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A CLASS OF THIRD-ORDER BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITION AT RESONANCE

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Abstract. In this paper, we consider third-order boundary value problem with, Dirichlet, Neumann and integral conditions at resonance case, where the kernel's dimension of the ordinary differential operator is equal to one and the ordinary differential equation which can be written as the abstract equation $Lu = Nu$, called semilinear form, where L is a linear Fredholm operator of index zero, and N is a nonlinear operator. First, we prove a priori estimates, and then we use Mawhin's coincidence degree theory to deduce the existence of solutions. One important ingredient to be able to apply this abstract results (Mawhin's coincidence degree theory) is proving the Fredholm property of the operator L. An example is also presented to illustrate the effectiveness of the main results.

1. Introduction

In this paper, we consider the following nonlinear third-order boundary value problem

$$u'''(t) = f(t, u(t), u'(t), u''(t)),\ t \in (0,1), \tag{1.1}$$

$$u(0) = u'(0) = 0,\ u(1) = \frac{3}{\eta^3}\int_0^\eta u(t)\,dt,\ \eta \in (0,1), \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, and $\eta \in (0,1)$. We say that the boundary value problem $(1.1) - (1.2)$ is a resonance problem if the linear equation $Lu = u''' = 0$, with the boundary value conditions (1.2) has non-trivial solution i.e. $dimKerL \geq 1$.

The theory of the boundary value problems with integral boundary conditions arises in different areas of applied mathematics and physics. For example, heat

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conduction, chemical engineering, underground water flow, thermo-elasticity and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. Recently, several authors have studied nonlocal boundary value problems at resonance and non-resonance for second-order, third-order and higher-order (in particular, third-order) ordinary differential equations, for instance see [2, 3, 5, 6, 8, 12, 14, 16, 18, 19, 21] and the references therein. However, to our knowledge the corresponding results for third-order with integral boundary conditions are rarely seen [1, 7, 9, 10, 11, 15, 17, 20, 22] and the references therein. In the most papers mentioned above, the coincidence degree theory of Mawhin was applied to establish existence theorem.

Inspired and motivation by works mentioned above, in the present article, we use the coincidence degree theory of Mawhin [13] to discuss the existence of solution for third-order nonlocal boundary value problem $(1.1)-(1.2)$ at resonance case, and establish an existence theorem. The paper is organized as follows. In Section 2 we give the background information from coincidence degree theory, we also define appropriate mappings and projectors that will be used in the sequel. We state and prove our main result in Section 3, and we give an example to illustrate Theorem 3.1.

2. Preliminaries

We first recall some notations and an abstract existence result (Mawhin 1979). Let Y, Z be two real Banach spaces and let $L : domL \subset Y \to Z$ be a linear operator which is Fredholm map of index zero (that is, ImL, the image of L, $KerL$, the kernel of L is finite dimensional with the same dimension as the Z/ImL), and $P : Y \to Y$, $Q : Z \to Z$ be continuous projections such that $ImP = KerL$, $KerQ = ImL$ and $Y = KerL \oplus KerP$, $Z = ImL \oplus ImQ$. It follows that $L \mid_{domL\cap KerP} \to ImL$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of Y such that $domL\cap\Omega \neq \phi$, the map $N : Y \to Z$ is said to be $L-compact$ on $\overline{\Omega}$ if the map $QN : \overline{\Omega} \to Z$ is bounded and $K_P\,(I-Q)\,N : \overline{\Omega} \to Y$ is compact.

We will formulate the boundary value problem $(1.1)-(1.2)$ as $Lu = Nu$ where L and N are appropriate operators. To obtain our existence results we use the following fixed point theorem of Mawhin.

Theorem 2.1. *(See* [13]*) Let L be a Fredholm operator of index zero and N be* $L-compact$ *on* $\overline{\Omega}$. *Assume that the following conditions are satisfied:*
(i) $Lu \neq \lambda Nu$ *for every* $(u,\lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0,1)$.
(ii) $Nu \notin ImL$ *for every* $u \in KerL \cap \partial\Omega$.
(iii) $deg\,(QN\mid_{KerL}, \Omega \cap KerL, 0) \neq 0$,
where $Q : Z \to Z$ *is a projection as above with* $ImL = KerQ$.
Then the abstract equation $Lu = Nu$ *has at least one solution in* $domL \cap \overline{\Omega}$.

In the following, we shall use the classical spaces $C\,[0,1]$, $C^1\,[0,1]$, $C^2\,[0,1]$ and $L^1\,[0,1]$. For $u \in C^2\,[0,1]$, we use the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$ where $\|u\|_\infty = \max\limits_{t\in[0,1]} |u\,(t)|$ and denote the norm in $L^1\,[0,1]$ by $\|\cdot\|_1$.
We will use the Sobolev space $W^{3,1}\,(0,1)$ which is defined by

$$W^{3,1}\,(0,1) = \left\{u : [0,1] \to \mathbb{R} : u, u', u''\,are\,absolutely\,continuous\,on\,[0,1]\,with\,u''' \in L^1\,[0,1]\right\}.$$

Let $Y = C^2[0,1]$, $Z = L^1[0,1]$, define the linear operator $L : domL \subset Y \to Z$ by

$$Lu = u''', \quad u \in domL,$$

where

$$domL = \left\{ u \in W^{3,1}(0,1) : u(0) = u'(0) = 0,\ u(1) = \frac{3}{\eta^3} \int_0^{\eta} u(t)\,dt \right\},$$

and define $N : Y \to Z$ by

$$Nu(t) = f(t, u(t), u'(t), u''(t)),\ t \in (0,1).$$

Then the boundary value problem $(1.1)-(1.2)$ can be written as $Lu = Nu$.

3. Existence results

We will assume that the following conditions hold and in all this paper let us set

$$Rf(s, u(s), u'(s), u''(s)) = \int_0^1 (1-s)^2 f(s, u(s), u'(s), u''(s))\,ds$$
$$-\frac{1}{\eta^3}\int_0^{\eta} (\eta - s)^3 f(s, u(s), u'(s), u''(s))\,ds.$$

(H_1) There exist functions $\alpha, \beta, \gamma, r \in L^1[0,1]$, such that for $(u,v,w) \in \mathbb{R}^3$, $t \in [0,1]$, it holds

$$|f(t,u,v,w)| \le \alpha(t)|u| + \beta(t)|v| + \gamma(t)|w| + r(t). \tag{3.1}$$

(H_2) There exists a constant $M > 0$ such that for $u \in domL$, if $|u''(t)| > M$ for all $t \in [0,1]$, then

$$Rf(s, u(s), u'(s), u''(s)) \neq 0. \tag{3.2}$$

(H_3) There exists a constant $M^* > 0$ such that for any $u(t) = \frac{b}{2}t^2 \in KerL$ with $\left|\frac{b}{2}\right| > M^*$, either

$$\frac{b}{2}\left[Rf(s, u(s), u'(s), u''(s))\right] < 0, \tag{3.3}$$

or else

$$\frac{b}{2}\left[Rf(s, u(s), u'(s), u''(s))\right] > 0. \tag{3.4}$$

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function, assume that conditions $(H_1)-(H_3)$ hold and that*

$$\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1 < \frac{1}{2}. \tag{3.5}$$

Then the boundary value problem $(1.1)-(1.2)$ has at least one solution in $C^2[0,1]$.

For the Proof of Theorem 3.1 we shall apply Theorem 2.1 and the following Lemmas. Before we state our lemmas, we say that L is a Fredholm operator of index zero, that is, ImL is closed and $\dim KerL = co\dim ImL$. This implies that there exist a continuous projections $P : Y \to Y$ and $Q : Z \to Z$ such that $ImP = KerL$ and $KerQ = ImL$. For this purpose, we must define P by (3.1) (see later), the linear continuous projector operator Q by

$$Qy(t) = \frac{1}{C}\left[\int_0^1 (1-s)^2 y(s)\,ds - \frac{1}{\eta^3}\int_0^\eta (\eta-s)^3 y(s)\,ds\right] t^2,$$

where $\frac{1}{C} = \frac{60}{2-\eta^3}$ and the linear operator $K_P : ImL \to domL \cap KerP$ by

$$K_P y(t) = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds, \quad \forall y \in ImL.$$

Lemma 3.2. (i) *The operator $L : domL \subset Y \to Z$ is a Fredholm operator of index zero.*
(ii) *For every $y \in ImL$, we have*

$$\|K_P y\| \le \|y\|_1 .$$

Proof. First, we prove (i). It is clear that

$$\begin{aligned} KerL &= \{u \in domL : \ Lu = 0\}, \\ &= \{u \in domL : \ u''' = 0\}, \\ &= \left\{u \in domL : \ u(t) = \frac{b}{2}t^2, \ b \in \mathbb{R}\right\} \simeq \mathbb{R}. \end{aligned}$$

Now, we show that

$$ImL = \left\{ y \in Z : \ \int_0^1 (1-s)^2 y(s)\,ds - \frac{1}{\eta^3}\int_0^\eta (\eta-s)^3 y(s)\,ds = 0 \right\}. \tag{3.6}$$

In fact

$$u''' = y, \tag{3.7}$$

has a solution $u(t)$ that satisfies the boundary value conditions (1.2), if and only if

$$\int_0^1 (1-s)^2 y(s)\,ds - \frac{1}{\eta^3}\int_0^\eta (\eta-s)^3 y(s)\,ds = 0. \tag{3.8}$$

From (3.7), we have

$$u(t) = u(0) + u'(0)\,t + u''(0)\,\frac{t^2}{2} + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds.$$

Thus from the condition $u(0) = u'(0) = 0$, we have

$$u(t) = u''(0)\,\frac{t^2}{2} + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds.$$

According to $u(1) = \frac{3}{\eta^3}\int_0^\eta u(t)\,dt$, we have

$$\frac{1}{2}u''(0) + \frac{1}{2}\int_0^1 (1-s)^2 y(s)\,ds = \frac{3}{\eta^3}\left[\int_0^\eta u''(0)\frac{t^2}{2}dt + \frac{1}{2}\int_0^\eta\int_0^t (t-s)^2 y(s)\,dsdt\right],$$

i.e.

$$\int_0^1 (1-s)^2 y(s)\,ds - \frac{1}{\eta^3}\int_0^\eta (\eta-s)^3 y(s)\,ds = 0.$$

Hence

$$ImL = \left\{ y \in Z \;:\; \int_0^1 (1-s)^2 y(s)\,ds - \frac{1}{\eta^3}\int_0^\eta (\eta-s)^3 y(s)\,ds = 0 \right\}.$$

On the other hand, if (3.8) holds, setting

$$u(t) = \frac{b}{2}t^2 + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds,$$

where b is an arbitrary constant, then $u(t)$ is a solution of (3.7). Hence (3.6) holds. For simplicity of notation in the definition of the projector operator Q, we set

$$Ry = \int_0^1 (1-s)^2 y(s)\,ds - \frac{1}{\eta^3}\int_0^\eta (\eta-s)^3 y(s)\,ds.$$

Let $C = \int_0^1 (1-t)^2 t^2 dt - \frac{1}{\eta^3}\int_0^\eta (\eta-t)^3 t^2 dt \neq 0,\ t \in (0,1]$. By simple calculation, we get $C = \frac{2-\eta^3}{60}$.
Now, we need to show that the operator Q is projector. From $Qy(t) = \frac{1}{C}\cdot(Ry)\cdot t^2$, we have

$$\begin{aligned}
\left(Q^2 y\right)(t) &= \left(Q\left(Qy\right)\right)(t),\\
&= \frac{1}{C}\left(\frac{1}{C}Ry\right)\left(\int_0^1 (1-t)^2 t^2 ds - \frac{1}{\eta^3}\int_0^\eta (\eta-t)^3 t^2 ds\right)t^2,\\
&= \frac{1}{C}(Ry)\,t^2\\
&= (Qy)(t),
\end{aligned}$$

which implies that the operator Q is a projector. Furthermore, $ImL = KerQ$.
In order, to show $Z = ImL \oplus ImQ$, it remains to shows two following steps.

Step 1. For $y \in Z$, let $y = (y - Qy) + Qy$, since $Q(y - Qy) = Qy - Q^2 y = 0$, we know $(y - Qy) \in KerQ = ImL$ and $Qy \in ImQ$. Thus

$$Z = ImL + ImQ.$$

Step 2. Let $y \in ImL \cap ImQ$. Since $y \in ImQ$, then there exists $\rho \in \mathbb{R}$ such that $y(t) = \rho t^2,\ t \in [0,1]$. Since $y \in ImL = KerQ$, then

$$0 = \rho(Ry)(t) = \rho\left(\int_0^1 (1-t)^2 t^2 ds - \frac{1}{\eta^3}\int_0^\eta (\eta-t)^3 t^2 ds\right) = \rho C.$$

Since $C \neq 0$, then $\rho = 0$, so we have $y(t) = 0,\ t \in [0,1]$, which implies

$$ImL \cap ImQ = \{0\}.$$

As consequence of Step 1 and Step 2, we deduce that

$$Z = ImL \oplus ImQ,$$

and so

$$\dim KerL = co\dim ImL = \dim ImQ = 1.$$

Thus L is Fredholm operator of index zero.
We are now ready to give the other projector employed in the proof of (ii). Define $P : Y \to Y$ by

$$(Pu)(t) = u''(0)\frac{t^2}{2}. \tag{3.9}$$

Note that $KerP = \left\{u \in Y :\ u''(0)\frac{t^2}{2} = 0\right\} = \{u \in Y\ :\ u''(0) = 0\}$ and $ImP = KerL$.
Similarly, we shall prove that the operator P is projector and $Y = KerP \oplus KerL$.
Fistly, since $(Pu)''(t) = u''(0)$, then $\left(P^2u\right)(t) = Pu(t),\ t \in [0,1]$.
Secondly, for all $u \in Y$ and $t \in [0,1]$, we have

$$\begin{aligned} u(t) &= (u(t) - Pu(t)) + Pu(t) \\ &= \left(u(t) - u''(0)\frac{t^2}{2}\right) + u''(0)\frac{t^2}{2}, \end{aligned}$$

that is $Y = KerP + KerL$. By simple calculation we can get $KerL \cap KerP = \{0\}$. Then $Y = KerP \oplus KerL$.
Before, to estimate the supremum norm of the generalized inverse operator K_P. It remains to prove that the operator K_P is the generalized inverse of L. In fact, if $y \in ImL$, then

$$(LK_P)\,y(t) = [(K_Py)(t)]''' = y(t).$$

And for $u \in domL \cap KerP$, we know

$$(K_PL)\,u(t) = (K_P)\,u'''(t) = \frac{1}{2}\int_0^t (t-s)^2 u'''(s)\,ds = u(t) - u(0) - u'(0)\,t - u''(0)\frac{t^2}{2},$$

in view of $u \in domL \cap KerP$, $u(0) = u'(0) = 0$ and $Pu = 0$, it follows that

$$(K_PL)\,u(t) = u(t).$$

This shows that $K_P = \left(L\mid_{domL\cap KerP}\right)^{-1}$.
Lastly, we estimate the supremum norm of the generalized inverse operator K_P. From the definition of K_P, it follows that

$$\|K_Py\|_\infty \le \frac{1}{2}\int_0^1 (1-s)^2 |y(s)|\,ds \le \int_0^1 |y(s)|\,ds = \|y\|_1.$$

From $(K_Py)'(t) = \int_0^t (t-s)\,y(s)\,ds$, we obtain

$$\left\|(K_Py)'\right\|_\infty \le \int_0^1 (1-s)\,|y(s)|\,ds \le \int_0^1 |y(s)|\,ds = \|y\|_1,$$

and from $(K_P y)''(t) = \int_0^t y(s)\,ds$, we obtain

$$\left\|(K_P y)''\right\|_\infty \le \int_0^1 |y(s)|\,ds = \|y\|_1 ,$$

then

$$\|K_P y\| \le \|y\|_1 . \tag{3.10}$$

□

Lemma 3.3. *Let* $\Omega_1 = \{u \in domL \setminus KerL : \ Lu = \lambda Nu, \ for\ some\ \lambda \in [0,1]\}$*. Then* Ω_1 *is bounded.*

Proof. Suppose that $u \in \Omega_1$, and $Lu = \lambda Nu$. Thus $\lambda \neq 0$ and $QNu = 0$, so it yields

$$Rf(s, u(s), u'(s), u''(s)) = 0.$$

Thus, by condition (H_2), there exists $t_1 \in [0,1]$, such that $|u''(t_1)| \le M$. In view of

$$u''(0) = u''(t_1) - \int_0^{t_1} u'''(t)\,dt,$$

then, we have

$$|u''(0)| \le M + \int_0^1 |u'''(s)|\,ds = M + \|u'''\|_1 = M + \|Lu\|_1 \le M + \|Nu\|_1 . \tag{3.11}$$

Again for $u \in \Omega_1$, then $(I-P)u \in domL \cap KerP = ImK_P$ and $LPu = 0$, $0 < \lambda < 1$ and $Nu = \frac{1}{\lambda} Lu \in ImL$, thus from Lemma 3.2, we know

$$\|(I-P)u\| = \|K_P L (I-P)u\| \le \|L(I-P)u\|_1 = \|Lu\|_1 \le \|Nu\|_1 . \tag{3.12}$$

From (3.11),(3.12) and $\|Pu\| = |u''(0)|$, we have

$$\|u\| \le \|Pu\| + \|(I-P)u\| = |u''(0)| + \|(I-P)u\| \le M + 2\|Nu\|_1 . \tag{3.13}$$

From (3.1) and (3.13), we obtain

$$\|u\| \le 2\left[\|\alpha\|_1 \|u\|_\infty + \|\beta\|_1 \|u'\|_\infty + \|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2}\right]. \tag{3.14}$$

Thus, from $\|u\|_\infty \le \|u\|$ and (3.14), we have

$$\|u\|_\infty \le \frac{2}{1 - 2\|\alpha\|_1}\left[\|\beta\|_1 \|u'\|_\infty + \|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2}\right]. \tag{3.15}$$

From $\|u'\|_\infty \le \|u\|$, (3.14) and (3.15), we have

$$\|u'\|_\infty \le \|u\| ,$$

$$\begin{aligned} \|u'\|_\infty &\le 2\left[1 + \frac{2\|\alpha\|_1}{1 - 2\|\alpha\|_1}\right]\left[\|\beta\|_1 \|u'\|_\infty + \|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2}\right], \\ &= \frac{2}{1 - 2\|\alpha\|_1}\left[\|\beta\|_1 \|u'\|_\infty + \|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2}\right], \end{aligned}$$

i.e

$$\|u'\|_\infty \left[\frac{1 - 2\|\alpha\|_1 - 2\|\beta\|_1}{1 - 2\|\alpha\|_1}\right] \le \frac{2}{1 - 2\|\alpha\|_1}\left[\|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2}\right].$$

Therefore

$$\left\|u'\right\|_\infty \leq \frac{2}{1-2\left\|\alpha\right\|_1-2\left\|\beta\right\|_1}\left[\left\|\gamma\right\|_1\left\|u''\right\|_\infty+\left\|r\right\|_1+\frac{M}{2}\right]. \tag{3.16}$$

Again, from $\left\|u''\right\|_\infty \leq \left\|u\right\|, (3.14), (3.15)$ and (3.16), we have

$$\begin{aligned}\left\|u''\right\|_\infty &\leq \left[2\left\|\beta\right\|_1+\frac{4\left\|\beta\right\|_1\left\|\alpha\right\|_1}{1-2\left\|\alpha\right\|_1}\right]\left\|u'\right\|_\infty+\left[\frac{4\left\|\alpha\right\|_1}{1-2\left\|\alpha\right\|_1}+2\right]\left[\left\|\gamma\right\|_1\left\|u''\right\|_\infty+\left\|r\right\|_1+\frac{M}{2}\right],\\ &\leq \left[\frac{4\left\|\beta\right\|_1}{\left(1-2\left\|\alpha\right\|_1-2\left\|\beta\right\|_1\right)\left(1-2\left\|\alpha\right\|_1\right)}+\frac{2}{1-2\left\|\alpha\right\|_1}\right]\left[\left\|\gamma\right\|_1\left\|u''\right\|_\infty+\left\|r\right\|_1+\frac{M}{2}\right],\\ &= \frac{2}{\left(1-2\left\|\alpha\right\|_1-2\left\|\beta\right\|_1\right)}\left[\left\|\gamma\right\|_1\left\|u''\right\|_\infty+\left\|r\right\|_1+\frac{M}{2}\right],\end{aligned}$$

i.e

$$\left\|u''\right\|_\infty \leq \left[\frac{2\left(\left\|r\right\|_1+\frac{M}{2}\right)}{1-2\left\|\alpha\right\|_1-2\left\|\beta\right\|_1-2\left\|\gamma\right\|_1}\right], \tag{3.17}$$

thus, from (3.17), there exists $M_1 > 0$ such that

$$\left\|u''\right\|_\infty \leq M_1, \tag{3.18}$$

therefore, from (3.18) and (3.16), there exists $M_2 > 0$, such that

$$\left\|u'\right\|_\infty \leq M_2, \tag{3.19}$$

hence, from (3.19) and (3.15), there exists $M_3 > 0$, such that

$$\left\|u\right\|_\infty \leq M_3. \tag{3.20}$$

Consequently

$$\left\|u\right\| = \max\left\{\left\|u\right\|_\infty, \left\|u'\right\|_\infty, \left\|u''\right\|_\infty\right\} \leq \max\left\{M_1, M_2, M_3\right\}.$$

Again, from $(3.1), (3.18), (3.19)$ and (3.20), we have

$$\left\|u'''\right\|_1 = \left\|Lu\right\|_1 \leq \left\|Nu\right\|_1 \leq \left\|\alpha\right\|_1 M_3+\left\|\beta\right\|_1 M_2+\left\|\gamma\right\|_1 M_1+\left\|r\right\|_1.$$

So, Ω_1 is bounded. □

Lemma 3.4. *The set* $\Omega_2 = \{u \in KerL : \ Nu \in ImL\}$ *is bounded.*

Proof. Let $u \in \Omega_2$, then $u \in KerL = \left\{u \in domL : \ u = \frac{b}{2}t^2, \ b \in \mathbb{R}, \ t \in [0,1]\right\}$. Also, since $KerQ = ImL$, then $QNu = 0$, therefore

$$Rf\left(s, \frac{b}{2}s^2, bs, b\right) = 0.$$

From condition (H_2), $\left\|u\right\|_\infty = \left|\frac{b}{2}\right| \leq M$, so $\left\|u\right\| \leq M$, thus Ω_2 is bounded. □

Before we define the set Ω_3, we must state our isomorphism, $J : KerL \to ImQ$. Let

$$J\left(\frac{b}{2}t^2\right) = \frac{b}{2}t^2, \ \forall b \in \mathbb{R}, \ t \in [0,1],$$

and define

$$\Omega_3 = \left\{u \in KerL : \ -\lambda Ju+(1-\lambda)QNu = 0, \ \lambda \in [0,1]\right\}.$$

Lemma 3.5. *If the first part of condition* (H_3) *holds, then*

$$\left(\frac{b}{2}\right)\left(\frac{60}{2-\eta^3}\right)\left[Rf\left(s,\frac{b}{2}s^2,bs,b\right)ds\right]<0, \tag{3.21}$$

for all $\left|\frac{b}{2}\right|>M^*$ *and* Ω_3 *is bounded.*

Proof. Suppose that $u=\frac{b_0}{2}t^2\in\Omega_3$. Then we obtain

$$\lambda\left(\frac{b_0}{2}\right)=(1-\lambda)\left(\frac{60}{2-\eta^3}\right)\left(Rf\left(s,b_0\frac{s^2}{2},b_0s,b_0\right)\right).$$

If $\lambda=1$, then $b_0=0$, which gives Ω_3 bounded.
Otherwise, if $\lambda\neq1$, there exist $M^*>0$ such that $\left|\frac{b_0}{2}\right|>M^*$. Then in view of (3.21), we have

$$\lambda\left(\frac{b_0}{2}\right)^2=(1-\lambda)\frac{b_0}{2}\left(\frac{60}{2-\eta^3}\right)\left(Rf\left(s,b_0\frac{s^2}{2},b_0s,b_0\right)\right)<0,$$

which contradicts the fact that $\lambda\left(\frac{b_0}{2}\right)^2\geq0$. Then $|u|=\left|\frac{b_0}{2}t^2\right|\leq\left|\frac{b_0}{2}\right|\leq M^*$, we obtain $\|u\|\leq M^*$. Hence $\Omega_3\subset\{u\in KerL:\|u\|\leq M^*\}$ is bounded.
If $\lambda=0$, it yields

$$Rf\left(s,\frac{b_0}{2}s^2,b_0s,b_0\right)=0.$$

Taking condition (H_2) into account, we obtain $\|u\|=\left|\frac{b}{2}\right|\leq M^*$. □

Now, define Ω_3 by

$$\Omega_3=\{u\in KerL:\ \lambda Ju+(1-\lambda)QNu=0,\ \lambda\in[0,1]\}$$

Lemma 3.6. *If the second part of* (H_3) *holds, then*

$$\left(\frac{b}{2}\right)\left(\frac{60}{2-\eta^3}\right)\left[Rf\left(s,\frac{b}{2}s^2,bs,b\right)\right]>0, \tag{3.22}$$

for all $\left|\frac{b}{2}\right|>M^*$ *and* Ω_3 *is bounded.*

Proof. A similar argument as above shows that Ω_3 is bounded. □

The Proof of Theorem 3.1 is now an easy consequence of the above lemmas and Theorem 2.1.

Proof. of Theorem 3.1.
Let Ω to be an open bounded subset of Y such that $\overset{3}{\underset{i=1}{\cup}}\overline{\Omega}_i\subset\Omega$. By using the fact that u''' is bounded and the Arzela-Ascoli Theorem, we can prove that $K_P(I-Q)N:\overline{\Omega}\to Y$ is compact, thus N is $L-compact$ on $\overline{\Omega}$. Then by Lemmas 3.3 and 3.4, we have
(*i*) $Lu\neq\lambda Nu$ for every $(u,\lambda)\in[(domL\setminus KerL)\cap\partial\Omega]\times(0,1)$.
(*ii*) $Nu\notin ImL$ for every $u\in KerL\cap\partial\Omega$.
(*iii*) Let $H(u,\lambda)=\pm\lambda Ju+(1-\lambda)QNu=0,\ \lambda\in[0,1]$.
According to Lemmas 3.5 and 3.6, we know that $H(u,\lambda)\neq0$ for every $u\in KerL\cap\partial\Omega$. Thus, by the homotopy property of degree, we obtain

$$\begin{aligned}deg\left(QN\mid_{KerL},\Omega\cap KerL,0\right)&=deg\left(H(\cdot,0),\Omega\cap KerL,0\right),\\&=deg\left(H(\cdot,1),\Omega\cap KerL,0\right),\end{aligned}$$

$$= deg\left(\pm J, \Omega \cap KerL, 0\right) \neq 0.$$

Then, by Theorem 2.1, $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$, so the boundary value problem $(1.1) - (1.2)$ has at least one solution in $C^2[0,1]$. The proof is complete. □

We construct an example to illustrate the applicability of the results presented.

Example 3.1. *Consider the following boundary value problem*

$$u'''(t) = f\left(t, u(t), u'(t), u''(t)\right),\ t \in (0,1), \tag{3.23}$$

$$u(0) = u'(0) = 0,\ u(1) = \frac{3}{\eta^3}\int_0^\eta u(t)\,dt,\ \eta \in (0,1), \tag{3.24}$$

where

$$f\left(t, u(t), u'(t), u''(t)\right) = \frac{1}{3}u''(t) + \frac{1}{6}(1-t)\left(1 - \cos\left(u'(t)\right)\sin\left(u(t)\right),\ \ t \in (0,1).$$

Here we have

$$\left|\frac{1}{3}u''(t) + \frac{1}{6}(1-t)\left(1-\cos\left(u'(t)\right)\sin\left(u(t)\right)\right| \leq \frac{1}{3} + \frac{1}{3}\left|u''(t)\right|,$$

that is

$$\alpha(t) = 0,\ \beta(t) = 0,\ \gamma(t) = \frac{1}{3}\ \text{ and }\ r(t) = \frac{1}{3}.$$

So, condition (H_1) is satisfied, which gives

$$\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1 = \frac{1}{3} < \frac{1}{2}.$$

Set

$$I = Rf\left(s, u(s), u'(s), u''(s)\right)$$

$$= \int_0^1 (1-s)^2 f\left(s, u(s), u'(s), u''(s)\right) ds - \frac{1}{\eta^3}\int_0^\eta (\eta - s)^3 f\left(s, u(s), u'(s), u''(s)\right) ds.$$

If $u''(t) < -M = -10$, then

$$f\left(t, u(t), u'(t), u''(t)\right) < \frac{1}{3}(1-M) = -3 < 0.$$

In this case, we have $I < 0$, because

$$\int_0^1 (1-s)^2 f\left(s, u(s), u'(s), u''(s)\right) ds < \int_0^\eta \left(1 - \frac{s}{\eta}\right)^3 f\left(s, u(s), u'(s), u''(s)\right) ds.$$

If $u''(t) > M = 10$, then

$$f\left(t, u(t), u'(t), u''(t)\right) > \frac{1}{3}(1+M) = \frac{11}{3} > 0.$$

Hence, $I > 0$, because

$$\int_0^1 (1-s)^2 f\left(s, u(s), u'(s), u''(s)\right) ds > \int_0^\eta \left(1 - \frac{s}{\eta}\right)^3 f\left(s, u(s), u'(s), u''(s)\right) ds.$$

Therefore, the condition (H_2) *is satisfied.*
If $\frac{b}{2} < -M^* = -5$ *and* $u(t) = \frac{b}{2}t^2$ *, then*

$$f(t, u(t), u'(t), u''(t)) < \frac{1}{3}(1+b) = -3 < 0.$$

In this case, we have $I < 0$*, because*

$$\int_0^1 (1-s)^2 f(s, u(s), u'(s), u''(s))\, ds < \int_0^\eta \left(1 - \frac{s}{\eta}\right)^3 f(s, u(s), u'(s), u''(s))\, ds.$$

Hence

$$\frac{b}{2}\left[\int_0^1 (1-s)^2 f(s, u(s), u'(s), u''(s))\, ds - \frac{1}{\eta^3}\int_0^\eta (\eta - s)^3 f(s, u(s), u'(s), u''(s))\, ds\right] > 0.$$

Therefore $\frac{b}{2}I > 0$*. So condition* (H_3) *is satisfied.*
Thus, all the conditions of Theorem 3.1 *are satisfied, which implies that the boundary value problem* $(3.23) - (3.24)$ *has at least one solution* $u \in C^2[0,1]$*.*

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EXISTENCE RESULTS FOR HYBRID DIFFERENTIAL EQUATION WITH GENERALIZED FRACTIONAL DERIVATIVE

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Abstract. Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). In recent years, fractional differential equations(FDEs) arise naturally in various fields such as rheology, fractals, chaotic dynamics, modelling and control theory, signal processing, bioengineering and biomedical applications, etc. In this paper, we discuss the existence results for hybrid differential equation with Katugampola fractional derivative. The argument is based upon Dhage fixed point theorem. We also discuss the existence result for hybrid differential equation.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). In recent years, fractional differential equations(FDEs) arise naturally in various fields such as rheology, fractals, chaotic dynamics, modelling and control theory, signal processing, bioengineering and biomedical applications, etc. Detailed study on fractional differential equations can be seen in, see [1, 2, 3, 4, 16]. Theory of fractional hybrid differential equation has been extensively studied by many authors [5, 6, 7, 8, 9, 15]. Recently, U. N. Katugampola [10] introduced generalized fractional derivative and it has been studied extensively by some researchers [11, 12, 13, 14].

Consider the hybrid differential equation involving generalized fractional derivative of the form

$$\begin{cases} {}^{\rho}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J := [0,a], \\ \frac{x(t)}{f(t,x(t))}|_{t=0} = x_0, \end{cases} \tag{1.1}$$

where ${}^{\rho}D^{\alpha}_{a_+}$ is Katugampola fractional derivative of order α and $\rho > 0$. Here $f : J \times R \to R| \{0\}$ and $g : J \times R \to R$ are given continuous function.

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The paper is organized as follows. In Section 2, we present notations and definition used throughout the paper. In Section 3, we discuss the existence result for hybrid differential equation.

2. Preliminary

In this section, we recall some definitions and results from fractional calculus. The following observations are taken from [9, 11]. Throughout this paper, let $C(J)$ a space of continuous functions from J into R with the norm

$$\|x\| = \sup\{|x(t)| : t \in J\}.$$

Definition 2.1. *The generalized left-sided fractional integral* ${}^{\rho}I^{\alpha}_{a+}f$ *of order* α *is defined by*

$$\left({}^{\rho}I^{\alpha}_{a+}\right)f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}f(s)ds,\ t > a, \tag{2.1}$$

if the integral exists.

The generalized fractional derivative, corresponding to the generalized fractional integral (2.1), *is defined for* $0 \le a < t$, *by*

$$\left({}^{\rho}D^{\alpha}_{a+}f\right)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t (t^{\rho}-s^{\rho})^{n-\alpha+1}s^{\rho-1}f(s)ds, \tag{2.2}$$

if the integral exists.

Lemma 2.1. *A function* $x \in C(J)$ *is the solution of fractional initial value problem*

$$\begin{cases} {}^{\rho}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), t \in J, \\ \frac{x(t)}{f(t,x(t))}|_{t=0} = x_0, \end{cases}$$

if and only if x *satisfies the following Volterra integral equation*

$$x(t) = f(t,x(t))\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}g(s,x(s))ds\right). \tag{2.3}$$

Lemma 2.2. *Let* S *be a non-empty, closed convex and bounded subset of the Banach algebra* X *let* $A : X \to X$ *and* $B : S \to X$ *be two operators such that*

1. *A is Lipschitzian with a Lipschitz constant k,*
2. *B is completely continuous,*
3. *$x = AxBy \Rightarrow x \in S$ for all $y \in S$, and*
4. *$Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$,*

then the operators has a solution.

3. Existence results

We make the following hypotheses to prove our main results.

(H1) The function $f : J \times R \to R|\{0\}$, there exixts a constant $L > 0$, such that

$$|f(t,x(t)) - f(t,y(t))| \le L(|x(t)-y(t)|),$$

for $t \in J$ and for all $x, y \in R$.

(H2) There exists a function $h : J \to R$, such that

$$|g(t,x(t))| \le h(t),\ \forall\ t \in J,\ x \in R.$$

(H3)

$$r \geq K\left(x_0 + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{1}{\Gamma(\alpha+1)} \|h\|_C\right) \tag{3.1}$$

where $|f(t,x)| \leq K,\ \forall\ t \in J,\ x \in R$.

$$L\left(\frac{x_0}{\Gamma(\gamma)} + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \|h\|_C\right) < 1.$$

Theorem 3.1. *Assume that [H1]-[H3] are satisfied. Then,* (1.1) *has solution on* J.

Proof. We define a subset S of X by

$$S = \{x \in X : \|x\|_C \leq r\}$$

where r satisfies inequality

$$r \geq K\left(x_0 + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{1}{\Gamma(\alpha+1)} \|h\|_C\right),$$

where $|f(t,x)| \leq K$.
Clearly S is closed, convex and bounded subset of the Banach space X. By Lemma 2.1 the initial value problem (1.1)

$$x(t) = f(t,x(t))\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s,x(s))ds\right). \tag{3.2}$$

Define two operators $A : X \to X$ by

$$Ax(t) = f(t,x(t)), \tag{3.3}$$

and $B : S \to X$ by

$$Bx(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s,x(s))ds. \tag{3.4}$$

Then $x = AxBx$. We shall show that the operators A and B satisfy all the condition of Lemma 2.2. We split the proof into a sequence of steps.
Step 1. The operator A is a Lipschitz on X.

$$\begin{aligned}|(Ax(t) - Ay(t))| &= |f(t,x(t)) - f(t,y(t))| \\ &\leq L\,|(x(t) - y(t))| \\ &\leq L\,\|x - y\|_C\,,\end{aligned}$$

which implies

$$\|Ax - Ay\| \leq L\,\|x - y\|_C\,.$$

Step 2. The Operator B is completely continuous on S.
First, we show that B is continuous on S. Let $\{x_n\}$ be a sequence in S convergent

to a point $x \in S$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty}\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s, x_n(s))ds\right)\\ &= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \lim_{n\to\infty} g(s, x_n(s))ds\\ &= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s, x(s))ds\\ &= Bx(t).\end{aligned}$$

This shows that B is continuous on S. It is sufficient to show that B(S) is uniformly bounded and equicontinuous set in X. First we note that

$$\begin{aligned}|Bx(t)| &= \left|x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s, x(s))ds\right|\\ &\le x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left|g(s, x(s))\right| ds\\ &\le x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left|h(s)\right| ds\\ &\le x_0 + \left(\frac{a^\rho}{\rho}\right)^{\alpha} \frac{1}{\Gamma(\alpha+1)} \|h\|_C,\end{aligned}$$

for all $t \in J$.

$$\|Bx\|_C \le x_0 + \left(\frac{a^\rho}{\rho}\right)^{\alpha} \frac{1}{\Gamma(\alpha+1)} \|h\|_C.$$

This shows that B is uniformly bounded on S.
Next, we show that B is an equicontinuous set in X. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in S$. Then we have

$$|Bx(t_1) - Bx(t_2)| \le \frac{1}{\Gamma(\alpha+1)} \|h\|_C \left(\left(\frac{t_1^\rho}{\rho}\right)^{\alpha} - \left(\frac{t_2^\rho}{\rho}\right)^{\alpha}\right).$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in S$ as $t_1 - t_2 \to 0$. Therefore, it follows fom the Arzela-Ascoli theorem that B is a completely continuous operator on S.
Step 3. Next we prove that (3) of Lemma 2.2.

$$\begin{aligned}|x(t)| &= \left|f(t, x(t))\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s, x(s))ds\right)\right|\\ &\le |f(t, x(t))|\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left|g(s, x(s))\right| ds\right)\\ &\le K\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left|h(s)\right| ds\right)\\ &\le K\left(x_0 + \left(\frac{a^\rho}{\rho}\right)^{\alpha} \frac{1}{\Gamma(\alpha+1)} \|h\|_C\right).\end{aligned}$$

Thus, we obtain

$$\|x\|_C \leq K\left(x_0 + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{1}{\Gamma(\alpha+1)}\|h\|_C\right) \leq r.$$

Step 4. Now, we show that $Mk < 1$, that is (4) of Lemma 2.2 holds.
Thus we have

$$M = \|B(s)\| = \sup\{\|Bx : x \in S\|\} \leq x_0 + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{1}{\Gamma(\alpha+1)}\|h\|_C \leq r,$$

and $k = L$. Thus, all the conditions of Lemma 2.2 are satisfied and hence the operator equation $x = AxBx$ has a solution in S. In consequence, the problem (1.1) has a solution on J. This complete the proof. □

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SOFT SEPARATION AXIOMS AND SOFT PRODUCT OF SOFT TOPOLOGICAL SPACES

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Abstract. In this article, we deal with the soft separation axioms using soft points on soft topological space and discuss the characterizations and properties of them. We extend these separation axioms to the soft product of soft topological spaces. Also we provide correct examples for the wrong examples example:1, example:2 and example:3 given in article [8].

For the vagueness and uncertinity of real life problems, there are several mathematical tools such as fuzzy sets, intuitionistic fuzzy sets, rough sets, vague sets etc. There is one more mathematical tool named soft sets which was introduced by Molodsov[12] in 1999. After that it was developed and used in decision making problems by Maji et. al in [10] and [11]. Aktas and Cagman [1] introduced the applications of soft set theory in algebraic structures in 2007. Kharral and Ahmad [9] introduced and discussed several properties of soft mappings. Shabir and Naz [16] investigated soft seperation axioms defined for crisp points in 2011. Hussain and Ahmad [7] investigate the properties of soft interior, soft closure and soft boundary in 2011. Aygunoglu and Aygun [2] in 2012 generalize Alexander subbase theorem and Tychonoff theorem to the soft topological spaces by defining and using the product of soft topological spaces. Nazmul and Samanta [13] studied the neighbourhood properties of soft topological spaces in 2013. There are several articles related to the properties of soft topological spaces and soft mappings on soft topological spaces. Some of them are [4], [6], [14], [17], [19] [20], [21]. Four different types of sepereation axioms were defined and discussed in [5], [8], [16] and [18]. Singh and Noorie [17] derives the relation among these four types of $T_i,\ i = 1, 2, 3, 4$ spaces in 2017.

In the second section of this article, we give some basic definitions and preliminaries of soft topological spaces.

In the third section of this article, we deal with the soft separation axioms using soft points and discuss about the characterizations and properties of them. In fact

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these separation axioms are stronger than other separation axioms. We extend these separation axioms to the product of soft topological spaces. Also we provide correct examples for the wrong examples Example:1, Example:2 and Example:3 given in article [8]

Throughout this paper, X is the universe set, E is a set of parameters and $\mathcal{P}(X)$ is the set of all subsets of X.

1. Preliminaries

Definition 1.1. [12] A mapping $F : E \to \mathcal{P}(X)$ is called a soft set and is denoted by $(F,\ E)$. The family of all soft sets over X is denoted as $SS(X,\ E)$

Definition 1.2. [12] Let $(F,\ E)$ and $(G,\ E)$ be two soft sets over X. Then $(F,\ E)$ is a soft subset of $(G,\ E)$ written as $(F,\ E)\tilde{\subseteq}(G,\ E)$, if $F(e) \subseteq G(e)$, for all $e \in E$. Also the soft sets $(F,\ E)$ and $(G,\ E)$ are equal written as $(F,\ E)\tilde{=}(G,\ E)$, if $(F,\ E)\tilde{\subseteq}(G,\ E)$ and $(G,\ E)\tilde{\subseteq}(F,\ E)$.

Definition 1.3. [12] Let $\{(F_i,\ E) : i \in I\}\tilde{\subseteq}SS(X,\ E)$, where I is an arbitrary index set. Then

1. the soft union of $\{(F_i,\ E) : i \in I\}$ is the soft set $(F,\ E)$, where F is the mapping defined as $F(e) = \cup\{F_i(e) : i \in I\}$, for every $e \in E$ and is denoted as $(F,\ E) = \tilde{\cup}\{(F_i,\ E) : i \in I\}$.
2. the soft intersection of $\{(F_i,\ E) : i \in I\}$ is the soft set $(F,\ E)$, where F is the mapping defined as $F(e) = \cap\{F_i(e) : i \in I\}$, for every $e \in E$ and is denoted as $(F,\ E) = \tilde{\cap}\{(F_i,\ E) : i \in I\}$.

Definition 1.4. [21] Let $(F,\ E)$ be a soft set over X. Then the soft relative complement F^c of $(F,\ E)$ is the mapping from E to $\mathcal{P}(X)$ defined by $F^c(e) = X - F(e)$ for every $e \in E$ and is denoted as $(F,\ E)^c$ or $(F^c,\ E)$.

Definition 1.5. [12] Let $(F,\ E)$ be a soft set over X. Then

1. $(F,\ E)$ is called as null soft set, if $F(e) = \phi$, for every $e \in E$. We simply write it as $\tilde{\phi}$.
2. $(F,\ E)$ is called as absolute soft set, if $F(e) = X$, for every $e \in E$. We simply write it as $\tilde{X}$.

Definition 1.6. *(*[16], [21]*)* Let $\tau \subseteq SS(X,\ E)$. Then τ is a soft topology on X if it satifies the following three conditions

1. $\tilde{\phi},\ \tilde{X} \in \tau$.
2. The soft union of any number of soft sets in τ is in τ.
3. The soft intersection of finite number of soft sets in τ is in τ.

This soft topological space over X is written as $(X,\ \tau,\ E)$ and the members of τ are called as soft open sets in X. Also the soft complement of soft open sets are called soft closed sets.

Definition 1.7. [21] The soft set $(F,\ E)$ over X is called as a soft point in X, denoted by x_e, if $F(e') = \begin{cases} \{x\} & \text{if } e' = e \\ \phi & \text{if } e' \in E - \{e\} \end{cases}$

Definition 1.8. [2] Let $(X,\ \tau,\ E)$ be a soft topological space. A subcollection $\mathcal{B}$ of τ is said to be a base for τ if every member of τ can be expressed as a union of members of τ.

Definition 1.9. [2] Let $(X,\ \tau,\ E)$ be a soft topological space. A subcollection $\mathcal{S}$ of τ is said to be a subbase for τ if the family of all finite intersetions of members of $\mathcal{S}$ forms a base for τ.

Definition 1.10. [21] A soft set $(G,\ E)$ in a soft topological space $(X,\ \tau,\ E)$ is known as a soft neighbourhood of a soft set $(F,\ E)$ if there exists a soft open set $(H,\ E)$ such that $(F,\ E)\tilde{\subseteq}(H,\ E)\tilde{\subseteq}(G,\ E)$.

Definition 1.11. [16]
Let $(F,\ E)$ be a soft set in a soft topological space $(X,\ \tau,\ E)$. Then the soft closure of $(F,\ E)$ is denoted as $Cl(F,\ E)$ and defined as $Cl(F,\ E) = \tilde{\cap}\{(G,\ E) : (G,\ E)\tilde{\in}\tau^c \textit{ and } (G,\ E)\tilde{\supseteq}(F,\ E)\}$.

Definition 1.12. [16] Let Y be a nonempty soft subset of a soft topological space $(X,\ \tau,\ E)$. Then $\tau_Y = \{(F,\ E)\tilde{\cap}E_Y : (F,\ E)\tilde{\in}\tau\}$ is called a soft relative topology on Y and $(Y,\ \tau_Y, E)$ is called a soft subspace of $(X,\ \tau,\ E)$, where $E_Y : E \to \mathcal{P}(Y)$ such that $E_Y(e) = Y$, for every $e \in E$.

Proposition 1.1. [16] *Let $(Y,\ \tau_Y, E)$ be a soft subspace of a soft topological space $(X,\ \tau,\ E)$ and (F, A) be a soft set over Y. Then (F, A) is a soft open set in Y if and only if $(F,\ E) = (G,\ E)\tilde{\cap}E_Y$, for some $(G,\ E)\tilde{\in}\tau$.*

Theorem 1.2. [21] *A soft set $(F,\ E)$ is soft open set if and only if $(G,\ E)$ is a soft neighbourhood of a soft set $(F,\ E)$, for each soft set $(F,\ E)$ contained in $(G,\ E)$.*

Proposition 1.3. [16] *Let $(X,\ \tau,\ E)$ be a soft topological space over X. Then the collection $\tau_e = \{F(e) : (F,\ E)\tilde{\in}\tau\}$ defines a topology on X.*

Proposition 1.4. [16] *Let $(X,\ \tau,\ E)$ be a soft topological space over X and $Y \subseteq X$. Then $(Y,\ \tau_{Y_e})$ is a subspace of $(X,\ \tau_e)$.*

Definition 1.13. [3] Let $(F, E_1)\tilde{\in}\ SS(X_1,\ E_1)$ and $(G, E_2)\ \tilde{\in}SS(X_2, E_2)$. Then the cartesian product $(F,\ E_1) \times (G,\ E_2)$ is defined by $(F \times G)_{(E_1\times E_2)}$, where $(F \times G)_{(E_1\times E_2)}(e_{1_i},\ e_{2_j}) = F(e_{1_i}) \times G(e_{2_j}),\ \forall(e_{1_i}, e_{2_j}) \in E_1 \times E_2$.

Definition 1.14. [2] The soft mappings $(p_q)_i,\ i \in \{1,\ 2\}$ is called soft projection mappings from $X_1 \times X_2$ to X_i defined by $(p_q)_i((F,\ E)_1 \times (F,\ E)_2) = (p_q)_i((F_1 \times F_2)_{(E_1\times E_2)}) = p_i(F_1 \times F_2)_{q_i(E_1\times E_2)} = (F, E)_i$, where $(F,\ E)_1 \in SS(X_1,\ E_1)$, $(F,\ E)_2 \in SS(X_2,\ E_2)$ and $p_i : X_1 \times X_2 \to X_i,\ q_i : E_1 \times E_2 \to E_i$ are projection mappings in classical meaning.

Definition 1.15. [2] Let $\{(\phi_\psi)_i : S(X,\ E) \to (Y_i,\ \tau_i)\}_{i\in\Delta}$ be a family of soft mappings where $\{(Y_i,\ \tau_i)\}_{i\in\Delta}$ be a family of soft topological spaces. Then the topology τ generated from the subbase $\{(\phi_\psi)_i^{-1}((F, E)) : (F, E) \in \tau_i,\ i \in \Delta\}$ is called the initial soft topology induced by the family of soft mappings $\{(\phi_\psi)_i\}_{i\in\Delta}$.

Definition 1.16. [2] Let $\{(X_i, \tau_i)\}_{i\in\Delta}$ be a family of soft topological spaces. Then the initial soft topology on $X(= \prod_{i\in\Delta} X_i)$ generated by the family $\{(p_q)_i\}_{i\in\Delta}$ is called soft product topology on X, where $(p_q)_i$ are the soft projection mapping from X to X_i.

Theorem 1.5. [9] *Let X and Y be crisp sets, F_A, $(F_A)_i \in SS(X,\ E)$ and G_B, $(G_B)_i \in SS(Y,\ K)$, where $i \in \Delta$, an index set. Then*

(1) *If $(F_A)_1 \tilde{\subseteq} (F_A)_2$, then $\Phi_\psi((F_A)_1) \tilde{\subseteq} \Phi_\psi((F_A)_2)$.*
(2) *If $(G_B)_1 \tilde{\subseteq} (G_B)_2$, then $\Phi_\psi^{-1}((G_B)_1) \tilde{\subseteq} \Phi_\psi^{-1}((G_B)_2)$.*
(3) *$(F_A) \tilde{\subseteq} \Phi_\psi^{-1}(\Phi_\psi(F_A))$, the equality holds if Φ_ψ is injective.*
(4) *$\Phi_\psi(\Phi_\psi^{-1}(F_A)) \tilde{\subseteq} (F_A)$, the equality holds if Φ_ψ is surjective.*
(5) *$\Phi_\psi(\tilde{\bigcup}_{i\in\Delta} (F_A)_i) = \tilde{\bigcup}_{i\in\Delta} \Phi_\psi((F_A)_i)$.*
(6) *$\Phi_\psi(\bigcap_{i\in\Delta} (F_A)_i) \tilde{\subseteq} \bigcap_{i\in\Delta} \Phi_\psi((F_A)_i)$.*
(7) *$\Phi_\psi^{-1}(\bigcap_{i\in\Delta} (G_B)_i) = \bigcap_{i\in\Delta} \Phi_\psi^{-1}((G_B)_i)$.*
(8) *$\Phi_\psi^{-1}(\bigcap_{i\in\Delta} (G_B)_i) = \bigcap_{i\in\Delta} \Phi_\psi^{-1}((G_B)_i)$.*
(9) *$\Phi_\psi^{-1}(E_Y) = E_X$ and $\Phi_\psi^{-1}(\phi_Y) = \phi_X$.*
(10) *$\Phi_\psi(E_X) = E_Y$ if Φ_ψ is surjective.*
(11) *$\Phi_\psi(\phi_x) = \phi_Y$.*

2. Soft separation axioms and product soft topological spaces

Definition 2.1. [8] A soft topological space (X, τ, E) is said to be a soft T_0 -space if for every pair of soft points x_{e_1}, y_{e_2} such that $x_{e_1} \neq y_{e_2}$, there exists $(F, E) \in \tau$ such that $x_{e_1} \tilde{\in} (F, E)$, $y_{e_2} \tilde{\notin} (F, E)$ or there exists $(G, E) \in \tau$ such that $y_{e_2} \tilde{\in} (G, E)$, $x_{e_1} \tilde{\notin} (G, E)$.

Definition 2.2. [8] A soft topological space (X, τ, E) is said to be a soft T_1 -space if every pair of soft points x_{e_1}, y_{e_2}, such that $x_{e_1} \neq y_{e_2}$ there exist $(F, E), (G, E) \in \tau$ such that $x_{e_1} \tilde{\in} (F, E)$, $y_{e_2} \tilde{\notin} (F, E)$ and $x_{e_1} \tilde{\notin} (G, E)$, $y_{e_2} \tilde{\in} (G, E)$.

Example 2.1. *Example for T_0-space.*
Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{\phi}, \tilde{X}, (F_1, E), (F_2, E), (F_3, E),\ (F_4, E)\}$ where

$$F_1(e) = \begin{cases} \{x\} & \text{if } e = e_1 \\ \{y\} & \text{if } e = e_2 \end{cases},\ F_2(e) = \begin{cases} \{x\} & \text{if } e = e_1 \\ \{x\} & \text{if } e = e_2 \end{cases},$$
$$F_3(e) = \begin{cases} \{x\} & \text{if } e = e_1 \\ X & \text{if } e = e_2 \end{cases},\ F_4(e) = \begin{cases} \{x\} & \text{if } e = e_1 \\ \phi & \text{if } e = e_2 \end{cases},$$

For the soft points x_{e_1}, y_{e_1}, there is a soft open set $(F_1, E) \in \tau$ with $x_{e_1} \tilde{\in} (F_1, E)$ and $y_{e_1} \tilde{\notin} (F_1, E)$. For the soft points x_{e_2}, y_{e_2}, there is a $(F_1, E) \in \tau$ with $x_{e_2} \tilde{\notin} (F_1, E)$ and $y_{e_2} \tilde{\in}\ (F_1, E)$. For the soft points x_{e_1}, y_{e_2}, there is a $(F_2, E) \in \tau$ with $x_{e_1} \tilde{\in}$ (F_2, E) and $y_{e_2} \tilde{\notin} (F_2, E)$. For the soft points x_{e_2}, y_{e_1}, there is a $(F_2, E) \in \tau$ with $x_{e_2} \tilde{\in} (F_2, E)$ and $y_{e_1} \tilde{\notin} (F_2, E)$. For the soft points x_{e_1}, x_{e_2}, there is a $(F_1, E) \in \tau$ with $x_{e_1} \tilde{\in} (F_1, E)$ and $x_{e_2} \tilde{\notin} (F_1, E)$. For the soft points y_{e_1}, y_{e_2}, there is a $(F_1, E) \in \tau$ with $y_{e_1} \tilde{\notin}\ (F_1, E)$ and $y_{e_2} \tilde{\in} (F_1, E)$. Thus (X, τ, E) is a soft T_0-space.

Example 2.2. *Let $X = \mathbf{Z}$, the set of all integers and $E = \mathbf{N}$, the set of all natural numbers. Define a soft topology on X as $\tau = \{(F, E)^c : F(e_i)$ is finite for each $e_i \in E\} \cup \{\tilde{\phi}\}$.*

(1) *Clearly $\tilde{\phi} \in \tau$ and $\tilde{X} \in \tau$.*

(2) *If* $(F_\alpha, E) \in \tau$ *for some* $\alpha \in \Delta$, *where* Δ *is some index set, then* $F^c_\alpha(e_i)$ *is finite for each* $e_i \in E$. *Now* $\cap F^c_\alpha(e_i) = (\cup F_\alpha(e_i))^c$ *is finite for each* $e_i \in E$. *So that* $\cup(F_\alpha, E) \in \tau$.

(3) *If* $(F_1, E), (F_2, E) \in \tau$, $F^c_1(e_i)$ *and* $F^c_2(e_i)$ *are finite for each* $e_i \in E$. *Now* $F^c_1(e_i) \cup F^c_2(e_i) = (F_1(e_i) \cap F_2(e_i))^c = ((F_1 \cap F_2)(e_i))^c = (F_1 \cap F_2)^c(e_i)$ *is finite for each* $e_i \in E$. *So that* $(F_1, E) \cap (F_2, E) \in \tau$.

Thus (X, τ, E) *is a soft topological space. For any two distinct soft points* x_{e_i} *and* y_{e_j}, $x^c_{e_i}$ *and* $y^c_{e_j}$ *are soft open sets such that* $x_{e_i} \in y^c_{e_j}$, $y_{e_j} \notin y^c_{e_j}$ *and* $x_{e_i} \notin x^c_{e_i}$, $y_{e_j} \in x^c_{e_i}$. *Thus* (X, τ, E) *is a soft* T_1 *space.*

Theorem 2.1. *Every soft* T_1*-space is a soft* T_0*-space.*

Proof. Proof is straight forward □

Theorem 2.2. *Let* $(X,\ \tau,\ E)$ *be a soft topological space. Then* $(X,\ \tau,\ E)$ *is a soft* T_0 *space if and only if for any two distinct soft points* x_{e_i} *and* y_{e_j}, *there is soft closed set* $(H,\ E)$ *such that* $x_{e_i} \tilde{\in} (H,\ E)$, $y_{e_j} \tilde{\notin} (H,\ E)$ *or there is soft closed set* $(K,\ E)$ *such that* $x_{e_i} \tilde{\notin} (K,\ E)$, $y_{e_j} \tilde{\in} (K,\ E)$.

Proof. Let us consider two distinct soft points x_{e_i} and y_{e_j}. Since $(X,\ \tau,\ E)$ is a soft T_0 space, there is soft open set $(F,\ E)$ such that $x_{e_i} \tilde{\in} (F,\ E)$, $y_{e_j} \tilde{\notin} (F,\ E)$ or there is soft open set $(G,\ E)$ such that $x_{e_i} \tilde{\notin} (G,\ E)$, $y_{e_j} \tilde{\in} (G,\ E)$. Let $(H,\ E) = (G^c,\ E)$ and $(K,\ E) = (F^c,\ E)$. Then $(H,\ E)$ is a soft closed set such that $x_{e_i} \tilde{\in} (H,\ E)$, $y_{e_j} \tilde{\notin} (H,\ E)$ or $(K,\ E)$ is a soft closed set such that $x_{e_i} \tilde{\notin} (K,\ E)$, $y_{e_j} \tilde{\in} (K,\ E)$.

Conversely, for any two distinct soft points x_{e_i} and y_{e_j}, there is a soft closed set $(H,\ E)$ such that $x_{e_i} \tilde{\in} (H,\ E)$, $y_{e_j} \tilde{\notin} (H,\ E)$ or there is soft closed set $(K,\ E)$ such that $x_{e_i} \tilde{\notin} (K,\ E)$, $y_{e_j} \tilde{\in} (K,\ E)$. Then $(H^c,\ E)$ is a soft open set such that $x_{e_i} \tilde{\notin} (H^c,\ E)$, $y_{e_j} \tilde{\in} (H^c,\ E)$ or $(K^c,\ E)$ is a soft open set such that $x_{e_i} \tilde{\in} (K^c,\ E)$, $y_{e_j} \tilde{\notin} (K^c,\ E)$. This proves that $(X,\ \tau,\ E)$ is a soft T_0 space. □

Example:1 given in the artice [8] for soft T_1 space which is not a soft T_0 space is wrong. Because it is not a soft T_0 space too.

Example 2.3. [8] $X = \{x_1, x_2\}$, $A = \{e_1, e_2\}$ *and* $\tau = \{\tilde{\phi}, \tilde{X}, (F, A)\}$ *where* $F(e) = \begin{cases} \{x_1\} & \text{if } e = e_1 \\ \{x_2\} & \text{if } e = e_2 \end{cases}$ *This* (X, τ, A) *is verified as soft* T_0 *space in* [8].

consider two soft points $e_F = \begin{cases} \{x_2\} & \text{if } e = e_1 \\ \phi & \text{if } e = e_2 \end{cases}$ *and* $e_G = \begin{cases} \phi & \text{if } e = e_1 \\ \{x_1\} & \text{if } e = e_2 \end{cases}$, *then there is no soft open set* (F, A) *in* (X, τ, A) *such that* $e_F \tilde{\in} (F, A)$ *and* $e_G \tilde{\notin} (F, A)$. *Thus* (X, τ, A) *is not a soft* T_0 *space.*

The following example will be a correct example for example:1 of [8]. It also shows that the converse of above theorem 2.1 is not true in general.

Example 2.4. *Example for a soft* T_0*-space which is not a soft* T_1*-space.*
Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ *and* $\tau = \{\tilde{\phi},\ \tilde{X},\ (F_1, E),\ (F_2, E),\ (F_3, E),\ (F_4, E),\ (F_5, E)\}$ *where*

$F_1(e) = \begin{cases} \{x\} & \text{if } e = e_1 \\ \{y\} & \text{if } e = e_2 \end{cases}$, $F_2(e) = \begin{cases} \{x\} & \text{if } e = e_1 \\ \phi & \text{if } e = e_2 \end{cases}$, $F_3(e) = \begin{cases} \phi & \text{if } e = e_1 \\ \{x\} & \text{if } e = e_2 \end{cases}$,

$F_4(e)=\begin{cases}\{x\} & \text{if } e=e_1\\ \{x\} & \text{if } e=e_2\end{cases}$, $F_5(e)=\begin{cases}\{x\} & \text{if } e=e_1\\ X & \text{if } e=e_2\end{cases}$

For the soft points x_{e_1}, y_{e_1}, there is a $(F_2,E)\in\tau$ with $x_{e_1}\tilde{\in}(F_2,E)$ and $y_{e_1}\tilde{\notin}(F_2,E)$. For the soft points x_{e_2}, y_{e_2}, there is a $(F_3,E)\in\tau$ with $x_{e_2}\tilde{\in}(F_3,E)$ and $y_{e_2}\tilde{\notin}(F_3,E)$. For the soft points x_{e_1}, y_{e_2}, there is a $(F_2,E)\in\tau$ with $x_{e_1}\tilde{\in}(F_2,E)$ and $y_{e_2}\tilde{\notin}(F_2,E)$. For the soft points x_{e_2}, y_{e_1}, there is a $(F_3,E)\in\tau$ with $x_{e_2}\tilde{\in}(F_3,E)$ and $y_{e_1}\tilde{\notin}(F_3,E)$. For the soft points x_{e_1}, x_{e_2}, there is a $(F_2,E)\in\tau$ with $x_{e_1}\tilde{\in}(F_2,E)$ and $x_{e_2}\tilde{\notin}(F_2,E)$. For the soft points y_{e_1}, y_{e_2}, there is a $(F_1,E)\in\tau$ with $y_{e_2}\tilde{\in}(F_1,E)$ and $y_{e_1}\tilde{\notin}(F_1,E)$. Thus (X,τ,E) is a soft T_0-space. But for the pair of soft points y_{e_1}, y_{e_2}, we dont have $(K,E)\in\tau$ such that $y_{e_1}\tilde{\in}(K,E)$ and $y_{e_2}\tilde{\notin}(K,E)$. Thus $(X,\tau,\ E)$ is not a soft T_1-space.

Theorem 2.3. (1) *A subspace of a soft T_0-space is a soft T_0-space.*
(2) *A subspace of a soft T_1-space is a soft T_1-space*

Proof. (1) Let $(X,\ \tau,\ E)$ be a soft T_0-space and $(Y,\ \tau_Y,\ E)$ be a soft subspace. Let x_{e_i}, y_{e_j} be two soft points in $SS(Y,\ E)$. Then x_{e_i}, $y_{e_j}\tilde{\in}\ SS(X,\ E)$. Since (X,τ,E) is a soft T_0 space, there is a soft open set $(F,\ E)$ in (X,τ,E) such that $x_{e_i}\tilde{\in}(F,\ E)$, $y_{e_j}\tilde{\notin}(F,\ E)$ or there is a soft open set $(G,\ E)$ in $(X,\ \tau,\ E)$ such that $y_{e_j}\tilde{\in}(G,\ E)$, $x_{e_i}\tilde{\notin}(G,\ E)$. Then $(F,\ E)\tilde{\cap}E_Y$ is a soft open set in $(Y,\ \tau_Y,\ E)$ such that $x_{e_i}\tilde{\in}(F,\ E)\tilde{\cap}E_Y$ $y_{e_j}\tilde{\notin}(F,\ E)\tilde{\cap}E_Y$ or $(G,\ E)\tilde{\cap}E_Y$ is a soft open set in $(Y,\ \tau_Y,\ E)$ such that $y_{e_j}\tilde{\in}(G,\ E)\tilde{\cap}E_Y$, $x_{e_i}\tilde{\notin}(G,\ E)\tilde{\cap}E_Y$. Thus $(Y,\ \tau,\ E)$ is a soft T_0-space
(2) Proof is similar to (1)

□

Theorem 2.4. *Let $(X,\ \tau,\ E)$ be a soft topological space. Then $(X,\ \tau,\ E)$ is a soft T_1 space if and only if for any soft points x_{e_i} and y_{e_j}, there exist two soft closed sets $(H,\ E)$ and $(K,\ E)$ such that $x_{e_i}\tilde{\in}(H,\ E)$, $y_{e_j}\tilde{\notin}(H,\ E)$, $y_{e_j}\tilde{\in}(K,\ E)$ and $x_{e_i}\tilde{\notin}(K,\ E)$.*

Proof. Proof is similar to the theorem 2.2 □

The following example shows that the product of soft T_0-spaces need not be a soft T_0-space

Definition 2.3. Let $\{(X_i,\ \tau_i,\ E_i) : i\in I\}$ be a family of soft topological spaces and $(\prod X_i,\prod\tau_i,\ \prod E_i)$ be their product soft toplogical space. Then a soft point in $(\prod X_i,\prod\tau_i,\ \prod E_i)$ is denoted as $\mathbf{x_e}$, where $\mathbf{x}=<x_i>_{i\in I},\ x_i\in X_i$ and $\mathbf{e}=<e_i>_{i\in I},\ e_i\in E_i$.

Example 2.5. *Let $X_1=\{x_1,y_1\}$, $E_1=\{e_{11},e_{12}\}$ and $\tau_1=\{\tilde{\phi},\tilde{X}_1,\ (F_1,E_1),\ (F_2,E_1),\ (F_3,E_1),\ (F_4,E_1),\ (F_5,E_1),\ (F_6,E_1),\ (F_7,E_1)\}$. $X_2=\{x_2,y_2\}$, $E_2=\{e_{21},e_{22}\}$ and $\tau_2=\{\tilde{\phi},\tilde{X}_2,(G_1,E_2),(G_2,E_2),(G_3,E_2),(G_4,E_2),(G_5,E_2),(G_6,E_2),(G_7,E_2)\}$ where*

$F_1(e)=\begin{cases}\{x_1\} & \text{if } e=e_{11}\\ \phi & \text{if } e=e_{12}\end{cases}$, $G_1(e)=\begin{cases}\{x_2\} & \text{if } e=e_{21}\\ \phi & \text{if } e=e_{22}\end{cases}$, $F_2(e)=\begin{cases}\phi & \text{if } e=e_{11}\\ \{x_1\} & \text{if } e=e_{12}\end{cases}$,

$G_2(e)=\begin{cases}\phi & \text{if } e=e_{21}\\ \{x_2\} & \text{if } e=e_{22}\end{cases}$, $F_3(e)=\begin{cases}\{x_1\} & \text{if } e=e_{11}\\ \{x_1\} & \text{if } e=e_{12}\end{cases}$, $G_3(e)=\begin{cases}\{x_2\} & \text{if } e=e_{21}\\ \{x_2\} & \text{if } e=e_{22}\end{cases}$,

$F_4(e) = \begin{cases} \{y_1\} & \text{if } e = e_{11} \\ \{x_1\} & \text{if } e = e_{12} \end{cases}$, $G_4(e) = \begin{cases} \{x_2\} & \text{if } e = e_{21} \\ \{y_2\} & \text{if } e = e_{22} \end{cases}$, $F_5(e) = \begin{cases} \{y_1\} & \text{if } e = e_{11} \\ \phi & \text{if } e = e_{12} \end{cases}$,
$G_5(e) = \begin{cases} \phi & \text{if } e = e_{21} \\ \{y_2\} & \text{if } e = e_{22} \end{cases}$, $F_6(e) = \begin{cases} X_1 & \text{if } e = e_{11} \\ \phi & \text{if } e = e_{12} \end{cases}$, $G_6(e) = \begin{cases} \phi & \text{if } e = e_{21} \\ X_2 & \text{if } e = e_{22} \end{cases}$,
$F_7(e) = \begin{cases} X_1 & \text{if } e = e_{11} \\ \{x_1\} & \text{if } e = e_{12} \end{cases}$, $G_7(e) = \begin{cases} \{x_2\} & \text{if } e = e_{21} \\ X_2 & \text{if } e = e_{22} \end{cases}$.

For the soft points $x_{1_{e_{11}}}$, $y_{1_{e_{11}}}$, *there is a soft open set* $(F_1, E_1) \in \tau_1$ *with* $x_{1_{e_{11}}} \tilde{\in} (F_1, E_1)$ *and* $y_{1_{e_{11}}} \tilde{\notin} (F_1, E_1)$. *For the soft points* $x_{1_{e_{11}}}$, $y_{1_{e_{12}}}$, *there is a soft open set* $(F_1, E_1) \in \tau_1$ *with* $x_{1_{e_{11}}} \tilde{\in} (F_1, E_1)$ *and* $y_{1_{e_{12}}} \tilde{\notin} (F_1, E_1)$. *For the soft points* $x_{1_{e_{12}}}$, $y_{1_{e_{11}}}$, *there is* $(F_2, E_1) \in \tau_1$ *with* $x_{1_{e_{12}}} \tilde{\in} (F_2, E_1)$ *and* $y_{1_{e_{11}}} \tilde{\notin} (F_2, E_1)$. *For the soft points* $x_{1_{e_{12}}}$, $y_{1_{e_{12}}}$, *there is* $(F_2, E_1) \in \tau_1$ *with* $x_{1_{e_{12}}} \tilde{\in} (F_2, E_1)$ *and* $y_{1_{e_{12}}} \tilde{\notin} (F_2, E_1)$. *For the soft points* $x_{1_{e_{11}}}$, $x_{1_{e_{12}}}$, *there is a soft open set* $(F_1, E_1) \in \tau_1$ *with* $x_{1_{e_{11}}} \tilde{\in} (F_1, E_1)$ *and* $x_{1_{e_{12}}} \tilde{\notin} (F_1, E_1)$. *For the soft points* $y_{1_{e_{11}}}$, $y_{1_{e_{12}}}$, *there is a soft open set* $(F_5, E_1) \in \tau_1$ *with* $y_{1_{e_{11}}} \tilde{\in} (F_5, E_1)$ *and* $y_{1_{e_{12}}} \tilde{\notin} (F_5, E_1)$. *Thus* (X_1, τ_1, E_1) *is a soft* T_0*-space.*

For the soft points $x_{2_{e_{21}}}$, $y_{2_{e_{21}}}$, *there is a soft open set* $(G_1, E_2) \in \tau_2$ *with* $x_{2_{e_{21}}} \tilde{\in} (G_1, E_2)$ *and* $y_{2_{e_{21}}} \tilde{\notin} (G_1, E_2)$. *For the soft points* $x_{2_{e_{21}}}$, $y_{2_{e_{22}}}$, *there is a soft open set* $(G_1, E_2) \in \tau_2$ *with* $x_{2_{e_{21}}} \tilde{\in} (G_1, E_2)$ *and* $y_{2_{e_{22}}} \tilde{\notin} (G_1, E_2)$. *For the soft points* $x_{2_{e_{22}}}$, $y_{2_{e_{21}}}$, *there is a soft open set* $(G_2, E_2) \in \tau_2$ *with* $x_{2_{e_{22}}} \tilde{\in} (G_2, E_2)$ *and* $y_{2_{e_{21}}} \tilde{\notin} (G_2, E_2)$. *For the soft points* $x_{2_{e_{22}}}$, $y_{2_{e_{22}}}$, *there is a soft open set* $(G_2, E_2) \in \tau_2$ *with* $x_{2_{e_{22}}} \tilde{\in} (G_2, E_2)$ *and* $y_{2_{e_{22}}} \tilde{\notin} (G_2, E_2)$. *For the soft points* $x_{2_{e_{21}}}$, $x_{2_{e_{22}}}$, *there is a soft open set* $(G_1, E_2) \in \tau_2$ *with* $x_{2_{e_{21}}} \tilde{\in} (G_1, E_2)$ *and* $x_{2_{e_{22}}} \tilde{\notin} (G_1, E_2)$. *For the soft points* $y_{2_{e_{21}}}$, $y_{2_{e_{22}}}$, *there is a soft open set* $(G_4, E_2) \in \tau_2$ *with* $y_{2_{e_{21}}} \tilde{\notin} (G_4, E_2)$ *and* $y_{2_{e_{22}}} \tilde{\in} (G_4, E_2)$. *Thus* (X_2, τ_2, E_2) *is a soft* T_0*-space.*

Now $E_1 \times E_2 = \{(e_{11},\ e_{21}), (e_{11},\ e_{22}), (e_{12},\ e_{21}), (e_{12}\ e_{22})\}$ *and* $\tau_1 \times \tau_2 = \{\tilde{\phi}$, $X_1 \tilde{\times} X_2$, $(F_1 \times G_1, E_1 \times E_2)$, $(F_1 \times G_2, E_1 \times E_2)$, $(F_1 \times G_3, E_1 \times E_2)$, $(F_1 \times G_4, E_1 \times E_2)$, $(F_1 \times G_5, E_1 \times E_2)$, $(F_1 \times G_6, E_1 \times E_2)$, $(F_1 \times G_7, E_1 \times E_2)$, $(F_2 \times G_1, E_1 \times E_2)$, $(F_2 \times G_2, E_1 \times E_2)$, $(F_2 \times G_3, E_1 \times E_2)$, $(F_2 \times G_4, E_1 \times E_2)$, $(F_2 \times G_5, E_1 \times E_2)$, $(F_2 \times G_6, E_1 \times E_2)$, $(F_2 \times G_7, E_1 \times E_2)$, $(F_3 \times G_1, E_1 \times E_2)$, $(F_3 \times G_2, E_1 \times E_2)$, $(F_3 \times G_3, E_1 \times E_2)$, $(F_3 \times G_4, E_1 \times E_2)$, $(F_3 \times G_5, E_1 \times E_2)$, $(F_3 \times G_6, E_1 \times E_2)$, $(F_3 \times G_7, E_1 \times E_2)$, $(F_4 \times G_1, E_1 \times E_2)$, $(F_4 \times G_2, E_1 \times E_2)$, $(F_4 \times G_3, E_1 \times E_2)$, $(F_4 \times G_4, E_1 \times E_2)$, $(F_4 \times G_5, E_1 \times E_2)$, $(F_4 \times G_6, E_1 \times E_2)$, $(F_4 \times G_7, E_1 \times E_2)$, $(F_5 \times G_1, E_1 \times E_2)$, $(F_5 \times G_2, E_1 \times E_2)$, $(F_5 \times G_3, E_1 \times E_2)$, $(F_5 \times G_4, E_1 \times E_2)$, $(F_5 \times G_5, E_1 \times E_2)$, $(F_5 \times G_6, E_1 \times E_2)$, $(F_5 \times G_7, E_1 \times E_2)$, $(F_6 \times G_1, E_1 \times E_2)$, $(F_6 \times G_2, E_1 \times E_2)$, $(F_6 \times G_3, E_1 \times E_2)$, $(F_6 \times G_4, E_1 \times E_2)$, $(F_6 \times G_5, E_1 \times E_2)$, $(F_6 \times G_6, E_1 \times E_2)$, $(F_6 \times G_7, E_1 \times E_2)$, $(F_7 \times G_1, E_1 \times E_2)$, $(F_7 \times G_2, E_1 \times E_2)$, $(F_7 \times G_3, E_1 \times E_2)$, $(F_7 \times G_4, E_1 \times E_2)$, $(F_7 \times G_5, E_1 \times E_2)$, $(F_7 \times G_6, E_1 \times E_2)$, $(F_7 \times G_7, E_1 \times E_2)\}$.

Suppose if the soft product of (X_1, τ_1, E_1) *and* (X_2, τ_2, E_2) *is a soft* T_0 *space, then for any two distinct soft points* $(x_1,\ y_2)_{(e_{11},\ e_{21})} = \begin{cases} \{(x_1,\ y_2)\} & \text{if } e = (e_{11},\ e_{21}) \\ \phi & \text{if } e = (e_{11},\ e_{22}) \\ \phi & \text{if } e = (e_{12},\ e_{21}) \\ \phi & \text{if } e = (e_{12},\ e_{22}) \end{cases}$

and $(y_1,\ y_2)_{(e_{11},\ e_{21})} = \begin{cases} \{(y_1,\ y_2)\} & \text{if } e = (e_{11},\ e_{21}) \\ \phi & \text{if } e = (e_{11},\ e_{22}) \\ \phi & \text{if } e = (e_{12},\ e_{21}) \\ \phi & \text{if } e = (e_{12},\ e_{22}) \end{cases}$, *there is a soft open set* $(F_m \times G_n, E_1 \times E_2)$ *in* $\tau_1 \times \tau_2$ *such that* $(x_1,\ y_2)_{(e_{11},\ e_{21})} \tilde{\in} (F_m \times G_n, E_1 \times E_2)$ *and* $(y_1,\ y_2)_{(e_{11},\ e_{21})} \tilde{\notin} (F_m \times G_n, E_1 \times E_2)$, *for some* $m, n \in \{1, 2, 3 \ldots, 7\}$. *Now* $(p_q)_2((x_1,\ y_2)_{(e_{11},\ e_{21})}) \tilde{\in} (p_q)_2((F_m \times G_n, E_1 \times E_2))$. *That is* $p_2(x_1,\ y_2)_{q_2(e_{11},\ e_{21})} \tilde{\in} p_2(F_m \times G_n)_{q_2(E_1 \times E_2)}$. *This implies* $y_{2_{e_{21}}} \tilde{\in} (G_n, E_2)$, *for some* $m, n \in \{1, 2, 3 \ldots, 7\}$. *Since* $(p_q)_2$ *is a soft projection mapping and* $(F_m \times G_n, E_1 \times E_2)$ *is a soft open set in* $X_1 \times X_2$, (G_n, E_2) *is a soft open set in* (X_2, τ_2, E_2) *containing* $y_{2_{e_{21}}}$. *But there is no soft open set* (G_n, E_2) *in* (X_2, τ_2, E_2) *containing* $y_{2_{e_{21}}}$, *for any* $n \in \{1, 2, 3 \ldots, 7\}$ *and hence* $(X_1 \times X_2, \tau_1 \times \tau_2, E_1 \times E_2)$ *is not a soft* T_0 *space.*

Definition 2.4. Let $(X,\ \tau,\ E)$ be a soft topological space and $A = \{x_{e_i} : x_{e_i}$ is a soft points of $(X,\ \tau,\ E)\}$.

(1) If the number of elements of the set A is finite, then $(X,\ \tau,\ E)$ is called a finite soft topological space.
(2) If the number of elements of the set A is countable, then $(X,\ \tau,\ E)$ is called a countable soft topological space.

Theorem 2.5. *If* $(X,\ \tau,\ E)$ *is a finite soft* T_1 *space, then* $(X,\ \tau,\ E)$ *is a soft discrete space.*

Proof. Let x_{e_i} be a soft point, $x \in X$ and $e_i \in E$. $(X,\ \tau,\ E)$ is a soft T_1 space, for any soft point $y_{e_j} \neq x_{e_i}$, there is a soft open set $(F_{x_j},\ E)$ such that $x_{e_i} \tilde{\in} (F_{x_j},\ E)$ and $y_{e_j} \tilde{\notin} (F_{x_j},\ E)$. Since $(X,\ \tau,\ E)$ is a finite soft topological space, $\tilde{\bigcap}_{y_{e_j} \neq x_{e_i}} (F_{x_j},\ E)$ is a soft open set such that $\tilde{\bigcap}_{y_{e_j} \neq x_{e_i}} (F_{x_j},\ E) = \begin{cases} \{x\} & \text{if } e = e_i \\ \phi & \text{if } e \neq e_i \end{cases}$. Thus x_{e_i} is soft open and hence $(X,\ \tau,\ E)$ is a soft discrete space. □

Definition 2.5. Let $(X,\ \tau,\ E)$ be a soft topological space. Then the soft set $(F,\ E)$ is called a soft G_δ set if it is a countable intersection of soft open sets.

Theorem 2.6. *If* $(X,\ \tau,\ E)$ *is a countable soft* T_1 *space and if every soft* G_δ *set is soft open in* $(X,\ \tau,\ E)$, *then* $(X,\ \tau,\ E)$ *is a soft discrete space.*

Proof. Let x_{e_i} be a soft point. Since $(X,\ \tau,\ E)$ is a soft T_1 space, for any soft point $y_{e_j} \neq x_{e_i}$, there is a soft open set $(F_{x_j},\ E)$ such that $x_{e_i} \tilde{\in} (F_{x_j},\ E)$ and $y_{e_j} \tilde{\notin} (F_{x_j},\ E)$. Since every soft G_δ set is soft open and $(X,\ \tau,\ E)$ is a countable soft topological space, $\tilde{\bigcap}_{y_{e_j} \neq x_{e_i}} (F_{x_j},\ E)$ is a soft open set such that $\tilde{\bigcap}_{y_{e_j} \neq x_{e_i}} (F_{x_j},\ E) = \begin{cases} \{x\} & \text{if } e = e_i \\ \phi & \text{if } e \neq e_i \end{cases}$. Thus x_{e_i} is soft open and hence $(X,\ \tau,\ E)$ is a soft discrete space. □

Theorem 2.7. *Product of soft T_1-spaces is a soft T_1-space*

Proof. Let $\{(X_i,\ \tau_i,\ E_i) : i \in I\}$ be a family of soft topological spaces and $(\prod X_i, \prod \tau_i, \prod E_i)$ be their product soft toplogical space. Suppose $\mathbf{x_e}$ and $\mathbf{y_f}$ be two distinct soft points, where $\mathbf{x} =< x_i >_{i\in I},\ \mathbf{y} =< y_i >_{i\in I}\ \ x_i, y_i \in X_i$ and $\mathbf{e} =< e_i >_{i\in I},\ \mathbf{f} =< f_i >_{i\in I}, e_i, f_i \in E_i$. Then there exists atleast one $\beta \in I$ such that $x_\beta \neq y_\beta$ or there exist $e_{i_k},\ e_{i_m} \in E_i$ such that $e_{i_k} \neq e_{i_m}$.

Case: 1

If $x_\beta \neq y_\beta$, $(p_q)_\beta(\mathbf{x_e}) = (p_{\beta_{q_\beta}})(\mathbf{x_e}) = p_\beta(\mathbf{x})_{q_\beta(\mathbf{e})} = x_{\beta_{e_\beta}}$ and $(p_q)_\beta\,(\mathbf{y_f}) = (p_{\beta_{q_\beta}})(\mathbf{y_f}) = p_\beta(\mathbf{y})_{q_\beta(\mathbf{f})} = y_{\beta_{f_\beta}}$. Since X_β is a soft T_1 space, there exist soft open sets (F_β, E_β) and (G_β, E_β) such that $x_{\beta_{e_\beta}} \tilde{\in} (F_\beta, E_\beta),\ y_{\beta_{f_\beta}} \tilde{\notin} (F_\beta, E_\beta)$ and $y_{\beta_{f_\beta}} \tilde{\in} (G_\beta, E_\beta),\ x_{\beta_{e_\beta}} \tilde{\notin} (G_\beta, E_\beta)$. Then the soft subbasic members $(p_q)_\beta^{-1}(F_\beta, E_\beta)$ and $(p_q)_\beta^{-1}(G_\beta, E_\beta)$ are the soft open sets containing $\mathbf{x_e}$ and $\mathbf{y_f}$ respectively. Suppose if $\mathbf{y_f} \tilde{\in}\ (p_q)_\beta^{-1}(F_\beta, E_\beta)$, then $p_\beta(\mathbf{y})_{q_\beta(f)} = (p_q)_\beta(\mathbf{y_f}) \tilde{\in}\ (p_q)_\beta((p_q)_\beta^{-1}(F_\beta, E_\beta))$. That is $y_{\beta_{f_\beta}} \tilde{\in}\ (F_\beta,\ E_\beta)$ which is a contradiction. Similarly we can prove $\mathbf{x_e} \tilde{\notin} (p_q)_\beta^{-1}\ (G_\beta, E_\beta)$. Thus $(p_q)_\beta^{-1}(F_\beta, E_\beta)$ and $(p_q)_\beta^{-1}(G_\beta, E_\beta)$ are the soft open sets such that $\mathbf{x_e}\ \tilde{\in} (p_q)_\beta^{-1}(F_\beta, E_\beta)$, $\mathbf{y_f}\ \tilde{\notin} (p_q)_\beta^{-1}$ (F_β, E_β) and $\mathbf{y_f}\ \tilde{\in} (p_q)_\beta^{-1}\ (G_\beta, E_\beta)$, $\mathbf{x_e}\ \tilde{\notin} (p_q)_\beta^{-1}(G_\beta, E_\beta)$.

Case: 2

If $e_{i_k} \neq e_{i_m}$, there are soft open sets (F_{i_k}, E_i) and (F_{i_m}, E_i) in $(X_i,\ \tau_i,\ E_i)$ such that $x_{e_{i_k}} \tilde{\in} (F_{i_k}, E_i),\ x_{e_{i_m}}\ (= y_{e_{i_m}}) \tilde{\notin}\ (F_{i_k}, E_i)$ and $x_{e_{i_m}} \tilde{\in} (F_{i_m}, E_i)$, $x_{e_{i_k}}\ \tilde{\notin} (F_{i_m}, E_i)$. Then $(p_q)_i^{-1}\ (F_{i_k}, E_i)$ and $(p_q)_i^{-1}\ (F_{i_m}, E_i)$ are soft open sets such that $\mathbf{x_e}\ \tilde{\in}$ $(p_q)_i^{-1}(F_{i_k}, E_i)$ and $\mathbf{y_f} \tilde{\in}\ (p_q)_i^{-1}(F_{i_m}, E_i)$. We can prove $\mathbf{y_f}\ \tilde{\notin} (p_q)_i^{-1}(F_{i_k},\ E_i)$ and $\mathbf{x_e}\ \tilde{\notin} (p_q)_i^{-1}\ (F_{i_m}, E_i)$ as we proved in case:1. This completes the proof. □

Theorem 2.8. *Let (X, τ, E) be a soft topological space. Then the following are equivalent.*

(1) (X, τ, E) *is a soft* τ_1*-space*
(2) $x_{e_i} = \tilde{\cap}\{(G, E) : (G, E) \in \tau$ *and* $x_{e_i} \tilde{\in} (G, E)\}$
(3) $x_{e_i} = \tilde{\cap}\{(F, E) : (F, E) \in \tau^c$ *and* $x_{e_i} \tilde{\in} (F, E)\}$

Proof. $(i) \Rightarrow (ii)$. Clearly $x_{e_i} \tilde{\subseteq} \tilde{\cap}\{(G, E) : (G, E) \in \tau$ and $x_{e_i} \tilde{\in} (G, E)\}$. Suppose if $y_{e_j} \tilde{\in} \tilde{\cap}\{(G, E) : (G, E) \in \tau\ and\ x_{e_i} \tilde{\in} (G, E)\}$ such that $x_{e_i} \neq y_{e_j}$. Then $x \neq y$ or $e_i \neq e_j$. In either cases, by our assumption, there is a soft open set (G, E) such that $x_{e_i} \tilde{\in} (G, E)$ and $y_{e_j} \tilde{\notin} (G, E)$.So $y_{e_j} \tilde{\notin} \tilde{\cap}\{(G, E) : (G, E) \tilde{\in} \tau\ and\ x_{e_i} \tilde{\in} (G, E)\}$. Thus $x_{e_i} = \tilde{\cap}\{(G, E) : (G, E) \in \tau\ and\ x_{e_i} \tilde{\in} (G, E)\}$.
$(ii) \Rightarrow (iii)$. Clearly $x_{e_i} \tilde{\subseteq} \tilde{\cap}\{(F, E) : (F, E) \in \tau^c\ and\ x_{e_i} \tilde{\in} (F, E)\}$. Let $y_{e_j} \tilde{\in} \tilde{\cap}\{(F, E)$ $: (F, E) \in \tau^c\ and\ x_{e_i} \tilde{\in} (F, E)\}$ such that $x_{e_i} \neq y_{e_j}$. By (ii), there exists $(G, E) \in \tau$ such that $y_{e_j} \tilde{\in} (G, E)$ and $x_{e_i} \tilde{\notin} (G, E)$. Now $(G, E)^c \in \tau^c$ and $y_{e_j} \tilde{\notin} (G, E)^c$ and $x_{e_i} \tilde{\in} (G, E)^c$ and hence $y_{e_j} \tilde{\notin} \tilde{\cap}\{(F, E) : (F, E) \in \tau^c\ and\ x_{e_i} \tilde{\in} (F, E)\}$. Thus $x_{e_i} = \tilde{\cap}$ $\{(F, E) : (F, E) \in \tau^c\ and\ x_{e_i} \tilde{\in}\ (F, E)\}$.
$(iii) \Rightarrow (i)$. Let x_{e_i} and y_{e_j} be two distinct soft points. Then by (iii), $x_{e_i} \neq y_{e_j} = \tilde{\cap}\{(F, E) : (F, E) \in \tau^c\ and\ y_{e_j} \tilde{\in} (F, E)\}$. There is some soft closed set (F_1, E) such that $y_{e_j} \tilde{\in} (F_1, E)$ and $x_{e_i} \tilde{\notin} (F_1, E)$. Then $(F_1, E)^c$ is a soft open set such that $x_{e_i} \tilde{\in} (F_1, E)^c$ and $y_{e_j} \tilde{\notin} (F_1, E)^c$. Similarly, from $y_{e_j} \neq x_{e_i} = \tilde{\cap}\{(F, E) :$ $(F, E) \in \tau^c\ and\ y_{e_j} \tilde{\in} (F, E)\}$, we can find another soft open set $(F_2, E)^c$ such that $x_{e_i} \tilde{\notin} (F_2, E)^c$ and $y_{e_j} \tilde{\in} (F_2, E)^c$. This proves that (X, τ, E) is a soft τ_1-space. □

Remark. (1) *From (iii) of theorem 2.8, it is clear that each soft point x_{e_i} is a soft closed set in a soft T_1 space.*

(2) *Let T_i= Number of elements in $F(e_i), i \in I$ an indexed set of E. If $T = \sum_{i\in I} T_i$ is finite, then the soft set (F,E) can be written as a finite union of soft points. Each soft point is a soft closed set, we have (F,E) is a soft closed set.*

(3) *If $T = \sum_{i\in I} T_i$ is infinite, (F,E) need not be a closed set. Following example shows this.*

Example 2.6. *Let X be an infinite set and $E = \mathbf{N}$. Let $\tau = \{(F,E)^c : \{e_i : F(e_i) \neq \phi\}$ is $finite\} \cup \{\tilde{\phi}\}$.*

1. *Clearly $\tilde{\phi} \in \tau$ and $\tilde{X} \in \tau$.*
2. *If $(F_{\alpha_i}, E) \in \tau, \alpha_i \in I$, for some index set I, then $\{e_j : F^c_{\alpha_i}(e_j) \neq \phi\}$ is a finite set. Now $\{e_j : (\cup F_{\alpha_i})^c(e_j) \neq \phi\} = \{e_j : \tilde{\cap} F^c_{\alpha_i}(e_j) \neq \phi\} \subseteq \{e_j : F^c_{\alpha_k}(e_j) \neq \phi\}$, for all $\alpha_k \in I$. Since $\{e_j : F^c_{\alpha_k}(e_j) \neq \phi\}$ is a finite set, $\{e_j : (\cup F_{\alpha_i})^c(e_j) \neq \phi\}$ is a finite set and hence $\tilde{\cup}(F_{\alpha_i}, E) \in \tau$.*
3. *If (F_{α_1}, E) and $(F_{\alpha_2}, E) \in \tau$, then $\{e_j : F^c_{\alpha_1}(e_j) \neq \phi\}$ and $\{e_j : F^c_{\alpha_2}(e_j) \neq \phi\}$ are finite sets. Now $\{e_j : (F^c_{\alpha_1} \tilde{\cup} F^c_{\alpha_2})(e_j) \neq \phi\} = \{e_j : (F_{\alpha_1} \tilde{\cap} F_{\alpha_2})^c(e_j) \neq \phi\}$ is a finite set. Thus $(F_{\alpha_1}, E) \tilde{\cap} (F_{\alpha_2}, E) \in \tau$*

Thus (X, τ, E) is a soft topological space. Let us take two distinct soft points x_{e_i} and y_{e_j}. Then either $x \neq y$ or $e_i \neq e_j$. In either cases $x^c_{e_i}$ and $y^c_{e_j}$ are two soft open sets such that $x_{e_i} \tilde{\in}\ y^c_{e_j}, y_{e_j} \tilde{\notin}\ y^c_{e_j}$ and $x_{e_i} \tilde{\notin} x^c_{e_i}$, $y_{e_j} \tilde{\in}\ x^c_{e_i}$. This proves that (X, τ, E) is a soft T_1 space.

Let us consider a soft set (G,E) such that $G(e_i) = \begin{cases} \{x\} & \text{if } e_i \text{ is even} \\ \phi & \text{if } e_i \text{ is odd} \end{cases}$. Define $T(e_i) = \begin{cases} 1 & \text{if } e_i \text{ is even} \\ 0 & \text{if } e_i \text{ is odd} \end{cases}$. $T = \sum T(e_i) = \infty$, because $2\mathbf{N}$ is an infinite set. Since $\{e_j : G(e_j) \neq \phi\}$ is not a finite set, (G,E) is not a soft closed set.

Definition 2.6. [8] A soft topological space (X, τ, E) is said to be a soft T_2 -space if for every pair of soft points x_{e_i} and y_{e_j} such that $x_{e_i} \neq y_{e_j}$ there exist soft open sets (F,E) and (G,E) such that $x_{e_i} \tilde{\in} (F,E)$, $y_{e_j} \tilde{\in} (G,E)$ and $(F,E) \tilde{\cap} (G,E) = \tilde{\phi}$.

Example:2 given in the artice [8] for soft T_1 and soft T_2 space is wrong. Because it is neither soft T_1 nor soft T_2 space.

Example 2.7. [8] *$X = \{x_1, x_2\}$, $A = \{e_1, e_2\}$ and $\tau = \{\tilde{\phi}, \tilde{X}, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\}$ where*

$$F_1(e) = \begin{cases} \{x_2\} & \text{if } e = e_1 \\ \{x_1\} & \text{if } e = e_2 \end{cases}, F_2(e) = \begin{cases} \{x_1\} & \text{if } e = e_1 \\ \{x_2\} & \text{if } e = e_2 \end{cases} F_3(e) = \begin{cases} \{x_1\} & \text{if } e = e_1 \\ \phi & \text{if } e = e_2 \end{cases}$$

$$F_4(e) = \begin{cases} X & \text{if } e = e_1 \\ \{x_1\} & \text{if } e = e_2 \end{cases}$$

This (X, τ, A) is verified as soft T_1 and soft T_2 spaces in [8].

consider two soft points $e_F = \begin{cases} \{x_1\} & \text{if } e = e_1 \\ \phi & \text{if } e = e_2 \end{cases}$ and $e_G = \begin{cases} \phi & \text{if } e = e_1 \\ \{x_2\} & \text{if } e = e_2 \end{cases}$, then there is no soft open set (F_i, A), $i \in \{1,2,3,4\}$ in (X, τ, A) such that $e_G \tilde{\in} (F_i, A)$ and $e_F \tilde{\notin} (F_i, A)$. Thus (X, τ, A) is not a soft T_1 space.

Similarly, there is no two soft open sets (F_i, A) (F_j, A), $i, j \in \{1, 2, 3, 4\}$, $i \neq j$ *in* (X, τ, A) *such that* $e_F \tilde{\in} (F_i, A)$, $e_G \tilde{\in} (F_j, A)$ *and* $(F_i, A) \tilde{\cap} (F_j, A) = \tilde{\phi}$ *Thus* (X, τ, A) *is not a soft* T_2 *space too.*

Next the example:3 given in article [8] is wrong.

Example 2.8. [8] $X = \{x_1, x_2\}$, $A = \{e_1, e_2\}$ *and* $\tau = \{\tilde{\phi}, \tilde{X}, (F_1, A), (F_2, A), (F_3, A)\}$ *where*

$$F_1(e) = \begin{cases} \{x_1\} & \text{if } e = e_1 \\ \phi & \text{if } e = e_2 \end{cases},\ F_2(e) = \begin{cases} \phi & \text{if } e = e_1 \\ \{x_2\} & \text{if } e = e_2 \end{cases}\ F_3(e) = \begin{cases} \{x_1\} & \text{if } e = e_1 \\ \{x_2\} & \text{if } e = e_2 \end{cases}$$

This (X, τ, A) *is verified as soft* T_1 *and soft* T_0 *spaces in* [8].

consider two soft points $e_F = \begin{cases} \{x_2\} & \text{if } e = e_1 \\ \phi & \text{if } e = e_2 \end{cases}$ *and* $e_G = \begin{cases} \phi & \text{if } e = e_1 \\ \{x_1\} & \text{if } e = e_2 \end{cases}$, *then there is no soft open set* (F_i, A), $i \in \{1, 2, 3\}$ *in* (X, τ, A) *such that* $e_F \tilde{\in} (F_i, A)$ *and* $e_G \tilde{\notin} (F_i, A)$. *Thus* (X, τ, A) *is not a soft* T_1 *space. Also there is no soft open set* (F_i, A), $i \in \{1, 2, 3\}$ *in* (X, τ, A) *such that* $e_F \tilde{\notin} (F_i, A)$ *and* $e_G \tilde{\in} (F_i, A)$. *Hence* (X, τ, A) *is not a soft* T_0 *space too.*

Correct example for soft T_1 spae which is not a soft T_2 space is given below.

Example 2.9. *Consider a soft topological space* (X, τ, E) *discussed in Example: 2.6. It is a soft* T_1 *space.*

Let x_{e_i} *and* y_{e_j} *be two distinct soft points. Then either* $x \neq y$ *or* $e_i \neq e_j$. *Assume that there exists two soft open sets* (F, E) *and* (G, E) *such that* $x_{e_i} \tilde{\in} (F, E)$ *and* $y_{e_j} \tilde{\in} (G, E)$. *Since* (F, E) *and* (G, E) *are soft open sets,* $\{e_j : F^c(e_j) \neq \phi\}$ *and* $\{e_j : G^c(e_j) \neq \phi\}$ *are finite sets. Now* $E - \{e_j : (F(e_j) \cap G(e_j))^c \neq \phi\} \neq \phi$. *For any* $e_k \in E - \{e_j : (F(e_j) \cap G(e_j))^c \neq \phi\}$, $F^c(e_k) = \phi$ *and* $G^c(e_k) = \phi$. *That is* $F(e_k) \cap G(e_k) = X$ *and hence* $(F, E) \tilde{\cap} (G, E) \neq \phi$. *This proves that* (X, τ, E) *is not a soft* T_2 *space.*

Theorem 2.9. *Every soft* T_2 *space is a soft* T_1 *space*

Proof. Proof is straight forward. □

Theorem 2.10. *Soft subspace of soft* T_2*-space is a soft* T_2*-space.*

Proof. Let (X, τ, E) be a soft T_2-space and (Y, τ_Y, E) be a soft subspace. Let $x_{e_i},\ y_{e_j}$ be two soft points in (Y, τ, E). Then $x_{e_i},\ y_{e_j} \tilde{\in} SS(X,\ E)$. Since (X, τ, E) is a soft T_2 space, there exist two soft open sets $(F,\ E)$ and $(G,\ E)$ in (X, τ, E) such that $x_{e_i} \tilde{\in} (F,\ E)$, $y_{e_j} \tilde{\in}\ (G,\ E)$ and $(F,\ E) \tilde{\cap} (G,\ E) = \tilde{\phi}$. Now $(F,\ E) \tilde{\cap} E_Y$ and $(G,\ E) \tilde{\cap} E_Y$ are soft open sets in (Y, τ_Y, E) such that $x_{e_i} \tilde{\in} (F,\ E) \tilde{\cap} E_Y$, $y_{e_j} \tilde{\in} (G,\ E)$ $\tilde{\cap} E_Y$ and $((F,\ E) \tilde{\cap} E_Y) \tilde{\cap} ((G,\ E) \tilde{\cap} E_Y) \tilde{\subseteq}\ (F,\ E) \tilde{\cap} (G,\ E) = \tilde{\phi}$. Thus (Y, τ_Y, E) is a soft T_2 space. □

Lemma 2.11. *Let* $(X,\ \tau,\ E)$ *be a finite soft* T_2 *space. Then* $(X,\ \tau,\ E)$ *is a soft discrete space.*

Proof. Proof follows from theorem:2.9 and theorem:2.5. □

Lemma 2.12. *If* $(X,\ \tau,\ E)$ *is a countable soft* T_2 *space and if every soft* G_δ *set is soft open in* $(X,\ \tau,\ E)$*, then* $(X,\ \tau,\ E)$ *is a soft discrete space.*

Proof. Proof follows from theorem:2.9 and theorem:2.6. □

Theorem 2.13. *Let $(X,\ \tau,\ E)$ be a soft topological space. Then $(X,\ \tau,\ E)$ is a soft T_2 space if and only if for any soft points x_{e_i} and y_{e_j}, there exist two soft closed neighbourhoods $(H,\ E)$ and $(K,\ E)$ containing disjoint soft open sets containing x_{e_i} and y_{e_j} respectively such that $(H,\ E)\tilde{\cup}(K,\ E) = \tilde{X}$.*

Proof. Since $(X,\ \tau,\ E)$ is a soft T_2 space, for any two distinct soft points x_{e_i} and y_{e_j}, there exist two soft open sets $(F,\ E)$ and $(G,\ E)$ such that $x_{e_i}\tilde{\in}(F,\ E)$ and $y_{e_j}\tilde{\in}(G,\ E)$ such that $(F,\ E)\tilde{\cap}(G,\ E) = \tilde{\phi}$. Now $x_{e_i}\tilde{\in}(G^c,\ E)$, $y_{e_j}\tilde{\in}(F^c,\ E)$ and $(F^c,\ E)\tilde{\cup}(G^c,\ E) = \tilde{X}$. Note that $(F,\ E)\tilde{\subseteq}(G^c,\ E)$ and $(G,\ E)\tilde{\subseteq}(F^c,\ E)$. Let $(F^c,\ E) = (K,\ E)$ and $(G^c,\ E) = (H,\ E)$. Then we have two soft closed neighbourhoods $(H,\ E)$ and $(K,\ E)$ containing disjoint soft open sets (F,E) and (G,E) respectively, such that $x_{e_i}\tilde{\in}(F,E)$, $y_{e_j}\tilde{\in}(G,E)$ $(H,\ E)\tilde{\cup}(K,\ E) = \tilde{X}$.

Conversely let x_{e_i} and y_{e_j} be two distinct soft points. Then there exist two soft closed neighbourhoods $(H,\ E)$ and $(K,\ E)$ and two soft open sets (L,E) containing x_{e_i} and (M,E) containing y_{e_j} such that $(L,E)\tilde{\subseteq}(H,E)$, $(M,E)\tilde{\subseteq}(K,E)$, $(L,E)\tilde{\cap}(M,E) = \tilde{\phi}$ and $(H,\ E)\tilde{\cup}(K,\ E) = \tilde{X}$. This proves that (X,τ,E) is a soft T_2 space □

Theorem 2.14. *Product of soft T_2-spaces is a soft T_2-space*

Proof. Let $\{(X_i,\ \tau_i,\ E_i) : i \in I\}$ be the collection of soft topoloogical spaces and $(\prod X_i,\ \prod \tau_i,\ \prod E_i)$ be their product soft toplogical space. Suppose $\mathbf{x_e}$ and $\mathbf{y_f}$ be two distinct soft points, where $\mathbf{x} =< x_i >_{i\in I}$, $\mathbf{y} =< y_i >_{i\in I}$ $x_i, y_i \in X_i$ and $\mathbf{e} =< e_i >_{i\in I}$, $\mathbf{f} =< f_i >_{i\in I}, e_i, f_i \in E_i$. Then there exists atleast one $\beta \in I$ such that $x_\beta \neq y_\beta$ or there exist $e_{i_k},\ e_{i_m} \in E_i$ such that $e_{i_k} \neq e_{i_m}$.

Case: 1

If $x_\beta \neq y_\beta$, $(p_q)_\beta\ (\mathbf{x_e}) = (p_{\beta_{q_\beta}})\ (\mathbf{x_e}) = p_\beta(\mathbf{x})_{\mathbf{q}_\beta(\mathbf{e})} = x_{\beta_{e_\beta}}$ and $(p_q)_\beta\ (\mathbf{y_f}) = (p_{\beta_{q_\beta}})$ $(\mathbf{y_f}) = p_\beta(\mathbf{y})_{\mathbf{q}_\beta(\mathbf{f})} = y_{\beta_{f_\beta}}$. Since X_β is a soft T_2 space, there are disjoint soft open sets (F_β, E_β) and $(G_\beta,\ E_\beta)$ such that $x_{\beta_{e_\beta}}\ \tilde{\in}\ (F_\beta, E_\beta)$ and $y_{\beta_{f_\beta}}\ \tilde{\in}(G_\beta,\ E_\beta)$. Then the subbasic members $(p_q)_\beta^{-1}(F_\beta,\ E_\beta)$ and $(p_q)_\beta^{-1}\ (G_\beta,\ E_\beta)$ are the soft open sets such that $\mathbf{x_e}\ \tilde{\in}\ (p_q)_\beta^{-1}\ (F_\beta, E_\beta)$ and $\mathbf{y_f}\ \tilde{\in}\ (p_q)_\beta^{-1}\ (G_\beta,\ E_\beta)$. Let $\mathbf{z_g}\ \tilde{\in}\ (p_q)_\beta^{-1}\ (F_\beta,$ $E_\beta)\tilde{\cap}\ (p_q)_\beta^{-1}\ (G_\beta,\ E_\beta)$, where $\mathbf{z} =< z_i >_{i\in I}, z_i \in X_i$ and $\mathbf{g} =< g_i >_{i\in I}, g_i \in E_i$. Then $\mathbf{z_g}\tilde{\in}\ (p_q)_\beta^{-1}(F_\beta,\ E_\beta)$ and $\mathbf{z_g}\tilde{\in}\ (p_q)_\beta^{-1}\ (G_\beta, E_\beta)$. $(p_q)_\beta\ (\mathbf{z_g})\ \tilde{\in}\ (p_q)_\beta((p_q)_\beta^{-1}$ $(F_\beta, E_\beta))$ and $(p_q)_\beta\ (\mathbf{z_g})\ \tilde{\in}\ (p_q)_\beta((p_q)_\beta^{-1}\ (G_\beta, E_\beta))$. That is $z_{\beta_{g_\beta}}\ \tilde{\in}\ (F_\beta, E_\beta)$ and $z_{\beta_{g_\beta}}\ \tilde{\in}\ (G_\beta, E_\beta)$ which is a contradiction to our assumption of soft T_2 space.

Case: 2

If $e_{i_k} \neq e_{i_m}$, there are disjoint soft open sets (F_{i_k}, E_i) and (F_{i_m}, E_i) such that $x_{e_{i_k}}$ $\tilde{\in}\ (F_{i_k}, E_i)$ and $x_{e_{i_m}} (= y_{e_{i_m}})\ \tilde{\in}\ (F_{i_m}, E_i)$. Then $(p_q)_i^{-1}\ (F_{i_k}, E_i)$ and $(p_q)_i^{-1}\ (F_{i_m},$ $E_i)$ are disjoint soft open sets containing $\mathbf{x_e}$ and $\mathbf{y_f}$ respectively. Let $\mathbf{z_g}\ \tilde{\in}\ (p_q)_i^{-1}$ $(F_{i_k}, E_i)\tilde{\cap}\,(p_q)_i^{-1}\ (F_{i_m}, E_i)$, where $\mathbf{z} =< z_i >_{i\in I}, z_i \in X_i$ and $\mathbf{g} =< g_i >_{i\in I}, g_i \in E_i$. Then $\mathbf{z_g}\ \tilde{\in}\ (p_q)_i^{-1}(F_{i_k}, E_i)$ and $\mathbf{z_g}\tilde{\in}\ (p_q)_i^{-1}\ (F_{i_m}, E_i)$. $(p_q)_i\ (\mathbf{z_g})\ \tilde{\in}\ (p_q)_i((p_q)_i^{-1}$ $(F_{i_k}, E_i))$ and $(p_q)_i\ (\mathbf{z_g})\ \tilde{\in}\ (p_q)_i((p_q)_i^{-1}\ (F_{i_m}, E_i))$. That is $z_{i_{g_i}}\ \tilde{\in}\ (F_{i_k}, E_i)$ and $z_{i_{g_i}}$ $\tilde{\in}\ (F_{i_m}, E_i)$ which is a contradiction to our assumption of soft T_2 space. □

Definition 2.7. Let $(X,\ \tau,\ E)$ be a soft topological space. Then $(X,\ \tau,\ E)$ is a soft Urysohn space or soft $T_{2\frac{1}{2}}$ space if for any two soft points x_{e_i} and y_{e_j}, there exist two soft open sets $(F,\ E)$ and $(G,\ E)$ such that $x_{e_i}\tilde{\in}(F,\ E)$, $y_{e_j}\tilde{\in}(G,\ E)$ and $Cl(F,\ E)\tilde{\cap}Cl(G,\ E) = \tilde{\phi}$.

Theorem 2.15. *Every soft $T_{2\frac{1}{2}}$-space is a soft T_2-space.*

Proof. Proof is straight forward □

Theorem 2.16. *Soft subspace of soft $T_{2\frac{1}{2}}$-space is a soft $T_{2\frac{1}{2}}$-space.*

Proof. Proof is similar to theorem 2.10 □

Theorem 2.17. *Let $(X,\ \tau,\ E)$ be a soft topological space. Then $(X,\ \tau,\ E)$ is a soft $T_{2\frac{1}{2}}$ space if and only if for any two soft points x_{e_i} and y_{e_j}, there exist two soft open sets $(H,\ E)$ and $(K,\ E)$ such that $x_{e_i}\tilde{\in}(H,\ E)$, $y_{e_j}\tilde{\in}(K,\ E)$ and $(H,\ E)$ and $(K,\ E)$ containing the disjoint closed soft neighbourhoods of x_{e_i} and y_{e_j} respectively with $(H,\ E)\tilde{\cup}(K,\ E) = \tilde{X}$.*

Proof. Since $(X,\ \tau,\ E)$ be a soft $T_{2\frac{1}{2}}$ space, for any soft points x_{e_i} and y_{e_j}, there exist two soft open sets $(F,\ E)$ and $(G,\ E)$ such that $x_{e_i}\tilde{\in}(F,\ E)$ and $y_{e_j}\tilde{\in}(G,\ E)$ such that $Cl(F,\ E)\tilde{\cap}Cl(G,\ E) = \tilde{\phi}$. Now $x_{e_i}\tilde{\in}[Cl(G,\ E)]^c$ and $y_{e_j}\tilde{\in}[Cl(F,\ E)]^c$ and $[Cl(F,\ E)]^c\tilde{\cup}[Cl(G,\ E)]^c = \tilde{X}$. Note that $Cl(F,\ E)\tilde{\subseteq}[Cl(G,\ E)]^c$ and $Cl(G,\ E)\tilde{\subseteq}[Cl(F,\ E)]^c$. Let $[Cl(F,\ E)]^c = (K,\ E)$ and $[Cl(G,\ E)]^c = (H,\ E)$. Then we have two soft open sets $(H,\ E)$ and $(K,\ E)$ containing x_{e_i} and y_{e_j} respectively, such that $x_{e_i}\tilde{\in}(F,\ E)\tilde{\subseteq}Cl(F,\ E)\ \tilde{\subseteq}(H,\ E)$, $y_{e_j}\tilde{\in}(G,\ E)\tilde{\subseteq}\ Cl(G,\ E)\ \tilde{\subseteq}(K,\ E)$ and $(H,\ E)\tilde{\cup}(K,\ E) = \tilde{X}$. Thus $(H,\ E)$ and $(K,\ E)$ are soft open sets containing the disjoint closed neighbourhoods $Cl(F,\ E)$ and $Cl(G,\ E)$, respectively such that $x_{e_i}\tilde{\in}Cl(F,\ E)$, $y_{e_j}\tilde{\in}Cl(G,\ E)$ and $(H,E)\tilde{\cup}(K,E) = \tilde{X}$.

Conversely, let x_{e_i} and y_{e_j} be two distinct soft points. By our assumption, there exist two soft open sets (H,E) and (K,E) containing disjoint closed neighbourhoods (L,E) and (M,E) of x_{e_i} and y_{e_j} respectively such that $(H,E)\tilde{\cup}(K,E) = \tilde{X}$. Note that there are soft open sets (F,E) and (G,E) such that $(F,E)\tilde{\subseteq}Cl(F,E)\tilde{\subseteq}(L,E)\tilde{\subseteq}(H,E)$, $(G,E)\tilde{\subseteq}Cl(G,E)\tilde{\subseteq}(M,E)\ \tilde{\subseteq}(K,E)$ and $(L,E)\tilde{\cap}(M,E) = \tilde{\phi}$. So that $Cl(F,E)\tilde{\cap}Cl(G,E) = \tilde{\phi}$. That is (F,E) and (G,E) are soft open sets containing x_{e_i} and y_{e_j} such that $Cl(F,E)\tilde{\cap}Cl(G,E) = \tilde{\phi}$. Thus (X,τ,E) is a soft $T_{2\frac{1}{2}}$ space. □

Soft single point space discussed in [5] is not a soft T_0 or T_1 or T_2 or $T_{2\frac{1}{2}}$ space. Because for the soft points x_{e_i} and x_{e_j}, there is no soft open set containing x_{e_i} not containing x_{e_j}.

Theorem 2.18. *Product of soft $T_{2\frac{1}{2}}$-spaces is a soft $T_{2\frac{1}{2}}$-space*

Proof. Proof is similar to theorem 2.14 □

Lemma 2.19. *Let (X,τ,E) be a finite soft $T_{2\frac{1}{2}}$ space. Then (X,τ,E) is a soft discrete space.*

Proof. Proof follows from theorem:2.15, theorem:2.9 and theorem:2.5. □

Lemma 2.20. *If (X,τ,E) is a countable soft $T_{2\frac{1}{2}}$ space and if every soft G_δ set is soft open in $(X,\ \tau,\ E)$, then (X,τ,E) is a soft discrete space.*

Proof. Proof follows from theorem:2.15, theorem:2.9 and theorem:2.6. □

3. Conclusion

For the soft separation axioms of soft points defined on soft topological space, we discuss the characterizations and properties of soft T_0, T_1, T_2 and soft $T_{2\frac{1}{2}}$ spaces. Also it is verified that the product of soft T_i spaces, $i = 1, 2, 2\frac{1}{2}$ is a soft T_i space. But there is an example given here for the product of soft T_0 spaces need not be a soft T_0 space. Also we provide correct examples for the wrong examples example:1, example:2 and example:3 given in article [8].

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$\mathcal{I}-$ALMOST LACUNARY VECTOR VALUED SEQUENCE SPACES IN $2-$NORMED SPACES

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Abstract. One of the wide-ranging applications and research areas of Summability theory is the concept of statistical convergence. This concept was studied a related concept of convergence by using lacunary sequence by Fridy and Orhan. At the last quarter of the 20th century, lacunary statistical convergence has been discussed and captured significant aspect of creating the basis of several investigations conducted in many branches of mathematics. On the other hand, in 1961 Krasnoselskii and Rutisky presented the definition of Orlicz function. Also, in 1963 Gähler introduced the notion of 2-normed spaces. The main goal of this article is to introduce $\mathcal{I}-$ almost convergence of lacunary sequences with regard to an Orlicz function in 2-normed spaces and other sequence spaces by considering the concept of ideal that was presented by Kostyrko and others. Additionally, we examine the relationship between these sequence spaces and fundamental inclusion theorems are investigated.

1. Introduction

The concept of $2-$normed spaces was initially introduced by Gähler [3] in the 1960's. Since then, this concept has been studied by many authors (see, for instance ([11], [13], [20], [21]).

Recall in [8] that an Orlicz function $\lambda : [0, \infty) \to [0, \infty)$ is continuous, convex, non-decreasing function such that $\lambda(0) = 0$ and $\lambda(u) > 0$ for $u > 0$, and $\lambda(u) \to \infty$ as $u \to \infty$.

Subsequently the notion of Orlicz function was used by Mursaleen, Khan, Chishti [9], Parashar and B. Choudhary [10], Savaş and Savaş [18], Savaş([16], [17], [19]) and others.

If convexity of Orlicz function λ is replaced by $\lambda(x+y) \leq \lambda(x) + \lambda(y)$ then this function is called Modulus function, which was presented and discussed by Ruckle [12] and Maddox [5].

An Orlicz function is said to satisfy Δ_2- condition if there exists a positive constant K such that $\lambda\left(2u\right) \leq T\lambda\left(u\right)$ for all $u \geq 0$.

Note that if λ is an Orlicz function then $\lambda\left(\psi x\right) \leq \psi\lambda\left(u\right)$ for all ψ with $0 < \psi < 1$.

Let E be a real vector space of dimension d, where $2 \leq d < \infty$. A $2-$norm on E is a function $\|.,.\| : E \times E \to R$ which satisfies (**i**) $\|u, v\| = 0$ if and only if u and v

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are linearly dependent; (**ii**) $\|u,v\| = \|v,u\|$; (**iii**) $\|\beta u, v\| = |\beta| \|u,v\|$, $\beta \in \mathbb{R}$; (**iv**) $\|u, v+w\| \leq \|u,v\| + \|u,w\|$. The pair $(E, \|.,.\|)$ is then called a 2− normed space [4].

Recall that $(E, \|.,.\|)$ is a 2−Banach space if every Cauchy sequence in E is convergent to some u in E.

The notion of ideal convergence was introduced first by P. Kostyrko et al [6] as a generalization of statistical convergence.

A family $\mathcal{I} \subset 2^F$ of subsets a nonempty set F is said to be an ideal in F if (**i**) $\emptyset \in \mathcal{I}$; (**ii**) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; (**iii**) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of F further satisfies $\{u\} \in \mathcal{I}$ for each $u \in F$ (see, [6], [7]) .

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(u_n)_{n\in\mathbb{N}}$ in E is said to be $\mathcal{I}-$convergent to $u \in E$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|u_n - L\| \geq \varepsilon\}$ belongs to $\mathcal{I}$ ([1, 14, 15]).

By a lacunary sequence $\theta = (l_s)$; $s = 0, 1, 2, ...$ where $l_0 = 0$, we shall mean an increasing sequence of non-negative integers with $l_s - l_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by θ will be denoted by $I_s = (l_{s-1}, l_s]$ and $\mu_s = l_s - l_{s-1}$ ([2]).

2. Main Results

Let $\mathcal{I}$ be an admissible ideal, λ be an Orlicz function, $(E, \|.,.\|)$ be a 2−normed space and $r = (r_i)$ be a sequence of positive real numbers. By $S(2-E)$ we denote the space of all sequences defined over $(E, \|.,.\|)$. Now we define the following sequence spaces:

$$\hat{w}^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta = \left\{ \begin{array}{c} x \in S(2-X) : \left\{ s \in \mathbb{N} : \frac{1}{\mu_s^\gamma} \sum\limits_{i \in I_s} \left[\lambda \left(\left\| \frac{u_{i+j} - L}{\rho}, z \right\| \right) \right]^{r_i} \geq \varepsilon \right\} \in \mathcal{I} \\ \text{for some } \rho > 0, L > 0 \text{ and each } z \in E, \text{ uniformly in } j \end{array} \right\},$$

$$\hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta = \left\{ \begin{array}{c} x \in S(2-E) : \left\{ s \in \mathbb{N} : \frac{1}{\mu_s^\gamma} \sum\limits_{i \in I_s} \left[\lambda \left\| \frac{u_{i+j}}{\rho}, z \right\| \right]^{r_i} \geq \varepsilon \right\} \in \mathcal{I} \\ \text{for some } \rho > 0 \text{ and each } z \in E, \text{ uniformly in } j \end{array} \right\},$$

$$\hat{w}_\infty(\lambda, r, \|.,.\|)_\theta = \left\{ \begin{array}{c} x \in S(2-E) : \exists K > 0 \text{ s.t. } \sup\limits_{s \in \mathbb{N}} : \dfrac{1}{\mu_s^\gamma} \sum\limits_{i \in I_s} \left[\lambda \left(\left\| \dfrac{u_{i+j}}{\rho}, z \right\| \right) \right]^{r_i} \leq K \\ \text{for some } \rho > 0, \text{ and each } z \in E \end{array} \right\},$$

$$\hat{w}_\infty^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta = \left\{ \begin{array}{c} x \in S(2-E) : \exists K > 0 \ni \left\{ s \in \mathbb{N} : \frac{1}{\mu_s^\gamma} \sum\limits_{i \in I_s} \left[\lambda \left\| \frac{u_{i+j}}{\rho}, z \right\| \right]^{r_i} \geq K \right\} \in \mathcal{I} \\ \text{for some } \rho > 0 \text{ and each } z \in E, \text{ uniformly in } j \end{array} \right\}.$$

The following inequality will be used in the study which is well known.

$$0 \leq r_i \leq \sup r_i = H, \; D = \max\left(1, 2^{H-1}\right)$$

then

$$|u_i + v_i|^{r_i} \leq D \left\{ |u_i|^{r_i} + |v_i|^{r_i} \right\}$$

for all i and $u_i, v_i \in C$

Theorem 2.1. $\hat{w}^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta$, $\hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta$, $\hat{w}_\infty^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta$ *are linear spaces.*

Proof. We shall prove the assertion for $\hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta$only and the others can be proved similarly. Suppose that $u, v \in \hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta$ and $\alpha, \beta \in \mathbb{R}$. So

$$\left\{s \in \mathbb{N} : \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\lambda\left(\left\|\frac{u_{i+j}}{\rho_1}, z\right\|\right)\right]^{r_i} \geq \varepsilon\right\} \in \mathcal{I} \text{ for some } \rho_1 > 0$$

and

$$\left\{s \in \mathbb{N} : \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\lambda\left(\left\|\frac{u_{i+j}}{\rho_2}, z\right\|\right)\right]^{r_i} \geq \varepsilon\right\} \in \mathcal{I} \text{ for some } \rho_2 > 0.$$

uniformly in j. Since $\|.,.\|$ is a 2−norm, and λ is an Orlicz function the following inequality holds: for all j

$$\begin{aligned} & \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\lambda\left(\left\|\frac{(\alpha u_{i+j} + \beta v_{i+j})}{(|\alpha| \rho_1 + |\beta| \rho_2)}, z\right\|\right)\right]^{r_i} \\ \leq \quad & D \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\frac{|\alpha|}{(|\alpha| \rho_1 + |\beta| \rho_2)} \lambda\left(\left\|\frac{u_{i+j}}{\rho_1}, z\right\|\right)\right]^{r_i} \\ + \quad & D \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\frac{|\beta|}{(|\alpha| \rho_1 + |\beta| \rho_2)} \lambda\left(\left\|\frac{v_{i+j}}{\rho_2}, z\right\|\right)\right]^{r_i} \\ \leq \quad & DF \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\lambda\left(\left\|\frac{u_{i+j}}{\rho_1}, z\right\|\right)\right]^{r_i} \\ + \quad & DF \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\lambda\left(\left\|\frac{v_{i+j}}{\rho_2} v, z\right\|\right)\right]^{r_i} \end{aligned}$$

where

$$F = \max\left[1, \left(\frac{|\alpha|}{(|\alpha| \rho_1 + |\beta| \rho_2)}\right)^H, \left(\frac{|\beta|}{(|\alpha| \rho_1 + |\beta| \rho_2)}\right)^H\right].$$

From the above inequality we get

$$\begin{aligned} & \left\{s \in \mathbb{N} : \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\lambda\left(\left\|\frac{(\alpha u_{i+j} + \beta v_{i+j})}{(|\alpha| \rho_1 + |\beta| \rho_2)}, z\right\|\right)\right]^{r_i} \geq \varepsilon\right\} \\ \subseteq \quad & \left\{s \in \mathbb{N} : DF \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\lambda\left(\left\|\frac{u_{i+j}}{\rho_1}, z\right\|\right)\right]^{r_i} \geq \frac{\varepsilon}{2}\right\} \\ & \cup \left\{s \in \mathbb{N} : DF \frac{1}{\mu_s^\gamma} \sum_{i \in I_s} \left[\lambda\left(\left\|\frac{v_{i+j}}{\rho_2}, z\right\|\right)\right]^{r_i} \geq \frac{\varepsilon}{2}\right\}, \end{aligned}$$

uniformly in j. Two sets on the right hand side belong to $\mathcal{I}$ and this completes the proof. □

It is also easy to verify that the space $\hat{w}_\infty(\lambda, r, \|.,.\|)_\theta$ is also a linear space.

Theorem 2.2. *If λ is an Orlicz function and (r_i) is bounded sequence of strictly positive real numbers then $\hat{w}_\infty(\lambda, r, \|.,.\|)_\theta$ is a paranormed space with respect to paranorm g defined by*

$$g(x) = \sum_{i \in I_s} \|u_{i+j}, z\| + \inf\left\{\rho^{\frac{r_t}{H}} : \sup_k \left[\lambda\left(\left\|\frac{u_{i+j}}{\rho}, z\right\|\right)\right]^{r_i} \leq 1, \ \rho > 0, \ t = 1, 2, ...\right\}, \ \textit{each } z \in E$$

Corollary 1. If one considers the sequence space $\hat{w}_{\infty}^{\mathcal{I}}(\lambda, r, \|.,.\|)_{\theta}$ which is larger than the space $\hat{w}_{\infty}(\lambda, r, \|.,.\|)_{\theta}$ the construction of the paranorm is not clear and we leave it as an open problem.

Theorem 2.3. *Let λ, λ_1 and λ_2 be Orlicz functions. Then we have $\hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_{\theta} \subseteq \hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_{\theta}$ provided (r_i) is such that $H_0 = \inf r_i > 0$.*

Proof. (i) For given $\varepsilon > 0$, first choose $\varepsilon_0 > 0$ such that $\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon$. Now using the continuity of λ choose $0 < \delta < 1$ such that $0 < t < \delta \Rightarrow \lambda(t) < \varepsilon_0$. Let $(u_i) \in \hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_{\theta}$. Now from the definition,

$$A(\delta) = \left\{ s \in \mathbb{N} \ : \frac{1}{\mu_s^{\gamma}} \sum_{i \in I_s} \left[\lambda_1 \left(\left\| \frac{u_{i+j}}{\rho}, z \right\| \right) \right]^{p_i} \geq \delta^H \right\} \in \mathcal{I},$$

uniformly in j. Thus if $s \notin A(\delta)$ then

$$\frac{1}{\mu_s^{\gamma}} \sum_{i \in I_s} \left[\lambda_1 \left(\left\| \frac{u_{i+j}}{\rho}, z \right\| \right) \right]^{r_i} < \delta^H$$

$$\text{i.e.} \sum_{i \in I_s} \left[\lambda_1 \left(\left\| \frac{u_{i+j}}{\rho}, z \right\| \right) \right]^{r_i} < \mu_s^{\gamma} \delta^H$$

$$\text{i.e.} \left[\lambda_1 \left(\left\| \frac{u_{i+j}}{\rho}, z \right\| \right) \right]^{r_i} < \delta^H \text{ for all } i \in I_s$$

$$\text{i.e.} \lambda_1 \left(\left\| \frac{u_{i+j}}{\rho}, z \right\| \right) < \delta \text{ for all } i \in I_s.$$

Hence from above using the continuity of λ we must have

$$\lambda \left(\lambda_1 \left(\left\| \frac{u_i}{\rho}, z \right\| \right) \right) < \varepsilon_0 \text{ for all } i \in I_s$$

which consequently implies that

$$\sum_{i \in I_s} \left[\lambda \left(\lambda_1 \left(\left\| \frac{u_{i+j}}{\rho}, z \right\| \right) \right) \right]^{p_k} < \mu_s^{\gamma} \max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \mu_s^{\gamma} \varepsilon,$$

$$\text{i.e.} \frac{1}{\mu_s^{\gamma}} \sum_{i \in I_s} \left[\lambda \left(\lambda_1 \left(\left\| \frac{u_{i+j}}{\rho}, z \right\| \right) \right) \right]^{r_i} < \varepsilon, \text{ uniformly in } j.$$

This shows that

$$\left\{ s \in \mathbb{N} : \frac{1}{\mu_s^{\gamma}} \sum_{i \in I_s} \left[\lambda \left(\lambda_1 \left(\left\| \frac{u_{i+j}}{\rho}, z \right\| \right) \right) \right]^{p_k} \geq \varepsilon \right\} \subset A(\delta), \text{ uniformly in } j$$

and so belongs to $\mathcal{I}$. This proves the result. □

Theorem 2.4. *Let the sequence (r_i) be bounded, then $\hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_{\theta} \subseteq \hat{w}^{\mathcal{I}}(\lambda, r, \|.,.\|)_{\theta} \subseteq \hat{w}_{\infty}^{\mathcal{I}}(\lambda, r, \|.,.\|)_{\theta}$.*

Proof. Let $u = (u_i) \in \hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_{\theta}$. Then given $\varepsilon > 0$ we have

$$\left\{ s \in \mathbb{N} \ : \frac{1}{\mu_s^{\gamma}} \sum_{i \in I_s} \left[\lambda \left(\left\| \frac{u_{i+j}}{\rho}, z \right\| \right) \right]^{r_i} \geq \varepsilon \right\} \in \mathcal{I} \text{ for some } \rho > 0, \text{ uniformly in } j.$$

Since λ is non-decreasing and convex it follows that, for all j,

$$\frac{1}{\mu_s^{\gamma}}\sum_{i\in I_s}\left[\lambda\left(\left\|\frac{u_i}{2\rho},z\right\|\right)\right]^{r_i}\leq\frac{D}{\mu_s^{\gamma}}\sum_{i\in I_s}\frac{1}{2^{p_i}}\left[\lambda\left(\left\|\frac{u_{i+j}-u_0}{\rho},z\right\|\right)\right]^{r_i}+\frac{D}{\mu_s^{\gamma}}\sum_{i\in I_s}\frac{1}{2^{r_i}}\left[\lambda\left(\left\|\frac{u_0}{\rho},z\right\|\right)\right]^{r_i}$$

$$\leq\frac{D}{\mu_s^{\gamma}}\sum_{i\in I_s}\left[\lambda\left(\left\|\frac{u_{i+j}-u_0}{\rho},z\right\|\right)\right]^{r_i}+Dmax\left\{1,sup\left[\lambda\left(\left\|\frac{u_0}{\rho},z\right\|\right)\right]^{r_i}\right\}.$$

Hence we have

$$\left\{s\in\mathbb{N}\ :\frac{1}{\mu_s^{\gamma}}\sum_{i\in I_s}\left[\lambda\left(\left\|\frac{u_i}{2\rho},z\right\|\right)\right]^{r_i}\geq\varepsilon\right\}$$
$$\subseteq\quad\left\{s\in\mathbb{N}\ :\frac{D}{\mu_s^{\gamma}}\sum_{i\in I_s}\left[\lambda\left(\left\|\frac{u_{i+j}}{\rho},z\right\|\right)\right]^{r_i}\geq\frac{\varepsilon}{2}\right\}$$
$$\cup\quad\left\{s\in\mathbb{N}\ :max\left\{1,sup\left[\lambda\left(\left\|\frac{u_0}{\rho},z\right\|\right)\right]^{r_i}\right\}\geq\frac{\varepsilon}{2}\right\},$$

uniformly in j. Since the set on the right hand side belongs to $\mathcal{I}$ so does the left hand side. The inclusion $\hat{w}^{\mathcal{I}}(\lambda,r,\|.,.\|)_{\theta}\subseteq\hat{w}_{\infty}^{I}(\lambda,r,\|.,.\|)_{\theta}$ is obvious. □

Theorem 2.5. (1) *Let* $0<\inf r_i\leq r_i<1$. *Then*

$$\hat{w}^{\mathcal{I}}(\lambda,r,\|.,.\|)_{\theta}\subseteq\hat{w}^{\mathcal{I}}(\lambda,r,\|.,.\|)_{\theta}.$$

(2) *Let* $1\leq r_i\leq\sup r_i<\infty$. *Then*

$$\hat{w}^{\mathcal{I}}(\lambda,r,\|.,.\|)_{\theta}\subseteq\hat{w}^{\mathcal{I}}(\lambda,r,\|.,.\|)_{\theta}.$$

Proof. Let $u\in\hat{w}^{\mathcal{I}}(\lambda,r,\|.,.\|)_{\theta}$, since $0<\inf r_i\leq 1$, we obtain the following:

$$\left\{s\in\mathbb{N}\ :\frac{1}{\mu_s^{\gamma}}\sum_{i\in I_s}\left[\lambda\left(\left\|\frac{u_{i+j}-L}{\rho},z\right\|\right)\right]\geq\varepsilon\right\}\subseteq\left\{s\in\mathbb{N}\ :\frac{1}{\mu_s^{\gamma}}\sum_{i\in I_s}\left[\lambda\left(\left\|\frac{u_{i+j}-L}{\rho},z\right\|\right)\right]^{r_i}\geq\varepsilon\right\}\in\mathcal{I},$$

uniformly in j. Thus $u\in\hat{w}^{I}(\lambda,\|.,.\|)_{\theta}$. Let us establish part (2). Let $r_i>1$ for each i, and $\sup r_i<\infty$. Let $x\in\hat{w}^{\mathcal{I}}(\lambda,\|.,.\|)_{\theta}$. Then for each $0<\epsilon<1$ there exists a positive integer N such that

$$\mu_s^{\gamma}\sum_{i\in I_s}\left[\lambda\left(\left\|\frac{u_i-L}{\rho},z\right\|\right)\right]\leq\epsilon<1,$$

uniformly in j, for all $s\geq N$. This implies that

$$\left\{s\in\mathbb{N}\ :\frac{1}{\mu_s^{\gamma}}\sum_{i\in I_s}\left[\lambda\left(\left\|\frac{u_{i+j}-L}{\rho},z\right\|\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\subseteq\left\{s\in\mathbb{N}\ :\frac{1}{\mu_s^{\gamma}}\sum_{i\in I_s}\left[\lambda\left(\left\|\frac{u_{i+j}-L}{\rho},z\right\|\right)\right]\geq\varepsilon\right\}\in\mathcal{I},$$

uniformly in j. Therefore $u\in\hat{w}^{\mathcal{I}}(\lambda,r,\|.,.\|)_{\theta}$. This completes the proof. □

Definition 2.1. *Let E be a sequence space. Then E is called solid if $(\alpha_i u_i)\in E$ whenever $(u_i)\in E$ for all sequences (α_i) of scalars with $|\alpha_i|\leq 1$ for all $i\in N$.*

We now have

Theorem 2.6. *The sequence spaces* $\hat{w}_0^{\mathcal{I}}(\lambda,r,\|.,.\|)_{\theta}$, $\hat{w}_{\infty}^{\mathcal{I}}(\lambda,r,\|.,.\|)_{\theta}$ *are solid.*

Proof. We give the proof for $\hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta$. Let $(u_i) \in \hat{w}_0^{I}(\lambda, r, \|.,.\|)_\theta$ and (α_i) be sequences of scalars such that $|\alpha_i| \leq 1$ for all $i \in N$. Then we have,

$$\left\{r \in \mathbb{N} \ : \frac{1}{\mu_s^{\gamma}} \sum_{i \in I_s} \left[\left(\lambda \left\| \frac{(\alpha_i u_{i+j})}{\rho}, z \right\| \right) \right]^{r_i} \geq \varepsilon \right\} \subseteq \left\{r \in \mathbb{N} \ : \frac{1}{\mu_s^{\gamma}} \sum_{i \in I_s} \left[\left(\lambda \left\| \frac{u_{i+j})}{\rho}, z \right\| \right) \right]^{r_i} \geq \varepsilon \right\} \in \mathcal{I},$$

uniformly in j. Hence $(\alpha_i u_i) \in \hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta$ for all sequences of scalars (α_i) with $|\alpha_i| \leq 1$ for all $i \in N$ whenever $(u_i) \in \hat{w}_0^{\mathcal{I}}(\lambda, r, \|.,.\|)_\theta$. □

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A NEW SUBCLASS OF UNIVALENT FUNCTIONS CONNECTED WITH CONVOLUTION DEFINED VIA EMPLOYING A LINEAR COMBINATION OF TWO GENERALIZED DIFFERENTIAL OPERATORS INVOLVING SIGMOID FUNCTION

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Abstract. By introducing an operator $E_{\mu}^{n}(\beta,\lambda,\omega,\varphi,t)f_{\gamma}(z)$ via a linear combination of two generalized differential operators involving modified Sigmoid function, we defined and studied certain geometric properties of a new subclass $T_{\gamma}D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$ of analytic functions in the open unit disk U. In particular, we give some properties of functions in this subclass such as; coefficient estimates, growth and distortion theorems, closure theorem and Fekete-Szego inequality for functions belonging to the subclass. Some earlier known results are special cases of results established for the new subclass defined.

1. INTRODUCTION AND PRELIMINARIES

Let $U = \{z \in c : |z| < 1\}$ be the unit disk. In the usual notation, let A denote the class of functions $f(z)$ which are analytic in the open unit disk and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which is analytic in the open unit disk U and let

$$\gamma(s) = \frac{2}{(1+e^{-s})}; \quad s \geq 0 \tag{1.2}$$

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with $\gamma(s) = 1$ for $s = 0$ be the modified Simoid function. (See details in [1, 2, 3, 4, 5]).
Also, we denote by T the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0 \tag{1.3}$$

as a subclass of A.

The class T of functions with negative coefficients from second term was first introduced by Silverman [6] and has since then opened up a prolific line of research in that direction among function theorists.
For $f_\gamma(z) \in T_\gamma$, Oluwayemi and Fadipe-Joseph [5] gave the following definition:

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \qquad a_k \geq 0 \tag{1.4}$$

as a consequence of (1.3). We note that $\gamma(s) = 1 + \frac{1}{2}s - \frac{1}{24}s^3 + \frac{1}{240}s^5 - \frac{17}{40320}s^7 + \ldots$ defined by (1.2). Furthermore, we define identity function for T_γ as

$$e_\gamma(z) = z. \tag{1.5}$$

2. Differential Operators

2.1. **Salagean Differential Operator.**

Definition 2.1. [7] *For $f \in A, n \in \aleph$, the operator D^n is defined by $D^n : A \to A$.*

$$\begin{aligned} &D^0 f(z) = f(z) \\ &D^1 f(z) = z f'(z) \\ &D^{n+1} f(z) = z(D^n f(z))', \quad z \in U \end{aligned} \tag{2.1}$$

Remark 1 : If $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k \in A$, then

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U \tag{2.2}$$

2.2. **Al-Oboudi Differential Operator.**

Definition 2.2. [8] *For $f \in A, n \in \mathbb{N} \cup 0$, the Al-Oboudi differential operator D_δ^n is defined by $D^n : A \to A$.*

$$\begin{aligned} &D^0 f(z) = f(z) \\ &D_\delta^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z) \\ &\cdots \\ &D_\delta^n f(z) = D_\delta(D_\delta^n f(z)), \quad z \in U. \end{aligned} \tag{2.3}$$

Remark 2 : If D_δ^n is a differential operator and for $f \in A$:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we have

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k, z \in U \tag{2.4}$$

and

$$\left(D_\delta^{n+1} f(z)\right)^{'} = \left(D_\delta^n f(z)\right)^{'} + \delta z \left(D_\delta^{n+1} f(z)\right)^{''}, \quad z \in U. \tag{2.5}$$

When $\delta = 1$, we get the Salagean differential operator (2.2).

2.3. **Opoola New Differential Operator.**

Definition 2.3. [9] *For a function $f \in A$ with*

$$D_t f(z) = 1 + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)t]^n a_k z^{k-1}, \quad 0 \le \mu \le \beta, t \ge 0.$$

Opoola defined the differential operator $D^n(\mu, \beta, t) f(z)$ such that

$$
\begin{aligned}
&D^0(\mu, \beta, t) f(z) = f(z) \\
&D^1(\mu, \beta, t) f(z) = z D_t f(z) = z t f^{'}(z) - z t(\mu - \beta) + (1 + (\mu - \beta - 1)t) f(z) \\
&\cdots \\
&D^n(\mu, \beta, t) f(z) = z D_t [D^{n-1}(\mu, \beta, t) f(z)], \quad n \in \mathbb{N} \cup 0.
\end{aligned}
\tag{2.6}
$$

Remark 3 : If $D^n(\mu, \beta, t) f(z)$ is a linear operator such that for $f \in A$,

$$D^n(\mu, \beta, t) f(z) = z + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)t]^n a_k z^k, \quad z \in U, 0 \le \mu \le \beta, t \ge 0. \tag{2.7}$$

It turns out that the differential operator $D^n(\mu, \beta, t) f(z)$ reduces to the Salagean and al-Oboudi differential operators (2.2) and (2.4) respectively for suitably varied parameters and by letting $t = \lambda$.

2.4. **Differential Operator Involving Modified Sigmoid Function.**

Definition 2.4. *[*[4], [5]*] Fadipe-Joseph et al. introduced Salagean differential operator $D^n f_\gamma(z)$ involving modified sigmoid function which is defined as follows: Let $f_\gamma(z) \in T_\gamma$, the Salagean differential operator denoted by $D^n f_\gamma(z)$ is defined by*

$$
\begin{aligned}
&D^0 f_\gamma(z) = f_\gamma(z) \\
&D^1 f_\gamma(z) = z D_t f(z) \\
&\cdots \\
&D^n f_\gamma(z) = D[D^{n-1} f_\gamma(z)] \\
&\gamma(s) z (D^{n-1} f_\gamma(z)), \quad z \in U.
\end{aligned}
\tag{2.8}
$$

Hence,

$$D^n f_\gamma(z) = \gamma^n(s)z + \sum_{k=2}^{\infty} \gamma^{n+1}(s)k^n a_k z^k, \quad z \in U. \tag{2.9}$$

2.5. **Darus and Ibrahim Generalized Differential Operator Involving Sigmoid Function.**

Definition 2.5. /[5] *Oluwayemi and fadipe-Joseph introduced the generalized differential operator $D^n_{\lambda,\omega} f_\gamma(z)$ involving sigmoid function as a consequence of* [10] *by following* (2.9)*:*

$$D^n_{\lambda,\omega} f_\gamma(z) = \gamma^n(s)z - \sum_{k=2}^{\infty} \gamma^{n+1}(s)[(k-1)(\lambda-\omega)+k]^n a_k z^k \tag{2.10}$$

for $\lambda, \omega \geq 0$. For more information on this, interested reader may see [4] *and* [10]

2.6. **Ruscheweyh Operator Involving Modified Sigmoid Function.**

Definition 2.6. [5] *Recently, Oluyemi and Fadipe-Joseph gave a Ruscheweyh Differential operator involving the modified Sigmoid function $R^n : T_\gamma \to T_\gamma$, with $n \in \mathbb{N} \cup 0$ such that*

$$R^n f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) C^n_{n+k-1} a_k z^k, \quad a_k \geq 0, \quad z \in U; \tag{2.11}$$

where $\gamma(s) = \frac{2}{1+e^{-s}}, \quad s \geq 0$ with $\gamma(s) = 1$ for $s = 0$.
Moreover,

$$\begin{aligned} C^n_{n+k-1} &= B_k(n) = B(n,k) = \binom{n+k-1}{n} \\ &= \frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!} \\ &= \frac{(n+1)_{k-1}}{(1)_{k-1}}. \end{aligned} \tag{2.12}$$

Hence, $B(0,k) = \binom{k-1}{0} = \frac{(1)_{k-1}}{(1)_{k-1}} = 1$.

2.7. **Linear Combination of a Generalized Salagean Differential Operator and Ruscheweyh Operator involving modified sigmoid function.**

Definition 2.7. [5] *By combining the generalized Salagean differential operator involving modified sigmoid and the Ruscheweyh operator involving modified sigmoid function, the following operator was defined in* [5] *by Oluwayemi and Fadipe-Joseph as:*

$$\begin{aligned} \Phi^n_{\lambda,\omega} f_\gamma(z) &= \mu D^n_{\lambda,\omega} f_\gamma(z) + (1-\mu) R^n f_\gamma(z) \\ &= [\mu\gamma^n(s) - \mu + 1]z - \sum_{k=2}^{\infty} \gamma(s)\{\mu\gamma^n(s)[(k-1)(\lambda-\omega)]^n + (1-\mu)B_k(n)\} a_k z^k, \end{aligned} \tag{2.13}$$

for $\lambda \in [0,1] \quad , \mu \in [0,1], \quad z \in U$.
Where $D^n_{\lambda,\omega} f_\gamma(z)$ and $R^n f_\gamma(z)$ are defined respectively in (2.10) and (2.11) respectively.

We note the following in respect of given by (2.13):
(i) Equation (2.13) corrects the one defined for $\Phi^n_{\lambda,\omega} f_\gamma(z)$ in [5].
(ii) That the operator defined in (2.13) is consequent upon a generalized different operator defined by Darus and Ibrahim [11].

2.8. **New Differential Operator Involving Modified Sigmoid Function.**

Definition 2.8. *Let $f_\gamma(z) \in T_\gamma$, then from (2.7) and (2.11) we obtain a generalized differential operator involving modified sigmoid function as follows:*

$$D^n(\varphi,\beta,t) f_\gamma(z) = \gamma^n(s)z + \sum_{k=2}^{\infty} \gamma^{n+1}(s)[1+(k+\beta-\varphi-1)t]^n a_k z^k, \quad z \in U, \tag{2.14}$$

for $0 \le \varphi \le \beta, \quad n \in \mathbb{N} \cup 0, t \ge 0$.

We note here that μ has been replaced by φ for convenience.

2.9. **New Differential Operator Involving Sigmoid Defined by Convolution.** For the purpose of defining our new differential operator of interest, the following definition is required:

Definition 2.9. ***(Hadamard product or convolution):*** *The Hadamard (or convolution) of two analytic functions $f(z)$ given by (1.1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is given by*

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in U. \tag{2.15}$$

Following (2.15) for (2.10) and (2.15), a certain new differential operator involving sigmoid function defined by convolution is defined as follows:

$$\begin{aligned} D^n_{\lambda,\omega}(\varphi,\beta,t) f_\gamma(z) &= (D^n_{\lambda,\omega} f_\gamma(z)) * (D^n(\varphi,\beta,t) f_\gamma(z)) \\ &= \gamma^n(s)z + \sum_{k=2}^{\infty} \gamma^{n+1}(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n a_k z^k \end{aligned} \tag{2.16}$$

2.10. **Linear Combination of the New Differential Operator Involving Sigmoid defined by Convolution and Ruscheweyh Operator involving modified sigmoid function.** Following (2.15), we combined equations (2.11) and (2.16) above to obtain a certain operator define as follows:

$$E_{\mu}^{n}(\beta,\lambda,\omega,\varphi,t)f_{\gamma}(z)$$
$$= \mu D_{\lambda,\omega}^{n}(\varphi,\beta,t)f_{\gamma}(z)+(1-\mu)R^{n}f_{\gamma}(z)$$
$$= [\mu\gamma^{n}(s)-\mu+1]z-\sum_{k=2}^{\infty}\gamma(s)\{\mu\gamma^{n}(s)[1+(k+\beta-\varphi-1)t]^{n}[(k-1)(\lambda-\omega)+k]^{n}$$
$$+(1-\mu)B_{k}(n)\}a_{k}z^{k} \tag{2.17}$$

Remark 4 : (i) For $n=0,\mu=1$ in (2.17) we have,

$$f_{\gamma}(z)=z-\sum_{k=2}^{\infty}\gamma(s)a_{k}z^{k}, a_{k}\geq 0,$$

defined by (1.4).
(ii) For $t=0,\mu=1$ in (2.17) we have,

$$D^{n}(\varphi,\beta,t)f_{\gamma}(z)=\gamma^{n}(s)z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)[1+(k+\beta-\varphi-1)t]^{n}a_{k}z^{k}$$

defined by (2.16).
(iii) For $\mu=0$ in (2.17) we have,

$$R^{n}f_{\gamma}(z)=z-\sum_{k=2}^{\infty}\gamma(s)C_{n+k-1}^{n}a_{k}z^{k},\quad a_{k}\leq 0,\quad z\in U$$

defined by (2.11).
(1v) For $t=0$ in (2.17) we have,

$$\Phi_{\lambda,\omega}^{n}f_{\gamma}(z)=[\mu\gamma^{n}(s)-\mu+1]z-\sum_{k=2}^{\infty}\gamma(s)\{\mu\gamma^{n}(s)[(k-1)(\lambda-\omega)]^{n}+(1-\mu)B_{k}(n)\}a_{k}z^{k},$$

as defined in (2.13) and corrects the one defined in [5].
(v) For $s=0,\mu=1,t=\delta,\beta=\varphi=0,\lambda=1$ and $\omega=2$ in (2.17) we have,

$$D_{\delta}^{n}f(z)=z-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n}a_{k}z^{k}, z\in U$$

which is Al-Oboudi differential operator for function $f\in T$ of the form (1.3).
(vi) For $s=0,\mu=1,t=0,\lambda=\omega=0$ in (2.17) we have,

$$D^{n}f(z)=z-\sum_{k=2}^{\infty}k^{n}a_{k}z^{k},\quad z\in U,$$

which is Salagean differential operator for functions $f\in T$.

In the field of geometric function theory, various subclasses of the normalized analytic functions which are univalent have been studied from different viewpoints. Many authors such as [[3],[4],[5],[10],[11],[12],[13],[14],[15],[16][17],[18],[19],[20],[21],[22]] have successfully defined and studied various subclasses of univalent functions. Particularly, Joshi and Sangle[13] introduced and investigated subclass $T_{\gamma}D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;n)$

of univalent functions by using Al-Oboudi operator as a generalized Salagean differential operator in the unit disk U. This was followed by the work of oluwayemi and Fadipe-Joseph[5] wherein they introduced and investigated subclass $T_\gamma D_{\lambda,\omega} f_\gamma(\alpha,\beta,\xi,\mu;p:n)$ by using the generalized differential operator $\Phi^n_{\lambda,\omega} f_\gamma(z)$. The motivation for this present work are the works of both Joshi and Sangle and Oluwayemi and Fadipe-Joseph. In particular, we introduce and investigate the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$ as a subclass of univalent functions by using the generalized differential operator $E^n_\mu(\beta,\lambda,\omega,\varphi,t)f_\gamma(z)$.

2.11. The class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$.

Definition 2.10. $\{T_\gamma D_{\lambda,\omega}(\alpha,\beta,\omega,\varphi,\lambda,\eta,\xi,t;p:n):\}$ *A function $f_\gamma(z)\in T_\gamma$ defined by* (1.4) *is in the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$ if*

$$\left|\frac{[E^n_\mu(\beta,\lambda,\omega,\varphi,t)f_\gamma(z)]' - [\mu\gamma^n(s)-\mu+1]}{p\xi[(E^n_\mu(\beta,\lambda,\omega,\varphi,t)f_\gamma(z))'] - [(E^n_\mu(\beta,\lambda,\omega,\varphi,t)f_\gamma(z))' - [\mu\gamma^n(s)-\mu+1]]}\right| < \eta,$$

where $\alpha\in[0,\frac{1}{2}), \eta\in(0,1), \frac{1}{2}\le\xi\le1, \mu\in[0,1], 0\ge\varphi\leqq\beta, n\in\mathbb{N}\cup0$, $n,t\ge0, p\ge2$ and $z\in U$.

3. Main Results

In this section we find the coefficient estimates for the functions in the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$. Our main characterization theorem for this function class is stated as Theorem 3.1 below.

3.1. Coefficient Estimates for class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$.

Theorem 3.1. *If a function $f_\gamma(z)$ belongs to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$,*
then

$$\sum_{k=2}^{\infty} k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n$$
$$+(1-\mu)B_k(n)\}a_k z^k$$
$$\le p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]$$

Proof. Suppose $f_\gamma(z)\in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$, by equation (2.17) and definition 2.10, we have that

$$\Bigg|-\sum_{k=2}^{\infty} k\gamma(s)\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n$$
$$+(1-\mu)B_k(n)\}a_k z^{k-1}\Bigg|$$
$$\le\eta\Bigg|-\sum_{k=2}^{\infty} p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]k\gamma(s)[1+\eta(1-p\xi)]\{\mu\gamma^n(s)$$
$$[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}a_k z^{k-1}\Bigg|$$

$|z| \leq r$ and as $r \to 1^+$, then

$$\sum_{k=2}^{\infty} k\gamma(s)\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}a_k$$
$$\leq \eta p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]+\sum_{k=2}^{\infty}\eta k\gamma(s)(1-p\xi)\{\mu\gamma^n(s)$$
$$[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}a_k$$
$$\Rightarrow \sum_{k=2}^{\infty} k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n$$
$$+(1-\mu)B_k(n)\}a_k$$
$$\leq p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha] \tag{3.1}$$

Hence,

$$\sum_{k=2}^{\infty} a_k \leq \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}} \tag{3.2}$$

The result is sharp for

$$f(z)=z-\frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}}z^k$$

□

Corollary 3.2. *Let a function $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,0,\lambda,\eta,\xi,t;p:n)$, then*

$$\sum_{k=2}^{\infty} k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}a_k z^k$$
$$\leq p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha],$$

which is the correct form of Theorem 3.1 in [5] *when $\eta=\beta$.*

Corollary 3.3. *Let $s=0$, then we have that a function $f_\gamma(z)$ belongs to the class $T_1D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$, then*

$$\sum_{k=2}^{\infty} k[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n$$
$$+(1-\mu)B_k(n)\}a_k z^k$$
$$\leq p\xi\eta(1-\alpha),$$

Corollary 3.4. *If $t=0$, in corollary 3.3,then we have the following:*
Let a function $f_\gamma(z)$ belongs to the class $T_1D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,0;p:n)$, then

$$\sum_{k=2}^{\infty} k[1+\eta(p\xi-1)]\{\mu[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}a_k z^k$$
$$\leq p\xi\eta(1-\alpha),$$

which is Corollary 3.2 in [5] *when* $\eta = \beta$.

Corollary 3.5. *If* $\mu = 1$, *in corollary 3.4,then we have the following: Let a function* $f_\gamma(z)$ *belongs to the class* $T_1 D_{\lambda,\omega}(\alpha, \beta, 1, \omega, \varphi, \lambda, \eta, \xi, 0; p : n)$, *then*

$$\sum_{k=2}^{\infty} k[1 + \eta(p\xi - 1)]\{[(k-1)(\lambda - \omega) + k]^n + (1-\mu)B_k(n)\}a_k z^k \leq p\xi\eta(1-\alpha),$$

which is Corollary 3.3 in [5] *when* $\eta = \beta$.

4. Growth and Distortion Theorems for the class $T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$

Theorem 4.1. *If a function* $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, *then for* $|z| \leq r < 1$, *we have*

$$r - \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1-\mu)B_2(n)\}} r^2 \leq |f_\gamma(z)|$$

$$r + \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1-\mu)B_2(n)\}} r^2$$

and

$$1 - \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1-\mu)B_2(n)\}} r \leq |f_\gamma(z)|$$

$$1 + \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1-\mu)B_2(n)\}} r$$

Proof. Since $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, Theorem 3.1 readily yields the inequality

$$\sum_{k=2}^{\infty} a_k \leq \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1-\mu)B_2(n)\}} \tag{4.1}$$

Thus, for $|z| = r < 1$, and by making use of (4.2) we have

$$|f_\gamma(z)| \leq |z| + \sum_{k=2}^{\infty} \gamma(s)a_k|z^k| \leq r + \gamma(s)r^2 \sum_{k=2}^{\infty} a_k$$
$$\leq r + \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1-\mu)B_2(n)\}} r^2$$

and

$$|f_\gamma(z)| \geq |z| - \sum_{k=2}^{\infty}\gamma(s)a_k|z^k| \geq r - \gamma(s)r^2\sum_{k=2}^{\infty}a_k$$

$$\geq r - \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{2[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\}}r^2$$

Also from Theorem 3.1, it follows that

$$\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(\lambda-\omega)+2]^n$$
$$+(1-\mu)B_2(n)\}\sum_{k=2}^{\infty}ka_k$$
$$\sum_{k=2}^{\infty}k\gamma(s)[1+(k+\beta-\varphi-1)t]^n[(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\}a_k$$
$$p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha].$$

Hence,

$$|f^{'}(z)| \leq 1 + \sum_{k=2}^{\infty}\gamma(s)ka_k|z^k| \leq 1+\gamma(s)r\sum_{k=2}^{\infty}a_k$$

$$\leq 1 + \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\}}r$$

and

$$|f^{'}(z)| \geq 1 - \sum_{k=2}^{\infty}\gamma(s)ka_k|z^k| \geq 1-\gamma(s)r\sum_{k=2}^{\infty}a_k$$

$$\geq 1 - \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\}}r.$$

This completes the proof of Theorem 4.1. □

4.1. **Closure Theorem.**

Theorem 4.2. *If a function* $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$. *Let* $f_1(z) = z$ *and*

$$f_\gamma(z) = z - \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\}}z^k,$$

$k \geq 2$.

Then the function $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$ *if and only if it can be expressed in the form*

$$f_\gamma(z) = \sum_{k=2}^{\infty}\mu_k f_k(z), \tag{4.2}$$

where $\mu_k \geq 0$ *and* $\sum_{k=1}^{\infty}\mu_k = 1$.

Proof. Let $f_\gamma(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \mu_k \geq 0, k = 1, 2, \cdots$, and $\sum_{k=1}^{\infty} \mu_k = 1$.
Thus

$$f_\gamma(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z).$$

Therefore,

$$\begin{aligned} f_\gamma(z) &= \mu_1 f_1(z) \\ &+ \sum_{k=2}^{\infty} \mu_k \{z - \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\}}\} z^k \\ &= (\mu_1 + \mu_2 + \mu_3 + \cdots)z \\ &- \sum_{k=2}^{\infty} \mu_k \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\}} z^k \end{aligned}$$

$$\mu_1(z) + \mu_2(z) + \mu_3(z) + \cdots = \mu_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z)$$

where $\mu_1 + \mu_1 + \mu_1 + \cdots = \sum_{k=1}^{\infty} \mu_k = 1$. Then

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \mu_k \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\}}.$$

It follows that

$$\begin{aligned} &\sum_{k=2}^{\infty} \mu_k \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\}} \\ &\times \frac{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\}}{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]} \\ &\sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1. \end{aligned}$$

In other words,

$$f_\gamma(z) = \mu_1 + \sum_{k=2}^{\infty} \mu_k = 1 \Rightarrow 1 - \mu_1 \leq 1.$$

By Theorem 3.1 therefore,
$f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$.
Conversely, if $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, then by Theorem 3.1,

$$a_k \leq \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\}}$$

By setting

$$\mu_k \le \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\}}a_k$$

and

$$\mu_1 = 1-\sum_{k=2}^{\infty}\mu_k.$$

So that

$$\mu_k = \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)}{[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]k\gamma(s).}$$

Consequently, f_k can be expressed in the form (5.1). Hence , $f_\gamma(z)=\sum_{k=1}^{\infty}\mu_k f_k(z)$. Thus the proof is complete. □

5. Fekete-Szego inequality for the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$

In this section, Fekete-Szego inequality for functions $f_\gamma(z)$ belonging to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$ was established.

Theorem 5.1. *If a function $f_\gamma(z)$ belongs to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n)$, and $\Delta\in\mathbb{N}$. Then*

$$\left|a_3-\Delta a_2^2\right| \le \left|\frac{AB^2-\Delta A^2C}{CB^2}\right|.$$

Proof. From (**??**),

$$a_k \le \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n [(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}},\quad a_k\ge 2. \tag{5.1}$$

From (5.1),

$$a_2 = \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{2\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(1+\beta-\varphi)t]^n [(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\}},\quad (k=2),$$

and

$$a_3 = \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{3\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(2+\beta-\varphi)t]^n [2(\lambda-\omega)+3]^n+(1-\mu)B_3(n)\}},\quad (k=2).$$

So that

$$a_3-\Delta a_2^2 = \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{3\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(2+\beta-\varphi)t]^n [2(\lambda-\omega)+3]^n+(1-\mu)B_3(n)\}}$$
$$-\Delta\left\{\frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{2\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(1+\beta-\varphi)t]^n [(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\}}\right\}^2$$

Such that

$$A = p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]$$

$$B = \frac{3\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(2+\beta-\varphi)t]^n}{[2(\lambda-\omega)+3]^n+(1-\mu)B_3(n)\}}$$

$$C = \frac{2\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(1+\beta-\varphi)t]^n}{[(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\}}.$$

Therefore,

$$\left|a_3 - \Delta a_2^2\right| \leq \left|\frac{AB^2 - \Delta A^2 C}{CB^2}\right|.$$

□

Let $t = 0$ in Theorem 5.1 we have the following:

Corollary 5.2. *If a function $f_\gamma(z)$ belongs to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,0;p:n)$, and $\varphi \in \mathbb{N}$. Then*

$$\left|a_3 - \Delta a_2^2\right| \leq \left|\frac{AB_1^2 - \Delta A^2 C_1}{C_1 B_1^2}\right| = \left|\frac{R\Omega_2^2 - \varphi R^2 \Omega_1}{\Omega_1 \Omega_2^2}\right|.$$

For

$$A = p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha] = R$$

$$B_1 = \frac{3\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)}{[2(\lambda-\omega)+3]^n+(1-\mu)B_3(n)\}} = \Omega_2$$

$$C_1 = \frac{2\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(1+\beta-\varphi)t]^n}{[(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\}} = \Omega_1.$$

Remark* 5*:

$$\left|a_3 - \Delta a_2^2\right| \leq \left|\frac{R\Omega_2^2 - \varphi R^2 \Omega_1}{\Omega_1 \Omega_2^2}\right|$$

where $\Delta = \varphi$ is the result in [5] that is due to Oluwayemi and Fadipe-Joseph.

6. Conclusion

This work is a generalization of some earlier well-known (defined) differential operators, some of which were illustrated in this work. Particularly in this work, we studied some geometrical properties of functions in the class

$$T_\gamma D_{\lambda,\omega}(\alpha,\beta,\mu,\omega,\varphi,\lambda,\eta,\xi,t;p:n),$$

and when $t = 0$ we obtained the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ studied in [5]. Furthermore, by suitably specializing the parameters involved, we obtained some of the results in [5] as special cases of our own results. Finally, by suitably varying the parameters involved in the results obtained in this new work, one is guaranteed of some other existing results and presumably new ones.

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