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Some New Results in Partial Cone b-Metric Space

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Abstract
In this paper, we introduce the concepts of the Ulam-Hyers-Rassias stability and the limit shadowing property of a fixed point problem and the P-property of a mapping in partial cone b-metric space. Also, we give such results by using the mapping which is studied by Fernandez et al. (Filomat **30**(10) (2016)) in partial cone b-metric space and provide some numerical examples to support our results. The results presented here extend and improve some recent results announced in the current literature.

Keywords: Fixed point, Limit shadowing property, P-property, Partial cone b-metric space, Ulam-Hyers-Rassias stability.
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1. Introduction

Fixed point theory plays an important role in applications of many branches of mathematics. There has been a number of generalizations of metric spaces. One of them is a b-metric space which is introduced by Czerwik [1]. After that a series of articles has been dedicated to the improvement of fixed point theory. In 2011, Hussain and Shah [2] introduced the concept of cone b-metric space and studied some topological properties. At the same year, Sönmez [3] introduced the concept of partial cone metric space and proved some important fixed point theorems in such spaces. In 2016, Fernandez et al. [4] introduced the concept of partial cone b-metric space which is a generalization of cone b-metric space and partial cone metric space. They also established the following fixed point result for asymptotically regular sequences in the setting of partial cone b-metric space.

Theorem 1.1. *(see [4, Theorem 5.1]) Let (X, p_b) be a complete partial cone b-metric space, P be a normal cone with the normal constant K and $T : X \to X$ be a mapping satisfying the inequality*

$$p_b(Tx,Ty) \leq a_1 p_b(x,Tx) + a_2 p_b(y,Ty) + a_3 p_b(x,Ty) + a_4 p_b(y,Tx) + a_5 p_b(x,y) \tag{1.1}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are non-negative real numbers and satisfy the condition $a_3 + a_4 + a_5 < 1$. If there exists an asymptotically T-regular sequence in X, then T has a unique fixed point.

In this paper, we consider the mapping satisfying (1.1) in partial cone b-metric space. This paper contains four sections. In section 2, we give basic definitions and a detailed overview of the fundamental results. In section 3, we prove the Ulam-Hyers-Rassias stability and the limit shadowing property of the fixed point problem. In section 4, we present the P-property result of the mapping. Our results can be viewed as refinement and generalization of several well-known results in partial cone metric space and cone b-metric space.

2. Preliminaries

Let $(E, \|.\|)$ be a real Banach space. A subset P of E is called a cone if and only if
(1) P is closed, nonempty and $P \neq \{\theta\}$;
(2) $ax+by \in P$ for all $x,y \in P$ and $a,b \geq 0$;
(3) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y-x \in intP$ (the interior of P). A cone P is called normal if there is a number $K > 0$ such that for all $x,y \in E$, $\theta \leq x \leq y$ implies that

$$\|x\| \leq K\|y\|. \tag{2.1}$$

The least positive number satisfying (2.1) is called the normal constant of P. It is clear that $K \geq 1$.

Definition 2.1. (see [2]) Let X be a nonempty set, and let P be a cone in a real Banach space E. A vector-valued function $d : X \times X \to P$ is said to be cone b-metric with the constant $s \geq 1$ if the following conditions are satisfied:
(1) $\theta \leq d(x,y)$, for all $x,y \in X$, and $d(x,y) = \theta$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(3) $d(x,y) \leq s[d(x,z)+d(z,y)]$ for all $x,y,z \in X$.
Then the pair (X,d) is called a cone b-metric space.

Definition 2.2. (see [3]) Let X be a nonempty set, and let P be a cone in a real Banach space E. A partial cone metric on X is a function $p : X \times X \to P$ such that, for all $x,y,z \in X$:
(1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
(2) $\theta \leq p(x,x) \leq p(x,y)$;
(3) $p(x,y) = p(y,x)$;
(4) $p(x,y) \leq p(x,z)+p(z,y)-p(z,z)$.
In this case, the pair (X, p) is called a partial cone metric space.

Definition 2.3. (see [4, Definition 3.1]) Let X be a nonempty set, and let P be a cone in a real Banach space E. A partial cone b-metric on X is a function $p_b : X \times X \to P$ such that, for all $x,y,z \in X$:
(1) $x = y \Longleftrightarrow p_b(x,x) = p_b(x,y) = p_b(y,y)$;
(2) $\theta \leq p_b(x,x) \leq p_b(x,y)$;
(3) $p_b(x,y) = p_b(y,x)$;
(4) $p_b(x,y) \leq s[p_b(x,z)+p_b(z,y)]-p_b(z,z)$.
Then the pair (X, p_b) is called a partial cone b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

In partial cone b-metric space (X, p_b), if $x,y \in X$ and $p_b(x,y) = \theta$, then $x = y$, but the converse may not be true. It is clear that every partial cone metric space is a partial cone b-metric space with the coefficient $s = 1$ and every cone b-metric space is a partial cone b-metric space with the same coefficient and zero self distance. However, the converse of these facts does not necessarily hold.

Example 2.4. (see [4]) (i) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x,y \geq 0\}$, $X = [0,\infty)$, $p > 1$ be a constant and $p_b : X \times X \to P$ be defined by

$$p_b(x,y) = ((\max\{x,y\})^p + |x-y|^p, \alpha(\max\{x,y\})^p) + |x-y|^p))$$

for all $x,y \in X$, where $\alpha \geq 0$ is a constant. Then (X, p_b) is a partial cone b-metric space with coefficient $s = 2^p > 1$. But it is not a partial cone metric space.

(ii) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x,y \geq 0\}$, $X = [0,\infty)$, $p > 1$ be a constant and $p_b : X \times X \to P$ be defined by

$$p_b(x,y) = ((\max\{x,y\})^p, \alpha(\max\{x,y\})^p)$$

for all $x,y \in X$, where $\alpha \geq 0$ is a constant. Then (X, p_b) is a partial cone b-metric space which is not a cone b-metric space.

Definition 2.5. (see [4]) Let (X, p_b) be a partial cone b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is:
(i) convergent to x and x is called a limit of $\{x_n\}$ if

$$\lim_{n\to\infty} p_b(x_n,x) = \lim_{n\to\infty} p_b(x_n,x_n) = p_b(x,x).$$

(ii) Cauchy sequence if there is $a \in P$ such that for every $\varepsilon > 0$ there is N such that for all $n,m > N$, $\|p_b(x_n,x_m)-a\| < \varepsilon$.

Definition 2.6. (see [4]) A partial cone b-metric space (X, p_b) is said to be complete if every Cauchy sequence in (X, p_b) is convergent in (X, p_b).

Theorem 2.7. *(see [4]) Let (X, p_b) be a partial cone b-metric space and P be a normal cone with a normal constant K. Let $x \in X$ and $\{x_n\}$ be a sequence in X. Then*
(i) $\{x_n\}$ converges to x if and only if $p_b(x_n, x) \to p_b(x, x)$ as $n \to \infty$.
(ii) $p_b(x_n, x_n) \to p_b(x, x)$ as $n \to \infty$ if $p_b(x_n, x) \to p_b(x, x)$ as $n \to \infty$.

Definition 2.8. (see [4, Definition 4.1]) Let (X, p_b) be a partial cone b-metric space. A sequence $\{x_n\}$ in X is said to be asymptotically T-regular if $\lim_{n\to\infty} p_b(x_n, Tx_n) = \theta$.

3. The Ulam-Hyers-Rassias stability and the limit shadowing property results

Speaking of the stability problem of functional equations, we follow a question raised in 1940 by Ulam, concerning approximate homomorphisms of groups (see [5]). Hyers [6] gave the first affirmative partial answer to the question of Ulam for Banach spaces in 1941 and after the fact, this type of stability is called the Ulam-Hyers stability. Hyers's theorem was generalized by Aoki [7] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. Rassias [8] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

and derived Hyers's theorem. The work of Rassias has influenced a number of mathematicians to develop the notion what is now a days referred to as Ulam-Hyers-Rassias stability of linear mappings. Since then, stability of other functional equations, differential equations, and of various integral equations has been extensively investigated by many mathematicians (see [9, 10, 11, 12, 13]).

Now, we introduce the concept of Ulam-Hyers-Rassias stability of a fixed point problem in partial cone b-metric space.

Definition 3.1. Let (X, p_b) be a partial cone b-metric space and $T : X \to X$ be a mapping. A fixed point problem

$$Tx = x \tag{3.1}$$

has Ulam-Hyers-Rassias stability if and only if there exists the function $\sigma : [0,\infty) \to [0,\infty)$ which is increasing, continuous at 0 and $\sigma(0) = 0$ such that for $\varepsilon > 0$ and $y^* \in X$ which is an ε-solution of the fixed point equation (3.1), that is, y^* satisfied the inequality

$$\|p_b(y^*, Ty^*)\| \leq \sigma(t),$$

there exists a solution $x^* \in X$ of (3.1) such that

$$\|p_b(x^*, y^*)\| \leq c_1.\sigma(t)$$

for some $c_1 > 0$.

Remark 3.2. If the function σ is defined by $\sigma(t) = \varepsilon$ for all $t \geq 0$ where $\varepsilon > 0$, then the fixed point equation (3.1) has Ulam-Hyers stability.

Next, we prove that the fixed point equation (3.1) has the Ulam-Hyers-Rassias stability.

Theorem 3.3. *Let (X, p_b) be a complete partial cone b-metric space, P be a normal cone with the normal constant K and $T : X \to X$ be a mapping satisfying the inequality*

$$p_b(Tx, Ty) \leq a_1 p_b(x, Tx) + a_2 p_b(y, Ty) + a_3 p_b(x, Ty) + a_4 p_b(y, Tx) + a_5 p_b(x, y) \tag{3.2}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are non-negative real numbers such that the condition $s(a_1 + a_3 s + a_4 + a_5) < 1$ holds. If there exists an asymptotically T-regular sequence in X, then the fixed point problem (3.1) has the Ulam-Hyers-Rassias stability.

Proof. Since $a_3 + a_4 + a_5 < s(a_1 + a_3 s + a_4 + a_5) < 1$, then all hypotheses of Theorem 1.1 are satisfied. Hence, we can say that the mapping T has a unique fixed point $x^* \in X$. Let $\varepsilon > 0$ and $y^* \in X$ be a ε-solution of (3.1), that is,

$$\|p_b(y^*, Ty^*)\| \leq \sigma(t).$$

Now we have

$$\begin{aligned} p_b(x^*,y^*) &= p_b(Tx^*,y^*) \\ &\leq s[p_b(Tx^*,Ty^*)+p_b(Ty^*,y^*)]-p_b(Ty^*,Ty^*) \\ &\leq sp_b(Tx^*,Ty^*)+sp_b(Ty^*,y^*). \end{aligned} \tag{3.3}$$

Also, we obtain

$$\begin{aligned} & sp_b(Tx^*,Ty^*) \\ \leq\ & s[a_1p_b(x^*,Tx^*)+a_2p_b(y^*,Ty^*)+a_3p_b(x^*,Ty^*)+a_4p_b(y^*,Tx^*)+a_5p_b(x^*,y^*)] \\ \leq\ & a_1sp_b(x^*,y^*)+a_2sp_b(y^*,Ty^*)+a_3s[s(p_b(x^*,y^*)+p_b(y^*,Ty^*))-p_b(y^*,y^*)]+a_4sp_b(y^*,x^*)+a_5sp_b(x^*,y^*) \\ \leq\ & a_1sp_b(x^*,y^*)+a_2sp_b(y^*,Ty^*)+a_3s^2p_b(x^*,y^*)+a_3s^2p_b(y^*,Ty^*)+a_4sp_b(y^*,x^*)+a_5sp_b(x^*,y^*). \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\left[1-(a_1s+a_3s^2+a_4s+a_5s)\right]p_b(x^*,y^*) \leq \left(a_2s+a_3s^2+s\right)p_b(y^*,Ty^*).$$

Hence, we get

$$\|p_b(x^*,y^*)\| \leq K.\frac{a_2s+a_3s^2+s}{1-s(a_1+a_3s+a_4+a_5)}\|p_b(y^*,Ty^*)\|.$$

Therefore, we obtain

$$\|p_b(x^*,y^*)\| \leq c_1\sigma(t)$$

where

$$c_1 = K.\frac{a_2s+a_3s^2+s}{1-s(a_1+a_3s+a_4+a_5)} > 0.$$

This completes the proof. □

The following example illustrates Theorem 3.3.

Example 3.4. Let (X,p_b) be a complete partial cone b-metric space which is defined as in Example 2.4 (i) such that $p=2$ and $s=4$. Let T be a self mapping of X such that $Tx=\frac{2x}{5}$ for all $x\in X$. Then, the mapping T satisfies the contractive condition (3.2) with $a_1=a_2=a_3=a_4=0$ and $a_5=\frac{1}{5}$. It is clearly seen that 0 is the unique fixed point of T. Assume that $\varepsilon>0$ and $y^*\in X$ is an ε-solution of the fixed point problem of T, that is,

$$\|p_b(y^*,Ty^*)\| \leq \sigma(t).$$

If we take $K=1$, we get

$$\|p_b(0,y^*)\| \leq 20.\sigma(t),$$

and so the fixed point problem (3.1) has the Ulam-Hyers-Rassias stability.

Corollary 3.5. *Under the assumptions of Theorem 3.3, the fixed point problem (3.1) has the Ulam-Hyers stability, that is, for every $y^*\in X$ and $\varepsilon>0$ with $\|p_b(y^*,Ty^*)\|\leq\varepsilon$, there exists a unique $x^*\in X$ such that*

$$Tx^*=x^* \quad \text{and} \quad \|p_b(x^*,y^*)\|\leq c_1\varepsilon$$

for some $c_1>0$.

The following example demonstrates Corollary 3.5.

Example 3.6. Let (X,p_b) be a complete partial cone b-metric space which is defined as in Example 2.4 (ii) such that $p=2$, and let T be a self mapping of X such that $Tx=\frac{x}{4}$ for all $x\in X$. Then, the mapping T satisfies the contractive condition (3.2) with $a_1=a_2=a_3=a_4=0$ and $a_5=\frac{1}{3}$. It is clearly seen that 0 is the unique fixed point of T. If we take $K=1$, we get

$$\|p_b(0,y^*)\| \leq 6.\varepsilon,$$

and so the fixed point problem (3.1) has the Ulam-Hyers stability.

The limit shadowing property of a fixed point problem have evoked much interest to many researchers, for example, Sintunavarat [12], Pilyugin [14].

In 2014, Sintunavarat [12] introduced the limit shadowing property of a fixed point problem in metric spaces.

Definition 3.7. (see [12]) Let (X,d) be a metric space and $T: X \to X$ be a mapping. We say that the fixed point problem of T has the limit shadowing property in X if for any sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, it follows that there exists $x^* \in X$ such that $\lim_{n\to\infty} d(T^n x^*, x_n) = 0$.

Similarly, we define the limit shadowing property of a fixed point problem in partial cone b-metric space.

Definition 3.8. Let (X, p_b) be a partial cone b-metric space and $T: X \to X$ be a mapping. We say that the fixed point problem of T has the limit shadowing property in X if for any sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} p_b(x_n, Tx_n) = \theta$, it follows that there exists $x^* \in X$ such that $\lim_{n\to\infty} p_b(T^n x^*, x_n) = \theta$.

Now, we prove that the fixed point equation (3.1) has the limit shadowing property.

Theorem 3.9. *Let (X, p_b) be a complete partial cone b-metric space, P be a normal cone and $T: X \to X$ be a mapping satisfying (3.2) with $a_3 + a_4 + a_5 < 1$. If there exists an asymptotically T-regular sequence in X, then the fixed point problem of T has the limit shadowing property in X.*

Proof. Let $\{x_n\}$ is an asymptotically T-regular sequence in X. Then we say that

$$\lim_{n\to\infty} p_b(x_n, Tx_n) = \theta.$$

Also, from Theorem 1.1, the mapping T has a unique fixed point $x^* \in X$ and the sequence $\{x_n\}$ converges to x^*. Therefore, we can write

$$\lim_{n\to\infty} p_b(x_n, T^n x^*) = \lim_{n\to\infty} p_b(x_n, x^*) = \theta.$$

This completes the proof. □

The following example illustrates Theorem 3.9.

Example 3.10. Let (X, p_b) and T be defined as in Example 3.6. Choose a sequence $\{x_n\}$, $x_n \neq 0$ for any positive integer n, which converges to zero. Then $\{x_n\}$ is an asymptotically T-regular sequence in (X, p_b). We can see that there is $x^* = 0 \in X$ such that

$$\begin{aligned} \lim_{n\to\infty} p_b(T^n x^*, x_n) &= \lim_{n\to\infty} p_b(0, x_n) = \lim_{n\to\infty} (x_n^2, \alpha x_n^2) \\ &= (0, \alpha 0) \\ &= \theta. \end{aligned}$$

Hence the fixed point problem of T has the limit shadowing property.

4. The P-property result

Rhoades defined the P-property on metric spaces in his works [15], [16] and [17]. Denote, as usual, by $F(T)$ the set of fixed points of the mapping $T: X \to X$. We say that a self-mapping T has the P-property whenever $F(T) = F(T^n)$ for all $n \geq 1$, that is, it has no periodic points. Note that $F(T) \subseteq F(T^n)$ for all $n \geq 1$. It is clear that if T is a mapping which has a fixed point x^*, then x^* is also a fixed point of T^n for all $n \geq 1$. It is well known that the converse is not true. However if a mapping T satisfies $F(T^n) \subseteq F(T)$ for all $n \geq 1$, then it is said to have the P-property.

In 2018, Huang et al. [18] gave a characterization for the P-property in b-metric space.

Theorem 4.1. *(see [18]) Let (X,d) be a b-metric space with coefficient $s \geq 1$. Let $T: X \to X$ be a mapping such that $F(T) \neq \emptyset$ and*

$$d(Tx, T^2 x) \leq \lambda d(x, Tx)$$

for all $x \in X$, where $0 \leq \lambda < 1$ is a constant. Then the mapping T has the P-property.

Now, we generalize Theorem 4.1 to partial cone b-metric space.

Theorem 4.2. *Let (X, p_b) be a partial cone b-metric space, P be a normal cone with the normal constant K and $T : X \to X$ be a mapping such that $F(T) \neq \emptyset$. Then T has the P-property if it is satisfied the following inequality*

$$p_b(Tx, T^2x) \leq \lambda p_b(x, Tx)$$

where $0 \leq \lambda < 1$.

Proof. We always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $x^* \in F(T^n)$. By the hypotheses, it is clear that

$$\begin{aligned} p_b(x^*, Tx^*) &= p_b(TT^{n-1}x^*, T^2T^{n-1}x^*) \leq \lambda p_b(T^{n-1}x^*, T^n x^*) \\ &= \lambda p_b(TT^{n-2}x^*, T^2T^{n-2}x^*) \\ &\leq \lambda^2 p_b(T^{n-2}x^*, T^{n-1}x^*) \leq \ldots \leq \lambda^n p_b(x^*, Tx^*). \end{aligned}$$

Since P is a normal cone with the normal constant K, then we have

$$\|p_b(x^*, Tx^*)\| \leq K\lambda^n \|p_b(x^*, Tx^*)\| \to 0 \text{ as } n \to \infty.$$

Hence, we get $p_b(x^*, Tx^*) = \theta$, that is, $x^* \in F(T)$. ☐

Next we prove that the mapping T has the P-property.

Theorem 4.3. *Let (X, p_b) be a complete partial cone b-metric space, P be a normal cone and $T : X \to X$ be a mapping satisfying the inequality (3.2) with $a_1 + a_2 + 2sa_3 + a_4 + a_5 < 1$. Then the mapping T has the P-property.*

Proof. Noting $a_3 + a_4 + a_5 < a_1 + a_2 + 2sa_3 + a_4 + a_5 < 1$, by Theorem 1.1, we get $x^* \in F(T)$. Using (3.2), we obtain

$$\begin{aligned} & p_b(Tx, T^2x) \\ = \; & p_b(Tx, TTx) \\ \leq \; & a_1 p_b(x, Tx) + a_2 p_b(Tx, T^2x) + a_3 p_b(x, T^2x) + a_4 p_b(Tx, Tx) + a_5 p_b(x, Tx) \\ \leq \; & a_1 p_b(x, Tx) + a_2 p_b(Tx, T^2x) + a_3[s(p_b(x, Tx) + p_b(Tx, T^2x)) - p_b(Tx, Tx)] + a_4 p_b(Tx, Tx) + a_5 p_b(x, Tx) \\ \leq \; & a_1 p_b(x, Tx) + a_2 p_b(Tx, T^2x) + a_3 s p_b(x, Tx) + a_3 s p_b(Tx, T^2x) + a_4 p_b(Tx, x) + a_5 p_b(x, Tx). \end{aligned}$$

Hence, we have

$$p_b(Tx, T^2x) \leq \frac{a_1 + sa_3 + a_4 + a_5}{1 - (a_2 + sa_3)} p_b(x, Tx).$$

Therefore, we obtain

$$p_b(Tx, T^2x) \leq \lambda . p_b(x, Tx)$$

where $\lambda = \frac{a_1 + sa_3 + a_4 + a_5}{1 - (a_2 + sa_3)} < 1$. Consequently, by Theorem 4.2, the mapping T has the P-property. ☐

Finally, we give an example to support Theorem 4.3.

Example 4.4. Let (X, p_b) and T be the same as in Example 3.6. If we take $a_1 = a_2 = a_3 = a_4 = 0$ and $a_5 = \frac{1}{16}$, then we get

$$p_b(Tx, T^2x) = p_b\left(\frac{x}{4}, \frac{x}{16}\right) = \left(\frac{x^2}{16}, \alpha\frac{x^2}{16}\right) = \frac{1}{16} p_b\left(x, \frac{x}{4}\right) = \frac{1}{16} p_b(x, Tx)$$

and so the mapping T has the P-property.

Conclusion

In this paper, based on the class of mappings studied by Fernandez et al. [4], we have proved the Ulam-Hyers-Rassias stability and the limit shadowing property results of a fixed point problem and the P-property of a mapping in partial cone b-metric space. If $P = [0, \infty)$ and $s = 1$ are taken in our results, the similar results are obtained in partial metric space.

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The Generating Functions for Special Pringsheim Continued Fractions

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Abstract
In previous works, some relations between Pringsheim continued fractions and vertices of the paths of minimal length on the suborbital graphs $\mathbf{F}_{u,N}$ were investigated. Then, for special vertices, the relations between these vertices and Fibonacci numbers were examined. On the other hand, Koshy studied relation between recurrence relations of Fibonacci numbers, Pell numbers and generating functions. In this work, it is showed that every vertex on the path of minimal length of suborbital graph $\mathbf{F}_{u,N}$ has a Pringsheim continued fraction. Then, by Koshy's motivation, the generating function of the recurrence relation of these pringsheim continued fractions are examined.

Keywords: Fibonacci Sequence, Generating Functions, Pell Sequence, Continued Fractions, Suborbital Graphs
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1. Introduction

The idea of suborbital graphs is used by Jones, Singerman and Wicks [1] for finite permutation groups, described by Sims [2] for the congruence subgroup $\Gamma_0(N)$ of the modular group Γ. The $\mathbf{G}_{u,N}$ and $\mathbf{F}_{u,N}$ suborbital graphs are consisted at these works. In [3] and [8], the results are extended. The definition of minimal length path for $\mathbf{F}_{u,N}$ suborbital graphs is given by Deger in [4]. In [4], it has been showed that; there is a integer t, which provides $u^2+tu+1\equiv 0\,(mod\,N)$ congruence equation for $\mathbf{F}_{u,N}$ suborbital graphs. By using integer t, the vertices on the minimal length path of $\mathbf{F}_{u,N}$ suborbital graphs are consisted with Pringsheim continued fractions, according to edge conditions of $\mathbf{F}_{u,N}$ suborbital graphs [3]. When $t=3$, the continued fractions give the even term index Fibonacci numbers [4].

On the other hand, the relations between Fibonacci sequences, Pell sequences and Generating Functions have been examined by Koshy in [5] and [6]. The author has used recurrence relations of Fibonacci and Pell sequences to write generating functions. In this paper, we examine the generating functions for vertices on the minimal length path of $\mathbf{F}_{u,N}$ suborbital graphs. Particularly, for $t=3$ and $t=6$, we find the relations between Fibonacci and Pell numbers with generating functions of $\mathbf{F}_{u,N}$ suborbital graphs. The interval of convergence of the series obtained by generating functions are examined.

1.1 Fibonacci, Lucas, Pell, Pell-Lucas Sequences

The numbers $1,1,2,3,5,8,\ldots$ are called Fibonacci Numbers and the sequence of these numbers is called Fibonacci sequence. The recurrence relation of the Fibonacci sequence is

$$F_n=F_{n-1}+F_{n-2};n\geq 3$$

with $F_1 = F_2 = 1$; initial conditions for n^{th} Fibonacci number. When initial conditions are chanced, the same relation gives Lucas numbers which are $1,3,4,7,\ldots$. So, the recurrence relation for n^{th} Lucas number is

$$L_n = L_{n-1} + L_{n-2}; n \geq 3$$

with $L_1 = 1, L_2 = 3$ initial conditions.

The numbers $1,2,5,12,\ldots$ are called Pell numbers and the sequence of these numbers is called Pell Sequence. The recurrence relation of n^{th} Pell Number is

$$P_n = 2P_{n-1} + P_{n-2}; n \geq 3$$

with $P_1 = P_2 = 2$; initial conditions . When the initial conditions are chanced, the recurrence relation consists Pell-Lucas numbers which are $1,3,7,17,\ldots$. The recurrence relation for n^{th} Pell-Lucas number is

$$Q_n = 2Q_{n-1} + Q_{n-2}; n \geq 3$$

with $Q_1 = 1, Q_2 = 3$ initial conditions.

Theorem 1.1. *[5]*

$$F_{n-1}F_{n+1} - {F_n}^2 = (-1)^n, n \geq 1.$$

Theorem 1.2. *[5] For all $n \geq 1$, let F_n is the n^{th} Fibonacci number, L_n is the n^{th} Lucas number, P_n is the n^{th} Pell number and Q_n is the n^{th} Pell-Lucas number,then the following identities are written;*

i) $F_{2n} = F_n L_n$

ii) $L_n = F_{n-1} + F_{n+1}$

iii) $5F_n = L_{n-1} + L_{n+1}$

iv) ${L_n}^2 = 5{F_n}^2 + 4(-1)^n$

v) $L_{4n} = 5{F_{2n}}^2 + 2$

vi) $P_n + P_{n-1} = Q_n$

vii) $2Q_n + 3P_n = P_{n+2}$.

1.2 Continued Fractions and Recurrence Relations

For all $m \in \mathbb{Z}^+ \cup \{0\}$,

$$y_0 + \mathrm{K}_{m=1}^{\infty} = y_0 + \cfrac{x_1}{y_1 + \cfrac{x_2}{y_2 + \ddots}} \tag{1.1}$$

is called (infinite) continued fraction, where $x_m \in \mathbb{Z} - \{0\}$ and $y_m \in \mathbb{Z}$ [7]. The n^{th} approximant of the continued fraction is $f_n = y_0 + \mathrm{K}_{m=1}^{n}$. The Möbius transformation of this continued fraction is $T_n(z) = \frac{x_m}{y_m + z}$. If $|y_m| \geq 1 + |x_m|$, we call this transformation Pringsheim transformation and the continued fraction (1.1) is called Pringsheim continued fraction.

The n^{th} numerator X_n and the n^{th} denominator Y_n of a continued fraction $x_0 + \mathrm{K}(x_m/y_m)$ are defined by the recurrence relations

$$\begin{bmatrix} X_n \\ Y_n \end{bmatrix} := y_n \begin{bmatrix} X_{n-1} \\ Y_{n-1} \end{bmatrix} + a_n \begin{bmatrix} X_{n-2} \\ Y_{n-2} \end{bmatrix}, \tag{1.2}$$

where $n = 1,2,3,\ldots$ and initial conditions $X_{-1} := 1, Y_{-1} := 0, X_0 := x_0, Y_0 := 1$. The modified approximant $T_n(z_n)$ can be written as $T_n(z_n) = \frac{X_n + X_{n-1}z_n}{Y_n + Y_{n-1}z_n}$, where $n = 0,1,2,3,\ldots$ and so, for the n^{th} approximant f_n we have $f_n = T_n(0) = \frac{X_n}{Y_n}, f_{n-1} = T_n(\infty) = \frac{X_{n-1}}{Y_{n-1}}$.

1.3 Solving Recurrence Relations

In this section, especially we will introduce a method for solving a large and important class of recurrence relations as following;
Linear Homogeneous Recurrence Relations with Constant Coefficients
Let take

$$y_n = c_1 y_{n-1} + c_2 y_{n-2} + ... + c_k y_{n-k}, \tag{1.3}$$

recurrence relation, where $c_1, c_2, ..., c_k \in \mathbb{R}$ and $c_k \neq 0$. The equation (1.3) is called k^{th}-order linear homogeneous recurrence relation with constant coefficients (LHRRWCCs). Here the term linear means that every term on the right-hand side (RHS) of the equation (1.3) contains at most the first power of each predecessor y_i. A recurrence relation is homogeneous if every term on the RHS is a multiple of some y_i, namely the relation is satisfied by the sequence 0, that is, $y_n = 0$ for every n. All coefficients c_i are constants. Since y_n depends on its k immediate predecessors, the order of the recurrence relation is k . Accordingly, to solve a k^{th} order LHRRWCC, we will need k initial conditions, say, $y_0 = c_0, y_1 = c_1, ..., y_{k-1} = c_k$.
Now, we will examine the second order LHRRWCCs

$$y_n = \alpha y_{n-1} + \beta y_{n-2}, \tag{1.4}$$

where α and β are nonzero solution of the form $c\phi^n$, then $c\phi^n = \alpha c\phi^{n-1} + \beta c\phi^{n-2}$. Since $c\phi^n \neq 0$, this yields $\phi^2 = \alpha\phi + \beta$, that is, $\phi^2 - \alpha\phi - \beta = 0$, so ϕ must be a solution of the characteristic equation

$$x^2 - \alpha x - \beta = 0 \tag{1.5}$$

of the recurrence relation (1.4). The roots of equation (1.4) are called the characteristic roots of recurrence relation (1.5).
The following theorem gives us how characteristic roots help solve LHRRWCCs.

Theorem 1.3. *[5] Let ϕ and δ be the different solutions of the equation $x^2 - \alpha x - \beta = 0$ where $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. Then every solution of the LHRRWCCs $y_n = \alpha y_{n-1} + \beta y_{n-2}$, where $y_0 = c_0$ and $y_1 = c_1$, is of the form $y_n = A\phi^n + B\delta^n$ for some constants A and B.*

1.4 Generating Functions

Let take $\alpha_0, \alpha_1, \alpha_2, ...$ sequence, where $\alpha_0, \alpha_1, ..$ are real numbers. Then, the function

$$h(x) = \alpha_0 + \alpha_1 x + a_2 x^2 + ... + \alpha_n x^n + ... \tag{1.6}$$

is called the generating function for the sequence α_n. When we want to write the generating function for finite sequence $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n$ where $\alpha_i = 0$ for $i > n$, that is;

$$h(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + ... + \alpha_n x^n. \tag{1.7}$$

In this paper, we will use generating functions as above;

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + ... + \alpha^n x^n + ... = \sum_{n=0}^{\infty} \alpha^n x^n, \tag{1.8}$$

where α_n is real number for each $n \in \mathbf{N}$.

Example 1.4. *Let use generating functions to solve the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1 = F_2$. Here, from the two initial conditions and Fibonacci recurrence relation, we get $F_0 = 0$. Let*

$$h(x) = F_0 + F_1 x + F_2 x^2 + ... + F_n x^n + ...$$

be the generating function of the Fibonacci sequence. Because of the orders of F_{n-1} and F_{n-2} are 1 and 2 less than the order of F_n, respectively, find $xh(x)$ and $x^2h(x)$:

$$\begin{aligned}
xh(x) &= F_1 x^2 + F_2 x^3 + F_3 x^4 + ... + F_{n-1} x^n + ... \\
x^2 h(x) &= F_1 x^3 + F_2 x^4 + F_3 x^5 + ... + F_{n-2} x^n + ... \\
h(x) - xh(x) - x^2 h(x) &= F_1 x + (F_2 - F_1) x^2 + (F_3 - F_2 - F_1) x^3 + ... + (F_n - F_{n-1} - F_{n-2}) x^n + ... = x
\end{aligned}$$

since $F_1 = F_2$ *and* $F_n = F_{n-1} + F_{n-2}$. *So;*

$$\begin{aligned}
&(1-x-x^2)h(x) = x \\
&h(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right] \\
&\sqrt{5}h(x) = \frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \\
&\qquad = \sum_0^\infty \alpha^n x^n - \sum_0^\infty \beta^n x^n = \sum_0^\infty (\alpha^n - \beta^n)x^n \\
&h(x) = \sum_0^\infty \frac{\alpha^n - \beta^n}{\sqrt{5}} x^n.
\end{aligned}$$

Then, by the equality of generating functions, the Binet formula for F_n *is :*

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

Corollary 1.5. *[5] Let* F_n *is the* n^{th} *Fibonacci number;*

$$\frac{1-x}{1-3x+x^2} = \sum_{n=0}^\infty F_{2n+1}x^n. \tag{1.9}$$

Corollary 1.6. *[5] Let* F_n *is the* n^{th} *Fibonacci number;*

$$\frac{x}{1-3x+x^2} = \sum_{n=0}^\infty F_{2n}x^n. \tag{1.10}$$

Corollary 1.7. *[6] Let* P_n *is the* n^{th} *Pell number;*

$$\frac{2x^2}{1-6x^2+x^4} = \sum_{n=0}^\infty P_{2n}x^{2n}. \tag{1.11}$$

Corollary 1.8. *[6] Let* Q_n *is the* n^{th} *Pell-Lucas number;*

$$\frac{1-3x^2}{1-6x^2+x^4} = \sum_{n=0}^\infty Q_{2n}x^{2n}. \tag{1.12}$$

1.5 Suborbital Graphs

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0(mod N)\right\}$$

is the subgroup of the well known modular group Γ. So, on $\hat{\mathbb{Q}} := \mathbb{Q} \cup \infty$ we have that $T(\infty) = v \approx w = S(\infty) \Longleftrightarrow T^{-1}S \in \Gamma_0(N)$. This is a Γ equivalence invariant on $\hat{\mathbb{Q}}$ transitively but imprimitively. By this action, we can form suborbital graphs for Γ on $\hat{\mathbb{Q}}$. By this relation, we can write $\frac{r}{s} \approx \frac{x}{y} \Longleftrightarrow ry - sx \equiv 0 mod(N)$. For details, see [8],[9],[3].

These ideas firstly were used by Sims [2], and got important in books by Tsukuzu [10], Biggs and White [11] with applying to finite groups. For example $O(\alpha,\alpha) := \{(\gamma,\gamma) | \gamma \in \Omega\}$ is the diagonal of $\Omega \times \Omega$. For $\forall \alpha \in \hat{\mathbb{Q}}$, graph $G(\alpha,\alpha)$ contains only one loup. This graph is self paired and is called trivial suborbital graph. In this work, we study with non-trivial graphs. Because Γ modular group acts on $\hat{\mathbb{Q}}$ by transitively, for $v \in \mathbb{Q}$ every suborbit contains (∞, v). If $v := \frac{u}{N}$ is taken where $N > 1, (u,N) = 1$, this suborbit is showed with $O(u,N)$ and the suborbital graph $G(\infty, v)$ which correspond that suborbit is showed with $G_{u,N}$.

Theorem 1.9. *[1]* $\mathbf{G}_{u,N} = \mathbf{G}_{u',N'}$ *iff* $N = N'$ *and* $u \equiv u' \,(mod\, N)$.

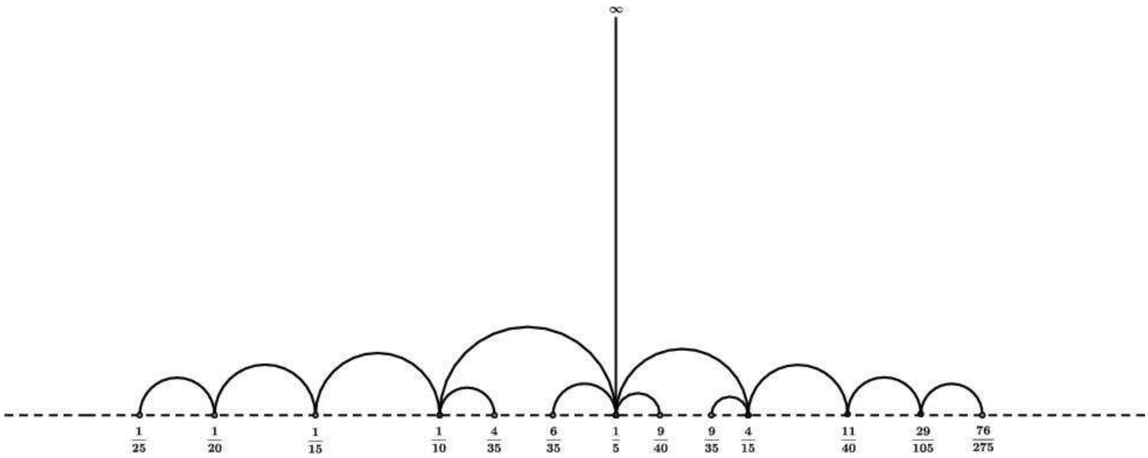

Figure 1.1. Some vertices in the suborbital graph $\mathbf{F}_{1,5}$

Theorem 1.10. *[1] On $G_{u,N}$ suborbital graph there is $\frac{r}{s} \rightarrow \frac{x}{y}$ edge iff $x \equiv (modN), y \equiv (modN)$ and $ry - sx = N$.* ■

Theorem 1.11. *[1] The suborbital graph paired with $\mathbf{G}_{u,N}$ is $\mathbf{G}_{-\bar{u},N}$ where $\bar{u}$ satisfies $u\bar{u} \equiv 1\,(mod\,N)$.*

Theorem 1.12. *[1] $\mathbf{G}_{u,N}$ is self-paired iff $u^2 \equiv -1\,(mod\,N)$.*

We let $\mathbf{F}_{u,N}$ be the subgraph of $\mathbf{G}_{u,N}$ whose vertices for the block $[\infty] = \{x/y \in \hat{\mathbb{Q}} | y \equiv 0mod(N)\}$ containing ∞. Hence, $\mathbf{G}_{u,N}$ consist of $\Psi(N)$ disjoint copies of $\mathbf{F}_{u,N}$. The following theorem gives us the edge conditions for $\mathbf{F}_{u,N}$ suborbital graphs.

Theorem 1.13. *[1] $\frac{r}{s} \rightarrow \frac{x}{y} \in \mathbf{F}_{u,N}$ if and only if $x \equiv \mp ur\,(mod\,N)$, $ry - sx = \mp N$.*

Definition 1.14. *The main definitions used in our paper:*

(i) *Let take different vertices of the graph $\mathbf{F}_{u,N}$ as a sequence $w_0, w_1, \ldots, w_m$. When $m \geq 2$, $w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_m \rightarrow w_0$ is called a* directed circuit *(*closed path*). If at least one arrow (not all) is overturned in this form, we will called it* undirected circuit. *When $m = 2$, whether the circuit is directed or not,it is called a* triangle. *When $m = 1$, the form $w_0 \rightarrow w_1 \rightarrow w_0$ is called a* self paired edge.

(ii) *The configurations $w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_m$ and $w_0 \rightarrow w_1 \rightarrow \ldots$ are called a* path *and an* infinite path *in $\mathbf{F}_{u,N}$, respectively.*

(iii) *When $\frac{r}{s} \overset{\leq}{\longrightarrow} \frac{x}{y} \in \mathbf{F}_{u,N}$ $\left(or \frac{x}{y} \overset{\leq}{\longleftarrow} \frac{r}{s} \in \mathbf{F}_{u,N}\right)$ according to edge conditions from Theorem (1.13), we call $\frac{x}{y}$ the* farthest vertex *for $\frac{r}{s}$ vertex, which means that for $\frac{r}{s}$ vertex , there is no vertex which has greater (or smaller) value than $\frac{x}{y}$ in the suborbital graph $\mathbf{F}_{u,N}$.*

(iv) *The path $w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_m$ is called of* minimal length *if and only if $w_i \nleftrightarrow w_j$, where $i < j-1$, $i \in \{0,1,2,3,...,m-2\}$, $j \in \{2,3,...,m\}$ and w_{i+1} must be the farthest vertex which can be joined with the vertex w_i in $\mathbf{F}_{u,N}$.*

(v) *If $\mathbf{F}_{u,N}$ does not have any circuits, we call it a* forest. *If $\mathbf{F}_{u,N}$ is a connected non-empty graph without circuits ,it is called a* tree.

2. Main Results

Theorem 2.1. *[4] If $(u,N) = 1$, then exist an integer t such that $u^2 + tu + 1 \equiv 0\,(mod\,N)$, for $t \geq 2$.*

On $\mathbf{F}_{u,N}$, $\varphi = \begin{pmatrix} -u & (u^2 - tu + 1)/N \\ -N & u - t \end{pmatrix} \in \Gamma_0(N)$ is a transformation which join the vertices to each other by respectively

$$\infty = \frac{1}{0} \longrightarrow \frac{u}{N} \longrightarrow \frac{u + \frac{1}{t}}{N} \longrightarrow \frac{u + \frac{1}{t - \frac{1}{t}}}{N} \longrightarrow \frac{u + \frac{1}{t - \frac{1}{t - \frac{1}{t}}}}{N} \longrightarrow \cdots$$

on the infinite minimal length path. So, this transformation forms the edges with a continued fractions construction for every edge. Thus, if $u+\frac{x}{y}$ is a vertex on the minimal length path of $\mathbf{F}_{u,N}$, the farthest vertex which can be joined with this vertex is $\varphi\left(\frac{u+\frac{x}{y}}{N}\right)=\frac{u+\frac{y}{ky-x}}{N}$. If the initial vertex is taken $w_0=\frac{u}{N}$, for any $q\in\mathbb{Z}^+$ $w_q=\varphi^q(w_0)$ equality is held. Also, if recurrence relations are used, then the n^{th} vertex which is on the minimal length path of $\mathbf{F}_{u,N}$ is given by ,$\frac{u+T_n(0)}{N}=\frac{u+\frac{X_n}{Y_n}}{N}=\frac{X_{n+1}u-X_n}{X_{n+1}N}$, where for each $n\geq 0, n\in\mathbf{N}x_n:=-1, y_n:=-t$ and $Y_n:=-X_{n+1}$. From matrix relations, we can write

$$\begin{pmatrix} X_{n-1} & X_n \\ -X_n & -X_{n+1}\end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 1 & -t\end{pmatrix}^n.$$

Corollary 2.2. *In right direction, we can write the vertices of the paths of minimal length in $\mathbf{F}_{u,N}$ as above;*

$$\frac{1}{0}\longrightarrow\frac{u}{N}\longrightarrow\frac{u+\frac{1}{t}}{N}\longrightarrow\frac{u+\frac{1}{t-\frac{1}{t}}}{N}\longrightarrow\frac{u+\frac{1}{t-\frac{1}{t-\frac{1}{t}}}}{N}\longrightarrow\cdots.$$

Here, $w_0=\frac{u}{N}$, $w_1=\frac{u+\frac{1}{t}}{N}$, $w_2=\frac{u+\frac{1}{t-\frac{1}{t}}}{N},\cdots$. It is clear that for every vertex we can obtain continued fraction . That is $\frac{1}{t-\frac{1}{t-\frac{1}{t-\ddots}}}$. Now, if we write the Möbius transformation of this continued fraction, we get $T_m(z)=\frac{-1}{-t+z}$, for all m. According to Pringsheim continued fraction's definition; if we take $x_m=-1$ and $y_m=-t$, then $|y_m|\leq 1+|x_m|$ holds. So this continued fraction is Pringsheim continued fraction. Here, we can symbolize this continued fraction with a fraction; for n^{th} vertex $\frac{H_n}{H_{n+1}}$. Thus, we can write a recurrence relation

$$H_n=tH_{n-1}-H_{n-2} \tag{2.1}$$

with $H_1=1, H_2=t$, $n\geq 3$ initial conditions.
If we solve this recurrence relation according to Theorem (1.3), we can obtain above corollary;

Corollary 2.3. *For $n\geq 1$;*

$$H_n=\left(-\frac{1}{2}\right)^n\frac{1}{\sqrt{t^2-4}}\left[(t+\sqrt{t^2-4})^n-(t-\sqrt{t^2-4})^n\right].$$

Lemma 2.4. *From recurrence relation (2.1), the generating function of this relation is*

$$h(x)=\frac{1}{1-tx+x^2}.$$

So;

$$h(x)=\frac{1}{\sqrt{t^2-4}}\sum_{k=0}^{\infty}\left[\left(\frac{2}{t-\sqrt{t^2-4}}\right)^n-\left(\frac{2}{t+\sqrt{t^2-4}}\right)^n\right]x^n.$$

Proof. Let recurrence relation $H_n=tH_{n-1}-H_{n-2}$ where $H_1=1, H_2=t$, $n\geq 3$. So, we shall define the generating function of H_n as following;

$$\begin{aligned}
&h(x)=H_0+H_1x+H_2x^2+\ldots+H_nx^n+\cdots\\
&txh(x)=tH_0x+tH_1x^2+\ldots+tH_{n-1}x^n+tH_nx^{n+1}+\cdots\\
&x^2h(x)=H_0x^2+\ldots+H_{n-2}x^n+H_{n-1}x^{n+1}+H_nx^{n+2}+\cdots\\
&h(x)-txh(x)+x^2h(x)=H_0+(H_1-tH_0)x+(H_2-tH_1+H_0)x^2+\cdots+(H_n-tH_{n-1}+H_{n-2})x^n+\cdots=1\\
&h(x)=\frac{1}{1-tx+x^2}.
\end{aligned}$$

Now, if we rewrite $h(x)=\frac{1}{1-tx+x^2}$ as a sum of partial fractions, where $\Delta=t^2-4$. Then,

$$\begin{aligned}h(x)&=\frac{1}{\sqrt{t^2-4}}\frac{1}{x-\frac{t+\sqrt{t^2-4}}{2}}-\frac{1}{\sqrt{t^2-4}}\frac{1}{x-\frac{t-\sqrt{t^2-4}}{2}}\\&=\frac{1}{\sqrt{t^2-4}}\left[\frac{-1}{1-\frac{2x}{t+\sqrt{t^2-4}}}+\frac{1}{1-\frac{2x}{t-\sqrt{t^2-4}}}\right]\end{aligned}$$

holds. From equation (1.8), we have the generating function as;

$$\begin{aligned}h(x)&=\frac{1}{\sqrt{t^2-4}}\left[\sum_{n=0}^{\infty}\left(\frac{2}{t-\sqrt{t^2-4}}\right)^n x^n-\sum_{n=0}^{\infty}\left(\frac{2}{t+\sqrt{t^2-4}}\right)^n x^n\right]\\&=\frac{1}{\sqrt{t^2-4}}\sum_{n=0}^{\infty}\left[\left(\frac{2}{t-\sqrt{t^2-4}}\right)^n-\left(\frac{2}{t+\sqrt{t^2-4}}\right)^n\right]x^n.\end{aligned}$$

□

Theorem 2.5. *From the $h(x)$ generating function, for $t=3$;*

$$h(x)=\frac{1}{1-3x+x^2}=\sum_{n=0}^{\infty}F_{2n+2}x^n. \tag{2.2}$$

Proof. When we take $t=3$ in Lemma 2.4, we can write $h(x)=\frac{1}{1-3x+x^2}$. We rewrite $h(x)=\frac{1}{1-3x+x^2}$ as a sum of partial functions as above;

$$\frac{1}{1-3x+x^2}=\frac{x}{1-3x+x^2}+\frac{1-x}{1-3x+x^2}.$$

From the equation (1.9) and the equation (1.10),

$$\frac{1}{1-3x+x^2}=\sum_{n=0}^{\infty}F_{2n+1}x^n+\sum_{n=0}^{\infty}F_{2n}x^n$$

is written. If we use Fibonacci recurrence relation, we can obtain $g(x)=\frac{1}{1-3x+x^2}=\sum_{n=0}^{\infty}F_{2n+2}x^n$. □

Theorem 2.6. *From the generating function $h(x)$, if $t=6$, then we have*

$$h(x)=\frac{1}{1-6x+x^2}=\frac{1}{2}\sum_{n=0}^{\infty}P_{2n+2}x^n. \tag{2.3}$$

Proof. If $t=6$, then from Lemma 2.4, we can write $h(x)=\frac{1}{1-6x+x^2}$. So, if we rewrite $h(x)=\frac{1}{1-6x+x^2}$ as a sum of partial functions as above;

$$\frac{1}{1-6x+x^2}=\frac{1}{2}\left(3\frac{2x}{1-3x+x^2}+2\frac{1-3x}{1-3x+x^2}\right),$$

then from the equation (1.11) and the equation (1.12), we get

$$\frac{1}{1-6x+x^2}=\frac{1}{2}\left(\sum_{n=0}^{\infty}3P_{2n}x^n+\sum_{n=0}^{\infty}2Q_{2n}x^n\right).$$

If we use Pell-Lucas identity at Theorem (1.2)(vii), then we have $h(x)=\frac{1}{1-6x+x^2}=\frac{1}{2}\sum_{n=0}^{\infty}P_{2n+2}x^n$. □

Corollary 2.7. *The interval of convergence of* $\sum_{n=0}^{\infty} F_{2n+2}x^n$ *is* $|x| < \frac{1}{\alpha^2}$, *where* $\alpha = \frac{1+\sqrt{5}}{2}$.

Proof. According to golden ratio test of power series and from the equation of $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$;

$$\lim_{n\to\infty} \left| \frac{F_{2n+4}x^{n+1}}{F_{2n+2}x^n} \right| = |x| \frac{F_{2n+4}}{F_{2n+2}} = |x|\alpha^2$$

is written. For the series to be convergent, it is necessary that $|x| < \frac{1}{\alpha^2}$. □

Corollary 2.8. *The interval of convergence of* $\sum_{n=0}^{\infty} P_{2n+2}x^n$ *is* $|x| < \frac{1}{\gamma^2}$, *where* $\gamma = 1+\sqrt{2}$.

Proof. According to golden ratio test of power series and from the equation of $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \gamma$;

$$\lim_{n\to\infty} \left| \frac{P_{2n+4}x^{n+1}}{P_{2n+2}x^n} \right| = |x| \frac{P_{2n+4}}{P_{2n+2}} = |x|\gamma^2$$

is written. For the series to be convergent, it is necessary that $|x| < \frac{1}{\gamma^2}$. □

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Norm Properties of S-Universal Operators

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Abstract
We investigate the norm properties of a generalized derivation on a norm ideal $\mathscr{J}$ in $\mathscr{B}(H)$, the algebra of bounded linear operators on a Hilbert space H. Specifically, we extend the concept of $S-$universality from the inner derivation to the generalized derivation context, establish the necessary conditions for the attainment of the optimal value of the circumdiameters of numerical ranges and the spectra of two bounded linear operators on H. Moreover, we characterize the antidistance from an operator to its similarity orbit in terms of the circumdiameters, norms, numerical and spectra radii of a pair of S-universal operators.

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1. Introduction

A derivation δ on an algebra $\mathscr{A}$ is a linear map $\delta:\mathscr{A}\to\mathscr{A}$ such that for all $A,B\in\mathscr{A}$, $\delta(AB)=\delta(A)B+A\delta(B)$. Fix $A,B\in\mathscr{A}$ and define a mapping of $\mathscr{A}$ into $\mathscr{A}$ by $\delta_{A,B}(X)=AX-XB$ for all $X\in\mathscr{A}$. Then $\delta_{A,B}$ is called a generalized derivation on $\mathscr{A}$. In the case that $A=B$, we have an inner derivation $\delta_A:=\delta_{A,A}$. That is, $\delta_A(X)=\delta_{A,A}(X)=AX-XA$ for all $X\in\mathscr{A}$. Now, for a fixed $A\in\mathscr{A}$, the mappings R_A and L_A of $\mathscr{A}$ into $\mathscr{A}$ defined by $L_A(X)=AX$ and $R_A(X)=XA$, for all $X\in\mathscr{A}$, are called the left and the right multiplications by an operator A, respectively.
Let H be a complex Hilbert space and let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on H. Stampfli [1] computed the norms of both the inner and generalized derivation on $\mathscr{B}(H)$; in particular, he proved that for fixed $A,B\in\mathscr{B}(H)$,

$$\|\delta_A\|=2d(A), \tag{1.1}$$

where $d(A)=\inf\{\|A-\lambda I\|:\lambda\in\mathbb{C}\}$, and

$$\|\delta_{A,B}\|=\inf\{\|A-\lambda I\|+\|B-\lambda I\|:\lambda\in\mathbb{C}\}. \tag{1.2}$$

Following [2], a norm ideal $(\mathscr{J},\|\cdot\|_{\mathscr{J}})$ in $\mathscr{B}(H)$ consists of a proper two-sided ideal $\mathscr{J}$ together with the norm $\|\cdot\|_{\mathscr{J}}$ satisfying the following conditions;

(i) $(\mathscr{J},\|\cdot\|_{\mathscr{J}})$ is a Banach space;

(ii) $\|AXB\|_{\mathscr{J}}\le\|A\|\|X\|_{\mathscr{J}}\|B\|$ for all $X\in\mathscr{J}$ and all operators $A,B\in\mathscr{B}(H)$.

For a good account of the theory of norm ideals, we refer to [2]. An example of such an ideal is the Schatten p-ideal, $C_p(H)$, $1 \leq p \leq \infty$, see for instance [2]. The space $C_p(H)$ consists of the compact operators X such that $\sum_j S_j^p(X) < \infty$, where $\{S_j(X)\}_j$ denotes the sequence of singular values of X. For $X \in C_p(H)$ where $1 \leq p \leq \infty$, we set $\|X\|_p = (\sum_j S_j^p(X))^{\frac{1}{p}}$, where, by convention, $\|X\|_\infty = S_1(X)$ is the usual operator norm of X. Then $(C_p(H), \|\cdot\|_p)$ is a norm ideal. Moreover, $C_1(H), C_2(H)$ and $C_\infty(H)$ are the trace class, the Hilbert-Schmidt class and the class of compact operators respectively.
For $A, B \in \mathscr{B}(H)$, if $X \in \mathscr{J}$, then

$$\begin{aligned}\|\delta_{A,B}(X)\|_{\mathscr{J}} &= \|AX - XB\|_{\mathscr{J}} \\ &= \|(A-\lambda)X - X(B-\lambda)\|_{\mathscr{J}} \\ &\leq (\|A-\lambda\| + \|B-\lambda\|)\,\|X\|_{\mathscr{J}}.\end{aligned}$$

Taking supremum over all $X \in \mathscr{J}$, we get $\|\delta_{A,B}|\mathscr{J}\| \leq \|A-\lambda\| + \|B-\lambda\|$, and from equation (1.2), it follows that the restriction $\delta_{A,B}|\mathscr{J}$ of $\delta_{A,B}$ to $\mathscr{J}$ is a bounded linear operator on $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ and

$$\|\delta_{A,B}|\mathscr{J}\| \leq \|\delta_{A,B}\| \tag{1.3}$$

for each norm ideal $\mathscr{J}$ in $\mathscr{B}(H)$. If $A = B$ in (1.3), then

$$\|\delta_A|\mathscr{J}\| \leq \|\delta_A\| = 2d(A). \tag{1.4}$$

The question as to when the equality is attained in (1.4) was considered by Fialkow [3] who introduced the concept of $S-$universal operators. An operator $A \in \mathscr{B}(H)$ is said to be $S-$universal if $\|\delta_A|\mathscr{J}\| = 2d(A)$. Having introduced the concept of $S-$universal operators, Fialkow in [3] studied the criteria of $S-$universality for a subnormal operator and posed several questions. Barraa and Boumazgour [4] later characterized $S-$universality for arbitrary hyponormal operators thereby answering a question posed by [3] in the affirmative. Motivated by the work [4], the current second author and his co-authors gave a number of results on the properties of these operators in [5, 6]. In the current paper, we extend $S-$universality to the setting of generalized derivations thereby giving a condition for a pair of operators on H to be $S-$universal.
Given an algebra $\mathscr{A}$ with a unit, let Inv$(\mathscr{A})$ be the set of invertible elements of $\mathscr{A}$, and $A \in$ Inv$(\mathscr{A})$ be fixed; then the mapping α_A of $\mathscr{A}$ into $\mathscr{A}$ given by $\alpha_A(X) = A^{-1}XA$, for all $X \in \mathscr{A}$, is an automorphism on $\mathscr{A}$ and is called an inner automorphism on $\mathscr{A}$. It is clear that $\alpha_A = I$ if A belongs to the centre of $\mathscr{A}$. In particular, if $\mathscr{A}$ is commutative, then I is the only inner automorphism. We refer to [7, 8] for details on inner automorphisms. Now, for fixed $A, B \in$ Inv$(\mathscr{A})$, we define a mapping $\alpha_{A,B} : \mathscr{A} \to \mathscr{A}$ by $\alpha_{A,B}(X) = A^{-1}XB$ for all $X \in \mathscr{A}$. We shall call $\alpha_{A,B}$ a generalized inner automorphism on $\mathscr{A}$. It can be easily proved that $\alpha_{A,B}$ is indeed an automorphism on $\mathscr{A}$.
Let $A \in \mathscr{B}(H)$, we denote by $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$, $W(A)$, $r(A)$ and $\omega(A)$; the spectrum, the point spectrum, the approximate point spectrum, the numerical range, the spectral and the numerical radii of A, respectively. We refer to [7, 12] for basic properties of numerical ranges and spectra of bounded linear operators. The numerical range and spectrum of generalized derivations on $\mathscr{B}(H)$ and their restrictions to a norm ideal $\mathscr{J}$ have been determined in literature. See for instance [9] and references therein. It was proved that

$$\sigma(\delta_{A,B}) \subseteq \sigma(A) - \sigma(B) \text{ and } \overline{W}(\delta_{A,B}) \subseteq \overline{W(A)} - \overline{W(B)}, \tag{1.5}$$

while for the restriction on $\mathscr{J}$,

$$\sigma(\delta_{A,B}|\mathscr{J}) = \sigma(A) - \sigma(B) \text{ and } \overline{W}(\delta_{A,B}|\mathscr{J}) = \overline{W(A)} - \overline{W(B)}. \tag{1.6}$$

Let S_1 and S_2 be two nonempty sets. We call the set $\text{diam}_c(S_1, S_2) = \sup\{|\alpha - \beta| : \alpha \in S_1, \beta \in S_2\}$ the circumdiameter of the sets S_1 and S_2. If $S_1 = S_2$, then we simply obtain the usual diameter of S_1, $\text{diam}(S_1) = \sup\{|\alpha - \beta| : \alpha, \beta \in S_1\}$. In this study, we shall consider two circumdiameters $\text{diam}_c(\overline{W}(A;B)) := \text{diam}_c(\overline{W}(A), \overline{W}(B))$ and $\text{diam}_c(\sigma(A;B)) := \text{diam}_c(\sigma(A), \sigma(B))$. It is important to note that when $A = B$, then the circumdiameters $\text{diam}_c(W(A;A))$ and $\text{diam}_c(\sigma(A;A))$ turn out to be the diameters of the numerical range and the spectrum of A, respectively, and whose relationships with the norms of derivations were well studied in [5, 6].

2. Algebraic Properties of Generalized Derivations

In this section, we study various properties of the generalized derivation acting on an algebra $\mathscr{A}$.

Proposition 2.1. *A generalized derivation $\delta_{A,B}$ is linear but fails to be a derivation on an algebra $\mathscr{A}$ while an inner derivation $\delta_{A,A}$ is a derivation on $\mathscr{A}$.*

Proof. First we prove that $\delta_{A,B}$ is linear. Fix $A,B \in \mathscr{A}$ and let $\alpha,\beta \in \mathbb{C}$. Then for all $X,Y \in \mathscr{A}$, we have

$$\begin{aligned}\delta_{A,B}(\alpha X+\beta Y) &= A(\alpha X+\beta Y)-(\alpha X+\beta Y)B\\ &= \alpha(AX-XB)+\beta(AY-YB)\\ &= \alpha\delta_{A,B}(X)+\beta\delta_{A,B}(Y).\end{aligned}$$

Next we show that $\delta_{A,B}$ fails to be a derivation on $\mathscr{A}$. Indeed, for all $X,Y \in \mathscr{A}$, we have;

$$\begin{aligned}\delta_{A,B}(XY) &= A(XY)-(XY)B\\ &= AXY-XYB+XBY-XBY\\ &= (AX-XB)Y+X(BY-YB)\\ &= \delta_{A,B}(X)Y+X\delta_{B,B}(Y).\end{aligned}$$

Since $\delta_{A,B}(X)Y+X\delta_{B,B}(Y)$ is not equal to $\delta_{A,B}(X)Y+X\delta_{A,B}(Y)$, it follows that $\delta_{A,B}$ fails to be a derivation on $\mathscr{A}$. On the other hand, an inner derivation $\delta_{A,A}$ turns out to be a derivation. Indeed, the linearity of δ_A follows from the linearity of $\delta_{A,B}$. Now for a fixed $A \in \mathscr{A}$, we have for all $X,Y \in \mathscr{A}$,

$$\begin{aligned}\delta_A(XY) &= A(XY)-(XY)A.\\ &= (AX-XA)Y+X(AY-YA)\\ &= \delta_A(X)Y+X\delta_A(Y), \quad \text{as desired.}\end{aligned}$$

This completes the proof. □

In the next proposition, we prove that the sum of two generalized derivations is a generalized derivation

Proposition 2.2. *The sum of two generalized derivations on $\mathscr{A}$ is a generalized derivation on $\mathscr{A}$. In particular, for fixed $A,B,C,D \in \mathscr{A}$,*

$$\delta_{A,B}+\delta_{C,D}=\delta_{A+C,B+D}.$$

Proof. For fixed $A,B,C,D \in \mathscr{A}$ and for all $X \in \mathscr{A}$, it follows from the linearity of a generalized derivation that,

$$\begin{aligned}(\delta_{A,B}+\delta_{C,D})(X) &= \delta_{A,B}(X)+\delta_{C,D}(X)\\ &= AX-XB+CX-XD\\ &= (A+C)X-X(B+D)\\ &= \delta_{A+C,B+D}(X).\end{aligned}$$

□

The following is an immediate consequence of proposition 2.2 above.

Corollary 2.3. *For fixed $A,C \in \mathscr{A}$, we have $\delta_A+\delta_C=\delta_{A+C}$.*

Remark 2.4. The question of when the product of two derivations is a derivation has been considered by a number of authors. For instance [11] characterized when the product $\delta_{C,D}\delta_{A,B}$ is a generalized derivation in the cases when $\mathscr{A}$ is the algebra of all bounded operators on a Banach space and when $\mathscr{A}$ is a C^*-algebra.

Proposition 2.5. *Let $\delta_{A,B}$ be a generalized derivation on an algebra $\mathscr{A}$, then for each $n \in \mathbb{N}$,*

$$\delta^n_{A,B}(X)=\sum_{r=0}^{n}(-1)^r\binom{n}{r}A^{n-r}XB^r \tag{2.1}$$

for all $X \in \mathscr{A}$.

Proof. By mathematical induction, let $p(n)$ be the statement that for all $n \in \mathbb{N}$, equation (2.1) holds. Then clearly $p(1)$ is true. Now, suppose that $p(k)$ is true for $k \in \mathbb{N}$. This means that for all $X \in \mathscr{A}$, $\delta_{A,B}^{k}(X) = \sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^r$. Then, for $p(k+1)$, we have

$$\begin{aligned}
\delta_{A,B}^{k+1}(X) &= \delta_{A,B}(\delta_{A,B}^{k}(X)) \\
&= A\delta_{A,B}^{k}(X) - \delta_{A,B}^{k}(X)B \\
&= A\left(\sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^r\right) - \left(\sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^r\right)B \\
&= \sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r+1}XB^r - \sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^{r+1} \\
&= \sum_{r=0}^{k+1}(-1)^r\binom{k}{r}A^{k-r+1}XB^r - \sum_{r=1}^{k+1}(-1)^{r-1}\binom{k}{r-1}A^{k-r+1}XB^r \\
&= \sum_{r=0}^{k+1}(-1)^r\binom{k}{r}A^{k-r+1}XB^r + \sum_{r=0}^{k+1}(-1)^r\binom{k}{r-1}A^{k-r+1}XB^r \\
&= \sum_{r=0}^{k+1}(-1)^r\left(\binom{k}{r}+\binom{k}{r-1}\right)A^{k-r+1}XB^r \\
&= \sum_{r=0}^{k+1}(-1)^r\binom{k+1}{r}A^{k-r+1}XB^r.
\end{aligned}$$

Thus $p(k+1)$ is true. Hence $p(k)$ implies $p(k+1)$ and therefore by the principle of mathematical induction, it follows that $p(n)$ is true for all $n \in \mathbb{N}$. □

For inner automorphisms, if δ is a continuous derivation on a Banach algebra $\mathscr{A}$, then $\exp(\delta)$ is a continuous automorphism on $\mathscr{A}$ and if A is an element of a Banach algebra $\mathscr{A}$ with unit, then $\exp(\delta_A) = \alpha_{\exp A}$, see [7, 8]. In the next result, we extend these relations to the setting of the generalized derivation $\delta_{A,B}$ and generalized inner automorphism $\alpha_{\exp A, \exp B}(X)$.

Proposition 2.6. *Let $\delta_{A,B}$ be a generalized derivation on a Banach algebra $\mathscr{A}$. Then,* $\exp\delta_{A,B}(X) = \exp(A)X\exp(-B) = \alpha_{\exp A,\exp B}(X)$.

Proof. Using equation (2.1), we have

$$\begin{aligned}
\exp\delta_{A,B}(X) &= \sum_{n=0}^{\infty}\frac{1}{n!}\delta_{A,B}^{n}(X) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}A^{n-r}XB^r \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\frac{1}{n!}\frac{n!}{(n-r)!r!}A^{n-r}XB^r \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\frac{1}{(n-r)!}\frac{1}{r!}A^{n-r}XB^r \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left((-1)^r\frac{1}{(n-r)!}A^{n-r}\right)X\left(\frac{1}{r!}B^r\right) \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\frac{1}{(n-r)!}(A)^{n-r}\right)X(-1)^r\left(\frac{1}{r!}B^r\right) \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\frac{1}{(n-r)!}(A)^{n-r}\right)X\left((-1)^r\frac{1}{r!}B^r\right) \\
&= \exp(A)X\exp(-B) \\
&= \alpha_{\exp A,\exp B}(X), \text{ as claimed.}
\end{aligned}$$

□

3. $S-$universality and Generalized Derivations

In this section, we consider $\mathscr{A} = \mathscr{B}(H)$, the algebra of bounded linear operators on H and study the norm properties of generalized derivations restricted to norm ideals $\mathscr{J}$ in $\mathscr{B}(H)$. Most importantly, we extend the concept of $S-$universal operators to the setting of generalized derivations.

Theorem 3.1. *Let $A, B \in \mathscr{B}(H)$ be $S-$universal operators and $\mathscr{J}$ a norm ideal in $\mathscr{B}(H)$. Then $\|\delta_{A,B}\| = \|\delta_{A,B}|\mathscr{J}\|$.*

Proof. For fixed $A, B \in \mathscr{B}(H)$, we have that $\|\delta_A\| = 2d(A)$ and $\|\delta_B\| = 2d(B)$. Since A, B are $S-$universal it follows that $\|\delta_A\| = 2\|A\|$ and $\|\delta_B\| = 2\|B\|$, see [6]. Thus, $d(A) + d(B) = \|A\| + \|B\|$. That is; $\inf_{\lambda\in\mathbb{C}} \|A-\lambda\| + \inf_{\lambda\in\mathbb{C}} \|B-\lambda\| = \|A\| + \|B\|$. This implies that $\inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|) = \|A\| + \|B\|$. But $\inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|) = \|\delta_{A,B}\|$ so that $\|\delta_{A,B}\| = \|A\| + \|B\|$. This is equivalent to $W_N(A) \cap W_N(-B) \neq \emptyset$ which by [6] further implies that $\|\delta_{A,B}\| = \frac{1}{2}(\|\delta_A\| + \|\delta_B\|) = \frac{1}{2}(\|\delta_A|\mathscr{J}\| + \|\delta_B|\mathscr{J}\|) = \|\delta_{A,B}|\mathscr{J}\|$, where $W_N(A)$ is the normalized maximal numerical range of A. This completes the proof. □

The theorem 3.1 above extends the notion of S-universality from the setting of inner derivation to the setting of a generalized derivation. In particular, give the following definition;

Definition 3.2. Let $A, B \in \mathscr{B}(H)$. The pair (A, B) is said to be $S-$universal if $\|\delta_{A,B}|\mathscr{J}\| = \|\delta_{A,B}\|$.

As noted earlier, a special class of norm ideals is the Schatten p-ideal $C_p(H)$.

Theorem 3.3. *Let $A, B \in B(H)$ be $S-$universal, then*

$$\|\delta_{A,B}|C_p\| = \|A\| + \|B\|.$$

Proof. Since A, B are $S-$universal and $C_p(H)$ is a norm ideal in $\mathscr{B}(H)$, it follows that $\|\delta_{A,B}|C_p\| = \|\delta_{A,B}\| = \inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|)$. By a compactness argument, $\exists\mu \in \mathbb{C}$ such that $\inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|) = \|A-\mu\| + \|B-\mu\|$. We note that $\delta_{A,B}|C_p = \delta_{A-\mu,B-\mu}|C_p = L_{A-\mu}|C_p - R_{B-\mu}|C_p$. Thus $\|L_{A-\mu}|C_p - R_{B-\mu}|C_p\| = \|A-\mu\| + \|B-\mu\|$. On the other hand, since $\|L_{A-\mu}\| = \|A-\mu\|$ and $\|R_{B-\mu}\| = \|B-\mu\|$, it follows that $\|L_{A-\mu}|C_p - R_{B-\mu}|C_p\| = \|L_{A-\mu}|C_p\| + \|R_{B-\mu}|C_p\|$. Without loss of generality, we may assume that $\mu = 0$. Then $\|L_A|C_p - R_B|C_p\| = \|L_A|C_p\| + \|R_B|C_p\| = \|A\| + \|B\|$. This completes the proof. □

The following are immediate from Theorem 3.3 above;

Corollary 3.4. *Let $A, B \in \mathscr{B}(H)$ be $S-$universal operators, then $\|\delta_{A,B}|\mathscr{J}\| = \|A\| + \|B\|$*

Proof. Since $C_p(H) \subseteq \mathscr{J}$, it follows by Theorem 3.3 that
$\|\delta_{A,B}|\mathscr{J}\| \geq \|\delta_{A,B}|C_p\| = \|A\| + \|B\|$. The rest of the proof follows from the fact that $\|\delta_{A,B}|\mathscr{J}\| \leq \|A\| + \|B\|$. □

Remark 3.5. For $A, B \in \mathscr{B}(H)$, the equation

$$\|A-B\| = \|A\| + \|B\| \tag{3.1}$$

was studied by many authors, see for instance [4, 10] and references therein. In [10], it is shown that if A and B satisfy equation (3.1), then 0 must be in the approximate point spectrum of the operator $\|B\|A + \|A\|B$. Moreover, Lin proved that the converse holds if either A or B is an isometric operator. Another result in this direction as provided in [4] asserts that non-zero A and B in $\mathscr{B}(H)$ satisfy the equation (3.1), if and only if $\|A\|\|B\|$ is in the closure of the numerical range of the operator $-A*B$.

We now give further consequences of Theorem 3.3,

Corollary 3.6. *Let $A, B \in \mathscr{B}(H)$. If the operators A, B are S-universal and L_A, R_B are defined on $C_p(H)$. Then $0 \in \sigma_{ap}(\|A\|R_B + \|B\|L_A)$. Moreover, the converse holds if either A or B is isometric.*

Proof. By Theorem 3.3, we have that for $A, B \in \mathscr{B}(H)$ $S-$universal, $\|L_A|C_p - R_B|C_p\| = \|L_A|C_p\| + \|R_B|C_p\| = \|A\| + \|B\|$. The result now follows from Remark 3.5. □

Corollary 3.7. *Let $A, B \in \mathscr{B}(H)$. If the operators A, B are $S-$universal, then $\|L_A|C_p\|\|R_B|C_p \in \overline{W(-L_{A^*}|C_pR_B|C_p)}$*

Another consequence which follows from the fact that $\|A\| \in \sigma(A)$ if and only if $\|A\| \in \overline{W(A)}$ and Corollary 3.7 is the following;

Corollary 3.8. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal, then*

$$\|L_A|C_p\|\|R_B|C_p\| \in \sigma(-L_{A^*}|C_pR_B|C_p).$$

In the next results, we consider the pair of $S-$universal operators $A,B \in \mathscr{B}(H)$ and establish the relationship between the circumdiameter
$\mathrm{diam}_c(\overline{W}(A;B))$ and the norm of a generalized derivation.

Theorem 3.9. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal, then $diam_c(\overline{W}(A;B)) = \|A\| + \|B\|$.*

Proof. If the pair A,B are $S-$universal, then by corollary 3.8,
we have $\|L_A|C_p\|\|R_B|C_p\| \in \sigma(-L_{A^*}|C_pR_B|C_p)$. But $\sigma(-L_{A^*}|C_pR_B|C_p) = -\sigma(A^*)\sigma(B)$, and $\|L_A|C_p\|\|R_B|C_p\| = \|A\|\|B\|$. So there exists $\alpha \in \sigma(A), \beta \in \sigma(B)$ such that $\|A\|\|B\| = -\overline{\alpha}\beta$. Since $|\alpha| \leq \|A\|$ and $|\beta| \leq \|B\|$, there exists $\theta \in \mathbb{R}$ such that $\alpha = \|A\|e^{i\theta}$ and $\beta = -\|B\|e^{i\theta}$. Also since $\sigma(\delta_{A,B}|C_p) = \sigma(A) - \sigma(B)$, it follows that $r(\delta_{A,B}|C_p) = \mathrm{diam}_c(\sigma(A;B)) \geq |\alpha - \beta| = |\|A\|e^{i\theta} + \|B\|e^{i\theta}| = \|A\| + \|B\|$. By the spectral inclusion, it follows that $\mathrm{diam}_c(\overline{W}(A;B)) \geq \mathrm{diam}_c(\sigma(A;B)) \geq \|A\| + \|B\|$. For the reverse inequality, we have

$$\begin{aligned}\mathrm{diam}_c(\overline{W}(A;B)) &= \sup\{|\alpha - \beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} \\ &\leq \sup\{|\alpha| + |\beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} \\ &\leq \sup\{|\alpha| : \alpha \in \overline{W(A)}\} + \sup\{|\beta| : \beta \in \overline{W(B)}\} \\ &\leq \|A\| + \|B\|, \text{ as desired.}\end{aligned}$$

□

The following consequences are immediate;

Corollary 3.10. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal, then*

$$diam_c(\overline{W}(A;B)) = \|\delta_{A,B}|\mathscr{J}\|.$$

Proof. Follows from Theorem 3.9 and Corollary 3.4. □

Corollary 3.11. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal operators, then*

$$diam_c(\overline{W}(A;B)) = \|\delta_{A,B}\|.$$

Another consequence of Theorem 3.9 which interestingly coincides with and is a summary of the results obtained in [6] is the following;

Corollary 3.12. *Let $A \in \mathscr{B}(H)$ be $S-$universal operator, $\mathscr{J}$ be a norm ideal in $\mathscr{B}(H)$ and $C_p(H)$ be the Schatten norm ideal in $\mathscr{B}(H)$. Then,*
$diam(\overline{W}(A)) = \|\delta_A\| = \|\delta_A|\mathscr{J}\| = \|\delta_A|C_p\| = 2\|A\|$.

Proof. From Theorem 3.9, Corollaries 3.10 and 3.11 above, we have that for $A = B$,

$$\mathrm{diam}_c(\overline{W}(A;A)) = \|\delta_{A,A}|\mathscr{J}\| = \|\delta_{A,A}|\mathscr{B}(H)\| = 2\|A\|.$$

□

For arbitrary operators $A,B \in \mathscr{B}(H)$, the circumdiameters $\mathrm{diam}_c(\overline{W}(A,B))$ and $\mathrm{diam}_c(\sigma(A;B))$ are related to the sum of numerical and spectral radii of A and B, respectively. In fact for $A,B \in \mathscr{B}(H)$,

$$\begin{aligned}\mathrm{diam}_c(\overline{W}(A;B)) &= \sup\{|\alpha - \beta| : \alpha \in \overline{W}(A), \beta \in \overline{W}(B)\} \\ &\leq \sup\{|\alpha| + |\beta| : \alpha \in \overline{W}(A), \beta \in \overline{W}(B)\} \\ &\leq \omega(A) + \omega(B). \end{aligned} \tag{3.2}$$

Similarly, it can be shown that

$$\mathrm{diam}_c(\sigma(A;B)) \leq r(A) + r(B). \tag{3.3}$$

Remark 3.13. A natural question then arises: When can equalities be obtained in (3.2) and (3.3)? In the next results, we answer this question in the affirmative in the case that the operators are $S-$universal.

Theorem 3.14. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal, then*

1. $diam_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$
2. $diam_c(\sigma(A;B)) = r(A) + r(B)$.

Proof. It is clear from equation (3.2) that for arbitrary operators $A,B \in \mathscr{B}(H)$, we have $\mathrm{diam}_c(W(A;B)) \le \omega(A)+\omega(B)$. To prove the reverse inequality, we have for A,B $S-$universal, $\mathrm{diam}_c(\overline{W}(A;B)) = \|\delta_{A,B}\| = \|A\| + \|B\| \ge \omega(A)+\omega(B)$, which proves (1). The proof of (2) is similar. □

The diameters $\mathrm{diam}_c(\overline{W}(A;B))$ and $\mathrm{diam}_c(\sigma(A;B))$ are respectively related to the numerical and spectral radii of a generalized derivation. In fact for a generalized derivation on a norm ideal $\mathscr{J}$, $\mathrm{diam}_c(\overline{W}(A;B))$ and $\mathrm{diam}_c(\sigma(A;B))$ turn out to be exactly the numerical and spectral radii of the generalized derivation respectively, as we give in the following theorem;

Theorem 3.15. *For $A,B \in \mathscr{B}(H)$, we have;*

1. $\omega(\delta_{A,B}) \le diam_c(\overline{W}(A;B))$
2. $r(\delta_{A,B}) \le diam_c(\sigma(A;B))$. *Moreover, if $\mathscr{J}$ is a norm ideal in $\mathscr{B}(H)$, we have;*
3. $\omega(\delta_{A,B}|\mathscr{J}) = diam_c(\overline{W}(A;B))$
4. $r(\delta_{A,B}|\mathscr{J}) = diam_c(\sigma(A;B))$.

Proof. As remarked in the introduction, we have that for $A,B \in \mathscr{B}(H)$, $\overline{W}(\delta_{A,B}) \subseteq \overline{W(A)} - \overline{W(B)}$. Let $\lambda \in \overline{W}(\delta_{A,B})$. Then $\exists$ $\alpha \in \overline{W(A)}$ and $\beta \in \overline{W(B)}$ such that $|\lambda| \le |\alpha-\beta|$. Taking supremum over all $\lambda \in \overline{W}(\delta_{A,B})$, we obtain $\omega(\delta_{A,B}) \le |\alpha-\beta|$. Now, taking supremum over all $\alpha \in \overline{W(A)}$ and $\beta \in \overline{W(B)}$, we get, $\omega(\delta_{A,B}) \le \sup\{|\alpha-\beta| : \ \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} = \mathrm{diam}_c(\overline{W}(A,B))$. On the other hand, we have; $\sigma(\delta_{A,B}) \subseteq \sigma(A)-\sigma(B)$. Now, by letting $\lambda \in \sigma(\delta_{A,B})$, it follows that $\exists \lambda_1 \in \sigma(A), \lambda_2 \in \sigma(B)$ such that $|\lambda| \le |\lambda_1 - \lambda_2|$. Taking supremum over all $\lambda \in \sigma(\delta_{A,B})$ and then over all $\lambda_1 \in \sigma(A), \lambda_2 \in \sigma(B)$, we obtain $r(\delta_{A,B}) \le \mathrm{diam}_c(\sigma(A;B))$. This proves assertions 1 and 2. To prove assertions 3 and 4, we have that the restriction of $\delta_{A,B}$ to a norm ideal $\mathscr{J}$ yields the equalities; $\overline{W}(\delta_{A,B}) = \overline{W(A)} - \overline{W(B)}$ and $\sigma(\delta_{A,B}) = \sigma(A)-\sigma(B)$. Now by similar arguments as above, we obtain the assertions 3 and 4. □

As an immediate consequence, we give the following;

Corollary 3.16. *For $A,B \in \mathscr{B}(H)$, we have*

1. $\omega(\delta_{A,B}) \le \omega(A)+\omega(B)$
2. $r(\delta_{A,B}) \le r(A)+r(B)$
   Moreover, if $A,B \in \mathscr{B}(H)$ are $S-$universal, then
3. $\omega(\delta_{A,B}|\mathscr{J}) = \omega(A)+\omega(B)$
4. $r(\delta_{A,B}|\mathscr{J}) = r(A)+r(B)$

Proof. Following Theorems 3.14 and 3.15, we have;

$$\omega(\delta_{A,B}) \le \mathrm{diam}_c(\overline{W}(A;B)) \le \omega(A)+\omega(B)$$

and

$$r(\delta_{A,B}) \le \mathrm{diam}_c(\sigma(A;B)) \le r(A)+r(B).$$

Now, assume that $A,B \in \mathscr{B}(H)$ are $S-$universal operators. Then, Theorems 3.14 and 3.15 yield

$$\omega(\delta_{A,B}|\mathscr{J}) = \mathrm{diam}_c(\overline{W}(A;B)) = \omega(A)+\omega(B)$$

and

$$r(\delta_{A,B}|\mathscr{J}) = \mathrm{diam}_c(\sigma(A;B)) = r(A)+r(B),$$

as desired. □

4. Normaloid and Spectraloid operators

In this section, we explore other special classes of operators for which we obtain the equalities $\text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$ and $\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$ without the operators $A, B \in \mathscr{B}(H)$ being necessarily S-universal. Recall that an operator $A \in \mathscr{B}(H)$ is said to be **normaloid** if $\omega(A) = \|A\|$, while it is said to be **spectraloid** if $r(A) = \omega(A)$. Note that a normaloid operator is a spectraloid operator. We refer to [12] for details on these operators. We give the following result;

Theorem 4.1. *If $A, B \in \mathscr{B}(H)$ are both normaloid operators, then;*

1. *$diam_c(\sigma(A;B)) = r(A) + r(B)$*
2. *$diam_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$.*

Proof. By equation (3.3), we have that for arbitrary $A, B \in \mathscr{B}(H)$,
$\text{diam}_c(\sigma(A,B)) \le r(A) + r(B)$. Now, we suppose that both $A, B \in \mathscr{B}(H)$ are normaloid and prove the reverse inequality. By definition; $\text{diam}_c(\sigma(A,B)) = \sup\{|\alpha - \beta| : \alpha \in \sigma(A), \beta \in \sigma(B)\} \ge |\alpha - \beta|$ for all $\alpha \in \sigma(A), \beta \in \sigma(B)$. For $\alpha \in \sigma(A)$ and $\beta \in \sigma(B)$, we have that $|\alpha| \le \|A\|$ and $|\beta| \le \|B\|$. Let $\theta \in \mathbb{R}$ such that $\alpha = \|A\|e^{i\theta}$ and $\beta = -\|B\|e^{i\theta}$. Then since A, B are normaloid, it follows that

$$|\alpha - \beta| = |\|A\|e^{i\theta} + \|B\|e^{i\theta}| = \|A\| + \|B\| = \omega(A) + \omega(B) \ge r(A) + r(B).$$

Therefore $\text{diam}_c(\sigma(A;B)) \ge r(A) + r(B)$ and hence $\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$, as desired. This proves 1. To prove assertion 2, recall from equation (3.2) that $\text{diam}_c(\overline{W}(A;B)) \le \omega(A) + \omega(B)$ for arbitrary $A, B \in \mathscr{B}(H)$. Now, by the spectral inclusion, the definition of a normaloid operator as well as the proof of assertion 1 above, we have:

$$\text{diam}_c(\overline{W}(A;B)) \ge \text{diam}_c(\sigma(A;B)) = \|A\| + \|B\| = \omega(A) + \omega(B).$$

This completes the proof. □

Theorem 4.2. *Let $A, B \in \mathscr{B}(H)$. Then the following are equivalent:*

1. *Both A and B are normaloid.*
2. *Both A and B are spectraloid.*
3. *$diam_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$.*
4. *$diam_c(\sigma(A;B)) = r(A) + r(B)$.*
5. *The pair (A, B) is $S-$universal.*

Proof. $(1) \Rightarrow (2)$: By the fact that a normaloid operator is a spectraloid. Clearly, $(2) \Rightarrow (3)$. From the proof of Theorem 4.1, we have that $\text{diam}_c(\sigma(A;B)) \ge \|A\| + \|B\|$. But we know that $\text{diam}_c(\sigma(A;B)) \le \|A\| + \|B\|$. Hence $\text{diam}_c(\sigma(A;B)) = \|A\| + \|B\|$. This implies that $\text{diam}_c(\overline{W}(A;B)) = \|A\| + \|B\|$ since it is obvious that $\text{diam}_c(\overline{W}(A;B)) \le \|A\| + \|B\|$ and $\text{diam}_c(\overline{W}(A;B)) \ge \text{diam}_c(\sigma(A;B)) = \|A\| + \|B\|$. Thus $\text{diam}_c(\sigma(A;B)) = \text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B) \ge r(A) + r(B)$. So $\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$. Hence $(3) \Rightarrow (4)$. $(4) \Rightarrow (5)$: Now, $\text{diam}_c(\sigma(A;B)) = r(A) + r(B) = r(\delta_{A,B}|\mathscr{J})$ which implies that A, B are $S-$universal by Corollary 3.16. $(5) \Rightarrow (1)$: If A, B are $S-$universal, then by Theorem 3.15 and Corollary 3.16, we have $\text{diam}_c(\overline{W}(A;B)) = \omega(\delta_{A,B}|\mathscr{J}) = \omega(A) + \omega(B)$ which is only true for the class of normaloid operators. □

5. Anti-distance and Similarity orbit

A unitary operator on a Hilbert space H is a bounded linear operator $U : H \to H$ that satisfies $U^*U = UU^* = I$, where U^* is the adjoint of U and $I : H \to H$ is the identity operator. Let $B \in \mathscr{B}(H)$. A unitary similarity orbit through B is defined as the set $U_S = \{U^*BU : U \ \text{unitary}\}$. The anti-distance from A to the orbit U_S with respect to the norm $\|\cdot\|$ is given by $\sup\{\|A - U^*BU\| : U \ \text{unitary}\}$. In [13], T. Ando determined the upper and lower bounds for the anti-distance $\sup\{\|A - U^*BU\|_\infty : U \ \text{unitary}\}$, where U runs over the set of unitary matrices.
Just like in the case of a generalized derivation, two operators $A, B \in \mathscr{B}(H)$ must be fixed in order to define the anti-distance from A to the unitary similarity orbit through B, U_S.
Therefore, the question about the relation between the norms of a generalized derivation and the anti-distance is apparent. From the available literature, very little attempt has been made towards addressing questions in this direction. It is clear that

$\sup\{\|A-U^*BU\| : U \text{ unitary}\} \leq \|\delta_{A,B}\|$.
In [**?**], Boumazgour established that for any $A,B \in \mathscr{B}(H)$,

$$\|\delta_{A,B}\| = \sup\{\|A-U^*BU\| : \mathscr{U} \text{ unitary}\}. \tag{5.1}$$

Moreover, he proved the following;

1. If $A,B \in \mathscr{B}(H)$, then for $1 \leq p \leq \infty$,

$$\sup\{\|A-U^*BU\| : U \text{ unitary}\} \leq 2^{\frac{1}{p}}\|\delta_{A,B}|C_p\|. \tag{5.2}$$

2. If A,B are hyponormal and cohyponormal operators respectively, (If, in particular both of them are normal) then,

$$\sup\{\|A-U^*BU\| : U \text{ unitary}\} \leq \sqrt{2}\text{diam}_c(\sigma(A;B)) \tag{5.3}$$

In this section, we give some of results in the same direction. In the next result, we characterize the anti-distance in terms of the circumdiameters, norms, numerical and spectral radii of pair of S-universal operators.

Theorem 5.1. *For S-universal operators $A,B \in \mathscr{B}(H)$,*

1. $\sup\{\|A-U^*BU\| : U \text{ unitary}\} = diam_c(\overline{W}(A;B))$,
2. $\sup\{\|A-U^*BU\| : U \text{ unitary}\} = r(A)+r(B)$,
3. $\sup\{\|A-U^*BU\| : U \text{ unitary}\} = \|A\|+\|B\|$,
4. $\sup\{\|A-U^*BU\| : U \text{ unitary}\} = \omega(A)+\omega(B)$,
5. $\sup\{\|A-U^*BU\| : U \text{ unitary}\} = \omega(\delta_{A,B}|\mathscr{J})$, *and*
6. $\sup\{\|A-U^*BU\| : U \text{ unitary}\} = r(\delta_{A,B}|\mathscr{J})$,

where $\mathscr{J}$ is a norm ideal in $\mathscr{B}(H)$.

Proof. Let $A,B \in \mathscr{B}(H)$ be S-universal, then by equation (5.1) we get;$\sup\{\|A-U^*BU\| : U \text{ unitary}\} = \|\delta_{A,B}|B(H)\| = \|A\|+\|B\|$. Clearly, $\text{diam}_c(\overline{W}(A;B)) = \|\delta_{A,B}\|$ for A,B S-universal. This proves assertions 1 and 3. By Theorem 3.14, we have that $\text{diam}_c(\overline{W}(A;B)) = \omega(A)+\omega(B)$ and $\text{diam}_c(\sigma(A;B)) = r(A)+r(B)$ which implies that $\sup\{\|A-U^*BU\| : U \text{ unitary}\} = \omega(A)+\omega(B)$ and that $\sup\{\|A-U^*BU\| : U \text{ unitary}\} = r(A)+r(B)$ proving the assertion 2 and 4. The proves for assertions 5 and 6 is clear from Corollary 3.16. ☐

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Tauberian Theorems for Statistical Logarithmic Summability of Strongly Measurable Fuzzy Valued Functions

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Abstract
We define statistical logarithmic summability of strongly measurable fuzzy valued functions and we give slowly decreasing type Tauberian conditions under which statistical limit at infinity and statistical logarithmic summability of strongly measurable fuzzy valued functions imply ordinary limit at infinity in one dimensional fuzzy number space E^1. Besides, we give slowly oscillating type Tauberian conditions for statistical limit and statistical logarithmic summability of strongly measurable fuzzy valued functions in $n-$dimensional fuzzy number space E^n.

Keywords: fuzzy set theory, logarithmic summability, statistical limits, Tauberian theorems
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1. Preliminaries

Let $\mathscr{K}_c(\mathbb{R}^n)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$. If $A,B \in \mathscr{K}_c(\mathbb{R}^n)$ and $k \in \mathbb{R}$ then the operations of addition and scalar multiplication are defined as

$$A+B=\{a+b : a \in A, b \in B\} \text{ and } kA=\{ka : a \in A\}.$$

The Hausdorff metric on $\mathscr{K}_c(\mathbb{R}^n)$ is defined by

$$d(A,B)=\max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\},$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

A fuzzy number is a mapping $u:\mathbb{R}^n \to [0,1]$ which satisfies the following four conditions:
(i) u is normal, i.e. there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0)=1$.
(ii) u is fuzzy convex, i.e. $u[\lambda x+(1-\lambda)y] \geq \min\{u(x),u(y)\}$, for all $x,y \in \mathbb{R}^n$ and for all $\lambda \in [0,1]$.
(iii) u is upper semi-continuous.
(iv) The set $[u]_0 := \overline{\{x \in \mathbb{R}^n : u(x)>0\}}$ is compact.[1]

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The set of all fuzzy numbers is denoted by E^n and E^n is called fuzzy number space. If $u \in E^n$, then α-level set $[u]_\alpha$ of u, defined by

$$[u]_\alpha := \begin{cases} \{x \in \mathbb{R}^n : u(x) \geq \alpha\} & , \quad (0 < \alpha \leq 1), \\ \overline{\{x \in \mathbb{R}^n : u(x) > 0\}} & , \quad (\alpha = 0), \end{cases}$$

is a nonempty compact convex subset of $\mathbb{R}^n$.

Let $r \in \mathbb{R}^n$. We say that $\bar{r}$ is a crisp fuzzy number if

$$\bar{r}(x) := \begin{cases} 1 & , \quad \text{if} x = r \\ 0 & , \quad \text{otherwise.} \end{cases}$$

The operations addition and scalar multiplication on fuzzy numbers are defined by

$$u + v = w \quad \Longleftrightarrow \quad [w]_\alpha = [u]_\alpha + [v]_\alpha, \text{ for all } \alpha \in [0,1]$$

and

$$[ku]_\alpha = k[u]_\alpha, \text{ for all } \alpha \in [0,1].$$

Lemma 1.1. *[2]*

(i) $\overline{0} \in E^n$ *is neutral element with respect to* $+$, *i.e.,* $u + \overline{0} = \overline{0} + u = u$, *for all* $u \in E^n$.

(ii) For any $a, b \in \mathbb{R}$ *with* $a, b \geq 0$ *or* $a, b \leq 0$, *and any* $u \in E^n$, *we have* $(a+b)u = au + bu$. *For general* $a, b \in \mathbb{R}$, *the above property does not hold.*

(iii) For any $a \in \mathbb{R}$ *and any* $u, v \in E^n$, *we have* $a(u+v) = au + av$.

(iv) For any $a, b \in \mathbb{R}$ *and any* $u \in E^n$, *we have* $a(bu) = (ab)u$.

The metric D on E^n is defined as follows:

$$D(u,v) := \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha).$$

From [2], we have the following lemma.

Lemma 1.2. *Let* $u, v, w, z \in E^n$ *and* $k \in \mathbb{R}$.

(i) (E^n, D) *is a complete metric space.*

(ii) $D(ku, kv) = |k| D(u,v)$.

(iii) $D(u+v, w+v) = D(u,w)$.

(iv) $D(u+v, w+z) \leq D(u,w) + D(v,z)$.

(v) $|D(u,\overline{0}) - D(v,\overline{0})| \leq D(u,v) \leq D(u,\overline{0}) + D(v,\overline{0})$.

We recall the concepts of measurability and integrability for fuzzy valued function.

Definition 1.3. *[3] Let* $T = [a,b] \subset \mathbb{R}$. *A function* $s : T \to E^n$ *is strongly measurable if for all* $\alpha \in [0,1]$ *the set valued function* $s_\alpha : T \to \mathscr{K}_c(\mathbb{R}^n)$ *defined by*

$$s_\alpha(x) = [s(x)]_\alpha$$

is Lebesgue measurable, when $\mathscr{K}_c(\mathbb{R}^n)$ *is endowed with the topology generated by Hausdorff metric* d.

Theorem 1.4. *[3] If fuzzy valued function* s *is strongly measurable, then it is measurable with respect to the topology generated by* D.

Definition 1.5. *[3] Let* $s : T \to E^n$. *The integral of* s *over* T *is defined by the following:*

$$\left[\int_T s(x)dx\right]_\alpha = \int_T [s(x)]_\alpha dx = \left\{\int_T f(x)dx \mid f : T \to \mathbb{R}^n \text{ is a measurable selection of } s_\alpha\right\},$$

for $\alpha \in (0,1]$.

A function $s:T\to E^n$ is called integrably bounded if there exists an integrable function $h:T\to\mathbb{R}^+$ such that $D(s(t),\bar{0})\leq h(t)$, for all $t\in T$.

A strongly measurable and integrably bounded function $s:T\to E^n$ is said to be integrable over T if

$$\int_T s(x)dx\in E^n.$$

Theorem 1.6. *[3] If $s:T\to E^n$ is strongly measurable and integrably bounded, then s is integrable.*

Definition 1.7. *A fuzzy valued function $s:T\to E^n$ is said to be continuous at $x_0\in T$ if for each $\varepsilon>0$ there is a $\delta>0$ such that $D(s(x),s(x_0))<\varepsilon$, whenever $x\in T$ with $|x-x_0|<\delta$. If s is continuous at each $x\in T$, then we say s is continuous on T.*

Theorem 1.8. *[3] If $f:T\to E^n$ is continuous then it is strongly measurable.*

Strong measurability of fuzzy valued functions does not imply continuity by the following example.

Example 1.9. *Let $\mu,\nu\in E^n$ with $\mu\neq\nu$ and define $s:T\to E^n$ by*

$$s(x):=\begin{cases}\mu & , \quad if x\in\mathbb{Q}\\ \nu & , \quad otherwise.\end{cases}$$

Fuzzy valued function s is strongly measurable but it is not continuous.

Theorem 1.10. *[3] If $s:T\to E^n$ is continuous, then s is integrable.*

Theorem 1.11. *[3] If $s:T\to E^n$ is continuous, $g(x)=\int_a^x s(t)dt$ is Lipschitz continuous on T.*

Theorem 1.12. *[3] Let $f,g:T\to E^n$ be integrable and $\lambda\in\mathbb{R}$. Then,*

(i) $\int_T(f(x)+g(x))dx=\int_T f(x)dx+\int_T g(x)dx;$

(ii) $\int_T\lambda f(x)dx=\lambda\int_T f(x)dx;$

(iii) $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$, *where $a<c<b$.*

(iv) The function $F:T\to\mathbb{R}_+$ defined by $F(x)=D(f(x),g(x))$ is integrable on T and

$$D\left(\int_T f(x)dx,\int_T g(x)dx\right)\leq\int_T D(f(x),g(x))dx.$$

Lemma 1.13. *[2] Suppose $\mu\in E^n$ and define $s:T\to E^n$ by $s(x)=\mu$, for all $x\in[a,b]$. Then,*

$$\int_a^b s(x)dx=(b-a)\mu.$$

If $u\in E^1$, then α*-level set* $[u]_\alpha$ of u is closed, bounded and non-empty interval and we can write $[u]_\alpha:=[u^-(\alpha),u^+(\alpha)]$. The partial ordering relation on E^1 is defined as follows:

$$u\preceq v\Longleftrightarrow[u]_\alpha\preceq[v]_\alpha\Longleftrightarrow u^-(\alpha)\leq v^-(\alpha)\ \text{ and } u^+(\alpha)\leq v^+(\alpha),\ \text{for all }\alpha\in[0,1].$$

Combining the results of Lemma 6 in [4], Lemma 5 in [5], Lemma 3.4, Theorem 4.9 in [6] and Lemma 14 in[7], following lemma is obtained.

Lemma 1.14. *Let $u,v,w,e\in E^1$ and $\varepsilon>0$. The following statements hold:*

(i) $D(u,v)\leq\varepsilon$ *if and only if* $u-\overline{\varepsilon}\preceq v\preceq u+\overline{\varepsilon}$

(ii) If $u\preceq v+\overline{\varepsilon}$ *for every* $\varepsilon>0$, *then* $u\preceq v$.

(iii) If $u\preceq v$ *and* $v\preceq w$, *then* $u\preceq w$.

(iv) If $u\preceq w$ *and* $v\preceq e$, *then* $u+v\preceq w+e$.

(v) If $u+w\preceq v+w$ *then* $u\preceq v$.

A fuzzy valued function $s : T \to E^1$ has the parametric representation

$$[s(x)]_\alpha = [s_\alpha^-(x), s_\alpha^+(x)],$$

where $s_\alpha^+, s_\alpha^- : T \to \mathbb{R}$, for all $\alpha \in [0,1]$.

Theorem 1.15. *[8] Fuzzy valued function $s : T \to E^1$ is strongly measurable if and only if s_α^+ and s_α^- are measurable for all $\alpha \in [0,1]$.*

Lemma 1.16. *[9] Fuzzy valued function $s : T \to E^1$ is integrable if and only if s_α^+, s_α^- are integrable over T and*

$$\left[\int_T s(x)dx\right]_\alpha = \left[\int_T s_\alpha^-(x)dx, \int_T s_\alpha^+(x)dx\right], \tag{1.1}$$

for all $\alpha \in [0,1]$.

Lemma 1.17. *[9] Let $f, g : T \to E^1$ be integrable and $f(x) \preceq g(x)$, for all $x \in T$. Then, $\int_T f(x)dx \preceq \int_T g(x)dx$.*

Talo et al.[9] and Belen[10] defined statistical limits of fuzzy valued functions at ∞ independently. Talo et al.[9] took the case of strongly measurable fuzzy valued functions while Belen[10] took the case of continuous fuzzy valued functions. In view of Theorem 1.8 and Example 1.9, definition of Talo et al.[9] is more general and hence we prefer to use that definiton.

Definition 1.18. *[9] A strongly measurable fuzzy valued function $s : [a,\infty) \to E^n$ has statistical limit at ∞ if there exists a fuzzy number μ such that for every $\varepsilon > 0$,*

$$\lim_{b\to\infty} \frac{1}{b-a} |\{x \in (a,b) : D(s(x),\mu) > \varepsilon\}| = 0, \tag{1.2}$$

where by $|\{.\}|$ we denote the Lebesgue measure of the set $\{.\}$. In this case, we write $st\text{-}\lim_{x\to\infty} s(x) = \mu$.

Remark 1.19. *In (1.2), the set $\{x \in (a,b) : D(s(x),\mu) > \varepsilon\}$ is Lebesgue measurable by Theorem 1.4.*

Theorem 1.20. *[9] Let s be strongly measurable fuzzy valued function. Then,*

$$\lim_{x\to\infty} s(x) = \mu \Rightarrow st - \lim_{x\to\infty} s(x) = \mu. \tag{1.3}$$

The converse of Theorem 1.20 does not hold in general. As a counter example, we can give Example 1.9.

Definition 1.21. *[11] A fuzzy valued function $s : [1,\infty) \to E^1$ is said to be slowly decreasing with respect to logarithmic summability if for every $\varepsilon > 0$ there exist $x_0 > 1$ and $\lambda > 1$ such that*

$$s(t) \succeq s(x) - \overline{\varepsilon} \tag{1.4}$$

whenever $x_0 \le x < t \le x^\lambda$.

Definition 1.22. *A fuzzy valued function $s : [1,\infty) \to E^n$ is said to be slowly oscillating with respect to logarithmic summability if for every $\varepsilon > 0$ there exist $x_0 > 1$ and $\lambda > 1$ such that*

$$D(s(t), s(x)) \le \varepsilon \tag{1.5}$$

whenever $x_0 \le x < t \le x^\lambda$.

2. Main Results

By $L_{loc}([a,\infty), E^n)$, we denote the set of fuzzy valued functions $s : [a,\infty) \to E^n$ such that s is integrable on every bounded interval $[a,x]$, $x > a$. We define statistical logarithmic summability of fuzzy valued functions as the following.

Definition 2.1. *Let $s \in L_{loc}([1,\infty), E^n)$. Logarithmic average $\tau(x)$ of s is defined by*

$$\tau(x) = \frac{1}{\log x} \int_1^x \frac{s(u)}{u} du, \quad x \in (1,\infty).$$

We say that s is statistically logarithmic summable to a fuzzy number μ if $st\text{-}\lim_{x\to\infty} \tau(x) = \mu$.

By [11, Theorem 3.2] and Theorem 1.20, ordinary limit of fuzzy valued functions at infinity implies statistical logarithmic summability. But the converse is not true in general which can be seen by the following example.

Example 2.2. *Take fuzzy valued function $s : [1,\infty) \to E^1$ such that*

$$(s(x))(t) = \begin{cases} 1-(t-\sin x)^2, & if \quad \sin x - 1 \le t \le \sin x + 1, \\ 0, & otherwise. \end{cases}$$

Then, we have

$$\begin{aligned} & s^-_\alpha(x) = \sin x - \sqrt{1-\alpha} \quad , \quad s^+_\alpha(x) = \sin x + \sqrt{1-\alpha}, \\ & \lim_{x\to\infty} \tau^-_\alpha(x) = \lim_{x\to\infty} \frac{1}{\log x}\int_1^x \frac{s^-_\alpha(u)}{u}du = -\sqrt{1-\alpha}, \\ & \lim_{x\to\infty} \tau^+_\alpha(x) = \lim_{x\to\infty} \frac{1}{\log x}\int_1^x \frac{s^+_\alpha(u)}{u}du = \sqrt{1-\alpha}. \end{aligned}$$

Since $\lim_{x\to\infty} D(\tau(x),\mu) = 0$ where $[\mu]_\alpha = [-\sqrt{1-\alpha}, \sqrt{1-\alpha}]$, by Theorem 1.20 fuzzy valued function s is statistical logarithmic summable to fuzzy number

$$\mu(t) = \begin{cases} 1-t^2 & if \quad -1 \le t \le 1, \\ 0 & otherwise. \end{cases}$$

But ordinary limit of s at infinity does not exist.

In the following two theorems, we give slowly decreasing and slowly oscillating type conditions under which statistical limits of fuzzy valued functions implies ordinary limit at infinity.

Theorem 2.3. *If a strongly measurable fuzzy valued function $s : [1,\infty) \to E^1$ is slowly decreasing with respect to logarithmic summability, then $st\text{-}\lim\limits_{x\to\infty} s(x) = \mu$ implies $\lim\limits_{x\to\infty} s(x) = \mu$.*

Proof. Let strongly measurable fuzzy valued function $s : [1,\infty) \to E^1$ be slowly decreasing with respect to logarithmic summability and $\text{st-}\lim\limits_{x\to\infty} s(x) = \mu$. Then for given an $\varepsilon > 0$ there exist $x_0 > 1$ and $\lambda > 1$ such that slow decrease condition (1.4) is satisfied. Also, as in [12, Proof of Theorem 1], since $\text{st-}\lim\limits_{x\to\infty} s(x) = \mu$ there exists a sequence $b_n \uparrow \infty$ of real numbers such that

$$D(s(b_n),\mu) \le \varepsilon, \qquad n = 1,2,\dots. \tag{2.1}$$

and for some n_0 we have

$$b_{n+1} < b_n^\lambda, \qquad n = n_0+1, n_0+2, \dots \tag{2.2}$$

Now consider $t \in (b_n, b_{n+1}]$ for $n > n_0$. In view of (2.2) and monotonicity of sequence (b_n) we get

$$b_n < t \le b_{n+1} < b_n^\lambda < t^\lambda.$$

So by slow decrease condition (1.4) and by (2.1), for every $n > n_0$ and $t \in (b_n, b_{n+1}]$ we have

$$s(t) \succeq s(b_n) - \bar{\varepsilon} \succeq \mu - 2\bar{\varepsilon}. \tag{2.3}$$

Again for every $n > n_0$ and $t \in (b_n, b_{n+1}]$ we have

$$s(t) \preceq s(b_{n+1}) + \bar{\varepsilon} \preceq \mu + 2\bar{\varepsilon}. \tag{2.4}$$

Then combining (2.3) and (2.4) we get

$$D(s(t),\mu) \le 2\varepsilon \quad for\ every \quad t \in \bigcup_{n=n_0+1}^{\infty} (b_n, b_{n+1}] = (b_{n_0+1}, \infty).$$

This proves $\lim\limits_{x\to\infty} s(x) = \mu$. □

Theorem 2.4. *If a strongly measurable fuzzy valued function $s : [1,\infty) \to E^n$ is slowly oscillating with respect to logarithmic summability, then $st\text{-}\lim\limits_{x\to\infty} s(x) = \mu$ implies $\lim\limits_{x\to\infty} s(x) = \mu$.*

Proof. Let $\operatorname{st-lim}_{x\to\infty} s(x) = \mu$ and s be slowly oscillating with respect to logarithmic summability. Then as in the proof of Theorem 2.3, for given $\varepsilon > 0$ and $\lambda > 1$ there exists a sequence $b_n \uparrow \infty$ such that (2.1) and (2.2) are satisfied. By condition (2.2) and by condition of slow oscillation we have

$$D(s(t), s(b_n)) \le \varepsilon \qquad \text{whenever} \quad x_0 \le b_n < t < b_{n+1} \tag{2.5}$$

for large enough n, say $n > n_1$. From (2.1) and (2.5) it follows that

$$D(s(t), \mu) \le D(s(t), s(b_n)) + D(s(b_n), \mu) \le 2\varepsilon$$

for every $t \in \bigcup_{n=n_1+1}^{\infty} (b_n, b_{n+1}] = (b_{n_1+1}, \infty)$. This means that $\lim_{x\to\infty} s(x) = \mu$. □

Now we aim to replace logarithmic summability with statistical logarithmic summability in Theorem 2.8.

Lemma 2.5. *If $s : [1,\infty) \to E^1$ is a fuzzy valued function such that slow decrease condition with respect to logarithmic summability (1.4) is satisfied for $\varepsilon := 1$ where $x_0 > 1$ and $\lambda > 1$, then there exists a constant $B_1 > 0$ such that*

$$s(t) \succeq s(x) - B_1 \ln\left(\frac{\ln t}{\ln x}\right) \qquad \textit{whenever} \qquad x_0 \le x < t^{1/\lambda}. \tag{2.6}$$

Proof. Let $s : [1,\infty) \to E^1$ be a fuzzy valued function such that slow decrease condition with respect to logarithmic summability (1.4) is satisfied only for $\varepsilon := 1$ where $x_0 > 1$ and $\lambda > 1$, and let $x_0 \le x < t^{1/\lambda}$ be given. Then consider the sequence

$$t_0 := t, \qquad t_p := t_{p-1}^{1/\lambda}, \qquad p = 1, 2, \ldots, q+1,$$

where q is defined by the condition $t_{q+1} \le x < t_q$. Since (1.4) is satisfied for $\varepsilon := 1$, we get

$$s(t) \succeq s(t_1) - 1 \succeq s(t_2) - 2 \succeq \cdots \succeq s(t_q) - q \succeq s(x) - q - 1.$$

Then by the calculations regarding q in [12, Proof of Lemma 1], we get

$$s(t) \succeq s(x) - 1 - \frac{1}{\ln\lambda} \ln\left(\frac{\ln t}{\ln x}\right) \qquad \textit{whenever} \qquad x_0 \le x < t^{1/\lambda}. \tag{2.7}$$

Then in view of $x < t^{1/\lambda}$, we have $\ln\lambda < \ln\left(\frac{\ln t}{\ln x}\right)$ and as result we conclude

$$s(t) \succeq s(x) - B_1 \ln\left(\frac{\ln t}{\ln x}\right) \qquad \textit{whenever} \qquad x_0 \le x < t^{1/\lambda}$$

with $B_1 := 2/\ln\lambda$. □

Lemma 2.6. $s \in L_{loc}([1,\infty), E^1)$. *Under the assumptions of Lemma 2.5, there exists a constant $B_2 > 0$ such that*

$$\frac{1}{\ln t}\int_{x_0}^{t} \frac{s(t)}{x} dx \succeq \frac{1}{\ln t}\int_{x_0}^{t} \frac{s(x)}{x} dx - B_2 \qquad \textit{whenever} \qquad t > x_0^{\lambda}. \tag{2.8}$$

Proof. Let the fuzzy valued function s satisfy slow decrease condition only for $\varepsilon := 1$ where this time assume $x_0 > e$. Then by (2.6), we get the following:

$$\begin{aligned}
\int_{x_0}^{t} \frac{s(t)}{x} dx &= \int_{x_0}^{t^{1/\lambda}} \frac{s(t)}{x} dx + \int_{t^{1/\lambda}}^{t} \frac{s(t)}{x} dx \\
&\succeq \int_{x_0}^{t^{1/\lambda}} \frac{s(x)}{x} dx - B_3 \int_{x_0}^{t^{1/\lambda}} \frac{1}{x} \ln\left(\frac{\ln t}{\ln x}\right) dx + \int_{t^{1/\lambda}}^{t} \frac{s(x)}{x} dx - \int_{t^{1/\lambda}}^{t} \frac{dx}{x} \\
&= \int_{x_0}^{t} \frac{s(x)}{x} dx - B_3 \int_{x_0}^{t^{1/\lambda}} \frac{1}{x} \ln\left(\frac{\ln t}{\ln x}\right) dx - \int_{t^{1/\lambda}}^{t} \frac{dx}{x} \\
&\succeq \int_{x_0}^{t} \frac{s(x)}{x} dx - B_3 (\ln t)\left(\frac{\ln\lambda}{\lambda} + \frac{(\ln\ln x_0)}{\lambda} + \frac{1}{\lambda}\right).
\end{aligned}$$

where we took into account the calculations in [12, Proof of Lemma 3]. If we take

$$B_2 := \frac{B_1}{\lambda}(\ln\lambda + \ln\ln x_0 + 1),$$

this proves (2.8). □

Lemma 2.7. *If $s \in L_{loc}([1,\infty),E^1)$ is slowly decreasing with respect to logarithmic summability, then logarithmic mean τ is slowly decreasing with respect to logarithmic summability.*

Proof. Let $s \in L_{loc}([1,\infty),E^1)$ and be s be slowly decreasing with respect to logarithmic summability. We aim to show that logarithmic mean τ of s is also slowly decreasing.

Let some $0 < \varepsilon < 1$ be given. Then consider $x_0 \leq x < t \leq x^\lambda$ that in slow decrease condition (1.4), where

$$1 < \lambda \leq 1 + \frac{\varepsilon}{\max\{1, B_2\}}$$

and B_2 is from (2.8).

By the following equality

$$\begin{aligned}
&\tau(t) + \left(1 - \frac{\ln x}{\ln t}\right)\frac{1}{\ln x}\int_1^{x_0}\frac{s(u)}{u}du + \left(1 - \frac{\ln x}{\ln t}\right)\frac{1}{\ln x}\int_{x_0}^{x}\frac{s(u)}{u}du + \left(1 - \frac{\ln x}{\ln t}\right)\frac{\ln x_0}{\ln x}s(x)\\
&= \tau(t) + \left(1 - \frac{\ln x}{\ln t}\right)\tau(x) + \left(1 - \frac{\ln x}{\ln t}\right)\frac{\ln x_0}{\ln x}s(x)\\
&= \tau(x) + \frac{1}{\ln t}\int_x^t \frac{s(u)}{u}du + \left(1 - \frac{\ln x}{\ln t}\right)\frac{\ln x_0}{\ln x}s(x),
\end{aligned}$$

we have

$$\begin{aligned}
&\tau(t) + \left(1 - \frac{\ln x}{\ln t}\right)\frac{1}{\ln x}\int_1^{x_0}\frac{s(u)}{u}du + \left(1 - \frac{\ln x}{\ln t}\right)\frac{1}{\ln x}\int_{x_0}^{x}\frac{s(u)}{u}du + \left(1 - \frac{\ln x}{\ln t}\right)\frac{\ln x_0}{\ln x}s(x)\\
&= \tau(x) + \frac{1}{\ln t}\int_x^t \frac{s(u)}{u}du + \left(1 - \frac{\ln x}{\ln t}\right)\frac{\ln x_0}{\ln x}s(x).
\end{aligned} \tag{2.9}$$

Then by Lemma 2.6 and from slow decrease condition (1.4) we get

$$\begin{aligned}
&\tau(t) + \left(1 - \frac{\ln x}{\ln t}\right)\frac{1}{\ln x}\int_1^{x_0}\frac{s(u)}{u}du + \left(1 - \frac{\ln x}{\ln t}\right)\left\{\frac{1}{\ln x}\int_{x_0}^{x}\frac{s(x)}{u}du + B_2\right\} + \left(1 - \frac{\ln x}{\ln t}\right)\frac{\ln x_0}{\ln x}s(x)\\
&\succeq \tau(x) + \frac{1}{\ln t}\int_x^t \frac{s(x)-1}{u}du + \left(1 - \frac{\ln x}{\ln t}\right)\frac{\ln x_0}{\ln x}s(x),
\end{aligned}$$

which yields

$$\tau(t) + \left(1 - \frac{\ln x}{\ln t}\right)\frac{1}{\ln x}\int_1^{x_0}\frac{s(u)}{u}du + \left(1 - \frac{\ln x}{\ln t}\right)B_2 \succeq \tau(x) - \left(1 - \frac{\ln x}{\ln t}\right) + \left(1 - \frac{\ln x}{\ln t}\right)\frac{\ln x_0}{\ln x}s(x). \tag{2.10}$$

At this point there exists $x_1 > x_0^\lambda$ such that

$$\left(1 - \frac{\ln x}{\ln t}\right)\frac{\ln x_0}{\ln x}s(x) \succeq -\bar{\varepsilon} \qquad \textit{whenever} \qquad x > x_1 \tag{2.11}$$

holds since by (2.7) we have

$$\frac{s(x)}{\ln x} \succeq \frac{s(x_0)-1}{\ln x} - \frac{1}{\ln x \ln \lambda}\ln\left(\frac{\ln x}{\ln x_0}\right) \to 0 \quad as \quad x \to \infty.$$

Besides there exists x_2 such that

$$\left(1 - \frac{\ln x}{\ln t}\right)\frac{1}{\ln x}\int_1^{x_0}\frac{s(u)}{u}du \preceq -\bar{\varepsilon} \qquad \textit{whenever} \qquad x > x_2, \tag{2.12}$$

since

$$\lim_{x\to\infty}\left(1 - \frac{\ln x}{\ln t}\right)\frac{1}{\ln x}\int_1^{x_0}\frac{s(u)}{u}du = \bar{0}.$$

Also from the fact $\frac{1}{\lambda} \leq \frac{\ln x}{\ln t}$ we have

$$\left(1 - \frac{\ln x}{\ln t}\right)B_2 \leq \left(1 - \frac{1}{\lambda}\right)B_2 \leq (\lambda - 1)B_2 \leq \varepsilon, \tag{2.13}$$

and again from the fact that $\frac{1}{\lambda} \leq \frac{\ln x}{\ln t}$ we get

$$-\left(1-\frac{\ln x}{\ln t}\right) \geq -\left(1-\frac{1}{\lambda}\right) \geq -(\lambda-1) \geq -\varepsilon. \tag{2.14}$$

Then inserting the expressions (2.11)–(2.14) in equality (2.10) we get

$$\tau(t) \succeq \tau(x) - 4\bar{\varepsilon} \qquad \textit{whenever} \qquad x_3 \leq x < t \leq x^{\lambda},$$

where $x_3 = \max\{x_0, x_1, x_2\}$. This proves that τ is slowly decreasing with respect to logarithmic summability. □

Analogous of Corollary 3.8 in [11] can be given as the following.

Theorem 2.8. *If $s \in L_{loc}([1,\infty), E^n)$ is logarithmic summable to a fuzzy number μ and is slowly decreasing with respect to logarithmic summability then $\lim\limits_{x\to\infty} s(x) = \mu$.*

In view of Theorem 2.8, Theorem 2.3 and Lemma 2.7 we give the following result.

Theorem 2.9. *If $s \in L_{loc}([1,\infty), E^1)$ is slowly decreasing with respect to logarithmic summability, then st-$\lim\limits_{x\to\infty} \tau(x) = \mu$ implies $\lim\limits_{x\to\infty} s(x) = \mu$.*

Replacing absolute value with metric D in Lemma 2 and Lemma 4 in [12] we obtain the following lemmas in fuzzy setting.

Lemma 2.10. *If $s : [1,\infty) \to E^n$ is a fuzzy valued function such that slow oscillation condition with respect to logarithmic summability (1.5) is satisfied for $\varepsilon := 1$ where $x_0 > 1$ and $\lambda > 1$, then there exists a constant $B_3 > 0$ such that*

$$D(s(t), s(x)) \leq B_3 \ln\left(\frac{\ln t}{\ln x}\right) \qquad \textit{whenever} \qquad x_0 \leq x < t^{1/\lambda}.$$

Lemma 2.11. *Let $s \in L_{loc}([1,\infty), E^n)$. Under the assumptions of Lemma 2.10, there exists a constant $B_4 > 0$ such that*

$$\frac{1}{\ln t}\int_{x_0}^{t} \frac{D(s(t), s(x))}{x} dx \leq B_4 \qquad \textit{whenever} \qquad t > x_0^{\lambda}.$$

Lemma 2.12. *If $s \in L_{loc}([1,\infty), E^n)$ is slowly oscillating with respect to logarithmic summability, then logarithmic mean τ is also slowly oscillating.*

Proof. As in the proof of Lemma 2.7, for given $0 < \varepsilon < 1$ consider $x_0 \leq x < t \leq x^{\lambda}$ that in slow oscillation condition (1.5), where

$$1 < \lambda \leq 1 + \frac{\varepsilon}{\max\{1, B_4\}}$$

and B_4 is from Lemma 2.11. Adding $2\left(1-\frac{\ln x}{\ln t}\right)\left(1-\frac{\ln x_0}{\ln x}\right)s(x)$ to both sides of the equation (2.9) we get

$$\begin{aligned}
&\tau(t) + \frac{\ln t - \ln x}{\ln t \ln x}\int_1^{x_0} \frac{s(u)}{u} du + \frac{\ln t - \ln x}{\ln t \ln x}\int_{x_0}^{x} \frac{s(u)}{u} du + \frac{1}{\ln t}\int_x^t \frac{s(x)}{u} du + \frac{\ln t - \ln x}{\ln t \ln x}(\ln x - \ln x_0)s(x)\\
&= \tau(x) + \frac{1}{\ln t}\int_x^t \frac{s(u)}{u} du + \frac{\ln t - \ln x}{\ln t} s(x) + \frac{\ln t - \ln x}{\ln t \ln x}\int_{x_0}^{x} \frac{s(x)}{u} du.
\end{aligned}$$

Then by the properties given in Lemma 1.2 and Theorem 1.12 we have

$$\begin{aligned}
D(\tau(t), \tau(x)) &= D\Bigg(\frac{\ln t - \ln x}{\ln t \ln x}\int_1^{x_0} \frac{s(u)}{u} du + \frac{\ln t - \ln x}{\ln t \ln x}\int_{x_0}^{x} \frac{s(u)}{u} du + \frac{1}{\ln t}\int_x^t \frac{s(x)}{u} du + \frac{\ln t - \ln x}{\ln t \ln x}(\ln x - \ln x_0)s(x),\\
&\qquad \frac{1}{\ln t}\int_x^t \frac{s(u)}{u} du + \frac{\ln t - \ln x}{\ln t} s(x) + \frac{\ln t - \ln x}{\ln t \ln x}\int_{x_0}^{x} \frac{s(x)}{u} du\Bigg)\\
&\leq \frac{\ln t - \ln x}{\ln t \ln x}\ln x_0 D(s(x), \bar{0}) + \frac{\ln t - \ln x}{\ln t \ln x}\int_1^{x_0} \frac{D(s(u), \bar{0})}{u} du + \frac{\ln t - \ln x}{\ln t \ln x}\int_{x_0}^{x} \frac{D(s(u), s(x))}{u} du\\
&\qquad + \frac{1}{\ln t}\int_x^t \frac{D(s(u), s(x))}{u} du\\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

By Lemma 2.10, there exists $x_1 > x_0^\lambda$ such that $J_1 \leq \varepsilon$ for $x > x_1$ in view of the fact that

$$\frac{D(s(x),\bar{0})}{\ln x} \leq \frac{D(s(x),s(x_0))}{\ln x} + \frac{D(s(x_0),\bar{0})}{\ln x} \leq B_3 \frac{\ln(\ln x/\ln x_0)}{\ln x} + \frac{D(s(x_0),\bar{0})}{\ln x} \to 0 \quad (\text{as } x \to \infty).$$

Besides, since

$$\lim_{x\to\infty} \frac{\ln t - \ln x}{\ln t \ln x} \int_1^{x_0} \frac{D(s(u),\bar{0})}{u} du = 0$$

there exists x_2 such that $J_2 \leq \varepsilon$ for $x > x_2$.
Furthermore, from the fact that $\frac{1}{\lambda} \leq \frac{\ln x}{\ln t}$ and by Lemma 2.11 we have $J_3 \leq (\lambda - 1)B_4 \leq \varepsilon$ for $x > x_0^\lambda$.
Again from the fact that $\frac{1}{\lambda} \leq \frac{\ln x}{\ln t}$ and by slow oscillation condition we have $J_4 \leq \varepsilon$.

Hence combining all findings we have

$$D(\tau(t),\tau(x)) \leq J_1 + J_2 + J_3 + J_4 \leq 4\varepsilon \quad \text{whenever} \quad x_3 \leq x < t \leq x^\lambda,$$

where $x_3 = \max\{x_1, x_2\}$, and this completes the proof. □

Analogous of Corollary 2.1 in [13] may be given for $s \in L_{loc}([1,\infty),E^n)$ as the following. The proof is similar and hence omitted.

Theorem 2.13. *If $s \in L_{loc}([1,\infty),E^n)$ is logarithmic summable to a fuzzy number μ and is slowly oscillating with respect to logarithmic summability, then $\lim_{x\to\infty} s(x) = \mu$.*

In view of Theorem 2.4, Lemma 2.12 and Theorem 2.13 we give the following result.

Theorem 2.14. *If $s \in L_{loc}([1,\infty),E^n)$ is slowly oscillating with respect to logarithmic summability, then st-$\lim_{x\to\infty} \tau(x) = \mu$ implies $\lim_{x\to\infty} s(x) = \mu$.*

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An Approximate Technique for Solving Lagerstrom Equation

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Abstract
The Lagerstrom's equation has been solved by an approximate technique combining both homotopy perturbation and variational iteration method. By this technique the solution of Lagerstrom's equation can be determined for viscous flow past a solid at low Reynolds number where a significance mater is the occurrence of logarithmic term. In this technique ExpIntegralEi function has been used for simplifying the calculation. The results have been calculated by this technique shows a good agreement with those obtained by numerical method.

Keywords: ExpIntegralEi, Homotopy perturbation method, Lagerstrom's equation.
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1. Introduction

The perturbation methods [1]-[2] are widely used for solving nonlinear problems mostly in week nonlinear problems. For Lagerstrom's equation a logarithmic singularity arises and the straight forward perturbation method fails to give uniformly valid solution. Several methods are used to solve Lagerstrom's model equation. Earlier P. A. Lagerstrom [3] used Matched asymptotic expansions. A geometric analysis and Rigorous asymptotic expansion methods have introduced by N. Popovic, P. Szmolyan [4]-[5]. S. Kaplun and P. A. Lagerstrom [6] have presented asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers. N. Fenichel [7], K. K. Alymkulov and D. A. Tursunov [8], P. A. Lagerstrom and R. G. Casten [9] used singular perturbation technique for solving ordinary differential equation. Lagerstrom's model equation is given by the non-autonomous second-order boundary value problem

$$u'' + \frac{n-1}{x}u' + uu' = 0, \tag{1.1}$$

where

$$u(\varepsilon) = 0, u(\infty) = 1, \tag{1.2}$$

with $n \in N$, $0 < \varepsilon \leq \infty$ and prime denotes differentiating with respect to x.

Let $\xi = \frac{x}{\varepsilon}$ then Eq. (1.1) and Eq. (1.2) can be written as

$$u'' + \frac{n-1}{\xi}u' + \varepsilon u u' = 0, \tag{1.3}$$

and

$$u(1) = 0, u(\infty) = 1, \tag{1.4}$$

with $1 < \xi < \infty$ and prime denotes differentiating with respect to ξ. Eq. (1.1) and Eq. (1.3) are the model which was first introduce in [6] and [10] for viscous flow past a solid at low Reynolds number. When $\varepsilon = 0$ the solution of Eq. (1.3) becomes $u = 1 - \frac{1}{\xi}$ and it is known as unperturbed solution. In this paper a new technique has been presented to solve Eq. (1.3) with boundary conditions in equation Eq. (1.4) based on combined homotopy perturbation method [11] and variational iteration method [12].

2. The Method

First we consider the Lagerstrom's model equation is given in Eq. (1.3) as of the form

$$u'' + \frac{n-1}{\xi}u' + ku' + p(u-k)u' = 0, \tag{2.1}$$

where

$$u(1) = 0, u(\infty) = 1. \tag{2.2}$$

When $p = 0$ Eq. (2.1) has a solution that one can be determined easily. When $p \leq 1$ we consider the approximate solution of the form [11]

$$u = u_0 + pu_1 + p^2u_2 + \mathscr{O}(p^3). \tag{2.3}$$

On substituting the value of u in Eq. (2.1) we get

$$u_0'' + \frac{n-1}{\xi}u_0' + ku_0' + \left(u_1'' + \frac{n-1}{\xi}u_1' + ku_1' + (u_0 - k)u_0'\right)p + \left(u_2'' + \frac{n-1}{\xi}u_2' + ku_1' + (u_0 - k)u_1' + u_1u_0'\right)p^2 + \mathscr{O}(p^3) = 0. \tag{2.4}$$

Equating the coefficient of like power of p we obtain

$$u_0'' + \frac{n-1}{\xi}u_0' + ku_0' = 0, \tag{2.5}$$

$$u_1'' + \frac{n-1}{\xi}u_1' + ku_1' + (u_0 - k)u_0' = 0, \tag{2.6}$$

$$u_2'' + \frac{n-1}{\xi}u_2' + ku_1' + (u_0 - k)u_1' + u_1u_0' = 0. \tag{2.7}$$

Solving above set of linear equations we get $u_0, u_1, u_2, \cdots, etc.$ and substituting these in Eq. (2.3) we obtain an approximate solution

3. Examples

3.1 Example 1

Let us first consider $n=3$ then Eq. (2.5), Eq. (2.6) and Eq. (2.7) turned into the flowing form

$$u_0'' + \frac{2}{\xi}u_0' + ku_0' = 0, \tag{3.1}$$

$$u_1'' + \frac{2}{\xi}u_1' + ku_1' + (u_0 - k)u_0' = 0, \tag{3.2}$$

$$u_2'' + \frac{2}{\xi}u_2' + ku_1' + (u_0 - k)u_1' + u_1 u_0' = 0. \tag{3.3}$$

Obviously Eq. (3.1) can be rewritten as

$$\frac{d}{d\xi}\left(\xi^2 e^{k\xi} u_0'\right) = 0, \tag{3.4}$$

where $\xi^2 e^{k\xi}$ is an integrating factor of Eq. (3.1).

Integrating Eq. (3.4) with respect to ξ we get

$$u_0' = \frac{Be^{-k\xi}}{\xi^2}. \tag{3.5}$$

Again integrating Eq. (3.5) with respect to ξ we get

$$u_0 = A + B\left(-\frac{e^{-k\xi}}{\xi} - k\, ExpIntegralEi(-k\xi)\right), \tag{3.6}$$

where $ExpIntegralEi$ function is defined as

$$Ei(x) = \gamma + \ln x + \exp\left(\frac{x}{2}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n!2^{n-1}}\sum_{k=0}^{\frac{n-1}{2}}\frac{1}{2k+1}, \tag{3.7}$$

and γ is Euler Gama constant.

Using condition $u_0(1) = 0$ and from Eq. (3.6) we obtain

$$A + B\left(-e^{-k} - k\, Ei(-k)\right) = 0. \tag{3.8}$$

Since at $\xi \to \infty, -\frac{e^{-k\xi}}{\xi} - k\, Ei(-k\xi) \to 0$ and for satisfying the condition $u_0(\infty) = 1, A$ must be considered as 1 then solving Eq. (3.8) for B we obtain

$$B = \frac{e^k}{1 + e^k k\, Ei(-k)}. \tag{3.9}$$

Substituting the values of A and B in Eq. (3.6) we obtain

$$u_0 = 1 + \frac{e^k}{1 + e^k k\, Ei(-k)}\left(-\frac{e^{-k\xi}}{\xi} - k\, ExpIntegralEi(-k\xi)\right), \tag{3.10}$$

where k is constant to be determined.

We consider $\int_0^\infty (u_0 - k)u_0' d\xi \cong 0$ and we have obtain $k \cong 0.5$. For solving u_1' Eq. (3.2) can be written as

$$\frac{d}{d\xi}\left(\xi^2 e^{k\xi} u_1'\right) = \left(\xi^2 e^{k\xi}(k-u_0)u_0'\right), \tag{3.11}$$

where $\xi^2 e^{k\xi}$ is an integrating factor. Substituting the values of u_0 in Eq. (3.11) and then integrating we obtain

$$\xi^2 e^{k\xi} u_1' = \frac{e^k\left(e^{k(1-\xi)} - \xi + k\xi + e^k(k-1)k\xi\, Ei(-k) + e^k(1+k\xi)Ei(-k\xi)\right)}{\left(1+e^k k\, Ei(-k)\right)^2} + C. \tag{3.12}$$

Using condition $u_1'(1) = 0$ then from Eq. (3.12) we obtain

$$C = -\frac{e^k\left(k + e^k(k-1)k\, Ei(-k) + e^k(1+k)Ei(-k)\right)}{\left(1+e^k k\, Ei(-k)\right)^2}. \tag{3.13}$$

Substituting the values of C in Eq. (3.12) we obtain

$$u_1' = \frac{g}{\xi^2 e^{k\xi}}, \tag{3.14}$$

where

$$g = \frac{e^k\left(e^{k(1-\xi)} - \xi + k\xi + e^k(k-1)k\xi\, Ei(-k) + e^k(1+k\xi)\, Ei(-k\xi)\right)}{\left(1+e^k k\, Ei(-k)\right)^2} - \frac{e^k\left(k + e^k(k-1)k\, Ei(-k) + e^k(1+k)\, Ei(-k)\right)}{\left(1+e^k k\, Ei(-k)\right)^2}. \tag{3.15}$$

We have calculated u_1 by numerical integration as

$$u_1 = \int_1^\xi \frac{g}{\xi^2 e^{k\xi}} d\xi. \tag{3.16}$$

Thus the approximate solution can be written for $p = 1$

$$u = u_0 + u_1, \tag{3.17}$$

where u_0 and u_1 are obtain from Eq. (3.10) and Eq. (3.16).

3.2 Example 2

Consider $n = 2$ and proceeding in a similar way as example 1 then we obtain

$$u_0 = 1 - \frac{Ei(-k\xi)}{Ei(-k)}, \tag{3.18}$$

and

$$u_1 = \int_1^\xi \frac{f}{\xi e^{k\xi}} d\xi, \tag{3.19}$$

where

$$f = \frac{e^{-k\xi} + k\xi\left(Ei(-k) - k\, Ei(-k) - Ei(-k\xi)\right)}{k\, Ei(-k)^2} + \frac{e^{-k} + k\, Ei(-k)^2}{k\, Ei(-k)^2}. \tag{3.20}$$

Thus the approximate solution can be written for $p = 1$

$$u = u_0 + u_1, \tag{3.21}$$

where u_0 and u_1 are obtain from Eq. (3.18) and Eq. (3.19).

4. Results and Discussion

In the present work a new technique has been presented for solving Lagerstrom's model equation based on combined homotopy perturbation method [11] and variational iteration method [12]. First the approximate solution obtained by present procedure by Eq. (3.17) compared with numerical solution for $n = 3, k = 0.544$ have presented in Table 1 also we have presented the absolute percentage error in Table 1. Next the approximate solution obtained by present procedure by Eq. (3.21) compared with numerical solution for $n = 2, k = 0.49$ have presented in Table 2 also we have presented the absolute percentage error in Table 2.

ξ	Numerical Result	Present Result $Er(\%)$
1.00	0.00000	0.00000 0.000000
1.25	0.376978	0.376613 0.096
1.50	0.60695	0.606553 0.065
1.75	0.748508	0.749424 0.122
2.00	0.836696	0.839742 0.364
2.25	0.892385	0.897609 0.585
2.50	0.928041	0.935012 0.751
2.75	0.951176	0.959282 0.852
3.00	0.966376	0.975006 0.893
3.50	0.983257	0.991504 0.838
4.00	0.990994	0.997712 0.677
4.50	0.994656	0.999413 0.478

Table 1. Comparison between the numerical results and present results for $n = 3, k = 0.544$ and $Er(\%)$ denote the absolute percentage error with numerical result.

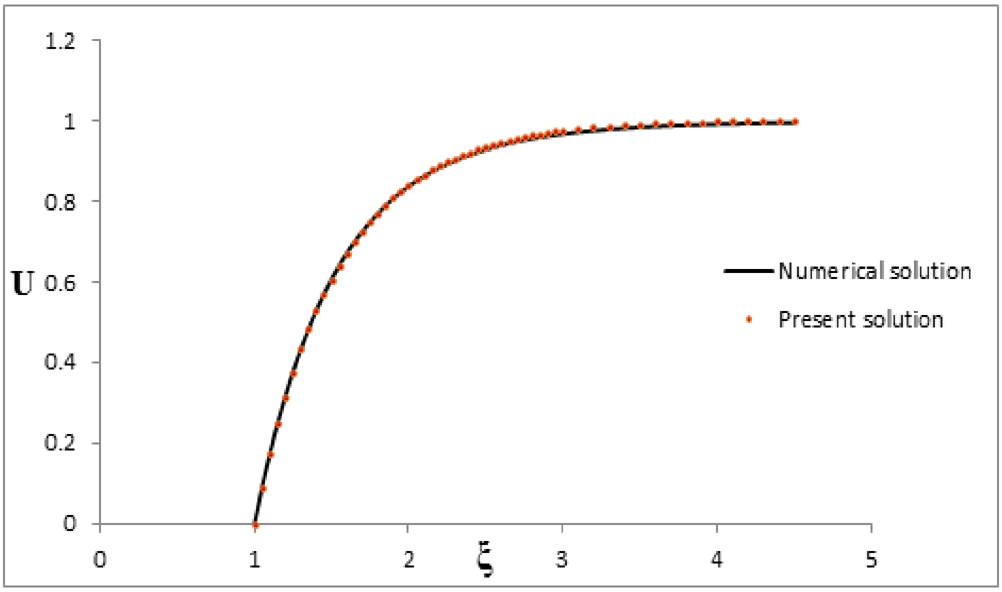

Figure 4.1. Present method solution has been presented by red colour circle line and black colour solid line represents numerical solution when $n = 3, k = 0.544$.

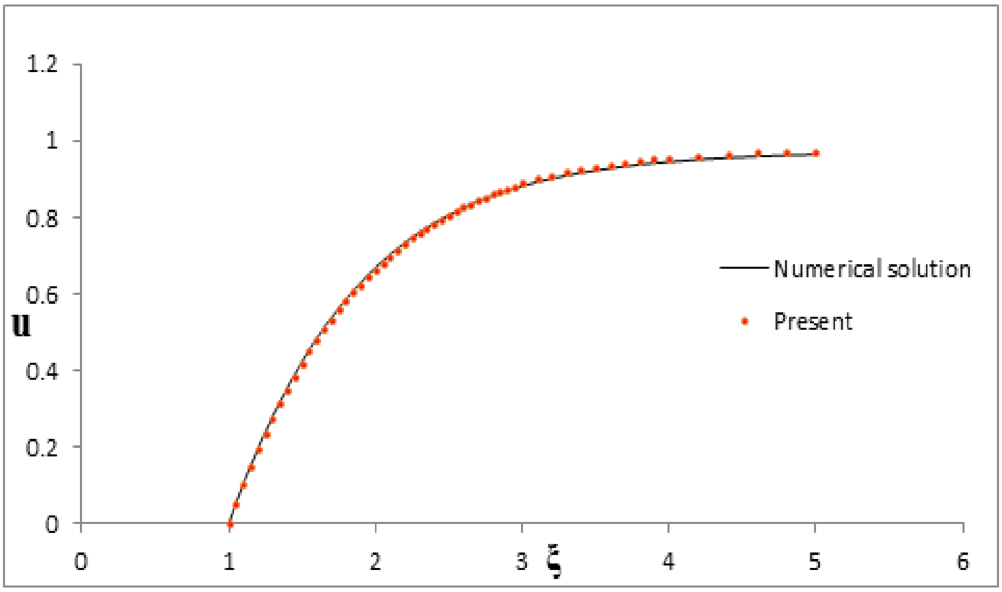

Figure 4.2. Present method solution has been presented by red colour circle line and black colour solid line represents numerical solution when $n = 2, k = 0.49$.

ξ	Numerical Result	Present Result $Er(\%)$
1.00	0.00000	0.00000 0.000000
1.25	0.23860	0.23631 0.959
1.50	0.42265	0.41829 1.031
1.75	0.56288	0.55769 0.922
2.00	0.66853	0.66414 0.656
2.25	0.74745	0.74519 0.302
2.50	0.80605	0.80667 0.076
2.75	0.84941	0.85308 0.432
3.00	0.88143	0.88788 0.731
3.50	0.92438	0.93268 0.898
4.00	0.94681	0.95595 0.965
4.50	0.95913	0.96661 0.779
5.00	0.96595	0.97006 0.425

Table 2. Comparison between the numerical results and present results for $n = 2, k = 0.49$ and $Er(\%)$ denote the absolute percentage error with numerical result.

Then we have presented the numerical solution by black color solid line and present method solution by red color circle line for $n = 3, k = 0.544$ in figure 4.1 and in figure 4.2 we have presented the numerical solution by black color solid line and present method solution by red color circle line for $n = 2, k = 0.49$. We observe that in all tables and figure the present method solution shows a good coincide with numerical results.

5. Conclusion

A new method has been presented for solving Lagerstrom's equation having a significance mater is the occurrence of logarithmic term. The results obtained by the present paper are nicely shows a good agreement with corresponding numerical solutions for several values of n .The method is useful for solving nonlinear logarithmic singularity arising in science and engineering.

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A Family of Arbitrary High-Order Iterative Methods for Approximating Inverse and the Moore–Penrose Inverse

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Abstract
In this work, a family of iterative algorithms for approximating the inverse of a square matrix and the Moore-Penrose inverse of a non-square one is proposed. These methods are based on arbitrary high-order iterative techniques which are used for computing roots of a nonlinear function. Therefore the presented techniques occupy any high-order convergence. The proposed methods are convenient and self-explanatory, achieve satisfactory results, and also require less and easy computations compared to some current schemes. Experimental results are provided to illustrate the reliability and robustness of the techniques.

Keywords: Approximate inverse, Iterative method, Moore–Penrose.
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1. Introduction

For every matrix $A \in \mathbb{C}^{m\times n}$, an $m \times n$-matrix of mn complex variables, the Moore-Penrose inverse of A denoted by $A^{\dagger}$ exists and is a unique matrix for which all the following conditions hold:

$$1. AA^{\dagger}A = A, \qquad 2. A^{\dagger}AA^{\dagger} = A^{\dagger}, \qquad 3. \left(AA^{\dagger}\right)^{*} = AA^{\dagger}, \qquad 4. \left(A^{\dagger}A\right)^{*} = A^{\dagger}A,$$

in which A^{*} denotes complex conjugate transpose of the matrix A. The Moore-Penrose pseudo-inverse has a significant role in matrix computation as well as in other fields of computational and applied mathematics. As a result of this, continuous efforts are carried out in order to improve algorithms approximating generalized inverse of a non-square matrix. In particular, some remarkable fields of applications of the concept broadly lie in digital image processing, optimization problems, solution of the system $Ax = b$ and other industrial applications.

Accordingly, finding methods for computing the Moore-Penrose inverse received a meaningful attention of many researchers [1, 2, 3, 4]. For $A \in \mathbb{C}^{n\times n}$, one of the primary and enough efficient and stable computational methods for approximating A^{-1} was introduced by Schulz[5]:

$$X_{k+1} = X_k\left(2I_n - X_kA\right), \qquad k = 0,1,2,\dots. \tag{1.1}$$

where I_n is the identity matrix, and X_0 is an initial value for approximating A^{-1}. This iterative method will quadratically converge to A^{-1}, after enough iterations provided that all eigenvalues of $I_n - X_0A$ are less than 1, i.e., for a matrix norm it holds that $\|I_n - X_0A\| < 1$[6].

Anyway, a family of iterative schemes for approximating the inverse of a square nonsingular matrix proposed by Li et al.[7] as follows:

$$X_{k+1} = X_k \left(pI - \frac{p(p-1)}{2} AX_k + \ldots + (AX_k)^{p-1} \right); \qquad p = 2, 3, \ldots.$$

Each member of the family under the condition $\|I_n - X_0 A\| < 1$ is a convergent iterative method whose order is related to p. It is easy to verify that for $p = 2$ the above relation yields the iterative method (1.1). Furthermore, for $p = 3$ it turns into the following

$$X_{k+1} = X_k \left(3I - 3AX_k + (AX_k)^2 \right), \qquad k = 0, 1, 2, \ldots, \tag{1.2}$$

that is third-order convergence iterative method[7]. Then, Chen and Wang [8] extended the results in [7] for the case of non-square matrices straightforwardly and approximated the Moore-Penrose inverse of them.

In this paper, we are dealing with iterative algorithms that every one of them can find roots of a linear or nonlinear function. Then, it is shown that based on each of which we can establish an iterative technique that is able to approximate inverse of a square matrix and the Moore-Penrose inverse of a non-square one. The presented schemes have different orders of convergence that can arbitrarily increase. Consequently, it yields a matrix inversion method having arbitrary high order convergence.

2. A family of iterative algorithms for solving nonlinear equations

In this section, three well-known iterative algorithms for solving nonlinear equations are reviewed, and a self-explanatory way to extend them is given. Getting along the next member of the family of algorithms, one can achieve a higher order convergence iterative method which of course requires much computational cost. However, for all introduced algorithms, it is needed to compute the first derivate of the given function only. Therefore, it depends on user and the task which iterative technique is more suitable.

Algorithm 2.1. *For a given function $f(x)$ and an initial guess x_0 sufficiently close to a simple root of $f(x)$, the following iterative scheme is known as Newton method and has a quadratic convergence*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots.$$

For a proof of its convergence and more details see, for instance, [9].

Algorithm 2.2. *For every function $f(x)$ and an initial guess x_0 sufficiently close to a simple root of it, the two step Newton method is of the form*

$$x_n^2 = x_n^1 - \frac{f(x_n^1)}{f'(x_n^1)},$$

$$x_{n+1}^1 = x_n^2 - \frac{f(x_n^2)}{f'(x_n^2)}, \qquad n = 0, 1, 2, \ldots,$$

which is of fourth order and proposed by Traub [10].

Algorithm 2.3. *Consider the function $f(x)$ and assume that x_0 is an initial guess sufficiently close to a simple root of $f(x)$. Then*

$$x_n^2 = x_n^1 - \frac{f(x_n^1)}{f'(x_n^1)},$$

$$x_n^3 = x_n^2 - \frac{f(x_n^2)}{f'(x_n^2)},$$

$$x_{n+1}^1 = x_n^3 - \frac{f(x_n^3)}{f'(x_n^3)}, \qquad n = 0, 1, 2, \ldots,$$

is eighth order method for computing a simple root of $f(x)$ [11].

Continuing the above process, it is fairly straightforward to verify that the kth algorithm is as follows. Moreover, pursuing the uncomplicated calculations similar to what is performed in the case of each of the above algorithms will show that the below one has 2^k order convergence.

Algorithm 2.4. *Consider a function $f(x)$ and an initial guess x_0 sufficiently close to a simple root of $f(x)$. Then for every $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers that is the set of all positive integers, the following iterative scheme converges to the simple zero of the given function*

$$x_n^2 = x_n^1 - \frac{f(x_n^1)}{f'(x_n^1)},$$

$$x_n^3 = x_n^2 - \frac{f(x_n^2)}{f'(x_n^2)},$$

$$\vdots$$

$$x_n^k = x_n^{k-1} - \frac{f(x_n^{k-1})}{f'(x_n^{k-1})},$$

$$x_{n+1}^1 = x_n^k - \frac{f(x_n^k)}{f'(x_n^k)}, \qquad n = 0, 1, 2, \ldots.$$

It is worth noting that applying kth $(k = 1, 2, \ldots)$ algorithm when k increases, yields a faster converges but requires much computations, i.e., the more rate of convergence, the more computation cost. Anyway, on the positive side, for every $k = 1, 2, \ldots$ it is needed to compute only first derivative of the given function.

3. A family of iterative methods to approximate the inverse and Moore-Penrose inverse

In the previous section a class of iterative methods for solving $f(x) = 0$ was introduced. Here, it is shown that each member of the family can give us an iterative method to approximate the inverse and the Moore-Penrose inverse. Inheriting its essential properties and corresponding technique, the obtained method for matrix inversion has the same order of convergence. It means that we have iterative methods of arbitrary high order.

To begin with, consider the function $f(x) = \frac{1}{x} - a$ and its first derivative that is $f'(x) = \frac{-1}{x^2}$. Also, in what follows suppose that $A \in \mathbb{C}^{m\times n}, (m \geq n)$ and I and X_0 are the identity matrix and an initial approximation of desired inverse of the appropriate size, respectively. Then, it is straightforward to verify that

- Algorithm 2.1 easily concludes $x_{n+1} = x_n(2 - ax_n)$ that in its matrix form becomes the following iterative method for computing the inverse or Moore-Penrose inverse of A:

  $$X_{k+1} = X_k(2I_n - AX_k), \qquad k = 0, 1, 2, \ldots,$$

  that is a second-order convergent method.

- Algorithm 2.2 yields

  $$x_{n+1} = x_n\left(4 - 6ax + 4a^2x^2 - a^3x^3\right), \tag{3.1}$$

  which leads to the following iterative method for computing the inverse or the Moore-Penrose inverse of matrix A:

  $$X_{k+1} = X_k\left(4I - 6(AX_k) + 4(AX_k)^2 - (AX_k)^3\right); \qquad k = 0, 1, 2, \ldots. \tag{3.2}$$

  The above iterative method has fourth order convergence.

- Applying Algorithm 2.3 produces

  $$x_{n+1} = x_n\left(8 - 28ax + 56a^2x^2 - 70a^3x^3 + 56a^4x^4 - 28a^5x^5 + 8a^6x^6 - a^7x^7\right),$$

  from which we have the following eight order iterative method for computing the inverse or Moore-Penrose inverse of A:

  $$X_{k+1} = X_k\left(8I - 28(AX_k) + 56(AX_k)^2 - 70(AX_k)^3 + 56(AX_k)^4 - 28(AX_k)^5 + 8(AX_k)^6 - (AX_k)^7\right), \tag{3.3}$$

  where $k = 0, 1, 2, \ldots$.

Certainly, one can continue using Algorithm 2.4 and for greater positive integer k obtain higher order iterative schemes and accordingly higher order iterative methods for matrix inversion. In what follows, in order to avoid unnecessary prolix computations, we restrict ourselves in proving that (3.2) meets all required conditions to be a fourth order iterative method for computing the inverse or the Moore-Penrose inverse of a given matrix. The extension to the remaining iterative schemes is straightforward.

Theorem 3.1. *Let $A \in \mathbb{C}^{m\times n}, (m \geq n)$ and an initial approximation $X_0 \in \mathbb{C}^{n\times m}$ of desired inverse be given. If $\|I - AX_0\| < 1$, then (3.2) converges to the inverse of A and has fourth order convergence.*

Proof. Assume that $\|E_0\| = \|I - AX_0\| < 1$ and similarly for every $k = 1,2,\ldots$ it holds that $\|E_k\| = \|I - AX_k\|$. Then,

$$\begin{aligned} \|E_{k+1}\| &= \|I - AX_{k+1}\| = \left\|I - AX_k\left(4I - 6(AX_k) + 4(AX_k)^2 - (AX_k)^3\right)\right\| \\ &= \left\|(I - AX_k)^4\right\| = \left\|{E_k}^4\right\| \leq \|E_k\|^4 \leq \|E_{k-1}\|^{16} \leq \ldots \leq \|E_0\|^{4^{k+1}}. \end{aligned}$$

Consequently, $\|E_0\| < 1$ concludes that $\|I - AX_0\| \to 0$ as $k \to \infty$ which implies the convergence of the sequence $X_k, k = 0,1,2,\ldots$.

Besides, to find the order of convergence of (3.2) let Y_k be difference between the desired inverse and X_k. In other words, if the desired inverse is denoted by V, then $Y_k = V - X_k, (k = 0,1,2,\ldots)$. Therefore,

$$AY_{k+1} = I - AX_{k+1} = E_{k+1} = E_k^4 = (AY_k)^4,$$

which for $\alpha = \|A\|^3$ concludes

$$\|Y_{k+1}\| = \left\|Y_k(AY_k)^3\right\| \leq \alpha\|Y_k\|^4,$$

and it means that (3.2) is at least a fourth order convergence. □

Remark 3.2. *It should be noted that for a given matrix A, several initial values for X_0 are introduced. Ben-Israel and Greville [1] proposed one of the most interesting and useful starting matrix as follows*

$$X_0 = \beta A^*, \qquad \beta \in \left(0, \frac{2}{\|A\|_2^2}\right).$$

In addition, $X_0 = \frac{A^T}{\|A\|_1\|A\|_\infty}$ and $X_0 = \mu I, \mu \in \mathbb{R}$ provided that $\|I - \mu A\| < 1$ are some other conceivable starting values. It is also worth mentioning that, for some special matrices, employing particular forms of the stating point may leads to more accurate results. For example,

$$X_0 = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix},$$

is a proper choice when the A is strictly diagonally dominant [12].

Remark 3.3. *The following expression that is clearly equivalent to (3.2), can yield a more accurate approximate inverse or cause rapidly converge to the inverse, especially when are dealing with ill-conditioned matrices*

$$X_{k+1} = X_k\left(4I - AX_k\left(6I - AX_k\left(4I - AX_k\right)\right)\right).$$

Remark 3.4. *For $A \in \mathbb{C}^{m\times n}, (m \leq n)$ there are discussions and calculations completely similar to what already performed in case of $m \geq n$. The only change that must be taken into account is that (3.1) as an algebraic expression can obviously be written as $x_{n+1} = \left(4 - 6xa + 4x^2a^2 - x^3a^3\right)x_n$. Consequently, the iterative scheme (3.2) turns into the following*

$$X_{k+1} = \left(4I - 6(X_kA) + 4(X_kA)^2 - (X_kA)^3\right)X_k; \qquad k = 0,1,2,\ldots.$$

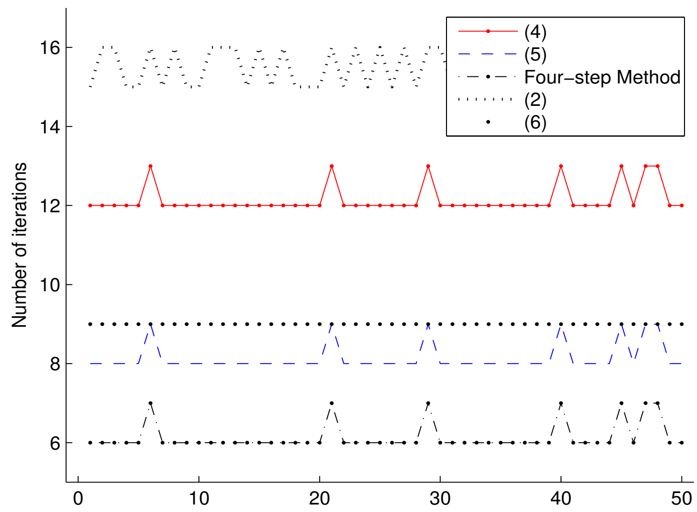

Figure 4.2. Comparison of the number of iterations for 50 matrices

4. Numerical experiments

In this section, the validity and the influence of the proposed methods are examined by a comprehensive example. In the presented experiment iterative methods given by (3.2) and (3.3) are considered. Also, for $k = 4$ Algorithm 2.4, by facile computations analogous to what already discussed in Section 3, will lead to a sixteenth order iterative method for matrix inversion. Denoting it by *Four-step Method*, we also test it in our evaluations. The results obtained by these techniques are compared to the what achieved by the third order method (1.2) and the following seventh-order iterative scheme suggested by Soleymani[13]

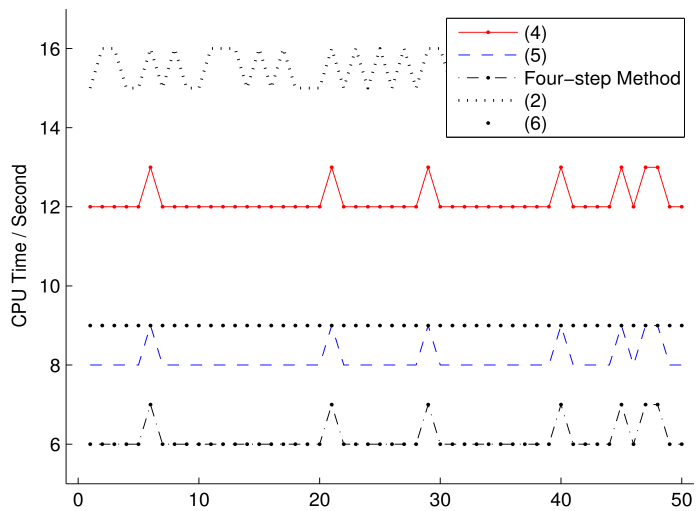

Figure 4.1. Comparison of CPU time for 50 matrices

$$X_{k+1} = \frac{1}{16}X_k \quad [\quad 120I - 3939AX_k + 735(AX_k)^2 - 861(AX_k)^3 + 651(AX_k)^4 - 315(AX_k)^5 + \quad 93(AX_k)^6 - 15(AX_k)^7 + (AX_k)^8], \qquad k = 0,1,2,\ldots. \tag{4.1}$$

We examined these five iterative methods employing 50 matrices of size 500×450 randomly generated by MATLAB. In our experiments, starting point and the stopping criterion are considered as $X_0 = \frac{1}{2\|A\|_2^2}A^*$ and $\|I - X_kA\| < 10^{-7}$, respectively. The CPU time and number of iterations for these 50 matrices and the five selected iterative methods are shown in Figures 4.1 and 4.2, in the order already mentioned.

5. Conclusion

A family of iterative algorithms to compute either the inverse of an $n \times n$ matrix or the Moore-Penrose inverse of a non-square one was studied. The main point of the illustrated method was the fact that they are arbitrary high-order iterative techniques that can also be used to compute the roots of a both linear and nonlinear function. Finding numerical technique to find other kinds of inverses, particularly those used in real world problems, is the subject of our future research.

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