2020

Volume : 2
Issue : 1

Ikonion Journal of MATHEMATICS

https://dergipark.org.tr/ikjm
ISSN: 2687-6531

Volume 2 2020
Issue 1 https://dergipark.org.tr/tr/pub/ikjm

IKONION

Journal of Mathematics

ISSN: 2687-6531

CONTENTS

IKONION JOURNAL OF MATHEMATICS
YEAR: 2020 VOLUME :2 ISSUE: 1
ISSN: 2687-6531

FUZZY DIFFERENTIAL SUBORDINATIONS FOR ANALYTIC FUNCTIONS INVOLVING WANAS OPERATOR

Abbas Kareem Wanas*

Department of Mathematics, College of Science, University of Al-Qadisiyah, Iraq,
E-mail: abbas.kareem.w@qu.edu.iq

(Received: 30.11.2019, Accepted: 02.06.2020, Published Online: 30.07.2020)

Abstract

The purpose of the present paper is to establish some properties of fuzzy subordination of analytic functions associated with Wanas differential operator which defined in the open unit disk. Further, we obtain results related to fractional derivative (Riemann-Liouville derivative).

Keywords: Fuzzy set; Fuzzy differential subordination; Wanas differential operator; Fractional derivative.

1. Introduction

Denote by $\mathcal{M}_\lambda$ the class of functions f which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^{n-\lambda} \quad (0 \le \lambda < 1), \tag{1.1}$$

For functions $f_j \in \mathcal{M}_\lambda$ $(j = 1,2)$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^{n-\lambda} \ (j = 1,2),$$

we define the Hadamard product (convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^{n-\lambda} = (f_2 * f_1)(z)\,.$$

A function $f \in \mathcal{M}_\lambda$ is said to be univalent starlike of order ρ $(0 \le \rho < 1)$, if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (z \in U).$$

Denote this class by $S(\rho)$.

Wanas [20] introduced the differential operator $W_{\alpha,\beta}^{k,\eta} : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ as follows

$$W_{\alpha,\beta}^{k,\eta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\eta} a_n z^n ,$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $m, \eta \in \mathbb{N}_0 = \{0,1,2,3, \dots\}$.

It is easily verified that if $f \in \mathcal{M}_\lambda$, then we have

$$W_{\alpha,\beta}^{k,\eta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\eta} a_n z^{n-\lambda} , \tag{1.2}$$

It follows from (1.2) that

$$z \left(W_{\alpha,\beta}^{k,\eta} f(z) \right)' = \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \right] W_{\alpha,\beta}^{k,\eta+1} f(z)$$

$$- \left[\lambda + \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right] W_{\alpha,\beta}^{k,\eta} f(z). \tag{1.3}$$

Some of the special cases of the operator defined by (1.2) can be found in [1,3,4,16,19]. For more details see [22].

Definition 1.1 [23]. Let X be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:

A pair (A, F_A), where $F_A : X \to [0,1]$ and $A = \{x \in X : 0 < F_A(x) \leq 1\} = supp(A, F_A)$

is called fuzzy subset. The function F_A is called membership function of the fuzzy subset (A, F_A).

Definition 1.2 [13]. Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x), x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \leq F_N(x), x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Let $D \subseteq \mathbb{C}$ and f, g analytic functions. We denote by

$$f(D) = supp\big(f(D), F_{f(D)}\big) = \big\{f(z) : 0 < F_{f(D)}\big(f(z)\big) \leq 1, z \in D\big\}$$

and

$$g(D) = supp\big(g(D), F_{g(D)}\big) = \big\{g(z) : 0 < F_{g(D)}\big(g(z)\big) \leq 1, z \in D\big\}.$$

Definition 1.3 [13]. Let $D \subseteq \mathbb{C}$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied:

1) $f(z_0) = g(z_0)$,
2) $F_{f(D)}\big(f(z)\big) \leq F_{g(D)}\big(g(z)\big), z \in D$.

Definition 1.4 [14]. Let $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) fuzzy differential subordination

$$F_{\psi(\mathbb{C}^3\times U)}\big(\psi(p(z), zp'(z), z^2p''(z); z)\big) \leq F_{h(U)}\big(h(z)\big), \tag{1.4}$$

i.e. $\psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z)\ , z \in U,$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $p(z) \prec_F q(z), z \in U$ for all p satisfying (1.4). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z), z \in U$ for all fuzzy dominant q of (1.4) is said to be the fuzzy best dominant of (1.4).

In order to prove our main results, we need the following lemma.

Lemma 1.1 [6]. Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi\big(q(z)\big)$ and $h(z) = \theta\big(q(z)\big) + Q(z)$. Suppose that

1) $Q(z)$ is starlike in U,

2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p is analytic in U, with $p(0) = q(0), p(U) \subset D$ and $\psi: \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi\big(p(z), zp'(z)\big) = \theta\big(p(z)\big) + zp'(z).\phi\big(p(z)\big)$ is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}\left[\theta\big(p(z)\big) + zp'(z).\phi\big(p(z)\big)\right] \leq F_{h(U)}h(z),$$

implies $F_{p(U)}p(z) \leq F_{q(U)}q(z)$,

i.e. $p(z) \prec_F q(z)$ and q is the fuzzy best dominant, where

$$\psi(\mathbb{C}^2 \times U) = supp\left(\mathbb{C}^2 \times U, F_{\psi(\mathbb{C}^2\times U)}\psi\big(p(z), zp'(z)\big)\right)$$

$$= \left\{z \in \mathbb{C} : 0 < F_{\psi(\mathbb{C}^2\times U)}\psi\big(p(z), zp'(z)\big) \leq 1\right\},$$

and $$h(U) = supp\left(U, F_{h(U)}h(z)\right) = \left\{z \in \mathbb{C} : 0 < F_{h(U)}h(z) \leq 1\right\}.$$

Recently, Oros and Oros [14,15], Lupaş [7-11], Lupaş and Oros [12], Wanas and Majeed [21] and Altınkaya and [2] have obtained fuzzy differential subordination results for certain classes of analytic functions.

2. Fuzzy Subordination Results

Theorem 2.1. Let $\gamma, \delta, \mu \in \mathbb{C}$, $t \in \mathbb{C} \setminus \{0\}$, $\tau > 0$ and q be univalent function in U with $q(0) = 1, q(z) \neq 0$ and assume that

$$Re\left\{\frac{\gamma\mu}{t}q(z) + (\mu - 2)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} + 1 + \frac{\delta}{t}(\mu - 1)\right\} > 0. \tag{2.1}$$

Suppose that $z(q(z))^{\mu-2}q'(z)$ is starlike in U. If $f \in \mathcal{M}_\lambda$ and $\chi(\gamma, \delta, \mu, \tau, k, \eta, \alpha, \beta; z)$ is analytic in U, where

$$\chi(\gamma,\delta,\mu,\tau,k,\eta,\alpha,\beta;z)=\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\mu\tau}\left[\gamma+\delta\left(\frac{W_{\alpha,\beta}^{k,\eta}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)}\right)^{\tau}\right.$$

$$\left.+\,t\tau\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\eta}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)}\right)^{\tau}\left(\frac{W_{\alpha,\beta}^{k,\eta+2}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)}-\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)\right]. \quad (2.2)$$

then

$$F_{\psi(\mathbb{C}^2\times U)}[\chi(\gamma,\delta,\mu,\tau,k,\eta,\alpha,\beta;z)]\le F_{\psi(\mathbb{C}^2\times U)}\left[(q(z))^{\mu}\left(\gamma+\frac{\delta}{q(z)}+t\frac{zq'(z)}{(q(z))^{2}}\right)\right]$$

$$=F_{h(U)}h(z), \quad (2.3)$$

implies

$$F_{\left(\frac{W_{\alpha,\beta}^{k,\eta+1}}{W_{\alpha,\beta}^{k,\eta}}\right)^{\tau}(U)}\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau}\le F_{q(U)}q(z),$$

i.e.

$$\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau}\prec_F q(z)$$

and q is the fuzzy best dominant.

Proof. Define p by

$$p(z)=\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau}=\left(\frac{1+\sum_{n=2}^{\infty}\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha^m+n\beta^m}{\alpha^m+\beta^m}\right)\right]^{\eta+1}a_nz^{n-\lambda-1}}{1+\sum_{n=2}^{\infty}\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha^m+n\beta^m}{\alpha^m+\beta^m}\right)\right]^{\eta}a_nz^{n-\lambda-1}}\right)^{\tau}. \quad (2.4)$$

Then the function p is analytic in U and $p(0)=1$. After simple computation we have

$$(p(z))^{\mu}\left(\gamma+\frac{\delta}{p(z)}+t\frac{zp'(z)}{(p(z))^2}\right)=\chi(\gamma,\delta,\mu,\tau,k,\eta,\alpha,\beta;z), \quad (2.5)$$

where $\chi(\gamma,\delta,\mu,\tau,k,\eta,\alpha,\beta;z)$ is given by (2.2).

From (2.3) and (2.5), we obtain

$$F_{\psi(\mathbb{C}^2\times U)}\left[(p(z))^{\mu}\left(\gamma+\frac{\delta}{p(z)}+t\frac{zp'(z)}{(p(z))^2}\right)\right]\le F_{\psi(\mathbb{C}^2\times U)}\left[(q(z))^{\mu}\left(\gamma+\frac{\delta}{q(z)}+t\frac{zq'(z)}{(q(z))^2}\right)\right].$$

Define the functions θ and ϕ by

$$\theta(w)=(\gamma w+\delta)w^{\mu-1}\quad and\quad \phi(w)=t\,w^{\mu-2}.$$

Obviously, the functions θ and ϕ are analytic in $D=\mathbb{C}\setminus\{0\}$ and $\phi(w)\neq 0, w\in D$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = tz(q(z))^{\mu-2}q'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = (q(z))^{\mu}\left(\gamma + \frac{\delta}{q(z)} + t\frac{zq'(z)}{(q(z))^2}\right).$$

Since $z(q(z))^{\mu-2}q'(z)$ is starlike univalent in U, we find that Q is starlike univalent in U.

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{\gamma\mu}{t}q(z) + (\mu-2)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} + 1 + \frac{\delta}{t}(\mu-1)\right\}. \quad (2.6)$$

Using (2.1), (2.6) becomes

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0.$$

Therefore, by Lemma 1.1, we get $F_{p(U)}p(z) \leq F_{q(U)}q(z)$. By using (2.4), we obtain

$$F_{\left(\frac{W_{\alpha,\beta}^{k,\eta+1}}{W_{\alpha,\beta}^{k,\eta}}\right)^{\tau}(U)}\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau} \leq F_{q(U)}q(z),$$

i.e. $\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau} \prec_F q(z)$ and q is the fuzzy best dominant.

By taking the fuzzy dominant $q(z) = \frac{1+z}{1-z}, \mu = t = 1$ and $\gamma = \delta = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. Let $Re\left\{\frac{1+z^2}{1-z^2}\right\} > 0$. If $f \in \mathcal{M}_{\lambda}$ and

$$\tau\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\eta+2}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)} - \frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}\left[\tau\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\eta+2}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)} - \frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)\right]$$
$$\leq F_{\psi(\mathbb{C}^2\times U)}\left[\frac{2z}{1-z^2}\right],$$

implies

$$\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau} \prec_F \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the fuzzy best dominant.

By fixing $\eta = 0$ in Corollary 2.1, we obtain the following corollary:

Corollary 2.2. Let $Re\left\{\frac{1+z^2}{1-z^2}\right\} > 0$. If $f \in \mathcal{M}_\lambda$ and $\tau\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)$ is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}\left[\tau\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \le F_{\psi(\mathbb{C}^2\times U)}\left[\frac{2z}{1-z^2}\right],$$

implies

$$\left(\frac{zf'(z)}{f(z)}\right)^\tau \prec_F \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the fuzzy best dominant.

3. Fractional Derivative Operator Results

In this section, we introduce some applications of section 2 containing fractional derivative operators (Riemann-Liouville derivative).

Definition 3.1 [16]. The fractional derivative of order $\lambda, (0 \le \lambda < 1)$ of a function f is defined by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\epsilon)}{(z-\epsilon)^\lambda}d\epsilon, \tag{3.1}$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\epsilon)^{-\lambda}$ is removed by requiring $\log(z-\epsilon)$ to be real, when $(z-\epsilon) > 0$.

Let $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \dots$. The Gaussian hypergeometric function ${}_2F_1$ (see [17]) is defined by

$${}_2F_1(a,b,c;z) = {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!},$$

where $(x)_n$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n=0) \\ x(x+1)\dots(x+n-1) & (n \in \mathbb{N}) . \end{cases}$$

Definition 3.2 [4]. Let $0 \le \lambda < 1$ and $u, v \in \mathbb{R}$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,u,v}$ of a function f is defined by:

$$J_{0,z}^{\lambda,u,v}f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\left\{z^{\lambda-u}\int_o^z (z-\epsilon)^{-\lambda}f(\epsilon).\ {}_2F_1\left(u-\lambda,-v;1-\lambda;1-\frac{\epsilon}{z}\right)d\epsilon\right\}, \\ \qquad (0 \le \lambda < 1) \\ \frac{d^n}{dz^n}J_{0,z}^{\lambda-n,u,v}f(z), \qquad (n \le \lambda < n+1, n \in \mathbb{N}) , \end{cases} \tag{3.2}$$

where the function f is analytic in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\varepsilon}) \,, (z \to 0),$$

for $\varepsilon > max\ \{0, u - v\} - 1$, and the multiplicity of $(z - \epsilon)^{-\lambda}$ is removed by requiring $\log(z - \epsilon)$ to be real, when $(z - \epsilon) > 0$.

By comparing (3.1) with (3.2), we find

$$J_{0,z}^{\lambda,\lambda,v} f(z) = D_z^{\lambda} f(z), \ \ (0 \le \lambda < 1).$$

In terms of gamma function, we have

$$J_{0,z}^{\lambda,u,v} z^n = \frac{\Gamma(n+1)\Gamma(n-u+v+1)}{\Gamma(n-u+1)\Gamma(n-\lambda+v+1)} z^{n-u} \,, \tag{3.3}$$

$$(\ 0 \le \lambda < 1, u, v \in \mathbb{R}\ and\ n > max\{0, u - v\} - 1\ ).$$

Now, we define

$$\Omega(z) = \sum_{n=2}^{\infty} \sigma_n z^n \,.$$

By Definition 3.1, we have

$$D_z^{\lambda} \Omega(z) = \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} \sigma_n z^{n-\lambda} = \sum_{n=2}^{\infty} a_n \, z^{n-\lambda} \,,$$

where

$$a_n = \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} \sigma_n \,, \qquad n = 2,3, \dots \ .$$

Thus $G_1(z) = z + D_z^{\lambda} \Omega(z) \in \mathcal{M}_{\lambda}$, then we obtain the following result:

Theorem 3.1. Let the assumptions of Theorem 2.1 hold. Then

$$\left(\frac{W_{\alpha,\beta}^{k,\eta+1} G_1(z)}{W_{\alpha,\beta}^{k,\eta} G_1(z)} \right)^{\tau} \prec_F q(z)$$

and q is the fuzzy best dominant.

Proof. It can easily observed that $G_1(z) = z + D_z^{\lambda} \Omega(z) \in \mathcal{M}_{\lambda}$. Thus by using Theorem 2.1, we obtain the result.

Also, by using (3.3), we have

$$J_{0,z}^{\lambda,u,v} \Omega(z) = \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-u+v+1)}{\Gamma(n-u+1)\Gamma(n-\lambda+v+1)} \sigma_n z^{n-u} = \sum_{n=2}^{\infty} a_n z^{n-u} \ ,$$

where

$$a_n = \frac{\Gamma(n+1)\Gamma(n-u+v+1)}{\Gamma(n-u+1)\Gamma(n-\lambda+v+1)} \sigma_n \,, \qquad n = 2,3, \dots \ .$$

Let $u = \lambda$. Then $G_2(z) = z + J_{0,z}^{\lambda,u,v} \Omega(z) \in \mathcal{M}_{\lambda}$, then we obtain the following result:

Theorem 3.2. Let the assumptions of Theorem 2.1 hold. Then

$$\left(\frac{W_{\alpha,\beta}^{k,\eta+1}G_2(z)}{W_{\alpha,\beta}^{k,\eta}G_2(z)}\right)^{\tau} \prec_F q(z)$$

and q is the fuzzy best dominant.

Proof. It can easily observed that $G_2(z) = z + J_{0,z}^{\lambda,u,v}\Omega(z) \in \mathcal{M}_\lambda$. Thus by using Theorem 2.1, we obtain the result.

4. Conclusions

In the present work, we have introduced some properties of fuzzy differential subordination of analytic functions by using Wanas differential operator. Further, fractional derivative (Riemann-Liouville derivative) is investigated in this study and therefore it may be considered as a useful tool for those who are interested in the above-mentioned topics for further research.

Acknowledgement

The author would like to thank the referee(s) for their helpful comments and suggestions.

References

[1] Al-Oboudi, F.M. (2004) On univalent functions defined by a generalized Salagean operator. Int. J. Math. Math. Sci., 27: 1429-1436.

[2] Altınkaya, Ş., Wanas, A.K. (2020) Some properties for fuzzy differential subordination defined by Wanas operator. Earthline Journal of Mathematical Sciences, 4(1): 51-62.

[3] Cho, N.E., Kim, T.H. (2003) Multiplier transformations and strongly close-to-convex functions. Bull. Korean Math. Soc., 40(3): 399-410.

[4] Cho, N.E., Srivastava, H.M. (2003) Argument estimates of certain analytic functions defined by a class of multiplier transformations. Math. Comput. Modelling, 37(1-2): 39-49.

[5] Goyal, S.P., Goyal, R. (2005) On a class of multivalent functions defined by a generalized Ruscheweyh derivatives involving a general fractional derivative operator. J. Indian Acad. Math., 27(2): 439-456.

[6] Haydar, A. (2015) On fuzzy differential subordination. Mthematica Moravica, 19(1): 123-129.

[7] Lupaş, A.A. (2013) A note on special fuzzy differential subordinations using generalized Sălăgean operator and Ruscheweyh derivative. J. Comp. Anal. Appl., 15(8): 1476-1483.

[8] Lupaş, A.A. (2016) A note on special fuzzy differential subordinations using multiplier transformation. Analele Universitatii Oradea, Fasc. Matematica, XXIII(2): 183-191.

[9] Lupaş, A.A. (2017) On special fuzzy differential subordinations using generalized Salagean operator and Ruscheweyh derivative. J. Adv. Appl. Comp. Math., 4: 26-34.

[10] Lupaş, A.A. (2017) On special fuzzy differential subordinations using multiplier transformation. J. Comp. Anal. Appl., 23(6): 1029-1035.

[11] Lupaş, A.A. (2018) A note on special fuzzy differential subordinations using multiplier transformation and Ruschewehy derivative. J. Comp. Anal. Appl., 25(6): 1125-1131.

[12] Lupaş, A.A., Oros, Gh. (2015) On special fuzzy differential subordinations using Sălăgean and Ruscheweyh operators. Appl. Math. Comp., 261: 119-127.
[13] Oros, G.I., Oros, Gh. (2011) The notion of subordination in fuzzy set theory. General Mathematics, 19(4): 97-103.
[14] Oros, G.I., Oros, Gh. (2012) Fuzzy differential subordination. Acta Universitatis Apulensis, 30: 55-64.
[15] Oros, G.I., Oros, Gh. (2012) Dominants and best dominants in fuzzy differential subordinations. Stud. Univ. Babeş-Bolyai Math., 57(2): 239-248.
[16] Salagean, G.S. (1983) Subclasses of univalent functions. Lecture Notes in Math., Springer Verlag, Berlin, 1013: 362-372.
[17] Srivastava H.M., Owa, S. (1989) Univalent Functions, Fractional Calculus and Their Applications, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane and Toronto.
[18] Srivastava, H.M., Owa S. (Eds.) (1992) Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.
[19] S. R. Swamy, Inclusion properties of certain subclasses of analytic functions, Int. Math. Forum, 7(36)(2012), 1751–1760.
[20] Wanas, A.K. (2019) New differential operator for holomorphic functions. Earthline Journal of Mathematical Sciences, 2(2): 527-537.
[21] Wanas, A.K., Majeed, A.H. (2018) Fuzzy differential subordination properties of analytic functions involving generalized differential operator. Sci. Int. (Lahore), 30(2): 297-302.
[22] Wanas, A.K., Murugusundaramoorthy, G. (2020) Differential sandwich results for Wanas operator of analytic functions. Mathematica Moravica, 24(1): 17-28.
[23] Zadeh, L.A. (1965) Fuzzy sets. Information and control, 8: 338-353.

IKONION JOURNAL OF MATHEMATICS
YEAR: 2020 VOLUME :2 ISSUE: 1
ISSN: 2687-6531

A SOLVABLE SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

A. Furkan Şahinkaya[1], İbrahim Yalçınkaya[1], D. Turgut Tollu[1,*]

[1] Department of Mathematics and Computer Sciences, Faculty of Science,
Necmettin Erbakan University, Konya, TURKEY
*E-mail: dttollu@erbakan.edu.tr (corresponding author)

(Received: 31.01.2020, Accepted: 28.07.2020, Published Online: 30.07.2020)

Abstract

In this paper, we show that the following systems of nonlinear difference equations

$$x_{n+1}=\frac{x_ny_n+a}{x_n+y_n}, y_{n+1}=\frac{y_nz_n+a}{y_n+z_n}, z_{n+1}=\frac{z_nx_n+a}{z_n+x_n} \text{ for } n\in\mathbb{N}_0$$

where $a\in[0,\infty)$ and the initial values x_0, y_0, z_0 are real numbers, can be solved in explicit form. Also, we investigate the asymptotic behavior of the solutions by using these formulae and give some numerical examples which verify our theoretical result.

Keywords: Asymptotic behavior; explicit solution; nonlinear difference equation; system.
MSC 2010: 39A10.

1 Introduction

Recently, studying nonlinear difference equations and their systems have taken much attention see [1-22] and the references therein. That is because nonlinear difference equations and their systems have appeared in many scientific areas such as biology, physics, economics, etc.

Li and Zhu [10] studied the globally asymptotic stability of the nonlinear difference equation

$$x_{n+1}=\frac{x_nx_{n-1}+a}{x_n+x_{n-1}} \text{ for } n\in\mathbb{N}_0, \tag{1}$$

where $a\in[0,\infty)$ and the initial values are positive real numbers.

Abu-Saris et al. [1] investigated the globally asymptotically stability of the nonlinear difference equation

$$x_{n+1}=\frac{x_nx_{n-k}+a}{x_n+x_{n-k}} \text{ for } n\in\mathbb{N}_0, \tag{2}$$

where k is a nonnegative integer, $a\in[0,\infty)$ and the initial values are positive real numbers.

Motivated by all above mentioned study, in this paper, we show that the following systems of nonlinear difference equations

$$x_{n+1}=\frac{x_ny_n+a}{x_n+y_n},\ y_{n+1}=\frac{y_nz_n+a}{y_n+z_n},\ z_{n+1}=\frac{z_nx_n+a}{z_n+x_n} \text{ for } n\in\mathbb{N}_0, \tag{3}$$

where $a\in[0,\infty)$ and the initial values x_0, y_0, z_0 are real numbers, can be solved in explicit form. Also, we investigate the asymptotic behavior of the solutions by using these formulae and give some numerical examples which verify our theoretical result.

2 Solvability and the general solution of the system

In this section, we show that system (3) can be solved for both the case $a = 0$ and the case $a > 0$. We also obtain its general solution in explicit form.

2.1 Case $a = 0$

In this case, system (3) is in the form of

$$x_{n+1} = \frac{x_n y_n}{x_n + y_n},\ y_{n+1} = \frac{y_n z_n}{y_n + z_n},\ z_{n+1} = \frac{z_n x_n}{z_n + x_n} \text{ for } n \in \mathbb{N}_0. \tag{4}$$

The changes of variables

$$x_n = \frac{1}{u_n},\ y_n = \frac{1}{v_n},\ z_n = \frac{1}{w_n}, \tag{5}$$

where $x_n y_n z_n \neq 0$ for every $n \in \mathbb{N}_0$, reduce system (3) to the linear system of difference equations

$$u_{n+1} = u_n + v_n,\ v_{n+1} = v_n + w_n,\ w_{n+1} = w_n + u_n \text{ for } n \in \mathbb{N}_0. \tag{6}$$

By summing the equations of (6), we have

$$u_{n+1} + v_{n+1} + w_{n+1} = 2\left(u_n + v_n + w_n\right) \text{ for } n \in \mathbb{N}_0, \tag{7}$$

whose solution is

$$u_n + v_n + w_n = 2^n K_0 \text{ for } n \in \mathbb{N}_0, \tag{8}$$

where $K_0 = u_0 + v_0 + w_0$, from (6) and (8) we can write

$$v_n + w_{n+1} = 2^n K_0. \tag{9}$$

From (6) and (9), one can obtain the equations

$$v_{n+2} - v_{n+1} + v_n = 2^n K_0, \tag{10}$$

and

$$v_{n+1} = v_n - v_{n-1} + 2^{n-1} K_0. \tag{11}$$

A particular solution of (11) is

$$\frac{2^n K_0}{3}. \tag{12}$$

From (11) and (12), we have

$$v_{n+1} - \frac{2^{n+1} K_0}{3} = v_n - \frac{2^n K_0}{3} - \left(v_{n-1} - \frac{2^{n-1} K_0}{3}\right). \tag{13}$$

Let

$$r_n = v_n - \frac{2^n K_0}{3} \text{ for } n \in \mathbb{N}_0. \tag{14}$$

Then, from (13) and (14), we obtain the equation

$$r_{n+1} = r_n - r_{n-1} \text{ for } n \in \mathbb{N}_1, \tag{15}$$

whose solution is given by

$$r_{6n+i} = r_i \quad \text{for} \quad i \in \{0, 1, 2, 3, 4, 5\} \tag{16}$$

which is periodic with period 6. Therefore, (14) and (16) imply that

$$r_0 = v_0 - \frac{K_0}{3} = -\frac{u_0 - 2v_0 + w_0}{3}, \tag{17}$$

$$r_1 = v_1 - \frac{2K_0}{3} = v_0 + w_0 - \frac{2K_0}{3} = -\frac{2u_0 - v_0 - w_0}{3}, \tag{18}$$

$$r_2 = r_1 - r_0 = -\frac{u_0 + v_0 - 2w_0}{3}, \tag{19}$$

$$r_3 = r_2 - r_1 = -r_0 = \frac{u_0 - 2v_0 + w_0}{3}, \tag{20}$$

$$r_4 = r_3 - r_2 = -r_1 = \frac{2u_0 - v_0 - w_0}{3}, \tag{21}$$

$$r_5 = r_4 - r_3 = -(r_1 - r_0) = \frac{u_0 + v_0 - 2w_0}{3}, \tag{22}$$

and so

$$v_{6n} = 2^{6n}\frac{K_0}{3} + r_0 = u_0\left(\frac{2^{6n}-1}{3}\right) + v_0\left(\frac{2^{6n}+2}{3}\right) + w_0\left(\frac{2^{6n}-1}{3}\right), \tag{23}$$

$$v_{6n+1} = 2^{6n+1}\frac{K_0}{3} + r_1 = u_0\left(\frac{2^{6n+1}-2}{3}\right) + v_0\left(\frac{2^{6n+1}+1}{3}\right) + w_0\left(\frac{2^{6n+1}+1}{3}\right), \tag{24}$$

$$v_{6n+2} = 2^{6n+2}\frac{K_0}{3} + r_2 = u_0\left(\frac{2^{6n+2}-1}{3}\right) + v_0\left(\frac{2^{6n+2}-1}{3}\right) + w_0\left(\frac{2^{6n+2}+2}{3}\right), \tag{25}$$

$$v_{6n+3} = 2^{6n+3}\frac{K_0}{3} + r_3 = u_0\left(\frac{2^{6n+3}+1}{3}\right) + v_0\left(\frac{2^{6n+3}-2}{3}\right) + w_0\left(\frac{2^{6n+3}+1}{3}\right), \tag{26}$$

$$v_{6n+4} = 2^{6n+4}\frac{K_0}{3} + r_4 = u_0\left(\frac{2^{6n+4}+2}{3}\right) + v_0\left(\frac{2^{6n+4}-1}{3}\right) + w_0\left(\frac{2^{6n+4}-1}{3}\right), \tag{27}$$

$$v_{6n+5} = 2^{6n+5}\frac{K_0}{3} + r_5 = u_0\left(\frac{2^{6n+5}+1}{3}\right) + v_0\left(\frac{2^{6n+5}+1}{3}\right) + w_0\left(\frac{2^{6n+5}-2}{3}\right). \tag{28}$$

By using the formulae (23)-(28) in the second equation of (6), one can find the formulae

$$w_{6n} = u_0\left(\frac{2^{6n}-1}{3}\right) + v_0\left(\frac{2^{6n}-1}{3}\right) + w_0\left(\frac{2^{6n}+2}{3}\right), \tag{29}$$

$$w_{6n+1} = u_0\left(\frac{2^{6n+1}+1}{3}\right) + v_0\left(\frac{2^{6n+1}-2}{3}\right) + w_0\left(\frac{2^{6n+1}+1}{3}\right), \tag{30}$$

$$w_{6n+2} = u_0\left(\frac{2^{6n+2}+2}{3}\right) + v_0\left(\frac{2^{6n+2}-1}{3}\right) + w_0\left(\frac{2^{6n+2}-1}{3}\right), \tag{31}$$

$$w_{6n+3} = u_0\left(\frac{2^{6n+3}+1}{3}\right) + v_0\left(\frac{2^{6n+3}+1}{3}\right) + w_0\left(\frac{2^{6n+3}-2}{3}\right), \tag{32}$$

$$w_{6n+4} = u_0\left(\frac{2^{6n+4}-1}{3}\right) + v_0\left(\frac{2^{6n+4}+2}{3}\right) + w_0\left(\frac{2^{6n+4}-1}{3}\right) \tag{33}$$

and

$$w_{6n+5} = u_0\left(\frac{2^{6n+5}-2}{3}\right) + v_0\left(\frac{2^{6n+5}+1}{3}\right) + w_0\left(\frac{2^{6n+5}+1}{3}\right). \tag{34}$$

Similarly, by using the formulae (29)-(34) in the third equation of (6), one can obtain the formulae

$$u_{6n} = u_0\left(\frac{2^{6n}+2}{3}\right) + v_0\left(\frac{2^{6n}-1}{3}\right) + w_0\left(\frac{2^{6n}-1}{3}\right), \tag{35}$$

$$u_{6n+1} = u_0\left(\frac{2^{6n+1}+1}{3}\right) + v_0\left(\frac{2^{6n+1}+1}{3}\right) + w_0\left(\frac{2^{6n+1}-2}{3}\right), \tag{36}$$

$$u_{6n+2} = u_0\left(\frac{2^{6n+2}-1}{3}\right) + v_0\left(\frac{2^{6n+2}+2}{3}\right) + w_0\left(\frac{2^{6n+2}-1}{3}\right), \tag{37}$$

$$u_{6n+3} = u_0\left(\frac{2^{6n+3}-2}{3}\right) + v_0\left(\frac{2^{6n+3}+1}{3}\right) + w_0\left(\frac{2^{6n+3}+1}{3}\right), \tag{38}$$

$$u_{6n+4} = u_0\left(\frac{2^{6n+4}-1}{3}\right) + v_0\left(\frac{2^{6n+4}-1}{3}\right) + w_0\left(\frac{2^{6n+4}+2}{3}\right) \tag{39}$$

and

$$u_{6n+5} = u_0\left(\frac{2^{6n+5}+1}{3}\right) + v_0\left(\frac{2^{6n+5}-2}{3}\right) + w_0\left(\frac{2^{6n+5}+1}{3}\right). \tag{40}$$

Now, by using the formulae in (23)-(40) into (5), we get the general solution of (4) as follows:

$$x_{6n} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n}+2\right) + x_0z_0\left(2^{6n}-1\right) + x_0y_0\left(2^{6n}-1\right)}, \tag{41}$$

$$x_{6n+1} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+1}+1\right) + x_0z_0\left(2^{6n+1}+1\right) + x_0y_0\left(2^{6n+1}-2\right)}, \tag{42}$$

$$x_{6n+2} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+2}-1\right) + x_0z_0\left(2^{6n+2}+2\right) + x_0y_0\left(2^{6n+2}-1\right)}, \tag{43}$$

$$x_{6n+3} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+3}-2\right) + x_0z_0\left(2^{6n+3}+1\right) + x_0y_0\left(2^{6n+3}+1\right)}, \tag{44}$$

$$x_{6n+4} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+4}-1\right) + x_0z_0\left(2^{6n+4}-1\right) + x_0y_0\left(2^{6n+4}+2\right)}, \tag{45}$$

$$x_{6n+5} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+5}+1\right) + x_0z_0\left(2^{6n+5}-2\right) + x_0y_0\left(2^{6n+5}+1\right)}, \tag{46}$$

$$y_{6n} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n}-1\right) + x_0z_0\left(2^{6n}+2\right) + x_0y_0\left(2^{6n}-1\right)}, \tag{47}$$

$$y_{6n+1} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+1}-2\right) + x_0z_0\left(2^{6n+1}+1\right) + x_0y_0\left(2^{6n+1}+1\right)}, \tag{48}$$

$$y_{6n+2} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+2}-1\right) + x_0z_0\left(2^{6n+2}-1\right) + x_0y_0\left(2^{6n+2}+2\right)}, \tag{49}$$

$$y_{6n+3} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+3}+1\right) + x_0z_0\left(2^{6n+3}-2\right) + x_0y_0\left(2^{6n+3}+1\right)}, \tag{50}$$

$$y_{6n+4} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+4}+2\right) + x_0z_0\left(2^{6n+4}-1\right) + x_0y_0\left(2^{6n+4}-1\right)}, \tag{51}$$

$$y_{6n+5} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+5}+1\right) + x_0z_0\left(2^{6n+5}+1\right) + x_0y_0\left(2^{6n+5}-2\right)}, \tag{52}$$

$$z_{6n} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n}-1\right) + x_0z_0\left(2^{6n}-1\right) + x_0y_0\left(2^{6n}+2\right)}, \tag{53}$$

$$z_{6n+1} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+1}+1\right) + x_0z_0\left(2^{6n+1}-2\right) + x_0y_0\left(2^{6n+1}+1\right)}, \tag{54}$$

$$z_{6n+2} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+2}+2\right) + x_0z_0\left(2^{6n+2}-1\right) + x_0y_0\left(2^{6n+2}-1\right)}, \tag{55}$$

$$z_{6n+3} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+3}+1\right) + x_0z_0\left(2^{6n+3}+1\right) + x_0y_0\left(2^{6n+3}-2\right)}, \tag{56}$$

$$z_{6n+4} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+4}-1\right) + x_0z_0\left(2^{6n+4}+2\right) + x_0y_0\left(2^{6n+4}-1\right)}, \tag{57}$$

$$z_{6n+5} = \frac{3x_0y_0z_0}{y_0z_0\left(2^{6n+5}-2\right) + x_0z_0\left(2^{6n+5}+1\right) + x_0y_0\left(2^{6n+5}+1\right)}. \tag{58}$$

2.2 Case $a > 0$

In this case, system (3) can be written in the forms of

$$x_{n+1} + \sqrt{a} = \frac{x_n y_n + a + \sqrt{a} x_n + \sqrt{a} y_n}{x_n + y_n}, \tag{59}$$
$$y_{n+1} + \sqrt{a} = \frac{y_n z_n + a + \sqrt{a} y_n + \sqrt{a} z_n}{y_n + z_n}, \tag{60}$$
$$z_{n+1} + \sqrt{a} = \frac{z_n x_n + a + \sqrt{a} z_n + \sqrt{a} x_n}{z_n + x_n} \tag{61}$$

and

$$x_{n+1} - \sqrt{a} = \frac{x_n y_n + a - \sqrt{a} x_n - \sqrt{a} y_n}{x_n + y_n}, \tag{62}$$
$$y_{n+1} - \sqrt{a} = \frac{y_n z_n + a - \sqrt{a} y_n - \sqrt{a} z_n}{y_n + z_n}, \tag{63}$$
$$z_{n+1} - \sqrt{a} = \frac{z_n x_n + a - \sqrt{a} z_n - \sqrt{a} x_n}{z_n + x_n}. \tag{64}$$

From (59)-(61) and (62)-(64), we get the system

$$\frac{X_{n+1}^{+}}{X_{n+1}^{-}} = \frac{X_n^{+}}{X_n^{-}} \frac{Y_n^{+}}{Y_n^{-}}, \ \frac{Y_{n+1}^{+}}{Y_{n+1}^{-}} = \frac{Y_n^{+}}{Y_n^{-}} \frac{Z_n^{+}}{Z_n^{-}}, \ \frac{Z_{n+1}^{+}}{Z_{n+1}^{-}} = \frac{Z_n^{+}}{Z_n^{-}} \frac{X_n^{+}}{X_n^{-}} \tag{65}$$

where

$$x_n + \sqrt{a} = X_n^{+}, \ y_n + \sqrt{a} = Y_n^{+}, \ z_n + \sqrt{a} = Z_n^{+}, \tag{66}$$

and

$$x_n - \sqrt{a} = X_n^{-}, \ y_n - \sqrt{a} = Y_n^{-}, \ z_n - \sqrt{a} = Z_n^{-}. \tag{67}$$

for $(x_n + y_n)(y_n + z_n)(z_n + x_n) \neq 0$ and $(x_n \pm \sqrt{a})(y_n \pm \sqrt{a})(z_n \pm \sqrt{a}) \neq 0$ for $n \in \mathbb{N}_0$. System (65) can easily be solved. By iterating (65) for $n \geq 0$, we get

$$\begin{aligned}
&\frac{X_1^{+}}{X_1^{-}} = \frac{X_0^{+}}{X_0^{-}} \frac{Y_0^{+}}{Y_0^{-}}, \frac{Y_1^{+}}{Y_1^{-}} = \frac{Y_0^{+}}{Y_0^{-}} \frac{Z_0^{+}}{Z_0^{-}}, \frac{Z_1^{+}}{Z_1^{-}} = \frac{Z_0^{+}}{Z_0^{-}} \frac{X_0^{+}}{X_0^{-}} \\
&\frac{X_2^{+}}{X_2^{-}} = \frac{X_0^{+}}{X_0^{-}} \left(\frac{Y_0^{+}}{Y_0^{-}}\right)^2 \frac{Z_0^{+}}{Z_0^{-}}, \frac{Y_2^{+}}{Y_2^{-}} = \frac{Y_0^{+}}{Y_0^{-}} \left(\frac{Z_0^{+}}{Z_0^{-}}\right)^2 \frac{X_0^{+}}{X_0^{-}}, \frac{Z_2^{+}}{Z_2^{-}} = \frac{Z_0^{+}}{Z_0^{-}} \left(\frac{X_0^{+}}{X_0^{-}}\right)^2 \frac{Y_0^{+}}{Y_0^{-}} \\
&\frac{X_3^{+}}{X_3^{-}} = \left(\frac{X_0^{+}}{X_0^{-}}\right)^2 \left(\frac{Y_0^{+}}{Y_0^{-}}\right)^3 \left(\frac{Z_0^{+}}{Z_0^{-}}\right)^3, \frac{Y_3^{+}}{Y_3^{-}} = \left(\frac{Y_0^{+}}{Y_0^{-}}\right)^2 \left(\frac{Z_0^{+}}{Z_0^{-}}\right)^3 \left(\frac{X_0^{+}}{X_0^{-}}\right)^3, \frac{Z_3^{+}}{Z_3^{-}} = \left(\frac{Z_0^{+}}{Z_0^{-}}\right)^2 \left(\frac{X_0^{+}}{X_0^{-}}\right)^3 \left(\frac{Y_0^{+}}{Y_0^{-}}\right)^3 \\
&\quad \vdots
\end{aligned} \tag{68}$$

From (68), we conclude that the solution of system (65) is in the form of

$$\frac{X_n^{+}}{X_n^{-}} = \left(\frac{X_0^{+}}{X_0^{-}}\right)^{a_n} \left(\frac{Y_0^{+}}{Y_0^{-}}\right)^{b_n} \left(\frac{Z_0^{+}}{Z_0^{-}}\right)^{c_n}, \tag{69}$$
$$\frac{Y_n^{+}}{Y_n^{-}} = \left(\frac{Y_0^{+}}{Y_0^{-}}\right)^{a_n} \left(\frac{Z_0^{+}}{Z_0^{-}}\right)^{b_n} \left(\frac{X_0^{+}}{X_0^{-}}\right)^{c_n}, \tag{70}$$
$$\frac{Z_n^{+}}{Z_n^{-}} = \left(\frac{Z_0^{+}}{Z_0^{-}}\right)^{a_n} \left(\frac{X_0^{+}}{X_0^{-}}\right)^{b_n} \left(\frac{Y_0^{+}}{Y_0^{-}}\right)^{c_n}. \tag{71}$$

From (69)-(71), we obtain

$$\frac{X_0^+}{X_0^-}=\left(\frac{X_0^+}{X_0^-}\right)^{a_0}\left(\frac{Y_0^+}{Y_0^-}\right)^{b_0}\left(\frac{Z_0^+}{Z_0^-}\right)^{c_0}, \tag{72}$$

$$\frac{Y_0^+}{Y_0^-}=\left(\frac{Y_0^+}{Y_0^-}\right)^{a_0}\left(\frac{Z_0^+}{Z_0^-}\right)^{b_0}\left(\frac{X_0^+}{X_0^-}\right)^{c_0}, \tag{73}$$

$$\frac{Z_0^+}{Z_0^-}=\left(\frac{Z_0^+}{Z_0^-}\right)^{a_0}\left(\frac{X_0^+}{X_0^-}\right)^{b_0}\left(\frac{Y_0^+}{Y_0^-}\right)^{c_0} \tag{74}$$

from which it follows that $a_0=1, b_0=c_0=0$. By using (69)-(71) in (65), we have the system

$$\left(\frac{X_0^+}{X_0^-}\right)^{a_{n+1}}\left(\frac{Y_0^+}{Y_0^-}\right)^{b_{n+1}}\left(\frac{Z_0^+}{Z_0^-}\right)^{c_{n+1}}=\left(\frac{X_0^+}{X_0^-}\right)^{a_n+c_n}\left(\frac{Y_0^+}{Y_0^-}\right)^{a_n+b_n}\left(\frac{Z_0^+}{Z_0^-}\right)^{b_n+c_n} \tag{75}$$

for $n\in\mathbb{N}_0$, which implies that

$$a_{n+1}=a_n+c_n,\ c_{n+1}=c_n+b_n,\ b_{n+1}=b_n+a_n. \tag{76}$$

By taking $a_n=u_n$, $c_n=v_n$, $b_n=w_n$ and $a_0=1, b_0=c_0=0$, the solution of (76) can be obtained from the solution of (6) as follows:

$$a_{6n}=\frac{2^{6n}+2}{3},\ a_{6n+1}=\frac{2^{6n+1}+1}{3},\ a_{6n+2}=\frac{2^{6n+2}-1}{3}, \tag{77}$$

$$a_{6n+3}=\frac{2^{6n+3}-2}{3},\ a_{6n+4}=\frac{2^{6n+4}-1}{3},\ a_{6n+5}=\frac{2^{6n+5}+1}{3}. \tag{78}$$

$$b_{6n}=\frac{2^{6n}-1}{3},\ b_{6n+1}=\frac{2^{6n+1}+1}{3},\ b_{6n+2}=\frac{2^{6n+2}+2}{3}, \tag{79}$$

$$b_{6n+3}=\frac{2^{6n+3}+1}{3},\ b_{6n+4}=\frac{2^{6n+4}-1}{3},\ b_{6n+5}=\frac{2^{6n+5}-2}{3}, \tag{80}$$

$$c_{6n}=\frac{2^{6n}-1}{3},\ c_{6n+1}=\frac{2^{6n+1}-2}{3},\ c_{6n+2}=\frac{2^{6n+2}-1}{3}, \tag{81}$$

$$c_{6n+3}=\frac{2^{6n+3}+1}{3},\ c_{6n+4}=\frac{2^{6n+4}+2}{3},\ c_{6n+5}=\frac{2^{6n+5}+1}{3}. \tag{82}$$

Consequently, from (69)-(71) and (77)-(82), we have the following formulae:

$$x_{6n}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n}+2}{3}}\left(Y_0^+\right)^{\frac{2^{6n}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n}-1}{3}}+\left(X_0^-\right)^{\frac{2^{6n}+2}{3}}\left(Y_0^-\right)^{\frac{2^{6n}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n}-1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n}+2}{3}}\left(Y_0^+\right)^{\frac{2^{6n}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n}-1}{3}}-\left(X_0^-\right)^{\frac{2^{6n}+2}{3}}\left(Y_0^-\right)^{\frac{2^{6n}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n}-1}{3}}}, \tag{83}$$

$$x_{6n+1}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+1}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+1}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+1}-2}{3}}+\left(X_0^-\right)^{\frac{2^{6n+1}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+1}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+1}-2}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+1}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+1}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+1}-2}{3}}-\left(X_0^-\right)^{\frac{2^{6n+1}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+1}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+1}-2}{3}}}, \tag{84}$$

$$x_{6n+2}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+2}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+2}+2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+2}-1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+2}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+2}+2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+2}-1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+2}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+2}+2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+2}-1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+2}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+2}+2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+2}-1}{3}}}, \tag{85}$$

$$x_{6n+3}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+3}-2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+3}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+3}+1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+3}-2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+3}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+3}+1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+3}-2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+3}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+3}+1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+3}-2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+3}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+3}+1}{3}}}, \tag{86}$$

$$x_{6n+4}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+4}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+4}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+4}+2}{3}}+\left(X_0^-\right)^{\frac{2^{6n+4}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+4}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+4}+2}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+4}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+4}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+4}+2}{3}}-\left(X_0^-\right)^{\frac{2^{6n+4}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+4}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+4}+2}{3}}}, \tag{87}$$

$$x_{6n+5}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+5}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+5}-2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+5}+1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+5}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+5}-2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+5}+1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+5}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+5}-2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+5}+1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+5}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+5}-2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+5}+1}{3}}}, \tag{88}$$

$$y_{6n}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n}+2}{3}}\left(Z_0^+\right)^{\frac{2^{6n}-1}{3}}+\left(X_0^-\right)^{\frac{2^{6n}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n}+2}{3}}\left(Z_0^-\right)^{\frac{2^{6n}-1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n}+2}{3}}\left(Z_0^+\right)^{\frac{2^{6n}-1}{3}}-\left(X_0^-\right)^{\frac{2^{6n}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n}+2}{3}}\left(Z_0^-\right)^{\frac{2^{6n}-1}{3}}}, \tag{89}$$

$$y_{6n+1}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+1}-2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+1}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+1}+1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+1}-2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+1}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+1}+1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+1}-2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+1}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+1}+1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+1}-2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+1}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+1}+1}{3}}}, \tag{90}$$

$$y_{6n+2}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+2}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+2}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+2}+2}{3}}+\left(X_0^-\right)^{\frac{2^{6n+2}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+2}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+2}+2}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+2}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+2}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+2}+2}{3}}-\left(X_0^-\right)^{\frac{2^{6n+2}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+2}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+2}+2}{3}}}, \tag{91}$$

$$y_{6n+3}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+3}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+3}-2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+3}+1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+3}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+3}-2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+3}+1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+3}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+3}-2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+3}+1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+3}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+3}-2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+3}+1}{3}}}, \tag{92}$$

$$y_{6n+4}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+4}+2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+4}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+4}-1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+4}+2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+4}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+4}-1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+4}+2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+4}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+4}-1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+4}+2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+4}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+4}-1}{3}}}, \tag{93}$$

$$y_{6n+5}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+5}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+5}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+5}-2}{3}}+\left(X_0^-\right)^{\frac{2^{6n+5}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+5}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+5}-2}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+5}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+5}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+5}-2}{3}}-\left(X_0^-\right)^{\frac{2^{6n+5}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+5}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+5}-2}{3}}}, \tag{94}$$

$$z_{6n}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n}+2}{3}}+\left(X_0^-\right)^{\frac{2^{6n}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n}+2}{3}}}{\left(X_0^+\right)^{\frac{2^{6n}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n}+2}{3}}-\left(X_0^-\right)^{\frac{2^{6n}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n}+2}{3}}}, \tag{95}$$

$$z_{6n+1}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+1}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+1}-2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+1}+1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+1}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+1}-2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+1}+1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+1}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+1}-2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+1}+1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+1}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+1}-2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+1}+1}{3}}}, \tag{96}$$

$$z_{6n+2}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+2}+2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+2}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+2}-1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+2}+2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+2}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+2}-1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+2}+2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+2}-1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+2}-1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+2}+2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+2}-1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+2}-1}{3}}}, \tag{97}$$

$$z_{6n+3}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+3}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+3}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+3}-2}{3}}+\left(X_0^-\right)^{\frac{2^{6n+3}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+3}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+3}-2}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+3}+1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+3}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+3}-2}{3}}-\left(X_0^-\right)^{\frac{2^{6n+3}+1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+3}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+3}-2}{3}}}, \tag{98}$$

$$z_{6n+4}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+4}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+4}+2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+4}-1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+4}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+4}+2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+4}-1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+4}-1}{3}}\left(Y_0^+\right)^{\frac{2^{6n+4}+2}{3}}\left(Z_0^+\right)^{\frac{2^{6n+4}-1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+4}-1}{3}}\left(Y_0^-\right)^{\frac{2^{6n+4}+2}{3}}\left(Z_0^-\right)^{\frac{2^{6n+4}-1}{3}}}, \tag{99}$$

$$z_{6n+5}=\sqrt{a}\frac{\left(X_0^+\right)^{\frac{2^{6n+5}-2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+5}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+5}+1}{3}}+\left(X_0^-\right)^{\frac{2^{6n+5}-2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+5}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+5}+1}{3}}}{\left(X_0^+\right)^{\frac{2^{6n+5}-2}{3}}\left(Y_0^+\right)^{\frac{2^{6n+5}+1}{3}}\left(Z_0^+\right)^{\frac{2^{6n+5}+1}{3}}-\left(X_0^-\right)^{\frac{2^{6n+5}-2}{3}}\left(Y_0^-\right)^{\frac{2^{6n+5}+1}{3}}\left(Z_0^-\right)^{\frac{2^{6n+5}+1}{3}}}. \tag{100}$$

2.3 Behavior of the solutions of the system

In this section, we investigate the asymptotic behavior of the solutions of system (3) and give some numerical examples which verify our theoretical result. The main result of this subsection is the following theorem:

Theorem 1 *The following statements are true*
(i) If $a = 0$, then $(x_n, y_n, z_n) \to (0, 0, 0)$ as $n \to \infty$.
(ii) If $a > 0$, then $(|x_n|, |y_n|, |z_n|) \to (\sqrt{a}, \sqrt{a}, \sqrt{a})$ as $n \to \infty$.

Proof.
(i) From the formulae (41)-(58) the desired result immediately follows.
(ii) We prove (ii) for only x_{6n}, since the proof is similar for the other subsequences of x_n, y_n and z_n. Note that the formula (83) can be written as follows:

$$x_{6n} = \sqrt{a}\left(1 + \frac{2}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\frac{2^{6n}+2}{3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{6n}-1}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{6n}-1}{3}} - 1}\right), n \in \mathbb{N}_0. \quad (101)$$

Hence, we consider the function

$$f(x) = \frac{x+\sqrt{a}}{x-\sqrt{a}}$$

which supplies the property

$$\begin{cases} |f(x)| < 1, \text{if } x < 0, \\ |f(x)| > 1, \text{if } x > 0. \end{cases} \quad (102)$$

That is, it is arise two specific cases from the formula (101) and the property (102):
(a) If $x_0 < 0$, $y_0 < 0$ and $z_0 < 0$, then $(x_n, y_n, z_n) \to (-\sqrt{a}, -\sqrt{a}, -\sqrt{a})$ as $n \to \infty$.
(b) If $x_0 > 0$, $y_0 > 0$ and $z_0 > 0$, then $(x_n, y_n, z_n) \to (\sqrt{a}, \sqrt{a}, \sqrt{a})$ as $n \to \infty$.
As to the other cases, we consider the sequence

$$(s_n)_{n\ge 0} = \left(\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\frac{2^{6n}+2}{3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{6n}-1}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{6n}-1}{3}}\right)_{n\ge 0}.$$

If $s_n \to 0$ as $n \to \infty$, then $(x_n, y_n, z_n) \to (-\sqrt{a}, -\sqrt{a}, -\sqrt{a})$ as $n \to \infty$. If $s_n \to \infty$ as $n \to \infty$, then $(x_n, y_n, z_n) \to (\sqrt{a}, \sqrt{a}, \sqrt{a})$ as $n \to \infty$. ■

Now, we give some numerical examples to support our theoretical results related to system (3) with some restrictions on the parameter a.

Example 2 *We visualize the solutions of system (4) in figures (1)-(3) for $a = 0$ and for the sets of initial values: $\{x_0 = 5.2, y_0 = 0.7, z_0 = 3.1\}$, $\{x_0 = -5.2, y_0 = -0.7, z_0 = -3.1\}$, $\{x_0 = 5.2, y_0 = -0.7, z_0 = 3.1\}$, respectively.*

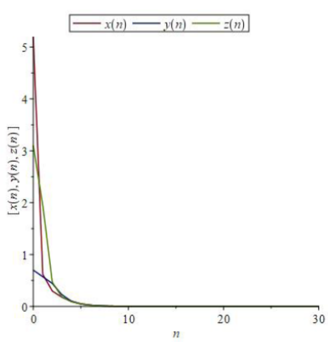

Figure 1

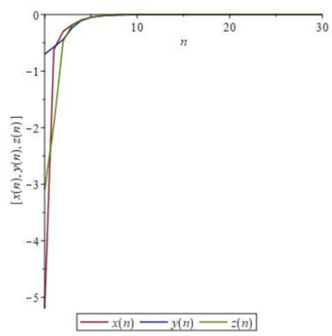

Figure 2

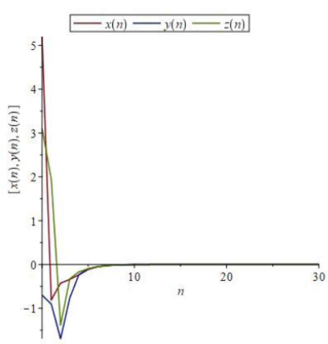

Figure 3

Example 3 *We visualize the solutions of system (3) in figures (4)-(6) for* $a = 3.14$ *and for the sets of initial values:* $\{x_0 = 0.9, y_0 = 0.7, z_0 = 2.5\}$, $\{x_0 = -0.9, y_0 = -0.7, z_0 = -2.5\}$, $\{x_0 = 0.9,\ y_0 = 0.7,$ $z_0 = -2.5\}$, *respectively.*

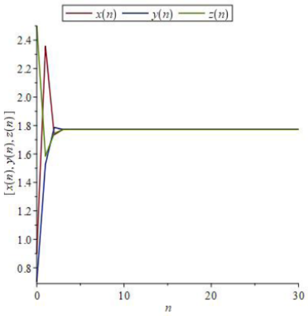

Figure 4

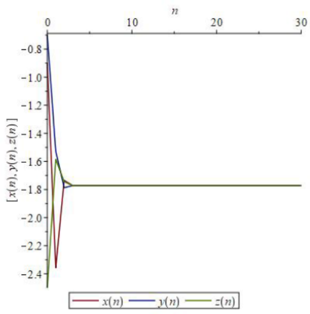

Figure 5

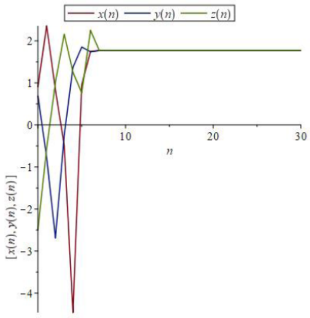

Figure 6

Example 4 *We visualize the solutions of system (3) in figures (7)-(9) for* $a = 100$ *and for the sets of initial values:* $\{x_0 = 2.9, y_0 = 5.1, z_0 = 7.8\}$, $\{x_0 = -2.9, y_0 = -5.1, z_0 = -7.8\}$, $\{x_0 = -2.9,$ $y_0 = -5.1,\ z_0 = 7.8\}$, *respectively.*

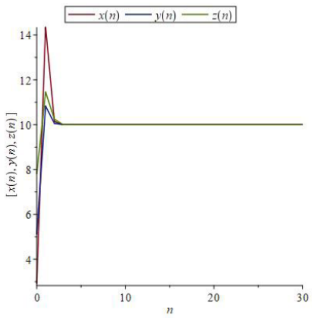

Figure 7

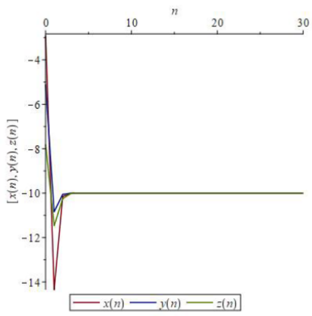

Figure 8

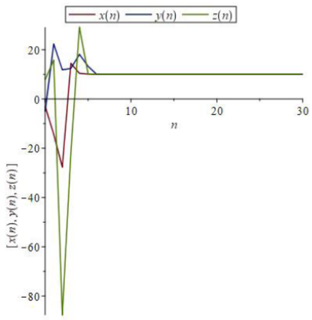

Figure 9

Example 5 *We visualize the solutions of system (3) in figures (10)-(12) for* $a = 2018$ *and for the sets of initial values:* $\{x_0 = 1.5, y_0 = 3.6, z_0 = 2.4\}$, $\{\ x_0 = -1.5, y_0 = -3.6, z_0 = -2.4\}$, $\{x_0 = -1.5,$ $y_0 = 3.6,\ z_0 = -2.4\}$, *respectively.*

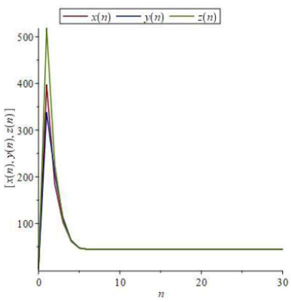

Figure 10

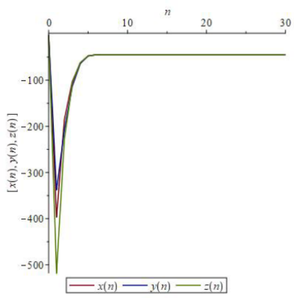

Figure 11

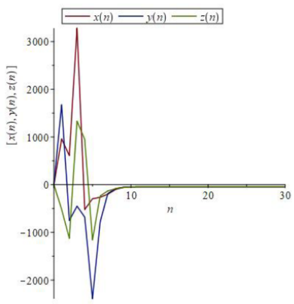

Figure 12

References

[1] Abu-Saris, R., Cinar, C., Yalcinkaya, İ. (2008) On the asymptotic stability of $x_{n+1} = \frac{x_n x_{n-k}+a}{x_n+x_{n-k}}$. Computers & Mathematics with Applications, 56(5): 1172–1175.

[2] Akgunes, N., Kurbanlı, A. S. (2014) On the system of rational difference equations $x_n = f\left(x_{n-a_1}, y_{n-b_1}\right),\ y_n = g\left(y_{n-b_2}, z_{n-c_1}\right),\ z_n = g\left(z_{n-c_2}, x_{n-a_2}\right)$. Selcuk Journal of Applied Mathematics, 15(1): 8 pages.

[3] Dekkar, I., Touafek, N. Yazlik, Y. (2017) Global stability of a third-order nonlinear system of difference equations with period-two coefficients. Revista de la real academia de ciencias exactas, f ísicas y naturales. Serie A. Matemáticas, 111: 325-347.

[4] Gümüş, M. Soykan, Y. (2016) Global character of a six-dimensional nonlinear system of difference equations. Discrete Dynamics in Nature and Society, Article ID 6842521: 7 pages.

[5] Haddad, N., Touafek, N., Rabago, J.F.T. (2017) Solution form of a higher-order system of difference equations and dynamical behavior of its special case. Mathematical Methods in the Applied Sciences, 40: 3599-3607.

[6] Haddad, N., Touafek, N., Rabago, J.F.T. (2018) Well-defined solutions of a system of difference equations. Journal of Applied Mathematics and Computing, 56: 439-458.

[7] Kara, M. Yazlik, Y. (2019) Solvability of a system of nonlinear difference equations of higher order. Turkish Journal of Mathematics, 43 (3): 1533-1565.

[8] Kurbanli, A.S., Çinar, C., Şimşek, D., (2011) On the periodicity of solutions of the system of rational difference equations. Applied Mathematics, 2: 410-413.

[9] Kurbanli, A. S., Çinar, C. Şimşek, D.(2011) On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1}+1}$, $y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1}+1}$. Mathematical and Computer Modelling, 53(5-6): 1261-1267.

[10] Li, X., Zhu, D. (2003) Global asymptotic stability in a rational equation. Journal of Difference Equations and Applications, 9(9): 833-839.

[11] Ozkan, O., Kurbanli, A. S. (2013) On a system of difference equations. Discrete Dynamics in Nature and Society, Article ID 970316: 7 pages.

[12] Stević, S., Alghamdi,M. A.,Alotaibi, A. Elsayed, E.M. (2015) Solvable product-type system of difference equations of second order. Electronic Journal of Differential Equations, No:169: 1-20.

[13] Stević, S. (2017) New class of solvable systems of difference equations. Applied Mathematics Letters, 63: 137-144.

[14] Tollu, D.T., Yazlik, Y., Taskara, N. (2014) On fourteen solvable systems of difference equations. Applied Mathematics and Computation, 233: 310-319.

[15] Yalcinkaya, I., Cinar, C. Simsek, D. (2008) Global asymptotic stability of a system of difference equations. Applicable Analysis, 87(6): 677-687, DOI: 10.1080/00036810802140657.

[16] Yalcinkaya, I. (2008) On the global asymptotic stability of a second-order system of difference equations. Discrete Dynamics in Nature and Society, vol. 2008, Article ID 860152:12 pages, doi:10.1155/2008/860152.

[17] Yalcinkaya, I., Tollu, D.T. (2016) Global behavior of a second-order system of difference equations. Advanced Studies in Contemporary Mathematics, 26(4): 653-667.

[18] Yazlik, Y., Elsayed E.M., Taskara, N. (2014) On the behaviour of the solutions of difference equation systems. Journal of Computational Analysis & Applications, 16(5): 932-941.

[19] Yazlik, Y.,Tollu, D.T., Taskara, N. (2015) On the behaviour of solutions for some systems of difference equations, Journal of Computational Analysis & Applications, 18(1): 166-178.

[20] Yazlik, Y.,Tollu, D.T., Taskara, N. (2015) On the solutions of a max-type difference equation system. Mathematical Methods in the Applied Sciences, 38(17): 4388-4410.

[21] Yazlik, Y.,Tollu, D.T., Taskara, N. (2016) On the solutions of a three-dimensional system of difference equations. Kuwait Journal of Science 43(1): 95-111.

[22] Yazlik, Y., Kara, M. (2019) On a solvable system of difference equations of higher-order with period two coefficients. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(2): 1675-1693.

IKONION JOURNAL OF MATHEMATICS

YEAR: 2020 VOLUME :2 ISSUE: 1

ISSN: 2687-6531

ON THE STABILITY OF COMPOSITE PLASMA IN POROUS MEDIUM

Pardeep KUMAR[1,*] , Hari MOHAN [1]

[1]Department of Mathematics, ICDEOL, Himachal Pradesh University, Summerhill, SHIMLA-5, INDIA

[*] E-mail: pkdureja@gmail.com (corresponding author)

(Received: 30.05.2020, Accepted: 24.08.2020, Published Online: 02.09.2020)

Abstract

Rayleigh-Taylor instability of a composite plasma in porous medium is considered to include the frictional effect of collisions of ionized with neutral atoms in the presence of a variable magnetic field. The system is found to be stable for stable density stratification. The magnetic field can stabilize a system which was unstable in its absence. The medium permeability has a decreasing or an increasing effect on the growth rates. With the increase in collisional frequency, the growth rates decrease but may have increasing influence in certain region.

Keywords: Composite Plasma, Porous Medium, Rayleigh-Taylor Instability, Variable Magnetic Field.

1. Introduction

A detailed treatment of Rayleigh-Taylor instability under varying assumptions of hydrodynamics and hydromagnetics, together with the possible extensions in various domains of interest has been given by Chandrasekhar [1]. The medium has been considered to be fully ionized. Quite often the plasma is not fully ionized and is, instead, partially ionized. In cosmic physics, there are several situations such as chromoshere, solar photosphere, and in cool interstellar cloud where the plasma are frequently not fully ionized but may instead be partially ionized. A partially-ionized plasma represents a state which often exists in the Universe and there are several situations when the interaction between the ionized and neutral gas components becomes important in cosmic physics. Ionized hydrogen is limited to certain rather sharply bounded regions in space surrounding, for example, O-type stars and clusters of such stars and that the gas outside these regions is essentially non-ionized has been reported by Stromgren [2]. Other examples of the existence of such situations are given by Alfven's [3] theory on the origin of the planetary system, in which a high-ionization rate is suggested to appear from collisions between a plasma and a neutral-gas cloud and by the absorption of plasma waves due to ion-neutral collisions such as in the solar photosphere and chromosphere and in cool interstellar clouds (Piddington [4]; Lehnert [5]). Hans [6] and Bhatia [7] has shown that the medium may be idealized as a composite mixture of a hydromagnetic (ionized) component and a neutral component, the two interacting through mutual collisional effects. A stabilizing effect of collisionals on Rayleigh-Taylor configuration has been shown by Hans [6] and Bhatia [7].

The medium has been considered to be non-porous in all the above studies. In recent years, the investigations of flow of fluids through porous media have become an important topic due to the recovery of crude oil from the pores of reservoir rocks. A great number of applications in geophysics may be found in the books by Phillips [8], Ingham and Pop [9], and Nield and Bejan [10]. When the fluid slowly permeates through the pores of a macroscopically homogeneous and isotropic porous medium, the gross effect is represented by Darcy's law according to which the usual viscous term in the equations of fluid motion is replaced by the resistance term $-\left(\frac{\mu}{k_1}\right)\vec{q}$, where μ is the viscosity of the fluid, k_1 is the medium permeability and $\vec{q}$ is the Darcian (filter) velocity of the fluid. Lapwood [11] has studied the stability of convective flow in hydrodynamics in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through porous medium has been considered by Wooding [12]. Generally, it is accepted that comets consists of a dusty 'snowball' of a mixture of frozen gases which in the process of their journey changes from solid to gas and vice versa. The physical properties of comets, meteorites, and interplanetary dust strongly suggest the importance of porosity in astrophysical context (Mcdonnel [13]).

Diaz et al. [14] studied the modification of the classical criterion for the linear onset and growth rate of the Rayleigh-Taylor instability (RTI) in a partially ionized (PI) plasma in the one-fluid description by considering a generalized induction equation. The occurrence conditions of the condensation instability in heat-releasing partially ionized plasma in an external magnetic field with an induction vector normal to the direction along which a perturbation occurs are considered by Molevich et al. [15].

The problem of the hydromagnetic stability of conducting fluid of variable density and variable magnetic field in porous medium may play an important role in astrophysics (stability of stellar atmosphere in magnetic field, heating of solar corona, theories, and sunspot magnetic field) and geophysics (stability of Earth's core and geothermal regions). Keeping in mind such astrophysical situations, a study has been made of the Raleigh-Taylor instability of a partially-ionized plasma in porous medium in presence of variable horizontal magnetic field, in the present paper.

2. Formulation of the Problem and Perturbation Equations

Here we consider an incompressible composite plasma layer consisting of an infinitely conducting hydromagnetic fluid of density ρ, permeated with neutrals of density ρ_d in porous medium, arranged in horizontal strata and acted on by the gravity force $\vec{g}(0,0,-g)$ and the variable horizontal magnetic field $\vec{H}\,(H_0\,(z),0,0)$. We assume that both the fluid and the neutral gas behave like continuum fluids and that effects on the neutral component resulting from the fields of gravity and pressure are neglected. The magnetic field interacts with the hydromagnetic component only.

Let $\vec{q}(u, v, w), \vec{h}\,(h_x, h_y, h_z), \delta\rho$ and δp denote, respectively, the perturbations in velocity, magnetic field $\vec{H}$, density ρ and pressure p; $\overrightarrow{q_d}, \rho_d, v_c$, μ, μ_e, ε and k_1 denote the velocity of the neutral gas, density of neutral gas, mutual collisional (frictional) frequency between the two components of the composite medium, viscosity of the hydromagnetic fluid, magnetic permeability, medium porosity and medium permeability, respectively. Then the linearized perturbation equations governing the motion of the composite plasma are

$$\frac{\rho}{\varepsilon}\frac{\partial \vec{q}}{\partial t} = -\nabla\delta p + \frac{\mu_e}{4\pi}\left[(\nabla\times\vec{h})\times\vec{H} + (\nabla\times\vec{H})\times\vec{h}\right] + \vec{g}\,\delta\rho + \frac{\rho_d v_c}{\varepsilon}(\overrightarrow{q_d} - \vec{q}) - \frac{\mu}{k_1}\vec{q}, \tag{1}$$

$$\frac{\partial \overrightarrow{q_d}}{\partial t} = -v_c(\overrightarrow{q_d} - \vec{q}) \tag{2}$$

$$(\nabla.\vec{q}) = 0, \tag{3}$$

$$\varepsilon\frac{\partial}{\partial t}\,\delta\rho = -w\frac{d\rho}{dz}\,, \tag{4}$$

$$\nabla.\vec{h} = 0, \tag{5}$$

$$\varepsilon\frac{\partial \vec{h}}{\partial t} = (\vec{H}.\nabla)\vec{q} - (\vec{q}\,.\nabla)\vec{H}\,, \tag{6}$$

3. Dispersion relation

We seek solutions of the above equations, in terms of normal modes, whose dependence on space-time co-ordinates is of the form

$$f(z)exp\left[ik_x x + ik_y y + nt\right], \tag{7}$$

where $f(z)$ is some function of z only, k_x and k_y $(k^2 = k_x^2 + k_y^2)$ are horizontal wave numbers and n is the frequency of the harmonic disturbance.

Eliminating $\overrightarrow{q_d}$ between (1) and (2) and using (7), (1)-(6) yield

$$\left(\frac{n'\rho}{\varepsilon} + \frac{\mu}{k_1}\right)u = -ik_x\delta p + \frac{\mu_e}{4\pi}h_x(DH_0)\,, \tag{8}$$

$$\left(\frac{n'\rho}{\varepsilon} + \frac{\mu}{k_1}\right)v = -ik_y\delta p + \frac{\mu_e\ H_0}{4\pi}(ik_x h_y - ik_y h_x)\,, \tag{9}$$

$$\left(\frac{n'\rho}{\varepsilon} + \frac{\mu}{k_1}\right)w = -D\delta p + \frac{g}{n\varepsilon}(D\rho)w + \frac{\mu_e\ H_0}{4\pi}\left(ik_x h_z - Dh_x - h_x\frac{DH_0}{H_0}\right), \tag{10}$$

$$ik_x u + ik_y v + Dw = 0, \tag{11}$$

$$ik_x h_x + ik_y h_y + Dh_z = 0, \tag{12}$$

$$n\delta\rho = -wD\rho, \qquad (12)\,**$$

$$\varepsilon n h_x = ik_x H_0 u - wDH_0, \tag{13}$$

$$\varepsilon n h_y = i k_x H_0 v, \tag{14}$$

$$\varepsilon n h_z = i k_x H_0 w, \tag{15}$$

where

$$n' = n\left(1 + \frac{\alpha_0 v_c}{n + v_c}\right), \alpha_0 = \frac{\rho_d}{\rho} \ and \ D = \frac{d}{dz}.$$

Eliminating u, v, h_x, h_y, h_z and δp between (8) - (10) and using (11) - (15), after a certain amount of algebra, we get

$$\frac{1}{\varepsilon}\left[D\left(n'\rho D w\right) - n'k^2 \rho w\right] + \frac{g k^2 (D\rho) w}{n\varepsilon} + \frac{\mu}{k_1}(D^2 - k^2) w + \frac{(D\mu)(Dw)}{k_1} +$$

$$+ \frac{\mu_e k_x^2}{4\pi n \varepsilon}\left[H_0^2 (D^2 - k^2) w + D(H_0^2) D w\right] = 0 \,. \tag{16}$$

Assume the stratifications in density, viscosity and magnetic field of the form

$$\rho = \rho_0 exp[\beta z], \ \ \mu = \mu_0 exp[\beta z], \ \rho_d = \rho_{d_0} exp[\beta z], \ H_0^2 = H_1^2 exp[\beta z], \tag{17}$$

where $\rho_0, \rho_{d_0}, \mu_0$, H_1 and β are constants. Equations (17) imply that the kinematic viscosity $\nu(= \mu/\rho = \mu_0/\rho_0)$ and the Alfven velocity V_A $(= \sqrt{\mu_e H_0^2/4\pi\rho} = \sqrt{\mu_e H_1^2/4\pi\rho_0}$ are constant everywhere.

Here we consider the case of two free boundaries. Let us assume that $\beta d \ll 1$, i.e. the variation of density at two neighbouring points in the velocity field which is much less than the average density has a negligible effect on the inertia of the fluid. The boundary conditions for the case of two free surfaces are

$$w = D^2 w = 0 \ \text{ at } z = 0 \text{ and } z = d \tag{18}$$

The solution of (16) satisfying the boundary conditions (18) is

$$w = A \sin\frac{m\pi z}{d} \ , \tag{19}$$

where A is a constant and m is any integer. Substituting (19) in (16) and neglecting the effect of heterogeneity on the inertia, we get

$$\left[\left(\frac{m\pi}{d}\right)^2 + k^2\right] - \frac{g k^2 \beta}{n\left[\frac{k_x^2 V_A^2}{n} + n' + \frac{\nu_0 \varepsilon}{k_1}\right]} = 0 \,. \tag{20}$$

Equation (20), on simplification, gives

$$n^3 + n^2\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + n\left[\left(k_x^2 V_A^2 - \frac{g\beta k^2}{L}\right) + \frac{\varepsilon}{k_1} v_c\right] + v_c\left[k_x^2 V_A^2 - \frac{g\beta k^2}{L}\right] = 0, \tag{21}$$

where $V_A^2 = \frac{\mu_e H_1^2}{4\pi\rho_0}$ and $L = (\frac{m\pi}{d})^2 + k^2$.

4. Discussion

Theorem 1: System is stable for stable density stratification and for unstable density stratification the system is stable or unstable under a condition.

Proof: For the **stable density stratification** ($\boldsymbol{\beta} < 0$) , (21) does not have any positive root and this implies the stability of the system. For **unstable density stratification** ($\boldsymbol{\beta} > 0$), the system is stable or unstable according as

$$k_x^2 V_A^2 \gtrless \frac{g\beta k^2}{L} \ . \tag{22}$$

Theorem 2: For unstable density stratification, in the absence of magnetic field, the system is unstable.

Proof: The system is clearly unstable in the absence of a magnetic field as can be seen from (21). However, the system can be completely stabilized by a large enough magnetic field as can be seen from (21), if

$$V_A^2 > \frac{g\beta k^2}{k_x^2 L} \ .$$

Theorem 3: The medium permeability has decreasing or increasing effect on the growth rates of the Taylor instability of a partially-ionized plasma in porous medium in the presence of a variable horizontal magnetic field.

Proof: If $\beta > 0$ and $k_x^2 V_A^2 < g\beta k^2/L$, (21) has at least one positive root. Let n_0 denotes the positive root of (21). Then

$$n_0^3 + n_0^2\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + n_0\left[\left(k_x^2V_A^2 - \frac{g\beta k^2}{L}\right) + \frac{\varepsilon}{k_1}v_c\right] + v_c\left[k_x^2V_A^2 - \frac{g\beta k^2}{L}\right] = 0, \tag{23}$$

To find the role of medium permeability on the growth rate of unstable modes, we examine the nature of dn_0/dk_1 . Equation (23) yields

$$\frac{dn_0}{dk_1} = \frac{n_0\varepsilon(n_0+v_c)}{k_1^2\left[3n_0^2 + 2n_0\left\{v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right\} + \left\{k_x^2V_A^2 - \frac{g\beta k^2}{L} + \frac{\varepsilon}{k_1}v_c\right\}\right]} \ . \tag{24}$$

For

$$\left|k_x^2V_A^2 - \frac{g\beta k^2}{L}\right| > 3n_0^2 + 2n_0\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + \frac{\varepsilon}{k_1}v_c \,,$$

dn_0/dk_1 , is negative. The growth rates, therefore, decrease with the increase in medium permeability. However, for

$$\left|k_x^2V_A^2 - \frac{g\beta k^2}{L}\right| < 3n_0^2 + 2n_0\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + \frac{\varepsilon}{k_1}v_c \,,$$

dn_0/dk_1 is positive and the growth rates increase with the increase in medium permeability. **The medium permeability, thus, has decreasing or increasing effect on the growth rates of the Taylor instability of a partially-ionized plasma in porous medium in the presence of a variable horizontal magnetic field.**

Theorem 4: The growth rates decrease with the increase in collisional frequency and the growth rates increase with the increase in collisional frequency under a region.

Proof: To find the role of collisional frequency on the growth rate of unstable modes, we examine the nature of dn_0/dv_c .

Equation (23) gives

$$\frac{dn_0}{dv_c} = -\frac{n_0^2(\alpha_0+1) + n_0\frac{\varepsilon}{k_1} + (k_x^2V_A^2 - g\beta k^2/L)}{3n_0^2 + 2n_0\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + \left[k_x^2V_A^2 - g\beta k^2/L + \frac{\varepsilon}{k_1}v_c\right]} . \tag{25}$$

Therefore, if, in addition to $k^2 > k_x^2V_A^2L/g\beta$, which is a sufficient condition for instability, we have either of the condition

$$\left|k_x^2V_A^2 - \frac{g\beta k^2}{L}\right| > 3n_0^2 + 2n_0\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + \frac{\varepsilon}{k_1}v_c , \tag{26}$$

or

$$\left|k_x^2V_A^2 - \frac{g\beta k^2}{L}\right| < (1+\alpha_0)n_0^2 + \frac{\varepsilon}{k_1}n_0 , \tag{27}$$

dn_0/dv_c is always negative. **The growth rates, therefore, decrease with the increase in collisional frequency. However, the growth rates increase with the increase in collisional frequency for the region**

$$\begin{aligned}(\mathbf{1}+\boldsymbol{\alpha}_0)\boldsymbol{n}_0^2 + \frac{\boldsymbol{\varepsilon}}{\boldsymbol{k}_1}\boldsymbol{n}_0 &< \left|\left(\boldsymbol{k}_x^2\boldsymbol{V}_A^2 - \frac{\boldsymbol{g\beta k^2}}{\boldsymbol{L}}\right)\right| \\ &< \mathbf{3}\boldsymbol{n}_0^2 + \mathbf{2}\boldsymbol{n}_0\left[\boldsymbol{v}_c(\mathbf{1}+\boldsymbol{\alpha}_0) + \frac{\boldsymbol{\varepsilon}}{\boldsymbol{k}_1}\right] + \frac{\boldsymbol{\varepsilon}}{\boldsymbol{k}_1}\boldsymbol{v}_c ,\end{aligned} \tag{28}$$

5. Results

The main results of the present study are:

(i) The system is found to be stable for stable stratification.

(ii) For unstable density stratification, the system is stable or unstable according as

$$\boldsymbol{k}_x^2\boldsymbol{V}_A^2 \gtrless \frac{\boldsymbol{g\beta k^2}}{\boldsymbol{L}} .$$

(iii) For unstable density stratification, in the absence of magnetic field, the system is unstable. However, the system can be completely stabilized by a large enough magnetic field if $\boldsymbol{V}_A^2 > \frac{\boldsymbol{g\beta k^2}}{\boldsymbol{k}_x^2\boldsymbol{L}}$.

(iv) The medium permeability has decreasing or increasing effect on the growth rates of the Taylor instability of a partially-ionized plasma in porous medium in the presence of a variable horizontal magnetic field.

(v) The growth rates decrease with the increase in collisional frequency and the growth rates increase with the increase in collisional frequency for the region

$$(1+\alpha_0)n_0^2+\frac{\varepsilon}{k_1}n_0<\left|\left(k_x^2V_A^2-\frac{g\beta k^2}{L}\right)\right|<3n_0^2+2n_0\left[v_c(1+\alpha_0)+\frac{\varepsilon}{k_1}\right]+\frac{\varepsilon}{k_1}v_c.$$

References

[1] Chandrasekhar, S. (1981) *Hydrodynamic and hydromagnetic stability*. Dover Publication, New York.

[2] Stromgren, B. (1939) The physical state of interstellar hydrogen. *Astrophysics J.*, 89: 526-547.

[3] Alfven, H. (1954) *The origin of the solar system.* Clarendon Press, Oxford.

[4] Piddington, J.H. (1954) Electromagnetic field equations for a moving medium with hall conductivity. *Monthly Notices Roy. Astron. Soc.*, 14: 638.

[5] Lehnert, B. (1959) Plasma of cosmic and laboratory scales. *Suppl. Nuovo Cimento*, 13: 59.

[6] Hans, H.K. (1968) Larmor radius and collisional effects on the combined Taylor and Kelvin instabilities in a composite medium. *Nucl. Fusion*, 8: 89-92.

[7] Bhatia, P.K. (1970) Collisional effects on the Rayleigh-Taylor instability in a composite medium. *Nucl. Fusion*, 10: 383-386.

[8] Phillips, O.M. (1991) *Flow and reaction in permeable rocks.* Cambridge University Press, Cambridge, U.K..

[9] Ingham, D.B., Pop, I. (1998) *Transport phenomena in porous medium.* Pergamon Press, Oxford, U.K.

[10] Nield, D.A. .and Bejan, A. (1999)*Convection in porous medium.* 2nd Edition, Springer, New York.

[11] Lapwood, E.R. (1948) Convection of a fluid in a porous medium. *Proc. Camb. Phil. Soc.*, 44:508-521.

[12] Wooding, R.A. (1960) Rayleigh instability of a thermal boundary layer in flow through a porous medium. *J. Fluid Mech.*, 9: 183-192.

[13] McDonnel, J.W. (1978) *Cosmic dust.* John Wiley and Sons, Toronto.

[14] Diaz, A.J., Khomenko, E., Collados, M. (2014) Rayleigh Taylor instability in partially ionized compressible plasmas: One fluid approach. *Astronomy & Astrophysics*, A97: 564-567.

[15] Molevich, N.E., Pichugin, S.Yu, Ryaschchikov, D.S. Zavershinskii, D.I. (2018) Condensation instability in partially ionized plasma in a magnetic field. *Bull. Lebedev Physics Institute*, 45(9): 267-271.

IKONION JOURNAL OF MATHEMATICS
YEAR: 2020 VOLUME :2 ISSUE: 1
ISSN: 2687-6531

A NOTE ON THE GENERALIZATIONS OF JACOBTHAL AND JACOBTHAL-LUCAS SEQUENCES

Yashwant K. PANWAR [1,*] , V. K. GUPTA [1]

[1] Department of Mathematics, Govt. Model College, Jhabua, INDIA
[2]Department of Mathematics, Govt. Madhav Science College, Ujjain, INDIA
[*] E-mail: yashwantpanwar@gmail.com (corresponding author)

(Received: 22.05.2020, Accepted: 26.08.2020, Published Online: 05.09.2020)

Abstract

In this paper, we present the generalization of Jacobsthal and Jacobsthal-Lucas sequences by the recurrence relations $J_n = 2aJ_{n-1} + (b-a^2)J_{n-2}$ & $j_n = 2aj_{n-1} + (b-a^2)j_{n-2}$, $n \geq 2$ with the initial conditions $J_0 = 0,\ J_1 = 1$ *and* $j_0 = 2,\ j_1 = 2a$. We establish some of the interesting properties of involving them. Also we describe and derive sums, connection formulae and Generating function. We have used their Binet's formula to derive the identities.

Keywords: Generalized Jacobsthal sequence, Generalized Jacobsthal-Lucas sequence, Binet's formula and Generating function.

1. Introduction

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are most prominent examples of recursive sequences. The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.

Kalman and Mena [10] generalize the Fibonacci sequence by

$$F_n = aF_{n-1} + bF_{n-2},\ n \geq 2 \quad with \quad F_0 = 0,\ F_1 = 1 \tag{1.1}$$

Horadam [8] defined generalized Fibonacci sequence $\{H_n\}$ by

$$H_n = H_{n-1} + H_{n-2},\ n \geq 3\ with\ H_1 = p,\ H_2 = p+q \tag{1.2}$$

where p and q are arbitrary integers.

The k-Fibonacci numbers defined by Falco'n and Plaza [4], for any positive real number k, the k-Fibonacci sequence is defined recurrently by

$$F_{k,n} = k\,F_{k,n-1} + F_{k,n-2},\ n \geq 2 \quad with \quad F_{k,0} = 0,\ F_{k,1} = 1 \tag{1.3}$$

The k-Lucas numbers defined by Falco'n [2],

$$L_{k,n} = k\,L_{k,n-1} + L_{k,n-2}\,,\; n \geq 2 \quad with \quad L_{k,0} = 2,\; L_{k,1} = k \tag{1.4}$$

Most of the authors introduced Fibonacci pattern based sequences in many ways which are known as Fibonacci-Like sequences and k-Fibonacci-like sequences [13, 17, 22, 28, 29].

Generalized Fibonacci sequence [7], is defined as

$$F_k = pF_{k-1} + qF_{k-2}\,,\; k \geq 2 \quad with \quad F_0 = a,\; F_1 = b \tag{1.5}$$

where p, q, a and b are positive integer.

(p,q) - Fibonacci numbers [19], is defined as

$$F_{p,q,n} = pF_{p,q,n} + bF_{p,q,n}\,,\; n \geq 2 \quad with \quad F_{p,q,0} = 0,\; F_{p,q,n} = 1 \tag{1.6}$$

(p,q) - Lucas numbers [20], is defined as

$$L_{p,q,n} = pL_{p,q,n} + bL_{p,q,n}\,,\; n \geq 2 \quad with \quad L_{p,q,0} = 2,\; L_{p,q,n} = p \tag{1.7}$$

Generalized (p,q) -Fibonacci-Like sequence [21], is defined by recurrence relation

$$S_{p,q,n} = pS_{p,q,n} + qS_{p,q,n}\,,\; n \geq 2 \quad with \quad S_{p,q,0} = 2k,\; S_{p,q,n} = 1 + kp \tag{1.8}$$

Goksal Bilgici [1], defined new generalizations of Fibonacci and Lucas sequences

$$f_k = 2af_{k-1} + (b-a^2)\,f_{k-2}\,,\; k \geq 2 \;\; with \;\; f_0 = 0,\; f_1 = 1 \tag{1.9}$$

$$l_k = 2al_{k-1} + (b-a^2)l_{k-2}\,,\; k \geq 2 \;\; with \;\; l_0 = 2,\; l_1 = 2a \tag{1.10}$$

Tulay Yagmur [30], defined generalizations of Pell and Pell-Lucas sequences

$$p_k = 2ap_{k-1} + (b-a^2)\,p_{k-2}\,,\; k \geq 2 \;\; with \;\; p_0 = 0,\; p_1 = 1 \tag{1.11}$$

$$q_k = 2aq_{k-1} + (b-a^2)q_{k-2}\,,\; k \geq 2 \;\; with \;\; q_0 = 2,\; q_1 = 2a \tag{1.12}$$

In this study, we present the generalization of Jacobsthal and Jacobsthal-Lucas sequences, in much the same way that Bilgici did for Fibonacci and Lucas sequences in [1] and Yagmur did for Pell and Pell-Lucas sequences in [30]. We prove the Catalan, Cassini, and d'Ocagne identities for this sequence. Moreover, we introduce the special sums of the generalized Jacobsthal and Jacobsthal-Lucas sequences and prove them using Binet's formula.

2. Generalized Jacobsthal and Jacobsthal-Lucas Sequences

In this section, we review basic definitions and introduce relevant facts.

For $n \geq 2$, The generalized Jacobsthal sequence is defined by

$$J_n = 2aJ_{n-1} + (b-a^2)J_{n-2} \tag{2.1}$$

with initial conditions $J_0 = 0$, $J_1 = 1$.

First few generalized Jacobsthal numbers are

$$\{J_n\} = \{0, 1, 2a, 3a^2 + b, 4a^3 + 4ab, 5a^4 + 10a^2b + b^2, ...\}$$

It is well known that the Jacobsthal and Jacobsthal-Lucas sequences are closely related.

For $n \geq 2$, The generalized Jacobsthal-Lucas sequence is defined by

$$j_n = 2a\, j_{n-1} + (b - a^2)\, j_{n-2} \tag{2.2}$$

with initial conditions $j_0 = 2$, $j_1 = 2a$.

First few generalized Jacobsthal-Lucas numbers are

$$\{j_n\} = \{2, 2a, 2a^2 + 2b, 2a^3 + 6ab, 2a^4 + 12a^2b + 2b^2, 2a^5 + 20a^3b + 10ab^2, ...\}$$

In (2.1) and (2.2), $a \,\&\, b$ are any nonzero real numbers.

If $a = \frac{1}{2}$ & $b = \frac{9}{4}$, then we obtained classical Jacobsthal and Jacobsthal-Lucas sequences,
If $a = \frac{1}{2}$ & $b = \frac{5}{4}$, then we obtained classical Fibonacci and Lucas sequences,
If $a = 1$ & $b = 3$, then we obtained classical Pell sequence and Pell-Lucas sequences,
If $a = \frac{3}{2}$ & $b = \frac{1}{4}$, then we obtained classical Mersenne and Fermat sequences.

For any positive integer k,
If $a = \frac{k}{2}$ & $b = \left(\frac{4+k^2}{4}\right)$, then we obtained k-Fibonacci and k-Lucas sequences,
If $a = 1$ & $b = (1+k)$, then we obtained k-Pell and k-Pell-Lucas sequences,
If $a = \frac{k}{2}$ & $b = \left(\frac{8+k^2}{4}\right)$, then we obtained k- Jacobsthal and k- Jacobsthal-Lucas sequences.

2.1. Explicit sum formulae of generalized Jacobsthal and Jacobsthal-Lucas sequences

Theorem 2.1. Explicit sum Formula for generalized Jacobsthal sequence is given by

$$J_n = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-i-1}{i} (2a)^{n-2i-1} (b-a^2)^i \tag{2.3}$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence, the proof is clear.

Theorem 2.2. Explicit sum Formula for new generalized Jacobsthal-Lucas sequence is given by

$$j_n = 2\sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i} (2a)^{n-2i} (b-a^2)^i - \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-i-1}{i} (2a)^{n-2i} (b-a^2)^i \tag{2.4}$$

Proof. Applying Binet's formula of generalized Jacobsthal-Lucas sequence, the proof is clear

2.2. Binet's formula of Generalized Jacobsthal and Jacobsthal-Lucas sequences

In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers. In our case, Binet's formula allows us to express the generalized Jacobsthal and Jacobsthal-Lucas sequences in function of the roots of the following characteristic equation, associated to the recurrence relation (2.1) & (2.2):

$$x^2 = 2ax + (b - a^2) \tag{2.5}$$

Theorem 2.3. ***(Binet's formula).*** The nth terms of the generalized Jacobsthal sequence is given by

$$J_n = \frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2} \tag{2.6}$$

where $\Re_1$ & $\Re_2$ are the roots of the characteristic equation (2.5), with

$\Re_1 = a + \sqrt{b}$, $\Re_2 = a - \sqrt{b}$.

Proof. We use the Principle of Mathematical Induction (PMI) on n. It is clear the result is true for $n = 0$ & $n = 1$ by hypothesis. Assume that it is true for i such that $0 \le i \le r + 1$, then

$$J_i = \frac{\Re_1^i - \Re_2^i}{\Re_1 - \Re_2}$$

It follows from definition generalized Jacobsthal sequence (2.1) and equation (2.6)

$$J_{r+2} = 2aJ_{r+1} + (b - a^2)J_r = \frac{\Re_1^{r+2} - \Re_2^{r+2}}{\Re_1 - \Re_2}$$

Thus, the formula is true for any positive integer n.

Theorem 2.4. ***(Binet's formula).*** The nth terms of the generalized Jacobsthal-Lucas sequence is given by

$$j_n = \Re_1^n + \Re_2^n \tag{2.7}$$

Proof. It can be proved same as Theorem 2.3.

Theorem 2.5. For every integer n, we have

(i) $$J_{-n} = \frac{-J_n}{(a^2 - b)^n} \tag{2.8}$$

(ii) $$j_{-n} = \frac{j_n}{(a^2 - b)^n} \tag{2.9}$$

Theorem 2.6. For every integer n, we have

$$J_n j_n = J_{2n} \tag{2.10}$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence and generalized Jacobsthal-Lucas sequence, the proof is clear.

2.3. Identities of Generalized Jacobsthal and Jacobsthal-Lucas sequences

In this section, we introduce Catalan, Cassini and d'Ocagne identities for the generalized Jacobsthal and Jacobsthal-Lucas sequences and prove them using Binet's formula stated in the previous section.

2.3.1. Catalan's Identity

Catalan's identity for Fibonacci numbers was found in 1879 by Eugene Charles Catalan a Belgian mathematician who worked for the Belgian Academy of Science in the field of number theory:

Theorem 2.7. ***(Catalan's identity).*** For every integers $n \; and \, r$, we have

$$J_{n+r}J_{n+r} - J_n^2 = -(a^2-b)^{n-r} J_r^2 \tag{2.11}$$

and

$$j_{n+r}\, j_{n+r} - j_n^2 = 4b(a^2-b)^{n-r}\, j_r^2 \tag{2.12}$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence and generalized Jacobsthal-Lucas sequence completes the proof of Catalan's identity.

2.3.2. Cassini's Identity

This is one of the oldest identities involving the Fibonacci numbers. It was discovered in 1680 by Jean-Dominique Cassini a French astronomer:

Theorem 2.8. ***(Cassini's identity).*** For every integers n, we have

$$J_{n+1}J_{n+1} - J_n^2 = -(a^2-b)^{n-1} \tag{2.13}$$

and

$$j_{n+1}j_{n+1} - j_1^2 = 4b(a^2-b)^{n-1} \tag{2.14}$$

Proof. Taking $r=1$ in Catalan's identity (2.11) & (2.12) the proof is completed.

2.3.3. d'Ocagne's identity

Theorem 2.9. ***(d'Ocagne's identity).*** For every integers $n \; and \, m$, we have

$$J_m J_{n+1} - J_n J_{m+1} = (a^2-b)^n J_{m-n} \tag{2.15}$$

and

$$j_m\, j_{n+1} - j_n\, j_{m+1} = -4ab(a^2-b)^n J_{m-n} \tag{2.16}$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence and generalized Jacobsthal-Lucas sequence completes the proof of d'Ocagne's identity.

3. The Sums of the Generalized Jacobsthal and Jacobsthal-Lucas Sequences

Binet's formula allows us to express the sum of generalized Jacobsthal and Jacobsthal-Lucas sequences.

3.1. Sums of Generailzed Jacobsthal Sequence

Theorem 3.1. For fixed integers $p,\ q\ with\ 0 \le q \le p-1$, the following equality holds

$$J_{p(n+2)+q} = j_p\, J_{p(n+1)+q} - (a^2-b)^p J_{pn+q} \tag{3.1}$$

Proof. From the the Binet's formula of generalized Jacobsthal and Jacobsthal-Lucas sequences,

$$\begin{aligned}
j_p\, J_{p(n+1)+q} &= \left(\Re_1^p + \Re_2^p\right)\left(\frac{\Re_1^{p(n+1)+q} - \Re_2^{p(n+1)+q}}{\Re_1 - \Re_2}\right)\\
&= \frac{1}{\Re_1-\Re_2}\left[\Re_1^{p(n+2)+q} + (a^2-b)^p \Re_1^{pn+q} - (a^2-b)^p \Re_2^{pn+q} - \Re_2^{p(n+2)+q}\right]\\
&= \frac{1}{\Re_1-\Re_2}\left[\left\{\Re_1^{p(n+2)+q} - \Re_2^{p(n+2)+q}\right\} + (a^2-b)^p\left(\Re_1^{pn+q} - \Re_2^{pn+q}\right)\right]\\
&= J_{p(n+2)+q} + (a^2-b)^p J_{pn+q}
\end{aligned}$$

then, the equality becomes,

$$J_{p(n+2)+q} = j_p\, J_{p(n+1)+q} - (a^2-b)^p J_{pn+q}$$

Theorem 3.2. For fixed integers $p,\ q\ with\ 0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} J_{pi+q} = \frac{J_{p(n+1)+q} - (a^2-b)^q J_{p-q} - J_q - (a^2-b)^p J_{pn+q}}{j_p - (a^2-b)^p - 1} \tag{3.2}$$

Proof. From the the Binet's formula of generalized Jacobsthal sequence,

$$\begin{aligned}
\sum_{i=0}^{n} J_{pi+q} &= \sum_{i=0}^{n} \frac{\Re_1^{pi+q} - \Re_2^{pi+q}}{\Re_1 - \Re_2}\\
&= \frac{1}{\Re_1-\Re_2}\left[\sum_{i=0}^{n} \Re_1^{pi+q} - \sum_{i=0}^{n} \Re_2^{pi+q}\right]\\
&= \frac{1}{\Re_1-\Re_2}\left[\frac{\Re_1^{pn+q+p} - \Re_1^q}{\Re_1^p - 1} - \frac{\Re_2^{pn+q+p} - \Re_2^q}{\Re_2^p - 1}\right]\\
&= \frac{1}{(a^2-b)^p - j_p + 1}\left[(a^2-b)^p J_{pn+q} - J_{p(n+1)+q} + J_q + (a^2-b)^q J_{p-q}\right]\\
&= \frac{J_{p(n+1)+q} - (a^2-b)^q J_{p-q} - J_q - (a^2-b)^p J_{pn+q}}{j_p - (a^2-b)^p - 1}
\end{aligned}$$

This completes the proof.

Corollary 3.3. Sum of odd generalized Jacobsthal sequence, If $p = 2m+1$ then Eq. (3.2) is

$$\sum_{i=0}^{n} J_{(2m+1)i+q} = \frac{J_{(2m+1)(n+1)+q} - (a^2-b)^q J_{2m+1-q} - J_q - (a^2-b)^{(2m+1)} J_{(2m+1)n+q}}{j_{(2m+1)} - (a^2-b)^{(2m+1)} - 1} \tag{3.3}$$

For example

(1) If $m=0$ then $p=1$: $\sum_{i=0}^{n} J_{i+q} = \frac{J_{n+q+1} - (a^2-b)^q J_{1-q} - J_q - (a^2-b) J_{n+q}}{2a - (a^2-b) - 1}$ (3.4)

(i) For $q=0$: $\sum_{i=0}^{n} J_i = \frac{J_{n+1} - 1 - (a^2-b) J_n}{2a - (a^2-b) - 1}$

(2) If $m=1$ then $p=3$: $\sum_{i=0}^{n} J_{3i+q} = \frac{J_{3n+q+3} - (a^2-b)^q J_{3-q} - J_q - (a^2-b)^3 J_{3n+q}}{a^3(2-a^3) + 3ab(2-ab) + b(3a^4+b^2) - 1}$ (3.5)

(i) For $q=0$: $\sum_{i=0}^{n} J_{3i} = \frac{J_{3n+3} - (3a^2+b) - (a^2-b)^3 J_{3n}}{a^3(2-a^3) + 3ab(2-ab) + b(3a^4+b^2) - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{3i+1} = \frac{J_{3n+4} - 2a(a^2+b) - 1 - (a^2-b)^3 J_{3n+1}}{a^3(2-a^3) + 3ab(2-ab) + b(3a^4+b^2) - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{3i+2} = \frac{J_{3n+5} - (a^2-b)^2 - 2a - (a^2-b)^3 J_{3n+2}}{a^3(2-a^3) + 3ab(2-ab) + b(3a^4+b^2) - 1}$

(3) If $m=2$ then $p=5$: $\sum_{i=0}^{n} J_{5i+q} = \frac{J_{5n+q+5} - (a^2-b)^q J_{5-q} - J_q - (a^2-b)^5 J_{5n+q}}{j_5 - (a^2-b)^5 - 1}$ (3.6)

(i) For $q=0$: $\sum_{i=0}^{n} J_{5i} = \frac{J_{5n+5} - J_5 - (a^2-b)^5 J_{5n}}{j_5 - (a^2-b)^5 - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{5i+1} = \frac{J_{5n+6} - (a^2-b) J_4 - 1 - (a^2-b)^5 J_{5n+1}}{j_5 - (a^2-b)^5 - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{5i+2} = \frac{J_{5n+7} - (a^2-b)^2 J_3 - 2a - (a^2-b)^5 J_{5n+2}}{j_5 - (a^2-b)^5 - 1}$

(iv) For $q=3$: $\sum_{i=0}^{n} J_{5i+3} = \frac{J_{5n+8} - (a^2-b)^3 2a - (3a^2+b) - (a^2-b)^5 J_{5n+3}}{j_5 - (a^2-b)^5 - 1}$

(v) For $q=4$: $\sum_{i=0}^{n} J_{5i+4} = \frac{J_{5n+9} - (a^2-b)^4 - 4a(a^2+b) - (a^2-b)^5 J_{5n+4}}{j_5 - (a^2-b)^5 - 1}$

(vi) For $q=5$: $\sum_{i=0}^{n} J_{5i+5} = \frac{J_{5n+10} - J_5 - (a^2-b)^5 J_{5n+5}}{j_5 - (a^2-b)^5 - 1}$

Corollary 3.4. Sum of even generalized Jacobsthal sequence, If $p=2m$ then Eq. (3.2) is

$$\sum_{i=0}^{n} J_{2mi+q} = \frac{J_{2m(n+1)+q} - (a^2-b)^q J_{2m-q} - J_q - (a^2-b)^{2m} J_{2mn+q}}{j_{2m} - (a^2-b)^{2m} - 1} \tag{3.7}$$

For example

(1) If $m=1$ then $p=2$: $\sum_{i=0}^{n} J_{2i+q} = \frac{J_{2n+2+q} - (a^2-b)^q J_{2-q} - J_q - (a^2-b)^2 J_{2n+q}}{j_2 - (a^2-b)^2 - 1}$ (3.8)

(i) For $q=0$: $\sum_{i=0}^{n} J_{2i} = \frac{J_{2n+2} - 2a - (a^2-b)^2 J_{2n}}{(2a^2+2b) - (a^2-b)^2 - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{2i+1} = \frac{J_{2n+3} - (a^2-b) - 1 - (a^2-b)^2 J_{2n+1}}{(2a^2+2b) - (a^2-b)^2 - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{2i+2} = \frac{J_{2n+4} - 2a - (a^2-b)^2 J_{2n+2}}{(2a^2+2b) - (a^2-b)^2 - 1}$

(2) If $m=2$ then $p=4$: $\sum_{i=0}^{n} J_{4i+q} = \frac{J_{4n+4+q} - (a^2-b)^q J_{4-q} - J_q - (a^2-b)^4 J_{4n+q}}{j_4 - (a^2-b)^4 - 1}$ (3.9)

(i) For $q=0$: $\sum_{i=0}^{n} J_{4i} = \frac{J_{4n+4} - J_4 - (a^2-b)^4 J_{4n}}{j_4 - (a^2-b)^4 - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{4i+1} = \frac{J_{4n+5} - (a^2-b) J_3 - 1 - (a^2-b)^4 J_{4n+1}}{j_4 - (a^2-b)^4 - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{4i+2} = \frac{J_{4n+6} - (a^2-b)^2 2a - 2a - (a^2-b)^4 J_{4n+2}}{j_4 - (a^2-b)^4 - 1}$

(iv) For $q=3$: $\sum_{i=0}^{n} J_{4i+3} = \frac{J_{4n+7} - (a^2-b)^3 - J_3 - (a^2-b)^4 J_{4n+3}}{j_4 - (a^2-b)^4 - 1}$

(v) For $q=4$: $\sum_{i=0}^{n} J_{4i+4} = \frac{J_{4n+8} - J_4 - (a^2-b)^4 J_{4n+4}}{j_4 - (a^2-b)^4 - 1}$

(3) If $m=3$ then $p=6$: $\sum_{i=0}^{n} J_{6i+q} = \frac{J_{6n+6+q} - (a^2-b)^q J_{6-q} - J_q - (a^2-b)^6 J_{6n+q}}{j_6 - (a^2-b)^6 - 1}$ (3.10)

(i) For $q=0$: $\sum_{i=0}^{n} J_{6i} = \frac{J_{6n+6} - J_6 - (a^2-b)^6 J_{6n}}{j_6 - (a^2-b)^6 - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{6i+1} = \frac{J_{6n+7} - (a^2-b) J_5 - 1 - (a^2-b)^6 J_{6n+1}}{j_6 - (a^2-b)^6 - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{6i+2} = \frac{J_{6n+8} - (a^2-b)^2 J_4 - 2a - (a^2-b)^6 J_{6n+2}}{j_6 - (a^2-b)^6 - 1}$

(iv) For $q=3$: $\sum_{i=0}^{n} J_{6i+3} = \frac{J_{6n+9} - (a^2-b)^3 J_3 - J_3 - (a^2-b)^6 J_{6n+3}}{j_6 - (a^2-b)^6 - 1}$

Theorem 3.5. For fixed integers p, q *with* $0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} (-1)^i J_{pi+q} = \frac{(-1)^n J_{p(n+1)+q} + (-1)^n (a^2-b)^p J_{pn+q} - (a^2-b)^q J_{p-q} + J_q}{j_p + (a^2-b)^p + 1} \quad (3.11)$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence, the proof is clear.

For different values of $p \& q$:

(i) $\sum_{i=0}^{n}(-1)^i J_i = \frac{(-1)^n J_{n+1} + (-1)^n (a^2-b) J_n - 1}{2a + a^2 - b + 1}$

(ii) $\sum_{i=0}^{n}(-1)^i J_{2i} = \frac{(-1)^n J_{2n+2} + (-1)^n (a^2-b)^2 J_{2n} - 2a}{(2a^2+2b) + (a^2-b)^2 + 1}$

(iii) $\sum_{i=0}^{n}(-1)^i J_{2i+1} = \frac{(-1)^n J_{2n+3} + (-1)^n (a^2-b)^2 J_{2n+1} - (a^2-b) + 1}{(2a^2+2b) + (a^2-b)^2 + 1}$

(iv) $\sum_{i=0}^{n}(-1)^i J_{4i} = \frac{(-1)^n J_{4n+4} + (-1)^n (a^2-b)^4 J_{4n} - J_4}{j_4 + (a^2-b)^4 + 1}$

(v) $\sum_{i=0}^{n}(-1)^i J_{4i+1} = \frac{(-1)^n J_{4n+5} + (-1)^n (a^2-b)^4 J_{4n+1} - (a^2-b) J_3 + 1}{j_4 + (a^2-b)^4 + 1}$

(vi) $\sum_{i=0}^{n}(-1)^i J_{4i+2} = \frac{(-1)^n J_{4n+6} + (-1)^n (a^2-b)^4 J_{4n+2} - (a^2-b)^2 2a + 2a}{j_4 + (a^2-b)^4 + 1}$

(vii) $\sum_{i=0}^{n}(-1)^i J_{4i+3} = \frac{(-1)^n J_{4n+7} + (-1)^n (a^2-b)^4 J_{4n+3} - (a^2-b)^3 + J_3}{j_4 + (a^2-b)^4 + 1}$

3.2. Sums of Generailzed Jacobsthal-Lucas Sequence

Theorem 3.6. For fixed integers p, q *with* $0 \le q \le p-1$, the following equality holds

$$j_{p(n+1)+q} = j_p j_{pn+q} - (a^2-b)^p j_{p(n-1)+q} \tag{3.12}$$

Theorem 3.7. For fixed integers p, q *with* $0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} j_{pi+q}(x,y) = \frac{j_{p(n+1)+q} + (a^2-b)^q j_{p-q} - j_q - (a^2-b)^p j_{pn+q}}{j_p - (a^2-b)^p - 1} \tag{3.13}$$

Corollary 3.8. Sum of odd generalized Jacobsthal-Lucas sequence, If $p = 2m+1$ then Eq. (3.13) is

$$\sum_{i=0}^{n} j_{(2m+1)i+q} = \frac{j_{(2m+1)(n+1)+q} + (a^2-b)^q j_{2m+1-q} - j_q - (a^2-b)^{(2m+1)} j_{(2m+1)n+q}}{j_{(2m+1)} - (a^2-b)^{(2m+1)} - 1} \tag{3.14}$$

For example

(1) If $m=0$ then $p=1$: $\sum_{i=0}^{n} j_{i+q} = \frac{j_{n+q+1} + (a^2-b)^q j_{1-q} - j_q - (a^2-b) j_{n+q}}{2a - (a^2-b) - 1}$ (3.15)

(i) For $q=0$: $\sum_{i=0}^{n} j_i = \frac{j_{n+1} + 2a - 2 - (a^2-b) j_n}{2a - (a^2-b) - 1}$

(2) If $m=1$ then $p=3$: $\sum_{i=0}^{n} j_{3i+q} = \frac{j_{3n+q+3} + (a^2-b)^q j_{3-q} - j_q - (a^2-b)^3 j_{3n+q}}{2(a^3+3ab) - (a^2-b)^3 - 1}$ (3.16)

(3) If $m=2$ then $p=5$: $\sum_{i=0}^{n} j_{5i+q} = \frac{j_{5n+q+5} + (a^2-b)^q j_{5-q} - j_q - (a^2-b)^5 j_{5n+q}}{j_5 - (a^2-b)^5 - 1}$ (3.17)

Corollary 3.9. Sum of even generalized Jacobsthal-Lucas sequence, If $p = 2m$ then Eq. (3.13) is

$$\sum_{i=0}^{n} j_{2mi+q} = \frac{j_{2m(n+1)+q} + (a^2-b)^q j_{2m-q} - j_q - (a^2-b)^{2m} j_{2mn+q}}{j_{2m} - (a^2-b)^{2m} - 1} \tag{3.18}$$

For example

(1) If $m=1$ then $p=2$: $\sum_{i=0}^{n} j_{2i+q} = \frac{j_{2n+2+q} + (a^2-b)^q j_{2-q} - j_q - (a^2-b)^2 j_{2n+q}}{j_2 - (a^2-b)^2 - 1}$ (3.19)

(2) If $m=2$ then $p=4$: $\sum_{i=0}^{n} j_{4i+q} = \frac{j_{4n+4+q} + (a^2-b)^q j_{4-q} - j_q - (a^2-b)^4 j_{4n+q}}{j_4 - (a^2-b)^4 - 1}$ (3.20)

(3) If $m=3$ then $p=6$: $\sum_{i=0}^{n} j_{6i+q} = \frac{j_{6n+6+q} + (a^2-b)^q j_{6-q} - j_q - (a^2-b)^6 j_{6n+q}}{j_6 - (a^2-b)^6 - 1}$ (3.21)

Theorem 3.10. For fixed integers p, q *with* $0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} (-1)^i j_{pi+q} = \frac{(-1)^n j_{p(n+1)+q} + (-1)^n (a^2-b)^p j_{pn+q} + (a^2-b)^q j_{p-q} + j_q}{j_p + (a^2-b)^p + 1} \tag{3.22}$$

Proof. Applying Binet's formula of generalized Jacobsthal-Lucas sequence, the proof is clear. For different values of $p \& q$:

(i) $$\sum_{i=0}^{n} (-1)^i j_i = \frac{(-1)^n \{ j_{n+1} + (a^2-b) j_n \} + 2(a+1)}{2a + (a^2-b) + 1}$$

(ii) $$\sum_{i=0}^{n} (-1)^i j_{2i} = \frac{(-1)^n \{ j_{2n+2} + (a^2-b)^2 j_{2n} \} + 2(a^2+b+1)}{2(a^2+b) + (a^2-b)^2 + 1}$$

(iii) $$\sum_{i=0}^{n} (-1)^i j_{2i+1} = \frac{(-1)^n \{ j_{2n+3} + (a^2-b)^2 j_{2n+1} \} + 2a(a^2-b+1)}{2(a^2+b) + (a^2-b)^2 + 1}$$

(iv) $$\sum_{i=0}^{n} (-1)^i j_{4i} = \frac{(-1)^n \{ j_{4n+4} + (a^2-b)^4 j_{4n} \} + j_4 + 2}{j_4 + (a^2-b)^4 + 1}$$

(v) $$\sum_{i=0}^{n} (-1)^i j_{4i+1} = \frac{(-1)^n \{ j_{4n+5} + (a^2-b)^4 j_{4n+1} \} + (a^2-b) j_3 + 2a}{j_4 + (a^2-b)^4 + 1}$$

(vi) $$\sum_{i=0}^{n} (-1)^i j_{4i+2} = \frac{(-1)^n \{ j_{4n+6} + (a^2-b)^4 j_{4n+2} \} + \{ (a^2-b)^2 + 1 \} (2a^2 + 2b)}{j_4 + (a^2-b)^4 + 1}$$

4. Generalized Identities of the Product of Generalized Jacobsthal and Jacobsthal-Lucas Sequences

Thongmoon [24, 25], defined various identities of Fibonacci and Lucas numbers. Singh, Bhadouria and Sikhwal [16], present some generalized identities involving common factors of Fibonacci and Lucas numbers. Gupta and Panwar [6], present identities involving common factors of generalized Fibonacci, Jacobsthal and jacobsthal-Lucas numbers. Panwar, Singh and Gupta ([14, 15]), present Generalized Identities Involving Common factors of generalized Fibonacci, Jacobsthal and jacobsthal-Lucas numbers. Singh, Sisodiya and Ahmed [18], investigate some products of k-Fibonacci and k-Lucas numbers, also present some generalized identities on the products of k-Fibonacci and k-Lucas numbers to establish connection formulas between them with the help of Binet's formula. Pakapongpun [12], present Jacobsthal-like sequence ([7]), also provide generalized identities on the products of Jacobsthal-like and Jacobsthal-Lucas numbers. Thongkam, Butsuwan and Bunya [23], present the investigation of products of (p, q)-Fibonacci-like and (p, q)-Lucas numbers. In this section, we present identities involving product of generalized Jacobsthal and Jacobsthal-Lucas sequences and related identities.

Theorem 4.1. If J_k *and* j_k are generalized Jacobsthal and Jacobsthal-Lucas sequences, then

$$J_{2k+p}\, j_{2k+1} = J_{4k+p+1} + (a^2-b)^{2k+1} J_{p-1}, \text{ where } k \ge 0 \ \& \ p \le 0 \tag{4.1}$$

Proof. Applying Binet's formula of generalized Jacobsthal and Jacobsthal-Lucas sequences,

$$J_{2k+p}\, j_{2k+1} = \left(\frac{\Re_1^{2k+p} - \Re_2^{2k+p}}{\Re_1 - \Re_2}\right)\left(\Re_1^{2k+1} + \Re_2^{2k+1}\right) \tag{4.2}$$

$$= \left(\frac{\Re_1^{4k+p+1} - \Re_2^{4k+p+1}}{\Re_1 - \Re_2}\right) + \frac{\left(\Re_1\Re_2\right)^{2k}}{\left(\Re_1 - \Re_2\right)}\left(\Re_1^{p}\Re_2 - \Re_2^{p}\Re_1\right)$$

$$= \left(\frac{\Re_1^{4k+p+1} - \Re_2^{4k+p+1}}{\Re_1 - \Re_2}\right) + \left(\Re_1\Re_2\right)^{2k}(a^2-b)\left(\frac{\Re_1^{p-1} - \Re_2^{p-1}}{\Re_1 - \Re_2}\right)$$

$$= J_{4k+p+1} + (a^2-b)^{2k+1} J_{p-1}$$

This completes the proof.

Corollary 4.2. *For different values of p, (4.1) can be expressed as:*

(i) *If* $p=-3$, *then*: $J_{2k-3}\, j_{2k+1} = J_{4k-2} - (a^2-b)^{2k-3}(4a^3+4ab)$ (4.3)

(ii) *If* $p=-2$, *then*: $J_{2k-2}\, j_{2k+1} = J_{4k-1} - (a^2-b)^{2k-2}(3a^2+b)$ (4.4)

(iii) *If* $p=-1$, *then*: $J_{2k+2}\, j_{2k+1} = J_{4k} - 2a(a^2-b)^{2k-1}$ (4.5)

Following theorems can be solved by Binet's formula of new generalization of Fibonacci and Lucas numbers.

Theorem 4.3. $J_{2k+p}\, j_{2k+2} = J_{4k+p+2} + (a^2-b)^{2k+2} J_{p-2}$, *where* $k \ge 0 \ \& \ p \le 0$ (4.6)

Corollary 4.4. *For different values of p, (4.6) can be expressed as:*

(*i*) *If* $p=-3$, *then*: $J_{2k-3}j_{2k+2}=J_{4k-1}-(a^2-b)^{2k-3}J_5$ (4.7)

(*ii*) *If* $p=-2$, *then*: $J_{2k-2}j_{2k+2}=J_{4k}-(a^2-b)^{2k-2}J_4$ (4.8)

(*iii*) *If* $p=-1$, *then*: $J_{2k-1}j_{2k+2}=J_{4k+1}-(a^2-b)^{2n-1}J_3$ (4.9)

Theorem 4.5. $J_{2k+p}j_{2k}=J_{4k+p}+(a^2-b)^{2k}J_p$, *where* $k\geq 0$ & $p\leq 0$ (4.10)

Corollary 4.6. *For different values of p, (4.10) can be expressed as:*

(*i*) *If* $p=-3$, *then*: $J_{2k-3}j_{2k}=J_{4k-3}-(a^2-b)^{2n-3}J_3$ (4.11)

(*ii*) *If* $p=-2$, *then*: $J_{2k-2}j_{2k}=J_{4k-3}-2a(a^2-b)^{2k-2}$ (4.12)

(*iii*) *If* $p=-1$, *then*: $J_{2k-1}j_{2k}=J_{4k-1}-(a^2-b)^{2k-1}$ (4.13)

Theorem 4.7. $J_{2k-p}j_{2k+1}=J_{4k-p+1}+(a^2-b)^{2k+1}J_{-p-1}$, *where* $k\geq 0$ & $p\leq 0$ (4.14)

Corollary 4.8. *For different values of p, (4.14) can be expressed as:*

(*i*) *If* $p=-3$, *then*: $J_{2k+3}j_{2k+1}=J_{4k+4}-2a(a^2-b)^{2k-1}$ (4.15)

(*ii*) *If* $p=-2$, *then*: $J_{2k+2}j_{2k+1}=J_{4k+3}-(a^2-b)^{2k}$ (4.16)

(*iii*) *If* $p=-1$, *then*: $J_{2k+1}j_{2k+1}=J_{4k+2}$ (4.17)

Theorem 4.9. $J_{2k-p}j_{2k-1}=J_{4k-p-1}+(a^2-b)^{2k-1}J_{1-p}$, *where* $k\geq 0$ & $p\leq 0$ (4.18)

Corollary 4.10. *For different values of p, (4.18) can be expressed as:*

(*i*) *If* $p=-3$, *then*: $J_{2k+3}j_{2k-1}=J_{4k+2}+(a^2-b)^{2k-1}(4a^3+4ab)$ (4.19)

(*ii*) *If* $p=-2$, *then*: $J_{2k+2}j_{2k-1}=J_{4k+1}+(a^2-b)^{2k-1}(3a^2+b)$ (4.20)

(*iii*) *If* $p=-1$, *then*: $J_{2k+1}j_{2k-1}=J_{4k}+2a(a^2-b)^{2k-1}$ (4.21)

Theorem 4.11. $J_{2k-p}j_{2k}=J_{4k-p}+(a^2-b)^{2k}J_{-p}$, *where* $k\geq 0$ & $p\leq 0$ (4.22)

Corollary 4.12. *For different values of p, (4.22) can be expressed as:*

(*i*) *If* $p=-3$, *then*: $J_{2k+3}j_{2k}=J_{4k+3}+(a^2-b)^{2k}(3a^2+b)$ (4.23)

(*ii*) *If* $p=-2$, *then*: $J_{2k+2}j_{2k}=J_{4k+2}+2a(a^2-b)^{2k}$ (4.24)

(*iii*) *If* $p=-1$, *then*: $J_{2k+1}j_{2k}=J_{4k+1}+(a^2-b)^{2k}$ (4.25)

Theorem 4.13. $J_{2k} j_{2k+p} = J_{4k+p} - (a^2 - b)^{2k} J_p$, *where* $k \geq 0$ & $p \leq 0$ (4.26)

Corollary 4.14. *For different values of p, (4.26) can be expressed as:*

(i) *If* $p = -3$, *then*: $J_{2k} j_{2k-3} = J_{4n-3} + (a^2 - b)^{2k-3}(3a^2 + b)$ (4.27)

(ii) *If* $p = -2$, *then*: $J_{2k} j_{2k-2} = J_{4k-2} + 2a(a^2 - b)^{2k-2}$ (4.28)

(iii) *If* $p = -3$, *then*: $J_{2k} j_{2k-1} = J_{4k-1} + (a^2 - b)^{2k-1}$ (4.29)

Theorem 4.15. $4bJ_{2k} J_{2k+p} = j_{4k+p} - (a^2 - b)^{2k} j_p$, *where* $k \geq 0$ & $p \leq 0$ (4.30)

Corollary 4.16. *For different values of p, (4.30) can be expressed as:*

(i) *If* $p = -3$, *then*: $4bJ_{2k} J_{2k-3} = j_{4k-3} - (2a^3 + 6ab)(a^2 - b)^{2k-3}$ (4.31)

(ii) *If* $p = -2$, *then*: $4bJ_{2k} J_{2k-2} = j_{4k-2} - (2a^2 + 2b)(a^2 - b)^{2k-2}$ (4.32)

(iii) *If* $p = -1$, *then*: $4bJ_{2k} J_{2k-1} = j_{4k-1} - 2a(a^2 - b)^{2k-1}$ (4.33)

Theorem 4.17. $j_{2k} j_{2k+p} = j_{4k+p} + (a^2 - b)^{2k} j_p$, *where* $k \geq 0$ & $p \leq 0$ (4.34)

Corollary 4.18. *For different values of p, (4.34) can be expressed as:*

(i) *If* $p = -3$, *then*: $j_{2k} j_{2k-3} = j_{4k-3} + (2a^3 + 6ab)(a^2 - b)^{2k-3}$ (4.35)

(ii) *If* $p = -2$, *then*: $j_{2k} j_{2k-2} = j_{4k-2} + (2a^2 + 2b)(a^2 - b)^{2k-2}$ (4.36)

(iii) *If* $p = -1$, *then*: $j_{2k} j_{2k-1} = j_{4k-1} + 2a(a^2 - b)^{2k-1}$ (4.37)

5. Generating function of the Generalized Jacobsthal and Jacobsthal-Lucas Sequences

Generating functions provide a powerful technique for solving linear homogeneous recurrence relations. Even though generating functions are typically used in conjunction with linear recurrence relations with constant coefficients, we will systematically make use of them for linear recurrence relations with non constant coefficients. In this paragraph, the generating function for generalized Jacobsthal and Jacobsthal-Lucas Sequences are given. As a result, generalized Jacobsthal and Jacobsthal-Lucas Sequences are seen as the coefficients of the corresponding generating function. Function defined in such a way is called the generating function of the generalized Jacobsthal and Jacobsthal-Lucas Sequences. So,

Theorem 5.1. The generating functions of the generalized Jacobsthal and Jacobsthal-Lucas sequences are given, respectively, by

(i) $$J(x) = \sum_{n=0}^{\infty} J_n x^n = \frac{x}{1 - 2ax - (b - a^2)x^2} \quad (5.1)$$

(ii) $$j(x)=\sum_{n=0}^{\infty} j_n x^n=\frac{2-2ax}{1-2ax-(b-a^2)x^2} \tag{5.2}$$

Proof. Applying the generating functions $J(x)$ and $j(x)$ can be written as $J(x)=\sum_{n=0}^{\infty} J_n x^n$ and $j(x)=\sum_{n=0}^{\infty} j_n x^n$. Then, we write

$$J(x)=J_0+xJ_1+x^2J_2+x^3J_3+...+x^nJ_n+...$$

and then $$2axJ(x)=2axJ_0+2ax^2J_1+2ax^3J_2+2ax^4J_3+...+2ax^{n+1}J_n+...$$

$$(b-a^2)x^2J(x)=(b-a^2)x^2J_0+(b-a^2)x^3J_1+(b-a^2)x^4J_2+...+(b-a^2)x^{n+2}J_n+...$$

$$\rightarrow \quad \{1-2ax-(b-a^2)x^2\}J(x)=x$$

$$\rightarrow \quad J(x)=\sum_{n=0}^{\infty} J_n x^n=\frac{x}{1-2ax-(b-a^2)x^2}$$

Similarly, we have

$$j(x)=j_0+xj_1+x^2j_2+x^3j_3+...+x^nj_n+...$$

and then $$2axj(x)=2axj_0+2ax^2j_1+2ax^3j_2+2ax^4j_3+...+2ax^{n+1}j_n+...$$

$$(b-a^2)x^2J(x)=(b-a^2)x^2j_0+(b-a^2)x^3j_1+(b-a^2)x^4j_2+...+(b-a^2)x^{n+2}j_n+...$$

$$\rightarrow \quad \{1-2ax-(b-a^2)x^2\}j(x)=2-2ax$$

$$\rightarrow \quad j(x)=\sum_{n=0}^{\infty} j_n x^n=\frac{2-2ax}{1-2ax-(b-a^2)x^2}$$

This completes the proof.

6. Conclusion

In this paper generalized Jacobsthal and Jacobsthal-Lucas sequences have been studied. Many of the properties of these sequences like Catalan’s identity, Cassini’s identity or Simpson’s identity, d’ocagnes’s identity are proved by simple algebra and Binet’s formula. We describe sums of generalized Jacobsthal and Jacobsthal-Lucas sequences. This enables us to give in a straightforward way several formulas for the sums of such generalized numbers. These identities can be used to develop new identities of numbers and polynomials. Also we present some generalized identities involving product of Jacobsthal and Jacobsthal-Lucas sequences. Finally we present the generating function of Jacobsthal and Jacobsthal-Lucas sequences.

References

[1] Bilgici, G. (2014) New Generalizations of Fibonacci and Lucas Sequences. Applied Mathematical Sciences, 8(29): 1429-1437.

[2] Falcon, S. (2001) On the k-Lucas Numbers. International Journal of Contemporary Mathematical Sciences, 6(21): 1039-1050.
[3] Falcon, S. (2012) On k-Lucas Numbers of Arithmetic Indexes. Applied Mathematics, 3: 1202-1206.
[4] Falcon, S., Plaza, A. (2007) On the k-Fibonacci Numbers. Chaos,Solitons and Fractals, 32(5): 1615-1624.
[5] Falcon, S., Plaza, A. (2009) On k-Fibonacci Numbers of Arithmetic Indexes. Applied Mathematics and Computation, 208: 180-185.
[6] Gupta, V. K., Panwar, Y. K. (2012) Common Factors of Generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers. International Journal of Applied Mathematical Research, 1(4): 377-382.
[7] Gupta, V. K., Panwar, Y. K., Sikhwal, O. (2012) Generalized Fibonacci Sequences. Theoretical Mathematics & Applications, 2(2): 115-124.
[8] Horadam, A. F., (1961) A Generalized Fibonacci Sequence. American Mathematical Monthly, 68(5): 455-459.
[9] Horadam, A. F., (1996) Jacobsthal representation numbers. The Fibonacci Quarterly, 34(1): 40-54.
[10] Kalman, D., Mena, R. (2002) The Fibonacci Numbers–Exposed. The Mathematical Magazine, 2.
[11] Koshy, T. (2001) Fibonacci and Lucas numbers with applications. New York, Wiley-Interscience.
[12] Pakapongpun, A. (2020) Identities on the product of Jacobsthal-like and Jacobsthal-Lucas numbers. Notes on Number Theory and Discrete Mathematics, 26(1): 209-215.
[13] Panwar, Y. K., Rathore, G. P. S., Chawla, R. (2014) On the k-Fibonacci-like numbers. Turkish J. Anal. Number Theory, 2(1): 9-12.
[14] Panwar, Y. K., Singh, B., Gupta, V. K. (2013) Generalized Identities Involving Common Factors of Generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers. International journal of Analysis and Application, 3(1): 53-59.
[15] Panwar, Y. K., Singh, B., Gupta, V. K. (2013) Identities Involving Common Factors of Generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers. Applied Mathematics and and Physic, 1(4): 126-128.
[16] Singh, B., Bhadouria, P., Sikhwal, O. (2013) Generalized Identities Involving Common Factors of Fibonacci and Lucas Numbers. International Journal of Algebra, 5(13): 637-645.
[17] Singh, B., Sikhwal, O., Bhatnagar, S. (2010) Fibonacci-Like Sequence and its Properties. Int. J. Contemp. Math. Sciences, 5(18): 859-868.
[18] Singh, B., Sisodiya, K., Ahmed, F. (2014) On the Products of k-Fibonacci Numbers and k-Lucas Numbers. International Journal of Mathematics and Mathematical Sciences. Article ID 505798, 4 pages. http://dx.doi.org/10.1155/2014/505798
[19] Suvarnamani, A., Tatong, M. (2015) Some Properties of (p, q)-Fibonnacci Numbers. Science and Technology RMUTT Journal, 5(2): 17-21.
[20] Suvarnamani, A., Tatong, M. (2016) Some Properties of (p, q)-Lucas Numbers. Kyungpook Mathematical Journal, 56(2): 367-370.
[21] Taşyurdu, Y. (2019) Generalized (p, q)-Fibonacci-Like Sequences and Their Properties. Journal of Research, 11(6): 43-52.
[22] Taşyurdu, Y., Cobanoğlu, N., Dilmen, Z. (2016) On The a New Family of k-Fibonacci Numbers. Erzincan University Journal of Science and Thechnology, 9(1): 95-101.
[23] Thongkam, B., Butsuwan, K., Bunya, P. (2020) Some properties of (p, q)-Fibonacci-like and (p, q)-Lucas numbers. Notes on Number Theory and Discrete Mathematics, 26(1): 216-224.

[24] Thongmoon, M. (2009) Identities for the common factors of Fibonacci and Lucas numbers. International Mathematical Forum, 4(7): 303–308.
[25] Thongmoon, M. (2009) New identities for the even and odd Fibonacci and Lucas numbers. International Journal of Contemporary Mathematical Sciences, 4(14): 671–676.
[26] Uygun, S., Owusu, E. (2016) A new generalization of Jacobsthal numbers (Bi-Periodic Jacobsthal Sequences). Journal of Mathematical Analysis, 7(4): 28-39.
[27] Uygun, S., Owusu, E. (2019) A new generalization of Jacobsthal numbers (Bi-Periodic Jacobsthal Lucas Sequences). Journal of Advances in Mathematics and Computer Science, 34(5): 1-13.
[28] Wani, A. A., Catarino, P., Rafiq, R. U. (2018) On the Properties of k-Fibonacci-Like Sequence. International Journal of Mathematics And its Applications, 6(1-A): 187-198.
[29] Wani, A. A., Rathore, G. P. S., Sisodiya, K. (2016) On The Properties of Fibonacci-Like Sequence. International Journal of Mathematics Trends and Technology, 29(2): 80-86.
[30] Yagmur, T. (2019) New Approach to Pell and Pell-Lucas Sequence. Kyungpook Mathematical Journal, 59(1): 23-34.

IKONION JOURNAL OF MATHEMATICS

YEAR: 2020 VOLUME :2 ISSUE: 1

ISSN: 2687-6531

A NEW CONCEPT FOR FRACTIONAL QUANTUM CALCULUS: $(\beta, q)-$CALCULUS AND ITS PROPERTIES

Ali Kurt[1,*], Orkun Tasbozan[2]

[1] Department of Mathematics, Pamukkale University, Denizli, TURKEY,
[2] Department of Mathematics, Hatay Mustafa Kemal University, Hatay, TURKEY
*E-mail: akurt@pau.edu.tr (corresponding author)

(Received: 04.08.2020, Accepted: 06.10.2020, Published Online: 16.10.2020)

Abstract

In this article, authors expressed a new definition called $(\beta, q)-$derivative and integral which is a new type of fractional quantum derivative and integral. Also theorems and proofs on some basic properties of $(\beta, q)-$derivative and integral such as product rule, quotient rule, linearity and etc. are stated. These new definitions can be used in many different models arising in physical applications.

Keywords: $q-$calculus; $(\beta, q)-$derivative; $(\beta, q)-$integral.

1 Introduction

Fractional calculus which has drawn attention of scientists, has gained considerable interest from both theoretical and the applied points of view in recent years. There are numerous applications in many fields such as electrical networks, chemical physics, fluid flow, economics, signal and image processing, viscoelasticity, porous media, aerodynamics, modeling for physical phenomena exhibiting anomalous diffusion, and so on. In contrast to integer-order differential and integral operators, fractional-order differential operators give us chance to model nonlinear and complex phenomenons in nature and make understand the hereditary properties of several processes. The monographs [1, 2, 3] are commonly cited for the theory of fractional derivatives and integrals and applications to differential equations of fractional order. Indeed, studies on the theory of non-integer derivatives and integrals are not new. There are many different definitions of fractional derivatives and integrals. Among these Riemann-Liouville, Caputo, Grünwald-Letnikov, Riesz-Fischer derivatives and integrals are known commonly. But recently a new fractional derivative and its anti-derivative definition is presented by Atangana [4] called "Beta derivative" and "Atangana Beta integral" respectively.

Definition 1 *Let $f : [0, \infty) \to \mathbb{R}$ be a function. "Beta derivative" of f is defined by*

$$ {}_0^A D_t^{\beta}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon\,(t + 1/\Gamma(\beta))^{1-\beta}) - f(t)}{\varepsilon} \tag{1} $$

for all $t > 0, \beta \in (0, 1)$. The anti-derivative of a function $f : [a, \infty) \to$ is defined as

$$ {}_0^A I_t^{\beta}(f)(t) = \int_a^t f(x) \left(x + \frac{1}{\Gamma(\beta)} \right)^{\beta - 1} dx $$

where $\beta \in (0, 1]$.

Quantum calculus or q-calculus which was firstly declared by FH Jackson in the early twentieth century is known as calculus without limits, but this kind of calculus had already been worked out by Euler and Jacobi. Absence of limit lets us to study on sets of non-differentiable functions. Recently it arose great interest in the mathematical modeling in quantum calculus. Hence $q-$calculus is seen as a connection between mathematics and physics, scientists have been paying great attention to it because of its different and huge application areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions, quantum theory, mechanics and the theory of relativity. Kac and Cheung [5] gave many of the basic features of quantum calculus in their book.

Let us give some basic definitions of q-calculus.

Definition 2 *[6] Let f be a function defined on a q-geometric set I, i.e., $qt \in I$ for all $t \in I$. For $0 < q < 1$, we define the q-derivative as*

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in I \setminus \{0\}, \tag{2}$$

$$D_q(f)(0) = \lim_{t \to 0} D_q f(t). \tag{3}$$

Note that

$$\lim_{q \to 1} D_q f(t) = \lim_{q \to 1} \frac{f(t) - f(qt)}{(1-q)t} = \frac{df}{dt}.$$

Definition 3 *[6] For $t \geq 0$, we set $J_t = \{tq^n : n \in N \cup \{0\}\} \cup \{0\}$ and define the definite q-integral of a function $f : J_t \to \mathbb{R}$ by*

$$I_q f(t) = \int_0^t f(s) d_q s = (1-q)t \sum_{n=0}^{\infty} q^n f(q^n t).$$

2 Properties of $(\beta, q)-$Derivative and Integral

Definition 4 ($(\beta, q)-$ Derivative) *$(\beta, q)-$ Derivative of a function f can be expressed as*

$$T^{(\beta,q)} f(t) = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{f(qt) - f(t)}{(q-1)t} \tag{4}$$

where

$$q = 1 + \frac{\varepsilon(t + 1/\Gamma(\beta))^{1-\beta}}{t}.$$

Example 5 *Assume that $f(t) = t^n$. The $(\beta, q)-$derivative of function f can be evaluated as follows.*

$$\begin{aligned} T^{(\beta,q)}(f)(t) &= \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{t^n - (qt)^n}{(1-q)t} \\ &= \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{t^n - q^n t^n}{(1-q)t} \\ &= \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} t^{n-1} \; [n]_q \end{aligned}$$

where $[n]_q = \frac{1-q^n}{1-q}$.

Theorem 6 *Let $f(t), g(t)$ are $(\beta, q)-$differentiable. $T^{(\beta,q)}$ operator satisfies following properties.*

1. *$T^{(\beta,q)}(af(t) + bg(t)) = aT^{(\beta,q)} f(t) + bT^{(\beta,q)} g(t), \; a, b \in \mathbb{R}$.*
2. *$T^{(\beta,q)}(fg)(t) = g(t) \; T^{(\beta,q)} f(t) + f(qt) T^{(\beta,q)} g(t)$.*

3. $T^{(\beta,q)}\left(\frac{f}{g}\right)(t)=\frac{g(t)T^{(\beta,q)}f(t)-f(t)T^{(\beta,q)}g(t)}{g(t)g(qt)}$.

Proof.

1. *The proof can be easily seen from the definition of* $(\beta,q)-$ *derivative.*
2. *From the definition of* (β,q)*-derivative,*

$$\begin{aligned}
T^{(\beta,q)}(fg)(t) &= (t+1/\Gamma(\beta))^{1-\beta}\frac{f(t)g(t)-f(qt)g(qt)}{(1-q)t} \\
&= \frac{(t+1/\Gamma(\beta))^{1-\beta}}{(1-q)t}\left[f(t)g(t)-f(qt)g(qt)+f(qt)g(t)-f(qt)g(t)\right] \\
&= g(t)(t+1/\Gamma(\beta))^{1-\beta}\frac{f(t)-f(qt)}{(1-q)t}+f(qt)(t+1/\Gamma(\beta))^{1-\beta}\frac{g(t)-g(qt)}{(1-q)t} \\
&= g(t)T^{(\beta,q)}f(t)+f(qt)T^{(\beta,q)}g(t)
\end{aligned}$$

can be obtained.

3. *By using the definition of* $(\beta,q)-$*derivative we acquire*

$$\begin{aligned}
T^{(\beta,q)}\left(\frac{f}{g}\right)(t) &= (t+1/\Gamma(\beta))^{1-\beta}\frac{\frac{f(t)}{g(t)}-\frac{f(qt)}{g(qt)}}{(1-q)t} \\
&= \frac{(t+1/\Gamma(\beta))^{1-\beta}}{g(t)g(qt)}\left(g(t)\frac{f(t)-f(qt)}{(1-q)t}-f(t)\frac{g(t)-g(qt)}{(1-q)t}\right) \\
&= \frac{g(t)T^{(\beta,q)}f(t)-f(t)T^{(\beta,q)}g(t)}{g(t)g(qt)}.
\end{aligned}$$

■

Definition 7 *Let* $f:[0,\infty)\to\mathbb{R}$ *be a continuous function. Thus the* $(\beta,q)-$*integral of function* f *can be declared as follows.*

$$I^{(\beta,q)}f(t)=\int_0^t f(s)d_q^{\beta}s=\int_0^t (s+1/\Gamma(\beta))^{\beta-1}f(s)d_qs=(1-q)t\sum_{n=0}^{\infty}q^nf(q^nt)(q^nt+1/\Gamma(\beta))^{\beta-1}.$$

Theorem 8 *Let function* f *is* $(\beta,q)-$ *differentiable and integrable function. The following properties are satisfied for* $t>0$*.*

1. $T^{(\beta,q)}(I^{(\beta,q)}f(t))=f(t)$.
2. $I^{(\beta,q)}(T^{(\beta,q)}f(t))=f(t)$.
3. $\int_a^t T^{(\beta,q)}f(s)d_q^{\beta}s=f(t)-f(a)$ *for* $a\in(a,t)$.

Proof.

1. From definition of $(\beta, q)-$derivative and integral we obtain

$$\begin{aligned}
T^{(\beta,q)}(I^{(\beta,q)}f(t)) &= T^{(\beta,q)}\int_0^t f(s)d_q^\beta s \\
&= T^{(\beta,q)}\int_0^t (s+1/\Gamma(\beta))^{\beta-1}f(s)d_q s \\
&= T^{(\beta,q)}\left[(1-q)t\sum_{n=0}^{\infty}q^n f(q^n t)\left(q^n t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\right] \\
&= \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}}{t}\left[t\sum_{n=0}^{\infty}q^n f(q^n t)\left(q^n t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} - qt\sum_{n=0}^{\infty}q^n f(q^n(qt))\left(q^n(qt)+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\right] \\
&= \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left[\sum_{n=0}^{\infty}q^n f(q^n t)\left(q^n t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} - q\sum_{n=0}^{\infty}q^n f(q^{n+1}t)\left(q^{n+1}t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\right] \\
&= \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left[\sum_{n=0}^{\infty}q^n f(q^n t)\left(q^n t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} - \sum_{n=0}^{\infty}q^{n+1} f(q^{n+1}t)\left(q^{n+1}t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\right] \\
&= f(t).
\end{aligned}$$

2. By using the definition of $(\beta, q)-$derivative and integral we get

$$\begin{aligned}
I^{(\beta,q)}(T^{(\beta,q)}f(t)) &= \int_0^t T^{(\beta,q)}f(s)d_q^\beta s \\
&= \int_0^t \left(s+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{f(s)-f(qs)}{(1-q)s}d_q^\beta s \\
&= \int_0^t \frac{f(s)-f(qs)}{(1-q)s}d_q s \\
&= (1-q)t\sum_{n=0}^{\infty}q^n\left(\frac{f(q^n t)}{(1-q)q^n t}-\frac{f(q(q^n t))}{(1-q)q^n t}\right) \\
&= \sum_{n=0}^{\infty}\left[f(q^n t)-f(q^{n+1}t)\right] \\
&= f(t).
\end{aligned}$$

3. It can easily seen by the help of (2).

■

Theorem 9 *Let $f, g : [0, \infty) \to \mathbb{R}$ are continuous and $(\beta, q)-$differentiable and integrable functions. For $t \in [0, \infty)$,*

1. $\int_0^t [af(s)+bg(s)]\,d_q^\beta s = a\int_0^t f(s)d_q^\beta s + b\int_0^t g(s)d_q^\beta s,\ a, b \in \mathbb{R}.$

2. $\int_a^t f(s)T^{(\beta,q)}g(s)d_q^\beta s = (fg)(t) - \int_a^t g(qs)T^{(\beta,q)}f(s)d_q^\beta s.$

Proof.

1. One can easily see the proof from the definition.

2. From Theorem 6 (2), it can be deducted

$$T^{(\beta,q)}(fg)(t) = f(t)T^{(\beta,q)}g(t) + g(qt)Df(t).$$

By taking $(\beta, q)-$integral of two sides; we have

$$\int_0^t T^{(\beta,q)}(fg)(s)d_q^\beta s = \int_0^t f(s)T^{(\beta,q)}g(s)d_q^\beta s + \int_0^t g(qs)T^{(\beta,q)}f(s)d_q^\beta s.$$

Then using Theorem 8 (2), we get

$$\int_0^t f(s)T^{(\beta,q)}g(s)d_q^\beta s = f(t)g(t) - \int_0^t g(qs)T^{(\beta,q)}f(s)d_q^\beta s.$$

■

3 Conclusion

In this paper the authors give a new definition entitled "$(\beta, q)-$derivative and integral" by establishing relation between with quantum calculus and beta fractional calculus. By using these new definitions, many applications on different branches of science can be made. Possible applications by using "$(\beta, q)-$derivative and integral" are left as open problem.

References

[1] Oldham, K., & Spanier, J. (1974). The fractional calculus theory and applications of differentiation and integration to arbitrary order. Elsevier.

[2] Podlubny, I. (1999). Fractional differential equations, vol. 198 of Mathematics in Science and Engineering.

[3] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and applications of fractional differential equations (Vol. 204). Elsevier.

[4] Atangana, A., & Oukouomi Noutchie, S. C. (2014). Model of break-bone fever via beta-derivatives. BioMed research international, 2014.

[5] Kac, V., & Cheung, P. (2002). Euler-Maclaurin Formula. In Quantum Calculus (pp. 92-98). Springer, New York, NY.

[6] Tariboon, J., & Ntouyas, S. K. (2013). Quantum calculus on finite intervals and applications to impulsive difference equations. Advances in Difference Equations, 2013(1), 282.