Universal Journal of Mathematics and Applications

VOLUME III
ISSUE III

ISSN 2619-9653

http://dergipark.gov.tr/ujma

VOLUME III ISSUE III
ISSN 2619-9653

September 2020
http://dergipark.gov.tr/ujma

UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

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Universal Journal of Mathematics and Applications, 3 (3) (2020) 93-101
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.702033

Application of the Aboodh Transform for Solving Fractional Delay Differential Equations

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Article Info

Keywords: *Aboodh Transform, Fractional Delay differential Equation, Partial derivatives, function series.*
2010 AMS: *65D15, 65L05*
Received: *11 March 2020*
Accepted: *13 July 2020*
Available online: *29 September 2020*

Abstract

In this article, we extend the concept of the Aboodh transform to the solution of partial differential equations of fractional order using Caputo's fractional derivative. The transformation concerned is applicable to solve many classes of partial differential equations with derivatives of order and integrals. Consequently, the fractional Delay Differential Equations (DDEs) which we are going to study in this work. The resulting numerical proofs show that the method converges favorably towards the analytical solution.

1. Introduction

Many physical problems can be described by mathematical models that involve ordinary or partial differential equations. A mathematical model is a simplified description of physical reality expressed in mathematical terms. Thus, the investigation of the exact or approximation solution helps us to understand the means of these mathematical models. Several numerical methods were developed for solving ordinary or partial differential equations. the Aboodh transform method is used to solve the linear and nonlinear fractional delay differential equations (FDDEs). New integral transform Aboodh transform is particularly useful for finding solutions for fractional delay differential equation. Aboodh transform is a useful technique for solving these equations but this transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. [1, 3–7]. Fractional delay differential equations (FDDEs) are a very recent topic. Although it seems natural to model certain processes and systems in engineering and other science (with memory and heritage properties) with this kind of equations, only in the last few years has the attention of the scientific community been devoted to them. The purpose of this work is to find the approximate solution of delay differential equations of fractional order using aboodh transform and adomian decomposition methods [9].
This work consists of three sections, as well as, this introduction. In Section 2, definition and properties of the Aboodh transform and method of solution is presented. Section 3, basic concepts of delay differential equations and fractional calculus are given. Finally, the application of the Aboodh transform method for solving delay differential equations of fractional order, with illustrative examples have been given.

2. The Aboodh Transform

The Aboodh transform is a new integral transform similar to the Laplace transform and other integral transforms that are defined in the time domain $t \geq 0$, such as the Sumudu transform, the Natural transform and the Elzaki transform, respectively [4].
An Aboodh transform is defined for functions of exponential order. We consider functions in the set F defined by:

$$F = \left\{ f(t) : \left| f(t) \right| < Me^{-vt},\ if\ t \in \left[0;\infty\right[,\ M, k_1, k_2 > 0;\ k_1 \leq v \leq k_2 \right\}.$$

For a given function in the set F, the constant M must be finite number and k_1, k_2 may be infinite or finite with variable v define as $k_1 \leq v \leq k_2$.

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$f(t)$	$T(v)=A[f(t)]$
1	$\frac{1}{v^2}$
t	$\frac{1}{v^3}$
$t^n,\ n\geq 1$	$\frac{n!}{v^{n+2}}$
e^{at}	$\frac{1}{v^2-av}$
$sin(at)$	$\frac{1}{v(v^2+a^2)}$
$cos(at)$	$\frac{1}{v^2+a^2}$
$sinh(at)$	$\frac{1}{v(v^2-a^2)}$
$t\,cosh(at)$	$\frac{1}{v^2-a^2}$

Table 1: Aboodh transform of some functions

Then, the Aboodh integral transform denoted by the operator $A(:)$ is defined by the integral equation:

$$T(v)=A[f(t)]=\frac{1}{v}\int_0^{\infty} f(t)e^{-vt}dt, \quad t\geq 0,\ k_1\leq v\leq k_2. \tag{2.1}$$

Standard Aboodh transform for some special functions found are given below in Table 1.

Theorem 2.1. *Aboodh transform of some partial derivatives :*

$$(i)-A[f'(t)]=vT(v)-\frac{f(0)}{v},$$

$$(ii)-A[f''(t)]=v^2T(v)-\frac{f'(0)}{v}-f(0),$$

$$(iii)-T^n(v)=A\left[f^{(n)}(t)\right]=v^nT(v)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2-n+k}}. \tag{2.2}$$

Remark 2.2. *The Aboodh transform is linear, i.e., if α and β are any constants and $f(t)$ and $g(t)$ are functions defined over the set F above, then :*

$$A[\alpha f(t)]\pm\beta g(t)]=\alpha A[f(t)]\pm\beta A[g(t)].$$

2.1. Aboodh transform method

Let us consider the general nonlinear ordinary differential equation (ODE) of the form [8]:

$$\frac{d^ny(t)}{dt^n}+P(y)+Q(t-\tau)=g(t),\ n=1,2,3,..... \tag{2.3}$$

with initial condition:

$$y^{(k)}(0)=y_0^k, \tag{2.4}$$

where $\frac{d^ny}{dt^n}$ is the derivative of y of order n , P is the linear bounded operator , Q is a nonlinear bounded operator and $g(t)$ is a given continuous function, and $y=y(t)$.

Inspired by Wu [4] method, if we take Abooth transform on both sides of Eqs((2.3)-(2.4)), the linear part with constant coefficients is then transferred into an algebraic one, so that we can identify the Lagrange multiplier in a more straightforward way. Now, we extend this idea to find the unknown Lagrange multiplier. Taking the above Aboodh transform to both sides of eq(2.3),and (2.2), then the linear part is transformed into an algebraic equation as follows:

Applying the Aboodh transform , we obtain

$$A\left[\frac{d^ny(t)}{dt^n}\right]+A[P(y)]+A[Q(t-\tau)]=A[g(t)] \tag{2.5}$$

.

But

$$A\left[\frac{d^ny(t)}{dt^n}\right]=v^nA(y(t))-\frac{E}{v^{2-n+k}} \tag{2.6}$$

where $E=\sum_{k=0}^{n-1} g^{(k)}(0)$,

$$A\left(y(t)\right)=\frac{E}{v^{2+k}}-v^{-n}A\left[P(y)\right]-v^{-n}A\left[Q\left(t-\tau\right)\right]+v^{-n}A\left[g(t)\right]. \tag{2.7}$$

The standard Aboodh decomposition method defines the solution $y(t)$ by the series:

$$y(t)=\sum_{n=0}^{\infty} y_n(t), \tag{2.8}$$

the nonlinear operator is decomposed as:

$$Q\left(t-\tau\right)=\sum_{n=0}^{\infty} B_n, \tag{2.9}$$

where B_n is the a domain polynomial of y_0,y_1, y_2, ...,$y_n$that are given by:

$$B_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[Q\left(\sum_{n=0}^{\infty}\lambda^n y_n\right)\right],$$

then the Adomian series reads :

$$\begin{cases} B_0=f\left(y_0\right) \\ B_1=y_1 f'\left(y_0\right) \\ B_2=y_2 f'\left(y_0\right)+\frac{1}{2}y_1^2 f''\left(y_0\right) \\ B_3=y_3 f'\left(y_0\right)+y_1 y_2 f''\left(y_0\right)+\frac{1}{3!}y_1^3 f'''\left(y_0\right) \\ \vdots \end{cases} \tag{2.10}$$

Applying (2.8) and (2.9) into (2.7), we obtain

$$A\left(\sum_{n=0}^{\infty} y_n\right)=\frac{E}{v^{2+k}}-v^{-n}A\left[P(\sum_{n=0}^{\infty} y_n)\right]-v^{-n}A\left[\sum_{n=0}^{\infty} B_n\right]+v^{-n}A\left[g(t)\right], \tag{2.11}$$

comparing both side of (2.11):

$$A\left[y_0\right]=\frac{E}{v^{2+k}}+v^{-n}A\left[g(t)\right], \tag{2.12}$$

$$A\left[y_1\right]=-v^{-n}A\left[Py_0\right]-v^{-n}A\left[B_0\right], \tag{2.13}$$

$$A\left[y_2\right]=-v^{-n}A\left[Py_1\right]-v^{-n}A\left[B_1\right]. \tag{2.14}$$

In general the recursive relation is given by:

$$A\left[y_n\right]=-v^{-n}A\left[Py_{n-1}\right]-v^{-n}A\left[B_{n-1}\right], \quad n\geqslant 1 \tag{2.15}$$

By the Aboodh transformed inverse method to(2.12)-(2.15), we get :

$$y_0=K(t)$$

$$y_n=-A^{-1}\left[v^{-n}A\left[Py_{n-1}\right]\right]-A^{-1}\left[v^{-n}A\left[B_{n-1}\right]\right], \quad n\geqslant 1.$$

where $K(t)$ is a function that satisfies the initial conditions.

2.2. Illustrative examples

Example 2.3. *Consider the following nonlinear delay differential equation (NDDE) of first order:*

$$y'(t) = 1 + 2y^2\left(\tfrac{t}{2}\right) \quad , 0 \le t \le 1 \ , \tag{2.16}$$

with the initial condition $y(0) = 0$. *The exact solution of the problem is :*

$$y(t) = sinh(t).$$

Applying the Aboodh transform on both sides, we have:

$$A\left[y'(t)\right] = A\left[1\right] + 2A\left[y^2\left(\frac{t}{2}\right)\right],$$

By definition (ii) *in theorem (2.1), we have :*

$$A\left[y'(t)\right] = vT(v) - \frac{f(0)}{v} = \frac{1}{v^2} + 2A\left[f^2\left(\frac{t}{2}\right)\right].$$

So, we have :

$$T(v) = A(y(t)) = \frac{1}{v^3} + \frac{2}{v}A\left[y^2\left(\frac{t}{2}\right)\right]. \tag{2.17}$$

Applying the Aboodh inverse operator, A^{-1} *on both sides of (2.17), we obtain:*

$$y(t) = A^{-1}\left[\frac{1}{v^3}\right] + 2A^{-1}\left[\frac{1}{v}A\left[y^2(\frac{t}{2})\right]\right]. \tag{2.18}$$

Using the Table 1, we have: $A^{-1}\left[\frac{1}{v^3}\right] = t$, *hence:*

$$y_0(t) = A^{-1}\left[\frac{1}{v^3}\right] = t,$$

so,

$$y_0\left(\frac{t}{2}\right) = \frac{t}{2}$$

$$y_{n+1} = 2A^{-1}\left[\frac{1}{v}A\left[B_n\right]\right], \tag{2.19}$$

from equation (2.10), we have :

$$\begin{cases} B_0 = f(y_0) = y_0^2\left(\frac{t}{2}\right) = \frac{t^2}{4} \\ B_1 = y_1 f'(y_0) = 2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right) \\ B_2 = y_2\left(\frac{t}{2}\right)2y_0\left(\frac{t}{2}\right) + \frac{1}{2}y_1^2 \\ \vdots \end{cases}.$$

For $n = 0$, *the equation (2.19), become :*

$$y_1(t) = 2A^{-1}\left[\frac{1}{v}A\left[B_0\right]\right] = 2A^{-1}\left[\frac{1}{v}A\left[\frac{t^2}{4}\right]\right] = 2A^{-1}\left[\frac{1}{2v^5}\right] = \frac{t^3}{3!}$$

So,

$$y_1\left(\frac{t}{2}\right) = \frac{t^3}{48}.$$

For $n = 1$, *the equation (2.19), become :*

$$y_2(t) = 2A^{-1}\left[\frac{1}{v}A\left[B_1\right]\right] = 2A^{-1}\left[\frac{1}{v}A\left[2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\right]\right] = 2A^{-1}\left[\frac{1}{v}A\left[\frac{t^4}{48}\right]\right] = \frac{t^5}{5!}$$

So,

$$y_2\left(\frac{t}{2}\right) = \frac{t^5}{3840}.$$

For $n = 2$, *the equation (2.19), become:*

$$y_3(t) = 2A^{-1}\left[\frac{1}{v}A\left[B_2\right]\right] = 2A^{-1}\left[\frac{1}{v}A\left[2y_2\left(\frac{t}{2}\right)y_0\left(\frac{t}{2}\right) + y_1^2\left(\frac{t}{2}\right)\right]\right]$$

$$= 2A^{-1}\left[\frac{1}{v}A\left[\frac{t^6}{3840} + \frac{t^6}{2304}\right]\right] = 2A^{-1}\left[\frac{1}{v}\left[\frac{6!}{3840v^8} + \frac{6!}{2304v^8}\right]\right] = \frac{t^7}{7!}.$$

Therefore the approximate solution is given as:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + ... = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!}...... = sinh(t).$$

Example 2.4. *Consider the following linear delay differential equation (NDDE) of first order:*

$$y'(t) = e^t y\left(\tfrac{t}{2}\right) + y(t) \quad , 0 \le t \le 1$$

,
with the initial condition $y(0) = 1$.
The exact solution of the problem is :

$$y(t) = e^{2t}.$$

Applying the Aboodh transform on both sides, we have :

$$A\left[y'(t)\right] = A\left[e^t y\left(\frac{t}{2}\right) + y(t)\right].$$

By definition (ii) *in theorem (2.1), we have :*

$$vA[y(t)] - \frac{y(0)}{v} = A\left[e^t y\left(\frac{t}{2}\right) + y(t)\right],$$

$$A[y(t)] = \frac{1}{v^2} + \frac{1}{v}A\left[e^t y\left(\frac{t}{2}\right) + y(t)\right]. \tag{2.20}$$

Applying the Aboodh inverse operator, A^{-1}*on both sides of (2.20), we obtain :*

$$y(t) = A^{-1}\left[\frac{1}{v^2}\right] + A^{-1}\left[\frac{1}{v}A\left[e^t y\left(\frac{t}{2}\right) + y(t)\right]\right]. \tag{2.21}$$

By the Aboodh transform method, equation (2.21), can be written as :

$$\begin{aligned} &y_0(t) = 1 \\ y_{n+1}(x) = A^{-1}&\left[\frac{1}{v}A\left[e^t y_n\left(\tfrac{t}{2}\right) + y_n(t)\right]\right], \; n \ge 0. \end{aligned}$$

For $n = 0$*, we have :*

$$y_1(t) = A^{-1}\left[\frac{1}{v}A\left[e^t y_0\left(\frac{t}{2}\right) + y_0(t)\right]\right] = A^{-1}\left[\frac{1}{v}\left[A\left[e^t\right] + A[1]\right]\right]. \tag{2.22}$$

But,

$$A\left[e^t\right] = \frac{1}{v^2 - v} = \frac{1}{v^2}\left(\frac{1}{1 - \frac{1}{v}}\right) = \frac{1}{v^2}\left(1 + \frac{1}{v} + \frac{1}{v^2} + \frac{1}{v^3} + \ldots\right) = \frac{1}{v^2} + \frac{1}{v^3} + \frac{1}{v^4} + \ldots$$

Hence, equation (2.22), can be written as:

$$y_1(t) = A^{-1}\left[\frac{1}{v^3} + \frac{1}{v^4} + \frac{1}{v^5} + \ldots\ldots + \frac{1}{v^3}\right] = 2t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots\ldots$$

we obtain the following approximation for $n \ge 1$*:*

$$y_2(t) = \frac{3t^2}{2} + \frac{5t^3}{12} + \frac{11t^4}{24}. + \ldots\ldots$$

$$y_3(t) = \frac{9t^3}{2} + \frac{15t^4}{128} + \frac{259t^5}{1920} + \ldots\ldots$$

$$\vdots$$

Thus the approximate becomes :

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \ldots$$

$$= 1 + 2t + 2t^2 + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \ldots\ldots\ldots = e^{2t}.$$

3. Fractional Delay Differential Equation

In this section we apply the Aboodh decomposition method to solve linear and nonlinear fractional delay differential equation.

Definition 3.1. *The Aboodh transform of the Caputo fractional derivative is defined as follows [2]:*

$$i) - A\left[D^\alpha f(t)\right] = v^\alpha A[f(t)] - \sum_{k=0}^{n-1} v^{\alpha-2-k} f^{(k)}(0), \quad n-1 < \alpha \leqslant n.$$

$$ii) - A\left(t^\alpha\right) = \frac{\Gamma(\alpha+1)}{v^{\alpha+2}}.$$

3.1. Analysis of the method

Let us consider the general nonlinear ordinary differential equation of the form:

$$D^{\alpha}y(t)+P(y)+Q(t-\tau)=g(t), \quad \tau\in\mathbb{R}, \quad t<\tau, \quad n-1<\alpha\leqslant n. \tag{3.1}$$

With initial condition:

$$y^{(k)}(0)=y_0^k,$$

where $D^{\alpha}y(t)$ is the term of the fractional order derivative , P is the linear bounded operator , Q is a nonlinear bounded operator and $g(t)$ is a given continuous function, and $y=y(t)$.
The Aboodh decomposition method consists of applying the Aboodh transform first on both side of(3.1), to give:

$$A\left[D^{\alpha}y(t)\right]+A\left[P(y)\right]+A\left[Q(t-\tau)\right]=A\left[g(t)\right],$$

by definition (3.1)

$$A(y(t))=\frac{E}{v^{2+k}}-v^{-\alpha}A\left[P(y)\right]-v^{-\alpha}A\left[Q(t-\tau)\right]+v^{-\alpha}A\left[g(t)\right],, \tag{3.2}$$

where $E=\sum_{k=0}^{n-1}g^{(k)}(0)$.
The standard Aboodh decomposition method defines the solution $y(t)$ by the series:

$$y(t)=\sum_{n=0}^{\infty}y_n(t),, \tag{3.3}$$

the nonlinear operator is decomposed as:

$$Q(t-\tau)=\sum_{n=0}^{\infty}B_n. \tag{3.4}$$

Where B_n as in (2.9). The first a domain polynomials are given as in (2.10). Apply (3.3) and (3.4) in (3.2), we have:

$$A\left[\sum_{n=0}^{\infty}y_n(t)\right]=\frac{E}{v^{2+k}}-v^{-\alpha}A\left[P\sum_{n=0}^{\infty}y_n(t)\right]-v^{-\alpha}A\left[\sum_{n=0}^{\infty}B_n\right]+v^{-\alpha}A\left[g(t)\right]. \tag{3.5}$$

Comparing both side of (3.5):

$$A\left[y_0\right]=\frac{E}{v^{2+k}}+v^{-\alpha}A\left[g(t)\right], \tag{3.6}$$

$$A\left[y_1\right]=-v^{-\alpha}A\left[Py_0\right]-v^{-\alpha}A\left[B_0\right],$$

$$A\left[y_2\right]=-v^{-\alpha}A\left[Py_1\right]-v^{-\alpha}A\left[B_1\right].. \tag{3.7}$$

In general the recursive relation is given by:

$$A\left[y_n\right]=-v^{-\alpha}A\left[Py_{n-1}\right]-v^{-\alpha}A\left[B_{n-1}\right], \quad n\geqslant 1, \tag{3.8}$$

applying inverse Aboodh transform to(3.6)-(3.8), then:

$$y_0=K(t),$$

$$y_n=-A^{-1}\left[v^{-\alpha}A\left[Py_{n-1}\right]\right]-A^{-1}\left[v^{-\alpha}A\left[B_{n-1}\right]\right], \quad n\geqslant 1, \tag{3.9}$$

where $K(t)$ is a function that satisfies the initial conditions.

Example 3.2. *Consider the nonlinear delay differential equation of first order:*

$$D^{\alpha}y(t)=1+2y^2\left(\frac{t}{2}\right), \quad 0\leqslant t\leqslant 1, \quad 0<\alpha\leqslant 1.$$

with initial condition:

$$y(0)=0,, \tag{3.10}$$

apply Aboodh transform to both side of equation(3.9), we obtain :

$$A\left[D^{\alpha}y(t)\right]=A\left[1+2y^2\left(\frac{t}{2}\right)\right]$$

by using definition (3.1) and initial condition(3.10), we get :

$$v^{\alpha}A\left[y(t)\right]=\frac{1}{v^{2}}+A\left[2y^{2}\left(\frac{t}{2}\right)\right],$$

$$A\left[y(t)\right]=\frac{1}{v^{2+\alpha}}+\frac{1}{v^{\alpha}}A\left[2y^{2}\left(\frac{t}{2}\right)\right]. \tag{3.11}$$

Applying the inverse Aboodh transform to (3.11), we obtain :

$$y(t)=A^{-1}\left[\frac{1}{v^{2+\alpha}}\right]+A^{-1}\left[\frac{1}{v^{\alpha}}A\left[2y^{2}\left(\frac{t}{2}\right)\right]\right],$$

where,

$$y_{0}\left(t\right)=A^{-1}\left[\frac{1}{v^{2+\alpha}}\right]=\frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)}.$$

So, we have :

$$y_{0}\left(\frac{t}{2}\right)==\frac{t^{\alpha}}{2^{\alpha}\Gamma\left(\alpha+1\right)}.$$

And,

$$y_{n+1}(t)=A^{-1}\left[\frac{1}{v^{\alpha}}A\left[2B_{n}\right]\right]. \tag{3.12}$$

From equation (2.10), we have:

$$\begin{aligned} B_{0}&=y_{0}^{2}\left(\frac{t}{2}\right)\\ B_{1}&=2y_{0}\left(\frac{t}{2}\right)y_{1}\left(\frac{t}{2}\right)\\ B_{2}&=y_{2}\left(\frac{t}{2}\right)2y_{0}\left(\frac{t}{2}\right)+\frac{1}{2}y_{1}^{2}. \end{aligned}$$

For $n=0$, *the equation (3.12), become*

$$\begin{cases} y_{1}(t)=A^{-1}\left[\frac{1}{v^{\alpha}}A\left[2B_{0}\right]\right]=A^{-1}\left[\frac{1}{v^{\alpha}}A\left[2y_{0}^{2}\left(\frac{t}{2}\right)\right]\right]\\ =A^{-1}\left[\frac{1}{v^{\alpha}}A\left[2\left(\frac{t^{\alpha}}{2^{\alpha}\Gamma\left(\alpha+1\right)}\right)^{2}\right]\right]=A^{-1}\left[\frac{1}{v^{\alpha}}A\left[\left(\frac{t^{2\alpha}}{2^{2\alpha-1}\Gamma^{2}\left(\alpha+1\right)}\right)\right]\right]\\ =A^{-1}\left[\frac{1}{v^{3\alpha+2}}\frac{\Gamma\left(2\alpha+1\right)}{2^{2\alpha-1}\Gamma^{2}\left(\alpha+1\right)}\right]=\frac{\Gamma\left(2\alpha+1\right)}{2^{2\alpha-1}\Gamma^{2}\left(\alpha+1\right)}A^{-1}\left[\frac{1}{v^{3\alpha+2}}\right]\\ =\frac{\Gamma\left(2\alpha+1\right)}{2^{2\alpha-1}\Gamma^{2}\left(\alpha+1\right)}\times\frac{t^{3\alpha}}{\Gamma\left(3\alpha+1\right)}, \end{cases}$$

we choose $C=\frac{\Gamma\left(2\alpha+1\right)}{2^{2\alpha-1}\Gamma^{2}\left(\alpha+1\right)}$, *so, we have :*

$$y_{1}(t)=C\frac{t^{3\alpha}}{\Gamma\left(3\alpha+1\right)},$$

and,

$$y_{1}\left(\frac{t}{2}\right)=C\frac{t^{3\alpha}}{2^{3\alpha}\Gamma\left(3\alpha+1\right)},$$

for $n=0$, the equation (3.12), become :

$$\begin{cases} y_2(t)=A^{-1}\left[\frac{1}{v^{\alpha}}A\left[2B_1\right]\right]=A^{-1}\left[\frac{1}{v^{\alpha}}A\left[4y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\right]\right] \\ =A^{-1}\left[\frac{1}{v^{\alpha}}A\left[4\left(\frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}\right)\left(C\frac{t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)}\right),\right]\right] \\ =A^{-1}\left[\frac{1}{v^{\alpha}}A\left[C\left(\frac{t^{4\alpha}}{2^{4\alpha-2}\Gamma(\alpha+1)\Gamma(3\alpha+1)}\right)\right]\right] \\ =A^{-1}\left[\frac{1}{v^{5\alpha+2}}\frac{C.\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(\alpha+1)\Gamma(3\alpha+1)}\right]=\frac{C.\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(\alpha+1)\Gamma(3\alpha+1)}A^{-1}\left[\frac{1}{v^{5\alpha+2}}\right] \\ =\frac{C.\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(\alpha+1)\Gamma(3\alpha+1)}\times\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}. \end{cases}$$

The series solution is given by:

$$y(t)=y_0(t)+y_1(t)+y_2(t)+y_3(t)+...$$

In particular case $\alpha=1$, then we obtain:

$$\begin{cases} y_0(t)=\frac{t}{\Gamma(2)}=t \\ y_1(t)=\frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)}\times\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}=\frac{\Gamma(3)}{2^1\Gamma^2(2)}\times\frac{t^3}{\Gamma(4)}=\frac{t^3}{3!} \\ y_2(t)=\frac{C.\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(\alpha+1)\Gamma(3\alpha+1)}\times\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}==\frac{C.\Gamma(5)}{2^2\Gamma(2)\Gamma(4)}\times\frac{t^5}{\Gamma(6)}=\frac{t^5}{5!}. \end{cases}$$

Recall that,

$$\Gamma(n+1)=n!,\quad \forall n\in\mathbb{N}.$$

The exact solution when $\alpha=1$ is given by :

$$y(t)=t+\frac{t^3}{3!}+\frac{t^5}{5!}+...=sinh(t).$$

Example 3.3. *Consider the nonlinear delay differential equation of first order$(n=2)$*

$$D^{\alpha}y(t)=1-2y^2\left(\frac{t}{2}\right),\quad 0\leqslant t\leqslant 1,\quad 1<\alpha\leqslant 2. \tag{3.13}$$

with initial condition

$$y(0)=1,\quad y'(0)=0.$$

Apply Aboodh transform to both side of (3.13), we get :

$$A\left[D^{\alpha}y(t)\right]=A\left[1-2y^2\left(\frac{t}{2}\right)\right].$$

Using definition (3.1) and initial condition :

$$v^{\alpha}A\left[y(t)\right]-v^{\alpha-2}y(0)-v^{\alpha-3}y'(0)=\frac{1}{v^2}-2A\left[y^2\left(\frac{t}{2}\right)\right]$$

$$A\left[y(t)\right]=\frac{1}{v^2}+\frac{1}{v^{2+\alpha}}-2v^{-\alpha}A\left[2y\left(\frac{t}{2}\right)\right]. \tag{3.14}$$

Applying the inverse Aboodh transform to(3.14), we get :

$$y(t)=A^{-1}\left[\frac{1}{v^2}\right]+A^{-1}\left[\frac{1}{v^{2+\alpha}}\right]-2A^{-1}\left[v^{-\alpha}A\left[y^2\left(\frac{t}{2}\right)\right]\right]$$

$$y_0(t)=A^{-1}\left[\frac{1}{v^2}\right]+A^{-1}\left[\frac{1}{v^{2+\alpha}}\right]=1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$y_0\left(\frac{t}{2}\right)=1+\frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}$$

$$y_{n+1} = -2A^{-1}\left[v^{-\alpha}A\left[B_n\right]\right], \tag{3.15}$$

from equation (2.10), we have :

$$\begin{cases} B_0 = f(y_0) = y_0^2\left(\frac{t}{2}\right) = \left(1+\frac{t^\alpha}{2^\alpha\Gamma(\alpha+1)}\right)^2 = 1+\frac{t^\alpha}{2^{\alpha-1}\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{2^{2\alpha}\Gamma^2(\alpha+1)} \\ B_1 = y_1 f'(y_0) = 2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right) \\ B_2 = y_2\left(\frac{t}{2}\right)2y_0\left(\frac{t}{2}\right)+\frac{1}{2}y_1^2. \\ \vdots \end{cases}$$

For $n=0$, the equation (3.15), become:

$$\begin{cases} y_1 = -2A^{-1}\left[v^{-\alpha}A\left[B_0\right]\right] = -2A^{-1}\left[v^{-\alpha}A\left[1+\frac{t^\alpha}{2^{\alpha-1}\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{2^{2\alpha}\Gamma^2(\alpha+1)}\right]\right] \\ = -2A^{-1}\left[\frac{1}{v^{\alpha+2}}+\frac{1}{2^{\alpha-1}v^{2\alpha+2}}+\frac{\Gamma(2\alpha+1)}{2^{2\alpha}v^{3\alpha+2}\Gamma^2(\alpha+1)}\right] \\ = \frac{-2t^\alpha}{\Gamma(\alpha+1)}-\frac{t^{2\alpha}}{2^{\alpha-2}\Gamma(2\alpha+1)}-\frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}. \end{cases}$$

The series solution is given by :

$$\begin{cases} y(t) = y_0(t)+y_1(t)+y_2(t)+y_3(t)+\ldots \\ = 1+\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{2t^\alpha}{\Gamma(\alpha+1)}-\frac{t^{2\alpha}}{2^{\alpha-2}\Gamma(2\alpha+1)}-\frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}+\ldots.. \end{cases}$$

In particular case $\alpha=2$, then we obtain :

$$\begin{cases} y_0(t) = 1+\frac{t^2}{\Gamma(3)} = 1+\frac{t^2}{2!} = 1+\frac{t^2}{2} \\ y_1(t) = \frac{-2t^\alpha}{\Gamma(\alpha+1)}-\frac{t^{2\alpha}}{2^{\alpha-2}\Gamma(2\alpha+1)}-\frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} \\ = -\frac{2t^2}{2!}-\frac{t^4}{2^0\Gamma(5)}-\frac{t^6\Gamma(5)}{2^3\Gamma^2(3)\Gamma(7)} = -t^2-\frac{t^4}{4!}-\frac{3t^6}{4\times 6!}. \end{cases}$$

The exact solution when $\alpha=2$ is given by :

$$y(t) = y_0(t)+y_1(t)+\ldots = 1-\frac{t^2}{2}+\frac{t^4}{4!}-\ldots = cos(t).$$

4. Conclusion

This study aims to propose an efficient algorithm for the solution of nonlinear fractional equations. The adomian decomposition method has been recognized as a powerful technique to solve many nonlinear differential equations. In this work, a combined method which groups together the transform (2.1) and the adomian decomposition are discussed to find an explicit approximate solution for fractional Delay Differential Equations (DDEs).

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Universal Journal of Mathematics and Applications, 3 (3) (2020) 102-108
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.669276

On Geometric Circulant Matrices Whose Entries are Bi-Periodic Fibonacci and Bi-Periodic Lucas Numbers

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Article Info

Keywords: *Bi–periodic Fibonacci numbers, Bi–periodic Lucas numbers, Geometric circulant matrices, Spectral norm.*
2010 AMS: *15A60, 11B39, 15B05.*
Received: *2 February 2020*
Accepted: *13 July 2020*
Available online: *29 September 2020*

Abstract

In this study, we obtain upper and lower bounds for the spectral norms of the geometric circulant matrices with the bi–periodic Fibonacci numbers and bi–periodic Lucas numbers, respectively. Then we give some bounds for the spectral norms of Kronecker and Hadamard products of these matrices.

1. Introduction

The well-known Fibonacci and Lucas sequences are given by the following recursive equations: for $n \geq 0$,

$$F_0 = 0,\ F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n$$

and

$$L_0 = 2,\ L_1 = 1, \qquad L_{n+2} = L_{n+1} + L_n,$$

respectively.

Many researchers gave various generalizations of the Fibonacci sequence in past fifty years. An interesting one, called bi–periodic Fibonacci sequence, was introduced by Edson and Yayenie in [5] as follows:

$$q_0 = 0,\ q_1 = 1, \text{ and } q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even;} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd.} \end{cases} \qquad (n \geq 2),$$

where a and b are nonzero real numbers. They obtained many identities for the sequence $\{q_n\}_{n=0}^{\infty}$. For instance, they gave the following extended Binet formula

$$q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \right) \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad (n \geq 0)$$

where $\alpha = \left(ab + \sqrt{a^2b^2 + 4ab}\right)/2$ and $\beta = \left(ab - \sqrt{a^2b^2 + 4ab}\right)/2$. Here, $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$ is the parity function.

In [4], Bilgici gave a general form of the Lucas sequence similar to the generalized Fibonacci sequence $\{q_n\}_{n=0}^{\infty}$ as follows:

$$l_0 = 2,\ l_1 = a, \text{ and } l_n = \begin{cases} al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd;} \\ bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even.} \end{cases} \qquad (n \geq 2),$$

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where a and b are nonzero real numbers. He also derived many identities for the sequence $\{l_n\}_{n=0}^{\infty}$. For example, he gave the following extended Binet formula

$$l_n = \left(\frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right)(\alpha^n + \beta^n), \quad (n \geq 0).$$

The $n \times n$ r–circulant matrix, C_r, associated with the numbers $c_0, c_1, ..., c_{n-1}$ is of the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i \\ rc_{n+j-i}, & j < i \end{cases}$$

that is

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}.$$

For $r = 1$, the r–circulant matrix C_r reduces to circulant matrix C , i.e.,

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

Circulant matrices, r–circulant matrices, and their versions have been studied in many papers. For example, in [20], Solak found some bounds for the spectral norms of circulant matrices with the Fibonacci and Lucas number entries. Afterwards, Shen and Cen [16] developed Solak's results. Later, many researchers studied different types of these matrices. For more details, we refer the interested reader to [1–3, 6, 8, 9, 12, 15, 17–19, 21–23, 25].
In [10], Kızılateş and Tuğlu defined the $n \times n$ geometric circulant matrix, C_{s^*}, associated with the numbers $c_0, c_1, ..., c_{n-1}$ as

$$C_{s^*} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ sc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ s^2c_{n-2} & sc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s^{n-2}c_2 & s^{n-3}c_3 & s^{n-4}c_4 & \cdots & c_0 & c_1 \\ s^{n-1}c_1 & s^{n-2}c_2 & s^{n-3}c_3 & \cdots & sc_{n-1} & c_0 \end{pmatrix}.$$

They calculated bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci numbers and hyperharmonic Fibonacci numbers. Same authors [11] also found the norms of geometric and symmetric geometric circulant matrices with the Tribonacci numbers. In [13], Köme and Yazlık presented some bounds for the spectral norms of the r-circulant matrices with the bi–periodic Fibonacci and Lucas numbers.
The purpose of this paper is to find some new upper and lower bounds for the spectral norms of the geometric circulant matrices with the bi-periodic Fibonacci numbers and bi–periodic Lucas numbers, respectively.
Now we need the following definitions and lemmas to derive new bounds.
The Euclidean (Frobenius) norm of matrix A $\left(A = (a_{ij})\right.$ be any $m \times n$ matrix$\left.\right)$ is defined as

$$\|A\|_E = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}.$$

Let A^H is conjugate transpose of the matrix A. Then the spectral norm of matrix A is defined as

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i (A^H A)}.$$

The following inequality [7] holds:

$$\frac{1}{\sqrt{n}}\|A\|_E \leq \|A\|_2 \leq \|A\|_E \tag{1.1}$$

Lemma 1.1 ([7]). *Let $A = (a_{ij})$ and $B = (b_{ij})$ be any $m \times n$ matrices and let $A \circ B$ is the Hadamard product of A and B. Then*

$$\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2 .$$

Lemma 1.2 ([14]). *Let $A=(a_{ij})$ and $B=(b_{ij})$ be any $m\times n$ matrices. Then*

$$\|A\circ B\|_2 \le r_1(A)\,c_1(B)$$

where

$$r_1(A)=\max_{1\le i\le m}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2} \quad \text{and} \quad c_1(B)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{m}|b_{ij}|^2}.$$

Lemma 1.3 ([7]). *Let $A=(a_{ij})$ and $B=(b_{ij})$ be any $m\times n$ matrices and let $A\otimes B$ is the Kronecker product of A and B. Then*

$$\|A\otimes B\|_2=\|A\|_2\,\|B\|_2.$$

2. Main Results

Theorem 2.1. *Let $F=C_{s^*}\left(\left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}}q_0,\left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}}q_1,\ldots,\left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}}q_{n-1}\right)$ be an $n\times n$ geometric circulant matrix where a and b are nonzero positive real numbers and $s\in\mathbb{C}$. Then*

(i) If $|s|>1$, we have

$$\sqrt{\frac{q_{n-1}q_n}{b}}\le\|F\|_2\le|s|\sqrt{\left(\frac{|s|^{2n-2}-1}{|s|^2-1}\right)\frac{q_{n-1}q_n}{b}}.$$

(ii) If $|s|<1$, we have

$$\frac{|s|\sqrt{ab}}{b\sqrt{ab+4}}\sqrt{\frac{2|s|^{2n+2}-|s|^{2n}(ab+2)-|s|^2 l_{2n}+l_{2n-2}}{|s|^4-|s|^2(ab+2)+1}-2\left(\frac{|s|^{2n}-(-1)^n}{|s|^2+1}\right)}\le\|F\|_2\le\sqrt{\frac{(n-1)q_{n-1}q_n}{b}}.$$

Proof. If we consider the definition of $F=C_{s^*}\left(\left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}}q_0,\left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}}q_1,\ldots,\left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}}q_{n-1}\right)$, then we have the following matrix:

$$F=\begin{pmatrix}
\left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}}q_0 & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}}q_1 & \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}}q_2 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}}q_{n-1}\\
s\left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}}q_{n-1} & \left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}}q_0 & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}}q_1 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-2)}{2}}q_{n-2}\\
s^2\left(\frac{a}{b}\right)^{\frac{\xi(n-2)}{2}}q_{n-2} & s\left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}}q_{n-1} & \left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}}q_0 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-3)}{2}}q_{n-3}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
s^{n-1}\left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}}q_1 & s^{n-2}\left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}}q_2 & s^{n-3}\left(\frac{a}{b}\right)^{\frac{\xi(3)}{2}}q_3 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}}q_0
\end{pmatrix}.$$

Thus we get the Euclidean norm of the matrix F as

$$\|F\|_E^2=\sum_{k=0}^{n-1}(n-k)\left(\frac{a}{b}\right)^{\xi(k)}q_k^2+\sum_{k=1}^{n-1}k\left|s^{n-k}\right|^2\left(\frac{a}{b}\right)^{\xi(k)}q_k^2.$$

(i) If $|s|>1$, from [24, Theorem 2.3], we obtain

$$\begin{aligned}
\|F\|_E^2&\ge\sum_{k=0}^{n-1}(n-k)\left(\frac{a}{b}\right)^{\xi(k)}q_k^2+\sum_{k=1}^{n-1}k\left(\frac{a}{b}\right)^{\xi(k)}q_k^2\\
&=n\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^{\xi(k)}q_k^2\\
&=n\frac{q_{n-1}q_n}{b}.
\end{aligned}$$

So we have

$$\frac{1}{\sqrt{n}}\|F\|_E\ge\sqrt{\frac{q_{n-1}q_n}{b}}$$

that is

$$\sqrt{\frac{q_{n-1}q_n}{b}}\le\|F\|_2.$$

Now, let us choose the matrices

$$A=\begin{pmatrix} q_0 & 1 & 1 & \cdots & 1 & 1\\ s & q_0 & 1 & \cdots & 1 & 1\\ s^2 & s & q_0 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ s^{n-1} & s^{n-2} & s^{n-3} & \cdots & s & q_0 \end{pmatrix} \tag{2.1}$$

and

$$B=\begin{pmatrix} \left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}} q_0 & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} q_1 & \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} q_2 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} q_{n-1}\\ \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} q_{n-1} & \left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}} q_0 & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} q_1 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-2)}{2}} q_{n-2}\\ \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} q_2 & \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} q_{n-1} & \left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}} q_0 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-3)}{2}} q_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} q_1 & \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} q_2 & \left(\frac{a}{b}\right)^{\frac{\xi(3)}{2}} q_3 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}} q_0 \end{pmatrix} \tag{2.2}$$

such that $F=A\circ B$. Therefore we obtain

$$r_1(A)=\max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|a_{ij}\right|^2}=\sqrt{q_0^2+\sum_{k=1}^{n-1}|s|^{2k}}=|s|\sqrt{\frac{|s|^{2n-2}-1}{|s|^2-1}}$$

and

$$c_1(B)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|b_{ij}\right|^2}=\sqrt{\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^{\xi(k)}q_k^2}=\sqrt{\frac{q_{n-1}q_n}{b}}$$

By Lemma 1.2, we get

$$\sqrt{\frac{q_{n-1}q_n}{b}}\le\|F\|_2\le|s|\sqrt{\left(\frac{|s|^{2n-2}-1}{|s|^2-1}\right)\frac{q_{n-1}q_n}{b}}.$$

(ii) If $|s|<1$, then we have

$$\begin{aligned}\|F\|_E^2&\ge\sum_{k=0}^{n-1}(n-k)\left|s^{n-k}\right|^2\left(\frac{a}{b}\right)^{\xi(k)}q_k^2+\sum_{k=1}^{n-1}k\left|s^{n-k}\right|^2\left(\frac{a}{b}\right)^{\xi(k)}q_k^2\\ &=n|s|^{2n}\sum_{k=0}^{n-1}\left(\frac{\left(\frac{a}{b}\right)^{\frac{\xi(k)}{2}}q_k}{|s|^k}\right)^2\\ &=\frac{an|s|^{2n}}{b(ab+4)}\left(\sum_{k=0}^{n-1}\left(\frac{\alpha^2}{|s|^2ab}\right)^k+\sum_{k=0}^{n-1}\left(\frac{\beta^2}{|s|^2ab}\right)^k-2\sum_{k=0}^{n-1}\left(\frac{-1}{|s|^2}\right)^k\right)\\ &=\frac{an|s|^2}{b(ab+4)}\left(\frac{2|s|^{2n+2}-|s|^{2n}(ab+2)-|s|^2l_{2n}+l_{2n-2}}{|s|^4-|s|^2(ab+2)+1}-2\left(\frac{|s|^{2n}-(-1)^n}{|s|^2+1}\right)\right).\end{aligned}$$

Therefore we obtain the following lower bound:

$$\frac{|s|\sqrt{ab}}{b\sqrt{ab+4}}\sqrt{\frac{2|s|^{2n+2}-|s|^{2n}(ab+2)-|s|^2l_{2n}+l_{2n-2}}{|s|^4-|s|^2(ab+2)+1}-2\left(\frac{|s|^{2n}-(-1)^n}{|s|^2+1}\right)}\le\|F\|_2.$$

In the meantime, let the matrices A and B be given in (2.1) and (2.2) such that $F=A\circ B$. Then we have

$$r_1(A)=\max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|a_{ij}\right|^2}=\sqrt{n-1}$$

and

$$c_1(B)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|b_{ij}\right|^2}=\sqrt{\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^{\xi(k)}q_k^2}=\sqrt{\frac{q_{n-1}q_n}{b}}.$$

Combining the above equations, we get the following inequality:

$$\frac{|s|\sqrt{ab}}{b\sqrt{ab+4}}\sqrt{\frac{2|s|^{2n+2}-|s|^{2n}(ab+2)-|s|^2l_{2n}+l_{2n-2}}{|s|^4-|s|^2(ab+2)+1}-2\left(\frac{|s|^{2n}-(-1)^n}{|s|^2+1}\right)}\le\|F\|_2\le\sqrt{\frac{(n-1)q_{n-1}q_n}{b}}.$$

□

Theorem 2.2. *Let* $L = C_{s^*}\left(\left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} l_0, \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} l_1, \ldots, \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} l_{n-1}\right)$ *be an* $n \times n$ *geometric circulant matrix where* a *and* b *are nonzero positive real numbers and* $s \in \mathbb{C}$*. Then*

(i) If $|s| > 1$, we have

$$\sqrt{\frac{l_{n-1}l_n + 2a}{b}} \leq \|L\|_2 \leq \sqrt{\left(\frac{|s|^{2n} - 1}{|s|^2 - 1}\right) \frac{l_{n-1}l_n + 2a}{b}}$$

(ii) If $|s| < 1$, we have

$$\frac{|s|\sqrt{ab}}{b} \sqrt{\frac{2|s|^{2n+2} - |s|^{2n}(ab+2) - |s|^2 l_{2n} + l_{2n-2}}{|s|^4 - |s|^2(ab+2) + 1} + 2\left(\frac{|s|^{2n} - (-1)^n}{|s|^2 + 1}\right)} \leq \|L\|_2 \leq \sqrt{\frac{n}{b}(l_{n-1}l_n + 2a)}$$

Proof. Firstly, we have the following matrix:

$$L = \begin{pmatrix} \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} l_0 & \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} l_1 & \left(\frac{a}{b}\right)^{\frac{\xi(3)}{2}} l_2 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} l_{n-1} \\ s\left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} l_{n-1} & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} l_0 & \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} l_1 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} l_{n-2} \\ s^2\left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} l_{n-2} & s\left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} l_{n-1} & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} l_0 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-2)}{2}} l_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s^{n-1}\left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} l_1 & s^{n-2}\left(\frac{a}{b}\right)^{\frac{\xi(3)}{2}} l_2 & s^{n-3}\left(\frac{a}{b}\right)^{\frac{\xi(4)}{2}} l_3 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} l_0 \end{pmatrix}.$$

Thus we get the Euclidean norm of the matrix L as

$$\|L\|_E^2 = \sum_{k=0}^{n-1}(n-k)\left(\frac{a}{b}\right)^{\xi(k+1)} l_k^2 + \sum_{k=1}^{n-1} k\left|s^{n-k}\right|^2 \left(\frac{a}{b}\right)^{\xi(k+1)} l_k^2.$$

(i) If $|s| > 1$, from [13, Theorem 2.1], we get

$$\begin{aligned} \|L\|_E^2 &\geq \sum_{k=0}^{n-1}(n-k)\left(\frac{a}{b}\right)^{\xi(k+1)} l_k^2 + \sum_{k=1}^{n-1} k\left(\frac{a}{b}\right)^{\xi(k+1)} l_k^2 \\ &= n\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^{\xi(k+1)} l_k^2 \\ &= \frac{n}{b}(l_{n-1}l_n + 2a). \end{aligned}$$

So we obtain

$$\frac{1}{\sqrt{n}}\|L\|_E \geq \sqrt{\frac{l_{n-1}l_n + 2a}{b}}$$

that is

$$\sqrt{\frac{l_{n-1}l_n + 2a}{b}} \leq \|L\|_2.$$

In the meantime, let us choose the matrices

$$C = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ s & 1 & 1 & \cdots & 1 & 1 \\ s^2 & s & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s^{n-1} & s^{n-2} & s^{n-3} & \cdots & s & 1 \end{pmatrix} \tag{2.3}$$

and

$$D = \begin{pmatrix} \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} l_0 & \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} l_1 & \left(\frac{a}{b}\right)^{\frac{\xi(3)}{2}} l_2 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} l_{n-1} \\ \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} l_{n-1} & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} l_0 & \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} q_1 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} l_{n-2} \\ \left(\frac{a}{b}\right)^{\frac{\xi(3)}{2}} l_2 & \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} l_{n-1} & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} l_0 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(n-2)}{2}} l_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} l_1 & \left(\frac{a}{b}\right)^{\frac{\xi(3)}{2}} l_2 & \left(\frac{a}{b}\right)^{\frac{\xi(4)}{2}} l_3 & \cdots & \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} l_0 \end{pmatrix} \tag{2.4}$$

such that $L = C \circ D$. Therefore we have

$$r_1(C) = \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|c_{ij}\right|^2} = \sqrt{\sum_{k=0}^{n-1}|s|^{2k}} = \sqrt{\frac{|s|^{2n}-1}{|s|^2-1}}$$

and

$$c_1(D) = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|d_{ij}\right|^2} = \sqrt{\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^{\xi(k+1)} l_k^2} = \sqrt{\frac{l_{n-1}l_n+2a}{b}}.$$

By Lemma 1.2, we obtain

$$\sqrt{\frac{l_{n-1}l_n+2a}{b}} \le \|L\|_2 \le \sqrt{\left(\frac{|s|^{2n}-1}{|s|^2-1}\right)\left(\frac{l_{n-1}l_n+2a}{b}\right)}.$$

(ii) If $|s|<1$, then we get

$$\begin{aligned}
\|L\|_E^2 &\ge \sum_{k=0}^{n-1}(n-k)\left|s^{n-k}\right|^2\left(\frac{a}{b}\right)^{\xi(k+1)} l_k^2 + \sum_{k=1}^{n-1}k\left|s^{n-k}\right|^2\left(\frac{a}{b}\right)^{\xi(k+1)} l_k^2 \\
&= n|s|^{2n}\sum_{k=0}^{n-1}\left(\frac{\left(\frac{a}{b}\right)^{\frac{\xi(k+1)}{2}} l_k}{|s|^k}\right)^2 \\
&= \frac{an|s|^{2n}}{b}\left(\sum_{k=0}^{n-1}\left(\frac{\alpha^2}{|s|^2ab}\right)^k + \sum_{k=0}^{n-1}\left(\frac{\beta^2}{|s|^2ab}\right)^k + 2\sum_{k=0}^{n-1}\left(\frac{-1}{|s|^2}\right)^k\right) \\
&= \frac{an|s|^2}{b}\left(\frac{2|s|^{2n+2}-|s|^{2n}(ab+2)-|s|^2 l_{2n}+l_{2n-2}}{|s|^4-|s|^2(ab+2)+1} + 2\frac{|s|^{2n}-(-1)^n}{|s|^2+1}\right).
\end{aligned}$$

Thus we get the following inequality:

$$\frac{|s|\sqrt{ab}}{b}\sqrt{\frac{2|s|^{2n+2}-|s|^{2n}(ab+2)-|s|^2 l_{2n}+l_{2n-2}}{|s|^4-|s|^2(ab+2)+1} + 2\left(\frac{|s|^{2n}-(-1)^n}{|s|^2+1}\right)} \le \|L\|_2.$$

In the meantime, let the matrices C and D be given in (2.3) and (2.4) such that $L = C \circ D$. Then we have

$$r_1(C) = \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|c_{ij}\right|^2} = \sqrt{n}$$

and

$$c_1(D) = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|d_{ij}\right|^2} = \sqrt{\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^{\xi(k+1)} l_k^2} = \sqrt{\frac{l_{n-1}l_n+2a}{b}}.$$

Combining the above equations, we have the following inequality:

$$\frac{|s|\sqrt{ab}}{b}\sqrt{\frac{2|s|^{2n+2}-|s|^{2n}(ab+2)-|s|^2 l_{2n}+l_{2n-2}}{|s|^4-|s|^2(ab+2)+1} + 2\left(\frac{|s|^{2n}-(-1)^n}{|s|^2+1}\right)} \le \|L\|_2 \le \sqrt{\frac{n}{b}(l_{n-1}l_n+2a)}.$$

□

Corollary 2.3. *Let a and b be nonzero positive real numbers and the matrices F and L be as in Theorem 2.1 and Theorem 2.2, respectively.*

(i) *If $|s|>1$, then we have*

$$\|F\circ L\|_2 \le \frac{|s|}{b\left(|s|^2-1\right)}\sqrt{\left(|s|^{2n-2}-1\right)\left(|s|^{2n}-1\right)q_{n-1}q_n\left(l_{n-1}l_n+2a\right)}.$$

(ii) *If $|s|<1$, then we have*

$$\|F\circ L\|_2 \le \frac{1}{b}\sqrt{n(n-1)q_{n-1}q_n\left(l_{n-1}l_n+2a\right)}.$$

Proof. The proof follows from Lemma 1.1, Theorem 2.1 and Theorem 2.2. □

Corollary 2.4. *Let a and b be nonzero positive real numbers and the matrices F and L be as in Theorem 2.1 and Theorem 2.2, respectively.*

(*i*) *If* $|s|>1$*, then we have*

$$\|F\otimes L\|_2 \geq \frac{1}{b}\sqrt{q_{n-1}q_n\left(l_{n-1}l_n+2a\right)}$$

and

$$\|F\otimes L\|_2 \leq \frac{|s|}{b\left(|s|^2-1\right)}\sqrt{\left(|s|^{2n-2}-1\right)\left(|s|^{2n}-1\right)q_{n-1}q_n\left(l_{n-1}l_n+2a\right)}.$$

(*ii*) *If* $|s|<1$*, then we have*

$$\|F\otimes L\|_2 \geq \frac{a\,|s|^2}{b\sqrt{ab+4}}\sqrt{\left(\frac{2\,|s|^{2n+2}-|s|^{2n}(ab+2)-|s|^2\,l_{2n}+l_{2n-2}}{|s|^4-|s|^2(ab+2)+1}\right)^2-4\left(\frac{|s|^{2n}-(-1)^n}{|s|^2+1}\right)^2}$$

and

$$\|F\otimes L\|_2 \leq \frac{1}{b}\sqrt{n\,(n-1)\,q_{n-1}q_n\left(l_{n-1}l_n+2a\right)}.$$

Proof. The proof follows from Lemma 1.3, Theorem 2.1 and Theorem 2.2. □

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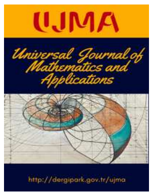
Universal Journal of Mathematics and Applications, 3 (3) (2020) 109-114
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: *www.dergipark.gov.tr/ujma*
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.652513



Some New Integral Inequalities for n-Times Differentiable Trigonometrically Convex Functions

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Article Info

Keywords: *Convex function, trigonometrically convex function, Hölder Integral inequality, Power-Mean Integral inequality and Hölder-İşcan integral inequality.*
2010 AMS: *26A51, 26D10, 26D15.*
Received: *28 November 2019*
Accepted: *14 September 2020*
Available online: *29 September 2020*

Abstract

In this manuscript, by using an integral identity together with both the Hölder, Hölder-İşcan and the Power-mean integral inequalities we obtain several new inequalities for n-time differentiable trigonometrically convex functions.

1. Preliminaries

$\Omega : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $r,s \in I$ with $r < s$. The inequality

$$\Omega\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Omega(u)\,du \leq \frac{\Omega(r)+\Omega(s)}{2}$$

is well known in the literature as Hermite-Hadamard's (H-H) integral inequality for convex functions [13]. The classical H-H inequality provides estimates of the mean value of a continuous convex or concave function. In recent years, significant improvements and generalizations have been found on convexity theory and H-H inequality; see for example [1–6, 8, 13].

Definition 1.1. *A function $\Omega : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$\Omega(\varepsilon r + (1-\varepsilon)s) \leq \varepsilon\Omega(r) + (1-\varepsilon)\Omega(s)$$

is valid for all $r,s \in I$ and $\varepsilon \in [0,1]$. If this inequality reverses, then Ω is said to be concave on interval $I \neq \varnothing$.

For some inequalities, generalizations and applications concerning convexity see [2–4, 6, 11–15]. Recently, in the literature there are so many papers about n-times differentiable functions on several kinds of convexities. In references [2, 4, 8, 14], readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of convex functions see for instance the recent papers [1, 3, 5, 6] and the references within these papers.
In [9], Kadakal gave the concept of the trigonometrically convex functions and related Hermite-Hadamard type inequalities.

Definition 1.2 ([9]). *A non-negative function $\Omega : I \to \mathbb{R}$ is called trigonometrically convex function on interval $[r,s]$, if for each $r,s \in I$ and $\varepsilon \in [0,1]$,*

$$\Omega(\varepsilon r + (1-\varepsilon)s) \leq \left(\sin\frac{\pi\varepsilon}{2}\right)\Omega(r) + \left(\cos\frac{\pi\varepsilon}{2}\right)\Omega(s).$$

If this inequality reveresed, then the function is called trigonometrically concave.

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Theorem 1.3 ([9]). *Let the function $\Omega : [r,s] \to \mathbb{R},\ s>0$, be a trigonometrically convex function. If $0 \le r < s$ and $\Omega \in L[r,s]$, then the following inequality holds:*

$$\frac{1}{s-r}\int_r^s \Omega(x)dx \le \frac{2}{\pi}\left[\Omega(r)+\Omega(s)\right].$$

Remark 1.4. *It is easily seen that, if the function $\Omega : [r,s] \to \mathbb{R},\ s>0$, be a trigonometrically concave function, then for $0 \le r < s$ and $\Omega \in L[r,s]$, then the following inequality holds:*

$$\frac{1}{s-r}\int_r^s \Omega(x)dx \ge \frac{2}{\pi}\left[\Omega(r)+\Omega(s)\right].$$

Theorem 1.5 ([9]). *Let the function $\Omega : [r,s] \to \mathbb{R},\ s>0$,be a trigonometrically convex function. If $0 \le r < s$ and $\Omega \in L[r,s]$, then the following inequalities holds:*

$$\Omega\left(\frac{s+r}{2}\right) \le \frac{\sqrt{2}}{s-r}\int_r^s \Omega(x)dx.$$

Remark 1.6. *It is easily seen that, if the function $\Omega : [r,s] \to \mathbb{R},\ s>0$, be a trigonometrically concave function, then for $0 \le r < s$ and $\Omega \in L[r,s]$, then the following inequality holds:*

$$\Omega\left(\frac{a+b}{2}\right) \ge \frac{\sqrt{2}}{s-r}\int_r^s \Omega(x)dx.$$

A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 1.7 (Hölder-İşcan Integral Inequality [7]). *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[r,s]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[r,s]$ then*

$$\int_r^s |f(x)g(x)|\,dx \le \frac{1}{s-r}\left\{\left(\int_r^s (s-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_r^s (s-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}+\left(\int_r^s (x-r)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_r^s (x-r)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}$$

Let $0<r<s$, throughout this paper we will use

$$A(r,s)=\frac{r+s}{2}$$
$$L_p(r,s)=\left(\frac{s^{p+1}-r^{p+1}}{(p+1)(s-r)}\right)^{\frac{1}{p}},\ \ r\neq s,\ p\in\mathbb{R},\ \ p\neq -1,0$$

for the arithmetic and generalized logarithmic mean, respectively.

2. Main Results

We will use the following Lemma for obtain our main results.

Lemma 2.1 ([10]). *Let $\Omega : I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable mapping on I° for $n \in \mathbb{N}$ and $\Omega^{(n)} \in L[r,s]$, where $r,s \in I^\circ$ with $r<s$, we have the identity*

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s \Omega(x)dx=\frac{(-1)^{n+1}}{n!}\int_r^s x^n\Omega^{(n)}(x)dx \tag{2.1}$$

where an empty sum is understood to be nil.

Theorem 2.2. *For $n\in\mathbb{N}$; let $\Omega : I \subseteq (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I° and $r,s \in I^\circ$ with $r<s$. If $\Omega^{(n)} \in L[r,s]$ and $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically convex function on the interval $[r,s]$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s \Omega(x)dx\right| \le \frac{s-r}{n!}\left(\frac{4}{\pi}\right)^{\frac{1}{q}} L_{np}^n(s,r)A^{\frac{1}{q}}\left(\left|\Omega^{(n)}(r)\right|^q,\left|\Omega^{(n)}(s)\right|^q\right). \tag{2.2}$$

Proof. If the function $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically convex on the interval $[r,s]$, using Lemma 2.1, the Hölder integral inequality and

$$\left|\Omega^{(n)}(x)\right|^q=\left|\Omega^{(n)}\left(\frac{s-x}{s-r}r+\frac{x-r}{s-r}s\right)\right|^q \le \sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(r)\right|^q+\cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(s)\right|^q,$$

we get

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right| \\
&\leq \frac{1}{n!}\int_r^s x^n\left|\Omega^{(n)}(x)\right|dx \\
&\leq \frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s\left|\Omega^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\leq \frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s\left[\sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(a)\right|^q+\cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(b)\right|^q\right]dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\left|\Omega^{(n)}(r)\right|^q\int_r^s\sin\frac{\pi(s-x)}{2(s-r)}dx+\left|\Omega^{(n)}(s)\right|^q\int_r^s\cos\frac{\pi(s-x)}{2(s-r)}dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}\left(\frac{s^{np+1}-r^{np+1}}{np+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}(s-r)\left|\Omega^{(n)}(r)\right|^q+\frac{2}{\pi}(s-r)\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(s-r)^{\frac{1}{p}}(s-r)^{\frac{1}{q}}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\left(\frac{s^{np+1}-r^{np+1}}{(np+1)(s-r)}\right)^{\frac{1}{p}}\left[\frac{\left|\Omega^{(n)}(r)\right|^q+\left|\Omega^{(n)}(s)\right|^q}{2}\right]^{\frac{1}{q}} \\
&= \frac{s-r}{n!}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\left[\frac{s^{np+1}-r^{np+1}}{(np+1)(s-r)}\right]^{\frac{1}{p}}\left[\frac{\left|\Omega^{(n)}(r)\right|^q+\left|\Omega^{(n)}(s)\right|^q}{2}\right]^{\frac{1}{q}} \\
&= \frac{s-r}{n!}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}L_{np}^n(r,s)A^{\frac{1}{q}}\left(\left|\Omega^{(n)}(r)\right|^q,\left|\Omega^{(n)}(s)\right|^q\right).
\end{aligned}
$$

□

Corollary 2.3. *Under the conditions Theorem 2.2 for $n=1$ we have the following inequality:*

$$
\left|\frac{\Omega(s)s-\Omega(s)s}{s-r}-\frac{1}{s-r}\int_r^s\Omega(x)dx\right|\leq\left(\frac{4}{\pi}\right)^{\frac{1}{q}}L_p(r,s)\left[\frac{|\Omega'(r)|^q+|\Omega'(s)|^q}{2}\right]^{\frac{1}{q}}.
$$

Proposition 2.4. *Let $r,s\in(0,\infty)$ with $r<s$, $q>1$ and $m\in(-\infty,0]\cup[1,\infty)\setminus\{-2q,-q\}$, we have*

$$
L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(r,s)\leq\left(\frac{4}{\pi}\right)^{\frac{1}{q}}L_p(r,s)A^{\frac{1}{q}}(r^m,s^m)
$$

Proof. Under the assumption of the Proposition, let $\Omega(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}$, $x\in(0,\infty)$. Then

$$
\left|\Omega'(x)\right|^q=x^m
$$

is trigonometrically convex on $(0,\infty)$ and the result follows directly from Corollary 2.3. □

Theorem 2.5. *For $n\in\mathbb{N}$; let $\Omega:I\subseteq(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $r,s\in I^\circ$ with $r<s$. If $\Omega^{(n)}\in L[r,s]$ and $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically convex function on the interval $[r,s]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right| \\
&\leq\frac{(s-r)^{\frac{1}{q}}}{n!}\left(\left[sL_{np}^{np}(r,s)-L_{np+1}^{np+1}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{2(\pi-2)}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{(s-r)^{\frac{1}{q}}}{n!}\left(\left[L_{np+1}^{np+1}(r,s)-aL_{np}^{np}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{2(\pi-2)}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{4}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
\tag{2.3}
$$

Proof. If the function $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically convex on the interval $[r,s]$, using Lemma 2.1, the Hölder-İşcan integral inequality and

$$
\left|\Omega^{(n)}(x)\right|^q=\left|\Omega^{(n)}\left(\frac{s-x}{s-r}r+\frac{x-r}{s-r}s\right)\right|^q\leq\sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(r)\right|^q+\cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(s)\right|^q,
$$

we get

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right| \\
&\le \frac{1}{n!}\int_r^s x^n\left|\Omega^{(n)}(x)\right|dx \\
&\le \frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s\left|\Omega^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\le \frac{1}{n!\,(s-r)}\left(\int_r^s (s-x)x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s (s-x)\left[\sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(a)\right|^q+\cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(b)\right|^q\right]dx\right)^{\frac{1}{q}} \\
&+\frac{1}{n!\,(s-r)}\left(\int_r^s (x-r)x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s (x-r)\left[\sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(r)\right|^q+\cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(s)\right|^q\right]dx\right)^{\frac{1}{q}} \\
&=\frac{1}{n!\,(s-r)}\left(\int_r^s (s-x)x^{np}dx\right)^{\frac{1}{p}}\left(\left|\Omega^{(n)}(r)\right|^q\int_r^s (s-x)\sin\frac{\pi(s-x)}{2(s-r)}dx+\left|\Omega^{(n)}(s)\right|^q\int_r^s (b-x)\cos\frac{\pi(s-x)}{2(s-r)}dx\right)^{\frac{1}{q}} \\
&+\frac{1}{n!\,(s-r)}\left(\int_r^s (x-r)x^{np}dx\right)^{\frac{1}{p}}\left(\left|\Omega^{(n)}(r)\right|^q\int_r^s (x-r)\sin\frac{\pi(s-x)}{2(s-r)}dx+\left|\Omega^{(n)}(s)\right|^q\int_r^s (x-r)\cos\frac{\pi(s-x)}{2(s-r)}dx\right)^{\frac{1}{q}} \\
&=\frac{1}{n!\,(s-r)}\left((s-r)\left[sL_{np}^{np}(r,s)-L_{np+1}^{np+1}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{4(s-r)^2}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{2(\pi-2)(s-r)^2}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{1}{n!\,(s-r)}\left((s-r)\left[L_{np+1}^{np+1}(r,s)-rL_{np}^{np}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{2(\pi-2)(s-r)^2}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{4(s-r)^2}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}} \\
&=\frac{(s-r)^{\frac{1}{q}}}{n!}\left(\left[sL_{np}^{np}(r,s)-L_{np+1}^{np+1}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{2(\pi-2)}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{(s-r)^{\frac{1}{q}}}{n!}\left(\left[L_{np+1}^{np+1}(r,s)-rL_{np}^{np}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{2(\pi-2)}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{4}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

□

Theorem 2.6. *For $n\in\mathbb{N}$; let $\Omega: I\subseteq(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $r,s\in I^\circ$ with $r<s$. If $\Omega^{(n)}\in L[r,s]$ and $\left|\Omega^{(n)}\right|^q$ for $q\ge 1$ is trigonometrically convex on the interval $[r,s]$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right|\le\frac{1}{n!}(s-r)^{1-\frac{1}{q}}L_n^{n\left(1-\frac{1}{q}\right)}\left\{\left|\Omega^{(n)}(r)\right|^q S_1(r,s)+\left|\Omega^{(n)}(s)\right|^q S_2(r,s)\right\}^{\frac{1}{q}},$$

where

$$S_1(r,s)=\int_r^s x^n\sin\frac{\pi(s-x)}{2(s-r)}dx,\quad S_2(r,s)=\int_r^s x^n\cos\frac{\pi(s-x)}{2(s-r)}dx.$$

Proof. From Lemma 2.1 and Power-mean integral inequality, we have

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right| \\
&\le \frac{1}{n!}\int_r^s x^n\left|\Omega^{(n)}(x)\right|dx \\
&\le \frac{1}{n!}\left(\int_r^s x^n dx\right)^{1-\frac{1}{q}}\left(\int_r^s x^n\left|\Omega^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\le \frac{1}{n!}\left(\int_r^s x^n dx\right)^{1-\frac{1}{q}}\left(\int_r^s x^n\left[\sin\frac{\pi(s-x)}{2(s-r))}\left|\Omega^{(n)}(r)\right|^q+\cos\frac{\pi(s-x)}{2(s-r))}\left|\Omega^{(n)}(s)\right|^q\right]dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}\left(\int_r^s x^n dx\right)^{1-\frac{1}{q}}\left(\left|\Omega^{(n)}(r)\right|^q\int_r^s x^n\sin\frac{\pi(s-x)}{2(s-r))}dx+\left|\Omega^{(n)}(s)\right|^q\int_r^s x^n\cos\frac{\pi(s-x)}{2(s-r))}dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(s-r))^{1-\frac{1}{q}}\left[\frac{s^{n+1}-r^{n+1}}{(n+1)(s-r)}\right]^{1-\frac{1}{q}}\left\{\left|\Omega^{(n)}(s)\right|^q S_1(r,s)+\left|\Omega^{(n)}(r)\right|^q S_2(r,s)\right\}^{\frac{1}{q}} \\
&= \frac{1}{n!}(s-r)^{1-\frac{1}{q}}L_n^{n\left(1-\frac{1}{q}\right)}\left\{\left|\Omega^{(n)}(r)\right|^q S_1(r,s)+\left|\Omega^{(n)}(s)\right|^q S_2(r,s)\right\}^{\frac{1}{q}}.
\end{aligned}
$$

□

Corollary 2.7. *Under the conditions Theorem 2.6 for $n=1$ we have the following inequality:*

$$\left|\frac{\Omega(s)s-\Omega(s)s}{s-r}-\frac{1}{s-r}\int_r^s \Omega(x)dx\right| \leq \left(\frac{r+s}{2}\right)^{1-\frac{1}{q}}\left[\frac{2\pi s-4(s-r)}{\pi^2}\left|\Omega'(r)\right|^q+\frac{4(s-r)-2\pi r}{\pi^2}\left|\Omega'(s)\right|^q\right]^{\frac{1}{q}}.$$

Proposition 2.8. *Let $r,s\in(0,\infty)$ with $r<s$, $q>1$ and $m\in(-\infty,0]\cup[1,\infty)\backslash\{-2q,-q\}$, we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(r,s)\leq A^{1-\frac{1}{q}}(r,s)\left[\frac{2\pi s-4(s-r)}{\pi^2}r^m+\frac{4(s-r)-2\pi r}{\pi^2}s^m\right]^{\frac{1}{q}}.$$

Proof. The result follows directly from Corollary 2.7 for the function

$$\Omega(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}, x\in(0,\infty).$$

This completes the proof of Proposition. □

Corollary 2.9. *Using Proposition 2.8. for $m=1$, we have following inequality:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(r,s)\leq A^{1-\frac{1}{q}}(r,s)\left[\frac{4(s-r)^2}{\pi^2}\right]^{\frac{1}{q}}.$$

Corollary 2.10. *Using Proposition 2.8 for $q=1$, we have following inequality:*

$$L_{m+1}^{m+1}(r,s)\leq\frac{2\pi s-4(s-r)}{\pi^2}r^m+\frac{4(s-r)-2\pi r}{\pi^2}s^m.$$

Corollary 2.11. *Using Corollary 2.10 for $m=1$, we have following inequality:*

$$L_2^2(r,s)\leq\frac{4(s-r)^2}{\pi^2}.$$

Corollary 2.12. *With the conditions of the Theorem 2.6 for $q=1$ we have the following inequality:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right|\leq\frac{1}{n!}\left\{\left|\Omega^{(n)}(r)\right|S_1(r,s)+\left|\Omega^{(n)}(s)\right|S_2(r,s)\right\}$$

Theorem 2.13. *For $n\in\mathbb{N}$; let $\Omega: I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $r,s\in I^\circ$ with $r<s$. If $\Omega^{(n)}\in L[r,s]$ and $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically concave on the interval $[a,b]$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right|\leq\frac{s-r}{n!}\left(\frac{1}{2}\right)^{\frac{1}{2q}}L_{np}^n(r,s)\left|\Omega^{(n)}\left(\frac{r+s}{2}\right)\right|.$$

Proof. Since $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically concave on the interval $[r,s]$, with respect to Hermite-Hadamard inequality we can write

$$\int_r^s\left|\Omega^{(n)}(x)\right|^q dx\leq\frac{s-r}{\sqrt{2}}\left|\Omega^{(n)}\left(\frac{r+s}{2}\right)\right|^q.$$

Using Lemma 2.1 and the Hölder integral inequality we have

$$\begin{aligned}\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right| &\leq\frac{1}{n!}\int_r^s x^n\left|\Omega^{(n)}(x)\right|dx\\ &\leq\frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s\left|\Omega^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\\ &\leq\frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\frac{s-r}{\sqrt{2}}\left|f^{(n)}\left(\frac{r+s}{2}\right)\right|^q\right)^{\frac{1}{q}}\\ &=\frac{s-r}{n!}\left(\frac{1}{2}\right)^{\frac{1}{2q}}\left[\frac{s^{np+1}-r^{np+1}}{(np+1)(s-r)}\right]^{\frac{1}{p}}\left|\Omega^{(n)}\left(\frac{r+s}{2}\right)\right|\\ &=\frac{s-r}{n!}\left(\frac{1}{2}\right)^{\frac{1}{2q}}L_{np}^n(r,s)\left|\Omega^{(n)}\left(\frac{r+s}{2}\right)\right|.\end{aligned}$$

□

Corollary 2.14. *With the conditions of the Theorem 2.13 for $n=1$ we have the following inequality:*

$$\left|\frac{\Omega(s)s-\Omega(s)s}{s-r}-\frac{1}{s-r}\int_r^s\Omega(x)dx\right|\leq\left(\frac{1}{2}\right)^{\frac{1}{2q}}L_p(r,s)\left|\Omega'\left(\frac{r+s}{2}\right)\right|.$$

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Universal Journal of Mathematics and Applications, 3 (3) (2020) 115-120
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.588104

Approximate Controllability for Time-Dependent Impulsive Neutral Stochastic Partial Differential Equations with Fractional Brownian Motion and Memory

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Article Info

Keywords: *Approximate controllability, Impulsive systems, Fractional Brownian motion, Delay differential equations.*
2010 AMS: *34K50, 93B05, 34A37, 93E03, 60H20.*
Received: *7 July 2019*
Accepted: *13 July 2020*
Available online: *29 September 2020*

Abstract

In this manuscript, we investigate the approximate controllability for time-dependent impulsive neutral stochastic partial differential equations with fractional Brownian motion and memory in Hilbert space. By using semigroup theory, stochastic analysis techniques and fixed point approach, we derive a new set of sufficient conditions for the approximate controllability of nonlinear stochastic system under the assumption that the corresponding linear system is approximately controllable. Finally, an example is provided to illustrate our results.

1. Introduction

Approximate controllability is one of the important fundamental concepts in mathematical control theory and plays an important role in both deterministic and stochastic control systems. Controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. The controllability of nonlinear stochastic systems in infinite dimensional spaces has recently received a lot of attentions see [1–5, 10, 11, 13] and the references therein. Moreover, the approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. Further, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications see [7, 8] and the references cited therein.

Likewise the theory of impulsive differential equations plays a major role in investigation of qualitative theory. Impulsive differential equations are differential equations involving impulse effect, appear as a natural description of observed evolution phenomena of several real life problems, for detail refer [6]. In other way, many dynamical systems (Physical, Social, Biological, Engineering etc.) can be conveniently expressed in the form of differential equations. In case of physical systems such as air crafts, some external forces act which are not continuous with respect to time and the duration of their effect is near negligible as compared with total duration of original process. Same phenomena's are observed in case of biological systems (e.g. heart beat, blood-flow, pulse frequency), social systems (e.g. price-index frequency, demand and supply of goods) and in many other dynamical systems also such effects are called impulsive effects.

In recent years, stochastic differential equations driven by fractional Brownian motion have attracted much attention due to its a wide applications in a verity of physical phenomena, such as in economic and finance, biology and communication networks. The fractional Brownian motion was introduced by Kolmogorov in 1940 in [9], and later studied by Mandelbrot and Van Ness, who in 1968 provided in [10] a stochastic integral representation of this process in terms of a standard Brownian motion. There has been some recent interest in studying evolution equations driven by fractional Brownian motion. Recently, Lakhel [11] studied controllability results of neutral stochastic delay partial functional integrodifferential equations perturbed by fractional Brownian motion by using the theory of semigroup. Very recently, many interesting works have been done on stochastic differential equations driven by fractional Brownian motion see [3, 11, 12, 15] and the references cited therein.

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However, the study of the approximate controllability for time-dependent impulsive neutral stochastic partial differential equations with fractional Brownian motion and memory has not been discussed in the standard literature. Motivated by the above consideration, the goal of this paper is to study the approximate controllability for time-dependent impulsive neutral stochastic partial differential equations with fractional Brownian motion and memory:

$$d[x(t)-g(t,x(t-r))]=A(t)[x(t)-g(t,x(t-r))]dt+[f(t,x(t),x(t-r))+Bu(t)]dt+\sigma(t)d\mathtt{B}^{\mathtt{H}}(t),\quad t\neq t_k,\ t\in J:=[0,T],\tag{1.1}$$

$$\Delta x(t_k)=I_k(x(t_k^-)),\qquad k\in\{1,2,...,m\},\tag{1.2}$$

$$x(t)=\varphi(t)\in\mathscr{C}_r=\mathscr{C}^b_{\Im_0}([-r,0];\mathrm{H}),\quad -r\le t\le 0,\ r>0.\tag{1.3}$$

where $x(\cdot)$ is a stochastic process taking values in a real separable Hilbert space H; $A(t):\mathscr{D}\subset\mathrm{H}\to\mathrm{H}$ is a family of unbounded operators defined on a common domain $\mathscr{D}$, which is dense in the space H and generates a strong evolution operator $\mathscr{U}(s,t)$, $0\le t\le s\le T$ and $g,f:J\times\mathrm{H}\to\mathrm{H}$, $\sigma:J\to\mathscr{L}_2^0$ are Borel measurable functions and $I_k:\mathrm{H}\to\mathrm{H}$, $k=1,2,...,m$ are continuous functions. The control function $u(\cdot)$ takes values in $\mathscr{L}_2^{\Im}(J,\mathrm{U})$ of admissible control functions for a separable Hilbert space U and B is a bounded linear operator from U into H. Furthermore, let $0=t_0<t_1<\cdots t_m<t_{m+1}=T$ be prefixed points, and $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, represents the jump of the function x at time with I_k determining the size of the jump, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively. Let $\varphi(t):[-r,0]\to\mathrm{H}$ is an $\Im_0$-measurable random variables independent of the Wiener process W with $\mathbf{E}\left[\sup_{-r\le s\le 0}\|\varphi\|^2_{\mathrm{H}}\right]<\infty$.

2. Preliminaries

Let $(\Omega,\Im,\mathbb{P})$ be a complete probability space. A standard fractional Brownian motion $\{\beta^{\mathtt{H}}(t),t\in\mathbb{R}\}$ with Hurst parameter $\mathtt{H}\in(0,1)$ is a zero mean Gaussian process with the covariance function

$$R_{\mathtt{H}(t,s)}=\mathbf{E}\left[\beta^{\mathtt{H}}(t)\beta^{\mathtt{H}}(s)\right]=\frac{1}{2}\left(|t|^{2\mathtt{H}}+|s|^{2\mathtt{H}}-|t-s|^{2\mathtt{H}}\right),\quad t,s\in\mathbb{R}.$$

Let H and K be two real separable Hilbert spaces and let $\mathscr{L}(\mathrm{K},\mathrm{H})$ be the space of bounded linear operator from K to H. Let $Q\in\mathscr{L}(\mathrm{H},\mathrm{K})$ be an operator defined by $Qe_n=\lambda_ne_n$ with finite trace $\mathrm{tr}Q=\sum_{n=1}^\infty\lambda_n<\infty$. where $\lambda_n\ge0$ $(n=1,2,...)$ are non-negative real numbers and $\{e_n\}$ $(n=1,2,...)$ is a complete orthonormal basis in K. We define the infinite dimensional fractional Brownian motion on K with covariance Q as $\mathtt{B}^{\mathtt{H}}(t)=\mathtt{B}^{\mathtt{H}}_Q(t)=\sum_{n=1}^\infty\sqrt{\lambda_n}e_n\beta_n^{\mathtt{H}}(t)$. where $\beta_n^{\mathtt{H}}$ are real, independent fractional Brownian motion's. This process is Gaussian, it starts from 0, has zero mean and covariance

$$\mathbf{E}\left\langle\mathtt{B}^{\mathtt{H}}(t),x\right\rangle\left\langle\mathtt{B}^{\mathtt{H}}(s),y\right\rangle=R(s,t)\langle Q(x),y\rangle\quad\text{for }x,y\in\mathtt{Y}\text{ and }t,s\in[0,T]$$

Now, define the Weiner integrals with respect to the Q-fractional Brownian motion, we introduce the space $\mathscr{L}_2^0=\mathscr{L}_2^0(\mathrm{K},\mathrm{H})$ of all Q-Hilbert-Schmidt operators $\zeta:\mathrm{K}\to\mathrm{H}$. We recall that $\zeta\in\mathscr{L}(\mathrm{K},\mathrm{H})$ is called a Q-Hilbert-Schmidt operator, if

$$\|\zeta\|^2_{\mathscr{L}_2^0}=\sum_{n=1}^\infty\left\|\sqrt{\lambda_n}\zeta e_n\right\|^2<\infty,$$

and that the space $\mathscr{L}_2^0$ equipped with the inner product $<\varphi,\zeta>_{\mathscr{L}_2^0}=\sum_{n=1}^\infty<\varphi e_n,\zeta e_n>$ is a separable Hilbert space. Let $\phi(s):s\in[0,T]$ be a function with values in $\mathscr{L}_2^0(\mathrm{K},\mathrm{H})$ such that $\sum_{n=1}^\infty\left\|K^*\phi Q^{1/2}e_n\right\|^2_{\mathscr{L}_2^0}<\infty$. The Weiner integral of ϕ with respect to $\mathtt{B}^{\mathtt{H}}$ is defined by

$$\int_0^t\phi(s)d\mathtt{B}^{\mathtt{H}}=\sum_{n=1}^\infty\int_0^t\sqrt{\lambda_n}\phi(s)e_nd\beta_n^{\mathtt{H}}(s).\tag{2.1}$$

Lemma 2.1. *If $\zeta:[0,T]\to\mathscr{L}_2^0(\mathrm{K},\mathrm{H})$ satisfies $\int_0^t\|\zeta(s)\|^2_{\mathscr{L}_2^0}ds<\infty$, then (2.1) is well defined as an H-valued random variable and*

$$\mathbf{E}\left\|\int_0^t\zeta(s)d\mathtt{B}^{\mathtt{H}}(s)\right\|^2\le2\mathtt{H}t^{2\mathtt{H}-1}\int_0^t\|\zeta\|^2_{\mathscr{L}_2^0}ds.$$

Let $r>0$ and $\mathscr{C}:=\mathscr{C}([-r,0];\mathrm{H})$ denotes the family of all continuous functions from $[-r,0]$ to H. The space $\mathscr{C}$ is assumed to be equipped with the norm

$$\|\zeta\|_{\mathscr{C}}:=\sup_{-r<t<0}\|\zeta(t)\|_{\mathrm{H}},\quad\zeta(t)\in\mathscr{C}.$$

We also assume that $\mathscr{C}^b_{\Im_0}([-r,0];\mathrm{H})$ denotes the family of all almost surely bounded, $\Im_0$-measurable, $\mathscr{C}([-r,0];\mathrm{H})$-value random variables. For all $t\ge0$, $x_t=\{x(t+\theta):-r\le\theta\le0\}$ is regarded as $\mathscr{C}([-r,0];\mathrm{H})$-valued stochastic process. Further, let $\mathscr{PC}(J,\mathscr{L}^2(\Omega,\mathrm{H}))=\Big\{x(t)$ is continuous everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-)=x(t_k)$, $k=1,2,...,m\Big\}$ be the Banach space of piece-wise continuous function from J into $\mathscr{L}^2(\Omega,\mathrm{H})$ with the norm

$$\|x\|_{\mathscr{PC}}=\sup_{t\in J}|x(t)|<\infty.$$

Let $\mathscr{PC} = \mathscr{PC}(J, \mathscr{L}^2)$ be the closed subspace of $\mathscr{PC}(J, \mathscr{L}^2(\Omega, \mathrm{H}))$ consisting of measurable and $\Im_t$-adapted H-valued process $x(\cdot) \in \mathscr{PC}(J, \mathscr{L}^2(\Omega, \mathrm{H}))$ endowed with the norm

$$\|x\|^2_{\mathscr{PC}} = \mathbf{E} \sup_{t \in J} \|x(t)\|^2_{\mathrm{H}}.$$

In what follows, we assume that $\{A(t), t \geq 0\}$ is a family of closed densely defined linear unbounded operators on H and with domain $\mathscr{D} = \mathscr{D}(A(t))$ independent of t.

Definition 2.2. *A family of bounded linear operators $\{\mathscr{U}(t,s)\}_{(t,s)\in\Delta} : \mathscr{U}(t,s) : \mathrm{H} \to \mathrm{H}$ for $(t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leq s \leq t \leq T\}$ is called an evolution system if the following properties are satisfied*
1. $\mathscr{U}(t,t) = I$ where I is the identity operator in H.
2. $\mathscr{U}(t,s)\mathscr{U}(s,r) = \mathscr{U}(t,r)$ for $0 \leq r \leq s \leq t \leq T$.
3. $\mathscr{U}(t,s) \in \mathscr{L}(\mathrm{H})$ the space of bounded linear operators on H, *where for every $(t,s) \in \Delta$ and for each $x \in \mathrm{H}$, the mapping $(t,s) \to \mathscr{U}(t,s)x$ is strongly continuous.*

Remark 2.3. *If $A(t)$, $t \geq 0$ is a second order differential operator A, i.e. $A(t) = A$ for each $t \geq 0$. Then, A generates a $\mathscr{C}_0$-semigroup $\{e^{At}, t \geq 0\}$.*

More details on evolution systems and their properties could be found on the books of Pazy [5].
It is convenient to introduce the relevant operators and the basic controllability condition.

(i) The operator $\mathscr{L}_0^T \in \mathrm{L}(\mathscr{L}_2^{\Im})(J, \mathrm{H}, \mathscr{L}_2(\Omega, \Im_T, \mathrm{H}))$ is defined by

$$\mathscr{L}_0^T u = \int_0^T \mathscr{U}(T,s)Bu(s)ds,$$

where $\mathscr{L}_2^{\Im}(J, \mathrm{H})$ is the space of all $\Im_t$-adapted, H-valued measurable square integrable processes on $J \times \Omega$.

(ii) The linear controllability operator Π_0^T which is associated with the operator $\mathscr{L}_0^T$ is defined by

$$\Pi_0^T\{\cdot\} = \mathscr{L}_0^T(\mathscr{L}_0^T)^*\{\cdot\} = \int_0^T \mathscr{U}(T,t)BB^*\mathscr{U}^*(T,t)\mathbf{E}\{\cdot|\Im_t\}dt.$$

which belongs to $\mathrm{L}(\mathscr{L}_2^{\Im})(J, \mathrm{H}, \mathscr{L}_2(\Omega, \Im_T, \mathrm{H}))$ and the controllability operator $\Gamma_s^T \in \mathrm{L}(\mathrm{H}, \mathrm{H})$ is

$$\Gamma_s^T = \int_s^T \mathscr{U}(T,t)BB^*\mathscr{U}^*(T,t)dt, \quad 0 \leq s \leq t.$$

Lemma 2.4. *For any $z \in \mathscr{L}_2(\Omega, \Im_T, \mathrm{H})$, there exists $\bar{\phi} \in \mathscr{L}_2^{\Im}(J, \mathrm{L}_2^0)$ such that*

$$z = \mathbf{E}z + \int_0^T \bar{\phi}(s)d\mathrm{B}^{\mathtt{H}}(s).$$

Let $x(t; \varphi, u)$ be the state value of the system (1.1)-(1.3) at terminal time T corresponding to the control u and the initial value φ. Introduce the set $\Re(T, \varphi) = \{x(T; \varphi, u) : u(\cdot) \in \mathscr{L}_2^{\Im}(J, \mathrm{U})\}$ is called the reachable set of the system (1.1)-(1.3).

Definition 2.5. *The system (1.1)-(1.3) is said to be approximately controllable on the interval J if $\overline{\Re(T,x)} = \mathscr{L}_2(\Omega, \Im_T, \mathrm{H})$*

Definition 2.6. *An $\Im_t$-adapted stochastic process $x : J \to \mathrm{X}$ is called a mild solution of (1.1)-(1.3) if for each $u \in \mathscr{L}_2^{\Im}(J, \mathrm{U})$ and for $t \in J$, $\mathbf{P}\{\omega : \int_J \|x(s)\|^2_{\mathrm{X}} ds < +\infty\} = 1$ it satisfies the integral equation*

$$x(t) = \mathscr{U}(t,0)[\varphi(0) - g(0,\varphi)] + g(t, x(t-r)) + \int_0^t \mathscr{U}(t,s)f(s, x(s), x(s-r))ds$$
$$+ \int_0^t \mathscr{U}(t,s)Bu(s)ds + \int_0^t \mathscr{U}(t,s)\sigma(s)d\mathrm{B}^{\mathtt{H}}(s) + \sum_{0<t_k<t} \mathscr{U}(t,t_k)I_k(x(t_k^-)), \quad t \in [0,T].$$

We improve the following hypotheses to prove our results:

(H1) $\mathscr{U}(t,s)$ is a compact operator for $t - s > 0$ and there exists a constant $k \geq 1$ such that

$$\|\mathscr{U}(t,s)\|_{\mathscr{L}(\mathrm{H})} \leq k, \text{ for } (t,s) \in \Delta.$$

(H2) There exists a positive constant k_0 such that $t \in J$, $x, y \in \mathrm{H}$

$$\|g(t,x) - g(t,y)\|^2_{\mathrm{H}} \leq k_0 \|x - y\|^2_{\mathrm{H}}.$$

(H3) There exists a positive constant k_1 such that for all $x_1, x_2, y_1, y_2 \in \mathrm{H}$

$$\|f(t,x) - f(t,y)\|^2_{\mathrm{H}} \leq k_1 \left(\|x_1 - x_2\|^2_{\mathrm{H}} + \|y_1 - y_2\|^2_{\mathrm{H}}\right).$$

(H4) There exists some positive constants Q_k, $k = 1, 2, \ldots, m$ such that for $x, y \in \mathrm{H}$

$$\|I_k(x) - I_k(y)\|^2_{\mathrm{H}} \leq Q_i \|x - y\|^2_{\mathrm{H}}.$$

(H5) For all $t \in J$, there exists a positive constant l such that

$$\|g(t,0)\|_{\mathrm{H}}^2 \vee \|f(t,0)\|_{\mathrm{H}}^2 \vee \|I_k(0)\|_{\mathrm{H}}^2 \le l.$$

(H6) The function $\sigma : [0,T] \to \mathscr{L}_2^0(\mathrm{K},\mathrm{H})$ satisfies

$$\int_0^t \|\sigma(s)\|_{\mathscr{L}_2^0}^2 ds < \infty, \ \text{ for } \ T > 0.$$

(H7) For $0 \le t \le T$, the operator $aR(a,\Gamma_t^T) := a(aI + \Gamma_t^T)^{-1} \to 0$ as $a \to 0^+$ in the strong operator topology.

(H8) The functions f, g are uniformly bounded.

Remark 2.7. *The assumption* **(H7)** *is equivalent to the linear system of (1.1)-(1.3) is approximately controllable.*

3. Controllability Result

Let $a > 0$ and $z \in \mathscr{L}_2(\Omega, \Im_T, \mathrm{H})$. Define the control function

$$u_a(t,x) = B^* \mathscr{U}^*(T,t)\Bigg[R(a,\Pi_0^T)\Bigg(\mathbf{E}z - \mathscr{U}(T,0)\,[\varphi(0) - g(0,\varphi)] - g(t,x(t-r)) - \sum_{0<t_k<T} \mathscr{U}(T,t_k) I_k(x(t_k^-))\Bigg) + \int_0^T R(a,\Pi_s^T) z(s) d\mathrm{B}^{\mathrm{H}}(s) - \int_0^T R(a,\Pi_s^T)\mathscr{U}(T,s) f(s,x(s),x(s-r)) ds\Bigg]$$

Theorem 3.1. *Suppose that* **(H1)-(H8)** *hold. Then the operator* Θ *has a fixed point in* $\mathscr{PC}$ *provided that*

$$4\left(\frac{a^2 + 3T^2k^4k_B^2}{a^2}\right)\left[k_0 + 2T^2k^2k_1 + mk^2\sum_{k=1}^m Q_k\right]\|x-y\|_{\mathscr{PC}}^2 < 1. \tag{3.1}$$

Proof. Consider the operator $\Theta : \mathscr{PC}(J, \mathscr{L}^2(\Omega;\mathrm{H})) \to \mathscr{PC}(J, \mathscr{L}^2(\Omega;\mathrm{H}))$ defined by

$$(\Theta x)(t) = \mathscr{U}(t,0)\,[\varphi(0) - g(0,\varphi)] + g(t,x(t-r)) + \int_0^t \mathscr{U}(t,s) f(s,x(s),x(t-r)) ds + \int_0^t \mathscr{U}(t,s) B u_a(s,x) ds + \int_0^t \mathscr{U}(t,s)\sigma(s) d\mathrm{B}^{\mathrm{H}}(s) + \sum_{0<t_k<t} \mathscr{U}(t,t_k) I_k(x(t_k^-)).$$

In what follows, we shall show that system (1.1)-(1.3) is approximately controllable if for all $a > 0$ there exists a fixed point of the operator Θ. By our assumptions, Holder's inequality and the Doob martingale inequality, we obtain

$$\begin{aligned}\mathbf{E}\|u_a(t,x)\|^2 \le \frac{7}{a^2}k^2k_B^2\Bigg[&\|\mathbf{E}z\|^2 + 2k^2[1+k_0]\mathbf{E}\|\varphi\|_{\mathscr{PC}}^2 + 2[M+k_0](\mathbf{E}\|\varphi\|_{\mathscr{PC}}^2 + \|x\|_{\mathscr{PC}}^2) \\ &+ 2mk^2\sum_{k=1}^m Q_k(M + \|x\|_{\mathscr{PC}}^2) + k^2 2\mathrm{H}t^{2\mathrm{H}-1}\int_0^t \|\sigma(s)\|_{\mathscr{L}_2^0}^2 ds \\ &+ 2k^2T[MT + k_1(r\mathbf{E}\|\varphi\|_{\mathscr{PC}}^2 + 2T\|x\|_{\mathscr{PC}}^2)]\Bigg].\end{aligned}$$

Step 1: We claim that $\Theta(\mathscr{PC}) \subset \mathscr{PC}$. Then we have

$$\begin{aligned}\|(\Theta x)(t)\|_{\mathscr{PC}}^2 \le 6\Bigg[&k^2(\|\varphi(0) - g(0,\varphi)\|^2) + \mathbf{E}\|g(t,x(t-r)) - g(t,0) + g(t,0)\|^2 \\ &+ k^2T\mathbf{E}\int_0^t \|f(s,x(s),x(s-r)) - f(t,0,0) + f(t,0,0)\|^2 ds + k^2k_B^2T^2\mathbf{E}\|u_a(s,x)\|^2 \\ &+ 2k^2\mathrm{H}t^{2\mathrm{H}-1}\int_0^t \|\sigma(s)\|_{\mathscr{L}_2^0}^2 ds + k^2\mathbf{E}\sum_{0<t_k<t}\left\|I_x(x(t_k^-)) - I_k(0) + I_k(0)\right\|^2\Bigg] \\ \le 6\Bigg[&\frac{7}{a^2}T^2k^4k_B^4(\|\mathbf{E}z\|^2) + [1 + \frac{7}{a^2}T^2k^4k_B^4]\Bigg(2k^2(1+k_0)\mathbf{E}\|\varphi\|_{\mathscr{PC}}^2 \\ &+ 2[M + k_0(\mathbf{E}\|\varphi\|_{\mathscr{PC}}^2 + \|x\|_{\mathscr{PC}}^2)] + 2mk^2\sum_{k=1}^m Q_k(M + \|x\|_{\mathscr{PC}}^2) \\ &+ 2k^2T[MT + k_1(\mathbf{E}\|\varphi\|_{\mathscr{PC}}^2 + \|x\|_{\mathscr{PC}}^2)] + 2k^2\mathrm{H}t^{2\mathrm{H}-1}\int_0^t \|\sigma(s)\|_{\mathscr{L}_2^0}^2 ds\Bigg)\Bigg] < \infty.\end{aligned}$$

Hence, $\Theta(\mathscr{PC}) \subset \mathscr{PC}$.

Now, we are going to use the Banach fixed point theorem to prove that Θ has a unique fixed point in $\mathscr{PC}$.

Step 2: We claim that Θ is a contraction on $\mathscr{PC}$.

For any $x, y \in \mathscr{PC}$, $t \in J$ then we have

$$\begin{aligned}\|(\Theta x)(t) - (\Theta y)(t)\|_{\mathscr{PC}}^2 &\le 4\left[k_0 + 2T^2k^2k_1 + mk^2\right]\|x-y\|_{\mathscr{PC}}^2 + 4T^2k^2k_B^2\mathbf{E}\|u_a(t,x) - u_a(t,y)\|_{\mathrm{H}}^2 \\ &\le 4\left(\frac{a^2 + 3T^2k^4k_B^2}{a^2}\right)\left[k_0 + 2T^2k^2k_1 + mk^2\sum_{k=1}^m Q_k\right]\|x-y\|_{\mathscr{PC}}^2.\end{aligned}$$

By (3.1), we conclude that Θ is a contraction mapping on $\mathscr{PC}$. Thus by the Banach fixed point theorem, has a unique fixed point $x(\cdot) \in \mathscr{PC}$. Hence the proof. $\square$

Theorem 3.2. *Assume the condition in Theorem 3.1 and* **(H8)** *are satisfied, then system (1.1)-(1.3) is approximately controllable on* $[0,T]$.

Proof. By Theorem 3.1, Θ has a unique fixed point x_a^* in $\mathscr{PC}$. By using the stochastic Fubini theorem, it can easily be seen that

$$\begin{aligned} x_a^*(T) = z - aR(a,\Pi_0^T)\Big[\mathbf{E}z - \mathscr{U}(T,0)\left[\varphi(0) - g(0,\varphi)\right] - g(t,x^*(t-r)) + \int_0^T \bar{\phi}(s)d\mathtt{B}^{\mathtt{H}}(s)\Big] \\ + \int_0^T aR(a,\Pi_s^T)\mathscr{U}(T,s)f(s,x^*,x^*(s-r))ds \\ + \int_0^T aR(a,\Pi_s^T)\mathscr{U}(T,s)\sigma(s)d\mathtt{B}^{\mathtt{H}}(s) + \sum_{0<t_k<t} \mathscr{U}(T,t_k)I_k(x^*(t_k^-)). \end{aligned}$$

It follows from the assumption **(H8)** that there exists $\overline{\mathscr{K}}$ such that

$$\|f(s,x^*(s),x^*(s-r))\|_{\mathrm{H}}^2 + \|g(s,x^*(t-r))\|_{\mathscr{L}_2^0}^2.$$

Then there is a subsequence still denoted by $\{f(s,x^*(s),x^*(s-r)), g(s,x^*(s-r))\}$ which converges weakly to say, $\{f(s),g(s)\}$ in $\mathrm{H}\times\mathscr{L}_2^0$. On the other hand, by **(H7)**, for all $o \le t \le T$, $aR(a,\Pi_s^T) \longrightarrow 0$ as $a \to 0^+$ strongly and moreover $\|aR(a,\Pi_s^T)\| \le 1$. Therefore, by the Lebesgue dominated convergence theorem and the compactness of $\mathscr{U}(\cdot,\cdot)$ it follows that

$$\begin{aligned} \mathbf{E}\|x_a^*(T) - z\|^2 \le 5\mathbf{E}\Big\|aR(a,\Pi_0^T)\Big(\mathbf{E}z - aR(a,\Pi_0^T)\Big)\Big[\mathbf{E}z - \mathscr{U}(T,0)\left[\varphi(0)-g(0,\varphi)\right] - g(t,x^*(t-r)) \\ + \int_0^T \bar{\phi}(s)d\mathtt{B}^{\mathtt{H}}(s)\Big]\Big\|^2 + 5\mathbf{E}\left(\int_0^T \Big\|aR(a,\Pi_s^T)\mathscr{U}(T,s)[f(s,x^*(s),x^*(s-r)) - f(s)]\Big\| ds\right)^2 \\ + 5\mathbf{E}\left(\int_0^T \Big\|aR(a,\Pi_s^T)\mathscr{U}(T,s)f(s)\Big\| ds\right)^2 + 10\mathtt{H}T^{2\mathtt{H}-1}\int_0^T \Big\|aR(a,\Pi_s^T)\mathscr{U}(T,s)\sigma(s)\Big\|_{\mathscr{L}_2^0}^2 ds \\ + 5\mathbf{E}\Big\|\sum_{0<t_k<T} aR(a,\Pi_s^T)\mathscr{U}(T,t_k)I_k(x^*(t_k^-))\Big\|^2 \longrightarrow 0 \text{ as } a \to 0^+. \end{aligned}$$

Thus, $x_a^*(T) \to z$ holds in H and consequently we obtain the approximate controllability of system (1.1)-(1.3). $\square$

4. An Example

We consider the following stochastic classical heat equation with memory of the form:

$$\begin{aligned} d\left[u(t,\zeta) - \bar{g}(t,u(t-r),\zeta)\right] = \left[\frac{\partial^2}{\partial\zeta^2}u(t,\zeta) + a(t,\zeta)u(t,\zeta) - \bar{g}(t,u(t-r),\zeta)\right]dt \\ + f(t,u(t,\zeta),u(t-r),\zeta)dt + \mu(t,\zeta)dt + \sigma(t)d\mathtt{B}^{\mathtt{H}}(t), \zeta \in [0,\pi], t \ne t_k, J = [0,T] \\ u(t_k^+,\zeta) - u(t_k^-,\zeta) = I_k(u(t_k^-,\zeta)), \quad k \in \{1,2,...,m\}, \\ u(t,0) = u(t,\pi) = 0, \ t \in [0,T], \\ u(\theta,\cdot) = \varphi(\theta,\cdot) \in \mathrm{H} = \mathscr{L}^2[0,\pi], \quad \varphi(\cdot,\zeta) \in \mathscr{C}([-r,0];\mathbb{R}), \quad \theta \in [0,\pi]. \end{aligned} \tag{4.1}$$

where $\mathtt{B}^{\mathtt{H}}$ is a fractional Brownian motion and let $\mathrm{H} = \mathrm{K} = \mathrm{U} = \mathscr{L}^2([0,\pi])$. Define $A : \mathrm{H} \to \mathrm{H}$ by $Ax = x''$ with domain $\mathscr{D}(A) = \Big\{x \in \mathrm{H}:$ x, x' are absolutely continuous $x'' \in \mathrm{H}, x(0) = x(\pi) = 0\Big\}$. The spectrum of A consists of the eigenvalues $-n^2$ for $n \in \mathbb{N}$, with associated eigenvectors $e_n = \sqrt{\frac{2}{\pi}}\sin nx$, $n = 1,2,3,..$ It is wellknown that A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}$, $t \ge 0$ on H is given by

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2} < x, e_n > e_n, \quad x \in \mathrm{H}.$$

Now, define the fractional Brownian motion in K by

$$\mathtt{B}^{\mathtt{H}}(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta^{\mathtt{H}}(t)e_n,$$

where $\mathtt{H} \in (1/2,1)$ and $\{\beta_n^{\mathtt{H}}\}_{n\in\mathbb{N}}$ is a sequence of one dimensional fractional Brownian motions mutually independent. On the domain $\mathscr{D}(A)$, we define the operators $A(t) : \mathscr{D}(A) \subset \mathrm{H} \to \mathrm{H}$ by $A(t)z(\zeta) = Az(\zeta) + a(t,\zeta)z(\zeta)$. Let $a(\cdot,\cdot)$ be continuous and $a(t,\zeta) \le -\lambda, \lambda > 0$ for all $t \in J$, $\zeta \in [0,\pi]$, it follows that the system

$$\begin{cases} du(t) = A(t)u(t)dt, & t \ge s, \\ u(s) = z \in \mathrm{H}. \end{cases}$$

has an associated evolution family $\left\{\mathscr{U}(t,s)\right\}_{t\geq s}$ as $\mathscr{U}(t,s)z(\zeta)=\left(S(t-s)e^{\int_s^t a(p,\zeta)dt}z\right)(\zeta)$
From the above expression, it follows that $\mathscr{U}(t,s)$ is a compact operator and every $t,s\in J$ with $t>s$

$$\|\mathscr{U}(t,s)\|\leq e^{-(1+\lambda)(t-s)}.$$

Thus, **(H1)** is true.
Now we define the linear continuous mapping $B:\mathrm{U}\rightarrow\mathrm{H}$ by $u(t)=\mu(t,\zeta)$ where $\mu(t,\zeta):J\times[0,\pi]\rightarrow[0,\pi]$ is continuous.
Define $x(t)(\cdot)=u(t,\cdot)$, $f(t,u,u)(\cdot)=\bar{f}(t,u,u)(\cdot), g(t,u)(\cdot)=\bar{g}(t,u)(\cdot)$. Then, under the above conditions, we can represent the stochastic control system (4.1) in the abstract from (1.1)-(1.3). Thus we can conclude that the stochastic control system (4.1) is approximately controllable on $[0,T]$ provided that all the conditions of Theorem 3.2 are satisfied.

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Universal Journal of Mathematics and Applications, 3 (3) (2020) 121-128
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.756304

Existence and Uniqueness of Generalised Fractional Cauchy-Type Problem

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Article Info

Keywords: *Generalized Fractional Cauchy-type problem, Existence and uniqueness, Hilfer-Hadamard-type fractional derivative, Equivalence with Volterra integral equation.*
2010 AMS: *34A08, 34A34, 34A12.*
Received: *22 June 2020*
Accepted: *22 September 2020*
Available online: *29 September 2020*

Abstract

In this paper, we study the existence and uniqueness of Generalized Fractional Cauchy-type problem involving Hilfer-Hadamard-type fractional derivative for a nonlinear fractional differential equation. Also, we prove an equivalence between the Cauchy-type problem and Volterra integral equation(VIE).

1. Introduction

We consider the Cauchy-type problem

$$\begin{cases} {}_HD_{a+}^{\alpha,\beta}x(t) = \varphi(t,x(t)), \quad n-1<\alpha<n, 0\leq\beta\leq 1, \\ {}_HD_{a+}^{\gamma-j}x(t)\big|_{t=a} = x_{a_j}, \quad (j=1,2,...,n), \quad \gamma=\alpha+\beta(n-\alpha). \end{cases} \tag{1.1}$$

From the above initial condition and by definition 2.3(in this paper), it is clear that

$${}_HD_{a+}^{\gamma-j}x(t) = \delta^{n-j}\ {}_HI_{a+}^{n-\gamma}x(t),$$

where ${}_HD_{a+}^{\alpha,\beta}$ is the Hilfer-Hadamard-type fractional derivative of order α and type β [1,2] Fractional differential equations have numerous applications in science, physics, chemistry, and engineering [3,6].
Recently, the theory and applications of fractional derivatives have received considerable attention by researchers. They have studied some results of the existence and uniqueness of solutions for fractional differential equations on the different finite intervals such as the examples in [1,21] and references therein.
In this paper, we find a variety of results for the initial values problem (1.1), which are equivalent with (VIE), existence and uniqueness. In section 2, we present some preliminaries. In section 3, we establish the equivalence of the Cauchy-type problem (1.1) and (VIE). In section 4, we prove the existence and uniqueness results for a solution of the Cauchy-type problem (1.1) in the weighted space.

2. Preliminaries

In this section, we introduce some notations, Lemmas, definitions and weighted spaces, which are important for developing some theories in this paper. For further explanations, see [5].
Let $0<a<b<+\infty$. Assume that $C[a,b], AC[a,b], C^n[a,b]$ and $C_\mu^n[a,b]$ be the spaces of continuous, absolutely continuous, n-times continuous and continuously differentiable functions on $[a,b]$ respectively. And let $L^p(a,b)$ with $p\geq 1$ be the space of Lebesgue integrable

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functions on (a,b). Moreover, we recall some of weighted spaces [5] in definition 2.1.
Definition 2.1 [5] Let $\Omega=[a,b]$ $(0<a<b<+\infty)$ is a finite interval and $0\le\mu<1$. We introduce the weighted space $C_{\mu,\log}[a,b]$ of continuous functions φ on $(a,b]$

$$C_{\mu,\log}[a,b]=\{\varphi:(a,b]\to\mathbb{R}:[\log(t/a)]^{\mu}\varphi(t)\in C[a,b]\}$$

with the norm

$$\|\varphi\|_{C_{\mu,\log}}=\left\|[\log(t/a)]^{\mu}\varphi(t)\right\|_{C},\quad C_{0,\log}[a,b]=C[a,b].$$

And for $n\in\mathbb{N}$ and $\delta=t\frac{d}{dt}$, we have

$$C^{n}_{\delta,\mu}[a,b]=\left\{\varphi:\|\varphi\|_{C^{n}_{\delta,\mu}}=\sum_{k=0}^{n-1}\|\delta^{k}\varphi\|_{C}+\|\delta^{n}\varphi\|_{C_{\mu,\log}}\right\},C^{0}_{\delta,\mu}[a,b]=C_{\mu,\log}[a,b].$$

The space $C_{\mu,\log}[a,b]$ is the complete metric space defined with the distance as

$$d(x_1,x_2)=\|x_1-x_2\|_{C_{\mu,\log}}[a,b]:=\max_{t\in[a,b]}\left|[\log(t/a)]^{\mu}\left[x_1(t)-x_2(t)\right]\right|,$$

where $\log(.)=\log_{e}(.)$.
Definition 2.2 [4,5] Let $0<a<b<+\infty$. The Hadamard fractional integral of order $\alpha\in\mathbb{R}^{+}$ for a function $\varphi:(a,+\infty)\to\mathbb{R}$ is defined as

$${}_{H}I^{\alpha}_{a+}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(\log\frac{t}{\tau})^{\alpha-1}\quad\frac{\varphi(\tau)}{\tau}d\tau,\qquad(t>a).$$

Definition 2.3 [4,5] Let $0<a<b<+\infty$. The Hadamard fractional derivative of order α applied to the function $\varphi:(a,+\infty)\to\mathbb{R}$ is defined as

$${}_{H}D^{\alpha}_{a+}\varphi(t)=\delta^{n}({}_{H}I^{n-\alpha}_{a+}\varphi(t)),\quad n-1<\alpha<n,\quad n=[\alpha]+1,$$

where $\delta^{n}=(t\frac{d}{dt})^{n}$, and $[\alpha]$ denotes the integer part of the real number α.
Lemma 2.4 [5] Let $n\in\mathbb{N}_0=\{0,1,2,...\}$ and let $\mu_1,\mu_2\in\mathbb{R}$ such that $0\le\mu_1\le\mu_2<1$. The following embeddings hold:

$$C^{n}_{\delta}[a,b]\longrightarrow C^{n}_{\delta,\mu_1}[a,b]\longrightarrow C^{n}_{\delta,\mu_2}[a,b],$$

with

$$\|\varphi\|_{C^{n}_{\delta,\mu_2}}\le K_{\delta}\|\varphi\|_{C^{n}_{\delta,\mu_1}},\quad K_{\delta}=\min\left[1,\left(\log(b/a)\right)^{\mu_2-\mu_1}\right],\ a\neq0.$$

In particular,

$$C[a,b]\longrightarrow C_{\mu_1,\log}[a,b]\longrightarrow C_{\mu_2,\log}[a,b]$$

with

$$\|\varphi\|_{C_{\mu_2,\log}}\le\left(\log(b/a)\right)^{\mu_2-\mu_1}\|\varphi\|_{C_{\mu_1,\log}},\ a\neq0.$$

Lemma 2.5 [5]
(a_1) If $\Re(\alpha)\ge0,\Re(\beta)\ge0$ and $0<a<b<+\infty$, then

$$\left[{}_{H}I^{\alpha}_{a+}\left(\log(\tau/a)\right)^{\beta-1}\right](x)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\log(t/a))^{\alpha+\beta-1},\quad x>a,$$
$$\left[{}_{H}D^{\alpha}_{a+}\left(\log(\tau/a)\right)^{\beta-1}\right](x)=\frac{\Gamma(\beta)}{\Gamma(\alpha-\beta)}(\log(t/a))^{\alpha-\beta-1},\quad x>a.$$

(a_2) Let $\Re(\alpha)\ge0,n=[\Re(\alpha)]+1$ and $0<a<b<+\infty$. The equality $({}_{H}D^{\alpha}_{a+}x)(t)$ $=0$ is valid if and only if

$$x(t)=\sum_{k=1}^{n}c_k(\log(t/a))^{\alpha-k},$$

where $c_k\in\mathbb{R}(k=1,2,...,n)$ are arbitrary constants.
(a_3) Let $\Re(\alpha)\ge0,\Re(\beta)\ge0$ and $0\le\mu<1$. If $0<a<b<+\infty$, then for $\varphi\in C_{\mu,\log}[a,b]$

$${}_{H}I^{\alpha}_{a+}\,{}_{H}I^{\beta}_{a+}\varphi={}_{H}I^{\alpha+\beta}_{a+}\varphi$$

holds at any point $t\in(a,b]$. When $\varphi\in C[a,b]$, then this relation will be valid at any point $t \in (a,b]$.

Theorem 2.6 [5] Let $\Re(\alpha) \geq 0, n = [\Re(\alpha)] + 1$, and $0 < a < b < +\infty$. Also, let ${}_HI_{a+}^{n-\alpha}\varphi$ be the Hadamard-type fractional integral of order $n-\alpha$ of the function φ. If $\varphi \in C_{\mu,\log}[a,b]$ $(0 \leq \mu < 1)$ and ${}_HI_{a+}^{n-\alpha}\varphi \in C_{\delta,\mu}^n[a,b]$, then

$$({}_HI_{a+}^{\alpha}\, {}_HD_{a+}^{\alpha}\varphi)(t) = \varphi(t) - \sum_{k=1}^{n} \frac{(\delta^{n-k}({}_HI_{a+}^{n-\alpha}\varphi))(a)}{\Gamma(\alpha-k+1)}(\log\frac{t}{a})^{\alpha-k}.$$

Lemma 2.7 [5] Let $0 < a < b < +\infty, \Re(\alpha) \geq 0, n = [\Re(\alpha)] + 1$ and $0 \leq \Re(\mu) < 1$.

(a) If $\Re(\mu) > \Re(\alpha) > 0$, then the fractional integration operator ${}_HI_{a+}^{\alpha}$ is bounded from $C_{\mu,\log}[a,b]$ into $C_{\mu-\alpha,\log}[a,b]$:

$$\| {}_HI_{a+}^{\alpha}\varphi\|_{C_{\mu-\alpha,\log}} \leq k_1\|\varphi\|_{C_{\mu,\log}},$$

where

$$k_1 = \Big(\log(b/a)\Big)^{\Re(\alpha)} \frac{\Gamma[\Re(\alpha)]|\Gamma(1-\Re(\mu))|}{|\Gamma(\alpha)|\Gamma(1+\Re(\alpha-\mu))}.$$

In particular, ${}_HI_{a+}^{\alpha}$ is bounded in $C_{\mu,\log}[a,b]$.

(b) If $\Re(\mu) \leq \Re(\alpha)$, then the fractional integration operator ${}_HI_{a+}^{\alpha}$ is bounded from $C_{\mu,\log}[a,b]$ into $C[a,b]$:

$$\| {}_HI_{a+}^{\alpha}\varphi\|_{C} \leq k_2\|\varphi\|_{C_{\mu,\log}},$$

where

$$k_2 = \Big(\log(b/a)\Big)^{\Re(\alpha-\mu)} \frac{\Gamma[\Re(\alpha)]|\Gamma(1-\Re(\mu))|}{|\Gamma(\alpha)|\Gamma(1+\Re(\alpha-\mu))}.$$

In particular, ${}_HI_{a+}^{\alpha}$ is bounded in $C_{\mu,\log}[a,b]$.

Definition 2.8 [2] Let $n-1 < \alpha < n, 0 \leq \beta \leq 1$, and $\varphi \in L^1(a,b)$. The Hilfer-Hadamard fractional derivative ${}_HD^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}({}_HD^{\alpha,\beta}\varphi)(t) &= \big({}_HI^{\beta(n-\alpha)}(\delta)^n\, {}_HI^{(n-\alpha)(1-\beta)}\varphi\big)(t)\\ &= \big({}_HI^{\beta(n-\alpha)}(\delta)^n\, {}_HI^{n-\gamma}\varphi\big)(t); \quad \gamma = \alpha + n\beta - \alpha\beta\\ &= \big({}_HI^{\beta(n-\alpha)}\, {}_HD^{\gamma}\varphi\big)(t),\end{aligned}$$

where ${}_HI^{(\cdot)}$ and ${}_HD^{(\cdot)}$ is the Hadamard fractional integral and derivative defined by definitions 2.2 and 2.3 respectively.

Definition 2.9 [5, 13] Assume that $\varphi(x,y)$ is defined on set $(a,b] \times G, G \subset \mathbb{R}$. The function $\varphi(x,y)$ satisfies Lipschitz condition with respect to y, if for all $x \in (a,b]$ and for all $y_1, y_2 \in G$,

$$|\varphi(x,y_1) - \varphi(x,y_2)| \leq L|y_1 - y_2|,$$

where $L > 0$ is Lipschitz constant.

Definition 2.10 [1, 12] Let $0 < \alpha < 1, 0 \leq \beta \leq 1$. The weighted space $C_{1-\gamma}^{\alpha,\beta}[a,b]$ is defined by

$$C_{1-\gamma}^{\alpha,\beta}[a,b] = \{\varphi \in C_{1-\gamma}[a,b] : D_{a+}^{\alpha,\beta}\varphi \in C_{1-\gamma}[a,b]\}, \gamma = \alpha + \beta - \alpha\beta.$$

Lemma 2.11 [9] Let $0 < a < b < +\infty, \alpha > 0, 0 \leq \mu < 1$ and $\varphi \in C_{\mu,\log}[a,b]$. If $\alpha > \mu$, then ${}_HI_{a+}^{\alpha}\varphi$ is continuous on $[a,b]$ and

$${}_HI_{a+}^{\alpha}\varphi(a) = \lim_{t\to a^+} {}_HI_{a+}^{\alpha}\varphi(t) = 0.$$

Lemma 2.12 [2] Let $\Re(\alpha) > 0, 0 \leq \beta \leq 1, \gamma = \alpha + n\beta - \alpha\beta, n-1 < \gamma \leq n,\ n = [\Re(\alpha)] + 1$ and $0 < a < b < \infty$. If $\varphi \in L^1(a,b)$ and $({}_HI_{a+}^{n-\gamma}\varphi)(t) \in AC_{\delta}^n[a,b]$, then

$${}_HI_{a+}^{\alpha}\,({}_HD_{a+}^{\alpha,\beta}\varphi)(t) = {}_HI_{a+}^{\gamma}\,({}_HD_{a+}^{\gamma}\varphi)(t) = \varphi(t) - \sum_{j=1}^{n} \frac{(\delta^{(n-j)}({}_HI_{a+}^{n-\gamma}\varphi))(a)}{\Gamma(\gamma-j+1)}(\log\frac{t}{a})^{\gamma-j}.$$

Lemma 2.13 [13] Let $0 < a < b < +\infty, 0 \leq \mu < 1, \varphi \in C_{\mu,\log}[a,c]$ and $\varphi \in C_{\mu,\log}[c,b]$. Then, $\varphi \in C_{\mu,\log}[a,b]$ and

$$\|\varphi\|_{C_{\mu,\log}[a,b]} \leq \max\left\{\|\varphi\|_{C_{\mu,\log}[a,c]}, \Big(\log(b/a)\Big)^{\mu}\|\varphi\|_{C[c,b]}\right\}.$$

Theorem 2.14 [5] Let (U,d) be a non-empty complete metric space. Let $0 \leq \omega < 1$ and let $\mathrm{T}:\mathrm{U} \to \mathrm{U}$ be the map such that, for every $u,v \in \mathrm{U}$, the relation

$$d(\mathrm{T}u,\mathrm{T}v) \leq \omega\, d(u,v), \qquad 0 \leq \omega < 1$$

holds. Then, the operator T has a unique fixed point $u^* \in \mathrm{U}$.

Furthermore, if $\mathrm{T}^k (k \in \mathbb{N})$ is the sequence of operators defined by

$$\mathrm{T}^1 = \mathrm{T}, \qquad \mathrm{T}^k = \mathrm{T}\mathrm{T}^{k-1} \in \mathbb{N}\setminus\{1\},$$

then, for any $u_0 \in \mathrm{U}$, the sequence $\{\mathrm{T}^k u_0\}_{k=1}^{+\infty}$ converges to the above fixed point u^*.

3. Equivalence of the Cauchy-Type Problem (1.1) and (VIE)

In this section, we are going to prove the equivalence of the Cauchy-type problem (1.1) and (VIE). So, we need the following definition:
Definition 3.1 Let $n-1<\alpha<n,\ 0\leq\beta\leq 1, \gamma=\alpha+n\beta-\alpha\beta$ and $0\leq\mu<1$. We consider the underlying spaces defined by

$$C^{\alpha,\beta}_{\delta;n-\gamma,\mu}[a,b]=\{\varphi\in C_{n-\gamma,\log}[a,b]:\ {}_HD^{\alpha,\beta}_{a+}\varphi\in C_{\mu,\log}[a,b]\}$$

and

$$C^{\gamma}_{n-\gamma,\log}[a,b]=\{\varphi\in C_{n-\gamma,\log}[a,b]:\ {}_HD^{\gamma}_{a+}\varphi\in C_{n-\gamma,\log}[a,b]\},$$

where $C_{n-\gamma,\log}[a,b]$ and $C_{\mu,\log}[a,b]$ are weighted spaces of continuous functions on $(a,b]$ defined by

$$C_{\gamma,\log}[a,b]=\left\{\varphi:(a,b]\to\mathbb{R}:\left(\log t/a\right)^{\gamma}\varphi(t)\in C[a,b]\right\}.$$

In the next theorem, we studied the equivalence between the Cauchy-type problem (1.1) and (VIE) of the second kind

$$x(t)=\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log(t/a))^{\gamma-k}+\frac{1}{\Gamma(\alpha)}\int_a^t(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau},\quad t>a. \tag{3.1}$$

Theorem 3.2 Let $n-1<\alpha<n, 0\leq\beta\leq 1, \gamma=\alpha+\beta(n-\alpha)$, and assume that $\varphi(.,x(.))\in C_{\mu,\log}[a,b]$, where $\varphi:(a,b]\times\mathbb{R}\to\mathbb{R}$ be a function for any $x\in C_{\mu,\log}[a,b](n-\gamma\leq\mu<n-\beta(n-\alpha))$. If $x\in C^{\gamma}_{n-\gamma,\log}[a,b]$, then x satisfies (1.1) if and only if x satisfies the integral equation (3.1).

Proof. First part, we will prove the necessity.
Assume that $x\in C^{\gamma}_{n-\gamma,\log}[a,b]$ is a solution of (1.1). We prove that x is a solution of (3.1) as follows:
By the definition 3.1 of $C^{\gamma}_{n-\gamma,\log}[a,b]$, Lemma 2.7 (b) and definition 2.3, we have

$${}_HI^{n-\gamma}_{a+}x\in C[a,b],\qquad {}_HD^{\gamma}_{a+}x=\delta^n\ {}_HI^{n-\gamma}_{a+}x\in C_{n-\gamma,\log}[a,b].$$

Thus, by definition 2.1, we get

$${}_HI^{n-\gamma}_{a+}x\in C^{n}_{\delta,n-\gamma}[a,b].$$

Now, by applying Theorem 2.6, we obtain

$${}_HI^{\gamma}_{a+}\ {}_HD^{\gamma}_{a+}x(t)=x(t)-\sum_{k=1}^{n}\frac{(\delta^{n-k}({}_HI^{n-\gamma}_{a+}\varphi))(a)}{\Gamma(\gamma-k+1)}(\log\frac{t}{a})^{\gamma-k},\quad t\in(a,b],$$

or

$${}_HI^{\gamma}_{a+}\ {}_HD^{\gamma}_{a+}x(t)=x(t)-\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log\frac{t}{a})^{\gamma-k},\quad t\in(a,b], \tag{3.2}$$

where x_{a_k} comes from the initial condition of (1.1). By our hypothesis $\varphi(.,x(.))\in C_{\mu,\log}[a,b]$ and since $x\in C_{n-\gamma,\log}[a,b]\subset C_{\mu,\log}[a,b]$, Lemma 2.7, we can see that the integral ${}_HI^{\alpha}_{a+}\varphi(.,x(.))\in C_{\mu-\alpha,\log}[a,b]$ for $\mu>\alpha$ and ${}_HI^{\alpha}_{a+}\varphi(.,x(.))\in C[a,b]$ for $\mu\leq\alpha$. By applying the operator ${}_HI^{\alpha}_{a+}$ to both sides of the problem of Cauchy-type (1.1) and Lemma 2.12 we obtain

$${}_HI^{\gamma}_{a+}\ {}_HD^{\gamma}_{a+}x=\ {}_HI^{\alpha}_{a+}\ {}_HD^{\alpha,\beta}_{a+}x=\ {}_HI^{\alpha}_{a+}(\ {}_HD^{\alpha,\beta}_{a+}x)=\ {}_HI^{\alpha}_{a+}\varphi. \tag{3.3}$$

From (3.2) and (3.3) we get

$$x(t)=\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log\frac{t}{a})^{\gamma-k}+\ {}_HI^{\alpha}_{a+}[\varphi(\tau,x(\tau))](t),\quad t\in(a,b], \tag{3.4}$$

which is the (VIE)(3.1).

Second part, we will prove the sufficiency.
Assume that $x\in C^{\gamma}_{n-\gamma,\log}[a,b]$ satisfies (3.1) which is written as (3.4). Then, ${}_HD^{\gamma}_{a+}x$ exists and ${}_HD^{\gamma}_{a+}x\in C_{n-\gamma,\log}[a,b]$. Now, by applying the operator ${}_HD^{\gamma}_{a+}$ to both sides of (3.4), we get

$${}_HD^{\gamma}_{a+}x(t)=\ {}_HD^{\gamma}_{a+}\left[\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log\frac{t}{a})^{\gamma-k}+\ {}_HI^{\alpha}_{a+}[\varphi(\tau,x(\tau))](t)\right].$$

By using Lemma 2.5 (a_2) and (a_3), and definition 2.3, we obtain

$$\begin{aligned}{}_HD^{\gamma}_{a+}x&=\ {}_HD^{\gamma}_{a+}\left[{}_HI^{\alpha}_{a+}\varphi\right]\\&=\delta^n({}_HI^{n-\gamma}_{a+}\ {}_HI^{\alpha}_{a+}\varphi)\\&=\delta^n({}_HI^{n-\beta(n-\alpha)}\varphi)\\&=\ {}_HD^{\beta(n-\alpha)}_{a+}\varphi\end{aligned} \tag{3.5}$$

From (3.5) and the hypothesis ${}_HD^{\gamma}_{a+}x \in C_{n-\gamma,\log}[a,b]$, we have

$$ {}_HD^{\beta(n-\alpha)}_{a+}\varphi \in C_{n-\gamma,\log}[a,b]. $$

Now, by applying ${}_HI^{\beta(n-\alpha)}_{a+}$ to both sides of (3.5) we obtain

$$ (\,{}_HI^{\beta(n-\alpha)}_{a+}\;{}_HD^{\gamma}_{a+}x)(t) = (\,{}_HI^{\beta(n-\alpha)}_{a+}\;{}_HD^{\beta(n-\alpha)}_{a+}\varphi(\tau,x(\tau)))(t); $$

that is,

$$ {}_HI^{\beta(n-\alpha)}_{a+}\delta^n({}_HI^{n-\gamma}_{a+}x)(t) = (\,{}_HI^{\beta(n-\alpha)}_{a+}\;{}_HD^{\beta(n-\alpha)}_{a+}\varphi(\tau,x(\tau)))(t). $$

Since

$$ \delta^n({}_HI^{n-\beta(n-\alpha)}_{a+}\varphi(t,x(t))) = {}_HD^{\beta(n-\alpha)}_{a+}\varphi(.,x(.)) \in C_{n-\gamma,\log}[a,b], $$

and $\gamma > \beta(n-\alpha)$ and by definition 2.1, we have ${}_HI^{n-\beta(n-\alpha)}_{a+}\varphi \in C^n_{\delta;n-\gamma}[a,b]$ (also that which is found in the first part of this proof, or by Lemma 2.7 (b) with $\mu < n-\beta(n-\alpha)$, for a continuity of ${}_HI^{n-\beta(n-\alpha)}_{a+}\varphi$). Then, Theorem 2.6 with definition 2.8 allow us to write

$$ {}_HD^{\alpha,\beta}_{a+}x(t) = \varphi(t,x(t)) - \sum_{k=1}^{n}\frac{(\delta^{n-k}({}_HI^{n-\beta(n-\alpha)}_{a+}\varphi))(a)}{\Gamma(\beta(k-\alpha))}(\log\frac{t}{a})^{\beta(n-\alpha)-k}, \tag{3.6} $$

since $\mu < n-\beta(n-\alpha)$. Then, it follows by Lemma 2.11 that

$$ \left[\,{}_HI^{n-\beta(n-\alpha)}_{a+}\varphi)\right](a) = 0. $$

Therefore, we can write the relation (3.6) as

$$ {}_HD^{\alpha,\beta}_{a+}x(t) = \varphi(t,x(t)), \qquad t\in(a,b]. $$

Finally, we will show that the initial condition of (1.1) also holds. For that, we apply ${}_HD^{\gamma-j}_{a+} = \delta^{n-j}\;{}_HI^{n-\gamma}_{a+}(j=1,2,...,n)$ to both sides of (3.4) and by using Lemma 2.5 (a_1) and (a_3), we obtain

$$ {}_HD^{\gamma-j}_{a+}x(t) = x_{a_j} + \left[\delta^{n-j}(\,{}_HI^{n-\beta(n-\alpha)}_{a+}\varphi(\tau,x(\tau)))\right](t) \tag{3.7} $$

Now, taking the limit as $t\to a$, in (3.7), we get

$$ {}_HD^{\gamma-j}_{a+}x(t)\big|_{t=a} = x_{a_j}, \qquad (j=1,2,...,n). $$

The proof of this theorem is complete.

Remark 3.3 For $0<\alpha<1$, Theorem 3.2 is reduced to Theorem 21 (see[9]).

4. Existence and Uniqueness

In this section, we will prove the existence and uniqueness results for a solution of the Cauchy-type problem (1.1) in the weighted space $C^{\alpha,\beta}_{n-\gamma,\log}[a,b]$ by using the Banach fixed point theorem. For that, we need the following Lemma.

Lemma 4.1 If $\mu\in\mathbb{R}(0\le\mu<1)$, then the Hadamard-type fractional integral operator ${}_HI^{\alpha}_{a+}$ *with* $\alpha\in\mathbb{C}(\Re(\alpha)>0)$ is bounded from $C_{\mu,\log}[a,b]$ *into* $C_{\mu,\log}[a,b]$ such that,

$$ \|\,{}_HI^{\alpha}_{a+}\varphi\|_{C_{\mu,\log}[a,b]} \le \frac{\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)}(\log(t/a))^{\alpha}\|\varphi\|_{C_{\mu,\log}[a,b]}. \tag{4.1} $$

Proof. By Lemma 2.7, the result of this Lemma follows. Now, we will prove the inequality (4.1). By definition 2.1 of the weighted space $C_{\mu,\log}[a,b]$, we have

$$ \begin{aligned} \|\,{}_HI^{\alpha}_{a+}\varphi\|_{C_{\mu,\log}[a,b]} &= \left\|(\log(t/a))^{\mu}\;{}_HI^{\alpha}_{a+}\varphi\right\|_{C[a,b]} \\ &\le \left\|\varphi\right\|_{C_{\mu,\log}[a,b]}\left\|{}_HI^{\alpha}_{a+}(\log(t/a))^{-\mu}\right\|_{C_{\mu,\log}[a,b]}. \end{aligned} $$

Now, by using Lemma 2.5 (a_1) (with β replaced by $1-\mu$) we obtain

$$ \|\,{}_HI^{\alpha}_{a+}\varphi\|_{C_{\mu,\log}[a,b]} \le \frac{\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)}(\log(t/a))^{\alpha}\|\varphi\|_{C_{\mu,\log}[a,b]}. $$

Hence, the proof of this Lemma is complete.

Theorem 4.2 Let $n-1<\alpha<n, 0\le\beta\le1, \gamma=\alpha+\beta(n-\alpha)$, and assume that $\varphi(.,x(.))\in C_{\mu,\log}[a,b]$, where $\varphi:(a,b]\times\mathbb{R}\to\mathbb{R}$ be

a function for any $x \in C_{\mu,\log}[a,b](n-\gamma \leq \mu < n-\beta(n-\alpha))$ and satisfies the Lipschitz condition given in definition 2.9 with respect to x. Then, there exists a unique solution $x(t)$ for the Cauchy-type problem (1.1) in the weighted space $C^{\alpha,\beta}_{\delta;n-\gamma,\mu}[a,b]$.

Proof. First, we will prove the existence of the unique solution $x(t) \in C_{n-\gamma,\log}[a,b]$. According to Theorem 3.2, it is sufficient to prove the existence of the unique solution $x(t) \in C_{n-\gamma,\log}[a,b]$ to the nonlinear (VIE)(3.1) and that is based on Theorem 2.14 (Banach fixed point theorem). Since the equation (3.1) makes sense in any interval $[a,t_1] \subset [a,b]$, then we choose $t_1 \in (a,b]$ such that the following estimate holds

$$\omega_1 := L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_1/a))^{\alpha} \;<\; 1, \tag{4.2}$$

where $L>0$ is a Lipschitz constant. So, we will prove the existence of the unique solution $x(t) \in C_{n-\gamma,\log}[a,t_1]$ to the equation (3.1) on the interval $(a,t_1]$. For this we know that the space $C_{n-\gamma,\log}[a,t_1]$ is a complete metric space defined with the distance as

$$d(x_1,x_2) = \|x_1-x_2\|_{C_{n-\gamma,\log}}[a,t_1] := \max_{t\in[a,t_1]}\left|[\log(t/a)]^{n-\gamma}\big[x_1(t)-x_2(t)\big]\right|.$$

The equation (3.1) we be rewritten as the following:

$$x(t) = (\mathrm{T}x)(t),$$

where T is the operator defined by

$$(\mathrm{T}x)(t) = x_0(t) + \left[\,{}_HI^{\alpha}_{a+}\varphi(\tau,x(\tau))\right](t). \tag{4.3}$$

with

$$x_0(t) = \sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log(t/a))^{\gamma-k}. \tag{4.4}$$

Now, we claim that T maps from $C_{n-\gamma,\log}[a,t_1]$ into $C_{n-\gamma,\log}[a,t_1]$. In fact, it is clear from (4.4) that $x_0(t) \in C_{n-\gamma,\log}[a,t_1]$. And since $\varphi(t,x(t)) \in C_{n-\gamma,\log}[a,t_1]$, then, by Lemma 2.7 and Lemma 4.1 [with $\mu = n-\gamma, b = t_1$ and $\varphi(.) = \varphi(.,x(.))$], the integral in the right-hand side of (4.1) is relevant to $C_{n-\gamma,\log}[a,t_1]$. Thus, $(\mathrm{T}x)(t) \in C_{n-\gamma,\log}[a,t_1]$.
Next, we will prove that T is the contraction. That is, we will prove that the following estimate holds:

$$\left\|\mathrm{T}x_1-\mathrm{T}x_2\right\|_{C_{n-\gamma,\log}[a,t_1]} \leq \omega_1\left\|x_1-x_2\right\|_{C_{n-\gamma,\log}[a,t_1]},\ 0<\omega_1<1. \tag{4.5}$$

By equations (4.1) and (4.4), and using the Lipschitz condition given in definition 2.9 and applying the estimate (4.1) [with $\mu = n-\gamma, b = t_1$ and $\varphi(t) = \varphi(t,x_1(t)) - \varphi(t,x_2(t))$], we get

$$\begin{aligned}
\left\|\mathrm{T}x_1-\mathrm{T}x_2\right\|_{C_{n-\gamma,\log}[a,t_1]} &= \left\|\,{}_HI^{\alpha}_{a+}\varphi(t,x_1(t)) - {}_HI^{\alpha}_{a+}\varphi(t,x_2(t))\right\|_{C_{n-\gamma,\log}[a,t_1]}\\
&\leq \left\|\,{}_HI^{\alpha}_{a+}\big[|\varphi(t,x_1(t))-\varphi(t,x_2(t))|\big]\right\|_{C_{n-\gamma,\log}[a,t_1]}\\
&\leq L\left\|\,{}_HI^{\alpha}_{a+}\big[|x_1(t))-x_2(t))|\big]\right\|_{C_{n-\gamma,\log}[a,t_1]}\\
&\leq L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_1/a))^{\alpha}\|x_1-x_2\|_{C_{n-\gamma,\log}[a,t_1]}\\
&= \omega_1\|x_1-x_2\|_{C_{n-\gamma,\log}[a,t_1]},
\end{aligned}$$

which yields (4.5), $0<\omega_1<1$. According to (4.2) and by applying the Theorem 2.14 (Banach fixed point theorem), we obtain a unique solution $x^* \in C_{n-\gamma,\log}[a,t_1]$ to (VIE)(3.1) on the interval $(a,t_1]$.
This solution x^* is given from a limit of the convergent sequence $(\mathrm{T}^m x_0^*)(t)$:

$$\lim_{m\to\infty}\left\|\mathrm{T}^m x_0^* - x^*\right\|_{C_{n-\gamma,\log}[a,t_1]} = 0,$$

where x_0^* is any function in $C_{n-\gamma,\log}[a,t_1]$ and

$$\begin{aligned}
(\mathrm{T}^m x_0^*)(t) &= (\mathrm{T}\mathrm{T}^{m-1}x_0^*)(t)\\
&= x_0(t) + \left[\,{}_HI^{\alpha}_{a+}\varphi(\tau,(\mathrm{T}^{m-1}x_0^*)(\tau))\right](t),
\end{aligned}$$

Let us put $x_0^*(t) = x_0(t)$ with $x_0(t)$, which is defined by (4.4).
If we indicate $x_m(t) := (\mathrm{T}^m x_0^*)(t)$, then it is clear that

$$\lim_{m\to+\infty}\left\|x_m(t)-x^*\right\|_{C_{n-\gamma,\log}[a,t_1]} = 0. \tag{4.6}$$

Next, we consider the interval $[t_1,b]$. From the (VIE)(3.1) we have

$$\begin{aligned}
x(t) &= \sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log(t/a))^{\gamma-k} + \frac{1}{\Gamma(\alpha)}\int_a^{t_1}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau}\\
&= x_{01} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau},
\end{aligned} \tag{4.7}$$

where x_{01} is defined by

$$x_{01}=\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log(t/a))^{\gamma-k}+\frac{1}{\Gamma(\alpha)}\int_a^{t_1}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau}, \tag{4.8}$$

and is the known function. We note that $x_{01}\in C_{n-\gamma,\log}[t_1,b]$. Now, we will prove the existence of the unique solution $x(t)\in C_{n-\gamma,\log}[t_1,b]$ to the equation (3.1) on the interval $(t_1,b]$. Also, we use Theorem 2.14 (Banach fixed point theorem) for the space $C_{n-\gamma,\log}[t_1,t_2]$, where $t_2\in(t_1,b]$ (with $t_2=t_1+h_1,\ h_1>0,\ t_2\le b$) satisfies

$$\omega_2:=L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_2/t_1))^{\alpha}<1.$$

The space $C_{n-\gamma,\log}[t_1,t_2]$ is a complete metric space defined with the distance as

$$d(x_1,x_2)=\|x_1-x_2\|_{C_{n-\gamma,\log}}[t_1,t_2]=\max_{t\in[t_1,t_2]}\left|[\log(t/a)]^{n-\gamma}\big[x_1(t)-x_2(t)\big]\right|.$$

Also, we can rewrite equation (4.6) as the following:

$$x(t)=(\mathrm{T}x)(t), \tag{4.9}$$

where T is the operator given by

$$(\mathrm{T}x)(t)=x_{01}(t)+\left[\,{}_HI^{\alpha}_{t_1+}\varphi(\tau,x(\tau))\right](t).$$

As in the beginning part of this proof, since $x_{01}(t)\in C_{n-\gamma,\log}[t_1,t_2]$ and $\varphi(t,x(t))\in C_{n-\gamma,\log}[t_1,t_2]$, then, by Lemma 2.7 and Lemma 4.1 [with $\mu=n-\gamma, b=t_2$ and $\varphi(.)=\varphi(.,x(.))$], the integral in the right-hand side of (4.9) also belongs to $C_{n-\gamma,\log}[t_1,t_2]$. Thus, $(\mathrm{T}x)(t)\in C_{n-\gamma,\log}[t_1,t_2]$.
Furthermore, using the Lipschitz condition given in definition 2.9 and applying the estimate (4.1) [with $\mu=n-\gamma, b=t_2$ and $\varphi(t)=\varphi(t,x_1(t))-\varphi(t,x_2(t))$], we get

$$\begin{aligned}\big\|\mathrm{T}x_1-\mathrm{T}x_2\big\|_{C_{n-\gamma,\log}[t_1,t_2]}&=\big\|\,{}_HI^{\alpha}_{t_1+}\varphi(t,x_1(t))-\,{}_HI^{\alpha}_{t_1+}\varphi(t,x_2(t))\big\|_{C_{n-\gamma,\log}[t_1,t_2]}\\&\le\big\|\,{}_HI^{\alpha}_{t_1+}\big[|\varphi(t,x_1(t))-\varphi(t,x_2(t))|\big]\big\|_{C_{n-\gamma,\log}[t_1,t_2]}\\&\le L\,\big\|\,{}_HI^{\alpha}_{t_1+}\big[|x_1(t))-x_2(t))|\big]\big\|_{C_{n-\gamma,\log}[t_1,t_2]}\\&\le\ \omega_2\|x_1-x_2\|_{C_{n-\gamma,\log}[t_1,t_2]}.\end{aligned}$$

This, together with (4.8), $0<\omega_2<1$, indicates that T is a contraction. And by applying the Theorem 2.14 (Banach fixed point theorem), we obtain a unique solution $x_1^*\in C_{n-\gamma,\log}[t_1,t_2]$ to (VIE)(3.1) on the interval $(t_1,t_2]$. Moreover, this solution x_1^* is given from a limit of the convergent sequence $(\mathrm{T}^m x_{01}^*)(t)$:

$$\lim_{m\to+\infty}\big\|\mathrm{T}^m x_{01}^*-x_1^*\big\|_{C_{n-\gamma,\log}[t_1,t_2]}=0,$$

where x_{01}^* is any function in $C_{n-\gamma,\log}[t_1,t_2]$. Again, we can put $x_{01}^*(t)=x_{01}(t)$ defined by (4.7). Hence,

$$\lim_{m\to+\infty}\big\|x_m(t)-x_1^*\big\|_{C_{n-\gamma,\log}[t_1,t_2]}=0,$$

where

$$\begin{aligned}x_m(t)&=(\mathrm{T}^m x_{01}^*)(t)\\&=x_{01}(t)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau}.\end{aligned}$$

Next, if $t_2\neq b$, we consider the interval $[t_2,t_3]$ such that $t_3=t_2+h_2$ with $h_2>0,\ t_3\le b$ and

$$\omega_3:=L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_3/t_2))^{\alpha}<1.$$

By using the same argument as above, we conclude that there exists a unique solution $x_2^*\in C_{n-\gamma,\log}[t_2,t_3]$ to (VIE)(3.1) on $[t_2,t_3]$. If $t_3\neq b$, then we continue the previous process until we get a unique solution $x(t)$ to the (VIE)(3.1) and $x(t)=x_i^*$ such that $x_i^*\in C_{n-\gamma,\log}[t_{i-1},t_i]$ for $i=1,2,...,L$, where $a=t_0<t_1<t_2<...<t_L=b$ and

$$\omega_i:=L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_i/t_{i-1}))^{\alpha}<1.$$

Thus, by using Lemma 2.13, it yields that there exists a unique solution $x(t)\in C_{n-\gamma,\log}[a,b]$ to the (VIE)(3.1) on the whole interval $(a,b]$. Therefore, $x(t)\in C_{n-\gamma,\log}[a,b]$ is a unique solution to the Cauchy-type problem (1.1).

Finally, we will show that such unique solution $x(t)\in C_{n-\gamma,\log}[a,b]$ is in the weighted space $C^{\alpha,\beta}_{n-\gamma,\mu}[a,b]$. By definition 3.1, it is sufficient to prove that ${}_HD^{\alpha,\beta}_{a+}x\in C_{\mu,\log}[a,b]$. From the above proof, a solution $x(t)\in C_{n-\gamma,\log}[a,b]$ is a limit of the sequence $x_m(t)\in C_{n-\gamma,\log}[a,b]$ such that

$$\lim_{m\to+\infty}\big\|x_m-x\big\|_{C_{n-\gamma,\log}[a,b]}=0. \tag{4.10}$$

Hence, by using equation (1.1), Lipschitz condition given in definition 2.9 and Lemma 2.4, we have

$$\begin{aligned} \left\| {}_{H}D_{a+}^{\alpha,\beta} x_m(t) - {}_{H}D_{a+}^{\alpha,\beta} x(t) \right\|_{C_{\mu,\log}[a,b]} &= \left\| \varphi(t,x_m(t)) - \varphi(t,x(t)) \right\|_{C_{\mu,\log}[a,b]} \\ &\leq L\,(\log(b/a))^{\mu-n+\gamma} \left\| x_m(t) - x(t) \right\|_{C_{n-\gamma,\log}[a,b]}. \end{aligned} \tag{4.11}$$

Clearly, the equations (4.10) and (4.10) yield that

$$\lim_{m\to+\infty} \left\| {}_{H}D_{a+}^{\alpha,\beta} x_m(t) - {}_{H}D_{a+}^{\alpha,\beta} x(t) \right\|_{C_{\mu,\log}[a,b]} = 0,$$

and, hence, $({}_{H}D_{a+}^{\alpha,\beta} x) \in C_{\mu,\log}[a,b]$. Thus, the proof of this theorem is complete.

Remark 4.3 For $0 < \alpha < 1$, Theorem 4.2 is reduced to Theorem 22 (see[9]).

Acknowledgement

The authors are grateful to the referee for his/her important remarks and suggestions.

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Universal Journal of Mathematics and Applications, 3 (3) (2020) 129-132
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.760899

New Analytical Solutions of Fractional Complex Ginzburg-Landau Equation

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Article Info

Keywords: $(1/G')$*-Expansion Method, Complex Ginzburg-Landau Equation, Nonlinear Phenomena, Optical Solutions.*
2010 AMS: *35R11, 34A08, 35A20, 26A33.*
Received: *30 June 2020*
Accepted: *22 September 2020*
Available online: *29 September 2020*

Abstract

In recent years, nonlinear concepts have attracted a lot of attention due to the deep mathematics and physics they contain. In explaining these concepts, nonlinear differential equations appear as an inevitable tool. In the past century, considerable efforts have been made and will continue to be made to solve many nonlinear differential equations. This study is also a step towards analytical solution of the complex Ginzburg-Landau equation (CGLE) used to describe many phenomena on a wide scale. In this study, the CGLE was solved analytically by $(1/G')$-expansion method.

1. Introduction

In recent years mathematical and physical aspects of nonlinear phenomena draw much attention [1–4]. Since true laws of nature are drawn by nonlinear interactions. And, one of the inevitable tools for translating these laws into a mathematical language are nonlinear differential equations. It would not be an exaggeration to say that the past century was a century of nonlinear equations. Many different nonlinear differential equations have been the subject of studies to explain various nonlinear phenomena. Some of the most famous of these equations are Korteweg - de Vries (KdV) [5], Boussinesq [6], Cahn-Hilliard [7], nonlinear Schrödinger [8] and Ginzburg- Landau [9], etc. Especially complex form of Ginzburg-Landau equation (CGLE) is very interesting due to its capability of explaining very complex events in physics such as superconductivity, superfluidity [10], strings in field theory [11], Bose-Einstein condensation [12], etc. Due to its flexibility CGLE has been studied extensively by physicist and mathematicians.

In recent years, analytical and numerical solutions of fractional differential partial differential equations have been obtained by different methods [7, 13–16]. $(1/G')$-expansion method has been widely used to obtain analytical solutions of partial differential equations [17–19]. This method stands out for its flexibility, reliability and convenience. In this study, new wave solutions of conformable time fractional CGLE were obtained by $(1/G')$-expansion method.

2. Governing Equation

In this study, conformable time fractional CGLE is taken account as the governing equation which is in the form of;

$$iD_t^{\eta}+aq_{xx}+bG(|q|^2)q=\frac{1}{|q|^2q^*}\left[\alpha|q|^2(|q|^2)_{xx}-\beta\{(|q|^2)_x\}^2\right]+\gamma q \tag{2.1}$$

where $\eta\in(0,1)$, x represents the distance along the fibers, while t represents the time; a,b,α,β and γ are constants. The coefficients a and b arrise from the group velocity dispersion (GVD) and nonlinearity. The terms α,β and γ arrise from the perturbation effects in particular, γ occurs from the debasement effect. In Eq. (2.1), function G must possess the uniformity of the complex function $G(|q|^2)q$ which is k times continuously differentiable, consequently

$$G(|q|^2)q\in\bigcup_{m,n=l}^{\infty}C^k((-n,n)\times(-m,m);R^2). \tag{2.2}$$

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To obtain the solution of Eq. (2.1), the usual decomposition into phase-amplitude components produces:

$$q(x,t) = H(\xi)e^{(-\kappa x+\omega\frac{t^{\eta}}{\eta}+\theta)} \tag{2.3}$$

where the ξ is defined as

$$\xi = k(x - v\frac{t^{\eta}}{\eta}). \tag{2.4}$$

The function H denotes the pulse shape, v implies the speed of the soliton, κ denotes the soliton frequency; ω represents the soliton wave number, θ represents the phase constant. When the amplitude-phase decomposition subrogated into Eq. (2.1) and separating into real and imaginary parts, the following equations yields:

$$-\omega H + a(k^2 H_{\xi\xi} - \kappa^2 H) + bG(H^2)H = 2k^2(\alpha - 2\beta)\frac{(H_{\xi})^2}{H} + 2k^2\alpha H_{\xi\xi} + \gamma H \tag{2.5}$$

and

$$v = -2a\kappa.$$

By decides on

$$\alpha = 2\beta$$

the first term on the right-hand side of Eq. (2.5) set to zero. Thus Eq. (2.1) becomes

$$iD_t^{\eta} + aq_{xx} + bG(|q|^2)q = \frac{\beta}{|q|^2 q^*}\left[2|q|^2(|q|^2)_{xx} - \{(|q|^2)_x\}^2\right] + \gamma q \tag{2.6}$$

and Eq. (2.5) becomes

$$k^2(a - 4\beta)H_{\xi\xi} - (\omega + a\kappa^2 + \gamma)H + bG(H^2)H = 0. \tag{2.7}$$

3. $(1/G')$-Expansion Method

The $(1/G')$-expansion method is implemented to various partial differential equations (PDEs) [17–19]. This method is a powerful analtical method for the computation of analytical solutions of PDEs. Now, lets deal with the nonlinear conformable time fractional partial differential equation for $\varphi(x,t)$ in the form

$$H\left(\varphi, \frac{\partial^{\eta}\varphi}{\partial t^{\eta}}, \frac{\partial\varphi}{\partial x}, \frac{\partial^2\varphi}{\partial t^2}, \frac{\partial^2\varphi}{\partial x^2}, \ldots\right) = 0 \tag{3.1}$$

where $\varphi(x,t)$ is the unknown function and H is the polynomial of $\varphi(x,t)$ and its partial derivatives.
Presentation the wave variable as

$$\varphi(x,t) = \varphi(\xi), \xi = k(x - v\frac{t^{\eta}}{\eta}). \tag{3.2}$$

where k and c are parameters. Using Eq. (3.2), we get Eq. (3.1) becomes an ordinary differential equation for $\varphi = \varphi(\xi)$

$$F\left(\varphi, \varphi', \varphi'', \varphi''', \ldots\right) = 0. \tag{3.3}$$

where prime implies derivative respect to ξ. According to $(1/G')$-expansion method, it is supposed that the analytical solutions of Eq. (3.3) can be expressed as a polynomial of $(1/G')$ as

$$\varphi(\xi) = \sum_{i=0}^{n} a_i \left(\frac{1}{G'}\right)^i, \quad a_n \neq 0 \tag{3.4}$$

where $G = G(\xi)$ satisfies the second order ordinary differential equation

$$G'' + \lambda G' + \mu = 0 \tag{3.5}$$

and $a_i (i = 0, \ldots, n), \lambda, \mu$ are constants to be determined later. To obtain the solution of Eq. (3.5) with $G = G(\xi)$, the Eq. (3.4) will contain the following equation

$$\frac{1}{G'(\xi)} = \frac{1}{-\frac{\mu}{\lambda} + A\tanh(\lambda\xi) - A\sinh(\lambda\xi)} \tag{3.6}$$

where A is integral constant.
Step1.
The positive integer n in Eq. (3.4) can be stated by figuring out the homogeneous balance between the highest order derivatives and the highest nonlinear terms of $\varphi(\xi)$ in Eq. (3.3).
Step2.
Replacing (3.4) with Eq. (3.5) into Eq. (3.3) and simplifying by collecting together all the same powered terms of $(1/G')$, the left hand side of Eq. (3.3) is turns into a polynomial. After equalizing each coefficient of this polynomial to zero, we get a set of algebraic equations in terms of $a_i (i = 0, \ldots, n), \lambda, \mu, c, k$.
Step3.
By solving the system by symbolic computer software, then replacing the results with the solutions of Eq. (3.5) into Eq. (3.4) leads to analytical solutions of Eq. (3.3).

4. Analytical Solutions of Complex Ginzburg-Landau Equation

As it can be seen Kerr law nonlinearity can be applied to Eq. (4.2). Since, non-harmonic motion of electrons with an external electric field cause to nonlinear responses in the optical fiber. Due to the Kerr law nonlinearity, $G(u)$ is can be taken as u, hence Eq. (2.6) becomes

$$iD_t^{\eta} + aq_{xx} + b|q|^2 q = \frac{\beta}{|q|^2 q^*}\left[2|q|^2(|q|^2)_{xx} - \{(|q|^2)_x\}^2\right] + \gamma q \tag{4.1}$$

and Eq. (4.2) turns into

$$k^2(a-4\beta)H_{\xi\xi} - (\omega + a\kappa^2 + \gamma)H + bH^3 = 0. \tag{4.2}$$

According to the balance principle, we obtain $n=1$. Consequently, the analytical solution of Eq.(4.2) can be obtanied as

$$H(\xi) = a_0 + a_1\left(\frac{1}{G'}\right). \tag{4.3}$$

and thus

$$\varphi''(\xi) = 2a_1\mu^2\left(\frac{1}{G'(\xi)}\right)^3 + 3a_1\lambda\mu\lambda\left(\frac{1}{G'(\xi)}\right)^2 + a_1\lambda^2\left(\frac{1}{G'(\xi)}\right). \tag{4.4}$$

By replacing Eqs. (4.3)-(4.4) by Eq. (4.2) and gathering together all the same powered terms of $(1/G')$, the left hand side of Eq.(4.2) is turned into another polynomial in $(1/G')$. Equalizing each coefficient of this polynomial to zero, gets a set of algebraic equations as follows:

$$\begin{aligned}
&\left(\frac{1}{G'(\xi)}\right)^0 : -a\kappa^2 a_0 - \omega a_0 + ba_0^3 - \gamma a_0 = 0,\\
&\left(\frac{1}{G'(\xi)}\right)^1 : -4k^2\beta a_1\lambda^2 - \omega a_1 - a\kappa^2 a_1 - \gamma a_1 + 3ba_0^2 a_1 + k^2 a a_1\lambda^2 = 0,\\
&\left(\frac{1}{G'(\xi)}\right)^2 : 3k^2 a a_1\lambda\mu - 12k^2\beta a_1\lambda\mu + 3ba_0 a_1^2 = 0,\\
&\left(\frac{1}{G'(\xi)}\right)^3 : -8k^2\beta a_1\mu^2 + 2k^2 a a_1\mu^2 + ba_1^3 = 0.
\end{aligned} \tag{4.5}$$

Solving the system above, gets

$$k = \pm\frac{\sqrt{2a\kappa^2 + 2\gamma + 2\omega}}{\lambda\sqrt{4\beta - a}}, \quad a_0 = \pm\frac{\sqrt{a\kappa^2 + \gamma + \omega}}{\sqrt{b}}, \quad a_1 = \pm\frac{2\mu\sqrt{a\kappa^2 + \gamma + \omega}}{\sqrt{b}\lambda}. \tag{4.6}$$

By the help of the statements (4.6), (4.3) and (3.6), we obtain analytical solutions of Eq. (4.1) as follows:

$$q_{1,2}(x,t) = \pm\frac{\sqrt{a\kappa^2 + \gamma + \omega}}{\sqrt{b}}\left(\frac{2\mu}{\lambda\left(\mp A\sinh(\delta\xi) + A\cosh(\delta\xi) - \frac{\mu}{\lambda}\right)} + 1\right)e^{i\left(\theta + \frac{\omega t^{\eta}}{\eta} - \kappa x\right)}$$

where

$$\xi = \frac{2a\kappa t^{\eta}}{\eta} + x$$

and

$$\delta = \frac{\sqrt{2a\kappa^2 + 2\gamma + 2\omega}}{\sqrt{4\beta - a}}.$$

5. Conclusions

In this study, the complex Ginzburg-Landau equation (CGLE) used in the evaluation of many physical phenomena such as Bose-Einstein condensation, superconductivity, super-fluidity, semiconductor laser excitations was solved analytically. In order to solve the CGLE, the $(1/G')$-expansion method, which mathematicians have being used in analytical solution of nonlinear partial differential equations in recent years, has been used. In this study, it has been shown that the $(1/G')$-expansion method can be successfully applied in the analytical solutions of the CGLE. In addition, the flexibility, reliability and convenience of the method have been demonstrated with a new study.

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