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ON THE PERIODICITY OF SOLUTIONS OF A SYSTEM OF RATIONAL DIFFERENCE EQUATIONS

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Abstract

In this paper, we have investigated the periodicity of the well-defined solutions of the system of difference equations

$$u_{n+1}=\frac{u_{n-1}+v_n}{\alpha v_n u_{n-1}-1},\ v_{n+1}=\frac{v_{n-1}+u_n}{\alpha u_n v_{n-1}-1},\ w_{n+1}=\frac{u_n}{v_n}$$

where $u_0,u_{-1},v_0,v_{-1},w_0,w_{-1}\in\mathbb{R}\setminus\{0\}$ and $\alpha>0$.

Keywords: Difference equation; system; solutions; periodicity.

1. Introduction

In recent years, there has been a lot of interest in studying difference equations an their systems [1-24]. One of the reasons for this is a necessity for some techniques which can be used in investigating difference equations and their systems arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics etc. There are many papers with related to the systems of difference equations for example,

In [3] Cinar studied the solutions of the systems of the difference equations

$$x_{n+1}=\frac{1}{y_n},\ y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}.$$

In [2] Camouzis and Papaschinnopoulos studied the global asymptotic behavior of positive solutions of the system of rational difference equations

$$x_{n+1}=1+\frac{x_n}{y_{n-m}},\ y_{n+1}=1+\frac{y_n}{x_{n-m}}.$$

In [11] Kulenović and Nurkanović studied the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1}=\frac{a+x_n}{b+y_n},\ y_{n+1}=\frac{c+y_n}{d+z_n},\ z_{n+1}=\frac{e+z_n}{f+x_n}.$$

In [22] Yalcinkaya and Cinar studied the global asmptotic stability of the system of difference equations

$$z_{n+1}=\frac{t_n z_{n-1}+a}{t_n+z_{n-1}},\ t_{n+1}=\frac{z_n t_{n-1}+a}{z_n+t_{n-1}}.$$

In [12] Kurbanli et al. studied the periodicity of solutions of the system of rational difference equations

$$x_{n+1}=\frac{x_{n-1}+y_n}{y_n x_{n-1}-1},\ y_{n+1}=\frac{y_{n-1}+x_n}{x_n y_{n-1}-1}.$$

In [13] Kurbanli et al. studied the behavaior of positive solutions of the system of rational difference equations

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}+1},\ y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1}.$$

In this paper, we investigated the periodicity of the well-defined solutions of the difference equation system

$$u_{n+1}=\frac{u_{n-1}+v_n}{\alpha v_n u_{n-1}-1},\ v_{n+1}=\frac{v_{n-1}+u_n}{\alpha u_n v_{n-1}-1},\ w_{n+1}=\frac{u_n}{v_n} \tag{1}$$

where $u_0,u_{-1},v_0,v_{-1},w_0,w_{-1}\in\mathbb{R}\setminus\{0\}$ and $\alpha>0.$ Note that system (1) can be written as

$$x_{n+1}=\frac{x_{n-1}+y_n}{y_n x_{n-1}-1},\ y_{n+1}=\frac{y_{n-1}+x_n}{x_n y_{n-1}-1},\ z_{n+1}=\frac{x_n}{y_n} \tag{2}$$

by the change of variables $u_n=\frac{x_n}{\sqrt{\alpha}},\ v_n=\frac{y_n}{\sqrt{\alpha}},\ w_n=z_n.$ That's why, we will consider system (2) instead of system (1) for the remaining part of the paper.

2. Main Result

Our main result in this paper is the following:

Theorem 1. Let $y_0=a,\ y_{-1}=b,\ x_0=c,\ x_{-1}=d,\ z_0=e,\ z_{-1}=f$ be nonzero arbitrary real numbers and $\{x_n,y_n,z_n\}$ be a solution of system (2). Also, assume that $ad\neq 1$, $bc\neq 1$, $(b+c)\neq 0$ and $(d+a)\neq 0$. Then, all solutions of system (2) are as following:

$$x_n=\begin{cases}\frac{d+a}{ad-1}, & n=6k+1\\ b, & n=6k+2\\ a, & n=6k+3\\ \frac{b+c}{cb-1}, & n=6k+4\\ d, & n=6k+5\\ c, & n=6k+6\end{cases}\quad \text{for } k\in\mathbb{N}_0,$$

$$y_n=\begin{cases}\frac{b+c}{cb-1}, & n=6k+1\\ d, & n=6k+2\\ c, & n=6k+3\\ \frac{d+a}{ad-1}, & n=6k+4\\ b, & n=6k+5\\ a, & n=6k+6\end{cases}\quad \text{for } k\in\mathbb{N}_0,$$

$$z_n=\begin{cases}\dfrac{c}{a}, & n=6k+1\\ \dfrac{(d+a)(cb-1)}{(ad-1)(b+c)}, & n=6k+2\\ \dfrac{b}{d}, & n=6k+3\\ \dfrac{a}{c}, & n=6k+4\\ \dfrac{(b+c)(ad-1)}{(cb-1)(d+a)}, & n=6k+5\\ \dfrac{d}{b}, & n=6k+6\end{cases}\quad \text{for } k\in\mathbb{N}_0.$$

Proof. We prove the theorem by induction for k. If $k=0$, from system (2) we have

$$x_1=\frac{x_{-1}+y_0}{y_0x_{-1}-1}=\frac{d+a}{ad-1},$$

$$y_1=\frac{y_{-1}+x_0}{x_0y_{-1}-1}=\frac{b+c}{cb-1},$$

$$z_1=\frac{x_0}{y_0}=\frac{c}{a},$$

$$x_2=\frac{x_0+y_1}{y_1x_0-1}=\frac{c+\frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c-1}=\frac{bc^2-c+b+c}{bc+c^2-bc+1}=b,$$

$$y_2=\frac{y_0+x_1}{x_1y_0-1}=\frac{a+\frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a-1}=\frac{a^2d-a+d+a}{da+a^2-ad+1}=d,$$

$$z_2=\frac{x_1}{y_1}=\frac{\frac{d+a}{ad-1}}{\frac{b+c}{cb-1}}=\frac{(d+a)(cb-1)}{(ad-1)(b+c)},$$

$$x_3=\frac{x_1+y_2}{y_2x_1-1}=\frac{\frac{d+a}{ad-1}+d}{d\frac{d+a}{ad-1}-1}=\frac{d+a+ad^2-d}{d^2+da-ad+1}=a,$$

$$y_3=\frac{y_1+x_2}{x_2y_1-1}=\frac{\frac{b+c}{cb-1}+b}{b\frac{b+c}{cb-1}-1}=\frac{b+c+b^2c-b}{b^2+bc-cb+1}=c,$$

$$z_3=\frac{x_2}{y_2}=\frac{b}{d},$$

$$x_4=\frac{x_2+y_3}{y_3x_2-1}=\frac{b+c}{cb-1},$$

$$y_4=\frac{y_2+x_3}{x_3y_2-1}=\frac{d+a}{ad-1},$$

$$z_4=\frac{x_3}{y_3}=\frac{a}{c},$$

$$x_5=\frac{x_3+y_4}{y_4x_3-1}=\frac{a+\frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a-1}=\frac{a^2d-a+d+a}{da+a^2-ad+1}=d,$$

$$y_5=\frac{y_3+x_4}{x_4y_3-1}=\frac{c+\frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c-1}=\frac{bc^2-c+b+c}{bc+c^2-cb+1}=b,$$

$$z_5=\frac{x_4}{y_4}=\frac{\frac{b+c}{cb-1}}{\frac{d+a}{ad-1}}=\frac{(b+c)(ad-1)}{(cb-1)(d+a)},$$

$$x_6=\frac{x_4+y_5}{y_5x_4-1}=\frac{\frac{b+c}{cb-1}+b}{b\frac{b+c}{cb-1}-1}=\frac{b+c+b^2c-b}{b^2+bc-cb+1}=c,$$

$$y_6=\frac{y_4+x_5}{x_5y_4-1}=\frac{\frac{d+a}{ad-1}+d}{d\frac{d+a}{ad-1}-1}=\frac{d+a+ad^2-d}{d^2+da-ad+1}=a,$$

$$z_6=\frac{x_5}{y_5}=\frac{d}{b}.$$

Now, suppose that $k>0$ and that our assumption holds for $k=n-1$. That is;

$$x_{6(n-1)+1}=x_{6n-5}=\frac{d+a}{ad-1},\qquad y_{6(n-1)+1}=y_{6n-5}=\frac{b+c}{cb-1},\qquad z_{6(n-1)+1}=z_{6n-5}==\frac{c}{a},$$

$$x_{6(n-1)+2}=x_{6n-4}=b,\qquad y_{6(n-1)+2}=y_{6n-4}=d,\qquad z_{6(n-1)+2}=z_{6n-4}=\frac{(d+a)(cb-1)}{(ad-1)(b+c)},$$

$$x_{6(n-1)+3}=x_{6n-3}=a,\qquad y_{6(n-1)+3}=y_{6n-3}=c,\qquad z_{6(n-1)+3}=z_{6n-3}=\frac{b}{d},$$

$$x_{6(n-1)+4}=x_{6n-2}=\frac{b+c}{cb-1},\qquad y_{6(n-1)+4}=y_{6n-2}=\frac{d+a}{ad-1},\qquad z_{6(n-1)+4}=z_{6n-2}=\frac{a}{c},$$

$$x_{6(n-1)+5}=x_{6n-1}=d,\qquad y_{6(n-1)+5}=y_{6n-1}=b,\qquad z_{6(n-1)+5}=z_{6n-1}=\frac{(b+c)(ad-1)}{(cb-1)(d+a)},$$

$$x_{6(n-1)+6}=x_{6n}=c,\qquad y_{6(n-1)+6}=y_{6n}=a,\qquad z_{6(n-1)+6}=z_{6n}=\frac{d}{b}.$$

From system (2), we have the following for $k=n$:

$$x_{6n+1}=\frac{x_{6n-1}+y_{6n}}{y_{6n}x_{6n-1}-1}=\frac{d+a}{ad-1},$$

$$y_{6n+1}=\frac{y_{6n-1}+x_{6n}}{x_{6n}y_{6n-1}-1}=\frac{b+c}{cb-1},$$

$$z_{6n+1}=\frac{x_{6n}}{y_{6n}}=\frac{c}{a},$$

$$x_{6n+2}=\frac{x_{6n}+y_{6n+1}}{y_{6n+1}x_{6n}-1}=\frac{c+\frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c-1}=\frac{bc^2-c+b+c}{bc+c^2-cb+1}=b,$$

$$y_{6n+2}=\frac{y_{6n}+x_{6n+1}}{x_{6n+1}y_{6n}-1}=\frac{a+\frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a-1}=\frac{a^2d-a+d+a}{ad+a^2-ad+1}=d,$$

$$z_{6n+2}=\frac{x_{6n+1}}{y_{6n+1}}=\frac{\frac{d+a}{ad-1}}{\frac{b+c}{cb-1}}=\frac{(d+a)(cb-1)}{(ad-1)(b+c)},$$

$$x_{6n+3}=\frac{x_{6n+1}+y_{6n+2}}{y_{6n+2}x_{6n+1}-1}=\frac{\frac{d+a}{ad-1}+d}{d\frac{d+a}{ad-1}-1}=\frac{d+a+ad^2-d}{d^2+da-ad+1}=a,$$

$$y_{6n+3}=\frac{y_{6n+1}+x_{6n+2}}{x_{6n+2}y_{6n+1}-1}=\frac{\frac{b+c}{cb-1}+b}{b\frac{b+c}{cb-1}-1}=\frac{b+c+b^2c-b}{b^2+bc-cb+1}=c,$$

$$z_{6n+3}=\frac{x_{6n+2}}{y_{6n+2}}=\frac{b}{d},$$

$$x_{6n+4}=\frac{x_{6n+2}+y_{6n+3}}{y_{6n+3}x_{6n+2}-1}=\frac{b+c}{cb-1},$$

$$y_{6n+4}=\frac{y_{6n+2}+x_{6n+3}}{x_{6n+3}y_{6n+2}-1}=\frac{d+a}{ad-1},$$

$$z_{6n+4}=\frac{x_{6n+3}}{y_{6n+3}}=\frac{a}{c},$$

$$x_{6n+5}=\frac{x_{6n+3}+y_{6n+4}}{y_{6n+4}x_{6n+3}-1}=\frac{a+\frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a-1}=\frac{a^2d-a+d+a}{da+a^2-ad+1}=d,$$

$$y_{6n+5}=\frac{y_{6n+3}+x_{6n+4}}{x_{6n+4}y_{6n+3}-1}=\frac{c+\dfrac{b+c}{cb-1}}{\dfrac{b+c}{cb-1}c-1}=\frac{c^2b-c+b+c}{bc+c^2-cb+1}=b,$$

$$z_{6n+5}=\frac{x_{6n+4}}{y_{6n+4}}=\frac{\dfrac{b+c}{cb-1}}{\dfrac{d+a}{ad-1}}=\frac{(b+c)(ad-1)}{(cb-1)(d+a)}$$

and

$$x_{6n+6}=\frac{x_{6n+4}+y_{6n+5}}{y_{6n+5}x_{6n+4}-1}=\frac{\dfrac{b+c}{cb-1}+b}{b\dfrac{b+c}{cb-1}-1}=\frac{b+c+b^2c-b}{b^2+bc-cb+1}=c,$$

$$y_{6n+6}=\frac{y_{6n+4}+x_{6n+5}}{x_{6n+5}y_{6n+4}-1}=\frac{\dfrac{d+a}{ad-1}+d}{d\dfrac{d+a}{ad-1}-1}=\frac{d+a+ad^2-d}{d^2+da-ad+1}=a,$$

$$z_{6n+6}=\frac{x_{6n+5}}{y_{6n+5}}=\frac{d}{b}.$$

Therefore, the proof is completed by induction.

The followig Corollary is a natural result of Theorem 1:

Corollary 1. Let $y_0=a,\ y_{-1}=b,\ x_0=c,\ x_{-1}=d,\ z_0=e,\ z_{-1}=f$ be nonzero arbitrary real numbers and $\{x_n,y_n,z_n\}$ be a solutions of the system (2). Also, assume that $ad\neq 1$, $bc\neq 1$, $(b+c)\neq 0$ and $(d+a)\neq 0$. Then the sequences (x_n), (y_n) and (z_n) are six periodic.

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MULTIPLICATIVE VOLTERRA INTEGRAL EQUATIONS AND THE RELATIONSHIP BETWEEN MULTIPLICATIVE DIFFERENTIAL EQUATIONS

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Abstract

In this study, the multiplicative Volterra integral equation is defined by using the concept of multiplicative integral. The solution of multiplicative Volterra integral equation of the second kind is researched by using the successive approximations method with respect to the multiplicative calculus and the necessary conditions for the continuity and uniqueness of the solution are given. The main purpose of this study is to investigate the relationship of the multiplicative integral equations with the multiplicative differential equations.

Keywords: Multiplicative calculus; Multiplicative differential equations; Multiplicative Volterra integral equations; Successive approximations method.

1. Introduction

Grossman and Katz [10] have built non-Newtonian calculus between years 1967-1970 as an alternative to classic calculus. They have set an infinite family of calculus, including classic, geometric, harmonic, quadratic, bigeometric, biharmonic and biquadratic calculus. Also, they defined a new kind of derivative and integral by using multiplication and division operations instead of addition and subtraction operations. Later, the new calculus that establish in this way is named multiplicative calculus by Stanley [16]. Multiplicative calculus provide different point of view for applications in science and engineering. It is discussed and developed by many researchers. Stanley [16] developed multiplicative calculus, gave some basic theorems about derivatives, integrals and proved infinite products in this calculus. Aniszewska [1] used the multiplicative version of Runge-Kutta method for solving multiplicative differential equations. Bashirov, Mısırlı and Özyapıcı [2] demonstrated some applications and usefulness of multiplicative calculus for the attention of researchers in the branch of analysis. Rıza, Özyapıcı and Mısırlı [14] studied the finite difference methods for the numerical solutions of multiplicative differential equations and Volterra integral equations. Mısırlı and Gurefe [13] developed multiplicative Adams Bashforth-Moulton methods to obtain the numerical solution of multiplicative differential equations. Bashirov, Rıza [4] discussed multiplicative differentiation for complex valued functions and Bashirov, Norozpour [6] extended the multiplicative integral to complex valued functions. Bashirov [5] studied double integrals in the sense of multiplicative calculus. Bhat et al. [7] defined multiplicative Fourier transform and found the solution of multiplicative differential equations by applying multiplicative Fourier transform. Bhat et al. [8] defined multiplicative Sumudu transform and solved some multiplicative differential equations by using multiplicative Sumudu transform. For more details see in [1-10, 13, 14, 16-20].

Integral equations have used for the solution of many problems in applied mathematics, mathematical physics and engineering since the 18th century. The integral equations have begun to enter the problems of engineering and other fields because of the relationship with differential equations and so their importance has increased in recent years. The reader may refer for relevant terminology on the integral equations to [11, 12, 15, 21, 22].

Now, we will give some necessary definitions and theorems in multiplicative calculus as follows:

Definition 1. Let f be a function whose domain is $\mathbb{R}$ the set of real numbers and whose range is a subset of $\mathbb{R}$. The multiplicative derivative of the f at x is defined as the limit

$$\frac{d^*f(x)}{dx} = f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}.$$

The limit is also called $*$-derivative of f at x, briefly.

If f is a positive function on an open set $A \subseteq \mathbb{R}$ and its classical derivative $f'(x)$ exists, then its multiplicative derivative also exists and

$$f^*(x) = e^{\left[\frac{f'(x)}{f(x)}\right]} = e^{(ln\circ f)'(x)}$$

where $ln \circ f(x) = lnf(x)$. Moreover, if f is multiplicative differentiable and $f^*(x) \neq 0$, then its classical derivative exists and

$f'(x) = f(x) \cdot lnf^*(x)$ [16].

The multiplicative derivative of f^* is called the second multiplicative derivative and it is denoted by f^{**}. Likewise, the n-th multiplicative derivative can be defined of f and denoted by $f^{*(n)}$ for $n = 0,1,2,\ldots$. If n-th derivative $f^{(n)}(x)$ exists, then its n-th multiplicative derivative $f^{*(n)}(x)$ also exists and

$f^{*(n)}(x) = e^{(ln\circ f)^{(n)}(x)},\ n = 0,1,2,\ldots$ [2].

Definition 2. The multiplicative absolute value of $x \in \mathbb{R}$ denoted with the symbol $|x|^*$ and defined by

$$|x|_* = \begin{cases} x, & x \geq 1 \\ \frac{1}{x}, & x < 1. \end{cases}$$

Theorem 1. Let f and g be multiplicative differentiable functions. Then the functions $c.f, f.g, f+g, f/g, f^g$ are multiplicative differentiable where c is an arbitrary constant and their multiplicative derivative can be shown as

(1) $(cf)^*(x) = f^*(x)$
(2) $(fg)^*(x) = f^*(x)g^*(x)$
(3) $(f+g)^*(x) = f^*(x)^{\frac{f(x)}{f(x)+g(x)}}g^*(x)^{\frac{g(x)}{f(x)+g(x)}}$
(4) $\left(\frac{f}{g}\right)^*(x) = \frac{f^*(x)}{g^*(x)}$
(5) $(f^g)^*(x) = f^*(x)^{g(x)}f(x)^{g'(x)}$
(6) $[f^*(x)]^n = [f^n(x)]^*$ for $n \in \mathbb{R}$ [16].

Theorem 2. *(Multiplicative Mean Value Theorem)* If the function f is continuous on $[a,b]$ and is *-differentiable on (a,b), then there exits $a<c<b$ such that

$f^*(c)=\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{b-a}}$ [3].

Definition 3. Let f be a function with two variables, then its multiplicative partial derivatives are defined as

$$\frac{\partial^* f(x,y)}{\partial x}=f_x^*(x,y)=e^{\frac{\partial}{\partial x}ln(f(x,y))} \quad \text{and} \quad \frac{\partial^* f(x,y)}{\partial y}=f_y^*(x,y)=e^{\frac{\partial}{\partial y}ln(f(x,y))} \text{ [5].}$$

Theorem 3. (*Multiplicative Chain Rule*) Suppose that f be a function of two variables y and z with continuous multiplicative partial derivatives. If y and z are differentiable functions on (a,b) such that $f(y(x),z(x))$ is defined for every $x\in(a,b)$, then

$$\frac{d^* f(y(x),z(x))}{dx}=f_y^*(y(x),z(x))^{y'(x)}f_z^*(y(x),z(x))^{z'(x)} \text{ [2].}$$

Definition 4. Let f be a positive function and continuous on the interval $[a,b]$, then it is multiplicative integrable or briefly $*$-integrable on $[a,b]$ and

$$*\int_a^b f(x)^{dx}=e^{\int_a^b \ln(f(x))dx} \text{ [16].}$$

Theorem 4. If f and g are integrable functions on $[a,b]$ in the sense of multiplicative, then

(1) $*\int_a^b\left(f(x)^k\right)^{dx}=\left(*\int_a^b(f(x))^{dx}\right)^k$

(2) $*\int_a^b\left(f(x)g(x)\right)^{dx}=*\int_a^b(f(x))^{dx}*\int_a^b(g(x))^{dx}$

(3) $*\int_a^b\left(\frac{f(x)}{g(x)}\right)^{dx}=\frac{*\int_a^b(f(x))^{dx}}{*\int_a^b(g(x))^{dx}}$

(4) $*\int_a^b(f(x))^{dx}=*\int_a^c(f(x))^{dx}*\int_c^b(f(x))^{dx}$

where $k\in\mathbb{R}$ and $a\le c\le b$ [2,3].

Theorem 5. *(Fundamental Theorem of Multiplicative Calculus)* If the function f has multiplicative derivative on $[a,b]$ and f^* is multiplicative integrable on $[a,b]$, then

$$*\int_a^b f^*(x)^{dx}=\frac{f(b)}{f(a)} \text{ [2,3].}$$

Definition 5. The equation of the form

$y^*(x)=f(x,y(x))$

including the multiplicative derivative of y is called first order multiplicative differential equation. It is equivalent to the ordinary differential equation $y'(x)=y(x)\ln f(x,y(x))$. Similarly, n-th order multiplicative differential equation is defined by $F\left(x,y,y^*,\dots,y^{*(n-1)},y^{*(n)}(x)\right)=1,\ (x,y)\in\mathbb{R}\times\mathbb{R}^+$ [2,3]. The equation of the form

$\left(y^{*(n)}\right)^{a_n(x)}\left(y^{*(n-1)}\right)^{a_{n-1}(x)} \dots (y^{**})^{a_2(x)}(y^{*})^{a_1(x)}y^{a_0(x)} = f(x)$
that f is a positive function, is called multiplicative linear differential equation. If the exponentials $a_n(x)$ are constants, then the equation called as multiplicative linear differential equation with constant exponentials; if not it is called as multiplicative linear differential equation with variable exponentials [17].

2. Multiplicative Volterra Integral Equations

An equation in which an unknown function appears under one or more signs of multiplicative integration is called a multiplicative integral equation (MIE), if the multiplicative integral exists. The equation

$$u(x) = f(x) \, * \int_a^x [u(t)]^{K(x,t)^{dt}}$$

where $f(x)$ and $K(x,t)$ are known functions, $u(x)$ is unknown function, is called linear multiplicative Volterra integral equation (LMVIE) of the second kind. The function $K(x,t)$ is the kernel of multiplicative Volterra integral equation. If $f(x) = 1$, then the equation takes the form

$$u(x) = * \int_a^x [u(t)]^{K(x,t)^{dt}}$$

and it is called LMVIE of the first kind.

Example 1. Show that the function $u(x) = e^{2x}$ is a solution of the MVIE $u(x) = e^x * \int_0^x \left[\left(u(t)\right)^{\frac{1}{x}}\right]^{dt}$.

Solution. Substituting the function e^{2x} in place of $u(x)$ into the right side of the equation, we obtain

$$e^x \, * \int_0^x \left[\left(u(t)\right)^{\frac{1}{x}}\right]^{dt} = e^x \, * \int_0^x \left[(e^{2t})^{\frac{1}{x}}\right]^{dt} = e^x \, e^{\int_0^x \ln e^{\frac{2t}{x}} dt} = e^x \, e^{\int_0^x \frac{2t}{x} dt} = e^x \, e^{\frac{2}{x}\cdot\frac{t^2}{2}\Big|_0^x} = e^{2x} = u(x)$$

So, this means that the function $u(x) = e^{2x}$ is a solution of the MVIE.

2.1. The Successive Approximation Method For Solving Multiplicative Volterra Integral Equations

Theorem 6. Consider LMVIE of the second kind as

$$u(x) = f(x) \, * \int_0^x [u(t)]^{K(x,t)^{dt}}. \tag{1}$$

If $f(x)$ is positive and continuous on $[0,a]$ and $K(x,t)$ is continuous on the rectangle $0 \le t \le x$ and $0 \le x \le a$, then there exists an unique continuous solution of (1) as

$$u(x) = \prod_{n=0}^{\infty} \varphi_n(x) = e^{\sum_{n=0}^{\infty} ln\varphi_n(x)}$$

such that the series $\sum_{n=0}^{\infty} ln\varphi_n(x)$ is absolute and uniform convergent where

$$\varphi_0(x) = f(x) \, , \, \varphi_n(x) = * \int_0^x [\, \varphi_{n-1}(t)]^{K(x,t)^{dt}}, \, n = 1,2, \dots .$$

Proof: Take the initial approximation as

$$u_0(x) = f(x) = \varphi_0(x) \, . \tag{2}$$

If we write $u_0(x)$ instead of $u(x)$ in equation (1), then we get the new function showed with $u_1(x)$ as

$$u_1(x) = f(x) \ *\int_0^x [u_0(t)]^{K(x,t)^{dt}}. \tag{3}$$

Since the multiplicative integral which is in equation (3) depends on variable x, we can show it with

$$\varphi_1(x) = *\int_0^x [u_0(t)]^{K(x,t)^{dt}} = *\int_0^x [\varphi_0(t)]^{K(x,t)^{dt}}$$

and write the equation (3) as follow

$$u_1(x) = f(x)\varphi_1(x) = \varphi_0(x)\varphi_1(x) \tag{4}$$

by using (2). Therefore the third approximation is obtained as

$$u_2(x) = f(x) \ *\int_0^x [u_1(t)]^{K(x,t)^{dt}}.$$

By the equation (4), we find

$$\begin{aligned} u_2(x) &= f(x) \ *\int_0^x [\varphi_0(t).\varphi_1(t)]^{K(x,t)^{dt}} \\ &= f(x) \ *\int_0^x \left([\varphi_0(t)]^{K(x,t)}[\varphi_1(t)]^{K(x,t)}\right)^{dt} \\ &= f(x) \ *\int_0^x [\varphi_0(t)]^{K(x,t)^{dt}} \ *\int_0^x [\varphi_1(t)]^{K(x,t)^{dt}}. \end{aligned}$$

If we set $\varphi_2(x) = *\int_0^x [\varphi_1(t)]^{K(x,t)^{dt}}$, then $u_2(x) = \varphi_0(x)\,\varphi_1(x)\,\varphi_2(x)$. In a similar way, we get

$$u_n(x) = \varphi_0(x)\,\varphi_1(x)\,\varphi_2(x)\ \dots\ \varphi_n(x) \tag{5}$$

where

$\varphi_0(x) = f(x)\,,\ \varphi_n(x) = *\int_0^x [\ \varphi_{n-1}(t)]^{K(x,t)^{dt}},\ n = 1,2,\dots .$

Continuing this process, we get the series

$$u(x) = \varphi_0(x)\,\varphi_1(x)\,\varphi_2(x)\ \dots\ \varphi_n(x).\dots = \prod_{n=0}^{\infty}\varphi_n(x) = e^{\sum_{n=0}^{\infty} ln\varphi_n(x)}. \tag{6}$$

From (5) and (6), it is clear that $\lim_{n\to\infty} u_n(x) = u(x)$.

Assume that $F = \max_{x\in[0,a]} f(x)$ and $K = \max_{0\le t\le x\le a}|K(x,t)|$. Then we find

$$\varphi_0(x) = f(x) \le F$$

$$\begin{aligned} \varphi_1(x) &= *\int_0^x [\varphi_0(t)]^{K(x,t)^{dt}} = e^{\int_0^x K(x,t)ln\varphi_0(t)dt} \le e^{\int_0^x |K(x,t)|\,|ln\varphi_0(t)|dt} = e^{\int_0^x |K(x,t)|\,ln|\varphi_0(t)|_* dt} \\ &\le e^{K.lnF.x} \end{aligned}$$

$$\varphi_2(x) = *\int_0^x [\varphi_1(t)]^{K(x,t)^{dt}} = e^{\int_0^x K(x,t)ln\varphi_1(t)dt} \le e^{K.lnF\int_0^x |K(x,t)|.tdt} \le e^{K^2.lnF\int_0^x tdt} = e^{lnF.K^2.\frac{x^2}{2!}}$$

$\vdots$

$$\varphi_n(x) \le e^{lnF.\frac{K^n x^n}{n!}}.$$

Also, we can see that $|ln\varphi_n(x)| = ln|\varphi_n(x)|_* \leq lnF\ \frac{K^n a^n}{n!}$ for $= 1,2,\ldots$. Because of $|ln\varphi_n(x)| \leq lnF\ \frac{K^n a^n}{n!}$, the series $\sum_{n=0}^{\infty} ln\varphi_n(x)$ is absolute and uniform convergence from the Weierstrass M-test. Since each terms of this series are continuous, the function which this series convergences uniformly is continuous. Hence $u(x)$ is continuous function. Now, we will show $u(x)$ is a solution of the equation (1). Since $\varphi_n(x) =* \int_0^x \varphi_{n-1}(t)^{K(x,t)^{dt}}$ and $\varphi_0(x) = f(x)$, we find

$$\varphi_0(x)\prod_{n=1}^{N}\varphi_n(x) = f(x)\prod_{n=1}^{N}\left(* \int_0^x \varphi_{n-1}(t)^{K(x,t)^{dt}} \right)$$

$$\prod_{n=0}^{N}\varphi_n(x) = f(x) * \int_0^x \left(\prod_{n=1}^{N-1}\varphi_{n-1}(t)\right)^{K(x,t)^{dt}}$$

$$\prod_{n=0}^{N}\varphi_n(x) = f(x) * \int_0^x \left(\prod_{n=0}^{N}\varphi_n(t)\right)^{K(x,t)^{dt}} \tag{7}$$

From (6) and (7), we obtain

$$u(x) = \lim_{N\to\infty}\prod_{n=0}^{N}\varphi_n(x) = \lim_{N\to\infty} f(x)\ e^{\int_0^x K(x,t)\sum_{n=0}^{N} ln\varphi_n(t)dt} = f(x)\ e^{\int_0^x K(x,t)\lim_{N\to\infty}(\sum_{n=0}^{N} ln\varphi_n(t))dt}$$

$$= f(x)\ e^{\int_0^x K(x,t) ln e^{\sum_{n=0}^{\infty} ln\varphi_n(t)} dt} = f(x)\ e^{\int_0^x K(x,t) ln u(t) dt} = f(x)\ * \int_0^x u(t)^{K(x,t)^{dt}}.$$

by using the uniform convergence of the series $\sum_{n=0}^{\infty} ln\varphi_n(x)$. This indicates that $u(x)$ is the solution of the equation (1). Now, we will show the uniqueness of the solution. Assume that $u(x)$ and $v(x)$ are different solutions of the equation (1). Since

$$u(x) = f(x)\ * \int_0^x [u(t)]^{K(x,t)^{dt}}$$

$$v(x) = f(x) * \int_0^x [v(t)]^{K(x,t)^{dt}}$$

we find

$$\frac{u(x)}{v(x)} = * \int_0^x \left[\frac{u(t)}{v(t)}\right]^{K(x,t)^{dt}} = e^{\int_0^x K(x,t)(ln u(t) - ln v(t))dt}.$$

If we set $\frac{u(x)}{v(x)} = \phi(x)$, we can write $\phi(x) = * \int_0^x [\phi(t)]^{K(x,t)^{dt}} = e^{\int_0^x K(x,t) ln\phi(t)dt}$. Because of $ln\phi(x) = \int_0^x K(x,t) ln\phi(t)dt$, we find

$$|ln\phi(x)| = \left|\int_0^x K(x,t) ln\phi(t)dt\right| \leq \int_0^x |K(x,t)||ln\phi(t)|dt \leq K\int_0^x |ln\phi(t)|dt.$$

It is taken as $h(x) = \int_0^x |ln\phi(t)|dt$, then we write

$|ln\phi(x)| \leq Kh(x)$

$|ln\phi(x)| - Kh(x) \leq 0.$

By multiplication with e^{-Kx} both sides of the inequality, then

$e^{-Kx}|ln\phi(x)| - e^{-Kx}Kh(x) \leq 0$

$\frac{d}{dx}\left(e^{-Kx}h(x)\right) \leq 0$

and by integration both sides of this inequality from 0 to x we find

$e^{-Kx}h(x) - e^{-K0}h(0) \leq 0$

$e^{-Kx}h(x) \leq 0.$

Since $h(x) \leq 0$ and $h(x) \geq 0$, we find $h(x) = 0$. Therefore $|ln\phi(x)| = 0$ for every $x \in [0, a]$, i.e., $\phi(x) = 1$ for every $x \in [0, a]$. Thus $\phi(x) = \frac{u(x)}{v(x)} = 1$ and we obtain $u(x) = v(x)$. This completes the proof.

Remark 1. If the following iterations of method of successive approximations are set by

$u_0(x) = f(x)$

$$u_n(x) = f(x) * \int_0^x [u_{n-1}(t)]^{K(x,t)^{dt}}, \quad n = 1,2,3, ...$$

for the multiplicative integral equation

$u(x) = f(x) * \int_0^x [u(t)]^{K(x,t)^{dt}}$

where $f(x)$ is positive and continuous on $[0, a]$ and $K(x,t)$ is continuous for $0 \leq x \leq a$, $0 \leq t \leq x$, then the sequence of successive approximations $u_n(x)$ converges to the solution $u(x)$.

Example 2. Solve the multiplicative Volterra integral equation

$$u(x) = e^x * \int_0^x \left[\left(u(t)\right)^{(t-x)}\right]^{dt}$$

with using the successive approximations method.

Solution. Let taken $u_0(x) = e^x$, then the first approximation is obtained as

$$u_1(x) = e^x * \int_0^x \left[(e^t)^{(t-x)}\right]^{dt} = e^x e^{\int_0^x ln\left(e^{t.(t-x)}\right)dt} = e^x e^{\int_0^x t.(t-x)dt} = e^{\left(x - \frac{x^3}{3!}\right)}$$

and by using this approximation it can be obtained as

$$u_2(x) = e^x * \int_0^x \left[\left(e^{\left(t - \frac{t^3}{3!}\right)}\right)^{(t-x)}\right]^{dt} = e^{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)}.$$

By proceeding similarly, the n^{th} approximation is

$$u_n(x) = e^{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)}.$$

Since the expression $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$ is the Maclaurin series of $sinx$, $\lim_{n\to\infty} u_n(x) = e^{sinx}$. Therefore the solution of the equation $u(x) = e^{sinx}$.

3. The Relationship Between Multiplicative Differential Equations

We will investigate the relationship of the multiplicative Volterra integral equations with the multiplicative differential equations.

3.1. The Conversion of the Multiplicative Volterra Integral Equations to Multiplicative Differential Equations

In this section, we demonstrate the method of converting a multiplicative Volterra integral equation into a multiplicative differential equation. For this, we need the Leibniz Formula in the sense of multiplicative calculus.

Firstly, we will give necessary lemma with using proof of multiplicative Leibniz formula.

Lemma 1. Let Ω be an open set in $\mathbb{R}^2$. Suppose that $f:\Omega \to \mathbb{R}$ be a function such that the multiplicative partial derivatives $f_{xy}^{**}(x,y)$, $f_{yx}^{**}(x,y)$ exists in Ω and are continuous, then we have

$$\frac{\partial^*}{\partial x}\left(\frac{\partial^*}{\partial y} f(x,y)\right) = \frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial x} f(x,y)\right).$$

Proof. Fix x and y. $F(h,k)$ is taken as

$$F(h,k) = \left(\frac{f(x+h,y+k)\, f(x,y)}{f(x,y+k)\, f(x+h,y)}\right)^{\frac{1}{hk}}$$

By using the multiplicative mean value theorem, we find

$$F(h,k) = \left(\frac{f(x+h,y+k)\, f(x,y)}{f(x,y+k)\, f(x+h,y)}\right)^{\frac{1}{hk}} = \left(\left(\frac{\frac{f(x+h,y+k)}{f(x,y+k)}}{\frac{f(x+h,y)}{f(x,y)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{h}} = \left(\frac{\partial^*}{\partial y}\left(\frac{f(x+h,y+\lambda_1 k)}{f(x,y+\lambda_1 k)}\right)\right)^{\frac{1}{h}}$$

$$= \frac{\partial^*}{\partial y}\left(\left(\frac{f(x+h,y+\lambda_1 k)}{f(x,y+\lambda_1 k)}\right)^{\frac{1}{h}}\right) = \frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial x} f(x+\lambda_2 h, y+\lambda_1 k)\right)$$

and

$$F(h,k) = \left(\frac{f(x+h,y+k)\, f(x,y)}{f(x,y+k)\, f(x+h,y)}\right)^{\frac{1}{hk}} = \left(\left(\frac{\frac{f(x+h,y+k)}{f(x+h,y)}}{\frac{f(x,y+k)}{f(x,y)}}\right)^{\frac{1}{h}}\right)^{\frac{1}{k}} = \left(\frac{\partial^*}{\partial x}\left(\frac{f(x+\lambda_3 h,y+k)}{f(x+\lambda_3 h,y)}\right)\right)^{\frac{1}{k}}$$

$$= \frac{\partial^*}{\partial x}\left(\left(\frac{f(x+\lambda_3 h,y+k)}{f(x+\lambda_3 h,y)}\right)^{\frac{1}{k}}\right) = \frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial y} f(x+\lambda_3 h, y+\lambda_4 k)\right)$$

for some $0 < \lambda_1, \lambda_2, \lambda_3, \lambda_4 < 1$ which all of them depend on x, y, h, k. Therefore,

$$\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial x} f(x+\lambda_2 h, y+\lambda_1 k)\right) = \frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial y} f(x+\lambda_3 h, y+\lambda_4 k)\right)$$

for all h and k. Taking the limit $h, k \to 0$ and using the assumed continuity of both partial derivatives, it gives

$$\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial x}f(x,y)\right)=\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial y}f(x,y)\right).$$

Theorem 7. *(Multiplicative Leibniz Formula)* Let $A, I \subseteq \mathbb{R}$ be open set and f be a continuous function on $A \times I$ into $\mathbb{R}$. If f_x^* exists and is continuous on $A \times I$, $h(x), v(x)$ are continuously differentiable functions of A into I, then we have

$$\frac{d^*}{dx}\left(*\int_{h(x)}^{v(x)} f(x,t)^{dt}\right)=*\int_{h(x)}^{v(x)} f_x^*(x,t)^{dt}\ \frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}$$

Proof. Let $f(x,t)=\frac{\partial^*}{\partial t}F(x,t)=F_t^*(x,t)$. Hence we can write $*\int_{h(x)}^{v(x)} f(x,t)^{dt}=*\int_{h(x)}^{v(x)} F_t^*(x,t)^{dt}$. Since $*\int_{h(x)}^{v(x)} f(x,t)^{dt}=\frac{F(x,v(x))}{F(x,h(x))}$, we find

$$\frac{d^*}{dx}\left(*\int_{h(x)}^{v(x)} f(x,t)^{dt}\right)=\frac{d^*}{dx}\left(\frac{F\big(x,v(x)\big)}{F\big(x,h(x)\big)}\right)=\frac{\frac{d^*}{dx}F\big(x,v(x)\big)}{\frac{d^*}{dx}F\big(x,h(x)\big)}$$

by using properties of multiplicative derivative. Therefore we get

$$\frac{d^*}{dx}\left(*\int_{h(x)}^{v(x)} f(x,t)^{dt}\right)=\frac{F_x^*\big(x,v(x)\big)^1\left[F_{v(x)}^*\big(x,v(x)\big)\right]^{v'(x)}}{F_x^*\big(x,h(x)\big)^1\left[F_{h(x)}^*\big(x,h(x)\big)\right]^{h'(x)}} \tag{8}$$

with multiplicative chain rule. By using Lemma 1, we obtain

$$\begin{aligned}
\frac{d^*}{dx^*}\left(*\int_{h(x)}^{v(x)} f(x,t)^{dt}\right) &=*\int_{h(x)}^{v(x)}\left(\frac{\partial^*}{\partial t}F_x^*(x,t)\right)^{dt}\frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}\\
&=*\int_{h(x)}^{v(x)}\left(\frac{\partial^*}{\partial t}\left(\frac{\partial^*}{\partial x}F(x,t)\right)\right)^{dt}\frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}\\
&=*\int_{h(x)}^{v(x)}\frac{\partial^*}{\partial x}\left(\frac{\partial^*}{\partial t}F(x,t)\right)^{dt}\frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}\\
&=*\int_{h(x)}^{v(x)}\frac{\partial^*}{\partial x}\big(F_t^*(x,t)\big)^{dt}\frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}\\
&=*\int_{v(x)}^{h(x)} f_x^*(x,t)^{dt}\ \frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}.
\end{aligned}$$

from the equality (8). This completes the proof.

Example 3. Show that the multiplicative integral equation $u(x) = \sin x * \int_0^x ([u(t)]^{x\tan t})^{dt}$ can be transformed to a multiplicative differential equation.

Solution. If we consider the equation $u(x) = \sin x\ * \int_0^x ([u(t)]^{x\tan t})^{dt}$ and differentiate it by using multiplicative Leibniz formula, we write

$$u^*(x) = \frac{d^*}{dx}(\sin x)\ \frac{d^*}{dx}\left(* \int_0^x ([u(t)]^{x\tan t})^{dt} \right)$$

$$= e^{\frac{\cos x}{\sin x}} * \int_0^x \left[\frac{\partial^*}{\partial x}([u(t)]^{x\tan t})\right]^{dt} \frac{(u(x)^{x\tan x})^{x'}}{(u(0)^{x\tan 0})^{0'}}$$

$$= e^{\cot x} * \int_0^x [u(t)^{\tan t}]^{dt}\, u(x)^{x\tan x}$$

To take derivative is continued until the expression gets rid of the integral sign. Hence, we obtain

$$u^{**}(x) = \frac{d^*}{dx}(e^{\cot x})\frac{d^*}{dx}\left(* \int_0^x [u(t)^{\tan t}]^{dt} \right)\frac{d^*}{dx}(u(x)^{x\tan x})$$

$$= e^{-cosec^2 x} * \int_0^x [1]^{dt} \frac{(u(x)^{\tan x})^{x'}}{(u(0)^{\tan 0})^{0'}} e^{\left((x.tanx)'\, lnu(x) + \frac{u'(x)}{u(x)} x\, tanx\right)}$$

$$= e^{-cosec^2 x}\ [u(x)]^{\tan x}\ e^{\left(lnu(x)^{(x\, tanx)'} + \frac{u'(x)}{u(x)} x\, tanx\right)}$$

$$= e^{-cosec^2 x}\ [u(x)]^{(2\tan x + x\, sec^2 x)}\ \left(e^{\frac{u'(x)}{u(x)}}\right)^{xtanx}$$

$$= e^{-cosec^2 x}\ [u(x)]^{(2\tan x + x\, sec^2 x)}\ [u^*(x)]^{x\ \tan x}.$$

Thus the multiplicative integral equation is equivalent to the multiplicative differential equation $u^{**}(x) = e^{-cosec^2 x}\ u(x)^{(2\tan x + x\, sec^2 x)}.[u^*(x)]^{x\tan x}$.

3.2. The Conversion of the Multiplicative Linear Differential Equations to Multiplicative Integral Equations

In this section, we prove that the multiplicative linear differential equation with constant or variable exponentials is converted to MVIE. We need to following theorem for converting n^{th} order multiplicative differential equation to MVIE.

Theorem 8. If n is a positive integer and a is a constant with $x \geq a$, then we have

$$* \int_a^x \ldots (n) \ldots * \int_a^x u(t)^{dt \ldots dt} = * \int_a^x \left[\left(u(t)\right)^{\frac{(x-t)^{(n-1)}}{(n-1)!}}\right]^{dt}$$

Proof. Let

$$I_n =* \int_a^x [u(t)]^{(x-t)^{(n-1)}dt} . \tag{9}$$

If it is taken $F(x,t) = [u(t)]^{(x-t)^{(n-1)}}$, we can write that

$$\begin{aligned}\frac{d^* I_n}{dx} &=* \int_a^x F_x^*(x,t)^{dt} \frac{[F(x,x)]^1}{[F(x,a)]^0} \\ &=* \int_a^x F_x^*(x,t)^{dt}\end{aligned}$$

by using the multiplicative Leibniz formula to equation (9). Then we find

$$\begin{aligned}\frac{d^* I_n}{dx} &=* \int_a^x \left(e^{\frac{\partial}{\partial x} \ln F(x,t)}\right)^{dt} \\ &=* \int_a^x \left(e^{(n-1)(x-t)^{(n-2)} \ln u(t)}\right)^{dt} \\ &=* \int_a^x \left(e^{\ln\left([u(t)]^{(n-1)(x-t)^{(n-2)}}\right)}\right)^{dt} \\ &=* \int_a^x \left([u(t)]^{(n-1)(x-t)^{(n-2)}}\right)^{dt}.\end{aligned}$$

Hence we get

$$\frac{d^* I_n}{dx} = \left(* \int_a^x \left([u(t)]^{(x-t)^{(n-2)}}\right)^{dt}\right)^{(n-1)} = (I_{n-1})^{(n-1)} \tag{10}$$

where $n > 1$. Since $I_1(x) =* \int_a^x u(t)^{dt}$ for $n = 1$, then we can write

$$\frac{d^* I_1}{dx} = \frac{d^*}{dx}\left(* \int_a^x \left(u(t)\right)^{dt}\right) = u(x). \tag{11}$$

If it is taken multiplicative derivative of the equation (10) by using multiplicative Leibniz formula, then

$$\begin{aligned}\frac{d^{**} I_n}{dx^{(2)}} &= \frac{d^*}{dx}\left(* \int_a^x \left([u(t)]^{(x-t)^{(n-2)}}\right)^{dt}\right)^{(n-1)} \\ &= \left(\frac{d^*}{dx}\left(* \int_a^x \left([u(t)]^{(x-t)^{(n-2)}}\right)^{dt}\right)\right)^{(n-1)}\end{aligned}$$

$$=\left(*\int_a^x\left[\frac{\partial^*}{\partial x}\left([u(t)]^{(x-t)^{(n-2)}}\right)\right]^{dt}\frac{\left[[u(x)]^{(x-x)^{(n-2)}}\right]^1}{\left[[u(a)]^{(x-a)^{n-2}}\right]^0}\right)^{(n-1)}$$

$$=\left(*\int_a^x\left(\frac{\partial^*}{\partial x}\left([u(t)]^{(x-t)^{(n-2)}}\right)\right)^{dt}\right)^{(n-1)}$$

$$=\left(*\int_a^x\left(e^{\frac{\partial}{\partial x}\left(ln\left([u(t)]^{(x-t)^{(n-2)}}\right)\right)}\right)^{dt}\right)^{(n-1)}$$

$$=\left(*\int_a^x\left[e^{\left(ln\left((u(t))^{(n-2)(x-t)^{(n-3)}}\right)\right)}\right]^{dt}\right)^{(n-1)}$$

$$=\left(*\int_a^x\left[(u(t))^{(n-2)(x-t)^{(n-3)}}\right]^{dt}\right)^{(n-1)}$$

$$=\left(*\int_a^x\left[(u(t))^{(x-t)^{(n-3)}}\right]^{dt}\right)^{(n-1).(n-2)}$$

$$=(I_{n-2})^{(n-1)(n-2)}.$$

By proceeding similarly, we obtain

$$\frac{d^{*(n-1)}I_n}{dx^{(n-1)}}=(I_1)^{(n-1)!}$$

Hence, we write

$$\frac{d^{*(n)}I_n}{dx^{(n)}}=\left(\frac{d^*I_1}{dx}\right)^{(n-1)!}=[u(x)]^{(n-1)!}$$

from the equation (11). Now, we will take multiplicative integral by considering the above relations.

From the equation (11), $I_1(x)=*\int_a^x u(t)^{dt}$. Also, we have

$$I_2(x)=*\int_a^x I_1(x_2)^{dx_2}=*\int_a^x *\int_a^{x_2} u(x_1)^{dx_1dx_2}$$

where x_1 and x_2 are parameters. By proceeding similarly, we obtain

$$I_n(x)=\left(*\int_a^x *\int_a^{x_n}\dots *\int_a^{x_3} *\int_a^{x_2} u(x_1)^{dx_1dx_2\dots dx_n}\right)^{(n-1)!}$$

where $x_1,x_2,\dots,x_n$ are parameters. If we write the equation (9) instead of the statement I_n, then it is find

$$* \int_a^x \left[\left(u(t)\right)^{(x-t)^{n-1}} \right]^{dt} = \left(* \int_a^x * \int_a^{x_n} \dots * \int_a^{x_3} * \int_a^{x_2} u(x_1)^{dx_1 dx_2 \dots dx_n} \right)^{(n-1)!}$$

Hence we can write

$$\left(* \int_a^x \left[\left(u(t)\right)^{(x-t)^{n-1}} \right]^{dt} \right)^{\frac{1}{(n-1)!}} = * \int_a^x * \int_a^{x_n} \dots * \int_a^{x_3} * \int_a^{x_2} u(x_1)^{dx_1 dx_2 \dots dx_n}$$

If it is taken $= x_1 = x_2 = \cdots = x_n$, therefore we obtain

$$* \int_a^x \dots (n) \dots * \int_a^x u(t)^{dt \dots dt} = * \int_a^x \left[\left(u(t)\right)^{\frac{(x-t)^{n-1}}{(n-1)!}} \right]^{dt}.$$

This completes the proof.

Let the n^{th}- order multiplicative linear differential equation

$$\frac{d^{*(n)}y}{dx^{(n)}} \left(\frac{d^{*(n-1)}y}{dx^{(n-1)}} \right)^{a_1(x)} \left(\frac{d^{*(n-2)}y}{dx^{(n-2)}} \right)^{a_2(x)} \dots \left(\frac{d^{*}y}{dx} \right)^{a_{n-1}(x)} (y)^{a_n(x)} = f(x) \tag{12}$$

that given the initial conditions

$$y(0) = c_0 \, , y^*(0) = c_1 \, , y^{*(n-1)}(0) = c_{n-1} \tag{13}$$

It can be transformed the multiplicative Volterra integral equation. Hence the solution of (12)-(13) may be reduced to a solution of some multiplicative Volterra integral equation.

Take $\frac{d^{*(n)}y}{dx^{(n)}} = u(x)$. By integrating both sides of the equality $\frac{d^*}{dx}\left(\frac{d^{*(n-1)}y}{dx^{(n-1)}} \right) = u(x)$, we write

$$* \int_0^x d^* \left(\frac{d^{*(n-1)}y}{dx^{(n-1)}} \right) = * \int_0^x u(t)^{dt}$$

$$\frac{y^{*(n-1)}(x)}{y^{*(n-1)}(0)} = * \int_0^x u(t)^{dt}$$

$$y^{*(n-1)}(x) = c_{n-1} * \int_0^x u(t)^{dt}$$

By proceeding similarly, we find

$$* \int_0^x d^* \left(\frac{d^{*(n-2)}y}{dx^{(n-2)}} \right) = * \int_0^x \left(c_{n-1} * \int_0^x u(t)^{dt} \right)^{dt}$$

$$\frac{y^{*(n-2)}(x)}{y^{*(n-2)}(0)} = * \int_0^x c_{n-1}{}^{dt} * \int_0^x * \int_0^x u(t)^{dtdt}$$

$$\frac{y^{*(n-2)}(x)}{c_{n-2}} = e^{\int_0^x \ln c_{n-1} dt} * \int_0^x * \int_0^x u(t)^{dtdt}$$

$$\frac{y^{*(n-2)}(x)}{c_{n-2}} = (c_{n-1})^{x} * \int_0^x * \int_0^x u(t)^{dtdt}$$

$$y^{*(n-2)}(x) = c_{n-2}\ (c_{n-1})^{x} * \int_0^x * \int_0^x u(t)^{dtdt}$$

$$* \int_0^x d^* \left(\frac{d^{*(n-3)}y}{dx^{(n-3)}} \right) =* \int_0^x [(c_{n-1})^{x}.c_{n-2}]^{dt} * \int_0^x * \int_0^x * \int_0^x u(t)^{dtdtdt}$$

$$y^{*(n-3)}(x) = c_{n-3}\ (c_{n-2})^{x}\ (c_{n-1})^{x^2} * \int_0^x * \int_0^x * \int_0^x u(t)^{dtdtdt}$$

$$\vdots$$

$$y^* = c_1\ (c_2)^x\ (c_3)^{x^2} \dots (c_{n-2})^{x^{(n-3)}}\ (c_{n-1})^{x^{(n-2)}} * \int_0^x \dots (n-1) \dots * \int_0^x u(t)^{dt \dots dt}$$

Hence, we get

$$y = c_0\ (c_1)^x\ (c_2)^{x^2} \dots (c_{n-1})^{x^{(n-1)}} * \int_0^x \dots (n) \dots * \int_0^x u(t)^{dt \dots dt}$$

If we take into account the above expressions, the multiplicative linear differential equation (12) is written as follows

$$u(x) \left[(c_{n-1})^{a_1(x)} \left(* \int_0^x u(t)^{dt} \right)^{a_1(x)} \right] \left[(c_{n-2})^{a_2(x)}\ (c_{n-1})^{x\, a_2(x)} \left(* \int_0^x * \int_0^x u(t)^{dtdt} \right)^{a_2(x)} \right] \dots$$

$$\left[(c_0)^{a_n(x)}\ (c_1)^{x a_n(x)}\ (c_2)^{x^2 a_n(x)} \cdot \dots \cdot (c_{n-1})^{x^{n-1} a_n(x)} \quad \left(* \int_0^x \dots (n) \dots * \int_0^x u(t)^{dt \dots dt} \right)^{a_n(x)} \right] = f(x)$$

$$u(x)\ (c_0)^{a_n(x)}\ (c_1)^{x\, a_n(x) + a_{n-1}(x)} \dots (c_{n-1})^{x^{n-1} a_n(x) + \dots + a_1(x)} \left(* \int_0^x u(t)^{dt} \right)^{a_1(x)} \left(* \int_0^x * \int_0^x u(t)^{dtdt} \right)^{a_2(x)} \dots$$

$$\left(* \int_0^x \dots (n) \dots * \int_0^x u(t)^{dt \dots dt} \right)^{a_n(x)} = f(x) \tag{14}$$

If we set

$a_1(x) + a_2(x)\, x + \dots + a_n(x)\, x^{n-1} = f_{n-1}(x)$

$a_2(x) + a_3(x)\, x + \dots + a_n(x)\, x^{n-2} = f_{n-2}(x)$

$\vdots$

$a_{n-1}(x) + a_n(x)\, x = f_1(x)$

$a_n(x) = f_0(x)$

and

$$F(x) = \frac{f(x)}{(c_0)^{f_0(x)}\,(c_1)^{f_1(x)} \dots (c_{n-1})^{f_{n-1}(x)}}$$

then we can edit the equation (14) in the form as follows

$$u(x)\left(*\int_0^x u(t)^{dt}\right)^{a_1(x)}\left(*\int_0^x *\int_0^x u(t)^{dtdt}\right)^{a_2(x)} \dots \left(*\int_0^x \dots (n) \dots *\int_0^x u(t)^{dt\dots dt}\right)^{a_n(x)} = F(x)\,.$$

By using Theorem 8, we get

$$u(x)\left(*\int_0^x u(t)^{dt}\right)^{a_1(x)}\left(*\int_0^x [u(t)^{x-t}]^{dt}\right)^{a_2(x)} \dots \left(*\int_0^x u(t)^{\frac{(x-t)^{n-1}}{(n-1)!}dt}\right)^{a_n(x)} = F(x).$$

Then we find the equation

$$u(x) *\int_0^x \left(u(t)^{\left[a_1(x)+(x-t)a_2(x)+\cdots+a_n(x)\frac{(x-t)^{n-1}}{(n-1)!}\right]}\right)^{dt} = F(x).$$

If we put $K(x,t) = a_1(x) + (x-t)a_2(x) + \cdots + a_n(x)\frac{(x-t)^{n-1}}{(n-1)!}$ as the kernel function, then the equation (12) is turned into

$$u(x) *\int_0^x u(t)^{K(x,t)dt} = F(x)$$

which is a MVIE of the second kind.

Example 4. Form a multiplicative Volterra integral equation corresponding to the multiplicative differential equation $\frac{d^{*2}y(x)}{dx^{(2)}} = y(x)^{cosx}$ with the initial conditions $y(0) = 1, y^*(0) = 1$.

Solution. Let $\frac{d^{*2}y(x)}{dx^{(2)}} = u(x)$. Then we write

$$*\int_0^x d^*y^* = *\int_0^x u(t)^{dt}$$

$$\frac{y^*(x)}{y^*(0)} = *\int_0^x u(t)^{dt}$$

$$y^*(x) = *\int_0^x u(t)^{dt}.$$

Therefore we find

$$*\int_0^x y^*(t)^{dt} = *\int_0^x *\int_0^x u(t)^{dt}$$

$$\frac{y(x)}{y(0)} =* \int_0^x * \int_0^x u(t)^{dt}$$

$$y(x) =* \int_0^x \left[u(t)^{(x-t)}\right]^{dt}$$

If we replace the equation $y(x) =* \int_0^x \left[u(t)^{(x-t)}\right]^{dt}$ into the given multiplicative differential equation, we obtain $u(x) =* \int_0^x \left[u(t)^{cos\, x\, (x-t)}\right]^{dt}$.

4. Conclusion

In this paper, the multiplicative Voltterra integral equation is defined by using the concept of multiplicative integral. The solution of multiplicative Volterra integral equation is obtained with the successive approximations method. The multiplicative Leibniz formula is proved and the multiplicative Volterra integral equation is converted to a multiplicative differential equation by aid of multiplicative Leibniz formula. The multiplicative linear differential equation with constant or variable exponentials is converted to a multiplicative Volterra integral equation is proved.

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ON THE CIRCULAR INVERSION IN MAXIMUM PLANE

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Abstract

In this paper, we investigate general properties and basic concepts of circular inversions in the maximum plane. We delve into cross-ratio and harmonic conjugates under maximum circular inversion. Furthermore, we illustrated figures related to inversions obtained in the maximum plane via Mathematica.

Keywords: Inversion; maximum metric; cross-ratio; harmonic conjugates.

1. Introduction

Whatever we are working on these days, we think according to Euclidean geometry. Especially considering the distance between two points, the first thing that comes to mind is the Euclidean metric and its distance function. There are so many useful metrics for measuring distance. One of them is the maximum metric which is defined in maximum metric geometry. In [5], the maximum metric is defined as follows.

Definition 1.1. $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ are two points in the Cartesian plane, the maximum metric distance is given by

$$d_M(X,Y) = \max\{|x_2 - x_1|, |y_2 - y_1|\}. \tag{1}$$

We use this distance function to define circular inversion in the maximum plane. Inversion in geometry is a transformation, that is not an isometry and not even an affine transformation. Inversions have the property that they transform certain circles in lines and that they preserve the angles. According to [5], inversion, which is a kind of study of transformations in the Euclidean plane, is also called "circular inversion" since it is defined on a circle. Inversion can be thought of as a reflection in the circle. Inversion can map the circle into the circle, circle into the line, or line into the circle. It is possible to apply inversion which has different transformation examples from subjects previously studied, to the solution of many problems in geometry. For more details about concepts and properties of inversions in different planes, see [1], [2] and [4].

In this paper, we define an inversion in the maximum plane. After giving the definition, we examine basic concepts and general properties of circular inversions in this plane. Cross-ratio and harmonic conjugates under maximum circular inversion are also studied. Moreover, we draw figures related to the properties of inversions that we obtained during this study.

2. Basic Concepts

In this section, we briefly mention some basic concepts. By the maximum metric d_M , the shortest path between the points P_1 and P_2 is a line segment that is parallel to a coordinate axis.

Proposition 2.1. Every Euclidean translation preserves the distance in the maximum plane. Thence, each of them is an isometry in $\mathbb{R}^2_M$.

Proposition 2.2. Let d_E denote the Euclidean distance function and l be the line passing through the points P_1 and P_2 in the analytical plane. If l has the slope m, then

$$d_M(P_1, P_2) = \frac{max\{1,|m|\}}{\sqrt{1+m^2}} d_E(P_1, P_2). \quad (2)$$

Proposition 2.2 states that d_M-distance along any line is a positive constant multiple of Euclidean distance along the same line.

Corollary 2.3. Let P_1, P_2 and X be three collinear points in $\mathbb{R}^2$. Then, $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_M(P_1, X) = d_M(P_2, X)$.

Corollary 2.4. Let P_1, P_2 and X be three distinct collinear points in $\mathbb{R}^2$. Then,

$$d_E(P_1, X)/d_E(P_2, X) = d_M(P_1, X)/d_M(P_2, X). \quad (3)$$

That is the ratios of the Euclidean and d_M-distance along a line are the same.

Definition 2.5. Let $\mathcal{C}$ be a circle centered at a point O with radius r. If P is any point other than O, the inverse of P with respect to $\mathcal{C}$ is the point P' on the ray $\overrightarrow{OP}$ such that the product of the distances of P and P' from O is equal to r^2 , that is

$$d_E(O, P). d_E(O, P') = r^2, \quad (4)$$

see [3].

Clearly, if P' is the inverse point of P, then P is the inverse point of the P'. Note that if P is in the interior of $\mathcal{C}$, P' is exterior to $\mathcal{C}$; and vice-versa. So, the interior of $\mathcal{C}$ except for O is mapped to the exterior and the exterior to the interior. $\mathcal{C}$ itself is left by the inversion pointwise fixed. O has no image, and no point of the plane is mapped to O. However, points close to O are mapped to points far from O, and points far from O mapped to points close to O. By adjoining one "ideal point", or "point at infinity", to the Euclidean plane, we can include O in the domain and range of an inversion.

Now in $\mathbb{R}^2_M$, the definition of inversion with respect to a M-circle (maximum circle) can be given as follows.

Definition 2.6. Let $\mathcal{C}$ be a M-circle centered at point O with radius r in $\mathbb{R}^2_M$, and P_∞ be the ideal point adjoined to the maximum plane. In $\mathbb{R}^2_M$, the maximum circular inversion with respect to $\mathcal{C}$ is the transformation

$$I_M(O, r): \mathbb{R}^2_M \cup \{P_\infty\} \to \mathbb{R}^2_M \cup \{P_\infty\}$$

given by

$$d_M(O,P).d_M(O,P') = r^2, \tag{5}$$

where $I_M(O,r)(O) = P_\infty,\ I_M(O,r)(P_\infty) = O\ ,\ I_M(O,r)(P) = P'$ for $P \neq 0$ and P' is on the ray $\overrightarrow{OP}$.

Lemma 2.7. Let C be the M-circle which is centered at the origin and the radius is r. If the point P is in the interior of C, the point P' is in the exterior to C, and vice-versa.

Proof. Let us consider that the point P is in the interior of C. Thus,

$$d_M(O,P) < r\ .$$

Since $P' = I_M(O,r)$ and from Equation 5, then it is obtained

$$r^2 = d_M(O,P).d_M(O,P') < r.d_M(O,P')$$

and

$$d_M(O,P') > r.$$

So, the point P' is in the exterior of C.

Proposition 2.8. Let $I_M(O,r)$ be the maximum circular inversion, with respect to a M-circle, C centered at the orijin and the radius is r in $\mathbb{R}^2_M$. Therefore, the maximum circular inverse of the point $P = (x,y)$ is the point $P' = (x',y')$, whose coordinates are

$$x' = \frac{r^2 x}{(\max\{|x|,|y|\})^2}\ ,\ y' = \frac{r^2 y}{(\max\{|x|,|y|\})^2}\ . \tag{6}$$

Proof. The M-circle C, which is centered at the origin and has the radius r, is the set of points satisfies the equation $\max\{|x|,|y|\} = r$. Let $P = (x,y)$ and $P' = (x',y')$ are inverse points with respect to C. Since the points O, P and P' are collinear and the rays $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ have the same direction, then $\overrightarrow{OP'} = k.\overrightarrow{OP}$ for $k \in \mathbb{R}^+$, and $(x',y') = (kx,ky)$. Using by $d_M(O,P).d_M(O,P') = r^2$, $k = \frac{r^2}{(\max\{|x|,|y|\})^2}$ is obtained and by substituting the obtained value of k, the required results are obtained. Note that, if the point P' is inverse of P, then P is the inverse of P'. As a result of this, the equivalent form would be written as

$$x = \frac{r^2 x'}{(\max\{|x'|,|y'|\})^2}\ ,\ y = \frac{r^2 y'}{(\max\{|x'|,|y'|\})^2}\ . \tag{7}$$

Corollary 2.6. Let $I_M(O',r)$ be the maximum circular inversion, with respect to a M-circle centered at $O' = (a,b)$ and the radius is r in $\mathbb{R}^2_M$, then the maximum circular inverse of the point $P = (x,y)$ is the point $P' = (x',y')$, whose coordinates are

$$x' = a + \frac{r^2(x-a)}{(\max\{|x-a|,|y-b|\})^2}\ ,\ y' = b + \frac{r^2(y-b)}{(\max\{|x-a|,|y-b|\})^2}\ . \tag{8}$$

Proof. The proof is obvious by the fact that all translations are isometries of the maximum plane.

Remark 2.7. It is clear that the interior of C_M , except the center O, is mapped to the exterior and exterior to the interior under maximum circular inversion.

3. Circular Inversions in $\mathbb{R}_M^2$

In this section, the results and the definitions obtained by maximum circular inversion are given. First, inversions of lines and circles according to their positions in $\mathbb{R}_M^2$ are investigated. In addition, properties of inversions in the Euclidean and the Maximum planes are compared. First, the following properties of inversion in the Euclidean plane, which are well known, will be given as:

i. The inverse of a line through the center of inversion is the line itself.
ii. The inverse of a line not passing through the center of inversion is a circle passing through the center of inversion and conversely.
iii. The inverse of a circle not passing through the center of inversion is a circle not passing through the center of inversion.
iv. Circles with center of inversion are mapped into circles with center of inversion.

All of the properties of inversion in the Euclidean space which are given above are not valid in the maximum plane. We now give the theorem to show which properties given above are satisfied or not in the maximum plane.

Theorem 3.1.

i. Lines passing through the center O are mapped onto themselves under the maximum circular inversion $I_M(O,r)$.
ii. Lines not containing the center of the maximum circular inversion circle are not mapped onto maximum circles centered O under the maximum circular inversion $I_M(O,r)$.
iii. Maximum circles centered O are mapped onto maximum circles with the center O under the maximum circular inversion $I_M(O,r)$.
iv. Maximum circles not through O are not mapped onto any maximum circles under the maximum circular inversion $I_M(O,r)$.
v. Maximum circles containing the center of inversion circle are not mapped onto straight lines not containing the center O under the maximum circular inversion $I_M(O,r)$.

Proof. By examining all possible cases the properties in the Theorem 3.1. are obtained.
For i. and ii. let $ax+by+c=0$ be a line in the maximum plane. By using Equation 7, it is acquired that

$$\frac{ax'r^2}{(\max\{|x'|,|y'|\})^2}+\frac{by'r^2}{(\max\{|x'|,|y'|\})^2}+c=0. \tag{9}$$

Thus, it can be written as

$$ax'r^2+by'r^2+c(\max\{|x'|,|y'|\})^2=0. \tag{10}$$

Now, Equation 10 would be considered under cases which are given below:

Case 1. If $|x'| \geq |y'|$, then $c(x')^2 + ar^2x' + br^2y' = 0$.

1.1. If $c = 0$, the inverse of the line $ax + by = 0$ is $ax' + by' = 0$ that means both lines are the same.

1.2. If $c \neq 0, a = 0$, the inverse of the line $by + c = 0$ is the parabola $c(x')^2 + br^2y' = 0$

1.3. If $c \neq 0, b = 0$, the inverse of the line $ax + c = 0$ is the line $x' = 0$ or $x' = \frac{-ar^2}{c}$.

1.4. If $c \neq 0, a \neq 0$ and $b \neq 0$, the inverse of the line $ax + by + c = 0$ is the parabola $c(x')^2 + ar^2x' + br^2y' = 0$.

Case 2. If $|y'| \geq |x'|$, then $c(y')^2 + br^2y' + ar^2x' = 0$.

2.1. If $c = 0, a \neq 0$ and $b \neq 0$, the inverse of the line $ax + by = 0$ is $ax' + by' = 0$ which means both lines are the same.

2.2. If $c \neq 0, a = 0$, the inverse of the line $by + c = 0$ is the line $y' = 0$ or $y' = \frac{-br^2}{c}$.

2.3. If $c \neq 0, b = 0$, the inverse of the line $ax + c = 0$ is the parabola $c(y')^2 + ar^2x' = 0$.

2.4. If $c \neq 0, a \neq 0$ and $b \neq 0$, the inverse of the line $ax + by + c = 0$ is the parabola $c(y')^2 + br^2y' + ar^2x' = 0$.

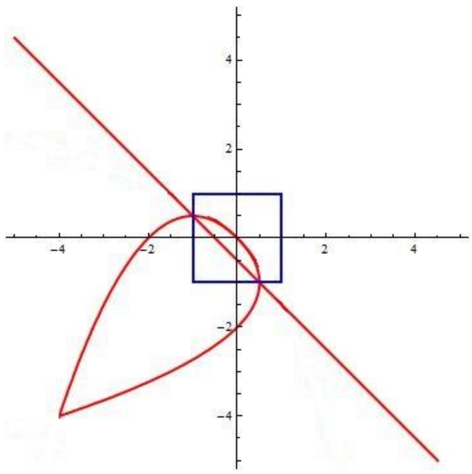

Figure 1. A line not passing through O isn't mapped onto a maximum circle with center O

For iii. let $\max\{|x|,|y|\} = r_1$ be the radius of the circle C' whose center is the same with C the maximum circular inversion circle. The inversion of this circle respect to C is

$$\max\left\{\left|\frac{|x'|r^2}{(\max\{|x'|,|y'|\})^2}\right|, \left|\frac{|y'|r^2}{(\max\{|x'|,|y'|\})^2}\right|\right\} = r_1 \tag{11}$$

and

$$\max\{|x'|,|y'|\} = \frac{r^2}{r_1} = r_2. \tag{12}$$

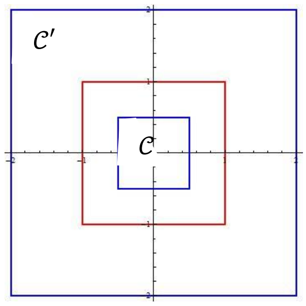

Figure 2. A maximum circle centered O is mapped onto a maximum circle with center O.

The other properties would be obtained similarly by using the definition of the maximum circular inversion and Proposition 2.8.

4. The Cross Ratio and Harmonic Conjugates in $\mathbb{R}_M^2$

The inversion in maximum plane is not an isometry. Thence, the distance is not preserved under maximum circular inversion. However, related to the concept of the distance, it can be shown that the cross-ratio is preserved under maximum circular inversion. Thus, in this section, the cross-ratio and harmonic conjugates in $\mathbb{R}_M^2$ are investigated.

Proposition 4.1. Let P, Q and O be three different collinear points in $\mathbb{R}_M^2$. If P' and Q' are inverses of P and Q respectively with respect to the maximum inversion circle $I_M(O,r)$, then

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)} \tag{13}$$

is obtained.

Proof. Let P, Q and O be three different collinear points, $P = (x_1, y_1), Q = (x_2, y_2), P' = (x'_1, y'_1)$ and $Q' = (x'_2, y'_2)$. Note that inverse points P' and Q' lies on the same line l with P, Q and O. If the slope of line l is m, then two cases would be considered; $|m| \geq 1$ and $|m| \leq 1$.

If $|m| \geq 1$, then

$$d_M(P',Q') = \max\{|x_2' - x_1'|, |y_2' - y_1'|\}.$$

$$=\max\left\{\left|\frac{r^2.x_2}{(\max\{|x_2|,|y_2|\})^2} - \frac{r^2.x_1}{(\max\{|x_1|,|y_1|\})^2}\right|, \left|\frac{r^2.y_2}{(\max\{|x_2|,|y_2|\})^2} - \frac{r^2.y_1}{(\max\{|x_1|,|y_1|\})^2}\right|\right\}$$

$$=\max\left\{\left|\frac{r^2.x_2}{(y_2)^2} - \frac{r^2.x_1}{(y_1)^2}\right|, \left|\frac{r^2}{y_2} - \frac{r^2}{y_1}\right|\right\}$$

$$= \left|\frac{r^2}{y_2} - \frac{r^2}{y_1}\right| = \frac{r^2|y_1 - y_2|}{|y_1||y_2|}$$

$$= \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)} \tag{14}$$

is acquired.

The case $|m| \leq 1$ can be easily shown with a similar method.

If P, Q and O are not collinear, then the equality in Proposition 3.2.1 is not valid for all P, Q in $\mathbb{R}_M^2$. For example, let $O = (0,0), P = (-1,1)$ and $Q = (1,2)$ and the radius is $r = 2$. The inversion $I_M(O, 2)$ maps P and Q onto $P' = (-4,4)$ and $Q' = (1,2)$, respectively. Then, it can be easily computed that $d_M(P,Q) = 2,\ d_M(P',Q') = 5, d_M(O,P) = 1$ and $d_M(O,Q) = 2$. So, the equality in Proposition 4.1 is obviously not valid for every points in $\mathbb{R}_M^2$. However, the following two propositions show some conditions that the equality in Proposition 4.1 is satisfied.

Proposition 4.2. Let C be the maximum inversion circle which is centered at origin and the radius is r. Let P, Q and O be any three distinct non-collinear points in $\mathbb{R}_M^2$. If P' and Q' are inverses of P and Q respectively and P and Q lie on the lines with slope $\{0, \infty\}$ or $\{1, -1\}$ passing through the origin, then

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)} \tag{15}$$

is obtained.

Proof. Note that $P = (p, 0)$ and $Q = (0, q)$ are mapped to $P' = \left(\frac{r^2}{p}, 0\right)$ and $Q' = \left(0, \frac{r^2}{q}\right)$ or $P = (p, -p)$ and $Q = (q, q)$ are mapped to $P' = \left(\frac{r^2}{p}, \frac{-r^2}{p}\right)$ and $Q' = \left(\frac{r^2}{q}, \frac{r^2}{q}\right)$. So, it can be easily shown that

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)}. \tag{16}$$

Proposition 4.3. Let C be the maximum inversion circle which is centered at origin and the radius is r and let P, Q and O be any three distinct non-collinear points in $\mathbb{R}_M^2$. If the slope of the line passing through P and Q is 1 and $x_P y_Q + y_P x_Q = 0$ where $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$, then

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)}. \tag{17}$$

Proof. Let the line passing through P and Q be $l: y = x + c$. Note that $P = (p, p + c)$ and $Q = (q, q + c)$ are mapped to $P' = \left(\frac{r^2}{p}, \frac{(p+c)r^2}{p}\right)$ and $Q' = \left(\frac{r^2}{q}, \frac{(q+c)r^2}{q}\right)$ respectively. Therefore, it can be easily shown that

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)}. \tag{18}$$

Let $d_M[PQ]$ denotes the maximum directed distance from P to Q along a line in the maximum plane. If the ray with initial point P containing Q has the positive direction of orientation, then $d_M[PQ] = d_M(P,Q)$. If the ray has the opposite direction, then $d_M[PQ] = -d_M(P,Q)$.

Now let P, Q, R and S are four distinct points on an oriented line in the maximum plane. Therefore, their maximum cross ratio $(PQ, RS)_M$ is defined by

$$(PQ, RS)_M = \frac{d_M[PR] d_M[QS]}{d_M[PS] d_M[QR]}. \tag{19}$$

Note that the maximum cross ratio is positive if both R and S are between P and Q or if neither R nor S is between P and Q, whereas the cross ratio is negative if pairs $\{P, Q\}$ and $\{R, S\}$ seperate each other. Also a maximum circular inversion with respect to $\mathcal{C}$ centered at origin which is different from P, Q, R and S preserve the maximum cross ratio.

Theorem 4.4. The maximum circular inversion preserves the maximum cross ratio.

Proof. Suppose that P, Q, R and S are four collinear points in the maximum plane. Let P', Q', R' and S' be inverse points of P, Q, R and S respectively according to the maximum circular inversion $I_M(O, r)$. Note that maximum circular inversion preserves the seperation or non-seperation of the pairs $\{P, Q\}$ and $\{R, S\}$ and also it reverses the maximum-directed distance from the point P to the point Q along a line l to maximum-directed distance from the point Q' to the point P'. The required result follows from Proposition 4.1 as

$$(P'Q', R'S')_M = \frac{d_M(P', R')d_M(Q'S')}{d_M(P'S')d_M(Q'R')}$$

$$= \frac{\frac{r^2 d_M(P,R)}{d_M(O,P)d_M(O,R)} \cdot \frac{r^2 d_M(Q,S)}{d_M(O,Q)d_M(O,S)}}{\frac{r^2 d_M(P,S)}{d_M(O,P)d_M(O,S)} \cdot \frac{r^2 d_M(Q,R)}{d_M(O,Q)d_M(O,R)}}$$

$$= \frac{d_M(P,R)d_M(Q,S)}{d_M(P,S)d_M(Q,R)}$$

$$= (PQ, RS)_M. \tag{20}$$

Let l be a line in $\mathbb{R}^2_M$. Suppose that P, Q, R and S are four points on l. It is called that P, Q, R and S form a harmonic set if $(PQ, RS)_M = -1$, and it is denoted by $H(PQ, RS)_M$. That is, any pair R and S on l for which

$$\frac{d_M[PR]d_M[QS]}{d_M[PS]d_M[QR]} = -1 \tag{21}$$

is said to divide P and Q harmonically. The points R and S are called maximum harmonic conjugates with respect to P and Q.

Theorem 4.5. Let $\mathcal{C}$ be a maximum circle with center O, and line segment $[PQ]$ a diameter of $\mathcal{C}$ in $\mathbb{R}^2_M$. Let R and S be distinct points of the ray $\overrightarrow{OP}$, which divide the segment $[PQ]$ internally and externally. Thus, R and S are maximum harmonic conjugates with respect to P and Q if and only if R and S are inverse points with respect to the maximum circular inversion $I_M(O, r)$.

Proof. Let R and S are maximum harmonic conjugates with respect to P and Q. Then,

$$(PQ, RS)_M = -1 \Rightarrow \frac{d_M[PR]d_M[QS]}{d_M[PS]d_M[QR]} = -1. \tag{22}$$

Since R divides the line segment $[PQ]$ internally and R is on the ray $\overrightarrow{OQ}$,

$$d_M(R,Q) = r - d_M(O,R) \text{ and } d_M(P,R) = r + d_M(O,R). \tag{23}$$

Since S divides the line segment $[PQ]$ externally and S is on the ray $\overrightarrow{OQ}$, it is obtained that

$$d_M(P,S) = r + d_M(O,S) \text{ and } d_M(Q,S) = d_M(O,S) - r. \tag{24}$$

Thus,

$$\frac{(r+d_M(O,R))(d_M(O,S)-r)}{(r+d_M(O,S))(r-d_M(O,R))} = -1 \tag{25}$$

$$\Rightarrow \big(r + d_M(O,R)\big)(d_M(O,S) - r) = \big(r + d_M(O,S)\big)(d_M(O,R) - r). \tag{26}$$

By simplifying the last equality, $d_M(O,R).\, d_M(O,S) = r^2$ is obtained. Then, R and S are the maximum inverse points with respect to the maximum circular inversion $I_M(O,r)$. For the other condition (S and R are on the ray $\overrightarrow{OP}$) with similar calculations, the same conclusion is obtained. Conversely, if R and S are maximum inverse points with respect to the maximum circular inversion $I_M(O,r)$, it can be proven with a similar method.

Conclusions

Inversions are not isometries. They transform distances and angles. We examine the way inversions transform the distance in the Maximum plane. We investigate general properties and basic concepts of circular inversions by means of maximum metric. We delve into cross-ratio and harmonic conjugates under maximum circular inversion. In addition, via Mathematica, we illustrated figures related to results we acquired. Drawing their figures reinforce the visualization of the results. Within the knowledge of the maximum metric is the special case of the alpha metric, which is mentioned in [4], we study the maximum circular inversions by examining the special cases in detail. Thence, it is expected to contribute to the literature on inversions.

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NEAR SOFT GROUPOID

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Abstract

In this article, firstly some concepts on the near soft set obtained by combining the near set and the soft set are given. In the previous studies in the literature, the definition of a soft element with binary operation in the set of all non-empty soft elements of a soft set and the definition of the concept of soft groupoid depending on the set of soft elements are given. In this study, starting from the concept of soft element, the concept of near soft groupoid is defined by using the near soft element with binary operation in the set of all non-empty soft near elements of a near soft set. In addition, properties related to the defined near soft groupoid are given with theorem and example.

Keywords: Near set; Soft Set; Near Soft Set; Near Soft Element; Near Soft Group, Near Soft Groupoid.

1 Introduction

The notion of near sets has been given by Peters [1, 2] and the concept of soft theory has been given by Molodtsov [4]. Then it was studied by many scientists [3, 4, 5, 6, 7]. The definition of soft element given by Wardowski[8] with a binary operation on the set of all nonempty soft elements of a given soft set. Then J.Ghosh[9, 10] defines soft groupoid based on the set of soft elements. In the rough set theory, which is another concept, the concepts of group and groupoid have been studied[11, 12]. Feng and Li [5] have investigated the problem of combining soft sets with rough sets, and introduced the notion of rough soft sets. Afterwards, Tasbozan [13, 14]combine near sets approach with soft set theory and introduced the notion of near soft sets. In this paper, we introduce the concept of near soft element and define near soft groupoid using the near soft element with a binary operation on the set of all nonempty near soft elements of a given near soft set.

2 Preliminary

In this section, we recall some descriptions and results presented and discussed in [13]. Also, we present the concepts of near soft sets, their fundamental properties, and operations such as near soft point, near soft elements. Then we define a binary composition on near soft sets and this form is called near soft groupoid over near soft set.

A nearness approximation space (NAS) is denoted by $NAS = (O, F, \sim_{B_r}, N_r, \upsilon_{N_r})$ which is defined with a set of perceived objects O, a set of probe functions F representing object features, an indiscernibility relation $\sim_{B_r} = \{(x, x') \in O \times O | \forall i \in B_r, i(x) = i(x')\}$ defined relative to $B_r \subseteq B \subseteq F$, a collection of partitions (families of neighbourhoods) $N_r(B)$, and a neighbourhood overlap function Nr . The relation $\sim_{B_r}$ is the usual indiscernibility relation from rough set theory restricted to a subset

$B_r \subseteq B$. The subscript r denotes the cardinality of the restricted subset B_r, where we consider $\binom{|B|}{r}$, i.e., $|B|$ functions $i \in F$ taken r at a time to define the relation $\sim_{B_r}$ .This relation defines a partition of O into non-empty, pairwise disjoint subsets that are equivalence classes denoted by $[x]_{B_r}$, where $[x]_{B_r} = \{x\prime \in O | x \sim_{B_r} x'\}$. These classes contitue a new set called the quotient set $O \diagup \sim_{Br}$, where $O \diagup \sim_{B_r} = \{[x]_{B_r} | x \in O\}$. And the overlap function υ_{N_r} is defined by $\upsilon_{N_r} : P\ (O) \times P\ (O) \to [0, 1]$, where $P\ (O)$ is the powerset of O.

Definition 1 *Let $NAS = (O, F, \sim_{Br}, N_r, \upsilon_{N_r})$ be a nearness approximation space and $\sigma = (F, B)$ be a soft set over O . The lower and upper near approximation of $\sigma = (F, B)$ with respect to NAS are denoted by $N_r * (\sigma) = (F_*, B)$ and $N_r^*(\sigma) = (F^*, B)$, which are soft sets over with the set-valued mappings given by*

$F_*(\phi) = N_r * (F(\phi))$=$\cup\{x \in O : [x]_{Br} \subseteq F(\phi)\}$ and

$F^*(\phi) = N_r^*(F(\phi))$=$\cup\{x \in O : [x]_{Br} \cap F(\phi) \neq \emptyset\}$ where all $\phi \in B$. The operators N_r* and N_r^* are called the lower and upper near approximation operators on soft sets, respectively. If $Bnd_{N_r(B)}(\sigma) \geqslant 0$, then the soft set σ is called a near soft set [13].

The collection of all near soft sets on O will be denoted $NSS(O)$.

Definition 2 *Let O be an initial universe set, E be the universe set of parameters and $B \subseteq E$.For a near soft set (F, B) over O , the set*

$$Supp(F, B) = \{\phi \in B : F(\phi) \neq \emptyset\}$$

is called the support of the near soft set (F, B).

1. *A near soft set (F, B) is called non-null near soft set (with respect to the parameters of B) if $Supp(F, B) \neq \emptyset$. Otherwise (F, B) is called null near soft set.*
2. *A near soft set (F, B) is called full null near soft set if $Supp(F, B) = B$. A collection of all full near soft sets on O will be denoted by $NS_f(O)$.*

Definition 3 *Let O be an initial universe set, E be the universe set of parameters and $B \subseteq E$ and $(F, B) \in NSS(O)$. We say that $(\phi, \{x_k\})$ is a nonempty near soft element of (F, B) if $\phi \in B$ and $x_k \in F(\phi)$. The pair $(\phi, \emptyset)$, where $\phi \in B$ will be called an empty near soft element of (F, B). Then $(\phi, \{x_k\})$ is a near soft element of (F, B) and denoted by F_B.*

Example 4 *Let $X = \{x_1, x_2, x_3\} \subseteq O = \{x_1, x_2, x_3, x_4, x_5\}, B = \{\phi_1, \phi_2\} \subseteq F = \{\phi_1, \phi_2, \phi_3\}$ denote a set of perceptual objects and a set of functions respectively. Let $(F, B) = \sigma$ defined by $(F, B) = \{(\phi_1, \{x_1, x_2\}), (\phi_2, \{x_3\})\}$. For $r = 1$*

$$[x_1]_{\phi_1} = \{x_1, x_2\},\ [x_3]_{\phi_1} = \{x_3, x_4\}$$
$$[x_1]_{\phi_2} = \{x_1, x_3\},\ [x_2]_{\phi_2} = \{x_2, x_4\}$$

$N_*(\sigma) = \{(\phi_1, \{x_1, x_2\})\}$, $N^*(\sigma) = \{(\phi_1, \{x_1, x_2\}), (\phi_2, O)\}$, *then σ is a near soft set. For $r = 2$*

$$[x_1]_{\phi_1,\phi_2} = \{x_1\},\ [x_2]_{\phi_1,\phi_2} = \{x_2\}, [x_3]_{\phi_1,\phi_2} = \{x_3\}, [x_4]_{\phi_1,\phi_2} = \{x_4\}$$

$N_*(\sigma) = \{(\phi_1, \{x_1, x_2\}), , (\phi_2, \{x_3\})\}$, $N^*(\sigma) = \{(\phi_1, \{x_1, x_2\}), , (\phi_2, \{x_3\})\}$, *then σ is a near soft set. Hence all the near soft elements of (F, B) are*

$$(\phi_1, \{x_1\}), (\phi_1, \{x_2\}), (\phi_2, \{x_3\})$$

Near Soft Groupoid

Let $(F, \circ)$ and $(O, *)$ be two groupoids, $(O, *)$ be a group with "$*$" operation, $(F, \circ)$ be a group with "$\circ$" operation and $B \subseteq F$. Also let $(F, B) \in NS_f(O)$, i.e., (F, B) be a full near soft set on O.,*i.e.*, for each parameter $\phi \in B$, there exists at least one nonempty near soft element of (F, B). We define a binary composition $*$ on (F, B) by

$$(\phi_i, \{x_a\}) * (\phi_j, \{x_b\}) = (\phi_i \circ \phi_j, \{x_a * x_b\})$$

for all $(\phi_i, \{x_a\}), (\phi_j, \{x_b\}) \in (F, B)$. (F, B) is said to be closed under the binary composition $*$ if and only if $(\phi_i \circ \phi_j, \{x_a * x_b\}) \in (F, B)$ for all $(\phi_i, \{x_a\}), (\phi_j, \{x_b\}) \in (F, B)$ i.e., if and only if $\phi_i \circ \phi_j \in B$ and $x_a * x_b \in F(\phi_i \circ \phi_j)$ for all $(\phi_i, \{x_a\}), (\phi_j, \{x_b\}) \in (F, B)$.

Definition 5 *If (F, B) is closed under the binary composition $*$, then the algebraic system $((F, B), *)$ is said to be a near soft groupoid over O.*

Theorem 6 *Let $(F, B) \in NS_f(O)$, then $((F, B), *)$ forms a near soft groupoid over O if and only if*

1. B is a subgroupoid of F i.e., $\phi_i \circ \phi_j \in B$ for all $\phi_i, \phi_j \in B$
2. for $\phi_i, \phi_j \in B$, $x_a \in F(\phi_i)$, $x_b \in F(\phi_j)$ then $x_a * x_b \in F(\phi_i \circ \phi_j)$.

Proof. Suppose $((F, B), *)$ is a near soft groupoid over (F, O). Let $\phi_i, \phi_j \in B$. Since $(F, B) \in NS_f(O)$, there exist some $x_a, x_b \in O$ such that $(\phi_i, \{x_a\}), (\phi_j, \{x_b\}) \in (F, B)$. Hence $(\phi_i, \{x_a\}) * (\phi_j, \{x_b\}) \in (F, B)$. This implies $(\phi_i \circ \phi_j, \{x_a * x_b\}) \in (F, B)$, $\phi_i \circ \phi_j \in B$ and $x_a * x_b \in F(\phi_i \circ \phi_j)$ by definition (near soft element). Therefore B is a subgroupoid of F and for $\phi_i, \phi_j \in B$, $x_a \in F(\phi_i)$, $x_b \in F(\phi_j)$ then $x_a * x_b \in F(\phi_i \circ \phi_j)$. Conversely, suppose that the given two conditions hold. Now let $(\phi_i, \{x_a\})$, $(\phi_j, \{x_b\}) \in (F, B)$. This implies that $\phi_i, \phi_j \in B$, $x_a \in F(\phi_i)$, $x_b \in F(\phi_j)$ by hypothesis (1), $\phi_i, \phi_j \in B$ then $\phi_i \circ \phi_j \in B$,by hypothesis (2), $x_a \in F(\phi_i)$, $x_b \in F(\phi_j)$ then $x_a * x_b \in F(\phi_i \circ \phi_j)$. Therefore $(\phi_i \circ \phi_j, \{x_a * x_b\}) \in (F, B)$. So (F, B) is closed under the binary composition $*$. Hence $((F, B), *)$ forms a near soft groupoid over O. ■

Example 7 *Let $O = \{0, 1, 4, 5\}$ be the set of objects which $(O, \cdot)$ be a group with "$\cdot$" operation being multiplication of O integers modulo 4 and $F = \{\phi_1, \phi_2\}$ be a set of quotient function*

$$\phi_i : \mathcal{O} \to \mathcal{O}/\sim_{\phi_i}$$

$$\phi_1 : 0 \to \phi(0) = \bar{0} = \{0, 4\}$$

$$\phi_2 : 1 \to \phi(1) = \bar{1} = \{1, 5\}$$

which $(F, +)$ be a group with "$+$" operation being addition the classes of residues of integers modulo 4 .

Take $B = \{\phi_1\} \subset F$ and define a near soft set $\sigma = (F, B) = \{\phi_1, \{0, 4\}\}$ with $[0]_{\phi_1} = \{0, 4\}, [1]_{\phi_1} = \{1, 5\}$,

$$N_*(\sigma) = \{\phi_1, \{0, 4\}\}, N^*(\sigma) = \{\phi_1, \{0, 4\}\}$$

Hence all the near soft elements of F_B are $\{\phi_1, \{0\}\}, \{\phi_1, \{4\}\}$. Then the binary composition "$\cdot$" is given by

$$\begin{aligned}
\{\phi_i, \{x_a\}\} \cdot \{\phi_j, \{x_b\}\} &= \{\phi_i + \phi_j, \{x_a \cdot x_b\}\} \\
\{\phi_1, \{0\}\} \cdot \{\phi_1, \{0\}\} &= \{\phi_1, \{0\}\} \\
\{\phi_1, \{0\}\} \cdot \{\phi_1, \{4\}\} &= \{\phi_1, \{0\}\} \\
\{\phi_1, \{4\}\} \cdot \{\phi_1, \{4\}\} &= \{\phi_1, \{0\}\}
\end{aligned}$$

Hence $(F_B, \cdot)$ is a near soft groupoid over O.

Definition 8 *Let $(F_B, *)$ be a near soft groupoid over (F, O) where the binary composition $*$ is defined. Then $*$ said to be*

1. commutative if $(\phi_i, \{x_a\}) * (\phi_j, \{x_b\}) = (\phi_j, \{x_b\}) * (\phi_i, \{x_a\})$

2. associative if $[(\phi_i, \{x_a\}) * (\phi_j, \{x_b\})] * (\phi_k, \{x_c\}) = (\phi_i, \{x_a\}) * [(\phi_j, \{x_b\}) * (\phi_k, \{x_c\})]$ for all $(\phi_i, \{x_a\}), (\phi_j, \{x_b\}), (\phi_k, \{x_c\}) \in F_B$

Definition 9 *A near soft element $(\phi, \{x\}) \in F_B$ is said to be a near soft identity element in a near soft groupoid $(F_B, *)$ if for all $(\phi_i, \{x_a\}) \in F_B$*

$$(\phi, \{x\}) * (\phi_i, \{x_a\}) = (\phi_i, \{x_a\}) = (\phi_i, \{x_a\}) * (\phi, \{x\})$$

Definition 10 *Let $(F_B, *)$ be a near soft groupoid over (F, O)*

1. *If the composition $\circ$ on B and the composition $*$ on O are associative (commutative) then the composition $*$ on F_B is associative (commutative).*
2. *If F_B contains the near soft identity element $(\phi, \{x\})$ then ϕ is the identity element of B and x is the identity element of $\cup_{\phi_i \in B} F(\phi_i)$.*
3. *If $*$ is associative then near soft groupoid $(F_B, *)$ is called near soft semigroup.*
4. *If the soft semigroup $(F_B, *)$ contains near soft identity element then called near soft monoid.*

Definition 11 *Let $(F_B, *)$ be a near soft groupoid with near soft identity element $(\phi, \{x\})$. A near soft element $(\phi_i, \{x_a\}) \in F_B$ is said to be invertible if there exists a near soft element $(\phi_i', \{x_a'\}) \in F_B$ such that*

$$(\phi_i, \{x_a\}) * (\phi_i', \{x_a'\}) = (\phi, \{x\}) = (\phi_i', \{x_a'\}) * (\phi_i, \{x_a\})$$

Then $(\phi_i', \{x_a'\})$ is called the near soft inverse of $(\phi_i, \{x_a\})$ and denoted by $(\phi_i, \{x_a\})^{-1}$.

Theorem 12 *Let $(F_B, *)$ be a near soft groupoid with near soft identity element $(\phi, \{x\})$. If a near soft element $(\phi_i, \{x_a\}) \in F_B$ is invertible then ϕ_i is invertible in F and $\{x_a\} \in F(\phi_i)$ is invertible in O.*

Proof. Suppose $(\phi_i, \{x_a\}) \in F_B$ is invertible. then there exist a near soft element $(\phi_i', \{x_a'\}) \in F_B$ such that

$$(\phi_i, \{x_a\}) * (\phi_i', \{x_a'\}) = (\phi, \{x\}) = (\phi_i', \{x_a'\}) * (\phi_i, \{x_a\})$$
$$(\phi_i \circ \phi_i', \{x_a * x_a'\}) = (\phi, \{x\}) = (\phi_i' \circ \phi_i, \{x_a' * x_a\})$$
$$\phi_i \circ \phi_i' = \phi = \phi_i' \circ \phi_i \text{ and } x_a * x_a' = x = x_a' * x_a$$

Since $(\phi, \{x\})$ is the near soft identity element of F_B then ϕ is the identity element of B and x is the identity element of $\cup_{\phi_i \in B} F(\phi_i)$. Also ϕ_i is invertible in $B \subseteq F$ and $\{x_a\} \in F(\phi_i)$ is invertible in $\cup_{\phi_i \in B} F(\phi_i) \subseteq O$. ■

Remark 13 *Converse of this theorem is not necessarily true. In a near soft groupoid $(F_B, *)$ with near soft identity element, if ϕ_i is invertible in B and $\{x_a\} \in F(\phi_i)$ is invertible in O then $(\phi_i, \{x_a\}) \in F_B$ is not necessarily invertible in F_B.*

Example 14 *Let $O = \{0, 1, 4, 5\}$ be the set of objects which $(O, \cdot)$ be a group with "$\cdot$" operation being multiplication of O integers modulo 4 and $F = \{\phi_1, \phi_2\}$ be a set of quotient function*

$$\phi_i : \mathcal{O} \to \mathcal{O}/R$$

$$\phi_1 : 0 \to \phi(0) = \bar{0} = \{0, 4\}$$

$$\phi_2 : 1 \to \phi(1) = \bar{1} = \{1, 5\}$$

which $(F, +)$ be a group with "$+$" operation being addition the classes of residues of integers modulo 4 . $\sigma = (F, B) = \{\phi_1, \{0, 4\}\}$ is a near soft set , it is all the near soft elements are $\{\phi_1, \{0\}\}, \{\phi_1, \{4\}\}$. Hence $(F_B, *)$ is a near soft groupoid with near soft identity element $\{\phi_1, \{0\}\}$. The near soft inverse of $\{\phi_1, \{4\}\}$ is $(\{\phi_1, \{4\}\})^{-1} = (\{\phi_1, \{0\}\}) \in F_B$. Therefore $(\{\phi_1, \{4\}\})$ is invertible in F_B.

Conclusion

As a result, the definition of near soft groupoid was given to the concept of near soft set obtained with the help of near set and soft set concepts in this study by applying the definition of soft element used in previous studies. This new concept can be used in new studies in set theory.

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ANALYTICAL SOLUTIONS OF CONFORMABLE BOUSSINESQ-DOUBLE-SINH-GORDON AND FIRST BOUSSINESQ-LIOUVILLE EQUATIONS WITH THE HELP OF AUXIULARY EQUATION METHOD

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Abstract

In this article, the analytical solutions of nonlinear fractional order Boussinesq-Double-Sinh-Gordon equation and first Boussinesq-Liouville equation are obtained with the aid of auxiulary equation method where the fractional derivatives are in conformable sense. Both equations were first converted to non-linear ordinary derivative differential equations with the help of wave transformation. auxiliary equation method was used to find analytical solutions of these ordinary derivative equations. Three dimensional graphics of the obtained results for nonlinear fractional order Boussinesq-Double-Sinh-Gordon equation and first Boussinesq-Liouville equation are given.

Keywords: Conformable Fractional Partial Differential Equations; Auxiulary Equation Method; Conformable Boussinesq-Double-Sinh-Gordon; Conformable First Boussinesq-Liouville.

1. Introduction

Obtaining analytical solutions of fractional order nonlinear partial differential equations is crucial for understanding the physical behavior and change process of the event under consideration.

Because the nonlinear partial differential equation models containing integer-order derivatives do not correspond exactly to events in nature, while the differential equations containing fractional order derivatives in which parameters are present correspond exactly. That is, the nonlinear partial differential that occurs when modeling a physical event according to the fractional computation integer computation it helps to express the equation more clearly. Analytical and numerical solutions of fractional order nonlinear partial differential equations including Riemann-Liouville, Caputo and conformable fractional derivative approach, which are frequently encountered in the literature, were obtained using various methods. Some of these methods are (G '/ G) expansion [9], first integral [7], exponential function [10], jacobi elliptical [12], homotopy analysis [13], finite elements [5], finite difference [6], functional change [3], auxiliary equation [2] Alhakim and Moussa, Methods such as tangent hyperbolic [4], separation of variables [11] were used.

In this article, non-linear Boussinesq-Double-Sinh-Gordon which contains conformable fractional order derivatives and analytical solutions of First Boussinesq-Liouville equations with auxiliary equation method was obtained.

2. Metarial and Method

In 2003, the auxiliary equation method was first used by S. Jion and Sirendaoreji to obtain complete solutions of nonlinear partial differential equations [8]. S. Jion and Sirendaoreji to obtain the exact solutions of the partial differential equations discussed in this study with the auxiliary equation method

$$\left(\frac{dz}{d\xi}\right)^2 = az^2(\xi) + bz^3(\xi) + cz^4(\xi) \tag{1}$$

they benefited from the solutions of ordinary differential equations. Then, in 2008, M.A. in his study, Abdou [1] gave a wider class of the solutions of his equation, Schrödinger, the nonlinear partial differential equation, obtained analytical solutions of the Whitham – Broer – Kaup and generalized Zakharov equations [1]. The following steps are followed to solve the analytical solution of the fractional order partial differential equation by the auxiliary equation method [8].

I. Step: General form of a partial differential equation containing a conformable fractional order derivative according to the nonlinear time variable

$$P\left(\frac{\partial^p u}{\partial t^p}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0 \tag{2}$$

it can be written as. Here P is a nonlinear function, $p \in (0,1)$ ve $\frac{\partial^p u}{\partial t^p}$ derivative, means the p-order conformable fractional derivative of the function $u(x,t)$.

II. Step: (2) to show the wave velocity of w, $u(x,t) = U(\xi)$, $\xi = x + w\frac{t^p}{p}$

if conversion is used (2) partial differential equation

$$G(U, U', U'', U''', \dots) = 0 \tag{3}$$

is transformed into ordinary differential differential equation.

III. Step: (3) given the ordinary differential equation

$$U(\xi) = \sum_{i=0}^{n} a_i z^i(\xi) \tag{4}$$

search for analytical solution. Here $a_i (i = 0,1, \dots, n))$ are the coefficients to be determined later, while the function $z(\xi)$ is the solutions of the differential equation (1). The positive n value in the equation given by (4) is found with the help of homogeneous balance. For this, for the term derivative which is linear from the highest digit in the equation given by (3)

$$\mathcal{O}\left(\frac{d^r U}{d\xi^r}\right) = n + r, \ r = 0,1,2, \dots$$

and for the highest non-linear term

$$\mathcal{O}\left(u^{q}\frac{d^{r}U}{d\xi^{r}}\right) = qn + r, \qquad r = 0,1,2,\dots, \qquad q = 0,1,2,\dots$$

the formulas written are synchronized [11].

IV. Step: As a last step, by writing the equation (4)and the necessary derivatives in the equation (3), an equation containing the forces of the expression $z(\xi)$ is obtained. The resulting equation is arranged according to the forces of the expression $z(\xi)$ and then a coefficient system of algebraic equation is created by synchronizing the coefficients of the forces of the expression $z(\xi)$ to zero. This algebraic system of equations containing a, b, c, w, a_i coefficients is solved with the help of the Mathematica program and coefficients are found. Analytical solutions of fractional order partial differential equation are obtained by using the results obtained by solving this system and using the formulas given in Table 1.

No	$z(\xi)$	Condition
1	$\dfrac{-ab\,sech^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b^2 - ac\left(1-\varepsilon\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}$	$a > 0$
2	$\dfrac{ab\,csch^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b^2 - ac\left(1+\varepsilon\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}$	$a > 0$
3	$\dfrac{2ab\,\mathrm{sech}\left(\sqrt{a}\xi\right)}{\varepsilon\sqrt{\Delta} - b\,sech\left(\sqrt{a}\xi\right)}$	$a > 0, \Delta > 0$
4	$\dfrac{2ab\sec\left(\sqrt{-a}\xi\right)}{\varepsilon\sqrt{\Delta} - b\sec\left(\sqrt{-a}\xi\right)}$	$a < 0, \Delta > 0$
5	$\dfrac{2ab\,\mathrm{csch}\left(\sqrt{a}\xi\right)}{\varepsilon\sqrt{-\Delta} - b\,csch\left(\sqrt{a}\xi\right)}$	$a > 0, \Delta < 0$
6	$\dfrac{2ab\csc\left(\sqrt{-a}\xi\right)}{\varepsilon\sqrt{\Delta} - b\csc\left(\sqrt{-a}\xi\right))}$	$a < 0, \Delta > 0$
7	$\dfrac{-a\,sech^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{ac}\tanh\left(\frac{\sqrt{a}}{2}\xi\right)}$	$a > 0, c > 0$
8	$\dfrac{-a\sec^2\left(\frac{\sqrt{-a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{-ac}\tan\left(\frac{\sqrt{-a}}{2}\xi\right)}$	$a < 0, c > 0$
9	$\dfrac{-a\,csch^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{ac}\coth\left(\frac{\sqrt{a}}{2}\xi\right)}$	$a > 0, c > 0$

10	$\dfrac{-acsc^2\left(\frac{\sqrt{-a}}{2}\xi\right)}{b+2\varepsilon\sqrt{-ac}cot\left(\frac{\sqrt{-a}}{2}\xi\right)}$	$a<0, c>0$
11	$-\frac{a}{b}\left(1+\varepsilon tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)$	$a>0, \Delta=0$
12	$-\frac{a}{b}\left(1+\varepsilon coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)$	$a>0, \Delta=0$
13	$\dfrac{4ae^{\varepsilon\sqrt{a}\xi}}{\left(e^{\varepsilon\sqrt{a}\xi}-b\right)^2-4ac}$	$a>0$
14	$\dfrac{\pm 4ae^{\varepsilon\sqrt{a}\xi}}{1-4ace^{2\varepsilon\sqrt{a}\xi}}$	$a>0, b=0$

Table 1: Solutions of equation (1) with $\Delta = b^2 - 4ac$ and $\mathcal{E} = \pm 1$ [1]

3. Aplications of the Method

In this section, we will show the solution applications of fractional order partial differential equations using the auxiliary equation method.

Example 1: Fractional Order Boussinesq-Double-Sinh-Gordon Equation Boussinesq-Double-Sinh-Gordon equation with conformable fractional order derivative according to time variable

$$\frac{\partial^{2p}u}{\partial t^{2p}}-\alpha\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}=\sinh(u)+\frac{3}{2}\sinh(2u) \tag{5}$$

be considered as. Here, α is real constant and $p\in(0,1)$. (5) Conformable fractional order Boussinesq-Double-Sinh-Gordon equation

$$u(x,t)=u(\xi)$$

İncluding

$$\xi=x+w\frac{t^p}{p} \tag{6}$$

wave transformation is applied

$$(w^2-\alpha)u_{\xi\xi}+\ u_{\xi\xi\xi\xi}=\sinh(u)+\frac{3}{2}\sinh(2u) \tag{7}$$

ordinary differential system of differential equations is obtained. Here

$$v(\xi)=e^{u(\xi)} \tag{8}$$

is transformed and with the help of this transformation

$\sinh(u) = \dfrac{v - v^{-1}}{2}$
and
$\sinh(2u) = \dfrac{v^2 - v^{-2}}{2}$

by writing down the found equations in the equation given by (7)

$$-4w^2v^3v_{\xi\xi} + 4w^2v^2\left(v_\xi\right)^2 + 4\alpha v^3 v_{\xi\xi} - 4v_{\xi\xi\xi\xi}v^3 - 4\alpha v^2\left(v_\xi\right)^2 + 16v_{\xi\xi\xi}v_\xi v^2 + 24\left(v_\xi\right)^4$$
$$-48v_{\xi\xi}\left(v_\xi\right)^2 v + 12\left(v_{\xi\xi}\right)^2 v^2 + 3v^6 + 2v^5 - 2v^3 - 3v^2 = 0 \tag{9}$$

equation is obtained. The equilibrium between the terms $v_{\xi\xi\xi\xi}v^3$and the highest order nonlinear v^6 containing the highest order derivative in the equation given by (9)

$$n + 4 + 3n = 6n$$

equality is obtained. This equation has the value of $n = 2$. So in the equation (9)

$$v(\xi) = a_0 + a_1 z(\xi) + a_2 z^2(\xi) \tag{10}$$

In the form of an analytical solution is sought. If the equation (10) is substituted in the equation (9) ,an equation based on the forces of the expression $z(\xi)$ is obtained. By equating the coefficients of $z(\xi)$ and forces in this equation to zero;

$z^0(\xi)\colon -3a_0^2 - 2a_0^3 + 2a_0^5 + 3a_0^6 = 0$
$z^1(\xi)\colon -6a_0a_1 - 6a_0^2a_1 - 4a^2a_0^3a_1 + 10a_0^4a_1 + 18a_0^5a_1 - 4aa_0^3a_1w^2 + 4aa_0^3a_1\alpha = 0$
$z^2(\xi)\colon -3a_1^2 - 6a_0a_1^2 + 16a^2a_0^2a_1^2 + 20a_0^3a_1^2 + 45a_0^4a_1^2 - 6a_0a_2 - 6a_0^2a_2 - 64a^2a_0^3a_2 + 10a_0^4a_2 +$
$18a_0^5a_2 - 30aa_0^3a_1b - 8aa_0^2a_1^2w^2 - 16aa_0^3a_2w^2 - 6a_0^3a_1bw^2 + 8aa_0^2a_1^2\alpha + 16aa_0^3a_2\alpha +$
$6a_0^3a_1b\alpha = 0$
$z^3(\xi)\colon -2a_1^3 - 4a^2a_0a_1^3 + 20a_0^2a_1^3 + 60a_0^3a_1^3 - 6a_1a_2 - 12a_0a_1a_2 + 52a^2a_0^2a_1a_2 + 40a_0^3a_1a_2 +$
$90a_0^4a_1a_2 + 10aa_0^2a_1^2b - 260aa_0^3a_2b - 30a_0^3a_1b^2 - 80aa_0^3a_1c - 4aa_0a_1^3w^2 - 44aa_0^2a_1a_2w^2 -$
$14a_0^2a_1^2bw^2 - 20a_0^3a_2bw^2 - 8a_0^3a_1cw^2 + 4aa_0a_1^3\alpha + 44aa_0^2a_1a_2\alpha + 14a_0^2a_1^2b\alpha + 20a_0^3a_2b\alpha +$
$8a_0^3a_1c\alpha = 0$
$z^4(\xi)\colon 10a_0a_1^4 + 45a_0^2a_1^4 - 6a_1^2a_2 - 32a^2a_0a_1^2a_2 + 60a_0^2a_1^2a_2 + 180a_0^3a_1^2a_2 - 3a_2^2 - 6a_0a_2^2 +$
$256a^2a_0^2a_2^2 + 20a_0^3a_2^2 + 45a_0^4a_2^2 - 10aa_0a_1^3b - 110aa_0^2a_1a_2b - 15a_0^2a_1^2b^2 - 210a_0^3a_2b^2 -$
$80aa_0^2a_1^2c - 480aa_0^3a_2c - 120a_0^3a_1bc - 32aa_0a_1^2a_2w^2 - 32aa_0^2a_2^2w^2 - 10a_0a_1^3bw^2 -$
$62a_0^2a_1a_2bw^2 - 20a_0^2a_1^2cw^2 - 24a_0^3a_2cw^2 + 32aa_0a_1^2a_2\alpha + 32aa_0^2a_2^2\alpha + 10a_0a_1^3b\alpha +$
$62a_0^2a_1a_2b\alpha + 20a_0^2a_1^2c\alpha + 24a_0^3a_2c\alpha = 0$
$z^5(\xi)\colon 2a_1^5 + 18a_0a_1^5 - 4a^2a_1^3a_2 + 40a_0a_1^3a_2 + 180a_0^2a_1^3a_2 - 6a_1a_2^2 + 52a^2a_0a_1a_2^2 + 60a_0^2a_1a_2^2 +$
$180a_0^3a_1a_2^2 - 2aa_1^4b - 152aa_0a_1^2a_2b + 436aa_0^2a_2^2b - 12a_0a_1^3b^2 - 204a_0^2a_1a_2b^2 - 64aa_0a_1^3c -$
$608aa_0^2a_1a_2c - 144a_0^2a_1^2bc - 672a_0^3a_2bc - 96a_0^3a_1c^2 - 4aa_1^3a_2w^2 - 44aa_0a_1a_2^2w^2 - 2a_1^4bw^2 -$
$56a_0a_1^2a_2bw^2 - 44a_0^2a_2^2bw^2 - 16a_0a_1^3cw^2 - 80a_0^2a_1a_2cw^2 + 4aa_1^3a_2\alpha + 44aa_0a_1a_2^2\alpha + 2a_1^4b\alpha +$
$56a_0a_1^2a_2b\alpha + 44a_0^2a_2^2b\alpha + 16a_0a_1^3c\alpha + 80a_0^2a_1a_2c\alpha = 0$
$z^6(\xi)\colon 3a_1^6 + 10a_1^4a_2 + 90a_0a_1^4a_2 + 16a^2a_1^2a_2^2 + 60a_0a_1^2a_2^2 + 270a_0^2a_1^2a_2^2 - 2a_2^3 - 64a^2a_0a_2^3 +$
$20a_0^2a_2^3 + 60a_0^3a_2^3 - 38aa_1^3a_2b + 94aa_0a_1a_2^2b - 3a_1^4b^2 - 156a_0a_1^2a_2b^2 + 150a_0^2a_2^2b^2 - 16aa_1^4c -$

$512a a_0 a_1^2 a_2 c + 160a a_0^2 a_2^2 c - 96a_0 a_1^3 bc - 1008a_0^2 a_1 a_2 bc - 144a_0^2 a_1^2 c^2 - 480a_0^3 a_2 c^2 - 8a a_1^2 a_2^2 w^2 - 16a a_0 a_2^3 w^2 - 14a_1^3 a_2 bw^2 - 74a_0 a_1 a_2^2 bw^2 - 4a_1^4 cw^2 - 80a_0 a_1^2 a_2 cw^2 - 56a_0^2 a_2^2 cw^2 + 8a a_1^2 a_2^2 \alpha + 16a a_0 a_2^3 \alpha + 14a_1^3 a_2 b\alpha + 74a_0 a_1 a_2^2 b\alpha + 4a_1^4 c\alpha + 80a_0 a_1^2 a_2 c\alpha + 56a_0^2 a_2^2 c\alpha = 0$

$z^7(\xi)$: $18a_1^5 a_2 + 20a_1^3 a_2^2 + 180a_0 a_1^3 a_2^2 - 4a^2 a_1 a_2^3 + 40a_0 a_1 a_2^3 + 180a_0^2 a_1 a_2^3 - 2a a_1^2 a_2^2 b - 76a a_0 a_2^3 b - 42a_1^3 a_2 b^2 - 6a_0 a_1 a_2^2 b^2 - 144a a_1^3 a_2 c - 272a a_0 a_1 a_2^2 c - 24a_1^4 bc - 768a_0 a_1^2 a_2 bc - 48a_0^2 a_2^2 bc - 96a_0 a_1^3 c^2 - 864a_0^2 a_1 a_2 c^2 - 4a a_1 a_2^3 w^2 - 26a_1^2 a_2^2 bw^2 - 28a_0 a_2^3 bw^2 - 24a_1^3 a_2 cw^2 - 104a_0 a_1 a_2^2 cw^2 + 4a a_1 a_2^3 \alpha + 26a_1^2 a_2^2 b\alpha + 28a_0 a_2^3 b\alpha + 24a_1^3 a_2 c\alpha + 104a_0 a_1 a_2^2 c\alpha = 0$

$z^8(\xi)$: $45a_1^4 a_2^2 + 20a_1^2 a_2^3 + 180a_0 a_1^2 a_2^3 + 10a_0 a_2^4 + 45a_0^2 a_2^4 - 18a a_1 a_2^3 b - 33a_1^2 a_2^2 b^2 - 30a_0 a_2^3 b^2 - 176a a_1^2 a_2^2 c - 160a a_0 a_2^3 c - 216a_1^3 a_2 bc - 504a_0 a_1 a_2^2 bc - 24a_1^4 c^2 - 672a_0 a_1^2 a_2 c^2 - 240a_0^2 a_2^2 c^2 - 18a_1 a_2^3 bw^2 - 44a_1^2 a_2^2 cw^2 - 40a_0 a_2^3 cw^2 + 18a_1 a_2^3 b\alpha + 44a_1^2 a_2^2 c\alpha + 40a_0 a_2^3 c\alpha = 0$

$z^9(\xi)$: $60a_1^3 a_2^3 + 10a_1 a_2^4 + 90a_0 a_1 a_2^4 - 4a a_2^4 b - 24a_1 a_2^3 b^2 - 128a a_1 a_2^3 c - 288a_1^2 a_2^2 bc - 192a_0 a_2^3 bc - 192a_1^3 a_2 c^2 - 576a_0 a_1 a_2^2 c^2 - 4a_2^4 bw^2 - 32a_1 a_2^3 cw^2 + 4a_2^4 b\alpha + 32a_1 a_2^3 c\alpha = 0$

$z^{10}(\xi)$: $45a_1^2 a_2^4 + 2a_2^5 + 18a_0 a_2^5 - 6a_2^4 b^2 - 32a a_2^4 c - 192a_1 a_2^3 bc - 288a_1^2 a_2^2 c^2 - 192a_0 a_2^3 c^2 - 8a_2^4 cw^2 + 8a_2^4 c\alpha = 0$

$z^{11}(\xi)$: $18a_1 a_2^5 - 48a_2^4 bc - 192a_1 a_2^3 c^2 = 0$

$z^{12}(\xi)$: $3a_2^6 - 48a_2^4 c^2 = 0$

algebraic equation system is obtained. By solving this system of equations with the help of Mathematica program

$$a = 1, \quad b = 2\sqrt{c}, \quad w = -\sqrt{1+\alpha}, \quad a_0 = -1, \quad a_1 = -4\sqrt{c}, \quad a_2 = -4c \tag{11}$$
$$a = 1, \quad b = 2\sqrt{c}, \quad w = -\sqrt{3+\alpha}, \quad a_0 = 1, \quad a_1 = 4\sqrt{c}, \quad a_2 = 4c \tag{12}$$
$$a = \frac{1}{4}, \quad b = 0, \quad w = \sqrt{3+\alpha}, \quad a_0 = 1, \quad a_1 = 0, \quad a_2 = 4c \tag{13}$$
$$a = \frac{1}{4}, \quad b = 0, \quad w = \sqrt{1+\alpha}, \quad a_0 = -1, \quad a_1 = 0, \quad a_2 = -4c \tag{14}$$

solution sets are obtained.

The values given by (11) are substituted in the equations (6) and (10) ,and the solutions given in Table 1 and (8) using the transformation are given by the (5) conformable fractional order Boussinesq-double-sinh-Gordon Equation $u(x,t)$ analytical solutions

$$u_{1,2}(x,t) = \log\left(1 - \frac{8\,\mathrm{sech}^2\left(\frac{1}{2}\xi\right)}{4 - \left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right)^2} + \frac{16\,\mathrm{sech}^4\left(\frac{1}{2}\xi\right)}{\left(4 - \left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right)^2\right)^2}\right)$$

$$u_{3,4}(x,t) = \log\left(1 + \frac{8\,\mathrm{csch}^2\left(\frac{1}{2}\xi\right)}{4 - \left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right)^2} + \frac{16\,\mathrm{csch}^4\left(\frac{1}{2}\xi\right)}{\left(4 - \left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right)^2\right)^2}\right)$$

$$u_{5,6}(\mathrm{x},\mathrm{t}) = \log\left(1 - \frac{2\,\mathrm{sech}^2\left(\frac{1}{2}\xi\right)}{1 \pm \tanh\left(\frac{1}{2}\xi\right)} + \frac{\mathrm{sech}^4\left(\frac{1}{2}\xi\right)}{\left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right)^2}\right)$$

$$u_{7,8}(\mathrm{x},\mathrm{t}) = \log\left(1 + \frac{2\,\mathrm{csch}^2\left(\frac{1}{2}\xi\right)}{1 \pm \coth\left(\frac{1}{2}\xi\right)} + \frac{\mathrm{csch}^4\left(\frac{1}{2}\xi\right)}{\left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right)^2}\right)$$

$$u_{9,10}(\mathrm{x},\mathrm{t}) = \log\left(1 - 2\left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right) + \left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right)^2\right)$$

$$u_{11,12}(\mathrm{x},\mathrm{t}) = \log\left(1 - 2\left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right) + \left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right)^2\right)$$

$$u_{13,14}(\mathrm{x},\mathrm{t}) = \log\left(1 + \frac{16\sqrt{c}e^{\pm\xi}}{-4c + \left(-2\sqrt{c} + e^{\pm\xi}\right)^2} + \frac{64ce^{\pm 2\xi}}{\left(-4c + \left(-2\sqrt{c} + e^{\pm\xi}\right)^2\right)^2}\right)$$

it is found as. Here $\xi = x - \sqrt{3+\alpha}\frac{t^p}{p}$.

The values given by (12) are substituted in the equations (6) and (10) ,and the solutions given in Table 1 and (8) using the transformation are given by the (5) conformable fractional order Boussinesq-double-sinh-Gordon Equation $u(x,t)$ analytical solutions

$$u_{15,16}(\mathrm{x},\mathrm{t}) = \log\left(-1 + \frac{8\,\mathrm{sech}^2\left(\frac{1}{2}\xi\right)}{4 - \left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right)^2} - \frac{16\,\mathrm{sech}^4\left(\frac{1}{2}\xi\right)}{\left(4 - \left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right)^2\right)^2}\right)$$

$$u_{17,18}(\mathrm{x},\mathrm{t}) = \log\left(-1 - \frac{8\,\mathrm{csch}^2\left(\frac{1}{2}\xi\right)}{4 - \left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right)^2} - \frac{16\,\mathrm{csch}^4\left(\frac{1}{2}\xi\right)}{\left(4 - \left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right)^2\right)^2}\right)$$

$$u_{19,20}(\mathrm{x},\mathrm{t}) = \log\left(-1 - \frac{2\,\mathrm{sech}^2\left(\frac{1}{2}\xi\right)}{1 \pm \tanh\left(\frac{1}{2}\xi\right)} + \frac{\mathrm{sech}^4\left(\frac{1}{2}\xi\right)}{\left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right)^2}\right)$$

$$u_{21,22}(\mathrm{x},\mathrm{t}) = \log\left(-1 - \frac{2\,\mathrm{csch}^2\left(\frac{1}{2}\xi\right)}{1 \pm \coth\left(\frac{1}{2}\xi\right)} - \frac{\mathrm{csch}^4\left(\frac{1}{2}\xi\right)}{\left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right)^2}\right)$$

$$u_{23,24}(\mathrm{x},\mathrm{t}) = \log\left(-1 + 2\left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right) - \left(1 \pm \tanh\left(\frac{1}{2}\xi\right)\right)^2\right)$$

$$u_{25,26}(\mathrm{x,t}) = \log\left(-1 + 2\left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right) - \left(1 \pm \coth\left(\frac{1}{2}\xi\right)\right)^2\right)$$

$$u_{27,28}(\mathrm{x,t}) = \log\left(-1 - \frac{16\sqrt{c}e^{\pm\xi}}{-4c + \left(-2\sqrt{c} + e^{\pm\xi}\right)^2} - \frac{64ce^{\pm 2\xi}}{\left(-4c + \left(-2\sqrt{c} + e^{\pm\xi}\right)^2\right)^2}\right)$$

it is found as. Here $\xi = x - \sqrt{1+\alpha}\frac{t^p}{p}$.

The values given by (13) are substituted in the equations (6) and (10) ,and the solutions given in Table 1 and (8) using the transformation are given by the (5) conformable fractional order Boussinesq-double-sinh-Gordon Equation $u(x,t)$ analytical solutions

$$u_{29}(\mathrm{x,t}) = \log\left(1 - \mathrm{sech}^2\left(\frac{1}{2}\xi\right)\right)$$

$$u_{30}(\mathrm{x,t}) = \log\left(1 + \mathrm{csch}^2\left(\frac{1}{2}\xi\right)\right)$$

$$u_{31}(\mathrm{x,t}) = \log\left(1 + \frac{1}{4}\mathrm{csch}^2\left(\frac{1}{4}\xi\right)\mathrm{sech}^2\left(\frac{1}{4}\xi\right)\right)$$

$$u_{32,33}(\mathrm{x,t}) = \log\left(1 + \frac{4ce^{\pm\xi}}{(-c + e^{\pm\xi})^2}\right)$$

$$u_{34,35}(\mathrm{x,t}) = \log\left(1 + \frac{4ce^{\pm\xi}}{(1 - ce^{\pm\xi})^2}\right)$$

it is found as. Here $\xi = x + \sqrt{3+\alpha}\frac{t^p}{p}$.

The values given by (14) are substituted in the equations (6) and (10), and the solutions given in Table 1 and (8) using the transformation are given by the (5) conformable fractional order Boussinesq-double-sinh-Gordon Equation $u(x,t)$ analytical solutions

$$u_{36}(\mathrm{x,t}) = \log\left(-1 - \mathrm{sech}^2\left(\frac{1}{2}\xi\right)\right)$$

$$u_{37}(\mathrm{x,t}) = \log\left(-1 + \mathrm{csch}^2\left(\frac{1}{2}\xi\right)\right)$$

$$u_{38}(\mathrm{x,t}) = \log\left(-1 + \frac{1}{4}\mathrm{csch}^2\left(\frac{1}{4}\xi\right)\mathrm{sech}^2\left(\frac{1}{4}\xi\right)\right)$$

$$u_{39,40}(\mathrm{x,t}) = \log\left(-1 + \frac{4ce^{\pm\xi}}{\left(-c+e^{\pm\xi}\right)^2}\right)$$

$$u_{41,42}(\mathrm{x,t}) = \log\left(-1 + \frac{4ce^{\pm\xi}}{(1 - ce^{\pm\xi})^2}\right)$$

it is found as. Here $\xi = x + \sqrt{1+\alpha}\frac{t^p}{p}$.

The surfaces of some analytical solutions of the conformable fractional order equation given in the following figures 1-3 are given.

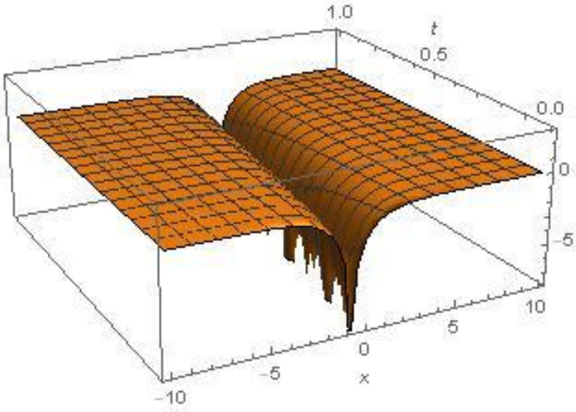

Figure 1: The surface of the solution $u_{15}(x,t)$ at $\alpha = 1, p = 0.75$

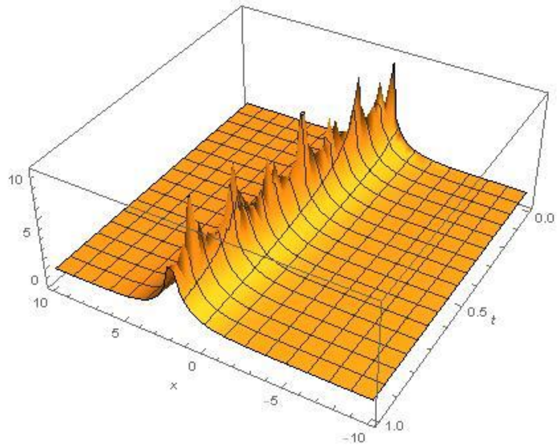

Figure 2: The surface of the solution $u_{17}(x,t)$ at $\alpha = 1, p = 0.75$

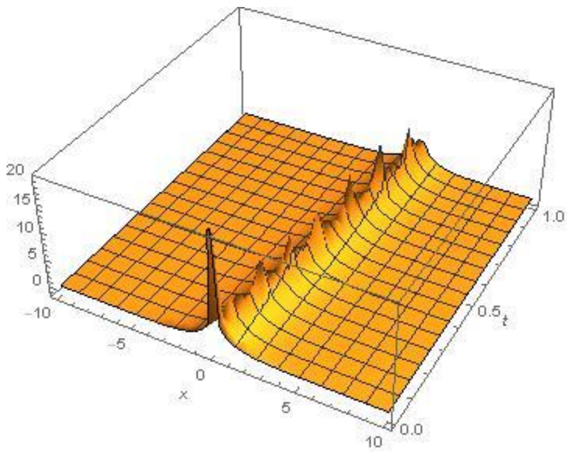

Figure 3: The surface of the solution $u_{21}(x,t)$ at $\alpha = 1, p = 0.75$

Example 2: *Fractional Order First Boussinesq-Liouville Equation*

Fractional Order First Boussinesq-Liouville Equation;

$$\frac{\partial^{2p}u}{\partial t^{2p}}-\alpha\frac{\partial^{2}u}{\partial x^{2}}+\frac{\partial^{4}u}{\partial x^{4}}=e^{u}+\frac{3}{4}e^{2u} \tag{15}$$

be considered as. Here, α is real constant and $p\in(0,1)$. Conformable fractional order given by (15) to the First Boussinesq-Lioville equation

$$u(x,t)=u(\xi)$$

İncluding

$$\xi=x+w\frac{t^{p}}{p} \tag{16}$$

wave transformation is applied;

$$(w^{2}-\alpha)u_{\xi\xi}+\ u_{\xi\xi\xi\xi}=e^{u}+\frac{3}{4}e^{2u} \tag{17}$$

ordinary differential differential equation is obtained. In the equation (17)

$$v(\xi)=e^{u(\xi)}$$

if transform is applied;

$$-4w^{2}v^{3}v''+4w^{2}v^{2}(v')^{2}+4\alpha v^{3}v''-4v^{(4)}v^{3}-4\alpha v^{2}(v')^{2}+16v'''v'v^{2}-48v''(v')^{2}v$$
$$+24(v')^{4}+12(v'')^{2}v^{2}+3v^{6}+4v^{5}=0 \tag{18}$$

differential equation is obtained. The value of $v^{(4)}v^{3}$, which contains the highest order derivative in the equation given by (18), and $n=2$ value from the homogeneous balance between the highest order nonlinear term and v^{6}. Thus, in the equation (18),

$$u(\xi)=\mathrm{a}_0+\mathrm{a}_1 z(\xi)+\ \mathrm{a}_2 z^{2}(\xi) \tag{19}$$

In the form of a full solution is sought. If the equation (19) and the necessary derivatives are substituted in the equation (18), an equation based on the forces of the expression $z(\xi)$ is obtained.
With this equation the coefficients of $z(\xi)$ and their forces equal to zero.

$z^{0}(\xi)$: $4\mathrm{a}_0^{5}+3\mathrm{a}_0^{6}=0$

$z^{1}(\xi)$: $-4a^{2}\mathrm{a}_0^{3}\mathrm{a}_1+20\mathrm{a}_0^{4}\mathrm{a}_1+18\mathrm{a}_0^{5}\mathrm{a}_1-4a\mathrm{a}_0^{3}\mathrm{a}_1 w^{2}+4a\mathrm{a}_0^{3}\mathrm{a}_1\alpha=0$

$z^{2}(\xi)$: $16a^{2}\mathrm{a}_0^{2}\mathrm{a}_1^{2}+40\mathrm{a}_0^{3}\mathrm{a}_1^{2}+45\mathrm{a}_0^{4}\mathrm{a}_1^{2}-64a^{2}\mathrm{a}_0^{3}\mathrm{a}_2+20\mathrm{a}_0^{4}\mathrm{a}_2+18\mathrm{a}_0^{5}\mathrm{a}_2-30a\mathrm{a}_0^{3}\mathrm{a}_1 b-8a\mathrm{a}_0^{2}\mathrm{a}_1^{2}w^{2}-16a\mathrm{a}_0^{3}\mathrm{a}_2 w^{2}-6\mathrm{a}_0^{3}\mathrm{a}_1 bw^{2}+8a\mathrm{a}_0^{2}\mathrm{a}_1^{2}\alpha+16a\mathrm{a}_0^{3}\mathrm{a}_2\alpha+6\mathrm{a}_0^{3}\mathrm{a}_1 b\alpha=0$

$z^{3}(\xi)$: $-4a^{2}\mathrm{a}_0\mathrm{a}_1^{3}+40\mathrm{a}_0^{2}\mathrm{a}_1^{3}+60\mathrm{a}_0^{3}\mathrm{a}_1^{3}+52a^{2}\mathrm{a}_0^{2}\mathrm{a}_1\mathrm{a}_2+80\mathrm{a}_0^{3}\mathrm{a}_1\mathrm{a}_2+90\mathrm{a}_0^{4}\mathrm{a}_1\mathrm{a}_2+10a\mathrm{a}_0^{2}\mathrm{a}_1^{2}b-260a\mathrm{a}_0^{3}\mathrm{a}_2 b-30\mathrm{a}_0^{3}\mathrm{a}_1 b^{2}-80a\mathrm{a}_0^{3}\mathrm{a}_1 c-4a\mathrm{a}_0\mathrm{a}_1^{3}w^{2}-44a\mathrm{a}_0^{2}\mathrm{a}_1\mathrm{a}_2 w^{2}-14\mathrm{a}_0^{2}\mathrm{a}_1^{2}bw^{2}-20\mathrm{a}_0^{3}\mathrm{a}_2 bw^{2}-8\mathrm{a}_0^{3}\mathrm{a}_1 cw^{2}+4a\mathrm{a}_0\mathrm{a}_1^{3}\alpha+44a\mathrm{a}_0^{2}\mathrm{a}_1\mathrm{a}_2\alpha+14\mathrm{a}_0^{2}\mathrm{a}_1^{2}b\alpha+20\mathrm{a}_0^{3}\mathrm{a}_2 b\alpha+8\mathrm{a}_0^{3}\mathrm{a}_1 c\alpha=0$

$z^4(\xi)$: $20a_0a_1^4 + 45a_0^2a_1^4 - 32a^2a_0a_1^2a_2 + 120a_0^2a_1^2a_2 + 180a_0^3a_1^2a_2 + 256a^2a_0^2a_2^2 + 40a_0^3a_2^2 + 45a_0^4a_2^2 - 10aa_0a_1^3b - 110aa_0^2a_1a_2b - 15a_0^2a_1^2b^2 - 210a_0^3a_2b^2 - 80aa_0^2a_1^2c - 480aa_0^3a_2c - 120a_0^3a_1bc - 32aa_0a_1^2a_2w^2 - 32aa_0^2a_2^2w^2 - 10a_0a_1^3bw^2 - 62a_0^2a_1a_2bw^2 - 20a_0^2a_1^2cw^2 - 24a_0^3a_2cw^2 + 32aa_0a_1^2a_2\alpha + 32aa_0^2a_2^2\alpha + 10a_0a_1^3b\alpha + 62a_0^2a_1a_2b\alpha + 20a_0^2a_1^2c\alpha + 24a_0^3a_2c\alpha = 0$

$z^5(\xi)$: $4a_1^5 + 18a_0a_1^5 - 4a^2a_1^3a_2 + 80a_0a_1^3a_2 + 180a_0^2a_1^3a_2 + 52a^2a_0a_1a_2^2 + 120a_0^2a_1a_2^2 + 180a_0^3a_1a_2^2 - 2aa_1^4b - 152aa_0a_1^2a_2b + 436aa_0^2a_2^2b - 12a_0a_1^3b^2 - 204a_0^2a_1a_2b^2 - 64aa_0a_1^3c - 608aa_0^2a_1a_2c - 144a_0^2a_1^2bc - 672a_0^3a_2bc - 96a_0^3a_1c^2 - 4aa_1^3a_2w^2 - 44aa_0a_1a_2^2w^2 - 2a_1^4bw^2 - 56a_0a_1^2a_2bw^2 - 44a_0^2a_2^2bw^2 - 16a_0a_1^3cw^2 - 80a_0^2a_1a_2cw^2 + 4aa_1^3a_2\alpha + 44aa_0a_1a_2^2\alpha + 2a_1^4b\alpha + 56a_0a_1^2a_2b\alpha + 44a_0^2a_2^2b\alpha + 16a_0a_1^3c\alpha + 80a_0^2a_1a_2c\alpha = 0$

$z^6(\xi)$: $3a_1^6 + 20a_1^4a_2 + 90a_0a_1^4a_2 + 16a^2a_1^2a_2^2 + 120a_0a_1^2a_2^2 + 270a_0^2a_1^2a_2^2 - 64a^2a_0a_2^3 + 40a_0^2a_2^3 + 60a_0^3a_2^3 - 38aa_1^3a_2b + 94aa_0a_1a_2^2b - 3a_1^4b^2 - 156a_0a_1^2a_2b^2 + 150a_0^2a_2^2b^2 - 16aa_1^4c - 512aa_0a_1^2a_2c + 160aa_0^2a_2^2c - 96a_0a_1^3bc - 1008a_0^2a_1a_2bc - 144a_0^2a_1^2c^2 - 480a_0^3a_2c^2 - 8aa_1^2a_2^2w^2 - 16aa_0a_2^3w^2 - 14a_1^3a_2bw^2 - 74a_0a_1a_2^2bw^2 - 4a_1^4cw^2 - 80a_0a_1^2a_2cw^2 - 56a_0^2a_2^2cw^2 + 8aa_1^2a_2^2\alpha + 16aa_0a_2^3\alpha + 14a_1^3a_2b\alpha + 74a_0a_1a_2^2b\alpha + 4a_1^4c\alpha + 80a_0a_1^2a_2c\alpha + 56a_0^2a_2^2c\alpha = 0$

$z^7(\xi)$: $18a_1^5a_2 + 40a_1^3a_2^2 + 180a_0a_1^3a_2^2 - 4a^2a_1a_2^3 + 80a_0a_1a_2^3 + 180a_0^2a_1a_2^3 - 2aa_1^2a_2^2b - 76aa_0a_2^3b - 42a_1^3a_2b^2 - 6a_0a_1a_2^2b^2 - 144aa_1^3a_2c - 272aa_0a_1a_2^2c - 24a_1^4bc - 768a_0a_1^2a_2bc - 48a_0^2a_2^2bc - 96a_0a_1^3c^2 - 864a_0^2a_1a_2c^2 - 4aa_1a_2^3w^2 - 26a_1^2a_2^2bw^2 - 28a_0a_2^3bw^2 - 24a_1^3a_2cw^2 - 104a_0a_1a_2^2cw^2 + 4aa_1a_2^3\alpha + 26a_1^2a_2^2b\alpha + 28a_0a_2^3b\alpha + 24a_1^3a_2c\alpha + 104a_0a_1a_2^2c\alpha = 0$

$z^8(\xi)$: $45a_1^4a_2^2 + 40a_1^2a_2^3 + 180a_0a_1^2a_2^3 + 20a_0a_2^4 + 45a_0^2a_2^4 - 18aa_1a_2^3b - 33a_1^2a_2^2b^2 - 30a_0a_2^3b^2 - 176aa_1^2a_2^2c - 160aa_0a_2^3c - 216a_1^3a_2bc - 504a_0a_1a_2^2bc - 24a_1^4c^2 - 672a_0a_1^2a_2c^2 - 240a_0^2a_2^2c^2 - 18a_1a_2^3bw^2 - 44a_1^2a_2^2cw^2 - 40a_0a_2^3cw^2 + 18a_1a_2^3b\alpha + 44a_1^2a_2^2c\alpha + 40a_0a_2^3c\alpha = 0$

$z^9(\xi)$: $60a_1^3a_2^3 + 20a_1a_2^4 + 90a_0a_1a_2^4 - 4aa_2^4b - 24a_1a_2^3b^2 - 128aa_1a_2^3c - 288a_1^2a_2^2bc - 192a_0a_2^3bc - 192a_1^3a_2c^2 - 576a_0a_1a_2^2c^2 - 4a_2^4bw^2 - 32a_1a_2^3cw^2 + 4a_2^4b\alpha + 32a_1a_2^3c\alpha = 0$

$z^{10}(\xi)$: $45a_1^2a_2^4 + 4a_2^5 + 18a_0a_2^5 - 6a_2^4b^2 - 32aa_2^4c - 192a_1a_2^3bc - 288a_1^2a_2^2c^2 - 192a_0a_2^3c^2 - 8a_2^4cw^2 + 8a_2^4c\alpha = 0$

$z^{11}(\xi)$: $18a_1a_2^5 - 48a_2^4bc - 192a_1a_2^3c^2 = 0$

$z^{12}(\xi)$: $3a_2^6 - 48a_2^4c^2 = 0$

algebraic equation system is obtained. By solving this system of equations with the help of Mathematica program

$$a = 2 - w^2 + \alpha, b = \frac{a_1}{2}, c = -\frac{a_1^2}{16(w^2 - 2 - \alpha)},\ a_0 = 0,\ a_2 = -\frac{a_1^2}{4(w^2 - 2 - \alpha)} \qquad (20)$$

$$a = \frac{1}{4}(2 - w^2 + \alpha), b = 0, c = \frac{a_2}{4}, a_0 = 0, a_1 = 0 \qquad (21)$$

solutions are obtained. $u(x,t)$ analytical solutions of the conformable fractional order first Boussinesq-Liouville equation given by using the solutions given in Table 1 with the values given in (20) in the equations (16) and (19).

$$u_{1,2}(\mathrm{x},\mathrm{t}) = \log\left(-\frac{8a\,\mathrm{sech}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{4-\left(1\pm+\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}+\frac{16a\,\mathrm{sech}^4\left(\frac{\sqrt{a}}{2}\xi\right)}{\left(4-\left(1\pm\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2\right)^2}\right), a>0$$

$$u_{3,4}(\mathrm{x},\mathrm{t}) = \log\left(\frac{8a\,\mathrm{csch}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{4-\left(1\pm\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}-\frac{16a\,\mathrm{csch}^4\left(\frac{\sqrt{a}}{2}\xi\right)}{\left(4-\left(1\pm\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2\right)^2}\right), a>0$$

$$u_{5,6}(\mathrm{x},\mathrm{t}) = \log\left(-\frac{2a\,\mathrm{sech}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{1\pm\tanh\left(\frac{\sqrt{a}}{2}\xi\right)}+\frac{a\,\mathrm{sech}^4\left(\frac{\sqrt{a}}{2}\xi\right)}{\left(1\pm\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}\right), a>0$$

$$u_{7,8}(\mathrm{x},\mathrm{t}) = \log\left(\frac{2a\,\mathrm{csch}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{1\pm\coth\left(\frac{\sqrt{a}}{2}\xi\right)}+\frac{a\,\mathrm{csch}^4\left(\frac{\sqrt{a}}{2}\xi\right)}{\left(1\pm\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}\right), a>0$$

$$u_{9,10}(\mathrm{x},\mathrm{t}) = \log\left(-2a\left(1\pm\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)+a\left(1\pm\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2\right), a>0$$

$$u_{11,12}(\mathrm{x},\mathrm{t}) = \log\left(-2a\left(1\pm\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)+a\left(1\pm\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2\right), a>0$$

$$u_{13,14}(\mathrm{x},\mathrm{t}) = \log\left(\frac{4a\mathrm{a}_1e^{\pm\sqrt{a}\xi}}{\left(-\frac{\mathrm{a}_1}{2}+e^{\pm\sqrt{a}\xi}\right)^2-\frac{\mathrm{a}_1{}^2}{4}}+\frac{4a\mathrm{a}_1{}^2e^{\pm2\sqrt{a}\xi}}{\left(\left(-\frac{\mathrm{a}_1}{2}+e^{\pm\sqrt{a}\xi}\right)^2-\frac{\mathrm{a}_1{}^2}{4}\right)^2}\right), a>0$$

it is found as. Here $\xi = x + w\frac{t^p}{p}$ ve $\mathrm{a} = 2 - w^2 + \alpha$.

$u(x,t)$ analytical solutions of the conformable fractional order first Boussinesq-Liouville equation given by using the solutions given in Table 1 with the values given in (21) in the equations (16) and (19).

$$u_{15,16}(\mathrm{x},\mathrm{t}) = \log\left(\pm\mu\,\mathrm{sech}^2\left(\frac{\sqrt{-\mu}}{2}\xi\right)\right), \mu<0$$

$$u_{17,18}(\mathrm{x},\mathrm{t}) = \log\left(\pm\mu \sec^2\left(\frac{\sqrt{\mu}}{2}\xi\right)\right), \mu > 0$$

$$u_{19,20}(\mathrm{x},\mathrm{t}) = \log\left(\pm\mu \operatorname{csch}^2\left(\frac{\sqrt{-\mu}}{2}\xi\right)\right), \mu < 0$$

$$u_{21,22}(\mathrm{x},\mathrm{t}) = \log\left(\pm\mu \csc^2\left(\frac{\sqrt{\mu}}{2}\xi\right)\right), \mu > 0$$

$$u_{23}(\mathrm{x},\mathrm{t}) = \log\left(-\frac{1}{4}\mu \operatorname{csch}^2\left(\frac{\sqrt{-\mu}}{4}\xi\right)\operatorname{sech}^2\left(\frac{\sqrt{-\mu}}{4}\xi\right)\right), \mu < 0$$

$$u_{24}(\mathrm{x},\mathrm{t}) = \log\left(\frac{1}{4}\mu \csc^2\left(\frac{\sqrt{\mu}}{4}\xi\right)\operatorname{sech}^2\left(\frac{\sqrt{\mu}}{4}\xi\right)\right), \mu > 0$$

$$u_{25,26}(\mathrm{x},\mathrm{t}) = \log\left(\frac{\mathrm{a}_2\mu^2 e^{\pm\sqrt{-\mu}\xi}}{\left(e^{\pm\sqrt{-\mu}\xi} + \frac{\mathrm{a}_2\mu}{4}\right)^2}\right), \mu < 0$$

$$u_{27,28}(\mathrm{x},\mathrm{t}) = \log\left(\frac{\mathrm{a}_2\mu^2 e^{\pm\sqrt{-\mu}\xi}}{\left(1 + \frac{\mathrm{a}_2\mu}{4} e^{\pm\sqrt{-\mu}\xi}\right)^2}\right), \mu < 0$$

it is found as. Here $\xi = x + w\frac{t^p}{p}$ ve $\mu = w^2 - 2 - \alpha$.

The surfaces of some analytical solutions of the conformable fractional order equation given in the following figures 4-6 are given.

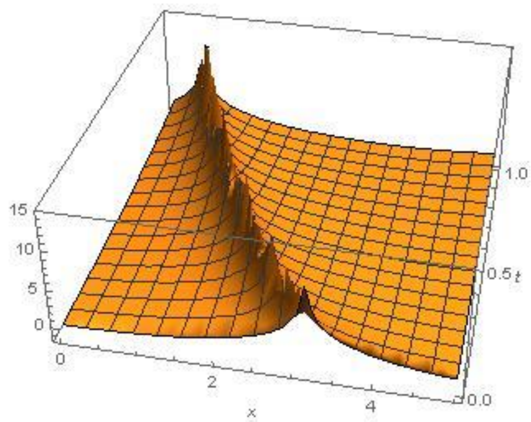

Figure 4: The surface of the solution $u_{17}(x,t)$ at $w = 2, \alpha = 1, p = 0.75$

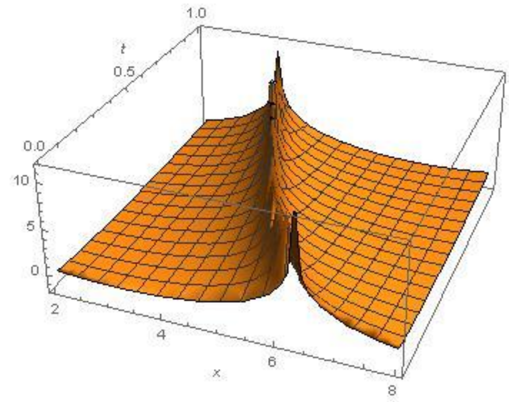

Figure 5: The surface of the solution $u_{19}(x,t)$ at $w=2, \alpha=3, p=0.75$

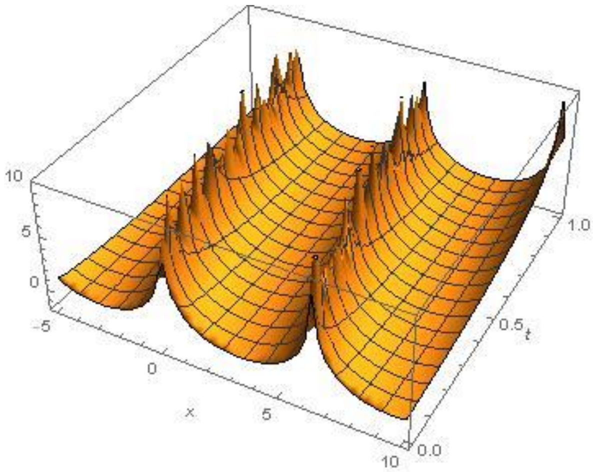

Figure 6: The surface of the solution $u_{21}(x,t)$ at $w=2, \alpha=1, p=0.75$

4. Results and Discussions

In this article, the Boussinesq-Double-Sinh-Gordon and First Boussinesq-Liouville equations, which are non-linear fractional order partial differential equations containing conformable fractional derivatives based on time, are discussed. Both equations were first converted to non-linear ordinary derivative differential equations with the help of wave transformation. auxiliary equation method was used to find analytical solutions of these ordinary derivative equations. For this, $z(\xi)$ consists of the forces of the expression

$$\sum_{i=0}^{n} a_i z^i(\xi)$$

searched for analytical solution in form. Substituting this analytical solution form into ordinary differential equations, an equation containing the powers of the expression $z(\xi)$ was found. The solutions of the algebraic equation system obtained by equating the coefficients of the forces of the $z(\xi)$ expression in this equation to zero were found with the help of the Mathematica program. Analytical solutions of the equations dealt with with the help of these values were found. As a result; it was seen by using Mathematica program that all analytical solutions obtained.

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