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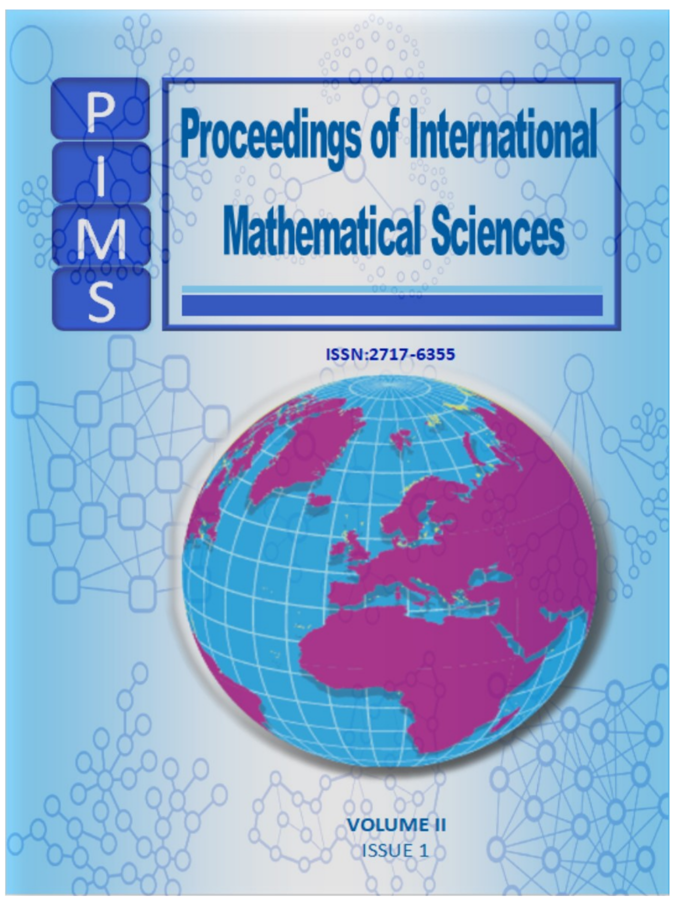

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ANALYTICAL SOLUTION FOR THE CONFORMABLE FRACTIONAL TELEGRAPH EQUATION BY FOURIER METHOD

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ABSTRACT. In this paper, the Fourier method is effectively implemented for solving a conformable fractional telegraph equation. We discuss and derive the analytical solution of the conformable fractional telegraph equation with nonhomogeneous Dirichlet boundary condition.

1. INTRODUCTION

The telegraph equation is better than the heat equation in modeling of physical phenomena, which has a parabolic behavior [3]. The one-dimensional telegraph equation can be written as follows:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \left(\frac{R}{L} + \frac{G}{C}\right)\frac{\partial u(x,t)}{\partial t} + \frac{RG}{LC}u(x,t) = \frac{1}{LC}\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (1.1)$$

where R and G are, respectively, the resistance and the conductance of resistor, C is the capacitance of capacitor, and L is the inductance of coil. Many concrete applications amount to replacing the time derivative in the telegraph equation with a fractional derivative. For example, in the works [4, 5], the authors have extensively studied the time-fractional telegraph equation with Caputo fractional derivative. For more details about the good effect of the fractional derivative, we refer to monographs [1, 2]. Recently, a new definition of fractional derivative, named "fractional conformable derivative", is introduced by Khalil et al. [6]. This novel fractional derivative is compatible with the classical derivative and it is excellent for studying nonregular solutions. The subject of the fractional conformable derivative has attracted the attention of many authors in domains such as mechanics, electronic, and anomalous diffusion. We are interested in studying in this paper the telegraph model (1.1) in framework of the time-fractional conformable derivative.

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Precisely, we will propose the following transformations:

$$\frac{\partial}{\partial t} \to \mathcal{D}_t^{(\alpha)} \quad \text{and} \quad \frac{\partial^2}{\partial t^2} \to \mathcal{D}_t^{(2\alpha)} = \mathcal{D}_t^{(\alpha)}\mathcal{D}_t^{(\alpha)}, \tag{1.2}$$

$$a = G/C, \quad b = R/L, \quad k^2 = 1/LC \tag{1.3}$$

where $\mathcal{D}_t^{(\alpha)}$ is the time-fractional conformable derivative operator [6]. Then, we get the fractional conformable telegraph model associated with the transformation (1.2) and (1.3) as follows:

$$\mathcal{D}_t^{(2\alpha)} u\left(x,t\right) + \left(a+b\right)\mathcal{D}_t^{(\alpha)} u\left(x,t\right) + abu\left(x,t\right) = k^2 \frac{\partial^2 u\left(x,t\right)}{\partial x^2} + f\left(x,t\right), \tag{1.4}$$

where x and t are the space and time variables, $f\left(x,t\right)$ is a sufficiently smooth function.

2. Preliminaries on conformable fractional calculus

We start recalling some concepts on the conformable fractional calculus.

Definition 2.1. *([6]). Let $\varphi : [0,+\infty[\to \mathbb{R}$ be a function. Then, the conformable fractional derivative of function φ of order α at $t > 0$ is defined by the following limit:*

$$\mathcal{D}_t^{(\alpha)}\left(\varphi\right)\left(t\right) = \lim_{\varepsilon \to 0} \frac{\varphi\left(t + \varepsilon t^{1-\alpha}\right) - \varphi\left(t\right)}{\varepsilon}, \tag{2.1}$$

when this limit exists and finished.

Definition 2.2. *([6]). Let $\alpha \in]0,1]$ and $\varphi : [0,+\infty[\to \mathbb{R}$ be real valued function. The conformable fractional integral of φ of order α from zero to t is defined by:*

$$\mathcal{I}_\alpha \varphi\left(t\right) := \int_0^t s^{\alpha-1}\varphi\left(s\right) ds, \quad t \geq 0, \tag{2.2}$$

Lemma. *Assume that φ is a continuous function on $]0,+\infty[$ and $0 < \alpha \leq 1$. Then, for all $t > 0$, we have $\mathcal{D}_t^{(\alpha)}\left[\mathcal{I}_\alpha\varphi\left(t\right)\right] = \varphi\left(t\right)$. According to [7], if φ is differentiable, then we have $\mathcal{I}_\alpha\left[\mathcal{D}_t^{(\alpha)}\left(\varphi\right)\left(t\right)\right] = \varphi\left(t\right) - \varphi\left(0\right)$.*

Definition 2.3. *([7]) Let $0 < \alpha \leq 1$ and $\varphi : [0,+\infty[\to \mathbb{R}$ be real valued function. Then, the fractional Laplace transform of order α starting from zero of φ is defined by:*

$$\mathcal{L}_\alpha\left[\varphi\left(t\right)\right]\left(s\right) = \int_0^{+\infty} t^{\alpha-1}\varphi\left(t\right) e^{-s\frac{t^\alpha}{\alpha}} dt. \tag{2.3}$$

Theorem. *([7]) Let $0 < \alpha \leq 1$ and $\varphi : [0,+\infty[\to \mathbb{R}$ be differentiable real valued function. Then, we have:*

$$\mathcal{L}_\alpha\left[\mathcal{D}_t^{(\alpha)}\varphi\left(t\right)\right]\left(s\right) = s\mathcal{L}_\alpha\left[\varphi\left(t\right)\right]\left(s\right) - \varphi\left(0\right). \tag{2.4}$$

We introduce the following theorem, which is used further in this paper.

Proposition. *Let λ and μ be positive constants with $4\mu > \lambda^2$ and $g : [0, +\infty\,[\to \mathbb{R}$ is a continuous function. For all $0 < \alpha \leq 1$, The initial value problem*

$$\begin{cases} \mathcal{D}_t^{(2\alpha)} y(t) + \lambda \mathcal{D}_t^{(\alpha)} y(t) + \mu y(t) = g(t), \\ y(0) = y_0, \quad \mathcal{D}_t^{(\alpha)} y(0) = y_\alpha. \end{cases} \tag{2.5}$$

admits a unique solution given by

$$\begin{aligned} y(t) = &\left[y_0 \cos\left(\sqrt{4\mu - \lambda^2} \frac{t^\alpha}{2\alpha} \right) + \frac{\lambda y_0 + 2y_\alpha}{\sqrt{4\mu - \lambda^2}} \sin\left(\sqrt{4\mu - \lambda^2} \frac{t^\alpha}{2\alpha} \right) \right] e^{-\frac{\lambda t^\alpha}{2\alpha}} \\ &+ \frac{2}{\sqrt{4\mu - \lambda^2}} \int_0^t e^{-\frac{\lambda \tau^\alpha}{2\alpha}} \sin\left(\sqrt{4\mu - \lambda^2} \frac{\tau^\alpha}{2\alpha} \right) g(t - \tau)\, d\tau. \end{aligned} \tag{2.6}$$

3. Nonhomogeneous conformable fractional telegraph equation with Dirichlet boundary condition

We determine the solution of conformable fractional telegraph equation (1.4) with the intial conditions

$$u(x, 0) = \phi(x), \quad \mathcal{D}_t^{(\alpha)} u(x, 0) = \psi(x), \quad 0 \leq x \leq \ell, \tag{3.1}$$

and the nonhomogeneous Dirichlet boundary conditions

$$u(0, t) = \mu_1(t), \quad u(\ell, t) = \mu_2(t), \quad t > 0, \tag{3.2}$$

where $\mu_1(t)$ and $\mu_2(t)$ are nonzero smooth functions with order-one continuous derivative, using the method of separating variables, in which $\phi(x)$, $\psi(x)$ are continuous functions satisfying

$$\phi(0) = \mu_1(0) \quad \text{and} \quad \phi(\ell) = \mu_2(0). \tag{3.3}$$

Other hand, we assume that

$$\frac{|a - b|}{2k} < \frac{\pi}{\ell}. \tag{3.4}$$

In order to solve the problem with nonhomogeneous boundary, we firstly transform the nonhomogeneous boundary into a homogeneous condition. Let

$$u(x, t) = W_1(x, t) + V_1(x, t),$$

where $W_1(x, t)$ is a new unknown function and

$$V_1(x, t) = \mu_1(t) + \frac{[\mu_2(t) - \mu_1(t)]\, x}{\ell} \tag{3.5}$$

which satisfies the boundary conditions

$$V_1(0, t) = \mu_1(t) \quad \text{and} \quad V_1(\ell, t) = \mu_2(t). \tag{3.6}$$

The function $W_1(x, t)$ then satisfies the problem with homogeneous boundary conditions:

$$\begin{cases} \mathcal{D}_t^{(2\alpha)} W_1(x, t) + (a + b)\, \mathcal{D}_t^{(\alpha)} W_1(x, t) + ab W_1(x, t) = k^2 \frac{\partial^2 W_1(x,t)}{\partial x^2} + \tilde{f}(x, t), \\ W_1(x, 0) = \phi_1(x), \quad \mathcal{D}_t^{(\alpha)} W_1(x, 0) = \psi_1(x), \\ W_1(0, t) = W_1(\ell, t) = 0, \end{cases} \tag{3.7}$$

in which

$$\begin{aligned}
\tilde{f}(x) &= -\mathcal{D}_t^{(2\alpha)} V_1(x,t) - (a+b)\,\mathcal{D}_t^{(\alpha)} V_1(x,t) - abV_1(x,t) + f(x,t)\,, \\
\phi_1(x) &= \phi(x) - \mu_1(0) - \frac{\mu_2(0) - \mu_1(0)}{\ell} x, \\
\psi_1(x) &= \psi(x) - \mathcal{D}_t^{(\alpha)} \mu_1(0) - \frac{\mathcal{D}_t^{(\alpha)} \mu_2(0) - \mathcal{D}_t^{(\alpha)} \mu_1(0)}{\ell} x.
\end{aligned} \tag{3.8}$$

We solve the corresponding homogeneous equation (3.7) ($\tilde{f}(x,t)$ being replaced by 0) with the boundary conditions by the method of separation of variables.

If we let $W_1(x,t) = X(x)\,Y(t)$ and substitute for $W_1(x,t)$ in (3.7), we obtain an ordinary linear differential equation for $X(x)$:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X(\ell) = 0, \end{cases} \tag{3.9}$$

and a fractional ordinary linear differential equation with the conformable derivative for $Y(t)$,

$$\mathcal{D}_t^{(2\alpha)} Y(t) + (a+b)\,\mathcal{D}_t^{(\alpha)} Y(t) + \left(ab + \lambda k^2\right) Y(t) = 0, \tag{3.10}$$

where the parameter λ is a positive constant.

The Sturm-Liouville problem given by (3.9) has eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{\ell^2}, \quad n \in \mathbb{N}^*, \tag{3.11}$$

and corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right), \quad n \in \mathbb{N}^*. \tag{3.12}$$

Now we seek a solution of the nonhomogeneous problem (3.7) in the following form

$$W_1(x,t) = \sum_{n=1}^{+\infty} B_n(t) \sin\left(\frac{n\pi x}{\ell}\right). \tag{3.13}$$

We assume that the series can be differentiated term by term. In order to determine $B_n(t)$, we expand $\tilde{f}(x,t)$ as a Fourier series by the eigenfunctions $\left\{\sin\left(\frac{n\pi x}{\ell}\right)\right\}$:

$$\tilde{f}(x,t) = \sum_{n=1}^{+\infty} \tilde{f}_n(t) \sin\left(\frac{n\pi x}{\ell}\right), \text{ where } \tilde{f}_n(t) = \frac{2}{\ell}\int_0^\ell \tilde{f}(x,t) \sin\left(\frac{n\pi x}{\ell}\right) dx. \tag{3.14}$$

Substituting (3.13), (3.14) into (3.7) yields

$$\mathcal{D}_t^{(2\alpha)} B_n(t) + (a+b)\,\mathcal{D}_t^{(\alpha)} B_n(t) + \left(ab + \lambda_n k^2\right) B_n(t) = \tilde{f}_n(t)\,. \tag{3.15}$$

Since $W_1(x,t)$ satisfies the initial conditions in (3.7), we must have

$$\begin{cases} \sum_{n=0}^{+\infty} B_n(0) \sin\left(\frac{n\pi x}{\ell}\right) = \phi_1(x)\,, \quad 0 < x < \ell, \\ \sum_{n=0}^{+\infty} \mathcal{D}_t^{(\alpha)} B_n(0) \sin\left(\frac{n\pi x}{\ell}\right) = \psi_1(x)\,, \quad 0 < x < \ell, \end{cases} \tag{3.16}$$

which yields

$$\begin{cases} B_n(0) = \frac{2}{\ell}\int_0^\ell \phi_1(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad n \in \mathbb{N}^*, \\ \mathcal{D}_t^{(\alpha)} B_n(0) = \frac{2}{\ell}\int_0^\ell \psi_1(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad n \in \mathbb{N}^*. \end{cases} \tag{3.17}$$

For each value of n, (3.15) and (3.17) make up a fractional initial value problem.

According to Proposition, we obtain the solution of problem (1.4), (3.1) and (3.2) as

$$
\begin{aligned}
u(x,t) = \sum_{n=1}^{+\infty} \Bigg[& B_n(0)\, e^{-\frac{(a+b)t^\alpha}{2\alpha}} \cos\left(\sqrt{4k^2\lambda_n - (a-b)^2}\,\frac{t^\alpha}{2\alpha}\right) \\
& + \frac{(a+b)\,B_n(0) + 2\mathcal{D}_t^{(\alpha)} B_n(0)}{\sqrt{4k^2\lambda_n - (a-b)^2}} e^{-\frac{(a+b)t^\alpha}{2\alpha}} \sin\left(\sqrt{4k^2\lambda_n - (a-b)^2}\,\frac{t^\alpha}{2\alpha}\right) \\
& + \frac{2}{\sqrt{4k^2\lambda_n - (a-b)^2}} \int_0^t e^{-\frac{(a+b)\tau^\alpha}{2\alpha}} \sin\left(\sqrt{4k^2\lambda_n - (a-b)^2}\,\frac{\tau^\alpha}{2\alpha}\right) \tilde{f}_n(t-\tau)\, d\tau \Bigg] \sin\left(\frac{n\pi x}{\ell}\right) \\
& + \mu_1(t) + \frac{(\mu_2(t) - \mu_1(t))\,x}{\ell}.
\end{aligned}
\tag{3.18}
$$

4. Conclusion

We have derived the analytical solution of the nonhomogeneous conformable fractional telegraph equation with Dirichlet boundary condition using Fourier method. The solution is given in the form of a series of functions with the use of the Fourier sine series and the conformable Lapalce transform.

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STABILITY RESULT FOR AN ABSTRACT TIME DELAYED EVOLUTION EQUATION WITH ARBITRARY DECAY OF VISCOELASTICITY

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Abstract. The paper is concerned with a second-order abstract semilinear evolution equation with infinite memory and time delay. With the help of the semigroup arguments and under suitable conditions on initial data and the kernel memory function, we state and prove the global existence of solution. Then, we establish the decay rates of the energy using the multiplier method by defining a suitable Lyapunov functional. This work extends previous works with time delay for a much wider class of kernels. We give also some applications to illustrate our results.

1. Introduction

Let H be a real Hilbert space with inner product and related norm denoted by $\langle .,. \rangle$ and $\| \, . \, \|$, respectively. Let $A : D(A) \longrightarrow H$ and $B : D(B) \longrightarrow H$ be a self-adjoint linear positive operator with domains $D(A) \subset D(B) \subset H$ such that the embeddings are dense and compact. Let $C : H \longrightarrow H$ is a self-adjoint linear operator and $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is the kernel of the memory term. $\tau > 0$ represents a time delay and $F \ : D(A^{\frac{1}{2}}) \to H$ is function satisfying some conditions to be specified later. We consider the following second-order abstract semilinear evolution equation with infinite memory and time delay

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s)Bu(t-s)ds + Cu_t(t-\tau) = F(u(t)), & t \in (0, +\infty), \\ u_t(t-\tau) = f_0(t-\tau) & t \in (0, \tau), \\ u(-t) = u_0(t), \quad u_t(0) = u_1, & t \in \mathbb{R}_+, \end{cases} \tag{1.1}$$

where the initial datum (u_0, u_1, f_0) belongs to a suitable spaces.

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In absence of time delay term, a large number of works are available, where various decay estimates were obtained, see [7, 14, 21]. For the particular case of the wave equation with finite memory, see [2, 24].

In many cases, delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay. Nicaise and Pignotti in [15] considered a wave equation with a linear damping and delay term and they proved that the energy is exponentially stable and some instability results are also given by constructing some sequences of delays for which the energy of some solutions does not tend to zero, see also [3, 17].

When the memory term is replaced by a frictional damping $Bu_t(t)$:

$$u_{tt}(t) + Au(t) + Bu_t(t) + \mu u_t(t-\tau) = 0, \qquad t > 0,$$

where μ, τ are fixed constants and B is a given operator, there exist in the literature different stability results. These results show that the damping $Bu_t(t)$ is strong enough to stabilize the system in presence of a time delay provided that $|\mu|$ is small enough, see [10, 16, 17].

Guesmia in [11] considered the following second-order abstract linear problem with infinite memory and time delay terms

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s)Au(t-s)ds + \mu u_t(t-\tau) = 0, & t > 0, \\ u(-t) = u_0(t), & t \in \mathbb{R}_+ \\ u_t(0) = u_1, \quad u_t(t-\tau) = f_0(t-\tau), & t \in (0,\tau), \end{cases}$$

He proved that the unique dissipation given by the memory term is strong enough to stabilize exponentially the system in presence of delay. In this work and others, the condition $h'(s) \leq -\delta h(s)$ for all $s \geq 0$ and some $\delta > 0$ is assumed to prove exponential decay of the energy, see [1, 4]. In [13], the previous condition is replaced by

$$h'(s) \leq -\zeta(t)h(s), \quad \forall s \geq 0, \tag{1.2}$$

where ζ is a positive nonincreasing differentiable function. The authors established the existence and the general decay results of the energy. Dai and Yang in [8] considered the same problem in [13] and solved the open problem proposed by Kirane and Said-Houari. Recently, Boukhatem and Benabderrahmane in [5] considered a variable coefficient viscoelastic equation with a time-varying delay in the boundary feedback and acoustic boundary conditions and nonlinear source term. They established a general decay results of the energy via suitable Lyapunov functionals and some properties of the convex functions where the kernel memory satisfies the equation (1.2). In [6], the same results have obtained in the case of constant delay.

Tatar in [23] introduced a new class of admissible kernels which lead to a wide range of possible decay rates. More precisely, He consider kernels satisfying

$$h(t-s) \geq \xi(t) \int_t^{+\infty} h(\pi - s)d\pi, \quad 0 \leq s \leq t,$$

for some $\xi(t) > 0$. This class contains the polynomial type functions and the exponential type. He proved that the last assumption on the relaxation in a viscoelastic problem ensuring uniform stability in an arbitrary rate.

For the case of distributed time delay, Guesmia and Tatar in [12] considered the following class of second-order linear hyperbolic equations

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s)Bu(t-s)ds + \int_0^{+\infty} f(s)u_t(t-s)ds = 0, & t > 0, \\ u(-t) = u_0(t), & t \in \mathbb{R}_+ \\ u_t(0) = u_1, & t \in \mathbb{R}_+, \end{cases}$$

where the function f is of class $C^1(\mathbb{R}_+, \mathbb{R})$ and satisfies, for some positive constant α,

$$|f(s)| \leq \alpha h(s), \quad \text{and} \quad |f'(s)| \leq \alpha h(s), \quad \forall s \in \mathbb{R}_+.$$

They given well-posedness and stability of the system and they proved that the infinite memory alone guarantees the asymptotic stability of the system and the decay rate of solutions is found explicitly in terms of the growth at infinity of the infinite memory and the distributed time delay convolution kernels.

Nicaise and Pignotti in [18] considered the following system

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F(U(t)) + k\mathcal{B}U(t-\tau), & t \in (0, +\infty), \\ U(0) = u_0, \mathcal{B}U(t-\tau) = f(t), & t \in (0, \tau), \end{cases}$$

where $\mathcal{A}$ generates a C_0-semigroup $(S(t))_{t\geq 0}$ that is exponentially stable, i.e., there exist two positive constants M and w such that

$$\|S(t)\|_{\mathcal{L}(H)} \leq Me^{-wt}, \quad \forall t \geq 0,$$

and $\mathcal{L}(H)$ denotes the space of bounded linear operators from H into itself. For a fixed delay parameter τ, a fixed bounded operator $\mathcal{B}$ from H into itself and for a real parameter k and $F : H \longrightarrow H$ satisfies some Lipschitz conditions, the initial datum U_0 belongs to H and $f \in C([0, \tau]; H)$. They showed that, if the C_0-semigroup describing the linear part of the model is exponentially stable, then the whole system retains this good property when a suitable smallness condition on the time-delay feedback is satisfied, see also [19].

Motivated by previous works, we study the well-posedness and the stability result of a semilinear abstract viscoelastic equation with infinite memory in presence of a time delayed damping and a nonlinear source term. Our results extend the decay results in previous works to kernels h which do not necessarily converge exponentially to zero at infinity. Moreover, our problem generalizes the linear problems to those with a nonlinear source term and to problems with more general time delayed damping term.

The paper is organized as follows. In Sect. 2, we prove the well-posedness by using the semigroup arguments under some assumptions on A, B, C, h and F. Then, we state and prove the stability result of solution by using the energy method to produce a suitable Lyapunov functional with arbitrary decay on h. Section 4 is devoted to some concrete examples in the aim to illustrate our abstract result.

2. Well-posedness

In this section, we state some assumptions on A, B, C and h and prove the well-posedness result by using semigroup theory.

For studying the problem (1.1), we introduce a new variable z as in [15]

$$z(\rho, t) = u_t(t - \rho\tau), \quad \rho \in (0, 1), \ t > 0.$$

Thus, we have

$$\tau z_t(\rho,t) + z_\rho(\rho,t) = 0, \quad \rho \in (0,1),\ t > 0.$$

Moreover, as in [9], we define

$$\eta^t(s) = u(t) - u(t-s), \quad t,s > 0.$$

Therefore, problem (1.1) takes the form

$$\begin{cases} u_{tt}(t) + Au(t) - h_0 Bu(t) + \int_0^{+\infty} h(s)B\eta^t(s)ds & \\ \qquad\qquad +Cz(1,t) = F(u(t)), & t \in (0,+\infty), \\ \tau z_t(\rho,t) + z_\rho(\rho,t) = 0, & \rho \in (0,1),\ t > 0, \\ \eta_t^t(s) = u_t(t) - \eta_s^t(s), & t,s > 0, \\ z(\rho,0) = f_0(-\rho\tau), & \rho \in (0,1), \\ z(0,t) = u_t(t), & t > 0, \\ u(-t) = u_0(t), \quad u_t(0) = u_1, & t \geq 0, \\ \eta^0(s) = u_0(0) - u_0(s), & s \geq 0. \end{cases} \tag{2.1}$$

We will need the following assumptions:

(A1) There exist positive constants a and b satisfying

$$b\|u\|^2 \leq \left\|B^{\frac{1}{2}}u\right\|^2 \leq a\left\|A^{\frac{1}{2}}u\right\|^2, \quad \forall u \in D(A^{\frac{1}{2}}). \tag{2.2}$$

(A2) The kernel function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is of class C^1 nonincreasing function satisfying

$$h_0 = \int_0^{+\infty} h(s)ds < \frac{1}{a}. \tag{2.3}$$

(A3) There exists $\mu \in \mathbb{R}^*$ such that

$$\|Cu\|^2 \leq |\mu|\|u\|^2, \quad \forall u \in H. \tag{2.4}$$

(A4) $F : D(A^{\frac{1}{2}}) \to H$ is globally Lipschitz continuous, namely

$$\exists \gamma > 0 \text{ such that } \|F(u) - F(v)\| \leq \gamma\left\|A^{\frac{1}{2}}(u-v)\right\|, \quad \forall u,v \in H$$

Let us denote $U = (u, u_t, \eta^t, z)^T$, the problem (2.1) can be rewritten:

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + \mathcal{F}(U(t)), \quad \forall t > 0, \\ U(0) = U_0 = \left(u_0, u_1, \eta^0, f_0(-\tau .)\right)^T, \end{cases} \tag{2.5}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \phi_2 \\ -(A - h_0 B)\phi_1 - \int_0^{+\infty} h(s)B\phi_3(s)ds - C\phi_4(1) \\ \phi_2 - \dfrac{\partial \phi_3}{\partial s} \\ \dfrac{-1}{\tau}\dfrac{\partial \phi_4}{\partial \rho} \end{pmatrix}$$

and

$$\mathcal{F}(\phi_1, \phi_2, \phi_3, \phi_4)^T = (0, F(\phi_1), 0, 0)^T$$

The domain $D(\mathcal{A})$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{c} (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathcal{H},\ (A - h_0 B)\phi_1 + \int_0^{+\infty} h(s)B\phi_3(s)ds \in H, \\ \phi_2 \in D(A^{\frac{1}{2}}), \dfrac{\partial \phi_3}{\partial s} \in L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}})), \\ \dfrac{\partial \phi_4}{\partial \rho} \in L^2(0,1;H),\ \phi_3(0) = 0,\ \phi_4(0) = \phi_2 \end{array} \right\}$$

where

$$\mathcal{H} = D(A^{\frac{1}{2}}) \times H \times L^2_h(\mathbb{R}_+, D(B^{\frac{1}{2}})) \times L^2(0,1;H).$$

The sets $L^2_h(\mathbb{R}_+, D(B^{\frac{1}{2}}))$ and $L^2(0,1;H)$ are respectively defined by

$$L^2_h(\mathbb{R}_+, D(B^{\frac{1}{2}})) = \left\{ \phi \, : \mathbb{R}_+ \to D(B^{\frac{1}{2}}), \quad \int_0^{+\infty} h(s) \, \left\| B^{\frac{1}{2}} \phi(s) \right\|^2 ds < +\infty \right\},$$

equipped with the inner product

$$\langle \phi_1, \phi_2 \rangle_{L^2_h(\mathbb{R}_+, D(B^{\frac{1}{2}}))} = \int_0^{+\infty} h(s) \, \left\langle B^{\frac{1}{2}} \phi_1(s), B^{\frac{1}{2}} \phi_2(s) \right\rangle ds.$$

And

$$L^2(0,1;H) = \left\{ \phi \, : (0,1) \to H, \quad \int_0^1 \|\phi(\rho)\|^2 d\rho < +\infty \right\},$$

equipped with the inner product

$$\langle \phi_1, \phi_2 \rangle_{L^2(0,1;H)} = \int_0^1 \langle \phi_1(\rho), \phi_2(\rho) \rangle \, d\rho.$$

The Hilbert space $\mathcal{H}$ equipped with the following inner product. For all $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and $W = (w_1, w_2, w_3, w_4)^T$ in $\mathcal{H}$, we have

$$\begin{aligned} \langle \Phi, W \rangle_{\mathcal{H}} &= \langle \phi_1, w_1 \rangle_{D(A^{\frac{1}{2}})} - h_0 \langle \phi_1, w_1 \rangle_{D(B^{\frac{1}{2}})} + \langle \phi_2, w_2 \rangle \\ &\quad + \langle \phi_3, w_3 \rangle_{L^2_h(\mathbb{R}_+, D(B^{\frac{1}{2}}))} + \tau\mu \langle \phi_4, w_4 \rangle_{L^2(0,1;H)}. \end{aligned}$$

The well-posedness of problem (2.5) is ensured by the following theorem:

Theorem 2.1. *Under the assumptions (A1)-(A4), for an initial datum $U_0 \in \mathcal{H}$, the system (2.5) has a unique mild solution $U \in C(\mathbb{R}_+, \mathcal{H})$ satisfies the following formula,*

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\mathcal{F}(U(s))ds.$$

Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$ and $\mathcal{F} \in C^1(\mathcal{H})$, then the solution of (2.5) satisfies (classical solution)

$$U \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Proof. To prove Theorem 2.1, we use the semigroup theory. The problem (2.5) can be seen as an inhomogeneous evolution problem. It's clear that $\mathcal{F}$ is globally lipschitz continuous, let show that the operator $\mathcal{A}$ generate a linear C_0-semigroup $(S(t))_{t\geq 0}$ on $\mathcal{H}$. Indeed,

• First, we prove that the linear operator $\mathcal{A}$ is dissipative.

Take $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathcal{D}(\mathcal{A})$, then

$$\begin{aligned} \left\langle \mathcal{A}\Phi, \Phi \right\rangle_{\mathcal{H}} &= \langle \phi_2, \phi_1 \rangle_{D(A^{\frac{1}{2}})} + \int_0^{+\infty} h(s) \Big\langle \phi_2 - \frac{\partial \phi_3}{\partial s}, \phi_3 \Big\rangle_{D(B^{\frac{1}{2}})} ds \\ &\quad - h_0 \langle \phi_2, \phi_1 \rangle_{D(B^{\frac{1}{2}})} + \tau|\mu| \int_0^1 \Big\langle \frac{-1}{\tau} \frac{\partial \phi_4}{\partial \rho}, \phi_4 \Big\rangle d\rho \\ &\quad - \Big\langle (A - h_0 B)\phi_1 + \int_0^{+\infty} h(s) B\phi_3(s) ds + C\phi_4(1), \phi_2 \Big\rangle. \end{aligned}$$

Using the definition of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ and the fact that H is a real Hilbert space, we conclude

$$\langle A - h_0 B\phi_1, \phi_2 \rangle = \langle A^{\frac{1}{2}} \phi_2, A^{\frac{1}{2}} \phi_1 \rangle - h_0 \langle B^{\frac{1}{2}} \phi_2, B^{\frac{1}{2}} \phi_1 \rangle \tag{2.6}$$

using the Cauchy-Schwarz and Young's inequalities and by (2.4), we have

$$-\left\langle C\phi_4(1),\phi_2\right\rangle \leq \frac{|\mu|}{2}\left(\|\phi_4(1)\|^2+\|\phi_2\|^2\right). \tag{2.7}$$

$$\left\langle \int_0^{+\infty} h(s)B\phi_3(s)ds,\phi_2\right\rangle = \int_0^{+\infty} h(s)\langle\phi_2,\phi_3\rangle_{D(B^{\frac{1}{2}})}ds.$$

Integrating by parts and using the definition of $\mathcal{D}(\mathcal{A})$ ($\phi_3(0)=0$), we obtain

$$\int_0^{+\infty} h(s)\Big\langle -\frac{\partial\phi_3}{\partial s},\phi_3\Big\rangle_{D(B^{\frac{1}{2}})}ds \leq \frac{1}{2}\int_0^{+\infty} h'(s)\|B^{\frac{1}{2}}\phi_3(s)\|^2ds. \tag{2.8}$$

Also using the fact that $\phi_4(0)=\phi_2$, we obtain

$$\tau|\mu|\int_0^1 \Big\langle \frac{-1}{\tau}\frac{\partial\phi_4}{\partial\rho},\phi_4\Big\rangle d\rho = \frac{|\mu|}{2}\Big(\|\phi_4(0)\|^2-\|\phi_4(1)\|^2\Big) = \frac{|\mu|}{2}\Big(\|\phi_2\|^2-\|\phi_4(1)\|^2\Big). \tag{2.9}$$

Consequently, inserting (2.6), (2.7), (2.8) and (2.9) in (2.6) and using the fact that h is nonincreasing, we find

$$\langle\mathcal{A}\Phi,\Phi\rangle_{\mathcal{H}} \leq \frac{1}{2}\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\phi_3(s)\right\|^2 ds + |\mu|\|u_t\|^2 \leq |\mu|\|\Phi\|^2, \tag{2.10}$$

which means that the operator $\mathcal{A}-|\mu|I$ is dissipative.

• Let us now prove that $\lambda I-\mathcal{A}$ is surjective. Indeed, let $(f_1,f_2,f_3,f_4)^T\in\mathcal{H}$, we show that there exists $\Phi=(\phi_1,\phi_2,\phi_3,\phi_4)^T\in\mathcal{D}(\mathcal{A})$ satisfying

$$(\lambda I-\mathcal{A})\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = \begin{pmatrix}f_1\\ f_2\\ f_3\\ f_4\end{pmatrix},$$

which is equivalent to

$$\begin{cases} \lambda\phi_1-\phi_2=f_1\\ \lambda\phi_2+(A-h_0B)\phi_1+\int_0^{+\infty}h(s)B\phi_3(s)ds+C\phi_4(1)=f_2\\ \lambda\phi_3-\phi_2+\dfrac{\partial\phi_3}{\partial s}=f_3\\ \lambda\phi_4+\dfrac{1}{\tau}\dfrac{\partial\phi_4}{\partial\rho}=f_4. \end{cases} \tag{2.11}$$

Suppose that we have found ϕ_1 with the appropriate regularity. Then, we have

$$\phi_2=\lambda\phi_1-f_1. \tag{2.12}$$

We note that the third equation in (2.11) with $\phi_3(0)=0$ has a unique solution

$$\phi_3(s)=e^{-\lambda s}\int_0^s e^{\lambda y}\left(f_3(y)-f_1+\lambda\phi_1\right)dy. \tag{2.13}$$

On the other hand, the fourth equation in (2.11) with $\phi_4(0)=\phi_2=\lambda\phi_1-f_1$ has a unique solution

$$\phi_4(\rho)=\left(\lambda\phi_1-f_1+\tau\int_0^\rho f_4(y)e^{\lambda\tau y}dy\right)e^{-\lambda\tau\rho},\quad \rho\in(0,1). \tag{2.14}$$

In particular,

$$\phi_4(1)=\left(\lambda\phi_1-f_1+\tau\int_0^1 f_4(y)e^{\lambda\tau y}dy\right)e^{-\lambda\tau}.$$

It remains only to determine ϕ_1.

Next, plugging (2.12) and (2.13) into the second equation in (2.11), we get

$$\left(A-\alpha B+\lambda e^{-\lambda\tau}C+\lambda^2 I\right)\phi_1=\tilde{f}, \tag{2.15}$$

where

$$\alpha=h_0-\lambda\int_0^\infty h(s)e^{-\lambda s}\left(\int_0^s e^{\lambda y}dy\right)ds=\int_0^\infty h(s)e^{-\lambda s}ds,$$

and

$$\begin{aligned}\tilde{f} &= f_2+\lambda f_1+e^{-\lambda\tau}C\left(f_1-\tau\int_0^1 f_4(y)e^{\tau y}dy\right)\\ &\quad-\int_0^\infty e^{-\lambda s}h(s)\int_0^s e^{-\lambda y}B\left(f_3(y)-f_1\right)dyds.\end{aligned}$$

We have just to prove that (2.15) has a solution $\phi_1\in D(A^{\frac{1}{2}})$ and replace in (2.12), (2.13) and (2.14) to obtain $\Phi\in\mathcal{D}(\mathcal{A})$ satisfying (2.11).
We have $\alpha<h_0$, by (2.3) and (2.2), we deduce that $A-\alpha B$ is a positive definite operator. Then, we take the duality brackets $\langle .,.\rangle_{D(A^{\frac{1}{2}})'\times D(A^{\frac{1}{2}})}$ with $w\in D(A^{\frac{1}{2}})$:

$$\left\langle\left(A-\alpha B+\lambda e^{-\lambda\tau}C+\lambda^2 I\right)\phi_1,w\right\rangle_{D(A^{\frac{1}{2}})'\times D(A^{\frac{1}{2}})}=\left\langle\tilde{f},w\right\rangle_{D(A^{\frac{1}{2}})'\times D(A^{\frac{1}{2}})}. \tag{2.16}$$

Consequently, the left-hand side of (2.16) is bilinear, continuous and coercive on $D(A^{\frac{1}{2}})$. Since, applying the Lax-Milgram theorem and classical regularity arguments, we conclude that (2.11) has a unique solution $\phi_1\in D(A^{\frac{1}{2}})$ satisfying. Using (2.13),

$$\left((A-h_0B)\phi_1+\int_0^{+\infty}h(s)B\phi_3(s)ds\right)\in H.$$

In conclusion, we have found $\Phi=(\phi_1,\phi_2,\phi_3,\phi_4)^T\in\mathcal{D}(\mathcal{A})$, which verifies (2.11), and thus $\lambda I-\mathcal{A}$ is surjective for all $\lambda>0$ and the same holds for the operator $\lambda I-(\mathcal{A}-|\mu|I)$.

Then, the Lumer-Phillips theorem implies that $|\mu|I-\mathcal{A}$ is a maximal monotone operator, $\mathcal{A}-|\mu|I$ is an infinitesimal generator of a strongly continuous semigroup of contraction in $\mathcal{H}$. Hence, the operator $\mathcal{A}$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ in $\mathcal{H}$. Consequently, by using Theorem 1.2, Ch. 6 of [22], the problem (2.5) has a unique solution $U\in C([0,+\infty),\mathcal{H})$. □

3. Stability result

The stability result of the solution of (2.1) holds under the following additional assumptions:

(A5) There exist a positive constant d satisfying

$$\left\|A^{\frac{1}{2}}u\right\|^2\leq d\left\|B^{\frac{1}{2}}u\right\|^2,\quad\forall u\in D(A^{\frac{1}{2}}). \tag{3.1}$$

(A6) Moreover, we assume that $F(0)=0$ and there exists a continuous and differentiable mapping $\psi: D(A^{\frac{1}{2}})\to\mathbb{R}$ satisfying

$$D_\psi=F\quad\text{and}\quad\langle F(u),u\rangle\geq 2\psi(u),\quad\forall u\in D(A^{\frac{1}{2}}). \tag{3.2}$$

(A7) The function h satisfies (A2) and there exists a positive function $\xi \in C(\mathbb{R}_+, \mathbb{R}_+^*)$ satisfying $\lim_{s\to+\infty} \xi(s)$ exists such that

$$\begin{cases} h(t-s) \geq \xi(t) \displaystyle\int_t^{+\infty} h(\pi - s)d\pi, & \forall t \in \mathbb{R}_+,\ \forall s \in [0,t], \\ h'(s) < 0, & \forall s \in \mathbb{R}_+. \end{cases} \tag{3.3}$$

The first inequality in (3.3), introduced in [25] and [23], implies that h converges to zero at least exponentially but it does not involve the derivative of h. This class contains the polynomial (or power) type ($h(t) = (1+t)^{-a}$, $a > 1$) functions and the exponential type ($h(t) = e^{-at}$, $a > 0$) functions.

Let establish some several Lemmas needed of our main result. We define the modified energy functional E associated to problem (2.1) by

$$\begin{aligned} E(t) \quad = \quad & \frac{1}{2}\Bigg(\left\|A^{\frac{1}{2}}u\right\|^2 - h_0 \left\|B^{\frac{1}{2}}u\right\|^2 + \|u_t\|^2 + \int_0^{+\infty} h(s) \left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\ & -2\psi(u) + \tau|\mu| \int_0^1 \|z(\rho,t)\|^2 d\rho \Bigg). \end{aligned} \tag{3.4}$$

Lemma 3.1. *Assume that (A1)-(A4) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the energy functional defined by (3.4) satisfies*

$$E'(t) \leq \frac{1}{2}\int_0^{+\infty} h'(s) \left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + |\mu|\|u_t\|^2. \tag{3.5}$$

Proof. Multiplying the first equation of (2.1) by u_t. Using (A6) and repeating exactly the same arguments to obtain (2.10). □

Remark. *Note that, from (3.5), the energy of solutions to problem (2.1) is not decreasing in general. Indeed, the second term in the right-hand side of (3.5), coming from the delay term, is nonnegative.*

Now, as in [20], for $n \in \mathbb{N}^*$, let consider the set

$$A_n = \{s \in \mathbb{R}_+,\ \ h(s) + nh'(s) \leq 0\},$$

and put $h_n = \int_{A_n^c} h(s)ds$. We have $h_n > 0$, otherwise, $A_n^c = \emptyset$. Furthermore, by the second inequality in (3.3), we have

$$\lim_{n\to+\infty} A_n^c = \cap_{n\in\mathbb{N}^*} A_n^c = \emptyset, \text{ and then } \lim_{n\to+\infty} h_n = 0.$$

In order to state our results, we need the following four lemmas.

Lemma 3.2. *Let U be solution of* (2.1)*. Then the functional*

$$I_1(t) = -\left\langle u_t(t), \int_0^{+\infty} h(s)\eta^t(s)ds \right\rangle, \tag{3.6}$$

satisfies, for $\varepsilon_1, \varepsilon_2 > 0$,

$$\begin{aligned} I_1'(t) \quad \leq \quad & -(h_0-\varepsilon_1)\|u_t\|^2 + \left(\varepsilon_2 + \frac{\sqrt{dh_n}}{2}\right)\left\|A^{\frac{1}{2}}u\right\|^2 - \frac{h_0^2}{2}\left\|B^{\frac{1}{2}}u\right|^2 \\ & + \left(2h_n - \frac{h_0}{2} + \frac{\sqrt{dh_n}}{2}\right)\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\ & + \frac{h_0}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2 ds \\ & - \left(2nh_0 + \frac{dnh_0}{4\varepsilon_2} + \frac{h(0)}{4b\varepsilon_1}\right)\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\ & + \left\langle Cz(1,t) - F(u), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle, \end{aligned} \tag{3.7}$$

Proof. Differentiating (3.6) with respect to t and using the third equation of 2.1, we find

$$I_1'(t) = -\left\langle u_{tt}(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle + \left\langle u_t(t), \int_0^{+\infty} h(s)\eta^t_s(s)ds\right\rangle - h_0\|u_t\|^2.$$

Integrating by parts with respect to s the second term in the right hand side of the previous equality and using the fact that $\lim_{s\to+\infty} h(s) = 0,\ \eta^t(0) = 0$, we obtain

$$I_1'(t) = -\left\langle u_{tt}(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle - \left\langle u_t(t), \int_0^{+\infty} h'(s)\eta^t(s)ds\right\rangle - h_0\|u_t\|^2.$$

On the other hand, by the first equation of (2.1), we have

$$\begin{aligned} & \left\langle u_{tt}(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle + \left\langle Au(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle \\ - \quad & h_0\left\langle Bu(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle + \left\langle \int_0^{+\infty} h(s)B\eta^t(s)ds, \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle \\ + \quad & \left\langle Cz(1,t) - F(u), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle = 0, \end{aligned}$$

using the definitions of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we get

$$\begin{aligned} I_1'(t) \quad = \quad & -h_0\|u_t\|^2 + \left\langle Cz(1,t) - F(u), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle \\ & -\left\langle u_t(t), \int_0^{+\infty} h'(s)\eta^t(s)ds\right\rangle + \left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)A^{\frac{1}{2}}\eta^t(s)ds\right\rangle \\ & \left\|\int_0^{+\infty} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 - h_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\rangle. \end{aligned} \tag{3.8}$$

Let estimate the last three terms in the right hand by using Cauchy-Schwarz and Young's inequalities and the definition of A_n. Then, using (2.2), (3.1) and (2.3), we get

$$-\left\langle u_t(t), \int_0^{+\infty} h'(s)\eta^t(s)ds\right\rangle \leq \varepsilon_1\|u_t\|^2 - \frac{h(0)}{4b\varepsilon_1}\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right\rangle,$$

$$\begin{aligned}
&\left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)A^{\frac{1}{2}}\eta^t(s)ds\right\rangle \\
= \quad & \left\langle A^{\frac{1}{2}}u(t), \int_{A_n} h(s)A^{\frac{1}{2}}\eta^t(s)ds\right\rangle + \left\langle A^{\frac{1}{2}}u(t), \int_{A_n^c} h(s)A^{\frac{1}{2}}\eta^t(s)ds\right\rangle. \\
\leq \quad & \varepsilon_2 \left\|A^{\frac{1}{2}}u\right\|^2 + \frac{dh_0}{4\varepsilon_2}\int_{A_n} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + \frac{\sqrt{dh_n}}{2}\left\|A^{\frac{1}{2}}u\right\|^2 \\
& + \frac{\sqrt{dh_n}}{2}\int_{A_n^c} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\
\leq \quad & \varepsilon_2 \left\|A^{\frac{1}{2}}u\right\|^2 - \frac{dnh_0}{4\varepsilon_2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + \frac{\sqrt{dh_n}}{2}\left\|A^{\frac{1}{2}}u\right\|^2 \\
& + \frac{\sqrt{dh_n}}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds,
\end{aligned}$$

$$\begin{aligned}
&\left\|\int_0^{+\infty} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 \\
= \quad & \left\|\int_{A_n} h(s)B^{\frac{1}{2}}\eta^t(s)ds + \int_{A_n^c} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 \\
\leq \quad & 2\left\|\int_{A_n} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 + 2\left\|\int_{A_n^c} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 \\
\leq \quad & 2h_0\int_{A_n} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + 2h_n\int_{A_n^c} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\
\leq \quad & -2nh_0\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + 2h_n\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds.
\end{aligned}$$

And for the last one, we have

$$\begin{aligned}
& -h_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\rangle \\
= \quad & -h_0^2\left\|B^{\frac{1}{2}}u\right\|^2 + h_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)B^{\frac{1}{2}}u(t-s)ds\right\rangle \\
= \quad & -\frac{h_0^2}{2}\left\|B^{\frac{1}{2}}u\right\|^2 + \frac{h_0}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2 ds \\
& -\frac{h_0}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds. \qquad (3.9)
\end{aligned}$$

Inserting these four inequalities in (3.8), we get (3.7). □

Lemma 3.3. *Let U be solution of* (2.1). *Then the functional*

$$I_2(t) = \langle u_t(t), u(t)\rangle, \qquad (3.10)$$

satisfies,

$$\begin{aligned}
I_2'(t) \quad = \quad & \|u_t\|^2 - \left\|A^{\frac{1}{2}}u\right\|^2 + \frac{h_0}{2}\left\|B^{\frac{1}{2}}u\right\|^2 + \frac{1}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2 ds \\
& -\frac{1}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds - \left\langle Cz(1,t) + F(u), u\right\rangle. \qquad (3.11)
\end{aligned}$$

Proof. Differentiating (3.10) with respect to t, we find

$$I_2'(t) = \|u_t\|^2 + \langle u_{tt}(t), u(t)\rangle.$$

On the other hand, multiplying the first equation of (2.1) by $u(t)$, we have

$$\langle u_{tt}(t), u(t)\rangle + \langle (A - h_0 B)u(t), u(t)\rangle + \left\langle \int_0^{+\infty} h(s)B\eta^t(s)ds, u(t) \right\rangle \\ + \langle Cz(1,t), u(t)\rangle = 0,$$

By the definitions of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we have

$$\langle u_{tt}(t), u(t)\rangle + \left\|A^{\frac{1}{2}}u\right\|^2 - h_0 \left\|B^{\frac{1}{2}}u\right\|^2 + \left\langle \int_0^{+\infty} h(s)B\eta^t(s)ds, u(t) \right\rangle \\ + \langle Cz(1,t), u(t)\rangle = 0.$$

Consequently,

$$I_2'(t) = \|u_t\|^2 - \left\|A^{\frac{1}{2}}u\right\|^2 + h_0 \left\|B^{\frac{1}{2}}u\right\|^2 - \left\langle \int_0^{+\infty} h(s)B\eta^t(s)ds, u(t) \right\rangle - \langle Cz(1,t), u(t)\rangle,$$

By using the inequality (3.9), we get (3.11). □

Similarly to [15], we introduce the following functional.

Lemma 3.4. *Let U be solution of (2.1). Then the functional*

$$I_3(t) = \tau e^{2\tau} \int_0^1 e^{-2\tau\rho} \|z(\rho,t)\|^2 ds, \tag{3.12}$$

satisfies,

$$I_3'(t) \le -2\tau \int_0^1 \|z(\rho,t)\|^2 ds + e^{2\tau}\|u_t\|^2 - \|z(1,t)\|^2. \tag{3.13}$$

Proof. By using the second equation of (2.1), we get

$$\begin{aligned} I_3'(t) &= 2\tau e^{2\tau} \int_0^1 e^{-2\tau\rho} \langle z_t(\rho,t), z(\rho,t)\rangle d\rho \\ &= -2e^{2\tau} \int_0^1 e^{-2\tau\rho} \langle z_\rho(\rho,t), z(\rho,t)\rangle d\rho \\ &= -2e^{2\tau} \int_0^1 e^{-2\tau\rho} \frac{\partial}{\partial\rho} \|z(\rho,t)\|^2 d\rho. \end{aligned}$$

Then, by integrating by parts and $z(0,t) = u_t(t)$, we get

$$I_3'(t) = -2\tau e^{2\tau} \int_0^1 e^{-2\tau\rho} \|z(\rho,t)\|^2 ds + e^{2\tau}\|u_t\|^2 - \|z(1,t)\|^2,$$

which is (3.13) by using the fact that $e^{-2\tau\rho} \ge e^{-2\tau}$, for any $\rho \in]0,1[$. □

Now, we consider two functionals J_1 and J_2 and we give their derivatives in the following lemma.

Lemma 3.5. *Let*

$$J_1(t) = \int_0^t \left(\int_t^{+\infty} h(\pi - s) d\pi \right) \left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 ds, \quad \forall t \in \mathbb{R}_+, \tag{3.14}$$

and

$$J_2(t) = \int_0^t \left(\int_t^{+\infty} h(\pi - s) d\pi \right) \left\| A^{\frac{1}{2}} \eta^t(s) \right\|^2 ds, \quad \forall t \in \mathbb{R}_+. \tag{3.15}$$

Then, for any $\lambda_1 \in]0, 1[$,

$$\begin{aligned} J_1'(t) \quad &\leq \quad h_0 \left\| B^{\frac{1}{2}} u \right\|^2 - (1 - \lambda_1)\xi(t) J_1(t) - \lambda_1 \int_0^t h(s) \left\| B^{\frac{1}{2}} u(t-s) \right\|^2 ds \\ &\quad + \lambda_1 \int_t^{+\infty} h(s) \left\| B^{\frac{1}{2}} u_0(s-t) \right\|^2 ds, \qquad \forall t \in \mathbb{R}_+, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} J_2'(t) \quad &\leq \quad h_0 \left\| A^{\frac{1}{2}} u \right\|^2 - (1 - \lambda_1)\xi(t) J_2(t) - \frac{\lambda_1}{a} \int_0^t h(s) \left\| B^{\frac{1}{2}} u(t-s) \right\|^2 ds \\ &\quad + d\lambda_1 \int_t^{+\infty} h(s) \left\| B^{\frac{1}{2}} u_0(s-t) \right\|^2 ds, \qquad \forall t \in \mathbb{R}_+. \end{aligned} \tag{3.17}$$

Proof. The functional J_1 is well-defined. Indeed, by using the fact that $\eta \in L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))$ and (3.3), we have

$$J_1(t) \leq \frac{1}{\xi(t)} \int_0^t h(t-s) \left\| B^{\frac{1}{2}} u(s) \right\|^2 ds \leq \frac{1}{\xi(t)} \int_0^t h(s) \left\| B^{\frac{1}{2}} u(t-s) \right\|^2 ds < +\infty.$$

By (3.1), we conclude that J_2 also is well defined.

Then, differentiating J_1 with respect to t and using the definition of u_0 and (3.3), we obtain

$$\begin{aligned} J_1'(t) \quad &= \quad \left(\int_t^{+\infty} h(\pi - s) d\pi \right) \left\| B^{\frac{1}{2}} u(t) \right\|^2 - \int_0^t h(t-s) \left\| B^{\frac{1}{2}} u(s) \right\|^2 ds \\ &= \quad h_0 \left\| B^{\frac{1}{2}} u \right\|^2 - (1 - \lambda_1) \int_0^t h(t-s) \left\| B^{\frac{1}{2}} u(s) \right\|^2 ds \\ &\quad - \lambda_1 \int_{-\infty}^t h(t-s) \left\| B^{\frac{1}{2}} u(s) \right\|^2 ds + \lambda_1 \int_{-\infty}^0 h(t-s) \left\| B^{\frac{1}{2}} u(s) \right\|^2 ds \\ &\leq \quad h_0 \left\| B^{\frac{1}{2}} u \right\|^2 - (1 - \lambda_1)\xi(t) J_1(t) - \lambda_1 \int_0^{+\infty} h(s) \left\| B^{\frac{1}{2}} u(t-s) \right\|^2 ds \\ &\quad + a\lambda_1 \int_t^{+\infty} h(s) \left\| B^{\frac{1}{2}} u_0(s-t) \right\|^2 ds, \end{aligned}$$

which is exactly (3.16). A similar argument yields the relation (3.17). $\square$

In this case, the Lyapunov functional L we will work with is

$$L(t) = E(t) + \epsilon\big(N_1 I_1(t) + N_2 I_2(t) + I_3(t)\big) + M_1 J_1(t) + a M_1 J_2(t), \tag{3.18}$$

where $\epsilon, N_1, N_2, M_1 > 0$ are positive constants to be chosen later.

Now we are in position to state and prove the decay result of solution of problem (2.1).

Theorem 3.6. *Assume that (A1)-(A7) hold. For any initial datum* $U_0 \in \mathcal{H}$*. Assume that* h *satisfies*

$$\int_0^{+\infty} h(s) ds < \frac{\gamma^2}{b}, \tag{3.19}$$

and there exists a positive constant δ_0 independent of μ such that, if

$$|\mu| < \delta_0, \tag{3.20}$$

then, for any $U_0 \in \mathcal{H}$, there exist positive constants δ_1 and δ_2 such that

$$E(t) \leq \delta_2 e^{-\delta_1 t}\left(1 + \int_0^t e^{\delta_1 s}\int_s^{+\infty} h(\pi)\left\|B^{\frac{1}{2}} u_0(\pi - s)\right\|^2 d\pi ds\right), \quad \forall t \in \mathbb{R}_+, \tag{3.21}$$

if $\lim_{t\to+\infty} \xi(t) > 0$, *and*

$$E(t) \leq \delta_2 e^{-\delta_1 \hat{\xi}(t)}\left(1 + \int_0^t e^{\delta_1 \hat{\xi}(s)}\int_s^{+\infty} h(\pi)\left\|B^{\frac{1}{2}} u_0(\pi - s)\right\|^2 d\pi ds\right), \quad \forall t \in \mathbb{R}_+, \tag{3.22}$$

if $\lim_{t\to+\infty} \xi(t) = 0$, *where*

$$\hat{\xi}(s) = \int_0^s \xi(\pi) d\pi, \quad \forall t \in \mathbb{R}_+. \tag{3.23}$$

Proof. In order to proof the decay estimates, we start by the derivative of the function L. On the other hand, by using (A6) and (2.2), we have

$$\begin{aligned} -\left\langle F(u), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle &\leq \frac{1}{b}\|F(u)\|^2 + \frac{b}{4}\left\|\int_0^{+\infty} h(s)\eta^t(s)ds\right\|^2 \\ &\leq \frac{\gamma^2}{b}\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{h_0}{4}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds, \end{aligned}$$

Combining (3.5), (3.7), (3.11), (3.13), (3.16) and (3.17), we obtain

$$\begin{aligned} L'(t) \leq\ & -\epsilon\Bigg[(C_1 - \frac{|\mu|}{\epsilon})\|u_t\|^2 + C_2\left\|A^{\frac{1}{2}}u\right\|^2 + C_3 h_0\left\|B^{\frac{1}{2}}u\right\|^2 - 2\tau\int_0^1 \|z(\rho,t)\|^2 d\rho \\ & + \int_0^{+\infty} h(s)\left(C_4\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 + C_5\left\|B^{\frac{1}{2}}u(t-s)\right\|^2\right)ds - 2N_2\psi(u)\Bigg] \\ & + \frac{\sqrt{dh_n}}{2}\epsilon N_1\left\|A^{\frac{1}{2}}u\right\|^2 + \left(2h_n + \frac{\sqrt{dh_n}}{2}\right)\epsilon N_1\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\ & + \left(\frac{1}{2} - \epsilon C_6\right)\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds - C_7\xi(t)(J_1(t) + J_2(t)) \\ & + C_8\int_t^{+\infty} h(s)\left\|B^{\frac{1}{2}}u_0(s-t)\right\|^2 ds - \epsilon\|z(1,t)\|^2 \\ & + \epsilon\left\langle Cz(1,t), N_1\int_0^{+\infty} h(s)\eta^t(s)ds - N_2 u\right\rangle, \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} &C_1 = (h_0 - \varepsilon_1)N_1 - N_2 - e^{2\tau}, && C_2 = N_2 - (\varepsilon_2 + \frac{\gamma_2}{b})N_1 - \frac{ah_0}{\epsilon}M_1, \\ &C_3 = \frac{h_0}{2}N_1 - \frac{N_2}{2} - \frac{M_1}{\epsilon}, && C_4 = \frac{h_0}{4}N_1 + \frac{N_2}{2}, \\ &C_5 = \frac{2\lambda_1}{\epsilon}M_1 - \frac{h_0}{2}N_1 - \frac{N_2}{2}, && C_6 = \left(2nh_0 + \frac{dnh_0}{4\varepsilon_2} + \frac{h(0)}{4b\varepsilon_1}\right)N_1, \\ &C_7 = (1-\lambda_1)M_1\min\{1,a\}, && C_8 = M_1\lambda_1(1+ad). \end{aligned} \tag{3.25}$$

At this point, we choose the different constants to obtain some results. First, we select $N_2 = (1 + ah_0)e^{2\tau}$ and we choose M_1, N_1 such that

$$\frac{\epsilon N_2}{2(1+ah_0)} < M_1 < \frac{e^{2\tau}}{2\epsilon}.$$

$$\max\left\{\frac{b}{bh_0 - 2\gamma^2}\left(2(1+ah_0)\frac{M_1}{\epsilon} - N_2\right), \frac{1}{h_0}(N_2 + e^{2\tau})\right\} < N_1 < \frac{1}{h_0}\left(N_2 + \frac{2M_1}{\epsilon}\right).$$

Note that M_1 exists as a result of the selection of N_2 for certain value of ϵ to be choose later and the choice of M_1 and N_2 guarantees the existence of N_1. Now, let pick ε_1, ε_2 and λ_1 such that

$$0 < \varepsilon_1 < h_0 - \frac{N_2 + e^{2\tau}}{N_1},$$

$$\varepsilon_2 = \frac{h_0}{2} - \frac{\gamma_2}{b} + \frac{1}{2N_1}\left(N_2 - 2(1+ah_0)\frac{M_1}{\epsilon}\right),$$

and

$$\frac{\epsilon}{4M_1}(N_2 + h_0 N_1) \leq \lambda_1 < 1,$$

ε_2 and λ_1 exist by the previous selection of N_1 and N_2. Consequently, it result that $C_1 > 0$, $C_2 = -C_3$, $C_3 < 0$ and $C_5 \geq 0$. Moreover, it's clear that $C_4 > 0$, so, we have

$$\begin{aligned}
&-\epsilon\Bigg[C_1\|u_t\|^2 + C_2\left(\left\|A^{\frac{1}{2}}u\right\|^2 - h_0\left\|B^{\frac{1}{2}}u\right\|^2\right) - 2N_2\psi(u) \\
&+\int_0^{+\infty} h(s)\left(C_4\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 + C_5\left\|B^{\frac{1}{2}}u(t-s)\right\|^2\right)ds\Bigg] \\
\leq\ & -\epsilon C_9\left(\|u_t\|^2 + \left\|A^{\frac{1}{2}}u\right\|^2 - h_0\left\|B^{\frac{1}{2}}u\right\|^2 - 2\psi(u) + \int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right)
\end{aligned}$$

where

$$C_9 = \frac{1}{N_2}\min\{C_1, C_2, C_4\}.$$

Observe that C_9 is positive and independent on μ. Next, using Cauchy-Schwarz's and Young's inequalities for estimate the last term in the right hand in (3.24). Then, by (2.2) and (2.4), we get

$$\begin{aligned}
&\epsilon\left\langle Cz(1,t), N_1\int_0^{+\infty} h(s)\eta^t(s)ds - N_2 u\right\rangle \\
\leq\ & \epsilon\|z(1,t)\|^2 + \epsilon|\mu|C_{10}\left(\left\|A^{\frac{1}{2}}u\right\|^2 + \int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right),
\end{aligned}$$

where

$$C_{10} = \frac{1}{2b}\max\{aN_2^2, h_0 N_1^2\}.$$

Inserting the above inequality and (3.26) in (3.24), we obtain

$$\begin{aligned} L'(t) &\leq -\epsilon C_{11}E(t)+\left(\frac{4h_n+\sqrt{dh_n}}{2}\right)\epsilon N_1 E(t) \\ &+\left(\frac{1}{2}-\epsilon C_6\right)\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds - C_7\xi(t)(J_1(t)+J_2(t)) \\ &+C_8\int_t^{+\infty} h(s)\left\|B^{\frac{1}{2}}u_0(s-t)\right\|^2 ds, \end{aligned} \tag{3.26}$$

where

$$C_{11}=2\min\left\{C_9-\frac{|\mu|}{\epsilon},\frac{2}{|\mu|},C_9-\epsilon|\mu|C_{10}\right\}.$$

Finally, we assume that $|\mu|$ satisfies (3.20) under the following choice of δ_0

$$\delta_0=\min\left\{\frac{C_9}{C_6},\frac{C_9\sqrt{2}}{\sqrt{C_{10}}}\right\}. \tag{3.27}$$

Then, we can choose n big enough and we fix ϵ such that

$$\frac{|\mu|}{2C_9}<\epsilon\leq\frac{1}{2C_6}<\frac{1}{M}, \tag{3.28}$$

where

$$M=N_1\max\left\{1,\frac{h_0}{b}\right\}+N_2\max\left\{1,\frac{a}{b}\right\}+\frac{2e^{2\tau}}{|\mu|}.$$

which imply that E is equivalent to $E+\epsilon(N_1I_1+N_2I_2+I_3)$. Indeed, by using Cauchy-Schwarz's and Young's inequalities, we have

$$\begin{aligned} |I_1(t)| &\leq \frac{1}{2}\left(\|u_t\|^2+\frac{h_0}{b}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right) &(3.29)\\ &\leq \max\left\{1,\frac{h_0}{b}\right\}E(t) &(3.30)\end{aligned}$$

and

$$|I_2(t)| \leq \frac{1}{2}\left(\|u_t\|^2+\frac{a}{b}\left\|A^{\frac{1}{2}}u\right\|^2\right)\leq\max\left\{1,\frac{a}{b}\right\}E(t). \tag{3.31}$$

From (3.12), it follows

$$|I_3(t)|=\tau e^{2\tau}\int_0^1 e^{-2\tau\rho}\|z(\rho,t)\|^2 ds\leq\tau e^{2\tau}\int_0^1 e^{-2\tau\rho}\|z(\rho,t)\|^2 ds\leq\frac{2e^{2\tau}}{|\mu|}E(t) \tag{3.32}$$

Combining (3.4), (3.29), (3.31) and (3.32) and by using (3.28), we have

$$E\sim E+\epsilon(N_1I_1+N_2I_2+I_3).$$

Moreover, the third term in the right hand of (3.26) is non-positive. Note that δ_0 is a positive constant independent of μ. Under the condition (3.20), we conclude that C_{11} is a positive constant and by using the fact that $\lim_{n\to+\infty}h_n=0$, we get

$$C_{12}=\epsilon C_{11}+\left(\frac{4h_n+\sqrt{dh_n}}{2}\right)\epsilon N_1>0.$$

Consequently, we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} L'(t) &\leq -C_{12}E(t) - C_7\xi(t)\Big(J_1(t) + J_2(t)\Big) \\ &\quad + C_8 \int_t^{+\infty} h(s) \left\| B^{\frac{1}{2}} u_0(s-t) \right\|^2 ds. \end{aligned} \tag{3.33}$$

Let distinguish two cases corresponding to the limit of ξ at infinity.

► If $\lim_{t\to+\infty} \xi(t) > 0$, there exist $t_0 \geq 0$ and $\xi_0 > 0$ such that $\xi(t) \geq \xi_0$, for all $t \geq t_0$. Therefore, using (3.18), we find

$$L'(t) \leq -\delta_1 L(t) + C_8 \int_t^{+\infty} h(s) \left\| B^{\frac{1}{2}} u_0(s-t) \right\|^2 ds, \quad \forall t \in \mathbb{R}_+, \tag{3.34}$$

where

$$\delta_1 = \min\left\{ \frac{C_{12}}{1+\epsilon M}, \frac{C_7\xi_0}{M_1}, \frac{C_7\xi_0}{aM_1} \right\}.$$

Then, integrating the differential inequality (3.34) over $[t_0, t]$, we obtain

$$L(t) \leq e^{-\delta_1 t}\left(e^{\delta_1 t_0} L(t_0) + C_7 \int_0^t e^{\delta_1 s} \int_s^{+\infty} h(\pi) \left\| B^{\frac{1}{2}} u_0(\pi - s) \right\|^2 d\pi ds \right), \ \forall t \in \mathbb{R}_+.$$

So, using (3.18) and (3.34), we get, for all $t \geq t_0$,

$$\begin{aligned} E(t) &\leq \frac{1}{1-\epsilon M} L(t) \\ &\leq \frac{1}{1-\epsilon M} \max\left\{ C_7, e^{\delta_1 t_0} L(t_0) \right\} \times \\ &\quad \times \left(1 + \int_0^t e^{\delta_1 s} \int_s^{+\infty} h(\pi) \left\| B^{\frac{1}{2}} u_0(\pi - s) \right\|^2 d\pi ds \right). \end{aligned} \tag{3.35}$$

For $t \in [0, t_0]$, we have

$$E(t) \leq \frac{1}{1-\epsilon M} L(t) e^{\delta_1 t} e^{-\delta_1 t} \leq \frac{1}{1-\epsilon M} \max_{s\in[0,t_0]} L(s) e^{\delta_1 t_0} e^{-\delta_1 t}. \tag{3.36}$$

Inequalities (3.35) and (3.36) gives (3.21) with

$$\delta_2 = \frac{1}{1-\epsilon M}\left\{ C_7, e^{\delta_1 t_0} \max_{s\in[0,t_0]} L(s) \right\}.$$

► If $\lim_{t\to+\infty} \xi(t) = 0$, there exist $t_0 \geq 0$ such that $\xi(t) \leq C_{12}$, for all $t \geq t_0$. Therefore, using (3.18), we obtain, for

$$\delta_1 = \min\left\{ \frac{1}{1+\epsilon M}, \frac{C_7}{M_1}, \frac{C_7}{aM_1} \right\},$$

$$L'(t) \leq -\delta_1 \xi(t) L(t) + C_8 \int_t^{+\infty} h(s) \left\| B^{\frac{1}{2}} u_0(s-t) \right\|^2 ds, \quad \forall t \in \mathbb{R}_+, \tag{3.37}$$

By integrating the above differential inequality over $[t_0, t]$, we get, for all $t \in \mathbb{R}_+$,

$$L(t) \leq e^{-\delta_1 \hat{\xi}(t)}\left(e^{\delta_1 \hat{\xi}(t_0)} L(t_0) + C_7 \int_0^t e^{\delta_1 \hat{\xi}(s)} \int_s^{+\infty} h(\pi) \left\| B^{\frac{1}{2}} u_0(\pi - s) \right\|^2 d\pi ds \right).$$

Then, using (3.18) and (3.37), we get, for all $t \geq t_0$,

$$\begin{aligned} E(t) &\leq \frac{1}{1-\epsilon M} \max\left\{C_7, e^{\delta_1 \hat{\xi}(t_0)} L(t_0)\right\} \times \\ &\times\left(1 + \int_0^t e^{\delta_1 s} \int_s^{+\infty} h(\pi) \left\| B^{\frac{1}{2}} u_0(\pi - s)\right\|^2 d\pi ds\right). \end{aligned} \tag{3.38}$$

For $t \in [0, t_0]$, we have

$$E(t) \leq \frac{1}{1-\epsilon M} L(t) e^{\delta_1 \hat{\xi}(t)} e^{-\delta_1 \hat{\xi}(t)} \leq \frac{1}{1-\epsilon M} \max_{s\in[0,t_0]} \left(L(s) e^{\delta_1 \hat{\xi}(s)}\right) e^{-\delta_1 \hat{\xi}(t)}. \tag{3.39}$$

Inequalities (3.38) and (3.39) gives (3.21) with

$$\delta_2 = \frac{1}{1-\epsilon M}\left\{C_7, \max_{s\in[0,t_0]}\left(L(s) e^{\delta_1 \hat{\xi}(s)}\right)\right\}.$$

Thus the proof of Theorem 3.6 is completed. □

4. Applications

We can seek our results in some problems. In this section, we consider only three illustrative problems. In the whole section, Ω is a bounded and regular domain of $\mathbb{R}^n$, with $n \geq 1$.

1-: Abstract linear problem

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s) Au(t-s) ds + C u_t(t-\tau) = 0, & t \in (0, +\infty), \\ u_t(t-\tau) = f_0(t-\tau), & t \in (0,\tau), \\ u(-t) = u_0(t), \quad u_t(0) = u_1, & t \geq 0, \end{cases} \tag{4.1}$$

where the operators A and C are a self-adjoint linear positive operators satisfy the assumptions (A1) and (A3), respectively. The memory kernel h satisfying (A2) and (A7).

2-: Let us consider the semilinear problem

$$\begin{cases} u_{tt}(t) + Au(t) + \int_0^{+\infty} h(s) \Delta u(t-s) ds + b(x) u_t(t-\tau) & \\ \qquad = F(u(t)), & t \in (0, +\infty), \\ u(x,t) = 0, & x \in \partial\Omega, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), & x \in \Omega,\ t \geq 0, \\ u_t(t-\tau) = f_0(t-\tau) & t \in (0,\tau), \end{cases} \tag{4.2}$$

with initial data $(u_0, u_1, f_0) \in [H^2(\Omega) \in \cap H_0^1(\Omega)] \times H_0^1(\Omega) \times H^1(0,\tau; L^2(\Omega))$. The constant $\beta > 0$ satisfies a suitable restriction to be specified below. The memory kernel h satisfying (A2) and (A7) and $b \in L^\infty(\Omega)$ is a function such that

$$b(x) \geq 0 \quad \text{a. e. in} \quad \Omega.$$

The source term F be globally Lipschitz continuous functional such that $F(0) = 0$ and satisfies (3.2).Our results hold with $H = L^2(\Omega)$ and the operators A, B are given by

$$A : D(A) \longrightarrow H \ : u \mapsto -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right), \quad B : D(B) \longrightarrow H \ : u \mapsto -\Delta u,$$

where $D(A) = D(B)) = H^2(\Omega) \cap H_0^1(\Omega)$. $a_{ij} \in C^1(\overline{\Omega})$, is symmetric and

$$\exists a_0 > 0, \quad \sum_{i,j=1}^{n} a_{ij}(x)\zeta_j\zeta_i \geq a_0|\zeta|^2, \quad x \in \overline{\Omega},\ \zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{R}^n.$$

The operators A and B are a linear, self-adjoint and positive operators in H such that $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$ with $\left\|A^{\frac{1}{2}}u\right\| = (a(u,u))^{1/2}$ and $\left\|B^{\frac{1}{2}}u\right\| = \|\nabla u\|_2$, where

$$a(u,u) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx.$$

Moreover, by using Poincare's inequality and the Sobolev's embedding theorem, we get (A1) and (A5). Then, the assumption (A3) holds with $Cu(x,t) = b(x)u(x,t)$.

3-: Coupled systems

$$\begin{cases} w_{tt}(t) - \alpha\Delta w(t) + \displaystyle\int_0^{+\infty} h(s) div(a_1(x)\nabla w(t-s))ds & \\ \qquad +\mu w_t(t-\tau) + dv(t) = f_1(w(t)), & t \in (0,+\infty), \\ v_{tt}(t) - \beta\Delta v(t) + \displaystyle\int_0^{+\infty} h(s) div(a_2(x)\nabla v(t-s))ds & \\ \qquad +\mu v_t(t-\tau) + dw(t) = f_2(v(t)), & t \in (0,+\infty), \\ w(x,t) = v(x,t) = 0, & x \in \partial\Omega, \\ w(x,-t) = w_0(x,t), \quad v(x,-t) = v_0(x,t), & x \in \Omega,\ t \geq 0, \\ w_t(x,0) = w_1(x), \quad v_t(x,0) = v_1(x), & x \in \Omega,\ t \geq 0, \\ w_t(t-\tau) = f_0(t-\tau), \quad v_t(t-\tau) = f_0(t-\tau), & t \in (0,\tau), \end{cases} \tag{4.3}$$

where α and β are positive constants, $a_1,\ a_2 \in C^1(\Omega)$, $a_1(x),\ a_2(x) > 0$ with The memory kernel h satisfying (A2) and (A7). The above system is equivalent to (1.1) where $u = (w,v)$, $f_0 = (l_0, m_0)$ and $H = (L^2(\Omega))^2$ with

$$\langle (w_1,v_1),(w_2,v_2)\rangle = \int_\Omega w_1w_2 + v_1v_2 dx.$$

We take $D(A) = D(B)) = (H^2(\Omega) \cap H_0^1(\Omega))^2$ and the operators A, B are given by

$$Au = -(\alpha\Delta w, \beta\Delta v) + d(v,w),$$
$$Bu = -(div(a_1(x)\nabla w), div(a_2(x)\nabla w)).$$

The function $F_2(u(t)) = (f_1(w(t)), f_2(v(t)))$ satisfies (A6).

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COMMON FIXED POINTS OF GERAGHTY-SUZUKI TYPE CONTRACTION MAPS IN b-METRIC SPACES

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Abstract. In this paper, we prove the existence and uniqueness of common fixed points for two pairs of selfmaps satisfying a Geraghty-Suzuki type contraction condition in which one pair is compatible, b-continous and the another one is weakly compatible in complete b-metric spaces. Further, we prove the same with different hypotheses on two pairs of selfmaps which satisfy b-(E.A)-property. We draw some corollaries from our results and provide examples in support of our results.

1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Banach contraction principle plays an important role in solving nonlinear equations, and it is one of the most useful results in fixed point theory. In the direction of generalization of contraction conditions, in 1973, Geraghty [17] proved a fixed point theorem, generalizing Banach contraction principle. In 1975, Dass and Gupta [14] extended contraction map to contraction map with rational expression and proved the existence of fixed points in complete metric spaces. In 2008, Suzuki [30] proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness.

The main idea of b-metric was initiated from the works of Bourbaki [10] and Bakhtin [6]. The concept of b-metric space or metric type space was introduced by Czerwik [12] as a generalization of metric space. Afterwards, many authors studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, we refer [2, 3, 8, 9, 13, 22, 28, 29].

In 2002, Aamari and Moutawakil [1] introduced the notion of property (E.A). Different authors applied this concept to prove the existence of common fixed points, we refer [4, 5, 25, 26, 27].

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We denote $\mathbb{N}$, the set of all natural numbers and $\mathbb{R}^+ = [0, \infty)$.

Definition 1.1. [12] *Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$*

(b_1) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 1.2. [9] *Let (X, d) be a b-metric space and $\{x_n\}$ a sequence in X.*

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(iii) A b-metric space (X, d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent.
(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

In general, a b-metric is not necessarily continuous.

Example 1.1. [19] *Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to \mathbb{R}^+$ as follows:*

$$d(m, n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 1.3. [9] *Let (X, d_X) and (Y, d_Y) be two b-metric spaces. A function $f : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e., whenever $\{x_n\}$ is b-convergent to x, fx_n is b-convergent to fx.*

Definition 1.4. [20] *A pair (A, B) of selfmaps on a metric space (X, d) is said to be compatible if $\lim\limits_{n\to\infty} d(BAx_n, ABx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.*

Definition 1.5. [1] *A pair (A, B) of selfmaps on a metric space (X, d) is said to be satisfy (E.A)-property if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.*

Definition 1.6. [25] *A pair (A, B) of selfmaps on a b-metric space (X, d) is said to be satisfy b-(E.A)-property if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.*

Definition 1.7. [21] *A pair (A, B) of selfmaps on a set X is said to be weakly compatible if $ABx = BAx$ whenever $Ax = Bx$ for any $x \in X$.*

In 1973, Geraghty [17] introduced a class of functions
$\mathfrak{S} = \{\beta : [0, \infty) \to [0, 1) / \lim\limits_{n\to\infty} \beta(t_n) = 1 \implies \lim\limits_{n\to\infty} t_n = 0\}$.

Theorem 1.1. [17] *Let (X,d) be a complete metric space. Let $T : X \to X$ be a selfmap satisfying the following: there exists $\beta \in \mathfrak{S}$ such that $d(Tx,Ty) \le \beta(d(x,y))d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point.*

We denote $\mathfrak{B} = \{\alpha : [0,\infty) \to [0,\frac{1}{s})/ \lim\limits_{n\to\infty} \alpha(t_n) = \frac{1}{s} \implies \lim\limits_{n\to\infty} t_n = 0\}$.

In 2011, Dukic, Kadelburg and Radenović [15] extended Theorem 1.9 to the case of b-metric spaces as follows.

Theorem 1.2. [15] *Let (X,d) ba a complete b-metric space with coefficient $s \ge 1$ and let $T : X \to X$ be a selfmap of X. Suppose that there exists $\alpha \in \mathfrak{B}$ such that $d(Tx,Ty) \le \alpha(d(x,y))d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point in X.*

The following lemmas are useful in proving our main results.

Lemma 1.3. [18] *Let (X,d) be a b-metric space with coefficient $s \ge 1$. Suppose that $\{x_n\}$ is a sequence in X such that $d(x_n,x_{n+1}) \le kd(x_{n-1},x_n)$ for all $n \in \mathbb{N}$, where $k \in [0,1)$ is a constant. Then $\{x_n\}$ is a b-Cauchy sequence in X.*

Lemma 1.4. [2] *Let (X,d) be a b-metric space with coefficient $s \ge 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively, then we have*
$$\frac{1}{s^2}d(x,y) \le \liminf_{n\to\infty} d(x_n,y_n) \le \limsup_{n\to\infty} d(x_n,y_n) \le s^2 d(x,y).$$
In particular, if $x = y$, then we have $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$. Moreover for each $z \in X$ we have
$$\frac{1}{s}d(x,z) \le \liminf_{n\to\infty} d(x_n,z) \le \limsup_{n\to\infty} d(x_n,z) \le sd(x,z).$$

In 2015, Latif, Parvaneh, Salimi and Al-Mazrooei [23] proved the existence and uniqueness of fixed points of a single selfmap satisfying Suzuki type contraction condition in b-metric spaces as follows.

Theorem 1.5. [23] *Let (X,d) be a complete b-metric space (with parameter $s > 1$) and let $f : X \to X, \alpha : X \times X \to [0,\infty)$ satisfying*
(a) $\alpha(x,y) \ge 1 \implies \alpha(fx,fy) \ge 1$,
(b) $\alpha(x,z) \ge 1, \alpha(z,y) \ge 1 \implies \alpha(x,y) \ge 1, x,y,z \in X$. Suppose that $\beta \in \mathfrak{B}$ such that $\frac{1}{2s}d(x,fx) \le d(x,y) \implies s\alpha(x,y)d(fx,fy) \le \beta(M(x,y))M(x,y)$ for all $x,y \in X$, where
$M(x,y) = \max\{d(x,y), \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s[d(x,y)+d(fx,fy)]}, \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)}\}$.
Also, suppose that the following assertions hold:
(i) there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \ge 1$;
(ii) for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n,x) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$.
Then, f has a fixed point.

The set $\{x_0, fx_0, f^2x_0, f^3x_0, \ldots\}$ is called an orbit of f at the point x_0 and is denoted by $O_f(x_0)$ [7].

Definition 1.8. [11] *A b-metric space X is said to be f-orbitally complete if every Cauchy sequence in $O_f(x_0)$ converges in X, where f is a selfmapping on X and $x_0 \in X$.*

Definition 1.9. [24] *Let X be any nonempty set and $\alpha : X \times X \to \mathbb{R}$. A selfmap $f : X \to X$ is said to have a property (H), if for any $x,y \in X$ with $x \ne y$, there exists $z \in X$ such that $\alpha(x,z) \ge 1, \alpha(y,z) \ge 1$ and $\alpha(z,fz) \ge 1$.*

Definition 1.10. [24] *Let (X,d) be a b-metric space with parameter $s \geq 1$ and $\alpha : X \times X \to \mathbb{R}$. A selfmap $f : X \to X$ is called a generalized α-Suzuki-Geraghty contraction if there exists a $\beta \in \mathfrak{B}$ such that for any $x, y \in X$,*
$\frac{1}{2s}d(x,fx) \leq sd(x,y) \implies d(fx,fy) \leq \beta(M(x,y))M(x,y)$,
where
$M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), d(f^2x,fx), d(f^2x,y), \frac{d(f^2x,fy)}{s}, \frac{d(f^2x,x)}{2s}, \frac{d(x,fy)+d(y,fx)}{2s}, \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s[d(x,y)+d(fx,fy)]}, \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)}\}$.

Theorem 1.6. [24] *Let (X,d) be a complete b-metric space with parameter $s \geq 1, \alpha : X \times X \to \mathbb{R}$ and $f : X \to X$. Assume that X is f-orbitally complete and the following conditions hold:*
(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(ii) *f is a generalized $\alpha - Suzuki - Geraghty$ contraction and a triangular α-orbital admissible;*
(iii) *either f is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*
Then f has a fixed point z in X and $\{f^n x_0\}$ converges to z. Moreover, f has a unique fixed point if condition (i) is replaced with the property (H).

Throughout this paper we denote
$\mathfrak{F} = \{\beta : [0,\infty) \to [0, \frac{1}{s}) / \limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \implies \lim_{n\to\infty} t_n = 0\}$.
In 2019, Faraji, Savić and Radenović [16] proved the following theorem.

Theorem 1.7. [16] *Let (X,d) be a complete b-metric space with parameter $s \geq 1$. Let $T, S : X \to X$ be selfmaps on X which satisfy: there exists $\beta \in \mathfrak{F}$ such that*
$$sd(Tx,Sy) \leq \beta(M(x,y))M(x,y) \text{ for all } x,y \in X,$$
where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Sy)\}$.
If either T or S is continuous, then T and S have a unique common fixed point.

Motivated by Theorem 1.5 and Theorem 1.6, in Section 2 of this paper, we prove the existence and uniqueness of common fixed points for two pairs of selfmaps satisfying a Geraghty-Suzuki type contraction condition in which one pair is compatible, b-continous and the another one is weakly compatible in complete b-metric spaces. Further, we prove the same with different hypotheses on two pairs of selfmaps which satisfy b-(E.A)-property. In Section 3, we draw some corollaries and examples in support of our results.

2. Main Results

Let A, B, S and T be mappings from a b-metric space (X,d) into itself and satisfying
$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X). \tag{2.1}$$
Now by (2.1), for any $x_0 \in X$, there exists $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$. In the same way for this x_1, we can choose a point $x_2 \in X$ such that $y_1 = Bx_1 = Sx_2$ and so on. In general, we define
$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0,1,2,\dots. \tag{2.2}$$

Proposition 2.1. *Let (X,d) be a b-metric space wuth coefficient $s \geq 1$. Assume that A, B, S and T are selfmappings of X which satisfy the following condition:*

there exists $\beta \in \mathfrak{F}$ *such that*

$$\frac{1}{2s}\min\{d(Sx,Ax),d(Ty,By)\} \leq \max\{d(Sx,Ty),d(Ax,By)\} \implies s^4 d(Ax,By) \leq \beta(M(x,y))M(x,y) \tag{2.3}$$

where

$M(x,y) = \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)}{2s}, \frac{d(Ty,Ax)}{2s}, \frac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \frac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}$,

for all $x, y \in X$. *Then we have the following:*

(*i*) *If* $A(X) \subseteq T(X)$ *and the pair* (B,T) *is weakly compatible and if* z *is a common fixed point of* A *and* S *then* z *is a common fixed point of* A, B, S *and* T *and it is unique.*

(*ii*) *If* $B(X) \subseteq S(X)$ *and the pair* (A,S) *is weakly compatible and if* z *is a common fixed point of* B *and* T *then* z *is a common fixed point of* A, B, S *and* T *and it is unique.*

Proof. First, we assume that (i) holds. Let z be a common fixed point of A and S. Then $Az = Sz = z$. Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = z$. Therefore $Az = Sz = Tu = z$.
We now prove that $Az = Bu$. Suppose that $Az \neq Bu$.
Since $\frac{1}{2s}\min\{d(Sz,Az),d(Tu,Bu)\} \leq \max\{d(Sz,Tu),d(Az,Bu)\}$.
From the inequality (2.3), we have

$$s^4 d(Az,Bu) \leq \beta(M(z,u))M(z,u) \tag{2.4}$$

where

$M(z,u) = \max\{d(Sz,Tu), d(Sz,Az), d(Tu,Bu), \frac{d(Sz,Bu)}{2s}, \frac{d(Tu,Az)}{2s}, \frac{d(Sz,Az)d(Tu,Bu)}{1+d(Sz,Tu)+d(Az,Bu)}, \frac{d(Sz,Bu)d(Tu,Az)}{1+s^4[d(Sz,Tu)+d(Az,Bu)]}\}$
$= \max\{0, 0, d(Az,Bu), \frac{d(Az,Bu)}{2s}, 0, 0, 0\} = d(Az,Bu)$.

From the inequality (2.4), we have
$s^4 d(Az,Bu) \leq \beta(d(z,u))d(z,u) \leq \frac{d(Az,Bu)}{s}$ so that $(s^5-1)d(Az,Bu) \leq 0$.
Since $(s^5-1) \geq 0$, it follows that $d(Az,Bu) = 0$.
Hence $Az = Bu$. Therefore $Az = Bu = Sz = Tu = z$.
Since the pair (B,T) is weakly compatible and $Bu = Tu$, we have $BTu = TBu$. i.e., $Bz = Tz$.
Now we show that $Bz = z$.
If $Bz \neq z$, then we have
$\frac{1}{2s}\min\{d(Sz,Az),d(Tz,Bz)\} \leq \max\{d(Sz,Tz),d(Az,Bz)\}$
From the inequality (2.3), we have

$$s^4 d(z,Bz) = s^4 d(Az,Bz) \leq \beta(M(z,z))M(z,z) \tag{2.5}$$

where

$M(z,z) = \max\{d(Sz,Tz), d(Sz,Az), d(Tz,Bz), \frac{d(Sz,Bz)}{2s}, \frac{d(Tz,Az)}{2s}, \frac{d(Sz,Az)d(Tz,Bz)}{1+d(Sz,Tz)+d(Az,Bz)}, \frac{d(Sz,Bz)d(Tz,Az)}{1+s^4[d(Sz,Tz)+d(Az,Bz)]}\}$
$= \max\{d(z,Bz), 0, 0, \frac{d(z,Bz)}{2s}, \frac{d(z,Bz)}{2s}, 0, \frac{[d(z,Bz)]^2}{1+2s^4[d(z,Bz)]}\} = d(z,Bz)$.

From the inequality (2.5), we have
$s^4 d(z,Bz) \leq \beta(M(z,z))M(z,z) = \beta(d(z,Bz))d(z,Bz) \leq \frac{d(z,Bz)}{s}$ so that $(s^5-1)d(z,Bz) \leq 0$.
Since $(s^5-1) \geq 0$, it follows that $d(z,Bz) = 0$.

Hence $Bz = z$. Therefore $Az = Bz = Sz = Tz = z$.
Therefore z is a common fixed point of A, B, S and T.

In a similar way, under the assumption (ii), the conclusion of the proposition follows.

Uniqueness follows from the inequality (2.3). $\square$

Remark. *Selfmaps A, B, S and T of a b-metric space X that satisfy* (2.3) *is said to be Geraghty-Suzuki type contraction maps on X.*

Proposition 2.2. *Let A, B, S and T be selfmaps of a b-metric space (X, d) and satisfy* (2.1) *and Geraghty-Suzuki type contraction maps. Then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by* (2.2) *is b-Cauchy in X.*

Proof. Let $x_0 \in X$ and let $\{y_n\}$ be defined by (2.2). Assume that $y_n = y_{n+1}$ for some n.
Case (i): n even.
We write $n = 2m$ for some $m \in \mathbb{N}$. Suppose that $d(y_{n+1}, y_{n+2}) > 0$. Since

$$\frac{1}{2s}\min\{d(Sx_{2m+2}, Ax_{2m+2}), d(Tx_{2m+1}, Bx_{2m+1})\} \leq \max\{d(Sx_{2m+2}, Tx_{2m+1}), d(Ax_{2m+2}, Bx_{2m+1})\}$$

From the inequality (2.3), we have

$$\begin{aligned} s^4 d(y_{n+1}, y_{n+2}) &= s^4 d(y_{2m+1}, y_{2m+2}) \\ &= s^4 d(y_{2m+2}, y_{2m+1}) \\ &= s^4 d(Ax_{2m+2}, Bx_{2m+1}) \\ &\leq \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+2}, x_{2m+1}) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} M(x_{2m+2}, x_{2m+1}) = \max\{&d(Sx_{2m+2}, Tx_{2m+1}), d(Sx_{2m+2}, Ax_{2m+2}), \\ &d(Tx_{2m+1}, Bx_{2m+1}), \tfrac{d(Sx_{2m+2}, Bx_{2m+1})}{2s}, \tfrac{d(Tx_{2m+1}, Ax_{2m+2})}{2s}, \\ &\tfrac{d(Sx_{2m+2}, Ax_{2m+2})d(Tx_{2m+1}, Bx_{2m+1})}{1+d(Sx_{2m+2}, Tx_{2m+1})+d(Ax_{2m+2}, Bx_{2m+1})}, \\ &\tfrac{d(Sx_{2m+2}, Bx_{2m+1})d(Tx_{2m+1}, Ax_{2m+2})}{1+s^4[d(Sx_{2m+2}, Tx_{2m+1})+d(Ax_{2m+2}, Bx_{2m+1})]}\} \\ = \max\{&0, d(y_{n+1}, y_{n+2}), 0, 0, \tfrac{d(y_n, y_{n+2})}{2s}, 0, 0\} = d(y_{n+1}, y_{n+2}). \end{aligned}$$

From the inequality (2.6), we have

$$\begin{aligned} s^4 d(y_{n+1}, y_{n+2}) &\leq \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+1}, x_{2m+1}) \\ &\leq \beta(d(y_{n+1}, y_{n+2}))d(y_{n+1}, y_{n+2}) \leq \frac{d(y_{n+1}, y_{n+2})}{s} \end{aligned}$$

which implies that $(s^5 - 1)d(y_{n+1}, y_{n+2}) \leq 0$.
Since $(s^5 - 1) \geq 0$, we have $d(y_{n+1}, y_{n+2}) \leq 0$.
Therefore $y_{n+2} = y_{n+1} = y_n$.
In general, we have $y_{n+k} = y_n$ for $k = 0, 1, 2, \ldots$.
Case (ii): n odd.
We write $n = 2m + 1$ for some $m \in \mathbb{N}$.
Since

$$\frac{1}{2s}\min\{d(Sx_{2m+2}, Ax_{2m+2}), d(Tx_{2m+3}, Bx_{2m+3})\} \leq \max\{d(Sx_{2m+2}, Tx_{2m+3}), d(Ax_{2m+2}, Bx_{2m+3})\},$$

from the inequality (2.3), we have

$$\begin{aligned} s^4 d(y_{n+1}, y_{n+2}) = s^4 d(y_{2m+2}, y_{2m+3}) &= d(Ax_{2m+2}, Bx_{2m+3}) \\ &\leq \beta(M(x_{2m+2}, x_{2m+3}))M(x_{2m+2}, x_{2m+3}) \end{aligned} \tag{2.7}$$

where

$$M(x_{2m+2}, x_{2m+3}) = \max\{d(Sx_{2m+2}, Tx_{2m+3}), d(Sx_{2m+2}, Ax_{2m+2}),$$

$$\begin{aligned}
&d(Tx_{2m+3}, Bx_{2m+3}), \tfrac{d(Sx_{2m+2},Bx_{2m+3})}{2s}, \tfrac{d(Tx_{2m+3},Ax_{2m+2})}{2s},\\
&\tfrac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+3},Bx_{2m+3})}{1+d(Sx_{2m+2},Tx_{2m+3})+d(Ax_{2m+2},Bx_{2m+3})},\\
&\tfrac{d(Sx_{2m+2},Bx_{2m+3})d(Tx_{2m+3},Ax_{2m+2})}{1+s^4[d(Sx_{2m+2},Tx_{2m+3})+d(Ax_{2m+2},Bx_{2m+3})]}\}\\
= \max\{0, 0, d(y_{n+1}, y_{n+2}), \tfrac{d(y_n,y_{n+2})}{2s}, 0, 0, 0\} = d(y_{n+1}, y_{n+2}).
\end{aligned}$$

From the inequality (2.7), we have

$$\begin{aligned}
s^4d(y_{n+1}, y_{n+2}) &\leq \beta(M(x_{2m+2}, x_{2m+3}))M(x_{2m+2}, x_{2m+3})\\
&\leq \beta(d(y_{n+1}, y_{n+2}))d(y_{n+1}, y_{n+2}) \leq \tfrac{d(y_{n+1},y_{n+2})}{s}
\end{aligned}$$

which implies that $(s^5 - 1)d(y_{n+1}, y_{n+2}) \leq 0$.
Since $(s^5 - 1) \geq 0$, we have $d(y_{n+1}, y_{n+2}) \leq 0$.
Therefore $y_{n+2} = y_{n+1} = y_n$.
In general, we have $y_{n+k} = y_n$ for $k = 0, 1, 2, \ldots$.
From Case (i) and Case (ii), we have $y_{n+k} = y_n$ for all $k = 0, 1, 2, \ldots$.
Hence $\{y_{n+k}\}$ is a constant sequence and hence $\{y_n\}$ is Cauchy.
Now we assume that $y_{n-1} \neq y_n$ for all $n \in \mathbb{N}$.
If n is odd, then $n = 2m + 1$ for some $m \in \mathbb{N}$.
Since

$$\tfrac{1}{2s} \min\{d(Sx_{2m+2}, Ax_{2m+2}), d(Tx_{2m+1}, Bx_{2m+1})\} \leq \max\{d(Sx_{2m+2}, Tx_{2m+1}), d(Ax_{2m+2}, Bx_{2m+1})\}.$$

From the inequality (2.3), we have

$$\begin{aligned}
s^4d(y_n, y_{n+1}) &= s^4d(y_{2m+1}, y_{2m+2}) = s^4d(y_{2m+2}, y_{2m+1})\\
&= s^4d(Ax_{2m+2}, Bx_{2m+1}) \leq \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+2}, x_{2m+1})
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
M(x_{2m+2}, x_{2m+1}) &= \max\{d(Sx_{2m+2}, Tx_{2m+1}), d(Sx_{2m+2}, Ax_{2m+2}),\\
&\quad d(Tx_{2m+1}, Bx_{2m+1}), \tfrac{d(Sx_{2m+2},Bx_{2m+1})}{2s}, \tfrac{d(Tx_{2m+1},Ax_{2m+2})}{2s},\\
&\quad \tfrac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+1},Bx_{2m+1})}{1+d(Sx_{2m+2},Tx_{2m+1})+d(Ax_{2m+2},Bx_{2m+1})},\\
&\quad \tfrac{d(Sx_{2m+2},Bx_{2m+1})d(Tx_{2m+1},Ax_{2m+2})}{1+s^4[d(Sx_{2m+2},Tx_{2m+1})+d(Ax_{2m+2},Bx_{2m+1})]}\}\\
&\leq \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_n), 0, \tfrac{d(y_n,y_n)+d(y_n,y_{n+1})}{2},\\
&\quad \tfrac{d(y_n,y_{n+1})d(y_{n-1},y_n)}{1+d(y_{n-1},y_n)+d(y_n,y_{n+1})}, 0\}\\
&\leq \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}.
\end{aligned}$$

Suppose $M(x_{2m+2}, x_{2m+1}) = d(y_n, y_{n+1})$.
Then from the inequality (2.8), we have

$$\begin{aligned}
s^4d(y_n, y_{n+1}) &\leq \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+2}, x_{2m+1})\\
&\leq \beta(d(y_n, y_{n+1}))d(y_n, y_{n+1}) \leq \tfrac{d(y_n,y_{n+1})}{s}
\end{aligned}$$

which implies that $(s^5 - 1)d(y_n, y_{n+1}) \leq 0$.
Since $(s^5 - 1) \geq 0$, we have $d(y_n, y_{n+1}) \leq 0$.
Therefore $M(x_{2m+2}, x_{2m+1}) = d(y_{n-1}, y_n)$.
From the inequality (2.8), we have

$$\begin{aligned}
s^4d(y_n, y_{n+1}) &\leq \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+2}, x_{2m+1})\\
&\leq \beta(d(y_{n-1}, y_n))d(y_{n-1}, y_n) \leq \tfrac{d(y_{n-1},y_n)}{s}.
\end{aligned} \tag{2.9}$$

Also, it is easy to see that (2.9) is valid when n is even.
Hence we have $d(y_n, y_{n+1}) \leq \frac{1}{s^5} d(y_{n-1}, y_n)$ for all $n \in \mathbb{N}$.
From Lemma 1.3, we have the sequence $\{y_n\}$ is a b-Cauchy sequence in X. □

The following is the main result of this paper.

Theorem 2.3. *Let A, B, S and T be selfmaps on a complete b-metric space (X, d) and satisfy* (2.1) *and Geraghty-Suzuki type contractive maps. If either*

(*i*) *the pair (A, S) compatible, A (or) S is b-continuous and the pair (B, T) is weakly compatible*

or

(*ii*) *the pair (B, T) compatible, B (or) T is b-continuous and the pair (A, S) is weakly compatible*

then A, B, S and T have a unique common fixed point in X.

Proof. By Proposition 2.2, the sequence $\{y_n\}$ is b-Cauchy in X.
Since X is b-complete, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Thus

$$\begin{cases} \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z \text{ and} \\ \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \end{cases} \tag{2.10}$$

Assume that (i) holds.
Since S is b-continuous, it follows that $\lim_{n\to\infty} SSx_{2n+2} = Sz$, $\lim_{n\to\infty} SAx_{2n} = Sz$.
By the b-triangle inequality, we have $d(ASx_{2n}, Sz) \leq s[d(ASx_{2n}, SAx_{2n}) + d(SAx_{2n}, Sz)]$.
Since the pair (A, S) is compatible, $\lim_{n\to\infty} d(ASx_{2n}, SAx_{2n}) = 0$.
Taking limit superior as $n \to \infty$, we have
$\limsup_{n\to\infty} d(ASx_{2n}, Sz) \leq s[\limsup_{n\to\infty} d(ASx_{2n}, SAx_{2n}) + \limsup_{n\to\infty} d(SAx_{2n}, Sz)] = 0$.
Therefore $\lim_{n\to\infty} ASx_{2n} = Sz$.
We now prove that $Sz = z$.
Suppose that $Sz \neq z$. Since

$$\frac{1}{2s}\min\{d(SSx_{2m+2}, ASx_{2m+2}), d(Tx_{2m+1}, Bx_{2m+1})\} \leq \max\{d(SSx_{2m+2}, Tx_{2m+1}), d(ASx_{2m+2}, Bx_{2m+1})\}$$

From the inequality (2.3), we have

$$s^4 d(ASx_{2n+2}, Bx_{2n+1}) \leq \beta(M(Sx_{2n+2}, x_{2n+1}))M(Sx_{2n+2}, x_{2n+1}) \tag{2.11}$$

where

$$\begin{aligned} M(Sx_{2n+2}, x_{2n+1}) = \max\{& d(SSx_{2n+2}, Tx_{2n+1}), d(SSx_{2n+2}, ASx_{2n+s}), \\ & d(Tx_{2n+1}, Bx_{2n+1}), \tfrac{d(SSx_{2n+2}, Bx_{2n+1})}{2s}, \tfrac{d(Tx_{2n+1}, ASx_{2n+2})}{2s}, \\ & \tfrac{d(SSx_{2n+2}, ASx_{2n+2})d(Tx_{2n+1}, Bx_{2n+1})}{1+d(SSx_{2n+2}, Tx_{2n+1})+d(ASx_{2n+2}, Bx_{2n+1})}, \\ & \tfrac{d(SSx_{2n+2}, Bx_{2n+1})d(Tx_{2n+1}, ASx_{2n+2})}{1+s^4[d(SSx_{2n+2}, Tx_{2n+1})+d(ASx_{2n+2}, Bx_{2n+1})]}\}. \end{aligned}$$

By taking limit superior as $n \to \infty$ on $M(Sx_{2n+2}, x_{2n+1})$ and using Lemma 1.4, we obtain

$$\begin{aligned} \limsup_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) &\leq \max\{s^2 d(Sz, z), 0, 0, \tfrac{s^2 d(Sz,z)}{2s}, \tfrac{s^2 d(Sz,z)}{2s}, 0, \tfrac{s^4[d(Sz,z)]^2}{1+2s^4 d(Sz,z)}\} \\ &= s^2 d(Sz, z). \end{aligned}$$

Therefore

$$\frac{1}{s^2} d(Sz, z) \leq \liminf_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) \leq \limsup_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) \leq s^2 d(Sz, z). \tag{2.12}$$

Taking limit superior as $n \to \infty$ in the inequality (2.11) and using Lemma 1.4, we get

$$\begin{aligned} s^4 \tfrac{1}{s^2} d(Sz, z) &\leq s^4 \limsup_{n\to\infty} d(ASx_{2n+2}, Bx_{2n+1}) \\ &= \limsup_{n\to\infty} s^4 d(ASx_{2n+2}, Bx_{2n+1}) \end{aligned}$$

$$\leq \limsup_{n\to\infty}[\beta(M(Sx_{2n+2}, x_{2n+1}))M(Sx_{2n+2}, x_{2n+1})]$$
$$= \limsup_{n\to\infty}\beta(M(Sx_{2n+2}, x_{2n+1}))\limsup_{n\to\infty}M(Sx_{2n+2}, x_{2n+1})$$
$$\leq \limsup_{n\to\infty}\beta(M(Sx_{2n+2}, x_{2n+1}))s^2 d(Sz, z).$$

Therefore
$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty}\beta(M(Sx_{2n+2}, x_{2n+1})) \leq \frac{1}{s}$ which implies that
$\limsup_{n\to\infty}\beta(M(Sx_{2n+2}, x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, it follows that $\lim_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) = 0$.
Therefore from the inequality (2.12), we have
$\frac{1}{s^2}d(Sz, z) \leq \lim_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) = 0$ which implies that $d(Sz, z) \leq 0$.
Therefore $Sz = z$.
We now show that $Az = z$. Suppose that $Az \neq z$.
Since
$\frac{1}{2s}\min\{d(Sz, Az), d(Tx_{2m+1}, Bx_{2m+1})\} \leq \max\{d(Sz, Tx_{2m+1}), d(Az, Bx_{2m+1})\}$
From the inequality (2.3), we have

$$s^4 d(Az, Bx_{2n+1}) \leq \beta(M(z, x_{2n+1}))M(z, x_{2n+1}) \tag{2.13}$$

where

$$M(z, x_{2n+1}) = \max\{d(Sz, Tx_{2n+1}), d(Sz, Az), d(Tx_{2n+1}, Bx_{2n+1}), \frac{d(Sz,Bx_{2n+1})}{2s}, \frac{d(Tx_{2n+1},Az)}{2s}, \frac{d(Sz,Az)d(Tx_{2n+1},Bx_{2n+1})}{1+d(Sz,Tx_{2n+1})+d(Az,Bx_{2n+1})}, \frac{d(Sz,Bx_{2n+1})d(Tx_{2n+1},Az)}{1+s^4[d(Sz,Tx_{2n+1})+d(Az,Bx_{2n+1})]}\}.$$

By taking limit superior as $n \to \infty$ on $M(z, x_{2n+1})$ and using Lemma 1.4, we obtain

$$\limsup_{n\to\infty} M(z, x_{2n+1}) \leq \max\{s^2 d(Az, z), 0, 0, \frac{s^2 d(Az,z)}{2s}, \frac{s^2 d(Az,z)}{2s}, 0, \frac{s^4[d(Az,z)]^2}{1+2s^4 d(Az,z)}\}$$
$$= s^2 d(Az, z).$$

Therefore

$$\frac{1}{s^2}d(Az, z) \leq \liminf_{n\to\infty} M(z, x_{2n+1}) \leq \limsup_{n\to\infty} M(z, x_{2n+1}) \leq s^2 d(Az, z). \tag{2.14}$$

Taking limit superior as $n \to \infty$ in the inequality (2.13) and using Lemma 1.4, we get

$$s^4 \frac{1}{s^2}d(Az, z) \leq s^4 \limsup_{n\to\infty} d(Az, Bx_{2n+1})$$
$$= \limsup_{n\to\infty} s^4 d(Az, Bx_{2n+1})$$
$$\leq \limsup_{n\to\infty}[\beta(M(z, x_{2n+1}))M(z, x_{2n+1})]$$
$$= \limsup_{n\to\infty}\beta(M(z, x_{2n+1}))\limsup_{n\to\infty} M(z, x_{2n+1})$$
$$\leq \limsup_{n\to\infty}\beta(M(z, x_{2n+1}))s^2 d(Az, z).$$

Hence
$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty}\beta(M(z, x_{2n+1})) \leq \frac{1}{s}$ which implies that
$\limsup_{n\to\infty}\beta(M(z, x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, it follows that $\lim_{n\to\infty} M(z, x_{2n+1}) = 0$.
Therefore from the inequality (2.14), we have
$\frac{1}{s^2}d(Az, z) \leq \lim_{n\to\infty} M(z, x_{2n+1}) = 0$ which implies that $d(Az, z) \leq 0$.
Therefore $Az = Sz = z$. Hence z is a common fixed point of A and S.

Now by Proposition 2.1, we have z is a unique common fixed point of A, B, S and T.
Assume that A is b-continuous, it follows that
$\lim_{n\to\infty} AAx_{2n} = Az$, $\lim_{n\to\infty} ASx_{2n+2} = Az$.
By the b-triangle inequality, we have
$d(SAx_{2n}, Az) \leq s[d(SAx_{2n}, ASx_{2n}) + d(ASx_{2n}, Az)]$.
Since the pair (A, S) is compatible, $\lim_{n\to\infty} d(ASx_{2n}, SAx_{2n}) = 0$.
Taking limit superior as $n \to \infty$, we have
$\limsup_{n\to\infty} d(SAx_{2n}, Az) \leq s[\limsup_{n\to\infty} d(SAx_{2n}, ASx_{2n}) + \limsup_{n\to\infty} d(ASx_{2n}, Az)] = 0$.
Therefore $\lim_{n\to\infty} SAx_{2n} = Az$.
Now we prove that $Az = z$. Suppose that $Az \neq z$.
Since

$$\frac{1}{2s}\min\{d(SAx_{2n}, AAx_{2n}), d(Tx_{2n+1}, Bx_{2n+1})\} \leq \max\{d(SAx_{2n}, Tx_{2n+1}), d(AAx_{2n}, Bx_{2n+1})\}$$

From the inequality (2.3), we have

$$s^4 d(ASx_{2n+2}, Bx_{2n+1}) \leq \beta(M(Sx_{2n+2}, x_{2n+1}))M(Sx_{2n+2}, x_{2n+1}) \tag{2.15}$$

where

$$M(Ax_{2n}, x_{2n+1}) = \max\{d(SAx_{2n}, Tx_{2n+1}), d(SAx_{2n}, AAx_{2n}), d(Tx_{2n+1}, Bx_{2n+1}), \frac{d(SAx_{2n}, Bx_{2n+1})}{2s}, \frac{d(Tx_{2n+1}, AAx_{2n})}{2s}, \frac{d(SAx_{2n}, AAx_{2n})d(Tx_{2n+1}, Bx_{2n+1})}{1+d(SAx_{2n}, Tx_{2n+1})+d(AAx_{2n}, Bx_{2n+1})}, \frac{d(SAx_{2n}, Bx_{2n+1})d(Tx_{2n+1}, AAx_{2n})}{1+s^4[d(SAx_{2n}, Tx_{2n+1})+d(AAx_{2n}, Bx_{2n+1})]}\}.$$

By taking limit superior as $n \to \infty$ on $M(Ax_{2n}, x_{2n+1})$ and using Lemma 1.4, we obtain

$$\begin{aligned}\limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1}) &\leq \max\{s^2 d(Az, z), 0, 0, \tfrac{s^2 d(Az,z)}{2s}, \tfrac{s^2 d(Az,z)}{2s}, 0, \tfrac{s^4[d(Az,z)]^2}{1+2s^2 d(Az,z)}\}\\ &= s^2 d(Az, z).\end{aligned}$$

Therefore

$$\frac{1}{s^2} d(Az, z) \leq \liminf_{n\to\infty} M(Ax_{2n}, x_{2n+1}) \leq \limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1}) \leq s^2 d(Az, z). \tag{2.16}$$

Taking limit superior as $n \to \infty$ in the inequality (2.15) and using Lemma 1.4, we get

$$\begin{aligned} s^4 \tfrac{1}{s^2} d(Az, z) &\leq s^4 \limsup_{n\to\infty} d(AAx_{2n}, Bx_{2n+1})\\ &= \limsup_{n\to\infty} s^4 d(AAx_{2n}, Bx_{2n+1})\\ &\leq \limsup_{n\to\infty} [\beta(M(Ax_{2n}, x_{2n+1}))M(Ax_{2n}, x_{2n+1})\\ &= \limsup_{n\to\infty} \beta(M(Ax_{2n}, x_{2n+1})) \limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1})\\ &\leq \limsup_{n\to\infty} \beta(M(Ax_{2n}, x_{2n+1}))s^2 d(Az, z).\end{aligned}$$

Thus
$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(Ax_{2n}, x_{2n+1})) \leq \frac{1}{s}$ which implies that
$\limsup_{n\to\infty} \beta(M(Ax_{2n}, x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, it follows that $\lim_{n\to\infty} M(Ax_{2n}, x_{2n+1}) = 0$.
Therefore from the inequality (2.16), we have
$\frac{1}{s^2} d(Az, z) \leq \lim_{n\to\infty} M(Ax_{2n}, x_{2n+1}) = 0$ which implies that $d(Az, z) \leq 0$.
Therefore $Az = z$.

Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Tu$.
We now show that $Bu = z$. Suppose that $Bu \neq z$.
Since
$\frac{1}{2s}\min\{d(Sx_{2n}, Ax_{2n}), d(Tu, Bu)\} \leq \max\{d(Sx_{2n}, Tu), d(Ax_{2n}, Bu)\}$
From the inequality (2.3), we have

$$s^4 d(Ax_{2n}, Bu) \leq \beta(M(x_{2n}, u))M(x_{2n}, u) \tag{2.17}$$

where
$M(x_{2n}, u) = \max\{d(Sx_{2n}, Tu), d(Sx_{2n}, Ax_{2n}), d(Tu, Bu), \frac{d(Sx_{2n},Bu)}{2s}, \frac{d(Tu,Ax_{2n})}{2s}, \frac{d(Sx_{2n},Ax_{2n})d(Tu,Bu)}{1+d(Sx_{2n},Tu)+d(Ax_{2n},Bu)}, \frac{d(Sx_{2n},Bu)d(Tu,Ax_{2n})}{1+s^4[d(Sx_{2n},Tu)+d(Ax_{2n},Bu)]}\}.$
By taking limit superior as $n \to \infty$ on $M(x_{2n}, u)$ and using Lemma 1.4, we obtain
$\limsup\limits_{n\to\infty} M(x_{2n}, u) \leq \max\{s^2 d(z, Bu), 0, 0, \frac{s^2 d(z,Bu)}{2s}, \frac{s^2 d(z,Bu)}{2s}, 0, \frac{s^4[d(z,Bu)]^2}{1+2s^2 d(z,Bu)}\}$
$= s^2 d(Az, z).$
Therefore

$$\frac{1}{s^2} d(z, Bu) \leq \liminf_{n\to\infty} M(x_{2n}, u) \leq \limsup_{n\to\infty} M(x_{2n}, u) \leq s^2 d(z, Bu). \tag{2.18}$$

Taking limit superior as $n \to \infty$ in the inequality (2.17) and using Lemma 1.4, we get

$$\begin{aligned} s^4 \tfrac{1}{s^2} d(z, Bu) &\leq s^4 \limsup_{n\to\infty} d(Ax_{2n}, Bu) \\ &= \limsup_{n\to\infty} s^4 d(Ax_{2n}, Bu) \\ &\leq \limsup_{n\to\infty} [\beta(M(x_{2n}, u))M(x_{2n}, u) \\ &= \limsup_{n\to\infty} \beta(M(x_{2n}, u)) \limsup_{n\to\infty} M(x_{2n}, u) \\ &\leq \limsup_{n\to\infty} \beta(M(x_{2n}, u)) s^2 d(z, Bu). \end{aligned}$$

Therefore
$\frac{1}{s} \leq 1 \leq \limsup\limits_{n\to\infty} \beta(M(x_{2n}, u)) \leq \frac{1}{s}$ which implies that $\limsup\limits_{n\to\infty} \beta(M(x_{2n}, u)) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, it follows that $\lim\limits_{n\to\infty} M(x_{2n}, u) = 0$.
Therefore from the inequality (2.18), we have
$\frac{1}{s^2} d(z, Bu) \leq \lim\limits_{n\to\infty} M(x_{2n}, u) = 0.$ implies that $d(z, Bu) \leq 0$.
Therefore $Bu = Tu = z$. Since the pair (B, T) is weakly compatible and $Bu = Tu$, we have
$BTu = TBu$. i.e., $Bz = Tz$.
We now show that $Bz = z$. Suppose that $Bz \neq z$.
Since
$\frac{1}{2s}\min\{d(Sx_{2n}, Ax_{2n}), d(Tz, Bz)\} \leq \max\{d(Sx_{2n}, Tz), d(Ax_{2n}, Bz)\}$
From the inequality (2.3), we have

$$s^4 d(Ax_{2n}, Bz) \leq \beta(M(x_{2n}, z))M(x_{2n}, z) \tag{2.19}$$

where
$M(x_{2n}, z) = \max\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz), \frac{d(Sx_{2n},Bz)}{2s}, \frac{d(Tz,Ax_{2n})}{2s}, \frac{d(Sx_{2n},Ax_{2n})d(Tz,Bz)}{1+d(Sx_{2n},Tz)+d(Ax_{2n},Bz)}, \frac{d(Sx_{2n},Bz)d(Tz,Ax_{2n})}{1+s^4 d(Sx_{2n},Tz)+d(Ax_{2n},Bz)}\}.$
By taking limit superior as $n \to \infty$ on $M(x_{2n}, z)$ and using Lemma 1.4, we obtain
$\limsup\limits_{n\to\infty} M(x_{2n}, z) \leq \max\{s^2 d(z, Bz), 0, 0, \frac{s^2 d(z,Bz)}{2s}, \frac{s^2 d(z,Bz)}{2s}, 0, \frac{s^6[d(z,Bz)]^2}{1+2s^2 d(z,Bz)}\}$

$$= s^2 d(Az, z).$$

Therefore

$$\frac{1}{s^2} d(z, Bz) \leq \liminf_{n\to\infty} M(x_{2n}, z) \leq \limsup_{n\to\infty} M(x_{2n}, z) \leq s^2 d(z, Bz). \tag{2.20}$$

Taking limit superior as $n \to \infty$ in the inequality (2.19) and using Lemma 1.4, we get

$$\begin{aligned} s^4 \tfrac{1}{s^2} d(z, Bz) &\leq s^4 \limsup_{n\to\infty} d(Ax_{2n}, Bz) \\ &= \limsup_{n\to\infty} s^4 d(Ax_{2n}, Bz) \\ &\leq \limsup_{n\to\infty} [\beta(M(x_{2n}, z)) M(x_{2n}, z) \\ &= \limsup_{n\to\infty} \beta(M(x_{2n}, z)) \limsup_{n\to\infty} M(x_{2n}, z) \\ &\leq \limsup_{n\to\infty} \beta(M(x_{2n}, z)) s^2 d(z, Bz). \end{aligned}$$

Therefore

$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(x_{2n}, z)) \leq \frac{1}{s}$ which implies that $\limsup_{n\to\infty} \beta(M(x_{2n}, z)) = \frac{1}{s}$.

Since $\beta \in \mathfrak{F}$, it follows that $\lim_{n\to\infty} M(x_{2n}, z) = 0$.

Therefore from the inequality (2.20), we have

$\frac{1}{s^2} d(z, Bz) \leq \lim_{n\to\infty} M(x_{2n}, z) = 0$. implies that $d(z, Bz) \leq 0$.

Hence $Bz = z$.

Therefore $Bz = Tz = z$.

Hence z is a common fixed point of A and S.

Now by Proposition 2.1, we have z is a unique common fixed point of A, B, S and T.

In a similar way, under the assumption (ii), the conclusion of the theorem holds. □

Theorem 2.4. *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $A, B, S, T : X \to X$ be selfmaps of X and satisfy* (2.1) *and Geraghty-Suzuki type contractive maps. Suppose that one of the pairs (A, S) and (B, T) satisfies the b-(E.A)-property and that one of the subspace $A(X), B(X), S(X)$ and $T(X)$ is b-closed in X. Then the pairs (A, S) and (B, T) have a point of coincidence in X. Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. We first assume that the pair (A, S) satisfies the b-(E.A)-property. So there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = q \tag{2.21}$$

for some $q \in X$.

Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$, and hence

$$\lim_{n\to\infty} Ty_n = q. \tag{2.22}$$

Now we show that $\lim_{n\to\infty} By_n = q$.

Since $\frac{1}{2s} \min\{d(Sx_n, Ax_n), d(Ty_n, By_n)\} \leq \max\{d(Sx_n, Ty_n), d(Ax_n, By_n)\}$.

From the inequality (2.3), we have

$$s^4 d(Ax_n, By_n) \leq \beta(M(x_n, y_n)) M(x_n, y_n) \tag{2.23}$$

where
$M(x_n,y_n)=\max\{d(Sx_n,Ty_n),d(Sx_n,Ax_n),d(Ty_n,By_n),\frac{d(Sx_n,By_n)}{2s},\frac{d(Ty_n,Ax_n)}{2s},$
$\frac{d(Sx_n,Ax_n)d(Ty_n,By_n)}{1+d(Sx_n,Ty_n)+d(Ax_n,By_n)},\frac{d(Sx_n,By_n)d(Ty_n,Ax_n)}{1+s^4[d(Sx_n,Ty_n)+d(Ax_n,By_n)]}\}.$
By taking limit superior as $n\to\infty$ on $M(x_n,y_n)$, and using (2.21) and (2.22), we obtain

$$\begin{cases}\limsup_{n\to\infty}M(x_n,y_n)=\max\{0,0,\limsup_{n\to\infty}d(Ax_n,By_n),\frac{\limsup_{n\to\infty}d(Ax_n,By_n)}{2s},0,0,0\}\\ \qquad\qquad\qquad\quad=\limsup_{n\to\infty}d(Ax_n,By_n).\end{cases}\tag{2.24}$$

On taking limit superior as $n\to\infty$ in (2.23), and using (2.24), we get
$s^4\limsup_{n\to\infty}d(Ax_n,By_n)=\limsup_{n\to\infty}[\beta(M(x_n,y_n))M(x_n,y_n)]$
$=\limsup_{n\to\infty}\beta(M(x_n,y_n))\limsup_{n\to\infty}M(x_n,y_n)$
$=\limsup_{n\to\infty}\beta(M(x_n,y_n))\limsup_{n\to\infty}d(Ax_n,By_n).$
Therefore
$\frac{1}{s}\leq 1\leq\limsup_{n\to\infty}\beta(M(x_n,y_n))\leq\frac{1}{s^5}\leq\frac{1}{s}$ which implies that
$\limsup_{n\to\infty}\beta(M(x_n,y_n))=\frac{1}{s}.$
Since $\beta\in\mathfrak{F}$, we have $\lim_{n\to\infty}M(x_n,y_n)=0$. i.e., $\limsup_{n\to\infty}d(Ax_n,By_n)=0$.
Therefore

$$\lim_{n\to\infty}d(Ax_n,By_n)=0.\tag{2.25}$$

By the b-triangular inequality, we have

$$d(q,By_n)\leq s[d(q,Ax_n)+d(Ax_n,By_n)].\tag{2.26}$$

On taking limits as $n\to\infty$ in (2.26), and using (2.21) and (2.25), we get
$\lim_{n\to\infty}d(q,By_n)\leq s[\lim_{n\to\infty}d(q,Ax_n)+\lim_{n\to\infty}d(Ax_n,By_n)]=0.$
Therefore $\lim_{n\to\infty}d(q,By_n)=0$.
Case (i): Assume that $T(X)$ is a b-closed subset of X.
In this case $q\in T(X)$, we can choose $r\in X$ such that $Tr=q$.
We now prove that $Br=q$. Suppose that $d(Br,q)>0$.
Since $\frac{1}{2s}\min\{d(Sx_n,Ax_n),d(Tr,Br)\}\leq\max\{d(Sx_n,Tr),d(Ax_n,Br)\}$
From the inequality (2.3), we have

$$s^4d(Ax_n,Br)\leq\beta(M(x_n,r))M(x_n,r)\tag{2.27}$$

where
$M(x_n,r)=\max\{d(Sx_n,Tr),d(Sx_n,Ax_n),d(Tr,Br),\frac{d(Sx_n,Br)}{2s},\frac{d(Tr,Ax_n)}{2s},$
$\frac{d(Sx_n,Ax_n)d(Tr,Br)}{1+d(Sx_n,Tr)+d(Ax_n,Br)},\frac{d(Sx_n,Br)d(Tr,Ax_n)}{1+s^4[d(Sx_n,Tr)+d(Ax_n,Br)]}\}.$
By taking limit superior as $n\to\infty$ on $M(x_n,r)$, and using (2.21), (2.22) and Lemma 1.4, we obtain

$$\limsup_{n\to\infty}M(x_n,r)\leq\max\{0,0,d(q,Br),\frac{d(q,Br)}{2},0,0,0\}=d(q,Br).\tag{2.28}$$

We have
$d(Br,q)\leq s[d(Br,Sx_n)+d(Sx_n,q)]$
$=2s^2[\frac{d(Br,Sx_n)}{2s}]+sd(Sx_n,q)\leq 2s^2M(x_n,r)+sd(Sx_n,q).$
On taking limit inferior as $n\to\infty$, we get

Therefore $\frac{1}{2s^2}d(Br,q) \leq \liminf_{n\to\infty} M(x_n,r)$.
Taking limit superior as $n \to \infty$ in (2.27) and using (2.28) and Lemma 1.4, we have

$$\begin{aligned} s^4(\tfrac{1}{s}d(q,Br)) &\leq s^4 \limsup_{n\to\infty} d(Ax_n, Br) \\ &= \limsup_{n\to\infty}[\beta(M(x_n,r))M(x_n,r)] \\ &= \limsup_{n\to\infty} \beta(M(x_n,r)) \limsup_{n\to\infty} M(x_n,r) \\ &\leq \limsup_{n\to\infty} \beta(M(x_n,r))d(q,Br). \end{aligned}$$

Therefore
$\frac{1}{s} \leq \limsup_{n\to\infty} \beta(M(x_n,r)) \leq \frac{1}{s}$ which implies that $\limsup_{n\to\infty} \beta(M(x_n,r)) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} M(x_n,r) = 0$.
Therefore $\frac{1}{2s^2}d(Br,q) \leq \lim_{n\to\infty} M(x_n,r) = 0$.
Thus $Br = q$. Hence $Br = Tr = q$, so that q is a coincidence point of B and T.
Since $B(X) \subseteq S(X)$, we have $q \in S(X)$, there exists $z \in X$ such that $Sz = q = Br$.
Now we show that $Az = q$. Suppose $Az \neq q$.
Since
$\frac{1}{2s}\min\{d(Sz,Az), d(Tr,Br)\} \leq \max\{d(Sz,Tr), d(Az,Br)\}$
From the inequality (2.3), we have

$$s^4 d(Az,q) = s^4 d(Az,Br) \leq \beta(M(z,r))M(z,r) \tag{2.29}$$

where

$$\begin{aligned} M(z,r) &= \max\{d(Sz,Tr), d(Sz,Az), d(Tr,Br), \tfrac{d(Sz,Br)}{2s}, \tfrac{d(Tr,Az)}{2s}, \\ &\qquad \tfrac{d(Sz,Az)d(Tr,Br)}{1+d(Sz,Tr)+d(Az,Br)}, \tfrac{d(Sz,Br)d(Tr,Az)}{1+s^4[d(Sz,Tr)+d(Az,Br)]}\} \\ &= \max\{0, d(q,Az), 0, 0, \tfrac{d(q,Az)}{2s}, 0, 0\} = d(q,Az). \end{aligned}$$

From the inequality (2.29), we have
$s^4 d(Az,q) \leq \beta(d(Az,q)d(Az,q)) < d(Az,q)$,
a contradiction.
Therefore $Az = Sz = q$ so that z is a coincidence point of A and S.
Since the pairs (A,S) and (B,T) are weakly compatible, we have $Aq = Sq$ and $Bq = Tq$.
Therefore q is also a coincidence point of the pairs (A,S) and (B,T).
We now show that q is a common fixed point of A, B, S and T.
Suppose $Aq \neq q$.
Since $\frac{1}{2s}\min\{d(Sq,Aq), d(Tr,Br)\} \leq \max\{d(Sq,Tr), d(Aq,Br)\}$,
from the inequality (2.3), we have

$$s^4 d(Aq,q) = s^4 d(Aq,Br) \leq \beta(M(q,r))M(q,r) \tag{2.30}$$

where

$$\begin{aligned} M(q,r) &= \max\{d(Sq,Tr), d(Sq,Aq), d(Tr,Br), \tfrac{d(Sq,Br)}{2s}, \tfrac{d(Tr,Aq)}{2s}, \\ &\qquad \tfrac{d(Sq,Aq)d(Tr,Br)}{1+d(Sq,Tr)+d(Aq,Br)}, \tfrac{d(Sq,Br)d(Tr,Aq)}{1+s^4[d(Sq,Tr)+d(Aq,Br)]}\} \\ &= \max\{d(Aq,q), 0, 0, \tfrac{d(Aq,q)}{2s}, \tfrac{d(Aq,q)}{2s}, 0, 0\} = d(Aq,q). \end{aligned}$$

Now, from the inequality (2.30), we have
$s^4 d(Aq,q) \leq \beta(d(Aq,q)d(Aq,q)) < d(Aq,q)$,
a contradiction.
Therefore $Aq = Sq = q$ so that q is a common fixed point of A and S.

By Proposition 2.1, we get that q is a unique common fixed point of A, B, S and T.
Case (ii): Suppose $A(X)$ is b-closed.
In this case, we have $q \in A(X)$ and since $A(X) \subseteq T(X)$, we choose $r \in X$ such that $q = Tr$.
The proof follows as in Case (i).
Case (iii): Suppose $S(X)$ is b-closed.
We follow the argument similar as Case (i) and we get conclusion.
Case (iv): Suppose $B(X)$ is b-closed. As in Case (ii), we get the conclusion.

For the case of (B, T) satisfies the b-(E.A)-property, we follow the argument similar to the case (A, S) satisfies the b-(E.A)-property. $\square$

3. Corollaries and examples

In this section we draw some corollaries from our main results and provide examples in support of our results.

If we take $A = B = f$ and $S = T = g$ in Theorem 2.3 and Theorem 2.4, we get Corollary 3.1 and Corollary 3.2, respectively.

Corollary 3.1. *Let (X, d) be a b-metric space and f and g be selfmaps of X. Assume that there exists $\beta \in \mathfrak{F}$ such that*

$$\begin{aligned} \tfrac{1}{2s} \min\{d(fx, gx), d(fy, gy)\} &\leq \max\{d(gx, gy), d(fx, fy)\} \\ &\implies s^4 d(fx, fy) \leq \beta(M(x, y))M(x, y) \end{aligned} \tag{3.1}$$

where
$$M(x, y) = \max\{d(gx, gy), d(gx, fx), d(gy, fy), \tfrac{d(gx,fy)}{2s}, \tfrac{d(gy,fx)}{2s}, \tfrac{d(gx,fx)d(gy,fy)}{1+d(gx,gy)+d(fx,fy)}, \tfrac{d(gx,fy)d(gy,fx)}{1+s^4[d(gx,gy)+d(fx,fy)]}\},$$
for all $x, y \in X$. If $f(X) \subseteq g(X)$, the pair (f, g) is compatible and f or g is b-continuous then f and g have a unique common fixed point in X.

Corollary 3.2. *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $f, g : X \to X$ be selfmaps of X and satisfy $f(X) \subseteq g(X)$ and the inequality (3.1). Suppose that the pair (f, g) satisfies the b-(E.A)-property and that one of the subspace $f(X)$ and $g(X)$ is b-closed in X. Then the pairs (f, g) have a point of coincidence in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.*

The following is an example in support of Theorem 2.3.

Example 3.1. Let $X = \mathbb{R}^+$ and let $d : X \times X \to \mathbb{R}^+$ defined by
$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in [0, 1], \\ 5 + \frac{1}{x+y} & \text{if } x, y \in (1, \infty), \\ \frac{27}{10} & \text{otherwise.} \end{cases}$$
The b-metric conditions (b_1) and (b_2) are trivially hold for this example.
Let us now check (b_3).
For this purpose we consider the following nontrivial case.
Let $y \in [0, 1]$ and $x, z \in (1, \infty)$.
Then $d(x, z) = 5 + \frac{1}{x+z}, d(x, y) = \frac{27}{10}, d(y, z) = \frac{27}{10}$.
We have
$2 \leq x + z \implies \frac{1}{x+z} \leq \frac{1}{2}$ so that $5 + \frac{1}{x+z} \leq 5 + \frac{1}{2} < \frac{489}{480}(\frac{27}{5})$.
Therefore $d(x, z) = 5 + \frac{1}{x+z} < \frac{489}{480}(\frac{27}{10} + \frac{27}{10}) = s[d(x, y) + d(y, z)]$ where $s = \frac{489}{480}$

so that (b_3) holds.
Thus d is a b-metric with $s = \frac{489}{480}$.
Clearly this d is complete so that (X, d) is a complete b-metric space.

Here we observe that when $x = \frac{101}{100}, z = \frac{102}{100} \in (1, \infty)$ and $y \in [0, 1)$, we have $d(x, z) = \frac{1115}{203} \nleq \frac{27}{5} = d(x, y) + d(y, z)$ so that d is not a metric.
We define $A, B, S, T : X \to X$ by

$$A(x) = 1 \text{ if } x \in [0, \infty),\ B(x) = \begin{cases} x^2 + 2 & \text{if } x \in [0, 1) \\ \frac{x^2+1}{2} & \text{if } x \in [1, \infty), \end{cases}$$

$$S(x) = \begin{cases} x + 2 & \text{if } x \in [0, 1) \\ \frac{x+1}{2} & \text{if } x \in [1, \infty), \end{cases} \text{ and } T(x) = \begin{cases} 3x^2 + 4 & \text{if } x \in [0, 1) \\ \frac{x(x+2)}{3} & \text{if } x \in [1, \infty). \end{cases}$$

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.
Here A is b-continuous.
We choose a sequence $\{x_n\}$ with $\{x_n\} = 1 + \frac{1}{2n}, n \geq 1$, we have
$ASx_n = A(\frac{1+\frac{1}{2n}+1}{2}) = 1$ and $SAx_n = S1 = 1$.
Therefore $\lim\limits_{n\to\infty} d(ASx_n, SAx_n) = 0$ so that the pair (A, S) is compatible and clearly the pair (B, T) is weakly compatible.
We define $\beta : [0, \infty) \to [0, \frac{1}{s})$ by $\beta(t) = \frac{480}{489} e^{\frac{-t}{100}}$. Then we have $\beta \in \mathfrak{F}$.

Case (i): $x, y \in [0, 1)$.
$d(Ax, By) = \frac{27}{10}, d(Sx, Ty) = 5 + \frac{1}{x+y}, d(Ty, By) = 5 + \frac{1}{x+y}, d(Sx, Ax) = \frac{27}{10}$,
$d(Ax, Ty) = \frac{27}{10}, d(Sx, By) = 5 + \frac{1}{x+y}$ and

$$\begin{aligned} M(x, y) &= \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \\ &= \max\{5 + \tfrac{1}{x+y}, \tfrac{27}{10}, 5 + \tfrac{1}{x+y}, (\tfrac{240}{489})(5 + \tfrac{1}{x+y}), (\tfrac{240}{489})(\tfrac{27}{10}), \tfrac{(\frac{27}{10})(5+\frac{1}{x+y})}{1+5+\frac{1}{x+y}+\frac{27}{10}}, \\ &\qquad \tfrac{(5+\frac{1}{x+y})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\} \\ &= 5 + \tfrac{1}{x+y}. \end{aligned}$$

Since

$$\begin{aligned} \tfrac{1}{2s} \min\{d(Sx, Ax), d(Ty, By)\} &= \tfrac{240}{489} \min\{\tfrac{27}{10}, 5 + \tfrac{1}{x+y}\} \\ &= (\tfrac{240}{489})(\tfrac{27}{10}) \\ &\leq \max\{5 + \tfrac{1}{x+y}, \tfrac{27}{10}\} \\ &= \max\{d(Sx, Ty), d(Ax, By)\}. \end{aligned}$$

Now we consider

$$s^4 d(Ax, By) = (\tfrac{489}{480})^4(\tfrac{27}{10}) \leq \tfrac{480}{489} e^{\frac{-(5+\frac{1}{x+y})}{100}} 5 + \tfrac{1}{x+y} = \beta(M(x, y))M(x, y).$$

Case (ii): $x, y \in (1, \infty)$.
$d(Ax, By) = \frac{27}{10}, d(Sx, Ty) = 5 + \frac{1}{x+y}, d(Ty, By) = 5 + \frac{1}{x+y}, d(Sx, Ax) = \frac{27}{10}$,
$d(Ax, Ty) = \frac{27}{10}, d(Sx, By) = 5 + \frac{1}{x+y}$ and

$$\begin{aligned} M(x, y) &= \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \\ &= \max\{5 + \tfrac{1}{x+y}, \tfrac{27}{10}, 5 + \tfrac{1}{x+y}, (\tfrac{240}{489})(5 + \tfrac{1}{x+y}), (\tfrac{240}{489})(\tfrac{27}{10}), \tfrac{(\frac{27}{10})(5+\frac{1}{x+y})}{1+5+\frac{1}{x+y}+\frac{27}{10}}, \\ &\qquad \tfrac{(5+\frac{1}{x+y})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\} \\ &= 5 + \tfrac{1}{x+y}. \end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax),d(Ty,By)\} &= \tfrac{240}{489}\min\{\tfrac{27}{10},5+\tfrac{1}{x+y}\}\\ &= (\tfrac{240}{489})(\tfrac{27}{10})\\ &\leq \max\{5+\tfrac{1}{x+y},\tfrac{27}{10}\}\\ &= \max\{d(Sx,Ty),d(Ax,By)\}.\end{aligned}$$

Now we consider

$s^4d(Ax,By)=(\frac{489}{480})^4(\frac{27}{10})\leq\frac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}5+\frac{1}{x+y}=\beta(M(x,y))M(x,y).$

Case (iii): $x\in[0,1),y\in(1,\infty)$.

$d(Ax,By)=\frac{27}{10},d(Sx,Ty)=5+\frac{1}{x+y},d(Ty,By)=5+\frac{1}{x+y},d(Sx,Ax)=\frac{27}{10}$, $d(Ax,Ty)=\frac{27}{10},d(Sx,By)=5+\frac{1}{x+y}$ and

$$\begin{aligned}M(x,y)&=\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\tfrac{d(Sx,By)}{2s},\tfrac{d(Ty,Ax)}{2s},\\&\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)},\tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\&=\max\{5+\tfrac{1}{x+y},\tfrac{27}{10},5+\tfrac{1}{x+y},(\tfrac{240}{489})(5+\tfrac{1}{x+y}),(\tfrac{240}{489})(\tfrac{27}{10}),\tfrac{(\frac{27}{10})(5+\frac{1}{x+y})}{1+5+\frac{1}{x+y}+\frac{27}{10}},\\&\qquad \tfrac{(5+\frac{1}{x+y})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\}\\&=5+\tfrac{1}{x+y}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax),d(Ty,By)\} &= \tfrac{240}{489}\min\{\tfrac{27}{10},5+\tfrac{1}{x+y}\}\\ &= (\tfrac{240}{489})(\tfrac{27}{10})\\ &\leq \max\{5+\tfrac{1}{x+y},\tfrac{27}{10}\}\\ &= \max\{d(Sx,Ty),d(Ax,By)\}.\end{aligned}$$

Now we consider

$s^4d(Ax,By)=(\frac{489}{480})^4(\frac{27}{10})\leq\frac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}(5+\frac{1}{x+y})=\beta(M(x,y))M(x,y).$

Case (iv): $x\in(1,\infty),y\in[0,1)$.

$d(Ax,By)=\frac{27}{10},d(Sx,Ty)=5+\frac{1}{x+y},d(Ty,By)=5+\frac{1}{x+y},d(Sx,Ax)=\frac{27}{10}$, $d(Ax,Ty)=\frac{27}{10},d(Sx,By)=5+\frac{1}{x+y}$ and

$$\begin{aligned}M(x,y)&=\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\tfrac{d(Sx,By)}{2s},\tfrac{d(Ty,Ax)}{2s},\\&\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)},\tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\&=\max\{5+\tfrac{1}{x+y},\tfrac{27}{10},5+\tfrac{1}{x+y},(\tfrac{240}{489})(5+\tfrac{1}{x+y}),(\tfrac{240}{489})(\tfrac{27}{10}),\tfrac{(\frac{27}{10})(5+\frac{1}{x+y})}{1+5+\frac{1}{x+y}+\frac{27}{10}},\\&\qquad \tfrac{(5+\frac{1}{x+y})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\}\\&=5+\tfrac{1}{x+y}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax),d(Ty,By)\} &= \tfrac{240}{489}\min\{\tfrac{27}{10},5+\tfrac{1}{x+y}\}\\ &= (\tfrac{240}{489})(\tfrac{27}{10})\\ &\leq \max\{5+\tfrac{1}{x+y},\tfrac{27}{10}\}\\ &= \max\{d(Sx,Ty),d(Ax,By)\}.\end{aligned}$$

Now we consider

$s^4d(Ax,By)=(\frac{489}{480})^4(\frac{27}{10})\leq\frac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}(5+\frac{1}{x+y})=\beta(M(x,y))M(x,y).$

Case (v): $x=1,y\in[0,1)$.

$d(Ax,By)=\frac{27}{10},d(Sx,Ty)=\frac{27}{10},d(Ty,By)=5+\frac{1}{x+y},d(Sx,Ax)=0$, $d(Ax,Ty)=\frac{27}{10},d(Sx,By)=\frac{27}{10}$ and

$M(x,y)=\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\frac{d(Sx,By)}{2s},\frac{d(Ty,Ax)}{2s},$

$$\frac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \frac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}$$
$$= \max\{\tfrac{27}{10}, 0, 5+\tfrac{1}{x+y}, (\tfrac{240}{489})(\tfrac{27}{10}), (\tfrac{240}{489})(\tfrac{27}{10}), 0, \frac{(\frac{27}{10})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\}$$
$$= 5+\tfrac{1}{x+y}.$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{240}{489}\min\{0, 5+\tfrac{1}{x+y}\}\\ &= 0 \le \max\{\tfrac{27}{10}, \tfrac{27}{10}\}\\ &= \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider

$$s^4 d(Ax,By) = (\tfrac{489}{480})^4(\tfrac{27}{10}) \le \tfrac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}(5+\tfrac{1}{x+y}) = \beta(M(x,y))M(x,y).$$

Case (vi): $x=1, y\in(1,\infty)$.

$d(Ax,By)=\frac{27}{10}, d(Sx,Ty)=\frac{27}{10}, d(Ty,By)=5+\frac{1}{x+y}, d(Sx,Ax)=0,$
$d(Ax,Ty)=\frac{27}{10}, d(Sx,By)=\frac{27}{10}$ and

$$M(x,y) = \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s},$$
$$\frac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \frac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}$$
$$= \max\{\tfrac{27}{10}, 0, 5+\tfrac{1}{x+y}, (\tfrac{240}{489})(\tfrac{27}{10}), (\tfrac{240}{489})(\tfrac{27}{10}), 0, \frac{(\frac{27}{10})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\}$$
$$= 5+\tfrac{1}{x+y}.$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{240}{489}\min\{0, 5+\tfrac{1}{x+y}\}\\ &= 0 \le \max\{\tfrac{27}{10}, \tfrac{27}{10}\}\\ &= \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider

$$s^4 d(Ax,By) = (\tfrac{489}{480})^4(\tfrac{27}{10}) \le \tfrac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}(5+\tfrac{1}{x+y}) = \beta(M(x,y))M(x,y).$$

Case (vii): $x\in[0,1), y=1$.

Here $d(Ax,By)=0$. Clearly the inequality (2.3) holds in this case.

Case (viii): $x\in[0,1), y=1$.

Here $d(Ax,By)=0$. In this case the inequality (2.3) holds clearly.

From all the above four cases, A, B, S and T are Geraghty-Suzuki type contraction maps. Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.3 and 1 is the unique common fixed point of A, B, S and T.

The following is an example in support of Theorem 2.4.

Example 3.2. Let $X=[0,1]$ and let $d: X\times X\to\mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x=y,\\ \frac{11}{15} & \text{if } x,y\in[0,\frac{2}{3}),\\ \frac{23}{25}+\frac{x+y}{26} & \text{if } x,y\in[\frac{2}{3},1],\\ \frac{121}{250} & \text{otherwise.}\end{cases}$$

The conditions (b_1) and (b_2) are trivially hold.

We now verify condition (b_3) for nontrivial case.

Let $y\in[0,\frac{2}{3})$ and $x,z\in[\frac{2}{3},1]$.

Then $d(x,z)=\frac{23}{25}+\frac{x+z}{26}, d(x,y)=\frac{121}{250}, d(y,z)=\frac{121}{250}$.

We have

$x+z\le 2 \implies \frac{x+z}{26}\le\frac{1}{13}$ so that $\frac{23}{25}+\frac{x+z}{26}\le\frac{23}{25}+\frac{1}{13}<\frac{51}{49}(\frac{121}{125})$.

Therefore $d(x,z)=\frac{23}{25}+\frac{x+z}{26}<\frac{51}{49}(\frac{121}{250}+\frac{121}{250})=s[d(x,y)+d(y,z)]$ where $s=\frac{51}{49}$.

The other cases also trivially hold with $s=\frac{51}{49}$ so that (b_3) holds and d is a b-metric.

Clearly this metric d is complete so that (X,d) is a complete b-metric space.

Here we observe that when $x = \frac{9}{10}, z = 1 \in [\frac{2}{3}, 1]$ and $y \in [0, \frac{2}{3})$, we have $d(x,z) = \frac{1291}{1300} \nleq \frac{121}{125} = d(x,y) + d(y,z)$ so that d is not a metric.
We define $A, B, S, T : X \to X$ by

$A(x) = \frac{2}{3}$ if $x \in [0,1]$, $B(x) = \begin{cases} \frac{1}{3} & \text{if } x \in [0, \frac{2}{3}) \\ 1 - \frac{x}{2} & \text{if } x \in [\frac{2}{3}, 1], \end{cases}$

$S(x) = \begin{cases} x & \text{if } x \in [0, \frac{2}{3}) \\ \frac{4}{3} - x & \text{if } x \in [\frac{2}{3}, 1], \end{cases}$ and $T(x) = \begin{cases} \frac{1}{4} & \text{if } x \in [0, \frac{2}{3}) \\ \frac{4}{3} - x & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. $A(X) = \{\frac{2}{3}\}$ is b-closed.
We choose a sequence $\{x_n\}$ with $\{x_n\} = \frac{2}{3} + \frac{1}{2n}, n \geq 2$ with $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{2}{3}$, hence the pair (A, S) satisfies the b-(E.A)-property.
Clearly the pairs (A, S) and (B, T) are weakly compatible.
We define $\beta : [0, \infty) \to [0, \frac{1}{s})$ by $\beta(t) = \frac{49}{51} e^{\frac{-t}{100}}$.
Then we have $\beta \in \mathfrak{F}$.

Case (i): $x, y \in [0, \frac{2}{3})$.

$d(Ax, By) = \frac{121}{250}, d(Sx, Ty) = \frac{11}{15}, d(Ty, By) = \frac{11}{15}, d(Sx, Ax) = \frac{121}{250}$,
$d(Ax, Ty) = \frac{121}{250}, d(Sx, By) = \frac{11}{15}$ and

$$\begin{aligned} M(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \\ &= \max\{\tfrac{11}{15}, \tfrac{121}{250}, \tfrac{11}{15}, (\tfrac{49}{102})(\tfrac{11}{15}), (\tfrac{49}{102})(\tfrac{121}{250}), \tfrac{(\frac{121}{250})(\frac{11}{15})}{1+\frac{11}{15}+\frac{121}{250}}, \tfrac{(\frac{11}{15})(\frac{121}{250})}{1+(\frac{51}{49})^4(\frac{11}{15}+\frac{121}{250})}\} = \tfrac{11}{15}. \end{aligned}$$

Since

$$\begin{aligned} \tfrac{1}{2s} \min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{49}{102} \min\{\tfrac{121}{250}, \tfrac{11}{15}\} \\ &= (\tfrac{49}{102})(\tfrac{121}{250}) \\ &\leq \max\{\tfrac{11}{15}, \tfrac{121}{250}\} \\ &= \max\{d(Sx,Ty), d(Ax,By)\}. \end{aligned}$$

Now we consider

$s^4 d(Ax, By) = (\frac{51}{49})^4 (\frac{121}{250}) \leq \frac{49}{51} e^{\frac{-(\frac{11}{15})}{100}} \frac{11}{15} = \beta(M(x,y))M(x,y)$.

Case (ii): $x, y \in (\frac{2}{3}, 1]$.

$d(Ax, By) = \frac{121}{250}, d(Sx, Ty) = \frac{11}{15}, d(Ty, By) = \frac{11}{15}, d(Sx, Ax) = \frac{121}{250}$,
$d(Ax, Ty) = \frac{121}{250}, d(Sx, By) = \frac{11}{15}$ and

$$\begin{aligned} M(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \\ &= \max\{\tfrac{11}{15}, \tfrac{121}{250}, \tfrac{11}{15}, (\tfrac{49}{102})(\tfrac{11}{15}), (\tfrac{49}{102})(\tfrac{121}{250}), \tfrac{(\frac{121}{250})(\frac{11}{15})}{1+\frac{11}{15}+\frac{121}{250}}, \tfrac{(\frac{11}{15})(\frac{121}{250})}{1+(\frac{51}{49})^4(\frac{11}{15}+\frac{121}{250})}\} = \tfrac{11}{15}. \end{aligned}$$

Since

$$\begin{aligned} \tfrac{1}{2s} \min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{49}{102} \min\{\tfrac{121}{250}, \tfrac{11}{15}\} \\ &= (\tfrac{49}{102})(\tfrac{121}{250}) \\ &\leq \max\{\tfrac{11}{15}, \tfrac{121}{250}\} \\ &= \max\{d(Sx,Ty), d(Ax,By)\}. \end{aligned}$$

Now we consider $s^4 d(Ax, By) = (\frac{51}{49})^4 (\frac{121}{250}) \leq \frac{49}{51} e^{\frac{-(\frac{11}{15})}{100}} \frac{11}{15} = \beta(M(x,y))M(x,y)$.

Case (iii): $x \in [0, \frac{2}{3}), y \in (\frac{2}{3}, 1]$.

$d(Ax, By) = \frac{121}{250}, d(Sx, Ty) = \frac{11}{15}, d(Ty, By) = \frac{11}{15}, d(Sx, Ax) = \frac{121}{250}$,
$d(Ax, Ty) = \frac{121}{250}, d(Sx, By) = \frac{11}{15}$ and

$$\begin{aligned} M(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \end{aligned}$$

$$= \max\{\tfrac{11}{15}, \tfrac{121}{250}, \tfrac{11}{15}, (\tfrac{49}{102})(\tfrac{11}{15}), (\tfrac{49}{102})(\tfrac{121}{250}), \tfrac{(\frac{121}{250})(\frac{11}{15})}{1+\frac{11}{15}+\frac{121}{250}}, \tfrac{(\frac{11}{15})(\frac{121}{250})}{(1+(\frac{51}{49})^4\frac{11}{15}+\frac{121}{250})}\} = \tfrac{11}{15}.$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{49}{102}\min\{\tfrac{121}{250}, \tfrac{11}{15}\}\\ &= (\tfrac{49}{102})(\tfrac{121}{250})\\ &\leq \max\{\tfrac{11}{15}, \tfrac{121}{250}\}\\ &= \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider

$$s^4 d(Ax,By) = (\tfrac{51}{49})^4(\tfrac{121}{250}) \leq \tfrac{49}{51}e^{\frac{-(\frac{11}{15})}{100}}\tfrac{11}{15} = \beta(M(x,y))M(x,y).$$

Case (iv): $x \in (\frac{2}{3}, 1], y \in [0, \frac{2}{3})$.

$d(Ax,By) = \frac{121}{250}, d(Sx,Ty) = \frac{11}{15}, d(Ty,By) = \frac{11}{15}, d(Sx,Ax) = \frac{121}{250}$,
$d(Ax,Ty) = \frac{121}{250}, d(Sx,By) = \frac{11}{15}$ and

$$\begin{aligned}M(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s},\\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\ &= \max\{\tfrac{11}{15}, \tfrac{121}{250}, \tfrac{11}{15}, (\tfrac{49}{102})(\tfrac{11}{15}), (\tfrac{49}{102})(\tfrac{121}{250}), \tfrac{(\frac{121}{250})(\frac{11}{15})}{1+\frac{11}{15}+\frac{121}{250}}, \tfrac{(\frac{11}{15})(\frac{121}{250})}{1+(\frac{51}{49})^4(\frac{11}{15}+\frac{121}{250})}\}\\ &= \tfrac{11}{15}.\end{aligned}$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{49}{102}\min\{\tfrac{121}{250}, \tfrac{11}{15}\}\\ &= (\tfrac{49}{102})(\tfrac{121}{250})\\ &\leq \max\{\tfrac{11}{15}, \tfrac{121}{250}\}\\ &= \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider $s^4 d(Ax,By) = (\frac{51}{49})^4(\frac{121}{250}) \leq \frac{49}{51}e^{\frac{-(\frac{11}{15})}{100}}\frac{11}{15} = \beta(M(x,y))M(x,y)$.

Case (v): $x = \frac{2}{3}, y \in [0, \frac{2}{3})$.

$d(Ax,By) = \frac{121}{250}, d(Sx,Ty) = \frac{121}{250}, d(Ty,By) = \frac{11}{15}, d(Sx,Ax) = 0$,
$d(Ax,Ty) = \frac{121}{250}, d(Sx,By) = \frac{121}{250}$ and

$$\begin{aligned}M(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s},\\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\ &= \max\{\tfrac{121}{250}, 0, \tfrac{11}{15}, (\tfrac{49}{102})(\tfrac{121}{250}), (\tfrac{49}{102})(\tfrac{121}{250}), 0, \tfrac{(\frac{121}{250})(\frac{121}{250})}{(1+(\frac{51}{49})^4\frac{11}{15}+\frac{121}{250})}\} = \tfrac{11}{15}.\end{aligned}$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{240}{489}\min\{0, \tfrac{11}{15}\}\\ &= 0 \leq \max\{\tfrac{121}{250}, \tfrac{121}{250}\}\\ &\quad = \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider

$$s^4 d(Ax,By) = (\tfrac{51}{49})^4(\tfrac{121}{250}) \leq \tfrac{49}{51}e^{\frac{-(\frac{11}{15})}{100}}\tfrac{11}{15} = \beta(M(x,y))M(x,y).$$

Case (vi): $x = \frac{2}{3}, y \in (\frac{2}{3}, 1]$.

$d(Ax,By) = \frac{121}{250}, d(Sx,Ty) = \frac{121}{250}, d(Ty,By) = \frac{11}{15}, d(Sx,Ax) = 0$,
$d(Ax,Ty) = \frac{121}{250}, d(Sx,By) = \frac{121}{250}$ and

$$\begin{aligned}M(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s},\\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\ &= \max\{\tfrac{121}{250}, 0, \tfrac{11}{15}, (\tfrac{49}{102})(\tfrac{121}{250}), (\tfrac{49}{102})(\tfrac{121}{250}), 0, \tfrac{(\frac{121}{250})(\frac{121}{250})}{1+(\frac{51}{49})^4(\frac{11}{15}+\frac{121}{250})}\} = \tfrac{11}{15}.\end{aligned}$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{240}{489}\min\{0, \tfrac{11}{15}\}\\ &= 0 \leq \max\{\tfrac{121}{250}, \tfrac{121}{250}\}\\ &\quad = \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider
$s^4 d(Ax, By) = (\frac{51}{49})^4(\frac{121}{250}) \leq \frac{49}{51} e^{\frac{-(\frac{11}{15})}{100}} \frac{11}{15} = \beta(M(x,y))M(x,y)$.
Case (vii): $x \in [0, \frac{2}{3}), y = \frac{2}{3}$.
Here $d(Ax, By) = 0$. Clearly the inequality (2.3) holds in this case.
Case (viii): $x \in [0, \frac{2}{3}), y = \frac{2}{3}$.
Here $d(Ax, By) = 0$. In this case the inequality (2.3) holds clearly.
From all the above four cases, A, B, S and T are Geraghty-Suzuki type contraction maps. Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.4 and $\frac{2}{3}$ is the unique common fixed point of A, B, S and T.

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EXISTENCE OF SOLUTION FOR A SYSTEM OF COUPLED FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract. This paper deals with the existence and uniqueness of solutions for a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. The existence results are obtained by using Leray-Schauder nonlinear alternative and Banach contraction principle. An illustrative example is presented at the end of the paper to illustrate the validity of our results.

1. Introduction

In this paper, we are interested in the existence of solutions for the nonlinear fractional differential equations

$$\begin{cases} {}^cD^{\alpha}u(t) = f(t,u(t),v(t)), & t\in[0,1],\ 2<\alpha\le 3, \\ {}^cD^{\beta}v(t) = g(t,u(t),v(t)), & t\in[0,1],\ 2<\beta\le 3, \end{cases} \tag{1.1}$$

subject to three-point coupled boundary conditions

$$\begin{cases} \lambda u(0)+\gamma u(1) = v(\eta),\ \lambda v(0)+\gamma v(1) = u(\eta), \\ u(0) = \int_0^{\eta} v(s)\,ds,\ u(0) = \int_0^{\eta} v(s)\,ds, \\ \lambda {}^C D^P u(0) + \gamma {}^C D^P u(1) = v(\eta),\ 1<p\le 2 \\ \lambda {}^C D^P v(0) + \gamma {}^C D^P v(1) = u(\eta),\ 1<p\le 2 \end{cases} \tag{1.2}$$

where $\gamma,\ \lambda\in\mathbb{R}^+,\ f,\ g\in C\left([0,1]\times\mathbb{R}^2,\mathbb{R}\right)$ and ${}^cD^{\alpha},\ {}^cD^{\beta}$ denote the Caputo fractional derivatives of order α and β respectively.

The concept of fractional calculus has played an important role in improving the work based on integer-order (classical) calculus in several diverse disciplines of science and engineering and the details of its basic notions, results and methods

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can be found in the texts ([2, 17]) and papers ([1, 21, 23]). The nonlocal nature of a fractional order differential operator, which take into account hereditary properties of various material and processes, has helped to improve the mathematical modeling of many natural phenomena and physical processes, see for example ([17, 22]). The increasing interest of fractional differential equations and inclusions are motivated by their applications in various fields of science such as physics chemistry, biology, economics, fluid mechanics, control theory, etc, we refer the reader to ([3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 19, 20, 27]) and the references therein.

Coupled systems of fractional-order differential equations constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion [25], disease models [12], synchronization of chaotic systems [24], etc. For some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of papers ([10, 15, 16, 26, 28, 29]).

The goal of this paper is to establish the existence and uniqueness results for the nonlocal boundary value problem (1.1) – (1.2) by using some well-known tools of fixed point theory such as Banach contraction principle and Leray-Schauder nonlinear alternative. The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel, for more details; see [17]. Section 3, deals with main results and we give an example to illustrate our results.

2. Preliminaries

In this section, we introduce some definitions and lemmas, see ([17, 18]).

Definition 2.1. *Let $\alpha > 0,\ n-1 < \alpha < n,\ n = [\alpha] + 1$ and $u \in C([0,\infty), \mathbb{R})$. The Caputo derivative of fractional order α for the function u is defined by*

$$^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds,$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2.2. *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $u : (0,\infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler Gamma function, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.1. [18]. *Let $\alpha > 0,\ n-1 < \alpha < n$ and the function $g : [0,T] \to \mathbb{R}$ be continuous for each $T > 0$. Then, the general solution of the fractional differential equation $^{c}D^{\alpha}g(t) = 0$ is given by*

$$g(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Also, in [8], authors have been proved that for each $T>0$ and $u\in C([0,T])$ we have
$$I^{\alpha c}D^{\alpha}u(t)=u(t)+c_0+c_1t+\cdots+c_{n-1}t^{n-1},$$
where $c_0,c_1,...,c_{n-1}$ are real constants and $n=[\alpha]+1$.

3. Existence results

Let $X=\{u(t):u(t)\in C([0,1],\mathbb{R})\}$ endowed with the norm $\|u\|=\sup\limits_{t\in[0,1]}|u(t)|$ such that $\|u\|<\infty$. Then $(X,\|.\|)$ is a Banach space and the product space $(X\times X,\|(u,v)\|)$ is also a Banach space equipped with the norm $\|(u,v)\|=\|u\|+\|v\|$.

Throughout the paper, we let
$$M=\frac{\Gamma(3-p)}{|\gamma-\eta^{2-p}|}\neq 0,\ |\lambda+\gamma-1|\neq 0,\ \left|\gamma-\eta^2\right|\neq 0,\ \ Q=\left|2(1-\eta)(\gamma-\eta)+\eta^2|\lambda+\gamma-1|\right|\neq 0,$$
$$A(t)=|\Lambda_1(t)|=|\lambda+\gamma-1|\left(\eta^2+2(1-\eta)t\right),$$
$$B(t)=|\Lambda_2(t)|=\left(\eta^3|\lambda+\gamma-1|+3\left|\gamma-\eta^2\right|(1-\eta)\right)\left(\eta^2+2(1-\eta)t\right)-Q\left(\eta^3+3(1-\eta)t^2\right),$$
and
$$Q=2(1-\eta)(\gamma-\eta)+\eta^2(\lambda+\gamma-1)\neq 0.$$

Lemma 3.1. *Let $y\in C([0,1],\mathbb{R})$. Then the solution of the linear differential system*
$$\begin{cases} {}^cD^{\alpha}u(t)=y(t),\ {}^cD^{\beta}v(t)=h(t),\ t\in[0,1],\ 2<\alpha,\ \beta\leq 3\\ \lambda u(0)+\gamma u(1)=v(\eta),\ \lambda v(0)+\gamma v(1)=u(\eta),\\ u(0)=\int_0^{\eta}v(s)\,ds,\ \ v(0)=\int_0^{\eta}u(s)\,ds,\\ \lambda^cD^pu(0)+\gamma^cD^pu(1)={}^cD^pv(\eta),\quad 1<p\leq 2,\\ \lambda^cD^pv(0)+\gamma^cD^pv(1)={}^cD^pu(\eta),\quad 1<p\leq 2,\end{cases}\tag{3.1}$$
is equivalent to the system of integral equations
$$\begin{aligned}u(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)\,d\tau\right)ds\\ &-\frac{\Lambda_1(t)}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)\,d\tau\right)ds\\ &-\frac{\Lambda_2(t)M}{6(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}h(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right]\\ &+\frac{\Lambda_1(t)}{Q(\lambda+\gamma-1)}\left[\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}h(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right]\end{aligned}\tag{3.2}$$

and

$$\begin{aligned} v(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)\,d\tau\right) ds \\ &\quad - \frac{\Lambda_1(t)}{Q(1-\eta)} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)\,d\tau\right) ds \\ &\quad - \frac{\Lambda_2(t) M}{6(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} h(s)\,ds - \gamma \int_0^1 \frac{(1-s)^{\beta-p-1}}{\Gamma(\alpha-p)} y(s)\,ds\right] \\ &\quad + \frac{\Lambda_1(t)}{Q(\lambda+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\alpha)} y(s)\,ds\right], \end{aligned} \tag{3.3}$$

where

$$\Lambda_1(t) = (\lambda+\gamma-1)\left(\eta^2 + 2(1-\eta)t\right),$$

and

$$\Lambda_2(t) = \left(\eta^3(\lambda+\gamma-1) + 3\left(\gamma-\eta^2\right)(1-\eta)\right)\left(\eta^2+2(1-\eta)t\right) - Q\left(\eta^3+3(1-\eta)t^2\right).$$

Proof. It is well known that the solution of equations ${}^cD^\alpha u(t) = y(t)$, ${}^cD^\beta v(t) = h(t)$ can be written as

$$u(t) = I^\alpha y(t) + c_0 + c_1 t + c_2 t^2, \tag{3.4}$$

$$v(t) = I^\beta h(t) + d_0 + d_1 t + d_2 t^2, \tag{3.5}$$

where $c_0, c_1, c_2 \in \mathbb{R}$ and $d_0, d_1, d_2 \in \mathbb{R}$ are arbitrary constants.
Then, from (3.4) we have

$$u'(t) = I^{\alpha-1} y(t) + c_1 + 2c_2 t,$$

and

$${}^cD^p u(t) = I^{\alpha-p} y(t) + c_2 \frac{2t^{2-p}}{\Gamma(3-p)}, \quad 1 < p \le 2.$$

By using the three-point boundary conditions, we obtain

$$c_2 = \frac{M}{2}\left(I^{\beta-p} y(\eta) - \gamma I^{\alpha-p} y(1)\right),$$

$$\begin{aligned} c_0 &= -\frac{2\eta^2(\lambda+\gamma-1)}{2(1-\eta)Q}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} h(\tau)\,d\tau\right) ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} h(\tau)\,d\tau\right) ds \\ &\quad - \frac{\left(\eta^2\left[\eta^3(\lambda+\gamma-1) + 3\left(\gamma-\eta^2\right)(1-\eta)\right] - \eta^3 Q\right) M}{2(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)} h(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds\right] \\ &\quad + \frac{\eta^2}{Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} h(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\right], \end{aligned}$$

and

$$c_1 = \frac{-2(\lambda+\gamma-1)}{Q}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)\,d\tau\right)ds$$

$$-\frac{\left(\eta^3(\lambda+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right)M}{3Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}h(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right]$$

$$+\frac{2(1-\eta)}{Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}y(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right].$$

Substituting the values of constants c_0, c_1 and c_2 in (3.4), we get solution (3.2). Similarly, we obtain solution (3.3). The proof is complete. □

The following relations hold:

$|A(t)| \le |\beta+\gamma-1|\left(\eta^2+2(1-\eta)\right) = A_1$,

and

$|B(t)| \le \left|\left(\eta^3|\beta+\gamma-1|+3\left|\gamma-\eta^2\right|(1-\eta)\right)\left(\eta^2+2(1-\eta)\right)-Q\left(\eta^3+3(1-\eta)\right)\right| = B_1$,

For the sake of brevity, we set

$\Delta_1 = \frac{\eta^{\beta+1}}{(1-\eta)\Gamma(\beta+2)}+\frac{A_1\eta^{\beta+1}}{Q(1-\eta)\Gamma(\beta+2)}+\frac{MB_1\eta^{\beta-p}}{(1-\eta)Q\Gamma(\lambda-p+1)}+\frac{A_1\eta^{\beta}}{Q|\beta+\gamma-1|\Gamma(\beta+1)}$,

$\Delta_2 = \frac{MB_1\gamma}{6(1-\eta)Q\Gamma(\alpha-p+1)}+\frac{A_1\gamma}{Q|\lambda+\gamma-1|\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}$,

$\Delta_3 = \frac{\eta^{\alpha+1}}{(1-\eta)\Gamma(\alpha+2)}+\frac{A_1\eta^{\alpha+1}}{Q(1-\eta)\Gamma(\alpha+2)}+\frac{MB_1\eta^{\alpha-p}}{(1-\eta)Q\Gamma(\alpha-p+1)}+\frac{A_1\eta^{\alpha}}{Q|\lambda+\gamma-1|\Gamma(\alpha+1)}$,

and

$\Delta_4 = \frac{MB_1\gamma}{6(1-\eta)Q\Gamma(\beta-p+1)}+\frac{A_1\gamma}{Q|\lambda+\gamma-1|\Gamma(\beta+1)}+\frac{1}{\Gamma(\beta+1)}$.

In view of Lemma 1.2, we define the operator $T : X\times X \to X\times X$ by

$$T(u,v)(t) = \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix},$$

where

$$T_1(u,v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),v(s))\,ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}g(\tau,u(\tau),v(\tau))\,d\tau\right)ds$$

$$-\frac{B(t)M}{6(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}g(s,u(s),v(s))\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}f(s,u(s),v(s))\,ds\right]$$

$$+\frac{A(t)}{Q|\beta+\gamma-1|}\left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}g(s,u(s),v(s))\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),v(s))\,ds.\right]$$

$$-\frac{A(t)}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}g(\tau,u(\tau),v(\tau))\,d\tau\right)ds$$

and

$$T_2(u,v)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s,u(s),v(s))\,ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,u(\tau),v(\tau))\,d\tau\right)ds$$

$$-\frac{B(t)M}{6(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}f(s,u(s),v(s))\,ds-\gamma\int_0^1 \frac{(1-s)^{\beta-p-1}}{\Gamma(\beta-p)}g(s,u(s),v(s))\,ds\right]$$
$$+\frac{A(t)}{Q|\beta+\gamma-1|}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),v(s))\,ds-\gamma\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}g(s,u(s),v(s))\,ds.\right]$$
$$-\frac{A(t)}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,u(\tau),v(\tau))\,d\tau\right)ds.$$

Observe that the boundary value problem $(1.1)-(1.2)$ has solutions if the operator equation $(u,v)=T(u,v)$ has fixed points.

Now we are in a position to present the first main results of this paper. The existence results is based on Leray-Schauder nonlinear alternative.

Lemma 3.2. [14] *(Leray-Schauder alternative). Let E be a Banach space and $T : E \to E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let*

$$\varepsilon(T)=\{(u,v)\in X\times X : (u,v)=\lambda T(u,v),\ for\ some\ 0<\lambda<1\}.$$

Then either the $\varepsilon(T)$ is unbounded or T has at least one fixed point.

Theorem 3.3. *Assume that $f, g : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are a continuous function and*
(H_1) *there exist constants $k_i>0,\ m_i>0,\ i=0,1,2$ such that $\forall u\in\mathbb{R}$, $\forall v\in\mathbb{R}$, we have*

$$|f(t,u,v)|\le k_0+k_1|u|+k_2|v|,$$

and

$$|g(t,u,v)|\le m_0+m_1|u|+m_2|v|.$$

If $(\Delta_2+\Delta_3)k_1+(\Delta_1+\Delta_4)m_1<1$ and $(\Delta_2+\Delta_3)k_2+(\Delta_1+\Delta_4)m_3<1$, where $\Delta_i,\ i=1,2,3,4$ are given above. Then the boundary value problem $(1.1)-(1.2)$ has at least one solution on $[0,1]$.

Proof. It is clear that T is a continuous operator where $T : X\times X\to X\times X$ is defined above. Now, we show that T is completely continuous. Let $\Omega\subset X\times X$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t,u(t),v(t))|\le L_1,\quad |g(t,u(t),v(t))|\le L_2,\quad \forall(u,v)\in\Omega.$$

Then for any $(u,v)\in\Omega$, we have

$$|T_1(u,v)(t)|\le\frac{L_2}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds$$
$$+\frac{|A(t)|L_2}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds+L_1\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds$$
$$+\frac{M|B(t)|}{6(1-\eta)Q}\left[L_2\int_0^\eta\frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds+\gamma L_1\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right]$$

$$+\frac{|A(t)|}{Q|\lambda+\gamma-1|}\left[L_2\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds+\gamma L_1\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]$$

$$\leq L_2\left\{\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds+\frac{A_1}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\alpha\beta)}d\tau\right)ds\right.$$

$$\left.+\frac{MB_1}{6(1-\eta)Q}\int_0^\eta\frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds+\frac{A_1}{6|\lambda+\gamma-1|}\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\right\}$$

$$+L_1\left\{\frac{M\gamma B_1}{6(1-\eta)Q}\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\frac{A_1\gamma}{Q|\lambda+\gamma-1|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.$$

$$\left.+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right\}$$

$$\leq L_2\Delta_1+L_1\Delta_2.$$

Hence

$$\|T_1(u,v)\|\leq L_2\Delta_1+L_1\Delta_2. \tag{3.6}$$

In the same way, we can obtain that

$$\|T_2(u,v)\|\leq L_1\Delta_3+L_2\Delta_4. \tag{3.7}$$

Thus, it follows from (3.6) and (3.7) that the operator T is uniformly bounded, since $\|T(u,v)\|\leq L_1(\Delta_1+\Delta_3)+L_2(\Delta_2+\Delta_4)$. Now, we show that T is equicontinuous. Let $t_1,\ t_2\in[0,1]$ with $t_1<t_2$. Then we have

$$|T_1(u(t_2),v(t_2))-T_1(u(t_1),v(t_1))|\leq L_1\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds$$

$$+L_1\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\frac{|A(t_2)-A(t_1)|L_2}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds$$

$$+\frac{(B(t_2)-B(t_1))M}{6(1-\eta)Q}\left[L_2\int_0^\eta\frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds+\gamma L_1\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right]$$

$$+\frac{A(t_2)-A(t_1)}{Q|\lambda+\gamma-1|}\left[L_2\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds-\gamma L_1\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right].$$

Obviously, the right-hand side of the above inequality tends to zero as $t_2\to t_1$. Similarly, we have

$$|T_2(u(t_2),v(t_2))-T_2(u(t_1),v(t_1))|\leq L_2\int_0^{t_1}\frac{(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}}{\Gamma(\beta)}ds$$

$$+L_2\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}ds+\frac{|A(t_2)-A(t_1)|\,L_1}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau\right)ds$$

$$+\frac{(B(t_2)-B(t_1))\,M}{6(1-\eta)\,Q}\left[L_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\gamma L_2\int_0^{1}\frac{(1-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds\right]$$

$$+\frac{A(t_2)-A(t_1)}{Q\,|\lambda+\gamma-1|}\left[L_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}ds-\gamma L_2\int_0^{1}\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}ds\right].$$

Again, it is seen that the right-hand side of the above inequality tends to zero as $t_2\to t_1$. Thus, the operator T is equicontinuous.
Therefore, the operator T is completely continuous.
Finally, it will be verified that the set $\varepsilon(T)=\{(u,v)\in X\times X\,:\,(u,v)=\lambda T(u,v)\,,\ 0\le\lambda\le1\}$ is bounded. Let $(u,v)\in\varepsilon(T)$, with $(u,v)=\lambda T(u,v)$ for any $t\in[0,1]$, we have

$$u(t)=\lambda T_1(u,v)(t)\,,\quad v(t)=\lambda T_2(u,v)(t)\,.$$

Then

$$\begin{aligned}|u(t)|&\le\Delta_2(k_0+k_1|u|+k_2|v|)+\Delta_1(m_0+m_1|u|+m_2|v|)\,,\\&=\Delta_2k_0+\Delta_1m_0+(\Delta_2k_1+\Delta_1m_1)|u|+(\Delta_2k_2+\Delta_1m_2)|v|\,,\end{aligned}$$

and

$$\begin{aligned}|v(t)|&\le\Delta_3(k_0+k_1|u|+k_2|v|)+\Delta_4(m_0+m_1|u|+m_2|v|)\,,\\&=\Delta_3k_0+\Delta_4m_0+(\Delta_3k_1+\Delta_4m_1)|u|+(\Delta_3k_2+\Delta_4m_2)|v|\,.\end{aligned}$$

Hence we have

$$\|u\|=\Delta_2k_0+\Delta_1m_0+(\Delta_2k_1+\Delta_1m_1)\|u\|+(\Delta_2k_2+\Delta_1m_2)\|v\|\,,$$

and

$$\|v\|=\Delta_3k_0+\Delta_4m_0+(\Delta_3k_1+\Delta_4m_1)|u|+(\Delta_3k_2+\Delta_4m_2)|v|\,,$$

which imply that

$$\begin{aligned}\|u\|+\|v\|=&(\Delta_2+\Delta_3)k_0+(\Delta_1+\Delta_4)m_0+[(\Delta_2+\Delta_3)k_1+(\Delta_1+\Delta_4)m_1]\|u\|\\&+[(\Delta_2+\Delta_3)k_2+(\Delta_1+\Delta_4)m_2]\|v\|\,.\end{aligned}$$

Consequently,

$$\|(u,v)\|=\frac{(\Delta_2+\Delta_3)k_0+(\Delta_1+\Delta_4)m_0}{\Delta_0},$$

where

$$\Delta_0=\min\{1-[(\Delta_2+\Delta_3)k_1+(\Delta_1+\Delta_4)m_1]\,,1-[(\Delta_2+\Delta_3)k_2+(\Delta_1+\Delta_4)m_2]\},$$

which proves that $\varepsilon(T)$ is bounded. Thus, by Lemma 3.2, the operator T has at least one fixed point. Hence boundary value problem (1.1) − (1.2) has at least one solution. The proof is complete. □

Now, we are in a position to present the second main results of this paper

Theorem 3.4. *Assume that f, g : $[0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist positive constants L_1 and L_2 such that for all $t \in [0,1]$ and u_i , $v_i \in \mathbb{R}$, $i = 1, 2$, we havre*
(1) $|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_1 (|u_1 - v_1| + |u_2 - v_2|)$,

(2) $|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq L_2 (|u_1 - v_1| + |u_2 - v_2|)$.
Then the boundary value problem (1.1)−(1.2) *has a unique solution on* $[0,1]$ *provided*

$$(\Delta_1 + \Delta_3) L_1 + (\Delta_2 + \Delta_4) L_2 < 1.$$

Proof. Let us set $\sup_{t\in[0,1]} |f(t,0,0)| = N_1 < \infty$ and $\sup_{t\in[0,1]} |g(t,0,0)| = N_2 < \infty$. For $u \in X$, we observe that

$$\begin{aligned} |f(t, u(t), v(t))| &\leq |f(t, u(t)) - f(t,0,0)| + |f(t,0,0)|, \\ &\leq L_1 (|u(t)| + |v(t)|) + N_1, \\ &\leq L_1 (\|u\| + \|v\|) + N_1, \end{aligned}$$

and

$$|g(t, u(t), v(t))| \leq |g(t, u(t)) - g(t,0,0)| + |g(t,0,0)| \leq L_2 \|u\| + N_2.$$

Then for $u \in X$, we have

$$|T_1(u,v)(t)| \leq \frac{1}{1-\eta} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} [L_2 \|(u,v)\| + N_2] \, d\tau \right) ds$$

$$+\frac{|A(t)|}{Q(1-\eta)} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} [L_2 \|(u,v)\| + N_2] \, d\tau \right) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [L_1 \|(u,v)\| + N_1] \, ds$$

$$+\frac{M|B(t)|}{6(1-\eta)Q} \left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)} [L_2 \|(u,v)\| + N_2] \, ds + \gamma \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} [L_1 \|(u,v)\| + N_1] \, ds \right]$$

$$+\frac{|A(t)|}{Q|\lambda+\gamma-1|} \left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} [L_2 \|(u,v)\| + N_2] \, ds + \gamma \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} [L_1 \|(u,v)\| + N_1] \, ds, \right]$$

$$\leq (L_2 \|(u,v)\| + N_2) \left\{ \frac{1}{1-\eta} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} d\tau \right) ds + \frac{A_1}{Q(1-\eta)} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} d\tau \right) ds \right.$$

$$\left. +\frac{MB_1}{6(1-\eta)Q} \int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)} ds + \frac{A_1}{6|\lambda+\gamma-1|} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} ds \right\}$$

$$\left. +\frac{MB_1}{6(1-\eta)Q} \int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)} ds + \frac{A_1}{6|\lambda+\gamma-1|} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} ds \right\}$$

$$+ (L_1 \|(u,v)\| + N_1) \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right.$$

$$+\frac{M\gamma B_1}{6(1-\eta)Q}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\frac{A_1\gamma}{Q|\lambda+\gamma-1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg\},$$

$$\leq (L_2 r+N_2)\Delta_1+(L_1 r+N_1)\Delta_2$$

Hence

$$\|T_1(u,v)\|\leq (L_2\Delta_1+L_1\Delta_2)r+N_2\Delta_1+N_1\Delta_2$$

In the same way, we can obtain that

$$\|T_2(u,v)\|\leq (L_1\Delta_3+L_2\Delta_4)r+N_2\Delta_4+N_1\Delta_3.$$

Consequently,

$$\|T(u,v)\|\leq ((\Delta_2+\Delta_3)L_1+(\Delta_1+\Delta_4)L_2)r+N_2(\Delta_1+\Delta_4)+N_1(\Delta_2+\Delta_3)\leq r.$$

Now, for (u_1,v_1), $(u_2,v_2)\in X\times X$ and for each $t\in[0,1]$, it follows from assumption (H_3) that

$$|T_1(u_2,v_2)(t)-T_1(u_1,v_1)(t)|\leq L_2(\|u_2-u_1\|+\|v_2-v_1\|)\Bigg\{\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds$$

$$+\frac{A_1}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds$$

$$+\frac{MB_1}{6(1-\eta)Q}\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds+\frac{A_1}{6|\lambda+\gamma-1|}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\Bigg\}$$

$$+L_1(\|u_2-u_1\|+\|v_2-v_1\|)\Bigg\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds$$

$$+\frac{M\gamma B_1}{6(1-\eta)Q}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\frac{A_1\gamma}{Q|\lambda+\gamma-1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg\}$$

$$\leq (L_2\Delta_1+L_1\Delta_2)(\|u_2-u_1\|+\|v_2-v_1\|).$$

Thus

$$\|T_1(u_2,v_2)-T_1(u_1,v_1)\|\leq (L_2\Delta_1+L_1\Delta_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{3.8}$$

. Similarly,

$$\|T_2(u_2,v_2)-T_2(u_1,v_1)\|\leq (L_2\Delta_3+L_1\Delta_4)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\|T(u_2,v_2)-T(u_1,v_1)\|\leq (L_2(\Delta_1+\Delta_3)+L_1(\Delta_2+\Delta_4))(\|u_2-u_1\|+\|v_2-v_1\|).$$

Since $L_2(\Delta_1+\Delta_3)+L_1(\Delta_2+\Delta_4)<1$, thus T is a contraction operator. Hence it follows by Banach's contraction principle that the boundary value problem (1.1) – (1.2) has a unique solution on $[0,1]$.

□

We construct an example to illustrate the applicability of the results presented.

Example 3.1. *Consider the following system fractional differential equation*

$$\begin{cases} {}^cD^3u(t) = \frac{t}{8}\left((cost)\,sin\left(\frac{|u(t)|+|v(t)|}{2}\right)\right) + \frac{e^{-(u(t)+v(t))^2}}{1+t^2}, & t \in [0,1], \\ {}^cD^3v(t) = \frac{1}{32}sin(2\pi u(t)) + \frac{|v(t)|}{16(1+|v(t)|)} + \frac{1}{2}, & t \in [0,1], \end{cases}$$

subject to the three-point coupled boundary conditions

$$\begin{cases} \frac{1}{100}u(0) + \frac{1}{10}u(1) = u\left(\frac{1}{2}\right), \\ u(0) = \int_0^{0,5} u(s)\,ds, \\ \frac{1}{100}{}^cD^{\frac{3}{2}}u(0) + \frac{1}{10}{}^cD^{\frac{3}{2}}u(1) = {}^cD^{\frac{3}{2}}u\left(\frac{1}{2}\right), \end{cases}$$

where $f(t,u,v) = \frac{t}{8}\left((cost)\,sin\left(\frac{|u|+|v|}{2}\right)\right) + \frac{e^{-(u+v)^2}}{1+t^2},\ t \in [0,1], \eta = 0,5, \lambda = 0,01, \gamma = 0,1, p = 1,5$ *and* $g(t,u,v) = \frac{1}{32\pi}sin(2\pi u(t)) + \frac{|v(t)|}{16(1+|v(t)|)} + \frac{1}{2}$.
It can be easily found that $M = \frac{20}{3}$ *and* $Q = \frac{9}{400}$.
Furthermore, by simple computation, for every $u_i,\ v_i \in \mathbb{R},\ i = 1,2,$ *we have*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L(|u_1 - v_1| + |u_2 - v_2|),$$

and

$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \leq L(|u_1 - v_1| + |u_2 - v_2|),$$

where $L_1 = L_2 = L = \frac{1}{16}$. *It can be easily found that* $\Delta_1 = \Delta_3 \cong 0,799562,\ \Delta_2 = \Delta_4 \cong 1,182808$.
Finally, since $L_1(\Delta_1 + \Delta_3) + L_2(\Delta_2 + \Delta_4) = 2L(\Delta_1 + \Delta_2) \cong 0,247796 < 1$, *thus all assumptions and conditions of Theorem* 3.4 *are satisfied. Hence, Theorem implies that the three-point boundary value problem* $(1.1) - (1.2)$ *has a unique solution*

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STRONG COUPLED FIXED POINTS OF CHATTERJEA TYPE (ψ, φ)-WEAKLY CYCLIC COUPLED MAPPINGS IN S-METRIC SPACES

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Abstract. In this paper, we introduce Chatterjea type (ψ, φ)-weakly cyclic coupled mapping in S-metric spaces and prove the existence and uniqueness of strong coupled fixed point of such mappings. We give an illustrative example to support of our result.

1. Introduction

In 1972, Chatterjea [8] introduced a contraction map which is not necessarily continuous and is known as Chatterjea contraction map or simply Chatterjea map and proved that every Chatterjea map has a unique fixed point in complete metric spaces. For more works on Chatterjea type mappings, we refer [7], [9], [10], [21], [31]. In 1997, Alber and Guerre-Delabriere [2] introduced the concept of weakly contractive mapping as a generalization of contractive map and proved the existence of fixed points for such mappings in Hilbert spaces. Rhoades [33] extended this study to metric space setting. In 2003, Kirk, Srinivasan and Veeramani [23] introduced cyclic contractions in metric spaces and proved the existence and uniqueness of cyclic contractions in complete metric spaces. After this, many authors introduced various types of cyclic contractions and cyclic weakly contractions and proved fixed point results, some of which are in [3], [5], [19], [20], [22], [24], [26], [27], [29], [30], [34]. Meanwhile, in 2006, Gnana Bhaskar and Lakshmikantham [14] introduced and developed coupled fixed point theory for mixed monotone operators. Later, coupled fixed point results were developed by [14], [18], [25], [28], [32], [37]. In 2013, Chandok and Postolache [7] introduced Chatterjea type cyclic weakly contractive maps and obtained fixed point results in complete metric spaces and in 2017, Choudhury, Maity and Konar [10], introduced Chatterjea type coupling and obtained the existence of strong unique coupled fixed points for such maps.

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Inspired by these works, in section 3 of this paper, we introduce Chatterjea type (ψ, φ)-weakly cyclic coupled mapping and prove the existence and uniqueness of strong coupled fixed point of such map in complete S-metric spaces. Also, we present an illustrative example in support of our result.

2. Preliminaries

We use the following propositions in proving our results.

Proposition 2.1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Then $\limsup_{n\to\infty} \max\{a_n, b_n\} = \max\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty} b_n\}$.

Proposition 2.2. (i) Let $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be real sequences then $\max\{c_n + d_n, e_n + f_n\} \leq \max\{c_n, e_n\} + \max\{d_n, f_n\}$.
(ii) Let $\{a_n\}$, $\{b_n\}$ be two real sequences, $\{b_n\}$ be bounded. Then $\liminf_{n\to\infty}(a_n + b_n) \leq \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

Proposition 2.3. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be nonnegative sequences satisfying $\max\{a_n, b_n\} \leq \max\{c_n + d_n, e_n + f_n\}$ with $\limsup_{n\to\infty} c_n = 0$ and $\limsup_{n\to\infty} e_n = 0$ then $\liminf_{n\to\infty} \max\{a_n, b_n\} \leq \liminf_{n\to\infty} \max\{d_n, f_n\}$.

Definition 2.1. [23] Let X be a nonempty set and $T : X \to X$ be an operator. If X_i, $i = 1, 2, ...m$ are nonempty subsets of X with $X = \bigcup_{i=1}^{m} X_i$ satisfying $T(X_1) \subset X_2$, ..., $T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$ is called a cyclic representation of X with respect to T.

Definition 2.2. [14] Let X be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of F if $F(x, y) = x$ and $F(y, x) = y$.

Throughout this paper, we denote the set of all reals by $\mathbb{R}$, the set of all natural numbers by $\mathbb{N}$, and
$\Psi = \{\psi : [0, \infty) \to [0, \infty)$ /(i) ψ is continuous (ii) ψ is nondecreasing (iii) $\psi(t) = 0$ if and only if $t = 0\}$.

Remark. For any $a, b \in [0, \infty)$, we have $\psi(\max\{a, b\}) = \max\{\psi(a), \psi(b)\}$ for any $\psi \in \Psi$.

Definition 2.3. [7] Let (X, d) be a metric space, m be a natural number, A_1, A_2, ..., A_m be nonempty subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. An operator $T : Y \to Y$ is called a *Chatterjea type cyclic weakly contraction* if
$\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T and if there exist $\psi \in \Psi$ and a function $\varphi : [0, \infty)^2 \to [0, \infty)$ with φ is lower semi continuous, $\varphi(t, t) > 0$ for $t \in (0, \infty)$ and $\varphi(0, 0) = 0$ such that
$\psi(d(Tx, Ty)) \leq \psi(\frac{1}{2}[d(x, Ty) + d(y, Tx)]) - \varphi(d(x, Ty), d(y, Tx))$,
for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, ..., m$, where $A_{m+1} = A_1$.

Theorem 2.4. [7] Let (X, d) be a complete metric space, $m \in N$, A_1, A_2,..., A_m be nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that T is a Chatterjea type cyclic weakly contraction. Then T has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Choudhury, Maity and Konar [10] extended the above notion of cyclic mapping to the case of mappings defined on $X \times X$ in the following definition.

Definition 2.4. [10] Let A and B be two nonempty subsets of X. A mapping $F : X \times X \to X$ is said to be *cyclic* with respect to A and B if $F(A, B) \subset B$ and $F(B, A) \subset A$. Such a function F is also said to be a coupling with respect to A and B.

Definition 2.5. [10] Let X be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x, x) \in X \times X$ is said to be a *strong coupled fixed point* of F if $F(x, x) = x$.

Definition 2.6. [10] Let A and B be two nonempty subsets of a metric space (X, d). A coupling $F : X \times X \to X$ is called a *Chatterjea type coupling* with respect to A and B if F is cyclic with respect to A and B satisfying, the inequality

$$d(F(x, y), F(u, v)) \leq k[d(x, F(u, v)) + d(u, F(x, y))], \tag{2.1}$$

where $x, v \in A$ and $y, u \in B$, for some $k \in (0, \frac{1}{2})$.

Theorem 2.5. [10] Let A and B be two nonempty closed subsets of a complete metric space (X, d). Let $F : X \times X \to X$ be a *Chatterjea type coupling* with respect to A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

In 2012, Sedghi, Shobe and Aliouche [35] introduced a new concept on metric spaces, namely S-metric spaces and studied some properties of these spaces. Subsequently, many authors developed coupled fixed point theorems and cyclic contractions on S-metric spaces. Some of them include [1], [12], [15], [16], [17], [25], [31], [37].

Definition 2.7. [35] *Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions: for each $x, y, z, a \in X$*

(S1) $S(x, y, z) \geq 0$,

(S2) $S(x, y, z) = 0$ *if and only if* $x = y = z$ *and*

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Example 2.1. [35] Let (X, d) be a metric space. Define $S : X^3 \to [0, \infty)$ by $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then S is an S-metric on X and S is called the S-metric induced by the metric d.

Example 2.2. [13] Let $X = \mathbb{R}$ and let $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$. Then (X, S) is an S-metric space.

Example 2.3. [36] Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric.

Lemma 2.6. [35] In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.

Lemma 2.7. [13] Let (X,S) be an S-metric space. Then
$S(x,x,z) \leq 2S(x,x,y) + S(y,y,z)$.

Definition 2.8. [35] Let (X,S) be an S-metric space.

(i) A sequence $\{x_n\} \subseteq X$ is said to converge to a point $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \epsilon$ and we denote it by $\lim\limits_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \epsilon$ for all $n,m \geq n_0$.

(iii) An S-metric space (X,S) is said to be complete if each Cauchy sequence in X is convergent.

Lemma 2.8. [35] Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 2.9. [35] Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$, then $\lim\limits_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y)$.

Lemma 2.10. [6] Let (X,S) be an S-metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, $\{x_n\}$ converges to x in X. Then $\lim\limits_{n\to\infty} S(x_n,x_n,y_n) = \lim\limits_{n\to\infty} S(x,x,y_n)$.

Lemma 2.11. *(*[4], [11]*)* Let (X,S) be an S-metric space and $\{x_n\}$ a sequence in X such that

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that
$S(x_{m_k},x_{m_k},x_{n_k}) \geq \epsilon$ with $S(x_{m_k-1},x_{m_k-1},x_{n_k}) < \epsilon$.
Also, we have the following:

$(i)\ \lim\limits_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k}) = \epsilon \qquad (ii)\ \lim\limits_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k}) = \epsilon$

$(iii)\ \lim\limits_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k-1}) = \epsilon \qquad (iv)\ \lim\limits_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \epsilon.$

We denote

$$\Phi = \{\varphi : [0,\infty)^2 \to [0,\infty) \text{ such that (i) } \varphi \text{ is continuous in each of its variables, and (ii) } \varphi(t_1,t_2) = 0 \text{ if and only if } t_1 = 0 \text{ and } t_2 = 0\}.$$

3. Chatterjea Type (ψ,φ)- Weakly cyclic Coupled Mapping

In the following, we define Chatterjea type (ψ,φ)-weakly cyclic coupled mapping.

Definition 3.1. Let (X,S) be an S-metric space. Let A and B be two nonempty subsets of X. Let $F : X \times X \to X$ be a mapping. If (i) F is cyclic with respect to A and B and (ii) there exist $\psi \in \Psi$, $\varphi \in \Phi$ such that

$$\begin{aligned}\psi(S(F(x,y),F(u,v),F(w,z))) \leq \psi(&\frac{1}{4}[\max\{S(x,x,F(w,z)),S(x,x,F(u,v))\}\\ &+\max\{S(w,w,F(x,y)),S(u,u,F(x,y))\}])\\ &-\varphi(\max\{S(x,x,F(w,z)),S(x,x,F(u,v))\},\\ &\max\{S(w,w,F(x,y)),S(u,u,F(x,y))\})\end{aligned} \tag{3.1}$$

for any $x, u, z \in A$ and $y, v, w \in B$, then we say that F is a *Chatterjea type* (ψ, φ)-*weakly cyclic coupled mapping with respect to* A *and* B.

Example 3.1. Let $X = [0, 1]$. We define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ x + y + z & \text{otherwise.} \end{cases}$$

Then (X, S) is an S-metric space.
Let $A = [0, \frac{1}{2}]$ and $B = [0, 1]$. We define $F : X \times X \to X$ by
$F(x, y) = \frac{xy}{16}$. Then $F(A, B) \subset B$ and $F(B, A) \subset A$ so that F is cyclic with respect to A and B. We define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$ and $\varphi : [0, \infty)^2 \to [0, \infty)$ by $\varphi(t_1, t_2) = \frac{1}{16}(t_1 + t_2)$. We now verify the inequality (3.1). Let $x, u, z \in A$ and $y, v, w \in B$. We now consider

$$\begin{aligned} \psi(S(F(x, y), F(u, v), F(w, z))) &= \psi(S(\tfrac{xy}{16}, \tfrac{uv}{16}, \tfrac{wz}{16})) \\ &= \tfrac{1}{2} S(\tfrac{xy}{16}, \tfrac{uv}{16}, \tfrac{wz}{16}) \\ &= \tfrac{xy}{32} + \tfrac{uv}{32} + \tfrac{wz}{32} \\ &\leq \tfrac{1}{32}[x + u + w] \\ &\leq \tfrac{1}{32}[S(x, x, F(w, z)) + S(x, x, F(u, v)) + \\ &\qquad + S(w, w, F(x, y)) + S(u, u, F(x, y))] \\ &\leq \tfrac{2}{32}[\max\{S(x, x, F(w, z)), S(x, x, F(u, v))\} \\ &\qquad + \max\{S(w, w, F(x, y)), S(u, u, F(x, y))\}] \\ &= \tfrac{1}{16}[t_1 + t_2] \\ &= \tfrac{1}{8}[t_1 + t_2] - \tfrac{1}{16}[t_1 + t_2] \\ &= \psi(\tfrac{1}{4}[t_1 + t_2]) - \varphi(t_1, t_2), \end{aligned}$$

where $t_1 = \max\{S(x, x, F(w, z)), S(x, x, F(u, v))\}$ *and*
$t_2 = \max\{S(w, w, F(x, y)), S(u, u, F(x, y))\}$.

Therefore F is a Chatterjea type (ψ, φ)-weakly cyclic coupled mapping with respect to A and B.

Lemma 3.1. Let (X, S) be an S-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim\limits_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0$ and $\lim\limits_{n\to\infty} S(y_n, y_n, y_{n+1}) = 0$. If either $\{x_n\}$ or $\{y_n\}$ is not Cauchy, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geq \epsilon. \tag{3.2}$$

We choose m_k as the smallest integer with $m_k > n_k$ satisfying (3.2).
i.e., $\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geq \epsilon$ with
$\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} < \epsilon$.
Also, the following limits hold.

(i) $\lim\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} = \epsilon$
(ii) $\lim\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\} = \epsilon$ and
(iii) $\lim\limits_{k\to\infty} \max\{S(x_{n_k}, x_{n_k}, x_{m_k-1}), S(y_{n_k}, y_{n_k}, y_{m_k-1})\} = \epsilon$.

Proof. (i) We consider

$$\begin{aligned} S(x_{m_k}, x_{m_k}, x_{n_k}) &\leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) \\ &< 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + \epsilon. \end{aligned}$$

Similarly, we have

$$\begin{aligned} S(y_{m_k}, y_{m_k}, y_{n_k}) &\leq 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + S(y_{m_k-1}, y_{m_k-1}, y_{n_k}) \\ &< 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + \epsilon. \end{aligned}$$

Hence
$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \max\{2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + \epsilon,$
$2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + \epsilon\}.$
Now, by applying Proposition 2.1, we have

$$\limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \epsilon. \tag{3.3}$$

We have $\epsilon \leq \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$. Hence
$\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$
$\leq \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \epsilon$ (from (3.3)).
Hence $\liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} = \epsilon$
$= \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}.$
Therefore $\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$ exists and
$\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} = \epsilon.$
Hence (i) holds.
(ii) We now consider
$S(x_{m_k}, x_{m_k}, x_{n_k}) = S(x_{n_k}, x_{n_k}, x_{m_k}) \leq 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1}).$
Similarly, we have
$S(y_{m_k}, y_{m_k}, y_{n_k}) = S(y_{n_k}, y_{n_k}, y_{m_k}) \leq 2S(y_{n_k}, y_{n_k}, y_{n_k-1}) + S(y_{m_k}, y_{m_k}, y_{n_k-1}).$
Then
$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \max\{2S(x_{n_k}, x_{n_k}, x_{n_k-1})$
$+ S(x_{m_k}, x_{m_k}, x_{n_k-1}),$
$2S(y_{n_k}, y_{n_k}, y_{n_k-1}) + S(y_{m_k}, y_{m_k}, y_{n_k-1})\}.$
On taking limit infimum as $k \to \infty$ and using Proposition 2.3, we get
$\liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$
$\leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}.$

By using (i), we get $\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$.
We now consider
$S(x_{m_k}, x_{m_k}, x_{n_k-1}) = S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) \leq 2S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k}).$
Similarly, we have
$S(y_{m_k}, y_{m_k}, y_{n_k-1}) = S(y_{n_k-1}, y_{n_k-1}, y_{m_k}) \leq 2S(y_{n_k-1}, y_{n_k-1}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k}).$
Now,
$\max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$
$\leq \max\{2S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k}),$
$2S(y_{n_k-1}, y_{n_k-1}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k})\}.$
On taking limit supremum as $k \to \infty$ and using Proposition 2.1, we get
$\limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$
$\leq \limsup_{k\to\infty} \max\{S(x_{n_k}, x_{n_k}, x_{m_k}), S(y_{n_k}, y_{n_k}, y_{m_k})\}$
$= \epsilon$ (by (i)).
Therefore we have
$\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$
$\leq \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\} \leq \epsilon.$

Thus, we have

$$\liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\} = \epsilon$$
$$= \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}.$$

Hence $\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$ exists and $\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\} = \epsilon$. Therefore (ii) holds.

(iii) We consider

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k})$$

and

$$S(y_{m_k}, y_{m_k}, y_{n_k}) \leq 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + S(y_{m_k-1}, y_{m_k-1}, y_{n_k}).$$

Now

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$$
$$\leq \max\{2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k}),$$
$$2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}.$$

On taking limit infimum as $k \to \infty$ and by Proposition 2.3, we get

$$\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}.$$

We have

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) \leq 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + S(x_{m_k}, x_{m_k}, x_{n_k})$$

and

$$S(y_{m_k-1}, y_{m_k-1}, y_{n_k}) \leq 2S(y_{m_k-1}, y_{m_k-1}, y_{m_k}) + S(y_{m_k}, y_{m_k}, y_{n_k}).$$

Then

$$\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}$$
$$\leq \max\{2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + S(x_{m_k}, x_{m_k}, x_{n_k}),$$
$$2S(y_{m_k-1}, y_{m_k-1}, y_{m_k}) + S(y_{m_k}, y_{m_k}, y_{n_k})\}.$$

On taking limit supremum as $k \to \infty$, we get

$$\limsup_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}$$
$$\leq \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$$
$$= \epsilon \text{ (by } (i))$$

so that $\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}$

$$\leq \limsup_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} \leq \epsilon.$$

Thus $\liminf_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} = \epsilon$

$$= \limsup_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}.$$

Hence $\lim_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}$ exists and $\lim_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} = \epsilon$.

This proves (iii). □

Theorem 3.2. Let (X, S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be a Chatterjea type (ψ, φ)-weakly cyclic coupled mapping with respect to A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Proof. Let $x_0 \in A$ and $y_0 \in B$ be arbitrary. We define the sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = F(y_n, x_n),\ y_{n+1} = F(x_n, y_n),\ n = 0, 1, 2, \ldots . \tag{3.4}$$

If $y_n = x_{n+1}$ and $x_n = y_{n+1}$ for some n, then we have

$$\begin{aligned}
\psi(S(x_n, x_n, y_n)) &= \psi(S(y_{n+1}, y_{n+1}, x_{n+1})) \\
&= \psi(S(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, F(y_n, x_n)), S(x_n, x_n, F(x_n, y_n))\} \\
&\qquad + \max\{S(y_n, y_n, F(x_n, y_n)), S(x_n, x_n, F(x_n, y_n))\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, F(y_n, x_n)), S(x_n, x_n, F(x_n, y_n))\}, \\
&\qquad \max\{S(y_n, y_n, F(x_n, y_n)), S(x_n, x_n, F(x_n, y_n))\}]) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\} \\
&\qquad + \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, y_n), S(x_n, x_n, x_n)\} \\
&\qquad + \max\{S(y_n, y_n, x_n), S(x_n, x_n, x_n)\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, y_n), S(x_n, x_n, x_n)\}, \\
&\qquad \max\{S(y_n, y_n, x_n), S(x_n, x_n, x_n)\}) \\
&= \psi(\tfrac{1}{2}S(x_n, x_n, y_n)) - \varphi(S(x_n, x_n, y_n), S(x_n, x_n, y_n)) \\
&\qquad \text{(by using Lemma 2.6)} \\
&\leq \psi(S(x_n, x_n, y_n)) - \varphi(S(x_n, x_n, y_n), S(x_n, x_n, y_n))
\end{aligned}$$

which implies that $\varphi(S(x_n, x_n, y_n), S(x_n, x_n, y_n)) = 0$ and hence $S(x_n, x_n, y_n) = 0$. Thus $x_n = y_n$ so that $A \cap B \neq \emptyset$ and (x_n, x_n) is a strong coupled fixed point of F and we are through.

If either $y_n \neq x_{n+1}$ or $x_n \neq y_{n+1}$ for all n, then we have the following. If $x_n = y_{n+1}$ and $y_n \neq x_{n+1}$ for all n, then we have

$$\begin{aligned}
\psi(S(y_{n+1}, y_{n+1}, x_{n+2})) &= \psi(S(F(x_n, y_n), F(x_n, y_n), F(y_{n+1}, x_{n+1}))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, F(y_{n+1}, x_{n+1})), \\
&\qquad\qquad S(x_n, x_n, F(x_n, y_n))\} \\
&\qquad + \max\{S(y_{n+1}, y_{n+1}, F(x_n, y_n)), \\
&\qquad\qquad S(x_n, x_n, F(x_n, y_n))\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, F(y_{n+1}, x_{n+1})), \\
&\qquad\qquad S(x_n, x_n, F(x_n, y_n))\}, \\
&\qquad \max\{S(y_{n+1}, y_{n+1}, F(x_n, y_n)), \\
&\qquad\qquad S(x_n, x_n, F(x_n, y_n))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, x_{n+2}), S(x_n, x_n, y_{n+1})\} \\
&\qquad + \max\{S(y_{n+1}, y_{n+1}, y_{n+1}), S(x_n, x_n, y_{n+1})\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+2}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad \max\{S(y_{n+1}, y_{n+1}, y_{n+1}), S(x_n, x_n, y_{n+1})\}) \\
&\leq \psi(\tfrac{1}{4}[\max\{2S(x_n, x_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, x_{n+2}), \\
&\qquad S(x_n, x_n, y_{n+1})\} + S(x_n, x_n, y_{n+1})]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+2}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad S(x_n, x_n, y_{n+1})) \\
&= \psi(\tfrac{1}{4}[2S(x_n, x_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, x_{n+2}) \\
&\qquad + S(x_n, x_n, y_{n+1})]) \\
&\quad -\varphi(\max\{S(x_n, x_n, x_{n+2}), S(x_n, x_n, y_{n+1})\}, S(x_n, x_n, y_{n+1})) \\
&= \psi(\tfrac{1}{4}S(y_{n+1}, y_{n+1}, x_{n+2})) - \varphi(S(x_n, x_n, x_{n+2}), 0) \\
&\leq \psi(S(y_{n+1}, y_{n+1}, x_{n+2})) - \varphi(S(x_n, x_n, x_{n+2}), 0)
\end{aligned}$$

which implies that $\varphi(S(x_n, x_n, x_{n+2}), 0) = 0$. Therefore $S(x_n, x_n, x_{n+2}) = 0$. Thus $x_n = x_{n+2}$. That is $y_{n+1} = x_{n+2}$ which is a contradiction. Hence this case does not arise.

Similar as above, the case $y_n = x_{n+1}$ and $x_n \neq y_{n+1}$ for all n does not arise.

Hence, we assume that $y_n \neq x_{n+1}$ and $x_n \neq y_{n+1}$ for all n. Now by using (3.1), we have

$$\begin{aligned}
\psi(S(x_1,x_1,y_2)) &= \psi(S(y_2,y_2,x_1))\\
&= \psi(S(F(x_1,y_1),F(x_1,y_1),F(y_0,x_0)))\\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_1,x_1,F(y_0,x_0)),S(x_1,x_1,F(x_1,y_1))\}\\
&\qquad +\max\{S(y_0,y_0,F(x_1,y_1)),S(x_1,x_1,F(x_1,y_1))\}])\\
&\quad -\varphi(\max\{S(x_1,x_1,F(y_0,x_0)),S(x_1,x_1,F(x_1,y_1))\},\\
&\qquad \max\{S(y_0,y_0,F(x_1,y_1)),S(x_1,x_1,F(x_1,y_1))\})\\
&= \psi(\tfrac{1}{4}[\max\{S(x_1,x_1,x_1),S(x_1,x_1,y_2)\}\\
&\qquad +\max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\}])\\
&\quad -\varphi(\max\{S(x_1,x_1,x_1),S(x_1,x_1,y_2)\},\\
&\qquad \max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\})\\
&= \psi(\tfrac{1}{4}[S(x_1,x_1,y_2)+\max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\}])\\
&\quad -\varphi(S(x_1,x_1,y_2),\max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\})\\
&\leq \psi(\tfrac{1}{4}[S(x_1,x_1,y_2)+\max\{2S(y_0,y_0,x_1)+S(x_1,x_1,y_2),\\
&\qquad\quad S(x_1,x_1,y_2)\}])\\
&\quad -\varphi(S(x_1,x_1,y_2),\max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\})\\
&= \psi(\tfrac{1}{4}[2S(x_1,x_1,y_2)+2S(y_0,y_0,x_1)])\\
&\quad -\varphi(S(x_1,x_1,y_2),\max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\})\\
&< \psi(\tfrac{1}{4}[2S(x_1,x_1,y_2)+2S(y_0,y_0,x_1)]).
\end{aligned}$$

Since ψ is monotonically increasing, it follows that $S(x_1,x_1,y_2) \leq \frac{1}{2}S(x_1,x_1,y_2)+\frac{1}{2}S(y_0,y_0,x_1)$ so that

$$S(x_1,x_1,y_2) \leq S(y_0,y_0,x_1). \tag{3.5}$$

Similarly, we have

$$\begin{aligned}
\psi(S(y_1,y_1,x_2)) &= \psi(S(F(x_0,y_0),F(x_0,y_0),F(y_1,x_1)))\\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_0,x_0,F(y_1,x_1)),S(x_0,x_0,F(x_0,y_0))\}\\
&\qquad +\max\{S(y_1,y_1,F(x_0,y_0)),S(x_0,x_0,F(x_0,y_0))\}])\\
&\quad -\varphi(\max\{S(x_0,x_0,F(y_1,x_1)),S(x_0,x_0,F(x_0,y_0))\},\\
&\qquad \max\{S(y_1,y_1,F(x_0,y_0)),S(x_0,x_0,F(x_0,y_0))\})\\
&= \psi(\tfrac{1}{4}[\max\{S(x_0,x_0,x_2),S(x_0,x_0,y_1)\}\\
&\qquad +\max\{S(y_1,y_1,y_1),S(x_0,x_0,y_1)\}])\\
&\quad -\varphi(\max\{S(x_0,x_0,x_2),S(x_0,x_0,y_1)\},\\
&\qquad \max\{S(y_1,y_1,y_1),S(x_0,x_0,y_1)\})\\
&\leq \psi(\tfrac{1}{4}[\max\{2S(x_0,x_0,y_1)+S(y_1,y_1,x_2),S(x_0,x_0,y_1)\}\\
&\qquad +S(x_0,x_0,y_1)])\\
&\quad -\varphi(\max\{S(x_0,x_0,x_2),S(x_0,x_0,y_1)\},S(x_0,x_0,y_1))\\
&= \psi(\tfrac{1}{4}[2S(x_0,x_0,y_1)+S(y_1,y_1,x_2)+S(x_0,x_0,y_1)])\\
&\quad -\varphi(\max\{S(x_0,x_0,x_2),S(x_0,x_0,y_1)\},S(x_0,x_0,y_1))\\
&< \psi(\tfrac{3}{4}S(x_0,x_0,y_1)+\tfrac{1}{4}S(y_1,y_1,x_2)),
\end{aligned}$$

and hence it follows that $S(y_1,y_1,x_2) \leq \frac{3}{4}S(x_0,x_0,y_1)+\frac{1}{4}S(y_1,y_1,x_2)$ so that

$$S(y_1,y_1,x_2) \leq S(x_0,x_0,y_1). \tag{3.6}$$

Again, by using (3.1), we have

$$\begin{aligned}
\psi(S(x_2,x_2,y_3)) &= \psi(S(y_3,y_3,x_2))\\
&= \psi(S(F(x_2,y_2),F(x_2,y_2),F(y_1,x_1)))\\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_2,x_2,F(y_1,x_1)),S(x_2,x_2,F(x_2,y_2))\}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \max\{S(y_1, y_1, F(x_2, y_2)), S(x_2, x_2, F(x_2, y_2))\}]) \\
&\quad - \varphi(\max\{S(x_2, x_2, F(y_1, x_1)), S(x_2, x_2, F(x_2, y_2))\}, \\
&\quad \max\{S(y_1, y_1, F(x_2, y_2)), S(x_2, x_2, F(x_2, y_2))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_2, x_2, x_2), S(x_2, x_2, y_3)\} \\
&\quad + \max\{S(y_1, y_1, y_3), S(x_2, x_2, y_3)\}]) \\
&\quad - \varphi(\max\{S(x_2, x_2, x_2), S(x_2, x_2, y_3)\}, \\
&\quad \max\{S(y_1, y_1, y_3), S(x_2, x_2, y_3)\}) \\
&\leq \psi(\tfrac{1}{4}[S(x_2, x_2, y_3) + \max\{2S(y_1, y_1, x_2) + S(x_2, x_2, y_3), \\
&\quad S(x_2, x_2, y_3)\}]) \\
&\quad - \varphi(S(x_2, x_2, y_3), \max\{S(y_1, y_1, y_3), S(x_2, x_2, y_3)\}) \\
&= \psi(\tfrac{1}{4}[S(x_2, x_2, y_3) + 2S(y_1, y_1, x_2) + S(x_2, x_2, y_3)]) \\
&\quad - \varphi(S(x_2, x_2, y_3), \max\{S(y_1, y_1, y_3), S(x_2, x_2, y_3)\}) \\
&< \psi(\tfrac{1}{2}S(x_2, x_2, y_3) + \tfrac{1}{2}S(y_1, y_1, x_2)),
\end{aligned}
$$

and hence we have $S(x_2, x_2, y_3) \leq \frac{1}{2}S(x_2, x_2, y_3) + \frac{1}{2}S(y_1, y_1, x_2)$
so that
$S(x_2, x_2, y_3) \leq S(y_1, y_1, x_2)$.
Similarly, we have

$$
\begin{aligned}
\psi(S(y_2, y_2, x_3)) &= \psi(S(F(x_1, y_1), F(x_1, y_1), F(y_2, x_2))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_1, x_1, F(y_2, x_2)), S(x_1, x_1, F(x_1, y_1))\} \\
&\quad + \max\{S(y_2, y_2, F(x_1, y_1)), S(x_1, x_1, F(x_1, y_1))\}]) \\
&\quad - \varphi(\max\{S(x_1, x_1, F(y_2, x_2)), S(x_1, x_1, F(x_1, y_1))\}, \\
&\quad \max\{S(y_2, y_2, F(x_1, y_1)), S(x_1, x_1, F(x_1, y_1))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_1, x_1, x_3), S(x_1, x_1, y_2)\} + S(x_1, x_1, y_2)]) \\
&\quad - \varphi(\max\{S(x_1, x_1, x_3), S(x_1, x_1, y_2)\}, S(x_1, x_1, y_2)) \\
&\leq \psi(\tfrac{1}{4}[\max\{2S(x_1, x_1, y_2) + S(y_2, y_2, x_3), S(x_1, x_1, y_2)\} \\
&\quad + S(x_1, x_1, y_2)]) \\
&\quad - \varphi(\max\{S(x_1, x_1, x_3), S(x_1, x_1, y_2)\}, S(x_1, x_1, y_2)) \\
&= \psi(\tfrac{1}{4}[2S(x_1, x_1, y_2) + S(y_2, y_2, x_3) + S(x_1, x_1, y_2)]) \\
&\quad - \varphi(\max\{S(x_1, x_1, x_3), S(x_1, x_1, y_2)\}, S(x_1, x_1, y_2)) \\
&< \psi(\tfrac{3}{4}S(x_1, x_1, y_2) + \tfrac{1}{4}S(y_2, y_2, x_3)), \text{ and hence}
\end{aligned}
$$

$S(y_2, y_2, x_3) \leq \frac{3}{4}S(x_1, x_1, y_2) + \frac{1}{4}S(y_2, y_2, x_3)$ so that
$S(y_2, y_2, x_3) \leq S(x_1, x_1, y_2)$.
In general, we have

$$
\begin{aligned}
\psi(S(x_{2n+1}, x_{2n+1}, y_{2n+2})) &= \psi(S(y_{2n+2}, y_{2n+2}, x_{2n+1})) \\
&= \psi(S(F(x_{2n+1}, y_{2n+1}), F(x_{2n+1}, y_{2n+1}), F(y_{2n}, x_{2n}))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_{2n+1}, x_{2n+1}, F(y_{2n}, x_{2n})), \\
&\quad S(x_{2n+1}, x_{2n+1}, F(x_{2n+1}, y_{2n+1}))\} \\
&\quad + \max\{S(y_{2n}, y_{2n}, F(x_{2n+1}, y_{2n+1})), \\
&\quad S(x_{2n+1}, x_{2n+1}, F(x_{2n+1}, y_{2n+1}))\}]) \\
&\quad - \varphi(\max\{S(x_{2n+1}, x_{2n+1}, F(y_{2n}, x_{2n})), \\
&\quad S(x_{2n+1}, x_{2n+1}, F(x_{2n+1}, y_{2n+1}))\}, \\
&\quad \max\{S(y_{2n}, y_{2n}, F(x_{2n+1}, y_{2n+1})), \\
&\quad S(x_{2n+1}, x_{2n+1}, F(x_{2n+1}, y_{2n+1}))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_{2n+1}, x_{2n+1}, x_{2n+1}), S(x_{2n+1}, x_{2n+1}, y_{2n+2})\} \\
&\quad + \max\{S(y_{2n}, y_{2n}, y_{2n+2}), S(x_{2n+1}, x_{2n+1}, y_{2n+2})\}]) \\
&\quad -\varphi(\max\{S(x_{2n+1}, x_{2n+1}, x_{2n+1}), S(x_{2n+1}, x_{2n+1}, y_{2n+2})\}, \\
&\quad \max\{S(y_{2n}, y_{2n}, y_{2n+2}), S(x_{2n+1}, x_{2n+1}, y_{2n+2})\}) \\
&\leq \psi(\tfrac{1}{4}[S(x_{2n+1}, x_{2n+1}, y_{2n+2}) + \max\{2S(y_{2n}, y_{2n}, x_{2n+1})
\end{aligned}
$$

$$+S(x_{2n+1}, x_{2n+1}, y_{2n+2}), S(x_{2n+1}, x_{2n+1}, y_{2n+2})\}])$$
$$- \varphi(S(x_{2n+1}, x_{2n+1}, y_{2n+2}), \max\{S(y_{2n}, y_{2n}, y_{2n+2}), S(x_{2n+1}, x_{2n+1}, y_{2n+2})\}).$$

That is

$$\psi(S(x_{2n+1}, x_{2n+1}, y_{2n+2})) \leq \psi(\frac{1}{4}[2S(x_{2n+1}, x_{2n+1}, y_{2n+2}) + 2S(y_{2n}, y_{2n}, x_{2n+1})]) -\varphi(S(x_{2n+1}, x_{2n+1}, y_{2n+2}), \max\{S(y_{2n}, y_{2n}, y_{2n+2}), S(x_{2n+1}, x_{2n+1}, y_{2n+2})\}) \tag{3.7}$$

$< \psi(\frac{1}{2}S(x_{2n+1}, x_{2n+1}, y_{2n+2}) + \frac{1}{2}S(y_{2n}, y_{2n}, x_{2n+1}))$ and hence $S(x_{2n+1}, x_{2n+1}, y_{2n+2}) \leq \frac{1}{2}S(x_{2n+1}, x_{2n+1}, y_{2n+2}) + \frac{1}{2}S(y_{2n}, y_{2n}, x_{2n+1})$ so that

$$S(x_{2n+1}, x_{2n+1}, y_{2n+2}) \leq S(y_{2n}, y_{2n}, x_{2n+1}), \text{ for each } n = 1, 2, 3, \dots . \tag{3.8}$$

Similarly, we have

$$\begin{aligned}
&\psi(S(y_{2n+1}, y_{2n+1}, x_{2n+2})) \\
&= \psi(S(F(x_{2n}, y_{2n}), F(x_{2n}, y_{2n}), F(y_{2n+1}, x_{2n+1}))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_{2n}, x_{2n}, F(y_{2n+1}, x_{2n+1})), S(x_{2n}, x_{2n}, F(x_{2n}, y_{2n}))\} \\
&\quad + \max\{S(y_{2n+1}, y_{2n+1}, F(x_{2n}, y_{2n})), S(x_{2n}, x_{2n}, F(x_{2n}, y_{2n}))\}]) \\
&\quad - \varphi(\max\{S(x_{2n}, x_{2n}, F(y_{2n+1}, x_{2n+1})), S(x_{2n}, x_{2n}, F(x_{2n}, y_{2n}))\}, \\
&\quad \max\{S(y_{2n+1}, y_{2n+1}, F(x_{2n}, y_{2n})), S(x_{2n}, x_{2n}, F(x_{2n}, y_{2n}))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_{2n}, x_{2n}, x_{2n+2}), S(x_{2n}, x_{2n}, y_{2n+1})\} + S(x_{2n}, x_{2n}, y_{2n+1})]) \\
&\quad -\varphi(\max\{S(x_{2n}, x_{2n}, x_{2n+2}), S(x_{2n}, x_{2n}, y_{2n+1})\}, S(x_{2n}, x_{2n}, y_{2n+1})) \\
&\leq \psi(\tfrac{1}{4}[\max\{2S(x_{2n}, x_{2n}, y_{2n+1}) + S(y_{2n+1}, y_{2n+1}, x_{2n+2}), \\
&\quad S(x_{2n}, x_{2n}, y_{2n+1})\} + S(x_{2n}, x_{2n}, y_{2n+1})]) \\
&\quad -\varphi(\max\{S(x_{2n}, x_{2n}, x_{2n+2}), S(x_{2n}, x_{2n}, y_{2n+1})\}, S(x_{2n}, x_{2n}, y_{2n+1})).
\end{aligned}$$

That is

$$\psi(S(y_{2n+1}, y_{2n+1}, x_{2n+2})) \leq \psi(\frac{3}{4}S(x_{2n}, x_{2n}, y_{2n+1}) + \frac{1}{4}S(y_{2n+1}, y_{2n+1}, x_{2n+2})) -\varphi(\max\{S(x_{2n}, x_{2n}, x_{2n+2}), S(x_{2n}, x_{2n}, y_{2n+1})\}, S(x_{2n}, x_{2n}, y_{2n+1})) \tag{3.9}$$

$$< \psi(\tfrac{3}{4}S(x_{2n}, x_{2n}, y_{2n+1}) + \tfrac{1}{4}S(y_{2n+1}, y_{2n+1}, x_{2n+2}))$$

which implies that
$S(y_{2n+1}, y_{2n+1}, x_{2n+2}) \leq \frac{3}{4}S(x_{2n}, x_{2n}, y_{2n+1}) + \frac{1}{4}S(y_{2n+1}, y_{2n+1}, x_{2n+2})$ and hence

$$S(y_{2n+1}, y_{2n+1}, x_{2n+2}) \leq S(x_{2n}, x_{2n}, y_{2n+1}) \text{ for each } n = 1, 2, 3, \dots . \tag{3.10}$$

Similarly, we get

$$S(x_{2n}, x_{2n}, y_{2n+1}) \leq S(y_{2n-1}, y_{2n-1}, x_{2n}) \text{ for each } n = 1, 2, 3, \dots ; \tag{3.11}$$

and

$$S(y_{2n}, y_{2n}, x_{2n+1}) \leq S(x_{2n-1}, x_{2n-1}, y_{2n}) \text{ for each } n = 1, 2, 3, \dots . \tag{3.12}$$

From (3.8) and (3.11) it follows that

$$S(x_n, x_n, y_{n+1}) \leq S(y_{n-1}, y_{n-1}, x_n) \text{ for } n = 1, 2, 3, \dots \tag{3.13}$$

and from (3.10) and (3.12) it follows that

$$S(y_n, y_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, y_n) \text{ for } n = 1, 2, 3, \dots . \tag{3.14}$$

Hence, from (3.13) and (3.14), it follows that $\{S(x_n, x_n, y_{n+1})\}$ is a decreasing sequence and converges to some $r \geq 0$ and $\{S(y_n, y_n, x_{n+1})\}$ is a decreasing sequence and hence converges to some $s \geq 0$.
From (3.13), we have $r \leq s$ and from (3.14), we have $s \leq r$. Therefore $r = s$.
Now on taking the limits as $n \to \infty$ in (3.7), we have
$\psi(r) \leq \psi(\frac{1}{2}r + \frac{1}{2}r) - \varphi(r, \max\{\lim\limits_{n\to\infty} S(y_{2n}, y_{2n}, y_{2n+2}), r\})$
$\quad = \psi(r) - \varphi(r, \max\{\lim\limits_{n\to\infty} S(y_{2n}, y_{2n}, y_{2n+2}), r\})$
which implies that $\varphi(r, \max\{\lim\limits_{n\to\infty} S(y_{2n}, y_{2n}, y_{2n+2}), r\}) = 0$ so that $r = 0$.

Therefore

$$\lim_{n\to\infty} S(x_n, x_n, y_{n+1}) = 0 \text{ and } \lim_{n\to\infty} S(y_n, y_n, x_{n+1}) = 0. \tag{3.15}$$

We now consider

$$\begin{aligned}
\psi(S(x_{n+1}, x_{n+1}, y_{n+1})) &= \psi(S(y_{n+1}, y_{n+1}, x_{n+1})) \\
&= \psi(S(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, F(y_n, x_n)), S(x_n, x_n, F(x_n, y_n))\} \\
&\qquad + \max\{S(y_n, y_n, F(x_n, y_n)), S(x_n, x_n, F(x_n, y_n))\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, F(y_n, x_n)), S(x_n, x_n, F(x_n, y_n))\}, \\
&\qquad \max\{S(y_n, y_n, F(x_n, y_n)), S(x_n, x_n, F(x_n, y_n))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\} \\
&\qquad + \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}) \\
&\leq \psi(\tfrac{1}{4}[\max\{2S(x_n, x_n, y_{n+1}) + S(x_{n+1}, x_{n+1}, y_{n+1}), \\
&\qquad S(x_n, x_n, y_{n+1})\} + S(y_n, y_n, y_{n+1}) + S(x_n, x_n, y_{n+1})]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}) \\
&= \psi(\tfrac{1}{4}[2S(x_n, x_n, y_{n+1}) + S(x_{n+1}, x_{n+1}, y_{n+1}) \\
&\qquad + 2S(y_n, y_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, y_{n+1}) \\
&\qquad\quad + S(x_n, x_n, y_{n+1})\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}) \\
&= \psi(\tfrac{1}{4}[3S(x_n, x_n, y_{n+1}) + 2S(y_n, y_n, x_{n+1}) \\
&\qquad + 2S(x_{n+1}, x_{n+1}, y_{n+1})]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}) \\
&< \psi(\tfrac{1}{4}[3S(x_n, x_n, y_{n+1}) + 2S(y_n, y_n, x_{n+1}) \\
&\qquad + 2S(x_{n+1}, x_{n+1}, y_{n+1})]) \text{ and hence}
\end{aligned}$$

$S(x_{n+1}, x_{n+1}, y_{n+1}) \leq \frac{3}{4}S(x_n, x_n, y_{n+1}) + \frac{1}{2}S(y_n, y_n, x_{n+1}) + \frac{1}{2}S(x_{n+1}, x_{n+1}, y_{n+1})$
which implies that
$S(x_{n+1}, x_{n+1}, y_{n+1}) \leq S(y_n, y_n, x_{n+1}) + \frac{3}{2}S(x_n, x_n, y_{n+1})$.
On taking limits as $n \to \infty$ and by using (3.15), we get

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+1}) = 0. \tag{3.16}$$

We now consider

$$\begin{aligned}
S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) &\leq 2S(x_n, x_n, y_n) + S(y_n, y_n, x_{n+1}) \\
&\qquad + 2S(y_n, y_n, x_n) + S(x_n, x_n, y_{n+1}) \\
&= 4S(x_n, x_n, y_n) + S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1}).
\end{aligned}$$

On taking limits as $n \to \infty$, we get
$\lim\limits_{n\to\infty} S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) = 0.$
We now prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose that either $\{x_n\}$ or $\{y_n\}$ is not Cauchy. Then there exist $\epsilon > 0$ and subsequences $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geq \epsilon. \tag{3.17}$$

We choose m_k as a smallest integer with $m_k > n_k$ satisfying (3.17).
That is $\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geq \epsilon$
with $\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} < \epsilon$.
We now prove the following.

$$\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} = \epsilon. \tag{3.18}$$

We consider $S(x_{m_k}, x_{m_k}, y_{n_k}) = S(y_{n_k}, y_{n_k}, x_{m_k})$
$\qquad \leq 2S(y_{n_k}, y_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k}).$
Also, we have
$S(y_{m_k}, y_{m_k}, x_{n_k}) \leq 2S(x_{n_k}, x_{n_k}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k}).$
Thus we have

$$\begin{aligned}\max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} \leq \max\{&2S(y_{n_k}, y_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k}),\\ &2S(x_{n_k}, x_{n_k}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k})\}.\end{aligned}$$

On taking limit supremum as $k \to \infty$, and using Proposition 2.1, we get

$$\begin{aligned}\limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} &\leq \limsup_{k\to\infty} \max\{S(x_{n_k}, x_{n_k}, x_{m_k}),\\ &\qquad S(y_{n_k}, y_{n_k}, y_{m_k})\}\\ &= \epsilon \text{ (by (i) of Lemma 3.1)}.\end{aligned}$$

We now consider

$$\begin{aligned}S(x_{m_k}, x_{m_k}, x_{n_k}) &= S(x_{n_k}, x_{n_k}, x_{m_k})\\ &\leq 2S(x_{n_k}, x_{n_k}, y_{n_k}) + S(y_{n_k}, y_{n_k}, x_{m_k})\end{aligned}$$

and

$$\begin{aligned}S(y_{m_k}, y_{m_k}, y_{n_k}) &= S(y_{n_k}, y_{n_k}, y_{m_k})\\ &\leq 2S(y_{n_k}, y_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k}, y_{m_k}) \text{ so that}\end{aligned}$$

$$\begin{aligned}\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \max\{&2S(x_{n_k}, x_{n_k}, y_{n_k}) + S(y_{n_k}, y_{n_k}, x_{m_k}),\\ &2S(y_{n_k}, y_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k}, y_{m_k})\}.\end{aligned}$$

On taking limit infimum as $k \to \infty$ and using (3.16),

$$\begin{aligned}\epsilon &\leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}\\ &\leq \liminf_{k\to\infty} \max\{S(y_{n_k}, y_{n_k}, x_{m_k}), S(x_{n_k}, x_{n_k}, y_{m_k})\} \text{ (by Proposition 2.3)}.\end{aligned}$$

From the above we have

$$\begin{aligned}\epsilon &\leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\}\\ &\leq \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} \leq \epsilon.\end{aligned}$$

Hence $\liminf\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\}$
$\qquad = \limsup\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} = \epsilon.$
Therefore $\lim\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\}$ exists and
$\lim\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} = \epsilon.$
Hence (3.18) is proved.

We now consider

$$\begin{aligned}
\psi(S(x_{m_k}, x_{m_k}, y_{n_k})) &= \psi(y_{n_k}, y_{n_k}, x_{m_k}) \\
&= \psi(S(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k-1}, y_{n_k-1}), F(y_{m_k-1}, x_{m_k-1}))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_{n_k-1}, x_{n_k-1}, F(y_{m_k-1}, x_{m_k-1})), \\
&\qquad\qquad S(x_{n_k-1}, x_{n_k-1}, F(x_{n_k-1}, y_{n_k-1}))\} \\
&\qquad + \max\{S(y_{m_k-1}, y_{m_k-1}, F(x_{n_k-1}, y_{n_k-1})), \\
&\qquad\qquad S(x_{n_k-1}, x_{n_k-1}, F(x_{n_k-1}, y_{n_k-1}))\}]) \\
&\qquad - \varphi(\max\{S(x_{n_k-1}, x_{n_k-1}, F(y_{m_k-1}, x_{m_k-1})), \\
&\qquad\qquad S(x_{n_k-1}, x_{n_k-1}, F(x_{n_k-1}, y_{n_k-1}))\}, \\
&\qquad\quad \max\{S(y_{m_k-1}, y_{m_k-1}, F(x_{n_k-1}, y_{n_k-1})), \\
&\qquad\qquad S(x_{n_k-1}, x_{n_k-1}, F(x_{n_k-1}, y_{n_k-1}))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\} \\
&\qquad\quad + \max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}]) \\
&\quad - \varphi(\max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}, \\
&\qquad\quad \max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\psi(S(y_{m_k}, y_{m_k}, x_{n_k})) &= \psi(S(F(x_{m_k-1}, y_{m_k-1}), F(x_{m_k-1}, y_{m_k-1}), F(y_{n_k-1}, x_{n_k-1}))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_{m_k-1}, x_{m_k-1}, F(y_{n_k-1}, x_{n_k-1})), \\
&\qquad\qquad S(x_{m_k-1}, x_{m_k-1}, F(x_{m_k-1}, y_{m_k-1}))\} \\
&\qquad + \max\{S(y_{n_k-1}, y_{n_k-1}, F(x_{m_k-1}, y_{m_k-1})), \\
&\qquad\qquad S(x_{m_k-1}, x_{m_k-1}, F(x_{m_k-1}, y_{m_k-1}))\}]) \\
&\qquad - \varphi(\max\{S(x_{m_k-1}, x_{m_k-1}, F(y_{n_k-1}, x_{n_k-1})), \\
&\qquad\qquad S(x_{m_k-1}, x_{m_k-1}, F(x_{m_k-1}, y_{m_k-1}))\}, \\
&\qquad\quad \max\{S(y_{n_k-1}, y_{n_k-1}, F(x_{m_k-1}, y_{m_k-1})), \\
&\qquad\qquad S(x_{m_k-1}, x_{m_k-1}, F(x_{m_k-1}, y_{m_k-1}))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\} \\
&\qquad\quad +\max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}]) \\
&\qquad -\varphi(\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}, \\
&\qquad\quad \max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}).
\end{aligned}$$

We now consider

$$\begin{aligned}
&\psi(\max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\}) \\
&\qquad = \max\{\psi(S(x_{m_k}, x_{m_k}, y_{n_k})), \psi(S(y_{m_k}, y_{m_k}, x_{n_k}))\} \\
&\qquad \leq \max\{\psi(\tfrac{1}{4}[\max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\} \\
&\qquad\qquad +\max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}]), \\
&\qquad\qquad \psi(\tfrac{1}{4}[\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\} \\
&\qquad\qquad +\max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}])\} \\
&\qquad\quad -\min\{\varphi(\max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}, \\
&\qquad\qquad \max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}), \\
&\qquad\qquad \varphi(\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}, \\
&\qquad\qquad \max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\})\}.
\end{aligned}$$

On letting $k \to \infty$ and by using (3.15), we get

$$\begin{aligned}
\psi(\epsilon) \leq \max\{&\psi(\tfrac{1}{4}[\lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) + \lim_{k\to\infty} S(y_{m_k-1}, y_{m_k-1}, y_{n_k})]), \\
&\psi(\tfrac{1}{4}[\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) + \lim_{k\to\infty} S(y_{n_k-1}, y_{n_k-1}, y_{m_k})])\} \\
- \min\{&\varphi(\lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), \lim_{k\to\infty} S(y_{m_k-1}, y_{m_k-1}, y_{n_k})),
\end{aligned}$$

$$\varphi(\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), \lim_{k\to\infty} S(y_{n_k-1}, y_{n_k-1}, y_{m_k}))\} \text{ so that}$$

$$\begin{aligned}
\psi(\epsilon) \leq \psi(\max\{&\tfrac{1}{4}[\lim_{k\to\infty} \max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(y_{n_k-1}, y_{n_k-1}, y_{m_k})\} \\
&\quad + \lim_{k\to\infty} \max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{m_k-1}, x_{m_k-1}, x_{n_k})\}], \\
&\tfrac{1}{4}[\lim_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} \\
&\quad + \lim_{k\to\infty} \max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{n_k-1}, x_{n_k-1}, x_{m_k})\}]\}) \\
&- \min\{\varphi(\lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), \lim_{k\to\infty} S(y_{m_k-1}, y_{m_k-1}, y_{n_k})), \\
&\quad \varphi(\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), \lim_{k\to\infty} S(y_{n_k-1}, y_{n_k-1}, y_{m_k}))\}. \\
&\quad \text{(By using (ii) and (iii) of Lemma 3.1)} \\
= \psi(\tfrac{\epsilon}{2}) &- \min\{\varphi(\lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), \lim_{k\to\infty} S(y_{m_k-1}, y_{m_k-1}, y_{n_k})), \\
&\quad \varphi(\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), \lim_{k\to\infty} S(y_{n_k-1}, y_{n_k-1}, y_{m_k}))\} \\
&\quad \text{(all these limits are positive by using Lemma 2.11)} \\
< \psi(\tfrac{\epsilon}{2})&, \text{ a contradiction.}
\end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences and hence convergent. Since A and B are closed subsets of X and $\{x_n\} \subset A$, $\{y_n\} \subset B$, there exist $x \in A$ and $y \in B$ such that

$$x_n \to x,\ y_n \to y \text{ as } n \to \infty. \tag{3.19}$$

By using (3.16), we get $\lim_{n\to\infty} S(x_n, x_n, y_n) = 0$. Now, by Lemma 2.9, we have $S(x, x, y) = 0$ and hence $x = y$ so that $A \cap B \neq \emptyset$ and $x \in A \cap B$.

Now, by (3.1) and (3.4), we have

$$\begin{aligned}
\psi(S(x_{n+1}, x_{n+1}, F(x,x)) &= \psi(S(F(x,x), F(x,x), F(y_n, x_n))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x, x, F(y_n, x_n)), S(x, x, F(x,x))\} \\
&\qquad + \max\{S(y_n, y_n, F(x,x)), S(x, x, F(x,x))\}]) \\
&\quad - \varphi(\max\{S(x, x, F(y_n, x_n)), S(x, x, F(x,x))\}, \\
&\qquad \max\{S(y_n, y_n, F(x,x)), S(x, x, F(x,x))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x, x, x_{n+1}), S(x, x, F(x,x))\} \\
&\qquad + \max\{S(y_n, y_n, F(x,x)), S(x, x, F(x,x))\}]) \\
&\quad - \varphi(\max\{S(x, x, x_{n+1}), S(x, x, F(x,x))\}, \\
&\qquad \max\{S(y_n, y_n, F(x,x)), S(x, x, F(x,x))\}).
\end{aligned}$$

On taking limits as $n \to \infty$, we get

$$\begin{aligned}
\psi(S(x, x, F(x,x))) &\leq \psi(\tfrac{1}{4}[\max\{S(x, x, x), S(x, x, F(x,x))\} \\
&\qquad + \max\{S(y, y, F(x,x)), S(x, x, F(x,x))\}]) \\
&\quad - \varphi(\max\{S(x, x, x), S(x, x, F(x,x))\}, \\
&\qquad \max\{S(y, y, F(x,x)), S(x, x, F(x,x))\}) \\
&\leq \psi(S(x, x, F(x,x))) - \varphi(S(x, x, F(x,x)), S(x, x, F(x,x))) \text{ and}
\end{aligned}$$

hence $\varphi(S(x, x, F(x,x)), S(x, x, F(x,x))) = 0$ so that

$S(x, x, F(x,x)) = 0$. Therefore $x = F(x,x)$ is a strong coupled fixed point of F.

We now prove the uniqueness of strong coupled fixed point of F. Suppose (x, x) and (y, y) are two strong coupled fixed points of F. We consider

$$\begin{aligned}
\psi(S(x, x, y)) &= \psi(S(F(x,x), F(x,x), F(y,y))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x, x, F(y,y)), S(x, x, F(x,x))\} \\
&\qquad + \max\{S(y, y, F(x,x)), S(x, x, F(x,x))\}]) \\
&\quad - \varphi(\max\{S(x, x, F(y,y)), S(x, x, F(x,x))\}, \\
&\qquad \max\{S(y, y, F(x,x)), S(x, x, F(x,x))\}) \\
&= \psi(\tfrac{1}{4}[S(x, x, y) + S(y, y, x)]) - \varphi(S(x, x, y), S(y, y, x))
\end{aligned}$$

$$\leq \psi(S(x,x,y)) - \varphi(S(x,x,y), S(y,y,x))$$

so that $\varphi(S(x,x,y), S(y,y,x)) = 0$. Thus $x = y$. $\square$

By choosing $\psi(t) = t$ in Theorem 3.2, then we have the following.

Corollary 3.3. Let (X, S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be mapping. If F is cyclic with respect to A and B and there exists $\varphi \in \Phi$ such that

$$\begin{aligned} S(F(x,y), F(u,v), F(w,z)) \leq \frac{1}{4}[&\max\{S(x,x,F(w,z)), S(x,x,F(u,v))\} \\ &+ \max\{S(w,w,F(x,y)), S(u,u,F(x,y))\}] \\ &-\varphi(\max\{S(x,x,F(w,z)), S(x,x,F(u,v))\}, \\ &\quad \max\{S(w,w,F(x,y)), S(u,u,F(x,y))\}) \end{aligned}$$

where $x, u, z \in A$ and $y, v, w \in B$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

By choosing $\varphi(t_1, t_2) = (\frac{1}{4} - k)(t_1 + t_2)$ in Corollary 3.3 then we have the following.

Corollary 3.4. Let (X, S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be mapping. If F is cyclic with respect to A and B and there exists $k \in (0, \frac{1}{4})$ such that

$$\begin{aligned} S(F(x,y), F(u,v), F(w,z)) \leq k[&\max\{S(x,x,F(w,z)), S(x,x,F(u,v))\} \\ &+ \max\{S(w,w,F(x,y)), S(u,u,F(x,y))\}] \end{aligned}$$

where $x, u, z \in A$ and $y, v, w \in B$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

By choosing $w = u$ and $z = v$ in Corollary 3.4, then we have the following.

Corollary 3.5. Let (X, S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be mapping. If F is cyclic with respect to A and B and there exists $k \in (0, \frac{1}{4})$ such that

$$S(F(x,y), F(u,v), F(u,v)) \leq k[S(x,x,F(u,v)) + S(u,u,F(x,y))$$

where $x, v \in A$ and $y, u \in B$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

The following example is in support of Theorem 3.2.

Example 3.2. Let $X = [0, 1]$. We define $S : X^3 \to [0, \infty)$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ x + y + z & \text{otherwise.} \end{cases}$$

Then (X, S) is a complete S-metric space.
Let $A = [0, \frac{1}{2}]$ and $B = [0, 1]$. We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x}{8(x+y+1)} & \text{if } x \in A \text{ and } y \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then $F(A, B) = [0, \frac{1}{16}] \subset B$ and $F(B, A) = \{0\} \subset A$ so that F is cyclic with respect to A and B. We define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$ and $\varphi : [0, \infty)^2 \to [0, \infty)$ by $\varphi(t_1, t_2) = \frac{1}{16}(t_1 + t_2)$. We now verify the inequality (3.1). Let $x, u, z \in A$ and $y, v, w \in B$. We now consider

$$\begin{aligned}\psi(S(F(x,y),F(u,v),F(w,z))) &= \psi(S(\tfrac{x}{8(x+y+1)},\tfrac{u}{8(u+v+1)},0))\\ &= \tfrac{1}{2}S(\tfrac{x}{8(x+y+1)},\tfrac{u}{8(u+v+1)},0)\\ &= \tfrac{x}{16(x+y+1)}+\tfrac{u}{16(u+v+1)}\\ &\leq \tfrac{1}{16}[x+u]\\ &\leq \tfrac{1}{16}[\max\{S(x,x,F(w,z)),S(x,x,F(u,v))\}\\ &\qquad +\max\{S(w,w,F(x,y)),S(u,u,F(x,y))\}]\\ &= \tfrac{1}{8}[t_1+t_2]-\tfrac{1}{16}[t_1+t_2]\\ &= \psi(\tfrac{1}{4}[t_1+t_2])-\varphi(t_1,t_2),\end{aligned}$$

where $t_1=\max\{S(x,x,F(w,z)),S(x,x,F(u,v))\}$ and

$t_2=\max\{S(w,w,F(x,y)),S(u,u,F(x,y))\}$.

Therefore F is a Chatterjea type (ψ,φ)-weakly cyclic coupled mapping with respect to A and B. Hence F satisfies all the hypotheses of Theorem 3.2 and $(0,0)$ is a unique strong coupled fixed point of F.

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