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Research Article

Ostrowski's Type Inequalities for the Complex Integral on Paths

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ABSTRACT. In this paper, we extend the Ostrowski inequality to the integral with respect to arc-length by providing upper bounds for the quantity

$$\left|f(v)\,\ell(\gamma)-\int_{\gamma}f(z)\,|dz|\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with the length $\ell(\gamma)$, $u=z(a)$, $v=z(x)$ with $x\in(a,b)$ and $w=z(b)$ while f is holomorphic in G, an open domain and $\gamma\subset G$. An application for circular paths is also given.

Keywords: Complex integral, continuous functions, holomorphic functions, Ostrowski inequality.

2020 Mathematics Subject Classification: 26D15, 26D10, 30A10, 30A86.

1. INTRODUCTION

In 1938, A. Ostrowski [8], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x\in[a,b]$.

Theorem 1.1 (Ostrowski, 1938 [8]). *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty(b-a) \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

In [6], S. S. Dragomir and S. Wang, by the use of the *Montgomery integral identity* [7, p. 565],

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt\,,\quad x\in[a,b]\,, \tag{1.2}$$

where $p:[a,b]^2\to\mathbb{R}$ is given by

$$p(x,t):=\begin{cases} t-a & \text{if}\quad t\in[a,x]\,,\\ t-b & \text{if}\quad t\in(x,b],\end{cases}$$

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gave a simple proof of Ostrowski's inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating the Riemann integral $\int_a^b f(t)\,dt$ by one arbitrary Riemann sum (see [6, Section 3]). For extensions of Ostrowski's inequality in terms of the p-norms of the derivative, see [1], [2] and [3]. For a recent survey on Ostrowski's inequality, see [4].

In order to extend this result for the complex integral, we need some preparations as follows. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that the actual choice of parametrization of γ does not matter. This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals. We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then, we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.3}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma), \tag{1.4}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$, we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality, we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

In the recent paper [5], we obtained the following result for functions of complex variable:

Theorem 1.2. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\begin{aligned} &\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\ &\le \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\,|dz| \\ &\le \left[\int_{\gamma_{u,v}} |z-u|\,|dz| + \int_{\gamma_{v,w}} |z-w|\,|dz| \right] \|f'\|_{\gamma_{u,w};\infty} \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\ &\le \max_{z\in\gamma_{u,v}} |z-u|\,\|f'\|_{\gamma_{u,v};1} + \max_{z\in\gamma_{v,w}} |z-w|\,\|f'\|_{\gamma_{v,w};1} \\ &\le \max\left\{ \max_{z\in\gamma_{u,v}} |z-u|, \max_{z\in\gamma_{v,w}} |z-w| \right\} \|f'\|_{\gamma_{u,w};1}. \end{aligned} \tag{1.6}$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} &\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\ &\le \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz| \right)^{1/q} \|f'\|_{\gamma_{u,v};p} + \left(\int_{\gamma_{v,w}} |z-w|^q\,|dz| \right)^{1/q} \|f'\|_{\gamma_{v,w};p} \\ &\le \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz| + \int_{\gamma_{v,w}} |z-w|^q\,|dz| \right)^{1/q} \|f'\|_{\gamma_{u,w};p}. \end{aligned} \tag{1.7}$$

Motivated by the above results, in this paper, we extend the Ostrowski inequality to the complex integral, by providing upper bounds for the quantity

$$\left| f(v)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$, with the length $\ell(\gamma)$, $u = z(a)$, $v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$. An application for circular paths is also given.

2. Ostrowski Type Results

We have the following result for functions of complex variable:

Theorem 2.3. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$ and*

$$\left| f(v)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right| \le \ell(\gamma_{u,v})\,\|f'\|_{\gamma_{u,v};1} + \ell(\gamma_{v,w})\,\|f'\|_{\gamma_{v,w};1} \tag{2.8}$$

$$\le \begin{cases} \frac{1}{2}\left[\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|\right]\|f'\|_{\gamma_{u,w};1}\,, \\ \\ \left[\ell^p(\gamma_{u,v}) + \ell^p(\gamma_{v,w})\right]^{1/p}\left(\|f'\|^q_{\gamma_{u,v};1} + \|f'\|^q_{\gamma_{v,w};1}\right)^{1/q} \\ p,\ q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2}\ell(\gamma_{u,w})\left[\|f'\|_{\gamma_{u,w};1} + \left|\|f'\|_{\gamma_{u,v};1} - \|f'\|_{\gamma_{v,w};1}\right|\right]. \end{cases}$$

Proof. Using the integration by parts formula, we have

$$\begin{aligned} \int_{\gamma_{u,v}} f(z)\,|dz| &= \int_a^x f(z(t))\,|z'(t)|\,dt \\ &= \int_a^x f(z(t))\,d\left(\int_a^t |z'(s)|\,ds\right) \\ &= f(z(t))\int_a^t |z'(s)|\,ds\Big|_a^x - \int_a^x \frac{df(z(t))}{dt}\left(\int_a^t |z'(s)|\,ds\right)dt \\ &= f(z(x))\int_a^x |z'(s)|\,ds - \int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right)z'(t)\,dt \\ &= f(v)\,\ell(\gamma_{u,v}) - \int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right)z'(t)\,dt \end{aligned}$$

and

$$\begin{aligned} \int_{\gamma_{v,w}} f(z)\,|dz| &= \int_x^b f(z(t))\,|z'(t)|\,dt \\ &= -\int_x^b f(z(t))\,d\left(\int_t^b |z'(s)|\,ds\right) \\ &= -\,f(z(t))\int_t^b |z'(s)|\,ds\Big|_x^b + \int_x^b \frac{df(z(t))}{dt}\left(\int_t^b |z'(s)|\,ds\right)dt \\ &= f(z(x))\int_x^b |z'(s)|\,ds + \int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right)z'(t)\,dt \\ &= f(v)\,\ell(\gamma_{v,w}) + \int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right)z'(t)\,dt\,. \end{aligned}$$

If we add these two equalities, we get

$$\begin{aligned}&\int_{\gamma_{u,v}} f(z)\,|dz| + \int_{\gamma_{v,w}} f(z)\,|dz|\\
=&f(v)\,\ell(\gamma_{u,v}) + f(v)\,\ell(\gamma_{v,w}) - \int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right) z'(t)\,dt\\
&+\int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right) z'(t)\,dt,\end{aligned}$$

which gives the following equality of interest

$$\begin{aligned}&f(v)\,\ell(\gamma_{u,w}) - \int_{\gamma_{u,w}} f(z)\,|dz|\\
&=\int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right) z'(t)\,dt - \int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right) z'(t)\,dt.\end{aligned}\tag{2.9}$$

By taking the modulus in (2.9) and using the properties of modulus, we get

$$\begin{aligned}&\left|f(v)\,\ell(\gamma_{u,w}) - \int_{\gamma_{u,w}} f(z)\,|dz|\right|\\
&\leq\left|\int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right) z'(t)\,dt\right|\\
&+\left|\int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right) z'(t)\,dt\right|\\
&\leq\int_a^x |f'(z(t))|\,|z'(t)|\left(\int_a^t |z'(s)|\,ds\right) dt\\
&+\int_x^b |f'(z(t))|\,|z'(t)|\left(\int_t^b |z'(s)|\,ds\right) dt =: B(x)\end{aligned}\tag{2.10}$$

for $x \in [a,b]$. We have

$$\int_a^t |z'(s)|\,ds \leq \int_a^x |z'(s)|\,ds \text{ for } t \in [a,x]$$

and

$$\int_t^b |z'(s)|\,ds \leq \int_x^b |z'(s)|\,ds \text{ for } t \in [x,b],$$

then

$$\begin{aligned}B(x) &\leq \int_a^x |z'(s)|\,ds \int_a^x |f'(z(t))|\,|z'(t)|\,dt\\
&+\int_x^b |z'(s)|\,ds \int_x^b |f'(z(t))|\,|z'(t)|\,dt\\
&=\ell(\gamma_{u,v})\int_{\gamma_{u,v}} |f'(z)|\,|dz| + \ell(\gamma_{v,w})\int_{\gamma_{v,w}} |f'(z)|\,|dz|\end{aligned}$$

and by (2.10), we get the first inequality in (2.8). The second part follows by Hölder's inequalities

$$mn+cd\leq\begin{cases}\max\{m,c\}(n+d)\\ (m^p+c^p)^{1/p}(n^q+d^q)^{1/q}, \text{ for } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases}$$

where $m,n,c,d\geq 0$. □

Corollary 2.1. *With the assumption of Theorem 2.3 and if there exists $m\in\gamma$ such that $\ell(\gamma_{u,m})=\ell(\gamma_{m,w})$, then*

$$\left|f(m)\ell(\gamma)-\int_\gamma f(z)|dz|\right|\leq\frac{1}{2}\ell(\gamma)\|f'\|_{\gamma_{u,w};1} \tag{2.11}$$

and if $s\in\gamma$ such that $\int_{\gamma_{u,s}}|f'(z)||dz|=\int_{\gamma_{s,w}}|f'(z)||dz|$, then

$$\left|f(s)\ell(\gamma)-\int_\gamma f(z)|dz|\right|\leq\frac{1}{2}\ell(\gamma)\|f'\|_{\gamma_{u,w};1}. \tag{2.12}$$

We have also the following result for p-norms:

Theorem 2.4. *With the assumption of Theorem 2.3, we have for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that*

$$\begin{aligned}&\left|f(v)\ell(\gamma)-\int_\gamma f(z)|dz|\right| \\ &\leq\frac{1}{q+1}\left[\ell^{1+1/q}(\gamma_{u,v})\|f'\|_{\gamma_{u,v};p}+\ell^{1+1/q}(\gamma_{v,w})\|f'\|_{\gamma_{v,w};p}\right] \\ &\leq\frac{1}{q+1}\begin{cases}\frac{1}{2^{1+1/q}}\left[\ell(\gamma_{u,w})+|\ell(\gamma_{u,v})-\ell(\gamma_{v,w})|\right]^{1+1/q}\\ \times\left[\|f'\|_{\gamma_{u,v};p}+\|f'\|_{\gamma_{v,w};p}\right],\\ \left[\ell^{q+1}(\gamma_{u,v})+\ell^{q+1}(\gamma_{v,w})\right]^{1/q}\|f'\|_{\gamma_{u,w};p},\\ \frac{1}{2^{1/p}}\left[\|f'\|^p_{\gamma_{u,w};p}+\left|\|f'\|^p_{\gamma_{u,v};p}-\|f'\|^p_{\gamma_{v,w};p}\right|\right]^{1/p}\\ \times\left[\ell^{1+1/q}(\gamma_{u,v})+\ell^{1+1/q}(\gamma_{v,w})\right].\end{cases}\end{aligned} \tag{2.13}$$

Proof. Using the weighted Hölder integral inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}&\int_a^x|f'(z(t))||z'(t)|\left(\int_a^t|z'(s)|ds\right)dt \\ &\leq\left(\int_a^x|f'(z(t))|^p|z'(t)|dt\right)^{1/p}\left(\int_a^x\left(\int_a^t|z'(s)|ds\right)^q|z'(t)|dt\right)^{1/q} \\ &=\left(\int_a^x|f'(z(t))|^p|z'(t)|dt\right)^{1/p}\left(\frac{\left(\int_a^x|z'(s)|ds\right)^{q+1}}{q+1}\right)^{1/q} \\ &=\frac{\left(\int_a^x|z'(s)|ds\right)^{1+1/q}}{(q+1)^{1/q}}\left(\int_a^x|f'(z(t))|^p|z'(t)|dt\right)^{1/p}\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b |f'(z(t))| |z'(t)| \left(\int_t^b |z'(s)| ds\right) dt \\
&\leq \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \left(\int_x^b \left(\int_t^b |z'(s)| ds\right)^q |z'(t)| dt\right)^{1/q} \\
&= \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \left(\frac{\left(\int_x^b |z'(s)| ds\right)^{q+1}}{q+1}\right)^{1/q} \\
&= \frac{\left(\int_x^b |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p}
\end{aligned}$$

for $x \in (a,b)$. If we add these two inequalities, we get

$$\begin{aligned}
B(x) &\leq \frac{\left(\int_a^x |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_a^x |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \\
&+ \frac{\left(\int_x^b |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \\
&= \frac{1}{q+1} \left[\ell^{1+1/q}(\gamma_{u,v}) \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right],
\end{aligned}$$

which proves the first inequality in (2.13). We also have

$$\begin{aligned}
&\ell^{1+1/q}(\gamma_{u,v}) \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p} \\
&\leq \max\left\{\ell^{1+1/q}(\gamma_{u,v}), \ell^{1+1/q}(\gamma_{v,w})\right\} \\
&\times \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right] \\
&= [\max\{\ell(\gamma_{u,v}), \ell(\gamma_{v,w})\}]^{1+1/q} \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right] \\
&= \frac{1}{2^{1+1/q}} [\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|]^{1+1/q} \\
&\times \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right]
\end{aligned}$$

and

$$
\begin{aligned}
&\ell^{1+1/q}\left(\gamma_{u,v}\right)\left(\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}+\ell^{1+1/q}\left(\gamma_{v,w}\right)\left(\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}\\
&\leq\max\left\{\left(\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p},\left(\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}\right\}\\
&\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)+\ell^{1+1/q}\left(\gamma_{v,w}\right)\right]\\
&=\left[\max\left\{\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|,\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right\}\right]^{1/p}\\
&\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)+\ell^{1+1/q}\left(\gamma_{v,w}\right)\right]\\
&=\frac{1}{2^{1/p}}\left[\int_{\gamma_{u,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|+\left|\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|-\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right|\right]^{1/p}\\
&\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)+\ell^{1+1/q}\left(\gamma_{v,w}\right)\right].
\end{aligned}
$$

By Hölder's discrete inequality, we have

$$
\begin{aligned}
&\ell^{1+1/q}\left(\gamma_{u,v}\right)\left(\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}+\ell^{1+1/q}\left(\gamma_{v,w}\right)\left(\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}\\
&\leq\left[\ell^{q+1}\left(\gamma_{u,v}\right)+\ell^{q+1}\left(\gamma_{v,w}\right)\right]^{1/q}\left[\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|+\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right]^{1/p}\\
&=\left[\ell^{q+1}\left(\gamma_{u,v}\right)+\ell^{q+1}\left(\gamma_{v,w}\right)\right]^{1/q}\left(\int_{\gamma_{u,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p},
\end{aligned}
$$

which prove the last part of (2.13). □

We have:

Corollary 2.2. *With the assumption of Theorem 2.4 and if there exists $m\in\gamma$ such that $\ell\left(\gamma_{u,m}\right)=\ell\left(\gamma_{m,w}\right)$, then*

$$
\left|f\left(m\right)\ell\left(\gamma\right)-\int_{\gamma}f\left(z\right)\left|dz\right|\right|\leq\frac{1}{2^{1+1/q}\left(q+1\right)}\ell^{1+1/q}\left(\gamma_{u,w}\right)\left[\left\|f'\right\|_{\gamma_{u,m};p}+\left\|f'\right\|_{\gamma_{m,w};p}\right]\tag{2.14}
$$

and if $s\in\gamma$ such that $\int_{\gamma_{u,s}}\left|f'\left(z\right)\right|^{p}\left|dz\right|=\int_{\gamma_{s,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|$, then

$$
\left|f\left(s\right)\ell\left(\gamma\right)-\int_{\gamma}f\left(z\right)\left|dz\right|\right|\leq\frac{1}{2^{1/p}\left(q+1\right)}\left[\ell^{1+1/q}\left(\gamma_{u,s}\right)+\ell^{1+1/q}\left(\gamma_{s,w}\right)\right]\left\|f'\right\|_{\gamma_{u,w};p}.\tag{2.15}
$$

Finally we have:

Theorem 2.5. *With the assumption of Theorem 2.3, we have*

$$\left|f(v)\ell(\gamma)-\int_\gamma f(z)|dz|\right| \tag{2.16}$$
$$\leq\frac{1}{2}\left[\ell^2(\gamma_{u,v})\|f'\|_{\gamma_{u,v};\infty}+\ell^2(\gamma_{v,w})\|f'\|_{\gamma_{v,w};\infty}\right]$$
$$\leq\frac{1}{2}\begin{cases}\frac{1}{4}\left[\ell(\gamma_{u,w})+|\ell(\gamma_{u,v})-\ell(\gamma_{v,w})|\right]^2\\ \times\left[\|f'\|_{\gamma_{u,v};\infty}+\|f'\|_{\gamma_{v,w};\infty}\right],\\ \\ \left[\ell^{2q}(\gamma_{u,v})+\ell^{2q}(\gamma_{v,w})\right]^{1/q}\left(\|f'\|^p_{\gamma_{u,v};\infty}+\|f'\|^p_{\gamma_{v,w};\infty}\right)^{1/p},\\ p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \left[\ell^2(\gamma_{u,v})+\ell^2(\gamma_{v,w})\right]\|f'\|_{\gamma_{u,w};\infty}.\end{cases}$$

Proof. We have

$$\int_a^x|f'(z(t))||z'(t)|\left(\int_a^t|z'(s)|\,ds\right)dt$$
$$\leq\max_{t\in[a,x]}|f'(z(t))|\int_a^x\left(\int_a^t|z'(s)|\,ds\right)|z'(t)|\,dt$$
$$=\frac{1}{2}\max_{t\in[a,x]}|f'(z(t))|\left(\int_a^x|z'(s)|\,ds\right)^2=\frac{1}{2}\ell^2(\gamma_{u,v})\|f'\|_{\gamma_{u,v};\infty}$$

and

$$\int_x^b|f'(z(t))||z'(t)|\left(\int_t^b|z'(s)|\,ds\right)dt$$
$$\leq\max_{t\in[x,b]}|f'(z(t))|\int_x^b\left(\int_t^b|z'(s)|\,ds\right)|z'(t)|\,dt$$
$$=\frac{1}{2}\max_{t\in[x,b]}|f'(z(t))|\left(\int_x^b|z'(s)|\,ds\right)^2=\frac{1}{2}\ell^2(\gamma_{v,w})\|f'\|_{\gamma_{v,w};\infty},$$

which by addition produce

$$B(x)\leq\frac{1}{2}\left[\ell^2(\gamma_{u,v})\|f'\|_{\gamma_{u,v};\infty}+\ell^2(\gamma_{v,w})\|f'\|_{\gamma_{v,w};\infty}\right]$$

and by utilising the inequality (2.10), we get the first part of (2.16).

The second part is obvious and we omit the details. □

Corollary 2.3. *With the assumption of Theorem 2.3 and if there exists $m\in\gamma$ such that $\ell(\gamma_{u,m})=\ell(\gamma_{m,w})$, then*

$$\left|f(m)\ell(\gamma)-\int_\gamma f(z)|dz|\right|\leq\frac{1}{8}\left[\|f'\|_{\gamma_{u,m};\infty}+\|f'\|_{\gamma_{m,w};\infty}\right]\ell^2(\gamma_{u,w})$$
$$\leq\frac{1}{4}\|f'\|_{\gamma_{u,w};\infty}\ell^2(\gamma_{u,w}).$$

3. Examples for Circular Paths

Let $[a,b]\subseteq[0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t)=R\exp(it)=R(\cos t+i\sin t)\,,\ t\in[a,b]\,.$$

If $[a,b]=[0,\pi]$, then we get a half circle, while for $[a,b]=[0,2\pi]$, we get the full circle. We have

$$z'(t)=Ri\exp(it)\,,\ t\in[a,b]$$

and $|z'(t)|=R$ for $t\in[a,b]$ giving that

$$\ell\left(\gamma_{[a,b],R}\right)=\int_a^b|z'(t)|\,dt=R(b-a)\,.$$

If $x\in[a,b]$ and $v=R\exp(ix)$, then by (2.8), we have

$$\begin{aligned}&\left|R(b-a)f(R\exp(ix))-R\int_a^b f(R\exp(it))\,dt\right|\\ \leq&R(x-a)R\int_a^x|f'(R\exp(it))|\,dt+R(b-x)R\int_x^b|f'(R\exp(it))|\,dt,\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\left|(b-a)f(R\exp(ix))-\int_a^b f(R\exp(it))\,dt\right|\\ \leq&R(x-a)\int_a^x|f'(R\exp(it))|\,dt+R(b-x)\int_x^b|f'(R\exp(it))|\,dt\end{aligned}\tag{3.17}$$

for $x\in[a,b]$. In particular, if we take $x=\frac{a+b}{2}$ in (3.17), then we get

$$\left|(b-a)f\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b f(R\exp(it))\,dt\right|\leq\frac{1}{2}R(b-a)\int_a^b|f'(R\exp(it))|\,dt.\tag{3.18}$$

If $m\in[a,b]$ is such that

$$\int_a^m|f'(R\exp(it))|\,dt=\int_m^b|f'(R\exp(it))|\,dt,$$

then from (3.17), we get

$$\begin{aligned}&\left|(b-a)f(R\exp(mi))-\int_a^b f(R\exp(it))\,dt\right|\\ \leq&\frac{1}{2}R(b-a)\int_a^b|f'(R\exp(it))|\,dt.\end{aligned}\tag{3.19}$$

By making use of (2.13), we get

$$\begin{aligned}&\left|R(b-a)f(R\exp(ix))-R\int_a^b f(R\exp(it))\,dt\right|\\ \leq&\frac{1}{q+1}\left[R^{1+1/q}(x-a)^{1+1/q}R^{1/p}\left(\int_a^x|f'(R\exp(it))|^p\,dt\right)^{1/p}\right.\\ &\left.+R^{1+1/q}(b-x)^{1+1/q}R^{1/p}\left(\int_x^b|f'(R\exp(it))|^p\,dt\right)^{1/p}\right],\end{aligned}$$

that is equivalent to

$$\begin{aligned} &\left|(b-a) f(R\exp(ix)) - \int_a^b f(R\exp(it))\,dt\right| \\ &\leq \frac{1}{q+1} R \left[(x-a)^{1+1/q} \left(\int_a^x |f'(R\exp(it))|^p\,dt\right)^{1/p}\right. \\ &\left. + (b-x)^{1+1/q} \left(\int_x^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right] \end{aligned} \tag{3.20}$$

for $x \in [a,b]$. If we take $x = \frac{a+b}{2}$ in (3.20), then we get

$$\begin{aligned} &\left|(b-a) f\left(R\exp\left(\frac{a+b}{2}i\right)\right) - \int_a^b f(R\exp(it))\,dt\right| \\ &\leq \frac{1}{(q+1)\,2^{1+1/q}} R (b-a)^{1+1/q} \left[\left(\int_a^{\frac{a+b}{2}} |f'(R\exp(it))|^p\,dt\right)^{1/p}\right. \\ &\left. + \left(\int_{\frac{a+b}{2}}^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right]. \end{aligned} \tag{3.21}$$

If $c \in [a,b]$ is such that

$$\int_a^c |f'(R\exp(it))|^p\,dt = \int_c^b |f'(R\exp(it))|^p\,dt,$$

then by (3.20), we get

$$\begin{aligned} &\left|(b-a) f(R\exp(ic)) - \int_a^b f(R\exp(it))\,dt\right| \\ &\leq \frac{1}{(q+1)\,2^{1/p}} R \left[(c-a)^{1+1/q} + (b-c)^{1+1/q}\right] \left(\int_a^b |f'(R\exp(it))|^p\,dt\right)^{1/p}. \end{aligned} \tag{3.22}$$

Further, if we use (2.16), then we have

$$\begin{aligned} &\left|R(b-a) f(R\exp(ix)) - R\int_a^b f(R\exp(it))\,dt\right| \\ &\leq \frac{1}{2} R^2 \left[(x-a)^2 \sup_{t\in[a,x]} |f'(R\exp(it))| + (b-x)^2 \sup_{t\in[x,b]} |f'(R\exp(it))|\right] \\ &\leq \frac{1}{2} R^2 \left[(x-a)^2 + (b-x)^2\right] \sup_{t\in[a,b]} |f'(R\exp(it))|, \end{aligned}$$

that is equivalent to

$$
\begin{aligned}
(3.23)\quad & \left|(b-a) f(R\exp(xi)) - \int_a^b f(R\exp(it))\,dt\right| \\
& \le \frac{1}{2} R\left[(x-a)^2 \sup_{t\in[a,x]} |f'(R\exp(it))| + (b-x)^2 \sup_{t\in[x,b]} |f'(R\exp(it))|\right] \\
& \le R\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right] \sup_{t\in[a,b]} |f'(R\exp(it))|
\end{aligned}
$$

for $x \in [a,b]$. In particular, we have

$$
\begin{aligned}
(3.24)\quad & \left|(b-a) f\left(R\exp\left(\frac{a+b}{2} i\right)\right) - \int_a^b f(R\exp(it))\,dt\right| \\
& \le \frac{1}{8} R(b-a)^2 \left[\sup_{t\in\left[a,\frac{a+b}{2}\right]} |f'(R\exp(it))| + \sup_{t\in\left[\frac{a+b}{2},b\right]} |f'(R\exp(it))|\right] \\
& \le \frac{1}{4} R(b-a)^2 \sup_{t\in[a,b]} |f'(R\exp(it))|.
\end{aligned}
$$

We give now examples for some fundamental complex functions. Consider the function $f(z) = z^n$, $z \in \mathbb{C}$ with $n \ge 1$. Then $f'(z) = nz^{n-1}$,

$$
f(R\exp(ix)) = R^n \exp(nxi),
$$

$$
|f'(R\exp(it))| = nR^{n-1} |\exp((n-1)it)| = nR^{n-1},\ t \in [a,b]
$$

and

$$
\int_a^b f(R\exp(it))\,dt = R^n \int_a^b \exp(nti)\,dt = R^n \frac{\exp(nbi) - \exp(nai)}{ni}.
$$

Making use of the inequality (3.23), we get

$$
\left|(b-a) R^n \exp(nxi) - R^n \frac{\exp(nbi) - \exp(nai)}{ni}\right| \le R\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right] nR^{n-1},
$$

which is equivalent to

$$
(3.25)\quad \left|(b-a)\exp(nxi) - \frac{\exp(nbi) - \exp(nai)}{ni}\right| \le n\left[\frac{1}{2}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]
$$

for $x \in [a,b]$. If we take in (3.25) $x = \frac{a+b}{2}$, then we get

$$
(3.26)\quad \left|(b-a)\exp\left(n\left(\frac{a+b}{2}\right)i\right) - \frac{\exp(nbi) - \exp(nai)}{ni}\right| \le \frac{1}{4} n(b-a)^2
$$

for $n \ge 1$. Consider the exponential function $f(z) = \exp(z)$, $z \in \mathbb{C}$. Then $f'(z) = \exp(z)$,

$$
|f'(R\exp(it))| = |\exp(R(\cos t + i\sin t))| = \exp(R\cos t),\ t \in [a,b]
$$

and by the inequality (3.17), we get

$$\begin{aligned}(3.27)\quad &\left|(b-a)\exp\left(R\exp\left(ix\right)\right)-\int_a^b \exp\left(R\exp\left(it\right)\right)dt\right| \\ &\le R\left[(x-a)\int_a^x \exp\left(R\cos t\right)dt+(b-x)\int_x^b \exp\left(R\cos t\right)dt\right],\end{aligned}$$

while from the inequality (3.23), we get

$$\begin{aligned}(3.28)\quad &\left|(b-a)\exp\left(R\exp\left(ix\right)\right)-\int_a^b \exp\left(R\exp\left(it\right)\right)dt\right| \\ &\le \frac{1}{2}R\left[(x-a)^2\sup_{t\in[a,x]}\exp\left(R\cos t\right)+(b-x)^2\sup_{t\in[x,b]}\exp\left(R\cos t\right)\right] \\ &\le R\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\exp\left(R\cos t\right)\end{aligned}$$

for $x\in[a,b]$. From the inequality (3.20), we get

$$\begin{aligned}(3.29)\quad &\left|(b-a)\exp\left(R\exp\left(ix\right)\right)-\int_a^b \exp\left(R\exp\left(it\right)\right)dt\right| \\ &\le \frac{1}{q+1}R\left[(x-a)^{1+1/q}\left(\int_a^x \exp\left(pR\cos t\right)dt\right)^{1/p}\right. \\ &\left.+(b-x)^{1+1/q}\left(\int_x^b \exp\left(pR\cos t\right)dt\right)^{1/p}\right]\end{aligned}$$

for $x\in[a,b]$, where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If in the inequality (3.27) we take $x=\frac{a+b}{2}$, then we get

$$\begin{aligned}(3.30)\quad &\left|(b-a)\exp\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b \exp\left(R\exp\left(it\right)\right)dt\right| \\ &\le \frac{1}{2}R\left(b-a\right)\int_a^b \exp\left(R\cos t\right)dt,\end{aligned}$$

while from the inequality (2.15), we get

$$\begin{aligned}(3.31)\quad &\left|(b-a)\exp\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b \exp\left(R\exp\left(it\right)\right)dt\right| \\ &\le \frac{1}{8}(b-a)^2R\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}\exp\left(R\cos t\right)+\sup_{t\in\left[\frac{a+b}{2},b\right]}\exp\left(R\cos t\right)\right] \\ &\le \frac{1}{4}R\left(b-a\right)^2\sup_{t\in[a,b]}\exp\left(R\cos t\right).\end{aligned}$$

From (3.29), we have

$$\left|(b-a)\exp\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b \exp\left(R\exp\left(it\right)\right)dt\right| \tag{3.32}$$
$$\leq \frac{1}{(q+1)\,2^{1+1/q}}R\left(b-a\right)^{1+1/q}$$
$$\times\left[\left(\int_a^{\frac{a+b}{2}}\exp\left(pR\cos t\right)dt\right)^{1/p}+\left(\int_{\frac{a+b}{2}}^b \exp\left(pR\cos t\right)dt\right)^{1/p}\right].$$

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Research Article

Certain Class of Bi-Bazilevič Functions with Bounded Boundary Rotation Involving Sălăgean Operator

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ABSTRACT. In the present paper, we consider certain classes of bi-univalent Bazilevič functions with bounded boundary rotation involving Sălăgean operator to obtain the estimates of their second and third coefficients. Further, certain special cases are also indicated. Some interesting remarks about the results presented here are also discussed.

Keywords: Analytic function, bi-univalent, Bazilevič functions, Salăgeăn operator.

2020 Mathematics Subject Classification: 30C45.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^2 - 5a_2 a_3 + a_4\right) w^4 + \dots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples in the class Σ (see [26]).

For functions $f \in \mathcal{A}$, Sălăgean [27] (see also [4] and [28]) defined the linear operator $\mathcal{D}^m : \mathcal{A} \to \mathcal{A}$ ($m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$) as follows:

$$\mathcal{D}^0 f(z) = f(z),$$

$$\mathcal{D}^1 f(z) = \mathcal{D} f(z) = z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

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and (in general)

$$\mathcal{D}^m f(z) = \mathcal{D}\left(\mathcal{D}^{m-1} f(z)\right) = z + \sum_{n=2}^{\infty} n^m a_n z^n. \tag{1.3}$$

From (1.3), we can easily deduce that

$$\mathcal{D}^{m+1} f(z) = z\left(\mathcal{D}^m f(z)\right)'. \tag{1.4}$$

Let $\mathcal{P}_k^{\lambda}(\alpha)$ be the class of analytic functions $p(z)$ in $\mathbb{U}$ normalized by $p(0) = 1$ and satisfying

$$\int_0^{2\pi} \left| \frac{\Re\left\{e^{i\lambda} p(z)\right\} - \alpha \cos\lambda}{1-\alpha} \right| d\theta \leq k\pi \cos\lambda, \tag{1.5}$$

where $z = re^{i\theta}, 0 \leq r < 1, |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1$ and $k \geq 2$. The class $\mathcal{P}_k^{\lambda}(\alpha)$ was introduced and studied by Moulis [16] (see also Aouf [3] and Noor et al. [21]). We note that
(i) $\mathcal{P}_k^0(0) = \mathcal{P}_k$, is the class of functions have their real parts bounded in the mean on $\mathbb{U}$, introduced by Robertson [25] and studied Pinchuk [24];
(ii) $\mathcal{P}_k^{\lambda}(0) = \mathcal{P}_k^{\lambda}$, is the class of functions introduced by Robertson [25] and he derived a variational formula for functions in this class;
(iii) $\mathcal{P}_k^0(\alpha) = \mathcal{P}_k(\alpha)$, is the class of functions introduced by Padmanabhan and Parvatham [23] (see also Umarani and Aouf [31]);
(iv) $\mathcal{P}_2^0(\alpha) = \mathcal{P}(\alpha)$, is the class of functions with positive real part of order α, $0 \leq \alpha < 1$;
(v) $\mathcal{P}_2^0(0) = \mathcal{P}$, is the class of functions having positive real part for $z \in \mathbb{U}$.

Using Salăgeăn operator $\mathcal{D}^m$ and the class $\mathcal{P}_k$, we now introduce the following subclass of Bi-Bazilevič̃ analytic functions of the class Σ as follows:

Definition 1.1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{B}_{\Sigma}^m(\gamma, \delta, b; k)$ if it satisfies the following subordination condition:*

$$1 + \frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta} + \gamma \frac{\mathcal{D}^{m+1} f(z)}{\mathcal{D}^m f(z)}\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta} - 1\right] \in \mathcal{P}_k \tag{1.6}$$

and

$$1 + \frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m g(w)}{w}\right)^{\delta} + \gamma \frac{\mathcal{D}^{m+1} g(w)}{\mathcal{D}^m g(w)}\left(\frac{\mathcal{D}^m g(w)}{w}\right)^{\delta} - 1\right] \in \mathcal{P}_k, \tag{1.7}$$

where $g = f^{-1}, b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, \gamma, \delta \in \mathbb{C}, m \in \mathbb{N}_0, k \geq 2$ and all powers are understood as principle values.

Taking additional choices of m, γ, δ, k and b, the class $\mathcal{B}_{\Sigma}^m(\gamma, \delta, b; k)$ reduces to the following subclasses of Σ:

(i) $\mathcal{B}_{\Sigma}^0(\gamma, \delta, 1; k) = \mathcal{B}_{\Sigma}(\gamma, \delta; k)$

$$= \left\{ f \in \Sigma : (1-\gamma)\left(\frac{f(z)}{z}\right)^{\delta} + \gamma \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\delta} \in \mathcal{P}_k \right.$$

$$\left. \text{and} \;\; (1-\gamma)\left(\frac{g(w)}{w}\right)^{\delta} + \gamma \frac{wg'(w)}{g(w)}\left(\frac{g(w)}{w}\right)^{\delta} \in \mathcal{P}_k \right\};$$

(ii) $\mathcal{B}_{\Sigma}^{0}(\gamma,\delta,1-\eta;2)=\mathcal{B}_{\Sigma}(\gamma,\delta,\eta)\,(0\le\eta<1)$ (see [15] for $f\in\mathcal{A}$)(see also [29])

$$=\left\{f\in\Sigma:\Re\left\{(1-\gamma)\left(\frac{f(z)}{z}\right)^{\delta}+\gamma\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\delta}\right\}>\eta\right.$$
$$\left.\text{and }\Re\left\{(1-\gamma)\left(\frac{g(w)}{w}\right)^{\delta}+\gamma\frac{wg'(w)}{g(w)}\left(\frac{g(w)}{w}\right)^{\delta}\right\}>\eta\right\};$$

(iii) $\mathcal{B}_{\Sigma}^{0}(\gamma,1,1;k)=\mathcal{B}_{\Sigma}(\gamma;k)$

$$=\left\{f\in\Sigma:(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)\in\mathcal{P}_k\text{ and }(1-\gamma)\frac{g(w)}{w}+\gamma g'(w)\in\mathcal{P}_k\right\};$$

(iv) $\mathcal{B}_{\Sigma}^{0}(\gamma,1,1-\eta;2)=\mathcal{B}_{\Sigma}(\gamma,\eta)\,(0\le\eta<1)$ (see [10] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\Re\left\{(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)\right\}>\eta\text{ and }\Re\left\{(1-\gamma)\frac{g(w)}{w}+\gamma g'(w)\right\}>\eta\right.$$

(v) $\mathcal{B}_{\Sigma}^{0}(1,\delta,1-\eta;2)=\mathcal{B}_{\Sigma}(\delta,\eta)\,(0\le\eta<1)$ (see [22] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\delta}\right\}>\eta\text{ and }\Re\left\{\frac{wg'(w)}{g(w)}\left(\frac{g(w)}{w}\right)^{\delta}\right\}>\eta\right\};$$

(vi) $\mathcal{B}_{\Sigma}^{0}(1,0,b;k)=\mathcal{S}_{\Sigma}(b;k)$ (see Nasr and Aouf [20] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\in\mathcal{P}_k\text{ and }1+\frac{1}{b}\left(\frac{wg'(w)}{g(w)}-1\right)\in\mathcal{P}_k\right\};$$

(vii) $\mathcal{B}_{\Sigma}^{0}(1,0,b;2)=\mathcal{S}_{\Sigma}(b)$ (see Nasr and Aouf [19] for $f\in\mathcal{A}$) (see also [5])

$$=\left\{f\in\Sigma:\Re\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0\text{ and }\Re\left\{1+\frac{1}{b}\left(\frac{wg'(w)}{g(w)}-1\right)\right\}>0\right\};$$

(viii) $\mathcal{B}_{\Sigma}^{1}(1,0,b;k)=\mathcal{C}_{\Sigma}(b;k)$ (see Nasr and Aouf [20] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\in\mathcal{P}_k\text{ and }1+\frac{1}{b}\frac{wg''(w)}{g'(w)}\in\mathcal{P}_k\right\};$$

(ix) $\mathcal{B}_{\Sigma}^{1}(1,0,b;2)=\mathcal{C}_{\Sigma}(b)$ (see Nasr and Aouf [18] for $f\in\mathcal{A}$) (see also [5])

$$=\left\{f\in\Sigma:\Re\left\{1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\right\}>0\text{ and }\Re\left\{1+\frac{1}{b}\frac{wg''(w)}{g'(w)}\right\}>0\right\};$$

(x) $\mathcal{B}_{\Sigma}^{0}(1,0,1;k)=\mathcal{S}_{\Sigma}(k)$ (see Pinchuk [24] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\frac{zf'(z)}{f(z)}\in\mathcal{P}_k\text{ and }\frac{wg'(w)}{g(w)}\in\mathcal{P}_k\right\};$$

(xi) $\mathcal{B}_{\Sigma}^{1}(1,0,1;k)=\mathcal{C}_{\Sigma}(k)$ (see Pinchuk [24] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:1+\frac{zf''(z)}{f'(z)}\in\mathcal{P}_k \text{ and } 1+\frac{wg''(w)}{g'(w)}\in\mathcal{P}_k\right\};$$

(xii) $\mathcal{B}_{\Sigma}^{0}(1,0,1-\eta;2)=\mathcal{S}_{\Sigma}(\eta)\,(0\le\eta<1)$ (see [9] and [30])

$$=\left\{f\in\Sigma:\Re\left(\frac{zf'(z)}{f(z)}\right)>\eta \text{ and } \Re\left(\frac{wg'(w)}{g(w)}\right)>\eta\right\};$$

(xiii) $\mathcal{B}_{\Sigma}^{1}(1,0,1-\eta;2)=\mathcal{C}_{\Sigma}(\eta)\,(0\le\eta<1)$ (see [9] and [30])

$$=\left\{f\in\Sigma:\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\eta \text{ and } \Re\left(1+\frac{wg''(w)}{g'(w)}\right)>\eta\right\};$$

(xiv) $\mathcal{B}_{\Sigma}^{0}\left(\gamma,\delta,(1-\alpha)e^{-i\lambda}\cos\lambda;k\right)=\mathcal{B}_{\Sigma}(\gamma,\delta,\alpha,\lambda;k)\left(|\lambda|<\frac{\pi}{2},0\le\alpha<1\right)$

$$=\left\{f\in\Sigma:\frac{e^{i\lambda}\left[(1-\gamma)\left(\frac{f(z)}{z}\right)^{\delta}+\gamma\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\delta}\right]-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right.$$

$$\left.\text{and }\frac{e^{i\lambda}\left[(1-\gamma)\left(\frac{g(w)}{w}\right)^{\delta}+\gamma\frac{wg'(w)}{g(w)}\left(\frac{g(w)}{w}\right)^{\delta}\right]-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right\}$$

or

$$=\left\{f\in\Sigma:(1-\gamma)\left(\frac{f(z)}{z}\right)^{\delta}+\gamma\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\delta}\in\mathcal{P}_k^{\lambda}(\alpha)\right.$$

$$\left.\text{and }(1-\gamma)\left(\frac{g(w)}{w}\right)^{\delta}+\gamma\frac{wg'(w)}{g(w)}\left(\frac{g(w)}{w}\right)^{\delta}\in\mathcal{P}_k^{\lambda}(\alpha)\right\};$$

(xv) $\mathcal{B}_{\Sigma}^{0}\left(1,0,be^{-i\lambda}\cos\lambda;2\right)=\mathcal{S}_{\Sigma}^{\lambda}(b)\left(|\lambda|<\frac{\pi}{2},b\in\mathbb{C}^*\right)$ (see Al-Oboudi and Haidan [2] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\Re\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0\right.$$

$$\left.\text{and }\Re\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{wg'(w)}{g(w)}-1\right)\right\}>0\right\};$$

(xvi) $\mathcal{B}_{\Sigma}^{1}\left(1,0,be^{-i\lambda}\cos\lambda;2\right)=\mathcal{C}_{\Sigma}^{\lambda}(b)\left(|\lambda|<\frac{\pi}{2},b\in\mathbb{C}^*\right)$ (see Al-Oboudi and Haidan [2] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\Re\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0\right.$$

$$\left.\text{and }\Re\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{wg''(w)}{g'(w)}\right)\right\}>0\right\};$$

(xvii) $\mathcal{B}_{\Sigma}^{0}\left(1,0,(1-\alpha)e^{-i\lambda}\cos\lambda;k\right)=\mathcal{S}_{\alpha}^{\lambda}(k)\left(|\lambda|<\frac{\pi}{2},0\leq\alpha<1\right)$

$$=\left\{f\in\Sigma:\frac{e^{i\lambda}\frac{zf'(z)}{f(z)}-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right.$$

$$\left.\text{and }\frac{e^{i\lambda}\frac{wg'(w)}{g(w)}-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right\}$$

or

$$=\left\{f\in\Sigma:\frac{zf'(z)}{f(z)}\in\mathcal{P}_k^{\lambda}(\alpha)\text{ and }\frac{wg'(w)}{g(w)}\in\mathcal{P}_k^{\lambda}(\alpha)\right\};$$

(xviii) $\mathcal{B}_{\Sigma}^{1}\left(1,0,(1-\alpha)e^{-i\lambda}\cos\lambda;k\right)=\mathcal{C}_{\alpha}^{\lambda}(k)\left(|\lambda|<\frac{\pi}{2},0\leq\alpha<1\right)$

$$=\left\{f\in\Sigma:\frac{e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right.$$

$$\left.\text{and }\frac{e^{i\lambda}\left(1+\frac{wg''(w)}{g'(w)}\right)-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right\}$$

or

$$=\left\{f\in\Sigma:1+\frac{zf''(z)}{f'(z)}\in\mathcal{P}_k^{\lambda}(\alpha)\text{ and }1+\frac{wg''(w)}{g'(w)}\in\mathcal{P}_k^{\lambda}(\alpha)\right\}.$$

In order to establish our main results, we need the following lemma:

Lemma 1.1. [3, Theorem 5 with $p=1$] *If* $p(z)=1+\sum_{n=1}^{\infty}c_nz^n\in\mathcal{P}_k^{\lambda}(\alpha)$ *in* $\mathbb{U}$, *then*

$$|c_n|\leq(1-\alpha)k\cos\lambda\qquad(n\in\mathbb{N}).\tag{1.8}$$

The result is sharp. Equality is attained for the odd coefficients and even coefficients, respectively, for the functions

$$p_1(z)=1+(1-\alpha)\cos\lambda\,e^{-i\lambda}\left[\left(\frac{k+2}{4}\right)\left(\frac{1-z}{1+z}\right)-\left(\frac{k-2}{4}\right)\left(\frac{1+z}{1-z}\right)-1\right],$$

$$p_2(z)=1+(1-\alpha)\cos\lambda\,e^{-i\lambda}\left[\left(\frac{k+2}{4}\right)\left(\frac{1-z^2}{1+z^2}\right)-\left(\frac{k-2}{4}\right)\left(\frac{1+z^2}{1-z^2}\right)-1\right].$$

Remark 1.1. *For* $\lambda=\alpha=0$ *in Lemma 1.1, we obtain the result obtained by Goswami et al.* [11] *for the class* $\mathcal{P}_k$.

Lewin [13] defined the class of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [9] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced the concept of bi-starlike functions and the bi-convex functions, and obtained estimates for the initial coefficients. Recently, Srivastava et al. [26], Ali et al. [1], Frasin and Aouf [10], Goyal and Goswami [12] and many others (see [6], [7], [8], [14], [17] and [32]) have introduced and investigated subclasses of bi-univalent functions and obtained non-sharp bounds for the initial coefficients.

In the present paper, we estimates on the coefficients for second and third coefficients for the functions in the subclass $\mathcal{B}_{\Sigma}^{m}(\gamma,\delta,b;k)$ and its special subclasses.

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $g=f^{-1}, b\in\mathbb{C}^{*}, \gamma, \delta\in\mathbb{C}, k\geq 2, m\in\mathbb{N}_0$ and all powers are understood as principle values.

Theorem 2.1. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}^{m}(\gamma,\delta,b;k)$ with $\delta\neq 1-\frac{3^m}{2^{2m-1}}$, $\delta\neq-\gamma$ and $\delta\neq-2\gamma$, then*

$$|a_2|\leq \min\left\{\sqrt{\frac{|b|\,k}{|(\delta-1)\,2^{2m-1}+3^m|\,|\delta+2\gamma|}},\ \frac{|b|\,k}{2^m\,|\delta+\gamma|}\right\} \tag{2.9}$$

and

$$|a_3|\leq \frac{|b|\,k}{3^m\,|\delta+2\gamma|}\min\left\{\begin{array}{c} 1+\frac{3^m}{|(\delta-1)2^{2m-1}+3^m|};\ 1+\frac{|\delta+2\gamma||1-\delta||b|k}{2|\delta+\gamma|^2};\\ 1+\frac{|\delta+2\gamma||\delta-1||b|k}{2|\delta+\gamma|^2}+\frac{3^m|\delta+2\gamma||b|k}{2^{2m-1}|\delta+\gamma|^2}\end{array}\right\}. \tag{2.10}$$

Proof. If $f\in\mathcal{B}_{\Sigma}^{m}(\gamma,\delta,b;k)$, according to the Definition 1.1, we have

$$1+\frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta}+\gamma\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^m f(z)}\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta}-1\right]=p(z) \tag{2.11}$$

and

$$1+\frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m g(w)}{w}\right)^{\delta}+\gamma\frac{\mathcal{D}^{m+1}g(w)}{\mathcal{D}^m g(w)}\left(\frac{\mathcal{D}^m g(w)}{w}\right)^{\delta}-1\right]=q(w), \tag{2.12}$$

where $p(z), q(w)\in\mathcal{P}_k$ and $g=f^{-1}$. Using the fact that the functions $p(z)$ and $q(w)$ have the following Taylor expansions

$$p(z)=1+p_1z+p_2z^2+... \tag{2.13}$$

and

$$q(w)=1+q_1w+q_2w^2+.... \tag{2.14}$$

Since

$$1+\frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta}+\gamma\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^m f(z)}\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta}-1\right]$$

$$=1+\left(\frac{\delta+\gamma}{b}\right)2^m a_2 z+\left(\frac{\delta+2\gamma}{b}\right)\left[3^m a_3+\frac{\delta-1}{2}2^{2m}a_2^2\right]z^2+... \tag{2.15}$$

and according to (1.2), we have

$$1+\frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m g(w)}{w}\right)^{\delta}+\gamma\frac{\mathcal{D}^{m+1}g(w)}{\mathcal{D}^m g(w)}\left(\frac{\mathcal{D}^m g(w)}{w}\right)^{\delta}-1\right]$$

$$=1-\left(\frac{\delta+\gamma}{b}\right)2^m a_2 w+\left(\frac{\delta+2\gamma}{b}\right)\left[\left(2a_2^2-a_3\right)3^m+\frac{\delta-1}{2}2^{2m}a_2^2\right]w^2+\ldots \tag{2.16}$$

from (2.13) and (2.14), combined with (2.15) and (2.16), it follows that

$$p_1 = \left(\frac{\delta+\gamma}{b}\right)2^m a_2, \tag{2.17}$$

$$p_2 = \left(\frac{\delta+2\gamma}{b}\right)\left[3^m a_3+\frac{\delta-1}{2}2^{2m}a_2^2\right], \tag{2.18}$$

$$q_1 = -\left(\frac{\delta+\gamma}{b}\right)2^m a_2, \tag{2.19}$$

$$q_2 = \left(\frac{\delta+2\gamma}{b}\right)\left[\left(2a_2^2-a_3\right)3^m+\frac{\delta-1}{2}2^{2m}a_2^2\right]. \tag{2.20}$$

Now, from (2.18) and (2.20), we deduce that

$$a_2^2=\frac{b\left(p_2+q_2\right)}{\left[(\delta-1)2^{2m}+(2)3^m\right](\delta+2\gamma)} \tag{2.21}$$

and

$$a_3-a_2^2=\frac{b\left(p_2-q_2\right)}{2(\delta+2\gamma)3^m}. \tag{2.22}$$

Using (2.21) in (2.22), we obtain

$$a_3=\frac{b}{\delta+2\gamma}\left[\frac{p_2-q_2}{(2)3^m}+\frac{p_2+q_2}{(\delta-1)2^{2m}+(2)3^m}\right]. \tag{2.23}$$

From (2.17) and (2.18), we get

$$a_3=\frac{b}{3^m(\delta+2\gamma)}\left[p_2+\frac{(\delta+2\gamma)(1-\delta)p_1^2 b}{2(\delta+\gamma)^2}\right], \tag{2.24}$$

while from (2.19) and (2.20), we deduce that

$$a_3=\frac{b}{3^m(\delta+2\gamma)}\left[-q_2+\frac{(\delta+2\gamma)(\delta-1)bq_1^2}{2(\delta+\gamma)^2}+\frac{2(\delta+2\gamma)3^m bq_1^2}{2^{2m}(\delta+\gamma)^2}\right]. \tag{2.25}$$

Combining (2.17) and (2.21) for the computation of the upper-bound of $|a_2|$, and (2.23), (2.24) and (2.25) for the computation of $|a_3|$, by using Lemma 1.1 (with $\alpha=\lambda=0$), we easily find the estimates of Theorem 2.1. This completes the proof of Theorem 2.1. □

Taking $m=0$ and $b=1$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{B}_{\Sigma}(\gamma,\delta;k)$.

Corollary 2.1. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}(\gamma,\delta;k)$ with $\delta\neq-1$, $\delta\neq-\gamma$ and $\delta\neq-2\gamma$, then*

$$|a_2|\leq\min\left\{\sqrt{\frac{2k}{|\delta+1|\,|\delta+2\gamma|}},\frac{k}{|\delta+\gamma|}\right\}$$

and

$$|a_3| \leq \frac{k}{|\delta+2\gamma|} \min\left\{1+\frac{2}{|\delta+1|}; 1+\frac{|\delta+2\gamma|\,|1-\delta|\,k}{2\,|\delta+\gamma|^2}; 1+\frac{|\delta+2\gamma|\,|\delta+3|\,k}{2\,|\delta+\gamma|^2}\right\}.$$

Taking $m=0, b=1-\eta\,(0\leq\eta<1)$ and $k=2$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{B}_\Sigma(\gamma,\delta,\eta)$.

Corollary 2.2. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_\Sigma(\gamma,\delta,\eta)$ with $0\leq\eta<1$, $\delta\neq-1$, $\delta\neq-\gamma$ and $\delta\neq-2\gamma$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{4(1-\eta)}{|\delta+1|\,|\delta+2\gamma|}}, \frac{2(1-\eta)}{|\delta+\gamma|}\right\}$$

and

$$|a_3| \leq \frac{2(1-\eta)}{|\delta+2\gamma|} \min\left\{1+\frac{2}{|\delta+1|}; 1+\frac{|\delta+2\gamma|\,|1-\delta|\,(1-\eta)}{|\delta+\gamma|^2}; 1+\frac{|\delta+2\gamma|\,|\delta+3|\,(1-\eta)}{|\delta+\gamma|^2}\right\}.$$

Taking $m=0, \delta=1, b=1-\eta\,(0\leq\eta<1)$ and $k=2$ in Theorem 2.1, we obtain the following corollary which improves the result of Frasin and Aouf [10, Theorem 3.2].

Corollary 2.3. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_\Sigma(\gamma,\eta)$ with $0\leq\eta<1$, $\gamma\neq-1$ and $\gamma\neq-\frac{1}{2}$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\eta)}{|2\gamma+1|}}, \frac{2(1-\eta)}{|\gamma+1|}\right\}$$

and

$$|a_3| \leq \frac{2(1-\eta)}{|2\gamma+1|} \min\left\{2, 1+\frac{4\,|2\gamma+1|\,(1-\eta)}{|\gamma+1|^2}\right\}.$$

Taking $m=0, \gamma=1, b=1-\eta\,(0\leq\eta<1)$ and $k=2$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{B}_\Sigma(\delta,\eta)$.

Corollary 2.4. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_\Sigma(\delta,\eta)$ with $\delta\neq-1$ and $\delta\neq-2$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{4(1-\eta)}{|\delta+1|\,|\delta+2|}}, \frac{2(1-\eta)}{|\delta+1|}\right\}$$

and

$$|a_3| \leq \frac{2(1-\eta)}{|\delta+2|} \min\left\{1+\frac{2}{|\delta+1|}; 1+\frac{|\delta+2|\,|1-\delta|\,(1-\eta)}{|\delta+1|^2}; 1+\frac{|\delta+2|\,|\delta+3|\,(1-\eta)}{|\delta+1|^2}\right\}.$$

Taking $\delta=m=0, \gamma=1$ and $k=2$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{S}_\Sigma(b)$.

Corollary 2.5. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_\Sigma(b)$, then*

$$|a_2| \leq \min\left\{\sqrt{2\,|b|}, 2\,|b|\right\}$$

and

$$|a_3| \leq |b| \min\{3, 1+2\,|b|\}.$$

Taking $\delta=0, m=1, \gamma=1$ and $k=2$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{C}_\Sigma(b)$.

Corollary 2.6. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{C}_{\Sigma}(b)$, then*

$$|a_2| \leq \min\left\{\sqrt{|b|}, |b|\right\}$$

and

$$|a_3| \leq \frac{|b|}{3} \min\left\{4, 1+2|b|\right\}.$$

Taking $m = 0$ and $b = (1-\alpha)e^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{B}_{\Sigma}(\gamma, \delta, \alpha, \lambda; k)$.

Corollary 2.7. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}(\gamma, \delta, \alpha, \lambda; k)$ with $\delta \neq -1$, $\delta \neq -\gamma$ and $\delta \neq -2\gamma$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2k(1-\alpha)\cos\lambda}{|\delta+1||\delta+2\gamma|}}, \frac{k(1-\alpha)\cos\lambda}{|\delta+\gamma|}\right\}$$

and

$$|a_3| \leq \frac{k(1-\alpha)\cos\lambda}{|\delta+2\gamma|} \min\left\{\begin{array}{c} 1+\frac{2}{|\delta+1|}; 1+\frac{|\delta+2\gamma||(1-\delta)|k(1-\alpha)\cos\lambda}{2|\delta+\gamma|^2}; \\ 1+\frac{|\delta+2\gamma||\delta+5|k(1-\alpha)\cos\lambda}{2|\delta+\gamma|^2} \end{array}\right\}.$$

Taking $m = \delta = 0, \gamma = 1, k = 2$ and $b \to be^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{S}_{\Sigma}^{\lambda}(b)$.

Corollary 2.8. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{\lambda}(b)$, then*

$$|a_2| \leq \min\left\{\sqrt{2|b|\cos\lambda}, 2|b|\cos\lambda\right\}$$

and

$$|a_3| \leq |b|\cos\lambda \, \min\left\{3, 1+2|b|\cos\lambda\right\}.$$

Taking $m = \gamma = 1, \delta = 0, k = 2$ and $b \to be^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{C}_{\Sigma}^{\lambda}(b)$.

Corollary 2.9. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{C}_{\Sigma}^{\lambda}(b)$, then*

$$|a_2| \leq \min\left\{\sqrt{|b|\cos\lambda}, |b|\cos\lambda\right\}$$

and

$$|a_3| \leq \frac{|b|\cos\lambda}{3} \min\left\{4, 1+2|b|\cos\lambda\right\}.$$

Taking $\delta = m = 0, \gamma = 1$ and $b = (1-\alpha)e^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{S}_{\alpha}^{\lambda}(k)$.

Corollary 2.10. *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\alpha}^{\lambda}(k)$ $\left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$, then*

$$|a_2| \leq \min\left\{\sqrt{k(1-\alpha)\cos\lambda}, k(1-\alpha)\cos\lambda\right\}$$

and

$$|a_3| \leq \frac{k(1-\alpha)\cos\lambda}{2} \min\left\{3, 1+k(1-\alpha)\cos\lambda\right\}.$$

Taking $\delta = 0, \gamma = m = 1$ and $b = (1-\alpha)e^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{C}_{\alpha}^{\lambda}(k)$.

Corollary 2.11. *Let* $f(z)$ *given by (1.1) belongs to the class* $\mathcal{C}_{\alpha}^{\lambda}(k)$ $\left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$, *then*

$$|a_2| \leq \min\left\{\sqrt{\frac{k(1-\alpha)\cos\lambda}{2}}, \frac{k(1-\alpha)\cos\lambda}{2}\right\}$$

and

$$|a_3| \leq \frac{k(1-\alpha)\cos\lambda}{6} \min\{4, 1 + k(1-\alpha)\cos\lambda\}.$$

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Research Article

Quantitative Voronovskaya-Type Theorems for Fejér-Korovkin Operators

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ABSTRACT. In recent times, quantitative Voronovskaya type theorems have been presented in spaces of non-periodic continuous functions. In this work, we proved similar results but for Fejér-Korovkin trigonometric operators. That is we measure the rate of convergence in the associated Voronovskaya type theorem. Recall that these operators provide the optimal rate in approximation by positive linear operators. For the proofs, we present new inequalities related with trigonometric polynomials as well as with the convergence factor of the Fejér-Korovkin operators. Our approach includes spaces of Lebesgue integrable functions.

Keywords: Fejér-Korovkin operators, quantitative Voronovskaya type theorems.

2020 Mathematics Subject Classification: 41A36, 41A25, 42A10.

1. INTRODUCTION

In recent times, there have been some interests in studying quantitative Voronovskaya-type theorems, but almost all the papers are concerned with positive linear operators in spaces of non-periodic functions. The methods used in those papers are not useful in dealing with periodic functions for two reasons (at least). First they use different kinds of Taylor's formula and second, in the non-periodical case do not appear conjugate functions.

It is known that the Voronovskaya-type theorems are related with the saturation class of some families of operators. We have noticed that in the case of trigonometric polynomial approximation process the Voronovskaya-type theorems depend on particular properties of the operators. In [1], the authors considered this kind of problem for Fejér sums. In this paper, we consider the Fejér-Korovkin operators.

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic, continuous functions f defined on the real line $\mathbb{R}$ with the sup norm

$$\|f\|_\infty = \max_{x\in[-\pi,\pi]} \mid f(x) \mid .$$

For $1 \le p < \infty$, the Banach space $\mathbb{L}^p$ consists of all 2π-periodic, p-th power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} \mid f(x) \mid^p dx\Big)^{1/p}.$$

In order to simplify, we write $X^p = \mathbb{L}^p$ for $1 \le p < \infty$ and $X^\infty = C_{2\pi}$. By W_p^r, we mean the family of all functions $f \in X^p$ such that $f, \dots, D^{r-1}(f)$ are absolutely continuous and

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$D^r(f) \in X^p$. Here, $D(f) = D^1(f) = f'$ and $D^{r+1}(f) = D(D^r(f))$. Recall that for $f \in \mathbb{L}^1$ and $k \in \mathbb{N}_0$, the Fourier coefficients are defined by

$$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt \qquad \text{and} \qquad b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt,$$

and the (formal) Fourier series is given by

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty}(a_k(f)\cos(kx) + b_k(f)\sin(kx)) := \sum_{k=0}^{\infty} A_k(f).$$

For $n \in \mathbb{N}$, the Fejér-Korovkin kernel is defined by

$$K_n(x) = \frac{1}{n+2}\sin^2\frac{\pi}{n+2}\frac{\cos^2((n+2)x/2)}{(\cos(\pi/(n+2)) - \cos x)^2}$$

for $x \neq \pm\pi/(n+2) + 2j\pi$, $j \in \mathbb{Z}$. For $f \in X^1$ and $n \in \mathbb{N}$, the Fejér-Korovkin operator is defined by

$$\mathbb{F}_n(f, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)K_n(t)dt.$$

Some Voronovskaya type theorems for the operators $\mathbb{F}_n$ are known.

Theorem 1.1. (Korovkin, [6]) *If $f \in C_{2\pi}$, $x \in [-\pi, \pi]$ and $f''(x)$ exists, then*

$$\mathbb{F}_n(f, x) - f(x) = \frac{\pi^2}{2n^2}f''(x) + o(n^{-2}).$$

Theorem 1.2. (Butzer and Görlich, [2, page 385]) *If $1 < p < \infty$ and $f \in W_p^2$, then*

$$\lim_{n\to\infty}\left|\left|n^2(\mathbb{F}_n(f) - f) - \frac{\pi^2}{2}f''\right|\right|_p = 0.$$

The main purpose of the paper is to present a quantitative Voronovskaya-type theorem for the operators $\mathbb{F}_n$. That is we want to estimate the rate of convergence to zero in the results presented above. This will be accomplished in the last section of the article, where the case $p = 1$ is also included.

The work is organized as follows. In Section 2, we include a collection of known definitions and results which will be used later. For instance, in the non-periodic case conjugate functions are not needed, but for our approach they are important. In Section 3, we prove some inequalities related with trigonometric polynomials (we think that they have independent interest). Section 4 is very technical. It involves complicated computations related with the convergence factors of Fejér-Korovkin operators. In Section 5, we include the main result (Theorem 5.7). The most important idea is to prove first a Voronovskaya theorem limited to polynomials (Proposition 5.8).

2. Known results

The convolution of $f, g \in \mathbb{L}^1$, g an even function, is defined by

$$(f * g)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)g(t)dt = \frac{1}{\pi}\int_{0}^{\pi}(f(x+t) + f(x-t))g(t)dt.$$

It is known that if $f \in X^p$ and $g \in X^1$, then $f * g \in X^p$ and

$$\|f * g\|_p \leq \|g\|_1 \|f\|_p. \tag{2.1}$$

For $f \in X^1$, the *conjugate function* is defined by

$$\widetilde{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \frac{f(x+t)-f(x-t)}{\tan(t/2)}dt = -\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\varepsilon}^{\pi} \frac{f(x+t)-f(x-t)}{\tan(t/2)}dt,$$

whenever the limit exists. It is known that if $f \in X^p$ with $1 < p < \infty$, then $\widetilde{f} \in X^p$, and that is not the case for $p = 1$ and $p = \infty$. Recall that for $n \in \mathbb{N}$ and $f \in X^1$, the Fejér sum of order n is defined by

$$\sigma_n(f,x) = \sum_{k=0}^{n}\Big(1 - \frac{k}{n+1}\Big)A_k(f,x).$$

Throughout the paper, we use the following notations. $\mathbb{T}_n$ denotes the family of all trigonometric polynomials of degree no greater that n, and $\mathbb{T}_n^0$ is the family of all $T \in \mathbb{T}_n$ with mean zero, that is

$$\int_{-\pi}^{\pi} T_n(x)dx = 0.$$

Proposition 2.1. ([1, Prop. 2.4]) *For each $n, r \in \mathbb{N}$ and $T \in \mathbb{T}_n$, one has*

$$(I-\sigma_n)^r(T) = \begin{cases} \dfrac{(-1)^{r/2}}{(n+1)^r}D^r(T), & r \text{ even}, \\[2ex] \dfrac{(-1)^{(r-1)/2}}{(n+1)^r}D^r(\widetilde{T}), & r \text{ odd}. \end{cases}$$

Theorem 2.3. ([9, p. 215]) *If $1 \le p \le \infty$, $r, n \in \mathbb{N}$ and $T \in \mathbb{T}_n$, then*

$$\|D^r(T)\|_p \le \Big(\frac{n}{2\sin(nh/2)}\Big)^r \|\Delta_h^r T\|_p \tag{2.2}$$

for any $h \in (0, 2\pi/n)$. Moreover $\|D^r(\widetilde{T})\|_p \le n^r \|T\|_p$.

For $r \in \mathbb{N}$, a function $f \in X^p$, and $h > 0$, the usual modulus of smoothness of order r is defined by

$$\omega_r(f,t)_p = \sup_{|h|\le t}\|(I-T_h)^r(f)\|_p, \tag{2.3}$$

where $T_h(f,x) = f(x+h)$ is the translation operator. We also use the notations $\Delta_h^r f(x) = (I-T_h)^r(f)$. For $1 \le p \le \infty$ and $f \in X^p$, the best approximation of f by elements of $\mathbb{T}_n$ is defined by

$$E_{n,p}(f) = \inf_{T\in\mathbb{T}_n}\|f-T\|_p.$$

Theorem 2.4. (Foucart et al, [5, Theorem 2.5]) *If $1 \le p \le \infty$, $f \in X^p$, and $n \in \mathbb{N}$, then*

$$E_{n,p}(f) \le 5\omega_1\Big(f, \frac{2\pi}{n+1}\Big)_p.$$

The following result is easy to prove (see [3, p. 77]).

Proposition 2.2. *Assume $1 \le p \le \infty$ and*

$$Q_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\lambda_{k,n}\cos(kx)$$

is a non-negative trigonometric polynomial. If $g \in W_p^2$, then

$$\|g - g * Q_n\|_p \le \frac{\pi^2}{2}(1-\lambda_{1,n})\|D^2(g)\|_p.$$

We need some results taken from the Zygmund book [10, pages 93 and 183].

Proposition 2.3. *If $\{c_n\}$ is a convex and bounded sequence, then $\{c_n\}$ decreases, $n\Delta^1 c_n \to 0$, and*

$$\sum_{n=0}^{\infty}(n+1)\Delta^2 c_n = c_0 - \lim_{n\to\infty} c_n.$$

Proposition 2.4. *If $\{c_n\}$ is a convex sequence which converges to zero, then the series*

$$\frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos(kx)$$

converges for all $x \neq 0$ to a nonnegative and integrable function.

3. Inequalities for trigonometric polynomials

Proposition 3.5. *The function*

$$\varphi(x) = \sum_{k=1}^{\infty}\frac{\cos(kx)}{k}, \qquad x \neq 0,$$

is integrable. Moreover

$$\|\varphi\|_1 \leq \frac{3}{2}.$$

Proof. If we consider Proposition 2.4 with the sequence $\{a_n\}$, given by $a_0 = 3/2$ and $a_k = 1/k$ for $k \in \mathbb{N}$, then

$$\frac{3}{4} + \sum_{k=1}^{\infty}\frac{\cos(kx)}{k} \geq 0$$

for $x \neq 0$. But

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k=1}^{\infty}\frac{\cos(kx)}{k}\Big|dx &\leq \frac{3}{4} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\frac{3}{4} + \sum_{k=1}^{\infty}\frac{\cos(kx)}{k}\Big|dx \\ &= \frac{3}{4} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{3}{4} + \sum_{k=1}^{\infty}\frac{\cos(kx)}{k}\Big)dx \\ &= \frac{3}{4} + \frac{1}{2}\sum_{k=0}^{\infty}(k+1)\Delta^2 a_k \\ &= \frac{3}{2},\end{aligned}$$

where we use Proposition 2.3. □

Remark 3.1. In the proof of the previous Proposition, we can not take $a_0 < 3/2$, because we need that $\Delta^2 a_0 = a_0 - 2a_1 + a_2 = a_0 - 3/2 \geq 0$.

Theorem 3.5. *For $1 \leq p \leq \infty$, each $n \in \mathbb{N}$ and $T \in \mathbb{T}_n$, one has*

$$\|(I-\sigma_n)(T)\|_p \leq \frac{3(n+1)}{2}\|(I-\sigma_n)^2(T)\|_p,$$

where σ_n denotes the Fejér sum. Moreover, if $T \in \mathbb{T}_n^0$, then

$$\|T\|_p \leq \frac{3(n+1)}{2}\|(I-\sigma_n)(T)\|_p.$$

Proof. Define $\tau_n : \mathbb{T}_n^0 \to \mathbb{T}_n^0$ by the equation

$$\tau_n(T) = (n+1)\sum_{k=1}^{n} \frac{A_k(x)}{k}, \tag{3.4}$$

where $T(x) = \sum_{k=1}^{n} A_k(x)$. Notice that, for each $T \in \mathbb{T}_n^0$, it follows from (2.1) that

$$\|\tau_n(T)\|_p \leq (n+1)\|\varphi\|_1\|T\|_p \leq \frac{3(n+1)}{2}\|T\|_p,$$

where φ is the function in Proposition 3.5. On the other hand, if T is given as in (3.4), then

$$\tau_n((I-\sigma_n)(T), x) = \tau_n\Big(\sum_{k=1}^{n} \frac{k}{n+1}A_k, x\Big) = \sum_{k=1}^{n} A_k(x) = T(x). \tag{3.5}$$

Therefore, for any $T \in \mathbb{T}_n$, if we set $T^* = T - A_0(T)/2$, then

$$\begin{aligned}\|(I-\sigma_n)(T)\|_p &= \|(I-\sigma_n)(T^*)\|_p \\ &= \|\tau_n((I-\sigma_n)^2(T^*)\|_p \\ &\leq \frac{3(n+1)}{2}\|(I-\sigma_n)^2(T^*)\|_p \\ &= \frac{3(n+1)}{2}\|(I-\sigma_n)^2(T)\|_p.\end{aligned}$$

Finally, if $T \in \mathbb{T}_n^0$, it follows from (3.5) that

$$\|T\|_p = \|\tau_n((I-\sigma_n)(T))\|_p \leq \frac{3(n+1)}{2}\|(I-\sigma_n)(T)\|_p.$$

□

Intermediate derivatives have been used by several authors. Here, we present some particular constants.

Corollary 3.1. *If $n \in \mathbb{N}$ and $T \in \mathbb{T}_n^0$, then*

$$\|T\|_p \leq \frac{9}{4}\|T''\|_p, \quad \|T\|_p \leq \frac{27}{8}\|D^3(\widetilde{T})\|_p,$$
$$\|\widetilde{T}'\|_p \leq \frac{3}{2}\|T''\|_p, \quad \|\widetilde{T}'\|_p \leq \frac{9}{4}\|D^3(\widetilde{T})\|_p$$

and

$$\|D^3(\widetilde{T})\|_p \leq 2(n+1)\|T''\|_p.$$

Proof. The result follows from Theorem 3.5 and Proposition 2.1. For instance,

$$\|T\|_p \leq \frac{3(n+1)}{2}\|(I-\sigma_n)(T)\|_p \leq \frac{9(n+1)^2}{4}\|(I-\sigma_n)^2(T)\|_p = \frac{9}{4}\|T''\|_p$$

and

$$\|T\|_p \leq \frac{27(n+1)^3}{8}\|(I-\sigma_n)^3(T)\|_p = \frac{27}{8}\|D^3(\widetilde{T})\|_p.$$

On the other hand,

$$\|\widetilde{T}'\|_p = (n+1)\|(I-\sigma_n)(T)\|_p \leq \frac{3}{2}(n+1)^2\|(I-\sigma_n)^2(T)\|_p = \frac{3}{2}\|T''\|_p$$

and

$$\|\widetilde{T}'\|_p \leq \frac{9}{4}(n+1)^3\|(I-\sigma_n)^3(T)\|_p = \frac{9}{4}\|D^3(\widetilde{T})\|_p.$$

Finally,

$$\|\widetilde{T}'''\|_p = \|(\widetilde{T''})'\|_p = (n+1)\|(I-\sigma_n)(T'')\|_p \leq 2(n+1)\|T''\|_p.$$

□

4. Estimates related with convergence factors

Proposition 4.6. *The Fejér-Korovkin kernel can be written in the form*

$$K_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \zeta_n\Big(\frac{k}{n+2}\Big)\cos(kx) = \frac{1}{2} + \sum_{k=1}^{n} \varrho_{k,n}\cos(kx).$$

The numbers $\varrho_{k,n}$ are usually called the convergence factors. Representations for the convergence factors of Fejér-Korovkin operators appeared in different places. For instance, see [8, p. 1098]. We set

$$\zeta_n(x) = (1-x)\cos(\pi x) + \frac{1}{n+2}\cot\frac{\pi}{n+2}\sin(\pi x). \tag{4.6}$$

Corollary 4.2. *For each $n > 1$, one has*

$$\varrho_{1,n} = \cos\frac{\pi}{n+2}, \qquad \varrho_{2,n} = 1 - 2(1-\cos^2\frac{\pi}{n+2})\Big(1-\frac{1}{n+2}\Big)$$

and

$$4 - \frac{1-\varrho_{2,n}}{1-\varrho_{1,n}} = 2 - 2\cos\frac{\pi}{n+2} + \frac{2(1+\cos(\pi/(n+2)))}{n+2}.$$

We need some estimates related with the convergence factors. In what follows, we set

$$a_n = \frac{1}{\pi}\Big(1 - \frac{\pi}{n+2}\cot\frac{\pi}{n+2}\Big),$$

$$H_{n,1}(x) = (1-x)\cos(\pi x) + \frac{\sin(\pi x)}{\pi} + \frac{\pi^2}{2}x^2 - \frac{\pi^2}{3}x^3, \tag{4.7}$$

$$H_{n,2}(x) = a_n(\sin(\pi x) - \pi x) \tag{4.8}$$

and

$$\lambda_{k,n} = 1 - \zeta_n\Big(\frac{k}{n+2}\Big) - \frac{\pi^2}{2}\frac{k^2}{(n+2)^2} + \frac{\pi^2 k^3}{3(n+2)^3} - \pi a_n \frac{k}{n+2}. \tag{4.9}$$

Lemma 4.1. *For $0 < x \leq \pi$, one has*

$$1 - \frac{x^2}{2} \leq x\cot x \leq 1 - \frac{x^2}{3}.$$

In particular,

$$a_n \leq \frac{\pi}{2(n+2)^2}.$$

Proof. The first assertion follows from standard arguments. For the second one,

$$a_n = \frac{1}{\pi}\Big(1 - \frac{\pi}{n+2}\cot\frac{\pi}{n+2}\Big) \leq \frac{1}{\pi}\frac{\pi^2}{2(n+2)^2}.$$

□

Lemma 4.2. *Let $x_0 \in (0, \pi/2)$ be defined by the Equation $3\tan(\pi x_0) = \pi(1 - x_0)$. For $x \in [0, x_0]$, one has*

$$\pi(1 - x)(1 - \cos(\pi x)) + \sin(\pi x) - \pi x \geq 0.$$

Moreover, $x_0 > \pi/6$.

Proof. For $0 < x \leq 1$,

$$-4\cos(\pi x) - \pi(1 - x)\sin(\pi x) \leq 0.$$

Therefore, the function $f_1(x) = -3\sin(\pi x) + \pi(1 - x)\cos(\pi x)$ decreases in $[0, 1]$. But $f_1(x_0) = 0$ if and only if

$$\tan(\pi x_0) = \frac{\pi}{3}(1 - x_0).$$

Hence,$f_1(x) \geq 0$ for $x \in [0, x_0]$.

With similar arguments, we verify that the function $f_2(x) = -2+2\cos(\pi x)+\pi(1-x)\sin(\pi x)$ increases in $[0, x_0]$. Thus, $f_2(x) \geq 0$ for $x \in [0, x_0]$.

If $f_3(x) = \pi(1 - x)(1 - \cos(\pi x)) + \sin(\pi x) - \pi x$, then $f_3'(x) = \pi f_2(x) \geq 0$, for $x \in [0, x_0]$. □

Lemma 4.3. *Suppose that $n, m \in \mathbb{N}$, $n \geq m \geq 5$ and $H_{n,1}$ is defined by (4.7), then*

$$\left|1 - H_{n,1}\Big(\frac{m}{n+2}\Big)\right| \leq \frac{\pi^4 m^4}{6(n+2)^4}\Big(\frac{1}{4} - \frac{m}{5(n+2)}\Big),$$

$$0 \leq H_{n,1}\Big(\frac{m}{n+2}\Big) - H_{n,1}\Big(\frac{m-1}{n+2}\Big) \leq \frac{\pi^3(m-1)^3}{4(n+2)^4}$$

and, for $0 \leq k \leq m - 2$,

$$\left|H_{n,1}\Big(\frac{k+2}{n+2}\Big) - 2H_{n,1}\Big(\frac{k+1}{n+2}\Big) + H_{n,1}\Big(\frac{k}{n+2}\Big)\right| \leq \frac{\pi^4(k+1)^2}{2(n+2)^4}.$$

Proof. Notice that

$$H'_{n,1}(x) = -\pi(1 - x)\sin(\pi x) + \pi^2 x(1 - x) = \pi(1 - x)(\pi x - \sin(\pi x))$$

and

$$H''_{n,1}(x) = \pi\Big(\pi(1 - x)(1 - \cos(\pi x)) + \sin(\pi x) - \pi x\Big).$$

(i) If $x_m = m/(n + 2)$, then

$$\begin{aligned} | 1 - H_{n,1}(x_m) | &= | H_{n,1}(0) - H_{n,1}(x_m) | \\ &= \left|\int_0^{x_m} H'_{n,1}(s)ds\right| \\ &= \pi\int_0^{x_m}(1 - s)(\pi s - \sin(\pi s))ds. \end{aligned}$$

For $0 \leq y \leq 1$, one has

$$0 \leq (1 - y)(\pi y - \sin(\pi y)) \leq \frac{\pi^3 y^3}{6}(1 - y),$$

hence, for $0 \leq z \leq n + 2$, the function

$$F(z) = \int_0^{z/(n+2)}(1 - s)(\pi s - \sin(\pi s))ds - \frac{\pi^3}{6(n+2)^4}\Big(\frac{z^4}{4} - \frac{z^5}{5(n+2)}\Big)$$

decreases. Taking into account that $F(0) = 0$,

$$| 1 - H_{n,1}(x_m) | = \pi F(m) \leq \frac{\pi^4 m^4}{6(n+2)^4}\Big(\frac{1}{4} - \frac{m}{5(n+2)}\Big).$$

(ii) On the other hand, if $y = (m-1)/(n+2)$,

$$\begin{aligned} 0 \leq H_{n,1}\Big(\frac{m}{n+2}\Big) - H_{n,1}\Big(\frac{m-1}{n+2}\Big) &= \int_y^{y+1/(n+2)} H'_{n,1}(s)ds \\ &= \pi \int_y^{y+1/(n+2)} (1-s)\Big(\pi s - \sin(\pi s)\Big)ds. \end{aligned}$$

As before, for $z \geq 0$, the function

$$\begin{aligned} G_y(z) &= \int_y^{y+z} (1-s)(\pi s - \sin(\pi s))ds - \frac{\pi^3((y+z)^4 - y^4)}{24(n+2)^4} \\ &= \int_0^z (1-(y+s))(\pi(y+s) - \sin(\pi(y+s))ds - \frac{\pi^3((y+z)^4 - y^4)}{24} \end{aligned}$$

decreases and $G_y(0) = 0$. Therefore,

$$\begin{aligned} H_{n,1}\Big(\frac{m}{n+2}\Big) - H_{n,1}\Big(\frac{m-1}{n+2}\Big) &\leq \frac{\pi^3((y+1/(n+2))^4 - y^4)}{24} \\ &= \frac{\pi^3}{24}\Big(4\frac{y^3}{(n+2)} + 6\frac{y^2}{(n+2)^2} + 4\frac{y}{(n+2)^3} + \frac{1}{(n+2)^4}\Big) \\ &= \frac{\pi^3}{24(n+2)^4}\Big(4(m-1)^3 + 6(m-1)^2 + 4(m-1) + 1\Big) \\ &= \frac{\pi^3(m-1)^3}{24(n+2)^4}\Big(4 + \frac{6}{(m-1)} + \frac{4}{(m-1)^2} + \frac{1}{(m-1)^3}\Big) \\ &\leq \frac{\pi^3(m-1)^3}{4(n+2)^4} \end{aligned}$$

for $m \geq 5$.
(iii) Let x_0 be given as in Lemma 4.2. Set $z = k/(n+2)$. Note that

$$0 \leq z < z + \frac{2}{n+2} \leq \frac{m-2+2}{n+2} \leq \frac{\pi}{6} < x_0.$$

Hence, if $0 \leq s \leq z + 2/(n+2)$,

$$\begin{aligned} 0 &\leq \frac{1}{\pi} H''_{n,1}(s) \\ &= \pi(1-s)(1-\cos(\pi s)) + \sin(\pi s) - \pi s \\ &\leq \pi(1-\cos(\pi s)) \\ &< \pi\frac{(\pi s)^2}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned}
0 &\le H_{n,1}(z) - 2H_{n,1}\Big(z + \frac{1}{n+2}\Big) + H_{n,1}\Big(z + \frac{2}{n+2}\Big) \\
&= \int_0^{1/(n+2)} (H'_{n,1}(z+s+1/(n+2)) - H'_{n,1}(z+s))ds \\
&= \int_0^{1/(n+2)} \int_0^{1/(n+2)} H''_{n,1}(z+s+t)dtds \\
&\le \frac{\pi^4}{2} \int_0^{1/(n+2)} \int_0^{1/(n+2)} (z+s+t)^2 dtds \\
&= \frac{\pi^4}{6} \int_0^{1/(n+2)} \Big(3\frac{(z+s)^2}{n+2} + 3\frac{(z+s)}{(n+2)^2} + \frac{1}{(n+2)^3}\Big)ds \\
&= \frac{\pi^4}{6}\Big(\frac{1}{n+2}\Big(\frac{3z^2}{n+2} + \frac{3z}{(n+2)^2} + \frac{1}{(n+2)^3}\Big) + \frac{3}{2(n+2)^3}\Big(2z + \frac{1}{n+2}\Big) + \frac{1}{(n+2)^4}\Big) \\
&= \frac{\pi^4}{6(n+2)^4}\Big(3k^2 + 3k + 1 + 3k + \frac{3}{2} + 1\Big) \\
&= \frac{\pi^4}{6(n+2)^4}\Big(3k^2 + 6k + \frac{5}{2}\Big) \\
&< \frac{\pi^4}{6(n+2)^4}\Big(3k^2 + 6k + 3\Big) \\
&= \frac{\pi^4}{2(n+2)^4}(k+1)^2.
\end{aligned}$$

□

Lemma 4.4. *If $n, m \in \mathbb{N}$, $n \ge m \ge 5$, and $H_{n,2}$ is defined by (4.7), then*

$$\Big|H_{n,2}\Big(\frac{m}{n+2}\Big)\Big| \le \frac{\pi^4}{12}\frac{m^3}{(n+2)^5},$$

$$\Big|H_{n,2}\Big(\frac{m-1}{n+2}\Big) - H_{n,2}\Big(\frac{m}{n+2}\Big)\Big| \le \frac{\pi^4}{4}\frac{m^2}{(n+2)^5},$$

and, for $0 \le k \le m-2$,

$$\Big|H_{n,2}\Big(\frac{k+2}{n+2}\Big) - 2H_{n,2}\Big(\frac{k-1}{n+2}\Big) + H_{n,2}\Big(\frac{k}{n+2}\Big)\Big| \le \frac{\pi^4(3k^2+9k+7)}{4(n+2)^5}.$$

Proof. (i) If $x = m/(n+2)$, one has

$$\begin{aligned}
|\ H_{n,2}(x)\ | &= \frac{1}{\pi}\Big(1 - \frac{\pi}{n+2}\cot\frac{\pi}{n+2}\Big)(\pi x - \sin(\pi x)) \\
&\le \frac{1}{\pi}\frac{\pi^2}{2(n+2)^2}\frac{(\pi x)^3}{6} \\
&= \frac{\pi^4}{12}\frac{m^3}{(n+2)^5}.
\end{aligned}$$

(ii) On the other hand, there exists $\theta \in (\pi(m-1)/(n+2), \pi m/(n+2))$ such that (see Lemma 4.1)

$$\begin{aligned}\Big|H_{n,2}\Big(\frac{m-1}{n+2}\Big) - H_{n,2}\Big(\frac{m}{n+2}\Big)\Big| &= a_n\Big|\sin\Big(\frac{\pi m}{n+2}\Big) - \sin\Big(\frac{\pi(m-1)}{n+2}\Big) - \frac{\pi}{n+2}\Big| \\ &= \frac{\pi a_n}{n+2}(1-\cos\theta) \\ &\le \frac{\pi a_n}{n+2}\frac{\theta^2}{2} \\ &\le \frac{\pi^4}{4(n+2)^3}\Big(\frac{m}{n+2}\Big)^2 \\ &\le \frac{\pi^4}{4}\frac{m^2}{(n+2)^5}.\end{aligned}$$

Finally, taking into account Lemma 4.1 and setting $z = k/(n+2)$, one has

$$\begin{aligned}&\mid H_{n,2}(z+2/(n+2)) - 2H_{n,2}(z+1/(n+2)) + H_{n,2}(z) \mid \\ =&a_n \mid \sin(\pi(z+2/(n+2)) - 2\sin(\pi(z+1/(n+2)) + \sin(\pi z)) \mid \\ =&\pi^2\, a_n \left|\int_0^{1/(n+2)}\int_0^{1/(n+2)} \sin(\pi(z+s+t))dsdt\right| \\ =&\pi\, a_n \int_0^{1/(n+2)} \Big(1-\cos\Big(\pi(z+s+\frac{1}{n+2})\Big)\Big)ds \\ \le&\frac{\pi^4}{4(n+2)^2}\int_0^{1/(n+2)} \Big(z+s+\frac{1}{n+2}\Big)^2 ds \\ =&\frac{\pi^4}{4(n+2)^2}\Big(\frac{3}{n+2}\Big(z+\frac{1}{n+2}\Big)^2 + \frac{3}{(n+2)^2}\Big(z+\frac{1}{n+2}\Big) + \frac{1}{(n+2)^3}\Big) \\ =&\frac{\pi^4}{4(n+2)^5}\Big(3(k+1)^2 + 3(k+1) + 1\Big) \\ =&\frac{\pi^4}{4(n+2)^5}\Big(3k^2 + 9k + 7\Big).\end{aligned}$$

□

Lemma 4.5. *If $n, m \in \mathbb{N}$, $5 \le m \le n$ and $\lambda_{k,n}$ is defined by (4.9), then*

$$\mid \lambda_{m,n} \mid \le \frac{\pi^4}{24}\frac{m^4}{(n+2)^4}, \qquad \mid m(\lambda_{m-1,n} - \lambda_{m,n}) \mid \le \frac{\pi^3 m^4}{4(n+2)^4}$$

and

$$\Big|\sum_{k=0}^{m-2}(k+1)\Delta^2\lambda_{k,n+2}\Big| < \frac{\pi^4 m^3(m-1)}{(n+2)^4}.$$

Proof. Notice that

$$\begin{aligned}1 - H_{n,1}(x) + H_{n,2}(x) &= 1 - (1-x)\cos(\pi x) - \frac{\sin(\pi x)}{\pi} - \frac{\pi^2}{2}x^2 + \frac{\pi^2}{3}x^3 + a_n(\sin(\pi x) - \pi x) \\ &= 1 - (1-x)\cos(\pi x) - \frac{\pi}{n}\cot\frac{\pi}{n}\sin(\pi x) - \frac{\pi^2}{2}x^2 + \frac{\pi^2}{3}x^3 - a_n\pi x.\end{aligned}$$

Therefore,

$$\lambda_{k,n} = 1 - H_{n,1}\Big(\frac{k}{n+2}\Big) + H_{n,2}\Big(\frac{k}{n+2}\Big).$$

(i) If $m \geq 5$, it follows from Lemmas 4.3 and 4.4 that

$$\begin{aligned}
|\, \lambda_{m,n} \,| &= \Big|1 - H_{n,1}\Big(\frac{m}{n+2}\Big) + H_{n,2}\Big(\frac{m}{n+2}\Big)\Big| \\
&\leq \frac{\pi^4}{6}\frac{m^4}{(n+2)^4}\Big(\frac{1}{4} - \frac{m}{5(n+2)}\Big) + \frac{\pi^4}{12}\frac{m^3}{(n+2)^5} \\
&\leq \frac{\pi^4}{24}\frac{m^4}{(n+2)^4}.
\end{aligned}$$

(ii) On the other hand,

$$\begin{aligned}
|\, m(\lambda_{m-1,n} - \lambda_{m,n}) \,| &\leq m\Big|H_{n,1}\Big(\frac{m}{n+2}\Big) - H_{n,1}\Big(\frac{m-1}{n+2}\Big)\Big| + m\Big|H_{n,2}\Big(\frac{m}{n+2}\Big) - H_{n,2}\Big(\frac{m-1}{n+2}\Big)\Big| \\
&\leq \frac{\pi^3 m(m-1)^3}{4(n+2)^4} + \frac{\pi^4}{4}\frac{m^3}{(n+2)^5} \\
&\leq \frac{\pi^3 m^3}{4(n+2)^4}\Big(m - 1 + \frac{\pi}{(n+2)}\Big) \\
&\leq \frac{\pi^3 m^4}{4(n+2)^4}.
\end{aligned}$$

(iii) Finally, for $0 \leq k \leq m-2$, then

$$\begin{aligned}
|\, \Delta^2 \lambda_{k,n+2} \,| &\leq \Big|H_{n,1}\Big(\frac{k+2}{n+2}\Big) - 2H_{n,1}\Big(\frac{k+1}{n+2}\Big) + H_{n,1}\Big(\frac{k}{n+2}\Big)\Big| \\
&\quad + \Big|H_{n,2}\Big(\frac{k+2}{n}\Big) - 2H_{n,2}\Big(\frac{k+1}{n}\Big) + H_{n,2}\Big(\frac{k}{n}\Big)\Big| \\
&\leq \frac{\pi^4(k+1)^2}{2(n+2)^4} + \frac{\pi^4(3k^2+9k+7)}{4(n+2)^5} \\
&\leq \frac{\pi^4}{(n+2)^4}(4k^2+11k+8).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{k=0}^{m-2}(k+1)(4k^2+11k+8) &= \sum_{k=0}^{m-2}(4k^3+15k^2+19k+8) \\
&= (m-1)(m-2)\Big((m-2)(m-1) + 15\frac{(2m-3)}{6} + \frac{19}{2} + \frac{8}{m-2}\Big) \\
&= m^4 - m^3.
\end{aligned}$$

Therefore,

$$\Big|\sum_{k=0}^{m-2}(k+1)\Delta^2\lambda_{k,n+2}\Big| \leq \frac{\pi^4 m^3(m-1)}{(n+2)^4}.$$

□

5. MAIN RESULTS

Theorem 5.6. *Assume* $1 \leq p \leq \infty$, $m \in \mathbb{N}$ *and* $m \geq 2$. *If* $g \in W_p^2$, $T \in \mathbb{T}_m$, *and* $\|g - T\|_p = E_{m,p}(g)$, *then*

$$\|g'' - T''\|_p \leq 30 \; E_{m,p}(g''). \tag{5.10}$$

Proof. It was proved in [1, Th. 3.4] that, for any $r \in \mathbb{N}$, $g \in W_p^r$, $T \in \mathbb{T}_m$, and $m \geq \max\{2, r\}$.

$$\|D^r(g) - D^r(S)\|_p \leq \Big(4 + \frac{1}{r} + \ln(2r)\Big)\Big(1 + \frac{e\pi}{2}\Big) E_{m,p}(D^r(g)),$$

when $\|g - S\|_p = E_{m,p}(g)$. Since

$$\Big(4 + \frac{1}{2} + \ln(4)\Big)\Big(1 + \frac{e\pi}{2}\Big) < 30,$$

one has

$$\|D^2(g) - D^2(T)\|_p \leq 30 E_{m,p}(D^2(g)).$$

□

Proposition 5.7. *If* $1 \leq p \leq \infty$ $g \in W_p^2$, *then*

$$\|g - \mathbb{F}_n(g)\|_p \leq \frac{\pi^4}{4(n+2)^2}\|D^2(g)\|_p.$$

Proof. It follows from Proposition 2.2 and Corollary 4.2 that

$$\begin{aligned} \|g - \mathbb{F}_n(g)\|_p &\leq \frac{\pi^2}{2}(1 - \varrho_{1,n})\|D^2(g)\|_p \\ &= \frac{\pi^2}{2}\Big(1 - \cos\frac{\pi}{n+2}\Big)\|D^2(g)\|_p \\ &\leq \frac{\pi^4}{4(n+2)^2}\|D^2(g)\|_p. \end{aligned}$$

□

Proposition 5.8. *For each* $1 \leq p \leq \infty$, *if* $n, m \in \mathbb{N}$, $n \geq m \geq 5$ *and* $T \in \mathbb{T}_m^0$, *then*

$$\Big\|\mathbb{F}_n(T) - T - \frac{\pi^2}{2(n+2)^2}T''\Big\|_p \leq \Big(\frac{9\pi^3 m^4}{(n+2)^4} + \frac{3\pi^2}{4(n+2)^3} + \frac{2\pi^2(m+1)}{3(n+2)^3}\Big)\|T''\|_p.$$

Proof. Let $\lambda_{k,n}$ be defined by (4.9). If $T = \sum_{k=0}^{m} A_k(x) \in \mathbb{T}_n$, with $A_k(x) = a_k \cos(kx) + b_k \sin(kx)$, then

$$\begin{aligned}
&T(x) - \mathbb{F}_n(T,x) + \frac{\pi^2}{2(n+2)^2} T''(x) \\
&= \sum_{k=0}^{m} \Big(1 - \zeta_{n+2}\Big(\frac{k}{n+2}\Big) - \frac{\pi^2 k^2}{2(n+2)^2}\Big) A_k(x) \\
&= \sum_{k=0}^{m} \Big(1 - \zeta_{n+2}\Big(\frac{k}{n+2}\Big) - \frac{\pi^2}{2}\frac{k^2}{(n+2)^2} + \frac{\pi^2 k^3}{3(n+2)^3} - \pi a_n \frac{k}{n+2}\Big) A_k(x) \\
&+ \sum_{k=1}^{m} \Big(\pi a_n \frac{k}{n+2} - \frac{\pi^2 k^3}{3(n+2)^3}\Big) A_k(x) \\
&= \sum_{k=0}^{m} \lambda_{k,n} A_k(x) + \frac{\pi a_n}{n+2}\widetilde{T}' - \frac{\pi^2}{3(n+2)^3}\widetilde{T}'''.
\end{aligned}$$

We apply twice the Abel transformation to obtain

$$\begin{aligned}
\sum_{k=0}^{m} \lambda_{k,n} A_k(x) &= \lambda_{m,n} \sum_{k=0}^{m} A_k(x) + \sum_{k=0}^{m-1} (\lambda_{k,n} - \lambda_{k+1,n}) \sum_{j=0}^{k} A_j(x) \\
&= \lambda_{m,n} T(x) + (\lambda_{m-1,n} - \lambda_{m,n}) \sum_{k=0}^{m-1}\sum_{j=0}^{k} A_k(x) + \sum_{k=0}^{m-2} \Delta^2 \lambda_{k,n} \sum_{j=0}^{k}\sum_{i=0}^{j} A_i(x) \\
&= \lambda_{m,n} T(x) + m(\Delta^1 \lambda_{m-1,n}) \sigma_m(T,x) + \sum_{k=0}^{m-2} (k+1)\Delta^2 \lambda_{k,n} \sigma_k(T,x),
\end{aligned}$$

where σ_k is the Fejér sum. It follows from Lemma 4.5 that

$$\begin{aligned}
\Big\|\sum_{k=0}^{n} \lambda_{k,n} A_k(x)\Big\|_p &\le \Big(\frac{\pi^4}{24}\frac{m^4}{(n+2)^4} + \frac{\pi^3 m^4}{4(n+2)^4} + \frac{\pi^4 m^3(m-1)}{(n+2)^4}\Big)\|T\|_p \\
&\le \Big(\frac{1}{24}\frac{16}{5} + \frac{1}{4} + \frac{16}{5}\Big)\frac{\pi^4 m^4}{(n+2)^4}\|T\|_p \\
&\le \frac{\pi^3 m^4}{(n+2)^4}\|T\|_p \\
&\le \frac{4\pi^3 m^4}{(n+2)^4}\|T\|_p.
\end{aligned}$$

Now, taking into account Lemma 4.1, we obtain

$$\begin{aligned}
\Big\|T - \mathbb{F}_n(T) + \frac{\pi^2}{2(n+2)^2} T''\Big\|_p &\le \Big\|\sum_{k=0}^{m} \lambda_{k,n} A_k(x)\Big\| + \frac{\pi a_n \|\widetilde{T}'\|_p}{n+2} - \frac{\pi^2 \|\widetilde{T}'''\|_p}{3(n+2)^3} \\
&\le \frac{4\pi^3 m^4}{(n+2)^4}\|T\|_p + \frac{\pi^2}{2(n+2)^3}\|\widetilde{T}'\|_p + \frac{\pi^2}{3(n+2)^3}\|\widetilde{T}'''\|_p.
\end{aligned}$$

Each one of the norms given above can be estimated with the help of Corollary 3.1 (here the condition $T \in \mathbb{T}_m^0$ is needed). That is

$$\Big\|T - \mathbb{F}_n(T) + \frac{\pi^2}{2(n+2)^2} T''\Big\|_p \le \Big(\frac{9\pi^3 m^4}{(n+2)^4} + \frac{3\pi^2}{4(n+2)^3} + \frac{2\pi^2(m+1)}{3(n+2)^3}\Big)\|T''\|_p.$$

□

Theorem 5.7. *Assume* $1 \le p \le \infty$ *and* $0 < \alpha < 1$. *If* $n+2 \ge 7^{2/\alpha}$ *and* $f \in W_p^2$, *then*

$$\left\|(n+2)^2(\mathbb{F}_{n-2}(f)-f)-\frac{\pi^2}{2}f''\right\|_p \le \frac{795\pi^2}{(n+2)^{2/3}}\|f''\|_p.$$

Proof. Fix $m \in \mathbb{N}$ such that

$$m+1 = [(n+2)^{\alpha/2}].$$

Note that

$$6 \le (n+2)^{\alpha/2} - 1 < [(n+2)^{\alpha/2}] = m+1 < n.$$

Let $T_m \in \mathbb{T}_m$ be given by the condition $E_{m.p}(f) = \|f - T_m\|_p$. Since

$$(n+2)^2(\mathbb{F}_n(f+c)-(f+c)) - \frac{\pi^2}{2}(f+c)'' = (n+2)^2(\mathbb{F}_n(f)-f) - \frac{\pi^2}{2}f''$$

and

$$E_{m,p}(f+c) = E_{m,p}(f),$$

for any real constant c, without losing generality, we can assume that T_m has mean zero. That is $T_m \in \mathbb{T}_m^0$. Taking into account Proposition 5.7 and (5.10),

$$\begin{aligned}\left\|(n+2)^2(\mathbb{F}_n(f)-f)-\frac{\pi^2}{2}f''\right\|_p &\le (n+2)^2\|\mathbb{F}_n(f-T_m)-(f-T_m)\|_p \\ &\quad + \frac{\pi^2}{2}\|f''-T_m''\|_p + \left\|(n+2)^2(\mathbb{F}_n(T_m)-T_m)-\frac{\pi^2}{2}T_m''\right\|_p \\ &\le \left(\frac{\pi^4}{4}+\frac{\pi^2}{2}\right)\|f''-T_m''\|_p + \left\|(n+2)^2(\mathbb{F}_n(T_m)-T_m)-\frac{\pi^2}{2}T_m''\right\|_p \\ &\le 30\left(\frac{\pi^4}{4}+\frac{\pi^2}{2}\right)E_{m,p}(f'') + \left\|(n+2)^2(\mathbb{F}_n(T_m)-T_m)-\frac{\pi^2}{2}T_m''\right\|_p.\end{aligned}$$

From Proposition 5.8, we know that

$$\begin{aligned}&\left\|(n+2)^2(\mathbb{F}_n(T_m)-T_m)-\frac{\pi^2}{2}T_m''\right\|_p \\ \le&\left(\frac{9\pi^3m^4}{(n+2)^2}+\frac{3\pi^2}{4(n+2)}+\frac{2\pi^2(m+1)}{3(n+2)}\right)\|T_m''\|_p \\ \le&\left(\frac{9\pi^3(n+2)^{2\alpha}}{(n+2)^2}+\frac{3\pi^2}{4(n+2)}+\frac{2\pi^2(n+2)^{\alpha/2}}{3(n+2)}\right)\left(\|T_m''-f''\|_p+\|f''\|_p\right) \\ \le&\left(\frac{9\pi^3}{(n+2)^{2(1-\alpha)}}+\frac{3\pi^2}{4(n+2)}+\frac{2\pi^2}{3(n+2)^{1-\alpha/2}}\right)\left(30E_{m,p}(f'')+\|f''\|_p\right) \\ \le&31\pi^2\left(9\pi+\frac{3}{4}+\frac{2}{3}\right)\frac{\|f''\|_p}{(n+2)^\alpha} \\ \le&(31\pi)^2\frac{\|f''\|_p}{(n+2)^\alpha}.\end{aligned}$$

Moreover, taking into account Theorem 2.4, one has

$$\begin{aligned} 30\Big(\frac{\pi^4}{4}+\frac{\pi^2}{2}\Big)E_{m,p}(f'') &\le 150\pi^2\Big(\frac{10}{4}+\frac{1}{2}\Big)\omega_1\Big(f'',\frac{2\pi}{m+1}\Big)_p \\ &= 450\pi^2\omega_1\Big(f'',\frac{2\pi}{m+1}\Big)_p \\ &\le 450\pi^2\omega_1\Big(f'',\frac{2\pi}{(n+2)^{\alpha/2}-1}\Big)_p. \end{aligned}$$

We have proved that

$$\Big\|(n+2)^2(\mathbb{F}_n(f)-f)-\frac{\pi^2}{2}f''\Big\|_p \le 450\pi^2\omega_1\Big(f,\frac{2\pi}{(n+2)^{\alpha/2}-1}\Big)_p+\frac{(31\pi)^2}{(n+2)^\alpha}\|f''\|_p.$$

This yields the result. □

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Research Article

Binomial Operator as a Hausdorff Operator of the Euler Type

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ABSTRACT. In this paper, we prove that the binomial operator is a Hausdorff operator of the Euler type and consequently, the binomial matrix domain associated with this operator is nothing new except an Euler sequence space. Therefore, all the results of published papers on the binomial sequence spaces like [4], can be extracted easily from [1] and the relation between the binomial and Euler operators that we introduce. Moreover, we compute the norm and the lower bound of the binomial operator on some sequence spaces.

Keywords: Binomial operator, Euler operator, norm, lower bound, Hausdorff matrix, sequence spaces.

2020 Mathematics Subject Classification: 26D15, 40C05, 40G05, 47B37, 47B39.

1. INTRODUCTION

Let $p \geq 1$ and ω denote the set of all real-valued sequences. The space ℓ_p is the set of all real sequences $x = (x_k) \in \omega$ such that

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty.$$

Definition 1.1. *The Hausdorff matrix $H^\mu = (h_{jk})_{j,k=0}^{\infty}$ is defined by*

$$h_{j,k} := \begin{cases} \binom{j}{k} \int_0^1 \theta^k (1-\theta)^{j-k} d\mu(\theta) & , \quad 0 \leq k \leq j, \\ 0 & , \quad k > j \end{cases}$$

for all $j, k \in \mathbb{N}_0$, where μ is a probability measure on $[0,1]$.

Theorem 1.1 (Hardy's formula, [9, Theorem 216]). *The Hausdorff matrix is a bounded operator on ℓ_p if and only if $\int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta) < \infty$ and*

$$\|H^\mu\|_{\ell_p} = \int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta) \qquad (1 < p < \infty). \tag{1.1}$$

Hausdorff operator has the following norm property.

Theorem 1.2 ([3, Theorem 9]). *Let $p \geq 1$ and H^μ, H^φ and H^ν be Hausdorff matrices such that $H^\mu = H^\varphi H^\nu$. Then, H^μ is bounded on ℓ_p if and only if both H^φ and H^ν are bounded on ℓ_p. Moreover, we have*

$$\|H^\mu\|_{\ell_p} = \|H^\varphi\|_{\ell_p} \|H^\nu\|_{\ell_p}.$$

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Definition 1.2. *For $0 < r < 1$ and $d\mu(\theta) =$ point evaluation at $\theta = r$, the associated Hausdorff matrix is the Euler matrix of order r, $E^r = (e^r_{j,k})$, who has the entries*

$$e^r_{j,k} = \begin{cases} \binom{j}{k}(1-r)^{j-k}r^k & , 0 \le k \le j \\ 0 & , otherwise \end{cases}$$

and the ℓ_p-norm $\|E^r\|_{\ell_p} = r^{\frac{-1}{p}}$.

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_n) \in \omega \ : \ Ax \in \lambda\}. \tag{1.2}$$

It is easy to see that for invertible matrix A and normed space λ, the matrix domain λ_A is a normed space with $\|x\|_{\lambda_A} := \|Ax\|_\lambda$. Let $1 \le p < \infty$. The matrix domains $e^r(p)$ and $e^r(\infty)$ associated by the Euler matrix E^r are

$$e^r(p) = \left\{ x = (x_n) \in w \ : \ \sum_j \left| \sum_{k=0}^{j} \binom{j}{k} (1-r)^{j-k} r^k \right|^p < \infty \right\}$$

and

$$e^r(\infty) = \left\{ x = (x_n) \in w \ : \ \sup_j \left| \sum_{k=0}^{j} \binom{j}{k} (1-r)^{j-k} r^k \right| < \infty \right\}.$$

By the notation of (1.2), the Euler sequence spaces $e^r(p)$ and $e^r(\infty)$ can be redefined by the matrix domain of

$$e^r(p) = (\ell_p)_{E^r} \qquad and \qquad e^r(\infty) = (\ell_\infty)_{E^r}.$$

Let r, s be two non-negative numbers that $r + s \neq 0$. The binomial matrix $B^{r,s} = (b^{r,s}_{j,k})$ is defined by

$$b^{r,s}_{j,k} = \begin{cases} \frac{1}{(r+s)^j}\binom{j}{k} s^{j-k} r^k & , 0 \le k \le j \\ 0 & , otherwise \end{cases}.$$

If $r + s = 1$, one can easily see that $B^{r,s} = E^r$. For $1 \le p < \infty$, the binomial sequence spaces $b^{r,s}(p)$ and $b^{r,s}(\infty)$ generated by $B^{r,s}$ are defined by

$$b^{r,s}(p) = \left\{ x = (x_n) \in w \ : \ \sum_j \left| \sum_{k=0}^{j} \frac{1}{(r+s)^j} \binom{j}{k} s^{j-k} r^k \right|^p < \infty \right\}$$

and

$$b^{r,s}(\infty) = \left\{ x = (x_n) \in w \ : \ \sup_j \left| \sum_{k=0}^{j} \frac{1}{(r+s)^j} \binom{j}{k} s^{j-k} r^k \right| < \infty \right\}.$$

The binomial sequence spaces $b^{r,s}(p)$ and $b^{r,s}(\infty)$ can be represented by the matrix domain of

$$b^{r,s}(p) = (\ell_p)_{B^{r,s}} \qquad and \qquad b^{r,s}(\infty) = (\ell_\infty)_{B^{r,s}}.$$

In this study, we investigate the norm and the lower bound of binomial operator $B^{r,s}$ from the sequence spaces A_p into the sequence spaces B_p and gain inequalities of the form

$$\|B^{r,s}x\|_{B_p} \le U\|x\|_{A_p} \quad and \quad \|B^{r,s}x\|_{B_p} \ge L\|x\|_{A_p}$$

for all sequences $x \in \ell_p$. The constants U and L are not depending on x, and the norm and the lower bound of T are the smallest and greatest possible value of U and L, respectively. The problem of finding the norm of matrix operators on the sequence space ℓ_p have been studied extensively by many mathematicians and abundant literature exists on the topic. Although topological properties and inclusion relations of $b^{r,s}(p)$ have largely been explored [4, 13, 12], computing the norm of binomial operators on sequence spaces has not been investigated to date. More recently, the author has computed the norm of operators on several sequence spaces, [7, 8, 21, 22, 14, 15, 16, 17, 18, 20, 19].

Several papers have published about the binomial sequence space, who is a matrix domain associated with the binomial operator or other spaces which obtain by this operator [4, 5, 11, 24, 23, 6]. Those are all have investigated the properties of this space such as inclusions, dual spaces, Schauder basis, compactness, matrix transformations etc. In this study, we reveal that this matrix is a Hausdorff one, of the Euler type, which is not worth wasting the mathematicians' time more. Moreover, the bounds of this operator has computed on some sequence spaces that has never done before.

2. Close relation of Binomial and Euler operators

In this section, we reveal the nature of binomial operator, its ℓ_p-norm and its relation with Euler operators. The following theorem is the main theorem of this study.

Theorem 2.3. *Suppose that r and s are two non-negative real numbers with $r+s\neq 0$. Then, $B^{r,s}$ is a Hausdorff matrix and*

- $B^{r,s}=E^{\frac{r}{r+s}}$, *where E^r is the Euler matrix of order r,*
- $E^r=B^{r,1-r}$,
- $\|B^{r,s}\|_{\ell_p}=\left(\frac{r+s}{r}\right)^{1/p}$,
- $B^{r,s}B^{t,u}=B^{rt,ru+st+su}$,
- *$B^{r,s}$ is invertible and its inverse is $B^{1+\frac{s}{r},\frac{-s}{r}}$.*

Proof. By letting $d\mu(\theta)=point\ evaluation\ at\ \theta=\frac{r}{r+s}$, the associated Hausdorff matrix is the binomial operator which accorollaryding to the Hardy's formula has the ℓ_p-norm $\left(\frac{r+s}{r}\right)^{1/p}$. This proves the first and the third parts. The second part is obvious. By applying the identity $E^rE^s=E^{rs}$ and part one, we have

$$\begin{aligned} B^{r,s}B^{t,u} &= E^{\frac{r}{r+s}}E^{\frac{t}{t+u}}=E^{\frac{rt}{(r+s)(t+u)}} \\ &= E^{\frac{rt}{rt+ru+st+su}}=B^{rt,ru+st+su} \end{aligned}$$

which results the fourth item. For obtaining the last part, since $(E^r)^{-1}=E^{\frac{1}{r}}$, applying the second part results in

$$(B^{r,s})^{-1}=(E^{\frac{r}{r+s}})^{-1}=E^{\frac{r+s}{r}}=B^{1+\frac{s}{r},\frac{-s}{r}}.$$

□

Remark 2.1. *One can verify the first result of the Theorem 2.3 directly by*

$$\begin{aligned} b^{r,s}_{j,k} &= \frac{1}{(r+s)^j}\binom{j}{k}s^{j-k}r^k=\binom{j}{k}\left(\frac{s}{r+s}\right)^{j-k}\left(\frac{r}{r+s}\right)^k \\ &= \binom{j}{k}\left(1-\frac{r}{r+s}\right)^{j-k}\left(\frac{r}{r+s}\right)^k \\ &= e^{\frac{r}{r+s}}_{j,k}. \end{aligned}$$

Remark 2.2. *One can also check the second part of Theorem 2.3 straightly by*

$$\begin{aligned}
(B^{r,s}B^{t,u})_{j,k} &= \frac{1}{(r+s)^j}\sum_{i=k}^{j}\binom{j}{i}\binom{i}{k}s^{j-i}r^i\frac{1}{(t+u)^i}t^{i-k}u^k \\
&= \frac{1}{(r+s)^j}(\frac{u}{t})^k s^j\frac{j!}{k!}\sum_{i=k}^{j}\frac{1}{(j-i)!(i-k)!}\left(\frac{rt}{s}\right)^i\frac{1}{(t+u)^i} \\
&= \frac{1}{(r+s)^j}(\frac{u}{t})^k s^j\frac{j!}{k!}\sum_{i=0}^{j-k}\frac{1}{(j-i-k)!i!}\left(\frac{rt}{s(t+u)}\right)^{i+k} \\
&= (\frac{s}{r+s})^j(\frac{ru}{st+su})^k\binom{j}{k}\sum_{i=0}^{j-k}\binom{j-k}{i}\left(\frac{rt}{s(t+u)}\right)^i \\
&= (\frac{s}{r+s})^j(\frac{ru}{st+su})^k\binom{j}{k}\left(1+\frac{rt}{s(t+u)}\right)^{j-k} \\
&= (\frac{1}{r+s})^j(\frac{ru}{t+u})^k\binom{j}{k}\left(s+\frac{rt}{t+u}\right)^{j-k} \\
&= \frac{1}{(r+s)^j}\frac{1}{(t+u)^j}(ru)^k\binom{j}{k}(st+su+rt)^{j-k} \\
&= B^{rt,ru+st+su}_{j,k}.
\end{aligned}$$

2.1. **Factorization of the Binomial operators and its applications.** In this part of study, we find some factorization for the binomial operator and obtain several inequalities and inclusions who are all the straightforward result of the Theorem 2.3.

Corollary 2.1. *Let r, s, t, u be positive numbers that $\frac{r}{s} < \frac{t}{u}$. The binomial operator $B^{r,s}$ has a factorization of the form*

$$B^{r,s} = E^{\frac{r(t+u)}{t(r+s)}}B^{t,u}.$$

In particular,

- $E^r = E^{\frac{r(t+u)}{t}}B^{t,u}, \quad r < \frac{t}{t+u}$,
- $B^{r,s} = E^{\frac{r}{t(r+s)}}E^t, \quad \frac{r}{r+s} < t$,
- $E^r = E^{\frac{r}{t}}E^t, \quad r < t$.

As a result of the above factorization, we have the following inequalities.

Corollary 2.2. *Let r, s, t, u be positive numbers that $\frac{r}{s} < \frac{t}{u}$ and $x \in \ell_p$. Then,*

$$\sum_{k=0}^{\infty}\left|\frac{1}{(r+s)^j}\binom{j}{k}s^{j-k}r^k x_k\right|^p \le \frac{t(r+s)}{r(t+u)}\sum_{k=0}^{\infty}\left|\frac{1}{(t+u)^j}\binom{j}{k}u^{j-k}t^k x_k\right|^p.$$

In particular,

$$\sum_{k=0}^{\infty}\left|\binom{j}{k}(1-r)^{j-k}r^k x_k\right|^p \le \frac{t}{r(t+u)}\sum_{k=0}^{\infty}\left|\frac{1}{(t+u)^j}\binom{j}{k}u^{j-k}t^k x_k\right|^p \quad , r < \frac{t}{t+u},$$

$$\sum_{k=0}^{\infty}\left|\frac{1}{(r+s)^j}\binom{j}{k}s^{j-k}r^k x_k\right|^p \le \frac{t(r+s)}{r}\sum_{k=0}^{\infty}\left|\binom{j}{k}(1-t)^{j-k}t^k x_k\right|^p \quad , \frac{r}{r+s} < t$$

and

$$\sum_{k=0}^{\infty}\left|\binom{j}{k}(1-r)^{j-k}r^k x_k\right|^p \leq \frac{t}{r}\sum_{k=0}^{\infty}\left|\binom{j}{k}(1-t)^{j-k}t^k x_k\right|^p \qquad , r<t.$$

Proof. The proof is obvious by Corollary 2.1. □

Theorem 2.4. *Let r,s,t,u be positive numbers that $\frac{r}{s}<\frac{t}{u}$. Then, $b^{t,u}(p)\subset b^{r,s}(p)$.*
In particular,

- $b^{t,u}(p)\subset e^r(p), \quad r<\frac{t}{t+u}$,
- $e^t(p)\subset b^{r,s}(p), \quad \frac{r}{r+s}<t$,
- $e^t(p)\subset e^r(p), \quad r<t$.

Proof. This is the straightforward result of Corollary 2.2. □

Remark 2.3. *The last part of previous theorem is the Theorem 3.4 of* [1].

Remark 2.4. *Since $\frac{r}{s}<\frac{t}{u}$, hence $\frac{r}{r+s}<\frac{t}{t+u}$. Now, accorollaryding to the Theorem 3.4 of* [1], $b^{t,u}(p)=e^{\frac{t}{t+u}}(p)\subset e^{\frac{r}{r+s}}(p)=b^{r,s}(p)$.

2.2. **The α-, β- and γ-dual of $b^{r,s}(p)$.** In this example, we show that Theorem 4.2 of [4] can be easily gained from Theorem 4.4 of [1]. Therefore, let us bring that theorem first

Theorem 2.5 ([1, Theorem 4.4])**.** *Define the sets A_q^r and A_∞^r as follows. For $1\leq p<\infty$,*

$$A_q^r=\left\{a=(a_k)\in w \ : \ \sup_{K\in F}\sum_k\left|\sum_{n\in K}\binom{n}{k}(r-1)^{n-k}r^{-n}a_n\right|^q<\infty\right\}$$

and

$$A_\infty^r=\left\{a=(a_k)\in w \ : \ \sup_{k\in\mathbb{N}}\sum_n\left|\binom{n}{k}(r-1)^{n-k}r^{-n}a_n\right|<\infty\right\}.$$

Then, $(E_1^r)^\alpha=A_\infty^r$ and $(e^r(p))^\alpha=A_q^r$, where $1<p\leq\infty$.

Now, we obtain the α-dual of $b^{r,s}(p)$ and $b^{r,s}(1)$. By applying the above theorem and the identity $B^{r,s}=E^{\frac{r}{r+s}}$ of Theorem 2.3,

$$\begin{aligned}(b^{r,s}(p))^\alpha &= (e^{\frac{r}{r+s}}(p))^\alpha=A_q^{\frac{r}{r+s}}\\ &= \left\{a=(a_k)\in w \ : \ \sup_{K\in F}\sum_k\left|\sum_{n\in K}\binom{n}{k}\left(\frac{r}{r+s}-1\right)^{n-k}\left(\frac{r}{r+s}\right)^{-n}a_n\right|^q<\infty\right\}\\ &= \left\{a=(a_k)\in w \ : \ \sup_{K\in F}\sum_k\left|\sum_{n\in K}\binom{n}{k}(-s)^{n-k}r^{-n}(r+s)^k a_n\right|^q<\infty\right\}=V_1^{r,s}\end{aligned}$$

and

$$\begin{aligned}(b^{r,s}(1))^\alpha &= (e^{\frac{r}{r+s}}(1))^\alpha=A_\infty^{\frac{r}{r+s}}\\ &= \left\{a=(a_k)\in w \ : \ \sup_{k\in\mathbb{N}}\sum_n\left|\binom{n}{k}\left(\frac{r}{r+s}-1\right)^{n-k}\left(\frac{r}{r+s}\right)^{-n}a_n\right|<\infty\right\}\\ &= \left\{a=(a_k)\in w \ : \ \sup_{k\in\mathbb{N}}\sum_n\left|\binom{n}{k}(-s)^{n-k}r^{-n}(r+s)^k a_n\right|<\infty\right\}=V_2^{r,s},\end{aligned}$$

where $V_1^{r,s}$ and $V_2^{r,s}$ are the α-duals of $b^{r,s}(p)$ and $b^{r,s}(1)$ respectively, as the author of [4] has proved in Theorem 4.2. Note that for obtaining the β- and γ-duals of $b^{r,s}(p)$ and $b^{r,s}(1)$ ([4], Theorem 4.3), we only need changing r to $\frac{r}{r+s}$ in Theorems 4.5 and 4.6 of [1].

3. Bounds of binomial operator on some sequence spaces

In this section, we investigate the bounds of binomial operator on some sequence spaces. In so doing, the following lemma is needed.

Lemma 3.1 ([15, Lemma 2.1]). *Let U is a bounded operator on ℓ_p, and A_p and B_p be two matrix domains such that $A_p \simeq \ell_p$. Then, the following statements hold:*

(i) If BT is a bounded operator on ℓ_p, then T is a bounded operator from ℓ_p into B_p and

$$\|T\|_{\ell_p,B_p} = \|T\|_{\ell_p} \quad and \quad L(T)_{\ell_p,B_p} = L(BT).$$

(ii) If T has a factorization of the form $T = UA$, then T is a bounded operator from the matrix domain A_p into ℓ_p and

$$\|T\|_{A_p,\ell_p} = \|U\|_{\ell_p} \quad and \quad L(T)_{A_p,\ell_p} = L(U).$$

(iii) If $BT = UA$, then T is a bounded operator from the matrix domain A_p into B_p and

$$\|T\|_{A_p,B_p} = \|U\|_{\ell_p} \quad and \quad L(T)_{A_p,B_p} = L(U).$$

In particular, if $AT = UA$, then T is a bounded operator from the matrix domain A_p into itself and $\|T\|_{A_p} = \|U\|_{\ell_p}$ and $L(T)_{A_p} = L(U)$. Also, if T and A commute, then $\|T\|_{A_p} = \|T\|_{\ell_p}$ and $L(T)_{A_p} = L(T)$.

Throughout this section, we use the notations $L(\cdot)$ for the lower bound of operators on ℓ_p and $L(\cdot)_{X,Y}$ for the lower bound of operators from the sequence space X into the sequence space Y.

3.1. Norm of binomial operator on difference sequence space.

The backward difference matrix $\Delta = (\delta_{j,k})$ is defined by

$$\delta_{j,k} = \begin{cases} 1 & , k = j \\ -1 & , k = j-1 \\ 0 & , otherwise \end{cases}$$

and the difference sequence space associated with this matrix is called bv_p

$$bv_p = \left\{ x = (x_n) \ : \ \sum_{n=1}^{\infty} |x_n - x_{n-1}|^p < \infty \right\}, \quad 1 \le p < \infty,$$

which has the norm $\|x\|_{bv_p} = \left(\sum_{n=1}^{\infty} |x_n - x_{n-1}|^p\right)^{1/p}$. The idea of difference sequence spaces was introduced by Kizmaz [10]. Recently, Roopaei in [14] has computed the norm of Hausdorff operators on bv_p sequence space.

Theorem 3.6 ([14, Theorem 2.4]). *The Hausdorff operator H^μ is a bounded operator on bv_p and*

$$\|H^\mu\|_{bv_p} = 1.$$

We have proved that the binomial operator is a Hausdorff operator of Euler type, hence

Corollary 3.3. *The binomial operator $B^{r,s}$ is a bounded operator on bv_p and $\|B^{r,s}\|_{bv_p} = 1$.*

3.2. **Bounds of binomial operator on the Hausdorff sequence space.** The Hausdorff matrix contains the famous classes of matrices. For $\alpha > 0$, some of these classes are as follows:

- The choice $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$ gives the Cesàro matrix of order α,
- The choice $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ gives the Gamma matrix of order α,
- The choice $d\mu(\theta) = \frac{|\log\theta|^{\alpha-1}}{\Gamma(\alpha)}d\theta$ gives the Hölder matrix of order α.

Theorem 3.7 ([3, Theorem 1]). *Let $p \geq 1$, and let H^μ is a bounded Hausdorff matrix on ℓ_p. Then,*

$$\|H^\mu x\|_{\ell_p} \geq L\|x\|_{\ell_p}$$

for every decreasing sequence x of non-negative terms, where

$$L^p = \sum_{k=0}^{\infty}\left(\int_0^1 (1-\theta)^k d\mu(\theta)\right)^p. \tag{3.3}$$

The constant in (3.3) is the best possible, and there is equality only when $x = 0$ or $p = 1$ or $d\mu(\theta)$ is the point mass at 1.

As an example of Theorem 3.7, we compute the lower bound of the Cesàro, Gamma and Euler operators by choosing their associated $d\mu(\theta)$.

- $L(C^\alpha) = \left\{\sum_{k=0}^{\infty}\left(\frac{\alpha}{\alpha+k}\right)^p\right\}^{1/p}$,
- $L(\Gamma^\alpha) = \left\{\sum_{k=0}^{\infty}\binom{\alpha+k}{k}^{-p}\right\}^{1/p}$,
- $L(E^\alpha) = \frac{1}{[1-(1-\alpha)^p]^{1/p}}, \quad 0 < \alpha < 1$,
- $L(B^{r,s}) = \frac{1}{[1-(\frac{s}{r+s})^p]^{1/p}}$ (by Theorem 2.3).

We use the notation $hau(p)$ as the set of all sequences whose H^μ-transforms are in the space ℓ_p, that is

$$hau(p) = \left\{x = (x_j) \in \omega : \sum_{j=0}^{\infty}\left|\sum_{k=0}^{j}\int_0^1 \binom{j}{k}\theta^k(1-\theta)^{j-k}d\mu(\theta)x_k\right|^p < \infty\right\},$$

where μ is a fixed probability measure on $[0,1]$.

Theorem 3.8. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $hau(p)$ and*

$$\|B^{r,s}\|_{\ell_p,hau(p)} = \left(\frac{r+s}{r}\right)^{1/p}\int_0^1 \theta^{\frac{-1}{p}}d\mu(\theta)$$

and

$$L(B^{r,s})_{\ell_p,hau(p)} \geq \left\{\frac{\sum_{k=0}^{\infty}\left(\int_0^1(1-\theta)^k d\mu(\theta)\right)^p}{1-(\frac{s}{r+s})^p}\right\}^{1/p}.$$

In particular, for $r+s=1$, the Euler operator E^r is a bounded operator from ℓ_p into $hau(p)$ and

$$\|E^r\|_{\ell_p,hau(p)} = r^{\frac{-1}{p}}\int_0^1 \theta^{\frac{-1}{p}}d\mu(\theta) \quad \text{and} \quad L(E^r)_{\ell_p,hau(p)} \geq \left\{\frac{\sum_{k=0}^{\infty}\left(\int_0^1(1-\theta)^k d\mu(\theta)\right)^p}{1-(1-r)^p}\right\}^{1/p}.$$

Proof. Applying Lemma 3.1 part (i) and Theorems 1.2 and 2.3, result that

$$\|B^{r,s}\|_{\ell_p,hau(p)} = \|H^\mu B^{r,s}\|_{\ell_p} = \|H^\mu\|_{\ell_p}\|B^{r,s}\|_{\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta).$$

Also, the identity $L(AB) \geq L(A)L(B)$ results in

$$L(B^{r,s})_{\ell_p,hau(p)} = L(B^{r,s}H^\mu) \geq \left\{\frac{1}{1-(\frac{s}{r+s})^p}\right\}^{1/p}\left\{\sum_{k=0}^{\infty}\left(\int_0^1(1-\theta)^k d\mu(\theta)\right)^p\right\}^{1/p}.$$

□

By letting $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$ in the definition of the Hausdorff matrix, the Cesàro matrix $C^\alpha = (C^\alpha_{jk})$ of order α is defined as follows

$$C^\alpha_{j,k} = \begin{cases} \frac{\binom{\alpha+j-k-1}{j-k}}{\binom{\alpha+j}{j}} & ,0 \leq k \leq j \\ 0 & ,\text{otherwise} \end{cases},$$

which accorollaryding to the Hardy's formula has the ℓ_p-norm

$$\|C^\alpha\|_{\ell_p} = \frac{\Gamma(\alpha+1)\Gamma(1/p^*)}{\Gamma(\alpha+1/p^*)}.$$

Note that, $C^0 = I$, where I is the identity matrix and C^1 is the well-known Cesàro matrix C which has the ℓ_p-norm $\|C\|_{\ell_p} = p^*$ and the lower bound $L(C) = \zeta(p)^{1/p}$. We use the notation $ces(\alpha,p)$ as the set of all sequences whose C^α-transforms are in the space ℓ_p, that is

$$ces(\alpha,p) = \left\{x = (x_j) \in \omega : \sum_{j=0}^{\infty}\left|\frac{1}{\binom{\alpha+j}{j}}\sum_{k=0}^{j}\binom{\alpha+j-k-1}{j-k}x_k\right|^p < \infty\right\}.$$

The space $ces(\alpha,p)$ is a Banach space which has the norm

$$\|x\|_{ces(\alpha,p)} = \left(\sum_{j=0}^{\infty}\left|\frac{1}{\binom{\alpha+j}{j}}\sum_{k=0}^{j}\binom{\alpha+j-k-1}{j-k}x_k\right|^p\right)^{1/p}.$$

We use the notation $ces(p)$ instead of $ces(1,p)$ as the sequence space associated with the well-known Cesàro matrix C. For more information about Cesàro matrix, the readers can refer to [20, 19].

Corollary 3.4. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $ces(\alpha,p)$ and*

$$\|B^{r,s}\|_{\ell_p,ces(\alpha,p)} = \frac{\left(\frac{r+s}{r}\right)^{1/p}\Gamma(\alpha+1)\Gamma(1/p^*)}{\Gamma(\alpha+1/p^*)}$$

and

$$L(B^{r,s})_{\ell_p,ces(\alpha,p)} \geq \left\{\frac{\sum_{k=0}^{\infty}\left(\frac{\alpha}{\alpha+k}\right)^p}{1-(\frac{s}{r+s})^p}\right\}^{1/p}.$$

In particular, for $r+s=1$ and $\alpha=1$, the Euler operator E^r is a bounded operator from ℓ_p into $ces(p)$ and $\|E^r\|_{\ell_p,ces(p)} = \frac{r^{-1/p}p}{p-1}$ and $L(E^r)_{\ell_p,ces(p)} \geq \left\{\frac{\zeta(p)}{1-(1-r)^p}\right\}^{1/p}$.

By letting $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ in the definition of the Hausdorff matrix, the Gamma matrix of order α, $\Gamma^\alpha = (\gamma^\alpha_{j,k})$, is

$$\gamma^\alpha_{j,k} = \begin{cases} \frac{\binom{\alpha+k-1}{k}}{\binom{\alpha+j}{j}} & , \ 0 \le k \le j \\ 0 & , \ otherwise \end{cases},$$

which accorollaryding to the Hardy's formula has the ℓ_p-norm $\|\Gamma^\alpha\|_{\ell_p} = \frac{\alpha p}{\alpha p-1}$. Note that, Γ^1 is the well-known Cesàro matrix. The Gamma space of order α, $gam(\alpha, p)$, is

$$gam(\alpha, p) = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{\alpha+j}{j}} \sum_{k=0}^{j} \binom{\alpha+k-1}{k} x_k \right|^p < \infty \right\},$$

which is a Banach spaces with the norm

$$\|x\|_{gam(\alpha,p)} = \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{\alpha+j}{j}} \sum_{k=0}^{j} \binom{\alpha+k-1}{k} x_k \right|^p \right)^{\frac{1}{p}}.$$

Note that $gam(1, p) = ces(p)$.

Corollary 3.5. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $gam(\alpha, p)$ and*

$$\|B^{r,s}\|_{\ell_p, gam(\alpha,p)} = \frac{\left(\frac{r+s}{r}\right)^{1/p} \alpha p}{\alpha p - 1}$$

and

$$L(B^{r,s})_{\ell_p, gam(\alpha,p)} \ge \left\{ \frac{\sum_{k=0}^{\infty} \binom{\alpha+k}{k}^{-p}}{1 - (\frac{s}{r+s})^p} \right\}^{1/p}.$$

In particular, for $r + s = 1$ and $\alpha = 1$, the Euler operator E^r is a bounded operator from ℓ_p into $ces(p)$ and $\|E^r\|_{\ell_p, ces(p)} = \frac{r^{-1/p}p}{p-1}$ and $L(E^r)_{\ell_p, ces(p)} \ge \left\{ \frac{\zeta(p)}{1-(1-r)^p} \right\}^{1/p}$.

Corollary 3.6. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $e^\alpha(p)$ and*

$$\|B^{r,s}\|_{\ell_p, e^\alpha(p)} = \left(\frac{r+s}{r\alpha} \right)^{1/p} \quad and \quad L(B^{r,s})_{\ell_p, e^\alpha(p)} \ge \frac{1}{[1-(1-\alpha)^p]^{1/p}[1-(\frac{s}{r+s})^p]^{1/p}}.$$

In particular, for $r + s = 1$, the Euler operator E^r is a bounded operator from ℓ_p into $e^\alpha(p)$ and $\|E^r\|_{\ell_p, e^\alpha(p)} = (r\alpha)^{-1/p}$ and $L(E^r)_{\ell_p, e^\alpha(p)} \ge \frac{1}{[1-(1-\alpha)^p]^{1/p}[1-(1-r)^p]^{1/p}}$.

Corollary 3.7. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $hol(\alpha, p)$ and*

$$\|B^{r,s}\|_{\ell_p, hol(\alpha,p)} = \left(\frac{r+s}{r} \right)^{1/p} \left(\frac{p}{p-1} \right)^\alpha.$$

In particular, for $r + s = 1$ and $\alpha = 1$, the Euler operator E^r is a bounded operator from ℓ_p into $ces(p)$ and $\|E^r\|_{\ell_p, ces(p)} = \frac{r^{-1/p}p}{p-1}$.

Corollary 3.8. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $b^{t,u}(p)$ and*

$$\|B^{r,s}\|_{\ell_p, b^{t,u}(p)} = \frac{(r+s)^{1/p}(t+u)^{1/p}}{(rt)^{1/p}} \quad and \quad L(B^{r,s})_{\ell_p, b^{t,u}(p)} \ge \frac{1}{[1-(\frac{u}{t+u})^p]^{1/p}[1-(\frac{s}{r+s})^p]^{1/p}}.$$

In particular,

- *for $r+s=1$, the Euler operator E^r is a bounded operator from ℓ_p into $b^{t,u}(p)$ and* $\|E^r\|_{\ell_p,b^{t,u}(p)} = \left(\frac{t+u}{rt}\right)^{1/p}$ *and* $L(E^r)_{\ell_p,b^{t,u}(p)} \geq \frac{1}{[1-(\frac{u}{t+u})^p]^{1/p}[1-(1-r)^p]^{1/p}}$,
- *for $t+u=1$, the binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $e^t(p)$ and* $\|B^{r,s}\|_{\ell_p,e^t(p)} = \left(\frac{r+s}{rt}\right)^{1/p}$ *and* $L(B^{r,s})_{\ell_p,e^t(p)} \geq \frac{1}{[1-(1-t)^p]^{1/p}[1-(\frac{s}{r+s})^p]^{1/p}}$,
- *for $r+s=t+u=1$, the Euler operator E^r is a bounded operator from ℓ_p into $e^t(p)$ and* $\|E^r\|_{\ell_p,e^t(p)} = (rt)^{-1/p}$ *and* $L(E^r)_{\ell_p,e^t(p)} \geq \frac{1}{[1-(1-t)^p]^{1/p}[1-(1-r)^p]^{1/p}}$.

We can also prove our results in the above corollary accorollaryding to Theorem 2.3.

Remark 3.5. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $b^{t,u}(p)$ and*

$$\begin{aligned}\|B^{r,s}\|_{\ell_p,b^{t,u}(p)} &= \|B^{r,s}B^{t,u}\|_{\ell_p} = \|B^{rt,ru+st+su}\|_{\ell_p}\\ &= \left(\frac{rt+ru+st+su}{rt}\right)^{1/p} = \frac{(r+s)^{1/p}(t+u)^{1/p}}{(rt)^{1/p}}.\end{aligned}$$

In particular,

- *for $r+s=1$, the Euler operator E^r is a bounded operator from ℓ_p into $b^{t,u}(p)$ and*

$$\begin{aligned}\|E^r\|_{\ell_p,b^{t,u}(p)} &= \|E^rB^{t,u}\|_{\ell_p} = \|B^{r,1-r}B^{t,u}\|_{\ell_p} = \|B^{rt,ru+(1-r)t+(1-r)u}\|_{\ell_p}\\ &= \|B^{rt,u+t-rt}\|_{\ell_p} = \left(\frac{u+t}{rt}\right)^{1/p},\end{aligned}$$

- *for $t+u=1$, the binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $e^t(p)$ and*

$$\begin{aligned}\|B^{r,s}\|_{\ell_p,e^t(p)} &= \|E^tB^{r,s}\|_{\ell_p} = \|B^{t,1-t}B^{r,s}\|_{\ell_p}\\ &= \|B^{rt,r+s-rt}\|_{\ell_p} = \left(\frac{r+s}{rt}\right)^{1/p},\end{aligned}$$

- *for $r+s=t+u=1$, the Euler operator E^r is a bounded operator from ℓ_p into $e^t(p)$ and*

$$\|E^r\|_{\ell_p,e^t(p)} = \|E^rE^t\|_{\ell_p} = \|B^{r,1-r}B^{t,1-t}\|_{\ell_p} = \|B^{rt,1-rt}\|_{\ell_p} = (rt)^{-1/p}.$$

Accorollarying to Lemma 3.1, for obtaining the bound of the operator T from the sequence space A_p into ℓ_p there is need that we have a factorization for T of the form $T=UA$. The existence of this factorization for the Hausdorff operators is a challenging problem.

Theorem 3.9. *If $B^{r,s}$ has a factorization of the form $B^{r,s}=UH^\mu$, then the binomial operator $B^{r,s}$ is a bounded operator from $hau(p)$ into ℓ_p and*

$$\|B^{r,s}\|_{hau(p),\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}\left(\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta)\right)^{-1}.$$

In particular, for $r+s=1$, the Euler operator E^r is a bounded operator from $hau(p)$ into ℓ_p and $\|E^r\|_{hau(p),\ell_p} = r^{\frac{-1}{p}}\left(\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta)\right)^{-1}$.

Proof. Similar to Bennett ([2, p. 120]), if $B^{r,s}$ has a factorization of the form $B^{r,s}=H^\omega H^\mu$, where ω is a quotient measure, then Lemma 3.1 part (ii) and Theorem 1.2 result in

$$\|B^{r,s}\|_{hau(p),\ell_p} = \|H^\omega\|_{\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}\left(\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta)\right)^{-1}.$$

□

Corollary 3.9. *The binomial operator $B^{r,s}$ is a bounded operator from $b^{t,u}(p)$ into ℓ_p and*

$$\|B^{r,s}\|_{b^{t,u}(p),\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}\left(\frac{t}{t+u}\right)^{1/p}.$$

In particular,

- *for $r+s=1$, the Euler operator E^r is a bounded operator from $b^{t,u}(p)$ into ℓ_p and $\|E^r\|_{b^{t,u}(p),\ell_p} = \left(\frac{t}{rt+ru}\right)^{1/p}$,*
- *for $t+u=1$, the binomial operator $B^{r,s}$ is a bounded operator from $e^t(p)$ into ℓ_p and $\|B^{r,s}\|_{e^t(p),\ell_p} = \left(\frac{rt+st}{r}\right)^{1/p}$,*
- *for $r+s=t+u=1$, the Euler operator E^r is a bounded operator from $e^t(p)$ into ℓ_p and $\|E^r\|_{e^t(p),\ell_p} = (\frac{t}{r})^{1/p}$.*

Proof. Let the binomial operator $B^{r,s}$ has a factorization of the form $B^{r,s} = UB^{t,u}$. Then, U is

$$U = B^{r,s}(B^{t,u})^{-1} = B^{r,s}B^{1+\frac{u}{t},\frac{-u}{t}} = B^{r+\frac{ru}{t},s-\frac{ru}{t}},$$

hence according to Lemma 3.1

$$\begin{aligned}\|B^{r,s}\|_{b^{t,u}(p),\ell_p} &= \|U\|_{\ell_p} = \|B^{r+\frac{ru}{t},s-\frac{ru}{t}}\|_{\ell_p} \\ &= \left(\frac{r+s}{r}\right)^{1/p}\left(\frac{t}{t+u}\right)^{1/p}.\end{aligned}$$

□

Corollary 3.10. *The binomial operator $B^{r,s}$ is a bounded operator on Hausdorff sequence space $hau(p)$ and*

$$\|B^{r,s}\|_{hau(p)} = \left(\frac{r+s}{r}\right)^{1/p} \quad \text{and} \quad L(B^{r,s})_{hau(p)} = \frac{1}{[1-(\frac{s}{r+s})^p]^{1/p}}.$$

In particular, for $r+s=1$, the Euler operator E^r is a bounded operator on $hau(p)$ and $\|E^r\|_{hau(p)} = r^{\frac{-1}{p}}$ and $L(E^r)_{hau(p)} = \frac{1}{[1-(1-r)^p]^{1/p}}$.

Proof. Since Hausdorff operators commute, hence by Lemma 3.1, we have

$$\|B^{r,s}\|_{hau(p)} = \|B^{r,s}\|_{\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}$$

and

$$L(E^r)_{hau(p)} = L(E^r) = \frac{1}{[1-(1-r)^p]^{1/p}}.$$

□

4. Lower bound of the transposed binomial operator on the transposed Hausdorff matrix domains

In this section, we intend to compute the lower bound of the transposed binomial operator $(B^{r,s})^t$ on the transposed Hausdorff sequence space $hau^t(p)$ for $0 < p < 1$. For this reason, we need the following theorem which is an analogy of Hardy's formula.

Theorem 4.10 ([2, Theorem 7.18]). *Fix p, $0 < p < 1$, and let $H^{\mu t}$ be the transposed Hausdorff matrix. Then,*

$$\|H^{\mu t}x\|_{\ell_p} \geq \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)\right) \|x\|_{\ell_p}$$

for every sequence x of non-negative terms. The constant is best possible, and there is equality only when $x = 0$ or $p = 1$ or $H = I$.

Theorem 4.11 ([2, Corollary 7.27]). *If $H^{\mu t}$ and $H^{\nu t}$ are two transposed Hausdorff matrices, then the lower bound (on $\ell_p, 0 < p < 1$) of their product is the product of their lower bounds.*

Theorem 4.12. *The transposed binomial operator is a bounded operator from ℓ_p into $hau^t(p)$ and*

$$L((B^{r,s})^t)_{\ell_p, hau^t(p)} = \left(\frac{r}{r+s}\right)^{1/p^*} \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta).$$

In particular, for $r+s = 1$, the transposed Euler operator E^{rt} is a bounded operator from ℓ_p into $hau^t(p)$ and $L(E^{rt})_{\ell_p, hau^t(p)} = r^{1/p^*} \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)$.

Proof. The proof is obvious accorollaryding to the Lemma 3.1 and Theorems 4.11 and 4.10. □

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Research Article

Existence and Multiplicity of Periodic Solutions for Nonautonomous Second-Order Discrete Hamiltonian Systems

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ABSTRACT. In this paper, we consider the periodic solutions of the following non-autonomous second order discrete system

$$\Delta^2 u(n-1) = \nabla F(n, u(n)), \quad n \in \mathbb{Z}.$$

When the nonlinear function $F(n,x)$ is like-quadratic for x, we obtain some existence and multiplicity results under twisting conditions by using the least action principle and a multiple critical point theorem.

Keywords: Periodic solution, second-order discrete Hamiltonian system, the least action principle, critical point theory.

2020 Mathematics Subject Classification: 35R02, 58J05.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the periodic solutions of the following nonautonomous second order discrete Hamiltonian system

$$\Delta^2 u(n-1) = \nabla F(n, u(n)),\ u(n) \in \mathbb{R}^N,\ n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta u(n) = u(n+1) - u(n), \Delta^2 u(n) = \Delta(\Delta u(n))$ and $\nabla F(n,x)$ denotes the gradient of the function F with respect to the second variable x. F satisfies the following condition:

$$(A) \quad \begin{array}{l} F(n,\cdot) \in C^1(\mathbb{R}^N, \mathbb{R}), \forall n \in \mathbb{Z}; \\ F(n+T, x) = F(n,x), \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N, T \in \mathbb{Z} \text{ and } T \geq 2. \end{array}$$

Historically, in 2003, Guo and Yu, first considered the existence of periodic solutions of difference equations as (1.1) via variational method and critical point theory in three papers [2, 3, 4]. In 2004, Zhou, Yu and Guo [11], further studied the existence and multiplicity of periodic solutions of the discrete Hamiltonian system (1.1). After that, the existence and multiplicity of periodic solutions for system (1.1) have been extensively studied and many interesting results were obtained. We refer the readers to [5, 6, 8, 9, 10] and the references therein for these topics. Among them, we should mention some work which have relation with our work of this paper. For the condition on F, Guo and Yu in [3], first required the nonlinearity $\nabla F(n,x)$ is sub-linear included the bounded case. We say that the nonlinearity $\nabla F(n,x)$ is growing sublinearly if there exist $M_1 > 0$, $M_2 > 0$ and $\alpha \in [0,1)$ such that

$$|\nabla F(t,x)| \leq M_1 |x|^\alpha + M_2, \quad \forall (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \tag{1.2}$$

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where $\mathbb{Z}[a,b] := \mathbb{Z} \cap [a,b]$ for all $a, b \in \mathbb{Z}$ with $a \le b$. Xue and Tang in [9] used the least action principle to verified that system (1.1) possesses at least one T-periodic solution with the assumption of

$$\lim_{|x|\to+\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) = +\infty. \tag{1.3}$$

We also refer the paper [3] for this topic. In the case of $\alpha = 1$, assumption (1.2) becomes to the following assumption in which the nonlinearity $\nabla F(n,x)$ does not exceed linear growth, that is, there are $M_3 > 0$ and $M_4 \ge 0$, such that

$$|\nabla F(n,x)| \le M_3|x| + M_4, \quad \forall (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N. \tag{1.4}$$

The case $\nabla F(n,x) = Ax + B$ satisfies the condition (1.4). It is well known that in this case the system (1.1) in general does not possess a solution. A twisting condition is required to avoid this case. Considering the nonlinearity $\nabla F(n,x)$ which is the sum of assumption (1.2) and (1.4), Hu [6] also used the least action principle to verify that system (1.1) possesses at least one T-periodic solution under a twisting condition which is included in the following case

$$\lim_{|x|\to+\infty} |x|^{-2} \sum_{n=1}^{T} F(n,x) > -\infty. \tag{1.5}$$

When the nonlinearity $\nabla F(n,x)$ meets the following assumption that there are $f, g : \mathbb{Z}[1,T] \to \mathbb{R}^+$ and $\alpha \in (0,1)$, such that

$$|\nabla F(t,x)| \le f(n)|x|^{\alpha} + g(n), \quad \forall\, (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N \tag{1.6}$$

or

$$|\nabla F(t,x)| \le f(n)|x| + g(n), \quad \forall\, (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N. \tag{1.7}$$

Tang and Zhang [8] obtained some existence results for the T-periodic solutions of system (1.1) under some different twisting conditions.

In this paper, we will further study the existence and multiplicity of T-periodic solutions of (1.5) with some different twisting conditions. The following are our main results.

Theorem 1.1. *Suppose that $F(n,x) = F_1(n,x) + F_2(n,x)$ with F_1 and F_2 satisfying the conditions (A) and the following three growing conditions:*
(B_1) There exist $f, g : \mathbb{Z}[1,T] \to \mathbb{R}^+$ and $\alpha \in [0,1)$, such that

$$|\nabla F_1(n,x)| \le f(n)|x|^{\alpha} + g(n).$$

(B_2) $F_2(n,x)$ satisfies condition (1.4), i.e., there exist constants $M_3, M_4 \in \mathbb{R}^+$ such that

$$|\nabla F_2(n,x)| \le M_3|x| + M_4, \;\; M_3 < \lambda_1 := \lambda_1 = 2 - 2\cos\frac{2\pi}{T}.$$

(B_3) F satisfies that

$$\lim_{|x|\to+\infty} \inf |x|^{-2} \sum_{n=1}^{T} F(n,x) > \frac{M_3^2 T}{2(\lambda_1 - M_3)}.$$

Then, system (1.1) possesses at least one T-periodic solution that minimizes the functional φ given by

$$\varphi(u) = \frac{1}{2}\sum_{n=1}^{T} |\Delta u(n)|^2 + \sum_{n=1}^{T} F(n, u(n)) \tag{1.8}$$

in the Hilbert space H_T defined by

$$H_T = \left\{u : \mathbb{Z} \to \mathbb{R}^N | u(n+T) = u(n),\, n \in \mathbb{Z}\right\}.$$

Remark 1.1. *The condition (B_3) is twisted with the conditions (B_1) and (B_2). Our condition (B_3) in Theorem 1.1 is different form the condition (A_4) in Theorem 1 of [6]. In Theorem 1.1, when $F_1(n,x) \equiv 0$, comparing with Theorem 1.3 of [8], the condition (B_3) in some sense is loose for some choices of T, one can check it for $T = 2, 3, 4, 5, 6$ and so on.*

Theorem 1.2. *Suppose $F(n,x) = F_1(n,x) + F_2(n,x)$ satisfying condition (A) with F_1 and F_2 satisfying the following three growing conditions:*
(B_4) F_1 satisfies the condition (1.7), i.e., there exist $f, g : \mathbb{Z}[1,T] \to \mathbb{R}^+$ such that

$$|\nabla F_1(n,x)| \leq f(n)|x| + g(n),\ \sum_{n=1}^{T} f(n) < \lambda_1.$$

(B_5) F_2 satisfies the condition (1.2), i.e., there are some constants $M_1, M_2 \in \mathbb{R}^+$ and $\alpha \in [0,1)$ such that

$$|\nabla F_2(n,x)| \leq M_1|x|^\alpha + M_2.$$

(B_6) F satisfies that

$$\lim_{|x|\to+\infty} \inf |x|^{-2} \sum_{n=1}^{T} F(n,x) > \frac{1}{2(\lambda_1 - \sum_{n=1}^{T} f(n))} \Big(\sum_{n=1}^{T} f(n)\Big)^2.$$

Then, system (1.1) possesses at least one T-periodic solution.

Remark 1.2. *Conditions of (B_4) and (B_6) in Theorem 1.2 are different form conditions of (A_3) and (A_4) in Theorem 1 of [6], respectively. Condition (B_6) in Theorem 1.2 is different form condition of (1.14) in Theorem 1.3 of [8].*

Theorem 1.3. *Suppose that $F(n,x)$ satisfies (A), (B_1), (B_2) , (B_3) and*
(A_1) there are some constants $\delta > 0$, $k \in \mathbb{Z}[0, [\frac{T}{2} - 1]]$ such that

$$-\frac{1}{2}\lambda_{k+1}|x|^2 \leq F(n,x) \leq -\frac{1}{2}\lambda_k|x|^2,$$

$\forall\, x \in \mathbb{R}^N$ with $|x| < \delta$ and $\forall n \in [1,T]$, where $\lambda_k = 2 - 2\cos k\omega$, $\omega = \frac{2\pi}{T}$, $T > 2$, $[a] = \max\{k \in \mathbb{Z} | k \leq a\}$ denotes the Gauss Function. Then, system (1.1) has at least two T-periodic solutions.

Parallelly, we have the following result.

Theorem 1.4. *Suppose that $F(n,x)$ satisfies (A), (B_4), (B_5) , (B_6) and (A_1). Then, system (1.1) has at least two T-periodic solutions.*

2. Some important lemmas

$H_T := \left\{u : \mathbb{Z} \to \mathbb{R}^N | u(n+T) = u(n),\, n \in \mathbb{Z}\right\}$ can be equipped with the inner product

$$\langle u, v\rangle = \sum_{n=1}^{T} \big(u(n), v(n)\big), \quad \forall u, v \in H_T, \tag{2.9}$$

so the norm $\|\cdot\|$ is

$$\|u\| = \Big(\sum_{n=1}^{T} |u(n)|^2\Big)^{\frac{1}{2}}, \quad \forall u \in H_T, \tag{2.10}$$

where $(\cdot,\cdot)$ and $|.|$ denotes the usual inner product and the usual norm in $\mathbb{R}^N$, respectively. It is easy to verify that $(H_T, \langle\cdot,\cdot\rangle)$ is a finite dimensional Hilbert space and linear homeomorphic to $\mathbb{R}^{NT}$.

For every positive number $r > 1$, we can equip H_T with another norm $\|u\|_r$, where

$$\|u\|_r = \Big(\sum_{n=1}^{T} |u(n)|^r\Big)^{\frac{1}{r}}, \qquad \forall u \in H_T.$$

Distinctly, $\|u\|_2 = \|u\|$ and $(H_T, \|u\|_2)$ is equivalent to $(H_T, \|u\|_r)$ for $r > 1$. Thus, there are two constants $C_2 \geq C_1 > 0$, such that $\forall u \in H_T$

(2.11) $$C_1\|u\|_r \leq \|u\| \leq C_2\|u\|_r.$$

For system (1.1), Xue and Tang [10] verify that the problem of seeking T-periodic solutions is equal to that of finding the critical points of $\varphi(u)$ defined in (1.8) on H_T.

To prove our results, we now give four useful lemmas.

Lemma 2.1. ([10]) *As a subspace of H_T, N_k is defined by:*

$$N_k := \{u \in H_T | -\Delta^2 u(n-1) = \lambda_k u(n)\},$$

where $\lambda_k = 2 - 2\cos k\omega$, $\omega = \frac{2\pi}{T}$, $k \in \mathbb{Z}[0, [\frac{T}{2}]]$. The following statements hold:
(i) $N_k \perp N_j$, $k \neq j$, $k, j \in \mathbb{Z}[0, [\frac{T}{2}]]$,
(ii) $H_T = \bigoplus_{k=0}^{[\frac{T}{2}]} N_k$.

Lemma 2.2 ([10]). *Define $H_k := \bigoplus_{j=0}^{k} N_j$, $H_k^{\perp} := \bigoplus_{j=k+1}^{[T/2]} N_j$, $k \in \mathbb{Z}[0, [T/2]-1]$, then one has*

$$\sum_{n=1}^{T} |\Delta u(n)|^2 \leq \lambda_k \|u\|^2, \qquad \forall u \in H_k;$$
$$\sum_{n=1}^{T} |\Delta u(n)|^2 \geq \lambda_{k+1} \|u\|^2, \qquad \forall u \in H_k^{\perp}.$$

Lemma 2.3 ([7]). *If φ is weakly lower semi continuous on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Lemma 2.4 ([1]). *Let φ be a C^1 function on $X = X_1 \bigoplus X_2$ with $\varphi(0) = 0$, satisfying (PS) condition and for some $\delta > 0$,*

$$\varphi(u) \geq 0 \quad for\, u \in X_1, \|u\| \leq \delta,$$
$$\varphi(u) \leq 0 \quad for\, u \in X_2, \|u\| \leq \delta.$$

Assume also that φ is bounded below and $\inf_X \varphi < 0$, then φ has at least two nonzero critical points.

By Lemma 2.1, one rewrites u as

(2.12) $$u = \bar{u} + \tilde{u} \in N_0 \bigoplus N_0^{\perp},$$

where $\bar{u} = (1/T)\sum_{n=1}^{T} u(n)$.

By (2.9), (2.10) and (2.12), one has

$$\begin{aligned}\|u\| &= \Big(\sum_{n=1}^{T}|u(n)|^2\Big)^{\frac{1}{2}} = \Big(\sum_{n=1}^{T}|\bar{u}+\tilde{u}(n)|^2\Big)^{\frac{1}{2}}\\ &= \Big(\sum_{n=1}^{T}\big(\bar{u}+\tilde{u}(n),\bar{u}+\tilde{u}(n)\big)\Big)^{\frac{1}{2}}\\ &= \Big(\sum_{n=1}^{T}\big(|\bar{u}|^2+|\tilde{u}(n)|^2\big)\Big)^{\frac{1}{2}} = \big(T|\bar{u}|^2+\|\tilde{u}\|^2\big)^{\frac{1}{2}}.\end{aligned}$$

Then, one has

$$\|u\| \le \sqrt{T+1}(|\bar{u}|^2+\|\tilde{u}\|^2)^{\frac{1}{2}} \quad \textit{and} \quad \|u\| \ge (|\bar{u}|^2+\|\tilde{u}\|^2)^{\frac{1}{2}}.$$

Therefore, one has that $\|u\| \to \infty$ if and only if $\big(|\bar{u}|+\|\tilde{u}\|^2\big)^{\frac{1}{2}} \to \infty$.

3. Proof of main results

Since the proof of Theorem 1.4 is similar to that of Theorem 1.3, we only prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 in this section.

For convenience, we denote

$$R_1 = \sum_{n=1}^{T} f(n), \quad R_2 = \sum_{n=1}^{T} g(n).$$

Proof of Theorem 1.1. According to (B_3), we can choose a positive constant a_1, such that

$$a_1 > \frac{\varepsilon + M_3}{\lambda_1 - M_3} > 0 \tag{3.13}$$

for a small number $\varepsilon > 0$ and

$$\lim_{|x|\to+\infty} \inf |x|^{-2}\sum_{n=1}^{T} F(n,x) > \frac{a_1}{2}M_3T. \tag{3.14}$$

By (B_1) , we obtain

$$\begin{aligned}&\Big|\sum_{n=1}^{T}\big[F_1(n,u(n)) - F_1(n,\bar{u})\big]\Big|\\ =&\Big|\sum_{n=1}^{T}\int_0^1 \big(\nabla F_1(n,\bar{u}+s\tilde{u}(n)),\tilde{u}(n)\big)ds\Big|\\ \le&\sum_{n=1}^{T}\int_0^1 f(n)|\bar{u}+s\tilde{u}(n)|^{\alpha}|\tilde{u}(n)|ds + \sum_{n=1}^{T}\int_0^1 g(n)|\tilde{u}(n)|ds\\ \le&\sum_{n=1}^{T} f(n)(|\bar{u}|^{\alpha}+|\tilde{u}(n)|^{\alpha})|\tilde{u}(n)| + \sum_{n=1}^{T} g(n)|\tilde{u}(n)|\end{aligned}$$

$$
\begin{aligned}
&\leq R_1|\bar{u}|^{\alpha}\|\tilde{u}\|_\infty + R_1\|\tilde{u}\|_\infty^{\alpha+1} + R_2\|\tilde{u}\|_\infty \\
&\leq \frac{\varepsilon}{2a_1}\|\tilde{u}\|_\infty^2 + \frac{a_1}{2\varepsilon}R_1^2|\bar{u}|^{2\alpha} + R_1\|\tilde{u}\|_\infty^{\alpha+1} + R_2\|\tilde{u}\|_\infty \\
&\leq \frac{\varepsilon}{2a_1}\|\tilde{u}\|^2 + \frac{a_1}{2\varepsilon}R_1^2|\bar{u}|^{2\alpha} + R_1\|\tilde{u}\|^{\alpha+1} + R_2\|\tilde{u}\| \qquad (3.15)
\end{aligned}
$$

for any $u \in H_T$ with $\|u\|_\infty := \max_{n\in\mathbb{Z}\cap[1,T]} |u(n)|$.

By (B_2), we have

$$
\begin{aligned}
&\left|\sum_{n=1}^{T}[F_2(u(n)) - F_2(\bar{u})]\right| \\
=&\left|\sum_{n=1}^{T}\int_0^1 (\nabla F_2(\bar{u}+s\tilde{u}(n)), \tilde{u}(n))\,ds\right| \\
\leq&\sum_{n=1}^{T}\int_0^1 M_3\big(|\bar{u}+s\tilde{u}(n)|\big)|\tilde{u}(n)|ds + \sum_{n=1}^{T}\int_0^1 M_4|\tilde{u}(n)|ds \\
\leq& M_3\sum_{n=1}^{T}\big(|\bar{u}| + \frac{1}{2}|\tilde{u}(n)|\big)|\tilde{u}(n)| + \sum_{n=1}^{T} M_4|\tilde{u}(n)| \\
\leq& M_3\sum_{n=1}^{T}|\bar{u}||\tilde{u}(n)| + \frac{M_3}{2}\sum_{n=1}^{T}|\tilde{u}|^2 + M_4\sum_{n=1}^{T}|\tilde{u}| \\
\leq& M_3\Big(\sum_{n=1}^{T}|\bar{u}|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{T}|\tilde{u}(n)|^2\Big)^{\frac{1}{2}} + \frac{M_3}{2}\sum_{n=1}^{T}|\tilde{u}|^2 + M_4\sum_{n=1}^{T}|\tilde{u}| \\
\leq& \frac{a_1}{2}M_3\sum_{n=1}^{T}|\bar{u}|^2 + \frac{M_3}{2a_1}\sum_{n=1}^{T}|\tilde{u}|^2 + \frac{M_3}{2}\sum_{n=1}^{T}|\tilde{u}|^2 + M_4\sum_{n=1}^{T}|\tilde{u}| \\
\leq& \frac{a_1}{2}M_3T|\bar{u}|^2 + \big(\frac{M_3}{2a_1} + \frac{M_3}{2}\big)\|\tilde{u}\|^2 + M_4\sqrt{T}\|\tilde{u}\| \qquad (3.16)
\end{aligned}
$$

for any $u \in H_T$.

Hence, by (1.8), (3.15), (3.16) and Lemma 2.2, we have

$$
\begin{aligned}
\varphi(u) \geq& \big(\frac{\lambda_1}{2} - \frac{\varepsilon}{2a_1} - \frac{M_3}{2a_1} - \frac{M_3}{2}\big)\|\tilde{u}\|^2 - (R_2 + M_4\sqrt{T})\|\tilde{u}\| - R_1\|\tilde{u}\|^{\alpha+1} \\
&+|\bar{u}|^2\left(|\bar{u}|^{-2}\sum_{n=1}^{T}F(n,\bar{u}) - \frac{a_1}{2}M_3T\right) - \frac{a_1}{2\varepsilon}R_1^2|\bar{u}|^{2\alpha}. \qquad (3.17)
\end{aligned}
$$

Since $u = \bar{u} + \tilde{u} \in N_0 \bigoplus N_0^{\perp}$, (3.13), (3.14) and (3.17) imply that

$$
\varphi(u) \to +\infty, \qquad \|u\| \to \infty.
$$

Thus, φ is coercive. Since φ is continuous, it possesses a bounded minimizing sequence in the finite dimensional Hilbert space H_T. Therefore, by Lemma 2.3, we obtain a critical point u which is a T-periodic solution of system (1.1) and the minimizer of the function φ. The proof is complete. □

By (3.15) in which the number $\frac{\varepsilon}{a_1}$ should be replaced by $\lambda_1 - \varepsilon$ and $M_3 = 0$ in (3.16), we have the following result.

Theorem 3.5. *Suppose that $F(n,x)$ with $F_2=0$ satisfying (A), (B_1) and*
(B_3') $\qquad \lim_{|x|\to+\infty}\inf |x|^{-2\alpha}\sum_{n=1}^{T}F(n,x) > \frac{M_1^2}{2\lambda_1}$.
Then, system (1.1) has at least one T-periodic solution that minimizes the functional φ in the Hilbert space H_T.

Comparing with Theorem 1.1 of [8], we see that the condition (B_3') is loose for some choices of T, for example $T=2,3,4,5$ and so on. In Theorem 1.1, when $F_1(n,x)\equiv 0$, comparing with Theorem 1.3 of [8], the condition (B_3) in some sense is loose for some choices of T, one can check it for $T=2,3,4,5,6$ and so on.

Now, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. By (B_6), we can choose a positive constant a_2 and $\varepsilon>0$, such that

$$a_2 > \frac{1}{\lambda_1 - R_1 - \varepsilon} \tag{3.18}$$

and

$$\lim_{|x|\to\infty}\inf |x|^{-2}\sum_{n=1}^{T}F(n) > \frac{a_2}{2}R_1^2. \tag{3.19}$$

By (B_4), $\forall u\in H_T$, we have

$$\begin{aligned}
&\Big|\sum_{n=1}^{T}[F_1(n,u(n)) - F_1(n,\bar{u})]\Big|\\
=&\Big|\sum_{n=1}^{T}\int_0^1(\nabla F_1(n,\bar{u}+s\tilde{u}(n)),\tilde{u}(n))ds\Big|\\
\le&\sum_{n=1}^{T}\int_0^1 f(n)|\bar{u}+s\tilde{u}(n)||\tilde{u}(n)|ds + \sum_{n=1}^{T}\int_0^1 g(n)|\tilde{u}(n)|ds\\
\le&\sum_{n=1}^{T}f(n)(|\bar{u}|+\frac{1}{2}|\tilde{u}(n)|)|\tilde{u}(n)| + \sum_{n=1}^{T}g(n)|\tilde{u}(n)|\\
\le&R_1|\bar{u}|\|\tilde{u}\|_\infty + \frac{R_1}{2}\|\tilde{u}\|_\infty^2 + R_2\|\tilde{u}\|_\infty\\
\le&\frac{1}{2a_2}\|\tilde{u}\|_\infty^2 + \frac{a_2}{2}R_1^2|\bar{u}|^2 + \frac{R_1}{2}\|\tilde{u}\|_\infty^2 + R_2\|\tilde{u}\|_\infty\\
=&(\frac{1}{2a_2}+\frac{R_1}{2})\|\tilde{u}\|_\infty^2 + R_2\|\tilde{u}\|_\infty + \frac{a_2}{2}R_1^2|\bar{u}|^2\\
\le&(\frac{1}{2a_2}+\frac{R_1}{2})\|\tilde{u}\|^2 + R_2\|\tilde{u}\| + \frac{a_2}{2}R_1^2|\bar{u}|^2.
\end{aligned} \tag{3.20}$$

By (B_5) , we have

$$\begin{aligned}
&\Big|\sum_{n=1}^{T}[F_2(n,u(n)) - F_2(n,\bar{u})]\Big|\\
=&\Big|\sum_{n=1}^{T}\int_0^1(\nabla F_2(n,\bar{u}+s\tilde{u}(n)),\tilde{u}(n))ds\Big|
\end{aligned}$$

$$\leq \sum_{n=1}^{T}\int_0^1 M_1|\bar{u}+s\tilde{u}(n)|^{\alpha}|\tilde{u}(n)|ds+\sum_{n=1}^{T}\int_0^1 M_2|\tilde{u}(n)|ds$$
$$\leq \sum_{n=1}^{T} M_1(|\bar{u}|^{\alpha}+|\tilde{u}(n)|^{\alpha})|\tilde{u}(n)|+\sum_{n=1}^{T} M_2|\tilde{u}(n)|$$
$$\leq M_1\sqrt{T}|\bar{u}|^{\alpha}\|\tilde{u}\|+M_1\sum_{n=1}^{T}|\tilde{u}|^{\alpha+1}+M_2\sum_{n=1}^{T}|\tilde{u}(n)|$$
$$\leq \frac{TM_1^2}{2\varepsilon}|\bar{u}|^{2\alpha}+\frac{\varepsilon}{2}\|\tilde{u}\|^2+C_1\|\tilde{u}\|^{\alpha+1}+C_2\|\tilde{u}\|. \tag{3.21}$$

Hence, by (1.8), (3.20), (3.21) and Lemma 2.2, we have

$$\varphi(u) \geq (\frac{\lambda_1}{2}-\frac{1}{2a_2}-\frac{R_1}{2}-\frac{\varepsilon}{2})\|\tilde{u}\|^2-C_1\|\tilde{u}\|^{\alpha+1}-(C_2+R_2)\|\tilde{u}(n)\|$$
$$+|\bar{u}|^2\left(|\bar{u}|^{-2}\sum_{n=1}^{T}F(n,\bar{u})-\frac{a_2}{2}R_1^2\right)-\frac{TM_1^2}{2\varepsilon}|\bar{u}|^{2\alpha}. \tag{3.22}$$

Since $u=\bar{u}+\tilde{u}\in N_0\bigoplus N_0^{\perp}$, (3.18), (3.19) and (3.22) imply that

$$\varphi(u)\to+\infty, \qquad \|u\|\to\infty,$$

that is, φ is coercive. It is easy to verify that there exists a bounded minimizing sequence which insures that φ possesses a minimal point in the finite dimensional Hilbert space H_T by Lemma 2.3. The proof is complete. □

Proof of Theorem 1.3. According to the proof of Theorem 1.1, we can implies that φ is bounded below and satisfies the (PS) condition. By (A_1) and Lemma 2.2, one has

$$\varphi(u)\leq \frac{1}{2}\lambda_k\|u\|^2+\sum_{n=1}^{T}\Big(-\frac{1}{2}\lambda_k|u|^2\Big)=0 \tag{3.23}$$

for any $u\in H_k$ with $\|u\|\leq\delta$ and

$$\varphi(u)\geq \frac{1}{2}\lambda_{k+1}\|u\|^2+\sum_{n=1}^{T}\Big(-\frac{1}{2}\lambda_{k+1}|u|^2\Big)=0 \tag{3.24}$$

for any $u\in H_k^{\perp}$ with $\|u\|\leq\delta$.
If $\inf_{u\in H_T}\varphi(u)<0$, we completed our proof of Theorem 1.3 by Lemma 2.4.
If $\inf_{u\in H_T}\varphi(u)\geq 0$, by (3.23) and (3.24), we have $\varphi(u)=\inf_{u\in H_T}\varphi(u)=0$ for any $u\in H_k$ with $\|u\|\leq\delta$, which implies that any $u\in H_k$ with $\|u\|\leq\delta$ are minimum points of φ. Thus, any $u\in H_k$ with $\|u\|\leq\delta$ are T-periodic solutions of systems (1.1) , and systems (1.1) has infinite T-periodic solutions in H_T. Hence, we complete the proof of our main results. □

The proof of Theorem 1.4 is almost the same as that in the proof of Theorem 1.3, so we omit it.

4. Examples

In this section, we will give two examples to illustrate our theorems.

Example 4.1. *Let $F(n+T,x) = F(n,x)$ for any $(n,x) \in (\mathbb{Z}, \mathbb{R}^N)$ and*

$$F(n,x) = \frac{\lambda_1}{16}|x|^2 + (\frac{T+1}{2} - n)|x|^{7/4} + (\frac{4}{3}T - n)|x|^{3/2}, \; n \in \mathbb{Z} \cap [1,T], \tag{4.25}$$

where

$$F_1(n,x) = (\frac{T+1}{2} - n)|x|^{7/4} + (2T-n)|x|^{3/2}, \; \forall (n,x) \in (\mathbb{Z} \cap [1,T], \mathbb{R}^N) \tag{4.26}$$

and

$$F_2(x) = \frac{\lambda_1}{16}|x|^2 - \frac{2}{3}T|x|^{3/2}, \; \forall x \in \mathbb{R}^N. \tag{4.27}$$

According to (4.26), one has

$$\begin{aligned} |\nabla F_1(n,x)| \le& \frac{7}{8}|T+1-2n||x|^{3/4} + \frac{3}{2}|2n-T||x|^{1/2} \\ \le& \frac{7}{8}(|T+1-2n| + \varepsilon)|x|^{3/4} + \frac{9T^3}{\varepsilon^2}, \; \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N, \end{aligned} \tag{4.28}$$

where $\varepsilon > 0$. Then, we obtained that (B_1) holds with $\alpha = 3/4$ and

$$f(n) = \frac{7}{8}(|T+1-2n| + \varepsilon), \quad g(n) = \frac{9T^3}{\varepsilon^2}. \tag{4.29}$$

According to (4.27), we have

$$\begin{aligned} |\nabla F_2(x)| \le& \frac{\lambda_1}{8}|x| + T|x|^{1/2} \\ \le& (\frac{\lambda_1}{8} + \varepsilon)|x| + \frac{T}{\varepsilon}, \; \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N, \end{aligned} \tag{4.30}$$

where $\varepsilon > 0$. Then, we obtained that (B_2) holds with

$$R_1 = \frac{\lambda_1}{8} + \varepsilon, \quad R_2 = \frac{T}{\varepsilon}. \tag{4.31}$$

Now, we verify that $F(n,x)$ satisfies (B_3). In fact,

$$\begin{aligned} &\lim_{|x| \to +\infty} \inf |x|^{-2} \sum_{n=1}^{T} F(n,x) \\ =& \lim_{|x| \to +\infty} \inf |x|^{-2} \sum_{n=1}^{T} [\frac{\lambda_1}{16}|x|^2 + (\frac{T+1}{2} - n)|x|^{7/4} + (\frac{4}{3}T - n)|x|^{3/2}] \\ =& \frac{\lambda_1}{16}T. \end{aligned}$$

It is easy to verify that $M_3 < \lambda_1$. If $T \in \{2,3,4,5,6,7,8,9,10\}$, we can choose $\varepsilon > 0$, such that

$$\lim_{|x| \to +\infty} \inf |x|^{-2} \sum_{n=1}^{T} F(n,x) = \frac{\lambda_1}{16}T > \frac{(\frac{\lambda_1}{8} + \varepsilon)^2 T}{2(\lambda_1 - \frac{\lambda_1}{8} - \varepsilon)} = \frac{M_3^2 T}{2(\lambda_1 - M_3)}.$$

Thus, the system (1.1) has at least one T-periodic solution by Theorem 1.1.

Example 4.2. *Let $F(n+T,x) = F(n,x)$, for any $(n,x) \in \mathbb{Z} \times \mathbb{R}^N$ and*

$$F(n,x) = \frac{T-n}{20}|x|^2 + \frac{4}{7}|x|^{7/4} - n|x|^{3/2} - |x| + (h(n),x). \tag{4.32}$$

Let

$$F_1(n,x)=\frac{T-n}{20}|x|^2-n|x|^{3/2}+(h(n),x), \tag{4.33}$$

$h:\mathbb{Z}\cap[1,T]\to\mathbb{R}^N$, $h(n+T)=h(n)$, *for* $n\in\mathbb{Z}\cap[1,T]$ *and*

$$F_2(x)=\frac{4}{7}|x|^{7/4}-|x|. \tag{4.34}$$

According to (4.33), one has

$$\begin{aligned}|\nabla F_2(n,x)|\leq&\frac{T-n}{10}|x|+\frac{3n}{2}+|h(n)|\\ \leq&(\frac{T-n}{10}+\varepsilon)|x|+\frac{T^2}{\varepsilon}+|h(n)|,\ \forall(n,x)\in\mathbb{Z}\times\mathbb{R}^N,\end{aligned}$$

where $\varepsilon>0$. *Then, we obtained that* (B_4) *holds with*

$$f(n)=\frac{T-n}{10}+\varepsilon,\quad g(n)=\frac{T^2}{\varepsilon}+|h(n)|. \tag{4.35}$$

It is easy to verify that (B_5) *holds with* $\alpha=3/4$ *and* $M_3=M_4=1$.
Now, we verify that $F(n,x)$ *satisfies* (B_6). *In fact, according to (4.32) and (4.35), we have*

$$\sum_{n=1}^{T}f(n)=\sum_{n=1}^{T}(\frac{T-n}{10}+\varepsilon)=T(\frac{T-1}{20}+\varepsilon)$$

and

$$\begin{aligned}&\lim_{|x|\to\infty}\inf|x|^{-2}\sum_{n=1}^{T}F(n,x)\\ =&\lim_{|x|\to\infty}\inf|x|^{-2}\sum_{n=1}^{T}\Big[\frac{T-n}{20}|x|^2+\frac{4}{7}|x|^{7/4}-n|x|^{3/2}-|x|+(h(n),x)\Big]\\ =&\frac{T(T-1)}{40}.\end{aligned}$$

When $T\in\{2,3,4\}$, *we can choose* $\varepsilon>0$, *such that*

$$\sum_{n=1}^{T}f(n)=T(\frac{T-1}{20}+\varepsilon)<\lambda_1$$

and

$$\begin{aligned}\lim_{|x|\to\infty}\inf|x|^{-2}\sum_{n=1}^{T}F(n,x)=&\frac{T(T-1)}{40}\\ >&\frac{1}{2(\lambda_1-\frac{T-1}{20}-\varepsilon)}\Big(\frac{T-1}{20}+\varepsilon\Big)^2\\ =&\frac{1}{2(\lambda_1-\sum_{n=1}^{T}f(n))}\Big(\sum_{n=1}^{T}f(n)\Big)^2.\end{aligned}$$

Thus, the system (1.1) has at least one T*-periodic solution by Theorem 1.2.*

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