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Exact Travelling Wave Solutions of the Nonlinear Evolution Equations by Improved F-Expansion in Mathematical Physics

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Abstract
With the assistance of representative calculation programming, the present paper examines the careful voyaging wave arrangements from the general (2+1)-dimensional nonlinear development conditions by utilizing the Improved F-expansion strategy. As a result, the used technique is effectively utilized and recently delivered some definite voyaging wave arrangements. The recently created arrangements have been communicated as far as trigonometric and hyperbolic capacities. The created arrangements have been returned into their relating condition with the guide of emblematic calculation programming Maple. Among the produced solutions, some solutions have been visualized by 3D and 2D line graphs under the choice of suitable arbitrary parameters to show their physical interpretation. The delivered arrangements show the intensity of the executed technique to evaluate the accurate arrangements of the nonlinear (2+1)-dimensional nonlinear advancement conditions, which are reasonably pertinent for using nonlinear science, scientific material science and designing. The Improved F-expansion method is a reliable treatment for searching essential nonlinear waves that enrich a variety of dynamic models that arise in engineering fields.

Keywords: The Improved F-expansion scheme, The general (2+1)-dimensional nonlinear evolution equation, Traveling wave solutions, Soliton solution.
2010 AMS: Primary 35C07, 35C08

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1. Introduction

As of late, nonlinear incomplete differential conditions (NPDEs) is comprehensively used to delineate various huge marvels and dynamic methodology in various fields of science and designing, particularly in liquid mechanics, hydrodynamics, numerical science, dissemination process, strong state material science, plasma material science, neural material science, substance energy and geo-optical filaments. In this article, we will study the generalized (2+1)-dimensional nonlinear evolution equations in the form

$$u_{xt}+u_{xxxy}+au_xu_{xy}+bu_{xx}u_y=0 \tag{1.1}$$

Recently, some special cases of Eq. (1.1) have been studied by several authors[1]-[4]. When setting a =4 and b = 2, we get:

$$u_{xt}+u_{xxxy}+4u_xu_{xy}+2u_{xx}u_y=0 \tag{1.2}$$

Eq. (1.2) known as the Calogero–Bogoyavlenskii–Schiff (CBS) equation with (2+1) dimensional.

where $u = u(x,y,t)$ is used for brevity. In the article, subscript occurring with a term denotes its partial derivative with respect to the subscript variable.in equ.(1.2) u_t describes the time evolution of the wave, while the terms $u_{xx}u_y$ and u_xu_{xy} are account for nonlinearity of the wave. The CBS equation has some physical situated history like it very well may be composed as potential structure [5]. The CBS equation was at first built by Bogoyavlenskii and Schiff in different ways [6]-[9]. The ongoing history of some past inquires about show that few powerful strategies for getting definite arrangements of the CBS condition are contributed by a differing gathering of specialists over the globe [10]-[13], for instance, the periodic and soliton solutions of the CBS equation were gotten by Gandarias et al.[10]. Its integrability has been demonstrated by Zhang et al.[11] and derived also the symmetry reductions of the equation. Li and Chen [12] found the exact solutions by using the generalized Raccati equation expansion method. Wang and Yang [13] employed the Hirota Bilinear strategy for construction of the quasi-periodic wave solutions in terms of theta functions for a Hirota bilinear equation.
When setting a = – 4 and b = – 2, we get:

$$u_{xt} + u_{xxxy} - 4u_xu_{xy} - 2u_{xx}u_y = 0 \tag{1.3}$$

Eq. (3) known as the breaking soliton equation with (2+1) dimensional.
When setting a = 4 and b = 4, we get:

$$u_{xt} + u_{xxxy} + 4u_xu_{xy} + 4u_{xx}u_y = 0 \tag{1.4}$$

Eq. (4) known as the Bogoyavlenskii equation with (2+1) dimensional.

Numerous researchers arranged through nonlinear evolution equations (NEEs) to build voyaging wave arrangement by executing a few techniques. The methods that are entrenched in ongoing writing, for example, the extended Kudryashov method[14], the modified simple equation method [15], the new extended (G'/G) expansion method [16]-[17], the Darboux transformation [18], the trial solution method [19], the Exp-Function Method [20], the multiple simplest equation method [21], $\exp(-\phi(\xi))$ -expansion method [22]-[26], Pseudo parabolic model [27]-[29], Sine-Gordon expansion method[30], Complex solitons in the conformable (2+1)-dimensional Ablowitz-Kaup-Newell-Segur equation [31], Modified auxiliary expansion method [32], Method of line [33], Bernoulli sub-equation function method [34]-[35], The modified exponential function method [36], Improved Bernoulli sub equation function method [37], Finite difference method [38] and so on.

The target of this article is to apply the Improved F-expansion technique to build the precise voyaging wave answers for nonlinear advancement conditions in scientific material science by means of the generalized (2+1)-dimensional nonlinear evolution equations.

The article is set up as pursues: In section 2, the Improved F-expansion scheme has been talked about. In segment 3, we apply this plan to the nonlinear development conditions raised previously. In section 4, represents Results and Discussion, In section and in section 5 ends are given.

2. Description of the Improved F-Expansion Method

In this segment, we portray in subtleties The Improved F-extension strategy technique for discovering traveling wave equations of nonlinear equations. Any nonlinear condition in two free factors x and t can be communicated in following structure:

$$Re(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}.........) = 0 \tag{2.1}$$

where, $u(\xi) = u(x,t)$ is an unknown function, $\Re$ is a polynomial of $u(\xi) = u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are included.
Step 1: The given PDE (2.1) can be changed into ODE utilizing the change $\xi = x \pm \omega t$ where ω is the speed of traveling wave such that $\omega \in R - \{0\}$
The traveling wave change grants us to diminish Eq. (2.1) to the following ODE:

$$\Re(u, u', u'', ...) = 0 \tag{2.2}$$

where $\Re$ is a polynomial in $u(\xi)$ and its derivatives, where $u'(\xi) = \frac{du}{d\xi}$, $u''(\xi) = \frac{d^2u}{d\xi^2}$, and so on.
step 2: Suppose the solution of Eq. (2.2) can be expressed by a polynomial in $\psi(\xi)$:

$$U = u(\xi) = \sum_{j=0}^{N} \alpha_j (\psi(\xi))^j + \sum_{j=1}^{N} \beta_j (\psi(\xi))^{-j} \tag{2.3}$$

where

$$\psi(\xi) = (m + \varphi(\xi)) \tag{2.4}$$

Here α_N or β_N may be zero, but both could not be zero simultaneously. $\alpha_j(j = 0,1,2\cdots N)$,$\beta_j(j = 1,2\cdots N)$ and m are constants to be determined later and $\Phi(\xi)$ satisfies the ODE in form:

$$\varphi'(\xi) = K + \varphi^2(\xi) \tag{2.5}$$

We now present three cases of the general solutions of the Riccati Equation (2.5) (Cruz, Schuch, Castanos and Rosas-Ortiz, 2015).
Case-I: When $K < 0$, we get following hyperbolic solution

$$\Phi(\xi) = -\sqrt{-K}\tanh(\sqrt{-K}\xi)$$
$$\Phi(\xi) = -\sqrt{-K}\coth(\sqrt{-K}\xi)$$

Case-II: When $K > 0$, we get following trigonometric solution

$$\Phi(\xi) = \sqrt{K}\tan(\sqrt{K}\xi)$$
$$\Phi(\xi) = -\sqrt{K}\cot(\sqrt{K}\xi)$$

Case-III: When $K = 0$, we get following solution

$$\Phi(\xi) = -\frac{1}{\xi}$$

Step 3: The value of the positive integer N can be determined by balancing the highest order linear terms with nonlinear terms of the highest order appearing in Eq. (2.2).
If the degree of $u(\xi)$ is $D[u(\xi)] = n$, then the degree of the other expressions will be as follows:

$$D\left[\frac{d^p u(\xi)}{d\xi^p}\right] = n + p, \quad D\left[u^p\left(\frac{d^q u(\xi)}{d\xi^q}\right)^s\right] = np + s(n+q)$$

Step 4: Substituting Eq. (2.3) along with Eqs. (2.4) and (2.5) into Eq. (2.2), we obtain polynomials in $(m + \varphi(\xi))^j$ and $(m + \varphi(\xi))^{-j}$, $(j = 0, 1, 2, \cdots, N)$.Gathering every coefficient of the came about polynomials to zero, yields an over-decided arrangement of arithmetical equations for α_j, β_j, ω and m.
Step 5: Assume the estimation of the constants can be gotten by fathoming the mathematical conditions got in step 4. Substituting the estimations of the constants together with the arrangements of Eq. (2.5), we will acquire new and far reaching precise traveling wave arrangements of the nonlinear development Eq. (2.1).

3. Application of the Method

In this section, we will exert the Improved F-expansion method to solve the equation (1.1).Now Using the traveling wave variable $\xi = x + y - \omega t$ and integrating with respect to ξ reduces Eq. (1.1) to the following ordinary differential equation for $u = u(\xi)$.

$$-\omega u' + \left(\frac{a+b}{2}\right)(u')^2 + u''' = 0 \tag{3.1}$$

Where, primes denote the differentiation with regard to ξ By balancing u''' and $(u')^2$ we obtain $N = 1$.Therefore the Improve F-expansion method admits to solution of (2.1) in the form

$$U(x,y,t) = \alpha_0 + \alpha_1(m + \varphi(\xi)) + \beta_1(m + \varphi(\xi))^{-1} \tag{3.2}$$

Now, substituting Eq. (3.2) into Eq. (3.1), and equating the coefficients of the powers $\varphi(\xi)$ then we obtain a system of algebraic equations. Solving this system of equations for $\alpha_0, \alpha_1, \beta_1, m$ and ω we obtain the following set values:

Set-1: $m = m, \omega = -4K, \alpha_0 = \alpha_0, \alpha_1 = 0, \beta_1 = \frac{12\left(m^2+K\right)}{a+b}$.

Set-2: $m = m, \omega = -4K, \alpha_0 = \alpha_0, \alpha_1 = -\frac{12}{a+b}, \beta_1 = 0$.

Set-3: $m = 0, \omega = -16K, \alpha_0 = \alpha_0, \alpha_1 = -\frac{12}{a+b}, \beta_1 = \frac{12K}{a+b}$.

Case-I: When $K < 0$, we get following hyperbolic solution

Family-1

$$U_1(x,y,t) = \alpha_0 + \frac{12\left(m^2+K\right)}{(a+b)\left(m-\sqrt{-K}\tanh(\sqrt{-K}\xi)\right)}$$
$$U_2(x,y,t) = \alpha_0 + \frac{12\left(m^2+K\right)}{(a+b)\left(m-\sqrt{-K}coth(\sqrt{-K}\xi)\right)}$$

where, $\omega = -4K$ and $\xi = x+y-\omega t$.
Family-2

$$U_3(x,y,t) = \alpha_0 - \frac{12\left(m-\sqrt{-K}\tanh(\sqrt{-K}\xi)\right)}{(a+b)}$$
$$U_4(x,y,t) = \alpha_0 - \frac{12\left(m-\sqrt{-K}coth(\sqrt{-K}\xi)\right)}{(a+b)}$$

where, $\omega = -4K$ and $\xi = x+y-\omega t$.
Family-3

$$U_5(x,y,t) = \alpha_0 + \frac{12\sqrt{-K}\tanh(\sqrt{-K}\xi)}{(a+b)} - \frac{12K}{(a+b)\sqrt{-K}\tanh(\sqrt{-K}\xi)}$$
$$U_6(x,y,t) = \alpha_0 + \frac{12\sqrt{-K}coth(\sqrt{-K}\xi)}{(a+b)} - \frac{12K}{(a+b)\sqrt{-K}coth(\sqrt{-K}\xi)}$$

where, $\omega = -16K$ and $\xi = x+y-\omega t$.
Case-II: When $K > 0$, we get following trigonometric solution

Family-4

$$U_7(x,y,t) = \alpha_0 + \frac{12\left(m^2+K\right)}{(a+b)\left(m+\sqrt{K}\tan(\sqrt{K}\xi)\right)}$$
$$U_8(x,y,t) = \alpha_0 + \frac{12\left(m^2+K\right)}{(a+b)\left(m-\sqrt{K}\cot(\sqrt{K}\xi)\right)}$$

where, $\omega = -4K$ and $\xi = x+y-\omega t$.
Family-5

$$U_9(x,y,t) = \alpha_0 - \frac{12\left(m+\sqrt{K}\tan(\sqrt{K}\xi)\right)}{(a+b)}$$
$$U_{10}(x,y,t) = \alpha_0 - \frac{12\left(m-\sqrt{K}\cot(\sqrt{K}\xi)\right)}{(a+b)}$$

where, $\omega = -4K$ and $\xi = x+y-\omega t$.
Family-6

$$U_{11}(x,y,t) = \alpha_0 + \frac{12\sqrt{K}\tan(\sqrt{K}\xi)}{(a+b)} + \frac{12K}{(a+b)\sqrt{K}\tan(\sqrt{K}\xi)}$$
$$U_{12}(x,y,t) = \alpha_0 + \frac{12\sqrt{K}\cot(\sqrt{K}\xi)}{(a+b)} - \frac{12K}{(a+b)\sqrt{K}\cot(\sqrt{K}\xi)}$$

where, $\omega = -16K$ and $\xi = x + y - \omega t$.
Case-III: When $K = 0$, we get following rational solution

Family-7

$$U_{13}(x,y,t) = \alpha_0 + \frac{12(m^2+K)}{(a+b)\left(m - \frac{1}{\xi}\right)}$$

where, $\omega = -4K$ and $\xi = x + y - \omega t$.
Family-8

$$U_{14}(x,y,t) = \alpha_0 - \frac{12\left(m - \frac{1}{\xi}\right)}{(a+b)}$$

where, $\omega = -4K$ and $\xi = x + y - \omega t$.
Family-9

$$U_{15}(x,y,t) = \alpha_0 + \frac{12}{(a+b)\xi} - \frac{12K\xi}{(a+b)}$$

where, $\omega = -16K$ and $\xi = x + y - \omega t$.

4. Results and Discussion

Around there, we will discuss the physical depiction of the procured careful and singular wave answer for the general (2+1)-dimensional nonlinear advancement condition. We address these arrangements in graphical and check about the kind of arrangement. Presently we pictorial some get arrangements acknowledge by applied techniques for the general (2+1)-dimensional nonlinear advancement condition.

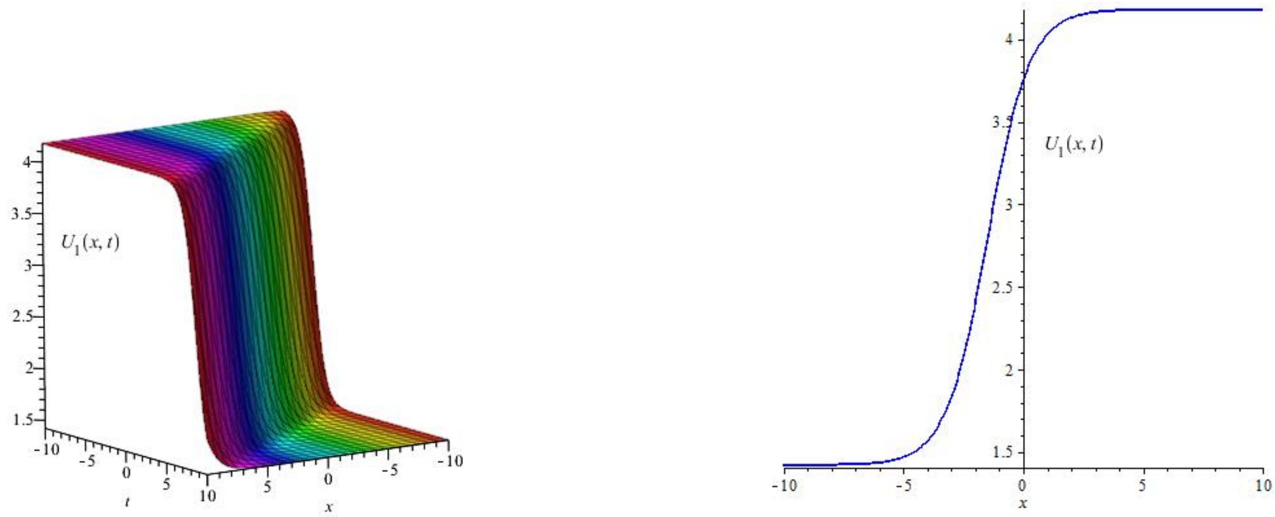

Figure 4.1. Kink Shape of $U_1(\xi)$ for $a_0 = -2, a = 2, b = 3, m = 2, K = -.33, y = 2$ within $-10 \le x, t \le 10$. The left-sided figure shows the 3D plot and the right-sided figure shows the 2D plot for $t = 0$

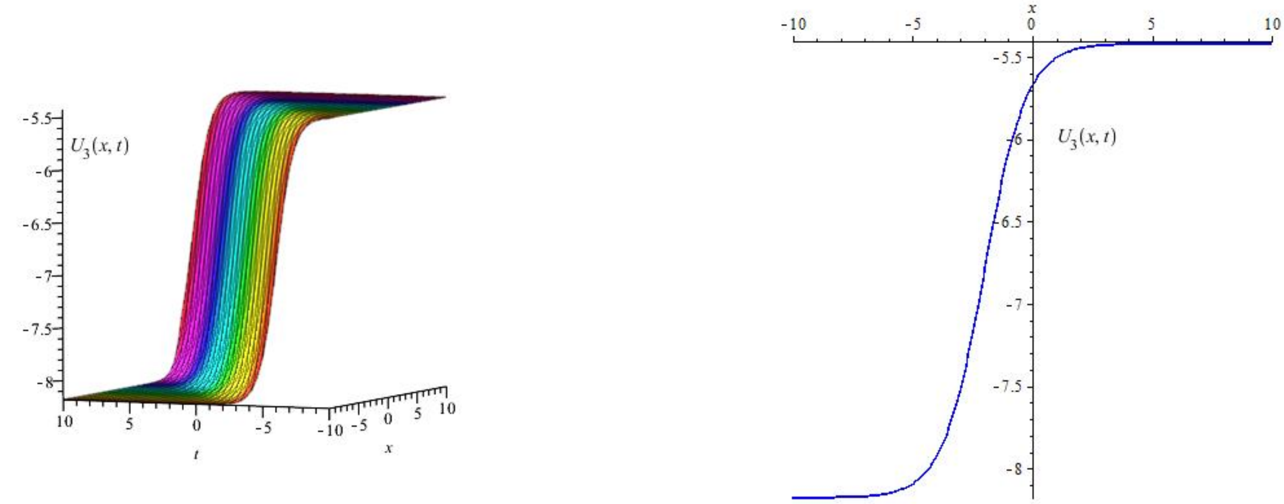

Figure 4.2. Kink Shape of $U_3(\xi)$ for $a_0 = -2, a = 2, b = 3, m = 2, K = -.33, y = 2$ within $-10 \leq x, t \leq 10$. The left-sided figure shows the 3D plot and the right-sided figure shows the 2D plot for $t = 0$

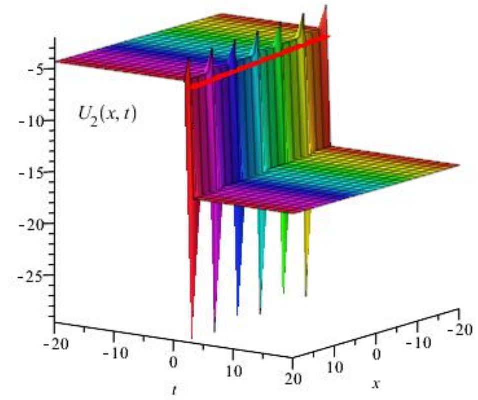

Figure 4.3. Kink Shape of $U_2(\xi)$ for $a_0 = -2, a = 2, b = 1, m = -2, K = -2, y = 0$ within $-10 \leq x, t \leq 10$.

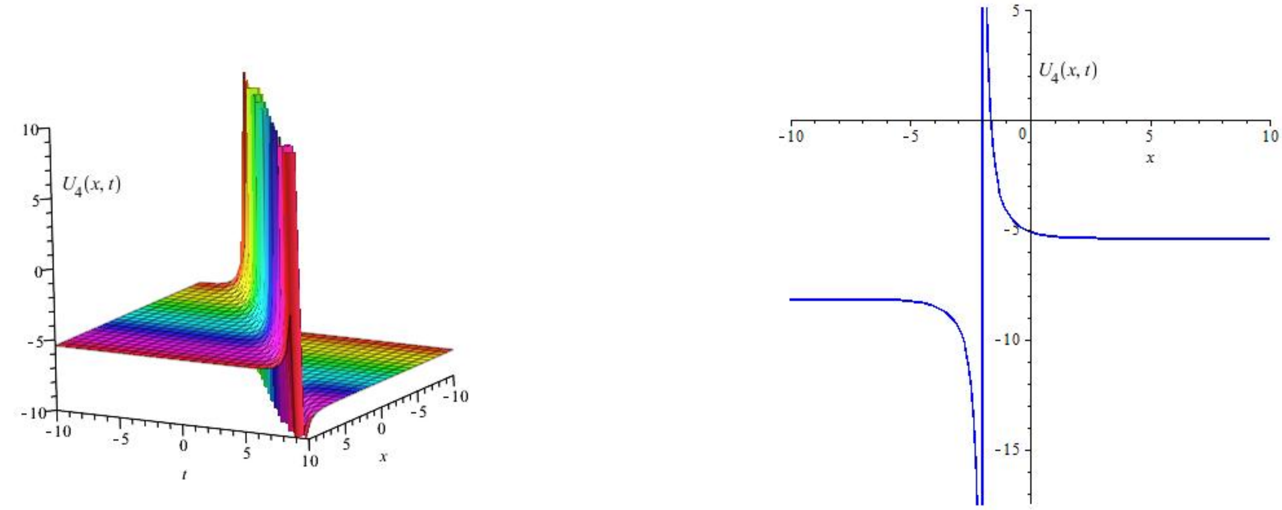

Figure 4.4. Singular Kink Shape of $U_4(\xi)$ for $a_0 = -2, a = 2, b = 3, m = 2, K = -.33, y = 2$ within $-10 \leq x,t \leq 10$. The left-sided figure shows the 3D plot and the right-sided figure shows the 2D plot for $t = 0$

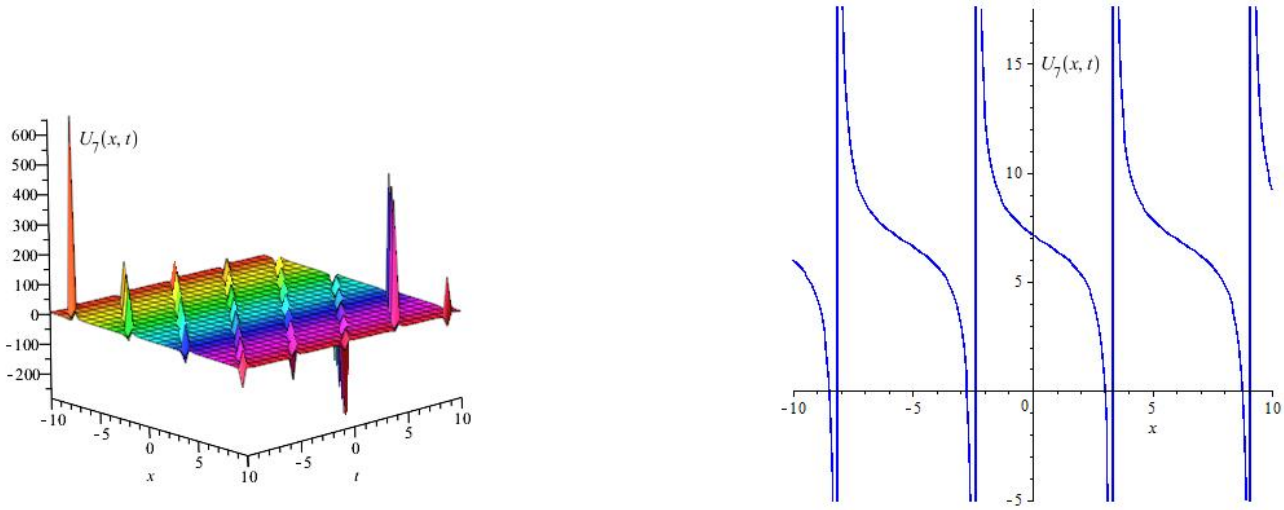

Figure 4.5. Periodic N soliton Shape of $U_7(\xi)$ for $a_0 = 2, a = 2, b = 3, m = 2, K = .3, y = 0$ within $-10 \leq x,t \leq 10$. The left-sided figure shows the 3D plot and the right-sided figure shows the 2D plot for $t = 0$

5. Conclusion

In this segment, we have seen that two kinds of traveling wave arrangements as far as hyperbolic and trigonometric capacities for the general (2+1)-dimensional nonlinear evolution equation is effectively discovered by utilizing the Improved F-expansion method. From our outcomes got in this paper, we finish up the Improved F-expansion scheme strategy is amazing, powerful and helpful. The exhibition of this technique is dependable, basic and gives numerous new arrangements. Likewise, the arrangements of the proposed nonlinear development conditions in this paper have numerous potential applications in atomic and molecule material science. At long last, this technique gives a ground-breaking scientific instrument to get increasingly broad accurate arrangements of a large number of nonlinear PDEs in numerical material science.

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Certain Subclass of Meromorphic Functions with Positive Coefficients Defined by Bessel Function

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Abstract
The aim of the present paper is to introduce a class $\Sigma_p^*(G,H,\tau,c)$ of meromorphic univalent functions in $E=\{0<|z|<1\}$ and investigate coefficient estimates, distortion properties and radius of convexity estimates for this class. Furthermore, it is shown that this class is closed under convex linear combinations, convolutions and integral transforms.

Keywords: Convolutions, distortion, meromorphic.
2010 AMS: 30C45

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1. Introduction

Let Σ symbolized the class of analytic functions, which are with a simple pole at the origin with residue 1 of the form in the punctured unit disc $E=\{z:0<|z|<1\}$ and of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n. \tag{1.1}$$

Let $\Sigma_s,\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ be the subclass of Σ consisting of univalent, meromorphically starlike of order α and meromorphically convex functions of α, $0\le\alpha<1$ respectively.

A function given by (1.1) is in the $\Sigma^*(\alpha)$

$$\Leftrightarrow\ \Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\alpha\ (z\in E) \tag{1.2}$$

and $f\in\Sigma_k(\alpha)$

$$\Leftrightarrow\ \Re\left\{-\frac{(1+zf''(z))}{f'(z)}\right\}>\alpha,\ (z\in E). \tag{1.3}$$

Recent years, many authors investigated the subcalss of meromorphic functions with positive coefficients (see [1, 2, 3, 4, 5].

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \ (a_n \geq 0) \tag{1.4}$$

that are analytic and univalent in E.

We recall here the generalized Bessel function of first kind of order γ (see [6]), denoted by

$$w(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma(\gamma+n+\frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+\gamma} (z \in U)$$

(where Γ stands for the Gamma Euler function) which is the particular solution of the second order linear homogeneous differential equation (see, for details, [7])

$$z^2 w''(z) + bzw'(z) + [cz^2 - \gamma^2 + (1-b)\gamma]w(z) = 0,$$

where $c, \gamma, b \in C$.

We introduce the function φ defined, in terms of the generalized Bessel function w by

$$\varphi(z) = 2^{\gamma}\Gamma\left(\gamma + \frac{b+1}{2}\right) z^{-(1+\frac{\gamma}{2})} w(\sqrt{z}).$$

By using the well-known Pochhammer symbol $(x)_\tau$ defined, for $x \in C$ and in terms of the Euler gamma function, by

$$(x)_\tau = \frac{\Gamma(x+\tau)}{\Gamma(x)} = \begin{cases} 1, & (\tau = 0); \\ x(x+1)(x+2)\cdots(x+n-1), & (\tau = n \in N = \{1,2,3\cdots\}). \end{cases}$$

We obtain the following series representation for the function $\varphi(z)$

$$\varphi(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-c)^{n+1}}{4^{n+1}(n+1)!(\tau)_{n+1}} z^n \left(\tau = \gamma + \frac{b+1}{2} \notin Z_0^- = \{0,-1,-2,\cdots\}\right).$$

Corresponding to the function φ define the Bessel operator S_τ^c by the following Hadamard product

$$\begin{aligned} S_\tau^c f(z) = (\varphi * f)(z) &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^{n+1} a_n}{(n+1)!(\tau)_{n+1}} z^n \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \phi(n,\tau,c) a_n z^n, \end{aligned} \tag{1.5}$$

where $\phi(n,\tau,c) = \frac{\left(\frac{-c}{4}\right)^n}{(n)!(\tau)_n}$

Definition 1.1. *Let $\Sigma_p^*(G,H,\tau,c)$ denote the subclass of Σ_p consisting of functions $f(z)$ in Σ_p which satisfy*

$$\left|\frac{z(S_\tau^c f(z))'}{S_\tau^c f(z)} + 1\right| < \left|G + H\frac{z(S_\tau^c f(z))'}{S_\tau^c f(z)}\right|, \tag{1.6}$$

for $-1 \leq G < H, \ 0 < H \leq 1$.

2. Coefficient Inequalities

Our first theorem gives a necessary and sufficient condition for a function to be in $\Sigma_p^*(G,H,\tau,c)$.

Theorem 2.1. *Let $f(z) \in \Sigma_p$ as given by (1.4). Then $f(z) \in \Sigma_p^*(G,H,\tau,c)$ if and only if*

$$\sum_{n=1}^{\infty} [(n+1) + (G+Hn)]\phi(n,\tau,c)a_n \leq H - G, \tag{2.1}$$

for $-1 \leq G < H, \ 0 < H \leq 1$.

Proof. Suppose $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n,\ \ a_n\geq 0,$ is in $\Sigma_p^*(G,H,\tau,c)$. Then

$$\left|\frac{\frac{z(S_\tau^c f(z))'}{S_\tau^c f(z)}+1}{G+H\frac{z(S_\tau^c f(z))'}{S_\tau^c f(z)}}\right|=\left|\frac{\sum_{n=1}^{\infty}(n+1)\phi(n,\tau,c)a_nz^n}{(H-G)\frac{1}{z}-\sum_{n=1}^{\infty}(G+Hn)\phi(n,\tau,c)a_nz^n}\right|<1 \tag{2.2}$$

for all $z\in E$. Since $Re(z)\leq|z|$ for all z, we have

$$Re\left\{\frac{\sum_{n=1}^{\infty}(n+1)\phi(n,\tau,c)a_nz^n}{(H-G)\frac{1}{z}-\sum_{n=1}^{\infty}(G+Hn)\phi(n,\tau,c)a_nz^n}\right\}<1,\ (z\in E).$$

Now choose the values of z on real axis so that $\frac{z(S_\tau^c f(z))'}{S_\tau^c f(z)}$ is real.
Upon clearing the denominator in (2.2) and letting $z\to 1$ through positive values, we obtain

$$\sum_{n=1}^{\infty}[(n+1)+(G+Hn)]\phi(n,\tau,c)a_n\leq H-G.$$

Conversely, suppose that (2.1) holds for all admissible values of G and H. We have

$$\begin{aligned}M(f,f')&=|z(S_\tau^c f(z))'+S_\tau^c f(z)|-|GS_\tau^c f(z)+Hz(S_\tau^c f(z))'|\\&=\Big|\sum_{n=1}^{\infty}(n+1)\phi(n,\tau,c)a_nz^n\Big|-\Big|(H-G)\frac{1}{z}-\sum_{n=1}^{\infty}(G+Hn)\phi(n,\tau,c)a_nz^n\Big|\end{aligned}$$

or

$$\begin{aligned}zM(f,f')&\leq\sum_{n=1}^{\infty}(n+1)\phi(n,\tau,c)a_n|z|^{n+1}-(H-G)+\sum_{n=1}^{\infty}(G+Hn)\phi(n,\tau,c)a_n|z|^{n+1}\\&=\sum_{n=1}^{\infty}[(n+1)+(G+Hn)]\phi(n,\tau,c)a_n|z|^{n+1}-(H-G).\end{aligned}$$

Since the above inequality holds for all $r=|z|,\ 0<r<1,$ letting $r\to 1$, we have

$$\sum_{n=1}^{\infty}[(n+1)+(G+Hn)]\phi(n,\tau,c)a_n\leq(H-G)$$

by (2.1). Hence it follows that $f(z)$ is in the class $\Sigma_p^*(G,H,\tau,c)$. □

Corollary 2.2. *If the function* $f(z)\in\Sigma_p^*(G,H,\tau,c)$ *then*

$$a_n\leq\frac{(H-G)}{[(n+1)+(G+Hn)]\phi(n,\tau,c)},\ (n\geq 1). \tag{2.3}$$

The result is sharp for the function

$$f_n(z)=\frac{1}{z}+\frac{(H-G)}{[(n+1)+(G+Hn)]\phi(n,\tau,c)}z^n,\ (n\geq 1). \tag{2.4}$$

3. Distortion Properties and Radius of Convexity Estimates

Theorem 3.1. *If the function* $f(z)\in\Sigma_p^*(G,H,\tau,c)$ *then for* $0\leq|z|=r<1$,

$$\frac{1}{r}-\frac{(H-G)}{(2+G+H)\phi(1,\tau,c)}r\leq|f(z)|\leq\frac{1}{r}+\frac{(H-G)}{(2+G+H)\phi(1,\tau,c)}r. \tag{3.1}$$

The result is sharp.

Proof. Suppose $f(z)$ is in $\Sigma_p^*(G,H,\tau,c)$. By Theorem 2.1, we have

$$\sum_{n=1}^{\infty} a_n \leq \frac{(H-G)}{[(n+1)+(G+Hn)]\phi(n,\tau,c)}, \quad (n \geq 1). \tag{3.2}$$

Thus $|f(z)| \leq \frac{1}{|z|} + |z| \sum_{n=1}^{\infty} a_n \leq \frac{1}{r} + \frac{(H-G)}{(2+G+H)\phi(1,\tau,c)} r.$
Also $|f(z)| \geq \frac{1}{|z|} - |z| \sum_{n=1}^{\infty} a_n \geq \frac{1}{r} - \frac{(H-G)}{(2+G+H)\phi(1,\tau,c)} r.$
Thus the result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(H-G)}{(2+G+H)\phi(1,\tau,c)} z.$$

□

Theorem 3.2. *If the function $f(z) \in \Sigma_p^*(G,H,\tau,c)$ then $f(z)$ is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $|z| < r = r(G,H,\tau,c,\delta)$, where*

$$r(G,H,\tau,c,\delta) = \inf_{n \geq 1} \left[\frac{(1-\delta)[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)n(n+2-\delta)} \right]^{\frac{1}{n+1}}.$$

The result is sharp.

Proof. Let $f(z)$ be in $\Sigma_p^*(G,H,\tau,c)$. Then by Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)} a_n \leq 1. \tag{3.3}$$

It is sufficient to show that $\left|2 + \frac{zf''(z)}{f'(z)}\right| \leq 1-\delta$ for $|z| \leq r(G,H,\tau,c,\delta)$, where $r(G,H,\tau,c,\delta)$ is as specified in the statement of the theorem. Then

$$\left|2 + \frac{zf''(z)}{f'(z)}\right| = \left| \frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n-1}}{\frac{-1}{z^2} + \sum_{n=1}^{\infty} n a_n z^{n-1}} \right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} n a_n |z|^{n+1}}.$$

This will be bounded by $1-\delta$ if

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n |z|^{n+1} \leq 1. \tag{3.4}$$

By (3.3), it follows that (3.4) is true if

$$\frac{n(n+2-\delta)}{1-\delta} |z|^{n+1} \leq \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{H-G}, \ (n \geq 1)$$

or

$$|z| \leq \left\{ \frac{(1-\delta)[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)n(n+2-\delta)} \right\}^{\frac{1}{n+1}}, \ (n \geq 1). \tag{3.5}$$

Setting $|z| = r(G,H,\tau,c,\delta)$ in (3.5), the result follows.
The result is sharp for the functions

$$f_n(z) = \frac{1}{z} + \frac{(H-G)}{[(n+1)+(G+Hn)]\phi(n,\tau,c)} z^n, \ (n \geq 1). \tag{3.6}$$

□

4. Convex Linear Combinations and Convolution Properties

We shall prove that the class $\Sigma_p^*(G,H,\tau,c)$ is closed under convex linear combinations and convolutions.

Theorem 4.1. *Let* $f_0(z) = \frac{1}{z}$ *and*

$$f_n(z) = \frac{1}{z} + \frac{(H-G)}{[(n+1)+(G+Hn)]\phi(n,\tau,c)}z^n, \ (n \geq 1). \tag{4.1}$$

Then $f(z) \in \Sigma_p^*(G,H,\tau,c)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$, *where* $\lambda_n \geq 0$ *and* $\sum_{n=0}^{\infty} \lambda_n = 1$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$ with $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$. Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(H-G)}{[(n+1)+(G+Hn)]\phi(n,\tau,c)}z^n. \tag{4.2}$$

Since

$$\frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)}\lambda_n \frac{(H-G)}{[(n+1)+(G+Hn)]\phi(n,\tau,c)}$$
$$= \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_n \leq 1,$$

by Theorem 2.1, $f(z)$ is in the class $\Sigma_p^*(G,H,\tau,c)$.

Conversely, suppose that the function $f(z)$ is the class $\Sigma_p^*(G,H,\tau,c)$. Setting

$$\lambda_n = \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)}a_n, \ n \geq 1$$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$, it follows that $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$.
Here, we see that $\lambda_n \geq 0$ $(n \geq 1)$ by definition and $\lambda_0 \geq 0$ in view of Theorem 2.1. This completes the proof of the theorem.

The result is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{(H-G)}{[(n+1)+(G+Hn)]\phi(n,\tau,c)}z^n, \ (n \geq 1). \tag{4.1}$$

□

Robertson [8], has shown that if $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ are in Σ_s then so their convolutions $(f*g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$.

Theorem 4.2. *If the function* $f(z)$ *and* $g(z)$ *are in the class* $\Sigma_p^*(G,H,\tau,c)$ *then* $(f*g)(z)$ *is the class* $\Sigma_p^*(G,H,\tau,c)$.

Proof. Suppose that $f(z)$ and $g(z)$ are in $\Sigma_p^*(G,H,\tau,c)$. By Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)}a_n \leq 1,$$

$$\sum_{n=1}^{\infty} \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)}b_n \leq 1.$$

Since $f(z)$ and $g(z)$ are regular in E, so $(f*g)(z)$. Furthermore

$$\sum_{n=1}^{\infty} \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)}a_n b_n \leq \left\{ \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)} \right\}^2 a_n b_n$$
$$\leq \sum_{n=1}^{\infty} \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)}a_n \left\{ \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)} \right\} b_n$$
$$\leq 1.$$

Hence by Theorem 2.1, $(f*g)(z)$ is in the class $\Sigma_p^*(G,H,\tau,c)$. □

5. Integral Transforms

In this section, we consider transforms of functions in the class $\Sigma_p^*(G,H,\tau,c)$ of the type considered by Bajpai [9].

Theorem 5.1. *If the function $f(z)$ is in the class $\Sigma_p^*(G,H,\tau,c)$ then the integral transforms*

$$F_c(z) = c\int_0^1 u^c f(uz)dz, \ (0 < c < \infty)$$

is in the class $\Sigma_p^*(G,H,\tau,c)$.

Proof. Suppose $f(z)$ is in $\Sigma_p^*(G,H,\tau,c)$. Then we have

$$F_c(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{n+c+1} a_n z^n.$$

Since

$$\sum_{n=1}^{\infty} \frac{[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)} \frac{ca_n}{n+c+1} \le \frac{\sum_{n=1}^{\infty}[(n+1)+(G+Hn)]\phi(n,\tau,c)}{(H-G)} a_n \le 1,$$

by Theorem 2.1, it follows that $F_c(z)$ is in the class $\Sigma_p^*(G,H,\tau,c)$. □

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Dual Jacobsthal Quaternions

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Abstract
In this paper, dual Jacobsthal quaternions were defined. Also, the relations between dual Jacobsthal quaternions which connected with Jacobsthal and Jacobsthal-Lucas numbers were investigated. Furthermore, Binet's formula, Honsberger identity, D'ocagne's identity, Cassini's identity and Catalan's identity for these quaternions were given.

Keywords: Jacobsthal number, Jacobsthal-Lucas number, Jacobsthal quaternion, dual Jacobsthal quaternion.
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1. Introduction

In 1843, Hamilton [1] introduced the set of quaternions which can be represented as

$$H = \{ q = q_0 + i\,q_1 + j\,q_2 + k\,q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \} \tag{1.1}$$

where

$$i^2 = j^2 = k^2 = -1, \quad i\,j = -j\,i = k, \quad j\,k = -k\,j = i, \quad k\,i = -i\,k = j.$$

After the work of Hamilton, several authors worked on different quaternions and their generalizations. ([2]-[22]).

In 1973, Sloane [23] introduced the set of Jacobsthal numbers.
Further, in 1988, Horadam [24]-[25] defined the Jacobsthal and Jacobsthal-Lucas sequences $\{J_n\}$ and $\{j_n\}$ with the recurrence relations respectively, as follows

$$J_0 = 0, \;\; J_1 = 1, \;\; J_n = J_{n-1} + 2J_{n-2}, \;\; for\; n \geq 2, \tag{1.2}$$

and

$$j_0 = 2, \;\; j_1 = 1, \;\; j_n = j_{n-1} + 2\,j_{n-2}, \;\; for\; n \geq 2, \tag{1.3}$$

In 1996, Horadam studied on the Jacobsthal and Jacobsthal-Lucas sequences and he gave Cassini-like formula as follows [26]

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n.2^{n-1} \tag{1.4}$$

$$j_{n+1}j_{n-1} - j_n^2 = 3^2.(-1)^{n+1}.2^{n-1} \tag{1.5}$$

The first eleven terms of Jacobsthal sequence $\{J_n\}$ are $0,1,1,3,5,11,21,43,85,171$ and 341. This sequence is given by the formula

$$J_n = \frac{2^n - (-1)^n}{3} \tag{1.6}$$

The first eleven terms of Jacobsthal-Lucas sequence $\{j_n\}$ are $2,1,5,7,17,31$ $65,127,257,511$ and 1025. This sequence is given by the formula

$$j_n = 2^n + (-1)^n \tag{1.7}$$

Also, we can see the matrix representations of Jacobsthal and Jacobsthal-Lucas numbers in [27],[28]. The members of these integer sequences can also be obtained in different ways: Binet formulae or matrix method by Köken and Bozkurt [27]-[28]. Several authors worked on Jacobsthal numbers and polynomials in [29]-[32].

In 2015, Szynal-Liana and Włoch [33] defined the Jacobsthal quaternions and the Jacobsthal- Lucas quaternions respectively as follows

$$JQ_n = J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}\,, \tag{1.8}$$

and

$$JLQ_n = j_n + i\,j_{n+1} + j\;j_{n+2} + k\,j_{n+3}\,. \tag{1.9}$$

where

$$i^2 = j^2 = k^2 = -1\,,\; i\,j = -j\,i = k\,,\quad j\,k = -k\,j = i\,,\quad k\,i = -i\,k = j.$$

In 2017, Torunbalcı Aydın and Yüce [34] given a new approach to Jacobsthal quaternions. Furthermore, some relations between Jacobsthal and Jacobsthal-Lucas quaternions are given in [34].

In 2017, Taşçı [35] defined k-Jacobsthal and k-Jacobsthal-Lucas quaternions as follows

$$QJ_{k,n} = J_{k,n} + i_1\,J_{k,n+1} + i_2\,J_{k,n+2} + i_3\,J_{k,n+3} \tag{1.10}$$

and

$$Q\,j_{k,n} = j_{k,n} + i_1\,j_{k,n+1} + i_2\,j_{k,n+2} + i_3\,j_{k,n+3} \tag{1.11}$$

where

$$i_1^2 = i_2^2 = i_3^2 = i_1\,i_2\,i_3 = -1\,.$$

In 2017, Cerda-Morales [36] worked on identities of third order Jacobsthal quaternions.
In 2018, Cerda-Morales [37] defined fourth-order Jacobsthal and Jacobsthal-Lucas quaternions as follows

$$QJ_n{}^{(4)} = J_n{}^{(4)} + i\,J_{n+1}{}^{(4)} + j\,J_{n+2}{}^{(4)} + k\,J_{n+3}{}^{(4)} \tag{1.12}$$

and

$$Q\,j_n{}^{(4)} = j_n{}^{(4)} + i\,j_{n+1}{}^{(4)} + j\,j_{n+2}{}^{(4)} + k\,j_{n+3}{}^{(4)} \tag{1.13}$$

In this paper, dual Jacobsthal and dual Jacobsthal-Lucas quaternions will be defined as follows

$$J_D = \{D_n^J = J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}\ \mid J_n,\, n-th\,\text{Jacobsthal number}\}$$

and

$$j_D = \{D_n^j = j_n + i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3} \mid j_n,\, n-th\text{Jacobsthal-Lucas number}\}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\;\; i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0\,.$$

All the studies on Jacobsthal quaternions are summarized in Table 1.

Table 1. Types of Jacobsthal quaternions [33]-[35].

	Definition	Multiplication Rules
Jacobsthal quaternion	$JQ_n = (J_n, J_{n+1}, J_{n+2}, J_{n+3})$ $J_n = J_{n-1} + 2J_{n-2},\, J_1 = J_2 = 1$	$(1, i, j, k),\; i^2 = j^2 = k^2 = -1$ $i\,j = -j\,i = k,\; j\,k = -k\,j = i$ $k\,i = -i\,k = j$
k-Jacobsthal quaternion	$QJ_{k,n} = (J_{k,n}, J_{k,n+1}, J_{k,n+2}, J_{k,n+3})$ $QJ_{k,n+2} = k\,QJ_{k,n+1} + 2\,QJ_{k,n}$	$(1, i_1, i_2, i_3)$, $i_1^2 = i_2^2 = i_3^2 = i_1\,i_2\,i_3 = -1$
Dual Jacobsthal quaternion	$D_n^J = (J_n, J_{n+1}, J_{n+2}, J_{n+3})$ $J_n = J_{n-1} + 2J_{n-2},\, J_1 = J_2 = 1$	$(1, i, j, k)\; i^2 = j^2 = k^2 = i\,j\,k = 0$ $i\,j = -j\,i = j\,k = -kj = k\,i = -i\,k = 0$

2. Dual Jacobsthal Quaternions

In this section, the dual Jacobsthal quaternions will be defined. Also, the relations between dual Jacobsthal quaternions which connected with Jacobsthal and Jacobsthal-Lucas numbers were investigated.

Dual Jacobsthal quaternions is defined by relation recurrence (1.2) as follows

$$J_D = \{D_n^J = J_n + iJ_{n+1} + j\,J_{n+2} + k\,J_{n+3} \,|\, J_n,\, n-th\text{Jacobsthal number}\} \tag{2.1}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\;\; i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0\,. \tag{2.2}$$

Also, the dual Jacobsthal-Lucas quaternion is defined by relation recurrence (1.3) as follows

$$j_D = \{D_n^j = j_n + i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3} \,|\, j_n,\, n-th\text{Jacobsthal-Lucas number}\}, \tag{2.3}$$

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\;\; i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0\,.$$

Let $D_n^{J_1}$ and $D_n^{J_2}$ be n-th terms of the dual Jacobsthal quaternion sequence $(D_n^{J_1})$ and $(D_n^{J_2})$ such that

$$D_n^{J_1} = J_n + iJ_{n+1} + j\,J_{n+2} + kJ_{n+3} \tag{2.4}$$

and

$$D_n^{J_2} = K_n + iK_{n+1} + j\,K_{n+2} + kK_{n+3} \tag{2.5}$$

Then, the addition and subtraction of the dual Jacobsthal quaternions is defined by

$$\begin{aligned} D_n^{J_1} \pm D_n^{J_2} = & \ (J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\ & \pm (K_n + iK_{n+1} + jK_{n+2} + kK_{n+3}) \\ = & \ (J_n \pm K_n) + i(J_{n+1} \pm K_{n+1}) + j(J_{n+2} \pm K_{n+2}) \\ & + k(J_{n+3} \pm K_{n+3}). \end{aligned} \tag{2.6}$$

Multiplication of the dual Jacobsthal quaternions is defined by

$$\begin{aligned} D_n^{J_1} D_n^{J_2} = & \ (J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\ & (K_n + iK_{n+1} + jK_{n+2} + kK_{n+3}) \\ = & \ (J_n K_n) + i(J_n K_{n+1} + J_{n+1} K_n) + j(J_n K_{n+2} + J_{n+2} K_n) \\ & + k(J_n K_{n+3} + J_{n+3} K_n). \end{aligned} \tag{2.7}$$

The scalar and the vector part of D_n^J which is the n-th term of the dual Jacobsthal quaternion (D_n^J) are denoted by

$$S_{D_n^J} = J_n \text{ and } V_{D_n^J} = iJ_{n+1} + jJ_{n+2} + kJ_{n+3}. \tag{2.8}$$

Thus, the dual Jacobsthal quaternion D_n^J is given by $D_n^J = S_{D_n^J} + V_{D_n^J}$.
Then, relation (2.7) is defined by

$$D_n^{J_1} D_n^{J_2} = S_{D_n^{J_1}} . S_{D_n^{J_2}} + S_{D_n^{J_1}} . V_{D_n^{J_2}} + S_{D_n^{J_2}} . V_{D_n^{J_1}} . \tag{2.9}$$

The conjugate of the dual Jacobsthal quaternion D_n^J is denoted by $\overline{D_n^J}$ and it is

$$\overline{D_n^J} = J_n - iJ_{n+1} - jJ_{n+2} - kJ_{n+3}. \tag{2.10}$$

The norm of D_n^J is defined as

$$N_{D_n^J} = \left\| D_n^J \right\|^2 = D_n^J \, \overline{D_n^J} = J_n^2. \tag{2.11}$$

Then, we give the following theorem using statements (2.1), (2.2) and

$$\begin{cases} J_n J_{n+1} + 2J_{n-1} J_n = J_{2n}, \\ J_n J_{m+1} + 2J_{n-1} J_m = J_{n+m}, \\ J_{n+1} + 2J_{n-1} = j_n, \\ J_n j_n = J_{2n}. \end{cases} \tag{2.12}$$

Theorem 2.1. *Let J_n and D_n^J be the n-th terms of the Jacobsthal sequence (J_n) and the dual Jacobsthal quaternion sequence (D_n^J), respectively. In this case, for $n \geq 1$ we can give the following relations:*

$$D_n^J + \overline{D_n^J} = 2J_n, \tag{2.13}$$

$$(D_n^J)^2 + D_n^J \overline{D_n^J} = 2J_n D_n^J, \tag{2.14}$$

$$D_{n+1}^J + 2D_n^J = D_{n+2}^J, \tag{2.15}$$

$$D_n^J - iD_{n+1}^J - jD_{n+2}^J - kD_{n+3}^J = J_n, \tag{2.16}$$

Proof. Proof of four equality can easily be done by the equations

$$D_n^J = J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}, \tag{2.17}$$

$$D_{n+1}^{J}=J_{n+1}+i\,J_{n+2}+j\,J_{n+3}+k\,J_{n+4} \tag{2.18}$$

(2.13):

$$\begin{aligned} D_n^J+\overline{D_n^J}= & \;(J_n+i\,J_{n+1}+j\,J_{n+2}+k\,J_{n+3}) \\ & +(J_n-i\,J_{n+1}-j\,J_{n+2}-k\,J_{n+3}) \\ = & \;(J_n+J_n)+i(J_{n+1}-J_{n+1})+j(J_{n+2}-J_{n+2}) \\ & +k(J_{n+3}-J_{n+3}) \\ = & \;2J_n. \end{aligned}$$

(2.14):

$$\begin{aligned} (D_n^J)^2+D_n^J\,\overline{D_n^J}= & \;(J_n+iJ_{n+1}+j\,J_{n+2}+kJ_{n+3}) \\ & \;(J_n+iJ_{n+1}+j\,J_{n+2}+kJ_{n+3}) \\ & +(J_n+i\,J_{n+1}+j\,J_{n+2}+kJ_{n+3}) \\ & \;(J_n-i\,J_{n+1}-j\,J_{n+2}-kJ_{n+3}) \\ = & \;(J_nJ_n)+i(J_nJ_{n+1}+J_{n+1}J_n)+j(J_nJ_{n+2}+J_{n+2}J_n) \\ & +k(J_nJ_{n+3}+J_{n+3}J_n) \\ & +J_nJ_n+i(-J_nJ_{n+1}+J_{n+1}J_n) \\ & +j(-J_nJ_{n+2}+J_{n+2}J_n) \\ & +k(-J_nJ_{n+3}+J_{n+3}J_n) \\ = & \;2J_nJ_n+2i\,J_nJ_{n+1}+2\,j\,J_nJ_{n+2}+2k\,J_nJ_{n+3} \\ = & \;2J_n(J_n+i\,J_{n+1}+j\,J_{n+2}+k\,J_{n+3}) \\ = & \;2J_n\,D_n^J \end{aligned}$$

(2.15):

$$\begin{aligned} D_{n+1}^J+2\,D_n^J= & \;(J_{n+1}+i\,J_{n+2}+j\,J_{n+3}+k\,J_{n+4}) \\ & +2(J_n+i\,J_{n+1}+j\,J_{n+2}+k\,J_{n+3}\,) \\ = & \;(J_{n+1}+2J_n)+i\,(J_{n+2}+2J_{n+1})+j\,(J_{n+3}+2J_{n+2}) \\ & +k\,(J_{n+4}+2J_{n+3}) \\ = & \;J_{n+2}+i\,J_{n+3}+j\,J_{n+4}+k\,J_{n+5} \\ = & \;D_{n+2}^J. \end{aligned}$$

(2.16):

$$\begin{aligned} D_n^J-i\,D_{n+1}^J-j\,D_{n+2}^J-k\,D_{n+3}^J= & \;(J_n+iJ_{n+1}+jJ_{n+2}+kJ_{n+3}) \\ - & \;i(J_{n+1}+iJ_{n+2}+jJ_{n+3}+kJ_{n+4}) \\ - & \;j(J_{n+2}+iJ_{n+3}+jJ_{n+4}+kJ_{n+5}) \\ - & \;k(J_{n+3}+iJ_{n+4}+jJ_{n+5}+kJ_{n+6}) \\ = & \;J_n\,. \end{aligned}$$

□

Theorem 2.2. *Let D_n^J and D_n^j be the n-th terms of the dual Jacobsthal quaternion sequence (D_n^J) and the dual Jacobsthal-Lucas quaternion sequence (D_n^j), respectively. The following relations are satisfied*

$$D_{n+1}^J+2\,D_{n-1}^J=D_n^j\,, \tag{2.19}$$

$$2\,D_{n+1}^J-D_n^J=D_n^j\,. \tag{2.20}$$

Proof. (2.19): From equations (2.17), (2.18) and identity between Jacobsthal number and Jacobsthal-Lucas number $j_n=J_{n+1}+2J_{n-1}$, it follows that

$$\begin{aligned} D_{n+1}^J+2\,D_{n-1}^J= & \;(J_{n+1}+i\,J_{n+2}+j\,J_{n+3}+k\,J_{n+4}) \\ & +2(J_{n-1}+i\,J_n+j\,J_{n+1}+k\,J_{n+2}) \\ = & \;(J_{n+1}+2J_{n-1})+i\,(J_{n+2}+2J_n) \\ & +j\,(J_{n+3}+2J_{n+1})+k\,(J_{n+4}+2J_{n+2}) \\ = & \;j_n+i\,j_{n+1}+j\,j_{n+2}+k\,j_{n+3} \\ = & \;D_n^j\,. \end{aligned}$$

(2.20): Using the identity between Jacobsthal number and Jacobsthal -Lucas number $J_n + j_n = 2J_{n+1}$, we get

$$\begin{aligned}
2D^J_{n+1} - D^J_n = & \ 2(J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4}) \\
& -(J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\
= & \ (2J_{n+1} - J_n) + i(J_{n+2} - J_{n+1}) \\
& + j(2J_{n+3} - J_{n+2}) + k(2J_{n+4} - J_{n+3}) \\
= & \ j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3} \\
= & \ D^j_n .
\end{aligned}$$

□

Theorem 2.3. *Let D^J_n be the n-th term of the dual Jacobsthal quaternion sequence (D^J_n) and $\overline{D^J_n}$ be conjugate of D^J_n. Then, we can give the following relations between these quaternions:*

$$\begin{aligned}
&(D^J_n)^2 = 2J_nD^J_n - J_n^2, \\
&(D^J_n)^2 + 2(D^J_{n-1})^2 = 2D^J_{2n-1} - J_{2n-1}, \\
&D^J_n\overline{D^J_n} + 2D^J_{n-1}\overline{D^J_{n-1}} = J_n^2 + 2J_{n-1}^2 = J_{2n-1}, \\
&D^J_{n+1}\overline{D^J_{n+1}} + 2D^J_n\overline{D^J_n} = J_{n+1}^2 + 2J_n^2 = J_{2n+1}, \\
&D^J_{n+1}\overline{D^J_{n+1}} - 2D^J_n\overline{D^J_n} = J_{n+1}^2 - 2J_n^2 = J_{2n+1} - 4J_n^2
\end{aligned} \tag{2.21}$$

Proof. It can be proved easily by using (2.10). Now, we will prove first two equalities

$$\begin{aligned}
(D^J_n)^2 = & \ J_nJ_n + i(J_nJ_{n+1} + J_{n+1}J_n) + j(J_nJ_{n+2} + J_{n+2}J_n) \\
& + k(J_nJ_{n+3} + J_{n+3}J_n) \\
= & \ 2J_n(J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) - J_nJ_n \\
= & \ 2J_nD^J_n - J_n^2.
\end{aligned}$$

$$\begin{aligned}
(D^J_n)^2 + 2(D^J_{n-1})^2 = & \ J_n^2 + 2i(J_nJ_{n+1}) + 2j(J_nJ_{n+2}) + 2k(J_nJ_{n+3}) \\
& + 2(J_{n-1}^2) + 4i(J_{n-1}J_n) + 4j(J_{n-1}J_{n+1}) \\
& + 4k(J_{n-1}J_{n+2}) \\
= & \ (J_n^2 + 2J_{n-1}^2) + i(2J_nJ_{n+1} + 4J_{n-1}J_n) \\
& + j(2J_nJ_{n+2} + 4J_{n-1}J_{n+1}) \\
& + k(2J_nJ_{n+3} + 4J_{n-1}J_{n+2}) \\
= & \ J_{2n-1} + 2iJ_{2n} + 2jJ_{2n+1} + 2kJ_{2n+2} \\
= & \ 2D^J_{2n-1} - J_{2n-1}.
\end{aligned}$$

We can prove last three equalities by using equation (2.12) as follows:

$$\begin{aligned}
D^J_n\overline{D^J_n} + 2D^J_{n-1}.\overline{D^J_{n-1}} &= J_n^2 + 2J_{n-1}^2 = J_{2n-1}, \\
D^J_{n+1}\overline{D^J_{n+1}} + 2D^J_n\overline{D^J_n} &= J_{n+1}^2 + 2J_n^2 = J_{2n+1}, \\
D^J_{n+1}\overline{D^J_{n+1}} - 2D^J_n\overline{D^J_n} &= J_{n+1}^2 - 2J_n^2 = J_{2n+1} - 4J_n^2
\end{aligned}$$

where identities $J_mJ_{n+1} + 2J_{m-1}J_n = J_{m+n}$ and $J_n^2 + 2J_{n-1}^2 = J_{2n-1}$ were used. □

Theorem 2.4. *Let D^J_n be the n-th term of dual Jacobsthal quaternion sequence (D^J_n). Then, we have the following identities*

$$\sum_{s=1}^{n} D^J_s = \frac{1}{2}[D^J_{n+2} - D^J_2], \tag{2.22}$$

$$\sum_{s=0}^{p} D^J_{n+s} = \frac{1}{2}[D^J_{n+p+2} - D^J_{n+1}], \tag{2.23}$$

$$\sum_{s=1}^{n} D^J_{2s-1} = \frac{2D^J_{2n}}{3} + \frac{1}{3}[n(2D^J_2 - D^J_3) - 2D^J_0], \tag{2.24}$$

$$\sum_{s=1}^{n} D_{2s}^{J} = \frac{2D_{2n+1}^{J}}{3} - \frac{1}{3}\left[n(2D_{2}^{J} - D_{3}^{J}) + 2D_{1}^{J}\right]. \tag{2.25}$$

Proof. (2.22) Hence, we can write

$$\begin{aligned}
\sum_{s=1}^{n} D_{s}^{J} &= \sum_{s=1}^{n} J_{s} + i\sum_{s=1}^{n} J_{s+1} + j\sum_{s=1}^{n} J_{s+2} + k\sum_{s=1}^{n} J_{s+3} \\
&= \frac{1}{2}[(J_{n+2}-1) + i(J_{n+3}-3) + j(J_{n+4}-5) + k(J_{n+5}-11)] \\
&= \frac{1}{2}[(J_{n+2}-J_{2}) + i(J_{n+3}-J_{3}) + j(J_{n+4}-J_{4}) + k(J_{n+5}-J_{5})] \\
&= \frac{1}{2}[J_{n+2} + iJ_{n+3} + jJ_{n+4} + kJ_{n+5} - (J_{2} + iJ_{3} + jJ_{4} + kJ_{5})] \\
&= \frac{1}{2}[D_{n+2}^{J} - D_{2}^{J}].
\end{aligned}$$

(2.23) Hence, we can write

$$\begin{aligned}
\sum_{s=0}^{p} D_{n+s}^{J} &= \sum_{s=0}^{p} J_{n+s} + i\sum_{s=0}^{p} J_{n+s+1} + j\sum_{s=0}^{p} J_{n+s+2} + k\sum_{s=0}^{p} J_{n+s+3} \\
&= \frac{1}{2}[(J_{n+p+2} - J_{n+1}) + i(J_{n+p+3} - J_{n+2}) + j(J_{n+p+4} - J_{n+3})] \\
&\quad + \frac{1}{2}[k(J_{n+p+5} - J_{n+4})] \\
&= \frac{1}{2}[J_{n+p+2} + iJ_{n+p+3} + jJ_{n+p+4} + kJ_{n+p+5} \\
&\quad - (J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4})] \\
&= \frac{1}{2}[D_{n+p+2}^{J} - D_{n+1}^{J}].
\end{aligned}$$

(2.24): Using $\sum_{i=0}^{n-1} J_{2i+1} = \frac{2J_{2n}+n}{3}$ and $\sum_{i=0}^{n} J_{2i} = \frac{2J_{2n+1}-n-2}{3}$, we get

$$\begin{aligned}
\sum_{s=1}^{n} D_{2s-1}^{J} &= (J_{1} + J_{3} + \ldots + J_{2n-1}) + i(J_{2} + J_{4} + \ldots + J_{2n}) \\
&\quad + j(J_{3} + J_{5} + \ldots + J_{2n+1}) + k(J_{4} + J_{6} + \ldots + J_{2n+2}) \\
&= \frac{(2J_{2n}+n)}{3} + i\,\frac{(2J_{2n+1}-n-2)}{3} + j\,\frac{(2J_{2n+2}+n-2)}{3} \\
&\quad + k\,\frac{(2J_{2n+3}-n-6)}{3} \\
&= \frac{2}{3}[J_{2n} + iJ_{2n+1} + jJ_{2n+2} + kJ_{2n+3}] \\
&\quad + \frac{1}{3}[n(1-i+j-k) - 2(i+j+3k)] \\
&= \frac{2D_{2n}^{J}}{3} + \frac{1}{3}[n(2D_{2}^{J} - D_{3}^{J}) - 2D_{0}^{J}].
\end{aligned}$$

(2.25): Using $\sum\limits_{i=0}^{n} J_{2i} = \frac{2J_{2n+1}-n-2}{3}$ we obtain

$$\begin{aligned}\sum_{s=1}^{n} D^J_{2s} &= (J_2+J_4+\ldots+J_{2n})+i(J_3+J_5+\ldots+J_{2n+1})\\ &\quad + j(J_4+J_6+\ldots+J_{2n+2})+k(J_5+J_7+\ldots+J_{2n+3})\\ &= \frac{(2J_{2n+1}-n-2)}{3}+i\,\frac{(2J_{2n+2}+n-2)}{3}+j\,\frac{(2J_{2n+3}-n-6)}{3}\\ &\quad +k\,\frac{(2J_{2n+4}+n-10)}{3}\\ &= \frac{2}{3}[J_{2n+1}+iJ_{2n+2}+jJ_{2n+3}+kJ_{2n+4}]\\ &\quad +\frac{1}{3}[-n(1-i+j-k)-2(1+i+3j+5k)]\\ &= \frac{2D^J_{2n+1}}{3}-\frac{1}{3}[n(2D^J_2-D^J_3)+2D^J_1].\end{aligned}$$

□

Theorem 2.5. *Let D^J_n and D^j_n be the n-th terms of the dual Jacobsthal quaternion sequence (D^J_n) and the dual Jacobsthal-Lucas quaternion sequence (D^j_n) ,respectively. Then, we have*

$$D^j_n\overline{D^J_n}-\overline{D^j_n}D^J_n = 2[J_nD^j_n-j_nD^J_n], \tag{2.26}$$

$$D^j_n\overline{D^J_n}+\overline{D^j_n}D^J_n = 2j_nJ_n = 2J_{2n}, \tag{2.27}$$

$$D^j_nD^J_n-\overline{D^j_n}\,\overline{D^J_n} = 2[D^j_nJ_n+D^J_nj_n-2J_{2n}], \tag{2.28}$$

$$D^j_nD^J_n+\overline{D^j_n}\,\overline{D^J_n} = 2J_{2n}. \tag{2.29}$$

Proof. (2.26):

$$\begin{aligned}D^j_n\overline{D^J_n}-\overline{D^j_n}D^J_n = \;& (j_n+ij_{n+1}+jj_{n+2}+kj_{n+3})\\ & (J_n-iJ_{n+1}-jJ_{n+2}-kJ_{n+3})\\ & -(j_n-ij_{n+1}-jj_{n+2}-kj_{n+3})\\ & (J_n+iJ_{n+1}+jJ_{n+2}+kJ_{n+3})\\ = \;& (j_nJ_n-j_nJ_n)\\ & +2i(j_{n+1}J_n-j_nJ_{n+1})\\ & +2j(j_{n+2}J_n-j_nJ_{n+2})\\ & +2k(j_{n+3}J_n-j_nJ_{n+3})\\ = \;& 2J_n(j_n+ij_{n+1}+jj_{n+2}+kj_{n+3})\\ & -2j_n(J_n+iJ_{n+1}+jJ_{n+2}+kJ_{n+3})\\ = \;& 2[J_nD^j_n-j_nD^J_n].\end{aligned}$$

(2.27):

$$\begin{aligned}D^j_n\overline{D^J_n}+\overline{D^j_n}D^J_n = \;& (j_n+ij_{n+1}+jj_{n+2}+kj_{n+3})\\ & (J_n-iJ_{n+1}-jJ_{n+2}-kJ_{n+3})\\ & +(j_n-ij_{n+1}-jj_{n+2}-kj_{n+3})\\ & (J_n+iJ_{n+1}+jJ_{n+2}+kJ_{n+3})\\ = \;& j_n[J_n-iJ_{n+1}-jJ_{n+2}-kJ_{n+3}]\\ & +(ij_{n+1}+jj_{n+2}+kj_{n+3})J_n\\ & +j_n[J_n+iJ_{n+1}+jJ_{n+2}+kJ_{n+3}]\\ & +(-ij_{n+1}-jj_{n+2}-kj_{n+3})J_n\\ = \;& 2j_nJ_n = 2J_{2n}.\end{aligned}$$

(2.28):

$$\begin{aligned}
D_n^j D_n^J - \overline{D_n^j}\,\overline{D_n^J} = \; & (j_n + i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3}) \\
& (J_n + i\, J_{n+1} + j\, J_{n+2} + k\, J_{n+3}) \\
& -(j_n - i\, j_{n+1} - j\, j_{n+2} - k\, j_{n+3}) \\
& (J_n - i\, J_{n+1} - j\, J_{n+2} - k\, J_{n+3}) \\
= \; & (j_n J_n - j_n J_n) + 2\,(i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3}) J_n \\
& +2\,(j_n + i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3}) J_n - 2\, j_n J_n \\
= \; & 2\,(D_n^j J_n + D_n^J j_n - 2\, j_n J_n) \\
= \; & 2\,(D_n^j J_n + D_n^J j_n - 2 J_{2n}).
\end{aligned}$$

(2.29):

$$\begin{aligned}
D_n^j D_n^J + \overline{D_n^j}\,\overline{D_n^J} = \; & (j_n + i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3}) \\
& (J_n + i\, J_{n+1} + j\, J_{n+2} + k\, J_{n+3}) \\
& +(j_n - i\, j_{n+1} - j\, j_{n+2} - k\, j_{n+3}) \\
& (J_n - i\, J_{n+1} - j\, J_{n+2} - k\, J_{n+3}) \\
= \; & 2\, j_n J_n = 2 J_{2n}.
\end{aligned}$$

In proofs, the identities of Jacobsthal and Jacobsthal-Lucas numbers given below were used, respectively,

$$J_m j_n - J_n j_m = (-1)^n\, 2^{n+1} J_{m-n},\; j_n J_n = J_{2n} \text{ and } j_{n+2} = j_{n+1} + 2\, j_n.$$

□

Theorem 2.6. ***(Binet's Formula)***. *Let D_n^J and D_n^j be $n-th$ terms of dual Jacobsthal quaternion sequence (D_n^J) and the dual Jacobsthal-Lucas quaternion sequence (D_n^j), respectively. For $n \geq 1$, Binet's formula for these quaternions are as follows respectively,*

$$D_n^J = \frac{1}{\alpha - \beta}\left[\underline{\alpha}\,\alpha^n - \underline{\beta}\,\beta^n\right] \tag{2.30}$$

and

$$D_n^j = (\underline{\underline{\alpha}}\,\alpha^n + \underline{\underline{\beta}}\,\beta^n) \tag{2.31}$$

where

$$\begin{aligned}
\underline{\alpha} &= 1 + i\,(1-\beta) + j\,(3-\beta) + k\,(5-3\beta),\; \alpha = 2, \\
\underline{\beta} &= 1 + i\,(1-\alpha) + j\,(3-\alpha) + k\,(5-3\alpha),\; \beta = -1, \\
\underline{\underline{\alpha}} &= (1-2\beta) + i\,(5-\beta) + j\,(7-5\beta) + k\,(17-7\beta),\; \alpha = 2, \\
\underline{\underline{\beta}} &= (2\alpha-1) + i\,(\alpha-5) + j\,(5\alpha-7) + k\,(7\alpha-17),\; \beta = -1.
\end{aligned}$$

Proof. The characteristic equation of recurrence relations $D_{n+2}^J = D_{n+1}^J + 2 D_n^J$ is

$$t^2 - t - 2 = 0.$$

The roots of this equation are $\alpha = 2$ and $\beta = -1$
where $\alpha + \beta = 1$, $\alpha - \beta = 3$, $\alpha\beta = -2$.
Using recurrence relation and initial values $D_0^J = (0, 1, 1, 3)$,
$D_1^J = (1, 1, 3, 5)$ the Binet's formula for D_n^J, we get

$$D_n^J = A\,\alpha^n + B\,\beta^n = \frac{1}{3}\left[\underline{\alpha}\,\alpha^n - \underline{\beta}\,\beta^n\right],$$

where $A = \frac{D_1^J - D_0^J \beta}{\alpha-\beta}$, $B = \frac{\alpha D_0^J - D_1^J}{\alpha-\beta}$ and

$\underline{\alpha} = 1 + i(1-\beta) + j(3-\beta) + k(5-3\beta)$, $\underline{\beta} = 1 + i(1-\alpha) + j(3-\alpha) + k(5-3\alpha)$. Similarly, using recurrence relations $D_{n+2}^j = D_{n+1}^j + 2D_n^j$, the Binet's formula for D_n^j is obtained as follows:

$$D_n^j = (\underline{\underline{\alpha}}\,\alpha^n + \underline{\underline{\beta}}\,\beta^n)$$

□

Theorem 2.7. ***(Honsberger Identity)*** *For $n, m \geq 0$ the Honsberger identity for the dual Jacobsthal quaternions D_n^J and D_m^J is given by*

$$D_n^J D_m^J + 2D_{n-1}^J D_{m-1}^J = 2D_{n+m-1}^J - J_{n+m-1}. \tag{2.32}$$

Proof. (2.32):

$$\begin{aligned} D_n^J D_m^J = \; & J_n J_m + i(J_n J_{m+1} + J_{n+1} J_m) + j(J_n J_{m+2} + J_{n+2} J_m) \\ & + k(J_n J_{m+3} + J_{n+3} J_m) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} 2D_{n-1}^J D_{m-1}^J = \; & 2(J_{n-1} J_{m-1}) + 2i(J_{n-1} J_m + J_n J_{m-1}) \\ & + 2j(J_{n-1} J_{m+1} + J_{n+1} J_{m-1}) \\ & + 2k(J_{n-1} J_{m+2} + J_{n+2} J_{m-1}) \end{aligned} \tag{2.34}$$

Finally, adding equations (2.33) and (2.34) side by side, we obtain

$$\begin{aligned} D_n^J D_m^J + 2D_{n-1}^J D_{m-1}^J = \; & J_{n+m-1} + i(2J_{n+m}) \\ & + j(2J_{n+m+1}) + k(2J_{n+m+2}) \\ = \; & 2D_{n+m-1}^J - J_{n+m-1}. \end{aligned}$$

where the identity $J_{n+m} = J_m J_{n+1} + 2J_{m-1} J_n$ was used [27] and [28]. □

Theorem 2.8. ***D'ocagne's Identity*** *For $n, m \geq 0$ the D'ocagne's identity for the dual-complex Jacobsthal quaternions D_n^J and D_m^J is given by*

$$D_m^J D_{n+1}^J - D_{m+1}^J D_n^J = (-1)^n 2^n J_{m-n}(1 + i + 5j + 7k). \tag{2.35}$$

Proof. (2.35):

$$\begin{aligned} D_m^J D_{n+1}^J - D_{m+1}^J D_n^J = \; & [(J_m J_{n+1} - J_{m+1} J_n)] \\ & + i[(J_m J_{n+2} - J_{m+1} J_{n+1}) + (J_{m+1} J_{n+1} - J_{m+2} J_n)] \\ & + j[(J_m J_{n+3} - J_{m+1} J_{n+2}) + (J_{m+2} J_{n+1} - J_{m+3} J_n)] \\ & + k[(J_m J_{n+4} - J_{m+1} J_{n+3}) + (J_{m+3} J_{n+1} - J_{m+4} J_n)] \\ = \; & (-1)^n 2^n J_{m-n}(1 + i + 5j + 7k). \end{aligned}$$

where the identity $J_m J_{n+1} - J_{m+1} J_n = (-1)^n 2^n J_{m-n}$ was used [27] and [28].

□

Theorem 2.9. ***(Cassini's Identity)****. Let D_n^J and D_n^j be $n-th$ terms of dual Jacobsthal quaternion sequence (D_n^J) and the dual Jacobsthal-Lucas quaternion sequence (D_n^j), respectively. Then, we have*

$$D_{n-1}^J D_{n+1}^J - (D_n^J)^2 = (-1)^n 2^{n-1}(1 + i + 5j + 7k). \tag{2.36}$$

$$D_{n-1}^j D_{n+1}^j - (D_n^j)^2 = (-2)^{n-1} 3^2(1 + i + 5j + 7k). \tag{2.37}$$

Proof. (2.36):

$$\begin{aligned}
D^J_{n-1} D^J_{n+1} - (D^J_n)^2 = &\quad (J_{n-1}J_{n+1} - J^2_n) \\
&+i\,(J_{n-1}J_{n+2} - J_nJ_{n+1}) \\
&+j\,(J_{n-1}J_{n+3} - 2J_nJ_{n+2} + J^2_{n+1}) \\
&+k\,(J_{n-1}J_{n+4} + J_{n+1}J_{n+2} - 2J_nJ_{n+3}) \\
= &\quad (J_{n-1}J_{n+1} - J^2_n) \\
&+i\,(J_{n-1}J_{n+1} - J^2_n) \\
&+5j\,(J_{n-1}J_{n+1} - J^2_n) \\
&+7k\,(J_{n-1}J_{n+1} - J^2_n) \\
= &\quad (-1)^n\, 2^{n-1}(1+i+5j+7k).
\end{aligned}$$

and (2.37):

$$\begin{aligned}
D^j_{n-1} D^j_{n+1} - (D^j_n)^2 = &\quad (j_{n-1}j_{n+1} - j^2_n) \\
&+i\,(j_{n-1}j_{n+2} - j_nj_{n+1}) \\
&+j\,(j_{n-1}j_{n+3} - 2j_nj_{n+2} + j^2_{n+1}) \\
&+k\,(j_{n-1}j_{n+4} + j_{n+1}j_{n+2} - 2j_nj_{n+3}) \\
= &\quad (j_{n-1}j_{n+1} - j^2_n) \\
&+i\,(j_{n-1}j_{n+1} - j^2_n) \\
&+5j\,(j_{n-1}j_{n+1} - j^2_n) \\
&+7k\,(j_{n-1}j_{n+1} - j^2_n) \\
= &\quad (-2)^{n-1}\, 3^2(1+i+5j+7k).
\end{aligned}$$

where identities of Jacobsthal numbers and Jacobsthal-Lucas numbers as follows:

$$\begin{aligned}
J_m J_{n-1} - J_{m-1} J_n &= (-1)^n\, 2^{n-1} J_{m-n}, \quad J_{n+2} = J_{n+1} + 2J_n \\
j_m j_{n-1} - j_{m-1} j_n &= (-2)^{n-1}\, 3^2\, j_{m-n}, \quad j_{n+2} = j_{n+1} + 2j_n.
\end{aligned}$$

were used [27] and [28]. □

Theorem 2.10. ***(Catalan's Identity)****. Let D^J_n and D^j_n be $n-th$ terms of dual Jacobsthal quaternion sequence (D^J_n) and the dual Jacobsthal-Lucas quaternion sequence (D^j_n), respectively. Then, we have*

$$D^J_{n+r} D^J_{n-r} - (D^J_n)^2 = -(-2)^{n-r}\, J^2_r\,(1+i+5j+7k). \tag{2.38}$$

$$D^j_{n+r} D^j_{n-r} - (D^j_n)^2 = -(-2)^{n-r}\, 3^2\, j^2_r\,(1+i+5j+7k). \tag{2.39}$$

Proof. (2.38):

$$\begin{aligned}
D^J_{n+r} D^J_{n-r} - (D^J_n)^2 = &\quad (J_{n+r}J_{n-r} - J^2_n) \\
&+i\,[(J_{n+r}J_{n-r+1} - J_nJ_{n+1}) \\
&+(J_{n+r+1}J_{n-r} - J_{n+1}J_n) \\
&+j\,[(J_{n+r}J_{n-r+2} - J_nJ_{n+2}) \\
&+(J_{n+r+2}J_{n-r} - J_{n+2}J_n)] \\
&+k\,[(J_{n+r}J_{n-r+3} - J_nJ_{n+3}) \\
&+(J_{n+r+3}J_{n-r} - J_{n+3}J_n)] \\
= &\quad -(-2)^{n-r}\, J^2_r\,(1+i+5j+7k).
\end{aligned}$$

and (2.39):

$$\begin{aligned}
D^j_{n+r} D^j_{n-r} - (D^j_n)^2 = &\quad (j_{n+r}j_{n-r} - j^2_n) \\
&+i\,[(j_{n+r}j_{n-r+1} - j_nj_{n+1}) + (j_{n+r+1}j_{n-r} - j_{n+1}j_n)] \\
&+j\,[(j_{n+r}j_{n-r+2} - j_nj_{n+2}) + (j_{n+r+2}j_{n-r} - j_{n+2}j_n)] \\
&+k\,[(j_{n+r}j_{n-r+3} - j_{n+}j_{n+3}) + (j_{n+r+3}j_{n-r} - j_{n+3}j_n)] \\
= &\quad -(-2)^{n-r}\, 3^2\, J^2_r\,(1+i+5j+7k).
\end{aligned}$$

where identities of Jacobsthal numbers and Jacobsthal-Lucas numbers as follows:

$$J_{n+r}J_{n-r} - J_nJ_n = -(-2)^{n-r}J_r^2$$
$$j_{n+r}j_{n-r} - j_nj_n = (-2)^{n-r}3^2J_r^2.$$

were used [29]. □

3. Conclusion

The difference between the Jacobsthal and the dual Jacobsthal quaternions originated from the quaternionic units, i.e., the quaternionic units for the Jacobsthal quaternion are

$$i^2 = j^2 = k^2 = -1,\ \ i\,j = -j\,i = k,\ \ j\,k = -k\,j = i,\ \ \ k\,i = -i\,k = j$$

whereas for the dual Jacobsthal quaternions they are

$$i^2 = j^2 = k^2 = i\,jk = 0,\ \ i\,j = -j\,i = jk = -k\,j = ki = -ik = 0.$$

The set J_D forms a commutative ring under the dual Jacobsthal quaternion multiplication and also it is a vector space of dimensions four on R and its basis is the set $\{1, i, j, k\}$. The interesting property of dual Jacobsthal quaternions is that by their means one can express the Galilean transformation in one quaternion equation. Since the multiplication and ratio of two dual Jacobsthal quaternions $D_n^{J_1}$ and $D_n^{J_2}$ is again a dual Jacobsthal quaternion, the set of dual Jacobsthal quaternions form a division algebra under addition and multiplication. There have been several studies on curve theory and magnetism by using the isomorphism between dual quaternion space and Galilean space G^4. Similar applications for dual Jacobsthal and dual Jacobsthal-Lucas quaternions can be applied to these areas.
Galilean transformation expressed by the dual four-component numbers shows the linkage between the space and time exists in the Newtonian physics. Moreover, it may have a considerable heuristic value for the study of the underlying mathematical formalism of physical laws. This study fills the gap in the literature by providing dual Jacobsthal quaternions.

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On the Trigonometric and p-Trigonometric Functions of Elliptical Complex Variables

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Abstract
In the early 2000s, the geometry of a one-parameter family of generalized complex number systems was studied (Math. Mag. **77**(2)(2004)). This family is denoted by $\mathbb{C}_p$. It is well known that $\mathbb{C}_p$ matches up with the elliptical complex number system when p is any negative real number. By using this system, Özen and Tosun expressed the elliptical complex valued trigonometric functions cosine, sine and p-trigonometric functions p-cosine, p-sine (Adv. Appl. Clifford Algebras **28**(3)(2018)). In this study, we introduce the remained elliptical complex valued trigonometric and p-trigonometric functions. Also we define the corresponding single-valued principal values of the inverse trigonometric and p-trigonometric functions by following the similar steps given in the literature.

Keywords: Generalized complex numbers, p-trigonometric functions, Elliptical complex numbers.
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1. Introduction

The generalized complex numbers were introduced by Yaglom [1] as in the following:

$$z = x + Iy \quad (x, y \in \mathbb{R}), \quad I^2 = Iq + p \quad (q, p \in \mathbb{R})$$

where I denotes a formal quantity which is subject to the relation indicated above.

In [2], Harkins studied the geometry of a one parameter family of generalized complex number systems. In this one parameter family, $q = 0$ and $I^2 = p \in \mathbb{R}$. It is denoted by

$$\mathbb{C}_p = \left\{ x + Iy \ : \ x, y \in \mathbb{R}, \, I^2 = p, \ p \in \mathbb{R} \right\}.$$

In the special case $p < 0$, $\mathbb{C}_p$ corresponds to the set of elliptical complex numbers. Let this set be denoted by $\mathbb{C}_p{}^*$. That is,

$$\mathbb{C}_p{}^* = \left\{ x + Iy \ : \ x, y \in \mathbb{R}, \, I^2 = p, \ p \in \mathbb{R}^- \right\}.$$

For $z_1 = (x_1 + Iy_1)$, $z_2 = (x_2 + Iy_2) \in \mathbb{C}_p{}^*$, addition and multiplication are defined by

$$\begin{aligned} z_1 + z_2 &= (x_1 + Iy_1) + (x_2 + Iy_2) = (x_1 + x_2) + I(y_1 + y_2) \\ z_1 z_2 &= (x_1 x_2 + p y_1 y_2) + I(x_1 y_2 + x_2 y_1). \end{aligned}$$

As it is well known, $\mathbb{C}_p{}^*$ is a field under these two operations [2].

On the other hand, the p-magnitude of $z = x + Iy \in \mathbb{C}_p{}^*$ is $\|z\|_p = \sqrt{x^2 - py^2}$. As a result of this case, the unit circle in $\mathbb{C}_p{}^*$ is an Euclidean ellipse which is given by the equation $x^2 - py^2 = 1$. Specially, if $p = -1$ this ellipse matches the Euclidean unit circle [2].

Let $z = x + Iy$ be a number in $\mathbb{C}_p{}^*$. This number can be expressed with a position vector (see Figure 1.1). The arc of ellipse between this vector and the real axis determines an elliptic angle θ_p. This angle is called p-argument of z.

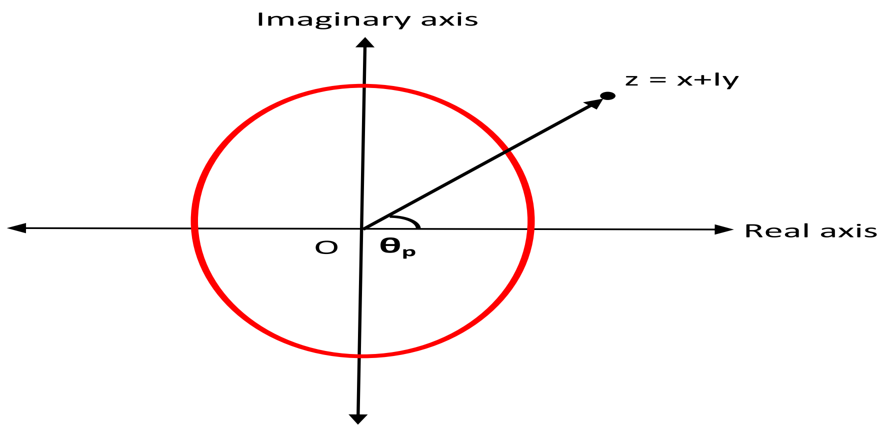

Figure 1.1. Elliptic angle in $\mathbb{C}_p{}^*$

On the other hand, the p-trigonometric functions p-cosine, p-sine and p-tangent are defined in $\mathbb{C}_p{}^*$ as follows [2]:

$$\cos_p(\theta_p) = \cos\left(\theta_p\sqrt{|p|}\right) \tag{1.1}$$

$$\sin_p(\theta_p) = \frac{1}{\sqrt{|p|}}\sin\left(\theta_p\sqrt{|p|}\right) \tag{1.2}$$

$$\tan_p(\theta_p) = \frac{\sin_p(\theta_p)}{\cos_p(\theta_p)}. \tag{1.3}$$

There can be found some interesting studies [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] on the generalized complex numbers and elliptical complex numbers in the literature.

Recently, Özen and Tosun have extended the trigonometric functions cosine, sine and p-trigonometric functions p-cosine, p-sine to the elliptical complex variables [3]. The functions cos, sin, $\cos_p$ and $\sin_p$ of an elliptical complex variable $\varphi_p = x + Iy \in \mathbb{C}_p$ are given as in the following

$$\cos(\varphi_p) = \cos(x)\cosh\left(y\sqrt{|p|}\right) - I\frac{1}{\sqrt{|p|}}\sin(x)\sinh\left(y\sqrt{|p|}\right) \tag{1.4}$$

$$\sin(\varphi_p) = \sin(x)\cosh\left(y\sqrt{|p|}\right) + I\frac{1}{\sqrt{|p|}}\cos(x)\sinh\left(y\sqrt{|p|}\right) \tag{1.5}$$

$$\cos_p(\varphi_p) = \cos_p(x)\cosh(py) + I\sin_p(x)\sinh(py) \tag{1.6}$$

$$\sin_p(\varphi_p) = \sin_p(x)\cosh(py) + I\frac{1}{p}\cos_p(x)\sinh(py) \tag{1.7}$$

in which case, φ_p is called elliptical complex angle. Also, these functions hold the following relations [3]:

$$\cos_p(\varphi_p) = \cos\left(\varphi_p\sqrt{|p|}\right)$$

$$\sin_p(\varphi_p) = \frac{1}{\sqrt{|p|}}\sin\left(\varphi_p\sqrt{|p|}\right).$$

Let the set of generalized complex numbers be showed with $\mathbb{C}_G$ in the case $I^2 = -q - rI\left(r^2 - 4q < 0\right)$. Thanks to Yaglom [1], it is known that there is an isomorphism between the set $\mathbb{C}_G$ and the set $\mathbb{C}$ as in the following:

$$\pi : \mathbb{C}_G \to \mathbb{C}$$

$$a_1 + b_1 I \to \pi(a_1 + b_1 I) = \left(a_1 - \frac{r}{2}b_1\right) + \left(\frac{b_1}{2}\sqrt{4q - r^2}\right)i.$$

If this isomorphism is restricted to the set of elliptical complex numbers, the following isomorphism

$$\begin{aligned} \pi^* \ : \ \mathbb{C}_p{}^* \ &\rightarrow \ \mathbb{C} \\ a_1 + b_1 I \rightarrow \ &\pi^*(a_1 + b_1 I) = a_1 + ib_1\sqrt{|p|} \end{aligned}$$

is immediately written by considering $r = 0$ and $q = -p$. Here the statement $\sqrt{|p|}$ represents the positive square root of the positive number $|p|$. Throughout the paper the statement $\sqrt{|p|}$ will be used in this sense.

Theorem 1.1. *[13] For the elliptical complex valued sine and cosine functions, the equalities*

1. $\sin(\pi^*(\varphi_p)) = \pi^*(\sin(\varphi_p))$
2. $\cos(\pi^*(\varphi_p)) = \pi^*(\cos(\varphi_p))$

are satisfied where $\varphi_p = x + Iy \in \mathbb{C}_p{}^*$.

The next two theorems, which reveal that the elliptical complex valued p-trigonometric functions $\cos_p(\varphi_p)$ and $\sin_p(\varphi_p)$ are surjective, can be given as consequences of the last theorem.

Theorem 1.2. *[3] For any elliptical complex number* $\psi_p = a + Ib \in \mathbb{C}_p{}^*$*, the equality* $\cos_p\left(\lambda_p^k\right) = \psi_p$ *is satisfied by the elliptical complex angles*

$$\lambda_p^k = \frac{Arg(u_k + iv_k)}{\sqrt{|p|}} + I\frac{\ln|u_k + iv_k|}{p}, \ \ k = 1,2$$

where $u_1 + iv_1, u_2 + iv_2 \in \mathbb{C}$ *are the complex numbers derived from the expression* $\left(a + ib\sqrt{|p|} + \sqrt{\left(a + ib\sqrt{|p|}\right)^2 - 1}\right)$.

Theorem 1.3. *[13] For any elliptical complex number* $\psi_p = a + Ib \in \mathbb{C}_p{}^*$*, the equality* $\sin_p\left(\chi_p^k\right) = \psi_p$ *is satisfied by the elliptical complex angles*

$$\chi_p^k = \frac{Arg(\varsigma_k + i\tau_k)}{\sqrt{|p|}} + I\frac{\ln|\varsigma_k + i\tau_k|}{p}, \ \ k = 1,2$$

where $\varsigma_1 + i\tau_1, \varsigma_2 + i\tau_2 \in \mathbb{C}$ *are complex numbers derived from the expression* $\left(i\left(a\sqrt{|p|} + ib|p|\right) + \sqrt{1 - \left(a\sqrt{|p|} + ib|p|\right)^2}\right)$.

Note that the last three theorems will be used to obtain single-valued principal values of the inverse cosine, sine, p-cosine and p-sine functions in Section 2.

Finally, we need to emphasize the principal square root of a complex number. Let $z = re^{i\varphi}$ be a complex number given by principal argument $-\pi < \varphi \leq \pi$ in the polar form. As it is well-known in the literature, the principal square root of z is defined as $\sqrt{z} = \sqrt{r}e^{i\frac{\varphi}{2}}$, $-\frac{\pi}{2} < \frac{\varphi}{2} \leq \frac{\pi}{2}$. We will use the statement "principle square root" in this sense throughout the rest of the paper.

2. Main Results

In this section, we obtain the elliptical complex valued tangent, cotangent, secant and cosecant functions. Then we define the corresponding single-valued principal values of the all inverse trigonometric functions by following the similar steps in [14]. Finally, we will repeat the same for $p-$trigonometric functions.

2.1 Results Related to Elliptical Complex-Valued Trigonometric Functions

In this subsection, firstly, we can give the following theorem by using the equations (1.4) and (1.5).

Theorem 2.1. *Tangent, cotangent, secant and cosecant functions of an elliptical complex variable* $\varphi_p = x + Iy \in \mathbb{C}_p{}^*$ *are given as in the following:*

1. $\tan(\varphi_p) = \frac{\sin(\varphi_p)}{\cos(\varphi_p)} = \frac{\sin(2x)}{\cos(2x) + \cosh\left(2y\sqrt{|p|}\right)} + I\frac{1}{\sqrt{|p|}}\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x) + \cosh\left(2y\sqrt{|p|}\right)}$,

2. $\cot(\varphi_p) = \frac{\cos(\varphi_p)}{\sin(\varphi_p)} = \frac{\sin(2x)\left(\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)\right)}{\sin^2(2x)+\sinh^2\left(2y\sqrt{|p|}\right)} - I\frac{1}{\sqrt{|p|}}\frac{\sinh\left(2y\sqrt{|p|}\right)\left(\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)\right)}{\sin^2(2x)+\sinh^2\left(2y\sqrt{|p|}\right)}$,

3. $\sec(\varphi_p) = \frac{1}{\cos(\varphi_p)} = \frac{2\cos(x)\cosh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)} + I\frac{2}{\sqrt{|p|}}\frac{\sin(x)\sinh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}$,

4. $\csc(\varphi_p) = \frac{1}{\sin(\varphi_p)} = \frac{2\sin(x)\cosh\left(y\sqrt{|p|}\right)}{\cosh\left(2y\sqrt{|p|}\right)-\cos(2x)} - I\frac{2}{\sqrt{|p|}}\frac{\cos(x)\sinh\left(y\sqrt{|p|}\right)}{\cosh\left(2y\sqrt{|p|}\right)-\cos(2x)}$.

Proof. We will prove the first item. The proofs of other items can be similarly completed.

1. By considering $|p| = -p$ and using some well-known trigonometric and hyperbolic identities, we get

$$\begin{aligned}
\tan(\varphi_p) &= \frac{\sin(\varphi_p)}{\cos(\varphi_p)} \\
&= \frac{\sin(x)\cosh\left(y\sqrt{|p|}\right) + I\frac{1}{\sqrt{|p|}}\cos(x)\sinh\left(y\sqrt{|p|}\right)}{\cos(x)\cosh\left(y\sqrt{|p|}\right) - I\frac{1}{\sqrt{|p|}}\sin(x)\sinh\left(y\sqrt{|p|}\right)} \\
&= \frac{\sin(x)\cos(x)\left(\cosh^2\left(y\sqrt{|p|}\right) - \sinh^2\left(y\sqrt{|p|}\right)\right)}{\cos^2(x)\cosh^2\left(y\sqrt{|p|}\right) + \sin^2(x)\sinh^2\left(y\sqrt{|p|}\right)} + \frac{I}{\sqrt{|p|}}\frac{\sinh\left(y\sqrt{|p|}\right)\cosh\left(y\sqrt{|p|}\right)\left(\cos^2(x)+\sin^2(x)\right)}{\cos^2(x)\cosh^2\left(y\sqrt{|p|}\right) + \sin^2(x)\sinh^2\left(y\sqrt{|p|}\right)} \\
&= \frac{2}{2}\frac{\sin(x)\cos(x)}{\cos^2(x)\cosh^2\left(y\sqrt{|p|}\right) + \sin^2(x)\sinh^2\left(y\sqrt{|p|}\right)} + \frac{I}{\sqrt{|p|}}\frac{2}{2}\frac{\sinh\left(y\sqrt{|p|}\right)\cosh\left(y\sqrt{|p|}\right)}{\cos^2(x)\cosh^2\left(y\sqrt{|p|}\right) + \sin^2(x)\sinh^2\left(y\sqrt{|p|}\right)} \\
&= \frac{\sin(2x)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)} + I\frac{1}{\sqrt{|p|}}\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}.
\end{aligned}$$

□

Lemma 2.2. *For the elliptical complex valued tangent, cotangent, secant and cosecant functions, the equalities*

1. $\tan(\pi^*(\varphi_p)) = \pi^*(\tan(\varphi_p))$,
2. $\cot(\pi^*(\varphi_p)) = \pi^*(\cot(\varphi_p))$,
3. $\sec(\pi^*(\varphi_p)) = \pi^*(\sec(\varphi_p))$,
4. $\csc(\pi^*(\varphi_p)) = \pi^*(\csc(\varphi_p))$.

are satisfied where π^* is the aforesaid isomorphism and $\varphi_p = x + Iy \in \mathbb{C}_p{}^*$.

Proof. We will prove the first and third item. Other items can be similarly proved.

1. It is very easy to see

$$\begin{aligned}
\pi^*(\tan(\varphi_p)) &= \pi^*\left(\left[\frac{\sin(2x)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right] + I\frac{1}{\sqrt{|p|}}\left[\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right]\right) \\
&= \left[\frac{\sin(2x)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right] + i\frac{1}{\sqrt{|p|}}\sqrt{|p|}\left[\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right] \\
&= \left[\frac{\sin(2x)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right] + i\left[\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right].
\end{aligned}$$

On the other hand, according to the theory of complex trigonometric functions (see [14, 15] for more details on the theory of complex trigonometric functions), it is clear that

$$\begin{aligned}\tan\left(\pi^*\left(\varphi_p\right)\right) &= \tan\left(x+iy\sqrt{|p|}\right)\\ &= \left[\frac{\sin(2x)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right]+i\left[\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right].\end{aligned}$$

So, the proof is completed.

3. Similarly above, we have the equalities

$$\begin{aligned}\pi^*\left(\sec\left(\varphi_p\right)\right) &= \pi^*\left(\left[\frac{2\cos(x)\cosh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right]+I\frac{2}{\sqrt{|p|}}\left[\frac{\sin(x)\sinh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right]\right)\\ &= \left[\frac{2\cos(x)\cosh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right]+i\frac{2}{\sqrt{|p|}}\sqrt{|p|}\left[\frac{\sin(x)\sinh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right]\\ &= \left[\frac{2\cos(x)\cosh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right]+i\left[\frac{2\sin(x)\sinh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right].\end{aligned}$$

and

$$\begin{aligned}\sec\left(\pi^*\left(\varphi_p\right)\right) &= \sec\left(x+iy\sqrt{|p|}\right)\\ &= \left[\frac{2\cos(x)\cosh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right]+i\left[\frac{2\sin(x)\sinh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right].\end{aligned}$$

Thus the desired equality holds.

□

Theorem 2.3. *For any elliptical complex number $\psi_p = a + Ib \in \mathbb{C}_p{}^*$, the equalities $\sin\varphi_p = \psi_p$, $\cos\alpha_p = \psi_p$, $\tan\beta_p = \psi_p$, $\cot\gamma_p = \psi_p$, $\sec\theta_p = \psi_p$ and $\csc\delta_p = \psi_p$ are satisfied by the principal elliptical complex angles*

1. $\varphi_p = Arg\left(\sigma + i\omega\right) - I\frac{\ln|\sigma+i\omega|}{\sqrt{|p|}}$,

2. $\alpha_p = Arg\left(\varepsilon + i\kappa\right) - I\frac{\ln|\varepsilon+i\kappa|}{\sqrt{|p|}}$,

3. $\beta_p = \dfrac{Arg\left(\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2} - i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right)}{-2} + I\dfrac{\ln\left|\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2} - i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right|}{2\sqrt{|p|}}$,

4. $\gamma_p = \dfrac{Arg\left(\frac{-1-pb^2+a^2}{1+2b\sqrt{|p|}+a^2-pb^2} - i\frac{2a}{1+2b\sqrt{|p|}+a^2-pb^2}\right)}{-2} + I\dfrac{\ln\left|\frac{-1-pb^2+a^2}{1+2b\sqrt{|p|}+a^2-pb^2} - i\frac{2a}{1+2b\sqrt{|p|}+a^2-pb^2}\right|}{2\sqrt{|p|}}$,

5. $\theta_p = Arg\left(\eta + i\zeta\right) - I\frac{\ln|\eta+i\zeta|}{\sqrt{|p|}}$,

6. $\delta_p = Arg\left(\Omega + i\mho\right) - I\frac{\ln|\Omega+i\mho|}{\sqrt{|p|}}$,

where $\sigma+i\omega\in\mathbb{C}$, $\varepsilon+i\kappa\in\mathbb{C}$, $\eta+i\zeta\in\mathbb{C}$ and $\Omega+i\mho\in\mathbb{C}$ are the principal complex values derived from the expressions $\left(i\left(a+ib\sqrt{|p|}\right)+\sqrt{1-\left(a+ib\sqrt{|p|}\right)^2}\right)$, $\left(a+ib\sqrt{|p|}+\sqrt{\left(a+ib\sqrt{|p|}\right)^2-1}\right)$, $\left(\frac{1}{a+ib\sqrt{|p|}}+\sqrt{\frac{1}{\left(a+ib\sqrt{|p|}\right)^2}-1}\right)$ *and* $\left(\sqrt{1-\frac{1}{\left(a+ib\sqrt{|p|}\right)^2}}+\frac{i}{a+ib\sqrt{|p|}}\right)$*, respectively.*

Proof. Now, we will show that the first and third equalities are satisfied. Similar steps can be followed for the other equalities.

1. By considering Theorem 1.1 and the theory of complex trigonometric functions (see [14, 15] for more details), we can write

$$\begin{aligned}
\sin(x+Iy)=a+Ib \quad &\Leftrightarrow \quad \pi^*(\sin(x+Iy))=\pi^*(a+Ib)\\
&\Leftrightarrow \quad \sin(\pi^*(x+Iy))=\pi^*(a+Ib)\\
&\Leftrightarrow \quad \sin\left(x+iy\sqrt{|p|}\right)=a+ib\sqrt{|p|}\\
&\Leftrightarrow \quad \arcsin\left(a+ib\sqrt{|p|}\right)=x+iy\sqrt{|p|}\\
&\Leftrightarrow \quad -i\log\left(i\left(a+ib\sqrt{|p|}\right)+\sqrt{1-\left(a+ib\sqrt{|p|}\right)^2}\right)=x+iy\sqrt{|p|}.
\end{aligned}$$

The purpose of us is to get unique solutions for x and y. To do so, we use the principal value of arcsine function. It is determined by employing the principal value of the logarithm function and the principal value of the square-root function. By keeping these situations in mind, let us denote by $\sigma+i\omega$ the principal complex value derived from the expression $\left(i\left(a+ib\sqrt{|p|}\right)+\sqrt{1-\left(a+ib\sqrt{|p|}\right)^2}\right)$. Then we have

$$-i\,\mathrm{Log}(\sigma+i\omega)=x+iy\sqrt{|p|}.$$

This equation yields the followings

$$\begin{aligned}
-i\left(\ln|\sigma+i\omega|+iArg(\sigma+i\omega)\right) &= x+iy\sqrt{|p|},\\
Arg(\sigma+i\omega)-i\ln|\sigma+i\omega| &= x+iy\sqrt{|p|}.
\end{aligned}$$

Then we get the unique solutions for x and y as

$$x=Arg(\sigma+i\omega),\ y=-\frac{\ln|\sigma+i\omega|}{\sqrt{|p|}}.$$

Thus, we can conclude

$$\varphi_p=Arg(\sigma+i\omega)-I\frac{\ln|\sigma+i\omega|}{\sqrt{|p|}}.$$

3. Similarly above, we can write

$$\begin{aligned}
\tan(x+Iy)=a+Ib \quad &\Leftrightarrow \quad \pi^*(\tan(x+Iy))=\pi^*(a+Ib)\\
&\Leftrightarrow \quad \tan(\pi^*(x+Iy))=\pi^*(a+Ib)\\
&\Leftrightarrow \quad \tan\left(x+iy\sqrt{|p|}\right)=a+ib\sqrt{|p|}\\
&\Leftrightarrow \quad \arctan\left(a+ib\sqrt{|p|}\right)=x+iy\sqrt{|p|}\\
&\Leftrightarrow \quad \frac{i}{2}\log\left(\frac{i+\left(a+ib\sqrt{|p|}\right)}{i-\left(a+ib\sqrt{|p|}\right)}\right)=x+iy\sqrt{|p|}.
\end{aligned}$$

We aim to obtain the unique solutions for x and y. To do so, if we use the principal value of arctangent function which is determined by employing the principal value of the logarithm function, we have

$$\frac{i}{2}\operatorname{Log}\left(\frac{a+i\left(1+b\sqrt{|p|}\right)}{-a+i\left(1-b\sqrt{|p|}\right)}\right)=x+iy\sqrt{|p|}.$$

This equation yields the followings

$$\begin{aligned}\frac{i}{2}\left(\ln\left|\frac{a+i\left(1+b\sqrt{|p|}\right)}{-a+i\left(1-b\sqrt{|p|}\right)}\right|+iArg\left(\frac{a+i\left(1+b\sqrt{|p|}\right)}{-a+i\left(1-b\sqrt{|p|}\right)}\right)\right) &= x+iy\sqrt{|p|},\\ \frac{Arg\left(\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right)}{-2}+i\frac{\ln\left|\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right|}{2} &= x+iy\sqrt{|p|}.\end{aligned}$$

In this case, we obtain the unique solutions for x and y as follows

$$x=\frac{Arg\left(\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right)}{-2},\quad y=\frac{\ln\left|\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right|}{2\sqrt{|p|}}.$$

Therefore, we can conclude

$$\beta_p=\frac{Arg\left(\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right)}{-2}+I\frac{\ln\left|\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right|}{2\sqrt{|p|}}.\quad\square$$

By taking into consideration Theorem 2.3, we can give the following corollary.

Corollary 2.4. *For any elliptical complex number $\psi_p=a+Ib\in\mathbb{C}_p{}^*$, the principal values of the inverse trigonometric functions:*

$$\begin{aligned}\operatorname{Arcsin}(\psi_p) &= \varphi_p\\ \operatorname{Arccos}(\psi_p) &= \alpha_p\\ \operatorname{Arctan}(\psi_p) &= \beta_p\\ \operatorname{Arccot}(\psi_p) &= \gamma_p\\ \operatorname{Arcsec}(\psi_p) &= \theta_p\\ \operatorname{Arccsc}(\psi_p) &= \delta_p\end{aligned}$$

can be expressed.

2.2 Results Related to Elliptical Complex-Valued p-Trigonometric Functions

In this subsection, firstly, let us define the elliptical complex valued p-trigonometric functions:

$$\frac{\sin_p(\varphi_p)}{\cos_p(\varphi_p)}=\tan_p(\varphi_p),\quad \frac{\cos_p(\varphi_p)}{\sin_p(\varphi_p)}=\cot_p(\varphi_p),\quad \frac{1}{\cos_p(\varphi_p)}=\sec_p(\varphi_p),\quad \frac{1}{\sin_p(\varphi_p)}=\csc_p(\varphi_p)$$

by means of the elliptical complex valued p-trigonometric functions

$$\cos_p(\varphi_p)=\cos\left(\varphi_p\sqrt{|p|}\right)=\cos_p(x)\cosh(py)+I\sin_p(x)\sinh(py)$$

and

$$\sin_p(\varphi_p)=\frac{1}{\sqrt{|p|}}\sin\left(\varphi_p\sqrt{|p|}\right)=\sin_p(x)\cosh(py)+I\frac{1}{p}\cos_p(x)\sinh(py)$$

given in (1.6) and (1.7).

As mentioned earlier in Section 1, real-valued p-trigonometric functions p-cosine, p-sine and p-tangent are defined in [2]. There is no such definition for neither cotangent function, secant function nor cosecant function. While the elliptical complex-valued functions $\cos_p(\varphi_p)$, $\sin_p(\varphi_p)$ and $\tan_p(\varphi_p)$ are extensions of real-valued functions p-cosine, p-sine and p-tangent, we can not say the same for the elliptical complex-valued functions $\cot_p(\varphi_p)$, $\sec_p(\varphi_p)$ and $\csc_p(\varphi_p)$. So, to use the notations $\cos_p(\varphi_p)$, $\sin_p(\varphi_p)$, $\tan_p(\varphi_p)$ and to use the statement "p-trigonometric function" are very natural for these functions. But, the reason of maintaining this situation for other functions $\cot_p(\varphi_p)$, $\sec_p(\varphi_p)$ and $\csc_p(\varphi_p)$ is not obvious. This reason is based on the relationships of these functions with the elliptical complex-valued trigonometric functions cotangent, secant and cosecant. Now, we give the next theorem including these relationships.

Theorem 2.5. *For any elliptical complex angle $\varphi_p = x + Iy \in \mathbb{C}_p{}^*$, the following equalities hold:*

1. $\tan_p(\varphi_p) = \frac{1}{\sqrt{|p|}}\tan\left(\varphi_p\sqrt{|p|}\right) = \frac{\sin_p(2x)}{\cos_p(2x)+\cosh(2yp)} + I\frac{\sinh(2yp)}{p\cos_p(2x)+p\cosh(2yp)}$,
2. $\cot_p(\varphi_p) = \sqrt{|p|}\cot\left(\varphi_p\sqrt{|p|}\right) = \frac{\sin_p(2x)\left(\cos_p(2x)+\cosh(2yp)\right)}{\left(\sin_p{}^2(2x)-\frac{1}{p}\sinh^2(2yp)\right)} + I\frac{\sinh(2yp)\left(\cos_p(2x)+\cosh(2yp)\right)}{\left(\sinh^2(2yp)-p\sin_p{}^2(2x)\right)}$,
3. $\sec_p(\varphi_p) = \frac{1}{\sqrt{|p|}}\sec\left(\varphi_p\sqrt{|p|}\right) = \frac{2\cos_p(x)\cosh(yp)}{\cos_p(2x)+\cosh(2yp)} - I\frac{2\sin_p(x)\sinh(yp)}{\cos_p(2x)+\cosh(2yp)}$,
4. $\csc_p(\varphi_p) = \sqrt{|p|}\csc\left(\varphi_p\sqrt{|p|}\right) = \frac{-2p\sin_p(x)\cosh(yp)}{\cosh(2yp)-\cos_p(2x)} + I\frac{2\cos_p(x)\sinh(yp)}{\cosh(2yp)-\cos_p(2x)}$.

Proof. We will prove the second and last items. The other items can be proved similarly.

2. It is easy to see the equality

$$\cot_p(\varphi_p) = \frac{\cos_p(\varphi_p)}{\sin_p(\varphi_p)} = \frac{\cos\left(\varphi_p\sqrt{|p|}\right)}{\frac{1}{\sqrt{|p|}}\sin\left(\varphi_p\sqrt{|p|}\right)} = \sqrt{|p|}\cot\left(\varphi_p\sqrt{|p|}\right).$$

On the other hand, since $\varphi_p\sqrt{|p|} = x\sqrt{|p|} + Iy\sqrt{|p|}$,

$$\begin{aligned}
\sqrt{|p|}\cot\left(\varphi_p\sqrt{|p|}\right) &= \sqrt{|p|}\left[\frac{\sin\left(2x\sqrt{|p|}\right)\left(\cos\left(2x\sqrt{|p|}\right)+\cosh(2y|p|)\right)}{\sin^2\left(2x\sqrt{|p|}\right)+\sinh^2(2y|p|)} - \frac{I}{\sqrt{|p|}}\frac{\sinh(2y|p|)\left(\cos\left(2x\sqrt{|p|}\right)+\cosh(2y|p|)\right)}{\sin^2\left(2x\sqrt{|p|}\right)+\sinh^2(2y|p|)}\right] \\
&= \frac{\frac{1}{\sqrt{|p|}}\sin\left(2x\sqrt{|p|}\right)\left(\cos\left(2x\sqrt{|p|}\right)+\cosh(2y|p|)\right)}{\frac{1}{\left(\sqrt{|p|}\right)^2}\left(\sin^2\left(2x\sqrt{|p|}\right)+\sinh^2(2y|p|)\right)} - \frac{I}{|p|}\frac{\sinh(2y|p|)\left(\cos\left(2x\sqrt{|p|}\right)+\cosh(2y|p|)\right)}{\frac{1}{\left(\sqrt{|p|}\right)^2}\left(\sin^2\left(2x\sqrt{|p|}\right)+\sinh^2(2y|p|)\right)} \\
&= \frac{\sin_p(2x)\left(\cos_p(2x)+\cosh(2yp)\right)}{\left(\sin_p{}^2(2x)-\frac{1}{p}\sinh^2(2yp)\right)} + I\frac{\sinh(2yp)\left(\cos_p(2x)+\cosh(2yp)\right)}{\left(\sinh^2(2yp)-p\sin_p{}^2(2x)\right)}
\end{aligned}$$

can be written from the second item of Theorem 2.1. Then, we can immediately obtain the desired equality.

4. It is not difficult to find the equality

$$\csc_p(\varphi_p) = \frac{1}{\sin_p(\varphi_p)} = \frac{1}{\frac{1}{\sqrt{|p|}}\sin\left(\varphi_p\sqrt{|p|}\right)} = \sqrt{|p|}\frac{1}{\sin\left(\varphi_p\sqrt{|p|}\right)} = \sqrt{|p|}\csc\left(\varphi_p\sqrt{|p|}\right).$$

Also, from the fourth item of Theorem 2.1

$$\begin{aligned}
\sqrt{|p|}\csc\left(\varphi_p\sqrt{|p|}\right) &= \sqrt{|p|}\left[\frac{2\sin\left(x\sqrt{|p|}\right)\cosh(y|p|)}{\cosh(2y|p|)-\cos\left(2x\sqrt{|p|}\right)} - \frac{2I}{\sqrt{|p|}}\frac{\cos\left(x\sqrt{|p|}\right)\sinh(y|p|)}{\cosh(2y|p|)-\cos\left(2x\sqrt{|p|}\right)}\right] \\
&= \frac{2|p|\frac{1}{\sqrt{|p|}}\sin\left(x\sqrt{|p|}\right)\cosh(y|p|)}{\cosh(2y|p|)-\cos\left(2x\sqrt{|p|}\right)} - 2I\frac{\cos\left(x\sqrt{|p|}\right)\sinh(y|p|)}{\cosh(2y|p|)-\cos\left(2x\sqrt{|p|}\right)} \\
&= \frac{-2p\sin_p(x)\cosh(yp)}{\cosh(2yp)-\cos_p(2x)} + I\frac{2\cos_p(x)\sinh(yp)}{\cosh(2yp)-\cos_p(2x)}
\end{aligned}$$

can be written by keeping $\varphi_p\sqrt{|p|} = x\sqrt{|p|} + Iy\sqrt{|p|}$ in mind. From above, we immediately get

$$\csc_p(\varphi_p) = \sqrt{|p|}\csc\left(\varphi_p\sqrt{|p|}\right) = \frac{-2p\sin_p(x)\cosh(yp)}{\cosh(2yp) - \cos_p(2x)} + I\frac{2\cos_p(x)\sinh(yp)}{\cosh(2yp) - \cos_p(2x)}. \quad \square$$

Theorem 2.6. *For any elliptical complex number $\psi_p = a + Ib \in \mathbb{C}_p{}^*$, the equalities $\cos_p(\lambda_p) = \psi_p$, $\sin_p(\chi_p) = \psi_p$, $\tan_p(\Gamma_p) = \psi_p$, $\cot_p(\Lambda_p) = \psi_p$, $\sec_p(\Delta_p) = \psi_p$ and $\csc_p(\Upsilon_p) = \psi_p$ are satisfied by the principal elliptical complex angles*

1. $\lambda_p = \frac{Arg(u+iv)}{\sqrt{|p|}} + I\frac{\ln|u+iv|}{p}$,
2. $\chi_p = \frac{Arg(\varsigma+i\tau)}{\sqrt{|p|}} + I\frac{\ln|\varsigma+i\tau|}{p}$,
3. $\Gamma_p = \dfrac{Arg\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|} + i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)}{-2\sqrt{|p|}} + I\dfrac{\ln\left|\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|} + i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right|}{2|p|}$,
4. $\Lambda_p = \dfrac{Arg\left(\frac{-a^2+pb^2-p}{-a^2+pb^2+2bp+p} + i\frac{2a\sqrt{|p|}}{-a^2+pb^2+2bp+p}\right)}{-2\sqrt{|p|}} + I\dfrac{\ln\left|\frac{-a^2+pb^2-p}{-a^2+pb^2+2bp+p} + i\frac{2a\sqrt{|p|}}{-a^2+pb^2+2bp+p}\right|}{2|p|}$
5. $\Delta_p = \frac{Arg(c+id)}{\sqrt{|p|}} + I\,\frac{\ln|c+id|}{p}$,
6. $\Upsilon_p = \frac{Arg(e+if)}{\sqrt{|p|}} + I\,\frac{\ln|e+if|}{p}$,

where $u+iv \in \mathbb{C}$, $\varsigma + i\tau \in \mathbb{C}$, $c+id \in \mathbb{C}$ and $e+if \in \mathbb{C}$ are the principal complex values which are derived from the expressions $\left(a+ib\sqrt{|p|} + \sqrt{\left(a+ib\sqrt{|p|}\right)^2 - 1}\right)$, $\left(i\left(a\sqrt{|p|} + ib|p|\right) + \sqrt{1 - \left(a\sqrt{|p|} + ib|p|\right)^2}\right)$, $\left(\frac{1}{a+ib\sqrt{|p|}} + \sqrt{\frac{1}{\left(a+ib\sqrt{|p|}\right)^2} - 1}\right)$, $\left(\sqrt{1 - \frac{1}{\left(\frac{a}{\sqrt{|p|}}+ib\right)^2}} + \frac{i}{\left(\frac{a}{\sqrt{|p|}}+ib\right)}\right)$, *respectively.*

Proof. We will show that the first, third and last equalities are satisfied. Similar steps can be followed for the other equalities.

1. Let us take into consideration the principle value of $\sqrt{\left(a+ib\sqrt{|p|}\right)^2 - 1}$ and calculate the principle value of the statement $\left(a+ib\sqrt{|p|} + \sqrt{\left(a+ib\sqrt{|p|}\right)^2 - 1}\right)$. If we show this principle value with $u+iv$, Theorem 1.2 gives the proof of this item.

3. By considering Lemma 2.2 and the theory of complex trigonometric functions, we can write

$$\begin{aligned}
\tan_p(x+Iy) = a+Ib \quad &\Leftrightarrow \quad \frac{1}{\sqrt{|p|}}\tan\left(x\sqrt{|p|} + Iy\sqrt{|p|}\right) = a+Ib \\
&\Leftrightarrow \quad \tan\left(x\sqrt{|p|} + Iy\sqrt{|p|}\right) = a\sqrt{|p|} + Ib\sqrt{|p|} \\
&\Leftrightarrow \quad \pi^*\left(\tan\left(x\sqrt{|p|} + Iy\sqrt{|p|}\right)\right) = \pi^*\left(a\sqrt{|p|} + Ib\sqrt{|p|}\right) \\
&\Leftrightarrow \quad \tan\left(\pi^*\left(x\sqrt{|p|} + Iy\sqrt{|p|}\right)\right) = \pi^*\left(a\sqrt{|p|} + Ib\sqrt{|p|}\right) \\
&\Leftrightarrow \quad \tan\left(x\sqrt{|p|} + iy|p|\right) = a\sqrt{|p|} + ib|p| \\
&\Leftrightarrow \quad \arctan\left(a\sqrt{|p|} + ib|p|\right) = x\sqrt{|p|} + iy|p| \\
&\Leftrightarrow \quad \frac{i}{2}\log\left(\frac{i + \left(a\sqrt{|p|} + ib|p|\right)}{i - \left(a\sqrt{|p|} + ib|p|\right)}\right) = x\sqrt{|p|} + iy|p|.
\end{aligned}$$

To get unique solutions for x and y is our aim. To do so, if we use the principal value of arctangent function which is determined by employing the principal value of the logarithm function, we obtain

$$\frac{i}{2}\,\mathrm{Log}\left(\frac{a\sqrt{|p|}+i(1+b|p|)}{-a\sqrt{|p|}+i(1-b|p|)}\right)=x\sqrt{|p|}+iy|p|$$

and so

$$\frac{i}{2}\,\mathrm{Log}\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)=x\sqrt{|p|}+iy|p|.$$

From here, the equalities

$$\begin{aligned}\frac{i}{2}\left(\ln\left|\frac{\left(pa^2-p^2b^2+1\right)+i\left(-2a\sqrt{|p|}\right)}{-pa^2+p^2b^2+1-2b|p|}\right|+iArg\left(\frac{\left(pa^2-p^2b^2+1\right)+i\left(-2a\sqrt{|p|}\right)}{-pa^2+p^2b^2+1-2b|p|}\right)\right) &= x\sqrt{|p|}+iy|p| \\ \frac{Arg\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)}{-2}+i\frac{\ln\left|\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right|}{2} &= x\sqrt{|p|}+iy|p|\end{aligned}$$

can be written. Thus we find the unique solutions for x and y as follows

$$x=\frac{Arg\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)}{-2\sqrt{|p|}},\quad y=\frac{\ln\left|\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right|}{2|p|}.$$

Therefore,

$$\Gamma_p=\frac{Arg\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)}{-2\sqrt{|p|}}+I\frac{\ln\left|\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right|}{2|p|}$$

can be concluded.

6. By considering Lemma 2.2 and the theory of complex trigonometric functions, we can write

$$\begin{aligned}\csc_p(x+Iy)=a+Ib \quad &\Leftrightarrow\quad \sqrt{|p|}\csc\left(x\sqrt{|p|}+Iy\sqrt{|p|}\right)=a+Ib \\ &\Leftrightarrow\quad \csc\left(x\sqrt{|p|}+Iy\sqrt{|p|}\right)=\frac{a}{\sqrt{|p|}}+I\frac{b}{\sqrt{|p|}} \\ &\Leftrightarrow\quad \pi^*\left(\csc\left(x\sqrt{|p|}+Iy\sqrt{|p|}\right)\right)=\pi^*\left(\frac{a}{\sqrt{|p|}}+I\frac{b}{\sqrt{|p|}}\right) \\ &\Leftrightarrow\quad \csc\left(\pi^*\left(x\sqrt{|p|}+Iy\sqrt{|p|}\right)\right)=\pi^*\left(\frac{a}{\sqrt{|p|}}+I\frac{b}{\sqrt{|p|}}\right) \\ &\Leftrightarrow\quad \csc\left(x\sqrt{|p|}+iy|p|\right)=\frac{a}{\sqrt{|p|}}+ib \\ &\Leftrightarrow\quad \mathrm{arccsc}\left(\frac{a}{\sqrt{|p|}}+ib\right)=x\sqrt{|p|}+iy|p| \\ &\Leftrightarrow\quad -i\log\left(\sqrt{1-\frac{1}{\left(\frac{a}{\sqrt{|p|}}+ib\right)^2}}+\frac{i}{\left(\frac{a}{\sqrt{|p|}}+ib\right)}\right)=x\sqrt{|p|}+iy|p|.\end{aligned}$$

The aim of us is to obtain unique solutions for x and y. For this reason, we use the principal value of arccosecant function. It is determined by employing the principal value of the logarithm function and the principal value of square-root function. By considering these cases, let us denote by $e+if$ the principal complex value derived from the expression $\left(\sqrt{1-\frac{1}{\left(\frac{a}{\sqrt{|p|}}+ib\right)^2}}+\frac{i}{\left(\frac{a}{\sqrt{|p|}}+ib\right)}\right)$. In this case, we have

$$-i\,\mathrm{Log}\,(e+if)=x\sqrt{|p|}+iy\,|p|\,.$$

This equation yields the followings

$$\begin{aligned}-i\,(\ln|e+if|+iArg\,(e+if)) &= x\sqrt{|p|}+iy\,|p|\,,\\ Arg\,(e+if)-i\ln|e+if| &= x\sqrt{|p|}+iy\,|p|\,.\end{aligned}$$

Then we get the unique solutions for x and y as

$$x=\frac{Arg\,(e+if)}{\sqrt{|p|}},\quad y=\frac{\ln|e+if|}{p}.$$

Thus, we can conclude

$$\Upsilon_p=\frac{Arg\,(e+if)}{\sqrt{|p|}}+I\,\frac{\ln|e+if|}{p}.\quad \square$$

By taking into account of Theorem 2.6, the following corollary can be given.

Corollary 2.7. *For any elliptical complex number $\psi_p=a+Ib\in\mathbb{C}_p{}^*$, the principal values of the inverse p-trigonometric functions:*

$$\begin{aligned}\mathrm{Arccos_p}\,(\psi_p) &= \lambda_p\\ \mathrm{Arcsin_p}\,(\psi_p) &= \chi_p\\ \mathrm{Arctan_p}\,(\psi_p) &= \Gamma_p\\ \mathrm{Arccot_p}\,(\psi_p) &= \Lambda_p\\ \mathrm{Arcsec_p}\,(\psi_p) &= \Delta_p\\ \mathrm{Arccsc_p}\,(\psi_p) &= \Upsilon_p\end{aligned}$$

can be expressed.

3. Conclusion

In this paper, the trigonometric and $p-$trigonometric functions of elliptical complex variables are considered. Also, the corresponding single-valued principle values of the inverse trigonometric and $p-$trigonometric functions are defined.

In the case $p=-1$, elliptical complex numbers correspond to complex numbers. As a result of this case, the elliptical complex valued trigonometric functions can be seen as generalized form of the complex valued trigonometric functions which have important roles in many areas of science.

In the future, the results obtained here may be used as a valuable tool in many areas of science just like in the case of complex valued trigonometric functions.

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Norm of Operators on the Generalized Cesàro Matrix Domain

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Abstract
Roopaei in [13] has introduced some factorization for the infinite Hilbert matrix and the Cesàro matrix of order n based on the generalized Cesàro matrix. In this research, we investigate the norm of these two operators on the generalized Cesàro matrix domain. Moreover we introduce some factorizations for the Hilbert matrix. Hence the present study is a complement of Roopaei's research.

Keywords: Hilbert matrix, Cesàro matrix, Norm, Sequence space.
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1. Introduction

Let ω be the space of all real-valued sequences. The space ℓ_p consists all real sequences $x = (x_k)_{k=0}^{\infty} \in \omega$ such that $\sum_{k=0}^{\infty} |x_k|^p < \infty$ which a Banach space with the norm

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty,$$

where $1 \leq p < \infty$.

Let T is a matrix with non-negative entries, assumed to map ℓ_p into itself and satisfies the inequality

$$\|Tx\|_{\ell_p} \leq K\|x\|_{\ell_p},$$

where K is a constant which is not depending on x for every $x \in \ell_p$. The constant K is called an upper bound for operator T and the smallest possible value of K is called the norm of T.

For an infinite matrix A and sequence space X, we define the matrix domain $A(X)$ as the set

$$A(X) = \{x \in \omega : Ax \in X\}$$

which is also a sequence space. In this study, we use the notation A_p for the matrix domain associated with the matrix A on the space $X = \ell_p$. For an invertible matrix A, the matrix domain A_p is a normed space with $\|x\|_{A_p} := \|Ax\|_{\ell_p}$. There are several new Banach spaces who have introduced and studied by using matrix domains of special lower triangular matrices. For more references we encourage the readers to some papers [1, 3, 17, 18] and textbook [2]. Recently, several mathematicians have computed the bounds of operators on some matrix domains in [9, 11, 12, 15, 16, 17, 18, 19].

Cesàro matrix. The infinite Cesàro operator is defined by

$$c_{j,k} = \left\{ \begin{array}{ll} \frac{1}{j+1} & 0 \le k \le j \\ 0 & otherwise, \end{array} \right\}$$

for all $j,k \in \mathbb{N}$. It can be represented by its arrays as

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix has the ℓ_p-norm $\|C\|_{\ell_p} = \frac{p}{p-1}$. The inequality

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{|x_k|}{n+1} \right)^p \le \left(\frac{p}{p-1} \right)^p \sum_{k=0}^{\infty} |x_k|^p,$$

which is called Hardy's inequality is resulted from the boundedness of Cesàro operator.

The matrix domain associated with the Cesàro matrix is the set

$$C_p \quad = \quad \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{x_k}{j+1} \right|^p < \infty \right\},$$

which is a Banach space with norm

$$\|x\|_{C_p} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{x_k}{j+1} \right|^p \right)^{\frac{1}{p}}.$$

The Cesàro sequence space C_p is studied in [10, 20]. Recently, Roopaei et al. [16] have investigated the general case C_p^n, its inclusion relations, dual spaces, matrix transformations as well as computing the norm of operators on this matrix domain in the case $1 \le p < \infty$.

Generalized Cesàro matrix. Let $N \ge 1$ be a real number, the generalized Cesàro matrix, $C^N = (c_{j,k}^N)$, is defined by

$$c_{j,k}^N = \left\{ \begin{array}{ll} \frac{1}{j+N} & 0 \le k \le j \\ 0 & otherwise, \end{array} \right.$$

and has the ℓ_p-norm $\|C^N\|_{\ell_p} = \frac{p}{p-1}$ ([6], Lemma 2.3). That is

$$C^N = \begin{pmatrix} \frac{1}{N} & 0 & 0 & \cdots \\ \frac{1}{1+N} & \frac{1}{1+N} & 0 & \cdots \\ \frac{1}{2+N} & \frac{1}{2+N} & \frac{1}{2+N} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that, C^1 is the well-known Cesàro matrix C. For more examples

$$C^2 = \begin{pmatrix} 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad and \quad C^3 = \begin{pmatrix} 1/3 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 0 & \cdots \\ 1/5 & 1/5 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The sequence space associated with the generalized Cesàro matrix is the set

$$C(N,p) \quad = \quad \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{x_k}{j+N} \right|^p < \infty \right\},$$

who has the norm

$$\|x\|_{C(N,p)} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{x_k}{j+N} \right|^p \right)^{\frac{1}{p}}.$$

Note that for $N=1$ we use the notation C_p instead of $C(1,p)$.

Recall the infinite Hilbert matrix which is defined by $H=(h_{j,k})=\frac{1}{j+k+1}$ for all non-negative integers j and k and has the matrix representation

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

According to [8] Theorem 323, the Hilbert matrix is a bounded operator on ℓ_p with

$$\|H\|_{\ell_p} = \frac{\pi}{\sin(\pi/p)}.$$

It has proved by Bennett [5] that the Hilbert operator can be factorized of the form $H=BC$, where C is the Cesàro matrix and $B=(b_{j,k})$ is defined by

$$b_{j,k} = \frac{k+1}{(j+k+1)(j+k+2)} \qquad (j,k=0,1,\ldots). \tag{1.1}$$

The matrix B is also a bounded operator on ℓ_p, ([5], Proposition 2), and $\|B\|_{\ell_p} = \frac{\pi}{p^* \sin(\pi/p)}$, where p^* is the conjugate of p i.e. $1/p+1/p^*=1$.

More recently, Roopaei in [13, 14] has generalized Bennett's factorization to introduce several factorization for the Hilbert matrix. He has showed that H can be presented of the form $H=B^N C^N$, where C^N is the generalized Cesàro matrix of the form:

Theorem 1.1 ([13], Theorem 2.2)**.** *The Hilbert matrix H, admits a factorization of the form $H=B^N C^N$, where $B^N=(b^N_{j,k})$ has the entries*

$$b^N_{j,k} = \frac{k+N}{(j+k+1)(j+k+2)} \qquad (j,k=0,1,\ldots). \tag{1.2}$$

and is a bounded operator on ℓ_p with bounds

$$\frac{\pi}{p^* \sin(\pi/p)} \le \|B^N\|_{\ell_p} \le \frac{N\pi}{p^* \sin(\pi/p)}.$$

In particular, for $N=1$, $H=BC$ and $\|B\|_{\ell_p} = \frac{\pi}{p^* \sin(\pi/p)}$.

2. Norm of Hilbert operator on generalized Cesàro space

The main purpose of this section is computing the norm of Hilbert operator on the generalized Cesàro space. Meanwhile, we introduce some factorization for the Hilbert matrix.

In sequel, we need the definition of another Hilbert matrix, H^1, who has the same norm as the Hilbert matrix and is defined by

$$h^1_{j,k} = \frac{1}{j+k+2} \qquad (j,k=0,1,\ldots), \tag{2.1}$$

or

$$H^1 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 2.1. *The Hilbert operator is a bounded operator from ℓ_p into the generalized Cesàro space $C(N,p)$ and*

$$\|H\|_{\ell_p,C(N,p)} \le \frac{p^*\pi}{\sin(\pi/p)}.$$

Proof. We have

$$\begin{aligned}\|H\|_{\ell_p,C(N,p)} &= \sup_{x\in\ell_p}\frac{\|Hx\|_{C(N,p)}}{\|x\|_{\ell_p}} = \sup_{x\in\ell_p}\frac{\|C^N Hx\|_{\ell_p}}{\|x\|_{\ell_p}}\\ &= \|C^N H\|_{\ell_p} \le \pi p^* \csc(\pi/p).\end{aligned}$$

□

Theorem 2.2. *The Hilbert operator is a bounded operator from the generalized Cesàro space $C(N,p)$ into ℓ_p and*

$$\|H\|_{C(N,p),\ell_p} \le \frac{N\pi}{p^*\sin(\pi/p)}.$$

In particular, the Hilbert matrix is a bounded operator from the Cesàro sequence space into ℓ_p and

$$\|H\|_{C_p,\ell_p} = \frac{\pi}{p^*\sin(\pi/p)}.$$

Proof. According to Theorem 1.1, the Hilbert matrix can be written as $H = B^N C^N$, where B^N is a bounded operator on ℓ_p and

$$\frac{\pi}{p^*\sin(\pi/p)} \le \|B^N\|_{\ell_p} \le \frac{N\pi}{p^*\sin(\pi/p)}.$$

Since C_p^N and ℓ_p are isomorphic, hence

$$\begin{aligned}\|H\|_{C(N,p),\ell_p} &= \sup_{x\in C(N,p)}\frac{\|Hx\|_{\ell_p}}{\|x\|_{C(N,p)}} = \sup_{x\in C(N,p)}\frac{\|B^N C^N x\|_{\ell_p}}{\|C^N x\|_{\ell_p}} = \sup_{y\in\ell_p}\frac{\|B^N y\|_{\ell_p}}{\|y\|_{\ell_p}}\\ &= \|B^N\|_{\ell_p} \le \frac{N\pi}{p^*\sin(\pi/p)}.\end{aligned}$$

In particular, for the symbol $N=1$, $C^N = C$ and $B^N = B$, where B is the factor in the Bennett's factorization of the Hilbert operator. Now, we have the desired result. □

Theorem 2.3. *The Hilbert operator is a bounded operator on the generalized Cesàro space and*

$$\|H\|_{C(N,p)} \le \frac{N\pi}{\sin(\pi/p)}.$$

In special case, the Hilbert operator is a bounded operator on the Cesàro matrix domain and

$$\|H\|_{C_p} = \frac{\pi}{\sin(\pi/p)}.$$

Proof. Let $D^N = (d_{j,k}^N)$ be $C^N B^N$, where B^N was defined by the relation (1.2). Then

$$\begin{aligned}d_{i,k}^N &= \sum_{j=0}^{i}\frac{1}{i+N}\frac{k+N}{(j+k+1)(j+k+2)}\\ &= \left(\frac{k+N}{k+1}\right)\left(\frac{i+1}{i+N}\right)\frac{1}{i+k+2}.\end{aligned}$$

But, $\frac{k+N}{k+1} \le N$ and $\frac{i+1}{i+N} \le 1$, for all non-negative integers j,k. Hence, $d_{j,k}^N \le N h_{j,k}^1$ which results in

$$\|D^N\|_{\ell_p} \le N\|H^1\|_{\ell_p} = N\frac{\pi}{\sin(\pi/p)}.$$

The map $x \to C^N x$ shows that the two sequence spaces $C(N,p)$ and ℓ_p are isomorphic, hence

$$\begin{aligned} \|H\|_{C(N,p)} &= \sup_{x \in C(N,p)} \frac{\|Hx\|_{C(N,p)}}{\|x\|_{C(N,p)}} = \sup_{x \in C(N,p)} \frac{\|C^N Hx\|_{\ell_p}}{\|C^N x\|_{\ell_p}} \\ &= \sup_{x \in C(N,p)} \frac{\|D^N C^N x\|_{\ell_p}}{\|C^N x\|_{\ell_p}} = \sup_{y \in \ell_p} \frac{\|D^N y\|_{\ell_p}}{\|y\|_{\ell_p}} \\ &= \|D^N\|_{\ell_p} \leq \frac{N\pi}{\sin(\pi/p)}. \end{aligned}$$

In particular, for $N = 1$, $C^N = C$ and $D^N = H^1$ which results the desired result. □

Corollary 2.4. *The Hilbert operator is a bounded operator from the generalized Cesàro space $C(N,p)$ into Cesàro sequence space C_p and*

$$\|H\|_{C(N,p),C_p} \leq \frac{N\pi}{\sin(\pi/p)}.$$

In particular, the Hilbert matrix is a bounded operator on the Cesàro matrix domain and

$$\|H\|_{C_p} = \frac{\pi}{\sin(\pi/p)}.$$

Proof. Let $P^N = (p^N_{j,k})$ be CB^N, where B^N was defined by the relation (1.2). Then

$$\begin{aligned} p^N_{i,k} &= \sum_{j=0}^{i} \frac{1}{i+1} \frac{k+N}{(j+k+1)(j+k+2)} \\ &= \left(\frac{k+N}{k+1}\right) \frac{1}{i+k+2}. \end{aligned}$$

But, $\frac{k+N}{k+1} \leq N$ for all non-negative integer k. Hence, $p^N_{j,k} \leq N h^1_{j,k}$ which results in

$$\|P^N\|_{\ell_p} \leq N \|H^1\|_{\ell_p} = N \frac{\pi}{\sin(\pi/p)}.$$

Since C^N_p and ℓ_p are isomorphic, hence

$$\begin{aligned} \|H\|_{C(N,p),C_p} &= \sup_{x \in C(N,p)} \frac{\|Hx\|_{C_p}}{\|x\|_{C(N,p)}} = \sup_{x \in C(N,p)} \frac{\|CB^N C^N x\|_{\ell_p}}{\|C^N x\|_{\ell_p}} \\ &= \sup_{y \in \ell_p} \frac{\|P^N y\|_{\ell_p}}{\|y\|_{\ell_p}} = \|P^N\|_{\ell_p} \leq \frac{N\pi}{\sin(\pi/p)}. \end{aligned}$$

In particular, for the symbol $N = 1$, $C^N = C$ and $B^N = B$, where B is the factor in the Bennett's factorization of the Hilbert operator. Now, we have the desired result. □

Similar to the above corollary we have the following result.

Corollary 2.5. *The Hilbert operator is a bounded operator from the Cesàro sequence space C_p into the generalized Cesàro space $C(N,p)$ and*

$$\|H\|_{C_p,C(N,p)} \leq \frac{\pi}{\sin(\pi/p)}.$$

In particular, the Hilbert matrix is a bounded operator on the Cesàro sequence space and

$$\|H\|_{C_p} = \frac{\pi}{\sin(\pi/p)}.$$

Corollary 2.6. *The Hilbert matrix H, can be represented of the form $H = C^{-1}P^NC^N$, where $P^N = (p^N_{j,k})$ is defined by*

$$p^N_{j,k} = \frac{(k+N)}{(k+1)(j+k+2)} \qquad (j,k = 0,1,\ldots).$$

In particular, for $N = 1$, $\|P\|_{\ell_p} = \pi\csc(\pi/p)$.

Proof. By a simple calculation, $P^N = CB^N$. Therefore by applying Theorem 1.1, $C^{-1}P^NC^N = H$, which proves the factorization. Note that for $N = 1$, $P^1 = P = H^1$, where the Hilbert matrix H^1 is

$$h^1_{j,k} = \frac{1}{j+k+2} \qquad (j,k = 0,1,\ldots),$$

and has the norm $\|H^1\|_{\ell_p} = \frac{\pi}{\sin(\pi/p)}$. □

Theorem 2.7. *The Hilbert matrix H, has a factorization of the form $H = C^{-N}A^NC$, where $A^N = (a^N_{j,k})$ is defined by*

$$a^N_{j,k} = \frac{j+1}{(j+N)(j+k+2)} \qquad (j,k = 0,1,\ldots).$$

In particular, for $N = 1$, H has the factorization $H = C^{-1}AC$, where $\|A\|_{\ell_p} = \pi\csc(\pi/p)$.

Proof. It is not difficult to verify that $A^N = C^NB$, therefore by applying Theorem 1.1, $C^{-N}A^NC^N = H$, which proves the factorization. Note that for $N = 1$, $A^1 = A = H^1$ and has the norm $\|A\|_{\ell_p} = \|H^1\|_{\ell_p} = \frac{\pi}{\sin(\pi/p)}$. □

3. Norm of Cesàro operator on the generalized Cesàro space

In this section we intend to compute the norm of Cesàro operator of order n on the generalized Cesàro space.

For the probability measure μ on the interval $[0,1]$, the Hausdorff matrix $H^\mu = (h_{j,k})$, is defined by

$$h_{j,k} = \begin{cases} \int_0^1 \binom{j}{k}\theta^k(1-\theta)^{j-k}d\mu(\theta) & 0 \le k \le j \\ 0 & otherwise, \end{cases}$$

For $1 \le p < \infty$, by Hardy's formula ([7], Theorem 216) one can obtain the norm of Hausdorff matrices. These operators are bounded iff $\int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta) < \infty$ and

$$\|H^\mu\|_{\ell_p} = \int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta).$$

By inserting $d\mu(\theta) = n(1-\theta)^{n-1}d\theta$ in the definition of the Hausdorff matrix, the Cesàro matrix of order n, $C^n = (c^n_{j,k})$ is

$$c^n_{j,k} = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & j \ge k \ge 0 \\ 0 & otherwise. \end{cases}$$

This matrix has the ℓ_p-norm

$$\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)},$$

according to Hardy's formula. Note that, $C^1 = C$, where C is the well-known Cesàro matrix.

For computing the norm of Cesàro matrix of order n on the generalized Cesàro matrix domain we need the following theorem.

Theorem 3.1 ([13], Theorem 3.2)**.** *For $n \ge 1$, Cesàro matrix of order n, C^n, has a factorization of the form $C^n = R^{n,N}C^N$, where C^N is the generalized Cesàro matrix of order N and $R^{n,N}$ is a bounded operator on ℓ_p with*

$$\|R^{n,N}\|_{\ell_p} \le \frac{N\Gamma(n+1)\Gamma(1+1/p^*)}{\Gamma(n+1/p^*)}.$$

Corollary 3.2. *The Cesàro operator of order n is a bounded operator from the generalized Cesàro space $C(N,p)$ into sequence space ℓ_p and*

$$\|C^n\|_{C(N,p),\ell_p} \leq \frac{N\Gamma(n+1)\Gamma(1+1/p^*)}{\Gamma(n+1/p^*)}.$$

Proof. Since $C(N,p)$ and ℓ_p are isomorphic, hence according to the Theorem 3.1 we have

$$\begin{aligned}\|C^n\|_{C(N,p),\ell_p} &= \sup_{x\in C(N,p)} \frac{\|C^n x\|_{\ell_p}}{\|x\|_{C(N,p)}} = \sup_{x\in C(N,p)} \frac{\|R^{n,N}C^N x\|_{\ell_p}}{\|C^N x\|_{\ell_p}} \\ &= \sup_{y\in \ell_p} \frac{\|R^{n,N} y\|_{\ell_p}}{\|y\|_{\ell_p}} = \|R^{n,N}\|_{\ell_p} \leq \frac{N\Gamma(n+1)\Gamma(1+1/p^*)}{\Gamma(n+1/p^*)}.\end{aligned}$$

Now, we have the desired result. □

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Some Properties of Generalized Topologies in GTSs

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Abstract
In this article, we introduce one new generalized topology and investigate its properties in a generalized topological space. Also, we give various properties of some generalized topologies defined in a generalized topological space. Finally, we analyze the nature of some special spaces.

Keywords: Genralized topology, Baire space, Stack and p-stack.
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1. Introduction

The notion of a generalized topological space was introduced by Császár in [3]. Let X be any non-null set. A family $\mu \subset exp(X)$ is a *generalized topology* [9] in X if $\emptyset \in \mu$ and $\bigcup\limits_{t\in T} G_t \in \mu$ whenever $\{G_t \mid t \in T\} \subset \mu$ where $exp(X)$ is a power set of X. We call the pair (X,μ) as a *generalized topological space* (GTS) [9]. If $X \in \mu$, then the pair (X,μ) is called a *strong generalized topological space* (sGTS) [9]. Let $Y \subset X$. Then the *subspace generalized topology* [2] is defined by, $\mu_Y = \{Y \cap U \mid U \in \mu\}$ and the pair (Y,μ_Y) is called as the *subspace generalized topological space* [2].

Let (X,μ) be a GTS and $A \subset X$. The *interior of* A [9] denoted by iA, is the union of all μ-open sets contained in A and the *closure of* A [9] denoted by cA, is the intersection of all μ-closed sets containing A when no confusion can arise. The elements in μ are called the μ-open sets, the complement of a μ-open set is called the μ-closed set and the complement of μ is denoted by $\mu^{'}$. Denote $\{U \in \mu \mid U \neq \emptyset\}$ by $\tilde{\mu}$ [8] and denote $\{U \in \mu \mid x \in U\}$ by $\mu(x)$ [8].

Throughout this paper, $\mathbb{R},\mathbb{Z},\mathbb{Q}$ and $\mathbb{N}$ denote the set of all real numbers, integers, rational numbers and natural numbers, respectively. The notations X_3,X_4,X_5 and X_6 are mean the sets $\{a,b,c\},\{a,b,c,d\},\{a,b,c,d,e\}$ and $\{a,b,c,d,e,f\}$, respectively.

2. Preliminaries

In this section, we remember some basic definitions and lemmas which will be useful in the development of the next sections.

A subset A of a GTS (X,μ) is said to be a *μ-nowhere dense* [6] (resp. *μ-dense* [6], *μ-codense* [7]) set if $icA = \emptyset$ (resp. $cA = X$, $c(X-A) = X$). A is said to be a *μ-strongly nowhere dense* set if for every $V \in \tilde{\mu}$, there is $U \in \tilde{\mu}$ such that $U \subset V$ and $U \cap A = \emptyset$. Then A is said to be a *μ-meager* (or *μ-first category*) (resp. *μ-s-meager* (or *μ-s-first category*)) set [8] if $A = \bigcup_{n\in\mathbb{N}} A_n$ where A_n is μ-nowhere dense (resp. μ-strongly nowhere dense) for all $n \in \mathbb{N}$.

In a GTS, every subset of a μ-strongly nowhere dense set is μ-nowhere dense and every subset of a μ-meager (resp. μ-s-meager) set is μ-meager (resp. μ-s-meager) [8].

Let (X,μ) be a GTS and $A \subset X$. Then A is said to be a *μ-second category* (μ-II category) (resp. *μ-s-second category* (μ-s-II category)) set [8] if A is not μ-meager (resp. μ-s-meager). A is *μ-residual* (resp. *μ-s-residual*) [8] if $X - A$ is μ-meager (resp. μ-s-meager).

A GTS (X,μ) is said to be *μ-II category* (resp. *μ-s-II category*) if X is μ-II category (resp. μ-s-II category) as a subset. A space X is called a *Baire space* (BS) [8] if each $V \in \tilde{\mu}$ is of μ-II category in X. A space (X,μ) is a *strong Baire space* (sBS) [8] if $V_1 \cap V_2 \cap \ldots \cap V_n$ is of μ-II category set for all $V_1, V_2, \ldots, V_n \in \mu$ such that $V_1 \cap V_2 \cap \ldots \cap V_n \neq \emptyset$. Also, every sBS is a BS [8].

Define $\mu^\star = \{\bigcup_t (U_1^t \cap U_2^t \cap U_3^t \cap \ldots \cap U_{n_t}^t) \mid U_1^t, U_2^t, \ldots, U_{n_t}^t \in \mu\}$ and $\mu^{\star\star} = \{A \subset X \mid A \text{ is of } \mu\text{-II category set}\} \cup \{\emptyset\}$ [8]. Then $\mu \subset \mu^\star$ and $\mu \subset \mu^{\star\star}$ if (X,μ) is a Baire space [8]. Also, $\mu^\star \subset \mu^{\star\star}$ if (X,μ) is a sBS [11].

A space (X,μ) is called hyperconnected [6] if every non-null μ-open subset of X is μ-dense in X. A GTS (X,μ) is said to be a *generalized submaximal space* [7] if every μ-dense subset of X is a μ-open set in X.

The following lemmas will be useful in the sequel.

Lemma 2.1. [8, Property 2.3] Let (X,μ) be a GTS and $A \subset X$ be a μ-nowhere (resp. μ-strongly nowhere) dense set. Then the closure of A and any subset of A are μ-nowhere (resp. μ-strongly nowhere) dense sets.

Lemma 2.2. [8, Property 2.5] Let (X,μ) be a GTS and $A \subset X$. Then the following hold.
(a) If A is s-meager then it is meager.
(b) If A is of II category then it is of s-II category.
(c) If A is s-residual then it is residual.

Lemma 2.3. [9, Proposition 4.7] Let (X,μ) be a GTS. If F_n is a μ-meager set for each $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} F_n$ is a μ-meager set in X.

Lemma 2.4. [9, Theorem 5.3] Let (X,μ) be a GTS. The following are equivalent.
(a) X is Baire.
(b) If $A \neq \emptyset$ is μ-residual in X, then A is μ-dense in X.
(c) If $B \neq X$ is μ-meager in X, then B is μ-codense in X.
(d) Every $U \in \tilde{\mu}$ is μ-II category in X.
(e) $iF = \emptyset$, for every F is a μ-meager set in X.
(f) For every μ-closed set F_n with $iF_n = \emptyset, i(\bigcup_{n\in\mathbb{N}} F_n) = \emptyset$.

Lemma 2.5. [12, Theorem 3.3] Let (X,μ) be a GTS. Then the following hold.
(a) If G_n is μ-s-meager for each $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} G_n$ is μ-s-meager.
(b) If F_n is μ-s-residual for each $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} F_n$ is μ-s-residual.

3. Properties of Generalized Topology

In this section, we give some properties of generalized topologies defined in a generalized topological space. Also, we check some families are either satisfied with the stack property or not.

We start the study of various types of generalized topologies in a generalized topological space by reminding the well-known definitions in GTSs.

Let (X,μ) be a GTS. A collection $\mathscr{C}$ of subsets of X is called a *stack* [10] if $A \in \mathscr{C}$ whenever $B \in \mathscr{C}$ and $B \subset A$. A stack $\mathscr{H}$ on X is called a *p-stack* [10] if $A, B \in \mathscr{H}$, then $A \cap B \neq \emptyset$.

Theorem 3.1. *Let (X,μ) be a GTS. Then $\tilde{\mu}^{\star\star}$ is a stack.*

Proof. Suppose $B \in \tilde{\mu}^{\star\star}$ and $B \subset A \subset X$. Then B is of μ-II category set in X. Since subset of a μ-meager set is μ-meager, A is of μ-II category set in X implies that $A \in \tilde{\mu}^{\star\star}$. Therefore, $\tilde{\mu}^{\star\star}$ is a stack. □

The below Corollary 3.2 directly follows from Theorem 3.1 so the proof is omitted.

Corollary 3.2. *Let (X,μ) be a GTS and $A \subset X$. Then the following hold.*
(a) If $i_{\mu^{\star\star}} A \neq \emptyset$, then $A \in \mu^{\star\star}$.
(b) If (X,μ) is a BS and if $i_\mu A \neq \emptyset$, then $A \in \mu^{\star\star}$.
(c) If (X,μ) is a sBS and if $i_{\mu^\star} A \neq \emptyset$, then $A \in \mu^{\star\star}$.
(d) If $A \in \mu^{\star\star}$, then $c_\mu A, c_{\mu^\star} A, c_{\mu^{\star\star}} A \in \mu^{\star\star}$.

The following Example 3.3, (a) shows that the generalized topology $\tilde{\mu}$ is not a stack in a generalized topological space, (b) proves that there exist a topology μ, in which $\tilde{\mu}$ is not a stack and (c) proves that μ is not a topology even if $\tilde{\mu}$ is a stack. The generalized topology $\tilde{\mu^\star}$ is not a stack as shown by the below Example 3.4.

Example 3.3. (a) Consider the generalized topological space (X,μ) where $X=\mathbb{R}$ and μ is the $\mathbb{Z}$ forbidden generalized topology on $\mathbb{R}$, that is, $\mu=\{U\subset\mathbb{R}\mid U\subset\mathbb{R}-\mathbb{Z}\}$. Then $\tilde{\mu}$ is not a stack. Because, if $U=\{\frac{n}{n+1}\mid n\in\mathbb{N}\}$, then $U\in\tilde{\mu}$. Here $U\subset\mathbb{Q}$. But $\mathbb{Q}\notin\tilde{\mu}$.
(b) Consider the generalized topological space (X,μ) where $X=[0,5]$ and $\mu=\{\emptyset,[0,2),(1,2),(1,4],[0,4],X\}$. Then μ is a topology. But $\tilde{\mu}$ is not a stack. For, let $G=[0,2)$ and $H=[0,2]$. Then $G\subset H$ and $G\in\tilde{\mu}$. But $H\notin\tilde{\mu}$.
(c) Consider the generalized topological space (X_4,μ) where $\mu=\{\emptyset,\{a,b\},\{a,\ b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},X_4\}$. Then $\tilde{\mu}$ is a stack. But μ is not a topology.

Example 3.4. Consider the generalized topological space (X,μ) where $X=[0,5]$ and $\mu=\{\emptyset,[0,2),(1,\frac{3}{2}),(1,4],[0,4]\}$. Then $\mu^\star=\{\emptyset,[0,2),(1,4],(1,\frac{3}{2}),(1,2),[0,4]\}$. Let $A=(1,2)\in\tilde{\mu}^\star$ and $B=[1,2]$. Then $A\subset B$. But $B\notin\tilde{\mu}^\star$. Thus, $\tilde{\mu}^\star$ is not a stack.

Theorem 3.5. *Let (X,μ) be a GTS. Then $(X,\mu^{\star\star})$ is a hyperconnected space if and only if $\tilde{\mu}^{\star\star}$ is a p-stack.*

Proof. Suppose that $(X,\mu^{\star\star})$ is a hyperconnected space. By Theorem 3.1, $\tilde{\mu}^{\star\star}$ is a stack. Let $A,B\in\tilde{\mu}^{\star\star}$. Then A and B are non-null $\mu^{\star\star}$-open sets and so A and B are $\mu^{\star\star}$-dense sets in X, by hypothesis. Therefore, $A\cap B\neq\emptyset$. Hence $\tilde{\mu}^{\star\star}$ is a p-stack. The reverse implication is directly follows from the definition of p-stack so the proof is omitted. □

Example 3.6. Consider the generalized topological space (X,μ) where $X=[0,5]$ and $\mu=\{\emptyset,[0,2),(1,4],[0,4]\}$. Then $\mu^{\star\star}=\{\emptyset\}\cup\{A,B\subset X\mid A\in exp((1,2))-\{\emptyset\},A\subset B\}$ and so $(X,\mu^{\star\star})$ is not a hyperconnected space. Because, if we take $W=[0,\frac{3}{2})\cup(\frac{3}{2},5]$, then $W\in\mu^{\star\star}$ and hence $\{\frac{3}{2}\}$ is a $\mu^{\star\star}$-closed set but $\{\frac{3}{2}\}$ is a non-null $\mu^{\star\star}$-open set in X. Let $U=[0,1]\cup\{\frac{3}{2}\}$ and $V=\{\frac{17}{10}\}\cup[2,5]$. Then $U,V\in\tilde{\mu}^{\star\star}$. But $U\cap V=\emptyset$. Thus, $\tilde{\mu}^{\star\star}$ is not a p-stack.

Theorem 3.7. *Let (X,μ) be a BS. If every non-null μ-open set is a μ-residual set, then (X,μ) is a hyperconnected space.*

Proof. Let $G\in\tilde{\mu}$. Then by hypothesis, G is μ-residual in X. By Lemma 2.4, G is a μ-dense set in X. Hence (X,μ) is a hyperconnected space. □

Theorem 3.8. *Let (X,μ) be a BS and every non-null μ-open set is μ-residual in X. If (X,μ) is a generalized submaximal space, then the following hold.*
(a) $\mu^{\star\star}=\mu$.
(b) $\tilde{\mu}$ is a stack.

Proof. It is enough to prove (a) only. Since (X,μ) is a BS we have $\mu\subset\mu^{\star\star}$. Let $B\in\mu^{\star\star}$. If $B=\emptyset$, then there is nothing to prove. Suppose $B\in\tilde{\mu}^{\star\star}$. Then B is of μ-II category set and so B is not μ-meager so that B is not a μ-nowhere dense set. Thus, $i_\mu c_\mu B\neq\emptyset$. Take $V=i_\mu c_\mu B$. Then $V\in\tilde{\mu}$. By hypothesis and Theorem 3.7, (X,μ) is a hyperconnected space so that V is μ-dense set in X. Then B is μ-dense set in X. Since (X,μ) is a generalized submaximal space, B is a μ-open set. Therefore, $B\in\tilde{\mu}$ so that $B\in\mu$. Thus, $\mu^{\star\star}\subset\mu$. Hence $\mu^{\star\star}=\mu$. □

In Theorem 3.8, replace the condition "(X,μ) be a BS" by the condition "(X,μ) be a sBS", we get $\mu=\mu^\star=\mu^{\star\star}$ and then $\tilde{\mu}^\star$ is a stack.

The following Example 3.9 shows that the necessary conditions are can not be dropped in Theorem 3.8.

Example 3.9. (a) Consider the generalized topological space (X,μ) where $X=[0,5]$ and $\mu=\{\emptyset,[0,2),(1,\frac{3}{2}),(1,3],[0,3]\}$. Then (X,μ) is a BS and every non-null μ-open set is a μ-residual set in X. Let $A=[1,4]$ be a subset of X. Then $c_\mu A=X$ and so A is a μ-dense subset of X. But $A\notin\mu$. Thus, (X,μ) is not a generalized submaximal space. Here $\mu^{\star\star}=\{\emptyset\}\cup\{A,B\subset X\mid A\in exp((1,\frac{3}{2}))-\{\emptyset\},A\subset B\}$. Choose $W=[1,2]$. Then $W\in\mu^{\star\star}$. But $W\notin\mu$. Hence $\mu^{\star\star}\nsubseteq\mu$.
(b) Consider the generalized topological space (X_5,μ) where $\mu=\{\emptyset,\{a,b\},\{a,\ c\},\{a,d\},\{b,c\},\{a,b,c\},\{a,b,d\},\{a,b,e\},\{a,c,d\},\{a,c,e\},\{a,d,e\},\{b,c,d\},\{b,c,e\},\{a,b,c,d\},\{a,b,c,e\},\{a,b,d,e\},\{a,c,d,e\},\{b,c,d,e\},X_5\}$. Therefore, (X_5,μ) is a generalized submaximal space and every non-null μ-open set is a μ-residual set in X_5. But (X_5,μ) is not a BS. For, if we take $A=\{a,c,d\}$ is a subset of X_5. Now $i_\mu c_\mu(\{a\})=i_\mu(\{a\})=\{\emptyset\};i_\mu c_\mu(\{c\})=i_\mu(\{c\})=\{\emptyset\};i_\mu c_\mu(\{d\})=i_\mu(\{d\})=\{\emptyset\}$. Then $\{a,c,d\}$ is a μ-meager set and so A is not a μ-II category set in X_5. Here, $\mu^{\star\star}=\{\emptyset\}$. Hence $\mu\nsubseteq\mu^{\star\star}$.
(c) Consider the generalized topological space (X_5,μ) where $\mu=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{a,b,c\},\{a,b,d\},\{a,b,e\},\{b,c,d\},\{a,b,c,d\},\{a,b,c,e\},\{a,b,d,e\},X_5\}$. Then (X_5,μ) is a BS and generalized submaximal space. Let $A=\{a\}$ be a subset of X_5. Then $X_5-A=\{b,c,d,e\}$. Consider, $i_\mu c_\mu(\{b\})=i_\mu(\{b,d,e\})=\{b\}\neq\emptyset$. Thus, $\{b\}$ is of μ-II category set in X_5. Therefore, X_5-A is of μ-II category set in X_5 so that X_5-A is not a μ-meager set which implies that A is not a μ-residual set in X_5. Thus, there is a non-null μ-open set which is not a μ-residual set in X_5. Here, $\mu^{\star\star}=\{\emptyset\}\cup\{A\subset X_5\mid \text{either } a\in A \text{ or } b\in A\}$. Let $G=\{a,c,d\}$. Then $G\in\mu^{\star\star}$. But $G\notin\mu$. Hence $\mu^{\star\star}\nsubseteq\mu$.

Theorem 3.10. *Every collection of all non-null μ-residual sets in X is a stack where μ is a generalized topology on X.*

Proof. Let $\eta = \{A \subset X \mid A$ is a non-null μ-residual set$\}$. Suppose that $V \in \eta$ and $V \subset U$. Then V is μ-residual and so $X - V$ is μ-meager in X. Since $V \subset U, X - U \subset X - V$ so that $X - U$ is a μ-meager set in X, since subset of a meager set is meager. Thus, U is μ-residual in X. Therefore, $U \in \eta$. Hence η is a stack. □

Theorem 3.11. *Let (X, μ) be a GTS and $\eta = \{\emptyset\} \cup \{A \subset X \mid A$ is non-null μ-residual set$\}$. Then η is a topology on X.*

Proof. By Theorem 3.10, η is closed under arbitrary union. Also, η is closed under finite intersection, by Lemma 2.3. Let $A \subset X$ be a non-null μ-residual set. Then $\emptyset = (X - X) \subset (X - A)$ and so X is a non-null μ-residual set. Thus, $X \in \eta$. Therefore, η is a topology on X. □

The below Theorem 3.12 (a) follows from the similar arguments in Theorem 3.10, Theorem 3.12 (b) follows from Lemma 2.5 (b) and the same considerations in Theorem 3.11 so the proof is omitted.

Theorem 3.12. *Let (X, μ) be a GTS. Then the following hold.*
(a) Every collection of all μ-s-residual sets in X is a stack.
(b) If $\eta = \{\emptyset\} \cup \{A \subset X \mid A$ is non-null μ-s-residual set$\}$, then η is a topology on X.

Theorem 3.13. *Let (X, μ) be a GTS. Then every collection of all μ-dense sets in X is a stack.*

Theorem 3.14. *Let (X, μ) be a generalized submaximal space. Then every collection of all μ-dense sets in X is a p-stack.*

Proof. Let $\eta = \{A \mid A$ is μ-dense subset of $X\}$. By Theorem 3.13, η is a stack. Let $G, H \in \eta$. Then G and H are μ-dense sets in X. By hypothesis, G and H are non-null μ-open sets in X so that $G \cap H \neq \emptyset$. Hence η is a p-stack. □

The following Example 3.15 shows that the condition "(X, μ) be a generalized submaximal space" can not be dropped in Theorem 3.14. Also, this example shows that the collection of all μ-codense sets in X is not a stack.

Example 3.15. Consider the generalized topological space (X_4, μ) where $\mu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X_4\}$. Here, $\{b\}$ is a μ-dense set. But $\{b\} \notin \mu$. Thus, (X_4, μ) is not a generalized submaximal space. Take $\eta = \{A \mid A$ is μ-dense subset of $X_4\}$. Then $\eta = \{\{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X_4\}$. Take $A = \{a,c\}$ and $B = \{b,d\}$. Then $A, B \in \eta$. But $A \cap B = \emptyset$.
Take ζ is the collection of all μ-codense sets in X. Then $\zeta = \{\emptyset, \{a\}.\{b\}, \{c\}$,
$\{d\}, \{a,c\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,c,d\}, X_4\}$. Choose $A = \{b,d\}$ and $B = \{b,c,d\}$. Then $A \in \zeta$ and $A \subset B$. But $B \notin \zeta$. Thus, ζ is not a stack.

Next, Theorem 3.16 follows from Lemma 2.4 and Theorem 3.14 so the direct proof is omitted.

Theorem 3.16. *Let (X, μ) be a BS. If (X, μ) be a generalized submaximal space, then $\eta = \{A \subset X \mid A$ is a non-null μ-residual set$\}$ is a p-stack.*

The following Theorem 3.17 follows from the fact that "super set of a dense set is dense" and the trivial proof is omitted.

Theorem 3.17. *Let (X, μ) be a GTS and $\eta = \{\emptyset\} \cup \{A \subset X \mid A$ is μ-dense$\}$. Then η is a strong generalized topology on X.*

The collection η defined on the above Theorem 3.17 is not closed under the finite intersection as shown by the following Example 3.18.

Example 3.18. Consider the generalized topological space (X_4, μ) where $\mu = \{\emptyset, \{a,b\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, X_4\}$. Take $\eta = \{\emptyset\} \cup \{A \subset X_4 \mid A$ is μ-dense set$\}$. Then $\eta = \{\emptyset\} \cup \{\{b\}, \{a,b\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X_4\}$. Let $A = \{a,d\}$ and $B = \{b,d\}$. Then $A, B \in \eta$. But $A \cap B = \{d\} \notin \eta$. Thus, η is not closed under the finite intersection.

Theorem 3.19. *Let (X, μ) be a GTS. Then $\mu^{\star\star} \neq \{\emptyset\}$ if and only if $\mu^{\star\star^\star}$ is a strong GT in X where $\mu^{\star\star^\star} = \{\bigcup_t (W_1^t \cap W_2^t \cap \ldots \cap W_{n_t}^t) \mid W_1^t, W_2^t, \ldots, W_{n_t}^t \in \mu^{\star\star}\}$ and hence it is a topology.*

Proof. Suppose that $\mu^{\star\star} \neq \{\emptyset\}$. Then there exists a non-null $\mu^{\star\star}$-open set in X. Take G is the non-null $\mu^{\star\star}$-open set in X. Then G is of μ-II category set in X. Since subset of a μ-meager set is μ-meager, X is of μ-II category. Therefore, $X \in \mu^{\star\star}$ and so $X \in \mu^{\star\star^\star}$. Hence $\mu^{\star\star^\star}$ is a strong generalized topology. Also, $\mu^{\star\star^\star}$ is closed under finite intersection. Hence $\mu^{\star\star^\star}$ is a topology. Converse, follows from the definition of $\mu^{\star\star^\star}$. □

Theorem 3.20. *Let (X, μ) be a GTS. If $\tilde{\mu}$ is a stack, then the following hold.*
(a) μ is a strong GT.
(b) $\mu^\star$ is a topology.

Proof. It is enough to prove (a) only. Suppose $\mu \neq \{\emptyset\}$ and $\tilde{\mu}$ is a stack. Then we can choose a non-null open set in μ. Take G is a non-null μ-open set in X. If $G = X$, then there is nothing to prove. Assume that, $G \subset X$. By hypothesis, $X \in \tilde{\mu}$ so that $X \in \mu$. Hence μ is a strong GT. □

The converse implication of (a) in Theorem 3.20 is not true as shown by the following Example 3.21.

Example 3.21. Consider the generalized topological space (X,μ) where $X = [0,5]$ and $\mu = \{\emptyset, [0,2), (1,3], (1,5], [0,3], X\}$. Then (X,μ) is a strong generalized topological space. But $\tilde{\mu}$ is not a stack. For, $[0,2) \subset [0,2]$. Here $[0,2) \in \mu$ but $[0,2] \notin \mu$.

The following Theorem 3.22 is a direct consequence of the definition of the stack so the proof is omitted. If $\eta \subset exp(X) - \{\emptyset\}$ where X is a non-null set, $\mu \subset \eta \subset \gamma$ and if η is a stack, then neither γ nor μ is stack as shown by the following Example 3.23.

Theorem 3.22. *Let X be a non-null set and $\eta \subset exp(X)$. If η is a stack and generalized topology, then η is a strong GT.*

Example 3.23. Consider the non-null space X_4. Take $\eta = \{\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X_4\}$. Let $\gamma = \{\{a\},\{b\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X_4\}$ and $\mu = \{\{a\},\{a,b\},\{a,c\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X_4\}$. Then $\mu \subset \eta \subset \gamma$. Here η is a stack. But neither γ nor μ is stack. For, let $A_1 = \{b\}, A_2 = \{a\}$ and $B_1 = \{b,d\}, B_2 = \{a,d\}$. Then $A_1 \in \gamma, A_2 \in \mu$ and $A_1 \subset B_1, A_2 \subset B_2$. But $B_1 \notin \gamma, B_2 \notin \mu$.

Moreover, $\mu^{\star}$ is a topology if $\tilde{\mu^{\star}}$ is a stack.

4. Nature of a New GT

In this section, we define a new generalized topology and give some of its properties in a generalized topological space.

First of all, we recall some definitions and facts for the development of this section.

A GTS (X,μ) is said to be a *weak Baire space* (for short, wBS) [8] if for every $U \in \tilde{\mu}$ is of μ-s-II category set in X. Also, every BS is a wBS.

Let (X,μ) be a GTS and $A \subset X$. Then A is called *μ-semi-open* (resp. *μ-α-open*) if $A \subset ciA$ (resp. $A \subset iciA$) [5].

In [8], Korczak - Kubiak et al. introduced a new generalized topology, namely $\mu^{\star\star}$, defined by using μ-II category sets and gave some properties of this generalized topology in a generalized topological space.

Motivated by this, we will introduce a new generalized topology, namely $\mu^{\mathscr{V}}$, (dependent on GT μ) in a generalized topological space which will be a convenient tool for considerations in this section.

The GT $\mu^{\mathscr{V}}$ defined as in the following way:

Definition 4.1. Let (X,μ) be a GTS. Then $\mu^{\mathscr{V}} = \{\emptyset\} \cup \{A \subset X \mid A \text{ is of } \mu\text{-s-II category set}\}$.

The family $\mu^{\mathscr{V}}$ is a strong generalized topology if $\mu^{\mathscr{V}} \neq \{\emptyset\}$. The converse implication is always true.

Let (X,μ) be a GTS. If $\mu \neq \{\emptyset\}$, then X is of μ-s-II category and hence $\mu^{\mathscr{V}}$ is a sGTS. Also, the reverse implication is true.

The following Example 4.2 shows that the family $\mu^{\mathscr{V}}$ is not closed under the finite intersection.

Example 4.2. Consider the generalized topological space (X_4,μ) where $\mu = \{\emptyset,\{a\},\{a,d\},\{b,d\},\{a,b,c\},\{a,b,d\},X_4\}$. Then $\mu^{\mathscr{V}} = \{\emptyset\} \cup \{A \subset X_4 \mid a \in A \text{ or } b \in A \text{ or } d \in A\}$. Here, $\{a,c\}$ and $\{b,c\}$ are of μ-s-II category subsets in X_4. Take $A = \{a,c\} \cap \{b,c\} = \{c\}$. Then A is a μ-strongly nowhere dense set in X_4 and so A is not μ-s-II category in X_4 so that $A \notin \mu^{\mathscr{V}}$. Therefore, $\mu^{\mathscr{V}}$ is not closed under the finite intersection.

Lemma 4.3. [8, Lemma 2.12] Let (X,μ) be GTS and $A \subset X$. Then

$$c_{\mu^{\star\star}}(A) = \begin{cases} X & \text{if } A \text{ is } \mu\text{-residual,} \\ A & \text{if } A \text{ is not } \mu\text{-residual} \end{cases}$$

Lemma 4.4. [8, Lemma 2.13] Let (X,μ) be GTS, X be a μ-II category set and $A \subset X$. If A is a $\mu^{\star\star}$-nowhere dense set, then A is a μ-meager set.

Theorem 4.5. *Let (X,μ) be a GTS. Then $\tilde{\mu^{\mathscr{V}}}$ is a stack.*

Proof. Suppose $A \subset B$ and $A \in \tilde{\mu^{\mathscr{V}}}$. Then A is of μ-s-II category set and so B is of μ-s-II category set, since subset of a μ-s-meager set is μ-s-meager. Therefore, $B \in \tilde{\mu^{\mathscr{V}}}$. Hence $\tilde{\mu^{\mathscr{V}}}$ is a stack. □

The following Corollary 4.6 follows from the similar arguments in Theorem 3.5 and so the proof is omitted.

Corollary 4.6. *Let (X,μ) be a GTS. Then $\tilde{\mu^{\mathscr{V}}}$ is a p-stack if and only if $(X,\mu^{\mathscr{V}})$ is a hyperconnected space.*

The following Proposition 4.7 directly follows from the definition of weak Baire space, strong Baire space and Lemma 2.2 (b) so the proof is omitted. Theorem 4.8 directly follows from Theorem 4.5, Proposition 4.7 so the trivial proof is omitted.

Proposition 4.7. *Let (X,μ) be a GTS. Then the following hold.*
(a) If (X,μ) is a wBS, then $\mu \subset \mu^{\mathscr{V}}$.
(b) If (X,μ) is a sBS, then $\mu^{\star} \subset \mu^{\mathscr{V}}$.
(c) $\mu^{\star\star} \subset \mu^{\mathscr{V}}$.

Theorem 4.8. *Let (X,μ) be a GTS and $A \subset X$. Then the following hold.*
(a) If $i_{\mu^{\mathscr{V}}}A \neq \emptyset$, then $A \in \mu^{\mathscr{V}}$.
(b) If (X,μ) is a wBS and if $i_{\mu}A \neq \emptyset$, then $A \in \mu^{\mathscr{V}}$.
(c) If (X,μ) is a sBS and if $i_{\mu^{\star}}A \neq \emptyset$, then $A \in \mu^{\mathscr{V}}$.
(d) If $A \in \mu^{\mathscr{V}}$, then $c_{\mu}A, c_{\mu^{\star}}A, c_{\mu^{\star\star}}A, c_{\mu^{\mathscr{V}}}A \in \mu^{\mathscr{V}}$.

Theorem 4.9. *Let (X,μ) be a wBS-sGTS and $A \subset X$. Then the following hold.*
(a) If A is μ-α-open, then $A \in \mu^{\mathscr{V}}$.
(b) If A is μ-semi-open, then $A \in \mu^{\mathscr{V}}$.

Proof. We will present the detailed proof only for (a). Suppose A is a μ-α-open set in X. Then $A \subset iciA$. If $A = \emptyset$, then there is nothing to prove. Assume that, $A \neq \emptyset$. Then $iciA \neq \emptyset$ so that $ciA \neq \emptyset$ which implies that $iA \neq \emptyset$, since μ is a sGT. Thus, $i_{\mu}A \neq \emptyset$. By hypothesis and Theorem 4.8 (b), $A \in \mu^{\mathscr{V}}$. □

By using Theorem 4.9, immediately we get two Observations as follows.

Observation 4.10. *Let (X,μ) be a wBS-sGTS. If A is a $\mu^{\mathscr{V}}$-dense subset of X, then the following hold.*
(a) $A \cap U \neq \emptyset$ for every non-null μ-α-open set U.
(b) $A \cap V \neq \emptyset$ for every non-null μ-semi-open set V.

Observation 4.11. *Let (X,μ) be a wBS-sGTS and $A \subset X$. If A is a $\mu^{\mathscr{V}}$-nowhere dense set in X, then the following hold.*
(a) If G is a non-null μ-α-open set, then $G \not\subseteq A$.
(b) If H is a non-null μ-semi-open set, then $H \not\subseteq A$.

In Theorem 4.9, we replace the condition "wBS-sGTS" by "BS-sGTS" we get $A \in \mu^{\star\star}$, by Corollary 3.2 (b). Theorem 4.9 is not reversible as shown in the following Example 4.12.

Example 4.12. Consider the generalized topological space (X_5,μ) where $\mu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. Then $\mu^{\mathscr{V}} = \{\emptyset\} \cup \{A \subset X_5 \mid a \in A \text{ or } b \in A \text{ or } c \in A\}$.
Let $U = \{c,d\}$. Then $U \in \mu^{\mathscr{V}}$. But U is not a μ-α-open set in X_5. For, $i_{\mu}c_{\mu}i_{\mu}U = i_{\mu}c_{\mu}(\{\emptyset\}) = i_{\mu}(\{d,e\}) = \emptyset$. Thus, $U \not\subset i_{\mu}c_{\mu}i_{\mu}U$.
Let $V = \{a,d\}$. Then $V \in \mu^{\mathscr{V}}$. Here $c_{\mu}i_{\mu}V = c_{\mu}(\{\emptyset\}) = \{d,e\}$. Thus, $V \not\subset c_{\mu}i_{\mu}V$. Therefore, V is not a μ-semi-open set in X_5.

Theorem 4.13. *Let (X,μ) be a GTS and $A \subset X$. If A is a $\mu^{\mathscr{V}}$-nowhere dense set, then the following hold.*
(a) If (X,μ) is a wBS, then A is μ-codense.
(b) If (X,μ) is a sBS, then A is $\mu^{\star}$-codense.
(c) A is $\mu^{\star\star}$-codense set in X.

Proof. It is enough to prove (b) only. Suppose (X,μ) is a sBS and A is a $\mu^{\mathscr{V}}$-nowhere dense set. Then $i_{\mu^{\mathscr{V}}}c_{\mu^{\mathscr{V}}}A = \emptyset$ and so $c_{\mu^{\mathscr{V}}}(X-A) = X$. By hypothesis and Proposition 4.7 (b), $c_{\mu^{\star}}(X-A) = X$. Therefore, A is a $\mu^{\star}$-codense set in X. □

Proposition 4.14 and Proposition 4.15 are follows from the similar considerations in Lemma 4.3 and Lemma 4.4, respectively so the proofs are omitted.

Proposition 4.14. *Let (X,μ) be a GTS and $A \subset X$. Then*

$$c_{\mu^{\mathscr{V}}}(A) = \begin{cases} X & \text{if } A \text{ is } \mu\text{-s-residual}, \\ A & \text{if } A \text{ is not } \mu\text{-s-residual} \end{cases}$$

Proposition 4.15. *Let (X,μ) be a GTS, X be a μ-s-II category set and $A \subset X$. Then the following hold.*
(a) If A is a $\mu^{\mathscr{V}}$-nowhere dense set, then A is a μ-s-meager set.
(b) If A is a $\mu^{\mathscr{V}}$-meager set, then A is a μ-s-meager set.
(c) If A is a $\mu^{\mathscr{V}}$-residual set, then A is a μ-s-residual set.
(d) If A is of μ-s-II category, then it is of $\mu^{\mathscr{V}}$-II category.

Theorem 4.16. *Let (X,μ) be a wBS. Then every μ-strongly nowhere dense set is $\mu^{\mathscr{V}}$-nowhere dense in X.*

Proof. Suppose (X,μ) is a wBS. Let A be a μ-strongly nowhere dense set in X. Suppose $i_{\mu^{\mathscr{V}}}c_{\mu^{\mathscr{V}}}A \neq \emptyset$. Then $c_{\mu^{\mathscr{V}}}A$ contains a non-null $\mu^{\mathscr{V}}$-open set in X and so $c_{\mu^{\mathscr{V}}}A$ contains a μ-s-II category set in X. Thus, $c_{\mu}A$ contains a μ-s-II category set in X, since $\mu \subset \mu^{\mathscr{V}}$. But $c_{\mu}A$ is a μ-s-meager set in X, by Lemma 2.1. Therefore, $i_{\mu^{\mathscr{V}}}c_{\mu^{\mathscr{V}}}A = \emptyset$. Hence every μ-strongly nowhere dense set is a $\mu^{\mathscr{V}}$-nowhere dense set in X. □

The following Corollary 4.17 follows from Theorem 4.16 so the direct proof is omitted.

Corollary 4.17. *Let (X,μ) be a wBS and $A \subset X$. Then the following hold.*
(a) If A is μ-s-meager, then A is $\mu^{\mathscr{V}}$-meager in X.
(b) If A is of $\mu^{\mathscr{V}}$-II category, then A is of μ-s-II category in X.
(c) If A is μ-s-residual, then A is $\mu^{\mathscr{V}}$-residual in X.

Theorem 4.18. *Let (X,μ) be a GTS. If X is of μ-s-II category, then $(X,\mu^{\mathscr{V}})$ is a BS.*

Proof. Let $G \in \tilde{\mu^{\mathscr{V}}}$. Suppose G is a $\mu^{\mathscr{V}}$-meager set. Then by hypothesis and Proposition 4.15 (b), G is a μ-s-meager set, which is a contradiction to $G \in \tilde{\mu^{\mathscr{V}}}$. Therefore, G is of $\mu^{\mathscr{V}}$-II category in X. Hence $(X,\mu^{\mathscr{V}})$ is a BS. □

The following Theorem 4.19 follows from the similar considerations in Theorem 3.19 so the easy proof is omitted.

Theorem 4.19. *Let (X,μ) be a GTS. Then $\mu^{\mathscr{V}} \neq \{\emptyset\}$ if and only if $\mu^{\mathscr{V}\star}$ is a strong GT in X where $\mu^{\mathscr{V}\star} = \{\bigcup_t(W_1^t \cap W_2^t \cap \ldots \cap W_{n_t}^t) \mid W_1^t, W_2^t, \ldots, W_{n_t}^t \in \mu^{\mathscr{V}}\}$ and hence it is a topology.*

In the rest of this section, we give some relations between various types of generalized topology in a generalized topological space.

First of all, we remember some Lemmas which is useful in the sequel.

Lemma 4.20. [14, Theorem 3.4] Let (X,μ) be a sBS and $A \subset X$. Then the following hold.
(a) *If A is a μ-nowhere dense set, then A is a $\mu^{\star}$-nowhere dense set.*
(b) *If A is a μ-meager set, then A is a $\mu^{\star}$-meager set.*
(c) *If A is a $\mu^{\star}$-II category set, then A is a μ-II category set.*

Lemma 4.21. [14, Theorem 3.7] Let (X,μ) be a BS and $A \subset X$. Then the following hold.
(a) *If A is a μ-nowhere dense set, then A is a $\mu^{\star\star}$-nowhere dense set.*
(b) *If A is a μ-meager set, then A is a $\mu^{\star\star}$-meager set.*
(c) *If A is a $\mu^{\star\star}$-II category set, then A is a μ-II category set.*

Here, $\mu^{\star\star^{\star}} = \{\bigcup_t(W_1^t \cap W_2^t \cap \ldots \cap W_{n_t}^t) \mid W_1^t, W_2^t, \ldots, W_{n_t}^t \in \mu^{\star\star}\}$ and $\mu^{\mathscr{V}\star} = \{\bigcup_t(W_1^t \cap W_2^t \cap \ldots \cap W_{n_t}^t) \mid W_1^t, W_2^t, \ldots, W_{n_t}^t \in \mu^{\mathscr{V}}\}$.

Now we define two generalized topologies and give some properties of these generalized topologies.

Define $\mu^{\star^{\star\star}} = \{\emptyset\} \cup \{A \subset X \mid A \text{ is of } \mu^{\star}\text{-II category set}\}$ and $\mu^{\mathscr{V}^{\star\star}} = \{\emptyset\} \cup \{A \subset X \mid A \text{ is of } \mu^{\mathscr{V}}\text{-II category set}\}$.

It is easily seen that the families $\mu^{\star^{\star\star}}$ and $\mu^{\mathscr{V}^{\star\star}}$ are generalized topologies. Also, these two generalized topologies are satisfied with the stack property in a GTS.

Theorem 4.22. *Let (X,μ) be a generalized topological space. Then the following hold.*
(a) $\tilde{\mu}^{\star\star^{\star}}$ is a stack.
(b) $\tilde{\mu}^{\mathscr{V}\star}$ is a stack.

Proof. It is enough to prove (a) only. Suppose $A \subset B$ where $A \in \tilde{\mu}^{\star\star^{\star}}$. Then $A = \bigcup_t(A_1^t \cap A_2^t \cap \ldots \cap A_{n_t}^t)$ where $A_1^t, A_2^t, \ldots, A_{n_t}^t \in \mu^{\star\star}$. Take $A_k = A_1^k \cap A_2^k \cap \ldots \cap A_{n_k}^k$ such that $A_k \neq \emptyset$ where $A_1^k, A_2^k, \ldots, A_{n_k}^k \in \mu^{\star\star}$. By hypothesis, $A_k \subset B$ so that $B = A_k \cup (B - A_k)$. Thus, $B = (A_1^k \cap A_2^k \cap \ldots \cap A_{n_k}^k) \cup (B - A_k)$ where $A_1^k, A_2^k, \ldots, A_{n_k}^k \in \mu^{\star\star}$ which implies that $B = (A_1^k \cup (B - A_k)) \cap (A_2^k \cup (B - A_k)) \cap \ldots \cap (A_{n_k}^k \cup (B - A_k))$ where $A_1^k, A_2^k, \ldots, A_{n_k}^k \in \mu^{\star\star}$. Since $A_1^k, A_2^k, \ldots, A_{n_k}^k \in \tilde{\mu}^{\star\star}$ we have $A_1^k \cup (B - A_k), A_2^k \cup (B - A_k), \ldots, A_{n_k}^k \cup (B - A_k) \in \tilde{\mu}^{\star\star}$, since $\tilde{\mu}^{\star\star}$ is a stack. Therefore, $B \in \tilde{\mu}^{\star\star^{\star}}$. Hence $\tilde{\mu}^{\star\star^{\star}}$ is a stack. □

Corollary 4.23. *Let (X,μ) be a generalized topological space and $\tilde{\mu}$ is a stack. Then $\tilde{\mu}^{\star}$ is a stack.*

Obviously, $\mu^{\star\star} \subset \mu^{\star\star^{\star}}$ and $\mu^{\mathscr{V}} \subset \mu^{\mathscr{V}\star}$. The reverse implications are true as shown by the following Theorem 4.24.

Theorem 4.24. *Let (X,μ) be a generalized topological space. Then the following hold.*
(a) If $(X,\mu^{\star\star})$ is a sBS, then $\mu^{\star\star^\star} \subset \mu^{\star\star}$.
(b) If $(X,\mu^{\mathscr{V}})$ is a sBS, then $\mu^{\mathscr{V}^\star} \subset \mu^{\mathscr{V}}$.

Proof. (a) Suppose $(X,\mu^{\star\star})$ is a sBS. Let $G \in \mu^{\star\star^\star}$. If $G=\emptyset$, then there is nothing to prove. Assume that, $G \neq \emptyset$. Then $G=\bigcup_t (G_1^t \cap G_2^t \cap \ldots \cap G_{n_t}^t)$ where $G_i^t \in \mu^{\star\star}$ for $i=1,2,...,n_K$. Take $G_k = G_1^k \cap G_2^k \cap \ldots \cap G_{n_k}^k$ such that $G_k \neq \emptyset$ where $G_1^k, G_2^k, \ldots, G_{n_k}^k \in \mu^{\star\star}$. By hypothesis, G_k is of $\mu^{\star\star}$-II category set in X so that G_k is of μ-II category set in X, by Lemma 4.21(c). Thus, G is of μ-II category set in X. Therefore, $G \in \mu^{\star\star}$. Hence $\mu^{\star\star^\star} \subset \mu^{\star\star}$.
(b) It is follows from the similar arguments in above case and Corollary 4.17 (b). □

The condition "$(X,\mu^{\mathscr{V}})$ is a sBS" is necessary in Theorem 4.24 (b) as shown by the following Example 4.25

Example 4.25. Consider the generalized topological space (X_4,μ) where $\mu=\{\emptyset,\{a,b\},\{a,b,c\}\}$. Then $\mu^{\mathscr{V}}=\{\emptyset\}\cup\{A \subset X_4 \mid$ either $a \in A$ or $b \in A\}$. Then $(X,\mu^{\mathscr{V}})$ is not a sBS. For, let $U=\{b,d\}; V=\{a,d\}$. Then $U,V \in \mu^{\mathscr{V}}$. But $U \cap V=\{d\}$ which is a $\mu^{\mathscr{V}}$-nowhere dense set. Here $\mu^{\mathscr{V}^\star}=exp(X)$. Thus, $\mu^{\mathscr{V}^\star} \nsubseteq \mu^{\mathscr{V}}$.

Theorem 4.26. *Let (X,μ) be a generalized topological space. Then $\mu^{\star\star^\star} \subset \mu^{\mathscr{V}^\star}$.*

Proof. Follows from the fact that $\mu^{\star\star} \subset \mu^{\mathscr{V}}$. □

Theorem 4.27. *Let (X,μ) be a wBS. Then $\mu^\star \subset \mu^{\mathscr{V}^\star}$.*

Proof. Let $G \in \mu^\star$. If $G=\emptyset$, then there is nothing to prove. Assume that, $G \neq \emptyset$. Then $G=\bigcup_t (G_1^t \cap G_2^t \cap \ldots \cap G_{n_t}^t)$ where $G_1^t, G_2^t, \ldots, G_{n_t}^t \in \mu$. By hypothesis, $\mu \subset \mu^{\mathscr{V}}$. Thus, $G=\bigcup_t (G_1^t \cap G_2^t \cap \ldots \cap G_{n_t}^t)$ where $G_1^t, G_2^t, \ldots, G_{n_t}^t \in \mu^{\mathscr{V}}$. Therefore, $G \in \mu^{\mathscr{V}^\star}$. □

Theorem 4.28. *Let (X,μ) be a GTS and X be a μ-s-II category set. Then $\mu^{\mathscr{V}} \subset \mu^{\mathscr{V}^{\star\star}}$.*

Proof. Let $B \in \mu^{\mathscr{V}}$. Suppose $B=\emptyset$. Then there is nothing to prove. Assume that, $B \neq \emptyset$. Then B is of μ-s-II category set in X. By hypothesis and Proposition 4.15 (d), B is of $\mu^{\mathscr{V}}$-II category set in X. Hence $B \in \mu^{\mathscr{V}^{\star\star}}$. □

Theorem 4.29. *Let (X,μ) be a sBS. Then the following hold.*
(a) $\mu^{\star^{\star\star}} \subset \mu^{\star\star^\star}$.
(b) $\mu^{\star^{\star\star}} \subset \mu^{\mathscr{V}}$.

Proof. This follows from Lemma 2.2 and Lemma 4.20. □

The following Example 4.30 shows that the reverse implications of Theorem 4.26, Theorem 4.27 and Theorem 4.29 (b) are need not be true in a generalized topological space.

Example 4.30. Consider the generalized topological space (X_4,μ) where $\mu=\{\emptyset,\{a,b\},\{b,c\},\{a,b,c\}\}$. Then $\mu^\star=\{\emptyset,\{b\},\{a,b\},\{b,c\},\{a,b,c\}\}$; $\mu^{\star\star}=\{\emptyset\}\cup\{A \subset X \mid b \in A\}$; $\mu^{\mathscr{V}}=\{\emptyset\}\cup\{A \subset X \mid a \in A$ or $b \in A$ or $c \in A\}$; $\mu^{\star^{\star\star}}=\{\emptyset\}\cup\{A \subset X \mid b \in A\}$; $\mu^{\star\star^\star}=\{\emptyset,\{b\},\{a,b\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\},X\}$; $\mu^{\mathscr{V}^\star}=exp(X)$.
(a). Let $A=\{a\}$. Then $A \in \mu^{\mathscr{V}^\star}$. But $A \notin \mu^{\star\star^\star}$.
(b). Let $B=\{c\}$. Then $B \in \mu^{\mathscr{V}^\star}$. But $B \notin \mu^\star$.
(c). Let $C=\{a,d\}$. Then $C \in \mu^{\mathscr{V}}$. But $C \notin \mu^{\star^{\star\star}}$.

The reverse implications of Theorem 4.29 (a) is need not be true as shown by the following Example 4.31.

Example 4.31. Consider the generalized topological space (X_4,μ) where $\mu=\{\emptyset,\{a\},\{a,b\},\{b,c\},\{a,b,c\}\}$. Then $\mu^\star=\{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},\{a,b,c\}\}$; $\mu^{\star\star}=\{\emptyset\}\cup\{A \subset X \mid a \in A$ or $b \in A\}$ and so $\mu^{\star^{\star\star}}=\{\emptyset\}\cup\{B \subset X \mid a \in B$ or $b \in B\}$; $\mu^{\star\star^\star}=exp(X)$. Let $G=\{c\}$. Then $G \in \mu^{\star\star^\star}$. But $G \notin \mu^{\star^{\star\star}}$.

Theorem 4.32. *Let (X,μ) be a BS-sGTS. If $(X,\mu^{\star\star})$ is a sBS, then $\mu^{\star\star^\star} \subset \mu^{\mathscr{V}^{\star\star}}$.*

Proof. Let A be a non-null $\mu^{\star\star^\star}$-open set. Then $A=\bigcup_t (A_1^t \cap A_2^t \cap \ldots \cap A_{n_t}^t)$ where $A_1^t, A_2^t, \ldots, A_{n_t}^t \in \mu^{\star\star}$. Take $A_k = A_1^k \cap A_2^k \cap \ldots \cap A_{n_k}^k$ where $A_1^k, A_2^k, \ldots, A_{n_k}^k \in \mu^{\star\star}$ such that $A_k \neq \emptyset$ for some k. By hypothesis, A_k is of $\mu^{\star\star}$-II category set and so A_k is of μ-II category set, by hypothesis and Lemma 4.21. Thus, A is of μ-II category set so that A is of μ-s-II category set. Hence $A \in \mu^{\mathscr{V}}$. By Theorem 4.28, $A \in \mu^{\mathscr{V}^{\star\star}}$. Hence $\mu^{\star\star^\star} \subset \mu^{\mathscr{V}^{\star\star}}$. □

5. Some Special Spaces

In this section, we analyze the nature of extremally disconnected and submaximal spaces in a generalized topological space. Finally, we prove every μ-isolated point is a μ-II category set in a GTS.

A GTS (X,μ) is called μ-extremally disconnected or simply, extremally-disconnected [4] if the μ-closure of every μ-open set is μ-open.

A subset B of a generalized topological space (X,μ) is said to be a *μ-G_δ-set* [1] if $B = \bigcap\limits_{n\in\mathbb{N}} B_n$ where each B_n is a μ-open set.

A generalized topological space (X,μ) is said to be a *generalized G_δ-submaximal space* [1] if every μ-dense subset of X is a μ-G_δ-set in X.

Lemma 5.1. [1, Lemma 3.7] Let (X,μ) be a GTS. If (X,μ) is a generalized submaximal space, then (X,μ) is a generalized G_δ-submaximal space.

Lemma 5.2. [13, Theorem 3.2] Let (X,μ) be a GTS. Then the following hold.
(a) $\mu^{\star\star} \neq \{\emptyset\}$ *if and only if* $(X,\mu^{\star\star})$ *is a sGTS.*
(b) *If* (X,μ) *is a BS, then* $\mu^{\star\star} \neq \{\emptyset\}$.

Theorem 5.3. *Let (X,μ) be a GTS. If either (X,μ) is a BS or $\mu^{\star\star} \neq \{\emptyset\}$, then $(X,\mu^{\star\star})$ is a $\mu^{\star\star}$-extremally disconnected space.*

Proof. We will present the detailed proof only for the case, $\mu^{\star\star} \neq \{\emptyset\}$. Then $\mu^{\star\star}$ is a sGTS, by Lemma 5.2. Let $G \in \mu^{\star\star}$. If $G = \emptyset$, then $c_{\mu^{\star\star}}G = G$ and so $c_{\mu^{\star\star}}G \in \mu^{\star\star}$. Suppose that $G \neq \emptyset$. Then G is of μ-II category set in X. Since $G \subset c_{\mu^{\star\star}}G$ and subset of a μ-meager set is μ-meager we have $c_{\mu^{\star\star}}G$ is of μ-II category set in X. Thus, $c_{\mu^{\star\star}}G \in \mu^{\star\star}$. Hence $(X,\mu^{\star\star})$ is a $\mu^{\star\star}$-extremally disconnected space. □

The following Example 5.4 shows that the condition "either (X,μ) is a Baire space or $\mu^{\star\star} \neq \{\emptyset\}$" can not be dropped in Theorem 5.3.

Example 5.4. Consider the generalized topological space (X,μ) where $X = [0,3]$ and $\mu = \{\emptyset,[0,2),(1,3],[0,1]\cup[2,3],X\}$. Then (X,μ) is not a BS and $\mu^{\star\star} = \{\emptyset\}$. Choose $G \in \mu^{\star\star}$. Then $G = \emptyset$ and so $c_{\mu^{\star\star}}G = X$. But $X \notin \mu^{\star\star}$. Thus, $c_{\mu^{\star\star}}G \notin \mu^{\star\star}$. Hence $(X,\mu^{\star\star})$ is not a $\mu^{\star\star}$-extremally disconnected space.

Theorem 5.5. *Let (X,μ) be a GTS. Then the following hold.*
(a) If $\tilde{\mu}$ is a stack, then (X,μ) is a μ-extremally disconnected space.
(b) If $\tilde{\mu}$ is a p-stack, then (X,μ) is a μ-extremally disconnected space.

Proof. It is enough to prove that (a) only, since every p-stack is a stack. Suppose that, $\tilde{\mu}$ is a stack. Then (X,μ) is a sGTS. Let $U \in \mu$. If $U = \emptyset$, then $c_\mu U = \emptyset$, since μ is a sGT. Thus, $c_\mu U \in \mu$. Assume that, $U \neq \emptyset$. Since $U \subset c_\mu U$ and $\tilde{\mu}$ is a stack we have $c_\mu U \in \mu$. Then (X,μ) is a μ-extremally disconnected space. □

Next, Example 5.6 shows that the condition "$\tilde{\mu}$ is a stack" can not be dropped in the above Theorem 5.5 (a). The reverse implications of Theorem 5.5 is need not be true as shown by the below Example 5.7.

Example 5.6. (a) Consider the generalized topological space (X,μ) where $X = [0,3]$ and $\mu = \{\emptyset,[0,1),[0,2),(1,3],[0,1)\cup(1,3],[0,2)\cup[\frac{5}{2},3],X\}$. Let $A = (1,3]$ and $B = [1,3]$ be subsets of X. Here $A \in \tilde{\mu}$ and $A \subset B$. But $B \notin \tilde{\mu}$. Thus, $\tilde{\mu}$ is not a stack. Take $G = [0,1)$. Then $G \in \mu$ and $c_\mu G = [0,1]$. But $c_\mu G \notin \mu$. Hence (X,μ) is not a μ-extremally disconnected space.
(b) Consider the generalized topological space (X_6,μ) where $\mu = \{\emptyset,\{a,b\},\{b,c\},\{a,b,c\}\}$. Let $A = \{a,b\}$ and $B = \{a,b,c,d\}$ be subsets of X_6. Here $A \in \tilde{\mu}$ and $A \subset B$. But $B \notin \tilde{\mu}$. Thus, $\tilde{\mu}$ is not a stack. Take $G = \{\emptyset\}$. Then $G \in \mu$ and $c_\mu G = \{d,e,f\}$. But $c_\mu G \notin \mu$. Hence (X_6,μ) is not a μ-extremally disconnected space.

Example 5.7. (a) Consider the generalized topological space (X_5,μ) where $\mu = \{\emptyset,\{a,c\},\{b,c\},\{a,b,c\},X_5\}$. Then (X_5,μ) is a μ-extremally disconnected space. Let $A = \{a,c\}$ and $B = \{a,b,c,d\}$ be subsets of X_5. Here $A \in \tilde{\mu}$ and $A \subset B$. But $B \notin \tilde{\mu}$. Thus, $\tilde{\mu}$ is not a stack.
(b) Consider the generalized topological space (X_4,μ) where $\mu = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},X_4\}$. Then (X_4,μ) is a μ-extremally disconnected space. But $\tilde{\mu}$ is not a p-stack. For, let $A = \{a\}$ and $B = \{b,c,d\}$ be subsets of X_4. Here $A \in \tilde{\mu}$ and $B \in \tilde{\mu}$. But $A \cap B = \emptyset$.

The following Theorem 5.8 is directly follows from the fact that subset of a μ-s-meager set is μ-s-meager and the converse part is trivial so the proof is omitted.

Theorem 5.8. *Let (X,μ) be a GTS. Then $\mu^{\mathscr{V}} \neq \{\emptyset\}$ if and only if $(X,\mu^{\mathscr{V}})$ is a $\mu^{\mathscr{V}}$-extremally disconnected space.*

Theorem 5.9. *Let (X,μ) be a hyperconnected sGTS. Then (X,μ) is a μ-extremally disconnected space.*

Proof. This is a direct consequence of the definition of the hyperconnected space. □

By Lemma 2.1, immediately we get the following two observations so the proofs are omitted.

Observation 5.10. *Let (X,μ) be a GTS and $\mu = \{\emptyset\} \cup \{A \subset X \mid A$ is a μ-nowhere dense set$\}$. If μ is a GT, then (X,μ) is a μ-extremally disconnected space.*

Observation 5.11. *Let (X,μ) be a GTS and $\mu = \{\emptyset\} \cup \{A \subset X \mid A$ is a μ-strongly nowhere dense set$\}$. If μ is a GT, then (X,μ) is a μ-extremally disconnected space.*

Moreover, every GTS (X,μ) is both $\mu^{\star\star}$-extremally disconnected space and $\mu^{\mathscr{V}}$-extremally disconnected space.

Lemma 5.12. *Let (X,μ) be a GTS and $A \subset X$. Then the following hold.*
(a) If A is a $\mu^{\star\star}$-dense set in X, then $A \in \mu^{\star\star}$.
(b) If A is a $\mu^{\mathscr{V}}$-dense set in X, then $A \in \mu^{\mathscr{V}}$.

Proof. We will present the detailed proof only for (a). Let A be a $\mu^{\star\star}$-dense subset of X. Then $A \cap B_i \neq \emptyset$ for all $B_i \in \tilde{\mu}^{\star\star}$.
Case 1: First we prove this result for a singleton set in $\tilde{\mu}^{\star\star}$. Assume that, each C_i is a singleton set in $\tilde{\mu}^{\star\star}$. Since $A \cap C_i \neq \emptyset$ for all $C_i \in \tilde{\mu}^{\star\star}$ we have $C_i \subset A$ for all $C_i \in \tilde{\mu}^{\star\star}$. Therefore, A is of μ-II category set in X, since subset of a μ-meager set is μ-meager.
Case 2: Now we prove this result for other set in $\tilde{\mu}^{\star\star}$. Assume that, each B_i having more than one element. Then each B_i contains a non-null singleton set which is of μ-II category set in X. By Case 1, A is of μ-II category set in X. □

The reverse implication of Lemma 5.12 need not be true as shown by Example 5.13.

Example 5.13. (a) Consider the generalized topological space (X,μ) where $X = [0,3]$ and $\mu = \{\emptyset, [0,2), (1,3], [0,1) \cup (1,3], [0,2) \cup [\frac{5}{2},3], X\}$. Then $\mu^{\star\star} = \{\emptyset\} \cup \{A, B \subset X \mid A \in exp((1,2)) - \{\emptyset\}, A \subset B\}$. Let $A = \{\frac{3}{2}\}$ be a subset of X. Then $A \in \mu^{\star\star}$. But A is not a $\mu^{\star\star}$-dense set in X.
(b) Consider the generalized topological space (X_6,μ) where $\mu = \{\emptyset, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}, \{a,b,c,e\}\{a,b,c,d,e\}\}$. Then $\mu^{\mathscr{V}} = \{\emptyset\} \cup \{A \subset X_6 \mid$ either $a \in A$ or $b \in A\}$. Let $G = \{a,d\}$ be a subset of X_6. Then $G \in \mu^{\mathscr{V}}$. But G is not a $\mu^{\mathscr{V}}$-dense set in X_6.

Theorem 5.14. *Let (X,μ) be a GTS. Then $\mu^{\mathscr{V}} \neq \{\emptyset\}$ if and only if $(X,\mu^{\mathscr{V}})$ is a generalized submaximal space.*

Proof. Let A be a $\mu^{\mathscr{V}}$-dense set in X. Then A is a $\mu^{\mathscr{V}}$-open set in X, by Lemma 5.12(b). Therefore, $(X,\mu^{\mathscr{V}})$ is a generalized submaximal space. Converse implication is trivial. □

Corollary 5.15 is directly follows from Lemma 5.1 and Theorem 5.14 so the proof is omitted.

Corollary 5.15. *Let (X,μ) be a GTS. Then $\mu^{\mathscr{V}} \neq \{\emptyset\}$ if and only if $(X,\mu^{\mathscr{V}})$ is a generalized G_δ-submaximal space.*

Theorem 5.16. *Let (X,μ) be a GTS. Then $\mu^{\star\star} \neq \{\emptyset\}$ if and only if $(X,\mu^{\star\star})$ is a generalized submaximal space.*

Corollary 5.17. *Let (X,μ) be a GTS. Then $\mu^{\star\star} \neq \{\emptyset\}$ if and only if $(X,\mu^{\star\star})$ is a generalized G_δ-submaximal space.*

In the rest of this section, we analyze the nature of an isolated point in a GTS. First of all, we remind the definition for isolated point in a generalized topological space.

Let (X,μ) be a generalized topological space. Then $x \in X$ is called μ-isolated [1] if $\{x\}$ is μ-open. If every point of X is μ-isolated, then X is called μ-discrete [1].

Theorem 5.18. *Let (X,μ) be a GTS and $x \in X$. If x is a μ-isolated point, then the following hold.*
(a) $\{x\} \in \mu^{\star\star}$.
(b) $\{x\} \in \mu^{\mathscr{V}}$.
(c) X is of μ-II category.
(d) X is of μ-s-II category.

Proof. (a) Let $x \in X$. Suppose x is a μ-isolated point in X. Take $A = \{x\}$. Then A is a μ-open subset of X and so A is not a μ-nowhere dense set. Thus, A is not a μ-meager set so that A is of μ-II category set in X. Therefore, $\{x\} \in \mu^{\star\star}$.
(b) Since $\mu^{\star\star} \subset \mu^{\mathscr{V}}$ we have $\{x\} \in \mu^{\mathscr{V}}$, by (a).
(c) Superset of a μ-II category set is of μ-II category so that X is of μ-II category.
(d) Since every μ-s-meager set is μ-meager we have X is of μ-s-II category. □

Theorem 5.19 immediately follows from Theorem 5.18 so the trivial proof is removed. The reverse implications of the Theorem 5.19 is not true in general as shown in the below Example 5.20.

Theorem 5.19. *Let (X,μ) be a GTS and $A \subset X$. Then the following hold.*
(a) If X is μ-discrete, then (X,μ) is a sBS.
(b) If A contains a μ-isolated point, then $A \in \mu^{\star\star}$ and hence $A \in \mu^{\mathscr{V}}$.

Example 5.20. Consider the generalized topological space (X_4,μ) where $\mu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}, X_4\}$. Then (X_4,μ) is a sBS. But X_4 is not a μ-discrete space. For, let $a \in X_4$. Then $\{a\}$ is not μ-open and so a is not a μ-isolated point in X_4.

Theorem 5.21. *Let (X,μ) be a GTS and $A \subset X$. Then A is of μ-II category set in X if and only if it has a $\mu^{\star\star}$-isolated point in X.*

The following Example 5.22 proves that X is not μ-discrete even if X is $\mu^{\star\star}$-discrete.

Example 5.22. Consider the generalized topological space (X_4,μ) where $\mu = \{\emptyset, \{a,b\}, \{c,d\}, X_4\}$. Here $\{x\} \in \mu^{\star\star}$ for all $x \in X_4$. Therefore, X_4 is $\mu^{\star\star}$-discrete. But X_4 is not a μ-discrete space. For, let $b \in X_4$. Then $\{b\}$ is not μ-open and so b is not a μ-isolated point in X_4.

Lemma 5.23. [14, Theorem 4.3] Let (X,μ) be a hyperconnected space. If X is of μ-II category, then (X,μ) is a BS.

Theorem 5.24. *Let (X,μ) be a GTS and $\tilde{\mu}$ is a p-stack. If X is of μ-II category, then (X,μ) is a BS.*

Proof. Suppose $\tilde{\mu}$ is a p-stack. Then (X,μ) is a hyperconnected space. By Lemma 5.23, (X,μ) is a BS. □

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