [3, 4] by inverse scattering. It was introduced for the first time by Boussinesq [1, 2] in 1871. This equation appears in a wide range of physical problems dealing with propagation of nonlinear waves; for example, in one-dimensional nonlinear lattice-waves [5], vibrations in a nonlinear string [6], ion sound waves in a plasma [7],...
The first solutions were constructed by Hirota [8] in 1977 with Bäcklund transformations. Non singular rational solutions were constructed by Ablowitz and Satsuma by using the Hirota bilinear method [9] in 1978. Freemann and Nimmo [10] gave in 1983 wronskians representations of the solutions. Other approaches were used; in particular, an algebro-geometrical method was given by Matveev et al. [11] in 1987; Darboux transformations [12] was used by Matveev; the $\overline{\partial}$ dressing method [13] was considered by Bogdanov.
Clarkson obtained solutions in terms of particular polynomials in a series of papers [14, 15] and recently, in 2017 gives new solutions [16] as second derivatives of polynomials.

Solutions to the Boussinesq equation and the Kadomtsev Petviashvili equation are considered in this paper. We give solutions from elementary exponential functions depending on several parameters. Then we construct rational solution in performing a passage to the limit when one of these parameters goes to 0. We obtain rational solutions as a quotient of a polynomial of degree $N(N+1)-2$ in x and t by a polynomial of degree $N(N+1)$ in x and t, depending on $2N$ parameters. We give explicit solutions in the simplest cases where $N=1, 2, 3$. We deduce and give explicit expressions of rational solutions to the Kadomtsev-Petviashvili (KP) equation for the cases of orders from 1 to 3.

Email address and ORCID number: **pierre.gaillard@u-bourgogne.fr, https://orcid.org/0000-0002-7073-8284 (P. Gaillard)**

2. Solutions to the Boussinesq equation

2.1. Solutions to the Boussinesq equation in terms of elementary exponentials

We consider the Boussinesq equation

$$u_{tt}-u_{xx}+(u^2)_{xx}+\frac{1}{3}u_{xxxx}=0.$$

We define the following notations.
We consider e, a_j, c_j, d_j, $1\leq j\leq N$, arbitrary real numbers, and α_j, β_j the numbers defined by

$$\alpha_j=\frac{3}{2}a_je+\frac{1}{2}\sqrt{1-3a_j^2e^2} \tag{2.1}$$

and

$$\beta_j=-\frac{3}{2}a_je+\frac{1}{2}\sqrt{1-3a_j^2e^2}. \tag{2.2}$$

We consider the following elementary functions

$$f_{ij}(x,t)=\alpha_j^{i-1}\exp(\alpha_jx-\alpha_j^2t+c_je^{2N-1})-\beta_j^{i-1}\exp(\beta_jx-\beta_j^2t+d_je^{2N-1}), \tag{2.3}$$

for $1\leq i\leq N$.
Then, we have the following statement:

Theorem 2.1. *The function v defined by*

$$v(x,t)=2\partial_x^2\ln(\det(f_{ij})_{(i,j)\in[1,N]}) \tag{2.4}$$

is a solution to the Boussinesq equation (1.1) with e, a_j, c_j and d_j, $1\leq j\leq N$ arbitrarily real parameters.

Proof. The corresponding Lax pair to the Boussinesq equation (1.1) is

$$\begin{cases}\phi_{xxx}+\frac{3}{2}u\phi_x-\frac{3}{4}\phi_x+u\phi=\lambda\phi,\\ \phi_t=-\phi_{xx}-u\phi.\end{cases} \tag{2.5}$$

The compatibility condition of the preceding system can be written as [12]

$$\begin{cases}w_x=\frac{3}{4}u_{xx}-\frac{3}{4}u_t,\\ w_t=\frac{1}{4}u_{xxx}+\frac{3}{4}(u^2)_x-\frac{3}{4}u_x+\frac{3}{4}u_{xt}.\end{cases} \tag{2.6}$$

The Boussinesq equation is obtained by excluding w from the above equations.
This system is covariant by the Darboux transformation. If $\phi_1,\ldots,\phi_N$ are solutions of the system (2.6), then $\phi[N]$ defined by $\phi[N]=\frac{W(\phi_1,\ldots,\phi_N,\phi)}{W(\phi_1,\ldots,\phi_N)}$ is another solution of this system (2.6) where u is replaced by $u[N]=u+2(\ln W(\phi_1,\ldots,\phi_N)_{xx}$ [12].
We choose $u=0$. Then the functions $\phi_j=f_{1j}$ verify the following system

$$\begin{cases}\phi_{xxx}-\frac{3}{4}\phi_x=\lambda\phi,\\ \phi_t=-\phi_{xx}.\end{cases} \tag{2.7}$$

Then the solution of (1.1) can be written as $v(x,t)=2(\ln W(\phi_1,\ldots,\phi_N)_{xx}$which is nothing else that (2.4) $v(x,t)=2\partial_x^2\ln(\det(f_{ij})_{(i,j)\in[1,N]})$. □

2.2. Rational solutions to the Boussinesq equation

To obtain rational solutions to the Boussinesq equation, we are going to perform a limit when the parameter e tends to 0.

2.2.1. Rational solutions as a limit case

We get the following result :

Theorem 2.2. *The function v defined by*

$$v(x,t)=\lim_{e\to0}2\partial_x^2\ln(\det(f_{ij})_{(i,j)\in[1,N]}) \tag{2.8}$$

is a rational solution to the Boussinesq equation (1.1) depending on $3N$ parameters a_j, c_j and d_j, $1\leq j\leq N$; the numerator is a polynomial of degree $N(N+1)-2$ in x and t, the denominator a polynomial of degree $N(N+1)$ in x and t.

Proof. It is a consequence of the previous result. □

2.2.2. Degenerate rational solutions

A more precise result can be formulated in the following way.
We consider e, a_j, c_j, d_j, $1 \leq j \leq N$, arbitrary real numbers, and γ_j, δ_j the numbers defined by

$$\gamma_j = \frac{3}{2}\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right) + \frac{1}{2}\sqrt{1 - 3\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right)^2},$$
$$\delta_j = -\frac{3}{2}\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right) + \frac{1}{2}\sqrt{1 - 3\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right)^2} \tag{2.9}$$

We consider the following elementary functions

$$g_{ij}(x,t,e) = \gamma_j^{i-1}\exp\left(\gamma_j x - \gamma_j^2 t + \sum_{k=1}^{N} c_k (je)^{2k-1}\right) - \delta_j^{i-1}\exp\left(\delta_j x - \delta_j^2 t + \sum_{k=1}^{N} d_k (je)^{2k-1})\right), \tag{2.10}$$

$$\varphi_{ij}(x,t) = \frac{\partial^j g_{i1}(x,t,0)}{\partial e^j}, \text{ for } 1 \leq i \leq N, \quad 1 \leq j \leq N. \tag{2.11}$$

Then get the following result :

Theorem 2.3. *The function v defined by*

$$v(x,t) = 2\partial_x^2 \ln(\det(\varphi_{ij})_{(i,j)\in[1,N]} \tag{2.12}$$

is a rational solution to the Boussinesq equation (1.1) depending on 3N parameters a_j, c_j and d_j, $1 \leq j \leq N$; the numerator is a polynomial of degree $N(N+1)-2$ in x and t, the denominator a polynomial of degree $N(N+1)$ in x and t.

Proof. In the coefficients α_j and β_j defined in (2.1,2.2), we replace a_j by $\sum_{k=1}^{N} a_k (je)^{2k-1}$, and in the functions f_{ij} defined in (2.3), c_j by $\sum_{k=1}^{N} c_k (je)^{2k-1}$ and d_j by $\sum_{k=1}^{N} d_k (je)^{2k-1}$; this gives functions g_{ij} defined by (2.10). Then, it is sufficient to combine the columns of the determinant obtained from this defined by (2.8) by replacing f_{ij} by g_{ij} and to take a passage to the limit when e tends to 0. So we get the solution v given by (2.12). $\square$

So we obtain an infinite hierarchy of rational solutions to the Boussinesq equation depending on the integer N.
In the following we give some examples of rational solutions.
These results are consequences of the previous result (2.12).
But, it is also to possible to prove it directly in replacing the expressions of each of the solutions given in the corresponding equation and check that the relation is verified.

2.3. First order rational solutions

We have the following result at order $N = 1$:

Theorem 2.4. *The function v defined by*

$$v(x,t) = \frac{-18{a_1}^2}{(-3a_1 x - c_1 + 3ta_1 + d_1)^2}, \tag{2.13}$$

is a solution to the Boussinesq equation (1.1) with a_1, c_1, d_1 arbitrarily real parameters.

Remark 2.5. *If $a_1 = 0$, then the solution is the trivial solution* 0.

Remark 2.6. *The solution (2.13) can be simplified and be rewritten as a solution depending on one parameter C_1.*

$$v(x,t) = \frac{-18}{(-3x + 3t + C_1)^2}. \tag{2.14}$$

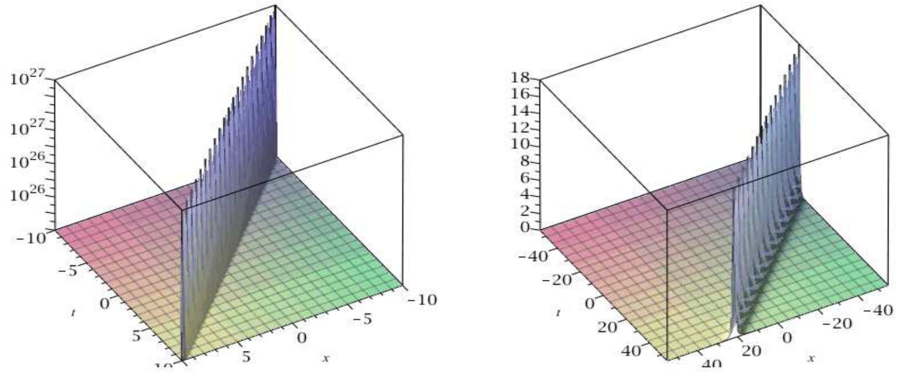

Figure 2.1: Solution of order 1 to (1.1), on the left $a_1 = 10^{13}$, $c_1 = 1$, $c_1 = 0$; on the right $a_1 = 1$, $c_1 = 10^2$, $d_1 = 0$.

Remark 2.7. *The case where $a_1 = 1$, $c_1 = 0$, $d_1 = 10^2$ gives the same figure as the case $a_1 = 1$, $c_1 = 10^2$, $d_1 = 0$. The roles played by the parameters c and d being the same, we only give the figures with parameters d equal to 0.*

2.4. Second order rational solutions

Theorem 2.8. *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^2}, \tag{2.15}$$

with

$n(x,t) = 9a_1a_2(27a_1{}^5a_2 + 243a_2{}^5a_1 - 162a_2{}^3a_1{}^3)x^4 + 9a_1a_2(-972a_2{}^5ta_1 - 324a_2{}^5a_1 + 216a_2{}^3a_1{}^3 + 648a_2{}^3a_1{}^3t - 36a_1{}^5a_2 - 108a_1{}^5ta_2)x^3$
$+9a_1a_2(972a_2{}^5ta_1 - 648a_2{}^3a_1{}^3t - 108a_2{}^3a_1{}^3 - 972a_2{}^3a_1{}^3t^2 + 162a_1{}^5t^2a_2 + 162a_2{}^5a_1 + 18a_1{}^5a_2 + 108a_1{}^5ta_2 + 1458a_2{}^5t^2a_1)x^2 + 9a_1a_2(-108a_1{}^5t^3a_2 +$
$216a_2{}^2c_2a_1 + 72d_2a_1{}^3 - 432a_2{}^3a_1{}^3t + 648a_2{}^3a_1{}^3t^3 - 24a_1{}^2d_1a_2 + 648a_2{}^3a_1{}^3t^2 + 648a_2{}^5ta_1 - 72a_1{}^3c_2 + 72d_1a_2{}^3 + 24a_1{}^2c_1a_2 - 972a_2{}^5a_1t^3 - 72a_2{}^3c_1 -$
$972a_2{}^5t^2a_1 + 72a_1{}^5ta_2 - 216a_2{}^2d_2a_1 - 108a_1{}^5t^2a_2)x + 9a_1a_2(24a_2{}^3c_1 + 24a_1{}^3c_2 + 324a_2{}^5a_1t^3 + 540a_2{}^3a_1{}^3t^2 + 216a_2{}^3a_1{}^3t - 216a_2{}^3a_1{}^3t^3 + 8a_1{}^2d_1a_2 -$
$90a_1{}^5t^2a_2 - 36a_1{}^5ta_2 + 36a_1{}^5t^3a_2 + 243t^4a_2{}^5a_1 - 162t^4a_2{}^3a_1{}^3 + 27a_1{}^5t^4a_2 + 72a_2{}^2d_2a_1 - 810a_2{}^5t^2a_1 - 324a_2{}^5ta_1 - 72ta_2{}^3d_1 - 72a_1{}^3td_2 + 216ta_2{}^2d_2a_1 +$
$24a_1{}^2td_1a_2 - 216ta_2{}^2c_2a_1 - 24a_1{}^2tc_1a_2 - 24d_2a_1{}^3 - 24d_1a_2{}^3 - 8a_1{}^2c_1a_2 + 72ta_2{}^3c_1 + 72a_1{}^3tc_2 - 72a_2{}^2c_2a_1)$,

and

$d(x,t) = (-9a_1{}^3a_2 + 27a_1a_2{}^3)x^3 + (27ta_1{}^3a_2 - 81ta_1a_2{}^3 + 9a_1{}^3a_2 - 27a_1a_2{}^3)x^2 + (-27t^2a_1{}^3a_2 + 81t^2a_1a_2{}^3 - 18ta_1{}^3a_2 + 54ta_1a_2{}^3)x + 9t^3a_1{}^3a_2 -$
$27t^3a_1a_2{}^3 + 9t^2a_1{}^3a_2 - 27t^2a_1a_2{}^3 + 18ta_1{}^3a_2 - 54ta_1a_2{}^3 - 12a_1c_2 + 12a_1d_2 + 4a_2c_1 - 4a_2d_1$,
is a rational solution to the Boussinesq equation (1.1), quotient of two polynomials with numerator of order 4 in x and t, denominator of degree 6 in x and t.

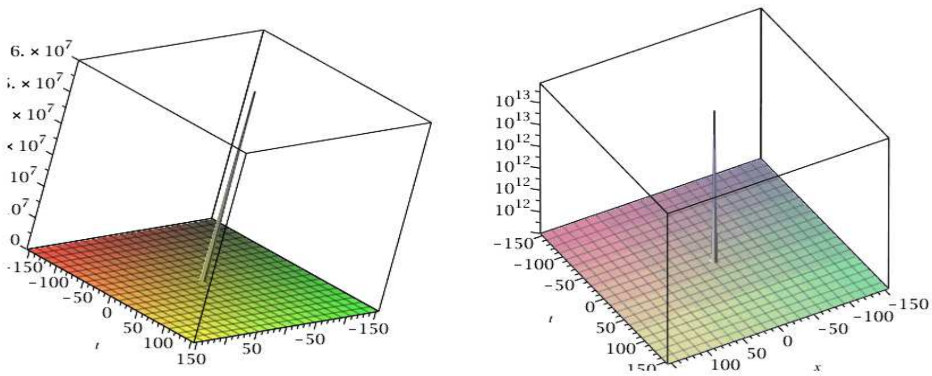

Figure 2.2: Solution of order 2 to (1.1); on the left, $a_1 = 10^7$, $a_2 = 1$, $c_1 = 1$, $c_2 = 1$, $d_1 = 0$, $d_2 = 0$; on the right, $a_1 = 1$, $a_2 = 10^7$, $c_1 = 1$, $c_2 = 1$, $d_1 = 0$, $d_2 = 0$.

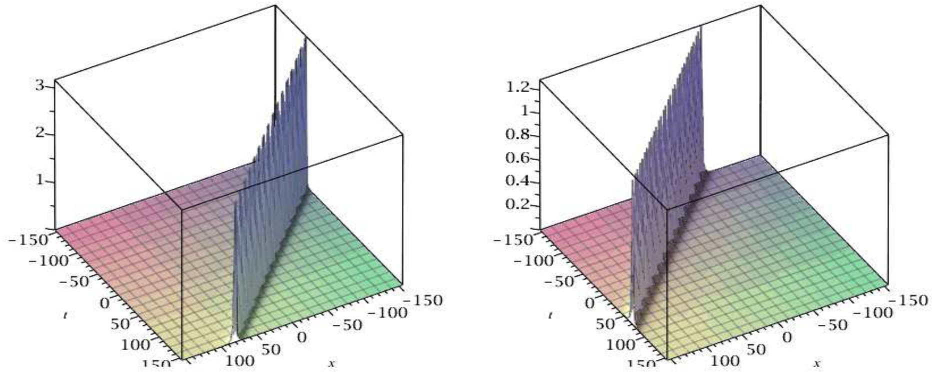

Figure 2.3: Solution of order 2 to (1.1); on the left, $a_1 = 1$, $a_2 = 1$, $c_1 = 10^7$, $c_2 = 0$, $d_1 = 1$, $d_2 = 1$; on the right, $a_1 = 1$, $a_2 = 1$, $c_1 = 0$, $c_2 = 10^7$, $d_1 = 0$, $d_2 = 0$.

2.5. Rational solutions of order three

We get the following rational solutions given by :

Theorem 2.9. *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^2}, \tag{2.16}$$

is a rational solution to the Boussinesq equation (1.1), quotient of two polynomials with numerator of order 10 *in x and t, denominator of degree* 12 *in x and t.*

Because of the length of the solution, we give it only in the appendix.

Remark 2.10. *If $c_1 = c_2 = c_3 = d_1 = d_2 = d_3 = 0$, then the determinant in the formula* (2.12) *can be simplified by* $\frac{(177147}{80}a_1a_2a_3^3(-80a_2^4 + 360a_3^2a_2^2 + a_1^4 - 30a_3^2a_1^2)$ *and the solution to the Boussinseq equation depends no longer on any parameter.*
If one of the parameters a_1, a_2 or a_3 is equal to 0 *then the solution of the Boussinesq equation is the trivial solution (equal to* 0*).*

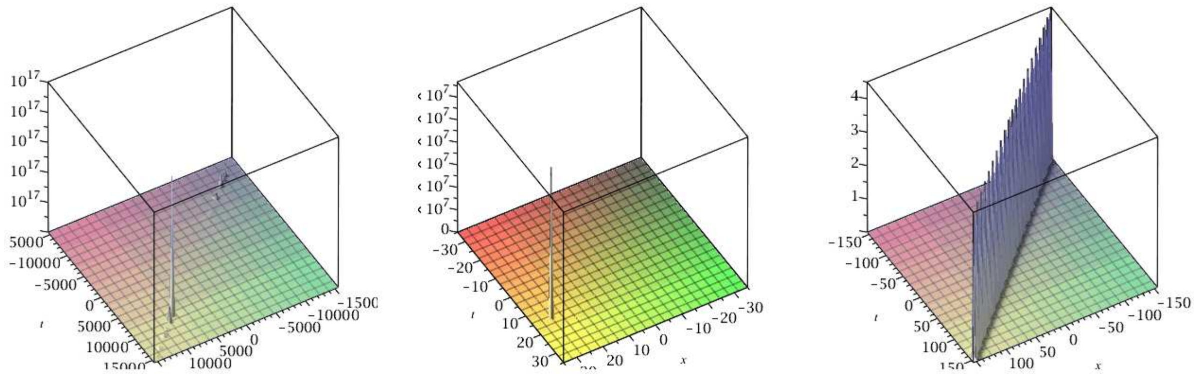

Figure 2.4: Solution of order 3 to (1.1); on the left, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $c_1 = 0$, $c_2 = 0$,$c_3 = 10^7$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; in the center, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $c_1 = 0$, $c_2 = 10^7$,$c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; on the right, $a_1 = 1$, $a_2 = 1$, $a_3 = 10^7$, $c_1 = 1$, $c_2 = 1$,$c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$.

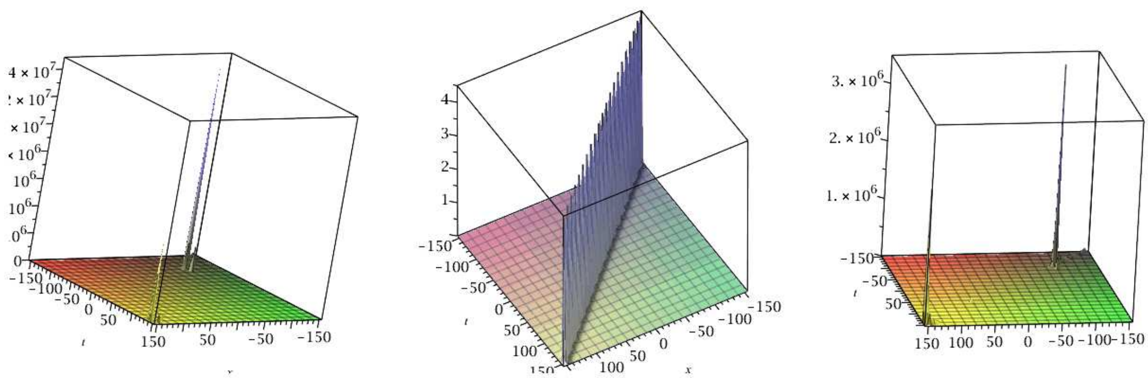

Figure 2.5: Solution of order 3 to (1.1)on the left, $a_1 = 1$, $a_2 = 10^7$, $a_3 = 1$, $c_1 = 1$, $c_2 = 1$,$c_3 = 10^7$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; in the center, $a_1 = 10^7$, $a_2 = 1$, $a_3 = 1$, $c_1 = 1$, $c_2 = 10^7$,$c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; on the right, $a_1 = 1$, $a_2 = 1$, $a_3 = 10^7$, $c_1 = 10^5$, $c_2 = 1$,$c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$.

3. Solutions to the Kadomtsev Petviashvili equation

We consider the Kadomtsev Petviashvili equation (KP) which can be written in the form

$$(4u_T - 6uu_X + u_{XXX})_X - 3u_{YY} = 0, \tag{3.1}$$

where subscripts X, Y and T denote as usual partial derivatives.
From the previous study, we can deduce easily solutions to the KP equation. It is sufficient for this, to use the following transformations $x = \imath X + \dfrac{3\imath T}{4}$, $t = \imath Y$ from the solutions to the Boussinesq equation to obtain solutions to the KP equation.

3.1. Solutions to the KP equation

3.2. First order rational solutions

We have the following result at order $N = 1$:

Theorem 3.1. *The function v defined by*

$$v(X.Y,T) = \frac{-288\,a_1{}^2}{(12\,ia_1X + 9\,ia_1T + 4\,c_1 - 12\,iYa_1 - 4\,d_1)^2}, \tag{3.2}$$

is a solution to the KP equation (3.1).

Remark 3.2. *The solution* (3.2) *can be simplified and be rewritten as depending on one parameter* $v(X.Y,T) = \dfrac{-288}{(12\,iX + 9\,iT + 4C_1 - 12\,iY)^2}$

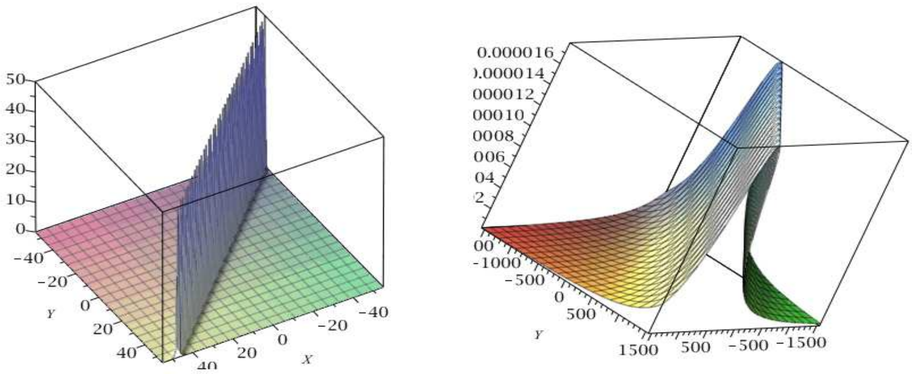

Figure 3.1: Solution of order 1 to (3.1), on the left $T = 10$, $a_1 = 10^6$, $c_1 = 1$, $d_1 = 1$; on the right $T = 10$, $a_1 = 1$, $c_1 = 10^3$, $d_1 = 1$.

Remark 3.3. *The case where* $T = 10$, $a_1 = 1$, $c_1 = 1$, $d_1 = 10^3$ *gives the same figure as the case* $T = 10$, $a_1 = 1$, $c_1 = 10^3$, $d_1 = 1$.

3.3. Second order rational solutions

We obtain the following solutions :

Theorem 3.4. *The function v defined by*

$$v(X.Y,T) = -2\,\frac{n(X.Y,T)}{d(X.Y,T)^2}, \tag{3.3}$$

with
$n(X.Y,T) = 144\,a_1a_2(-41472\,a_2{}^3a_1{}^3 + 62208\,a_2{}^5a_1 + 6912\,a_1{}^5a_2)X^4 + 144\,a_1a_2(-124416\,a_2{}^3a_1{}^3T - 27648\,a_1{}^5Ya_2 + 165888\,a_2{}^3a_1{}^3Y + 9216\,ia_1{}^5a_2$
$- 55296\,ia_2{}^3a_1{}^3 + 186624\,a_2{}^5a_1T + 82944\,ia_2{}^5a_1 + 20736\,a_1{}^5a_2T - 248832\,a_2{}^5Ya_1)X^3 + 144\,a_1a_2(-248832\,a_2{}^3a_1{}^3Y^2 - 27648\,ia_1{}^5Ya_2 + 23328\,a_1{}^5a_2T^2 +$
$373248\,a_2{}^3a_1{}^3YT + 165888\,ia_2{}^3a_1{}^3Y - 124416\,ia_2{}^3a_1{}^3T + 41472\,a_1{}^5Y^2a_2 - 62208\,a_1{}^5Ya_2T - 559872\,a_2{}^5Ya_1T - 139968\,a_2{}^3a_1{}^3T^2 - 41472\,a_2{}^5a_1 + 27648\,a_2{}^3a_1{}^3 +$
$209952\,a_2{}^5a_1T^2 + 186624\,ia_2{}^5a_1T + 373248\,a_2{}^5Y^2a_1 - 248832\,ia_2{}^5Ya_1 + 20736\,ia_1{}^5a_2T - 4608\,a_1{}^5a_2)X^2 + 144\,a_1a_2(-419904\,a_2{}^5Ya_1T^2 + 279936\,a_2{}^3a_1{}^3YT^2 -$
$46656\,a_1{}^5Ya_2T^2 + 62208\,a_1{}^5Y^2a_2T + 559872\,a_2{}^5Y^2a_1T - 373248\,a_2{}^3a_1{}^3Y^2T - 373248\,ia_2{}^5Ya_1T + 248832\,ia_2{}^3a_1{}^3YT - 41472\,ia_1{}^5Ya_2T - 18432\,ia_2{}^3c_1 +$
$41472\,a_2{}^3a_1{}^3T - 6912\,a_1{}^5a_2T - 62208\,a_2{}^5a_1T + 18432\,id_2a_1{}^3 - 18432\,ia_1{}^3c_2 + 18432\,id_1a_2{}^3 + 139968\,ia_2{}^5a_1T^2 - 93312\,ia_2{}^3a_1{}^3T^2 + 15552\,ia_1{}^5a_2T^2 +$
$27648\,ia_1{}^5Y^2a_2 + 248832\,ia_2{}^5Y^2a_1 - 165888\,ia_2{}^3a_1{}^3Y^2 - 55296\,ia_2{}^2d_2a_1 - 6144\,ia_1{}^2d_1a_2 + 55296\,ia_2{}^2c_2a_1 + 6144\,ia_1{}^2c_1a_2 - 69984\,a_2{}^3a_1{}^3T^3 + 11664\,a_1{}^5a_2T^3 +$
$104976\,a_2{}^5a_1T^3 - 248832\,a_2{}^5a_1Y^3 + 110592\,a_2{}^3a_1{}^3Y + 165888\,a_2{}^3a_1{}^3Y^3 - 18432\,a_1{}^5Ya_2 - 27648\,a_1{}^5Y^3a_2 - 165888\,a_2{}^5Ya_1)X + 144\,a_1a_2(6144\,a_2{}^3c_1 +$
$6144\,a_1{}^3c_2 + 41472\,ia_2{}^2c_2a_1T + 4608\,ia_1{}^2c_1a_2T - 104976\,a_2{}^5Ya_1T^3 + 69984\,a_2{}^3a_1{}^3YT^3 - 11664\,a_1{}^5Ya_2T^3 + 55296\,iYa_2{}^2d_2a_1 + 6144\,ia_1{}^2Yd_1a_2$
$- 55296\,iYa_2{}^2c_2a_1 - 6144\,ia_1{}^2Yc_1a_2 - 139968\,ia_2{}^5Ya_1T^2 + 93312\,ia_2{}^3a_1{}^3YT^2 - 15552\,ia_1{}^5Ya_2T^2 + 20736\,ia_1{}^5Y^2a_2T + 186624\,ia_2{}^5Y^2a_1T - 124416\,ia_2{}^3a_1{}^3Y^2T -$
$41472\,ia_2{}^2d_2a_1T - 4608\,ia_1{}^2d_1a_2T - 13122\,a_2{}^3a_1{}^3T^4 + 19683\,a_2{}^5a_1T^4 + 2187\,a_1{}^5a_2T^4 + 13824\,id_2a_1{}^3T - 13824\,ia_1{}^3c_2T + 13824\,id_1a_2{}^3T - 23328\,ia_2{}^3a_1{}^3T^3 +$
$3888\,ia_1{}^5a_2T^3 - 82944\,ia_2{}^5a_1Y^3 + 55296\,ia_2{}^3a_1{}^3Y + 55296\,ia_2{}^3a_1{}^3Y^3 - 9216\,ia_1{}^5Ya_2 - 9216\,ia_1{}^5Y^3a_2 - 82944\,ia_2{}^5Ya_1 - 18432\,iYa_2{}^3d_1 - 18432\,ia_1{}^3Yd_2 +$
$18432\,iYa_2{}^3c_1 + 18432\,ia_1{}^3Yc_2 + 34992\,ia_2{}^5a_1T^3 - 13824\,ia_2{}^3c_1T - 124416\,a_2{}^5Ya_1T + 82944\,a_2{}^3a_1{}^3YT - 186624\,a_2{}^5a_1Y^3T + 124416\,a_2{}^3a_1{}^3Y^3T - 13824\,a_1{}^5Ya_2T$

$-20736a_1{}^5Y^3a_2T+209952a_2{}^5Y^2a_1T^2-139968a_2{}^3a_1{}^3Y^2T^2+23328a_1{}^5Y^2a_2T^2+62208Y^4a_2{}^5a_1-41472Y^4a_2{}^3a_1{}^3+6912a_1{}^5Y^4a_2+2048a_1{}^2d_1a_2+18432a_2{}^2d_2a_1-6144d_2a_1{}^3-6144d_1a_2{}^3+15552a_2{}^3a_1{}^3T^2-2592a_1{}^5a_2T^2-23328a_2{}^5a_1T^2-138240a_2{}^3a_1{}^3Y^2+207360a_2{}^5Y^2a_1+23040a_1{}^5Y^2a_2-2048a_1{}^2c_1a_2-18432a_2{}^2c_2a_1)$,

and

$d(X.Y,T)=-1728ia_1a_2{}^3X^3+576ia_1{}^3a_2X^3+1728a_1a_2{}^3X^2-576a_1{}^3a_2X^2-3888ia_1a_2{}^3X^2T+5184iYa_1a_2{}^3X^2-1728iYa_1{}^3a_2X^2+1296ia_1{}^3a_2TX^2+7776iYa_1a_2{}^3TX+972ia_1{}^3a_2T^2X-2916ia_1a_2{}^3XT^2-5184iY^2a_1a_2{}^3X+1728iY^2a_1{}^3a_2X-2592iYa_1{}^3a_2XT-864a_1{}^3a_2TX-3456Ya_1a_2{}^3X+2592a_1a_2{}^3TX+1152Ya_1{}^3a_2X-768a_1c_2-256a_2d_1-3888iY^2a_1a_2{}^3T+2916iYa_1a_2{}^3T^2+1728iY^3a_1a_2{}^3-576iY^3a_1{}^3a_2-3456iYa_1a_2{}^3+1152iYa_1{}^3a_2-972iYa_1{}^3a_2T^2-729ia_1a_2{}^3T^3+256a_2c_1+243ia_1{}^3a_2T^3+768a_1d_2+1296iY^2a_1{}^3a_2T+972a_1a_2{}^3T^2-2592Ya_1a_2{}^3T-324a_1{}^3a_2T^2+864Ya_1{}^3a_2T+1728Y^2a_1a_2{}^3-576Y^2a_1{}^3a_2$,

is a rational solution to the KP equation (3.1), quotient of two polynomials with numerator of degree 4 *in x, y and t and denominator of degree* 6 *in x, y and t*.

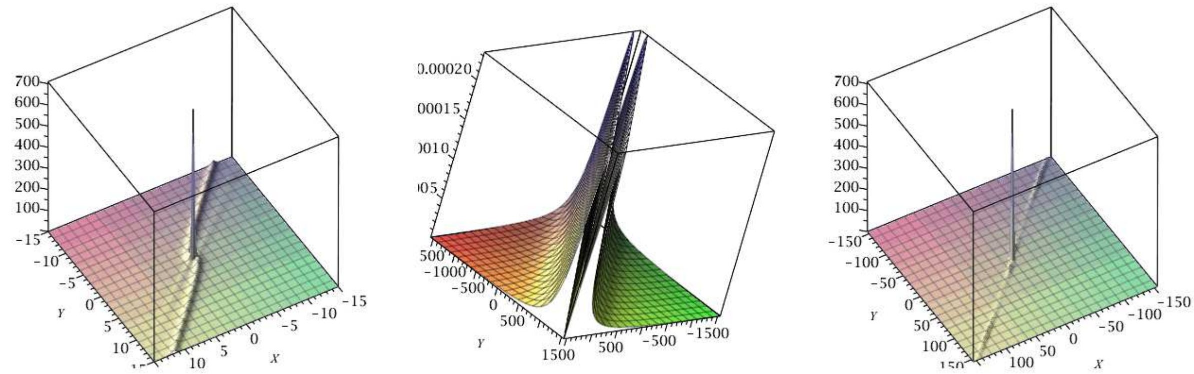

Figure 3.2: Solution of order 2 to (3.1); on the left $T=0,1$, $a_1=1$, $a_2=1$, $c_1=0$, $c_2=0$, $d_1=0$, $d_2=0$; in the center $T=0,1$, $a_1=1$, $a_2=1$, $c_1=0$, $c_2=10^8$, $d_1=0$, $d_2=0$; on the right $T=0,1$, $a_1=1$, $a_1=10^9$, $c_1=1$, $c_2=1$, $d_1=0$, $d_2=0$.

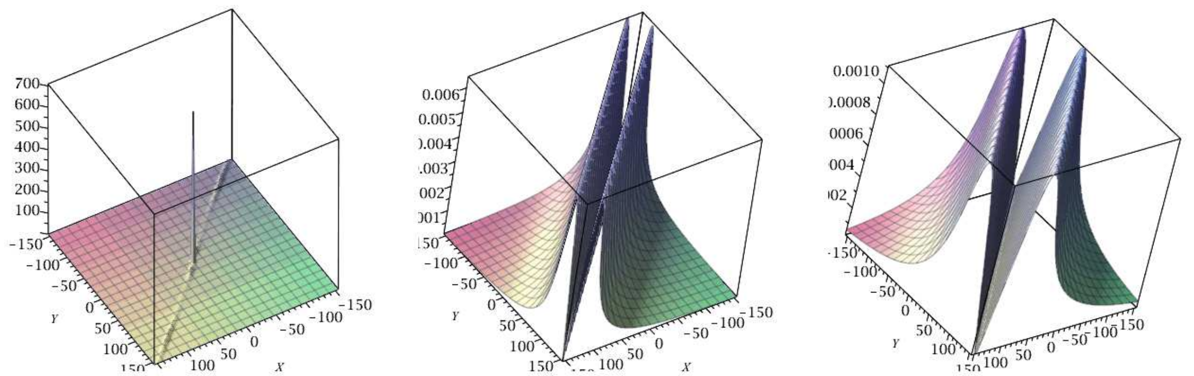

Figure 3.3: Solution of order 2 to (3.1); on the left $T=0,1$, $a_1=10^6$, $a_2=1$, $c_1=1$, $c_2=1$, $d_1=0$, $d_2=0$; in the center $T=0,1$, $a_1=1$, $a_2=1$, $c_1=10^6$, $c_2=0$, $d_1=0$, $d_2=0$; on the right $T=10$, $a_1=1$, $a_2=1$, $c_1=1$, $c_2=10^7$, $d_1=0$, $d_2=0$.

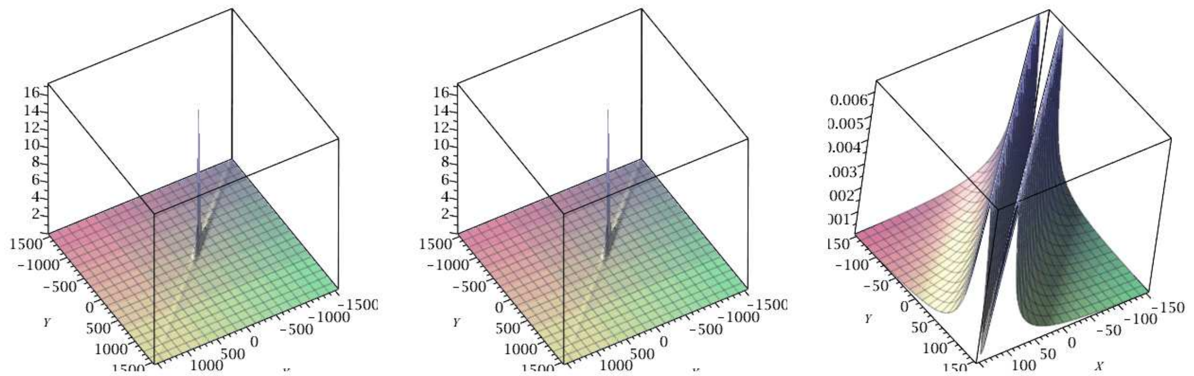

Figure 3.4: Solution of order 2 to (3.1); on the left $T=10$, $a_1=1$, $a_2=10^9$, $c_1=1$, $c_2=1$, $d_1=0$, $d_2=0$; in the center $T=10$, $a_1=10^{10}$, $a_2=1$, $c_1=1$, $c_2=1$, $d_1=0$, $d_2=0$; on the right $T=10$, $a_1=1$, $a_2=1$, $c_1=1$, $c_2=10^6$, $d_1=0$, $d_2=0$.

3.4. Rational solutions of order 3

We get the non singular rational solutions given by :

Theorem 3.5. *The function v defined by*

$$v(X.Y,T) = -2\,\frac{n(X.Y,T)}{d(X.Y,T)^2}, \tag{3.4}$$

is a rational solution to the KP equation (3.1), quotient of two polynomials with numerator of degree 10 *in* X, Y, T *and denominator of degree* 12 *in* X, Y *and* T.

Because of the length of the solution, we only give it in the appendix.

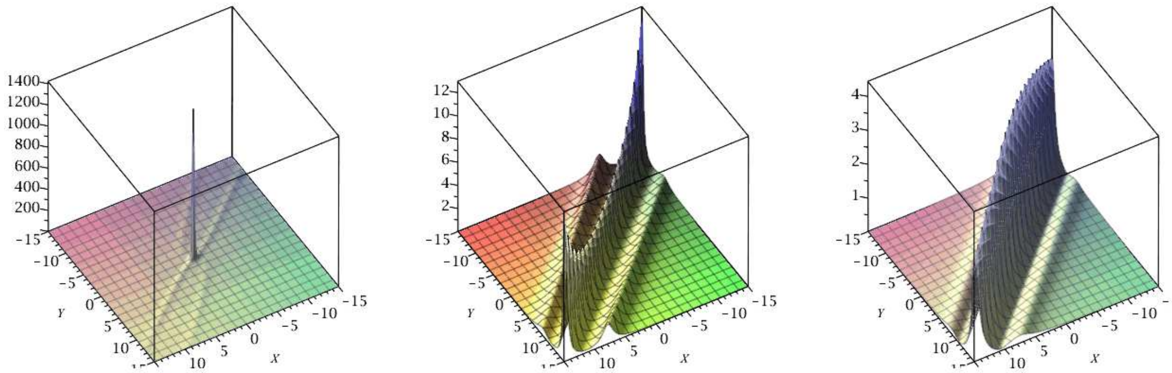

Figure 3.5: Solution of order 3 to (3.1); on the left $T = 0,1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $c_1 = 0$, $c_2 = 0$ $c_3 = 0$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; in the center $T = 0,1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $c_1 = 1$, $c_2 = 0$, $c_3 = 10^6$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; on the right $T = 0,1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $c_1 = 0$, $c_2 = 10^6$, $c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$.

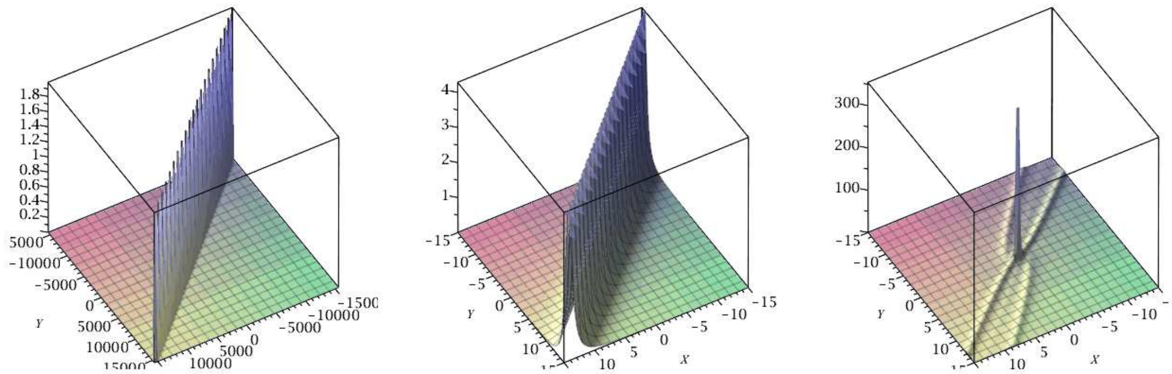

Figure 3.6: Solution of order 3 to (3.1); on the left $T = 0,1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 10^{24}$, $c_1 = 1$, $c_2 = 1$ $c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; in the center $T = 0,1$, $a_1 = 1$, $a_2 = 10^4$, $a_3 = 1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; on the right $T = 0,1$, $a_1 = 10$, $a_2 = 1$, $a_3 = 1$, $c_1 = 1$, $c_2 = 10$, $c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$.

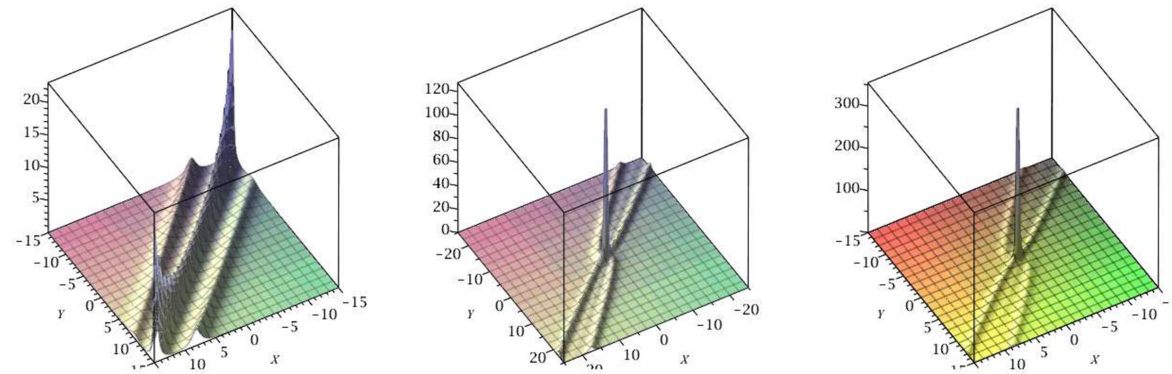

Figure 3.7: Solution of order 3 to (3.1); on the left $T = 0,1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $c_1 = 10^6$, $c_2 = 1$ $c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; in the center $T = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; on the right $T = 1$, $a_1 = 10^6$, $a_2 = 1$, $a_3 = 1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$.

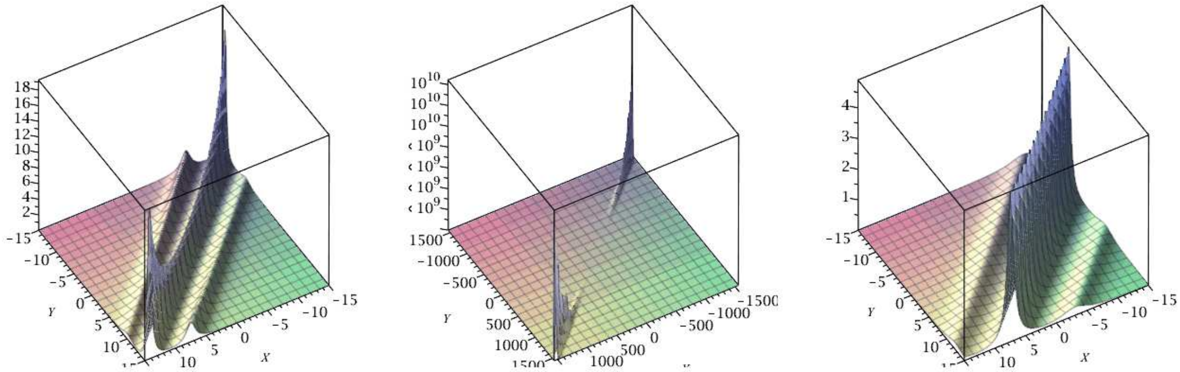

Figure 3.8: Solution of order 3 to (3.1); on the left $T=1$, $a_1=1$, $a_2=1$, $a_3=1$, $c_1=10^6$, $c_2=1$ $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$; in the center $T=10$, $a_1=10^6$, $a_2=1$, $a_3=1$, $c_1=1$, $c_2=1$, $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$; on the right $T=10$, $a_1=1$, $a_2=1$, $a_3=1$, $c_1=1$, $c_2=1$, $c_3=10^7$, $d_1=0$, $d_2=0$, $d_3=0$.

4. Conclusion

We have given three types of representations of solutions to the Boussinesq equation. First, solutions in terms of elementary exponential functions have been constructed. In particular, performing a passage to the limit when one parameter goes to 0 we get rational solutions to the Boussinesq equation. We give an other representation in terms of determinants without the presence of a limit. So we obtain an infinite hierarchy of multiparametric families of rational solutions to the Boussinesq equation as a quotient of a polynomial of degree $N(N+1)-2$ in x, t by a polynomial of degree $N(N+1)$ in x, t depending on $3N$ real parameters.
As a byproduct, we get easily similar rational solutions to the Kadomtsev Petviashvili equation as the quotient of determinants of polynomials, where the numerator is a polynomial of degree $N(N+1)-2$ in X, Y, T and the denominator is a polynomial of degree $N(N+1)$ in X, Y, T.
In particular, we construct explicit rational solutions to the Boussinesq equation of order 1, 2, 3.
Unlike other equations such as NLS, there are no specific structures that emerge as a function of the parameters.

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Universal Journal of Mathematics and Applications, 3 (2) (2020) 53-60
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.597667

Faster Convergent Modified Lindstedt-Poincare Solution of Nonlinear Oscillators

Nazmul Sharif[1*], M. S. Alam[1] and I. A. Yeasmin[1]

[1] *Department of Mathematics, Rajshahi University of Engineering & Technology (RUET), Kazla, Rajshahi, Bangladesh*
[*] *Corresponding author*

Article Info

Keywords: *Modified Lindstedt-Poincare method, Nonlinear oscillation, Perturbation method*
2010 AMS: *34A34, 34B15, 34C15*
Received: *28 July 2019*
Accepted: *25 March 2020*
Available online: *22 June 2020*

Abstract

The modified Lindstedt-Poincare method has been extended to obtain a faster convergent solution of nonlinear oscillators. First of all a classical type Lindstedt-Poincare solution has been determined and then a conversion formula has been used to find the desired solution. The solution has been compared and justified by corresponding numerical solution.

1. Introduction

Poincare [19] developed different methods to solve differential equations. Poincare and Lindstedt developed Lindstedt-Poincare method [1,2]. The Lindstedt-Poincare method [1,2] was originally developed for handling a weak nonlinear oscillator

$$\ddot{x}+\omega_0^2 x+\varepsilon f(x,\dot{x},\ddot{x})=0, \tag{1.1}$$

where ε is a small parameter, ω_0 is a constant, over dots denote differentiation with respect to t and $x(0)=a_0$, $\dot{x}(0)=0$ are the given initial conditions. Then Krylov-Bogoliubov's [3] and multiple time scale [1] methods were presented to investigate Eq. (1.1). The classical perturbation methods agree with numerical solutions (e. g. Runge-Kutta 4^{th} order method [19], finite elements method [5], etc.) when ε is very close to zero.

Several authors [4]- [6], [16] extended the Lindstedt-Poincare method to solve stronger nonlinear problems. Jones [4] presented an approximate technique by introducing a new parameter, $\alpha(\varepsilon)$ rather than the small parameter, ε. Such approximate solution is valid even for large value of ε. Burton [5] presented a modified version of the Lindstedt-Poincare method. Cheung et al. [6] further modified this method. However, all the approximate solutions obtained by approaches of [4]- [6] are effective for Duffing oscillator with cubical nonlinearity. The aim of this article is to present a new form of the modified Lindstedt-Poincare method of Cheung et al. [6] based on the conversion formula presented by Alam et al. [14] by introducing a parameter k. The solutions obtained for various nonlinear oscillators nicely agree with corresponding numerical solutions and provide better results than other existing solutions.

Besides the classical perturbation methods, many approximate techniques have been presented for solving the stronger nonlinear oscillators. Among them the asymptotic expansions [15,18], the homotopy perturbation [7], harmonic balance [8,9], energy balance [10] and iteration methods [11] are widely used. Singular differential equations are also solved using optimal successive complementary expansion method by F. Say [17].

2. The Lindstedt-Poincare method

Introducing a new variable, $\tau=\omega t$, t can be replaced and Eq. (1.1) is written as

$$\omega^2 x''+\omega_0^2 x+\varepsilon f(x,\omega x',\omega^2 x'')=0. \tag{2.1}$$

***Email addresses and ORCID numbers*:** **nazmul06024160@yahoo.com, https://orcid.org/0000-0003-4234-6651 (Nazmul Sharif), msalam1964@yahoo.com, https://orcid.org/0000-0002-6325-6797 (M. S. Alam), ismot.yeasmin@yahoo.com, https://orcid.org/0000-0002-0392-8547 (I. A. Yeasmin)**

Here ω is known as the frequency of the oscillator and the primes denote differentiation with respect to τ. According to Lindstedt-Poincare method [1,2], x and ω can be expanded in powers of ε as

$$x=\sum_{n=0}^{\infty} x_n\varepsilon^n, \tag{2.2}$$

and

$$\omega^2=\omega_0^2+\sum_{n=1}^{\infty}\omega_n\varepsilon^n. \tag{2.3}$$

Earlier it was chosen that $\omega=\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+\mathscr{O}(\varepsilon^3)$. But Veronis [12] and Burton [5] and Burton et al. [13] used series Eq. (2.3). In this article we have used the series in Eq. 2.3 for faster convergent solution.
By substituting x and ω into Eq. (2.1) and equating the coefficients of like powers of ε, we obtain the following equations:

$$\omega_0^2x_0''+\omega_0^2x_0=0, \tag{2.4}$$

$$\omega_0^2x_1''+\omega_0^2x_1=-2\omega_0\omega_1x_0''-f(x_0,x_0',x_0''), \tag{2.5}$$

$$\omega_0^2x_2''+\omega_0^2x_2=-2\left(\omega_0\omega_1+\omega_1^2\right)x_0''-2\omega_0\omega_1x_1''-x_1\frac{\partial f(x_0,x_0',x_0'')}{\partial x}$$

$$-\left(\omega_0x_1'+\frac{\omega_1x_0'}{2\omega_0}\right)\frac{\partial f(x_0,x_0',x_0'')}{\partial x'}-\left(\omega_0^2x_1''+\omega_1x_0''\right)\frac{\partial f(x_0,x_0',x_0'')}{\partial x''}. \tag{2.6}$$

The initial conditions are usually replaced by $x_0(0)=a_0$, $x_0'(0)=0$, $x_1(0)=x_1'(0)=x_2(0)=0\cdots$, and x_0,x_1 and ω_1,x_2 and ω_2 etc. are determined sequentially. In this article we only follow the initial conditions of $x_0'(0)=x_1'(0)=\cdots=0$, and

$$a_0=x_0(0)+\varepsilon x_1(0)+\varepsilon^2x_2(0)+\mathscr{O}(\varepsilon^3). \tag{2.7}$$

This assumption was introduced in [9] following [3].

3. Conversion formulae

Recently a conversion formula [14] has been presented to the modified Lindstedt-Poincare solution [6] from its classical version. This conversion formula can be used to obtain a faster convergent solution (concern of this article). Cheung et al. [6] reconsidered Eq. (2.3) to the following form

$$\omega^2=\left(\omega_0^2+\varepsilon\omega_1\right)\left(1+\frac{\varepsilon^2\omega_2}{\omega_0^2+\varepsilon\omega_1}+\frac{\varepsilon^3\omega_3}{\omega_0^2+\varepsilon\omega_1}+\mathscr{O}(\varepsilon^4)\right). \tag{3.1}$$

Then a new parameter α is chosen such as

$$\alpha(\varepsilon)=\frac{\varepsilon\omega_1}{\omega_0^2+\varepsilon\omega_1}. \tag{3.2}$$

Thus Eq. (3.1) can be rewritten in a series of α,

$$\omega^2=\frac{\omega_0^2}{(1-\alpha)}\left(1+\sum_{n=2}^{\infty}\delta_n\alpha^n\right). \tag{3.3}$$

Substituting the value of α from Eq. (3.2) into Eq. (3.3), we obtain a power series of ε,

$$\omega^2=\omega_0^2+\varepsilon\omega_1+\frac{\varepsilon^2\omega_1^2\delta_2}{\omega_0^2}+\frac{\varepsilon^3\omega_1^3(-\delta_2+\delta_3)}{\omega_0^4}+\frac{\varepsilon^4\omega_1^4(\delta_2-2\delta_3+\delta_4)}{\omega_0^6}+\mathscr{O}(\varepsilon^5). \tag{3.4}$$

Now Eq. (2.3) and Eq. (3.4) are identical. Therefore, we obtain

$$\frac{\omega_1^2\delta_2}{\omega_0^2}=\omega_2,\ \frac{\omega_1^3(-\delta_2+\delta_3)}{\omega_0^4}=\omega_3,\ \frac{\omega_1^4(\delta_2-2\delta_3+\delta_4)}{\omega_0^6}=\omega_4,\cdots, \tag{3.5}$$

or,

$$\delta_2=\frac{\omega_2\omega_0^2}{\omega_1^2},\ \delta_3=\frac{\omega_0^2\omega_1\omega_2+\omega_0^4\omega_3}{\omega_1^3},\ \delta_4=\frac{\omega_0^2\omega_1^2\omega_2+2\omega_0^4\omega_1\omega_3+\omega_0^6\omega_4}{\omega_1^4},\cdots. \tag{3.6}$$

The above relations measures the unknown coefficients $\delta_2,\delta_3,\cdots$ etc., where $\omega_0,\omega_1,\omega_2,\cdots$ etc. are calculated by classical Lindstedt-Poincare method [1,2]. Thus we can convert the frequency obtained by classical Lindstedt-Poincare method [1,2] to its modified form presented by Cheung et al. [6]. On the other hand transformation Eq. (3.2) makes Eq. (2.2) to the form

$$x=x_0+\alpha\tilde{x_1}+\alpha^2\tilde{x_2}+\mathscr{O}(\alpha^3). \tag{3.7}$$

The unknown coefficients $\tilde{x_1},\tilde{x_2},\cdots$ etc. still to be determined. We can substitute the value of α from Eq. (3.2) into Eq. (3.7) and obtain a series of ε,

$$x=x_0+\frac{\varepsilon\omega_1\tilde{x_1}}{\omega_0^2}+\frac{\varepsilon^2\omega_1^2\left(-\tilde{x_1}+\tilde{x_2}\right)}{\omega_0^4}+\frac{\varepsilon^3\omega_1^3\left(\tilde{x_1}-2\tilde{x_2}+\tilde{x_3}\right)}{\omega_0^6}+\mathscr{O}(\varepsilon^4). \tag{3.8}$$

Clearly that Eq. (2.2) is identical to Eq. (3.8). So, comparing equal powers of ε, we obtain the following algebraic equations:

$$\frac{\omega_1\tilde{x_1}}{\omega_0^2}=x_1,\ \frac{\omega_1^2\left(-\tilde{x_1}+\tilde{x_2}\right)}{\omega_0^4}=x_2,\ \frac{\omega_1^3\left(\tilde{x_1}-2\tilde{x_2}+\tilde{x_3}\right)}{\omega_0^6}=x_3,\cdots, \tag{3.9}$$

or,

$$\tilde{x_1}=\frac{\omega_0^2x_1}{\omega_1},\ \tilde{x_2}=\frac{\omega_0^2\omega_1x_1+\omega_0^4x_2}{\omega_1^2},\ \tilde{x_3}=\frac{\omega_0^2\omega_1^2x_1+2\omega_0^4\omega_1x_2+\omega_0^6x_3}{\omega_1^3},\cdots. \tag{3.10}$$

When $x_1,x_2,\cdots$ together with $\omega_0,\omega_1,\omega_2,\cdots$ are known, $\tilde{x_1},\tilde{x_2},\cdots$ are found by Eq. (3.10).

4. Example

Let us consider Duffing oscillator (cubical) $\ddot{x}+x+\varepsilon x^3=0$. For this problem, $\omega_0=1$ and $f(x,\dot{x},\ddot{x})=x^3$. Therefore, Eqs. (2.4)-(2.6) becomes

$$x_0''+x_0=0, \tag{4.1}$$

$$x_1''+x_1=-\omega_1x_0''-x_0^3, \tag{4.2}$$

$$x_2''+x_2=-3x_0^2x_1-x_1''\omega_1-x_0''\omega_2. \tag{4.3}$$

The solution of Eq. (4.1) is

$$x_0=a\cos\tau. \tag{4.4}$$

Substituting this value of x_0 in Eq. (4.2) and simplifying we obtain

$$x_1''+x_1=\omega_1a\cos\tau-\frac{3}{4}a^3\left(3\cos\tau+\cos3\tau\right). \tag{4.5}$$

It is noted that $x_1,x_2,\cdots$ do not contain the fundamental term to avoid secular terms. Therefore, the coefficient of $\cos\tau$ of Eq. (4.5) vanishes. Thus we obtain

$$\omega_1=\frac{3a^2}{4}. \tag{4.6}$$

The particular solution of Eq. (4.5) is

$$x_1=\frac{a^3\cos3\tau}{32}. \tag{4.7}$$

According to Lindstedt-Poincare method, $x_1(0)=x_1'(0)=0$. Therefore, the solution of Eq. (4.5) becomes

$$x_1=\frac{a^3\left(-\cos\tau+\cos3\tau\right)}{32}. \tag{4.8}$$

It has already been mentioned that we do strictly follow this rule. We may consider

$$x_1=\frac{a^3\left(-k\cos\tau+\cos3\tau\right)}{32}, \tag{4.9}$$

where k is a constant.
Alam et al. [9] was chosen a periodic solution of $\ddot{x}+\omega_0^2 x=\varepsilon f(x)$, $x(0)=a_0$, $\dot{x}(0)=0$, as

$$x=a\cos\varphi+a^3C_3(a)\cos 3\varphi+a^5C_5(a)\cos 5\varphi+\mathscr{O}(a^7),$$

where a and $\dot{\varphi}$ are constants. Alam et al. [9] considered above solution by choosing $k=0$.
$k=1$ is strictly followed by Cheung et al. [6] and various methods of perturbation for solving nonlinear oscillators. Thus the value of k can be considered as parameter. This will give us additional variation to find more accurate solutions of nonlinear oscillators. Determination of higher order solution will increase accuracy of the solution. But choosing k as a parameter we have found faster convergent solutions without finding higher order approximations. By finding a proper value of k, solution can be made more accurate with first few approximations. We have introduced k in the first approximate solution and consequently k appear in the second, third and fourth approximations.
Choosing a suitable value of k, we can find a series of ω which converge faster than that of obtained by Cheung et al. [6] and Alam et al. [14]. Carrying on a similar process, we have solved the higher order equations (e.g., Eq. (4.3),$\cdots$) and obtained the following results:

$$\omega_2=-\frac{3}{128}a^4\left(-1+2k\right),\ \omega_3=\frac{3a^6\left(-19+36k+7k^2\right)}{4096},\ \omega_4=-\frac{3a^8\left(-335+556k+342k^2+30k^3\right)}{131072}, \tag{4.10}$$

and

$$x_2=C_{2,1}\cos\tau+C_{2,3}\cos 3\tau+C_{2,5}\cos 5\tau,$$
$$x_3=C_{3,1}\cos\tau+C_{3,3}\cos 3\tau+C_{3,5}\cos 5\tau+C_{3,7}\cos 7\tau,$$

$$x_4=C_{4,1}\cos\tau+C_{4,3}\cos 3\tau+C_{4,5}\cos 5\tau+C_{4,7}\cos 7\tau+C_{4,9}\cos 9\tau, \tag{4.11}$$

where

$$C_{2,1}=\frac{a^5\left(20k+3k^2\right)}{1024},\ C_{2,3}=\frac{-a^5\left(21+3k\right)}{1024},\ C_{2,5}=\frac{a^5}{1024},\ C_{3,1}=-\frac{a^7k\left(375+160k+12k^2\right)}{32768},$$
$$C_{3,3}=\frac{3a^7\left(139+55k+4k^2\right)}{32768},\ C_{3,5}=-\frac{a^7\left(43+5k\right)}{32768},\ C_{3,7}=\frac{a^7}{32768},\ C_{4,1}=\frac{a^9k\left(6521+5750k+1100k^2+55k^3\right)}{1048576},$$

$$C_{4,3}=-\frac{a^9\left(7797+6144k+1125k^2+55k^3\right)}{1048576},\ C_{4,5}=\frac{a^9\left(1340+401k+25k^2\right)}{1048576},\ C_{4,7}=-\frac{a^9(65+7k)}{1048576},\ C_{4,9}=\frac{a^9}{1048576}. \tag{4.12}$$

Now utilizing the transformation formulae Eq. (3.6) and Eq. (3.10), we obtain respectively

$$\delta_2=\frac{1}{24}\left(1-2k\right),\delta_3=\frac{1}{576}\left(5-12k+7k^2\right),\delta_4=\frac{-1+20k-6k^2-30k^3}{13824}, \tag{4.13}$$

and

$$\tilde{x}_1=\tilde{C}_{1,1}\cos\tau+\tilde{C}_{1,3}\cos 3\tau,$$
$$\tilde{x}_2=\tilde{C}_{2,1}\cos\tau+\tilde{C}_{2,3}\cos 3\tau+\tilde{C}_{2,5}\cos 5\tau,$$
$$\tilde{x}_3=\tilde{C}_{3,1}\cos\tau+\tilde{C}_{3,3}\cos 3\tau+\tilde{C}_{3,5}\cos 5\tau+\tilde{C}_{3,7}\cos 7\tau,$$

$$\tilde{x}_4=\tilde{C}_{4,1}\cos\tau+\tilde{C}_{4,3}\cos 3\tau+\tilde{C}_{4,5}\cos 5\tau+\tilde{C}_{4,7}\cos 7\tau+\tilde{C}_{4,9}\cos 9\tau, \tag{4.14}$$

where

$$\tilde{C}_{1,1}=-\frac{ak}{24},\ \tilde{C}_{1,3}=\frac{a}{24},\ \tilde{C}_{2,1}=\frac{ak\left(-4+3k\right)}{576},\ \tilde{C}_{2,3}=\frac{a(1-k)}{192},\ \tilde{C}_{2,5}=\frac{a}{576},\ \tilde{C}_{3,1}=-\frac{ak\left(-9+16k+12k^2\right)}{13824},$$

$$\tilde{C}_{3,3}=\frac{a\left(-5+7k+4k^2\right)}{4608},\ \tilde{C}_{3,5}=-\frac{5a\left(-1+k\right)}{13824},\ \tilde{C}_{3,7}=\frac{a}{13824},\ \tilde{C}_{4,1}=\frac{ak\left(257-586k+236k^2+55k^3\right)}{331776},$$

$$\tilde{C}_{4,3}=-\frac{a\left(237-552k+261k^2+55k^3\right)}{331776},\ \tilde{C}_{4,5}=\frac{a\left(-28+41k+25k^2\right)}{331776},\ \tilde{C}_{4,7}=-\frac{7a\left(-1+k\right)}{331776},\ \tilde{C}_{4,9}=\frac{a}{331776}. \tag{4.15}$$

For the initial conditions, we obtain

$$\tilde{x}_1(0) = \frac{a(1-k)}{24},\ \tilde{x}_2(0) = \frac{a\left(4-7k+3k^2\right)}{576},\ \tilde{x}_3(0) = \frac{a\left(-9+25k-4k^2-12k^3\right)}{13824},$$

$$\tilde{x}_4(0) = \frac{a\left(-257+843k-822k^2+181k^3+55k^4\right)}{331776}. \tag{4.16}$$

It is clear that $\tilde{x}_1(0) = \tilde{x}_2(0) = \tilde{x}_3(0) = \tilde{x}_4(0) = 0$ when $k = 1$ and $x(0) = a_0 = a$. When $k \neq 1$, we obtain the following nonlinear algebraic equation

$$a_0 = a\left(1 + \frac{a(1-k)\alpha}{24} + \frac{a\left(4-7k+3k^2\right)\alpha^2}{576} + \frac{a\left(-9+25k-4k^2-12k^3\right)\alpha^3}{13824} + \frac{a\left(-257+843k-822k^2+181k^3+55k^4\right)\alpha^4}{331776}\right), \tag{4.17}$$

where $\alpha = \frac{\frac{3a^2}{4}}{1+\frac{3a^2}{4}}$. In general a_0 is given; so that a would be found solving Eq. (4.17) by an iteration method (numerical). It is noted that the higher order terms of α are small whatever the values of a and ε if we chose a suitable value of k. Therefore it requires one or two iterations to obtain a desired result.

5. Results and discussion

A faster convergent modified Lindstedt-Poincare solution has been determined. The solution is identical to that of Cheung et al. [6] and Alam et al. [14] for $k = 1$. When $k = 1$, then from Eq. (4.13) we get,

$$\delta_2 = -\frac{1}{24}, \delta_3 = 0, \delta_4 = -\frac{17}{13824}.$$

The above results are same as obtained by Cheung et al. [6] and Alam et al. [14]. When $k = \frac{5}{7}$, we obtain

$$\delta_2 = -\frac{1}{56}, \delta_3 = 0, \delta_4 = -\frac{9}{175616}.$$

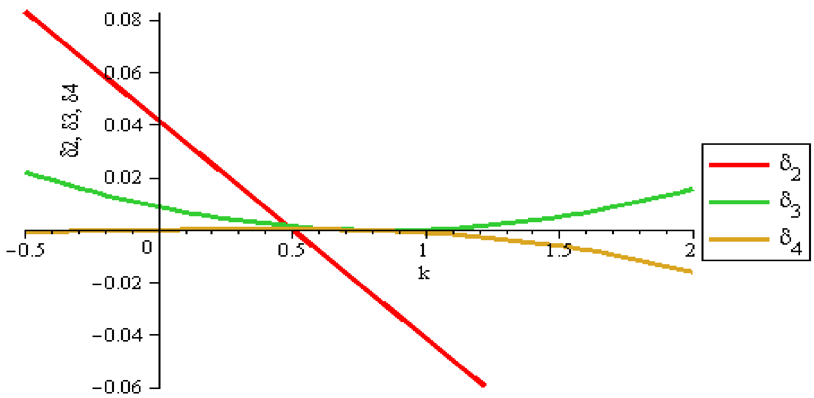

Figure 5.1: Variation of $\delta_2, \delta_3, \delta_4$ with k for duffing oscillator to determine small value of $\delta_2, \delta_3, \delta_4$.

It is clear that the α-series (Eq. (3.3)) converges faster when coefficients $|\delta_i|, i = 2, 3, \cdots$ etc. become small. We have plotted $\delta_2, \delta_3, \delta_4$ against k in the Fig. 5.1 for Duffing oscillator. We have found that $\delta_2, \delta_3, \delta_4$ all are small in the region $0.4 < k < 1$. The series (Eq. (3.3)) of frequency for the Duffing oscillator converges faster when $k = \frac{5}{7}$. For several values of a_0, the frequency ω have been calculated for both $k = 1$ (Alam et al. [14] and Cheung et al. [6]) and $k = \frac{5}{7}$, and presented in Table 1 together with numerical results obtained by Runge-Kutta 4^{th} order method.

It is hard to say what would be the suitable value of k for other nonlinear oscillators. We have plotted $\delta_2, \delta_3, \delta_4$ against k in the Fig. 5.2 for the quintic oscillator. We find from Fig. 5.2 that $\delta_2, \delta_3, \delta_4$ all are small in the region $0 < k < 1$. For the cubic Duffing oscillator, we see that δ_3 vanishes for both $k = 1$ and $k = \frac{5}{7}$. But for the quintic oscillator (i.e., $\ddot{x} + x + \varepsilon x^5 = 0$) δ_3 never vanishes. For this oscillator, we have obtained

$$\delta_2 = \frac{1}{120}(19-32k),\ \delta_3 = \frac{1}{14400}\left(1009-254k+1664k^2\right),$$
$$\delta_4 = \frac{1}{1728000}\left(14441-65806k+140552k^2-104448k^3\right).$$

We see from Fig. 5.2 that the values of these coefficients are opposite in sign when $\frac{19}{32} < k$. But all are positive when $k \leq \frac{19}{32}$ and δ_2 vanishes when $k = \frac{19}{32}$. Thus for $k = 1$ and $k = \frac{19}{32}$, we have obtained respectively

a_0	$\omega(k=1)$ $Er(\%)$	$\omega(k=\frac{5}{7})$ $Er(\%)$	ω_{nu}
1	1.31778 0.00000	1.31778 0.00000	1.31778
10	8.53390 0.003633	8.53351 0.000937	8.53359
100	84.7309 0.004013	84.7266 0.001062	84.7275
1000	847.248 0.004013	847.205 0.001062	847.214

Table 1: Comparison of the approximate frequencies obtained by present method with the numerical and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$) for the Duffing oscillator (where $Er(\%)$ denotes absolute percentage error).

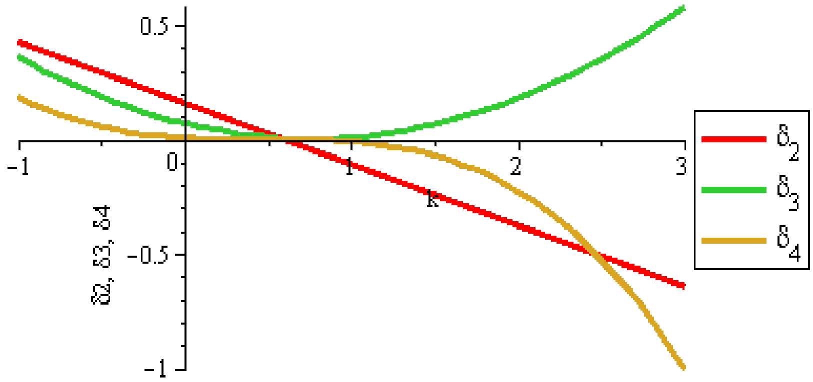

Figure 5.2: Variation of $\delta_2, \delta_3, \delta_4$ with k for quintic oscillator to determine small value of $\delta_2, \delta_3, \delta_4$.

$$\delta_2 = -\frac{13}{120},\ \delta_3 = \frac{2}{225},\ \delta_4 = -\frac{5087}{576000},$$

and

$$\delta_2 = 0,\ \delta_3 = \frac{541}{92160},\ \delta_4 = \frac{391129}{221184000}.$$

Comparing these results, we easily expect that α-series (Eq. (3.3)) converges faster for $k=\frac{19}{32}$. To verify this matter, we have calculated some results choosing $k=1$ (Alam et al. [14] and Cheung et al. [6]) and $k=\frac{19}{32}$ and presented in Table 2 together with corresponding numerical results.

a_0	$\omega(k=1)$ $Er(\%)$	$\omega(k=\frac{19}{32})$ $Er(\%)$	ω_{nu}
1	1.26470 0.000791	1.26471 0.000000	1.26471
10	74.6618 0.038961	74.6768 0.018878	74.6909
100	7465.44 0.038831	7466.93 0.018880	7468.34
1000	746531.22 0.040569	746701.04 0.0178304	746834.20

Table 2: Comparison of the approximate frequencies obtained by present method with the numerical and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$) for the quintic oscillator (where $Er(\%)$ denotes absolute percentage error).

For the nonlinear oscillator $\ddot{x}+x+\varepsilon\dot{x}^2x=0$ we have obtained

$$\delta_2 = \frac{1}{8}(3+2k),\ \delta_3 = \frac{1}{192}\left(63+76k+21k^2\right),$$

$$\delta_4 = \frac{1}{1563}\left(407+668k+426k^2+90k^3\right).$$

Thus for $k=1$ and $k=\frac{2}{5}$, we have obtained respectively

$$\delta_2 = \frac{5}{8},\ \delta_3 = \frac{5}{6},\ \delta_4 = \frac{1591}{1536},$$

and

$$\delta_2 = \frac{19}{40},\ \delta_3 = \frac{2419}{4800},\ \delta_4 = \frac{18703}{38400}.$$

We have calculated some results choosing $k=1$ (Alam et al. [14] and Cheung et al. [6]) and $k=\frac{2}{5}$ and presented in Table 3 together with corresponding numerical results. From Fig. 5.3 we see that $\delta_2, \delta_3, \delta_4$ are small near $k=\frac{-3}{2}$ but for $k=\frac{2}{5}$ obtained results are better for larger values of a_0.

a_0	$\omega(k=1)$ $Er(\%)$	$\omega(k=\frac{-3}{2})$ $Er(\%)$	$\omega(k=\frac{2}{5})$ $Er(\%)$	ω_{nu}
0.01	1.00001 0.000000	1.00001 0.000000	1.00001 0.000000	1.00001
0.1	1.00125 0.000000	1.00125 0.000000	1.00125 0.000000	1.00125
1	1.13651 0.023713	1.13682 0.0035187	1.13666 0.0105561	1.13678
10	9.12723 8.07624	10.3405 4.14306	9.95623 0.272934	9.92913
100	93.4396 6.55395	104.866 4.87324	101.947 1.95403	99.9931

Table 3: Comparison of the approximate frequencies obtained by present method with the numerical and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$) for the oscillator $\ddot{x}+x+\varepsilon\dot{x}^2x=0$ (where $Er(\%)$ denotes absolute percentage error).

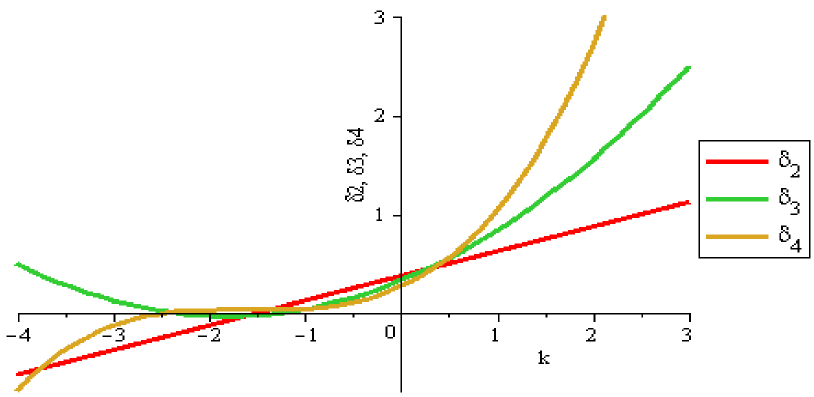

Figure 5.3: Variation of $\delta_2, \delta_3, \delta_4$ with k for the oscillator $\ddot{x}+x+\varepsilon\dot{x}^2x=0$ to determine small value of $\delta_2, \delta_3, \delta_4$.

For the nonlinear oscillator $\ddot{x}+x+\varepsilon\ddot{x}x^2=0$, we have obtained

$$\delta_2 = \frac{1}{72}(-11+6k),\ \delta_3 = \frac{1}{1728}\left(-17-36k+21k^2\right),$$
$$\delta_4 = \frac{1}{124416}\left(-3359+2812k-1242k^2+270k^3\right).$$

Thus for $k=1$, we have obtained

$$\delta_2 = \frac{-5}{72},\ \delta_3 = \frac{-1}{54},\ \delta_4 = \frac{1519}{124416},$$

and which are same as obtained in Alam et al. [14].

For different values of the unknown constant k we have calculated some results and presented in Table 4 together with corresponding numerical results and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$). From Table 4 it is clear that frequency of the oscillator depends on the parameter k and comparing various results suitable value of k can be determined. From Fig 5.4 we see the variation of $\delta_2, \delta_3, \delta_4$ with the unknown constant k, shows the region of convergence.

6. Conclusion

The modified Lindstedt-Poincare method of Cheung et al. [6] based on Alam et al. [14] has been presented in a new form introducing an unknown constant, k. All the coefficients related to the solution depend on this constant. When $k=1$, the solution is identical to that of Cheung et al. [6] and Alam et al. [14]. But a better result would be found for a particular value of k. Comparing various results of the unknown coefficients, $|\delta_i(k)|, i=2,3,\cdots$, the suitable value of k can be determined. The method is applied to obtain the approximate solution of Duffing oscillator, quintic oscillator and another two nonlinear equations whose nonlinear response is significant. All the solutions show a good agreement with numerical solutions obtained by Runge-Kutta 4^{th} order method and provide better results than other existing solutions. The results may be useful to the researches in the field of nonlinear mechanics for investigating periodic solution of some higher order nonlinear problems.

a_0	$\omega(k=1)$ $Er(\%)$	$\omega(k=2)$ $Er(\%)$	$\omega(k=3)$ $Er(\%)$	$\omega(k=5)$ $Er(\%)$	ω_{nu}
0.01	0.999963 0.000000	0.999963 0.000000	0.999963 0.000000	0.999963 0.000000	0.999963
0.1	0.996273 0.000000	0.996273 0.000000	0.996273 0.000000	0.996273 0.000000	0.996273
1	0.761518 0.00800967	0.761545 0.00446441	0.761568 0.00144438	0.761712 0.0174637	0.761579
10	0.120712 2.11720	0.121174 1.74258	0.121670 1.34038	0.124195 0.707086	0.123323
100	0.0121717 2.83240	0.0122225 2.42686	0.0122776 1.98699	0.0125643 0.30176	0.0125256
1000	0.00121728 2.87246	0.00122235 2.46792	0.00122788 2.02668	0.00125658 0.263309	0.00125328
10000	0.000121728 2.87479	0.000122235 2.47026	0.000122788 2.02903	0.000125658 0.260909	0.000125331

Table 4: Comparison of the approximate frequencies obtained by present method with the numerical and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$) for the oscillator $\ddot{x}+x+\varepsilon\ddot{x}x^2=0$ (where $Er(\%)$ denotes absolute percentage error).

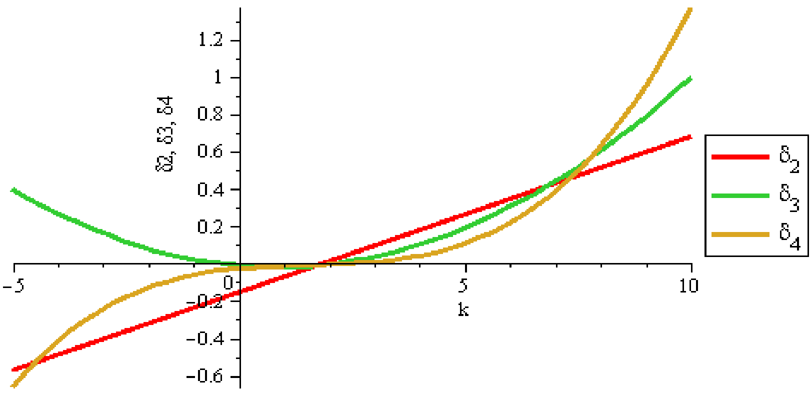

Figure 5.4: Variation of $\delta_2, \delta_3, \delta_4$ with k for the oscillator $\ddot{x}+x+\varepsilon\ddot{x}x^2=0$ to determine small value of $\delta_2, \delta_3, \delta_4$.

Acknowledgement

Authors are grateful to the Editor-In-Chief of the Journal and the anonymous reviewers for their valuable comments and suggestions which improved the quality and presentation of the paper.

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Universal Journal of Mathematics and Applications, 3 (2) (2020) 61-68
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: *www.dergipark.gov.tr/ujma*
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.636098

Approximate Analytical Solutions of Conformable Time Fractional Clannish Random Walker's Parabolic(CRWP) Equation and Modified Benjamin-Bona-Mahony(BBM) equation

Emrah Atilgan[1], Orkun Tasbozan[1], Ali Kurt[2*] and Syed Tauseef Mohyud-Din[3]

[1]*Hatay Mustafa Kemal University, Hatay, Türkiye*
[2]*Pamukkale University, Denizli, Türkiye*
[3]*Department of Mathematics, Faculty of Sciences, HITEC University, Taxila Cantt, Pakistan*
[*]*Corresponding author*

Article Info

Keywords: *Conformable Derivative, Fractional Clannish Random Walker's Parabolic(CRWP) Equation, Modified Benjamin-Bona-Mahony(BBM) equation.*
2010 AMS: *35R11, 34A08, 35A20, 26A33.*
Received: *22 October 2019*
Accepted: *3 June 2020*
Available online: *22 June 2020*

Abstract

In this paper, we propose the approximate analytical solutions of conformable time fractional Clannish Random Walker's Parabolic(CRWP) equation and Modified Benjamin-Bona-Mahony(BBM) equation with the aid of generalized homotopy analysis method (q-HAM). The h curves of approximate solutions for both equations are illustrated by graphics to determine the convergence interval. h values obtained from these graphics are used to compare approximate solutions with the analytical solutions. The results show that approximate solutions are consistent with the analytical solutions. Also it is understood that the method is reliable, applicable and efficient technique to get the exact solutions of fractional partial differential equations.

1. Introduction

Fractional derivative and fractional integration are challenging works in applied mathematics. They can also be used in different branches of science and engineering. Thus, there are many approaches in literature to define fractional derivative and integration. Among them, Riemann-Lioville definition and Caputo definition are the most used ones [1–3]. However, these definitions have some drawbacks in the following cases:

1. The Riemann-Lioville derivative does not satisfy $D_a^\alpha 1 = 0$ (Liouville-Caputo derivative satisfies), if α is not a natural number.
2. All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions.

$$D_a^\alpha(fg) = gD_a^\alpha(f) + fD_a^\alpha(g)$$

3. All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions.

$$D_a^\alpha\left(\frac{f}{g}\right) = \frac{fD_a^\alpha(f) - gD_a^\alpha(g)}{g^2}$$

4. All fractional derivatives do not satisfy the chain rule.

$$D_a^\alpha(fog)(t) = f^\alpha(g(t))g^\alpha(t)$$

5. All fractional derivatives do not satisfy $D^\alpha D^\beta = D^{\alpha+\beta}$ in general.
6. The Caputo definition assumes that the function f is differentiable.

Email addresses and ORCID numbers: **emrahatilgan@mku.edu.tr (E. Atilgan), otasbozan@mku.edu.tr (O .Tasbozan),pau.dr.alikurt@gmail.com, akurt@pau.edu.tr (A. Kurt), syedtauseefs@hotmail.com (S. T. Mohyud-Din)**

Because of these drawbacks of the existing definitions, researchers have been working on a better definition so that all the cases can be satisfied. In recent years, a new simple definition was introduced by Khalil *et.al* [4] called conformable fractional derivative. For the function $f:[0,\infty)\to\mathbb{R}$, the fractional derivative of f was defined by

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

where $t>0$ and $\alpha\in(0,1)$.
If f is α-differentiable in some $(0,a), a>0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t)$. The *"conformable fractional integral"* of a function f starting from $a\geq 0$ is defined as:

$$I_\alpha^a(f)(t)=\int_a^t \frac{f(x)}{x^{1-\alpha}}dx$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$. The following properties of conformable fractional derivative are given in [4].

Theorem 1.1. *Let $\alpha\in(0,1]$ and suppose f,g are α-differentiable at point $t>0$. Then*

1. $T_\alpha(cf+dg)=cT_\alpha(f)+cT_\alpha(g)$ *for all* $a,b\in\mathbb{R}$.
2. $T_\alpha(t^p)=pt^{p-\alpha}$ *for all* $p\in\mathbb{R}$.
3. $T_\alpha(\lambda)=0$ *for all constant functions* $f(t)=\lambda$.
4. $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$.
5. $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
6. *If, in addition f is differentiable, then* $T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}$.

Recently, Abdeljawad applied Khalil's definition on some fractional calculus functions, such as fractional versions of chain rule, exponential functions, Laplace transforms and so on [5]. Due to the simplicity and usefulness of this conformable version of fractional calculus, many researches applied this method in their field. Chung [6] applied conformable fractional derivatives on Newton's mechanics and constructed the fractional Euler-Lagrange equation. Neirameh [7] applied this method on a fractional order of extended biological population model. Benkhettou *et al.* [8] extended Khalil's definition to an arbitrary time scale. Similarly, Zhao and Li [9] introduced the conformable delta fractional derivative and delta fractional integral on time series with respect to Khalil's definition. Eslami and Rezazadeh [10] studied the first integral method for Wu-Zhang system with conformable time fractional derivative.
The q-Homotopy Analysis Method (q-HAM) was introduced by El-Tawil and Huseen [11] to solve non-linear differential equations. This is a more general method of Homotopy Analysis Method [17–20] (HAM) developed by Liao [12] and has one more parameter n. With the aid of this additional parameter, q-HAM provides more flexibility than HAM in controlling and adjusting the convergence region. q-HAM have been used to solve many mathematical problems in recent years. Iyiola *et.al.* [13] investigated an analytical solution of the time-fractional foam drainage equation using the advantages of q-HAM. In another study, q-HAM was used to find approximate series solutions of a fractional diffusion equation model [14].
The CRWP equation determines the behavior of two species A and B of random walkers who execute a concurrent one-dimensional random walk characterized by an intensification of the clannishness of the members of one species as the density of the other increases and the Benjamin-Bona-Mahony(BBM) equation is used for the analysis of the surface waves of long wavelength in liquids, hydromagnetic waves in cold plasma.

2. Description of q-Homotopy Analysis Method

We give an overview of the q-homotopy analysis method in this section and show how it is used in fractional differential equations.
Consider the differential equation below,

$$\mathscr{N}\left[D_t^\alpha u(x,t)\right]-g(x,t)=0 \tag{2.1}$$

where $\mathscr{N}$ is non-linear operator, D_t^α is the conformable fractional operator, g is the given function and $u(x,t)$ is the function which will be obtained after this solution procedure. The zeroth-order deformation equation for q-HAM is given as follows:

$$(1-nq)\mathscr{L}\left(\varphi(x,t;q)-u_0(x,t)\right)=qhH(x,t)\left(\mathscr{N}[D_t^\alpha\varphi(x,t;q)]-g(x,t)\right), \tag{2.2}$$

where $n\geqslant 1$, $q\in\left[0,\frac{1}{n}\right]$ is the embedded parameter, $h\neq 0$ is an auxiliary parameter and L is an auxiliary linear operator. $H(x,t)$ is a non-zero auxiliary function.
When $q=0$ and $q=\frac{1}{n}$, which are the boundary values, we have equation (2.2) to become

$$\varphi(x,t;0)=u_0(x,t)\quad \textit{and}\quad \varphi\left(x,t;\frac{1}{n}\right)=u(x,t) \tag{2.3}$$

respectively. When we increment q from 0 to $\frac{1}{n}$, the solution $\varphi(x,t;q)$ changes from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.
If $u_0(x,t)$, $\mathscr{L}$, h, $H(x,t)$ are chosen properly, then there always exists a solution $\varphi(x,t;q)$ of equation (2.2) for $q\in\left[0,\frac{1}{n}\right]$.
The Taylor series expansion of $\varphi(x,t;q)$ is

$$\varphi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m. \tag{2.4}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\Big|_{q=0}. \tag{2.5}$$

If the auxiliary linear operator L, the initial guess u_0, the auxiliary parameter h and $H(x,t)$ are properly chosen, then the Taylor series expansion of $\varphi(x,t;q)$ (2.4) converges at $q = \frac{1}{n}$. Hence, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{2.6}$$

Let the vector $\vec{u}_n$ is defined as follows:

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\}. \tag{2.7}$$

First, the equation (2.2) is differentiated m-times with respect to the embedding parameter q. Then, $q = 0$ is taken and placed in the differentiated equation. Finally, the whole equation is divided by $m!$, and the m^{th}-order deformation equation is obtained as follows:

$$\mathscr{L}\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = hH(x,t)\mathscr{R}_m(\vec{u}_{m-1}). \tag{2.8}$$

with initial conditions

$$u_m^{(k)}(x,0) = 0, \quad k = 0,1,2,...,m-1. \tag{2.9}$$

where

$$\mathscr{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\left(\mathscr{N}[Du_t^\alpha \varphi(x,t;q)] - g(x,t)\right)}{\partial q^{m-1}}\Big|_{q=0} \tag{2.10}$$

and

$$\chi_m^* = \begin{cases} 0 & m \leqslant 1 \\ n & otherwise. \end{cases} \tag{2.11}$$

3. Applications of the Method

3.1. Conformable Time-Fractional Clannish Random Walker's Parabolic(CRWP) Equation

Considering the conformable time-fractional CRWP equation as:

$$D_t^\alpha v + pv_x + svv_x + rv_{xx} = 0 \tag{3.1}$$

and initial condition

$$v(x,0) = \frac{r\ln(A) + \sqrt{r^2(\ln(A))^2 - 2sK}}{s} - \frac{2r\ln(A)}{s(1+dA^x)} \tag{3.2}$$

where $\alpha \in (0,1)$. The exact solution of CRWP was provided from ref. [15] as:

$$v(x,t) = \frac{r\ln(A) + \sqrt{r^2(\ln(A))^2 - 2sK}}{s} - \frac{2r\ln(A)}{s(1+dA^{x-(\sqrt{r^2(\ln(A))^2-2sK}+p)\frac{t^\alpha}{\alpha}})} \tag{3.3}$$

This analytical solution is used to compare with the numerical solutions obtained in this study. To obtain the approximate solution of Eq. (3.1) with initial condition (3.3), we can choose the linear operator is as

$$\mathscr{L}[\varphi(x,t;q)] = D_t^\alpha \varphi(x,t;q)$$

which satisfies the property

$$\mathscr{L}[m] = 0$$

where m is a constant. From Eq. (3.1), we define the nonlinear operator as follow:

$$\mathscr{N}[\varphi(x,t;q)] = \frac{\partial^\alpha \varphi(x,t;q)}{\partial t^\alpha} + p\frac{\partial \varphi(x,t;q)}{\partial x} + s\varphi(x,t;q)\frac{\partial \varphi(x,t;q)}{\partial x} + r\frac{\partial^2 \varphi(x,t;q)}{\partial x^2}.$$

From Theorem 1, the nonlinear operator can be re-written as follows:

$$\mathscr{N}[\varphi(x,t;q)] = t^{1-\alpha}\frac{\partial \varphi(x,t;q)}{\partial t} + p\frac{\partial \varphi(x,t;q)}{\partial x} + s\varphi(x,t;q)\frac{\partial \varphi(x,t;q)}{\partial x} + r\frac{\partial^2 \varphi(x,t;q)}{\partial x^2}.$$

The zeroth-order deformation equation is designed as:

$$(1-nq)\mathscr{L}\left[\varphi(x,t;q)-v_0(x,t)\right]=qh\mathscr{N}\left[\varphi(x,t;q)\right].$$

Considering $H(x,t)=1$, the m^{th}-order deformation equation

$$\mathscr{L}\left[v_m(x,t)-\chi_m^* v_{m-1}(x,t)\right]=hR_m\left(\mathbf{v}_{m-1}\right) \tag{3.4}$$

where

$$R_m\left(\mathbf{v}_{m-1}\right)=t^{1-\alpha}\frac{\partial v_{m-1}(x,t)}{\partial t}+s\sum_{n=0}^{m-1}v_n(x,t)\frac{\partial v_{m-1-n}(x,t)}{\partial x}+p\frac{\partial v_{m-1}(x,t)}{\partial x}+r\frac{\partial^2 v_{m-1}(x,t)}{\partial x^2}.$$

The solutions of the m^{th}-order deformation Eq. (3.4) for $m\geq 1$ is below

$$v_m(x,t)=\chi_m^* v_{m-1}(x,t)+h\mathscr{L}^{-1}\left[R_m\left(\mathbf{v}_{m-1}\right)\right]. \tag{3.5}$$

By using Eq.(3.5) with initial condition given by (3.3) we calculated $v_m(x,t)$ for $m=0,1$ respectively.

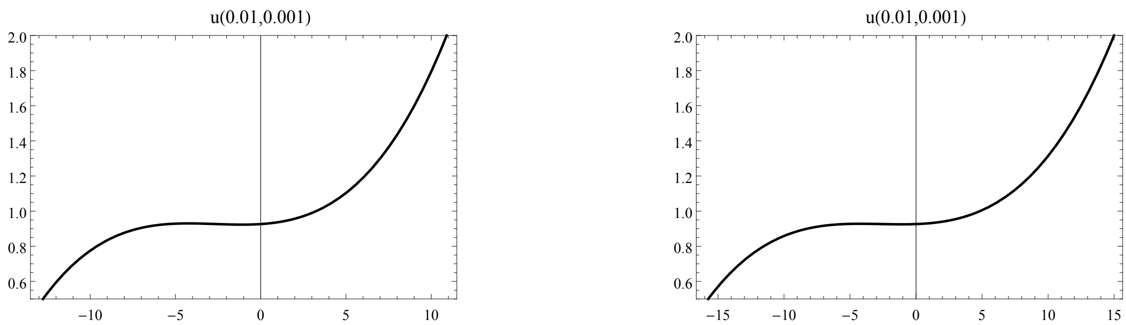
$$v_0(x,t)=\frac{r\ln(A)+\sqrt{r^2(\ln(A))^2-2sK}}{s}-\frac{2r\ln(A)}{s(1+dA^x)},$$

$$v_1(x,t)=\frac{2t^\alpha A^x dhr(\ln(A))^2(p+\sqrt{-2ks+r^2(\ln(A))^2})}{(1+A^x d)^2 s\alpha}.$$

One can obtain $v_m(x,t)$ for $m=2,3,\cdots$, following the same procedure using computer software such as Mathematica.
Finally, the series solution of equation (3.1) by applying q-HAM can be written in the form

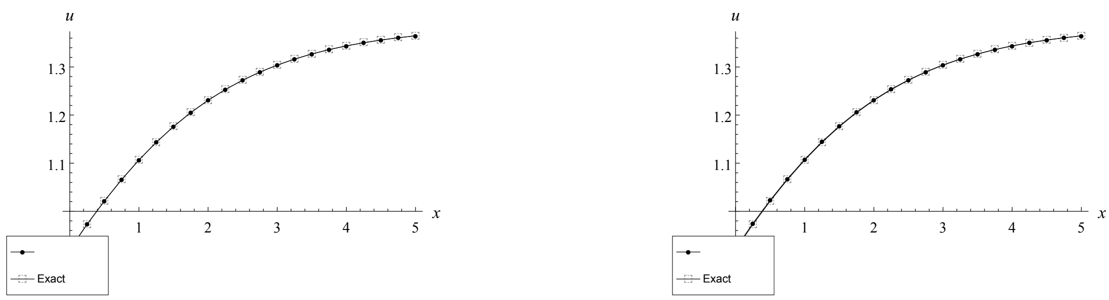
$$v(x,t,n,h)=v_0(x,t)+\sum_{n=1}^{\infty}v_i(x,t;n;h)\left(\frac{1}{n}\right)^i. \tag{3.6}$$

Equation (3.6) is an approximate solution to the problem (3.1) in terms of convergence parameter h and n. We take the parameters $p=2$, $q=2$, $r=2$, $d=2$, $A=2$ and $k=2$ are used for all calculations by 4^{th}-order q-HAM solution. Any other solutions can be calculated for different parameters.
The h-curves of the conformable time-fractional CRWP equation for $n=2$ and for $\alpha=0.8$ and $\alpha=0.9$ are given in Fig. 1. For $h=-1.11$, $t=0.001$ (fixed) and $\alpha=0.8$, $\alpha=0.9$, the approximate solution and the exact solution for the conformable time-fractional CRWP equation are compared in Fig. 2.



(a) $\alpha=0.8$ **(b)** $\alpha=0.9$

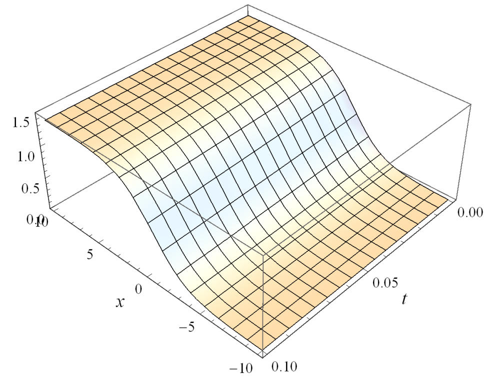
Figure 3.1: The h-curves of the conformable time-fractional CRWP equation for $n=2$ and different values of α.



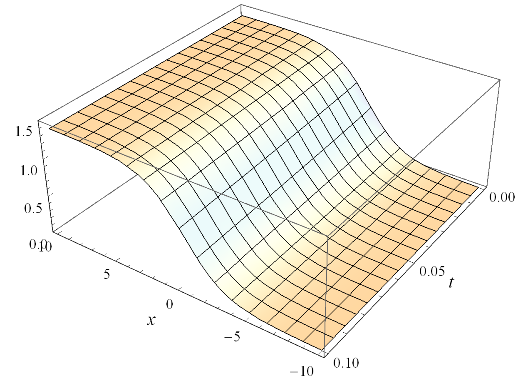
(a) Approximate solution for $\alpha=0.8$ **(b)** Approximate solution for $\alpha=0.9$

Figure 3.2: Comparison of the approximate solutions obtained for $h=-1.11$, $n=2$, $t=0.001$ and different values of α by the q-HAM with the exact solution for the conformable time-fractional CRWP equation

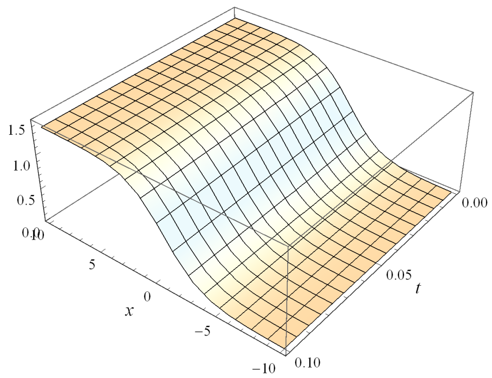
For $h=-1.11$ value which is selected from these graphics for both α values, the surfaces of the approximate solutions and the exact solutions are drawn in Fig. 3. As seen in Figure 3, the surfaces of the approximate solutions and the analytical solutions are so similar that cannot be distinguished.



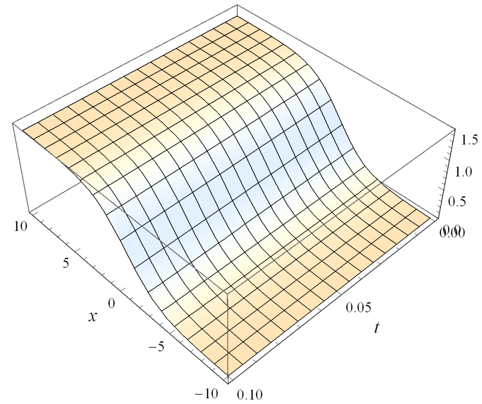
(a) Approximate solution for $\alpha = 0.8$



(b) Exact solution for $\alpha = 0.8$



(c) Approximate solution for $\alpha = 0.9$



(d) Exact solution for $\alpha = 0.9$

Figure 3.3: Comparison of the approximate solutions obtained for $h = -1.11$, $n = 2$ and different values of α by the q-HAM with the exact solution for the conformable time-fractional CRWP equation

3.2. Conformable Time-Fractional Modified Benjamin-Bona-Mahony(BBM) Equation

As the second application of q-HAM, we discuss the approximate solutions for BBM. The conformable time-fractional BBM equation is given as:

$$D_t^\alpha u + u_x - \nu u^2 u_x + u_{xxx} = 0 \tag{3.7}$$

and initial condition

$$u(x,0) = \frac{\sqrt{3}}{x+1} \tag{3.8}$$

where $\alpha \in (0,1)$. The exact solution was taken from [16]

$$u(x,t) = \frac{\sqrt{6}k}{\sqrt{\nu}(kx - k\frac{t^\alpha}{\alpha} + C)} \tag{3.9}$$

and $k = 1, C = 1, \nu = 2$ are chosen for all calculations by 4^{th}-order q-HAM solution. To find the series approximate solution of Eq. (3.7) with the initial condition (3.9), the linear operator is picked as

$$\mathscr{L}[\varphi(x,t;q)] = D_t^\alpha \varphi(x,t;q)$$

with the property

$$\mathscr{L}[m] = 0$$

where m is a constant. From Eq. (3.7), the nonlinear operator can be designed as following,

$$\mathscr{N}[\varphi(x,t;q)] = \frac{\partial^\alpha \varphi(x,t;q)}{\partial t^\alpha} + \frac{\partial \varphi(x,t;q)}{\partial x} - \nu \varphi(x,t;q)^2 \frac{\partial \varphi(x,t;q)}{\partial x} + \frac{\partial^3 \varphi(x,t;q)}{\partial x^3}.$$

Using the property of the conformable fractional derivative, the nonlinear operator can be re-written as follows,

$$\mathscr{N}[\varphi(x,t;q)] = t^{1-\alpha} \frac{\partial \varphi(x,t;q)}{\partial t} + \frac{\partial \varphi(x,t;q)}{\partial x} - \nu \varphi(x,t;q)^2 \frac{\partial \varphi(x,t;q)}{\partial x} + \frac{\partial^3 \varphi(x,t;q)}{\partial x^3}.$$

Hence, the zeroth-order deformation equation can be constructed as:

$$(1-nq)\mathscr{L}[\varphi(x,t;q)-u_0(x,t)] = qh\mathscr{N}[\varphi(x,t;q)].$$

For $H(x,t)=1$, the mth-order deformation equation turns into

$$\mathscr{L}[u_m(x,t)-\chi_m^* u_{m-1}(x,t)] = hR_m(\mathbf{u}_{m-1}) \tag{3.10}$$

where

$$R_m(\mathbf{u}_{m-1}) = t^{1-\alpha}\frac{\partial u_{m-1}(x,t)}{\partial t} + \frac{\partial u_{m-1}(x,t)}{\partial x} + \frac{\partial^3 u_{m-1}(x,t)}{\partial x^3} - \nu\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n} u_k(x,t)u_{n-k}(x,t)\right)\frac{\partial u_{m-1-n}(x,t)}{\partial x}.$$

The solutions of the mth-order deformation Eq. (3.10) for $m \geq 1$ is obtained as

$$v_m(x,t) = \chi_m^* v_{m-1}(x,t) + h\mathscr{L}^{-1}[R_m(\mathbf{v}_{m-1})]. \tag{3.11}$$

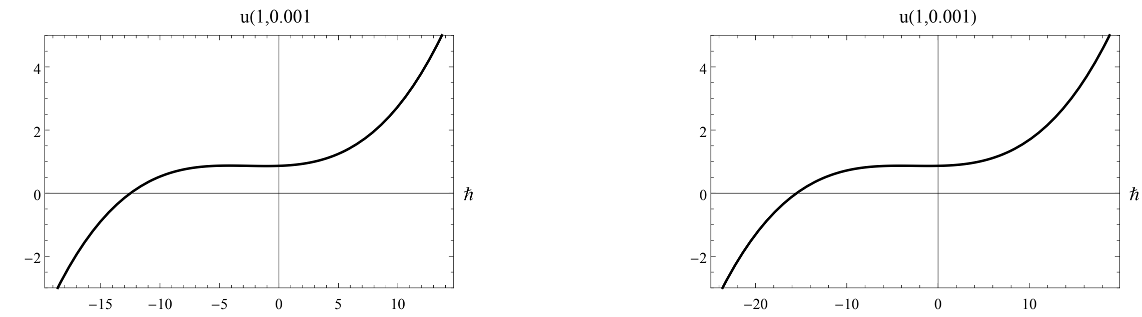
With the aid of Eq.(3.11) and the initial condition (3.9) we get the approximate solutions of $u_m(x,t)$ for $m=0,1$ as follows:

$$u_0(x,t) = \frac{\sqrt{3}}{x+1},$$
$$u_1(x,t) = \frac{\sqrt{3}ht^{\alpha}(5x^2+10x-1)}{(x+1)^4\alpha}.$$

The series solution of (3.7) for $m=0,1,\cdots$ expression by q-HAM can be expressed as:

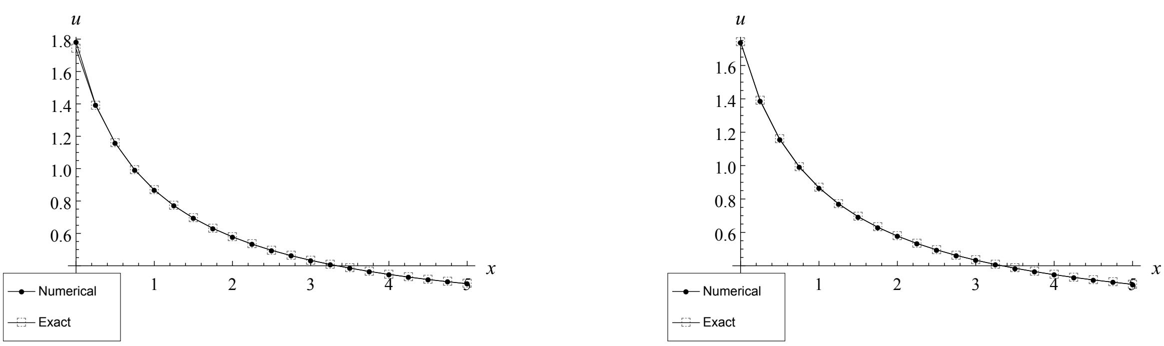
$$u(x,t,n,h) = \frac{\sqrt{3}}{x+1} + \sum_{n=1}^{\infty} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i. \tag{3.12}$$

The h-curves of the conformable time-fractional BBM equation for $n=2$ and for $\alpha=0.8$ and $\alpha=0.9$ are given in Fig. 4. In Fig. 5, the approximate solution and the exact solution for the conformable time-fractional BBM equation are compared, where $h=-2.787$, $t=0.001$ (fixed) and $\alpha=0.8$, $\alpha=0.9$. In Fig. 6, for $h=-2.787$ value which is chosen from these graphics for both α values, the surfaces of the approximate solutions and the exact solutions are illustrated.



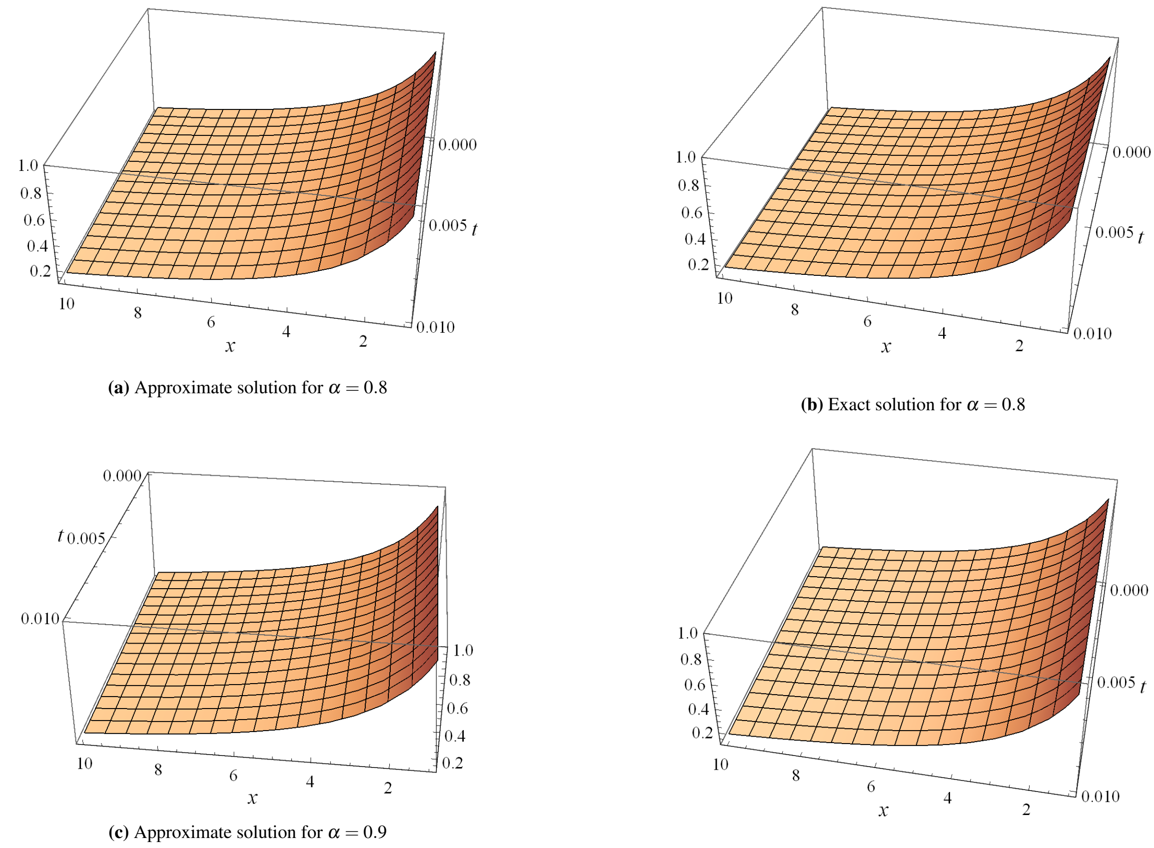
(a) $\alpha=0.8$ **(b)** $\alpha=0.9$

Figure 3.4: The h-curves of the conformable time-fractional BBM equation for $n=2$ and different values of α.



(a) Approximate solution for $\alpha=0.8$ **(b)** Approximate solution for $\alpha=0.9$

Figure 3.5: Approximate solutions obtained for $h=-2.787$, $n=2$, $t=0.001$ and different values of α by the q-HAM in comparison with exact solution for the conformable time-fractional BBM equation



(a) Approximate solution for $\alpha = 0.8$

(b) Exact solution for $\alpha = 0.8$

(c) Approximate solution for $\alpha = 0.9$

(d) Exact solution for $\alpha = 0.9$

Figure 3.6: Approximate solutions obtained for $h = -2.787$, $n = 2$ and different values of α by the q-HAM in comparison with exact solution for the conformable time-fractional BBM equation

4. Conclusions

In this paper we employ the q-Ham to obtain approximate analytical solutions of Modified Benjamin-Bona-Mahony(BBM) and Clannish Random Walker's Parabolic(CRWP) equations. q-HAM contains the parameter h which is used for adjusting the suitable convergence interval. The results show that the obtained approximate solutions are compatible with the exact solutions. As a result, q-HAM is an efficient and reliable technique to obtain the approximate analytical solutions of nonlinear conformable fractional partial differential equations. When $\mu = 1$ is chosen, it is seen that the solutions are the same of the non-fractional cases. In fact, this is the nature of fractional calculus. When fractional operations are converted to integer order operations, the results are consistent.

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Universal Journal of Mathematics and Applications, 3 (2) (2020) 69-77
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.649117

Differentiable Preinvex and Prequasiinvex Functions

Huriye Kadakal[1]

[1]*Ministry of Education, Hamdi Bozbağ Anatolian High School, Giresun, Turkey*

Article Info

Keywords: *Invex set, Preinvex function, Prequasiinvex function, Integral inequalities.*
2010 AMS: *26A51, 26D10, 26D15.*
Received: *20 November 2019*
Accepted: *11 May 2020*
Available online: *22 June 2020*

Abstract

In this paper, a new identity for functions defined on an open invex subset of set of real numbers is established, and by using this identity and the Hölder and Power mean integral inequalities we present new integral inequalities for functions whose powers of derivatives in absolute value are preinvex and prequasiinvex. We should especially mention that the results obtained in special cases coincide with the well-known results in the literature.

1. Introduction

Definition 1.1. *A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$.*

This definition is well known in the literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.
Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The celebrated double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2},$$

is well known in the literature as Hermite-Hadamard's integral inequality for convex functions [22]. Both inequalities hold in the reserved direction if the function f is concave. The classical Hermite-Hadamard integral inequality provides estimates of the mean value of a continuous convex or concave function. Hadamard's integral inequality for convex or concave functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; for example see [4, 9, 10, 18, 21, 22, 24]. Hermite-Hadamard (H-H) integral inequality (see [5]) has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from the above H-H inequality for particular choices of the function f.

Definition 1.2. *A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex if the inequality $f(tx+(1-t)y) \leq \max\{f(x), f(y)\}$ holds for all $x, y \in I$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex [6].*

Let us recall the notions of preinvexity and prequasiinvexity which are signicant generalizations of the notions of convexity and quasi-convexity respectively, and some related results.

Definition 1.3 ([25]). *Let K be a non-empty subset in $\mathbb{R}^n$ and $\eta: K \times K \to \mathbb{R}^n$. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if $x + t\eta(y,x) \in K$, $\forall x, y \in K \;\; t \in [0,1]$ .K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called η-connected set.*

This definition essentially says that there is a path starting from a point x which is contained in K. We do not require that the point y should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that y should be an end point of the path for every pair of points $x, y \in K$, then $\eta(y,x) = y - x$, and consequently invexity reduces to convexity [1]. Thus, it is true that every convex set is also an invex set with respect to $\eta(y,x) = y - x$, but the converse is not necessarily true, see [14, 26] and the references therein. For the sake of simplicity, we always assume that $K = [x, x + t\eta(y,x)]$, unless otherwise specified.

***Email address and ORCID number*: huriyekadakal@hotmail.com, https://orcid.org/0000-0002-0304-7192, (H. Kadakal)**

Definition 1.4 ([25]). *A function $f : K \to \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}$ is said to be preinvex with respect to η, if*

$$f(u+t\eta(v,u)) \leq (1-t)f(u)+tf(v), \quad \forall u,v \in K, \ \ t \in [0,1]$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(y,x) = x-y$ but the converse is not true see for instance.

Definition 1.5 ([2]). *A function $f : K \to \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}$ is said to be prequasiinvex with respect to η, if*

$$f(u+t\eta(v,u)) \leq \max\{f(u), f(v)\}, \quad \forall u,v \in K, \ \ t \in [0,1].$$

Also every quasi-convex function is a prequasiinvex with respect to the map $\eta(v,u) = v-u$ but the converse does not hold, see for example [2].
Mohan and Neogy [14] introduced Condition C as follows:

Definition 1.6 ([14]). *Let $S \subseteq \mathbb{R}$ be an open invex subset with respect to the map $\eta : S \times S \to \mathbb{R}$. We say that the function η satisfies the Condition C if, for any $x,y \in S$ and any $t \in [0,1]$,*

$$\eta(y, y+t\eta(x,y)) = -t\eta(x,y) \tag{1.1}$$
$$\eta(x, y+t\eta(x,y)) = (1-t)\eta(x,y). \tag{1.2}$$

Note that, from the Condition C, we have

$$\eta(y+t_2\eta(x,y), y+t_1\eta(x,y)) = (t_2-t_1)\eta(x,y)$$

for any $x,y \in S$ and any $t_1,t_2 \in [0,1]$.

In recent years, many mathematicians have been studying about preinvexity and types of preinvexity. These studies include, among others, the work of Ben-Israel and Mond [7], Pini [23], Noor [16, 17], Yang and Li [27] and Weir and Mond [25]. Ben-Israel and Mond , Weir and Mond and Noor have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems.
In a recent paper, Noor [15] has obtained the following Hermite-Hadamard type integral inequalities for the preinvex functions.

Theorem 1.7 ([15]). *Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and $a,b \in K^\circ$ with $\eta(b,a) > 0$. Then the following inequalities hold*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

For several recent results on inequalities for preinvex and prequasiinvex functions which are connected to (1.3), we refer the interested reader to [3, 11, 13, 19, 20] and the references therein.
Let $0 < a < b$, throughout this paper we will use

$$A(a,b) = \frac{a+b}{2}$$
$$L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad a \neq b, \ p \in \mathbb{R}, \ \ p \neq -1, 0$$

for the arithmetic and generalized logarithmic means, respectively.

2. Main results

In this section, using a general integral identity for a differentiable functions, we establish some new integral inequalities for mappings whose derivative in absolute value at certain powers are preinvex and prequasiinvex. We will use the following Lemma to prove our main results.

Lemma 2.1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot) : K \times K \to \mathbb{R}^n$ and $a,b \in K$ with $\eta(b,a) > 0$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function on K such that $f' \in L[a, a+\eta(b,a)]$. Then the following identity holds:*

$$f(a+\eta(b,a))[a+\eta(b,a)] - f(a)a - \int_a^{a+\eta(b,a)} f(x)dx = \eta(b,a)\int_0^1 (a+t\eta(b,a)) f'(a+t\eta(b,a))dt.$$

Proof. By changing the variable and integrating by parts, we have

$$\begin{aligned}\eta(b,a)\int_0^1 (a+t\eta(b,a)) f'(a+t\eta(b,a))dt &= \int_a^{a+\eta(b,a)} xf'(x)dx \\ &= f(a+\eta(b,a))[a+\eta(b,a)] - f(a)a - \int_a^{a+\eta(b,a)} f(x)dx.\end{aligned}$$

This completes the proof of lemma. □

Theorem 2.2. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot): K \times K \to \mathbb{R}^n$ and $a,b \in K$ with $\eta(b,a) > 0$. Suppose that $f: K \to \mathbb{R}$ is a differentiable function on K such that $f' \in L[a, a+\eta(b,a)]$. If $|f'|^q$ is preinvex on K for $q > 1$, then the following inequality holds:*

$$\left| f(a+\eta(b,a))[a+\eta(b,a)] - f(a)a - \int_a^{a+\eta(b,a)} f(x)dx \right| \le \eta^{1/q}(b,a) C_\eta^{\frac{1}{p}}(p,a,b) A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right). \tag{2.1}$$

where,

$$C_\eta(p,a,b) := \begin{cases} \eta(b,a) L_p^p[a+\eta(b,a), a], & a > 0,\ a+\eta(b,a) > 0, \\ \frac{2}{p+1} A\left((a+\eta(b,a))^{p+1}, (-a)^{p+1}\right), & a < 0,\ a+\eta(b,a) > 0, \\ \eta(b,a) L_p^p[-a, -(a+\eta(b,a))], & a < 0,\ a+\eta(b,a) < 0. \end{cases}$$

Proof. If $|f'|^q$ is preinvex on $[a, a+\eta(b,a)]$ for $q > 1$, using Lemma 2.1, the Hölder integral inequality and the inequality $|f'(a+t\eta(b,a))|^q \le t|f'(b)|^q + (1-t)|f'(a)|^q$,we have

$$\begin{aligned} \left| f(a+\eta(b,a))[a+\eta(b,a)] - f(a)a - \int_a^{a+\eta(b,a)} f(x)dx \right| &\le \eta(b,a) \int_0^1 |(a+t\eta(b,a))| \left|f'(a+t\eta(b,a))\right| dt \\ &\le \eta(b,a) \left(\int_0^1 |a+t\eta(b,a)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left|f'(a+t\eta(b,a))\right|^q dt \right)^{\frac{1}{q}} \\ &\le \eta^{\frac{1}{q}}(b,a) \left(\int_a^{a+\eta(b,a)} |x|^p dx \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \\ &= \eta^{\frac{1}{q}}(b,a) C_\eta^{\frac{1}{p}}(p,a,b) A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right). \end{aligned}$$

This completes the proof of theorem. □

Corollary 2.3. *Suppose that all the assumptions of Theorem 2.2 are satisfied. If we choose $\eta(b,a) = b-a$ then when $|f'|^q$ is convex on K for $q > 1$ we have*

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le (b-a)^{\frac{1}{q}-1} C^{\frac{1}{p}}(p,a,b) A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).$$

where,

$$C(p,a,b) = \begin{cases} (b-a)L_p^p(a,b), & a > 0, b > 0 \\ \frac{2}{p+1} A\left(b^{p+1}, (-a)^{p+1}\right), & a < 0,\ b > 0, \\ (b-a)L_p^p(-a,-b), & a < 0,\ b < 0. \end{cases}$$

We note that this result coincides with Corollary 2.3. in the special case $a,b > 0$ [12].

Remark 2.4. *If the mapping η satisfies condition C then by using of the preinvexity of $|f'|^q$ we get*

$$\left|f'(a+t\eta(b,a))\right|^q = \left|f'(a+\eta(b,a)+(1-t)\eta(a, a+\eta(b,a)))\right|^q \le t\left|f'(a+\eta(b,a))\right|^q + (1-t)\left|f'(a)\right|^q. \tag{2.2}$$

for every $t \in [0,1]$.
If we use the inequality (2.2) in the proof of Theorem 2.2, then the inequality (2.1) becomes to the following inequality:

$$\left| f(a+\eta(b,a))[a+\eta(b,a)] - f(a)a - \int_a^{a+\eta(b,a)} f(x)dx \right| \le \eta^{\frac{1}{q}}(b,a) C_\eta^{\frac{1}{p}}(p,a,b) A^{\frac{1}{q}}\left(|f'(a)|^q, \left|f'(a+\eta(b,a))\right|^q\right). \tag{2.3}$$

We note that by using of the preinvexity of $|f'|^q$ we have $|f'(a+\eta(b,a))|^q \le |f'(b)|^q$ .Therefore, the inequality (2.3) is better than the inequality (2.1).

Theorem 2.5. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot): K \times K \to \mathbb{R}^n$ and $a,b \in K$ with $\eta(b,a) > 0$. Suppose that $f: K \to \mathbb{R}$ is a differentiable function on K such that $f' \in L[a, a+\eta(b,a)]$. If $|f'|^q$ is preinvex on K for $q \ge 1$, then the following inequality holds:*

$$\left| f(a+\eta(b,a))[a+\eta(b,a)] - f(a)a - \int_a^{a+\eta(b,a)} f(x)dx \right| \le \eta^{-\frac{1}{q}}(b,a) D_{1,\eta}^{1-\frac{1}{q}}(a,b) \left[\left|f'(b)\right|^q D_{2,\eta}(a,b) + \left|f'(a)\right|^q D_{3,\eta}(a,b) \right]^{\frac{1}{q}} \tag{2.4}$$

where

$$D_{1,\eta}(a,b) := \begin{cases} \eta(b,a) A(a, a+\eta(b,a)), & a > 0,\ a+\eta(b,a) > 0 \\ A\left(a^2, [a+\eta(b,a)]^2\right), & a < 0,\ a+\eta(b,a) > 0 \\ -\eta(b,a) A(a, a+\eta(b,a)), & a < 0,\ a+\eta(b,a) < 0 \end{cases},$$

$$D_{2,\eta}(a,b) := \begin{cases} \frac{\eta(b,a)}{6}\left[(2a+\eta(b,a))(2\eta(b,a)-a) + 2a^2\right], & a > 0,\ a+\eta(b,a) > 0 \\ \frac{1}{6}\left[(a+\eta(b,a))^2(2\eta(b,a)-a) - a^3\right], & a < 0,\ a+\eta(b,a) > 0 \\ -\frac{\eta^2(b,a)}{6}(3a+2\eta(b,a)), & a < 0,\ a+\eta(b,a) < 0 \end{cases}$$

and

$$D_{3,\eta}(a,b) := \begin{cases} \frac{\eta(b,a)}{6}\left[(a+\eta(b,a))(2a+\eta(b,a))-2a^2\right], & a>0,\ a+\eta(b,a)>0 \\ \frac{1}{6}\left[(a+\eta(b,a))^3+a^2(a+3\eta(b,a))\right], & a<0,\ a+\eta(b,a)>0 \\ -\frac{\eta^2(b,a)}{6}(3a+\eta(b,a)), & a<0,\ a+\eta(b,a)<0. \end{cases}$$

Proof. From Lemma 2.1 and Power-mean integral inequality, we obtain

$$\begin{aligned} &\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\le \eta(b,a)\int_0^1 |(a+t\eta(b,a))|\left|f'(a+t\eta(b,a))\right|dt \\ &\le \eta(b,a)\left(\int_0^1 |a+t\eta(b,a)|\,dt\right)^{1-\frac{1}{q}} \times \left(\int_0^1 |a+t\eta(b,a)|\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}} \\ &\le \eta^{\frac{1}{q}}(b,a)\left(\int_a^{a+\eta(b,a)} |x|\,dx\right)^{1-\frac{1}{q}} \times \left(\int_0^1 |a+t\eta(b,a)|\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}} \\ &= \eta^{-\frac{1}{q}}(b,a)\left(\int_a^{a+\eta(b,a)} |x|\,dx\right)^{1-\frac{1}{q}} \left(\left|f'(b)\right|^q\int_a^{a+\eta(b,a)}(x-a)|x|\,dx+\left|f'(a)\right|^q\int_a^{a+\eta(b,a)}[\eta(b,a)-(x-a)]|x|\,dx\right)^{\frac{1}{q}} \\ &= \eta^{-\frac{1}{q}}(b,a)D_{1,\eta}^{1-\frac{1}{q}}(a,b)\left[\left|f'(b)\right|^q D_{2,\eta}(a,b)+\left|f'(a)\right|^q D_{3,\eta}(a,b)\right]^{\frac{1}{q}}. \end{aligned}$$

This completes the proof of theorem. □

Corollary 2.6. *Suppose that all the assumptions of Theorem 2.5 are satisfied. If we choose $\eta(b,a)=b-a$ then when $|f'|^q$ is convex on K for $q\ge 1$ we have*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le (b-a)^{-1-\frac{1}{q}}D_1^{1-\frac{1}{q}}(a,b)\left[\left|f'(b)\right|^q D_2(a,b)+\left|f'(a)\right|^q D_3(a,b)\right]^{\frac{1}{q}},$$

where

$$D_1(a,b) = \begin{cases} (b-a)A(a,b), & a>0,\ b>0 \\ A(a^2,b^2), & a<0,\ b>0 \\ -(b-a)A(a,b), & a<0,\ b<0 \end{cases},$$

$$D_2(a,b) = \begin{cases} \frac{1}{6}(b-a)^2(2b+a), & a>0,\ b>0 \\ \frac{1}{6}\left[b^2(2b-3a)-a^3\right], & a<0,\ b>0 \\ -\frac{1}{6}(b-a)^2(a+2b), & a<0,\ b<0 \end{cases}$$

and

$$D_3(a,b) = \begin{cases} \frac{1}{6}(b-a)^2(b+2a), & a>0,\ b>0 \\ \frac{1}{6}\left[b^3+a^2(3b-2a)\right], & a<0,\ b>0 \\ -\frac{1}{6}(b-a)^2(b+2a), & a<0,\ b<0. \end{cases}$$

We note that this result coincides with Corollary 2.6 in the special case $a,b>0$ [12].

Remark 2.7. *If the mapping η satisfies condition C then using the inequality (2.2) in the proof of Theorem 2.5, the inequality (2.4) becomes to the following inequality:*

$$\begin{aligned} &\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\le \eta^{-\frac{1}{q}}(b,a)D_1^{1-\frac{1}{q}}(a,b)\left[\left|f'(a+\eta(b,a))\right|^q D_2(a,b)+\left|f'(a)\right|^q D_3(a,b)\right]^{\frac{1}{q}}. \end{aligned} \tag{2.5}$$

We note that by using of the preinvexity of $|f'|^q$ we have $|f'(a+\eta(b,a))|^q \le |f'(b)|^q$. Therefore, the inequality (2.5) is better than the inequality (2.4).

Corollary 2.8. *If we take $q=1$ in Theorem 3, then we have the following inequality:*

$$\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{1}{\eta(b,a)}\left[\left|f'(b)\right|D_{2,\eta}(a,b)+\left|f'(a)\right|D_{3,\eta}(a,b)\right]$$

Theorem 2.9. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\to\mathbb{R}^n$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function on K such that $f'\in L[a,a+\eta(b,a)]$. If $|f'|^q$ is preinvex on K for $q>1$, then the following inequality holds:*

$$\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)} f(x)dx\right| \le \eta^{1-\frac{2}{q}}(b,a)\left[\left|f'(b)\right|^q E_{1,\eta}(q,a,b)+\left|f'(a)\right|^q E_{2,\eta}(q,a,b)\right]^{\frac{1}{q}} \tag{2.6}$$

where

$$E_{1,\eta}(q,a,b) := \begin{cases} \eta(b,a)L_{q+1}^{q+1}(a+\eta(b,a),a) - a\eta(b,a)L_q^q(a+\eta(b,a),a), & a>0, a+\eta(b,a)>0 \\ \eta(b,a)L_{q+1}^{q+1}(a+\eta(b,a),-a) - \frac{2a}{q+1}A\left((a+\eta(b,a))^{q+1},(-a)^{q+1}\right), & a<0,\ a+\eta(b,a)>0 \\ \eta(b,a)L_{q+1}^{q+1}[-(a+\eta(b,a)),-a] + \frac{2a}{q+1}A\left([-(a+\eta(b,a))]^{q+1},(-a)^{q+1}\right), & a<0,\ a+\eta(b,a)<0 \end{cases}$$

and

$$E_{2,\eta}(q,a,b) := \begin{cases} -\eta(b,a)L_{q+1}^{q+1}(a+\eta(b,a),a) + \eta(b,a)[\eta(b,a)-a]L_q^q(a+\eta(b,a),a), & a>0,\ a+\eta(b,a)>0 \\ -\eta(b,a)L_{q+1}^{q+1}(a+\eta(b,a),-a) + \frac{2a}{q+1}[\eta(b,a)+a]A\left[(a+\eta(b,a))^{q+1},(-a)^{q+1}\right], & a<0, a+\eta(b,a)>0 \\ -\eta(b,a)L_{q+1}^{q+1}[-(a+\eta(b,a)),-a] - \eta(b,a)[\eta(b,a)+a]L_q^q(-(a+\eta(b,a)),-a), & a<0,\ a+\eta(b,a)<0. \end{cases}$$

Proof. Since$|f'|^q$ is preinvex on $[a,a+\eta(b,a)]$ for $q>1$, using Lemma 2.1 and the Hölder integral inequality, we have the following inequality:

$$\begin{aligned} &\left|f(a+\eta(b,a))[a+\eta(b,a)] - f(a)a - \int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\le \eta(b,a)\left(\int_0^1 1^p dx\right)^{\frac{1}{p}}\left(\int_0^1 |a+t\eta(b,a)|^q\left|f'(a+t\eta(b,a))\right|^q dx\right)^{\frac{1}{q}} \\ &= \eta(b,a)\left(\int_0^1 |a+t\eta(b,a)|^q\left[t\left|f'(b)\right|^q + (1-t)\left|f'(a)\right|^q\right]dx\right)^{\frac{1}{q}} \\ &= \eta^{1-\frac{2}{q}}(b,a)\left(\left|f'(b)\right|^q\int_a^{a+\eta(b,a)}(x-a)|x|^q dx + \left|f'(a)\right|^q\int_a^{a+\eta(b,a)}[\eta(b,a)-x+a]|x|^q dx\right)^{\frac{1}{q}} \\ &= \eta^{1-\frac{2}{q}}(b,a)\left[\left|f'(b)\right|^q E_{1,\eta}(q,a,b) + \left|f'(a)\right|^q E_{2,\eta}(q,a,b)\right]^{\frac{1}{q}}. \end{aligned}$$

This completes the proof of theorem. □

Corollary 2.10. *Suppose that all the assumptions of Theorem 2.9 are satisfied. If we choose $\eta(b,a)=b-a$ then when $|f'|^q$ is convex on K for $q>1$ we have*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le (b-a)^{-\frac{2}{q}}\left[\left|f'(b)\right|^q E_1(q,a,b) + \left|f'(a)\right|^q E_2(q,a,b)\right]^{\frac{1}{q}},$$

where

$$E_1(q,a,b) = \begin{cases} (b-a)L_{q+1}^{q+1}(b,a) - a(b-a)L_q^q(b,a), & a>0,\ b>0 \\ (b-a)L_{q+1}^{q+1}(b,-a) - \frac{2a}{q+1}A\left(b^{q+1},(-a)^{q+1}\right), & a<0,\ b>0 \\ (b-a)L_{q+1}^{q+1}(-b,-a) + \frac{2a}{q+1}A\left([-(b)]^{q+1},(-a)^{q+1}\right), & a<0,\ b<0 \end{cases}$$

and

$$E_2(q,a,b) = \begin{cases} -(b-a)\left[L_{q+1}^{q+1}(b,a) - bL_q^q(b,a)\right], & a>0,\ b>0 \\ -(b-a)L_{q+1}^{q+1}(b,-a) + \frac{2ab}{q+1}A\left(b^{q+1},(-a)^{q+1}\right), & a<0,\ b>0 \\ -(b-a)\left[L_{q+1}^{q+1}(-b,-a) + bL_q^q(-b,-a)\right], & a<0,\ b<0 \end{cases}$$

We note that this result coincides with Corollary 2.13 in the special case $a,b>0$ [12].

Remark 2.11. *If the mapping η satisfies condition C then using the inequality (2.2) in the proof of Theorem 2.9, then the inequality (2.6) becomes to the following inequality:*

$$\left|f(a+\eta(b,a))[a+\eta(b,a)] - f(a)a - \int_a^{a+\eta(b,a)} f(x)dx\right| \le \eta^{1-\frac{2}{q}}(b,a)\left[\left|f'((a+\eta(b,a))\right|^q E_1(q,a,b) + \left|f'(a)\right|^q E_2(q,a,b)\right]^{\frac{1}{q}}. \tag{2.7}$$

Note that the inequality (2.7) is better than the inequality (2.6).

Theorem 2.12. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\to\mathbb{R}^n$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function on K such that $f'\in L[a,a+\eta(b,a)]$. If $|f'|^q$ is preconcave on K for $q>1$, then the following inequality holds:*

$$\left|f(a+\eta(b,a))[a+\eta(b,a)] - f(a)a - \int_a^{a+\eta(b,a)} f(x)dx\right| \le \eta(b,a)F_\eta^{\frac{1}{p}}(p,a,b)\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$

where

$$F_\eta(p,a,b) := \begin{cases} L_p^p[a+\eta(b,a),a], & a>0,\ a+\eta(b,a)>0, \\ \frac{2A\left((a+\eta(b,a))^{p+1},(-a)^{p+1}\right)}{(p+1)\eta(b,a)}, & a<0,\ a+\eta(b,a)>0, \\ L_p^p[-a,-(a+\eta(b,a))], & a<0,\ a+\eta(b,a)<0. \end{cases}$$

Proof. Since $|f'|^q$ is preconcave on $[a, a+\eta(b,a)]$ for $q>1$, with respect to Hermite-Hadamard inequality we get $\int_a^b |f'(x)|^q dx \leq \eta(b,a)\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q$. Using Lemma 2.1 and the Hölder integral inequality we have

$$\begin{aligned}
\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)} f(x)dx\right| &\leq \int_a^{a+\eta(b,a)} |x|\left|f'(x)\right|dx \\
&\leq \left(\int_a^{a+\eta(b,a)} |x|^p dx\right)^{\frac{1}{p}}\left(\int_a^{a+\eta(b,a)} \left|f'(x)\right|^q dx\right)^{\frac{1}{q}} \\
&= \eta(b,a)\left(\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} |x|^p dx\right)^{\frac{1}{p}}\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right| \\
&= \eta(b,a)F_\eta^{\frac{1}{p}}(p,a,b)\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 2.13. *Suppose that all the assumptions of Theorem 2.12 are satisfied. If we choose $\eta(b,a)=b-a$ then when $|f'|^q$ is preconcave on K for $q>1$ we have*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq F^{\frac{1}{p}}(p,a,b)\left|f'\left(\frac{a+b}{2}\right)\right|$$

where

$$F(p,a,b)=\begin{cases} L_p^p(b,a), & a>0,\ b>0, \\ \frac{2A\left(b^{p+1},(-a)^{p+1}\right)}{(p+1)\eta(b,a)}, & a<0,\ b>0, \\ L_p^p(-a,-b), & a<0,\ b<0. \end{cases}$$

We note that this result coincides with Corollary 2.17 in the special case $a,b>0$ [12].

Now we will give new results for prequasiinvex functions by using Lemma 2.1.

Theorem 2.14. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot): K\times K\to\mathbb{R}^n$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that $f: K\to\mathbb{R}$ is a differentiable function on K such that $f'\in L[a,a+\eta(b,a)]$. If $|f'|^q$ is prequasiinvex on K for $q>1$, then the following inequality holds:*

$$\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \eta(b,a)C_\eta(p,a,b)\left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} \tag{2.8}$$

where

$$C_\eta(p,a,b):=\begin{cases} L_p^p[a+\eta(b,a),a], & a>0,\ a+\eta(b,a)>0, \\ \frac{2A\left[(a+\eta(b,a))^{p+1},(-a)^{p+1}\right]}{(p+1)\eta(b,a)}, & a<0,\ a+\eta(b,a)>0, \\ L_p^p[-a,-(a+\eta(b,a))], & a<0,\ a+\eta(b,a)<0. \end{cases}$$

Proof. If $|f'|^q$ is prequasiinvex on $[a,a+\eta(b,a)]$ for $q>1$, using Lemma 2.1, the Hölder integral inequality and the inequality $|f'(a+t\eta(b,a))|^q \leq \max\left\{|f'(a)|^q,|f'(b)|^q\right\}$ we have

$$\begin{aligned}
\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)} f(x)dx\right| &\leq \eta(b,a)\int_0^1 |(a+t\eta(b,a))|\left|f'(a+t\eta(b,a))\right|dt \\
&\leq \eta(b,a)\left(\int_0^1 |a+t\eta(b,a)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \eta(b,a)\left(\int_0^1 |a+t\eta(b,a)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&= \eta(b,a)\left(\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} |x|^p dx\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} \\
&= \eta(b,a)C_\eta(p,a,b)\left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}$$

This completes the proof of theorem. □

Corollary 2.15. *Suppose that all the assumptions of Theorem 2.14 are satisfied. If we choose $\eta(b,a)=b-a$ then when $|f'|^q$ is prequasiinvex on K for $q>1$ we have*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq C(p,a,b)\left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}$$

where

$$C(p,a,b)=\begin{cases} L_p^p(b,a), & a>0,\ b>0,\\ \frac{2A\left(b^{p+1},(-a)^{p+1}\right)}{(p+1)(b-a)}, & a<0,\ b>0,\\ L_p^p(-a,-b), & a<0,\ b<0.\end{cases}$$

We note that this result coincides with Corollary 2.1 in the special case $a,b>0$ [8].

Remark 2.16. *If the mapping η satisfies condition C then by use of the prequasiinvexity of $|f'|^q$ we get*

$$\begin{aligned}\left|f'(a+t\eta(b,a))\right|^q &= \left|f'(a+\eta(b,a)+(1-t)\eta(a,a+\eta(b,a)))\right|^q\\ &\le \max\left\{\left|f'(a)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\end{aligned}\tag{2.9}$$

for every $t\in[0,1]$.
If we use the inequality (2.9) in the proof of Theorem 2.14, then the inequality (2.8) becomes to the following inequality:

$$\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)}f(x)dx\right|\le \eta(b,a)C_\eta(p,a,b)\left(\max\left\{\left|f'(a)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right)^{\frac{1}{q}}\tag{2.10}$$

We note that by use of the prequasiinvexity of $|f'|^q$ we have $|f'(a+\eta(b,a))|^q\le\max\left\{|f'(a)|^q,|f'(b)|^q\right\}$. Therefore, the inequality (2.10) is better than the inequality (2.8).

Theorem 2.17. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\to\mathbb{R}^n$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function on K such that $f'\in L[a,a+\eta(b,a)]$. If $|f'|^q$ is prequasiinvex on K for $q\ge1$, then the following inequality holds:*

$$\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)}f(x)dx\right|\le\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}D_{1,\eta}(a,b)\tag{2.11}$$

where

$$D_{1,\eta}(a,b):=\begin{cases}\eta(b,a)A(a,a+\eta(b,a)), & a>0,\ a+\eta(b,a)>0\\ A\left(a^2,[a+\eta(b,a)]^2\right), & a<0,\ a+\eta(b,a)>0\\ -\eta(b,a)A(a,a+\eta(b,a)), & a<0,\ a+\eta(b,a)<0\end{cases}.$$

Proof. From Lemma 2.1 and Power-mean integral inequality, we obtain

$$\begin{aligned}&\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)}f(x)dx\right|\\ &\le\eta(b,a)\int_0^1|(a+t\eta(b,a))|\left|f'(a+t\eta(b,a))\right|dt\\ &\le\eta(b,a)\left(\int_0^1|a+t\eta(b,a)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|a+t\eta(b,a)|\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\\ &\le\eta^{\frac{1}{q}}(b,a)\left(\int_a^{a+\eta(b,a)}|x|dx\right)^{1-\frac{1}{q}}\times\left(\int_0^1|a+t\eta(b,a)|\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]dt\right)^{\frac{1}{q}}\\ &=\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\int_a^{a+\eta(b,a)}|x|dx\\ &=\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}D_{1,\eta}(a,b).\end{aligned}$$

This completes the proof of theorem. □

Corollary 2.18. *Suppose that all the assumptions of Theorem 2.17 are satisfied. If we choose $\eta(b,a)=b-a$ then when $|f'|^q$ is prequasiinvex on K for $q\ge1$ we have*

$$\left|\frac{f(b)b-f(a)a-\int_a^b f(x)dx}{b-a}\right|\le\frac{D_1(a,b)}{b-a}\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}$$

where

$$D_1(a,b)=\begin{cases}(b-a)A(a,b), & a>0,\ b>0\\ A\left(a^2,b^2\right), & a<0,\ b>0\\ -(b-a)A(a,b), & a<0,\ b<0\end{cases}.$$

We note that this result coincides with Corollary 6 in the special case $a,b>0$ [8].

Remark 2.19. *If we use the inequality (2.9) in the proof of Theorem 2.17, then the inequality (2.11) becomes the following inequality:*

$$\left|f(a+\eta(b,a))[a+\eta(b,a)]-f(a)a-\int_a^{a+\eta(b,a)}f(x)dx\right|\le\left[\max\left\{\left|f'(a)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right]^{\frac{1}{q}}D_{1,\eta}(a,b)$$

This inequality is better than the inequality (2.11).

Corollary 2.20. *If we take $q=1$ in Theorem 7, then we have the following inequality:*

$$\left|f(a+\eta(b,a))\left[a+\eta(b,a)\right]-f(a)a-\int_a^{a+\eta(b,a)}f(x)dx\right|\leq\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}D_1(a,b)$$

Theorem 2.21. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\to\mathbb{R}^n$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function on K such that $f'\in L[a,a+\eta(b,a)]$. If $|f'|^q$ is prequasiinvex on K for $q>1$, then the following inequality holds:*

$$\left|f(a+\eta(b,a))\left[a+\eta(b,a)\right]-f(a)a-\int_a^{a+\eta(b,a)}f(x)dx\right|\leq\eta^{1-\frac{1}{q}}(b,a)\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}F_{1,\eta}^{\frac{1}{q}}(q,a,b)\tag{2.12}$$

where

$$F_{1,\eta}(p,a,b):=\begin{cases}\eta(b,a)L_q^q\left[a+\eta(b,a),a\right], & a>0,\ a+\eta(b,a)>0,\\ \frac{2}{q+1}A\left((a+\eta(b,a))^{p+1},(-a)^{p+1}\right), & a<0,\ a+\eta(b,a)>0,\\ \eta(b,a)L_q^q\left[-a,-(a+\eta(b,a))\right], & a<0,\ a+\eta(b,a)<0.\end{cases}$$

Proof. Since$|f'|^q$ for $q>1$ is prequasiinvex on $[a,b]$, using Lemma 2.1 and the Hölder integral inequality, we have the following inequality,

$$\begin{aligned}\left|f(a+\eta(b,a))\left[a+\eta(b,a)\right]-f(a)a-\int_a^{a+\eta(b,a)}f(x)dx\right|&\leq\eta(b,a)\left(\int_0^1 1^p dx\right)^{\frac{1}{p}}\left(\int_0^1|a+t\eta(b,a)|^q\left|f'(a+t\eta(b,a))\right|^q dx\right)^{\frac{1}{q}}\\&\leq\eta(b,a)\left(\int_0^1|a+t\eta(b,a)|^q\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]dx\right)^{\frac{1}{q}}\\&=\eta(b,a)\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(\int_0^1|a+t\eta(b,a)|^q dx\right)^{\frac{1}{q}}\\&=\eta^{1-\frac{1}{q}}(b,a)\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(\int_a^{a+\eta(b,a)}|x|^q dt\right)^{\frac{1}{q}}\\&=\eta^{1-\frac{1}{q}}(b,a)\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}F_{1,\eta}^{\frac{1}{q}}(q,a,b),\end{aligned}$$

This completes the proof of theorem. □

Corollary 2.22. *Suppose that all the assumptions of Theorem 2.21 are satisfied. If we choose $\eta(b,a)=b-a$ then when $|f'|^q$ is prequasiinvex on K for $q>1$ we have*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)^{-\frac{1}{q}}\left[\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}F_1^{\frac{1}{q}}(q,a,b)$$

where

$$F_1(p,a,b)=\begin{cases}(b-a)L_q^q(b,a), & a>0,\ b>0,\\ \frac{2}{q+1}A\left(b^{p+1},(-a)^{p+1}\right), & a<0,\ b>0,\\ (b-a)L_q^q(-a,-b), & a<0,\ b<0.\end{cases}$$

We note that this result coincides with Corollary 5 in the special case $a,b>0$ [8].

Remark 2.23. *If we use the inequality (2.9) in the proof of Theorem 2.21, then the inequality (2.12) becomes the following inequality:*

$$\left|f(a+\eta(b,a))\left[a+\eta(b,a)\right]-f(a)a-\int_a^{a+\eta(b,a)}f(x)dx\right|\leq\eta^{1-\frac{1}{q}}(b,a)\left[\max\left\{\left|f'(a)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right]^{\frac{1}{q}}F_{1,\eta}^{\frac{1}{q}}(q,a,b)$$

This inequality is better than the inequality (2.12).

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Universal Journal of Mathematics and Applications, 3 (2) (2020) 78-85
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.650209

Projective Curvature Tensor of Riemannian Manifolds Admitting A Projective Semi-Symmetric Connection

S. K. Chaubey[1], Pankaj R. B. Kanaujia[2*] and S. K. Yadav[3]

[1] *Math Section, Department of Information Technology, Shinas College of Technology, Shinas, Oman*
[2] *Math Section, Department of Information Technology, Higher College of Technology, Muscat, Oman*
[3] *Department of Mathematics, Poornima College of Engineering, Jaipur-302022, Rajasthan, India*
[*] *Corresponding author*

Article Info

Keywords: *Riemannian manifolds, Projective semi-symmetric connection, Different curvature tensors, nullity distribution, quasi Einstein manifolds*
2010 AMS: *53B15, 53C15, 53C25*
Received: *23 November 2019*
Accepted: *14 April 2020*
Available online: *22 June 2020*

Abstract

The aim of the present paper is to study the properties of Riemannian manifolds equipped with a projective semi-symmetric connection.

1. Introduction

At the foundation of Riemannian geometry there are three ideas. The first of these is the realization of the fact that a non-Euclidean geometry exists (N. I. Lobachevskii). The second is the concept of the interior geometry of surfaces by C. F. Gauss and third is the concept of an n-dimensional space by B. Riemann. Our present paper belongs to the study of third case. The idea of Riemannian geometry played an important role in the formulation of the general theory of relativity by A. Einstein. A non-flat n-dimensional Riemannian manifold (M_n, g), $(n > 2)$, is said to be quasi Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the tensorial expression

$$S(X,Y) = ag(X,Y) + b\pi(X)\pi(Y), \qquad X,Y \in TM$$

for smooth functions a and $b(\neq 0)$, where π is a non-zero 1-form associated with the Riemannian metric g and the associated unit vector field ξ [14]. The unit vector field ξ is called the generator and the 1-form π is called the associated 1-form of the manifold. It is observed that a collection of non-interacting pressure less perfect fluid of general relativity is a four dimensional semi-Riemannian quasi Einstein manifold whose associated scalars are $\frac{r}{2}$ and $\kappa\rho$, where κ is the gravitational constant, ρ and r are the energy density and scalar curvature, the generator of the manifold being the time like velocity vector field of the perfect fluid. If the generator of a quasi Einstein manifold is parallel vector field, then the manifold is locally a product manifold of one-dimensional distribution U and $(n-1)$ dimensional distribution $U^{\perp}$, where $U^{\perp}$ is involutive and integrable [23]. In an n-dimensional quasi Einstein manifold the Ricci tensor has precisely two distinct eigen values a and $a+b$, where the multiplicity of a is $n-1$ and $a+b$ is simple [14]. A proper η-Einstein contact metric manifold is a natural example of a quasi Einstein manifold ([7], [15]). Some of the physical and geometrical properties of quasi Einstein manifolds have been noticed in ([1], [16], [28], [45]- [47]).
The k-nullity distribution $N(k)$ of a Riemannian manifold M_n is defined by

$$N(k) : p \longrightarrow N_p(k) = \left\{Z \in T_pM : R(X,Y)Z = k\left[g(Y,Z)X - g(X,Z)Y\right]\right\}$$

for arbitrary vector fields X, Y and Z, where R represents the Riemannian curvature tensor and k is a smooth function on M_n [41]. The quasi Einstein manifold is called an $N(k)$-quasi Einstein manifold if the generator ξ of the manifold M_n belongs to k-nullity distribution [17].

Email addresses and ORCID numbers: **sk22_math@yahoo.co.in (S. K. Chaubey), pankaj.fellow@yahoo.co.in, https://orcid.org/0000-0002-3313-6006 (P. R. B. Kanaujla), prof_sky16@yahoo.com (K. Yadav)**

A space with constant curvature plays a central role in the development of differential geometry, Mathematical Physics and mechanics. Cartan ([8], [9]) developed the idea of a locally symmetric Riemannian manifold in 1926, which is a natural generalization of manifolds of constant curvature. The condition of local symmetry is equivalent to the fact that at every point $x \in M$, the local geodesic symmetry $F(x)$ is an isometry [6]. The idea of locally ϕ-symmetric Sasakian manifold was introduced by Takahashi [44] in 1977. Since then, the properties of such manifolds have been studied by several geometers on different spaces.
The notion of a semi-symmetric linear connection on a differentiable manifold has been introduced by Friedmann and Schouten [2] in 1924. Hayden [10] in 1932, introduced and studied the idea of semi-symmetric linear connection with torsion on a Riemannian manifold. After a long interval, Yano [13] started the systematic study of semi-symmetric metric connection on a Riemannian manifold in 1970. In this connection, the properties of semi-symmetric metric connection are studied in ([5], [19], [25], [27], [30], [36], [40], [42], [43], [48]) and others. P. Zhao and H. Song [21] defined and studied a projective semi-symmetric connection on Riemannian manifold in 2001. The properties of this connection has been studied by Zhao [22], Pal, Pandey and Singh [29] and others.
Motivated from the above studies, we start the study of Riemannian manifolds equipped with a projective semi-symmetric connection. The present paper is organized as follows: After introductory section, we brief the projective semi-symmetric connection in Section 2. It is proved that the curvature tensor with respect to projective semi-symmetric connection $\tilde{\nabla}$ coincide with the projective curvature tensor of the Levi-Civita connection ∇. We also prove that the manifold (M_n, g) endowed with $\tilde{\nabla}$ is a certain class of quasi Einstein manifold and the characteristic vector field ξ belongs to λ-nullity distribution with respect to the connection $\tilde{\nabla}$. Section 3 deals with the study of projective curvature tensor endowed with projective semi-symmetric connection $\tilde{\nabla}$ and prove that the projective curvature tensors with respect to connections $\tilde{\nabla}$ and ∇ coincide. The properties of semi-symmetric Riemannian manifold admitting a projective semi-symmetric connection $\tilde{\nabla}$ are given in Section 4. It is proved that the manifold is semi-symmetric for $\tilde{\nabla}$ if and only if it is flat. In Section 5, we study Riemannian manifold endowed with a projective semi-symmetric connection satisfying $\tilde{R}\cdot\tilde{P} = 0$ and prove some interesting results. In the last section, we construct an example which support the existence of such connection $\tilde{\nabla}$ and verify some results.

2. Projective semi-symmetric connection

Let M_n be an n-dimensional Riemannian manifold equipped with the Riemannian metric g and ∇ denotes the Levi-Civita connection on (M_n, g) and satisfy

$$\pi(X) = g(X,\xi) \quad and \quad g(\xi,\xi) = 1 \tag{2.1}$$

for arbitrary vector field X, where π is the first form associated with Riemannian metric g and ξ is a unit vector field of (M_n, g). A linear connection $\tilde{\nabla}$ defined on (M_n, g) is said to be semi-symmetric if the torsion tensor $\tilde{T}$ of the connection $\tilde{\nabla}$ defined as

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$$

and satisfies

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y,$$

i.e. $\tilde{T} \neq 0$ for arbitrary vector fields X and Y, otherwise it is symmetric. Also, a linear connection $\tilde{\nabla}$ on (M_n, g) is said to be metric if $\tilde{\nabla}g = 0$, otherwise non-metric. If the geodesics with respect to the linear connection $\tilde{\nabla}$ are consistent with those of Levi-Civita connection ∇, then $\tilde{\nabla}$ is called the projective equivalent connection with ∇. If we consider the linear connection $\tilde{\nabla}$ is semi-symmetric as well as projective equivalent, then it is called projective semi-symmetric connection [21]. The tensorial relation between projective semi-symmetric and Levi-Civita connections on Riemannian manifold (M_n, g) is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \psi(Y)X + \psi(X)Y + \phi(Y)X - \phi(X)Y \tag{2.2}$$

for arbitrary vector fields X and Y; where the 1-form ϕ and ψ are defined as

$$\phi(X) = \frac{1}{2}\pi(X) \quad and \quad \psi(X) = \frac{n-1}{2(n+1)}\pi(X) \tag{2.3}$$

for arbitrary vector fields X and Y [22]. From (2.1), (2.2) and (2.3), we have

$$(\tilde{\nabla}_X g)(Y,Z) = \frac{1}{n+1}[2\pi(X)g(Y,Z) - n\pi(Y)g(X,Z) - n\pi(Z)g(X,Y)] \tag{2.4}$$

for arbitrary vector fields X, Y and Z. Thus, the projective semi-symmetric connection $\tilde{\nabla}$ is non-metric. The properties of semi-symmetric non-metric connections have been noticed in ([3], [4], [18], [24], [26], [31]- [34], [37]- [39]) and many others. It can be easily seen from [22] that

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \beta(X,Y)Z + \theta(X,Z)Y - \theta(Y,Z)X, \tag{2.5}$$

where $\tilde{R}$ and R denote the curvature tensors with respect to the connections $\tilde{\nabla}$ and ∇, respectively and θ, β are $(0,2)$ type tensors satisfying the following relations

$$\theta(X,Y) = (\nabla_X\phi)(Y) + (\nabla_X\psi)(Y) - \psi(X)\psi(Y) - \phi(X)\phi(Y) - \psi(X)\phi(Y) - \phi(X)\psi(Y), \tag{2.6}$$

$$\beta(X,Y) = (\nabla_X\psi)(Y) - (\nabla_Y\psi)(X) + (\nabla_X\phi)(Y) - (\nabla_Y\phi)(X) \tag{2.7}$$

for arbitrary vector fields X, Y and Z. Let us suppose that the characteristic vector field ξ is a parallel unit vector field with respect to the Levi-Civita connection, *i.e.*, $\nabla\xi = 0$ and $||\xi|| = 1$. This expression is equivalent to

$$(\nabla_X \pi)(Y) = \nabla_X \pi(Y) - \pi(\nabla_X Y) = 0 \tag{2.8}$$

and therefore (2.2) and (2.3) assert that

$$\tilde{\nabla}_X \xi = \frac{1}{n+1}\{nX - \pi(X)\xi\}. \tag{2.9}$$

In consequence of (2.1), (2.3) and (2.8), equations (2.6) and (2.7) become

$$\beta(X,Y) = 0 \quad \textit{and} \quad \theta(X,Y) = \lambda\pi(X)\pi(Y), \tag{2.10}$$

where $\lambda = -\frac{n^2}{(n+1)^2}$. It can be easily observe from (2.5) and (2.10) that

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \lambda\{\pi(X)\pi(Z)Y - \pi(Y)\pi(Z)X\}. \tag{2.11}$$

Contracting (2.11) along X and then using (2.1), we have

$$\tilde{S}(Y,Z) = S(Y,Z) - \lambda(n-1)\pi(Y)\pi(Z), \tag{2.12}$$

which gives

$$\tilde{r} = r - \lambda(n-1).$$

Here $\tilde{S}$ and S; $\tilde{r}$ and r denote the Ricci tensors and scalar curvatures corresponding to the connections $\tilde{\nabla}$ and ∇, respectively.

Theorem 2.1. *An n-dimensional Riemannian manifold (M_n, g) equipped with a projective semi-symmetric connection $\tilde{\nabla}$ satisfying (2.8) holds the following curvature conditions:*

(i) $'\tilde{R}(X,Y,Z,U) = -'\tilde{R}(Y,X,Z,U)$,
(ii) $'\tilde{R}(X,Y,Z,U) \neq -'\tilde{R}(X,Y,U,Z)$,
(iii) $'\tilde{R}(X,Y,Z,U) \neq' \tilde{R}(Z,U,X,Y)$,
(iv) $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$,
(v) $(\tilde{\nabla}_X\tilde{R})(Y,Z)U + (\tilde{\nabla}_Y\tilde{R})(Z,X)U + (\tilde{\nabla}_Z\tilde{R})(X,Y)U = 2[\pi(X)R(Y,Z)U + \pi(Y)R(Z,X)U + \pi(Z)R(X,Y)U]$.

Proof. From (2.11), we have

$$'\tilde{R}(X,Y,Z,U) =' R(X,Y,Z,U) + \lambda\{\pi(X)\pi(Z)g(Y,U) - \pi(Y)\pi(Z)g(X,U)\} \tag{2.13}$$

for all vector fields $X,Y,Z,U \in T(M)$, where $'\tilde{R}(X,Y,Z,U) = g(\tilde{R}(X,Y)Z,U)$ and $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$. By considering (2.13) and curvature properties of R, we can easily verify the results (i), (ii) and (iii). With the help of (2.11) and Bianchi's first identity, we get

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0,$$

which shows that the Riemann curvature tensor with respect to the projective semi-symmetric connection $\tilde{\nabla}$ satisfies the Bianchi's first identity. Covariant derivative of (2.11) with respect to $\tilde{\nabla}$ gives

$$(\tilde{\nabla}_X\tilde{R})(Y,Z)U = (\tilde{\nabla}_X R)(Y,Z)U + \lambda\{\pi(U)(\tilde{\nabla}_X\pi)(Y)Z + \pi(Y)(\tilde{\nabla}_X\pi)(U)Z - \pi(U)(\tilde{\nabla}_X\pi)(Z)Y - \pi(Z)(\tilde{\nabla}_X\pi)(U)Y\}. \tag{2.14}$$

Also equations (2.1), (2.2), (2.3), (2.4), (2.8) and (2.9) yield

$$(\tilde{\nabla}_X\pi)(Y) = -\frac{n-1}{n+1}\pi(X)\pi(Y) \tag{2.15}$$

and

$$(\tilde{\nabla}_X R)(Y,Z)U = (\nabla_X R)(Y,Z)U + \frac{2}{n+1}\pi(X)R(Y,Z)U - \frac{n}{n+1}\{\pi(Y)R(X,Z)U + \pi(Z)R(Y,X)U + \pi(U)R(Y,Z)X\}. \tag{2.16}$$

In view of (2.15) and (2.16), (2.14) assumes the form

$$\begin{aligned}(\tilde{\nabla}_X\tilde{R})(Y,Z)U =& (\nabla_X R)(Y,Z)U + \frac{2}{n+1}\pi(X)R(Y,Z)U - \frac{n}{n+1}\{\pi(Y)R(X,Z)U + \pi(Z)R(Y,X)U + \pi(U)R(Y,Z)X\} \\ &- \frac{2\lambda(n-1)}{n+1}\{\pi(X)\pi(U)\pi(Y)Z - \pi(X)\pi(Z)\pi(U)Y\}.\end{aligned} \tag{2.17}$$

The cyclic sum of (2.17) for vector fields X, Y, Z and use of Bianchi's second identity for ∇ gives

$$(\tilde{\nabla}_X\tilde{R})(Y,Z)U + (\tilde{\nabla}_Y\tilde{R})(Z,X)U + (\tilde{\nabla}_Z\tilde{R})(X,Y)U = 2[\pi(X)R(Y,Z)U + \pi(Y)R(Z,X)U + \pi(Z)R(X,Y)U].$$

This shows that a Riemannian manifold (M_n, g) endowed with a projective semi-symmetric connection $\tilde{\nabla}$ satisfies the relation

$$(\tilde{\nabla}_X\tilde{R})(Y,Z)U + (\tilde{\nabla}_Y\tilde{R})(Z,X)U + (\tilde{\nabla}_Z\tilde{R})(X,Y)U = 0$$

if and only if

$$\pi(X)R(Y,Z)U + \pi(Y)R(Z,X)U + \pi(Z)R(X,Y)U = 0.$$

□

Let us suppose that the manifold (M_n,g) is Ricci flat with respect to the projective semi-symmetric connection $\tilde{\nabla}$, *i.e.*, $\tilde{S}=0$ and therefore equation (2.12) gives

$$S(Y,Z)=\lambda(n-1)\pi(Y)\pi(Z), \tag{2.18}$$

which shows that the manifold (M_n,g), $(n>2)$, is a certain class of quasi Einstein manifold with the associated scalars $a=0$ and $b=\lambda(n-1)$. In consequence of (2.18), (2.11) assumes the form

$$\tilde{R}(X,Y)Z=P(X,Y)Z, \tag{2.19}$$

where P denotes the Weyl projective curvature tensor with respect to the Levi-Civita connection ∇ and is given as

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{n-1}\{S(Y,Z)X-S(X,Z)Y\} \tag{2.20}$$

for all vector fields $X,Y,Z\in T(M)$. Thus we can conclude the results in the form of theorems as:

Theorem 2.2. *Let (M_n,g), $(n>2)$, be an n-dimensional Riemannian manifold equipped with a projective semi-symmetric connection $\tilde{\nabla}$ and the characteristic vector field ξ of the manifold is a parallel unit vector field. If (M_n,g) is Ricci flat with respect to the connection $\tilde{\nabla}$, then the projective curvature with respect to Levi-Civita connection ∇ coincide with the curvature tensor of $\tilde{\nabla}$.*

Theorem 2.3. *Let (M_n,g), $(n>2)$, be an n-dimensional Riemannian manifold endowed with a projective semi-symmetric connection $\tilde{\nabla}$ and ξ is a parallel unit vector field with respect to the Levi-Civita connection. If (M_n,g) is Ricci flat with respect to $\tilde{\nabla}$, then it is a certain class of quasi Einstein manifold.*

By our assumption, the characteristic vector field ξ is parallel unit vector field corresponding the Levi-Civita connection ∇ and therefore by equation (2.8) we can easily calculate that $R(X,Y)\xi=0$. After considering this fact and (2.1), equation (2.11) assumes the form

$$\tilde{R}(X,Y)\xi=\lambda\{\pi(X)Y-\pi(Y)X\}. \tag{2.21}$$

This shows that the characteristic vector field ξ belongs to the λ-nullity distribution with respect to the projective semi-symmetric connection $\tilde{\nabla}$. Thus we can state the following theorem:

Theorem 2.4. *If (M_n,g), $(n>2)$, be an n-dimensional Riemannian manifold admitting a projective semi-symmetric connection $\tilde{\nabla}$ and ξ is a parallel unit vector field with respect to ∇. Then the characteristic vector field of the manifold equipped with the projective semi-symmetric connection $\tilde{\nabla}$ belongs to λ-nullity distribution.*

In view of (2.1), (2.8), (2.11), (2.21) and symmetric and skew-symmetric properties of curvature tensor, we can state:

Lemma 2.5. *If an n-dimensional Riemannian manifold (M_n,g), $(n>2)$, admitting a projective semi-symmetric connection $\tilde{\nabla}$ and the characteristic vector field ξ is a parallel unit vector field, then the following relations satisfy*

(i) $\tilde{R}(\xi,X)Y=\lambda\{\pi(Y)X-\pi(X)\pi(Y)\xi\}$,
(ii) $\tilde{R}(X,\xi)Y=\lambda\pi(Y)\{\pi(X)\xi-X\}$,
(iii) $\pi(\tilde{R}(X,Y)Z)=0$ *for all vector fields* $X,Y,Z\in\chi(M)$.

Proof is obvious by straight forward calculations.
In view of (2.1), (2.8) and (2.12), we can compute that

$$(\tilde{\nabla}_X\tilde{S})(Y,Z)=(\nabla_X S)(Y,Z). \tag{2.22}$$

Hence we can state the theorem:

Theorem 2.6. *If (M_n,g), $(n>2)$, be an n-dimensional Riemannian manifold equipped with a projective semi-symmetric connection $\tilde{\nabla}$ and ξ is a parallel unit vector field. Then the manifold is Ricci-symmetric with respect to the projective semi-symmetric connection $\tilde{\nabla}$ if and only if it is Ricci-symmetric with respect to the Levi-Civita connection ∇.*

From equation (2.22), we can also observe that

$$(\tilde{\nabla}_X\tilde{S})(Y,Z)-(\tilde{\nabla}_Y\tilde{S})(X,Z)=(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)$$

and

$$(\tilde{\nabla}_X\tilde{S})(Y,Z)+(\tilde{\nabla}_Y\tilde{S})(Z,X)+(\tilde{\nabla}_Z\tilde{S})(X,Y)=(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)$$

and hence the following lemma:

Lemma 2.7. *Let (M_n,g), $(n>2)$, be an n-dimensional Riemannian manifold endowed with a projective semi-symmetric connection $\tilde{\nabla}$ and ξ is a parallel unit vector field. Then the Ricci tensor is of Codazzi type as well as cyclic parallel with respect to the projective semi-symmetric connection $\tilde{\nabla}$ if and only if it is Codazzi type as well cyclic parallel with respect to Levi-Civita connection ∇.*

3. Projective curvature tensor equipped with projective semi-symmetric connection

If $\tilde{P}$ denotes the Weyl projective curvature tensor with respect to the connection $\tilde{\nabla}$, then

$$'\tilde{P}(X,Y,Z,U) = '\tilde{R}(X,Y,Z,U) - \frac{1}{n-1}\{\tilde{S}(Y,Z)g(X,U) - \tilde{S}(X,Z)g(Y,U)\} \tag{3.1}$$

holds for arbitrary vector fields X, Y, Z and U, where $'\tilde{P}(X,Y,Z,U) = g(\tilde{P}(X,Y)Z,U)$. In consequence of (2.11) and (2.12), above equation becomes

$$'\tilde{P}(X,Y,Z,U) = ' P(X,Y,Z,U),$$

where P is the Weyl projective curvature tensor with respect to the Levi-Civita connection ∇ given in (2.20) and $'P(X,Y,Z,U) = g(P(X,Y)Z,U)$. From the above discussions, we can conclude the result in the form of theorem as:

Theorem 3.1. *If (M_n,g), $(n>2)$, be an n-dimensional Riemannian manifold admitting a projective semi-symmetric connection $\tilde{\nabla}$ and ξ is a parallel unit vector field, then the projective curvature tensors with respect to projective semi-symmetric and Levi-Civita connections coincide.*

Remark 3.2. *Zhao [22] considered the special projective semi-symmetric connection and proved that the Weyl projective curvature tensors are invariant with respect to the special projective semi-symmetric and Levi-Civita connections.*

From Theorems 2.2 and 3.1, we conclude the following:

Theorem 3.3. *Let (M_n,g), $(n>2)$, be an n-dimensional Riemannian manifold admitting a projective semi-symmetric connection $\tilde{\nabla}$ and ξ is a parallel unit vector field. If the Ricci tensor with respect to $\tilde{\nabla}$ is flat, then $\tilde{R} = \tilde{P} = P$.*

The Klein model of the Lobachevskii space and the central projection of a hemisphere on tangent space shows that the spaces of constant curvature are projectively flat and vice versa. Thus with the help of Theorem 3.1 and the above discussions, we can state:

Theorem 3.4. *If (M_n,g), $(n>2)$, be an n-dimensional Riemannian manifold admitting a projective semi-symmetric connection and ξ is a parallel unit vector field, then (M_n,g) is projectively flat with respect to the projective semi-symmetric connection if and only if it is of constant curvature.*

Now, we consider that the Riemannian manifold is flat with respect to the projective semi-symmetric connection, *i.e.*,$\tilde{R}=0$, then in consequence of Theorem 2.2 and equation (2.19), we obtain $P=0$. Therefore we can state the theorem:

Theorem 3.5. *If an n-dimensional Riemannian manifold (M_n,g), $(n>2)$, equipped with a projective semi-symmetric connection $\tilde{\nabla}$ and a parallel unit vector field ξ is flat with respect to $\tilde{\nabla}$, then it is a manifold of constant curvature although the converse part is also true.*

Remark 3.6. *The idea of constant curvature is playing a central role in the theory of relativity and cosmology. The simplest cosmological model can be constructed by assuming that the universe is isotropic and homogeneous. This is known as cosmological principle. When we translated this principle to Riemannian geometry, professes that the three dimensional position space is a space of maximal symmetry [11], i.e., a space of constant curvature whose curvature depends upon time. The cosmological solutions of Einstein field equations which contain a three dimensional space like surfaces of a constant curvature are the Robertson-Walker metric, while four dimensional space of constant curvature is the de Sitler model of the universe ([11], [12]).*

4. Semi-symmetric Riemannian manifold admitting projective semi-symmetric connection

A Riemannian manifold (M_n,g) is said to be semi-symmetric ([49], [50]) with respect to the Levi-Civita connection ∇ if its non-flat curvature tensor R satisfies the condition $R\cdot R=0$. Analogous to this, we can define:

Definition 4.1. *A non-flat Riemannian manifold (M_n,g), $(n>2)$, is said to be semi-symmetric with respect to the projective semi-symmetric connection $\tilde{\nabla}$ if $\tilde{R}\cdot\tilde{R}=0$.*

It is obvious that

$$(\tilde{R}(\xi,X)\cdot\tilde{R})(Y,Z)U = \tilde{R}(\xi,X)\tilde{R}(Y,Z)U - \tilde{R}(\tilde{R}(\xi,X)Y,Z)U - \tilde{R}(Y,\tilde{R}(\xi,X)Z)U - \tilde{R}(Y,Z)\tilde{R}(\xi,X)U.$$

In view of (2.1), (2.21) and Lemma 2.5, the last equation becomes

$$(\tilde{R}(\xi,X)\cdot\tilde{R})(Y,Z)U = -\lambda\{\pi(Y)\tilde{R}(X,Z)U + \pi(Z)\tilde{R}(Y,X)U + \pi(U)\tilde{R}(Y,Z)X\} + 2\lambda^2\{\pi(Y)Z - \pi(Z)Y\}\pi(X)\pi(U). \tag{4.1}$$

Let us suppose that $\tilde{R}\cdot\tilde{R}=0$, then equation (4.1) reflects that either $\lambda=0$ or $\pi(Y)\tilde{R}(X,Z)U + \pi(Z)\tilde{R}(Y,X)U + \pi(U)\tilde{R}(Y,Z)X = 2\lambda\{\pi(Y)Z - \pi(Z)Y\}\pi(X)\pi(U)$. Since $\lambda\neq 0$, therefore

$$\pi(Y)\tilde{R}(X,Z)U + \pi(Z)\tilde{R}(Y,X)U + \pi(U)\tilde{R}(Y,Z)X = 2\lambda\{\pi(Y)Z - \pi(Z)Y\}\pi(X)\pi(U).$$

Putting $U=\xi$ in last expression and then using (2.1) and (2.21), we find

$$\tilde{R}(Y,Z)X = \lambda\pi(X)\{\pi(Y)Z - \pi(Z)Y\}, \tag{4.2}$$

which is equivalent to $R(Y,Z)X=0$. Thus the manifold (M_n,g), $(n>2)$, equipped with a projective semi-symmetric connection $\tilde{\nabla}$ satisfying $\tilde{R}\cdot\tilde{R}=0$ is flat for Levi-Civita connection ∇. Conversely, if the manifold is flat, *i.e.*, $R=0$, then equation (2.11) assumes the form (4.2). With the help of (2.1), (2.21) and (4.2), (4.1) shows that $\tilde{R}\cdot\tilde{R}=0$. Hence we state the above result in the form of theorem as:

Theorem 4.2. *Let (M_n, g), $(n > 2)$, be an n-dimensional Riemannian manifold admitting a projective semi-symmetric connection $\tilde{\nabla}$ satisfying (2.8). Then the necessary and sufficient condition for a manifold (M_n, g) to be semi-symmetric with respect to the connection $\tilde{\nabla}$ is that the manifold is flat with respect to Levi-Civita connection ∇.*

In consequence of (2.11), (2.17), $\lambda = -\frac{n^2}{(n+1)^2}$ and Theorem 4.2, we find

$$(\tilde{\nabla}_X \tilde{R})(Y,Z)U = \rho(X)\tilde{R}(Y,Z)U,$$

where $\rho(X) = -\frac{2(n-1)}{n+1}\pi(X)$. Thus we can state:

Corollary 4.3. *Let (M_n, g), $(n > 2)$, be an n-dimensional Riemannian manifold admitting a projective semi-symmetric connection $\tilde{\nabla}$ and satisfies (2.8). If (M_n, g) is semi-symmetric with respect to the projective semi-symmetric connection $\tilde{\nabla}$, then it is recurrent.*

From Theorem 4.2, we can also conclude the following corollary:

Corollary 4.4. *A semi-symmetric Riemannian manifold (M_n, g), $(n > 2)$, endowed with a projective semi-symmetric connection $\tilde{\nabla}$ is projectively, conformally, concircularly, conharmonicaly, quasi-conformally and m-projectively flat for Levi-Civita connection ∇.*

In view of Theorem 3.1 and Corollary 4.4, we have

Corollary 4.5. *Every semi-symmetric Riemannian manifold (M_n, g), $(n > 2)$, equipped with a projective semi-symmetric connection $\tilde{\nabla}$ is projectively flat for $\tilde{\nabla}$.*

5. Riemannian manifold with a projective semi-symmetric connection satisfying $\tilde{R} \cdot \tilde{P} = 0$

We have

$$(\tilde{R}(X,Y)\cdot\tilde{P})(Z,U)V = \tilde{R}(X,Y)\tilde{P}(Z,U)V - \tilde{P}(\tilde{R}(X,Y)Z,U)V - \tilde{P}(Z,\tilde{R}(X,Y)U)V - \tilde{P}(Z,U)\tilde{R}(X,Y)V.$$

Replacing X by ξ in above equation and then using equation (2.1) and Lemma (2.5), we have

$$\begin{aligned}(\tilde{R}(\xi,Y)\cdot\tilde{P})(Z,U)V =&\lambda[\pi(\tilde{P}(Z,U)V)Y - \pi(\tilde{P}(Z,U)V)\pi(Y)\xi - \pi(Z)\tilde{P}(Y,U)V - \pi(U)\tilde{P}(Z,Y)V \\ &- \pi(V)\tilde{P}(Z,U)Y + \pi(Y)\pi(Z)\tilde{P}(\xi,U)V + \pi(U)\pi(Y)\tilde{P}(Z,\xi)V + \pi(Y)\pi(V)\tilde{P}(Z,U)\xi].\end{aligned} \tag{5.1}$$

Let us suppose that $\tilde{R}\cdot\tilde{P} = 0$, then we have from (5.1) $(\lambda \neq 0)$

$$\begin{aligned}\pi(Z)\tilde{P}(Y,U)V + \pi(U)\tilde{P}(Z,Y)V + \pi(V)\tilde{P}(Z,U)Y =&\pi(\tilde{P}(Z,U)V)Y - \pi(\tilde{P}(Z,U)V)\pi(Y)\xi \\ &+ \pi(Y)\pi(Z)\tilde{P}(\xi,U)V + \pi(U)\pi(Y)\tilde{P}(Z,\xi)V + \pi(Y)\pi(V)\tilde{P}(Z,U)\xi.\end{aligned}$$

Setting Z by ξ in the above equation and using (2.1), we get

$$\begin{aligned}\tilde{P}(Y,U)V + \pi(U)\tilde{P}(\xi,Y)V + \pi(V)\tilde{P}(\xi,U)Y =&\pi(\tilde{P}(\xi,U)V)Y - \pi(\tilde{P}(\xi,U)V)\pi(Y)\xi \\ &+ \pi(Y)\tilde{P}(\xi,U)V + \pi(U)\pi(Y)\tilde{P}(\xi,\xi)V + \pi(Y)\pi(V)\tilde{P}(\xi,U)\xi.\end{aligned} \tag{5.2}$$

In consequence of (2.1), (2.12), Lemma 2.5 and (3.1), we find that

$$\begin{aligned}&(i) \quad \tilde{P}(\xi,X)Y = \frac{1}{n-1}\{S(\xi,Y)X - S(X,Y)\xi\}, \\ &(ii) \quad \pi(\tilde{P}(X,Y)Z) = \frac{1}{n-1}\{\pi(Y)S(X,Z) - \pi(X)S(Y,Z)\}.\end{aligned} \tag{5.3}$$

In view of (5.2), (5.3) turns into the form

$$\begin{aligned}\tilde{P}(Y,U)V =&\frac{1}{n-1}\{\pi(U)S(V,Y)\xi - \pi(V)S(\xi,Y)U + \pi(V)S(U,Y)\xi - \pi(U)\pi(Y)S(\xi,V)\xi - S(U,V)Y \\ &+ \pi(Y)S(\xi,V)U + \pi(Y)\pi(V)S(\xi,\xi)U - \pi(Y)\pi(V)S(U,\xi)\xi\},\end{aligned} \tag{5.4}$$

which gives

$$\begin{aligned}\pi(\tilde{P}(Y,U)V) =&\frac{1}{n-1}\{\pi(U)S(V,Y) - \pi(V)S(\xi,Y)\pi(U) + \pi(V)S(U,Y) - \pi(U)\pi(Y)S(\xi,V) - S(U,V)\pi(Y) \\ &+ \pi(Y)S(\xi,V)\pi(U) + \pi(Y)\pi(V)S(\xi,\xi)\pi(U) - \pi(Y)\pi(V)S(U,\xi)\}.\end{aligned} \tag{5.5}$$

Using (5.3) (ii) in (5.5), we obtain

$$\pi(V)S(U,Y) = \pi(U)\pi(V)S(\xi,Y) + \pi(U)\pi(Y)S(\xi,V) - \pi(U)\pi(Y)S(\xi,V) - \pi(Y)\pi(V)\pi(U)S(\xi,\xi) + \pi(Y)\pi(V)S(U,\xi).$$

Setting $V = \xi$ in above equation and use of (2.1) gives

$$S(U,Y) = \pi(U)S(\xi,Y) + \pi(Y)S(U,\xi) - \pi(U)\pi(Y)S(\xi,\xi). \tag{5.6}$$

Using (2.1), (5.6) in (5.4), we find

$$\begin{aligned}{}'\tilde{P}(Y,U,V,X) =&\frac{1}{n-1}\{2\pi(U)\pi(X)\pi(V)S(\xi,Y) - 2\pi(X)\pi(U)\pi(V)\pi(Y)S(\xi,\xi) - \pi(V)S(\xi,Y)g(U,X) - \pi(U)S(\xi,V)g(Y,X) \\ &- \pi(V)S(U,\xi)g(X,Y) + \pi(U)\pi(V)g(X,Y)S(\xi,\xi) + \pi(Y)S(\xi,V)g(U,X) + \pi(Y)\pi(V)S(\xi,\xi)g(U,X)\}.\end{aligned}$$

$$(5.7)$$

Let $\{e_i, i = 1,2,...,n\}$ be an orthonormal basis of the tangent space at any point of the manifold (M_n, g). Then putting $X = Y = e_i$ in (5.7) and taking summation over i, $1 \leq i \leq n$, we get

$$\pi(U)S(\xi,V) = \frac{n+1}{n-1}\{\pi(U)\pi(V)S(\xi,\xi) - \pi(V)S(U,\xi)\}. \tag{5.8}$$

after considering equations (2.1), (3.1) and $\sum_{i=1}^{n}{}'\tilde{P}(e_i,U,V,e_i) = 0$. Replacing the vector field U with ξ in (5.8) and use of (2.1) gives $S(\xi,V) = 0$ and therefore equation (5.6) gives $S(U,Y) = 0$ for arbitrary vector fields U and Y. This equation shows that (M_n, g), $(n > 2)$, is Ricci flat with respect to the Levi-Civita connection. Thus we can state:

Theorem 5.1. *A Riemannian manifold (M_n, g), $(n > 2)$, equipped with $\tilde{\nabla}$ satisfying $\tilde{R}\cdot\tilde{P} = 0$ is Ricci flat.*

By the use of (2.1), (2.11), (2.12), (3.1), Theorems 3.1 and 5.1, we conclude that

$$\tilde{P}(Z,U)V = R(Z,U)V = P(Z,U)V.$$

Corollary 5.2. *If a Riemannian manifold (M_n, g) admits a projective semi-symmetric connection $\tilde{\nabla}$ and satisfies the condition $\tilde{R}\cdot\tilde{P} = 0$, then the curvature tensor R and projective curvature tensor P for ∇ coincides with projective curvature tensor of $\tilde{\nabla}$.*

From Theorems 4.2 and 5.1, we conclude the following corollary:

Corollary 5.3. *Let (M_n, g), $(n > 2)$, be an n-dimensional Riemannian manifold admit $\tilde{\nabla}$ and ξ is a parallel unit vector field. Then every semi-symmetric Riemannian manifold with respect to $\tilde{\nabla}$ satisfies $\tilde{R}\cdot\tilde{P} = 0$.*

6. Example

P. Alegre, D. E. Blair and A. Carriazo [20] introduced the idea of generalized Sasakian space form and they constructed many examples by using some different geometric techniques such as Riemannian submersions, warped products or conformal and related transformations in 2004. A Riemannian manifold M_n of dimension n equipped with a tensor field ϕ of type $(1,1)$, a structure vector field ξ and a covariant vector field η associated with the Riemannian metric g satisfies the relations

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \phi\xi = 0 \;\; \textit{and} \;\; g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y) \tag{6.1}$$

for arbitrary vector fields X and Y, is called an almost contact metric manifold $(M_n, \phi, \xi, \eta, g)$ [7]. An almost contact metric manifold $(M_n, \phi, \xi, \eta, g)$ is cosymplectic [7] if $\nabla\phi = 0$, which implies the following expressions

$$\nabla\xi = 0, \quad \nabla\eta = 0 \quad \textit{and} \quad R(X,Y)\xi = 0.$$

An almost contact metric manifold $(M_n, \phi, \xi, \eta, g)$ is said to be a generalized Sasakian space form [20] if the Riemannian curvature tensor R satisfies the tensorial relation

$$\begin{aligned} R(X,Y)Z =& f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{6.2}$$

for arbitrary vector fields X, Y and Z, where f_1, f_2 and f_3 are smooth functions on M_n. First author proved that the generalized Sasakian space form is a certain class of quasi Einstein manifold [35]. Here we suppose that the manifold M_n is cosymplectic as well as generalized Sasakian space form and $f_1 = f_3 \neq 0$. Replacing Z with the structure vector field ξ in (6.2) and then using equation (6.1), we get

$$R(X,Y)\xi = 0. \tag{6.3}$$

If we define the projective semi-symmetric connection $\tilde{\nabla}$ on M_n as

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{n-1}{2(n+1)}\{\eta(Y)X + \eta(X)Y\} + \frac{1}{2}\{\eta(Y)X - \eta(X)Y\}, \tag{6.4}$$

and torsion tensor of the connection $\tilde{\nabla}$

$$\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

for arbitrary vector fields X and Y, then with the help of equations (2.11), (6.3) and (6.4), we obtain the expression (2.21) and hence the Theorem 2.3 verified.

Acknowledgment

The authors express their sincere thanks to the referees and the Editor for their valuable comments in the improvement of the paper.

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Universal Journal of Mathematics and Applications, 3 (2) (2020) 86-92
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.641744

On Some Methods for Solution of Linear Diophantine Equations

Azam A. Imomov[1*] **and Yorqin T. Khodjaev**[1]

[1]*Karshi State University, 17 Kuchabag Street, 100180 Karshi City, Uzbekistan*
[*]*Corresponding author*

Article Info

Keywords: *Diophantine equation, partial solution, general solution.*
2010 AMS: *11D45, 11D72*
Received: *2 November 2020*
Accepted: *17 June 2020*
Available online: *22 June 2020*

Abstract

The paper considers a linear Diophantine equation. A method (algorithm) for finding a general class of solutions of equation is proposed. The proposed algorithm is explained by examples of equations with two and three variables, trying to direct the reader to a general idea that describes the essence of the method used.

1. Introduction

A Diophantine equation is an equation in several variables in which only integer solutions are allowed. One of its special cases is the linear Diophantine equation in $n \in \mathbb{N}$ variables (where $\mathbb{N} := \{1, 2, \ldots\}$), which is of the following general form

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = B, \tag{1.1}$$

whose solutions are required to be integers, where $\{A_i, B\} \subset \mathbb{Z}$, $n \geq 2$ and $i = 1, 2, \ldots, n$, where $\mathbb{Z}$ is the set of all integers. Equation (1.1) is named in honor of the Greek mathematician Diophantus of Alexandria (circa 300 c.e.).

A large number of works on the solution of Linear Diophantine equations are devoted. In these works, various methods and algorithms for solving equations are proposed and developed. For instance in [4], methods based on arguments of Euclidean algorithm are proposed. In this regard, we refer the reader, also to [1], [2], [3], [5], [6].

The aim of the paper is to attempt to give a general algorithm for finding the class of all solutions of equation (1.1), which, unlike the mentioned methods, would simplify the process of finding solutions.

In the second section, we give an algorithm for the method of finding the class of all solutions of an equation with two variables. Moreover, we will explain this algorithm using a typical example, trying to direct the reader to a general idea that describes the essence of the proposed method. In fact, the algorithm we proposed in this case, is based on arguments that differ significantly from the Euclidean algorithm. The latter is known as the algorithm for finding the greatest common divisor of integers.

In the Third and Fourth sections, the developed algorithm for an equation with two variables extends to cases of an equation with three and more variables.

2. Equation with two variables

Let $n = 2$ in equation (1.1). Then we consider the following equation with two variables x and y:

$$Ax + By = C, \tag{2.1}$$

where $\{A, B, C\} \subset \mathbb{Z}$ are the given numbers. The greatest common divisor of numbers $|A|$ and $|B|$ is denote by $\gcd(|A|, |B|)$. There are many sources devoted to finding solutions to equations (2.1). It is known that if the number $|C|$ is not divisible by $\gcd(|A|, |B|)$, then equation (2.1) has no solutions; see [4].

Now consider the case that the number C is divisible by $\gcd(|A|, |B|)$. Let's divide numbers A, B, and C by $\gcd(|A|, |B|)$. Then equation (2.1) be transformed to the following form:

$$ax + by = c, \tag{2.2}$$

Email address and ORCID number: **imomov_azam@mail.ru, (A. A. Imamov), khodjaev_yorqin@mail.ru, (Y. T. Khodjaev)**

where the numbers a, b, and c don't have a common divisor, that is $\gcd(a,b,c) = 1$. Integers with this property are hereinafter referred to as *prime numbers in common*.
It is known that to construct all the solutions of equation (2.2), one partial solution is needed.

2.1. General Algorithm for finding a partial solution.

As this is often done in order to find all solutions of equation (2.2), it is first necessary to somehow establish one partial solution. Typically, such a solution can be found primitively, using intuitive consideration. Then, using this solution, it is easy to construct the class of all solutions. In this case, a specific analytical method for finding a partial solution is not used. In this section, we put forward one algorithm for the method of finding a partial solution, based in fact, on the arguments essentially differing from Euclidian algorithm. Namely our algorithm is based on a very simple but a very important for our aim idea: searching for a linear representation of the unit using a finite number of so-called superpositions of the coefficients of the equation in question. And also, as will be shown below, this algorithm is applicable for cases of equations with an arbitrary number of variables.

Definition 2.1. *The number obtained as a result of eventual operations, which consist of the algebraic actions of addition and subtraction between several prime numbers in common, we call the* ***Superposition*** *of these numbers.*

Let

$$\mathbb{D} := \{a_1, a_2, \ldots, a_k\} \subset \mathbb{Z}, \quad k \in \mathbb{N}.$$

Henceforth we denote

$$\mathbf{Spos}\,(\mathbb{D}) \quad or \quad \mathbf{Spos}\left(a_{i_1}, a_{i_2}, \ldots, a_{i_j}\right), \ \ j \leq k,$$

any superposition of numbers from the set $\mathbb{D}$, where $a_{i_m} \in \mathbb{D}$, $i_m \in \mathbb{N}$ and $m = 1, 2, \ldots, j$.
The essence of the proposed algorithm will be based on the following simple axiom that describes the elementary properties of integers.

Main Axiom 2.1. *Using repeatedly superposition of two mutually prime numbers, one can construct any integer.*

First, we give an example.

Exercise 2.2. *Find a partial solution of the equation*

$$127x + 36y = 79. \tag{2.3}$$

Solution. The coefficients of the equation $a = 127$, $b = 36$, as seen, are mutual prime numbers. According to the Main axiom, using a superposition of the same numbers we find, for example, the following number:

1) $\mathbf{Spos}\,(a,b) = a - (b+b+b) = a - 3b = 127 - 3\times 36 = 19 =: a_1$.

Let continue to construct superpositions using the coefficients a, b and a_1:

2) $\mathbf{Spos}\,(a,a_1) = a - 5a_1 = -4a + 15b = 32 =: a_2$;
3) $\mathbf{Spos}\,(a_1,a_2) = a_2 - a_1 = -5a + 18b = 13 =: a_3$;
4) $\mathbf{Spos}\,(a_1,a_3) = a_1 - a_3 = 6a - 21b = 6 =: a_4$;
5) $\mathbf{Spos}\,(a_3,a_4) = a_3 - 2a_4 = -17a + 60b = 1$.

In the last step, we built the number 1 as follows:

$$-17a + 60b = 1. \tag{2.4}$$

Multiplying both sides of equality (2.4) by 79, we can easily get one solution to the equation (2.3). In fact, this follows from (2.4) that

$$-1343a + 4740b = 79,$$

so that $(x_0 = -1343; y_0 = 4740)$ is the partial solution of equation (2.3). □

Note that we built the superpositions so that the values of the sequence of $\{a_k, k \in \mathbb{N}\}$ step by step approached unity, and continued the process until they reached unity. It reached in step 5. In other words, we "crushed" the right side of equation (2.3) and as a result we got equality (2.4). Based on what has been done, we call the proposed method the ***Crushing method***.
We also note two more moments. Firstly, the set of steps is not unique, because superpositions of two numbers can be constructed as many time as desired, and the number of steps to the final equality of type (2.4) depends on the choice of these superpositions. Secondly, as a consequence of the first moment, the partial solution $(x = x_0; y = y_0)$ may be completely different.
The proposed Crushing method for finding one (partial) solution to equation (2.3) is universal in the sense that for any equation of type (3), a finite number of superposition steps can be performed to crush up the right-hand side to unity and thereby obtain an equality of type (2.4). In fact, for given coefficients $\{a,b,c\} \subset \mathbb{Z}$, one can always find the minimum number of steps of such superpositions.
Now we give a general description of our algorithm. We follow the procedure from the solution of equation (2.3). Let it be required to find one partial solution of equation (2.2) with the set of coefficients $\mathbb{D}_1 := \{a,b\} \subset \mathbb{Z}$. The following sequence of superpositions will be performed, until an appearance of the number 1:

$$\begin{array}{l}
\mathbf{Spos}\,(\mathbb{D}_1) =: a_1 \quad \text{and} \quad \text{define} \quad \mathbb{D}_2 := \{\mathbb{D}_1, a_1\}; \\
\mathbf{Spos}\,(\mathbb{D}_2) =: a_2 \quad \text{and} \quad \text{define} \quad \mathbb{D}_3 := \{\mathbb{D}_2, a_2\}; \\
\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\mathbf{Spos}\,(\mathbb{D}_k) = 1 \quad for \ \ some \ \ k \in \mathbb{N},
\end{array}$$

where $\mathbb{D}_k := \{\mathbb{D}_{k-1}, a_{k-1}\}$. From last equality we obtain

$$c \cdot \mathbf{Spos}\left(\mathbb{D}_k\right) = ax_0 + by_0 = c,$$

and needless to say we get $(x_0; y_0)$ – the solution of equation (2.3).
Thus, the above algorithm (Crushing method) can be admitted as a ***General Algorithm*** for finding a partial solution to the equation with two variables of the form (2.2).

2.2. General class of solutions.

Suppose that one pair of solutions $(x_0; y_0)$ is known. As we proved in the previous subsection, for equations that have a solution, one can always find one pair of solutions by using the Crushing method. Then it is obvious that equation (2.2) can be written as

$$ax + by = ax_0 + by_0. \tag{2.5}$$

Hence

$$a(x - x_0) = b(y_0 - y).$$

Since $\gcd(a, b) = 1$, it is necessary to be $x - x_0 = bk$, where $k \in \mathbb{Z}$. Then the general solution of equation (2.2) will have the following form:

$$\begin{cases} x = x_0 + bk, \\ y = y_0 - ak. \end{cases} \tag{2.6}$$

In view of the found general solution (2.6), we can write the equation (2.2) in the following equivalent form:

$$ax + by = a\left(x_0 + bk\right) + b\left(y_0 - ak\right), \tag{2.7}$$

and therefore

$$a\left(x - x_0\right) + b\left(y - y_0\right) - abk + abk = 0. \tag{2.8}$$

Relation (2.8) shows that equations (2.5) and (2.7) are the same. From here one can get a solution in the form of $x = x_0 - bk$ and $y = y_0 + ak$. Therefore, the general solution is finally written in the following form:

$$\begin{cases} x = x_0 \pm bk, \\ y = y_0 \mp ak, \end{cases} \tag{2.9}$$

where k is any number of $\mathbb{Z}$. From the last reasoning, we can conclude that in order to obtain a general solution to equation (2.5), it is sufficient to rewrite the equation (2.5) moving all expressions to the left-hand side and add zero in the form

$$-abk + abk.$$

In particular, according to (2.9), we obtain the general solution of equation (2.3) in the following form:

$$\begin{cases} x = -1343 \pm 36k, \\ y = 4740 \mp 127k, \end{cases}$$

where k is any number of $\mathbb{Z}$. Note that a particular solution can be minimized by choosing k. In this case, the general class of solutions of equation (2.3) can be written more simply:

$$\begin{cases} x = 25 \pm 36k, \\ y = -86 \mp 127k. \end{cases}$$

3. Equation with three variables

In this section, we demonstrate a solution to an equation with three unknowns. Without loss of generality, as in the case of two variables, we consider the equation

$$ax + by + cz = d, \tag{3.1}$$

where $\{a, b, c, d\} \subset \mathbb{Z}$ and $\gcd(a, b, c) = 1$.

3.1. Crushing method for three variables.

The previous section describes a general algorithm for finding one partial solution to the equation with two variables using the so-called Crushing method. Here we demonstrate the possibility of spreading this method to solve the equation (3.1). The concept of a superposition of three or more prime numbers in common is defined similarly to the case with two numbers. Further, we use the following statement, the proof of which follows from the Main Axiom.

Proposition 3.1. *Using the repeat superposition of three or more prime numbers in common, one can construct any integer.*

Exercise 3.2. *Find a partial solution of the equation*

$$30x+105y+56z=13. \tag{3.2}$$

Solution. The set coefficients of (3.2) is $\{a,b,c\}=\{30,105,56\}$. By virtue of Proposition above, using the superposition of these coefficients, we make the following steps:

1) $\mathbf{Spos}\,(a,c)=2a-c=4=:a_1;$
2) $\mathbf{Spos}\,(b,a_1)=b-26a_1=b-52a+26c=1.$

We have constructed number 1 only in the second step:

$$-52a+b+26c=1.$$

Multiplying both sides of the last equality by 13, we obtain

$$-676a+13b+338c=13,$$

and thereby we found one (partial) solution of equation (3.2):

$$x_0=-676, \quad y_0=13, \quad z_0=338.$$

The general class of solutions will be constructed below. □

As in the case of two variables, we note that there will be no obstacles to constructing the minimum number of steps of the corresponding superpositions for any given set of coefficients $\{a,b,c,d\}\subset\mathbb{Z}$ in equation (3.1), as a result of which we achieve one partial solution.

3.2. General class of solutions.

As noted in the previous subsection, for equations that have solutions, one can always find one solution $(x_0;y_0;z_0)$ using the crushing method. Now, using this solution, we intend to obtain a class of all solutions. For this, we extend the method from previous subsection to the case under consideration. Similarly to equality (2.8), we write the equation (3.1) in the following form:

$$a\,(x-x_0)+b\,(y-y_0)+c\,(z-z_0)-abk_1+abk_1-ack_2+ack_2-cbk_3+cbk_3=0. \tag{3.3}$$

Therefore, we can present the general solution in the form

$$\begin{cases} x=x_0\pm bk_1\pm ck_2, \\ y=y_0\mp ak_1\pm ck_3, \\ z=z_0\mp ak_2\mp bk_3, \end{cases} \tag{3.4}$$

where $\{k_1,k_2,k_3\}\subset\mathbb{Z}$.
The following matrix clearly indicates to the coefficients and their signs in each row of the system (3.4). Here, in front of the matrix, on the column we put the partial solution by the corresponding variable, on the rows of the matrix are the coefficients in the order specified in the equation (3.1).

$$\begin{array}{c} x_0 \longleftrightarrow \\ y_0 \longleftrightarrow \\ z_0 \longleftrightarrow \end{array} \begin{bmatrix} \mathbf{0}\cdot a & \pm b & \pm c \\ \mp a & \mathbf{0}\cdot b & \pm c \\ \mp a & \mp b & \mathbf{0}\cdot c \end{bmatrix}. \tag{3.5}$$

In each row of the matrix, multiplication by zero indicates the absence of the corresponding coefficient. For example, in the second row there is no second coefficient, which is consistent with the second row of system (3.4). The coefficients multiplied by zero form the diagonal of the matrix, which divides it into two parts. Coefficients in the upper part of the diagonal, as a rule, have a plus sign, and in the lower part - coefficients with a minus sign.
The constructed matrix which we call *the matrix of solution class*, visually specifies the form of the general solution of the equations in question.
In particular, based on (3.4) and (3.5), we can construct the class of general solutions of equation (3.2) in the following form:

$$\begin{cases} x=676\pm 105k_1\pm 56k_2, \\ y=13\mp 30k_1\pm 56k_3, \\ z=338\mp 30k_2\mp 105k_3, \end{cases}$$

where $\{k_1,k_2,k_3\}$ is any set inside $\mathbb{Z}$.

Remark 3.3. *In the book [4, pp. 23–33] proposed an algorithm for finding a solution of linear Diophantine equations, which requires a fairly lengthy process. The application of this algorithm was demonstrated by the example of one equation with three variables. By comparing, we can make sure that our proposed method looks slightly simpler.*

4. Equation with n variables

Now consider the equation (1.1) with n variables. Dividing by $\gcd(A_1, A_2, \ldots, A_n)$, we write it in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \tag{4.1}$$

where $\{a_i, b\} \subset \mathbb{Z}$, $i = 1, 2, \ldots, n$ and $\gcd(a_1, a_2, \ldots, a_n) = 1$.

4.1. General class of solutions.

In this subsection, we demonstrate a general description of the algorithm for constructing the class of all solutions of equation (4.1). Suppose that the set of numbers $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the partial solution to this equation. Such a solution can always be found using the general algorithm (crushing method) described in the previous sections. To find a general solution to the equation, we represent it in the form of type (3.3). After similar reasoning as in Subsection 3.2, we can construct the class of all solutions of equation (4.1) in the following form:

$$\begin{cases} x_1 = \alpha_1 \pm a_2k_{12} \pm a_3k_{13} \pm a_4k_{14} \pm \cdots \pm a_nk_{1n}, \\ x_2 = \alpha_2 \mp a_1k_{21} \pm a_3k_{23} \pm a_4k_{24} \pm \cdots \pm a_nk_{2n}, \\ x_3 = \alpha_3 \mp a_1k_{31} \mp a_2k_{32} \pm a_4k_{34} \pm \cdots \pm a_nk_{3n}, \\ x_4 = \alpha_4 \mp a_1k_{41} \mp a_2k_{42} \mp a_3k_{43} \pm \cdots \pm a_nk_{4n}, \\ \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ x_n = \alpha_n \mp a_1k_{n1} \mp a_2k_{n2} \mp a_4k_{n3} \mp \cdots \mp a_{n-1}k_{nn-1}, \end{cases} \tag{4.2}$$

where $\|k_{ij}\|_{i,j=1}^n$ is the quadratic matrix such that $k_{ij} \in \mathbb{Z}$, $k_{ii} = 0$ and $k_{ij} = k_{ji}$ for all $i, j = 1, 2, \ldots, n$.
The following matrix, as in the case of three variables, gives a clear picture of constructing a class of general solutions of the equation, indicating the coefficients and their signs in each row of the system (4.2):

$$\begin{array}{c} \alpha_1 \longleftrightarrow \\ \alpha_2 \longleftrightarrow \\ \alpha_3 \longleftrightarrow \\ \vdots \\ \alpha_{n-1} \longleftrightarrow \\ \alpha_n \longleftrightarrow \end{array} \begin{bmatrix} \mathbf{0} \cdot a_1 & \pm a_2 & \pm a_3 & \cdots & \pm a_{n-1} & \pm a_n \\ \mp a_1 & \mathbf{0} \cdot a_2 & \pm a_3 & \cdots & \pm a_{n-1} & \pm a_n \\ \mp a_1 & \mp a_2 & \mathbf{0} \cdot a_3 & \cdots & \pm a_{n-1} & \pm a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mp a_1 & \mp a_2 & \mp a_3 & \cdots & \mathbf{0} \cdot a_{n-1} & \pm a_n \\ \mp a_1 & \mp a_2 & \mp a_3 & \cdots & \mp a_{n-1} & \mathbf{0} \cdot a_n \end{bmatrix}. \tag{4.3}$$

Note that in the case of an equation with many variables, constructing the matrix of the form (4.3) greatly simplifies discussions in solving the equations.

Remark 4.1. *According to the fact that $k_{ij} = k_{ji}$, $i, j = 1, 2, \ldots, n$, the number of all different symbols k_{ij} is $\binom{n}{2}$, i.e.*

$$\binom{n}{2} = \frac{n(n-1)}{2}. \tag{4.4}$$

Let's get to the examples.

Exercise 4.2. *Find a partial solution of the equation*

$$30x + 42y + 105z + 70u = 17. \tag{4.5}$$

Solution. The coefficients of the equation are $a = 30$, $b = 42$, $c = 105$ and $d = 70$. Using the superpositions of the coefficients we obtain

1) **Spos** $(b, d) = 2b - d = 14 =: a_1$;
2) **Spos** $(a, a_1) = a - 2a_1 = a - 4b + 2d = 2 =: a_2$;
3) **Spos** $(c, a_2) = c - 52a_2 = c - 52a + 208b - 104d = 1$.

We obtained

$$-52a + 208b + c - 104d = 1.$$

Therefore

$$-884a + 3536b + 17c - 1768d = 17,$$

and thus we have a partial solution

$$x_0 = -884, \quad y_0 = 3536, \quad z_0 = 17, \quad u_0 = -1768.$$

Now we construct the matrix of solution class:

$$\begin{array}{r} -884 \longleftrightarrow \\ 3536 \longleftrightarrow \\ 17 \longleftrightarrow \\ -1768 \longleftrightarrow \end{array} \begin{bmatrix} \mathbf{0} \cdot 30 & \pm 42 & \pm 105 & \pm 70 \\ \mp 30 & \mathbf{0} \cdot 42 & \pm 105 & \pm 70 \\ \mp 30 & \mp 42 & \mathbf{0} \cdot 105 & \pm 70 \\ \mp 30 & \mp 42 & \mp 105 & \mathbf{0} \cdot 70 \end{bmatrix}.$$

Thereafter we can construct the class of all solutions in the form

$$\begin{cases} x = -884 \pm 42k_{12} \pm 105k_{13} \pm 70k_{14}, \\ y = 3536 \mp 30k_{21} \pm 105k_{23} \pm 70k_{24}, \\ z = 17 \mp 30k_{31} \mp 42k_{32} \pm 70k_{34}, \\ u = -1768 \mp 30k_{41} \mp 42k_{42} \mp 105k_{43}. \end{cases}$$

This system is a class of general solutions of equation (4.5) in accordance with the representation (4.2). Moreover, due to (4.4), the form of the general solution can be simplified by getting rid of double indices in numbers k_{ij}:

$$\begin{cases} x = -884 \pm 42k_1 \pm 105k_2 \pm 70k_3, \\ y = 3536 \mp 30k_1 \pm 105k_4 \pm 70k_5, \\ z = 17 \mp 30k_2 \mp 42k_4 \pm 70k_6, \\ u = -1768 \mp 30k_3 \mp 42k_5 \mp 105k_6, \end{cases}$$

where $\{k_i, i = 1, 2, \ldots, 6\} \subset \mathbb{Z}$. □

4.2. Special cases.

As in many mathematical theories, we can separate some cases related to the method of searching for partial solutions. In the following subsections, we highlight several special cases for which there is no need to use a general algorithm to find the partial solution.

4.2.1.

If in the equation $b = 0$, then the class of its general solutions can be constructed very simply, as in the following example.

Exercise 4.3. *Find a class of general solutions of the equation*

$$96x + 11y + 75z + 8u + 31v = 0.$$

Solution. It is seen that $x = y = z = u = v = 0$ is one of the partial solutions. By using (4.3), let's build the matrix

$$\begin{array}{c} 0 \longleftrightarrow \\ 0 \longleftrightarrow \\ 0 \longleftrightarrow \\ 0 \longleftrightarrow \\ 0 \longleftrightarrow \end{array} \begin{bmatrix} \mathbf{0} \cdot 96 & \pm 11 & \pm 75 & \pm 8 & \pm 31 \\ \mp 96 & \mathbf{0} \cdot 11 & \pm 75 & \pm 8 & \pm 31 \\ \mp 96 & \mp 11 & \mathbf{0} \cdot 75 & \pm 8 & \pm 31 \\ \mp 96 & \mp 11 & \mp 75 & \mathbf{0} \cdot 8 & \pm 31 \\ \mp 96 & \mp 11 & \mp 75 & \mp 8 & \mathbf{0} \cdot 31 \end{bmatrix}.$$

Then the class of general solutions to the equation can be constructed in the following form:

$$\begin{cases} x = 0 \pm 11k_{12} \pm 75k_{13} \pm 8k_{14} \pm 31k_{15}, \\ y = 0 \mp 96k_{21} \pm 75k_{23} \pm 8k_{24} \pm 31k_{25}, \\ z = 0 \mp 96k_{31} \mp 11k_{32} \pm 8k_{34} \pm 31k_{35}, \\ u = 0 \mp 96k_{41} \mp 11k_{42} \mp 75k_{43} \pm 31k_{45}, \\ v = 0 \mp 96k_{51} \mp 11k_{52} \mp 75k_{53} \mp 8k_{54}, \end{cases}$$

where numbers $k_{ij} \in \mathbb{Z}$, such that $k_{ii} = 0$ and $k_{ij} = k_{ji}$ for all $i, j = 1, 2, \ldots, 5$. □

4.2.2.

If the number b is divisible by some one of the coefficients $\{a_1, a_2, \ldots, a_n\}$, then the equation can be solved as in the following example.

Exercise 4.4. *Find a class of general solutions of the equation*

$$96x + 11y + 75z + 3u + 31v = 27.$$

Solution. We can put $x = y = z = v = 0$ and get $u = 9$. Hence by using (4.3), we obtain

$$\begin{cases} x = 0 \pm 11k_{12} \pm 75k_{13} \pm 3k_{14} \pm 31k_{15}, \\ y = 0 \mp 96k_{21} \pm 75k_{23} \pm 3k_{24} \pm 31k_{25}, \\ z = 0 \mp 96k_{31} \mp 11k_{32} \pm 3k_{34} \pm 31k_{35}, \\ u = 9 \mp 96k_{41} \mp 11k_{42} \mp 75k_{43} \pm 31k_{45}, \\ v = 0 \mp 96k_{51} \mp 11k_{52} \mp 75k_{53} \mp 3k_{54}, \end{cases}$$

herein $k_{ij} \in \mathbb{Z}$, such that $k_{ii} = 0$ and $k_{ij} = k_{ji}$ for all $i, j = 1, 2, \ldots, 5$. □

4.2.3.

If the set of coefficients $\{a_1, a_2, \ldots, a_n\}$ contains a pair of coefficients $\{a_{i1}, a_{i2}\}$ such that $\gcd(a_{i1}, a_{i2}) = 1$, then the class of general solutions of equation (4.1) can be constructed based on the solution of the equation with these coefficients and with the corresponding variables.

Exercise 4.5. *Find a class of general solutions of the equation*

$$96x + 11y + 75z + 8u + 31v = 27.$$

Solution. Putting $x = z = v = 0$ we have

$$11y + 8u = 27.$$

Now the solution to the last equation does not seem complicated. Thence we can easily construct the class of general solutions in the form of

$$\begin{cases} x = 0 \pm 11k_{12} \pm 75k_{13} \pm 8k_{14} \pm 31k_{15}, \\ y = 9 \mp 96k_{21} \pm 75k_{23} \pm 8k_{24} \pm 31k_{25}, \\ z = 0 \mp 96k_{31} \mp 11k_{32} \pm 8k_{34} \pm 31k_{35}, \\ u = -9 \mp 96k_{41} \mp 11k_{42} \mp 75k_{43} \pm 31k_{45}, \\ v = 0 \mp 96k_{51} \mp 11k_{52} \mp 75k_{53} \mp 8k_{54}, \end{cases}$$

where numbers $k_{ij} \in \mathbb{Z}$ are as above. □

Remark 4.6. *When it is necessary, as in Example 4.5, that the class of general solutions of equation* (4.1) *can be constructed based on the solution of the equation with three variables, if the set of coefficients* $\{a_1, a_2, \ldots, a_n\}$ *contains the subset* $\{a_{i1}, a_{i2}, a_{i3}\}$ *for which* $\gcd(a_{i1}, a_{i2}, a_{i3}) = 1$.

5. Concluding notes

In our opinion, the method proposed in this article is very simple and convenient for use by mathematicians with minimal mathematical skills. We believe that the Crushing method can be developed for a more general case. Subsequently, it will be possible to develop simplified solution methods for nonlinear equations.
So, in our subsequent researches, the proposed method will be modified and applied in other cases. We will pay special attention to minimizing the amount of numbers $k_{ij} \in \mathbb{Z}$. In particular, we will use it when searching for a solution to a system of linear Diophantine equations.
As an example, without details, here we give a general solution to the following system of equations found using the algorithm proposed above:

$$\begin{cases} 3x + 4y = 2, \\ 5x + 2z = 12, \\ y + 3u = 11. \end{cases}$$

Using the algorithm, we can find the solution in the following form:

$$\begin{cases} x = 2 + 12k, \\ y = -1 - 9k, \\ z = 1 - 30k, \\ u = 4 - 4k, \end{cases}$$

where $k \in \mathbb{Z}$.
In addition, we will be interested in solvability in integers of the following equation:

$$(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^p = b_1x_1 + b_2x_2 + \cdots + b_nx_n,$$

where $p \in \mathbb{N}$ and $\{a_i, b_i\} \subset \mathbb{Z}$, $i = 1, 2, \ldots, n$.

Acknowledgement

The authors thank the anonymous reviewer for carefully reading the manuscript and for the kind comments that contributed to the improvement of the paper.

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