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Projective Curvature Tensor on $N(\kappa)-$Contact Metric Manifold Admitting Semi-Symmetric Non-Metric Connection

Mustafa Altın

Technical Sciences Vocational School Bingol University, Bingol, Turkey

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Abstract

The object of the present paper is to classify $N(\kappa)$-contact metric manifolds admitting the semi-symmetric non-metric connection with certain curvature conditions the projectively curvature tensor. We studied projective flat, $\xi-$projectively flat, $\phi-$projectively flat $N(\kappa)$-contact metric manifolds admitting the semi-symmetric non-metric connection. Also, we examine such manifolds under some local symmetry conditions related to projective curvature tensor.

1. Introduction

An almost contact metric manifold is a $(2n+1)-$dimensional differentiable manifold with a structure (ϕ,ξ,η,g) such as

$$\phi^2(W_1)=-W_1+\eta(W_1)\xi,\eta(\xi)=1,\quad \phi(\xi)=0,\quad \eta(\phi(W_1))=0,g(\phi(W_1),\phi(W_2))=g(W_1,W_2)-\eta((W_1))\eta((W_2)) \tag{1.1}$$

for any vector fields $W_1,W_2\in\chi(M)$, where g is Riemannian metric, ϕ is a $(1,1)-$tensor field, ξ is a vector field and η is a $1-$form on M [1]. Blair, et al. [2] introduced the (κ,μ)-nullity distribution of an almost contact metric manifold M that is defined by

$$N(\kappa,\mu):p\longrightarrow N_p(\kappa,\mu)$$
$$N_p(\kappa,\mu)=\left\{W_3\in\Gamma(T_pM):R(W_1,W_2)W_3=(\kappa I+\mu h)\left[g(W_2,W_3)W_1-g(W_1,W_3)W_2\right]\right\}$$

for all $W_1,W_2\in\Gamma(TM)$, where κ and μ are real constants and $p\in M$. If $\xi\in N(\kappa,\mu)$, then M is called $(\kappa,\mu)-$contact metric manifold. If $\mu=0$, the (κ,μ)-nullity distribution reduces to κ-nullity distribution.

The idea of κ-nullity distribution on a contact metric manifold was firstly presented by Tanno in 1988 [3]. κ-nullity distribution of an almost contact metric manifold (M,ϕ,ξ,η,g) is a distribution defined as

$$N(\kappa):p\longrightarrow N_p(\kappa)=\left\{W_3\in\Gamma(T_pM):R(W_1,W_2)W_3=\kappa\left[g(W_2,W_3)W_1-g(W_1,W_3)W_2\right]\right\}$$

for any $W_1,\ W_2\in\Gamma(T_pM)$ and $\kappa\in\mathbb{R}$, where R is the Riemannian curvature tensor of M. If ξ belongs to $\kappa-$nullity distribution then M is called $N(\kappa)-$contact metric manifold. Thus on a $N(\kappa)$ contact metric manifold, we have

$$R(W_1,W_2)\xi=\kappa\left[\eta(W_2)W_1-\eta(W_1)W_2\right].$$

Email address and ORCID number: **maltin@bingol.edu.tr, https://orcid.org/0000-0001-5544-5910**

A $N(\kappa)$-contact metric manifold is Sasakian if and only if $k = 1$. Also, if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [4]. The Riemannian geometry of $N(\kappa)-$contact metric manifolds is studied in [2], [5]-[9]. Levi-Civita connection ∇ is a torsion free, i.e has zero torsion, and a metric connection, i.e $\nabla g = 0$. There are some kinds of linear connections except for Levi-Civita connection which is not need to be torsion free or metric. One of them is semi-symmetric non-metric connection [10]. Manifolds with semi-symmetric non-metric connection have been studied by many researchers [11]-[15]. In the Riemannian geometry of contact manifolds curvature tensors-such as conformal, concircular, projective curvature tensor etc.-have important applications. Some of geometric properties of structure on manifolds have been examined by the certain conditions on these curvature tensors. Many works on contact manifolds are stated in [16]-[23].

In this paper we study projective curvature tensor on $N(\kappa)$-contact metric manifolds with semi-symmetric non metric connection. In [24], Barman gave the curvature relations on such as manifolds. We use these properties and we examine flatness conditions of projective curvature tensor. Specifically, we given results for ξ-projectively flat, pseudo-quasi-projectively flat and $\phi-$projectively flat on $N(\kappa)$-contact metric manifolds with semi-symmetric non metric connection. After we investigate $\phi-$projectively semi-symmetric on $N(\kappa)$-contact manifolds admitting the semi-symmetric non-metric connection, we characterize this manifolds satisfying $\overset{\star}{Q}.\overset{\star}{P} = 0$ and $\overset{\star}{S}.\overset{\star}{P} = 0$, where $\overset{\star}{P}, \overset{\star}{Q}, \overset{\star}{Ric}$ are projective curvature tensor, Ricci tensor, Ricci curvature tensor, with a semi-symmetric non metric connection, respectively.

2. Preliminaries

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold. The $h = \frac{1}{2}\mathscr{L}_\xi \phi$, $\mathscr{L}_\xi$ denotes the Lie derivative along vector field ξ. For any $W_1 \in \Gamma(TM)$, we have

$$\nabla_{W_1}\xi = -\phi W_1 - \phi h W_1$$

An almost contact metric manifold M is called $K-$contact if ξ is Killing vector field. M is called normal contact metric manifold if $N_\phi + 2d\eta \otimes \xi = 0$, where N_ϕ is the Nijenhuis tensor of ϕ. A normal contact metric manifold is called Sasakian. On the other hand a contact metric manifold is Sasakian if and only if

$$R(W_1, W_2)\xi = [\eta(W_2)W_1 - \eta(W_1)W_2]$$

for all $W_1, W_2 \in \Gamma(TM)$. On a $K-$contact and Sasakian manifold $h = 0$.

A $N(\kappa)$-contact metric manifold is Sasakian if $\kappa = 1$. $N(\kappa)$-contact metric manifolds are characterized the different values of κ. As we mentioned in the introduction when $\kappa = 0$ then the manifold M is locally isometric to $E^{(n+1)}(0) \times S^{(n)}(4)$. On a $N(\kappa)$-contact metric manifold M^{2n+1}, we have following relations (for details see [1]):

$$(\nabla_{W_1}\phi)W_2 = g(W_1 + hW_1, W_2)\xi - \eta(W_2)(W_1 + hW_1),$$
$$(\nabla_{W_1}\eta)W_2 = g(W_1 + hW_1, \phi W_2).$$

The Riemannian curvature R of a $N(\kappa)-$contact metric manifold has following properties:

$$R(W_1, W_2)\xi = \kappa[\eta(W_2)W_1 - \eta(W_1)W_2], \tag{2.1}$$
$$R(\xi, W_1)W_2 = \kappa[g(W_1, W_2)\xi - \eta(W_2)W_1] \tag{2.2}$$

for all $W_1, W_2 \in \Gamma(TM)$. On the other hand the Ricci curvature of M is stated as [1];

$$Ric(W_1, W_2) = 2(n-1)g(W_1, W_2) + 2(n-1)g(hW_1, W_2) + 2(n\kappa - (n-1))\eta(W_1)\eta(W_2) \tag{2.3}$$
$$Ric(\phi W_1, \phi W_2) = Ric(W_1, W_2) - 2n\kappa\eta(W_1)\eta(W_2) - 4(n-1)g(hW_1, W_2) \tag{2.4}$$
$$Ric(W_1, \xi) = 2\kappa n\eta(W_1) \tag{2.5}$$

and the scalar curvature is given by

$$r = 2n(2n + \kappa - 2).$$

Example 2.1. *E. Boeckx [25] gave a classification for non-Sasakian $(\kappa, \mu)-$spaces. The number $I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-k}}$ is called by Boeckx invariant. D.E. Blair, et al. [26] gave an example of $N(\kappa)-$contact metric manifolds by using Boeckx invariant. They constructed $(2n+1)$-dimensional $N(1-\frac{1}{n})$-contact metric manifold, $n > 1$. For details see [26].*

Let define a map $\overset{\star}{\nabla}$on a Riemann manifold M as

$$\overset{\star}{\nabla}_{W_1}W_2 = \nabla_{W_1}W_2 + \eta(W_2)W_1$$

where ∇ is Levi-Civita connection on M. This map is a linear connection. The torsion of $\overset{\star}{\nabla}$ is given by

$$\overset{\star}{T}(W_1,W_2)=\eta(W_2)W_1-\eta(W_1)W_2$$

for all $W_1,W_2\in\Gamma(TM)$. Also we have

$$(\overset{\star}{\nabla}_U g)(W_1,W_2)=-\eta(W_1)g(W_2,U)-\eta(W_2)g(W_1,U)\neq 0.$$

Thus $\overset{\star}{\nabla}$ is not symmetric and not metric connection. This type of connection is called by semi-symmetric non-metric connection [10].

$N(\kappa)-$ contact metric manifolds with a semi-symmetric non-metric connection were studied by Barman [24]. For the sake of brevity we denote $(M,\overset{\star}{\nabla})$ by a $N(\kappa)-$contact metric manifolds with a semi-symmetric non-metric connection. Barman gave the curvature of $(M,\overset{\star}{\nabla})$ as follow:

$$\begin{aligned}\overset{\star}{R}(W_1,W_2)W_3=&R(W_1,W_2)W_3+g(W_1,\phi W_3)W_2+g(hW_1,\phi W_3)W_2-\eta(W_1)\eta(W_3)W_2-g(W_2,\phi W_3)W_1\\&-g(hW_2,\phi W_3)W_1+\eta(W_3)\eta(W_2)W_1.\end{aligned}\tag{2.6}$$

Thus, we have following curvature properties [24]:

$$\overset{\star}{R}(\xi,W_2)W_3=\kappa g(W_2,W_3)\xi-(\kappa+1)\eta(W_3)W_2-g(W_2,\phi W_3)\xi-g(hW_2,\phi W_3)\xi+\eta(W_3)\eta(W_2)\xi \tag{2.7}$$

$$\overset{\star}{R}(\xi,W_2)\xi=(\kappa+1)(\eta(W_2)\xi-W_2)$$

$$\overset{\star}{R}(W_1,W_2)\xi=(\kappa+1)(\eta(W_2)W_1-\eta(W_1)W_2).$$

The Ricci curvature of a $(M,\overset{\star}{\nabla})$ is given by

$$\overset{\star}{Ric}(W_2,W_3)=Ric(W_2,W_3)-2ng(W_2,\phi W_3)-2ng(hW_2,\phi W_3)+2n\eta(W_3)\eta(W_2).\tag{2.8}$$

Thus, we have

$$\overset{\star}{Ric}(W_2,\xi)=2n(\kappa+1)\eta(W_2)\tag{2.9}$$

$$\overset{\star}{r}=2n+r$$

where $\overset{\star}{Ric}$, $\overset{\star}{R}$ and $\overset{\star}{r}$ are the Ricci tensor, the Riemann curvature tensor and scalar curvature admitting the semi-symmetric non-metric connection respectively [24].

The projective curvature tensor P admitting the semi-symmetric non-metric connection is defined by

$$\overset{\star}{P}(W_1,W_2)W_3=\overset{\star}{R}(W_1,W_2)W_3-\frac{1}{2n}\left(\overset{\star}{Ric}(W_2,W_3)W_1-\overset{\star}{Ric}(W_1,W_3)W_2\right),\tag{2.10}$$

for all $W_1,W_2,W_3\in TM$.

3. Flatness conditions of projective curvature tensor on $(M,\overset{\star}{\nabla})$

In this section, we examine that a $(M,\overset{\star}{\nabla})$ is ξ-projectively flat, pseudo-quasi-projectively flat and $\phi-$projectively flat.

Definition 3.1. *A $(M,\overset{\star}{\nabla})$ is called*

- *ξ-projectively flat if we have $\overset{\star}{P}(W_1,W_2)\xi=0$ for all $W_1,W_2\in\Gamma(TM)$,*
- *pseudo-quasi-projectively flat if we have $g(\overset{\star}{P}(\phi W_1,W_2)W_3,\phi W_4)=0$ for all $W_1,W_2,W_3\in\Gamma(TM)$,*
- *ϕ-projectively flat if we have $g(\overset{\star}{P}(\phi W_1,\phi W_2)\phi W_3,\phi W_4)=0$ for all $W_1,W_2,W_3\in\Gamma(TM)$.*

Theorem 3.2. *A $(M,\overset{\star}{\nabla})$ is always ξ-projectively flat.*

Proof. By putting $W_3=\xi$ in (2.10), we obtain

$$\overset{\star}{P}(W_1,W_2)\xi=\overset{\star}{R}(W_1,W_2)\xi-\frac{1}{2n}\left(\overset{\star}{Ric}(W_2,\xi)W_1-\overset{\star}{Ric}(W_1,\xi)W_2\right).$$

Also from (2.7) and (2.9), we get

$$\overset{\star}{P}(W_1,W_2)\xi=R(W_1,W_2)\xi-\eta(W_1)W_2+\eta(W_2)W_1-\frac{1}{2n}\left(2n(k+1)\eta(W_2)W_1-2n(k+1)\eta(W_1)W_2\right).$$

and take into account (2.1), we have

$$\overset{\star}{P}(W_1,W_2)\xi=0 \tag{3.1}$$

for all $W_1,W_2\in\Gamma(TM)$. □

Theorem 3.3. *If $(M,\overset{\star}{\nabla})$ is pseudo-quasi-projectively flat, then M is an Einstein manifold admitting Levi-Civita connection.*

Proof. Using (2.10), we have

$$g(\overset{\star}{P}(\phi W_1,W_2)W_3,\phi W_4)=\overset{\star}{R}(\phi W_1,W_2,W_3,\phi W_4)-\frac{1}{2n}[\overset{\star}{Ric}(W_2,W_3)g(\phi W_1,\phi W_4)-\overset{\star}{Ric}(\phi W_1,W_3)g(W_2,\phi W_4)]. \tag{3.2}$$

Let $(M,\overset{\star}{\nabla})$ be a pseudo-quasi-projectively flat. Then, by using (2.8) in (3.2) , it follows that

$$\begin{aligned}\overset{\star}{R}(\phi W_1,W_2,W_3,\phi W_4)=\frac{1}{2n}[&(Ric(W_2,W_3)-2ng(W_2,\phi W_3)-2ng(hW_2,\phi W_3)+2n\eta(W_2)\eta(W_3))g(\phi W_1,\phi W_4)\\&-(Ric(\phi W_1,W_3)-2ng(\phi W_1,\phi W_3)-2ng(h\phi W_1,\phi W_3))\,g(W_2,\phi W_4)]\end{aligned}$$

and from (2.6) we get

$$R(\phi W_1,W_2,W_3,\phi W_4)=\frac{1}{2n}\left(Ric(W_2,W_3)g(\phi W_1,\phi W_4)-Ric(\phi W_1,W_3)g(W_2,\phi W_4)\right). \tag{3.3}$$

Take a local orthonormal basis set of M as $\{e_1,e_2,...,e_{2n},\xi\}$, then $\{\phi e_1,\phi e_2,...,\phi e_{2n},\xi\}$ is also a local orthonormal basis. Putting $W_1=W_4=e_i$ in (3.3) and summing over $i=1$ to $2n$, we get

$$\sum_{i=1}^{2n}R(\phi e_i,W_2,W_3,\phi e_i)=\frac{1}{2n}\left[\sum_{i=1}^{2n}\left(Ric(W_2,W_3)g(\phi e_i,\phi e_i)-Ric(\phi e_i,W_3)g(W_2,\phi e_i)\right)\right].$$

From (2.2) and (2.5), we obtain

$$Ric(W_2,W_3)=2n\kappa g(W_2,W_3).$$

□

Theorem 3.4. *Let a $(M,\overset{\star}{\nabla})$ be ϕ-projectively flat. If ξ is Killing vector field, then the manifold is an Einstein manifold.*

Proof. Firstly, putting $W_2=\phi W_2$ and $W_3=\phi W_3$ in (3.2), we get

$$g(\overset{\star}{P}(\phi W_1,\phi W_2)\phi W_3,\phi W_4)=\overset{\star}{R}(\phi W_1,\phi W_2,\phi W_3,\phi W_4)-\frac{1}{2n}\left(\overset{\star}{Ric}(\phi W_2,\phi W_3)g(\phi W_1,\phi W_4)-\overset{\star}{Ric}(\phi W_1,\phi W_3)g(\phi W_2,\phi W_4)\right). \tag{3.4}$$

Now, by using (2.8) in (3.4) and from definition of ϕ-projectively flat, it follows that

$$\begin{aligned}\overset{\star}{R}(\phi W_1,W_2,W_3,\phi W_4)=\frac{1}{2n}[&\left(Ric(\phi W_2,\phi W_3)-2ng(\phi W_2,\phi^2W_3)-2ng(h\phi W_2,\phi^2W_3)\right)g(\phi W_1,\phi W_4)\\&-\left(Ric(\phi W_1,\phi W_3)-2ng(\phi W_1,\phi^2W_3)-2ng(h\phi W_1,\phi^2W_3)\right)g(\phi W_2,\phi W_4)]\end{aligned}$$

and from (1.1), we get

$$R(\phi W_1,\phi W_2,\phi W_3,\phi W_4)=\frac{1}{2n}\left(Ric(\phi W_2,\phi W_3)g(\phi W_1,\phi W_4)\ -Ric(\phi W_1,\phi W_3)g(\phi W_2,\phi W_4)\right). \tag{3.5}$$

For local orthonormal basis $\{e_1,e_2,...,e_{2n},\phi e_1,\phi e_2,...,\phi e_{2n},\xi\}$ of M by putting $W_1=W_4=e_i$ in (3.5) and summing over $i=1$ to $2n$, we get

$$\sum_{i=1}^{2n}R(\phi e_i,\phi W_2,\phi W_3,\phi e_i)=\frac{1}{2n}\left[\sum_{i=1}^{2n}\left(Ric(\phi W_2,\phi W_3)g(\phi e_i,\phi e_i)-Ric(\phi e_i,\phi W_3)g(\phi W_2,\phi e_i)\right)\right]$$

From (2.2) and (2.5), we obtain

$$Ric(\phi W_2,\phi W_3)=2n\kappa g(\phi W_2,\phi W_3).$$

Also, from (2.4) we have

$$Ric(W_2,W_3)=2n\kappa g(W_2,W_3)+4(n-1)g(hW_2,W_3)$$

If ξ is Killing vector field then M is an Einstein manifold. □

4. Symmetry conditions admitting projective curvature tensor on $(M,\overset{\star}{\nabla})$

In this section, we study on a $(M,\overset{\star}{\nabla})$ under certain symmetry conditions. We firstly examine $\phi-$projectively semi-symmetric $(M,\overset{\star}{\nabla})$ and then we characterize this manifolds satisfying $\overset{\star}{Q}.\overset{\star}{P}=0$ and $\overset{\star}{Ric}.\overset{\star}{P}=0$, where $\overset{\star}{Q}$ is the Ricci operator defined by $\overset{\star}{Ric}(W_1,W_2)=g(\overset{\star}{Q}W_1,W_2)$.

Definition 4.1. *A $(M,\overset{\star}{\nabla})$ is said to be $\phi-$projectively semisymmetric if $\overset{\star}{P}(W_1,W_2)\phi=0$ for all $W_1,W_2\in\Gamma(M)$.*

Theorem 4.2. *A $\phi-$projectively $(M,\overset{\star}{\nabla})$ is isometric to Example 2.1.*

Proof. Suppose $(M,\overset{\star}{\nabla})$ be a $\phi-$projectively . Then, we have

$$\overset{\star}{P}(W_1,W_2)\phi W_3-\phi(\overset{\star}{P}(W_1,W_2)W_3)=0. \tag{4.1}$$

From (2.10), it follows that

$$\overset{\star}{P}(W_1,W_2)\phi W_3=\overset{\star}{R}(W_1,W_2)\phi W_3-\frac{1}{2n}\left(\overset{\star}{Ric}(W_2,\phi W_3)W_1-\overset{\star}{Ric}(W_1,\phi W_3)W_2\right). \tag{4.2}$$

Using (2.8) in (4.2), we obtain

$$\begin{aligned}\overset{\star}{P}(W_1,W_2)\phi W_3=&\overset{\star}{R}(W_1,W_2)\phi W_3-\frac{1}{2n}\left\{Ric(W_2,\phi W_3)W_1-2ng(W_2,\phi^2W_3)W_1-2ng(hW_2,\phi^2W_3)W_1\right\}\\&+\frac{1}{2n}\left\{Ric(W_1,\phi W_3)W_2-2ng(W_1,\phi^2W_3)W_2-2ng(hW_1,\phi^2W_3)W_2\right\}.\end{aligned}$$

From (2.1), (2.2) and (2.6), we have

$$\overset{\star}{P}(W_1,W_2)\phi W_3=\kappa g(W_2,\phi W_3)W_1-\kappa g(W_1,\phi W_3)W_2-\frac{1}{2n}Ric(W_2,\phi W_3)W_1+\frac{1}{2n}Ric(W_1,\phi W_3)W_2. \tag{4.3}$$

Also, by applying ϕ to $\overset{\star}{P}$, we get

$$\phi(\overset{\star}{P}(W_1,W_2)W_3)=\phi(\overset{\star}{R}(W_1,W_2)W_3)-\frac{1}{2n}\phi\left[\overset{\star}{Ric}(W_2,W_3)W_1-\overset{\star}{Ric}(W_1,W_3)W_2\right], \tag{4.4}$$

and using (2.8) in (4.4) yields

$$\begin{aligned}\phi(\overset{\star}{P}(W_1,W_2)W_3)=&\phi(\overset{\star}{R}(W_1,W_2)W_3)-\frac{1}{2n}\left\{Ric(W_2,W_3)-2ng(W_2,\phi W_3)-2ng(hW_2,\phi W_3)+2n\eta(W_2)\eta(W_3)\right\}\phi W_1\\&+\frac{1}{2n}\left\{Ric(W_1,W_3)-2ng(W_1,\phi W_3)-2ng(hW_1,\phi W_3)+2n\eta(W_1)\eta(W_3)\right\}\phi W_2\end{aligned}$$

Thus from (2.4) and (2.6), we have

$$\phi(\overset{\star}{P}(W_1,W_2)W_3)=\kappa g(W_2,W_3)\phi W_1-\kappa g(W_1,W_3)\phi W_2-\frac{1}{2n}Ric(W_2,W_3)\phi W_1+\frac{1}{2n}Ric(W_1,W_3)\phi W_2. \tag{4.5}$$

Putting (2.3), (4.3) and (4.5) in (4.1), we have

$$\begin{aligned}\overset{\star}{P}(W_1,W_2)\phi W_3-\phi(\overset{\star}{P}(W_1,W_2)W_3)=&\kappa g(W_2,\phi W_3)W_1-\kappa g(W_1,\phi W_3)W_2-\frac{2(n-1)}{2n}\left[g(W_2,\phi W_3)+g(hW_2,\phi W_3)\right]W_1\\&+\frac{2(n-1)}{2n}\left[g(W_1,\phi W_3)+g(hW_1,\phi W_3)\right]W_2-\kappa g(W_2,W_3)\phi W_1+\kappa g(W_1,W_3)\phi W_2\\&+\frac{1}{2n}[2(n-1)(g(W_2,W_3)+g(hW_2,W_3))+2(n\kappa-(n-1)\eta(W_2)\eta(W_3))]\phi W_1\\&-\frac{1}{2n}[2(n-1)(g(W_1,W_3)+g(hW_1,W_3))+2(n\kappa-(n-1)\eta(W_1)\eta(W_3))]\phi W_2\end{aligned} \tag{4.6}$$

Let take inner product with W_4 of (4.6) and then to contract W_2 and W_4, we obtain

$$\left\{2\kappa(1-n)+2(\frac{n^2-2n+1}{n})\right\}g(W_1,\phi W_3)+\{2(n-1)\}g(hW_1,\phi W_3)=0. \tag{4.7}$$

Now, putting $W_3 = \phi W_3$ in (4.7) and from (1.1), we get

$$\left\{2\kappa(1-n)+2(\frac{n^2-2n+1}{n})\right\}g(\phi W_1,\phi W_3)+\{2(n-1)\}g(hW_1,W_3)=0. \tag{4.8}$$

Taking trace in both sides of (4.8) and using $trh = 0$, we obtain

$$\kappa = \frac{n-1}{n}.$$

Thus M is isometric to Example 2.1. □

Theorem 4.3. *On a* $(M,\overset{\star}{\nabla})$*, we have* $\overset{\star}{Q}.\overset{\star}{P} = 0$.

Proof. For all $W_1, W_2, W_3 \in \Gamma(TM)$, we have

$$(\overset{\star}{Q}(W_1).\overset{\star}{P})(W_2,W_3) = \overset{\star}{Q}(\overset{\star}{P}(W_1,W_2)W_3) - \overset{\star}{P}(\overset{\star}{Q}W_1,W_2)W_3 - \overset{\star}{P}(W_1,\overset{\star}{Q}W_2)W_3 - \overset{\star}{P}(W_1,W_2)\overset{\star}{Q}W_3. \tag{4.9}$$

From (2.8) and (2.9), we have

$$\overset{\star}{Q}W_2 = 2(n-1)(W_2+hW_2)+2n(\phi W_2+\phi hW_2)+2(n\kappa+1)\eta(W_2)\xi \tag{4.10}$$

and, so

$$\overset{\star}{Q}\xi = 2n(\kappa+1)\xi. \tag{4.11}$$

Thus, for $W_3 = \xi$ in (4.9) we get

$$(\overset{\star}{Q}(W_1).\overset{\star}{P})(W_2,\xi) = \overset{\star}{Q}(\overset{\star}{P}(W_1,W_2)\xi) - \overset{\star}{P}(\overset{\star}{Q}W_1,W_2)\xi - \overset{\star}{P}(W_1,\overset{\star}{Q}W_2)\xi - \overset{\star}{P}(W_1,W_2)\overset{\star}{Q}\xi.$$

From (3.1), (4.10) and (4.11), it follows that

$$\overset{\star}{Q}.\overset{\star}{P} = 0.$$

□

Theorem 4.4. *A* $(M,\overset{\star}{\nabla})$ *satisfies* $\overset{\star}{P}.\overset{\star}{Ric} = 0$ *if and only if* M *is an Einstein manifold.*

Proof. Let $\overset{\star}{P}.\overset{\star}{Ric} = 0$ satisfies on $(M,\overset{\star}{\nabla})$, then we get

$$\overset{\star}{Ric}(\overset{\star}{P}(W_4,W_2)W_3,W_1)+\overset{\star}{Ric}(W_3,\overset{\star}{P}(W_4,W_2)W_1) = 0. \tag{4.12}$$

Putting $W_1 = W_4 = \xi$ in (4.12), we have

$$\overset{\star}{Ric}(\overset{\star}{P}(\xi,W_2)W_3,\xi)+\overset{\star}{Ric}(W_3,\overset{\star}{P}(\xi,W_2)\xi) = 0. \tag{4.13}$$

Also, from (2.10), we get

$$\overset{\star}{P}(\xi,W_2)W_3 = \overset{\star}{R}(\xi,W_2)W_3 - \frac{1}{2n}\left(\overset{\star}{Ric}(W_2,W_3)\xi - \overset{\star}{Ric}(\xi,W_3)W_2\right),$$

from (2.7), (2.8), (2.9), it follows that

$$\overset{\star}{P}(\xi,W_2)W_3 = \kappa g(W_2,W_3)\xi - \frac{1}{2n}Ric(W_2,W_3)\xi. \tag{4.14}$$

Again putting $W_3 = \xi$ in (4.14) and using (2.5), we obtain

$$\overset{\star}{P}(\xi,W_2)\xi = 0. \tag{4.15}$$

Using (2.9), (4.14) and (4.15) in (4.13), it follows that

$$Ric(W_2,W_3) = 2n\kappa g(W_2,W_3).$$

Conversely, let M be an Einstein manifold , i.e $Ric(W_2,W_3) = 2n\kappa g(W_2,W_3)$. Then, we get

$$\overset{\star}{P}(W_1,W_2)W_3 = \kappa(g(W_2,W_3)W_1 - g(W_1,W_3)W_2) - \frac{1}{2n}(2n\kappa g(W_2,W_3)W_1 - 2n\kappa g(W_1,W_3)W_2).$$

which implies $\overset{\star}{P}(W_1,W_2)W_3 = 0$. This also give us $\overset{\star}{P}.\overset{\star}{Ric} = 0$.

□

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On the Logarithmic Summability of Sequences in Intuitionistic Fuzzy Normed Spaces

Enes Yavuz

Department of Mathematics, Manisa Celal Bayar University, Manisa, Turkey

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Abstract

We introduce logarithmic summability in intuitionistic fuzzy normed spaces($IFNS$) and give some Tauberian conditions for which logarithmic summability yields convergence in $IFNS$. Besides, we define the concept of slow oscillation with respect to logarithmic summability in $IFNS$, investigate its relation with the concept of q-boundedness and give Tauberian theorems by means of q-boundedness and slow oscillation with respect to logarithmic summability. A comparison theorem between Cesàro summability method and logarithmic summability method in $IFNS$ is also proved in the paper.

1. Introduction and preliminaries

Fuzzy sets are put forward by Zadeh [1] in 1965 as a generalization of classical sets and have been studied by many mathematicians from varied branches. In classical sets, elements in the universal set are divided crisply into two groups as members and nonmembers, and partial membership is not allowed. Unlike the classical sets, fuzzy sets allow partial membership and take every elements in the universe into account by assigning degrees of membership between 1 and 0. Owing to the power in handling unclassifiable data, fuzzy sets are utilized in many real-world scenarios to cope with problems of uncertainty and indefiniteness. In 1983, inspired by fuzzy sets, Atanassov [2,3] considered also partial non-membership and extended fuzzy sets to intuitionistic fuzzy sets. Following Atanassov's introduction, concepts of intuitionistic fuzzy metric [4] and intuitionistic fuzzy norm (IF-norm) [5,6] are defined and related topics are studied. In particular, convergence of sequences in $IFNS$ is investigated and different types of convergence(e.g., statistical convergence and ideal convergence) are applied to sequences in $IFNS$ to grasp the convergence [7–11].

Recently Talo and Yavuz [12] introduced Cesàro summability of sequences in $IFNS$ and gave Tauberian theorems for Cesàro summability method in $IFNS$, by which they initiated summability theory and Tauberian theory in $IFNS$. In their study, they also defined the concept of slow oscillation in $IFNS$ and gave related theorems. Following their study, we now define logarithmic summability of sequences in $IFNS$ and prove a Tauberian theorem for logarithmic summability method. In the sequel, we define the notion of slow oscillation with respect to logarithmic summability in $IFNS$ and give slowly oscillating type Tauberian conditions for which logarithmic summability yields convergence in $IFNS$. Besides, we compare Cesàro summability and logarithmic summability in $IFNS$. Before continuing with main results we now give some preliminaries.

Definition 1.1. *[6] The triplicate (N,μ,ν) is said to be an $IFNS$ if N is a real vector space, and μ,ν are fuzzy sets on $N\times\mathbb{R}$ satisfying the following conditions for every $u,w\in N$ and $t,s\in\mathbb{R}$:*

(a) $\mu(u,t)=0$ *for* $t\leq 0$,

***Email address and ORCID number*:** enes.yavuz@cbu.edu.tr, https://orcid.org/0000-0002-4335-5210

(b) $\mu(u,t) = 1$ *for all* $t \in \mathbb{R}^+$ *if and only if* $u = \theta$
(c) $\mu(cu,t) = \mu\left(u, \frac{t}{|c|}\right)$ *for all* $t \in \mathbb{R}^+$ *and* $c \neq 0$,
(d) $\mu(u+w,t+s) \geq \min\{\mu(u,t), \mu(w,s)\}$,
(e) $\lim_{t\to\infty} \mu(u,t) = 1$ *and* $\lim_{t\to 0} \mu(u,t) = 0$,
(f) $\nu(u,t) = 1$ *for* $t \leq 0$,
(g) $\nu(u,t) = 0$ *for all* $t \in \mathbb{R}^+$ *if and only if* $u = \theta$
(h) $\nu(cu,t) = \nu\left(u, \frac{t}{|c|}\right)$ *for all* $t \in \mathbb{R}^+$ *and* $c \neq 0$,
(i) $\max\{\nu(u,t), \nu(w,s)\} \geq \nu(u+w,t+s)$,
(j) $\lim_{t\to\infty} \nu(u,t) = 0$ *and* $\lim_{t\to 0} \nu(u,t) = 1$.

We call (μ,ν) *an* $IF-$*norm on* N.

Example 1.2. *Let* $(N, \|\cdot\|)$ *be a normed space and* μ_0, ν_0 *be fuzzy sets on* $N \times \mathbb{R}$ *defined by*

$$\mu_0(u,t) = \begin{cases} 0, & t \leq 0, \\ \frac{t}{t+\|u\|}, & t > 0, \end{cases} \qquad \nu_0(u,t) = \begin{cases} 1, & t \leq 0, \\ \frac{\|u\|}{t+\|u\|}, & t > 0. \end{cases}$$

Then (μ_0, ν_0) *is* $IF-$*norm on* N.

Throughout the paper (N,μ,ν) will denote an $IFNS$.

Definition 1.3. *[6] A sequence* (u_n) *in* (N,μ,ν) *is said to be convergent to* $a \in N$ *and denoted by* $u_n \to a$ *if for every* $\varepsilon > 0$ *and* $t > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $\mu(u_n - a, t) > 1 - \varepsilon$ *and* $\nu(u_n - a, t) < \varepsilon$ *for all* $n \geq n_0$.

Definition 1.4. *[6] A sequence* (u_n) *in* (N,μ,ν) *is said to be Cauchy if for every* $\varepsilon > 0$ *and* $t > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $\mu(u_k - u_n, t) > 1 - \varepsilon$ *and* $\nu(u_k - u_n, t) < \varepsilon$ *for all* $k, n \geq n_0$.

Every convergent sequence is Cauchy in $IFNS$.

Definition 1.5. *[13] A sequence* (u_n) *in* (N,μ,ν) *is called q-bounded if* $\lim_{t\to\infty} \inf_{n\in\mathbb{N}} \mu(u_n,t) = 1$ *and* $\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \nu(u_n,t) = 0$.

2. Main results

Now we introduce logarithmic summability in $IFNS$ and prove corresponding Tauberian theorems. For some other studies concerning logarithmic summability and convergence methods in fuzzy setting see [14–26].

Definition 2.1. *Let sequence* (u_n) *be in* (N,μ,ν). *Logarithmic mean* τ_n *of* (u_n) *is defined by*

$$\tau_n = \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{u_k}{k} \quad \textit{where} \quad \ell_n = \sum_{k=1}^{n} \frac{1}{k}\,.$$

(u_n) *is said to be logarithmic summable to* $a \in N$ *if*

$$\lim_{n\to\infty} \tau_n = a.$$

Following theorem shows that convergence yields logarithmic summability in $IFNS$.

Theorem 2.2. *Let sequence* (u_n) *be in* (N,μ,ν). *If* (u_n) *is convergent to* $a \in N$, *then* (u_n) *is logarithmic summable to* a.

Proof. Let sequence (u_n) converge to $a \in N$. Fix $t > 0$. For $\varepsilon > 0$

- There exists $n_0 \in \mathbb{N}$ such that $\mu\left(u_n - a, \frac{t}{2}\right) > 1 - \varepsilon$ and $\nu\left(u_n - a, \frac{t}{2}\right) < \varepsilon$ for $n > n_0$.
- There exists $n_1 \in \mathbb{N}$ such that

$$\mu\left(\sum_{k=1}^{n_0} \frac{u_k - a}{k}, \frac{\ell_n t}{2}\right) > 1 - \varepsilon \quad \text{and} \quad \nu\left(\sum_{k=1}^{n_0} \frac{u_k - a}{k}, \frac{\ell_n t}{2}\right) < \varepsilon$$

for $n > n_1$, since we have

$$\lim_{n\to\infty} \mu\left(\sum_{k=1}^{n_0} \frac{u_k - a}{k}, \frac{\ell_n t}{2}\right) = 1 \quad \text{and} \quad \lim_{n\to\infty} \nu\left(\sum_{k=1}^{n_0} \frac{u_k - a}{k}, \frac{\ell_n t}{2}\right) = 0.$$

Hence we get

$$\begin{aligned}
\mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k}{k}-a,t\right) &= \mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k-a}{k},t\right)=\mu\left(\sum_{k=1}^{n}\frac{u_k-a}{k},\ell_n t\right)\\
&\geq \min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\mu\left(\sum_{k=n_0+1}^{n}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right)\right\}\\
&\geq \min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\mu\left(\sum_{k=n_0+1}^{n}\frac{u_k-a}{k},\frac{(\ell_n-\ell_{n_0})t}{2}\right)\right\}\\
&\geq \min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\mu\left(\frac{u_{n_0+1}-a}{n_0+1},\frac{t}{2(n_0+1)}\right),\cdots,\mu\left(\frac{u_n-a}{n},\frac{t}{2n}\right)\right\}\\
&= \min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\mu\left(u_{n_0+1}-a,\frac{t}{2}\right),\cdots,\mu\left(u_n-a,\frac{t}{2}\right)\right\}\\
&> 1-\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k}{k}-a,t\right) &< \max\left\{\nu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\nu\left(u_{n_0+1}-a,\frac{t}{2}\right),\cdots,\nu\left(u_n-a,\frac{t}{2}\right)\right\}\\
&< \varepsilon
\end{aligned}$$

whenever $n>\max\{n_0,n_1\}$, which completes the proof. □

Logarithmic summability does not imply convergence in *IFNS* by the next example.

Example 2.3. *Take $(u_n)=((-1)^n)$ in IF−normed space $(\mathbb{R},\mu_0,\nu_0)$ where μ_0 and ν_0 are as in Example 1.2. Sequence (u_n) is logarithmic summable to 0 in view of Theorem 2.13 and [12, Example 3.3], but it is not convergent.*

We now give some Tauberian conditions for which logarithmic summability yields convergence in *IFNS*.

Theorem 2.4. *Let sequence (u_n) be in (N,μ,ν). If (u_n) is logarithmic summable to $a\in N$, then it converges to a if and only if for each $t>0$*

$$\sup_{\lambda>1}\liminf_{n\to\infty}\mu\left(\frac{1}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^\lambda\rfloor}\frac{u_k-u_n}{k},t\right)=1 \tag{2.1}$$

and

$$\inf_{\lambda>1}\limsup_{n\to\infty}\nu\left(\frac{1}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^\lambda\rfloor}\frac{u_k-u_n}{k},t\right)=0. \tag{2.2}$$

Proof. Necessity. Let (u_n) converge to a. For all $\lambda>1$ and large enough n, that is when $\lfloor n^\lambda\rfloor>n$, we can write(see [27, Lemma 5.5(i)])

$$u_n-\tau_n=\frac{\ell_{\lfloor n^\lambda\rfloor}}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n\right)-\frac{1}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^\lambda\rfloor}\frac{u_k-u_n}{k}. \tag{2.3}$$

Since (τ_n) is Cauchy, for each $t>0$ we have

$$\lim_{n\to\infty}\mu\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n,t\right)=1 \quad \text{and} \quad \lim_{n\to\infty}\nu\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n,t\right)=0.$$

Hence, for sufficiently large n such that $\frac{\ell_{\lfloor n^\lambda\rfloor}}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\leq\frac{2\lambda}{\lambda-1}$ is satisfied, we have

$$\mu\left(\frac{\ell_{\lfloor n^\lambda\rfloor}}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n\right),t\right)=\mu\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n,\frac{t}{\frac{\ell_{\lfloor n^\lambda\rfloor}}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}}\right)\geq\mu\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n,\frac{t}{\frac{2\lambda}{\lambda-1}}\right)\to 1 \quad (n\to\infty)$$

and

$$\nu\left(\frac{\ell_{\lfloor n^{\lambda}\rfloor}}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n\right),t\right)=\nu\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n,\frac{t}{\frac{\ell_{\lfloor n^{\lambda}\rfloor}}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}}\right)\leq\mu\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n,\frac{t}{\frac{2\lambda}{\lambda-1}}\right)\to 0\quad(n\to\infty)$$

revealing that $\frac{\ell_{\lfloor n^{\lambda}\rfloor}}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n\right)\to 0$. So, by equation (2.3), we conclude

$$\lim_{n\to\infty}\mu\left(\frac{1}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^{\lambda}\rfloor}\frac{u_k-u_n}{k},t\right)=1\quad\text{and}\quad\lim_{n\to\infty}\nu\left(\frac{1}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^{\lambda}\rfloor}\frac{u_k-u_n}{k},t\right)=0,$$

which means that (2.1) and (2.2) are satisfied.

Sufficiency. Let conditions (2.1) and (2.2) be satisfied. Let $t>0$ be fixed. For $\varepsilon>0$ we have:

- There exist $\lambda>1$ and $n_0\in\mathbb{N}$ such that

$$\mu\left(\frac{1}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^{\lambda}\rfloor}\frac{u_k-u_n}{k},\frac{t}{3}\right)>1-\varepsilon\quad\text{and}\quad\mu\left(\frac{1}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^{\lambda}\rfloor}\frac{u_k-u_n}{k},\frac{t}{3}\right)<\varepsilon$$

  for $n>n_0$.
- There exists $n_1\in\mathbb{N}$ such that $\mu\left(\tau_n-a,\frac{t}{3}\right)>1-\varepsilon$ and $\nu\left(\tau_n-a,\frac{t}{3}\right)<\varepsilon$ for $n>n_1$.
- There exists $n_2\in\mathbb{N}$ such that

$$\mu\left(\frac{\ell_{\lfloor n^{\lambda}\rfloor}}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n\right),\frac{t}{3}\right)>1-\varepsilon\quad\text{and}\quad\nu\left(\frac{\ell_{\lfloor n^{\lambda}\rfloor}}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n\right),\frac{t}{3}\right)<\varepsilon,$$

  for $n>n_2$, since $\frac{\ell_{\lfloor n^{\lambda}\rfloor}}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n\right)\to 0$.

Hence, by equation (2.3), we get

$$\begin{aligned}\mu(u_n-a,t) &= \mu\left(u_n-\tau_n+\tau_n-a,t\right)\\ &= \mu\left(\frac{\ell_{\lfloor n^{\lambda}\rfloor}}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n\right)-\frac{1}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^{\lambda}\rfloor}\frac{u_k-u_n}{k}+\tau_n-a,t\right)\\ &\geq \min\left\{\mu\left(\frac{\ell_{\lfloor n^{\lambda}\rfloor}}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n\right),\frac{t}{3}\right),\mu\left(\frac{1}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^{\lambda}\rfloor}\frac{u_k-u_n}{k},\frac{t}{3}\right),\mu\left(\tau_n-a,\frac{t}{3}\right)\right\}\\ &> 1-\varepsilon\end{aligned}$$

and

$$\begin{aligned}\nu(u_n-a,t) &< \max\left\{\nu\left(\frac{\ell_{\lfloor n^{\lambda}\rfloor}}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\left(\tau_{\lfloor n^{\lambda}\rfloor}-\tau_n\right),\frac{t}{3}\right),\nu\left(\frac{1}{\ell_{\lfloor n^{\lambda}\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^{\lambda}\rfloor}\frac{u_k-u_n}{k},\frac{t}{3}\right),\nu\left(\tau_n-a,\frac{t}{3}\right)\right\}\\ &< \varepsilon\end{aligned}$$

for $n>\max\{n_0,n_1,n_2\}$, which completes the proof. □

Theorem 2.5. *Let sequence (u_n) be in (N,μ,ν). If (u_n) is logarithmic summable to $a\in N$, then it converges to a if and only if for each $t>0$*

$$\sup_{0<\lambda<1}\liminf_{n\to\infty}\mu\left(\frac{1}{\ell_n-\ell_{\lfloor n^{\lambda}\rfloor}}\sum_{k=\lfloor n^{\lambda}\rfloor+1}^{n}\frac{u_n-u_k}{k},t\right)=1$$

and

$$\inf_{0<\lambda<1}\limsup_{n\to\infty}\nu\left(\frac{1}{\ell_n-\ell_{\lfloor n^{\lambda}\rfloor}}\sum_{k=\lfloor n^{\lambda}\rfloor+1}^{n}\frac{u_n-u_k}{k},t\right)=0.$$

Proof. The proof is done similarly to that of Theorem 2.4 by using equation(see [27, Lemma 5.5(ii)])

$$u_n-\tau_n=\frac{\ell_{\lfloor n^\lambda\rfloor}}{\ell_n-\ell_{\lfloor n^\lambda\rfloor}}\left(\tau_n-\tau_{\lfloor n^\lambda\rfloor}\right)+\frac{1}{\ell_n-\ell_{\lfloor n^\lambda\rfloor}}\sum_{k=\lfloor n^\lambda\rfloor+1}^{n}\frac{u_n-u_k}{k}\quad(0<\lambda<1)$$

instead of (2.3). □

Now we introduce the concept of slow oscillation with respect to logarithmic summability in *IFNS*.

Definition 2.6. *(u_n) in (N,μ,ν) is said to be slowly oscillating with respect to logarithmic summability if*

$$\sup_{\lambda>1}\liminf_{n\to\infty}\min_{n<k\leq\lfloor n^\lambda\rfloor}\mu(u_k-u_n,t)=1 \tag{2.4}$$

and

$$\inf_{\lambda>1}\limsup_{n\to\infty}\max_{n<k\leq\lfloor n^\lambda\rfloor}\nu(u_k-u_n,t)=0, \tag{2.5}$$

for each $t>0$. "$\sup_{\lambda>1}$" in (2.4) and "$\inf_{\lambda>1}$" in (2.5) can be replaced by "$\lim_{\lambda\to1^+}$".

A sequence (u_n) in (N,μ,ν) is slowly oscillating with respect to logarithmic summability if for each $t>0$ and for all $\varepsilon>0$ there exist $\lambda>1$ and $n_0\in\mathbb{N}$ such that

$$\mu(u_k-u_n,t)>1-\varepsilon\quad\text{and}\quad\nu(u_k-u_n,t)<\varepsilon$$

whenever $n_0\leq n<k\leq\lfloor n^\lambda\rfloor$.

The proof of next theorem is analogous to that of Theorem 4.2 in [12] and hence omitted.

Theorem 2.7. *Let sequence (u_n) be in (N,μ,ν). For $t>0$, conditions (2.4) and (2.5) are equivalent to*

$$\sup_{0<\lambda<1}\liminf_{n\to\infty}\min_{\lfloor n^\lambda\rfloor<k\leq n}\mu(u_k-u_n,t)=1 \tag{2.6}$$

and

$$\inf_{0<\lambda<1}\limsup_{n\to\infty}\max_{\lfloor n^\lambda\rfloor<k\leq n}\nu(u_k-u_n,t)=0, \tag{2.7}$$

respectively. "$\sup_{0<\lambda<1}$" in (2.6) and "$\inf_{0<\lambda<1}$" in (2.7) can be replaced by "$\lim_{\lambda\to1^-}$".

Example 2.8. *Consider IF−normed space $(\mathbb{R},\mu_0,\nu_0)$ where μ_0 and ν_0 are as in Example 1.2. $u_n=\sum_{j=1}^{n}\frac{1}{j\ln j}$ is slowly oscillating with respect to logarithmic summability by the calculations below:*

Fix $t>0$. For $\varepsilon>0$ take $\lambda=e^{\frac{t\varepsilon}{1-\varepsilon}}$. Then for $1<n<k\leq\lfloor n^\lambda\rfloor$ we have

$$\mu_0(u_k-u_n,t)=\frac{t}{t+|u_k-u_n|}>\frac{t}{t+\frac{t\varepsilon}{1-\varepsilon}}=1-\varepsilon$$

and

$$\nu_0(u_k-u_n,t)=\frac{|u_k-u_n|}{|u_k-u_n|+t}<\frac{\frac{t\varepsilon}{1-\varepsilon}}{\frac{t\varepsilon}{1-\varepsilon}+t}=\varepsilon,$$

since $|u_k-u_n|=\sum_{j=n+1}^{k}\frac{1}{j\ln j}<\int_n^k\frac{du}{u\ln u}\leq\ln\left(\frac{\ln k}{\ln n}\right)\leq\ln\lambda=\frac{t\varepsilon}{1-\varepsilon}$.

Theorem 2.9. *Let sequence (u_n) be in (N,μ,ν). If (u_n) is slowly oscillating with respect to logarithmic summability then (2.1) and (2.2) are satisfied.*

Proof. Suppose that (u_n) is slowly oscillating with respect to logarithmic summability. Fix $t>0$. For $\varepsilon>0$ there exist $\lambda>1$ and $n_0\in\mathbb{N}$ such that

$$\mu(u_k-u_n,t)>1-\varepsilon\quad\text{and}\quad\nu(u_k-u_n,t)<\varepsilon$$

whenever $n_0 \leq n < k \leq \lfloor n^\lambda \rfloor$. Hence, we have

$$\begin{aligned}
\mu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k-u_n}{k},t\right) &= \mu\left(\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k-u_n}{k},(\ell_{\lfloor n^\lambda \rfloor}-\ell_n)t\right)\\
&\geq \min\left\{\mu\left(\frac{u_{n+1}-u_n}{n+1},\frac{t}{n+1}\right),\ldots,\mu\left(\frac{u_{\lfloor n^\lambda \rfloor}-u_n}{\lfloor n^\lambda \rfloor},\frac{t}{\lfloor n^\lambda \rfloor}\right)\right\}\\
&= \min\left\{\mu(u_{n+1}-u_n,t),\ldots,\mu(u_{\lfloor n^\lambda \rfloor}-u_n,t)\right\}\\
&> 1-\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\nu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k-u_n}{k},t\right) &\leq \max\left\{\nu(u_{n+1}-u_n,t),\ldots,\nu(u_{\lfloor n^\lambda \rfloor}-u_n,t)\right\}\\
&< \varepsilon
\end{aligned}$$

for $n \geq n_0$ and this completes the proof. □

In view of Theorem 2.4 and Theorem 2.9 we give the following Tauberian theorem.

Theorem 2.10. *Let sequence (u_n) be in (N,μ,ν). If (u_n) is logarithmic summable to $a \in N$ and slowly oscillating with respect to logarithmic summability, then (u_n) converges to a.*

Theorem 2.11. *Let sequence (u_n) be in (N,μ,ν). If $\{n\ln n(u_n-u_{n-1})\}$ is q-bounded, then (u_n) is slowly oscillating with respect to logarithmic summability.*

Proof. Let $\{n\ln n(u_n-u_{n-1})\}$ be q-bounded. In view of Definition 1.5, for given $\varepsilon>0$ there exists $M_\varepsilon>0$ so that

$$t > M_\varepsilon \Rightarrow \inf_{n\in\mathbb{N}}\mu(n\ln n(u_n-u_{n-1}),t) > 1-\varepsilon \quad \text{and} \quad \sup_{n\in\mathbb{N}}\nu(n\ln n(u_n-u_{n-1}),t) < \varepsilon.$$

For every $t>0$ choose $\lambda < 1+\frac{t}{M_\varepsilon}$. Then for $n_0 < n < k \leq \lfloor n^\lambda \rfloor$ we have

$$\begin{aligned}
\mu(u_k-u_n,t) &= \mu\left(\sum_{j=n+1}^{k}(u_j-u_{j-1}),t\right)\\
&\geq \min_{n+1\leq j\leq k}\mu\left(u_j-u_{j-1},\frac{t}{j(\ell_k-\ell_n)}\right)\\
&= \min_{n+1\leq j\leq k}\mu\left(j\ln j(u_j-u_{j-1}),\frac{t\ln j}{\ell_k-\ell_n}\right)\\
&\geq \min_{n+1\leq j\leq k}\mu\left(j\ln j(u_j-u_{j-1}),\frac{t\ln n}{\ell_k-\ell_n}\right)\\
&\geq \min_{n+1\leq j\leq k}\mu\left(j\ln j(u_j-u_{j-1}),\frac{t}{\frac{\ln k}{\ln n}-1}\right)\\
&\geq \min_{n+1\leq j\leq k}\mu\left(j\ln j(u_j-u_{j-1}),\frac{t}{\lambda-1}\right)\\
&\geq \inf_{n\in\mathbb{N}}\mu\left(n\ln n(u_n-u_{n-1}),\frac{t}{\lambda-1}\right)\\
&> 1-\varepsilon
\end{aligned}$$

and

$$\nu(u_k-u_n,t) < \sup_{n\in\mathbb{N}}\nu\left(n\ln n(u_n-u_{n-1}),\frac{t}{\lambda-1}\right) < \varepsilon.$$

Hence, (u_n) is slowly oscillating with respect to logarithmic summability. □

By Theorem 2.10 and Theorem 2.11, we conclude following Tauberian theorem.

Theorem 2.12. *Let sequence (u_n) be in (N,μ,ν). If (u_n) is logarithmic summable to $a\in N$ and $\{n\ln n(u_n-u_{n-1})\}$ is q-bounded, then (u_n) converges to a.*

Now we prove a comparison theorem.

Theorem 2.13. *Let sequence (u_n) be in (N,μ,ν). If (u_n) is Cesàro summable to $a\in N$, then (u_n) is logarithmic summable to a.*

Proof. Let (u_n) be Cesàro summable to $a\in N$. Then, Cesàro means $\sigma_n=\frac{1}{n}\sum_{k=1}^{n}u_k$ converges to a and $\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k}\to a$ by Theorem 2.2 with the agreement $\sigma_0=0$.

Fix $t>0$. For $\varepsilon>0$

- There exists $n_0\in\mathbb{N}$ such that $\mu\left(\sigma_n-a,\frac{t}{2}\right)>1-\varepsilon$ and $\nu\left(\sigma_n-a,\frac{t}{2}\right)<\varepsilon$ whenever $n>n_0$.
- There exists $n_1\in\mathbb{N}$ such that
$$\mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k}-a,\frac{t}{2}\right)>1-\varepsilon \quad\text{and}\quad \nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k}-a,\frac{t}{2}\right)<\varepsilon$$
whenever $n>n_1$.
- There exists $n_2\in\mathbb{N}$ such that $\mu\left(a,\frac{(\ell_n-1)t}{2}\right)>1-\varepsilon$ and $\nu\left(a,\frac{(\ell_n-1)t}{2}\right)<\varepsilon$ whenever $n>n_2$, since $\lim_{n\to\infty}\mu\left(a,\frac{(\ell_n-1)t}{2}\right)=1$ and $\lim_{n\to\infty}\nu\left(a,\frac{(\ell_n-1)t}{2}\right)=0$.

Then, we have(see [28])

$$\begin{aligned}
\mu(\tau_n-a,t) &= \mu\left(\frac{\sigma_n}{\ell_n}+\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k}-a,t\right)\\
&\geq \min\left\{\mu\left(\frac{\sigma_n}{\ell_n},\frac{t}{2}\right),\mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k}-a,\frac{t}{2}\right)\right\}\\
&= \min\left\{\mu\left(\sigma_n,\frac{\ell_n t}{2}\right),\mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k}-a,\frac{t}{2}\right)\right\}\\
&\geq \min\left\{\mu\left(\sigma_n-a,\frac{t}{2}\right),\mu\left(a,\frac{(\ell_n-1)t}{2}\right),\mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k}-a,\frac{t}{2}\right)\right\}\\
&> 1-\varepsilon
\end{aligned}$$

and

$$\nu(\tau_n-a,t)\leq\max\left\{\nu\left(\sigma_n-a,\frac{t}{2}\right),\nu\left(a,\frac{(\ell_n-1)t}{2}\right),\nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k}-a,\frac{t}{2}\right)\right\}<\varepsilon$$

whenever $n>\max\{n_0,n_1,n_2\}$, which completes the proof. □

Logarithmic summability does not imply Cesàro summability in *IFNS* by the next example.

Example 2.14. *Consider sequence $(u_n)=((-1)^n n)$ in $IF-$normed space $(\mathbb{R},\mu_0,\nu_0)$ where μ_0 and ν_0 are as in Example 1.2. Since*

$$\lim_{n\to\infty}\mu_0(\tau_{2n+1},t)=\lim_{n\to\infty}\mu_0\left(-\frac{1}{\ell_{2n+1}},t\right)=\lim_{n\to\infty}\frac{t}{t+\left|-\frac{1}{\ell_{2n+1}}\right|}=1$$

$$\lim_{n\to\infty}\nu_0(\tau_{2n+1},t)=\lim_{n\to\infty}\nu_0\left(-\frac{1}{\ell_{2n+1}},t\right)=\lim_{n\to\infty}\frac{\left|-\frac{1}{\ell_{2n+1}}\right|}{\left|-\frac{1}{\ell_{2n+1}}\right|+t}=0$$

we have $\tau_{2n+1}\to 0$, and since

$$\lim_{n\to\infty}\mu_0(\tau_{2n},t)=\lim_{n\to\infty}\mu_0(0,t)=\lim_{n\to\infty}\frac{t}{t+0}=1,\quad \lim_{n\to\infty}\nu_0(\tau_{2n},t)=\lim_{n\to\infty}\nu_0(0,t)=\lim_{n\to\infty}\frac{0}{0+t}=0$$

we have $\tau_{2n}\to 0$ which yields that $\lim_{n\to\infty}\tau_n=0$. So, (u_n) is logarithmic summable to 0. But, sequence (u_n) is not Cesàro summable.

We note that converse of Theorem 2.13 is true under the condition $\ln n(\tau_n - a) \to 0$, which can be seen by the following:

$$\begin{aligned}\mu\left(\sigma_n - a, t\right) &= \mu\left(\ell_n\left(\tau_n - a\right) - \frac{1}{n}\sum_{k=1}^{n-1}\ell_k\left(\tau_k - a\right), t\right) \\ &\geq \min\left\{\mu\left(\ell_n\left(\tau_n - a\right), \frac{t}{2}\right), \mu\left(\frac{1}{n}\sum_{k=1}^{n-1}\ell_k\left(\tau_k - a\right), \frac{t}{2}\right)\right\} \to 1 \quad as \quad n \to \infty\end{aligned}$$

and

$$\nu\left(\sigma_n - a, t\right) \leq \max\left\{\nu\left(\ell_n\left(\tau_n - a\right), \frac{t}{2}\right), \nu\left(\frac{1}{n}\sum_{k=1}^{n-1}\ell_k\left(\tau_k - a\right), \frac{t}{2}\right)\right\} \to 0 \quad as \quad n \to \infty.$$

By Theorem 2.13 and Example 2.14, we see that logarithmic summability method is stronger than Cesàro summability method in summing up sequences in $IFNS$.

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Coding Matrices for the Semi-Direct Product Groups

Amnah A. Alkinani[1*] and Ahmed A. Khammash[1]

[1]*Departement of Mathematical Sciences, Umm Al-Qura University, Makkah, Saudi Arabia*
[*]*Corresponding author*

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Abstract

We shall determine the coding matrix of the semi-direct product group $G = C_n \rtimes_\phi C_m$; $\phi : C_m \longrightarrow Aut(C_n)$ of two cyclic groups in order to generalize the known result for the dihedral group D_{2n}, which is known to be a semi-direct of the two cyclic groups C_n , C_2.

1. Introduction

An (n,k)-linear code C of length n over the finite field of q elements $\mathbb{F}_q$ is a k-dimensional subspace of $\mathbb{F}_q^n$. It gained more attention from the work of W. Hamming in 1950 [1]. The first connection between codes and group rings of finite groups appeared in the work of F. G. MacWilliams (1969) [2]. In (2006) T. Hurley [3] (starting with a coding matrix of the finite group G based on an appropriate listing of its elements) proved that the group ring RG of a finite group of order n over a ring R is isomorphic to certain well-defined ring of matrices, and hence gave a construction of codes from certain elements of the group ring such as units and zero divisors [4]. The coding matrices were determined for several classes of finite groups such as cyclic [3], elementary-abelian [3], dihedral groups D_{2n} [3], direct product [5] and the general linear group $GL(2,\mathbb{F})$ [6].
In this paper, we shall generalize Hurley's theorem in [3] to $C_n \rtimes_\phi C_2$ as a special case of $C_n \rtimes_\phi C_m$ and we will decide the form of the coding matrices of $C_n \rtimes_\phi C_m$.

The paper is organized as follows in section 2, we present some definitions and basic results with examples about group rings, coding matrices of group rings and codes. In section 3, we determine the coding matrix of the semi-direct product group of two cyclic groups with illustrative examples.

2. Preliminaries

Let G be a finite group of order n, and $\{g_1, g_2, ..., g_n\}$ be a fixed listing of the element of G. Consider the matrix of G relative to its listing, $M(G)$, which has the following form:

$$M(G) = \begin{pmatrix} g_1^{-1}g_1 & g_1^{-1}g_2 & \cdots & g_1^{-1}g_n \\ g_2^{-1}g_1 & g_2^{-1}g_2 & \cdots & g_2^{-1}g_n \\ \vdots & \vdots & \vdots & \vdots \\ g_n^{-1}g_1 & g_n^{-1}g_2 & \cdots & g_n^{-1}g_n \end{pmatrix}_{n\times n}.$$

Email addresses and ORCID numbers: **aaalkenane@uqu.edu.sa, https://orcid.org/0000-0001-8208-7995 (A. A. Alkinani), aakhammash@uqu.edu.sa, https://orcid.org/0000-0001-9404-1732 (A. A. Khammash)**

Then for each $u = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$, define the matrix $M(RG,u) \in M_n(R)$ as follows:

$$M(RG,u) = \begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \cdots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \cdots & \alpha_{g_2^{-1}g_n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \cdots & \alpha_{g_n^{-1}g_n} \end{pmatrix}_{n\times n}.$$

It is quite clear that the shape as well as the coefficients of the coding matrix $M(RG,u)$ depends on the listing of the group elements of the group G.

In [3], T. Hurley proved that the group ring RG of a group G of order n over a ring R is isomorphic to a certain ring of $(n\times n)$ matrices over R.

Theorem 2.1. *([3], Theorem 1)*
Let G be a group of order n with the given listing of the elements, then there is a bijective ring homomorphism is given by

$$\sigma : u \longrightarrow M(RG,u)$$

between RG and the ring of $(n\times n)$ G-matrices over R.

The coding matrices are known for several types of groups, for details see [3].

Definition 2.2.
- *Let R be a ring, a non zero element $u = \sum_{g\in G} \alpha_g g \in RG$ is called a zero-divisor if and only if there exists a non zero element $v \in RG$ such that $uv = 0$ or $vu = 0$.*
- *Let R be a ring with identity $I_R \neq 0$, an element $u \in RG$ is called a unit if and only if there exists an element $v \in RG$, such that $uv = 1 = vu$.*

Definition 2.3.
- *Let C be an (n,k)-code and let G be a $(k\times n)$-matrix whose rows are the basis for C, then G is called a generator matrix for C.*
- *A parity-check matrix H for an (n,k)-code C is a generator matrix of $C^{\perp}$, such that the dual code $C^{\perp}$ is defined by $C^{\perp} = \{u \in \mathbb{F}_q^n \mid u.v = 0 \text{ for all } v \in C\}$.*

Definition 2.4. *Let RG be the group ring of the group G over the ring R, where the listing of the elements of G is given by $\{g_1, g_2, \ldots, g_n\}$. Suppose W is a submodule of RG, $x \in W$ and $u \in RG$ is given. Then the group ring encoding is a mapping $f : W \longrightarrow RG$ such that $f(x) = xu$ or $f(x) = ux$. In the first case, f is a right group ring encoding and in the letter case is a left group ring encoding.*

Thus, a code C derived from a group ring encoding is the image of a group ring encoding, for a given $u \in RG$, either $C = \{ux : x \in W\}$ or $C = \{xu : x \in W\}$.

The map $\theta : RG \to R^n$, $\theta(\sum_{i=1}^{n} \alpha_{g_i} g_i) = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a ring isomorphism from RG to R^n. Thus every element in RG can be considered as n-tuple in R^n.

In the group ring the multiplication is not necessary be commute, and this allows the construction of non-commutative.

Definition 2.5. *If $xu = ux$ for all x, then the code $C = \{xu : x \in W\}$ is said to be commutative, and otherwise non-commutative codes.*

When u is a zero-divisor, it generates a zero-divisor code and when it is a unit, it generates a unit-derived code. The structure of codes from unit and zero-divisor in RG where done by P. Hurley and T. Hurley in [4] , [7].

Example 2.6. *Let $R = \mathbb{Z}_2 = \{0,1\}$ be the finite field of two elements and $G = S_3 = \prec a,b \mid a^3 = b^2 = 1, ba = a^2b \succ = \{1,a,a^2,b,ab,a^2b\}$ be the symmetric group of order 6. Then the coding matrices of S_3 is:*

$\times$	1	a	a^2	a^2b	ab	b
1	1	a	a^2	a^2b	ab	b
a^2	a^2	1	a	ab	b	a^2b
a	a	a^2	1	b	a^2b	ab
a^2b	a^2b	ab	b	1	a	a^2
ab	ab	b	a^2b	a^2	1	a
b	b	a^2b	ab	a	a^2	1

Thus,

$$M(S_3) = \begin{pmatrix} 1 & a & a^2 & a^2b & ab & b \\ a^2 & 1 & a & ab & b & a^2b \\ a & a^2 & 1 & b & a^2b & ab \\ a^2b & ab & b & 1 & a & a^2 \\ ab & b & a^2b & a^2 & 1 & a \\ b & a^2b & ab & a & a^2 & 1 \end{pmatrix}_{6\times 6}$$

And the group ring $RG = \mathbb{Z}_2S_3 = \sum_{g\in S_3} \alpha_g g \mid \alpha_g \in \mathbb{Z}_2 = \{c_0+c_1a+c_2a^2+c_3a^2b+c_4ab+c_5b\ ;\ c_i \in \mathbb{Z}_2\}$, *Such that* $(\mathbb{Z}_2S_3,+,.)$ *is* $\mathbb{F}$*-algebra. From T. Hurley's theorem :* $\mathbb{Z}_2S_3 \hookrightarrow M_{|S_3|\times|S_3|}(\mathbb{Z}_2)$. *So, if* $u \in \mathbb{Z}_2S_3\ ;\ u = c_0+c_1a+c_2a^2+c_3a^2b+c_4ab+c_5b$ *, then :*

$$M(\mathbb{Z}_2S_3,u) = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_2 & c_0 & c_1 & c_4 & c_5 & c_3 \\ c_1 & c_2 & c_0 & c_5 & c_3 & c_4 \\ c_3 & c_4 & c_5 & c_0 & c_1 & c_2 \\ c_4 & c_5 & c_3 & c_2 & c_0 & c_1 \\ c_5 & c_3 & c_4 & c_1 & c_2 & c_0 \end{pmatrix}_{6\times 6}$$

For the unit element $u = 1+a+a^2+ab+a^2b \in U(\mathbb{Z}_2S_3)$ *there exists* $u^{-1} = 1+a+a^2+ab+a^2b$ *such that* $uu^{-1} = 1$. *Then we have* $M(\mathbb{Z}_2S_3,u)$ *as follows :*

$$M(\mathbb{Z}_2S_3,u) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{6\times 6}$$

Also , from Hurley's theorems : If R has an identity 1_R, *then* $u \in RG$ *is a unit if and only if* $\sigma(u)$ *is a unit in* $R_{n\times n}$. *Hence we have the invertible matrix as follows :*
$U = \begin{pmatrix} A \\ B \end{pmatrix}$ *and* $V = \begin{pmatrix} C & D \end{pmatrix}$ *such that* $UV = 1_6$ *in* $R_{6\times 6}$.
Taking any r rows of U as a generator matrix define an (n,r)*-code. Then we have*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}_{3\times 6},\ B = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{3\times 6},$$

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}_{6\times 3} \quad \text{and } D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_{6\times 3}.$$

Such that

$$AC = BD = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3} \quad \text{and } AD = BC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3}.$$

Then,

$$UV = \begin{pmatrix} A \\ B \end{pmatrix}.\begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} AC & AD \\ BC & BD \end{pmatrix} = \begin{pmatrix} I_3 & O_3 \\ O_3 & I_3 \end{pmatrix} = I_{6\times 6}.$$

The linear code C of dimension $k = 3$, *generated by the matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}_{3\times 6},$$

is the unit derived code $C = \{ux \mid x \in W\}$, *where* $S = \{a\} \subset G$ *and* $W = \prec a \succ = \{1,a,a^2\}$. *The dual code* $C^\perp$ *is the linear code generated by the matrix*

$$D^T = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{3\times 6},$$

with dimension $n-k = 3$. *The dual code can be considered as the submodule* $C^\perp = \{(u^{-1})^T y \mid y \in W^\perp\}$, *where* $W^\perp = \prec G-S \succ = \{a^2b, ab, b\}$. *So,* $C = \{ux \mid x \in W\} = \{1+a+a^2+a^2b+ab, 1+a+a^2+b+a^2b, 1+a^2+a+ab+b\}$, $\theta(C) =$

$\{111110, 111101, 111011\}$, *and* $C^{\perp} = \{(u^{-1})^T y \mid y \in W^{\perp}\} = \{1+a^2b+ab+b+a, 1+ab+b+a^2b+a^2, b+a^2b+ab+b+a\}$, $\theta(C^{\perp}) = \{110111, 101111, 011111\}$. *Clearly, the matrix A is the generator matrix for an* $(6,3)$*-code, and* D^T *is the parity-check matrix for this code, since it is a generator matrix of* $C^{\perp}$ *as defined in (definition 2.3).*

3. Coding matrices of semi-direct product groups

Definition 3.1. *Let H and K be groups and let* ϕ *be a homomorphism,*

$$\phi : K \longrightarrow Aut(H)$$

Then the semi-direct product of H and K with respect the action ϕ *is the group G containing of ordered pairs* (h,k) *with* $h \in H$ *and* $k \in K$ *defined by:*

$$(h_1,k_1)(h_2,k_2) = (h_1\phi_{k_1}h_2, k_1k_2)$$

Where $\phi_k(h) = kh = khk^{-1}, \forall h \in H, k \in K$.

Denote of semi-direct product by $H \rtimes_{\phi} K$ (or simply, write $H \rtimes K$).

Example 3.2. *Let* $G = S_3$*, let N be the normal subgroup of order* 3 *generated by a* 3*-cycle, and let H be a subgroup of order* 2 *generated by a* 2*-cycle. Then* $G = N \rtimes H$. *This example generalizes a long two different lines:*

1 • Let $G = S_n$ *,* $N = A_n$ *and H a subgroup of order* 2 *generated by a* 2*-cycle. Then* $G = N \rtimes H$.
2 • Let $G = D_{2n}$*, the dihedral group of order* $2n$*. Then let* $N = C_n$ *and* $H = C_2$*. Then* $D_{2n} \cong C_n \rtimes C_2$.

We will decide the coding matrices of the semi-direct product groups $C_n \rtimes C_m$ as following:
Consider $G = C_n \rtimes C_m$; $C_n \lhd G$ of two groups $C_n =< x >= \{x \mid x^n = 1\}$ and $C_m =< y >= \{y \mid y^m = 1\}$. We may list the elements of the semi-direct product $C_n \rtimes C_m$ as follows: $x^i y^j$; $0 \leqslant i \leqslant n-1,\ 0 \leqslant j \leqslant m-1$:

$$1, x, x^2, \ldots, x^{n-1} \ ,\ y, xy, x^2y, \ldots, x^{n-1}y \ ,\ y^2, xy^2, x^2y^2, \ldots, x^{n-1}y^2 \ ,\ \ldots\ldots\ldots \ ,y^{m-1}, xy^{m-1}, x^2y^{m-1}, \ldots \ ,\ x^{n-1}y^{m-1} . \tag{3.1}$$

(m blocks each with n elements).
This product defined by the action of C_m on C_n (or group homomorphism) given by $\phi : C_m \longrightarrow Aut(C_n)$; $C_n \rtimes C_m = \{x^iy^j :$ $x^i \in C_n\ ,\ y^j \in C_m \mid x^iy^j.x^sy^t = x^i\phi_{y^j}x^s.y^jy^t\}$. The inverse of the element x^iy^j in $C_n \rtimes C_m$ is $\phi_{(m-j)}x^{n-i}.y^{m-j}$.
In fact, the automorphism group $Aut(C_n)$ is one to one correspondence with the set $\{x^r \mid hcf(n,r) = 1\}$ of generators of C_n, so $|Aut(C_n)| = \varphi(n)$, where φ is the Euler function.

Definition 3.3. *The Euler* φ*-function is defined as: for* $n \in Z^+$*, let* $\varphi(n)$ *be the number of positive integers* $a \leqslant n$ *with* $(a,n) = 1$.

Here, the non-identity element of C_2 acts on C_n by inverting elements; this is an automorphisms since C_n is an abelian, and the presentation for this group is: $< xy | x^n = y^m = 1, yxy^{-1} = x^{-1} >$.
More generally, a semi-direct product of any two cyclic groups C_n with generator x and C_m with generator y is given by one extra relation, $yxy^{-1} = x^k$, with $(k,n) = 1$, where $Aut(C_n) : x \longrightarrow x^k$ for some k; that is, the presentation: $< xy | x^n = y^m = 1, yxy^{-1} = x^k >$.
If y^r is a generator of C_m and $(r,m) = 1$, hence we have the presentation: $< xy | x^n = y^m = 1, y^rxy^{r^{-1}} = x^{k^r} >$.

Now, taking the trivial homomorphism $\phi : C_m \longrightarrow Aut(C_n)$; $C_m \mapsto I_{C_n}$ gives the direct product $G = C_n \rtimes C_m = C_n \times C_m$.
And consider $G = C_n \rtimes C_m$, we need to know when there is a non-trivial homomorphism $\phi : C_m \longrightarrow Aut(C_n)$ but since $Aut(C_n) \cong C_{\varphi(n)}$ and since $Hom(C_m, C_{\varphi(n)}) \cong C_{hcf(m,\varphi(n)\ )}$ we have the following:

Lemma 3.4. *There is a non-trivial homomorphism* $\phi : C_m \longrightarrow Aut(C_n)$ *iff* $hcf(m, \varphi(n)) \neq 1$.

Proof. We have $Hom(C_m, C_{\varphi(n)}) \cong C_{hcf(m,\varphi(n)\ )}$. If $hcf(\ m, \varphi(n)\ ) = 1$ then $Hom(C_m, C_{\varphi(n)}) \cong C_1$ the trivial subgroup and so the only element $\phi \in Hom(C_m, C_{\varphi(n)})$ is the trivial one given by $\phi(y) = I_{C_n}$. Conversely, suppose that $hcf(m, \varphi(n)) \neq 1$, to define $\phi \in Hom(C_m, C_{\varphi(n)})$ by $\phi(y) : x \longmapsto x^t$ (where $1 \leq t < \varphi(n)$ with $hcf(t, \varphi(n)) \neq 1$ in order for x^t to be a generator for $C_{\varphi(n)}$), we must have $order(\phi(y)) \mid m$ (as $y^m = 1$) and $order(\phi(y)) \mid \varphi(n)$ (as $\phi(y) \in C_{\varphi(n)}$). But this is possible since $hcf(m, \varphi(n)) \neq 1$. $\square$

So for example there will be no non-trivial semi-direct product $C_n \rtimes C_m$ (i.e. different from the direct product $C_n \times C_m$) if $hcf(m, \varphi(n)) = 1$, for instance $C_4 \rtimes C_3$ the only homomorphism $\phi : C_3 \longrightarrow Aut(C_4)$ is the one which takes $y \in C_m =< y >$ to the identity $I_{C_4} \in Aut(C_4) =< \theta_3 >= \{I_{C_4}, \theta_3\}$; $\theta_3 : x \longmapsto x^3 = x^{-1}$, therefore the only semi-direct product $C_4 \rtimes C_3$ is the direct product $C_4 \times C_3$.

Definition 3.5.
- *A circulant matrix is special type of Toeplitz matrix, which is one that is constant a long any diagonal running from upper left to lower right.*
- *A (general) Hankel matrix is one which is constant on any diagonal from upper right to lower left.*

In the following examples, we will clarify the coding matrices of $C_n \rtimes C_m$.

Example 3.6. *The semi-direct product of $C_3 \rtimes C_4$; $C_3 = \prec x \mid x^3 = 1 \succ = \{1, x, x^2\}$ and $C_4 = \prec y \mid y^4 = 1 \succ = \{1, y, y^2, y^3\}$. The listing of elements of $C_3 \rtimes C_4$ are : $1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3$. And it has non-trivial homomorphism since $(4, \varphi(3)\ ) = (4, 2) = 2 \neq 1$, the action of C_4 on C_3 given by $\phi : C_4 \to Aut(C_3)$, such that $Aut(C_3)$ is $\phi : C_3 \longrightarrow C_3$; $\mid Aut(C_3) \mid = \varphi(3) = 2$, hence it has $Aut(C_3) = \{\phi_1 : x \longrightarrow x\ ,\ \phi_2 : x \longrightarrow x^2\}$. At ϕ_1 give us the semi-direct product as a direct product, but at ϕ_2 give us the semi-direct product with the presentation $< xy | x^3 = y^4 = 1, yxy^{-1} = x^2 >$; $C_3 \rtimes C_4 = \{\ xy\ :\ x \in C_3\ ,\ y \in C_4\ :\ x_1y_1.x_2y_2 = x_1\phi_{y_1}(x_2).y_1y_2\ \}$ and the inverse of the element xy is $(\phi_{y^{-1}}(x^{-1}).y^{-1})$ as following:*

at ϕ_2

$\rtimes$	1	x	x^2	x^2y	xy	y	x^2y^2	xy^2	y^2	x^2y^3	xy^3	y^3
1	1	x	x^2	x^2y	xy	y	x^2y^2	xy^2	y^2	x^2y^3	xy^3	y^3
x^2	x^2	1	x	xy	y	x^2y	xy^2	y^2	x^2y^2	xy^3	y^3	x^2y^3
x	x	x^2	1	y	x^2y	xy	y^2	x^2y^2	xy^2	y^3	x^2y^3	xy^3
x^2y^3	x^2y^3	xy^3	y^3	1	x	x^2	y	xy	x^2y	y^2	xy^2	x^2y^2
xy^3	xy^3	y^3	x^2y^3	x^2	1	x	x^2y	y	xy	x^2y^2	y^2	xy^2
y^3	y^3	x^2y^3	xy^3	x	x^2	1	xy	x^2y	y	xy^2	x^2y^2	y^2
xy^2	xy^2	x^2y^2	y^2	y^3	x^2y^3	xy^3	1	x^2	x	y	x^2y	xy
x^2y^2	x^2y^2	y^2	xy^2	xy^3	y^3	x^2y^3	x	1	x^2	xy	y	x^2y
y^2	y^2	xy^2	x^2y^2	x^2y^3	xy^3	y^3	x^2	x	1	x^2y	xy	y
x^2y	x^2y	xy	y	y^2	xy^2	x^2y^2	y^3	xy^3	x^2y^3	1	x	x^2
xy	xy	y	x^2y	x^2y^2	y^2	xy^2	x^2y^3	y^3	xy^3	x^2	1	x
y	y	x^2y	xy	xy^2	x^2y^2	y^2	xy^3	x^2y^3	y^3	x	x^2	1

It follows that the coding matrix

$$M(C_3 \rtimes C_4) = \begin{pmatrix} T_0 & H_1 & H_2 & H_3 \\ H_4 & T_1 & T_2 & T_3 \\ H_5 & T_4 & T_5 & T_6 \\ H_6 & T_7 & T_8 & T_9 \end{pmatrix}_{12\times 12},$$

is a block matrix consisting of $16 = 4 \times 4$ matrices all are of size (3×3)-matrices from which $10 = (4-1)^2 + 1$ are circulant (Toeplitz) matrices and $6 = 2(4-1)$ Hankel-type-matrices.

Example 3.7. *Consider the semi-direct product $C_7 \rtimes C_3$, $C_7 = \prec x \mid x^7 = 1 \succ = \{1, x, x^2, x^3, x^4, x^5, x^6\}$ and $C_3 = \prec y \mid y^3 = 1 \succ = \{1, y, y^2\}$, where $\phi : C_3 \longrightarrow Aut(C_7) \cong C_6$. In fact $Aut(C_7) = \{\theta_i | i = 1, 2, 3, 4, 5, 6\} = < \theta_3 > = < \theta_5 > \cong C_6$; i.e. $order(\theta_3) = order(\theta_5) = 6$, while $order(\theta_2) = order(\theta_4) = 3$ and $order(\theta_6) = 2$. Therefore we may take $\phi_i : C_3 \longrightarrow Aut(C_7)$ to be the group homomorphism (or the action of C_3 on C_7) defined as $(\phi_i(y) = \theta_i; i = 1, 2, 4)$, since $order(\theta_i); i = 1, 2, 4 \mid order(y) = 3$. Clearly $\phi_1(y) = \theta_1 = I_{C_7}$ will induce the direct product $C_7 \times C_3$. (In fact it is easy to prove from the relations that $C_7 \rtimes_{\phi_4} C_3 \cong C_7 \rtimes_{\phi_2} C_3$). So we take $\phi_2(y) = \theta_2 : x \longmapsto x^2$ and consider $C_7 \rtimes_{\phi_2} C_3 = < xy \mid x^7 = y^3 = 1, yxy^{-1} = x^2 >$, generally $G = C_7 \rtimes_{\phi_i} C_3 = < xy \mid x^7 = y^3 = 1, yxy^{-1} = x^i; i = 1, 2, 4 >$. Therefore $C_7 \rtimes C_3$ has the listing: $1, x, x^2, x^3, x^4, x^5, x^6, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2, x^5y^2, x^6y^2$ subject to the above relations. From it we may deduce the product of different elements as $\{\ xy\ :\ x \in C_7\ ,\ y \in C_3\ :\ x_1y_1.x_2y_2 = x_1\phi_{y_1}(x_2).y_1y_2\ \}$ and the inverse of the element yx is $(\phi_{y^{-1}}(x^{-1}).y^{-1})$ as following:*

at ϕ_2

$\rtimes$	1	x	$\cdots$	x^5	x^6	x^6y	x^5y	$\cdots$	xy	y	x^6y^2	x^5y^2	$\cdots$	xy^2	y^2
1	1	x	$\cdots$	x^5	x^6	x^6y	x^5y	$\cdots$	xy	y	x^6y^2	x^5y^2	$\cdots$	xy^2	y^2
x^6	x^6	1	$\cdots$	x^4	x^5	x^5y	x^4y	$\cdots$	y	x^6y	x^5y^2	x^4y^2	$\cdots$	y^2	x^6y^2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x^2	x^2	x^3	$\cdots$	1	x	xy	y	$\cdots$	x^3y	x^2y	xy^2	y^2	$\cdots$	x^3y^2	x^2y^2
x	x	x^2	$\cdots$	x^6	1	y	x^6y	$\cdots$	x^2y	xy	y^2	x^6y^2	$\cdots$	x^2y^2	xy^2
x^4y^2	x^4y^2	xy^2	$\cdots$	x^3y^2	y^2	1	x^3	$\cdots$	x	x^4	y	x^3y	$\cdots$	xy	x^4y
xy^2	xy^2	x^5y^2	$\cdots$	y^2	x^4y^2	x^4	1	$\cdots$	x^5	x	x^4y	y	$\cdots$	x^5y	xy
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x^3y^2	x^3y^2	y^2	$\cdots$	x^2y^2	x^6y^2	x^6	x^2	$\cdots$	1	x^3	x^6y	x^2y	$\cdots$	y	x^3y
y^2	y^2	x^4y^2	$\cdots$	x^6y^2	x^3y^2	x^3	x^6	$\cdots$	x^4	1	x^3y	x^6y	$\cdots$	x^4y	y
x^2y	x^2y	x^4y	$\cdots$	x^5y	y	y^2	x^5y^2	$\cdots$	x^4y^2	x^2y^2	1	x^5	$\cdots$	x^4	x^2
x^4y	x^4y	x^6y	$\cdots$	y	x^2y	x^2y^2	y^2	$\cdots$	x^6y^2	x^4y^2	x^2	1	$\cdots$	x^6	x^4
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x^5y	x^5y	y	$\cdots$	xy	x^3y	x^3y^2	xy^2	$\cdots$	y^2	x^5y^2	x^3	x	$\cdots$	1	x^5
y	y	x^2y	$\cdots$	x^3y	x^5y	x^5y^2	x^3y^2	$\cdots$	x^2y^2	y^2	x^5	x^3	$\cdots$	x^2	1

It follows that the coding matrix

$$M(C_7 \rtimes C_3) = \begin{pmatrix} T_0 & H_1 & H_2 \\ H_3 & T_1 & T_2 \\ H_4 & T_3 & T_4 \end{pmatrix}_{21\times 21},$$

is a block matrix consisting of $9 = 3\times 3$ matrices all are of size (7×7)-matrices from which $5 = (3-1)^2+1$ are circulant (Toeplitz) matrices and $4 = 2(3-1)$ Hankel-type-matrices.

In general, we take $G = C_n \rtimes_\phi C_m$ with respect the action ϕ as previously and it has the elements listing (3.1). By inspecting each block sub-matrix provided by each sub-list in $(1)-(m)$ and there corresponding inverse elements, we conclude the following theorem:

Theorem 3.8. *With respect to the above elements listing (3.1) for the semi-direct product groups*

$$G = C_n \rtimes_\phi C_m =< xy|x^n = y^m = 1, yxy^{-1} = x^k >,$$

the coding matrix of this group is a block matrix

$$\begin{pmatrix} T_0 & H_1 & \cdots & H_{m-1} \\ H_m & T_1 & \cdots & T_{m-1} \\ \vdots & \vdots & \vdots & \vdots \\ H_{2(m-1)} & T_{(m-2)(m-1)} & \cdots & T_{(m-1)^2} \end{pmatrix}_{nm\times nm},$$

consisting of m^2 matrices all are of size $(n\times n)$ from which the $(m-1)^2+1$ matrices $T_0, T_1, \ldots, T_{(m-1)^2}$ are circulant (Toeplitz) and the $2(m-1)$ matrices $H_1, H_2, \ldots, H_{2(m-1)}$ are Hankel-type-matrices.

As a special case of this theorem, we deduce the coding matrices for the dihedral group $D_{2n} \cong C_n \rtimes C_2$ which was determined in [3].

Corollary 3.9. *The coding matrices for $C_n \rtimes C_2 \cong D_{2n}$ have the following form*

$$\begin{pmatrix} T_1 & H_1 \\ H_2 & T_2 \end{pmatrix}_{2n\times 2n},$$

where T_i, H_i ; $i = 1,2$ are circulant, Hankel-type $(n\times n)$-matrices, respectively.

Proof. Consider $C_n \rtimes C_2 \cong D_{2n}$ such that $C_n = \prec x \mid x^n = 1 \succ = \{1, x, x^2, \ldots, x^{n-1}\}$, $C_2 = \prec y \mid y^2 = 1 \succ = \{1, y\}$, the listing of elements of $C_n \rtimes C_2$ are : $1, x, x^2, \ldots, x^{n-1}, y, xy, x^2y, \ldots, x^{n-1}y$. And there is a non-trivial homomorphism since $(2, \varphi(n)) \neq 1$, so the action of C_2 on C_n given by $\phi : C_2 \longrightarrow Aut(C_n)$; $Aut(C_n) : \phi : C_n \longrightarrow C_n, |Aut(C_n)| = \varphi(n)$, hence we have $Aut(C_n) = \{\phi_1 : x \longrightarrow x\ ,\ \phi_{n-1} : x \longrightarrow x^{n-1}\}$.

$C_n \rtimes C_2 = \{ \ xy \ : \ x \in C_n \ , \ y \in C_2 \ : \ x_1 y_1 . x_2 y_2 = x_1 \phi_{y_1}(x_2) . y_1 y_2 \ \}$, and the inverse of the element yx is $(\phi_{y^{-1}}(x^{-1}).y^{-1})$. At ϕ_1 give us the semi-direct product as a direct product, but at ϕ_{n-1} give us the semi-direct product groups as following:

at ϕ_{n-1}

$\rtimes$	1	x	x^2	..	x^{n-1}	$x^{n-1}y$	..	x^2y	xy	y
1	1	x	x^2	..	x^{n-1}	$x^{n-1}y$	..	x^2y	xy	y
x^{n-1}	x^{n-1}	1	x	..	x^{n-2}	$x^{n-2}y$	..	xy	y	$x^{n-1}y$
x^{n-2}	x^{n-2}	x^{n-1}	1	..	x^{n-3}	$x^{n-3}y$	..	y	$x^{n-1}y$	$x^{n-2}y$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	x	x^2	x^3	..	1	y	..	x^3y	x^2y	xy
$x^{n-1}y$	$x^{n-1}y$	$x^{n-2}y$	$x^{n-3}y$	..	y	1	..	x^{n-3}	x^{n-2}	x^{n-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x^2	x^2	xy	y	..	x^3y	x^3	..	1	x	x^2
xy	xy	y	$x^{n-1}y$	..	x^2y	x^2	..	x^{n-1}	1	x
y	y	$x^{n-1}y$	x^2y	..	xy	x	..	x^{n-2}	x^{n-1}	1

□

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Covariant and Contravariant Symbols of Operators on $l^2(\mathbb{Z})$

Abdelhamid S. Elmabrok

Department of Mathematics, Faculty of Science, University of Benghazi, Benghazi, Libya

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Abstract

In this paper, we investigate covariant and contravariant symbols of operators generated by a representation of the integer group $\mathbb{Z}$. Then we describe some properties (Existence, Uniqueness, Boundedness, Compactnessi and Finite rank) of these operators and reformulated some know results in terms of wavelet transform (covariant and contravariant symbols).

1. Introduction

The notion of covariant and contravariant symbols of operators was introduced by Berezin in 1972 [1], as a generalization of a Wick and anti-Wick operator symbols [2]. Then a general theory of quantization was developed by F. A. Berezin in [3]. The construction of wavelet transform as covariant and contravariant symbols was realized with wavelets in Hilbert spaces [4]. Recently, in 2014 V. Kisil in his paper [5, Sec 4.2] studied Berezin covariant symbols as a special case of the covariant transform. Also, he applied wavelets on operator algebras by means of symbols of operators [6], which is an extension of the Berezin calculus. The purpose of the present paper is to describe some properties (existence, uniqueness, boundedness and compactness) of operators which have covariant and contravariant symbols and reformulated some know results on these operators.

The paper outline is as follows: In the second Section, we collects preliminary information from other works, which will be used here. In particular, the concepts of covariant and contravariant symbols of operators. In the third Section, we describe some proprieties of covariant and contravariant symbols of operators which generated by the representation of the integer group. Then, we reformulate some know results on existence, uniqueness, boundedness and compactness of linear operators in terms of wavelet transform. The final Section offers summary of our observations which lead to new directions for further research.

2. Preliminaries

In this section we present some fundamental concepts and known results on boundedness and compactness of linear operators in Hilbert spaces and wavelet transform on groups. We denoted by $B(\mathscr{H})$ the sets of all bounded linear operator A on Hilbert space $\mathscr{H}$. Let $\mathbb{G}$ be a group with a left Haar measure $d\mu$ and let π be a unitary irreducible representation of a group $\mathbb{G}$ by operators $\pi_g, g \in \mathbb{G}$ in a Hilbert space $\mathscr{H}$.

Definition 2.1. *[6] Let ψ_0 be a fixed vector in a space $\mathscr{H}$, it is called* a vacuum vector (mother wavelet). *Then the set of vectors $\psi_g = \pi(g)\psi_0$ for $g \in \mathbb{G}$ is called* a family of coherent states (wavelets). *We define* wavelet transform *as a mapping $\mathscr{W}$*

***Email address and ORCID number*: abdelhamid.elmabrok@uob.edu.ly, https://orcid.org/0000-0002-3393-0277**

from the Hilbert space to a space of functions over a group $\mathbb{G}$ *via its representational coefficients* $\mathscr{W}:\mathscr{H}\to L(\mathbb{G}): v\mapsto \hat{v}(g)$, *by*

$$\hat{v}(g)=\left\langle \pi(g^{-1})v,\psi_0\right\rangle=\langle v,\pi(g)\psi_0\rangle=\left\langle v,\psi_g\right\rangle. \tag{2.1}$$

The wavelet transform $\mathscr{W}$ *is a continuous linear mapping and the image of a vector is a bounded continuous function on* $\mathbb{G}$. *The linear space of all such images is denoted by* $W(\mathbb{G})$.

Definition 2.2. *[6] The* inverse wavelet transform *is a mapping* $\mathscr{M}: L_1(\mathbb{G})\to\mathscr{H}:\hat{v}(g)\mapsto\mathscr{M}[\hat{v}(g)]$ *given by the formula:*

$$\mathscr{M}[\hat{v}(g)]=\int_{\mathbb{G}}\hat{v}(g)\pi(g)\,d\mu(g)\psi_0=\int_{\mathbb{G}}\hat{v}(g)\psi_g\,d\mu(g), \tag{2.2}$$

where the integral expresses an operator acting on vector ψ_0.

An important observation [6] is that, two representations for groups $\mathbb{G}$ and $\mathbb{G}\times\mathbb{G}$ were defined correspondingly in the space $B(\mathscr{H})$ of bounded linear operators $\mathscr{H}\to\mathscr{H}$ as follows:

$$\hat{\pi}:\mathbb{G}\to B(B(\mathscr{H})): A\longmapsto \pi(g)^{-1}A\pi(g)$$

$$\breve{\pi}:\mathbb{G}\times\mathbb{G}\to B(B(\mathscr{H})): A\longmapsto \pi(g_1)^{-1}A\pi(g_2)$$

where $A\in B(\mathscr{H})$.
Let there be selected a vacuum vector $h_0\in\mathscr{H}$ and a test functional $l_0\in\mathscr{H}^*$ for π. Then there are canonically associated vacuum vector $p_0\in B(\mathscr{H})$ and test functional $f_0\in B^*(\mathscr{H})$ defined as follows:

$$p_0:\mathscr{H}\longrightarrow\mathscr{H}: h\longmapsto p_0h=\langle h,l_0\rangle h_0;$$

$$f_0: B(\mathscr{H})\longrightarrow\mathbb{C}: A\longmapsto\langle Ah_0,l_0\rangle.$$

They define the following coherent states and transformations of the test functional

$$p_g=\hat{\pi}(g)p_0=\left\langle\cdot,l_g\right\rangle h_g,\qquad p_{(g_1,g_2)}=\breve{\pi}(g_1,g_2)p_0=\left\langle\cdot,l_{g_1}\right\rangle h_{g_2},$$

$$f_g=\hat{\pi}^*(g)f_0=\left\langle\cdot h_g,l_g\right\rangle,\qquad f_{(g_1,g_2)}=\breve{\pi}^*(g_1,g_2)f_0=\left\langle\cdot h_{g_1},l_{g_2}\right\rangle,$$

where as usual we denote $h_g=\pi(g)h_0,\ l_g=\pi^*(g)l_0$.

Definition 2.3. *[6] The* covariant symbol $\tilde{a}(g)(\tilde{a}(g_1,g_2))$ *of an operator A acting on a Hilbert space* $\mathscr{H}$ *defined by* $h_0\in\mathscr{H}$ *and* $l_0\in\mathscr{H}^*$ *is its wavelet transform with respect to the representation* $\hat{\pi}(g),(\breve{\pi}(g_1,g_2))$ *respectively and the functional* f_0, *they are defined by the formulas*

$$\tilde{a}(g)=(\hat{\pi}(g)A,f_0)=\left\langle\pi(g)^{-1}A\pi(g)h_0,l_0\right\rangle=\left\langle Ah_g,l_g\right\rangle, \tag{2.3}$$

$$\tilde{a}(g_1,g_2)=(\breve{\pi}(g_1,g_2)A,f_0)=\left\langle\pi(g_1)^{-1}A\pi(g_2)h_0,l_0\right\rangle=\left\langle Ah_{g_2},l_{g_1}\right\rangle. \tag{2.4}$$

Definition 2.4. *[6] The* contravariant symbol *of an operator A is a function* $a(g)$ *and* $(a(g_1,g_2))$ *such that A is the inverse wavelet transform of* $a(g),a(g_1,g_2)$ *correspondingly with respect to* $\hat{\pi}(g),\breve{\pi}(g_1,g_2)$, *i.e.*

$$A=\int_G a(g)\hat{\pi}(g)p_0\,d\mu(g)=\int_G a(g)p_g\,d\mu(g). \tag{2.5}$$

$$\begin{aligned} A &= \int_{\mathbb{G}}\int_{\mathbb{G}} a(g_1,g_2)\breve{\pi}(g_1,g_2)p_0\,d\mu(g_1)\,d\mu(g_2)\\ &= \int_{\mathbb{G}}\int_{\mathbb{G}} a(g_1,g_2)p_{(g_1,g_2)}\,d\mu(g_1)\,d\mu(g_2), \end{aligned} \tag{2.6}$$

where the integral is defined in the weak sense.

Now, we turn to the separable Hilbert space $\mathscr{H}$ isomorphic to $l^2(\mathbb{Z})$ which is the space of all square-summable complex sequences on $\mathbb{Z}$. Formally

$$l^2(\mathbb{Z}) = \left\{ x(n), n \in \mathbb{Z} : x(n) \in \mathbb{C} \quad \texttt{and} \quad \sum_{n\in\mathbb{Z}} |x(n)|^2 < \infty \right\},$$

with the inner product and the norm, respectively,

$$\langle x, y\rangle = \sum_{n\in\mathbb{Z}} x(n)\overline{y(n)}, \qquad \|x\| = \left(\sum_{n\in\mathbb{Z}} |x(n)|^2 \right)^{\frac{1}{2}}.$$

Let $X^{'}$ be a *dual space (conjugate space)*, which is a normed vector space of all bounded linear functional from a normed space X into the field $\mathbb{C}$, with norm

$$\|f\| = \sup\{|f(x)| : \|x\| \le 1\}.$$

Also, if $X^{'}$ is the dual of a Banach space X. Then for all $x \in X$

$$\|x\| = \max\left\{ |f(x)| : f \in X^{'}, \|f\| = 1 \right\}.$$

Definition 2.5. *Let $F(S,X)$ is denoted to the collection of functions $S \to X$ for set S and a vector space X over $\mathbb{C}$. The* support *of $f \in F(S,X)$ is*

$$supp(f) := \{s \in S : f(s) \neq 0\} = f^{-1}(X \setminus \{0\}).$$

The collection of functions $S \to X$ having finite support *is denoted*

$$\mathscr{F}_{00}(S,X) := \{f \in F(S,X) : supp(f) \subset\subset S\},$$

where $\subset\subset$ denoted to a subset of finite cardinality. Also,

$$F(S) = F(S,\mathbb{C}) \quad \mathit{and} \quad \mathscr{F}_{00}(S) = \mathscr{F}_{00}(S,\mathbb{C}).$$

Proposition 2.6. *[7, p.98] The necessary and sufficient condition that there exist a bounded linear operator A defined on $\mathscr{H}$ such that $\langle Ae_n, e_m\rangle = a_{mn}$, is that, for any finite p and q and for arbitrary $\alpha_1, \alpha_2, \dots, \alpha_p$; $\beta_1, \beta_2, \dots, \beta_q$, the inequality*

$$\left| \sum_m^p \sum_n^q a_{mn} \alpha_m \bar{\beta}_n \right| \le M \sqrt{\sum_m^p |\alpha_m|^2} \sqrt{\sum_n^q |\beta_n|^2}, \tag{2.7}$$

holds, M being a fixed number.

Proposition 2.7. *[8] A bounded operator $A \in B(\mathscr{H})$ is compact if and only if satisfies:*

$$\lim_{n\to\infty} Ae_n = 0, \tag{2.8}$$

for each orthonormal basis for $\mathscr{H}$.

Proposition 2.8. *[9, p.91] Let A be a finite rank linear operator in $\mathscr{H}$ into itself, then A is compact.*

Definition 2.9. *[10, p.442] Let $\mathscr{H}$ be a Hilbert space and let $\{P_m\}$ be a resolution of the identity defined on $\mathscr{H}$. Further, let $\{\lambda_m\}$ be a sequence of scalars. A transformation of the form*

$$Av = \sum_{m=1}^{\infty} \lambda_m P_m v, \; v \in D_A, \tag{2.9}$$

where

$$D_A = \left\{ v \in \mathscr{H} : \lim_{N\to\infty} \sum_m^N \lambda_m P_m v, \; exists \right\}$$

is said to be a weighted sum of projections.

Theorem 2.10. *[10] Let A be a compact normal operator on a Hilbert space $\mathscr{H}$. Then there is a resolution of the identity $\{P_m\}$ and a sequence of complex numbers $\{\lambda_m\}$ such that $A = \sum_m \lambda_m P_m$, where the convergence is in terms of the uniform operator norm topology.*

3. Covariant and contravariant symbols of operators generated by representation of the integer group

One goal of this paper is to describe some proprieties of covariant and contravariant symbols of operators which generated by the representation of the integer group. We can reformulate some know results on existence, uniqueness, boundedness and compactness of linear operators in terms of covariant and contravariant symbols (wavelet transform).

3.1. Wavelet transforms for the integer group in $l^2(\mathbb{Z})$ space

The subject of wavelet transform has arisen many times in many applied areas and we are not able to give a comprehensive history and proper credit. One could mention important books [11, 12]. In the first part of this section, we look for wavelet transformation in a separable Hilbert space $l^2(\mathbb{Z})$) with an orthonormal basis $\{e_k\}, k \in \mathbb{Z}$. Then the group $\mathbb{Z}$ of integers has a unitary representation π on $l^2(\mathbb{Z})$, which is defined on the base as follows

$$\pi(m)e_k = e_{k+m},\ m \in \mathbb{Z}.$$

The adjoint representation is

$$\pi^*(m)e_k = e_{k-m},\ m \in \mathbb{Z}.$$

Hence, e_0 could be taken as a vacuum vector and test functional. Therefore, by Equation (2.1) the wavelet transform with a vacuum vector $\psi_0 = e_0$ is

$$\mathscr{W}(v) = \hat{v}(m) = \langle v, \pi(m)e_0\rangle = \langle v, e_m\rangle. \tag{3.1}$$

and by Equation (2.2) the inverse wavelet transform is

$$\mathscr{M}[\hat{v}(m)] = \sum_{\mathbb{Z}} \hat{v}(m)\pi(m)e_0 = \sum_{-\infty}^{\infty} \hat{v}(m)e_m. \tag{3.2}$$

This is the Fourier series.
Inspired by the corresponding propositions, [13, Sec 4.1], we now equivalently reformulate the following:

Remark 3.1. *The left regular representation $\Lambda(m)$ of the group $\mathbb{Z}$ is the unitary representation by left shifts in the space $l^2(\mathbb{Z})$ by*

$$\Lambda(m) : v(m) \longrightarrow v(-m+n). \tag{3.3}$$

Proposition 3.2. *The wavelet transform $\mathscr{W}$ intertwines π and the left regular representation Λ (3.3) of $\mathbb{Z}$:*

$$\mathscr{W}\pi(m) = \Lambda(m)\mathscr{W}.$$

Proof. By Equations (3.1) and (3.3). We have

$$\begin{aligned}
[\mathscr{W}\pi(m)v](n) &= \langle \pi(m)v, \pi(n)e_0\rangle, \\
&= \langle v, \pi^*(m)\pi(n)e_0\rangle, \\
&= \langle v, \pi^*(m)e_n\rangle, \\
&= \left\langle v, e_{(-m+n)}\right\rangle, \\
&= [\mathscr{W}v](-m+n), \\
&= \Lambda(m)[\mathscr{W}v](n).
\end{aligned}$$

□

Corollary 3.3. *The function space $W(\mathbb{Z})$ is invariant under the representation Λ of $\mathbb{Z}$.*

Proposition 3.4. *The inverse wavelet transform $\mathscr{M}$ intertwines Λ on $L_2(\mathbb{Z})$ and π on $\mathscr{H}$:*

$$\mathscr{M}\Lambda(m) = \pi(m)\mathscr{M}.$$

Proof. By Equations (3.2) and (3.3). We have

$$\begin{aligned}
\mathscr{M}[\Lambda(m)\hat{v}(n)] &= \mathscr{M}[\hat{v}(-m+n)], \\
&= \sum_n \hat{v}(-m+n)\pi(n)e_0, \\
&= \sum_n \hat{v}(-m+n)e_n, \\
&= \sum_k \hat{v}(k)e_{m+k}, \\
&= \sum_k \hat{v}(k)\pi(m)e_k, \\
&= \pi(m)\sum_k \hat{v}(k)e_k, \\
&= \pi(m)\mathscr{M}[\hat{v}(k)],
\end{aligned}$$

where, $k = n - m$. □

Corollary 3.5. *The image $\mathscr{M}(L_1(\mathbb{Z})) \subset \mathscr{H}$ of subspace under the inverse wavelet transform $\mathscr{M}$ is invariant under the representation π.*

Proposition 3.6. *The image $\mathscr{W}(\mathbb{Z})$ of the wavelet transform $\mathscr{W}$ has a reproducing kernel $K(m,n) = \langle w_m, w_n\rangle$. The reproducing formula is in fact a Discrete convolution:*

$$\hat{v}(n) = \sum_{m\in\mathbb{Z}} k(n,m)\hat{v}(m) = \sum_{m\in\mathbb{Z}} \hat{w}_0(n-m)\hat{v}(m).$$

with a wavelet transform of the vacuum vector $\hat{w}_0(n) = \langle w_0, \pi(n)w_0\rangle$.

Proof. By Equation (3.1) and since π is an irreducible square integrable representation defined by the same admissible vector w_0 ([13, Sec 8.2]) we have

$$\begin{aligned}
\hat{v}(n) &= \langle \pi(-n)v, w_0\rangle, \\
&= \sum_{m\in\mathbb{Z}} \langle \pi(-k)\pi(-n)v, w_0\rangle \overline{\langle \pi(-k)w_0, w_0\rangle}, \\
&= \sum_{m\in\mathbb{Z}} \langle \pi(-(k+n))v, w_0\rangle \langle \pi(k)w_0, w_0\rangle, \\
&= \sum_{m\in\mathbb{Z}} \hat{v}(n+k)\hat{w}_0(-k), \\
&= \sum_{m\in\mathbb{Z}} \hat{v}(m)\hat{w}_0(n-m).
\end{aligned}$$

□

3.2. Covariant and contravariant symbols

Berezin symbols and coherent states are a useful tool in quantum theory and have a lot of essentially diferent definitions [14]-[16]. In particular, they were described by Berezin, concerning so-called covariant and contravariant (or Wick and anti-Wick) symbols of operators (see, for example, [17]-[19]). In the second part of this section as the first applications, we describe the covariant and contravariant symbols of operators which realizes the unitary irreducible representations of $\mathbb{Z}$ on Hilbert spaces $l^2(\mathbb{Z})$. By Equations (2.3) and (2.4) the covariant symbol $\tilde{a}(m), \tilde{a}(m,n)$ of an operator $A \in l^2(\mathbb{Z})$ is its wavelet transform with respect to the representation $\hat{\pi}(m), \breve{\pi}(m,n)$, i.e.

$$\begin{aligned}
\tilde{a}(m) &= (\hat{\pi}(m)A, e_0) = \left\langle \pi(m)^{-1}A\pi(m)e_0, e_0\right\rangle \\
&= \langle Ae_m, e_m\rangle = a_{mm},
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\tilde{a}(m,n) &= (\breve{\pi}(m,n)A, e_0) = \left\langle \pi(m)^{-1}A\pi(n)e_0, e_0\right\rangle \\
&= \langle Ae_n, e_m\rangle = a_{mn},
\end{aligned} \tag{3.5}$$

where a_{mn} is a matrix representation in orthonormal basis e_k.
Now, by Equation (2.5) A is the inverse wavelet transform of $a(m)$ with respect to $\hat{\pi}(m)$, where the function $a(m)$ is the contravariant symbol of an operator A, i.e.

$$A = \sum_{m\in\mathbb{Z}} a(m)\hat{\pi}(m)p_0 = \sum_{m\in\mathbb{Z}} a(m)p_m, \tag{3.6}$$

$$Ae_k = \sum_{m\in\mathbb{Z}} a(m)p_m e_k = \sum_{m\in\mathbb{Z}} a(m)e_m \cdot \delta_{km} = a(k)e_k,$$

where δ_{km} is the Kronecker delta. Similarly, by (2.6) the inverse wavelet transform of $a(m,n)$ with respect to $\breve{\pi}(m,n)$ is

$$A = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a(m,n)\breve{\pi}(m,n)p_0 = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a(m,n)p_{(m,n)} \tag{3.7}$$

$$Ae_k = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a(m,n)p_{(m,n)}e_k = \sum_{n\in\mathbb{Z}} a(k,n)e_n.$$

Remark 3.7. *(i) The coherent states p_m and $p_{(m,n)}$ are a rank-one operators,*
(ii) The formula (3.5) yields a representation of A as an infinite matrix.

3.3. Some proprieties of covariant and contravariant symbols

The Berezin symbol of an operator provides important information about the operators, in particular, that, the Berezin symbol uniquely determines the operator (i.e., $A = 0$ if and only if $\tilde{A} = 0$), (see, for example, [20]-[24]). In the third part of this section we will classify some proprieties of covariant and contravariant symbols of operators in $B(l_2(\mathbb{Z}))$. Also, we will discuss further questions and reformulate some know results on existence, uniqueness, boundedness and compactness of these operators in terms of covariant and contravaraint symbols of operators: First, does every operator on $l^2(\mathbb{Z})$ have covariant symbols (3.4) and (3.5)? If yes, is it unique? If not, how to characterize operators which do have? This question is answered by the following proposition.

Proposition 3.8. *For every bounded operator A on $l^2(\mathbb{Z})$), there exists a unique covariant symbol given by formula (3.5).*

Proof. Let A be a bounded operator on $l^2(\mathbb{Z})$. Since $l^2(\mathbb{Z})$ contains an orthonormal basis $\{e_n\}$ (they span $l^2(\mathbb{Z})$). Also the set

$$\left\{ e_n \in l^2(\mathbb{Z}) : Ae_n = \sum_m a_{mn}e_m = b_m \in l^2(\mathbb{Z}) \right\},$$

which is subset of D_A domain of A. Therefore, any bounded operator on $l^2(\mathbb{Z})$ can be represent as a matrix $a_{mn} = \langle Ae_n, e_m\rangle = \tilde{a}(m,n)$, which is (3.5). To prove uniqueness of covariant symbols $\tilde{a}(m,n)$. From Riesz representation theorem on $l^2(\mathbb{Z})$, one simply notes that, there is one element $Ae_n \in l^2(\mathbb{Z})$, such that $\langle Ae_n, e_m\rangle = \tilde{a}(m,n)$. Hence, $\tilde{a}(m,n)$ is unique. □

Remark 3.9. *The formula (3.4) is a special case of (3.5) where $m = n$.*

The second question was about the contravariant symbol. Does any operator on $l^2(\mathbb{Z})$ have contravariant symbols (3.6) and (3.7)? If yes, is it unique? If not, how to characterize operator which do have?
Not every operator A on $l^2(\mathbb{Z})$ has a contravariant symbol satisfying (3.6). For example, let A be the operator of multiplication by t on $L^2[0,1]$ defined by $(Av)(t) = tv(t)$. A is a self-adjoint, bounded operator and has no eigenvalue. Therefore it does not have contravariant symbol $a(m)$ which satisfies (3.6).

Proposition 3.10. *For every compact normal operator A on $l^2(\mathbb{Z})$, there exists a unique contravariant symbol $a(m)$) satisfying formula (3.6).*

Proof. Let A be a compact normal operator on $l^2(\mathbb{Z})$. Then by Theorem 2.10 there is a resolution of the identity $\{P_m\}$ where P_m is an orthogonal projection and a measurable complex-valued function $a(m)$ on $\mathbb{Z}$ (weighted function) such that the operator A can be expressed as weighted sums (2.9)

$$A = \sum_{m\in\mathbb{Z}} a(m)P_m,$$

which is (3.6). It is easily to prove uniqueness of the contravariant symbol $a(m)$. We omit it here. □

Remark 3.11. *Formula (3.7) in the second question still needs more discussion.*

The third question, how to see from a covariant or contravariant symbol of operator if it is finite rank?
Not all operator on $l^2(\mathbb{Z})$ which have a covariant symbol satisfying (3.5) are of finite rank. For example, let A be the identity operator I which has the covariant symbol $I(m,n) = \langle Ie_n, e_m\rangle = \delta_{mn}$. But $Ie_n = e_n$. Therefore $rak(I) = \dim(Im(I)) = \infty$. Then I is not finite rank operator.

Proposition 3.12. *If the covariant symbol which satisfy (2.6) equal zero for any $m < m_1$ or $m > m_2$ and for all n, then A is a finite rank operator on $l^2(\mathbb{Z})$.*

Proof. Let the covariant symbol $\tilde{a}(m,n) = \langle Ae_n, e_m\rangle = a_{mn} = 0$, for any $m < m_1$ or $m > m_2$ and for all n. Since A define on $l^2(\mathbb{Z})$. Then there exists a complete orthonormal basis $e_m \in l^2(\mathbb{Z})$ such that the range of A is spanned by $e_{m_1}, e_{m_1+1}, \ldots., e_{m_2}$. So A has a finite rank. □

Also, not all operators on $l^2(\mathbb{Z})$ which have a contravariant symbol satisfying (3.7) are of finite rank. For example, let A be the identity operator I which has the contravariant symbol δ_{mn} such that

$$Iv = \sum_m \sum_n \delta_{mn} P_{mn} v = \sum_m \langle v, e_m\rangle e_m.$$

But $Iv = v$. Therefore $rak(I) = \dim(Im(I)) = \infty$. Then I is not a finite rank operator.

Proposition 3.13. *If an operator A on $l^2(\mathbb{Z})$ has the contravariant symbol $a(m,n)$ which satisfies (3.7) such that $a(m,n) = 0$ for any $m < m_1$ or $m < m_2$ and for all n, then A is a finite rank operator.*

Proof. Let A on $l^2(\mathbb{Z})$ have the contravariant symbol $a(m,n)$ satisfying (3.7) such that $a(m,n) = 0$ for any $m < m_1$ or $m < m_2$ and for all n. Then there exists a complete orthonormal basis $\{e_m\} \in l^2(\mathbb{Z})$ such that the range of A is spanned by $e_{m_1}, e_{m_1+1}, \ldots., e_{m_2}$. Now by (3.7)

$$A = \sum_{m=m_1}^{m_2} \sum_{n\in\mathbb{Z}} a(m,n) \langle ., e_n\rangle e_m.$$

Then A is a finite rank operator. □

Proposition 3.14. *A bounded operator $A \in B(\mathscr{H})$ is compact if and only if the covariant symbol a_{mn} for a fixed n makes a l^2 function on $\mathbb{Z}$ such that its norm tends to 0 as $n \longrightarrow \infty$.*

Proof. Let A be a compact linear operator from $l^2(\mathbb{Z})$ into itself and (e_n) a complete orthonormal set in $l^2(\mathbb{Z})$, therefore $Ae_n = \sum_{m\in\mathbb{Z}} a_{mn} e_m$.
Then by proposition 2.7 Equation (2.8).

$$\begin{aligned}
&\Leftrightarrow \lim_{n\to\infty} Ae_n = 0,\\
&\Leftrightarrow \lim_{n\to\infty} \sum_{m\in\mathbb{Z}} a_{mn} e_m = 0,\\
&\Leftrightarrow \lim_{n\to\infty} \sum_{m\in\mathbb{Z}} |a_{mn}|^2 = 0,\\
&\Leftrightarrow \lim_{n\to\infty} \|a_n\| = 0,
\end{aligned}$$

where $a_n = a_{mn}$, for a fixed n, with the norm $\|a_n\| = \left(\sum_{m\in\mathbb{Z}} |a_{mn}|^2\right)^{\frac{1}{2}}$. □

Proposition 3.15. *Let for an operator A, there exist a basis e_m such that the covariant symbol $a_{mn} = 0$, for any $m < m_1$ or $m > m_2$ and for all n, then A is compact.*

Proof. For an operator A let there exist a basis e_m such that the covariant symbol $a_{mn} \neq 0$ only for $m_1 \leq m \leq m_2$, then the range of A is spanned by $e_{m_1}, e_{m_1+1}, \ldots, e_{m_2}$. Therefore A has finite rank. Finally, by proposition 2.8 A is compact. □

Proposition 3.16. *Let A be an operator acting on a Hilbert space $\mathscr{H}$ and the covariant symbol a_{mn} of A belong to the Banach space $\mathscr{F}'_{00}(\mathbb{Z}\times\mathbb{Z})$ with norm*

$$\|a_{mn}\| := \sup\{|\langle a_{mn}, l_{mn}\rangle| : l_{mn} \in \mathscr{F}_{00}(\mathbb{Z}\times\mathbb{Z}), \|l_{mn}\|_2 = 1\}. \tag{3.8}$$

Then the operator A is bounded if and only if the covariant symbol a_{mn} is bounded in $\mathscr{F}'_{00}(\mathbb{Z}\times\mathbb{Z})$.

Proof. Let A be an operator acting on a Hilbert space $\mathscr{H}$. Let its covariant symbol a_{mn} be bounded in a Banach space $\mathscr{F}'_{00}(\mathbb{Z}\times\mathbb{Z})$ with norm $\|a_{mn}\|$ and

$$l_{mn} = \alpha\otimes\bar{\beta} \in \mathscr{F}_{00}(\mathbb{Z}\times\mathbb{Z}) \cong \mathscr{F}_{00}(\mathbb{Z})\otimes\mathscr{F}_{00}(\mathbb{Z}),$$

then by proposition 2.6 and Equation (3.8), we have

$$\begin{aligned}
|\langle a_{mn}, l_{mn}\rangle| &\leq \|a_{mn}\| \, \|l_{mn}\|_2\\
\left|\langle a_{mn}, \alpha\otimes\bar{\beta}\rangle\right| &\leq \|a_{mn}\| \, \|\alpha\otimes\beta\|_2\\
\left|\sum_{m,n} a_{mn}\alpha_m\bar{\beta}_n\right| &\leq M\|\alpha\|_2 \|\beta\|_2.
\end{aligned}$$

Thus, A is bounded.
Conversely, let A be a bounded linear operator acting on a Hilbert space $\mathscr{H}$ and a_{mn} its covariant symbol in a Banach space $\mathscr{F}'_{00}(\mathbb{Z}\times\mathbb{Z})$ with norm (3.8), then by necessary condition of proposition 2.6 a_{mn} satisfies the inequality (2.7), i.e

$$\left|\sum_{m}^{p}\sum_{n}^{q} a_{mn}\alpha_m\bar{\beta}_n\right| \leq M\|\alpha\|_2\|\beta\|_2,$$

therefore by equation (3.8)

$$\begin{array}{rcl} \left|\langle a_{mn},\alpha\otimes\bar{\beta}\rangle\right| & \leq & M\|\alpha\|_2\|\beta\|_2 \\ |\langle a_{mn},l_{mn}\rangle| & \leq & M\|l_{mn}\|_2. \end{array}$$

Then a_{mn} is bounded in $\mathscr{F}'_{00}(\mathbb{Z}\times\mathbb{Z})$. □

Proposition 3.17. *The mapping $\sigma : A \longmapsto \sigma_A(m,n)$ of operators to their covariant symbols is an algebra homomorphism from the algebra of operators on $\mathscr{H}$ to the algebra of infinite matrices on $W(\mathbb{Z})$, i.e.*

$$\sigma_{A_1A_2}(m,k) = \sum_{n\in\mathbb{Z}} \sigma_{A_1}(m,n)\sigma_{A_2}(n,k).$$

Proof. By (3.5), we have

$$\begin{array}{rcl} \sigma_{A_1A_2}(m,k) & = & \langle A_1A_2e_m,e_n\rangle, \\ & = & \langle A_1A_2\pi(m)h_0,\pi(n)l_0\rangle, \\ & = & \langle \pi(m)A_1A_2\pi^{-1}(n)h_0,l_0\rangle, \\ & = & \langle A_2\pi^{-1}(n)h_0,A_1^*\pi^*(m)l_0\rangle, \\ & = & \sum_n \langle \pi(k)A_2\pi^{-1}(n)h_0,l_0\rangle,\langle \pi(k)h_0,A_1^*\pi^*(m)l_0\rangle, \\ & = & \sum_n \langle \pi(m)A_1\pi^{-1}(k)h_0,l_0\rangle\langle \pi(k)A_2\pi^{-1}(n)h_0,l_0\rangle, \\ & = & \sum_n \langle A_1e_m,e_k\rangle\langle A_2e_k,e_n\rangle, \\ & = & \sum_n \sigma_{A_1}(m,n)\sigma_{A_2}(n,k). \end{array}$$

□

Finally, is there a relationship between the covariant and contravariant symbols of these operators?

Proposition 3.18. *If an operator A on $l^2(\mathbb{Z})$ has a contravariant symbol $a(m)$ which satisfies (3.6), then the covariant symbol of A which satisfies (3.4) is*

$$\tilde{a}(m) = a(m).$$

Proof. Let an operator A on $l^2(\mathbb{Z})$ have a contravariant symbol $a(m)$ which satisfies (3.6), that is

$$A = \sum_k a(k)p_k.$$

Then

$$Ae_m = \sum_k a(k)p_ke_m = \sum_k a(k)\langle e_m,e_k\rangle e_k = a(k)e_k$$

Now by (3.4)

$$\begin{array}{rcl} \tilde{a}(m) & = & \langle Ae_m,e_m\rangle = \langle a(k)e_k,e_m\rangle \\ & = & \sum_n a(k)e_k(n)\bar{e}_m(n) = a(m). \end{array}$$

□

Proposition 3.19. *If an operator A on $l^2(\mathbb{Z})$ has a contravariant symbol $a(m,n)$ which satisfies (3.7), then the covariant symbol of A which satisfies* (3.5) *is*

$$\tilde{a}(m,n) = a(n,m).$$

Proof. Let the operator A on $l^2(\mathbb{Z})$ have a contravariant symbol $a(m,n)$ which satisfies (3.7), that is

$$A = \sum_j \sum_i a(i,j) p_{(i,j)}.$$

Then

$$Ae_n = \sum_j \sum_i a(i,j) p_{(i,j)} e_n = \sum_j \sum_i a(i,j) \langle e_n, e_i \rangle e_j = \sum_j a(n,j) e_j$$

Now by (3.5)

$$\begin{aligned} \tilde{a}(m,n) &= \langle Ae_n, e_m \rangle = \left\langle \sum_j a(n,j) e_j, e_m \right\rangle = \sum_k \sum_j a(n,j) e_j(k) \bar{e}_m(k) \\ &= \sum_k a(n,k) e_k(m) = a(n,m). \end{aligned}$$

□

4. Conclusion

In this paper, we introduced the concepts of covariant and contravariant symbols of operators which generated by a representation of the integer group $\mathbb{Z}$. Then we described some properties of covariant and contravariant symbols in $B(l_2(\mathbb{Z}))$. Also, we reformulated some know results on (existence, uniqueness, boundedness and compactness) of these operators in terms of wavelet transform (covariant and contravariant symbols). Finally, the full investigation to find similar conditions for bounded and compact in term of covariant and contravariant symbols generated by another groups is left for further work.

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Fixed Point Formulation Using Exponential Logarithmic Transformations and Its Applications

C. Ganesa Moorthy

Department of Mathematics, Alagappa University, Karaikudi-630 003, India

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Abstract

The axioms for a metric D were transformed into axioms of the function $\exp D$, and a new generalized metric called multiplicative metric was introduced in 2008 based on these transformed axioms. A review of a method of converting metric fixed point results through logarithmic transformation to multiplicative metric fixed point results and converting multiplicative metric fixed point results through exponential transformation to metric fixed point results has been presented. Applications of this procedure have also been discussed.

1. Introduction

Fixed point theory accommodates many types of distance functions. On many occasions distance functions are introduced just to derive fixed point theorems and this is justified in the survey articles ([1],[2]). The survey article [2] presents conversions of metric fixed point results through exponential logarithmic transformations into fixed point results on a class of generalized metric spaces which are called multiplicative metric spaces. The concept of a multiplicative metric was just mentioned in the article [3] in the year 2008. But, other researchers used this concept to derive fixed point results by transforming arguments of proofs of results of metric fixed point theory. The survey article [2] applied transformations on statements instead of proofs, by referring to many research articles. The present article is to review these techniques of applying transformations, but by referring to books having collections of results with proofs. It is concluded as in [2] that there is a one to one correspondence between metric fixed point results and multiplicative metric fixed point results. Exact applications are also mentioned. The main purpose of applying transformations is to obtain many examples and to obtain new results. The main purpose of fixed point theory is to solve equations.

Only very few results are selected, mostly from the book [4], for transformation, and new articles like [5] are not considered for transformations. The statements of these results chosen for transformations are not presented for management of symbols. One more thing should be mentioned. The name "multiplicative metric" given in [3] is changed into "EL metric" to stress the usefulness of transformations. The usual notation used for a multiplicative metric is also changed in this article for convenience. It should be mentioned that apart from special nonlinear transformations there are special linear transformations. See for example [6]-[9].

2. Transformed metrics and topologies

Let us first recall the definition of a metric D.

Email address and ORCID number: ganesamoorthyc@gmail.com, https://orcid.org/0000-0003-3119-7531

Definition 2.1. *[1] Let X be a non empty set. Let D be a real valued function on $X \times X$ satisfying the following axioms.*

(I) $D(x,y) \geq 0$*, for all* $x,y \in X$
(II) $D(x,y) = D(y,x)$*, for all* $x,y \in X$
(III) $D(x,y) \leq D(x,z) + D(z,y)$*, for all* $x,y,z \in X$
(IV) $D(x,y) = 0$ *if and only if* $x = y$ *in* X.

Then D is called a metric on X. The pair (X,D) *is called a metric space.*

Definition 2.2. *[3] Let X be a non empty set. Let d be a real valued function on $X \times X$ satisfying the following axioms.*

(i) $d(x,y) \geq 1$*, for all* $x,y \in X$
(ii) $d(x,y) = d(y,x)$*, for all* $x,y \in X$
(iii) $d(x,y) \leq d(x,z)d(z,y)$*, for all* $x,y,z \in X$
(iv) $d(x,y) = 1$ *if and only if* $x = y$ *in* X.

Then d is called an EL metric on X. The pair (X,d) *is called an EL metric space.*

E and L are used for the words “Exponential” and “Logarithmic”. So, EL metrics are our transformed metrics. The word “generalized” is suppressed in giving name in Definition 2.2, because metrics are to be derived from EL metrics and EL metrics are to be derived from metrics.

Let (X,D) be a metric space. Let us use the notation “ $\exp D$” for the composite function which is defined by

$$(\exp D)(x,y) = \exp(D(x,y)) = e^{D(x,y)}, \text{ for all } x,y \in X.$$

Let $d = \exp D$. Then d satisfies the conditions (i),(ii),(iii) and (iv) given in Definition 2.2. Thus, (X,d) is an EL metric space, when $d = \exp D$.

On the other hand, let us consider an EL metric space (X,d). Let us use the notation “$\log d$” for the composite function which is defined by

$$(\log d)(x,y) = \log(d(x,y)), \text{ for all } x,y \in X.$$

Let $D = \log d$. Then D satisfies the conditions (I),(II),(III) and (IV) given in Definition 2.1. Thus (X,D) is a metric space, when $D = \log d$.

These two transformations provide fundamental techniques required to convert classical fixed point results for metric spaces into fixed point results for EL metric spaces. Let us first introduce technical terms required for transformations in results. Let us always associate D for a metric and d for an EL metric, even if it is not mentioned explicitly. Some times, the variations like D_X, D_Y, D_1, D_2, and d_X, d_Y, d_1, d_2 will be used for these purposes. Let us assume all classical definitions for open sets, closed sets, closures, topology induced by a metric, Cauchy sequences, convergent sequences, complete metric spaces, totally bounded metric spaces, compact spaces, etc., corresponding to metric spaces.

Notation 2.3. *Let* (X,D) *be a metric space. For each* $r \geq 0$*, and for each* $x \in X$*, let* $B_D(x,r) = \{y \in X : D(x,y) < r\}$ *and* $C_D(x,r) = \{y \in X : D(x,y) \leq r\}$ *denote open balls and closed balls.*

Definition 2.4. *[10] Let* (X,d) *be an EL metric space. For each* $r \geq 1$*, and for each* $x \in X$*, let* $B_d(x,r) = \{y \in X : d(x,y) < r\}$ *and* $C_d(x,r) = \{y \in X : d(x,y) \leq r\}$*. Let us call them also as open ball and closed ball having centre x and radius r. Let us say that* (X,d) *is totally bounded if for every* $r > 1$*, X is covered by finitely many open balls with radius r.*

Remark 2.5. *[10] Let d be an EL metric on X. Let* $D = \log d$ *so that* $d = \exp D$*. Then* $B_D(x,r) = B_d(x,\exp r)$*, because* $D(x,y) < r$ *if and only if* $d(x,y) < \exp r$*, for* $r > 0$*. Similarly,* $C_D(x,r) = C_d(x,\exp r)$*, for* $r > 0$*. So,* (X,d) *is totally bounded if and only if* (X,D) *is totally bounded.*

Definition 2.6. *[10] A subset A of an EL metric space* (X,d) *is said to be open, if for each* $x \in A$*, there is a number* $r > 1$ *such that* $B_d(x,r) \subseteq A$*. Let* $\tau_d = \{A \subseteq X : A \text{ is open in } (X,d)\}$*.*

Remark 2.7. *[10] Let A be a subset of an EL metric space* (X,d)*. Then* $A \in \tau_d$ *if and only if it is open with respect to the topology* τ_D *induced by* $D = \log d$*. This is a consequence of Remark 2.5. Thus* τ_d *is a topology and it coincides with* τ_D*. Let us say that* (X,d) *is compact, if* (X,τ_d) *is compact.*

Definition 2.8. *[10] Let* $(x_n)_{n=1}^{+\infty}$ *be a sequence in an EL metric space* (X,d)*. Then* $(x_n)_{n=1}^{+\infty}$ *is said to be Cauchy in* (X,d)*, if for every* $\varepsilon > 1$*, there is an integer* n_0 *such that* $d(x_n,x_m) < \varepsilon$*, for all* $n,m \geq n_0$*. The sequence* $(x_n)_{n=1}^{+\infty}$ *is said to be convergent in* (X,d)*, if for every* $\varepsilon > 1$*, there is an element* $x \in X$*, and there is an integer* n_0 *such that* $d(x_n,x) < \varepsilon$*, for all* $n \geq n_0$*. Let us say that* $(x_n)_{n=1}^{+\infty}$ *converges to a limit point x in* (X,d)*, in this case. An EL metric space is said to be complete, if every Cauchy sequence is convergent in it.*

Remark 2.9. *[10] Let $(x_n)_{n=1}^{+\infty}$ be a sequence in an EL metric space (X,d). Let $D = \log d$. By Remark 2.5, $(x_n)_{n=1}^{+\infty}$ is Cauchy in (X,d) if and only if it is Cauchy in (X,D). Also, $(x_n)_{n=1}^{+\infty}$ is convergent to x in (X,d) if and only if it is convergent to x in (X,D). In particular, every convergent sequence in (X,d) is Cauchy. Moreover, (X,d) is complete if and only if (X,D) is complete.*

A transformation method can be applied for arguments of proofs. The first illustration for this action is a proof of the following known fact.

Proposition 2.10. *Let (X,d) be an EL metric space. Let $(x_n)_{n=1}^{+\infty}$ be a sequence converging to x in (X,d). Then $(x_n)_{n=1}^{+\infty}$ is Cauchy in (X,d).*

Proof. Fix $\varepsilon > 1$. Then there is an integer n_0 such that $d(x_n,x) < \sqrt{\varepsilon}$, for all $n \geq n_0$. Then $d(x_n,x_m) \leq d(x_n,x)d(x,x_m) < \varepsilon$, for all $n,m \geq n_0$. This completes the proof. $\square$

Let us establish two more results which can also be obtained by applying transformation directly to the known results.

Lemma 2.11. *Let (X,d) be an EL metric space. Then*

$$1 \leq max\left\{\frac{d(u,v)}{d(x,y)}, \frac{d(x,y)}{d(u,v)}\right\} \leq d(u,x)d(v,y), \textit{ for all } x,y,u,v \in X.$$

Proof. $d(u,v) \leq d(u,x)d(x,y)d(y,v)$. So, $\frac{d(u,v)}{d(x,y)} \leq d(u,x)d(y,v)$. Similarly, $\frac{d(x,y)}{d(u,v)} \leq d(u,x)d(y,v)$. Now, the lemma follows. $\square$

Proposition 2.12. *Let (X,d) be an EL metric space. Let $(x_n)_{n=1}^{+\infty}$ converge to x and $(y_n)_{n=1}^{+\infty}$ converge to y in (X,d). Then $d(x_n,y_n) \to d(x,y)$ as $n \to +\infty$.*

Proof.

$$1 \leq max\left\{\frac{d(x_n,y_n)}{d(x,y)}, \frac{d(x,y)}{d(x_n,y_n)}\right\} \leq d(x_n,x)d(y_n,y), \text{ for all } n,$$

by Lemma 2.11. The result follows, because $d(x_n,x) \to 1$ and $d(y_n,y) \to 1$ as $n \to +\infty$. $\square$

Let us now transform arguments of the proof of the metrization lemma ([11], Chapter 6, Lemma 12).

Lemma 2.13. *Let $(U_n)_{n=0}^{+\infty}$ be a sequence of symmetric subsets of $X \times X$ such that $U_0 = X \times X$, each U_n contains the diagonal $\Delta = \{(x,x) : x \in X\}, U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$ for each n, and such that $\Delta = \bigcap_{n=1}^{+\infty} U_n$. Then there is an EL metric d on X (that is, a function on $X \times X$) such that $U_n \subseteq \{(x,y) \in X \times X : d(x,y) < 5^{2^{-n}}\} \subseteq U_{n-1}$, for each positive integer n.*

Proof. Define a function $f : X \times X \to [1,+\infty)$ by $f(x,y) = 5^{2^{-n}}$ if and only if $(x,y) \in U_{n-1} \setminus U_n$ and by $f(x,x) = 1$, for all $x \in X$. For every $x,y \in X$, let $d(x,y) = \inf \prod_{i=0}^{n} f(x_i,x_{i+1})$, where infimum is taken over all finite sequences (or chains) $x_0,x_1,...,x_{n+1}$ such that $x = x_0$ and $y = x_{n+1}$. Then $d(x,y) \geq 1$ and $d(x,y) \leq d(x,z)d(z,y)$, for all $x,y,z \in X$. Also, $f(x,y) = f(y,x)$, for all $x,y \in X$, because each U_n is symmetric. So, $d(x,y) = d(y,x)$, for all $x,y \in X$. Moreover, $d(x,x) = 1$, for all $x \in X$, because $f(x,x) = 1$, for all $x \in X$. Since $d(x,y) < 5^{2^{-n}}$, for all $(x,y) \in U_n$, then $U_n \subseteq \{(x,y) \in X \times X : d(x,y) < 5^{2^{-n}}\}$. Let us claim that, for any chain $x_0,x_1,...,x_{n+1}$, $f(x_0,x_{n+1}) \leq (\prod_{i=0}^{n} f(x_i,x_{i+1}))^2$. Let us establish the claim by induction on n. Let us call n as the length of the chain $x_0,x_1,...,x_{n+1}$. The claim is true for any chain with length $n = 0$, because $f(x,y) \geq 1$, for all $x,y \in X$. Let us assume for induction that the claim is true for any chain with length less than n, and $n \geq 1$. To complete the proof of our claim, let us consider a chain $x_0,x_1,...,x_{n+1}$ in X such that $f(x_i,x_{i+1}) > 1$, for all i. Let $a = \prod_{i=0}^{n} f(x_i,x_{i+1})$. Let k be the largest integer such that $\prod_{i=0}^{k} f(x_i,x_{i+1}) \leq \sqrt{a}$. Then $\prod_{i=k+1}^{n} f(x_i,x_{i+1}) \leq \sqrt{a}$. Hence, by our induction hypothesis, $f(x_0,x_k) \leq a$ and $f(x_{k+1},x_{n+1}) \leq a$. Moreover, $f(x_k,x_{k+1}) \leq a$. If m is the smallest integer such that $5^{2^{-m-1}} \leq a$, then $(x_0,x_k),(x_k,x_{k+1}),(x_{k+1},x_{m+1}) \in U_m$ and hence $(x_0,x_{n+1}) \in U_{m-1}$. Hence $f(x_0,x_{n+1}) \leq 5^{2^{-m}} \leq a^2$. Thus our claim is true. This claim proves that $\{(x,y) \in X \times X : d(x,y) < 5^{2^{-n}}\} \subseteq U_{n-1}$, for all $n = 1,2,,...$. This completes the proof. $\square$

One should compare the previous proof with the proof of ([11], Chapter 6, Lemma 12). This comparison is necessary to understand the technique of applying transformation in arguments of proofs. Note that the exponential function does not appear in the previous proof. This one happens, because $5^{d(x,y)}$ is also a metric, when d is an EL metric. One can also observe that the statement of ([11], Chapter 6, Lemma 12) can be directly converted into the statement of Lemma 2.13 of this article without going through proofs. Lemma 2.13 of this article and ([11], Chapter 6, Lemma 12) jointly imply the following. The topology induced by an EL metric is also induced by a metric, and the topology induced by a metric is also induced by an EL metric. But, this was already implied by Remark 2.7.

Conclusion 2.14. *If d is an EL metric and $D = \log d$, then topologies, convergent sequences, Cauchy sequences, continuity, compactness, and completeness are common for both d and D. Both d and D are jointly continuous. All these fundamental facts will be applied in this article.*

3. Advantage of transformations in arguments

Theorem 3.1. *Let (X,d) be a compact EL metric space. Let $T : X \to X$ be a mapping such that $d(T^2x,Tx) < d(Tx,x)$, whenever $Tx \neq x$ in X. Then T has a fixed point. Moreover, if $d(Tx.Ty) < d(x,y)$, whenever $x \neq y$ in X, then T has a unique fixed point x^*, and $(T^n x_0)_{n=1}^{+\infty}$ converges to x^*, for any fixed x_0 in X.*

Proof. Let $\phi(x) = d(Tx,x)$, for all $x \in X$. Then ϕ has a minimum at some $x^* \in X$. If $Tx^* \neq x^*$, then

$$1 \leq \phi(Tx^*) = d(T^2x^*,Tx^*) < d(Tx^*,x^*) = \phi(x^*).$$

So, $Tx^* = x^*$, because ϕ has a minimum at x^*.

Suppose further that $d(Tx,Ty) < d(x,y)$, whenever $x \neq y$ in X. Suppose $Tx^{**} = x^{**}$ and $Tx^* = x^*$ for two points x^*, x^{**} in X. If $x^* \neq x^{**}$, then

$$1 \leq d(x^{**},x^*) = d(Tx^{**},Tx^*) < d(x^{**},x^*).$$

So, $x^* = x^{**}$. Thus T has a unique fixed point x^* in X. Fix x_0 in X. Let $c_n = d(T^n x_0, x^*)$. If $c_m = 1$ for some m, then $(T^n x_0)_{n=1}^{+\infty}$ converges to x^* in (X,d). Suppose $c_n \neq 1$ for every n. Then

$$1 \leq c_{n+1} \leq d(T^{n+1}x_0,x^*) = d(T(T^n x_0),Tx^*) < d(T^n x_0,x^*) = c_n, \text{ for all } n.$$

Let $\lim_{n\to+\infty} c_n = c$. If $c = 1$, then $(T^n x_0)_{n=1}^{+\infty}$ converges to x^* in (X,d). Suppose $c > 1$. Let $(T^{n_i}x_0)_{i=1}^{+\infty}$ be a subsequence of $(T^n x_0)_{n=1}^{+\infty}$ such that the subsequence converges to some z in (X,d). Then

$$1 < c = \lim_{i\to+\infty} c_{n_i} = \lim_{i\to+\infty} d(T^{n_i}x_0,x^*) = d(z,x^*)$$

so that $x^* \neq z$. Since T is continuous,

$$1 < c = \lim_{i\to+\infty} d(T^{n_i+1}x_0,x^*) = d(Tz,x^*) = d(Tz,Tx^*) < d(z,x^*) = c.$$

This is impossible. Thus $c = 1$, and the result follows. □

Corollary 3.2. *Let (X,D) be a compact metric space. Let $T : (X,D) \to (X,D)$ be a mapping such that $D(Tx,Ty) < D(x,y)$, for $x \neq y$ in X. Then there is a unique fixed point x^* in X. Moreover, for any $x_0 \in X$, $(T^n x_0)_{n=1}^{+\infty}$ converges to x^* in (X,D).*

Proof. Define $d = \exp D$. Then (X,d) is a compact EL metric space. Also, $d(Tx,Ty) < d(x,y)$, for $x \neq y$ in X. So, the result follows from Theorem 3.1. □

The previous Corollary 3.2 is ([4], Theorem 3.5). The following Corollary 3.4 is the most fundamental theorem of metric fixed point theory. It is ([4], Theorem 3.1), which is the Banach contraction principle.

Theorem 3.3. *Let (X,d) be a complete EL metric space. Let $k \in (0,1)$. Let $T : X \to X$ be a mapping such that $d(Tx,Ty) \leq (d(x,y))^k$, for all $x,y \in X$. Then T has a unique fixed point x^*. Also, for each x_0 in X, the sequence $(T^n x_0)_{n=1}^{+\infty}$ converges to x^* in (X,d). Moreover,*

$$d(T^n x_0,x^*) \leq (d(x_0,Tx_0))^{\frac{k^n}{1-k}}, \textit{ for all } n.$$

Proof. Fix $x_0 \in X$. Let $x_n = T^n x_0$, for all $n = 1,2,3...$. Then $d(x_{n+1},x_n) \leq (d(x_n,x_{n-1}))^k \leq (d(x_1,x_0))^{k^n}$. For $m > n$,

$$\begin{aligned} 1 &\leq d(x_n,x_m) \\ &\leq d(x_n,x_{n+1})d(x_{n+1},x_{n+2})....d(x_{m-1},x_m) \\ &\leq (d(x_1,x_0))^{k^n+k^{n+1}+...+k^{m-1}} \\ &\leq (d(x_1,x_0))^{k^n+k^{n+1}+....} \\ &= (d(x_1,x_0))^{\frac{k^n}{1-k}}. \end{aligned}$$

Since the right hand side tends to 1 as $n \to +\infty$, $(x_n)_{n=1}^{+\infty}$ is a Cauchy sequence that converges to some element x^* in (X,d). Moreover, when m tends to infinity in the previous relation, it is obtained that

$$1 \leq d(x_n,x^*) \leq (d(Tx_0,x_0))^{\frac{k^n}{1-k}},$$

for every $n = 1,2,...$. If n tends to infinity in this relation, then it is concluded that $d(x_n,x^*) \to 1$ as $n \to +\infty$. Since

$$1 \leq d(Tx^*,x^*) = \lim_{n\to+\infty} d(x_{n+1},x_n) \leq \lim_{n\to+\infty} (d(x_1,x_0))^{k^n} = 1,$$

it follows that $Tx^* = x^*$. Moreover, if $Tx^{**} = x^{**}$, then

$$1 \leq d(x^*,x^{**}) = d(Tx^*,Tx^{**}) \leq (d(x^*,x^{**}))^k.$$

Since $k \in (0,1)$, it follows that $x^* = x^{**}$. That is, T has a unique fixed point x^* in X. □

Corollary 3.4. *Let (X,D) be a complete metric space. Let $k \in (0,1)$. Let $T : X \to X$ be a mapping such that $D(Tx,Ty) \leq kD(x,y)$, for all $x,y \in X$. Then T has a unique fixed point x^*. Moreover, for any x_0 in X, the sequence $(T^n x_0)_{n=1}^{+\infty}$ converges to x^*.*

Proof. Let $d = \exp D$. Then (X,d) is a complete EL metric space. Also, $d(Tx,Ty) \leq (d(x,y))^k$, for all $x,y \in X$. Corollary 3.4 now follows from Theorem 3.3. □

Conclusion 3.5. *One may understand that the proofs of Theorem 3.1 and Theorem 3.3 use arguments which resemble classical arguments and the arguments used in this article are obtained from classical arguments by means of logarithmic transformation. These types of transformed arguments should be developed, because one can derive fixed point results for metric spaces, as it is illustrated in Corollary 3.2 and Corollary 3.4.*

4. Transformation in statements

Sometimes, it would be convenient to establish existence of fixed points in particular examples, when transformed statements of results are applied. This is the main advantage of transformation of statements of results.

Example 4.1. *Let $X = \left[\frac{1}{2},2\right]$. Let $d(x,y) = \max\{xy^{-1},yx^{-1}\}$, for all $x,y \in X$. Then (X,d) is a compact EL metric space. Define $T : X \to X$ by $Tx = x^{\frac{1}{2}}$, for all $x \in X$. Then $d(Tx,Ty) < d(x,y)$ whenever $x \neq y$. This mapping T has a unique fixed point 1. For any fixed $x \in X$, $T^n x \to 1$ as $n \to +\infty$.*

One may understand that the following Corollary 4.2 can be associated with Example 4.1. Although Corollary 4.2 is a part of Theorem 3.1, let us derive it from Corollary 3.2 for an illustration of transformation techniques.

Corollary 4.2. *Let (X,d) be a compact EL metric space. Let $T : X \to X$ be a mapping such that $d(Tx,Ty) < d(x,y)$, for all $x,y \in X$. Then T has a unique fixed point x^*. Moreover, for any x_0 in X, $(T^n x_0)_{n=1}^{+\infty}$ converges to x^* in (X,d).*

Proof. Let $D = \log d$. Then (X,D) is a compact metric space, and $D(Tx,Ty) < D(x,y)$, whenever $x \neq y$ in X. Now, Corollary 3.2 implies Corollary 4.2. □

Corollary 4.3. *Let (X,d) be a complete EL metric space. Let $k \in (0,1)$. Let $T : X \to X$ be a mapping such that $d(Tx,Ty) \leq (d(x,y))^k$, for all $x,y \in X$. Then T has a unique fixed point.*

Proof. Let $D = \log d$. Then (X,D) is a complete metric space and $D(Tx,Ty) \leq kD(x,y)$, for all $x,y \in X$. Now, Corollary 3.4 implies Corollary 4.3. □

Corollaries 3.2, 3.4, 4.2 and 4.3 are evidences for existence of a one to one correspondence between fixed point results for metric spaces and fixed point results for EL metric spaces. Since fixed point results for metric spaces have already been established, let us go for transformations of fixed point results for metric spaces. Let us consider some results from the book [4] for transformation.

Theorem 4.4. *Let (X,d) be a complete EL metric space. Let $k \in (0,1)$. Let $T : X \to X$ be a mapping such that $d(Tx,Ty) \leq (d(x,y))^k$, for all $x,y \in X$. Let x^* be the unique fixed point of T. Let $(\varepsilon_n)_{n=1}^{+\infty}$ be a sequence of numbers such that $\lim\limits_{n\to+\infty} \varepsilon_n = 1$ and $\varepsilon_n > 1$, for all n. Let $(y_n)_{n=1}^{+\infty}$ be a sequence in X such that $d(y_{n+1},Ty_n) \leq \varepsilon_n$, for all $n = 1,2,\dots$. Then $\lim\limits_{n\to+\infty} y_n = x^*$.*

Proof. Let $D = \log d$. Then $D(Tx,Ty) \leq kD(x,y)$, for all $x,y \in X$. Also, $(\log\varepsilon_n)_{n=1}^{+\infty}$ is a sequence of positive numbers such that $D(y_{n+1},Ty_n) \leq \log\varepsilon_n$, for all $n = 1,2,\dots$, and such that $\log\varepsilon_n \to 0$ as $n \to +\infty$. Now, ([4], Theorem 3.2) implies the result. □

Theorem 4.5. *Let (X,d) be a complete EL metric space. Let $k \in (0,1)$. Let n be a positive integer. Let $T : X \to X$ be a mapping such that $d(T^n x,T^n y) \leq (d(x,y))^k$, for all $x,y \in X$. Then T has a unique fixed point.*

Proof. Let $D = \log d$. Then $D(T^n x,T^n y) \leq kD(x,y)$, for all $x,y \in X$. By ([4], Theorem 3.3), the result follows. □

Theorem 4.6. *Let (X,d) be a complete EL metric space. Let $\alpha : [1,+\infty) \to [0,1)$ be a function which satisfies the condition "$\alpha(t_n) \to 1$ implies $t_n \to 1$". Let $T : X \to X$ be a mapping such that $d(Tx,Ty) \leq (d(x,y))^{\alpha(d(x,y))}$, for all $x,y \in X$. Then T has a unique fixed point x^*. Also, for each x_0 in X, $(T^n x_0)_{n=1}^{+\infty}$ converges to x^*.*

Proof. Let $D = \log d$. Let $A : [0,+\infty) \to [0,1)$ be a mapping defined by $A(t) = \alpha(\exp t)$, for all $t \in (0,+\infty)$. If $A(t_n) \to 1$, then $\alpha(\exp t_n) \to 1$, $\exp t_n \to 1$, and $t_n \to 0$ as $n \to +\infty$. Moreover,

$$\begin{aligned} D(Tx,Ty) &\leq \alpha(d(x,y))D(x,y) \\ &= \alpha(\exp\log d(x,y))D(x,y) \\ &= A(D(x,y))D(x,y), \text{ for all } x,y \in X. \end{aligned}$$

The conditions required for ([4], Theorem 3.6) are satisfied for $T : (X,D) \to (X,D)$ in the complete metric space (X,D). Now, ([4], Theorem 3.6) implies the result. □

Definition 4.7. *Let (X,d) be an EL metric space. For a subset A of X, diameter of A is denoted by diam A and defined by diam $A = \sup\{d(x,y) : x,y \in A\}$. The subset A is said to be a bounded set, if diam A is finite. The EL metric space (X,d) is bounded, if diam X is finite.*

Let $D = \log d$ for an EL metric d on X. A subset A of X is bounded in the metric space (X,D) if and only if it is bounded in the EL metric space (X,d).

Theorem 4.8. *Let (X,d) be a bounded EL metric space. Let $\psi : [1,+\infty) \to [1,+\infty)$ be a function such that ψ is continuous from right at all points and such that $1 < \psi(r) < r$ for $r > 1$. Let $T : (X,d) \to (X,d)$ be a continuous mapping, with respect to the topology induced by d, such that $d(Tx,Ty) \leq \psi(d(x,y))$, for all $x,y \in X$. Then T has a unique fixed point x^*. Also, for each x_0 in X, $(T^n x_0)_{n=1}^{+\infty}$ converges to x^* as $n \to +\infty$.*

Proof. Let $D = \log d$ so that (X,D) is a complete metric space and $T : (X,D) \to (X,D)$ is continuous. Define $A : [0,+\infty) \to [0,+\infty)$ by $A(x) = \log(\psi(\exp x))$, for all $x \in [0,+\infty)$. Then A is continuous from right at all points such that $0 < A(r) = \log(\psi(\exp r)) < r$ for $r > 0$. Moreover,

$$D(Tx,Ty) \leq \log\psi(\exp(\log d(x,y))) = A(D(x,y)), \text{ for all } x,y \in X.$$

Then, the conditions of ([4], Theorem 3.7) are satisfied. So, by ([4], Theorem 3.7), the result follows. □

Let us recall from [4] that a real valued function ψ on $[a,+\infty)$ is upper semi continuous at $r \in [a,+\infty)$ from right, if $\limsup\limits_{j\to+\infty} \psi(r_j) \leq \psi(r)$, whenever $r_j \to r$.

Theorem 4.9. *Let (X,d) be a complete EL metric space. Let a mapping $\psi : [1,+\infty) \to [1,+\infty)$ be upper semi continuous from right at all points such that $1 < \psi(t) < t$ for $t > 1$. Let $T : (X,d) \to (X,d)$ be a mapping such that $d(Tx,Ty) \leq \psi(d(x,y))$, for all $x,y \in X$. Then T has a unique fixed point x^*. Also, for each x_0 in X, $(T^n x_0)_{n=1}^{+\infty}$ converges to x^*.*

Proof. Define D and A as in the proof of the previous theorem. Then $0 < A(t) < t$ for $t > 0$ and A is upper semi continuous from right at all points. Moreover,

$$D(Tx,Ty) \leq A(D(x,y)), \text{ for all } x,y \in X,$$

and (X,D) is a complete metric space. Then, by ([4], Theorem 3.8), the result follows. □

Theorem 4.10. *Let (X,d) be a complete EL metric space. Let $\psi : [1,+\infty) \to [1,+\infty)$ be a monotone non decreasing function such that $\lim_{n\to+\infty} \psi^n(t) = 1$ for $t > 1$. Let $T : (X,d) \to (X,d)$ be a mapping such that $d(Tx,Ty) \leq \psi(d(x,y))$, for all $x,y \in X$. Then T has a unique fixed point x^* and $(T^n x_0)_{n=1}^{+\infty}$ converges to x^*, for every $x_0 \in X$.*

Proof. Let $D = \log d$. Define $A : (0,+\infty) \to (0,+\infty)$ by $A(t) = \log(\psi(\exp t))$, for all $t \in (0,+\infty)$. Then A is a monotone non decreasing function such that $A^n(t) = \log(\psi^n(\exp t))$, for all $t > 0$ and for all $n = 1,2,\ldots$. So, $\lim_{n\to+\infty} A^n(t) = 0$, for every $t > 0$. Also,

$$D(Tx,Ty) \leq A(D(x,y)), \text{ for all } x,y \in X.$$

Now the result follows from ([4], Theorem 3.10). □

Notation 4.11. *For a mapping $T : (X,d) \to (X,d)$ on an EL metric space (X,d), let $k(T) = \sup\left\{\frac{\log d(Tx,Ty)}{\log d(x,y)} : x,y \in X, x \neq y\right\}$, and let $k_{+\infty}(T) = \limsup_{n\to+\infty} (k(T^n))^{\frac{1}{n}}$.*

Theorem 4.12. *Let (X,d) be a complete EL metric space. Let $T : (X,d) \to (X,d)$ be a continuous mapping for which $k_{+\infty}(T) < 1$. Then T has a unique fixed point x^*. Also, $(T^n x_0)_{n=1}^{+\infty}$ converges to x^*, for each $x_0 \in X$.*

Proof. Let $D = \log d$. Then the result follows from ([4], Theorem 3.11). □

Theorem 4.13. *Let (X,d) be a complete EL metric space. Let $\phi : X \to [1,+\infty)$ be a mapping. Let $T : (X,d) \to (X,d)$ be a continuous mapping which satisfies $d(x,Tx) \leq \frac{\phi(x)}{\phi(Tx)}$, for all $x \in X$. Then $(T^n x_0)_{n=1}^{+\infty}$ converges to a fixed point of T, for each $x_0 \in X$.*

Proof. Let $D = \log d$. Define $A : X \to [0,+\infty)$ by $A(x) = \log(\phi(x))$, for all $x \in X$. Then $T : (X,D) \to (X,D)$ is a continuous mapping such that

$$D(x,Tx) \leq A(x) - A(Tx), \text{ for all } x \in X.$$

Now the result follows from ([4], Theorem 3.13). □

Let us recall from [4] that a real valued function ϕ on a metric space (X,D) is lower semi continuous at x, if $\phi(x) \leq r$, whenever $\lim_{n\to+\infty} x_n = x$ in (X,D) and $\lim_{n\to+\infty} \phi(x_n) = r$ in the real line. In this case, it can also be stated that ϕ on (X,d) is lower semi continuous at x, where $d = \exp D$, in view of Remark 2.7.

Theorem 4.14. *Let (X,d) be a complete EL metric space. Let $\phi : (X,d) \to [1,+\infty)$ be a function which is lower semi continuous at every point of (X,d). Let $T : X \to X$ be a mapping which satisfies $d(x,Tx) \leq \frac{\phi(x)}{\phi(Tx)}$, for all $x \in X$. Then T has a fixed point.*

Proof. Let $D = \log d$. Define $A : (X,D) \to [0,+\infty)$ by $A(x) = \log(\phi(x))$, for all $x \in X$. Then A is lower semi continuous at every point of (X,d). Moreover,

$$D(x,Tx) \leq A(x) - A(Tx), \text{ for all } x \in X.$$

Now the result follows from ([4], Theorem 3.15). □

Theorem 4.15. *Let (X,d_X) and (Y,d_Y) be complete EL metric spaces. Let $T : X \to X$ be a mapping. Let $S : (X,d_X) \to (Y,d_Y)$ be a mapping having closed graph. Let $\phi : (S(X),d_Y) \to [1,+\infty)$ be a function which is lower semi continuous at every point of $(S(X),d_Y)$. Let $c > 0$ be a constant such that*

$$\max\{d_X(x,Tx),(d_Y(Sx,S(Tx)))^c\} \leq \frac{\phi(Sx)}{\phi(S(Tx))}, \textit{ for all } x \in X.$$

Then T has a fixed point.

Proof. Let $D_X = \log d_X$ and $D_Y = \log d_Y$. Define $A : (S(X), D_Y) \to [0, +\infty)$ by $A(y) = \log\phi(y)$, for all $y \in S(X)$. Then A is lower semi continuous at every point of $(S(X), D_Y)$. Also,

$$\max\{D_X(x,Tx), cD_Y(Sx, S(Tx))\} \leq A(Sx) - A(S(Tx)), \text{ for all } x \in X.$$

Now the result follows from ([4], Theorem 3.16). □

Definition 4.16. *Let (X,d) be an EL metric space. Let $D = \log d$. Let $CB(X)$ denote the collection of all nonempty closed and bounded subsets of (X,d) (or of (X,D)). For every $A, B \in CB(X)$, let*

$$P(A,B) = \max\left\{\sup_{y\in B}\inf_{x\in A} d(x,y), \sup_{y\in A}\inf_{x\in B} d(x,y)\right\},$$

and let

$$H(A,B) = \max\left\{\sup_{y\in B}\inf_{x\in A} D(x,y), \sup_{y\in A}\inf_{x\in B} D(x,y)\right\}.$$

Then H is the usual Hausdorff metric on $CB(X)$ derived from D. Then $P = \exp H$ on $CB(X)$ so that P is an EL metric on $CB(X)$. Let us call the EL metric P as Pompieu EL metric.

Let us recall that if (X,D) is a complete metric space, then $(CB(X), H)$ is also a complete metric space. The next theorem is a transform of the first fundamental fixed point theorem for set valued mappings.

Theorem 4.17. *Let (X,d) be a complete EL metric space. Let P be the Pompeiu EL metric on $CB(X)$ derived from d. Let $k \in (0,1)$. Let $T : X \to CB(X)$ be a mapping such that $P(Tx,Ty) \leq (d(x,y))^k$, for all $x,y \in X$. Then there exists an element $x^* \in X$ such that $x^* \in T(x^*)$.*

Proof. Let $D = \log d$ and $H = \log P$. Then $H(Tx,Ty) \leq kD(x,y)$, for all $x,y \in X$. Now, the result follows from ([4], Theorem 3.20). □

Let (X,d) be an EL metric space. For a subset A of X, let us use the notation diam A for the diameter $\sup\{d(x,y) : x,y \in A\}$ in (X,d). Let $D = \log d$. Let us use the notation Diam A for the diameter $\sup\{D(x,y) : x,y \in A\}$ in (X,D). Let us observe that Diam A = log diam A.

Theorem 4.18. *Let (X,d) be a bounded complete EL metric space. For every $x,y \in X$, let $O(x,y) = \{x, Tx, T^2(x), ..., y, Ty, T^2y, ...\}$. Let $\phi : [1,+\infty) \to [1,+\infty)$ be a non decreasing continuous function such that $\phi(s) < s$, for $s > 1$. Let $T : (X,d) \to (X,d)$ be a continuous mapping. Suppose for each $x \in X$, there exists a positive integer $n(x)$ such that $d(T^n x, T^n y) \leq \phi(diam\ O(x,y))$, for all $y \in X$, for all $n \geq n(x)$. Then there exists a unique fixed point x^* such that $(T^n x_0)_{n=1}^{+\infty}$ converges to x^*, for every $x_0 \in X$.*

Proof. Let $D = \log d$. Define $A : [0,+\infty) \to [0,+\infty)$ by $A(x) = \log(\phi(\exp x))$, for all $x \in [0,+\infty)$. Then A is a non decreasing continuous function such that $A(s) < s$ for $s > 0$. Also, for each $x \in X$ there exists a positive integer $n(x)$ such that

$$D(T^n x, T^n y) \leq \log(\phi(\exp(\log(diam\ O(x,y))))) = A(Diam\ O(x,y)), for all y \in X, \text{ for all } n \geq n(x).$$

Here $T : (X,D) \to (X,D)$ is continuous. Now the result follows from ([4], Theorem 3.22). □

The arguments used in the previous proof and ([4], Theorem 3.23) imply the following theorem.

Theorem 4.19. *Let (X,d), ϕ and $O(x,y)$ be as described in the previous theorem. Let $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \leq \phi(diam\ O(x,y)), \textit{ for all } x,y \in X.$$

Then T has a unique fixed point x^*. Also, for each $x_0 \in X$, the sequence $(T^n x_0)_{n=1}^{+\infty}$ converges to x^*.

Let us next convert ([4], Exercise 3.3), which is ([12], Chapter 1, Theorem 5.1). Let us recall the notations used in Definition 2.4 and let us use the transformation $D = \log d$ for this purpose.

Theorem 4.20. *Let (X,d) be a complete EL metric space and Let $T : X \to X$ be a mapping. Assume that for each $\varepsilon > 1$, there is a $\delta > 1$ such that $T(B_d(x,\varepsilon)) \subseteq B_d(x,\varepsilon)$, whenever $d(x,Tx) < \delta$. If $d(T^n x_0, T^{n+1} x_0) \to 1$, as $n \to +\infty$, for some $x_0 \in X$, then the sequence $(T^n x_0)_{n=1}^{+\infty}$ converges to a fixed point x^* of T in (X,d).*

Let us again use the notations used in Definition 2.4 to transform ([4], Definition 4.1).

Definition 4.21. *Let (X,d) be an EL metric space. Suppose, for any class $(C_d(x_i,r_i))_{i\in I}$ of closed balls in (X,d) which satisfy*

$$d(x_i,x_j) \leq r_i r_j, \text{ for all } i,j \in I,$$

the intersection $\bigcap_{i\in I} C_d(x_i,r_i)$ is non empty. Then (X,d) is said to be EL hyperconvex.

Remark 4.22. *Suppose (X,d) is EL hyperconvex. Let $D = \log d$. Then, for any class $(C_D(x_i,r_i))_{i\in I}$ of closed balls in (X,D) which satisfy*

$$D(x_i,x_j) \leq r_i + r_j, \text{ for all } i,j \in I,$$

the intersection $\bigcap_{i\in I} C_D(x_i,r_i)$ is non empty, because $C_D(x_i,r_i) = C_d(x_i,\exp r_i)$, for all $i \in I$, and

$$d(x_i,x_j) \leq (\exp r_i)(\exp r_j), \text{ for all } i,j \in I.$$

Then, by ([4], Definition 4.1), the metric space (X,D) is hyperconvex. On the other hand, if (X,D) is a hyperconvex metric space, then the EL metric space (X,d) is EL hyperconvex, where $d = \exp D$.

Example 4.23. *Consider the vector space l^∞ of all bounded real sequences with the usual metric D defined by*

$$D((x_n)_{n=1}^{+\infty},(y_n)_{n=1}^{+\infty}) = \sup_{n=1,2,\ldots} |x_n - y_n|.$$

Then by ([4], Proposition 4.2) and ([4], Theorem 4.2), (l^∞,D) is hyperconvex and hence (l^∞,d) is EL hyperconvex, where $d = \exp D$.

Theorem 4.24. *Let (X,d) be a bounded EL hyper convex EL metric space. Let $T : X \to X$ be a mapping such that $d(Tx,Ty) \leq d(x,y)$, for all $x,y \in X$. Then the fixed point set $\{x \in X : Tx = x\}$ is non empty and it is EL hyper convex with respect to d.*

Proof. Let $D = \log d$. Then $D(Tx,Ty) \leq D(x,y)$, for all $x,y \in X$. Now, the result follows from ([4], Theorem 4.8). □

Definition 4.25. *Let (X,d) be an EL metric space, and let $D = \log d$. Let A be a subset of X. Let $cov_d(A)$ (or, $cov_D(A)$) denote the intersection of all closed balls in (X,d) (or, in (X,D), respectively) which contain A. Since $C_D(x,\log r) = C_d(x,r)$, for $r \geq 1$ and $x \in X$, then $cov_d(A) = cov_D(A)$, for any subset A of X.*

A bounded set A in a metric space (X,D) is an admissible subset of X, according to ([4], Definition 4.2), if $A = cov_D(A)$. So, let us say that a bounded subset A of an EL metric space (X,d) is admissible when $A = cov_d(A)$, because $cov_d(A) = cov_D(A)$ for $D = \log d$. Following the book [4], let us write $A(X)$ for the collection of all admissible subsets of (X,d) or (X,D).

The next definition is ([4], Definition 5.1) for metric spaces.

Definition 4.26. *Let (X,D) be a metric space. Then $A(X)$ is said to be compact, if every descending chain non empty members of A(X) has non empty intersection. Let us follow the same terminology even for (X,d), where $d = \log D$.*

Let (X,D) be a metric space and let $d = \exp D$. Let A be a subset of X. Let $r_D(A) = \inf\left\{\sup\{D(x,y) : y \in A\} : x \in A\right\}$ and $r_d(A) = \inf\left\{\sup\{d(x,y) : y \in A\} : x \in A\right\}$. Then $r_d(A) = \exp r_D(A)$. Hence $r_d(A) < diam\,A$ if and only if $r_D(A) < Diam\,A$.

The next definition is ([4], Definition 5.2) given for metric spaces.

Definition 4.27. *Let (X,D) be a metric space. Then $A(X)$ is said to be normal if $r_D(A) < Diam\,A$, whenever $A \in A(X)$ and $Diam\,A > 0$. Equivalently, $A(X)$ is said to be normal if $r_d(A) < diam\,A$, whenever $A \in A(X)$ and $diam\,A > 1$, where $d = \log D$.*

The following is an obvious transformed form of ([4], Theorem 5.1).

Theorem 4.28. *Let (X,d) be a non empty bounded EL metric space for which $A(X)$ is compact and normal. Let $T : X \to X$ be a mapping such that $d(Tx,Ty) \leq d(x,y)$, for all $x,y \in X$. Then T has atleast one fixed point.*

From the results which have been derived, it can be observed that a transformation for a fixed point result requires transformations for some concepts. It has been established indirectly that there is an assurance for transformations of metric fixed point results to EL metric fixed point results. Only one thing should be observed: Results to be transformed should depend only on metrics. For example, let us consider ([4], Corollary 8.4) which states that if $T : A \to A$ is a mapping on an admissible subset A of $(l_1, ||\ ||_1)$ such that $||Tx - Ty||_1 \leq ||x - y||_1$, for all $x,y \in A$, then T has a fixed point in A. It seems to be a result for a normed space depending on algebraic structures; but it is a result for a metric space which is independent of algebraic structures. So, it is a result for the EL metric space (l_1, d) when

$$d((x_n)_{n=1}^{+\infty}, (y_n)_{n=1}^{+\infty}) = \prod_{n=1}^{+\infty} e^{|x_n - y_n|}, \text{ for all } (x_n)_{n=1}^{+\infty}, (y_n)_{n=1}^{+\infty} \in l_1.$$

For this metric d, ([4], Corollary 8.4) can be transformed to the following form.

Theorem 4.29. *Let A be an admissible subset of (l_1, d). Let $T : A \to A$ be a mapping such that $d(Tx, Ty) \le d(x, y)$, for all $x, y \in X$. Then T has a fixed point in A.*

A continuation is separated by the following conclusion of this section.

Conclusion 4.30. *Convert hypotheses of a metric fixed point result by means of exponential transform to get hypotheses for an EL metric fixed point result. For a proof of the EL metric fixed point result, use logarithmic transform.*

5. More transformed results

The next definition is proposed for transformed norms.

Definition 5.1. *Let X be a vector space over the field of real numbers or the field of complex numbers. Let $|| \ ||$ be a real valued function on X satisfying the following conditions, where $||x||$ is $|| \ ||(x)$:*

(i) $||x|| \ge 1$, for all $x \in X$, and $||x|| = 1$ if and only if $x = 0$ in X.
(ii) $||\lambda x|| = ||x||^{|\lambda|}$, for all $x \in X$, and for all scalars λ.
(iii) $||x + y|| \le ||x|| \, ||y||$, for all $x, y \in X$.

Then, let us call $|| \ ||$ as EL norm, and $(X, || \ ||)$ as EL normed space.

Example 5.2. *If $(X, || \ ||)$ is a normed space, then $|| \ ||_0$ is defined by $||x||_0 = 8^{\pi ||x||}$, for all $x \in X$, is an EL norm on X. On the other hand, if $(X, || \ ||)$ is an EL normed space, then $||x||_0$ defined by $||x||_0 = \log ||x||$, for all $x \in X$, is a norm on X.*

Let us discuss fixed point theorems again on metric spaces. Let us first transform the statement of ([4], Theorem 3.4), but let us present a direct proof (out line).

Corollary 5.3. *Let X be a non empty set with two EL metrics d_1 and d_2. Suppose $d_1(x, y) \le d_2(x, y)$, for all $x, y \in X$. Suppose (X, d_1) be complete. Let $T : (X, d_1) \to (X, d_1)$ be a continuous mapping such that $d_2(Tx, Ty) \le (d_2(x, y))^k$, for all $x, y \in X$, for some $k \in (0, 1)$. Then T has a unique fixed point x^*.*

Proof. Fix $x_0 \in X$. Then $(T^n x_0)_{n=1}^{+\infty}$ is a Cauchy sequence in (X, d_2), because $d_2(Tx, Ty) \le (d_2(x, y))^k$, for all $x, y \in X$. Since $d_1(x, y) \le d_2(x, y)$, for all $x, y \in X$, the sequence $(T^n x_0)_{n=1}^{+\infty}$ is Cauchy also in (X, d_1). Let $(T^n x_0)_{n=1}^{+\infty}$ converge to x^* in (X, d_1). Then x^* is a fixed point of T, because $T : (X, d_1) \to (X, d_1)$ is continuous. The uniqueness of x^* follows, because $k \in (0, 1)$. □

One can introduce $D_1 = \log d_1$, $D_2 = \log d_2$ and apply ([4], Theorem 3.4) to obtain the previous Theorem 5.3. But a direct proof has been presented just to observe that there is a technique of sharing conditions in hypotheses in a metric fixed point result between two metrics. This sharing technique is also a simple technique for converting results for generalizations. This sharing technique is also indirectly applied by using metric preserving functions; see [13]. For a good introduction about results on metric preserving functions, one may see the article [14].

Definition 5.4. *Let us say that a function $f : [1, +\infty) \to [1, +\infty)$ is EL metric preserving, if $f \circ d$ is an EL metric for every EL metric d. It is said to be strongly metric preserving, if the topologies induced by d and $f \circ d$ coincide, for every EL metric d.*

The exercise problem ([11], Chapter 4, Problem C) is converted to the following result.

Proposition 5.5. *Let $f : [1, +\infty) \to [1, +\infty)$ be a non decreasing continuous function such that $f(x) = 1$ if and only if $x = 1$ and such that $f(xy) \le f(x) f(y)$, for all $x, y \in [1, +\infty)$. Then f is a strongly EL metric preserving function.*

Proof. Define $F : [0, +\infty) \to [0, +\infty)$ by $F(x) = \log(f(\exp x))$. Then F is continuous, non decreasing, and $F(x) = 0$ if and only if $x = 0$ in $[0, +\infty)$. Moreover,

$$F(x + y) = \log(f(\exp(x + y))) \le \log(f(\exp x)) + \log(f(\exp y)) = F(x) + F(y), \text{ for all } x, y \in [0, +\infty).$$

Then F is a strongly metric preserving function (see [14], for definition). Then $F \circ \log d$ is a metric whenever d is an EL metric, and the metrics $\log d$, $F \circ \log d$ induce a common topology. Hence,

$$\exp \circ F \circ \log d = \exp \circ \log \circ f \circ \exp \circ \log d = f \circ d$$

is an EL metric whenever d is an EL metric, and the EL metrics $f \circ d$ and d induce the same topology. □

Let us now state a simplified version of ([4], Theorem 3.17) in transformed form.

Theorem 5.6. *Let (X,d) be a complete EL metric space. Let $r>0$. Let $\phi:(X,d)\to[r,+\infty)$ be a function. Let $\psi:[1,+\infty)\to[1,+\infty)$ be an EL metric preserving function, which is strictly increasing. Let $T:(X,d)\to(X,d)$ be a continuous mapping which satisfies $\psi(d(x,Tx))\le\frac{\phi(x)}{\phi(Tx)}$, for all $x\in X$. Then T has a fixed point.*

Proof. Let $d_1=\psi\circ d$. Fix $x_0\in X$. Let $x_n=T^n x_0$, for all $n=1,2,\dots$. Then

$$1\le d_1(x_n,x_{n+1})\le\frac{\phi(x_n)}{\phi(x_{n+1})},\text{ for all } n=1,2,\dots.$$

For $n>m$,

$$1\le d_1(x_m,x_n)\le d_1(x_m,x_{m+1})d_1(x_{m+1},x_{m+2})\dots d_1(x_{n-1},x_n)\le\frac{\phi(x_m)}{\phi(x_n)}.$$

So, $(\phi(x_n))_{n=1}^{+\infty}$ is a monotone non increasing sequence. Let $\phi(x_n)\to M\ge r>0$, as $n\to+\infty$. Then, by the previous inequality, $d_1(x_m,x_n)\to 1$ as $n,m\to+\infty$. That is, $(\psi\circ d)(x_m,x_n)\to 1$ as $n,m\to+\infty$. Therefore, $d(x_m,x_n)\to 1$ as $n,m\to+\infty$, because ψ is strictly increasing. Since (X,d) is complete, $(x_n)_{n=1}^{+\infty}$ converges to some point x^* in (X,d). Since $T:(X,d)\to(X,d)$ is continuous, and $(Tx_n)_{n=1}^{+\infty}$ converges to x^*, then $Tx^*=x^*$. □

Corollary 5.7. *Let (X,D) be a complete metric space. Let ϕ be a bounded below real valued function on (X,D). Let $\psi:[0,+\infty)\to[0,+\infty)$ be a strictly increasing metric preserving function. Let $T:(X,D)\to(X,D)$ be a continuous mapping which satisfies the condition $\psi(D(x,Tx))\le\phi(x)-\phi(Tx)$, for all $x\in X$. Then T has a fixed point.*

Proof. Let $d=\exp D$. Define a real valued function A on (X,d) by $A(x)=\exp(\phi(x))$, for all $x\in X$. Then, there is a number $r>0$ such that $A(x)\ge r$, for all $x\in X$. Define $B:[1,+\infty)\to[1,+\infty)$ by $B(s)=\exp(\psi(\log s))$, for all $s\ge 1$. Then B is a strictly increasing EL metric preserving function. Moreover,

$$\begin{aligned}B(d(x,Tx))=\exp(\psi(\log d(x,Tx)))=\exp(\psi(D(x,Tx)))&\le\exp(\phi(x)-\phi(Tx))\\&=\frac{A(x)}{A(Tx)},\text{ for all } x\in X\end{aligned}$$

Also, $T:(X,d)\to(X,d)$ is continuous, and (X,d) is complete. Now, the result follows from the previous theorem. □

Remark 5.8. *The function ψ given in Theorem 5.6 should be continuous at 1, and the function ψ given in Corollary 5.7 should be continuous at 0. So, by ([14], Theorem 3.4), ψ should be a strongly EL metric preserving function in Theorem 5.6 and ψ should be a strongly metric preserving function in Corollary 5.7. In both results, the conditions on ψ may be replaced by following simple conditions. For Theorem 5.6, assume that ψ is a strongly EL metric preserving function. For Corollary 5.7, assume that ψ is a strongly EL metric preserving function.*

Conclusion 5.9. *There are standard methods in literature to convert results for extensions and generalizations. Exponential-Logarithmic transformation method can be extended to derive many transformed results in analysis.*

6. Applications

It has been mentioned in Section 4 that the main advantage of transformed statements lies in verifying conditions in examples. Let us modify slightly Example 4.1 for justification of this sentence.

Example 6.1. *Let $X=\left[\frac{1}{2},+\infty\right]$. Let $d(x,y)=\max\{xy^{-1},yx^{-1}\}$, for all $x,y\in X$. Let $D(x,y)=|x-y|$, for all $x,y\in X$. Then (X,d) is a complete EL metric space and (X,D) is a complete metric space. Let $T:X\to X$ be defined by $Tx=x^{\frac{1}{2}}$, for all $x\in X$. Then,*

$$D(x,y)=|x-y|=\left|x^{\frac{1}{2}}-y^{\frac{1}{2}}\right|\left|x^{\frac{1}{2}}+y^{\frac{1}{2}}\right|=D(Tx,Ty)\left|x^{\frac{1}{2}}+y^{\frac{1}{2}}\right|,\textit{ for all } x,y\in X.$$

Thus, there is no constant $M>0$ such that

$$D(Tx,Ty)\le MD(x,y),\textit{ for all } x,y\in X.$$

Hence, the Banach contraction principle is not applicable in this case. However,

$$1\le d(Tx,Ty)=\max\left\{\left(\frac{x}{y}\right)^{\frac{1}{2}},\left(\frac{y}{x}\right)^{\frac{1}{2}}\right\}=(d(x,y))^{\frac{1}{2}}\le(d(x,y))^k,k\in\left[\frac{1}{2},1\right),\textit{ for all } x,y\in X.$$

By Theorem 3.3, for $x_0\in X$, the sequence $(T^n x_0)_{n=1}^{+\infty}$ converges to the unique fixed point 1.

Example 6.2. *Let X, d, D be as in the previous Example 6.1. Let $T : X \to X$ be defined by $Tx = \frac{x+1}{2}$, for all $x \in X$. Then*

$$d(Tx,Ty) = \max\left\{\frac{x+1}{y+1}, \frac{y+1}{x+1}\right\}$$

and

$$d(x,y) = \max\left\{\frac{x}{y}, \frac{y}{x}\right\}.$$

Hence, there is no $k \in (0,1]$ such that

$$d(Tx,Ty) \leq (d(x,y))^k, \text{ for all } x,y \in X.$$

Thus, Theorem 3.3 is not applicable in this case. However,

$$D(Tx,Ty) = \frac{1}{2}|x-y| = \frac{1}{2}D(x,y), \text{ for all } x,y \in X.$$

Then, by the Banach contraction principle, for each $x_0 \in X$, the sequence $(T^n x_0)_{n=1}^{+\infty}$ converges to the unique fixed point 1.

Conclusion 6.3. *Results in both metric fixed point theory and EL metric fixed point theory are applicable, and both of them should be developed.*

7. Final conclusions

Both metric fixed point theory and EL metric fixed point theory should be developed, as it has been explained by Example 6.1 and Example 6.2. Some more things have to be concluded for transformations.

The open mapping theorem and the closed graph theorem are equivalent in the sense that if one theorem is assumed then the other theorem can be derived from the first one. One may apply these theorems according to convenience in any particular format. A similar event happens in measure theory. Lebesgue monotone convergence theorem, Fatou's lemma and Lebesgue dominated convergence theorem are equivalent. They are applied according to convenience. So, there is a need to transform arguments and results of fixed point theory, because of the same reason.

The transformations used in this article are natural ones, but not trivial ones. The ultimate aim of transformation is to increase possibilities to derive new examples and results. The most expected application of fixed point theory is to increase possibilities to solve equations. The transform used in this article can be extended to many branches of analysis.

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Warped Translation Surfaces of Finite Type in Simply Isotropic 3-Spaces

Alev Kelleci Akbay

Department of Mathematics, Faculty of Science, Firat University, Elazig, Turkey

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Abstract

In this paper, we classify warped translation surfaces being invariant surfaces of i-type, that is, the generating curve has formed by the intersection of the surface with the isotropic xz-plane in the three-dimensional simply isotropic space $\mathbb{I}^3$ under the condition

$$\Delta^J x_i = \lambda_i x_i, \quad \text{with} \quad J = I, II.$$

Here, Δ^J is the Laplace operator with respect to first and second fundamental form and λ_i, $i = 1,2,3$ are some real numbers. Also, as an application, we give some examples for these surfaces and also some explicit graphics of them. All graphics have been plotted with Maple14.

1. Introduction

Let $\mathbb{E}^m$ denotes the m-dimensional Euclidean space and M^n be a connected n-dimensional submanifold in this space. An isometric immersion $\mathbf{x} : M \to \mathbb{E}^m$ is said to be of *k-type* if it can be expressed as a sum of eigenvectors of the Laplace-Beltrami operator of the induced metric Δ, corresponding to k distinct eigenvalues of Δ:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1 + \cdots + \mathbf{x}_k, \quad \text{such that} \quad \Delta\mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad i = 1, \ldots, k,$$

for a consant vector $\mathbf{x}_0$, smooth non-constant functions $\mathbf{x}_k$ and $\lambda_i \in \mathbb{R}$, [1]. If an isometric immersion $\mathbf{x}$ is of k-type, then the submanifold M is said to be of k-type [2, 3]. In [4], Chen gave a good survey related to finite type submanifolds. In [5], author proved that a submanifold M^n in $\mathbb{E}^m$ is of 1-type, that is, $\Delta\mathbf{x} = \lambda\mathbf{x}$, $\lambda \in \mathbb{R}^+$, if and only if it is either a minimal submanifold of $\mathbb{E}^m$ ($\lambda = 0$) or a minimal submanifold of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$ ($\lambda \neq 0$). In [6, 7], by generalizing of this, authors showed that if a hypersurface M^n of $\mathbb{E}^{n+1}$ satisfies

$$\Delta\mathbf{x} = A\mathbf{x}, \tag{1.1}$$

where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a diagonal matrix $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1})$. Moreover, Senoussi and Bekkar studied helicoidal surfaces in Euclidean 3-spaces satisfying the condition (1.1), [8]. Furthermore, in [9], Chen gave a detailed paper account of recent development about finite type submanifolds in Euclidean spaces.

On the other hand, very recently, the study on intrinsic (or extrinsic) properties of surfaces in (pseudo-) isotropic spaces has become a research subject for many researchers, see for examples [10]-[17]. Moreover, coordinate finite-type submanifolds

Email address and ORCID number: **alevkelleci@hotmail.com, https://orcid.org/0000-0003-2528-2131**

have been studied in isotropic spaces [18, 19]. Moreover, the study of finite type submanifolds was studied in simply isotropic spaces. In particular, Karacan et. al. studied translation surfaces and surfaces of revolution satisfying

$$\Delta^J x_i = \lambda_i x_i, \tag{1.2}$$

where $J = I, II$ and $i = 1,2,3$, in these spaces in [20, 21] and [22], respectively. Also, in [23], [24] and [25], authors studied affine translation surfaces, helicoidal surfaces and ruled surfaces satisfying the same condition.

In this paper, we are going to study on warped translation surfaces of finite type in three dimensional Isotropic space $\mathbb{I}^3$ satisfying (1.2).

2. Preliminaries

The simply isotropic 3-space $\mathbb{I}^3$ is a Cayley-Klein space defined from the 3-dimensional real projective space $\mathbb{P}^3(\mathbb{R})$ with the absolute figure as given $\{\omega, d_1, d_2, f\}$. In this space, the homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ are presented such that $\omega : x_0 = 0$ is a plane in $\mathbb{P}^3(\mathbb{R})$, $d_1 : x_0 = 0 = x_1 + \mathrm{i}x_2$ and $d_2 : x_0 = 0 = x_1 - \mathrm{i}x_2$ are two complex-conjugate straight lines in the plane, and also $f = [0 : 0 : 0 : 1]$ is a point in the intersection $d_1 \cap d_2$.

The group of motions of $\mathbb{I}^3$ is a six-parameter group given by [26]

$$\begin{aligned} \tilde{x} &= a_0 + x\cos\phi - y\sin\phi, \\ \tilde{y} &= a_1 + x\sin\phi + y\cos\phi, \\ \tilde{z} &= a_2 + c_1 x + c_2 y + z, \end{aligned} \tag{2.1}$$

where $\phi, a_0, a_1, a_2, c_1, c_2 \in \mathbb{R}$. Concerning this group of i-motions, it can be easily seen that these motions are indeed composed of an Euclidean motions onto the xy-plane and an affine shear transformation in z-direction. Thus, the projection of a point $S(x,y,z)$, in the z-direction onto $\mathbb{R}^2$, $\tilde{S}(x,y,0)$ is called the *top view* of S. Let $\vec{A} = (x_1, x_2, x_3)$ be a vector in $\mathbb{I}^3$. If $x_1 = x_2 = 0$, then $\vec{A}$ is called as *isotropic*, otherwise *non-isotropic*. A plane having an isotropic line is said to be an *isotropic plane* and a line with an isotropic director is an *isotropic line*.

Given two vectors $\vec{A} = (x_1, x_2, x_3)$ and $\vec{B} = (y_1, y_2, y_3)$, the *isotropic inner product* is calculated by [26]

$$\left\langle \vec{A}, \vec{B} \right\rangle = x_1 y_1 + x_2 y_2.$$

Moreover, M^2 is called as an *admissible surface* when the metric in M^2 induced by the isotropic scalar product has rank 2. More precisely, M^2 parameterized by a C^2 map $\mathbf{x}(u^1,u^2) = \left(x^1(u^1,u^2), x^2(u^1,u^2), x^3(u^1,u^2)\right)$, is admissible if and only if $X_{12} = x_1^1 x_2^2 - x_2^1 x_1^2 \neq 0$, where $x_k^i = \partial x^i / \partial u^k$ and

$$X_{ij} = \left| x_1^i x_2^j - x_1^j x_2^i \right|, \tag{2.2}$$

[17, 26]. As a result, every admissible C^2 surface M^2 can be locally parameterized as $\mathbf{x}(u^1,u^2) = \left(u^1, u^2, f(u^1,u^2)\right)$: one can say that M is in its *normal form*.

Furthermore, the *isotropic first and second fundamental forms* I and II, and also their coefficients of the *isotropic metric* tensor g_{ij} and h_{ij} are given by, respectively, [17]

$$\begin{aligned} \mathrm{I} &= g_{ij} u^i u^j \text{ and } g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle, \\ \mathrm{II} &= h_{ij} u^i u^j \text{ and } h_{ij} = \mathbf{x}_{ij} \cdot \mathbf{N}_m, \end{aligned} \tag{2.3}$$

where

$$\mathbf{N}_m = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{X_{12}} = \left(\frac{X_{23}}{X_{12}}, \frac{X_{31}}{X_{12}}, 1\right). \tag{2.4}$$

Here, we may call $\mathbf{N}_m$ the *minimal normal* because the trace of the Weingarten-like operator $-\mathbf{N}_m$ vanishes identically.

Also, the *isotropic Gaussian* and *mean curvatures* are given by [17]

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} \text{ and } H = \frac{1}{2} \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - g_{12}^2}. \tag{2.5}$$

Note that, a surface in $\mathbb{I}^3$ is called as isotropic flat (resp. isotropic minimal) if K (resp. H) vanishes.

Consequently, according to a local coordinate system, the Laplacian Δ^J, $J = I, II$ in terms of the first and second fundamental forms are defined as usual by, ([20, 21],[26])

$$\Delta^I = -\frac{1}{\sqrt{g_{11}g_{22}-g_{12}^2}}\left[\partial_1\left(\frac{g_{22}\partial_1 - g_{12}\partial_2}{\sqrt{g}}\right)+\partial_2\left(\frac{g_{11}\partial_2 - g_{12}\partial_1}{\sqrt{g}}\right)\right], \tag{2.6}$$

and

$$\Delta^{II} = -\frac{1}{\sqrt{h_{11}h_{22}-h_{12}^2}}\left[\partial_1\left(\frac{h_{22}\partial_1 - h_{12}\partial_2}{\sqrt{h_{11}h_{22}-h_{12}^2}}\right)+\partial_2\left(\frac{h_{11}\partial_2 - h_{12}\partial_1}{\sqrt{h_{11}h_{22}-h_{12}^2}}\right)\right], \tag{2.7}$$

where $\partial_i = \partial/\partial u^i$ and g^{ij} is the inverse of the metric, that is, $g^{ik}g_{kj} = \delta^i_j$. Moreover, throughout paper we will take as $g_{11}g_{22} - g_{12}^2 \neq 0$ and $h_{11}h_{22} - h_{12}^2 \neq 0$.

2.1. Warped translation surface in Simply Isotropic 3-space

In this work, we will be working on warped translation surfaces being one of the types of invariant surfaces in $\mathbb{I}^3$ and some algebraic equations in terms of the Laplacian operator and the coordinate functions of these surfaces. So, in this section, we will work to explain how warped translation surfaces in $\mathbb{I}^3$ are parameterized, (For more details, see [17].)

Let M^2 be a *warped translation* surface being invariant. So, M^2 can be parametrized as

$$M^2_{(a_0,a_1,0,c_1,c_2)} : \mathbf{P}(u,v) = (a_0v + x(u), a_1v, c_1vx(u) + z(u)), \tag{2.8}$$

such that $a_2 = (a_0c_1 + a_1c_2) = 0$, (a_0, a_1), $(c_1, c_2) \neq (0,0)$. Also, $\phi, a_0, a_1, a_2, c_1, c_2$ are the real constants as in Eq. (2.1).

Notice that since all simply isotropic invariant surfaces are admissible, throughout paper we will assume that warped translation surfaces are admissible, (see for more details, [17].)

3. Warped translation surfaces of finite type

As mentioned in the previous section, the warped translation surfaces can be parametrized as in (2.8) in Isotropic 3-spaces. In this section, we calculate the Laplacian operator Δ^J for these surfaces in $\mathbb{I}^3$. And then, we examine the warped translation surfaces satisfying the condition (1.2). Finally, we give the complete classification of these surfaces of finite type in $\mathbb{I}^3$.

Now, let us consider on a warped translation surface $M^2_{(a_0,a_1,0,c_1,c_2)}$ defined as in (2.8) with the generating curve α, $\alpha(u) = (u, 0, z(u))$, i.e.,

$$\mathbf{x}(u,v) = (a_0v + u, a_1v, c_1uv + z(u)),\ a_1 > 0. \tag{3.1}$$

Thus, we have

$$\begin{aligned}\mathbf{x}_u &= (1, 0, c_1v + z'),\\ \mathbf{x}_v &= (a_0, a_1, c_1u).\end{aligned}$$

Since $M^2_{(a_0,a_1,0,c_1,c_2)}$ is admissible, i.e., $a_1 \neq 0$ from (2.2), then $\mathbf{N}_m$ the minimal normal defined by (2.4) is computed as

$$\mathbf{N}_m = (-c_1v - z', \frac{a_0z' + c_1(a_0v - u)}{a_1}, 1).$$

By considering (2.3), we obtain the corresponding fundamental forms as [17]

$$\mathrm{I} = 1\mathrm{d}u^2 + 2a_0\mathrm{d}u\mathrm{d}v + ({a_0}^2 + {a_1}^2)\mathrm{d}v^2 \text{ and } \mathrm{II} = z''\mathrm{d}u^2 + 2c_1\mathrm{d}u\mathrm{d}v,$$

and from (2.5), the Gaussian and mean curvatures are

$$K = -\frac{c_1^2}{{a_1}^2} \text{ and } H = -\frac{a_0c_1}{{a_1}^2} + \frac{({a_0}^2 + {a_1}^2)}{2{a_1}^2}z''.$$

Finally, the Laplace-Beltrami operators defined as in (2.6) and (2.7) of a warped translation surface are obtained as, respectively,

$$\Delta^I = -\frac{{a_0}^2 + {a_1}^2}{{a_1}^2}\frac{\partial^2}{\partial u^2} + \frac{2a_0}{{a_1}^2}\frac{\partial^2}{\partial u\partial v} - \frac{1}{{a_1}^2}\frac{\partial^2}{\partial v^2}, \tag{3.2}$$

and

$$\Delta^{II} = -\frac{2}{c_1}\frac{\partial^2}{\partial u \partial v} + \frac{z''}{{c_1}^2}\frac{\partial^2}{\partial v^2},$$

where c_1 is a non-zero constant.

Now, firstly we would like to give the following theorem being the classification of parabolic revolution surfaces satisfying (3.2) in $\mathbb{I}^3$.

Theorem 3.1. *Let M^2 be a warped translation surface given by (3.1) in $\mathbb{I}^3$ such that it satisfies the condition $\Delta^I x_i = \lambda_i x_i$, where λ_i, $i = 1,2,3$ are some real constants. Then M^2 refers to one of the followings:*

1. *If $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 0$, then the function $z(u)$ is quadratic.*
2. *If $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$, then $(a_0, a_1, 0, c_1, c_2) = (a_0, a_1, 0, 0, c_2)$ and z is given by either*
   (a) $z(u) = z_1 \cosh(\sqrt{\Lambda_3}\, u) + z_2 \sinh(\sqrt{\Lambda_3}\, u)$, if $\lambda_3 > 0$, or
   (b) $z(u) = z_1 \cos(\sqrt{-\Lambda_3}\, u) + z_2 \sin(\sqrt{-\Lambda_3}\, u)$, if $\lambda_3 < 0$,

   where $\Lambda_3 = \frac{\lambda_3 {a_1}^2}{{a_0}^2 + {a_1}^2}$.

Proof. Assume that M^2 is a warped translation surface given by (3.1) and it satisfies the condition (1.2) for $J = I$. Let us take the expressions

$$\Delta^I x = \left(\Delta^I x_1, \Delta^I x_2, \Delta^I x_3\right),$$

(2.6) and (3.1) together. Thus by a straightforward computation, we get

$$\Delta^I x = \left(0, 0, -z''\frac{{a_0}^2 + {a_1}^2}{{a_1}^2} + \frac{2a_0 c_1}{{a_1}^2}\right).$$

So, as M^2 satisfies the condition $\Delta^I x_i = \lambda_i x_i$, where λ_i, $i = 1,2,3$ are some real constants, we have

$$0 = \lambda_1(a_0 v + u), \tag{3.3}$$

$$0 = \lambda_2 a_1 v, \tag{3.4}$$

$$-z''\frac{{a_0}^2 + {a_1}^2}{{a_1}^2} + \frac{2a_0 c_1}{{a_1}^2} = \lambda_3\left(c_1 uv + z\right). \tag{3.5}$$

So, from (3.3) and (3.4), we get directly $\lambda_1 = \lambda_2 = 0$. Now, we will consider on two possibilities coming from (3.5). First, if $\lambda_3 = 0$ then we get the following ODE

$$\frac{{a_0}^2 + {a_1}^2}{{a_1}^2} z'' - \frac{2a_0 c_1}{{a_1}^2} = 0, \tag{3.6}$$

whose solutions are given as in Case (1) in Theorem 3.1. Secondly, let $\lambda_3 \neq 0$. By considering $z = z(u)$ in (3.5), we obtain $c_1 = 0$ and

$$z'' + \frac{\lambda_3 {a_1}^2}{{a_0}^2 + {a_1}^2} z = 0.$$

By taking $\Lambda_3 = -\frac{\lambda_3 {a_1}^2}{{a_0}^2 + {a_1}^2}$, we can rewrite the last ODE as

$$z'' - \Lambda_3 z = 0,$$

whose solutions is given as in Case(2) in Theorem 3.1. □

Remark 3.2. *By comparing the second equality of (3.2) and (3.6), we conclude that the warped translation surface M^2 parametrized as in Case (1) in Theorem 3.1 is a isotropic minimal surface in $\mathbb{I}^3$.*

Remark 3.3. *By comparing the first equality of (3.2) and (3.6), we conclude that the warped translation surface M^2 parametrized as in Case (2) in Theorem 3.1 is a isotropic flat surface in $\mathbb{I}^3$.*

By considering the above Remark 3.2, we have the following:

Corollary 3.4. *A warped translation surface given by (3.1) in the three dimensional simply isotropic space $\mathbb{I}^3$ is harmonic if and only if the surface M^2 is isotropic minimal.*

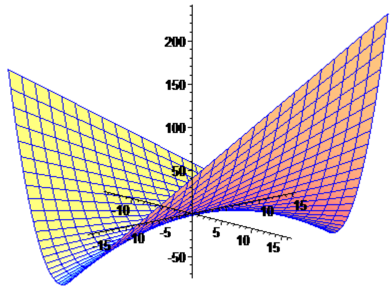

Figure 3.1: An isotropic minimal warped translation surface is parametrized as in Case (1) in Theorem 3.1.

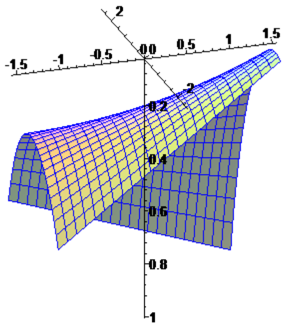

Figure 3.2: An isotropic flat warped translation surface is parametrized as in Case (2a) in Theorem 3.1.

Now, we would like to give some explicit examples of warped translation surfaces satisfing (1.2) for $J = I$ in $\mathbb{I}^3$:

Secondly, we would like to give the following theorem being the classification of warped translation surfaces satisfying (1.2) for $J = II$ in $\mathbb{I}^3$.

Theorem 3.5. *Let M^2 be a warped translation surface given by (3.1) in $\mathbb{I}^3$ such that it satisfies the condition $\Delta^{II} x_i = \lambda_i x_i$, where λ_i, $i = 1,2,3$ are some real constants. Then $M^2_{(a_0,a_1,0,0,c_2)}$ can be parametrized as*

$$\mathbf{x}(u,v) = (a_0 v + u, a_1 v, c_1 uv - \frac{2}{\lambda_3}),\ a_1 > 0.$$

Proof. Let M^2 be a warped translation surface given by (3.1) satisfying (1.2) and

$$\Delta^{II} x = \left(\Delta^{II} x_1, \Delta^{II} x_2, \Delta^{II} x_3\right).$$

By a straightforward computation, we get from (2.7)

$$\Delta^{II} x = \left(0, 0, -2\right).$$

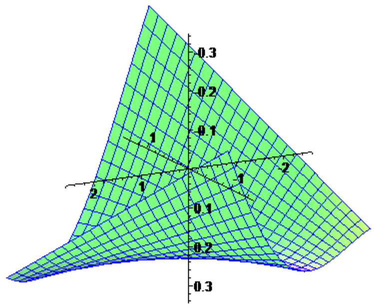

Figure 3.3: An isotropic flat warped translation surface is parametrized as in Case (2b) in Theorem 3.1.

So, as M^2 satisfies the condition $\Delta^{II} x_i = \lambda_i x_i$, where λ_i, $i = 1,2,3$ are some real constants, we have

$$\begin{aligned} 0 &= \lambda_1 (a_0 v + u), & (3.7)\\ 0 &= \lambda_2 a_1 v, & (3.8)\\ -2 &= \lambda_3 \Big(c_1 uv + z(u) \Big). & (3.9) \end{aligned}$$

From (3.7) and (3.8), we get directly, $\lambda_1 = \lambda_2 = 0$. Now by considering (3.9), we conclude directly that $\lambda_3 \neq 0$. And so we obtain $c_1 = 0$ and $z = -\frac{2}{\lambda_3}$. Thus M^2 can be parametrized as in Theorem 3.5. □

Definition 3.6. *A surface in a simply isotropic 3-space, $\mathbb{I}^3$ is called as $II-$harmonic if it satisfies the condition $\Delta^{II} x = 0$, [20].*

By considering the above Definition and the proof of Theorem 3.5, we have the following:

Corollary 3.7. *There are no $II-$harmonic warped translation surface in $\mathbb{I}^3$.*

Now, we would like to give some explicit examples of warped translation surfaces satisfying $\Delta^{II} x_i = \lambda_i x_i$ in $\mathbb{I}^3$:

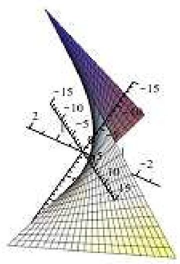

Figure 3.4: A warped translation surface is parametrized as in Theorem 3.5.

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A Comparative Study of the Numerical Approximations of the Quenching Time for a Nonlinear Reaction-Diffusion Equation

Frederick Jones[1] and He Yang[1*]

[1]*Department of Mathematics, Augusta University, Augusta, USA*
[*]*Corresponding author*

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Abstract

In this paper, we study the numerical methods for solving a nonlinear reaction-diffusion model for the polarization phenomena in ionic conductors. In particular, we propose three types of numerical methods, including the finite difference, cubic B-spline collocation, and local discontinuous Galerkin method, to approximate the quenching time of the model. We prove the conservation properties for all three numerical methods and compare their numerical performance.

1. Introduction

In this paper, we investigate some numerical approximations of the quenching time of the solutions to the following nonlinear reaction-diffusion equation

$$u_t(x,t) = u_{xx}(x,t) + \frac{1}{1-u(x,t)}, \quad 0<x<L,\ 0<t<T, \tag{1.1}$$

subject to the initial condition $u(x,0)=0$ and two types of boundary conditions, i.e., the zero Neumann boundary condition

$$u_x(0,t) = u_x(L,t) = 0, \tag{1.2}$$

and the zero Dirichlet boundary condition

$$u(0,t) = u(L,t) = 0. \tag{1.3}$$

Here $L>0$ is a given constant. The quenching time of the solutions to the equation (1.1) with some initial and boundary conditions is defined to be the finite time T (if it exists), such that

$$\lim_{t\to T^-} \max\{u(x,t) : 0 \le x \le L\} \to 1,$$

where $u(x,t)$ is the solution [1]. One can show that the initial-boundary value problem (1.1),(1.2) along with the zero initial condition leads to quenching at $T=1/2$. For the initial-boundary value problem (1.1),(1.3) with the zero initial condition,

Email addresses and ORCID numbers: **fjones19@augusta.edu, https://orcid.org/0000-0000-0000-0000 (F. Jones), hyang1@augusta.edu, https://orcid.org/0000-0001-9608-4920 (H. Yang)**

it has been shown that the quenching time exists if $L > 2\sqrt{2}$ (see [2]). However, there is no analytical formulation for the quenching time. The objective of this paper is to propose numerical methods to approximate the quenching time, and prove their conservation property and compare their numerical performance. Specifically, we design the finite difference, cubic B-spline collocation and local discontinuous Galerkin method for equation (1.1) with Dirichlet and Neumann boundary conditions.

Finite difference methods have been widely used to solve a great number of differential equation models due to its simplicity. For certain nonlinear acoustics problems, however, it has been shown that the local discontinuous Galerkin (LDG) methods lead to more accurate solution than the benchmark finite difference methods in literature [3]. The LDG methods, as a special type of finite element methods, are suitable for complicated geometry and parallel computing. It is also easy to design LDG methods with high order of accuracy and conservation properties [4]-[6]. The cubic B-spline collocation methods have also been applied to solve various partial differential equations [7]. In addition, the Galerkin approach with the cubic B-spline has been applied to solve the modified regularized long wave equation [8], the generalized regularized long wave equation [9, 10], and the Benjamin-Bona-Mahony-Burgers equation [11]. One of the advantages of the cubic B-spline collocation methods over the finite element type of methods is that the calculation of coefficient matrices does not require numerical quadrature [12]. In this work, we propose three numerical methods based on the finite difference, cubic B-spline collocation and LDG methods. We can prove that all three numerical methods satisfy certain conservation property. We also compute the quenching time of equation (1.1) with Dirichlet and Neumann boundary conditions using these methods. To make fair comparison, we compute the results using three numerical methods which have the same order accuracy in both time and space.

The remaining of the paper is organized as follows. In section 2, we present the numerical methods for the nonlinear reaction-diffusion equation. In section 3, we discuss the conservation properties of the proposed methods. In section 4, we show the numerical results for the approximation of quenching time.

2. Description of numerical methods

In this section, we describe our numerical methods for solving the nonlinear reaction-diffusion equation (1.1) with zero initial condition, and Neumann boundary condition (1.2) or Dirichlet boundary condition (1.3). In particular, we focus on the finite difference method (FDM), the cubic B-splines method and the local discontinuous Galerkin (LDG) method, and investigate the performance of these numerical methods.

2.1. Finite difference method

We partition the spatial domain $[0,L]$ into N elements uniformly using $(N+1)$ grid points: $0 = x_0 < x_1 < \ldots < x_N = L$. We denote the length of each element by h, thus we have $h = L/N$ and $x_i = ih$ for $i = 0,1,\ldots,N$. Suppose we want to solve the nonlinear reaction-diffusion equation (1.1) up to the final time T, then we discretize the temporal domain $[0,T]$ using $(M+1)$ uniformly spaced nodes $t_j := j\Delta t$ for $j = 0,1,\ldots,M$ with the step size $\Delta t = T/M$. Now we denote the numerical solution at $x = x_i$ and $t = t_j$ by U_i^j, then finite difference discretization of (1.1) can be written as

$$\frac{U_i^{j+1} - U_i^j}{\Delta t} = \frac{U_{i-1}^j - 2U_i^j + U_{i+1}^j}{h^2} + \frac{1}{1-U_i^j},$$

for $i = 1,2,\ldots,N-1$ and $j = 0,1,\ldots,M-1$. Here we have used the second-order centered difference in space and forward Euler method in time. That is,

$$u_t(x_i,t_j) \approx \frac{U_i^{j+1} - U_i^j}{\Delta t},$$

and

$$u_{xx}(x_i,t_j) \approx \frac{U_{i-1}^j - 2U_i^j + U_{i+1}^j}{h^2}.$$

For the case of homogeneous Dirichlet boundary condition, we have

$$U_0^j = U_N^j = 0,$$

for any j. Moreover, due to the initial condition, there is $U_i^0 = 0$ for any i. Therefore, at $(j+1)^{th}$ time level ($j \le M_1$), we update U_i^{j+1} by

$$U_i^{j+1} = rU_{i+1}^j + (1-2r)U_i^j + rU_{i-1}^j + \frac{\Delta t}{1-U_i^j}, \quad i = 1,2,\ldots,N-1, \tag{2.1}$$

and $U_0^{j+1} = U_N^{j+1} = 0$. Here $r = \Delta t/h^2$ has to be chosen small enough for the stability of the scheme.

For the case of homogeneous Neumann boundary condition, we introduce the ghost points $x_{-1} := x_0 - h$ and $x_{N+1} := x_N + h$, and approximate the Neumann boundary condition using the second-order centered difference, which leads to

$$U_{-1}^j = U_1^j, \quad U_{N+1}^j = U_{N-1}^j. \tag{2.2}$$

Then we update U_0^{j+1} and U_N^{j+1} by applying (2.1) at $i=0$ and $i=N$, as well as conditions in (2.2). Therefore, the scheme for this case can be written as

$$U_0^{j+1} = (1-2r)U_0^j + 2rU_1^j + \frac{\Delta t}{1-U_0^j}, \quad U_N^{j+1} = 2rU_{N-1}^j + (1-2r)U_N^j + \frac{\Delta t}{1-U_N^j}, \tag{2.3}$$

coupled with (2.1).

2.2. Cubic B-spline collocation method

The idea of the cubic B-spline collocation method is to look for the numerical solution represented using the cubic B-spline basis, such that equation (1.1) is satisfied at all the collocation points.

To describe the cubic B-spline collocation method, we use the same notation for the grid points in $[0,L]$ (see section 2.1), and define $x_j = jh$ for $j = 0, \pm 1, \pm 2, \ldots$. Let $B_j(x)$ be the cubic B-spline function defined as

$$B_j(x) = \frac{1}{h^3}\begin{cases} (x-x_{j-2})^3, & \text{if } x \in [x_{j-2}, x_{j-1}) \\ (x-x_{j-2})^3 - 4(x-x_{j-1})^3, & \text{if } x \in [x_{j-1}, x_j) \\ (x_{j+2}-x)^3 - 4(x_{j+1}-x)^3, & \text{if } x \in [x_j, x_{j+1}) \\ (x_{j+2}-x)^3, & \text{if } x \in [x_{j+1}, x_{j+2}] \\ 0, & \text{if } x \notin [x_{j-2}, x_{j+2}] \end{cases}$$

for any integer j. Note that each cubic B-spline basis function $B_j(x)$ is only nonzero when $x \in [x_{j-2}, x_{j+2}]$. Then the numerical solution $U(x,t)$ for $x \in [0,L]$ can be represented by

$$U(x,t) = \sum_{j=-1}^{N+1} \alpha_j(t) B_j(x), \tag{2.4}$$

since $B_j(x) = 0$ for $x \in [0,L]$ and any $j \le -2$ or $j \ge N+2$. Here $\alpha_j(t)$ for $j = -1, 0, \ldots, N+1$ are the unknown coefficients to be determined.

For the case of homogeneous Dirichlet boundary condition, we require $U(x,t)$ in (2.4) to satisfy equation (1.1) at x_j, for $j = 0, 1, \ldots, N$, and vanish at $x=0$ and $x=L$. These conditions lead to $(N+3)$ ordinary differential equations with $(N+3)$ unknowns, which can be further solved by any time discretization method. With the help of the results in Table 1, we can derive the formulation of the scheme as follows

$$\alpha'_{j-1} + 4\alpha'_j + \alpha'_{j+1} = \frac{6}{h^2}(\alpha_{j-1} - 2\alpha_j + \alpha_{j+1}) + \frac{1}{1-\alpha_{j-1} - 4\alpha_j - \alpha_{j+1}}, \tag{2.5}$$

for $j = 0, 1, \ldots, N$, coupled with

$$U(x_0,t) = \alpha_{-1} + 4\alpha_0 + \alpha_1 = 0, \tag{2.6}$$
$$U(x_N,t) = \alpha_{N-1} + 4\alpha_N + \alpha_{N+1} = 0. \tag{2.7}$$

If we apply (2.6) to (2.5) at $j=0$, then we can show that $\alpha_{-1} - 2\alpha_0 + \alpha_1 = -h^2/6$, which leads to $\alpha_0 = h^2/36$ and $\alpha_{-1} = -h^2/9 - \alpha_1$. Similarly, if we apply (2.7) to (2.5), we can show that $\alpha_N = h^2/36$ and $\alpha_{N+1} = -h^2/9 - \alpha_{N-1}$. If we further use these equations in (2.5), we can derive the following ODE system

$$A\frac{d\alpha}{dt} = \mathbf{f}, \tag{2.8}$$

where

$$A = \begin{bmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_{N-2} \\ \alpha_{N-1} \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_{N-2} \\ f_{N-1} \end{bmatrix},$$

	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$B_j(x)$	0	1	4	1	0
$B'_j(x)$	0	$3/h$	0	$-3/h$	0
$B''_j(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

Table 1: Evaluation of cubic B-spline basis function and its derivatives at grid points.

with $f_j = 6(\alpha_{j-1} - 2\alpha_j + \alpha_{j+1})/h^2 + 1/(1 - \alpha_{j-1} - 4\alpha_j - \alpha_{j+1})$. We then solve equation (2.8) using the forward Euler method.

For the case of homogeneous Neumann boundary condition, we impose the boundary condition for $U(x,t)$ defined in (2.4). Thus, we have

$$U_x(x_0,t) = \frac{3}{h}(\alpha_1 - \alpha_{-1}) = 0$$
$$U_x(x_N,t) = \frac{3}{h}(\alpha_{N+1} - \alpha_{N-1}) = 0,$$

which leads to $\alpha_{-1} = \alpha_1$ and $\alpha_{N+1} = \alpha_{N-1}$. We then apply these equations to (2.5) for $j = 0,1,\ldots,N$, and obtain the following ODE system

$$\tilde{A}\frac{d\tilde{\alpha}}{dt} = \tilde{\mathbf{f}}, \tag{2.9}$$

where

$$\tilde{A} = \begin{bmatrix} 4 & 2 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 2 & 4 \end{bmatrix}, \quad \tilde{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \vdots \\ \alpha_{N-1} \\ \alpha_N \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{f}} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}.$$

Note that we can compute f_0 and f_N using $f_0 = 12(\alpha_1 - \alpha_0)/h^2 + 1/(1 - 2\alpha_1 - 4\alpha_0)$ and $f_N = 12(\alpha_{N-1} - \alpha_N)/h^2 + 1/(1 - 2\alpha_{N-1} - 4\alpha_N)$. Similarly, we can then solve equation (2.9) using the forward Euler method.

2.3. Local discontinuous Galerkin method

To define the local discontinuous Galerkin (LDG) method for the initial-boundary value problem, we use $I_j := [x_{j-1}, x_j]$ to denote the j^{th} element, and rewrite (1.1) as

$$u_t = q_x + \frac{1}{1-u}, \quad q = u_x.$$

Let V_h^k be the piecewise defined polynomial space given by $V_h^k = \{v : v|_{I_j} \in P^k(I_j), \forall I_j\}$, where $P^k(I_j)$ is the space of polynomials of degree up to k on I_j. The LDG scheme is to look for u_h and $q_h \in V_h^k$, such that

$$\int_{I_j} \frac{\partial}{\partial t} u_h v\, dx = -\int_{I_j} q_h v_x\, dx + \widehat{q}_j v_j^- - \widehat{q}_{j-1} v_{j-1}^+ + \int_{I_j} \frac{v}{1-u_h} dx, \tag{2.10}$$
$$\int_{I_j} q_h w\, dx = -\int_{I_j} u_h w_x\, dx + \widehat{u}_j w_j^- - \widehat{u}_{j-1} w_{j-1}^+, \tag{2.11}$$

for any $v, w \in V_h^k$ and $j = 1,2,\ldots,N$. Here $v_j^- := \lim_{\varepsilon \to 0+} v(x_j - \varepsilon)$ and $v_{j-1}^+ := \lim_{\varepsilon \to 0+} v(x_{j-1} + \varepsilon)$ are the left- and right-hand limit of $v(x)$ at $x_{j+\frac{1}{2}}$ and $x_{j-\frac{1}{2}}$, respectively. Similar definition holds for w_j^- and w_{j-1}^+. Due to the discontinuity of the numerical solution at the grid points, we use $\widehat{q}_j$ and $\widehat{u}_j$ for $j = 0,1,\ldots,N$, to denote the so-called numerical fluxes, which will be carefully chosen for different boundary conditions.

For the initial-boundary value problem with the homogeneous Dirichlet boundary condition, we choose

$$\widehat{q}_j = (q_h)_j^+, \quad \widehat{u}_j = (u_h)_j^-, \quad \text{for } j = 1,2,\ldots,N-1. \tag{2.12}$$

That is, at any interior grid points x_j, the numerical fluxes $\widehat{q}_j$ and $\widehat{u}_j$ are computed using $q_h|_{I_{j+1}}$ (i.e., the restriction of the function q_h on the interval I_{j+1}) and $u_h|_{I_j}$ (i.e., the restriction of the function u_h on the interval I_j), respectively. At the left and right endpoints, we take

$$\widehat{q}_0 = (q_h)_0^+, \quad \widehat{u}_0 = 0, \quad \widehat{q}_N = (q_h)_N^- - \frac{1}{h}(u_h)_N^-, \quad \widehat{u}_N = 0. \tag{2.13}$$

Note that we have used the penalty term $(u_h)_N^-/h$ in the definition of $\widehat{q}_N$. Such a term is crucial when we design the numerical scheme [13].

For the problem with the homogeneous Neumann boundary condition, we use (2.12) along with the following choice at the boundary [14]:

$$\widehat{q}_0 = 0, \quad \widehat{u}_0 = (u_h)_0^+, \quad \widehat{q}_N = 0, \quad \widehat{u}_N = (u_h)_N^-. \tag{2.14}$$

Let $\{v_1^j(x), v_2^j(x), \ldots, v_{k+1}^j(x)\}$ be a local basis of V_h^k on I_j $(1 \le j \le N)$. We can express the numerical solution on I_j as

$$u_h|_{I_j} = \sum_{l=1}^{k+1} C_l^j(t) v_l^j(x), \quad q_h|_{I_j} = \sum_{l=1}^{k+1} D_l^j(t) v_l^j(x), \tag{2.15}$$

where $\{C_l^j(t)\}_{l=1,\ldots,k+1}$ and $\{D_l^j(t)\}_{l=1,\ldots,k+1}$ are the unknown coefficients to be determined. If we use (2.15) in equations (2.10)-(2.13) (for the case of Dirichlet boundary condition), or equations (2.10)-(2.12), (2.14) (for the case of Neumann boundary condition), we can derive an ODE system of $C_l^j(t)$ and $D_l^j(t)$ for $l = 1, 2, \ldots, k+1$ and $j = 1, 2, \ldots, N$, which will be solved by any time integration method.

3. Conservation properties of numerical methods

In this section, we present some properties of the three numerical methods described in section 2, when they are used to solve the initial-boundary value problem (1.1) with zero initial condition and the boundary condition (1.2). Throughout this section, we assume that the quenching time of such a problem is $T > 0$. Then it is easy to verify that its exact solution $u(x,t)$ satisfies the following equalities:

$$\int_0^L u_t \, dx = \int_0^L \frac{1}{1-u} dx, \tag{3.1}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2 dx + \int_0^L u_x^2 \, dx = \int_0^L \frac{u}{1-u} dx, \tag{3.2}$$

for any $t < T$. We will show that similar equalities also hold for the numerical solutions.

Theorem 3.1. *The numerical solution by the finite difference method in equation (2.1) and (2.3) satisfies the following equation*

$$h\sum_{i=0}^{N} d_i U_i^{j+1} = h\sum_{i=0}^{N} d_i U_i^j + h\Delta t \sum_{i=0}^{N} \frac{d_i}{1-U_i^j}, \tag{3.3}$$

if $U_i^j < 1$ for $i = 0, \ldots, N$. Here $d_i = 1$ for $i = 0$ or N, and $d_i = 2$ for $i = 1, 2, \ldots, N-1$.

Proof. We multiply each equation in (2.1) by 2 and sum over i for $i = 1, 2, \ldots, N-1$, to get

$$\begin{aligned}
\sum_{i=1}^{N-1} d_i U_i^{j+1} &= \sum_{i=1}^{N-1} d_i U_i^j + \sum_{i=1}^{N-1} 2rU_{i-1}^j + \sum_{i=1}^{N-1} 2rU_{i+1}^j - \sum_{i=1}^{N-1} 4rU_i^j + \Delta t \sum_{i=1}^{N-1} \frac{d_i}{1-U_i^j} \\
&= \sum_{i=1}^{N-1} d_i U_i^j + 2rU_0^j - 2rU_1^j + 2rU_N^j - 2rU_{N-1}^j + \Delta t \sum_{i=1}^{N-1} \frac{d_i}{1-U_i^j}.
\end{aligned} \tag{3.4}$$

We then add each of the equations in (2.3) to equation (3.4), and obtain

$$U_0^{j+1} + U_N^{j+1} + \sum_{i=1}^{N-1} d_i U_i^{j+1} = U_0^j + U_N^j + \sum_{i=1}^{N-1} d_i U_i^j + \Delta t \sum_{i=1}^{N} \frac{d_i}{1-U_i^j}. \tag{3.5}$$

Since equation (3.5) is equivalent to (3.3), we can conclude the proof. □

Note that $h\sum_{i=0}^{N} d_i U_i^{j+1}$ is the approximation of $\int_0^L u(x,t_{j+1})dx$ using the trapezoidal rule, and the right side of (3.3) is the approximation of $\int_0^L u(x,t_j)dx + \int_0^L \frac{1}{u(x,t_j)}dx$. Therefore, we can regard equation (3.3) as the discretization of the equality (3.1). Next, we consider the cubic B-spline method in (2.9) and we have the following theorem.

Theorem 3.2. *With the cubic B-spline method defined in (2.9), the following equality holds*

$$6h\sum_{i=0}^{N} d_i\alpha_i'(t) = \frac{h}{1-4\alpha_0-2\alpha_1} + \sum_{i=1}^{N-1}\frac{2h}{1-\alpha_{i-1}-4\alpha_i-\alpha_{i+1}} + \frac{h}{1-2\alpha_{N-1}-4\alpha_N}, \tag{3.6}$$

if $4\alpha_0+2\alpha_1<1$, $2\alpha_{N-1}+4\alpha_N<1$ and $\alpha_{i-1}+4\alpha_i+\alpha_{i+1}<1$ for $i=1,2,\dots,N-1$.

Proof. Equation (2.9) leads to

$$4\alpha_0' + 2\alpha_1' = \frac{12}{h^2}(\alpha_1-\alpha_0) + \frac{1}{1-4\alpha_0-2\alpha_1}, \tag{3.7}$$

$$\alpha_{j-1}' + 4\alpha_j' + \alpha_{j+1}' = \frac{6}{h^2}(\alpha_{j-1}-2\alpha_j+\alpha_{j+1}) + \frac{1}{1-\alpha_{j-1}-4\alpha_j-\alpha_{j+1}}, \tag{3.8}$$

$$2\alpha_{N-1}' + 4\alpha_N' = \frac{12}{h^2}(\alpha_{N-1}-\alpha_N) + \frac{1}{1-2\alpha_{N-1}-4\alpha_N}, \tag{3.9}$$

for $j=1,2,\dots,N-1$. We then multiply each equation in (3.8) by $2h$, sum over j and add (3.7) and (3.9), which leads to (3.6). □

Since $\int_0^L u_t\,dx \approx h\sum_{i=0}^{N} d_i U_t(x_i,t) = h\sum_{i=0}^{N} d_i(\alpha_{i-1}'+4\alpha_i'+\alpha_{i+1}') = 6h\sum_{i=0}^{N} d_i\alpha_i'$, and the right side of (3.6) is an approximation of $\int_0^L \frac{1}{1-u}dx$, equation (3.6) is also the discretization of the equality (3.1). The next theorem is about the conservation property of the local discontinuous Galerkin method (2.10), (2.11), (2.12) and (2.14).

Theorem 3.3. *Let u_h and q_h be the numerical solution to (2.10), (2.11), (2.12) and (2.14), then the following equality holds for sufficiently small $t>0$*

$$\frac{1}{2}\frac{d}{dt}\int_0^L u_h^2dx + \int_0^L q_h^2dx = \int_0^L \frac{u_h}{1-u_h}dx. \tag{3.10}$$

Proof. Let $v=u_h$ in (2.10), $w=q_h$ in (2.11) and add the resulting equations, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{I_j} u_h^2dx + \int_{I_j} q_h^2dx \qquad (3.11)\\
=\;& -\int_{I_j}(q_hu_h)_xdx + \widehat{q}_j(u_h)_j^- - \widehat{q}_{j-1}(u_h)_{j-1}^+ + \widehat{u}_j(q_h)_j^- - \widehat{u}_{j-1}(q_h)_{j-1}^+ + \int_{I_j}\frac{u_h}{1-u_h}dx\\
=\;& ((q_h)_{j-1}^+ - \widehat{q}_{j-1})(u_h)_{j-1}^+ + (\widehat{u}_j-(u_h)_j^-)(q_h)_j^- + \widehat{q}_j(u_h)_j^- - \widehat{u}_{j-1}(q_h)_{j-1}^+ + \int_{I_j}\frac{u_h}{1-u_h}dx,
\end{aligned}$$

for $j=1,2,\dots,N$. Using equation (2.12), we can show that equation (3.11) leads to

$$\frac{1}{2}\frac{d}{dt}\int_{I_j} u_h^2dx + \int_{I_j} q_h^2dx = (q_h)_j^+(u_h)_j^- - (q_h)_{j-1}^+(u_h)_{j-1}^- + \int_{I_j}\frac{u_h}{1-u_h}dx, \tag{3.12}$$

for any $j=2,3,\dots,N-1$. For $j=1$, equation (3.11) can be simplified as

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{I_1} u_h^2dx + \int_{I_1} q_h^2dx &= ((q_h)_0^+ - \widehat{q}_0)(u_h)_0^+ + (q_h)_1^+(u_h)_1^- - \widehat{u}_0(q_h)_0^+ + \int_{I_1}\frac{u_h}{1-u_h}dx\\
&= (q_h)_1^+(u_h)_1^- + \int_{I_1}\frac{u_h}{1-u_h}dx, \qquad (3.13)
\end{aligned}$$

where we have used (2.12) and (2.14) in the first and second equality above, respectively. For $j=N$, equation (3.11) can be written as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{I_N} u_h^2dx + \int_{I_N} q_h^2dx \qquad (3.14)\\
=\;& ((q_h)_{N-1}^+ - \widehat{q}_{N-1})(u_h)_{N-1}^+ + (\widehat{u}_N-(u_h)_N^-)(q_h)_N^- + \widehat{q}_N(u_h)_N^- - \widehat{u}_{N-1}(q_h)_{N-1}^+ + \int_{I_N}\frac{u_h}{1-u_h}dx,\\
=\;& -(u_h)_{N-1}^-(q_h)_{N-1}^+ + \int_{I_N}\frac{u_h}{1-u_h}dx.
\end{aligned}$$

Finally, we add (3.12)-(3.14) over all $j = 2,3,\dots,N-1$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L u_h^2 dx + \int_0^L q_h^2 dx \\
= &\sum_{j=2}^{N-1}\left((q_h)_j^+(u_h)_j^- - (q_h)_{j-1}^+(u_h)_{j-1}^-\right) + (q_h)_1^+(u_h)_1^- - (u_h)_{N-1}^-(q_h)_{N-1}^+ + \int_0^L \frac{u_h}{1-u_h}dx, \\
= &\int_0^L \frac{u_h}{1-u_h}dx.
\end{aligned}$$

□

Note that equation (3.10) is the discretized version of the conservation property (3.2) in the PDE level.

4. Numerical experiments

In this section, we present the numerical results of the quenching time for the nonlinear reaction-diffusion equation (1.1). We compare the accuracy of the finite difference method, the cubic B-spline collocation method and the local discontinuous Galerkin method for the quenching time. We choose the polynomial degree to be $k = 1$ for the LDG method, so that all three numerical methods have the second-order accuracy in space. Numerically, we compute the quenching time T in the following way

$$T := \min_t \left(\max_{x\in[0,L]} U(x,t) \right) \geq 1-\varepsilon,$$

where $U(x,t)$ is the numerical solution, and ε is chosen to be 10^{-7}.

We first consider equation (1.1) with zero initial condition and the homogeneous Neumann boundary condition, in which case the exact solution is

$$u(x,t) = 1-\sqrt{1-2t}, \quad t \leq 1/2.$$

Thus the exact quenching time is $t = 1/2$. We then use the above-mentioned numerical methods to approximate the quenching time. The results for the case of Neumann boundary condition are shown in Table 2. We have used $\Delta t = \lambda h^2$ with $\lambda = 0.16$ in all the simulations for the table. We start with the numerical simulations using $N = 80$ uniform elements, and then double the number of elements a few times until $N = 640$. Suppose the errors corresponding to N_1 and $N_2 := N_1/2$ are E_1 and E_2, respectively. We can approximate the convergence order by

$$\text{Convergence order} \approx \log_2(E_2/E_1).$$

From Table 2, we observe the second order convergence of the quenching time for all three numerical methods. Moreover, we find that the numerical quenching time of the local discontinuous Galerkin method is exactly the same as that of the finite difference method. However, the results of the cubic B-spline method is less accurate then other two methods. Overall speaking, all of the methods lead to the numerical quenching time with comparable accuracy. Figure 4.1 shows the numerical solutions at various time, i.e., $t = 0.1, 0.2, \dots, 0.5$, when the finite difference method with 80 elements is used. We can see that the method captures the quenching phenomena, although the numerical solution at $t = 0.5$ is slightly smaller than the exact solution.

	Finite Difference			Cubic B-Spline			Local Discontinuous Galerkin		
N	Time	Error	Order	Time	Error	Order	Time	Error	Order
80	0.50120	1.2E-3	-	0.50200	2.0E-3	-	0.50120	1.2E-3	-
160	0.50030	3.0E-4	2	0.50050	5.0E-4	2	0.50030	3.0E-4	2
320	0.50010	1.0E-4	1.585	0.50015	1.5E-4	1.737	0.50010	1.0E-4	1.585
640	0.50003	2.5E-5	2	0.50004	3.75E-5	2	0.50003	2.5E-5	2

Table 2: Numerical results about the quenching time of equation (1.1) with homogeneous Neumann boundary condition.

Next, we consider equation (1.1) with zero initial condition and the homogeneous Dirichlet boundary condition. In [2], the author proved that this problem leads to quenching in a finite time at $x = L/2$. However, the analytical formulation or any estimation of the quenching time was not discussed. Here we compute the quenching time using the aforementioned numerical methods. In this case, we start with $N = 80$ elements and keep refining the mesh until $N = 2560$ elements. Figure 4.2 shows the solutions by the finite difference method with $N = 80$ elements. We observe that the numerical solution at $x = L/2$ grows at an increasing rate. When we compare the numerical results in Figure 4.1 and 4.2, we notice that the problem with the Dirchlet boundary condition (1.3) leads to larger quenching time than the problem with the Neumann boundary condition (1.2). The numerical solution $u_h(x,t)$ and $q_h(x,t)$ by the LDG method right before quenching occurs is shown in Figure 4.3. We observe

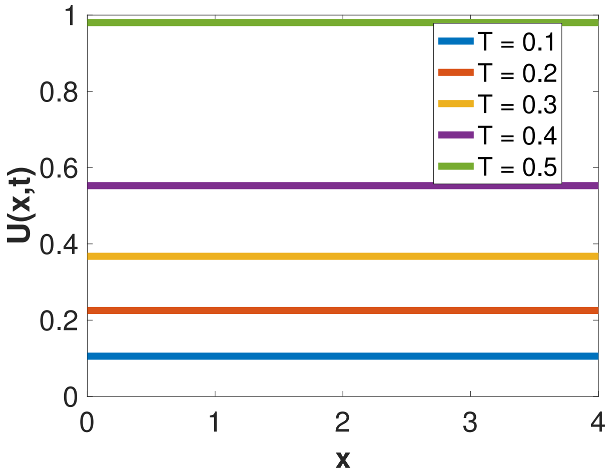

Figure 4.1: Solutions of equation (1.1) with homogeneous Neumann boundary condition by the finite difference method. $N = 80$ is used for the simulation.

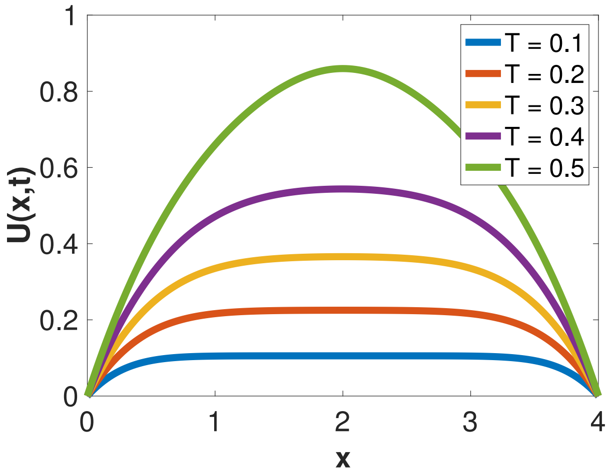

Figure 4.2: Solutions of equation (1.1) with homogeneous Dirichlet boundary condition by the finite difference method. $N = 80$ is used for the simulation.

that u_h is not very smooth at $x = 2$, and the quenching is about to happen at this point. Recall that q_h is the approximation of first derivative of the exact solution u. The right-hand figure 4.3 shows that the first derivative of the solution decreases, and jumps to a negative value at $x \to 2^+$.

Moreover, we compute the numerical quenching time for different numerical methods, and present the results in Table 3. We can see that the quenching time decreases monotonically at a decreasing rate for the finite difference and LDG methods, which indicates the convergence. However, the quenching time for the cubic B-spline method first decreases to 0.51035 and then increases to 0.51039. The results from the LDG method shows similar type of convergence as that from the finite difference method, i.e., the numerical quenching time converges to 0.51041 for both methods. Overall speaking, for both examples with the same number of elements, the LDG and finite difference method lead to slightly better results then the cubic B-spline method.

5. Conclusion

In this paper, we propose the finite difference, the cubic B-spline and the local discontinuous Galerkin methods to approximate the quenching time of a nonlinear reaction-diffusion equation with homogeneous Dirichlet or Neumann boundary condition. All of these numerical methods have displayed satisfactory results, with the second order convergence of the numerical quenching time. For the same number of elements in space, the finite difference and the local discontinuous Galerkin methods have slightly better performance than the cubic B-spline method. One can propose similar methods for higher dimensional system of reaction-diffusion equations. The theoretical analysis for the convergence of the numerical quenching time is currently under investigation.

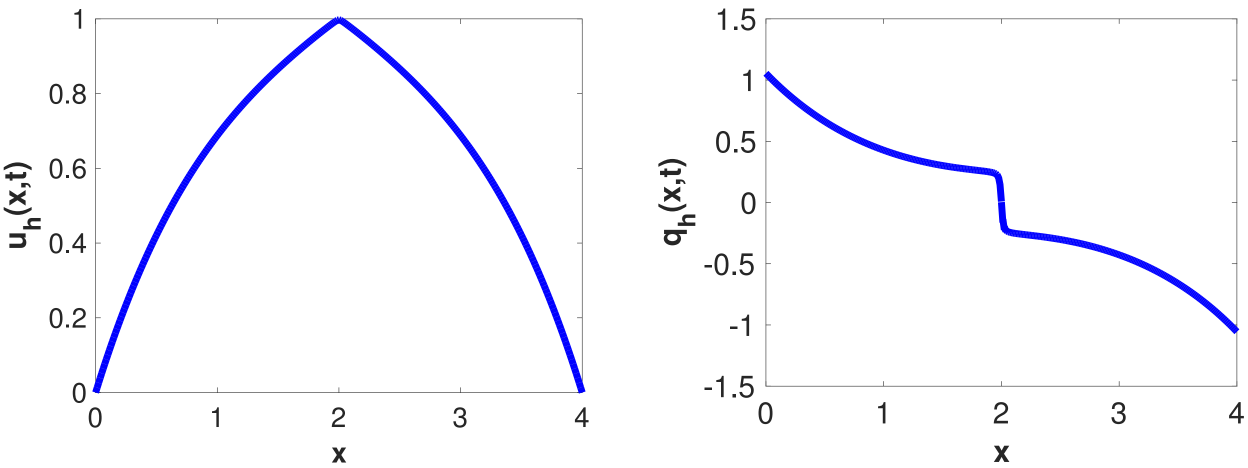

Figure 4.3: Numerical solution u_h and q_h of equation (1.1) with homogeneous Dirichlet boundary condition right before quenching occurs. Left: numerical solution u_h; right: numerical solution q_h The local discontinuous Galerkin method with $N = 320$ is used for the simulation.

N	Finite Difference	Cubic B-Spline	Local Discontinuous Galerkin
80	0.52250	0.51160	0.51070
160	0.51080	0.51050	0.51050
320	0.51053	0.51035	0.51043
640	0.51044	0.51035	0.51042
1280	0.51042	0.51037	0.51041
2560	0.51041	0.51039	0.51041

Table 3: Numerical results about the quenching time of equation (1.1) with Dirichlet boundary condition (1.3).

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On Solutions of a Higher Order Nonhomogeneous Ordinary Differential Equation

Elif Nuray Yıldırım[1*] and Ali Akgül[2]

[1]*Department of Mathematics, Faculty of Humanities and Social Sciences, İstanbul Commerce University, İstanbul, Turkey*
[2]*Department of Mathematics, Faculty of Science and Arts, Siirt University, Siirt, Turkey*

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Abstract

Higher order differential equations (ODE) has an important role in the modelling process. It is also much significant which the method is used for the solution. In this study, in order to get the approximate solution of a nonhomogeneous initial value problem, reproducing kernel Hilbert space method is used. Reproducing kernel functions have been obtained and the given problem transformed to the homogeneous form. The results have been presented with the graphics. Absolute errors and relative errors have been given in the tables.

1. Introduction

In this study, by using reproducing kernel method we aim to find the approximate solution of the problem in the form as:

$$(\mathbb{L}h)(x) = p(x)h'''(x) + q(x)h'(x) = f(x), \; a \leq x \leq b \tag{1.1}$$

The reproducing kernel method (RKM) have been used as an efficient way to solve different types of differential equations by many researcher for years. The theory of RKM was begin with the research of Aronszajn and Bergman [1, 2]. Since the method is very effective, many researcher applied the method to the several kind of problems. For instance Cui et al. [3] published a book about numerical analysis in the reproducing kernel space which is a comprehensive study. Syam et al. [4] used the method to solve a class of fractional Sturm-Liouville eigenvalue problems.

Jiang and Tian [5] solved the Volterra integro-differential equations of fractional order by the reproducing kernel method. Li et al. [6] applied the method for numerical solutions of fractional Riccati differential equations. For more details see [7]-[9].

In many models and problems, the equations need to be solved numerically. Therefore many approaches have been used and there have been lots of efforts for solving non-linear higher order ordinary differential equations in researches. For instance, Homotopy perturbation method [10], Adomian decomposition method [11], Chebyshev collocation method [12] used. Adomian decomposition method for solving initial value problems in second-order ordinary differential equations is given in [13]. Lu et al. Furthermore Runge- Kutta method [14], Predictor-Corrector method [15], decomposition method [16], direct block method [17], have been used for solving IVP. For a further reading and more details one can see [18]-[26].

Email addresses and ORCID numbers: ***Corresponding author: enuray@ticaret.edu.tr, https://orcid.org/0000-0002-2934-892X, aliakgul00727@gmail.com, https://orcid.org/0000-0001-9832-1424**

2. RKM for higher order nonhomogeneous ordinary differential equations

In this section, we will discuss the solution of a class of higher order ODEs for IVPs in the following form:

$$\begin{gathered}(\mathbb{L}h)(x) = p(x)h'''(x) + q(x)h'(x) = f(x),\ a \leq x \leq b \\ h^i(a) = \gamma_i,\ 0 \leq i \leq 2\end{gathered} \tag{2.1}$$

where $p(x)$ and $q(x)$ are continuous functions on the interval $[a,b]$. With the purpose of finding the solution of the problem (2.1) by using RKM (reproducing kernel method), we first need to define the RKS(reproducing kernel space) to which the equation belongs.The space construction is relevant to the order and conditions of the ODE that wanted to solved.

3. Construction of the method

To be able to solve the problem (2.1) using proposed method which we will denote RKM, we first construct reproducing kernel spaces. After the space construction, we will close to the approximate solution by obtaining the reproducing kernel functions belong to the given differential equation. In this section, we present some essential definitions and theorems in the theory of suggested method.

Definition 3.1 (Reproducing Kernel)**.** *[2] Let Q be a nonempty set. A function $R : Q \times Q \to \mathbb{F}$ is called a reproducing kernel of the Hilbert space $\mathscr{H}$ if and only if*

(a) $R(\cdot,x) \in \mathscr{H}, \quad \forall x \in Q,$

(b) $\langle\ \gamma\ , R(\cdot,x)\rangle = \gamma(x).$

The item (b) is called "reproducing property" of kernel R. The value of the function γ at the point x is reproduced by the inner product of γ with $R(\cdot,x)$.

Definition 3.2. *[3] The space $S_2^m[a,b]$ consist of the functions $h : [a,b] \to \mathbb{R}$ and define as follows:*

$$S_2^m[a,b] = \{h(x)|h^{(m-1)}(x) \in AC[a,b], \quad h^{(m)}(x) \in L^2[a,b], \quad x \in [a,b]\}. \tag{3.1}$$

$S_2^m[a,b]$ equipped with the inner product

$$< h,t >_{S_2^m} = \sum_{i=0}^{m-1} h^{(i)}(a)t^{(i)}(a) + \int_a^b h^{(m)}(x)t^{(m)}(x)dx.$$

Here we denote the vector space of absolutely continuous (real-valued) functions with $AC[a,b]$ and the quadratically integrable functions on the interval $[a,b]$ with $L^2[a,b]$.

Lemma 3.3. *If a Hilbert space has a reproducing kernel, it is called a reproducing kernel Hilbert space (RKHS).*

Lemma 3.4. *[3] $S_2^m[a,b]$ function space is a reproducing kernel space.*

The reproducing kernel function of the space S_2^m can be written as:

$$R_x(y) = \begin{cases} R(x,y) = \sum\limits_{i=1}^{2m} h_i(y)x^{i-1},\ x \leq y, \\ \\ R(y,x) = \sum\limits_{i=1}^{2m} t_i(y)x^{i-1},\ x > y. \end{cases}$$

For the proof of Lemma 3.4 one can see [3].

In the next subsection, we present a special reproducing kernel function space on the interval $[0,4\pi]$.

3.1. $S_2^4[0,4\pi]$ Reproducing kernel space

According to the reproducing kernel theory, we construct the space with concerning of the order of the derivative in the problem. For this reason in the equation (3.1) we choose $m=4$. Let we now define the function space

$$S_2^4[0,4\pi]=\{h(x)|h''(x)\in AC[0,4\pi],\quad h'''(x)\in L^2[0,4\pi],\quad x\in[0,4\pi]\}$$

with the inner product

$$\begin{aligned}<h,R_y>_{S_2^4[0,4\pi]}&=h(0)R_y(0)+h'(0)R'_y(0)+h''(0)R''_y(0)-h^{(3)}(0)R_y^{(3)}(0)\\&+\int_0^{4\pi}\left(h(x)+(4)(x)R_y^{(4)}(x)\right)dx.\end{aligned}$$

Integrating this equation by parts for four times, we have

$$\begin{aligned}<h,R_y>_{S_2^4[0,4\pi]}&=h(0)R_y(0)+h'(0)R'_y(0)+h''(0)R''_y(0)-h^{(3)}(0)R_y^{(3)}(0)\\&+h^{(3)}(4\pi)R_y^{(4)}(4\pi)-h^{(3)}(0)R_y^{(4)}(0)-h''(4\pi)R_y^{(5)}(4\pi)\\&+h''(0)R_y^{(5)}(0)+h'(4\pi)R_y^{(6)}(4\pi)-h'(0)R_y^{(6)}(0)\\&-h(4\pi)R_y^{(7)}(4\pi)+h(0)R_y^{(7)}(0)+\int_0^{4\pi}h(x)R_y^{(8)}(x)dx.\end{aligned}$$

Because of the conditions, we get the following equations:

1. $R_y(0)=0$
2. $R'_y(0)=0$
3. $R''_y(0)=0.$

With these three functions being zero we obtain:

$$\begin{aligned}<h,R_y>_{S_2^4[0,4\pi]}&=h^{(3)}(0)R_y^{(3)}(0)+h^{(3)}(4\pi)R_y^{(4)}(4\pi)-h^{(3)}(0)R_y^{(4)}(0)\\&-h''(4\pi)R_y^{(5)}(4\pi)+h'(4\pi)R_y^{(6)}(4\pi)-h(4\pi)R_y^{(7)}(4\pi)\\&+\int_0^{4\pi}h(x)R_y^{(8)}(x)dx.\end{aligned}$$

When the equation is rearranged we get the following equations:

4. $R_y^{(3)}(0)-R_y^{(4)}(0)=0$
5. $R_y^{(4)}(4\pi)=0$
6. $R_y^{(5)}(4\pi)=0$
7. $R_y^{(6)}(4\pi)=0$
8. $R_y^{(7)}(4\pi)=0.$

Then we will get:

$$<h,R_y>_{S_2^4[0,4\pi]}=\int_0^{4\pi}h(x)R_y^{(8)}(x)dx$$

With the knowledge of reproducing kernel property, the function $u(y)$ can be written in the form:

$$<h,R_y>_{S_2^4[0,4\pi]}=h(y).$$

For this reason, we reach

$$\int_0^{4\pi} h(x)R_y^{(8)}(x)dx = h(y). \tag{3.2}$$

Because of the definition of Dirac-Delta function, it is obvious that the equation (3.2) is equal to the $\delta(x-y)$. That gives us the following equation:

$$R_y^{(8)}(x) = \delta(x-y).$$

When $x \neq y$, the reproducing kernel function R_y can be written in the form as:

$$R_y(x) = \begin{cases} \sum\limits_{k=1}^{8} c_k x^{k-1}, \, x \leq y, \\ \sum\limits_{k=1}^{8} d_k x^{k-1}, \, x > y. \end{cases}$$

By using the feature of Dirac-Delta function, the following equations can be written:

9. $R_{y^+}(y) = R_{y^-}(y)$
10. $R'_{y^+}(y) = R'_{y^-}(y)$
11. $R''_{y^+}(y) = R''_{y^-}(y)$
12. $R'''_{y^+}(y) = R'''_{y^-}(y)$
13. $R^{(4)}_{y^+}(y) = R^{(4)}_{y^-}(y)$
14. $R^{(5)}_{y^+}(y) = R^{(5)}_{y^-}(y)$
15. $R^{(6)}_{y^+}(y) = R^{(6)}_{y^-}(y)$
16. $R^{(7)}_{y^+}(y) - R^{(7)}_{y^-}(y) = 1.$

In order to find the reproducing kernel function of the given space, we need to solve the differential equation system above. For this purpose, we needed sixteen equation since the (3.1) has sixteen coefficients and we obtained them.If we solve the system thus we get the reproducing kernel function as:

$$R_y(x) = \frac{1}{36}y^3x^3 + \frac{1}{144}y^3x^4 - \frac{1}{240}y^2x^5 + \frac{1}{720}yx^6 - \frac{1}{5040}x^7.$$

The proof of the following theorem is similar to the proof of the Lemma 3.4, so we omit it.

Theorem 3.5. *The function space $S_2^4[0,4\pi]$ is a reproducing kernel Hilbert space.*

Reproducing kernel function and the RKS has a vital role in the way to the solution. In the next section we give other essential part of the method.

4. Implementation of the method

Firstly, we define the linear operator as

$$\mathbb{L} : S_2^m[0,4\pi] \longrightarrow S_2^{m-n}[0,4\pi]$$

such that

$$\mathbb{L}h(x) = K(x, h(x), h'(x), ..., h^{(n)}(x)).$$

It is known that the operator $\mathbb{L}$ is bounded.

After the operator, we now construct the orthogonal system of the space $S_2^m[0,4\pi]$. Let $\eta_i(x) = \overline{R_{x_i}}(x)$ and $\psi_i(x) = \mathbb{L}^*\eta_i(x)$. Here, $\mathbb{L}^*$ is adjoint operator of $\mathbb{L}$ and the set of x_i which denoted by $\{x_i\}_{j=1}^{\infty}$ is dens in the interval $[0,4\pi]$.

Theorem 4.1. *If $\{x_i\}_{i=1}^{\infty}$ is dense in the $[0,4\pi]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $S_2^m[0,4\pi]$.*

Proof. Let $\{x_i\}_{j=1}^{\infty}$ is dense in the interval $[0,4\pi]$. By using adjoint operator and reproducing kernel properties we can write

$$\langle h(x),\psi_i(x)\rangle = \langle h(x),\mathbb{L}^*\eta_i(x)\rangle = \langle \mathbb{L}h(x),\eta_i(x)\rangle = h(x_i) = 0,\ i = 1,2,...$$

With the knowledge of density of $\{x_i\}_{j=1}^{\infty}$ and considering the continutiy of $h(x)$, we arrive that $h(x) = 0$. □

In order to obtain the approximate solution, with the help of Gram-Schimidt orthogonalization process, the orthonormal system $\{\psi_i(x)\}_{j=1}^{\infty}$ need to be construct. It can be denoted as:

$$\{\bar{\psi}_i(x)\}_{j=1}^{\infty} = \sum_{r=1}^{i}\zeta_{ir}\psi_r(x),\ \zeta_{ii} > 0,\ (i = 1,2,...)$$

Here the function ζ_{ik} represents the orthogonal coefficients.

Theorem 4.2. *Let $\{x_i\}_{j=1}^{\infty}$ be dense in $[0,4\pi]$.If the problem (2.1) has a uniqe solution then it can be denoted as follow:*

$$h(x) = \sum_{i=1}^{\infty}\sum_{r=1}^{i}\zeta_{ir}K(x_r,h(x_r),h'(x_r),...,h^{(n)}(x_r))\bar{\psi}_i(x).$$

Proof. Let we choose the solution of the equation (2.1) as $h(x)$. By knowing that $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ is the orthonormal basis of the space, we can write the following equality:

$$h(x) = \sum_{i=1}^{\infty}\langle h(x),\bar{\psi}_i(x)\rangle\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{r=1}^{i}\zeta_{ir}\langle h(x),\bar{\psi}_i(x)\rangle\bar{\psi}_i(x).$$

Let we now do apply the feature of adjoint operator at this step.

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{r=1}^{i}\zeta_{ir}\langle h(x),\mathbb{L}^*\eta_r(x)\rangle\bar{\psi}_i(x) &= \sum_{i=1}^{\infty}\sum_{r=1}^{i}\zeta_{ir}\langle \mathbb{L}h(x),\eta_r(x)\rangle\bar{\psi}_i(x)\\ &= \sum_{i=1}^{\infty}\sum_{r=1}^{i}\zeta_{ir}\langle K(x,h(x,)h'(x),...,h^{(n)},\eta_r(x)\rangle\bar{\psi}_i(x)\\ &= \sum_{i=1}^{\infty}\sum_{r=1}^{i}\zeta_{ir}K(x_r,h(x_r),h'(x_r),...,h^{(n)}(x_r))\bar{\psi}_i(x).\end{aligned}$$

With the last equation the proof is completed. □

5. Application and numerical results

In this section we will apply the proposed method to the problem which in the form (2.1). By aiming to find the approximate solution we will use suitable reproducing kernel Hilbert space and kernel functions which belongs to the space. Once we obtained the solution we will present the absolute and relative errors in the tables. Let us begin by considering the initial value problem given below:

$$\begin{gathered} h'''(x) + 4h'(x) = x,\ 0 \le x \le 4\pi \\ h(0) = h'(0) = 0,\ h''(0) = 1. \end{gathered} \tag{5.1}$$

The exact solution of the equation is

$$h(x) = \frac{3}{16}(1-\cos 2x) + \frac{1}{8}x^2. \tag{5.2}$$

We seek the solution function h of the form $h(x) = H(x) + S(x)$. This will ensure that the new boundary conditions are homogeneous. Here $S(x)$ denotes the transformation function which satisfies the initial conditions and $H(x)$ denotes the terms of new initial value problem which will be homogeneous. If we do the required arrangements, we will be obtaining the new homogeneous equation with the homogeneous conditions.

As a first step, let we start to transform the conditions of the equation (5.1) to the homogeneous one. For this purpose let we use the transformation function given follow as:

$$S(x) = \frac{x^2}{2}.$$

So we can write the h function as:

$$h(x) = H(x) + \frac{x^2}{2}$$

If we calculate the necessary derivatives then we get

$$\begin{aligned} h'(x) =& H'(x) + x \\ h''(x) =& H''(x) + 1 \\ h'''(x) =& H'''(x). \end{aligned} \tag{5.3}$$

When we put this equations into the main problem (5.1), the equation will transform to the new version which is homogeneous as:

$$H'''(x) + 4H'(x) = -3x, 0 \leq x \leq 4\pi$$

with the initial conditions

$$H(0) = H'(0) = H''(0) = 0.$$

By using the reproducing kernel function which found in section 3.1 and with the help of the programme Matlab we obtained the approximate solutions of the problem (5.1). By taking the dense point M=100, the exact solution and approximate solution compared. Besides, absolute and relative errors of the results are presented in the Table 1, Table 2 and graphics below.

x	**Exact**	**RKM (App.)**
0.1	0	0
0.2	0.004987516700	0.004986813148
0.3	0.01980106360	0.01979816350
0.4	0.04399957220	0.04399305315
0.5	0.07686749200	0.07685607747
0.6	0.1174433176	0.1174259428
0.7	0.1645579210	0.1645337905
0.8	0.2168811607	0.2168497896
0.9	0.2729749104	0.2729361522
1.0	0.3313503928	0.3313044491

Table 1: Exact solution and approximate solution.

x	RKM (AE)	RKM (RE)
0.1	0	0
0.2	$7.03552 \cdot 10^{-7}$	0.0001410625853
0.3	0.00000290010	0.0001464618295
0.4	0.00000651905	0.0001481616678
0.5	0.00001141453	0.0001484961939
0.6	0.0000173748	0.0001479420060
0.7	0.0000241305	0.0001466383378
0.8	0.0000313711	0.0001446464963
0.9	0.0000387582	0.0001419844774
1.0	0.0000459437	0.0001386559394

Table 2: Absolute Errors (AE) and Relative Errors (RE).

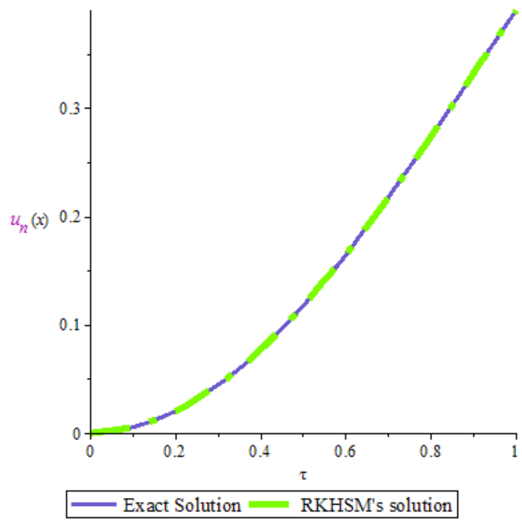

Figure 5.1: 2D Comparison between exact solution and RKM solution

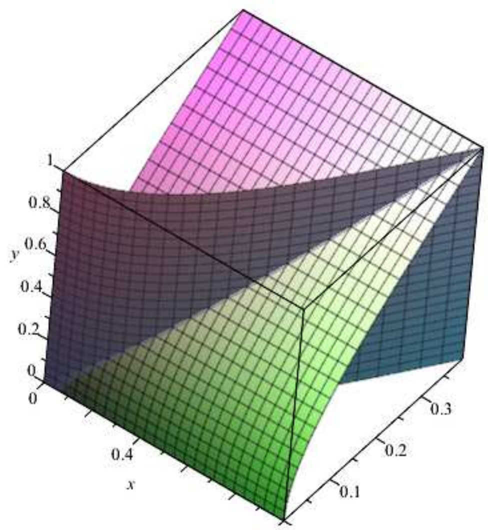

Figure 5.2: 3D Comparison

6. Conclusions

In this work, we presented the reproducing kernel method for solving an IVP which is a type of higher order differential equations. We found a new reproducing kernel space and used it to obtain the numerical results of the problem. When the exact solution and the approximate solution compared, it can easily seen that the method works quite well. The absolute and relative errors also a proof of that. As a result, it is clear that the method is effective and smooth. Moreover, it is an undeniable advantage that the method gives the result fast. This prevents waste of time and provides fast access to the solution.

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Robin Boundary Value Problem Depending on Parameters in a Ring Domain

İlker Gençtürk

Department of Mathematics, Faculty of Science and Arts, Kırıkkale University, Kırıkkale, Turkey

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Abstract

This study is devoted to give solvability conditions and solutions of the Robin boundary problem with constant coefficients for the homogeneous and the inhomogeneous Cauchy-Riemann equation in an annular domain. In order to get results, known representations and theorems in the literature are used. The representations for the solutions and solvability conditions are given in explicit form and here only a special Robin problem is considered. At the end of the paper, it is concluded that with some choices, boundary value problems for the Cauchy-Riemann equation reduce to some basic boundary problems in the ring domain.

1. Introduction and preliminaries

Recently, some complex model partial differential equations, which have important applications in some areas of applied sciences, were investigated in detail, especially for Robin problem see [1]-[7]. Also, the solvability and solutions of complex partial differential equations with boundary conditions were considered by many mathematicians. [8]-[11].

The Robin problem, called as third boundary problem, is a mixed form of the Dirichlet and the Neumann problems, which are basic boundary value problems in complex analysis.

The main aim of this paper is to give solvability conditions and solutions of Robin problem with real parameters for Cauchy-Riemann operators in an annular domain $R=\{z\in\mathbb{C}:0<r<|z|<1\}$. The results in this paper are obtained by using some integral representations in the annular domain [12]-[14], which are similar to ones in the unit disc. [15, 16].

For the convenience of the reader, we recall some relevant theorems without proofs:

Theorem 1.1 (The Complex Form of Gauss Theorem). *[15] Let $D\subset\mathbb{C}$ be a bounded domain with smooth boundary ∂D, and the closure $\overline{D}=D\cup\partial D$. Assume that $w\in C^1(D;\mathbb{C})\cap C(\overline{D};\mathbb{C})$. Then*

$$\int_D w_{\bar z}(z)dxdy=\frac{1}{2i}\int_{\partial D} w(z)dz,\ \int_D w_z(z)dxdy=-\frac{1}{2i}\int_{\partial D} w(z)d\bar z,$$

where

$$\partial_z=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\ \partial_{\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right);\ z=x+iy,\ x,y\ \in\mathbb{R}.$$

Email address and ORCID number: **ilkergencturk@gmail.com, https://orcid.org/0000-0002-0492-939X**

Theorem 1.2 (**Cauchy Integral Formula**). *Let γ be a simply closed smooth curve and D be the inner domain, bounded by γ. If w is an analytic function in D, continuous in $\overline{D}$ and $z \in D$, then*

$$w(z) = \frac{1}{2\pi i}\int_{\gamma} w(\zeta)\frac{d\zeta}{\zeta - z}. \tag{1.1}$$

Theorem 1.3 (**Cauchy-Pompeiu representation**). *[17] Under the assumptions of Theorem 1.1, we have for $z \in D$ that*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

where $\zeta = \xi + i\eta$.

The Dirichlet boundary value problem for analytic functions in R is

$$w_{\overline{z}} = 0, w = \gamma \text{ on } \partial R, \tag{1.2}$$

for a given function $\gamma \in C(\partial\mathbb{R}, \mathbb{C})$.

The following theorem is proved in [14]:

Theorem 1.4. *The Dirichlet problem (1.2) is solvable if and only if for $z \in R$*

$$\frac{1}{2\pi i}\int_{\partial R} \gamma(\zeta)\frac{\overline{z}d\zeta}{1 - \overline{z}\zeta} = \frac{1}{2\pi i}\int_{\partial R} \gamma(z)\frac{\overline{z}d\zeta}{r^2 - \overline{z}\zeta} = 0$$

in the class of analytic functions. Then the unique solution is given by the Cauchy type integral

$$w(z) = \frac{1}{2\pi i}\int_{\partial R} \gamma(\zeta)\frac{d\zeta}{\zeta - z}.$$

The normal derivative on the boundary of R is defined by

$$\partial_\nu = \begin{cases} z\partial_z + \overline{z}\partial_{\overline{z}}, & |z| = 1, \\ -\frac{z}{r}\partial_z - \frac{\overline{z}}{r}\partial_{\overline{z}}, & |z| = r. \end{cases}$$

The Robin boundary value problem for analytic functions in R is

$$w_{\overline{z}} = 0, w + \lambda|z|\partial_\nu w = \gamma \quad \text{on} \quad \partial R,\ \lambda = \begin{cases} 1, & |z| = 1, \\ -1, & |z| = r, \end{cases}$$

for a given function $\gamma \in C(\partial R, \mathbb{C})$.

2. The Robin boundary value problem depending on parameters for analytic functions

In this section, in R we investigate for $\alpha, \beta \in \mathbb{R}$, and $\gamma \in C(\partial R, \mathbb{C})$, the Robin boundary problem

$$w_{\overline{z}} = 0,\ z \in R, \tag{2.1}$$

$$(\alpha\, w + \beta\ \lambda|z|\partial_\nu w) = \gamma,\ z \in \partial R. \tag{2.2}$$

As a consequence of analyticity of w, the boundary condition (2.2) can be rewritten in the form

$$(\alpha w + \beta z w_z)|_{\partial R} = \gamma.$$

Introducing a new function

$$\varphi = \alpha w + \beta z w_z,$$

the boundary problem (2.1)-(2.2) turns out as the Dirichlet problem

$$\varphi_{\bar{z}} = 0 \text{ in } R,\ \varphi = \gamma \text{ on } \partial R. \tag{2.3}$$

On account of Theorem 1.4, boundary problem (2.3) can be uniquely solved if and only if for $z \in R$, the function γ satisfies that

$$\frac{1}{2\pi i}\int\limits_{\partial R} \gamma(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta} = \frac{1}{2\pi i}\int\limits_{\partial R} \gamma(z)\frac{\bar{z}d\zeta}{r^2-\bar{z}\zeta} = 0. \tag{2.4}$$

Then the unique solution of the problem (2.3) is obtained as

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_{\partial R} \gamma(\zeta)\frac{d\zeta}{\zeta - z}. \tag{2.5}$$

We note that as an analytic function in R, $w(z)$ has a unique representation by a Laurent series

$$w(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

which converges in R.

Then, we have

$$\begin{aligned}
\varphi(z) &= \alpha w(z) + \beta z w_z(z) \\
&= \alpha \sum_{n=-\infty}^{\infty} c_n z^n + \beta \sum_{n=-\infty}^{\infty} n c_n z^n \\
&= \sum_{n=-\infty}^{\infty} (\alpha + n\beta) c_n z^n.
\end{aligned}$$

Considering (2.5), it yields

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} (\alpha + n\beta) c_n z^n &= \frac{1}{2\pi i}\int\limits_{\partial R} \gamma(\zeta)\frac{d\zeta}{\zeta - z} \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}} z^n + \sum_{n=-\infty}^{-1} \frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}} z^n.
\end{aligned} \tag{2.6}$$

Comparing coefficients of both sides of (2.6), we have as long as $\alpha + n\beta \neq 0$,

$$\begin{aligned}
c_n &= \frac{1}{\alpha + n\beta}\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}},\ n = 0,1,2,\ldots; \\
c_n &= \frac{1}{\alpha + n\beta}\frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}},\ n = \ldots,-2,-1.
\end{aligned}$$

Therefore, we can assert that

$$w(z) = \sum_{n=0}^{\infty} \frac{1}{\alpha + n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}\right) z^n + \sum_{n=-\infty}^{-1} \frac{1}{\alpha + n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}\right) z^n. \tag{2.7}$$

Theorem 2.1. *For $\alpha, \beta \in \mathbb{R}$, and $\gamma \in C(\partial R, \mathbb{C})$, Robin boundary value problem (2.1)-(2.2) in R is solvable if and only if condition (2.4) is satisfied. In this case, solution of the problem if $\alpha + n\beta \neq 0$ for all $n \in \mathbb{Z}$ is given by (2.7).*

3. The Robin boundary value problem depending on parameters for inhomogeneous Cauchy-Riemann equation

In this section, we deal for $\alpha,\beta \in \mathbb{R}$, $\gamma \in C(\partial R,\mathbb{C})$ and $f \in C^{a}(\overline{R};\mathbb{C})$, $0<a<1$, with the Robin boundary problem

$$w_{\bar z} = f,\ z\in R, \tag{3.1}$$
$$(\alpha\, w+\beta\, \lambda|z|\partial_{\nu} w) = \gamma,\ z\in \partial R. \tag{3.2}$$

Solutions of equation $w_{\bar z}=f$ have the form

$$w(z)=\varphi(z)-\frac{1}{\pi}\int_{R}\frac{f(\zeta)}{\zeta-z}d\xi d\eta,$$

where $\varphi(z)$ is any analytic function in R, see [17].

By differentiating with respect to z implies

$$w_z=\varphi_z-\frac{1}{\pi}\int_{R}\frac{f(\zeta)}{(\zeta-z)^2}d\xi d\eta.$$

We note that the latter derivative is taken in distributional sense, see [15].

By introducing the new function

$$\varphi=w+\frac{1}{\pi}\int_{R}\frac{f(\zeta)}{\zeta-z}d\xi d\eta, \tag{3.3}$$

and using $w_{\bar z}=f$, the problem (3.1)-(3.2) is reduced to

$$\varphi_{\bar z} = 0,\ \text{in } R, \tag{3.4}$$
$$(\alpha\varphi+\beta z\varphi_z) = \left(\gamma+\frac{1}{\pi}\int_{R}\left[\frac{\beta z}{(\zeta-z)^2}-\frac{\alpha}{z-\zeta}\right]f(\zeta)d\xi d\eta-\beta\bar z f\right) := \widehat{\gamma},\ \text{on } \partial R, \tag{3.5}$$

the Robin problem in the previous section. By Theorem 2.1, (3.4)-(3.5) is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial R}\widehat{\gamma}(\zeta)\frac{\bar z d\zeta}{1-\bar z\zeta}=\frac{1}{2\pi i}\int_{\partial R}\widehat{\gamma}(z)\frac{\bar z d\zeta}{r^2-\bar z\zeta}=0. \tag{3.6}$$

In this case, solution of the problem if $\alpha+n\beta\neq 0$ for all $n\in\mathbb{Z}$ is given by

$$\varphi(z)=\sum_{n=0}^{\infty}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int_{|\zeta|=1}\widehat{\gamma}(\zeta)\frac{d\zeta}{\zeta^{n+1}}\right)z^n+\sum_{n=-\infty}^{-1}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int_{|\zeta|=r}\widehat{\gamma}(\zeta)\frac{d\zeta}{\zeta^{n+1}}\right)z^n. \tag{3.7}$$

It is clear that (3.5) for $z\in\partial R$ is equal to

$$\widehat{\gamma}(z)=\gamma(z)+\frac{1}{\pi}\int_{R}\left[\frac{\beta z}{(\zeta-z)^2}-\frac{\alpha}{z-\zeta}\right]f(\zeta)d\xi d\eta-\beta\bar z f(z). \tag{3.8}$$

So, by (3.8), the first boundary integral in (3.6) for $t=t_1+it_2$ can be written as

$$\frac{1}{2\pi i}\int_{\partial R}\widehat{\gamma}(\zeta)\frac{\bar z d\zeta}{1-\bar z\zeta}=\frac{1}{2\pi i}\int_{\partial R}\left(\gamma(\zeta)+\frac{1}{\pi}\int_{R}\left[\frac{\beta\zeta}{(\zeta-t)^2}-\frac{\alpha}{\zeta-t}\right]f(t)dt_1dt_2-\beta\overline{\zeta}f(\zeta)\right)\frac{\bar z d\zeta}{1-\bar z\zeta}.$$

By applying Fubini's theorem when changing the order of integrations, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial R}\widehat{\gamma}(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta} &= \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta}+\frac{1}{\pi}\int\limits_{R}f(t)\left(\frac{1}{2\pi i}\int\limits_{\partial R}\left[\frac{\beta\zeta}{(\zeta-t)^2}-\frac{\alpha}{\zeta-t}\right]\frac{\bar{z}d\zeta}{1-\bar{z}\zeta}\right)dt_1dt_2\\ &\quad -\frac{1}{2\pi i}\int\limits_{\partial R}\beta\overline{\zeta}f(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta}.\end{aligned}$$

By aid of the Cauchy integral formula (1.1),

$$\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\beta\zeta}{(\zeta-t)^2}\frac{d\zeta}{1-\bar{z}\zeta}=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\beta\zeta}{(\zeta-t)^2}\frac{d\zeta}{1-\bar{z}\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{\beta\zeta}{(\zeta-t)^2}\frac{d\zeta}{1-\bar{z}\zeta}=\frac{\beta}{(1-\bar{z}t)^2}$$

and

$$\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\alpha}{\zeta-t}\frac{d\zeta}{1-\bar{z}\zeta}=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\alpha}{\zeta-t}\frac{d\zeta}{1-\bar{z}\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{\alpha}{\zeta-t}\frac{d\zeta}{1-\bar{z}\zeta}=\frac{\alpha}{1-\bar{z}t},$$

hence it can be shown that

$$\frac{1}{2\pi i}\int\limits_{\partial R}\widehat{\gamma}(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta}=\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right]\frac{\bar{z}d\zeta}{1-\bar{z}\zeta}+\frac{1}{\pi}\int\limits_{R}\bar{z}f(\zeta)\frac{\beta-\alpha+\alpha\bar{z}\zeta}{(1-\bar{z}\zeta)^2}d\xi d\eta=0.$$

With similar calculations, for the second boundary integral in (3.6), we obtain

$$\frac{1}{2\pi i}\int\limits_{\partial R}\widehat{\gamma}(\zeta)\frac{\bar{z}d\zeta}{r^2-\bar{z}\zeta}=\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right]\frac{\bar{z}d\zeta}{r^2-\bar{z}\zeta}+\frac{1}{\pi}\int\limits_{R}\bar{z}f(\zeta)\frac{r^2(\beta-\alpha)+\alpha\bar{z}\zeta}{(r^2-\bar{z}\zeta)^2}d\xi d\eta=0.$$

If the value of (3.8) is substituted in (3.7), we can get for $\frac{\alpha}{\beta}\notin\mathbb{Z}$,

$$\begin{aligned}\varphi(z) &= \sum_{n=0}^{\infty}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\gamma(\zeta)+\frac{1}{\pi}\int\limits_{R}\left[\frac{\beta\zeta}{(\zeta-t)^2}-\frac{\alpha}{\zeta-t}\right]f(t)dt_1dt_2-\beta\overline{\zeta}f(\zeta)\right)\frac{d\zeta}{\zeta^{n+1}}\right)z^n\\ &\quad+\sum_{n=-\infty}^{-1}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left(\gamma(\zeta)+\frac{1}{\pi}\int\limits_{R}\left[\frac{\beta\zeta}{(\zeta-t)^2}-\frac{\alpha}{\zeta-t}\right]f(t)dt_1dt_2-\beta\overline{\zeta}f(\zeta)\right)\frac{d\zeta}{\zeta^{n+1}}\right)z^n.\end{aligned}$$

or equivalently

$$\begin{aligned}\varphi(z) &= \sum_{n=0}^{\infty}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\frac{1}{\pi}\int\limits_{R}f(t)\left[\frac{\beta\zeta}{(\zeta-t)^2}-\frac{\alpha}{\zeta-t}\right]dt_1dt_2\right)\frac{d\zeta}{\zeta^{n+1}}\right.\\ &\qquad\left.-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\beta\overline{\zeta}f(\zeta)\frac{d\zeta}{\zeta^{n+1}}\right)z^n\\ &\quad+\sum_{n=-\infty}^{-1}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}+\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left(\frac{1}{\pi}\int\limits_{R}f(t)\left[\frac{\beta\zeta}{(\zeta-t)^2}-\frac{\alpha}{\zeta-t}\right]dt_1dt_2\right)\frac{d\zeta}{\zeta^{n+1}}\right.\\ &\qquad\left.-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\beta\overline{\zeta}f(\zeta)\frac{d\zeta}{\zeta^{n+1}}\right)z^n.\end{aligned}$$

Because of

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\frac{1}{\pi}\int\limits_{R}f(t)\left[\frac{\beta\zeta}{(\zeta-t)^2}-\frac{\alpha}{\zeta-t}\right]dt_1dt_2\right)\frac{d\zeta}{\zeta^{n+1}}=0,\ \text{for } n=0,1,..$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left(\frac{1}{\pi}\int\limits_{R} f(t)\left[\frac{\beta\zeta}{(\zeta-t)^2}-\frac{\alpha}{\zeta-t}\right]dt_1dt_2\right)\frac{d\zeta}{\zeta^{n+1}}=0,\ \text{for } n=...,-2,-1,$$

we get

$$\begin{aligned}\varphi(z) &= \sum_{n=0}^{\infty}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right)\frac{d\zeta}{\zeta^{n+1}}\right)z^n\\ &\quad+\sum_{n=-\infty}^{-1}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left(\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right)\frac{d\zeta}{\zeta^{n+1}}\right)z^n.\end{aligned}$$

By using (3.3), solution of the problem (3.1)-(3.2) can be found as

$$\begin{aligned}w(z) &= \sum_{n=0}^{\infty}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right)\frac{d\zeta}{\zeta^{n+1}}\right)z^n\\ &\quad+\sum_{n=-\infty}^{-1}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left(\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right)\frac{d\zeta}{\zeta^{n+1}}\right)z^n\\ &\quad+\frac{1}{\pi}\int\limits_{R}\frac{1}{z-\zeta}f(\zeta)d\xi d\eta.\end{aligned}$$

Finally, we have just proved the following:

Theorem 3.1. *For $\alpha,\beta\in\mathbb{R}, f\in C^a(\overline{R};\mathbb{C}), 0<a<1, \gamma\in C(\partial R;\mathbb{C})$, the Robin problem*

$$w_{\overline{z}}=f \text{ in } R,\ \alpha w+\beta\lambda|z|\partial_\nu w=\gamma \text{ on } \partial R$$

is solvable if and only if for all $z\in R$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right]\frac{\overline{z}d\zeta}{1-\overline{z}\zeta}+\frac{1}{\pi}\int\limits_{R}\overline{z}f(\zeta)\frac{\beta-\alpha+\alpha\overline{z}\zeta}{(1-\overline{z}\zeta)^2}d\xi d\eta=0,$$

and

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right]\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta}+\frac{1}{\pi}\int\limits_{R}\overline{z}f(\zeta)\frac{r^2(\beta-\alpha)+\alpha\overline{z}\zeta}{(r^2-\overline{z}\zeta)^2}d\xi d\eta=0.$$

Then, the solution of the problem if $\alpha+n\beta\neq 0$ for all $n\in\mathbb{Z}$ is represented by

$$\begin{aligned}w(z) &= \sum_{n=0}^{\infty}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right)\frac{d\zeta}{\zeta^{n+1}}\right)z^n\\ &\quad+\sum_{n=-\infty}^{-1}\frac{1}{\alpha+n\beta}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left(\gamma(\zeta)-\beta\overline{\zeta}f(\zeta)\right)\frac{d\zeta}{\zeta^{n+1}}\right)z^n\\ &\quad+\frac{1}{\pi}\int\limits_{R}\frac{1}{z-\zeta}f(\zeta)d\xi d\eta.\end{aligned}$$

4. Conclusion

In this paper, a special kind of Robin problem for analytic functions (Theorem 2.1) and more generally for the inhomogeneous Cauchy–Riemann equation (Theorem 3.1) are investigated in a concentric ring domain. The representations of the solutions and solvability conditions are aimed for in explicit form.

Let us reconsider the Robin boundary condition

$$(\alpha w+\beta\lambda|z|\partial_\nu w)=\gamma \text{ on } R. \tag{4.1}$$

Under above boundary condition (4.1), with some special cases of α and β, the following results can be obtained:

- **i.)** By choosing $\alpha=\beta=1$, we have $(w+\lambda|z|\partial_\nu w)=\gamma$ on R. In this case, in (2.7), the coefficient of z^{-1}, c_{-1} may take arbitrary values from $\mathbb{C}$. Hence, for solvability of the problem, the condition $\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)d\zeta=0$ is needed. Furthermore, with an additional condition $z_0w(z_0)=c$, for some fixed point $z_0\in R, c\in\mathbb{C}$, the problem is uniquely solvable. This problem is another special kind of Robin problem and appears as Theorem 2.2.14 (for analytic functions) in [14]. As is the analytic case, by applying similar arguments, in the inhomogeneous case, the conditions $\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\Big(\gamma(\zeta)-\overline{\zeta}f(\zeta)\Big)d\zeta=0$ (for solvability) and $z_0w(z_0)=c$, for some fixed point $z_0\in R, c\in\mathbb{C}$ (for uniqueness of the solution) are needed. [14, Theorem 2.3.18]
- **ii.)** By choosing $\alpha=1$ and $\beta=0$, we have $w=\gamma$ on ∂R. Hence, these problems are reduced to the Dirichlet problems for analytic functions and the inhomogeneous Cauchy-Riemann equation, respectively, in [14, Theorem 2.2.12 and Theorem 2.3.16].
- **iii.)** By choosing $\alpha=0$, and $\beta=1$, we have $(\lambda|z|\partial_\nu w)=\gamma$ on R. Hence, these problems are reduced to the Neumann problems for analytic functions and the inhomogeneous Cauchy-Riemann equation, respectively in [14, Theorem 2.2.13 and Theorem 2.3.17], with additional conditions.

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On the Mean Flow Solutions of Related Rotating Disk Flows of the BEK System

Burhan Alveroğlu

Department of Mathematics, Faculty of Engineering and Natural Sciences, Bursa Technical University, 16330, Bursa, Turkey

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Abstract

This paper investigates the effects of the YHP roughness model on the mean flow solutions of some flows belong to the family of the rotating BEK system flows. The governing mean flow equations are formulated in the rotating frame of reference, therefore, they include terms arising from the centrifugal force. These mean flow equations are solved using the method of lines and the backward difference method. Then, obtained results are compared for specifically selected value of roughness parameters with the results of a fundamentally different roughness model, the MW model. The results of the YHP model reveal that applying surface roughness changes the characteristics of the mean flow components. Moreover, the comparison of the YHP and MW models points that these changes are notably different for each model. Therefore, possible future researches can be conducted to investigate the stability characteristics of the flows due to the selection of the roughness model.

1. Introduction

The studies on the rotating disk flows have grown significantly in the literature due to common characteristics of the rotating disk flows with the swept wing flows. Both flow types have inflectional mean flow components that cause a *crossflow instability* known as *the Type I instability mode* [1, 2]. The governing equations of the rotating disk flows are notably simplifier than those of the swept-wing flows due to its axisymmetric geometry. Moreover, exact similarity solutions of the Navier-Stokes equations can be found in case of the rotating disk flows [4].

Early studies on the instability properties of the rotating disk flow are established with a smooth disk configuration [4, 3]. However, the attention of the many researches have shifted to the rough disk configurations after three pioneering studies about the effect of surface roughness on the instability analysis have been published at *Nature* [7]-[6]. These studies have altered the common belief that rough surfaces trigger the flow instabilities and have revealed that surface roughness can be utilized to delay the onset of instabilities if the roughness is rightly sorted over the disk [5]. Therefore, determination of *the right sort of roughness* [5], has been a leading research field for numerous researchers over the past 30 years [8, 10, 13, 12].

An important family of rotating disk boundary layer flows are BEK system flows. This family of flows is driven by the difference of the rotation speeds of an impermeable rotating disk having an infinite radius, and the incompressible fluid rotating above the disk [4]. Both the disk and fluid rotate around the vertical axis passing through the center of disk. The angular velocities are denoted by Ω_D^* and Ω_F^* for the disk and fluid, respectively. This family includes three main types of flows: the von Kármán boundary layer flow, the Ekman boundary layer flow and the Bödewadt boundary layer flow. There are also

Email address and ORCID number: **burhan.alveroglu@btu.edu.tr, https://orcid.org/0000-0003-2699-9898**

infinitely many related flows between these three flows, in which the angular velocities of the disk and fluid are not equal to zero but also different.

Several attempts have been made to investigate the right sort of roughness for the BEK system flows [11, 12, 13]. In most recent studies, surface roughness has been modelled using two fundamentally different roughness models: *MW model* developed by Miklavcicv & Wang [14] and *YHP model* developed by Yoon, Hyun, and Park [15]. The first one uses partial-slip boundary conditions on the surface of the disk in order to apply roughness whereas the latter uses a fundamentally different approach. Despite easy implementation of the partial-slip boundary conditions in each direction of the disk, MW model sorts the roughness empirically. In YHP model, a new function of radial position r is introduced and the roughness is modelled as a wavy surface disk using this function along with assuming a radial symmetry . Therefore, this model can sort roughness in r direction only.

Many recent studies [11, 9, 12] have been conducted to determine the right sort of roughness for the BEK system flows using the YHP or MW models. The studies of Cooper *et al*. [12] and Garrett *et al* [9] are focused on von Kármán flow that is the most known member of the BEK system flows. They use both MW and YHP models, and their results reveal that two models have different effects on the instability characteristics of the flow. The study of Alveroğlu *et al*. [11] considered whole BEK system flows using the MW model to sort the roughness. However, the original theoretical research of the YHP model [15] conducted on the particular flows of the BEK system used a stationary frame of reference for the formulation. Therefore, it did not account the effects of the centrifugal terms appearing in a rotating frame formulation. This study uses the YHP roughness model formulated in the rotating frame and investigates the effects of the model on mean flow profiles of those particular flows of the BEK system, and compare the results with those of the previous study [11] that uses the MW model.

The overall structure of the study is stated as follows. Section 2 presents the governing steady mean-flow equations for the MW and YHP models for the entire BEK system. Computed mean flow profiles of the interested flows under the YHP model are presented in Section 3, and a comparison of the effects of the roughness models is also made in this section. Section 4 includes the conclusion.

2. The governing mean flow equations for MW and YHP models

The governing mean flow equations of the BEK system flows for both MW and YHP models are formulated in case of a steady fluid flow over an infinite disk. Both the disk and fluid are considered to rotate with constant angular velocities Ω_D^* and Ω_F^* around a joint axis passing through the centre of the disk. The coordinate system is assumed to be rotating with the disk. The von Kármán flow occurs if $\Omega_D^* \neq 0,\ \Omega_F^* = 0$, the Ekman flow occurs if $\Omega_D^* = \Omega_F^* \neq 0$, and the Bödewadt flow occurs if $\Omega_D^* = 0,\ \Omega_F^* \neq 0$.

The parameter defined related to the differential rotation rate between the disk and above fluid is used to distinguish the flows of the BEK system. This parameter are defined as

$$Ro = \frac{\Omega_F^* - \Omega_D^*}{\Omega^*}, \tag{2.1}$$

where Ω^* is the system rotation rate. It is called the Rossby number. The Rossby number for the von Kármán flow is $Ro = -1$, for the Ekman flow is $Ro = 0$, for the Bödewadt flow is $Ro = 1$. Furthermore, $-1 \leq Ro \leq 1$ for all the flows of the system. In this study, we will investigate the flows that occur when $-1 < Ro < 0$.

Using a rotating coordinate frame introduces Coriolis and centrifugal terms in the equations. The Coriolis number can be defined in terms of the Rossby number as

$$Co = 2\frac{\Omega_D^*}{\Omega^*} = 2 - Ro - Ro^2.$$

The nondimensional mean flow equations derived using the MW model are

$$\begin{aligned} Ro\Big(U^2 + U'W - (V^2-1)\Big) - Co\Big(V-1\Big) - U'' &= 0,\\ Ro\Big(2UV + V'W\Big) + CoU - V'' &= 0,\\ Ro\Big(WW' + P'\Big) - W'' &= 0,\\ 2U + W' &= 0. \end{aligned} \tag{2.2}$$

and the boundary conditions are given by

$$
\begin{array}{llll}
U(0)=\lambda U'(0), & V(0)=\eta V'(0) & \text{and} & W(0)=0, \\
U \longrightarrow 0, & V \longrightarrow 1, & \text{as} & z \longrightarrow \infty.
\end{array} \tag{2.3}
$$

The full discussion on the derivation of the equations (2.2) can be found in the pioneering study of Lingwood [4]. Here, it is sufficient to note that all derivatives denoted by primes are with respect to z, and U, V & W are the mean flow velocity components in radial, azimuthal and axial directions, respectively. The pressure term is denoted by P. The boundary conditions (2.3) on the disk surface are formulated using partial-slip conditions [11, 14]. The boundary conditions at infinity are the usual no-slip conditions.

The MW model imposes the roughness over the disk surface empirically using particular values for the parameters η and λ. The case $\lambda=\eta=0$ corresponds a smooth disk, $\eta>0,\ \lambda=0$ (concentric grooves) and $\eta=0,\ \lambda>0$ (radial grooves) correspond to *anisotropic* roughness, *radially* and *azimuthally*. The final case $\eta=\lambda\neq 0$ indicates an *isotropic* roughness. The effects of different values of these parameters on mean flow profiles of interested flows are discussed individually in the following section.

In YHP model, the governing mean flow equations are formulated in a new coordinate system (r,θ,η) using the Prandtl transformation [15]. Here, the new axial coordinate variable is set to $\eta=z-s(r)$, and $s(r)=\delta\cos(2\pi r/\gamma)$ is the dimensionless surface profile.

This surface profile is axisymmetric with respect to the axis of rotation. The governing mean flow equations in this coordinate system are stated as

$$
\begin{aligned}
Ro\left[rf\frac{\partial f}{\partial r}+h\frac{\partial f}{\partial \zeta}+f^2\left(1+r\frac{s's''}{1+s'^2}\right)\right] &= \frac{Ro+Co}{1+s'^2}+\left(1+s'^2\right)\frac{\partial^2 f}{\partial \zeta^2}+\frac{g}{1+s'^2}\left(Co+Rog\right), \\
Ro\left[rf\frac{\partial g}{\partial r}+h\frac{\partial g}{\partial \zeta}\right] &= \left(1+s'^2\right)\frac{\partial^2 g}{\partial \zeta^2}-f\left(Co+2gRo\right), \\
2f+r\frac{\partial f}{\partial r}+\frac{\partial h}{\partial \zeta} &= 0,
\end{aligned} \tag{2.4}
$$

and the boundary conditions are given by

$$
\begin{array}{l}
f(r,\zeta)=g(r,\zeta)=h(r,\zeta)=0, \quad \text{at } \zeta=0, \\
f(r,\zeta)=0, \quad g(r,\zeta)=1 \quad \text{as } \zeta\to\infty.
\end{array} \tag{2.5}
$$

Here, the steady-flow profiles in *radial*, *azimuthal* and *axial* directions are denoted $(f,\ g,\ h)$, respectively. A full description for the derivation of those equations can be found in the previous studies [9, 15]. However, this formulation has slight modifications to the original presentation of the YHP model [15]. The Coriolis and centrifugal terms appear in this new presentation due to formulating the model in a rotating frame.

The YHP model controls the roughness using the aspect ratio parameter $a=\delta/\gamma$. For $a=0$, those equations are coincide with the previous studies of Lingwood [4] and Alveroğlu [11]. However, it is only possible to apply surface roughness only in radial direction under the YHP model due to the definition of distribution function $s(r)$. Therefore, the YHP model can be compared with the MW model in only the case of concentric grooves of roughness profile.

3. Results and discussion

The governing equations (2.2)-(2.3) for the MW model are the system of nonlinear ODEs and solved in the previous study of Alveroğlu [11] for different values of the parameters η and λ using a fourth order Runga Kutta method. However, the governing equations (2.4)-(2.5) for the YHP model are highly nonlinear system of PDEs, and we provide the solutions of the YHP model for the flows of $-1<Ro<0$. In other words, we investigate the flows between the von Kármán and Ekman flows. The results arising from the YHP model for those flows are then compared with the results of the MW model.

In order to compute mean flow components those PDEs are reduced to ODEs with respect to ζ variable using method of lines. The required initial solution for the methods of lines is achieved at $r=0$ from the ODEs in ζ of the BEK system [4]. Then, the flow profiles at each incremental r value are computed using the the backward difference method. The computation grid at each r value is set from $\zeta=0$ to $\zeta=16$. In order to make a comparison with solutions of the MW model, the computed flow profiles of YHP model are averaged spatially. This averaged flow field is denoted by $(\bar{f}(\zeta),\ \bar{g}(\zeta),\ \bar{h}(\zeta))$ [9].

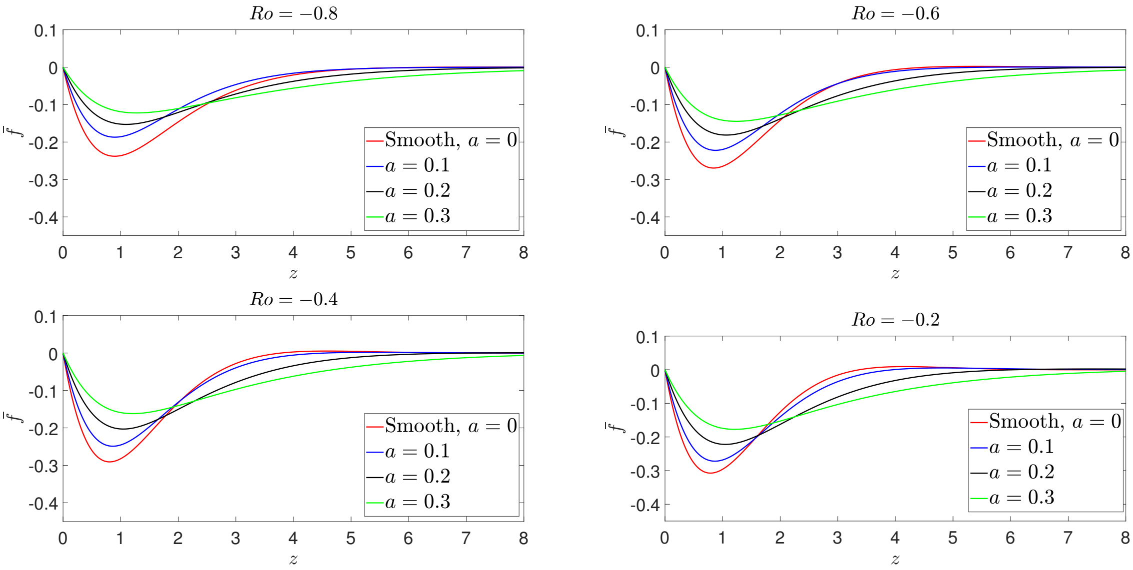


Figure 3.1: The mean flow components in *radial* direction at various values of roughness parameter *a*.

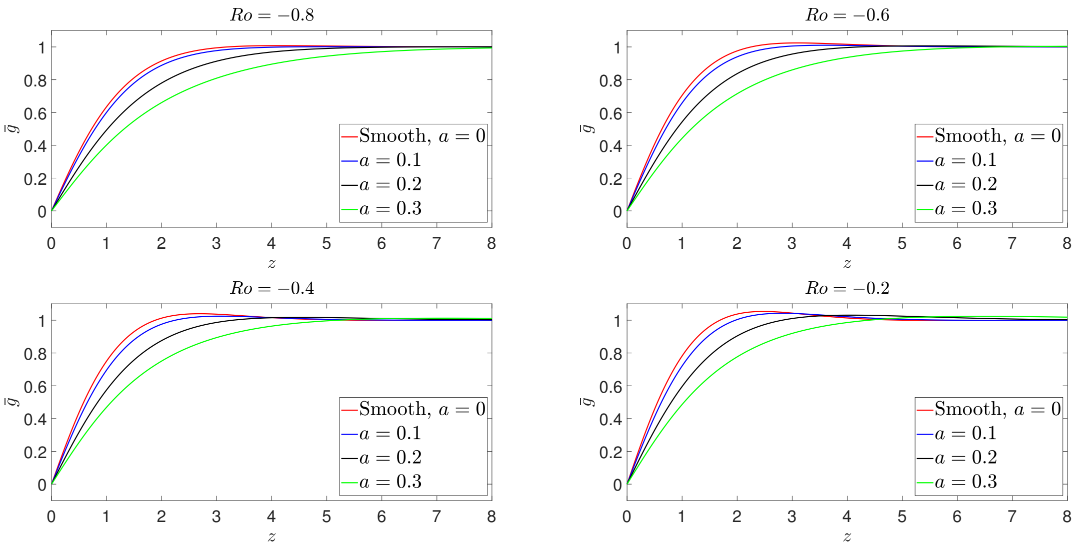


Figure 3.2: The mean flow components in *azimuthal* direction at various values of roughness parameter *a*.

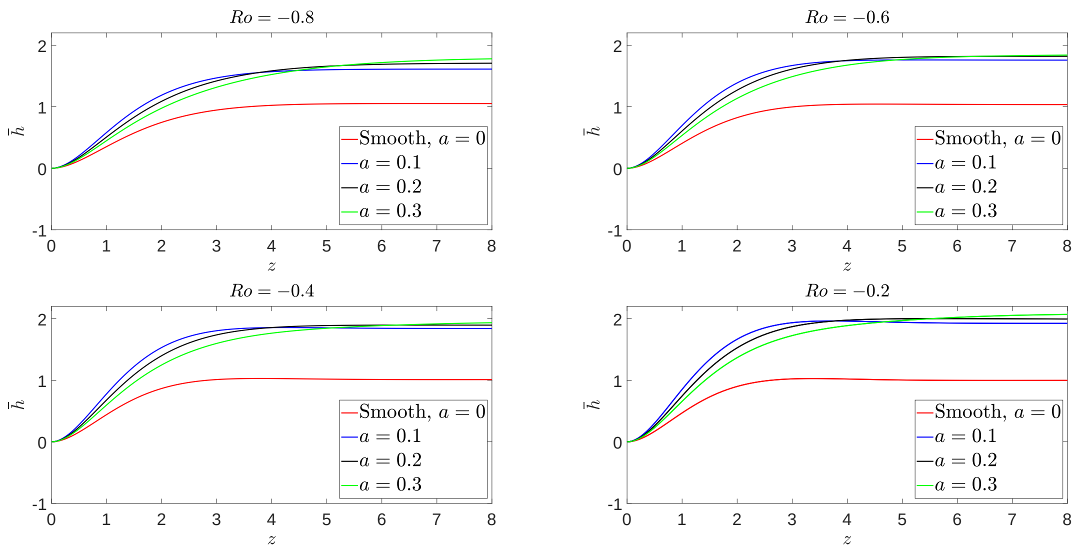


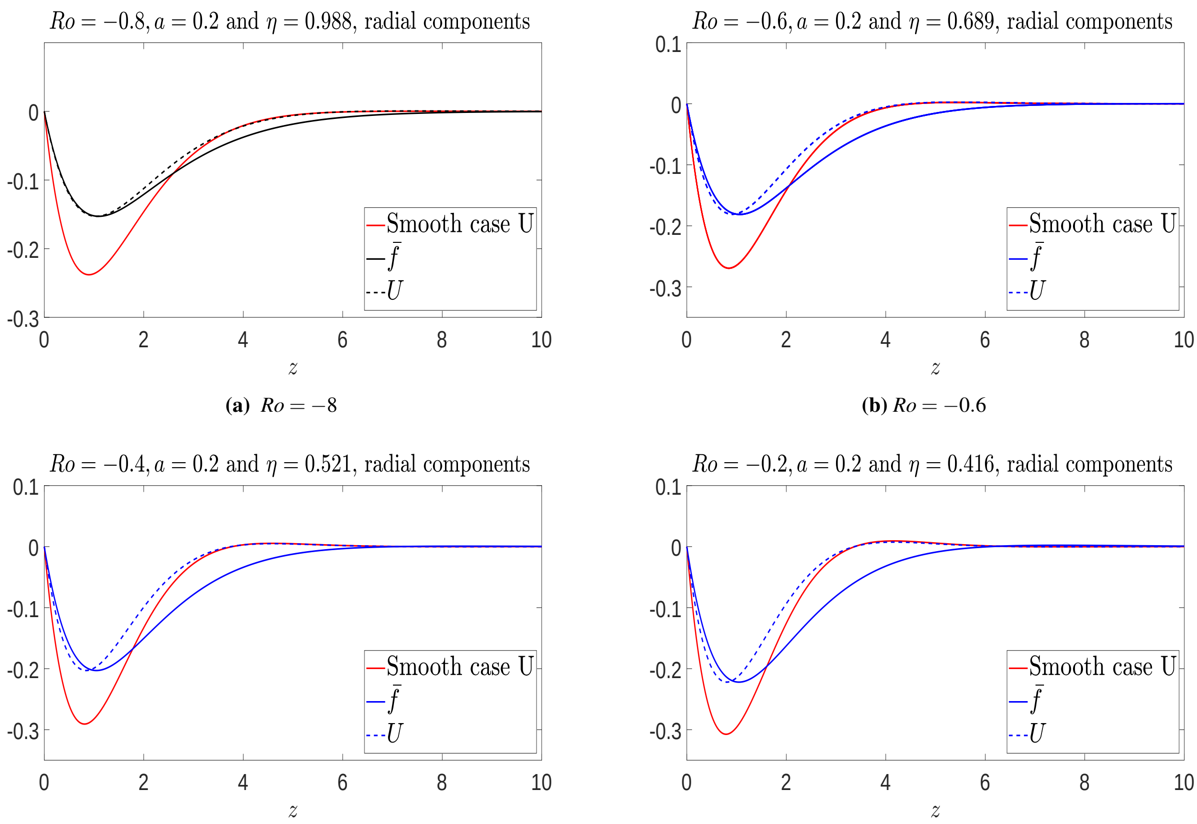
Figure 3.3: The mean flow components in *axial* direction at various values of roughness parameter *a*.

The spatially averaged flow components of the YHP model over one wavelength for roughness parameter $a = 0$ to $a = 0.3$ are given in Figures 3.1-3.3. The results indicate that increasing roughness level reduces the amount of the wall jet at vicinity of disk surface, i.e., it decreases the maximum radial velocity, $max(\bar{f})$, as seen in Figure 3.2(a)-(d) for each flow configuration. Also, increased roughness leads to an incremental widening of the azimuthal velocity component as shown in Figure 3.2(a)-(d) for each flow. In other words, increased roughness makes the boundary layer thicker. However, Figure 3.3(a)-(d) presents the increase in the magnitude of $\bar{h}$ component, i.e., the amount of the axial flow increases with the greater values of the roughness.

The MW and YHP roughness models are theoretically different as the latter one uses transformed coordinates. Therefore, it is not possible to make a *quantitative* comparison for equal values of roughness parameters. Instead, a *qualitative* approach is considered to compare the effects of increased roughness levels for both models. The moderate roughness value $a = 0.2$ is selected in YHP model and the maximum values of radial jets of flows between $-1 < Ro < 0$ are matched for both models. In other words, for each flow, the corresponding roughness parameter value of η in the MW model is determined such that the maximum values of radial mean flow components are same under both models. The matched mean flow profiles of the different flows are presented in Figure 3.4(a)-(d). The numerical values of matching parameters are also presented at Table 1.

Rossby Number Ro	$max(\bar{f})$	Roughness parameter η for MW Model
$Ro = -0.8$	-0.1528	0.988
$Ro = -0.6$	-0.1813	0.689
$Ro = -0.4$	-0.2030	0.521
$Ro = -0.2$	-0.2220	0.416

Table 1: Corresponding matching values of different flows for $a = 0.2$ of YHP model



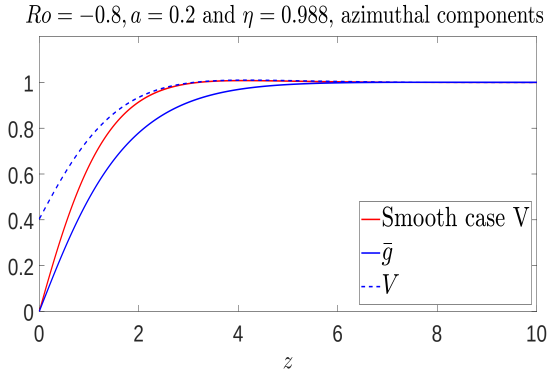
(a) $Ro = -8$

(b) $Ro = -0.6$

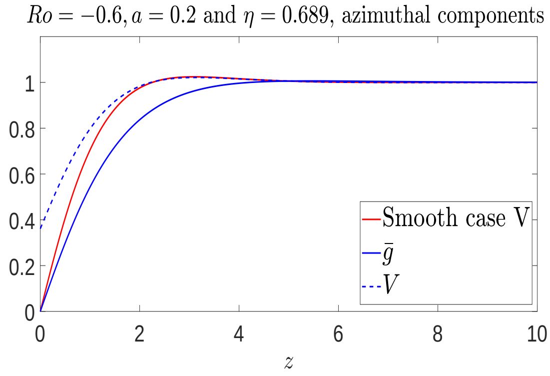
(c) $Ro = -0.4$

(d) $Ro = -0.2$

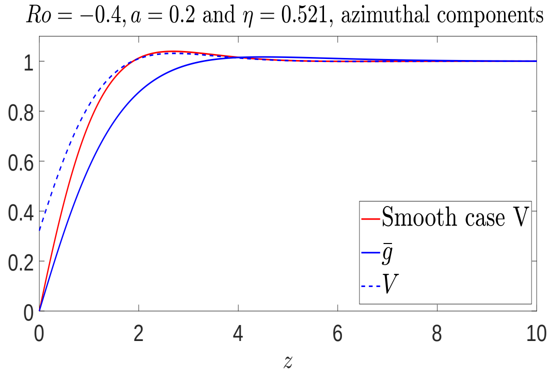
Figure 3.4: Compared mean flow profiles of different flows in case of the MW and YHP models in radial direction.



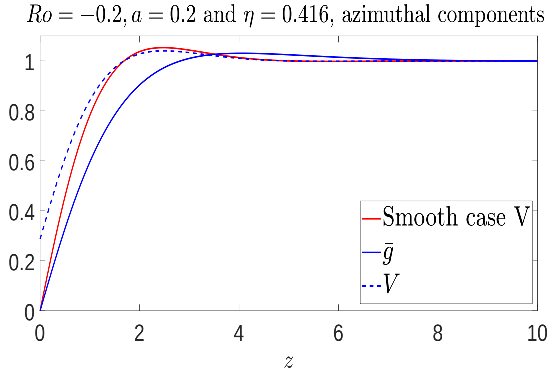
(a) $Ro = -8$



(b) $Ro = -0.6$

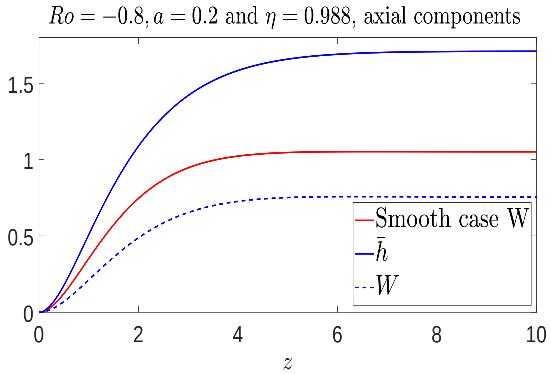


(c) $Ro = -0.4$

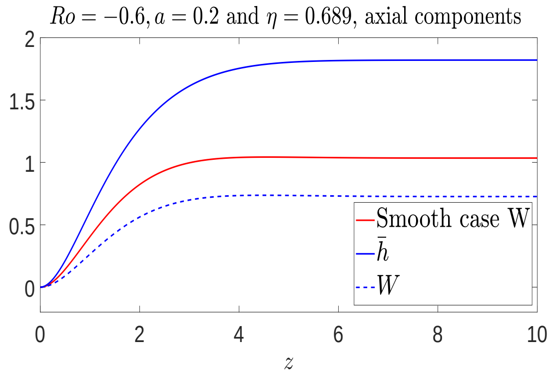


(d) $Ro = -0.2$

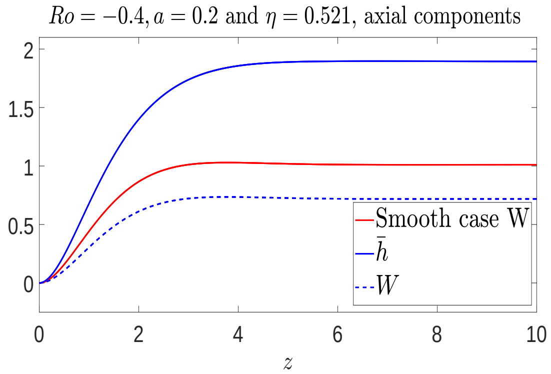
Figure 3.5: Compared mean flow profiles of different flows in case of the MW and YHP models in azimuthal direction.



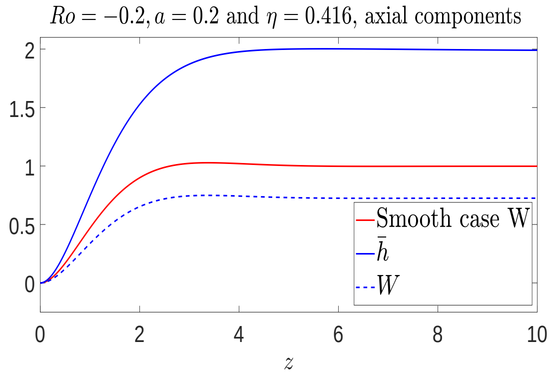
(a) $Ro = -8$



(b) $Ro = -0.6$



(c) $Ro = -0.4$



(d) $Ro = -0.2$

Figure 3.6: Compared mean flow profiles of different flows in case of the MW and YHP models in axial direction.

The difference of the effects of the MW and YHP models can be interpreted from Figures 3.5 - 3.6. The solid red lines in each parts shows the velocity components in absence of roughness. Figure 3.5 includes the velocity profiles in azimuthal direction and reveals that applying the MW model increases the value of azimuthal velocity profile for each flow, whereas applying YHP model causes a decrease. However, this difference is not huge and the values of this component become equal for each model at the far field of z domain. The main different effect of the models, on the other hand, has been observed at the axial flow components of each flow. The results for these components, $\bar{h}$ in the YHP model and W in the MW model, are represented in Figure 3.6. This figure reveals a substantial increase in the amount of axial flow in case of the YHP model, and a reduction in case of the MW model. Moreover, the presence of the roughness modelled with the YHP model seems to increase oscillatory behaviour of that component compared to the smooth case.

4. Conclusion

The aim of this study was to investigate the effects of the YHP model [15] in a rotating frame of reference on the mean flow solutions of some member of the BEK system flows, particularly the flows with $-1 < Ro < 0$. The mean flow profiles are computed using the method of lines and the backward difference from highly nonlinear governing equations (2.2). The obtained mean-flow components agree with the previous findings in case of a smooth disk [4, 11]. The results indicate that increased roughness for each flow vanishes the oscillations of the steady mean flows towards the boundary layer, and the magnitude of the wall jet, $\bar{f}$, is also reduced.

The computed profiles are also compared with the results of another theoretical roughness model, the MW model [14]. The comparison is made in each flow for selected the roughness parameter value $a = 0.2$ of YHP model and the corresponding roughness parameter η of the MW model. It seems that the results are changed substantially due to selection of the roughness model. This points possible future researches investigating the effects of the roughness models on the stability characteristics of the flows.

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Almost Para-Contact Metric Structures on 5-dimensional Nilpotent Lie Algebras

Nülifer Özdemir[1], Mehmet Solgun[2*] and Şirin Aktay[1]

[1]*Department of Mathematics, Eskişehir Technical University, 26470 Eskişehir, Turkey*
[2]*Department of Mathematics, Bilecik Seyh Edebali University, 11210 Bilecik, Turkey*
[*]*Corresponding author*

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Abstract

In this manuscript, almost para-contact metric structures on 5 dimensional nilpotent Lie algebras are studied. Some examples of para-Sasakian and para-contact structures on five-dimensional nilpotent Lie algebras are given.

1. Introduction

Almost paracontact structures were first studied by [1] and after the work of Zamkovoy in [2], many authors have made contribution. For recent studies, see [3]-[8]. In [9], almost paracontact metric structures were classified into 2^{12} classes taking into consideration the Levi-Civita covariant derivative of the fundamental 2-form of the structure. In this work, we study almost paracontact metric structures on 5-dimensional nilpotent Lie algebras.
In the literature, there are many researches on five dimensional Lie algebras equipped with an almost contact metric structure. Andrada et al., studied Sasakian structures on five dimensional Lie algebras [10]. Calvaruso and Fino introduced an approach on five dimensional K-contact Lie algebras [11]. Nilpotent Lie algebras having dimension 5 were classified in [12]. According to this classification, we examined the Lie algebras equipped with quasi-Sasakian structures in [13]. Also in [14], we studied some certain classes, such as $\alpha-$ Sasakian, $\beta-$ Kenmotsu, cosymplectic, nearly cosymplectic, on five dimensional nilpotent Lie algebras and obtained some results on the corresponding Lie groups. This paper is organised in a similar vein with almost paracontact metric structure. Under the light of the classifications given in [9] and [12], we investigate the existence of left invariant para-cosymplectic, nearly para-cosymplectic, α-para-Sasakian, β-para-Kenmotsu and paracontact structures on 5 dimensional nilpotent Lie algebras.

2. Preliminaries

A $2n+1$ dimensional differentiable manifold M has an almost paracontact structure (ϕ,ξ,η), if it has an endomorphism ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 = I - \eta\otimes\xi, \qquad \eta(\xi)=1, \phi(\xi)=0,$$

***Email addresses and ORCID numbers*: nozdemir@eskisehir.edu.tr (N. Özdemir), https://orcid.org/0000-0003-0507-2444 (M. Solgun), mehmet.solgun@bilecik.edu.tr, https://orcid.org/0000-0002-2275-7763, sirins@eskisehir.edu.tr, https://orcid.org/0000-0003-2792-3481 (Ş. Aktay)**

$$\text{there exists a disribution } \mathbb{D} : p \in M \longrightarrow \mathbb{D}_p = Ker\eta.$$

An almost paracontact manifold is one which has an almost paracontact structure and if in addition M has a semi-Riemannian metric g satisfying

$$g(\phi(X),\phi(Y)) = -g(X,Y) + \eta(X)\eta(Y)$$

for all vector fields X,Y, then M is called an almost paracontact metric manifold with an almost paracontact metric structure and a compatible metric g. The 2-form

$$\Phi(X,Y) = g(\phi(X),Y)$$

for all $X,Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M, is defined to be the fundamental 2-form of M. In [2], a classification of almost paracontact metric manifolds was obtained by using the covariant derivative of Φ.

In this work we focus on following almost paracontact structures.

Let M be a differentiable manifold with an almost paracontact metric structure (ϕ,ξ,η,g) and the fundamental 2-form Φ. (ϕ,ξ,η,g) is said to be

- para-cosymplectic if $\nabla_X\Phi(Y,Z) = 0$,
- nearly para-cosymplectic if $\nabla_X\Phi(X,Y) = 0$, or equivalently, $(\nabla_X\phi)(Y) + (\nabla_Y\phi)(X) = 0$,
- α-para-Sasakian if $\nabla_X\phi(Y) = \alpha(g(X,Y)\xi - \eta(Y)X)$ for a constant α,
- β-para-Kenmotsu if $\nabla_X\phi(Y) = \beta(g(X,\phi(Y))\xi + \eta(Y)\phi(X))$ for a constant β,
- α-paracontact if $\Phi = \alpha d\eta$, where $d\eta$ is the exterior derivative of η and α is a constant,
- paracontact if $\Phi = d\eta$

for all vector fields X, Y, Z on M.

An almost paracontact metric structure (ϕ,ξ,η,g) on a connected Lie group G uniquely induces an almost paracontact metric structure (ϕ,ξ,η,g) on the corresponding Lie algebra $\mathfrak{g}$.

In this manuscript, we investigate almost paracontact metric structures on 5-dimensional nilpotent Lie algebras. Nilpotent Lie algebras with dimensions ≤ 5 were classified in [12], see also [15, 16]. These are algebras denoted by $\mathfrak{g}_i$ with the corresponding basis $\{e_1,\ldots,e_5\}$ and non-zero brackets:

$$\begin{aligned}
\mathfrak{g}_1 &: [e_1,e_2] = e_5, [e_3,e_4] = e_5\\
\mathfrak{g}_2 &: [e_1,e_2] = e_3, [e_1,e_3] = e_5, [e_2,e_4] = e_5\\
\mathfrak{g}_3 &: [e_1,e_2] = e_3, [e_1,e_3] = e_4, [e_1,e_4] = e_5, [e_2,e_3] = e_5\\
\mathfrak{g}_4 &: [e_1,e_2] = e_3, [e_1,e_3] = e_4, [e_1,e_4] = e_5\\
\mathfrak{g}_5 &: [e_1,e_2] = e_4, [e_1,e_3] = e_5\\
\mathfrak{g}_6 &: [e_1,e_2] = e_3, [e_1,e_3] = e_4, [e_2,e_3] = e_5
\end{aligned}$$

3. Almost paracontact metric structures on $\mathfrak{g}_i$

Let (ϕ,ξ,η,g) be a left invariant a.p.c.m.s. (almost paracontact metric structure) on a connected Lie group G with corresponding Lie algebra $\mathfrak{g}_i$. We use the same notation for the corresponding a.p.c.m.s. on $\mathfrak{g}_i$.

We study each algebra $\mathfrak{g}_i$ seperately:

The algebra $\mathfrak{g}_1$: Consider the basis $\{e_1,\ldots,e_5\}$ with non-zero brackets

$$[e_1,e_2] = e_5, [e_3,e_4] = e_5.$$

- There is no para-cosymplectic structure on $\mathfrak{g}_1$.
  To see this, we show that $\mathfrak{g}_1$ does not have a non-zero parallel vector field. Let $\xi = \sum a_ie_i$ be a parallel vector field on $\mathfrak{g}_1$. Then for all basis elements, we have $g(\nabla_{e_i}\xi, e_j) = 0$. By Kozsul's formula,

  $$2g(\nabla_{e_1}\xi, e_2) = -g(e_1,[\xi,e_2]) + g(\xi,[e_2,e_1] + g(e_2,[e_1,\xi])) = -a_5g(e_5,e_5) = 0,$$

  implying $a_5 = 0$. Similarly, $2g(\nabla_{e_1}\xi,e_5) = a_2g(e_5,e_5) = 0$ gives $a_2 = 0$, $2g(\nabla_{e_2}\xi,e_5) = -a_1g(e_5,e_5) = 0$ yields $a_1 = 0$. From the equation $2g(\nabla_{e_3}\xi,e_5) = a_4g(e_5,e_5) = 0$, we get $a_4 = 0$ and $2g(\nabla_{e_4}\xi,e_5) = -a_3g(e_5,e_5) = 0$ gives $a_3 = 0$. Thus, a vector field $\xi = \sum a_ie_i$ is parallel if and only if $a_i = 0$. Since for a para-cosymplectic structure the characteristic vector field is parallel, there is no para-cosymplectic structure on $\mathfrak{g}_1$.

Similarly, there are no nonzero parallel vector fields and no para-cosymplectic structures on remaining Lie algebras $\mathfrak{g}_i$. Now we calculate covariant derivatives of basis elements as follows:

$$\nabla_{e_1} e_2 = \sum \varepsilon_i g(\nabla_{e_1} e_2, e_i) e_i, \quad \text{where} \quad \varepsilon_i = g(e_i, e_i)$$

We write $g(\nabla_{e_1} e_2, e_i)$ by Kozsul's formula. The nonzero covariant derivatives are:

$$\nabla_{e_1} e_2 = \frac{1}{2} e_5, \;\; \nabla_{e_1} e_5 = -\frac{1}{2} \varepsilon_2 \varepsilon_5 e_2,$$

$$\nabla_{e_2} e_1 = -\frac{1}{2} e_5, \;\; \nabla_{e_2} e_5 = \frac{1}{2} \varepsilon_1 \varepsilon_5 e_1,$$

$$\nabla_{e_3} e_4 = \frac{1}{2} e_5, \;\; \nabla_{e_3} e_5 = -\frac{1}{2} \varepsilon_4 \varepsilon_5 e_4,$$

$$\nabla_{e_4} e_3 = -\frac{1}{2} e_5, \;\; \nabla_{e_4} e_5 = \frac{1}{2} \varepsilon_3 \varepsilon_5 e_3,$$

$$\nabla_{e_5} e_1 = -\frac{1}{2} \varepsilon_2 \varepsilon_5 e_2, \;\; \nabla_{e_5} e_2 = \frac{1}{2} \varepsilon_1 \varepsilon_5 e_1, \;\; \nabla_{e_5} e_3 = -\frac{1}{2} \varepsilon_4 \varepsilon_5 e_4, \;\; \nabla_{e_5} e_4 = \frac{1}{2} \varepsilon_3 \varepsilon_5 e_3$$

- There is no nearly para-cosymplectic structure on $\mathfrak{g}_1$.
  Assume that (ϕ, ξ, η, g) is a nearly para-cosymplectic structure. Then we have $\nabla_{e_i} \phi(e_j) + \nabla_{e_j} \phi(e_i) = 0$.
  Let

$$\phi(e_1) = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5,$$

$$\phi(e_2) = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5,$$

$$\phi(e_3) = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5,$$

$$\phi(e_4) = d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4 + d_5 e_5,$$

$$\phi(e_5) = f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_5 e_5.$$

Since $0 = \Phi(e_i, e_i) = g(\phi(e_i), e_i)$, we have $a_1 = b_2 = c_3 = d_4 = f_5 = 0$.
From the equation $(\nabla_{e_1} \Phi)(e_1, e_5) = 0$, we obtain $0 = -\Phi(e_1, \nabla_{e_1} e_5) = -g(\phi(e_1), -\frac{1}{2}\varepsilon_2 \varepsilon_5 e_2)$, which implies $g(\phi(e_1), e_2) = -g(\phi(e_2), e_1) = 0$, thus $a_2 = b_1 = 0$.
Similarly, from $(\nabla_{e_1} \Phi)(e_1, e_2) = -\Phi(e_1, \nabla_{e_1} e_2) = -\frac{1}{2} g(\phi(e_1), e_5) = 0$, which implies $a_5 = f_1 = 0$.
$(\nabla_{e_4} \Phi)(e_4, e_3) = 0$ gives $g(\phi(e_4), e_5) = -g(\phi(e_5), e_4) = 0$ and so $d_5 = f_4 = 0$.
$(\nabla_{e_4} \Phi)(e_4, e_5) = 0$ gives $g(\phi(e_4), e_3) = -g(\phi(e_3), e_4) = 0$ and so $d_3 = c_4 = 0$.
$(\nabla_{e_2} \Phi)(e_2, e_1) = 0$ gives $g(\phi(e_2), e_5) = -g(\phi(e_5), e_2) = 0$ and so $b_5 = f_2 = 0$.
$(\nabla_{e_3} \Phi)(e_3, e_4) = 0$ gives $g(\phi(e_3), e_5) = -g(\phi(e_5), e_3) = 0$ and so $c_5 = f_3 = 0$.
Thus,

$$\phi(e_1) = a_3 e_3 + a_4 e_4,$$

$$\phi(e_2) = b_3 e_3 + b_4 e_4,$$

$$\phi(e_3) = c_1 e_1 + c_2 e_2,$$

$$\phi(e_4) = d_1 e_1 + d_2 e_2,$$

$$\phi(e_5) = 0.$$

Since

$$\begin{aligned} 0 &= (\nabla_{e_1} \phi)(e_5) + (\nabla_{e_5} \phi)(e_1) \\ &= e_3\{-\varepsilon_2 \varepsilon_5 b_3 + \tfrac{1}{2} \varepsilon_3 \varepsilon_5 a_4\} \\ &\quad + e_4\{-\varepsilon_2 \varepsilon_5 b_4 - \tfrac{1}{2} \varepsilon_4 \varepsilon_5 a_3\} \end{aligned}$$

and e_3, e_4 are linearly independent, we have

$$2\varepsilon_2 b_3 + \varepsilon_3 a_4 = 0,$$

$$2\varepsilon_2 b_4 - \varepsilon_4 a_3 = 0.$$

Similarly, from $(\nabla_{e_1} \phi)(e_5) + (\nabla_{e_5} \phi)(e_1) = 0$, we get

$$2\varepsilon_1 a_3 + \varepsilon_3 b_4 = 0,$$

$$-2\varepsilon_1 a_4 - \varepsilon_4 b_3 = 0.$$

Now, we have

$$2\varepsilon_2 b_3 + \varepsilon_3 a_4 = 0,$$

$$-\varepsilon_4 b_3 - 2\varepsilon_1 a_4 = 0.$$

Multiply the first equation by $2\varepsilon_3$ and the second equation by ε_1. Then we get $b_3 = 0$ and $a_4 = 0$. Similarly $a_3 = 0$ and $b_4 = 0$.
From the equation $(\nabla_{e_1}\phi)(e_3) + (\nabla_{e_3}\phi)(e_1) = 0$, we obtain $(c_2 + a_4)e_5 = 0$, that is, $c_2 = -a_4$ and since $a_4 = 0$, we have $c_2 = 0$.
From the equation $(\nabla_{e_2}\phi)(e_3) + (\nabla_{e_3}\phi)(e_2) = 0$, we obtain $c_1 = b_4$ and since $b_4 = 0$, we have $c_1 = 0$.
Similarly, $(\nabla_{e_2}\phi)(e_4) + (\nabla_{e_4}\phi)(e_2) = 0$ implies $d_1 = -b_3 = 0$ and $(\nabla_{e_1}\phi)(e_4) + (\nabla_{e_4}\phi)(e_1) = 0$ yields $d_2 = a_3 = 0$.
Therefore $\phi(e_i) = 0$ and there is no non-zero nearly para-cosymplectic structure on $\mathfrak{g}_1$.

- A vector field ξ on $\mathfrak{g}_1$ is Killing if and only if $\xi \in \langle e_5 \rangle$: For a Killing vector field $\xi = \sum_i \xi_i e_i$, we have $g(\nabla_{e_i}\xi, e_j) = -g(\nabla_{e_j}\xi, e_i)$. From $g(\nabla_{e_2}\xi, e_5) = -g(\nabla_{e_5}\xi, e_2)$, we have $\xi_1 = 0$.
$g(\nabla_{e_1}\xi, e_5) = -g(\nabla_{e_5}\xi, e_1)$ gives $\xi_2 = 0$.
$g(\nabla_{e_4}\xi, e_5) = -g(\nabla_{e_5}\xi, e_4)$ yields $\xi_3 = 0$.
$g(\nabla_{e_3}\xi, e_5) = -g(\nabla_{e_5}\xi, e_3)$ implies $\xi_4 = 0$ and we have no any other restriction on the coefficients of ξ.
By similar calculations, in $\mathfrak{g}_2$, $\mathfrak{g}_3$ and $\mathfrak{g}_4$, a vector field ξ is Killing if and only if $\xi = \xi_5 e_5$.
A vector field ξ in $\mathfrak{g}_5$ or $\mathfrak{g}_6$ is Killing on each of these algebras if and only if $\xi = \xi_4 e_4 + \xi_5 e_5$.
- There are α-para-Sasakian structures on $\mathfrak{g}_1$, where $\alpha = \pm\frac{1}{2}$.
For $y = \xi$, we get $-\phi(\nabla_x \xi) = \alpha\{g(x,\xi)\xi - x\}$. Thus, $\nabla_x \xi = \alpha\phi(x)$. The characteristic vector field of an α-para-Sasakian structure is Killing. Thus $\xi = \xi_5 e_5$ and

$$\phi(e_1) = \frac{1}{\alpha}\nabla_{e_1}\xi = \frac{1}{\alpha}\xi_5(-\frac{1}{2}\varepsilon_2\varepsilon_5 e_2),$$

$$\phi(e_2) = \frac{1}{\alpha}\nabla_{e_2}\xi = \frac{1}{\alpha}\xi_5(\frac{1}{2}\varepsilon_1\varepsilon_5 e_1),$$

$$\phi(e_3) = \frac{1}{\alpha}\nabla_{e_3}\xi = \frac{1}{\alpha}\xi_5(-\frac{1}{2}\varepsilon_4\varepsilon_5 e_4),$$

$$\phi(e_4) = \frac{1}{\alpha}\nabla_{e_4}\xi = \frac{1}{\alpha}\xi_5(\frac{1}{2}\varepsilon_3\varepsilon_5 e_3),$$

$$\phi(e_5) = 0.$$

Now we check the defining relation of an α-para-Sasakian structure (ϕ, ξ, η, g)

$$(\nabla_x \phi)(y) = \alpha\{g(x,y)\xi - \eta(y)x\}$$

for each pair of basis elements. For $x = y = e_1$, we should have

$$(\nabla_{e_1}\phi)(e_1) = \alpha\{g(e_1,e_1)\xi_5 e_5\},$$

which implies

$$-\frac{1}{4\alpha}\varepsilon_2\varepsilon_5 e_5 = \alpha\varepsilon_1 e_5.$$

Multiply both sides of the above equation by ε_1, we obtain $\varepsilon_1\varepsilon_2\varepsilon_5 = -4\alpha^2$. Thus $\varepsilon_1\varepsilon_2\varepsilon_5 = -1$ and $\alpha = \pm\frac{1}{2}$.
Similarly, for $x = y = e_3$, we get $\varepsilon_3\varepsilon_4\varepsilon_5 = -4\alpha^2$, which gives $\varepsilon_3\varepsilon_4\varepsilon_5 = -1$ and $\alpha = \pm\frac{1}{2}$. There is no any other restriction on ε_i or on α.
We have $\varepsilon_1\varepsilon_2\varepsilon_5 = -1$ and $\varepsilon_3\varepsilon_4\varepsilon_5 = -1$.
Case 1: If $\varepsilon_5 = -1$, then $\varepsilon_1\varepsilon_2 = 1$. Either $\varepsilon_1 = 1$ and $\varepsilon_2 = 1$; or $\varepsilon_1 = -1$ and $\varepsilon_2 = -1$. Also, since $\varepsilon_3\varepsilon_4 = 1$, $\varepsilon_3 = 1$ and $\varepsilon_4 = 1$; or $\varepsilon_3 = -1$ and $\varepsilon_4 = -1$. In these cases the signature is not $(3,2)$. Thus $\varepsilon_5 \neq -1$.
Case 2: If $\varepsilon_5 = 1$, then $\varepsilon_1\varepsilon_2 = -1$ and $\varepsilon_3\varepsilon_4 = -1$. There are four possibilities for the signature of the metric.

$$\varepsilon_1 = 1, \varepsilon_2 = -1, \varepsilon_3 = 1, \varepsilon_4 = -1, \varepsilon_5 = 1$$

$$\varepsilon_1 = 1, \varepsilon_2 = -1, \varepsilon_3 = -1, \varepsilon_4 = 1, \varepsilon_5 = 1$$

$$\varepsilon_1 = -1, \varepsilon_2 = 1, \varepsilon_3 = -1, \varepsilon_4 = 1, \varepsilon_5 = 1$$

$$\varepsilon_1 = -1, \varepsilon_2 = 1, \varepsilon_3 = 1, \varepsilon_4 = -1, \varepsilon_5 = 1.$$

One can check that (ϕ, ξ, η, g), where $\phi(e_i)$ are given as above and g has one of the signatures above, are α-para-Sasakian structures, where $\alpha = \pm\frac{1}{2}$.

- There is no β-para-Kenmotsu structure on $\mathfrak{g}_1$.
The characteristic vector field ξ of a β-para-Kenmotsu structure satisfies the property $g(\nabla_x\xi, y) = g(\nabla_y\xi, x)$. Checking for basis elements, we obtain that $\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4$. The definition of a β-para-Kenmotsu structure (ϕ, ξ, η, g) is
$$(\nabla_x\phi)(y) = -\beta\{g(x, \phi(y))\xi + \eta(y)\phi(x)\}.$$
For $y = \xi$, we get $\nabla_x\xi = \beta\phi^2(x) = \beta\{x - \eta(x)\xi\}$. Now
$$\nabla_{e_1}\xi = \nabla_{e_1}(\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4) = \frac{\xi_2}{2}e_5 = \beta\{e_1 - \varepsilon_1\xi_1(\xi_1 e_1 + ... + \xi_4 e_4)\}.$$
Since basis elements are linearly independent, we have $\xi_2 = 0$, $1 - \varepsilon_1\xi_1^2 = 0$, $\xi_1\xi_3 = 0$, $\xi_1\xi_4 = 0$. If $\xi_1 = 0$, then $1 - \varepsilon_1\xi_1^2 = 1 = 0$ and thus $\xi_1 \neq 0$. Therefore, $\xi_3 = \xi_4 = 0$ and $\xi = \xi_1 e_1$. Now for $x = e_2$, we have
$$\nabla_{e_2}\xi = -\frac{\xi_1}{2}e_5 = \beta\{e_2 - \eta(e_2)\xi\} = \beta e_2,$$
which implies $\xi_1 = 0$, that is $\xi = 0$.
- There are paracontact structures on $\mathfrak{g}_1$.
More generally, consider an α-paracontact structure (ϕ, ξ, η, g) with the fundamental 2-form Φ. Since $\Phi = \alpha d\eta$, we have
$$0 = \Phi(\xi, x) = \alpha d\eta(\xi, x) = \frac{1}{2}\{(\nabla_\xi\eta)(x) - (\nabla_x\eta)(\xi)\} = -(\nabla_\xi\eta)(x),$$
that is, $(\nabla_\xi\eta)(x) = g(\xi, \nabla_\xi x) = 0$. By Kozsul's formula,
$$0 = 2g(\nabla_\xi x, \xi) = -2g(\xi, [x, \xi]).$$
Thus for a paracontact structure, the characteristic vector field ξ satisfies $g(\xi, [x, \xi]) = 0$ for all vector fields x.
Let $\xi = \sum \xi_i e_i$. We have
$$0 = g(\xi, [e_1, \xi]) = g(\xi, \xi_2 e_5) = \xi_2\xi_5\varepsilon_5,$$
$$0 = g(\xi, [e_2, \xi]) = g(\xi, -\xi_1 e_5) = -\xi_1\xi_5\varepsilon_5,$$
$$0 = g(\xi, [e_3, \xi]) = g(\xi, \xi_4 e_5) = \xi_4\xi_5\varepsilon_5,$$
$$0 = g(\xi, [e_4, \xi]) = g(\xi, -\xi_3 e_5) = -\xi_3\xi_5\varepsilon_5.$$
It is easy to observe that the structure (ϕ, ξ, η, g), where $\xi = e_5$, g has signature $+,-,+,-,+$, $\phi(e_1) = e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = e_4$, $\phi(e_4) = e_3$, $\phi(e_5) = 0$ is paracontact.

The algebra $\mathfrak{g}_2$:

The nonzero brackets of basis elements are:
$$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5.$$
Assume that g is a semi Riemannian metric with signature $g(e_i, e_i) = \varepsilon_i$. Nonzero covariant derivatives of g calculated by the Kozsul's formula are:
$$\begin{array}{lll}
\nabla_{e_1}e_2 = \frac{1}{2}e_3, & \nabla_{e_1}e_3 = -\frac{1}{2}\varepsilon_2\varepsilon_3 e_2 + \frac{1}{2}e_5, & \nabla_{e_1}e_5 = -\frac{1}{2}\varepsilon_3\varepsilon_5 e_3, \\
\nabla_{e_2}e_1 = -\frac{1}{2}e_3, & \nabla_{e_2}e_3 = \frac{1}{2}\varepsilon_1\varepsilon_3 e_1, & \nabla_{e_2}e_4 = \frac{1}{2}e_5, \\
\nabla_{e_2}e_5 = -\frac{1}{2}\varepsilon_4\varepsilon_5 e_4, & \nabla_{e_3}e_1 = -\frac{1}{2}\varepsilon_2\varepsilon_3 e_2 - \frac{1}{2}e_5, & \nabla_{e_3}e_2 = \frac{1}{2}\varepsilon_1\varepsilon_3 e_1, \\
\nabla_{e_3}e_5 = \frac{1}{2}\varepsilon_1\varepsilon_5 e_1, & \nabla_{e_4}e_2 = -\frac{1}{2}e_5, & \nabla_{e_4}e_5 = \frac{1}{2}\varepsilon_2\varepsilon_5 e_2, \\
\nabla_{e_5}e_1 = -\frac{1}{2}\varepsilon_3\varepsilon_5 e_3, & \nabla_{e_5}e_2 = -\frac{1}{2}\varepsilon_4\varepsilon_5 e_4, & \\
\nabla_{e_5}e_3 = \frac{1}{2}\varepsilon_1\varepsilon_5 e_1, & \nabla_{e_5}e_4 = \frac{1}{2}\varepsilon_2\varepsilon_5 e_2. &
\end{array}$$

- There exists no nearly-para-cosymplectic structure.
Assume that (ϕ,ξ,η,g) is a nearly para-cosymplectic structure. Then we have $\nabla_{e_i}\phi(e_j)+\nabla_{e_j}\phi(e_i)=0$.
Let

$$\phi(e_1)=a_1e_1+a_2e_2+a_3e_3+a_4e_4+a_5e_5,$$
$$\phi(e_2)=b_1e_1+b_2e_2+b_3e_3+b_4e_4+b_5e_5,$$
$$\phi(e_3)=c_1e_1+c_2e_2+c_3e_3+c_4e_4+c_5e_5,$$
$$\phi(e_4)=d_1e_1+d_2e_2+d_3e_3+d_4e_4+d_5e_5,$$
$$\phi(e_5)=f_1e_1+f_2e_2+f_3e_3+f_4e_4+f_5e_5.$$

Since $0=\Phi(e_i,e_i)=g(\phi(e_i),e_i)$, we have $a_1=b_2=c_3=d_4=f_5=0$.
From the equation
$$\begin{aligned} 0 &= (\nabla_{e_2}\varphi)(e_2)=b_1\nabla_{e_2}e_1+b_3\nabla_{e_2}e_3+b_4\nabla_{e_2}e_4+b_5\nabla_{e_2}e_5 \\ &= b_1(-\tfrac{1}{2}e_3)+b_3(\tfrac{1}{2}\varepsilon_1\varepsilon_3)e_1+b_4(\tfrac{1}{2}e_5)+b_5(-\tfrac{1}{2}\varepsilon_4\varepsilon_5)e_4 \end{aligned}$$
we have, $b_1=b_3=b_4=b_5=0$. The equation $0=(\nabla_{e_5}\varphi)(e_5)$ gives $f_1=f_2=f_3=f_4=0$. Since $(\nabla_x\Phi)(x,y)=0$, we have
$$0=(\nabla_{e_1}\Phi)(e_1,e_2)=-\Phi(e_1,\nabla_{e_1}e_2)=-\frac{1}{2}g(\phi(e_1),e_3)=\frac{1}{2}g(\phi(e_3),e_1),$$
and thus $a_3=c_1=0$. In addition,
$$\begin{aligned} 0 &= (\nabla_{e_1}\Phi)(e_1,e_3)=-\Phi(e_1,\nabla_{e_1}e_3) \\ &= -g(\phi(e_1),-\tfrac{1}{2}\varepsilon_2\varepsilon_3e_2+\tfrac{1}{2}e_5) \\ &= \tfrac{1}{2}g(\phi(e_1),\varepsilon_2\varepsilon_3e_2)+\tfrac{1}{2}g(\phi(e_5),e_1) \\ &= \tfrac{1}{2}\varepsilon_2\varepsilon_3g(\phi(e_1),e_2) \end{aligned}$$
implies $a_2=b_1=0$. Since $\phi(e_5)=0$, we have $g(\phi(e_i),e_5)=-g(\phi(e_5),e_i)=0$ and as a result $a_5=b_5=c_5=d_5=0$.
Since
$$\begin{aligned} 0 &= (\nabla_{e_2}\Phi)(e_2,e_5)=-\Phi(e_2,\nabla_{e_2}e_5) \\ &= \tfrac{1}{2}\varepsilon_4\varepsilon_5g(\phi(e_2),e_4), \end{aligned}$$
we get $b_4=d_2=0$. Since
$$\begin{aligned} 0 &= (\nabla_{e_3}\Phi)(e_3,e_1)=-\Phi(\nabla_{e_3}e_1,e_3)=\Phi(e_3,\nabla_{e_3}e_1) \\ &= -\tfrac{1}{2}\varepsilon_2\varepsilon_3g(\phi(e_3),e_2), \end{aligned}$$
we have $c_2=b_3=0$. Now,
$$0=(\nabla_{e_1}\phi)(e_2)+(\nabla_{e_2}\phi)(e_1)=a_4\nabla_{e_2}e4=\frac{a_4}{2}e_5=0$$
gives $a_4=0$. Since $\phi(e_1)=0$, we obtain $0=g(\phi(e_1),e_4)=-g(\phi(e_4),e_1)$, that is $d_1=0$. From
$$0=(\nabla_{e_1}\phi)(e_5)+(\nabla_{e_5}\phi)(e_1)=\varepsilon_3\varepsilon_5c_4\phi(e_3)=0,$$
we get $c_4=0$ and this implies also $d_3=0$. To sum up $\phi(e_i)=0$ for all $i=1,\cdots,5$.
By similar calculations, there are no nearly-para-cosymplectic structures on the remaining Lie algebras $\mathfrak{g}_i$.
- There is no α-para-Sasakian structure.
Let (ϕ,ξ,η,g) be such a structure. We have $\xi=\xi_5e_5$, since the characteristic vector field is Killing. Since $g(\xi,\xi)=\xi_5^2\varepsilon_5=1$, $\varepsilon_5=1$. $\nabla_x\xi=\alpha\phi(x)$,
$$\phi(e_1)=\frac{1}{\alpha}\nabla_{e_1}e_5=-\frac{1}{2\alpha}\varepsilon_3\varepsilon_5e_3.$$
Now we check the defining relation $(\nabla_x\phi)(y)=\alpha\{g(x,y)\xi-\eta(y)x\}$ for basis elements.
Let $x=y=e_1$. In this case, the equation $(\nabla_{e_1}\phi)(e_1)=\alpha g(e_1,e_1)e_5$ implies
$$\frac{1}{4\alpha}\varepsilon_2\varepsilon_5e_2-\{\frac{1}{4\alpha}\varepsilon_3\varepsilon_5+\alpha\varepsilon_1\}e_5=0,$$
this is not possible since e_2 and e_5 are linearly independent.
- This algebra does not have a β-para-Kenmotsu structure.
From the equation $g(\nabla_x\xi,y)=g(\nabla_y\xi,x)$ in a β-para-Kenmotsu structure, ξ is obtained in the form $\xi=\xi_1e_1+\xi_2e_2+\xi_3e_3+\xi_4e_4$. Also for $x=e_3$ in the equation
$$\nabla_x\xi=\beta\phi^2(x)=\beta\{x-\eta(x)\xi\},$$

we get

$$\{\frac{\xi_2}{2}\varepsilon_1\varepsilon_3+\beta\varepsilon_3\xi_1\xi_3\}e_1-\{\frac{\xi_2}{2}\varepsilon_2\varepsilon_3+\beta\varepsilon_3\xi_2\xi_3\}e_2+\beta(\varepsilon_3\xi_3^2-1)e_3+\beta\varepsilon_3\xi_3\xi_4e_4-\frac{\xi_1}{2}e_5=0.$$

Linear independence of basis element yields $\xi_1=0$, $\xi_2=0$, $\xi_3=1$ and $\xi_4=0$. Thus $\xi=\xi_3e_3$. However, in this case,

$$\nabla_{e_2}\xi=\frac{\xi_3}{2}\varepsilon_1\varepsilon_3e_1\neq\beta\{e_2-\eta(e_2)\xi\}=\beta e_2.$$

- There are paracontact structures. Consider a paracontact structure (ϕ,ξ,η,g) with the fundamental 2-form Φ. Since $\Phi=d\eta$, the equation

$$g(\phi(e_i),e_j)=g(\nabla_{e_i}\xi,e_j)-g(\nabla_{e_j}\xi,e_i)$$

holds for all basis elements. Let $\xi=\sum\xi_ie_i$. Then,

$$g(\phi(e_1),e_2)=g(\nabla_{e_1}\xi,e_2)-g(\nabla_{e_2}\xi,e_1)=-\varepsilon_3\xi_3,$$
$$g(\phi(e_1),e_3)=g(\nabla_{e_1}\xi,e_3)-g(\nabla_{e_3}\xi,e_1)=-\varepsilon_5\xi_5,$$
$$g(\phi(e_1),e_4)=g(\nabla_{e_1}\xi,e_4)-g(\nabla_{e_4}\xi,e_1)=0,$$
$$g(\phi(e_1),e_5)=0,$$
$$g(\phi(e_2),e_3)=0,$$
$$g(\phi(e_2),e_4)=g(\nabla_{e_2}\xi,e_4)-g(\nabla_{e_4}\xi,e_2)=-\varepsilon_5\xi_5,$$
$$g(\phi(e_2),e_5)=g(\phi(e_3),e_4)=g(\phi(e_3),e_5)=g(\phi(e_4),e_5)=0.$$

Thus,

$$\phi(e_1)=-\xi_3\varepsilon_2\varepsilon_3e_2-\xi_5\varepsilon_3\varepsilon_5e_3,$$
$$\phi(e_2)=\xi_3\varepsilon_1\varepsilon_3e_1-\xi_5\varepsilon_4\varepsilon_5e_4,$$
$$\phi(e_3)=\xi_5\varepsilon_1\varepsilon_5e_1,$$
$$\phi(e_4)=\xi_5\varepsilon_2\varepsilon_5e_2,$$
$$\phi(e_5)=0.$$

Now the equation $\phi^2(e_3)=e_3-\eta(e_3)\xi$ and linear independence of basis elements imply

$$\xi_1\xi_3=0,\quad \xi_3\xi_4=0,\quad \xi_3\xi_5=0.$$

There are structures satisfying these properties. For example, the structure (ϕ,ξ,η,g), such that $\xi=e_5$, $\phi(e_1)=e_3$, $\phi(e_2)=e_4$, $\phi(e_3)=e_1$, $\phi(e_4)=e_2$ and the metric has signature $+,+,-,-,+$ is paracontact.

The algebra $\mathfrak{g}_3$: The nonzero brackets and nonzero covariant derivatives are as follows:

$$[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_1,e_4]=e_5,\quad [e_2,e_3]=e_5$$

$$\begin{array}{lll}
\nabla_{e_1}e_2=\frac{1}{2}e_3, & \nabla_{e_1}e_3=-\frac{1}{2}\varepsilon_2\varepsilon_3e_2+\frac{1}{2}e_4, & \nabla_{e_1}e_4=-\frac{1}{2}\varepsilon_3\varepsilon_4e_3+\frac{1}{2}e_5,\\
\nabla_{e_1}e_5=-\frac{1}{2}\varepsilon_4\varepsilon_5e_4, & \nabla_{e_2}e_1=-\frac{1}{2}e_3, & \nabla_{e_2}e_3=\frac{1}{2}\varepsilon_1\varepsilon_3e_1+\frac{1}{2}e_5,\\
\nabla_{e_2}e_5=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3, & \nabla_{e_3}e_1=-\frac{1}{2}\varepsilon_2\varepsilon_3e_2-\frac{1}{2}e_4, & \nabla_{e_3}e_2=\frac{1}{2}\varepsilon_1\varepsilon_3e_1-\frac{1}{2}e_5\\
\nabla_{e_3}e_4=\frac{1}{2}\varepsilon_1\varepsilon_4e_1, & \nabla_{e_3}e_5=\frac{1}{2}\varepsilon_2\varepsilon_5e_2, & \nabla_{e_4}e_1=-\frac{1}{2}\varepsilon_3\varepsilon_4e_3-\frac{1}{2}e_5\\
\nabla_{e_4}e_3=\frac{1}{2}\varepsilon_1\varepsilon_4e_1, & \nabla_{e_4}e_5=\frac{1}{2}\varepsilon_1\varepsilon_5e_1, & \nabla_{e_5}e_1=-\frac{1}{2}\varepsilon_4\varepsilon_5e_4,\\
\nabla_{e_5}e_2=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3, & \nabla_{e_5}e_3=\frac{1}{2}\varepsilon_2\varepsilon_5e_2, & \nabla_{e_5}e_4=\frac{1}{2}\varepsilon_1\varepsilon_5e_1.
\end{array}$$

- There is no α-para-Sasakian structure.
  The characteristic vector field of an α-para-Sasakian is Killing. Thus if (ϕ,ξ,η,g) is an α-para-Sasakian structure, $\xi=e_5$. Then
$$\phi(e_1)=\frac{1}{\alpha}\nabla_{e_1}e_5=-\frac{1}{2\alpha}\varepsilon_4\varepsilon_5e_4$$
  and the equation
$$(\nabla_{e_1}\phi)(e_1)=\alpha\{g(e_1,e_1)e_5-\eta(e_1)e_1\}$$
  result in the contradiction
$$\frac{1}{4\alpha}\varepsilon_3\varepsilon_5e_3-\{\frac{1}{4\alpha}\varepsilon_4\varepsilon_5-\alpha\varepsilon_1\}e_5=0.$$
- There is no β-para-Kenmotsu structure.
  Since ξ satisfies $g(\nabla_x\xi,y)=g(\nabla_y\xi,x)$, checking this condition for basis elements, we get that ξ is of the form $\xi=\xi_1e_1+\xi_2e_2$. For $x=e_1$, the equation $\nabla_x\xi=\beta\{x-\eta(x)\xi\}$ implies $(1-\beta\varepsilon_1\xi_1^2)e_1-\beta\varepsilon_1\xi_1\xi_2e_2-\frac{\xi_2}{2}e_3=0$. From linear independence, we have $\xi_2=0$ and so $\xi=\xi_1e_1$. Now for $x=e_2$, we get $\beta e_2+\frac{\xi_1}{2}e_3=0$, a contradiction.
- There are paracontact structures.
  By using the defining equation of an α-paracontact structure
$$\Phi(e_i,e_j)=g(\phi(e_i),e_j)=\alpha d\eta=\alpha\{g(\nabla_{e_i}\xi,e_j)-g(\nabla_{e_j}\xi,e_i)\},$$
  we write
$$\begin{aligned}\phi(e_1)&=-\alpha\{\xi_3\varepsilon_2\varepsilon_3e_2+\xi_4\varepsilon_3\varepsilon_4e_3+\xi_5\varepsilon_4\varepsilon_5e_4\},\\ \phi(e_2)&=\alpha\{\xi_3\varepsilon_1\varepsilon_3e_1-\xi_5\varepsilon_3\varepsilon_5e_3\},\\ \phi(e_3)&=\alpha\{\xi_4\varepsilon_1\varepsilon_4e_1+\xi_5\varepsilon_2\varepsilon_5e_2\},\\ \phi(e_4)&=\alpha\xi_5\varepsilon_1\varepsilon_5e_1,\\ \phi(e_5)&=0.\end{aligned}$$
  In addition, the relation $0=g(\phi(e_5),\phi(e_i))=-g(e_5,e_i)+\eta(e_5)\eta(e_i)$ gives $\xi_1\xi_5=\xi_2\xi_5=\xi_3\xi_5=\xi_4\xi_5=0$. We can find structures with these properties. For instance, (ϕ,ξ,η,g), where $\xi=e_5$, $\phi(e_1)e_4$, $\phi(e_2)=e_3$, $\phi(e_3)=e_2$, $\phi(e_4)=e_1$, $\phi(e_5)=0$ and g has the signature $+,+,-,-,+$ is a paracontact structure.

The algebra $\mathfrak{g}_4$: The nonzero brackets and nonzero covariant derivatives are:
$$[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_1,e_4]=e_5$$
$$\begin{array}{lll}\nabla_{e_1}e_2=\frac12e_3, & \nabla_{e_1}e_3=-\frac12\varepsilon_2\varepsilon_3e_2+\frac12e_4, & \nabla_{e_1}e_4=-\frac12\varepsilon_3\varepsilon_4e_3+\frac12e_5,\\ \nabla_{e_1}e_5=-\frac12\varepsilon_4\varepsilon_5e_4, & \nabla_{e_2}e_1=-\frac12e_3, & \nabla_{e_2}e_3=\frac12\varepsilon_1\varepsilon_3e_1,\\ \nabla_{e_3}e_1=-\frac12\varepsilon_2\varepsilon_3e_2-\frac12e_4, & \nabla_{e_3}e_2=\frac12\varepsilon_1\varepsilon_3e_1, & \nabla_{e_3}e_4=\frac12\varepsilon_1\varepsilon_4e_1,\\ \nabla_{e_4}e_1=-\frac12\varepsilon_3\varepsilon_4e_3-\frac12e_5 & \nabla_{e_4}e_3=\frac12\varepsilon_1\varepsilon_4e_1, & \nabla_{e_4}e_5=\frac12\varepsilon_1\varepsilon_5e_1,\\ \nabla_{e_5}e_1=-\frac12\varepsilon_4\varepsilon_5e_4, & \nabla_{e_5}e_4=\frac12\varepsilon_1\varepsilon_5e_1. & \end{array}$$

- This algebra does not admit an α-para-Sasakian structure.
  Let (ϕ,ξ,η,g) be an α-para-Sasakian structure. Since ξ is Killing, we have $\xi=e_5$ in $\mathfrak{g}_4$. From the equation $\nabla_{e_2}\xi=\alpha\phi(e_2)$, we get $\phi(e_2)=0$. On the other hand,
$$0=g(\phi(e_2),\phi(e_2))\neq-g(e_2,e_2)+\eta(e_2)\eta(e_2)=-\varepsilon_2.$$
- There exists no β-para-Kenmotsu structure.
  From the equation $g(\nabla_x\xi,y)=g(\nabla_y\xi,x)$, the Reeb vector field is obtained in the form $\xi=\xi_1e_1+\xi_2e_2+\xi_4e_4+\xi_5e_5$. Since $\nabla_{e_3}\xi=\beta\phi^2(e_3)$, we have
$$\frac12\varepsilon_1(\varepsilon_3\xi_2+\varepsilon_4\xi_4)e_1-\frac12\varepsilon_2\varepsilon_3\xi_1e_2-\beta e_3-\frac{\xi_1}{2}e_4=0.$$
  Since basis elements are linearly independent, there is no nonzero number β satisfying this equation.

- There is no paracontact structure. Since
$$\Phi(e_i,e_j)=g(\phi(e_i),e_j)=d\eta(e_i,e_j)=g(\nabla_{e_i}\xi,e_j)-g(\nabla_{e_j}\xi,e_i)$$
for a paracontact structure, we obtain $\phi(e_4)=\frac{\xi_5}{2}\varepsilon_1\varepsilon_5e_1$ and $\phi(e_5)=0$. On the other hand, $\phi^2(e_5)=e_5-\eta(e_5)\xi$ gives
$$\xi_1\xi_5\varepsilon_5e_1+\xi_2\xi_5\varepsilon_5e_2+\xi_3\xi_5\varepsilon_5e_3+\xi_4\xi_5\varepsilon_5e_4)+(\xi_5^2\varepsilon_5-1)e_5=0.$$
From linear independence of basis elements, we have
$$\xi_1\xi_5=\xi_2\xi_5=\xi_3\xi_5=\xi_4\xi_5=0,\quad \xi_5^2\varepsilon_5=1.$$
Since $\xi_5^2\neq 0$, we get $\xi_1=\xi_2=\xi_3=\xi_4$ and $\xi=\xi_5e_5$. Then, $0=\phi^2(e_4)\neq e_4-\eta(e_4)\xi=e_4$.

The algebra $\mathfrak{g}_5$:
$$[e_1,e_2]=e_4,\quad [e_1,e_3]=e_5$$

$$\begin{array}{lll}
\nabla_{e_1}e_2=\frac{1}{2}e_4, & \nabla_{e_1}e_3=\frac{1}{2}e_5, & \nabla_{e_1}e_4=-\frac{1}{2}\varepsilon_2\varepsilon_4e_2,\\
\nabla_{e_1}e_5=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3, & \nabla_{e_2}e_1=-\frac{1}{2}e_4, & \nabla_{e_2}e_4=\frac{1}{2}\varepsilon_1\varepsilon_4e_1,\\
\nabla_{e_3}e_1=-\frac{1}{2}e_5, & \nabla_{e_3}e_5=\frac{1}{2}\varepsilon_1\varepsilon_5e_1, & \nabla_{e_4}e_1=-\frac{1}{2}\varepsilon_2\varepsilon_4e_2,\\
\nabla_{e_4}e_2=\frac{1}{2}\varepsilon_1\varepsilon_4e_1, & \nabla_{e_5}e_1=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3\quad, & \nabla_{e_5}e_3=\frac{1}{2}\varepsilon_1\varepsilon_5e_1.
\end{array}$$

- There exists no α-para-Sasakian structure.
Let (ϕ,ξ,η,g) be an α-para-Sasakian structure. Since ξ is Killing, $\xi=\xi_4e_4+\xi_5e_5$. From the equation $\nabla_x\xi=\alpha\phi(x)$, we get $\phi(e_4)=\phi(e_5)=0$. In addition,
$$g(\phi(e_4),\phi(e_4))=-g(e_4,e_4)+\eta(e_4)\eta(e_4)$$
implies $0=-\varepsilon_4+\xi_4^2$. Thus, $\varepsilon_4=1$ and $\xi_4^2=1$. Similarly, $\xi_5^2=1$ and $\varepsilon_5=1$. However, in this case, $g(\xi,\xi)=\xi_4^2\varepsilon_4+\xi_5^2\varepsilon_5=2\neq 1$.
- There is no β-para-Kenmotsu structure.
The Reeb vector field ξ satisfies $g(\nabla_x\xi,y)=g(\nabla_y\xi,x)$. Checking for basis elements, ξ is obtained in the form $\xi=\xi_1e_1+\xi_2e_2+\xi_3e_3$. We also know that $\nabla_x\xi=\beta\phi^2(x)=\beta\{x-\eta(x)\xi\}$. For $x=e_4$, we have
$$\frac{\xi_2}{2}\varepsilon_1\varepsilon_4e_1-\frac{\xi_1}{2}\varepsilon_2\varepsilon_4e_2-\beta e_4=0.$$
Since basis elements are linearly independent, there is no nonzero number β satisfying this equation.
- There is no paracontact structure. Since
$$\Phi(e_i,e_j)=g(\phi(e_i),e_j)=d\eta(e_i,e_j)=g(\nabla_{e_i}\xi,e_j)-g(\nabla_{e_j}\xi,e_i)$$
for a paracontact structure, we obtain $\phi(e_4)=0$ and $\phi(e_5)=0$. On the other hand, $\phi^2(e_4)=e_4-\eta(e_4)\xi$ gives
$$\xi_1\xi_4\varepsilon_4e_1+\xi_2\xi_4\varepsilon_4e_2+\xi_3\xi_4\varepsilon_4e_3+(\xi_4^2\varepsilon_4-1)e_4)+\xi_5\xi_4\varepsilon_4e_5=0.$$
From linear independence of basis elements, we have
$$\xi_1\xi_4=\xi_2\xi_4=\xi_3\xi_4=\xi_5\xi_4=0,\quad \xi_4^2\varepsilon_4=1.$$
Since $\xi_4^2\neq 0$, we get $\xi_1=\xi_2=\xi_3=\xi_5$ and $\xi=\xi_4e_4$. In this case, $0=\phi^2(e_5)\neq e_5-\eta(e_5)\xi=e_5$.

The algebra $\mathfrak{g}_6$:
$$[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_2,e_3]=e_5$$

$$\begin{array}{lll}
\nabla_{e_1}e_2=\frac{1}{2}e_3, & \nabla_{e_1}e_3=-\frac{1}{2}\varepsilon_2\varepsilon_3e_2+\frac{1}{2}e_4, & \nabla_{e_1}e_4=-\frac{1}{2}\varepsilon_3\varepsilon_4e_3,\\
\nabla_{e_2}e_1=-\frac{1}{2}e_3, & \nabla_{e_2}e_3=\frac{1}{2}\varepsilon_1\varepsilon_3e_1+\frac{1}{2}e_5, & \nabla_{e_2}e_5=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3,\\
\nabla_{e_3}e_1=-\frac{1}{2}\varepsilon_2\varepsilon_3e_2-\frac{1}{2}e_4, & \nabla_{e_3}e_2=\frac{1}{2}\varepsilon_1\varepsilon_3e_1-\frac{1}{2}e_5, & \nabla_{e_3}e_4=\frac{1}{2}\varepsilon_1\varepsilon_4e_1,\\
\nabla_{e_3}e_5=\frac{1}{2}\varepsilon_2\varepsilon_5e_2 & \nabla_{e_4}e_1=-\frac{1}{2}\varepsilon_3\varepsilon_4e_3, & \nabla_{e_4}e_3=\frac{1}{2}\varepsilon_1\varepsilon_4e_1,\\
\nabla_{e_5}e_2=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3, & \nabla_{e_5}e_3=\frac{1}{2}\varepsilon_2\varepsilon_5e_2. &
\end{array}$$

- There exists no α-para-Sasakian structure.
  Since ξ is Killing, we have $\xi = \xi_4 e_4 + \xi_5 e_5$. From the equation $\nabla_x \xi = \alpha\phi(x)$ implies $\phi(e_4) = \phi(e_5) = 0$. In addition, $g(\phi(e_4), \phi(e_4)) = -g(e_4, e_4) + \eta(e_4)\eta(e_4)$ yields $\varepsilon_4 = 1$ and $\xi_4^2 = 1$. Similarly we have $\varepsilon_5 = 1$ and $\xi_5^2 = 1$, which contradicts with $g(\xi, \xi) = 1$.
- There is no β-para-Kenmotsu structure.
  The characteristic vector field of a β-para-Kenmotsu structure satisfies $g(\nabla_x \xi, y) = g(\nabla_y \xi, x)$. Then ξ should be of the form $\xi = \xi_1 e_1 + \xi_2 e_2$. Now since $\nabla_{e_4} \xi = \beta\phi^2(e_4) = \beta\{e_4 - \eta(e_4)\xi\}$, we have

$$\frac{\xi_1}{2}\varepsilon_3\varepsilon_4 e_3 + \beta e_4 = 0,$$

  and there is no nonzero β with this property.
- There is no paracontact structure.
  Since

$$\Phi(e_i, e_j) = g(\phi(e_i), e_j) = d\eta(e_i, e_j) = g(\nabla_{e_i}\xi, e_j) - g(\nabla_{e_j}\xi, e_i)$$

  for a paracontact structure, we obtain $\phi(e_4) = 0$ and $\phi(e_5) = 0$. On the other hand, $\phi^2(e_4) = e_4 - \eta(e_4)\xi$ gives

$$\xi_1\xi_4\varepsilon_4 e_1 + \xi_2\xi_4\varepsilon_4 e_2 + \xi_3\xi_4\varepsilon_4 e_3 + (\xi_4^2\varepsilon_4 - 1)e_4) + \xi_5\xi_4\varepsilon_4 e_5 = 0.$$

  From linear independence of basis elements, we have

$$\xi_1\xi_4 = \xi_2\xi_4 = \xi_3\xi_4 = \xi_5\xi_4 = 0, \quad \xi_4^2 = \varepsilon_4 = 1.$$

  Since $\xi_4^2 \neq 0$, we get $\xi_1 = \xi_2 = \xi_3 = \xi_5$ and $\xi = \xi_4 e_4$. In this case, $0 = \phi^2(e_5) \neq e_5 - \eta(e_5)\xi = e_5$.

After all, we state followings.

Theorem 3.1. *An almost paracontact metric structure on a five dimensional nilpotent Lie algebra $\mathfrak{g}$ is para-cosymplectic if and only if $\mathfrak{g}$ is abelian.*

Thus we may state

Corollary 3.2. *There is no para-cosymplectic left invariant almost paracontact metric structure on a five dimensional connected Lie group whose corresponding Lie algebra is nilpotent.*

In addition we deduce followings.

Theorem 3.3. *There is no left-invariant nearly para-cosymplectic structure on a five dimensional nilpotent Lie group.*

Theorem 3.4. *A 5-dimensional nilpotent Lie algebra has an α-para-Sasakian structure if it is isomorphic to $\mathfrak{g}_1$.*

Corollary 3.5. *A five dimensional nilpotent Lie group has a left-invariant α-para-Sasakian structure if its Lie algebra is isomorphic to $\mathfrak{g}_1$.*

Theorem 3.6. *There exists no β-para-Kenmotsu structure on a five dimensional nilpotent Lie algebra.*

Corollary 3.7. *There is no left-invariant β-para-Kenmotsu structure on a five dimensional nilpotent Lie group.*

Theorem 3.8. *A 5-dimensional nilpotent Lie algebra has a paracontact structure if it is isomorphic to $\mathfrak{g}_1$, $\mathfrak{g}_2$ or $\mathfrak{g}_3$.*

Corollary 3.9. *A 5-dimensional nilpotent Lie group has a left invariant paracontact structure if its Lie algebra is isomorphic to $\mathfrak{g}_1$, $\mathfrak{g}_2$ or $\mathfrak{g}_3$.*

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Disjunctive Total Domination of Some Shadow Distance Graphs

Canan Çiftçi

Department of Mathematics, Faculty of Arts and Sciences, Ordu University, Ordu, Turkey

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Abstract

Let G be a graph having vertex set $V(G)$. For $S \subseteq V(G)$, if each vertex is adjacent to a vertex in S or has at least two vertices in S at distance two from it, then the set S is a disjunctive total dominating set of G. The disjunctive total domination number is the minimum cardinality of such a set. In this work, we discuss the disjunctive total domination of shadow distance graphs of some graphs such as cycle, path, star, complete bipartite and wheel graphs.

1. Introduction

Domination in graphs [1] has received considerable attention in graph theory due to the various applications for real world problems such as the chess problem, communication network problems, location of radar stations, routing and coding theory [2]-[4]. There are several variations of domination; one of which is total domination [5]. Since implementations of dominating and total dominating sets in modern networks are expensive, some restrictions are added to them. Then Henning and Naicker [6] defined the disjunctive total domination as a relaxation of total domination. For a set $S \subseteq V(G)$, if each vertex is adjacent to a vertex in S or has at least two vertices in S at distance two from it, then the set S is a *disjunctive total dominating set*, briefly DTD-set, of G. When a vertex u satisfies one of these two conditions, it is known that u is *disjunctively totally dominated*, briefly DT-dominated, by vertices of S. Furthermore, when u satisfies the first condition (the second condition, respectively), it is known that u is *totally dominated* (*disjunctively dominated*, respectively) by vertices of S. The disjunctive total domination number, $\gamma_t^d(G)$, is the minimum cardinality of a DTD-set in G. A DTD-set which gives the value $\gamma_t^d(G)$ is called $\gamma_t^d(G)$-set. This parameter is studied on grids, trees, permutation graphs, claw-free graphs and it is applied on some graph modifications such as bondage and subdivision [6]-[12]. This paper is about disjunctive total domination number of shadow distance graph of some special graphs.

Let G be a graph having vertex set $V(G)$ and edge set $E(G)$. For two vertices u and v if there is an edge joining them, then they are adjacent (or neighbors). The *distance* $d_G(u,v)$ between u and v is the length of the shortest path joining them in G. The greatest distance between any pair of vertices of G is the *diameter* of G and denoted by $diam(G)$. We follow [1] for graph theory terminology and notation which are not defined here for simplicity.

The distance graph [13] $D(G,D_s)$ of G has vertex set $V(G)$ and two vertices u and v are neighbors in $D(G,D_s)$ if $d(u,v) \in D_s$, in which D is the set of all distances between distinct pairs of vertices in G and D_s is a subset of D. The shadow graph $D_2(G)$ [14] of a connected graph G is obtained by taking two copies of G and joining each vertex u in the first copy to the neighbors of the corresponding vertex v in the second copy.

The shadow distance graph $D_{sd}(G,D_s)$ of a connected graph G is defined by Kumar and Muralli [15] and is obtained from G

Email address and ORCID number: **cananciftci@odu.edu.tr, https://orcid.org/0000-0001-5397-0367**

with the following properties:

(i) The graph $D_{sd}(G,D_s)$ consists of two copies of G say G itself and G'.
(ii) For $v \in V(G)$, the corresponding vertex is denoted by $v' \in V(G')$.
(iii) $V(D_{sd}(G,D_s)) = V(G) \cup V(G')$.
(iv) $E(D_{sd}(G,D_s)) = E(G) \cup E(G') \cup E_{ds}$, in which E_{ds} is the set of all edges between two distinct vertices $v \in V(G)$ and $w' \in V(G')$ satisfying $d(v,w) \in D_s$ in G.

If $D_s = \{1\}$, then this gives the definition of shadow graph $D_2(G)$. The shadow graph $D_2(P_6)$ and shadow distance graphs $D_{sd}(P_6,\{2\})$, $D_{sd}(P_6,\{3\})$ are shown in Figure 1.1.

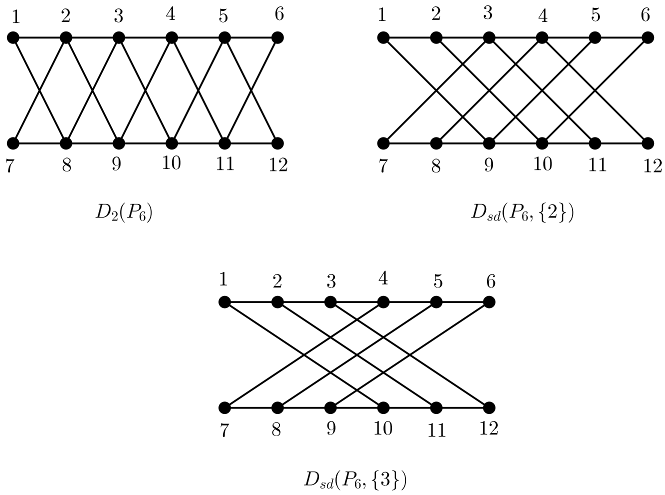

Figure 1.1: The shadow and shadow distance graphs of a path P_6

Now, we make use of the following known theorems in our results.

Theorem 1.1. *[6] Let G be a cycle with $n \geq 3$. Then $\gamma_t^d(G) = 2n/5$ when $n \equiv 0$ (mod 5) and $\gamma_t^d(G) = \lceil 2(n+1)/5 \rceil$ otherwise.*

Observation 1.2. *[11] If diam$(G) \in \{1,2\}$ for a connected graph G having at least two vertices, then $\gamma_t^d(G) = 2$.*

2. Disjunctive total domination of shadow distance graphs

We, in this section, determine the disjunctive total domination number of shadow distance graph of some special graphs such as cycle, path, star, complete bipartite and wheel graphs. Throughout the paper, we will label vertices of $D_2(G)$ and $D_{sd}(G,D_s)$ for $G \ncong W_{1,n}, K_{r,s}$ as the vertices in the first copy of G by $1,2,...,n$ and the vertices in the second copy of G by $n+1, n+2, ..., 2n$ starting from the left.

Theorem 2.1. *If $D_2(C_n)$ is a shadow graph of a cycle with $n \geq 3$, then*

$$\gamma_t^d(D_2(C_n)) = \begin{cases} \lceil \frac{2n}{5} \rceil + 1, & \textit{if } n \equiv 2 \textit{ (mod 5)} \\ \lceil \frac{2n}{5} \rceil, & \textit{otherwise}. \end{cases}$$

Proof. We first establish the upper bound for $\gamma_t^d(D_2(C_n))$. Let

$$S = \left\{ 5i+1 \mid 0 \leq i \leq \left\lceil \frac{n}{5} \right\rceil - 1 \right\} \cup \left\{ n+5i+2 \mid 0 \leq i \leq \left\lceil \frac{n-1}{5} \right\rceil - 1 \right\}.$$

In all cases of n based on mod 5, the set S is a DTD-set of $D_2(C_n)$. Thus, if $n \equiv 2$ (mod 5), then $|S| = \lceil \frac{2n}{5} \rceil + 1$ and for other cases $|S| = \lceil \frac{2n}{5} \rceil$. Therefore,

$$\gamma_t^d(D_2(C_n)) \leq |S| = \begin{cases} \lceil \frac{2n}{5} \rceil + 1, & \text{if } n \equiv 2 \text{ (mod 5)} \\ \lceil \frac{2n}{5} \rceil, & \text{otherwise}. \end{cases}$$

Now, we will prove the reverse inequality. Assume that $T = \{v_1, v_2, ..., v_i, ..., v_j, ..., v_t\}$ is a γ_t^d-set of $D_2(C_n)$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \leq v_i \leq n$ for $i \in \{1,2,...,m\}$

and $n+1 \leq v_j \leq 2n$ for $j \in \{m+1, m+2, ..., t\}$. Let $f_x = v_{x+1} - v_x$ for $x \in \{1, 2, ..., t-1\}$ with $x \neq m$. We must prove $f_x \leq 5$ for each $x \in \{1, 2, ..., t-1\}$ provided that $x \neq m$.

Let us suppose that $f_x \geq 6$ for every x. We claim that $f_x = 6$ for some $x \in \{1, 2, ..., t-1\}$ with $x \neq m$. In accordance with this claim, we construct the set

$$\left\{6i+1 \mid 0 \leq i \leq \left\lceil \frac{n}{6} \right\rceil - 1\right\} \cup \left\{n+6i+2 \mid 0 \leq i \leq \left\lceil \frac{n-1}{6} \right\rceil - 1\right\}.$$

However, some vertices, i.e. vertices 4 and 5 are not DT-dominated by this set. Thus, it is needed to add some new vertices. This makes $f_x < 6$ for some x, which contradicts our claim. Therefore, $f_x \leq 5$ for all $x \in \{1, 2, ..., t-1\}$ with $x \neq m$. Thus, it is clear that $\sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \leq 5(t-2)$. This yields

$$5\left(\left\lceil \frac{n}{5} \right\rceil - 1\right) + 5\left(\left\lceil \frac{n-1}{5} \right\rceil - 1\right) = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \leq 5(t-2).$$

Therefore, we have $|T| = t \geq \lceil \frac{2n}{5} \rceil + 1$ for $n \equiv 2 \pmod 5$ and $|T| = t \geq \lceil \frac{2n}{5} \rceil$ for the other cases of n. The proof is completed by combining the lower and upper bounds for $\gamma_t^d(D_2(C_n))$. □

Theorem 2.2. *If $D_2(P_n)$ is a shadow graph of a path with $n \geq 3$, then*

$$\gamma_t^d(D_2(P_n)) = \begin{cases} \lceil \frac{2n+2}{5} \rceil + 1, & \text{if } n \equiv 1 \text{ (mod 5)} \\ \lceil \frac{2n+2}{5} \rceil, & \text{otherwise.} \end{cases}$$

Proof. For the upper bound on $\gamma_t^d(D_2(P_n))$, let

$$S = \left\{5i+3 \mid 0 \leq i \leq \left\lceil \frac{n-2}{5} \right\rceil - 1\right\} \cup \left\{n+5i+2 \mid 0 \leq i \leq \left\lceil \frac{n-1}{5} \right\rceil - 1\right\}.$$

If $n \equiv 0, 2 \pmod 5$, then let $S' = S \cup \{n-1\}$; if $n \equiv 1 \pmod 5$, then let $S' = S \cup \{n, 2n-1\}$ and if $n \equiv 3, 4 \pmod 5$, then let $S' = S$. The set S' is a DTD-set of $D_2(P_n)$ in all cases. Thus, if $n \equiv 1 \pmod 5$, then $\gamma_t^d(D_2(P_n)) \leq |S'| = \lceil \frac{2n+2}{5} \rceil + 1$ and for other cases $\gamma_t^d(D_2(P_n)) \leq |S'| = \lceil \frac{2n+2}{5} \rceil$.

We now prove the lower bound on $\gamma_t^d(D_2(P_n))$. Assume that $T = \{v_1, v_2, ..., v_i, ..., v_j, ..., v_t\}$ is a γ_t^d-set of $D_2(P_n)$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \leq v_i \leq n$ for $i \in \{1, 2, ..., m\}$ and $n+1 \leq v_j \leq 2n$ for $j \in \{m+1, m+2, ..., t\}$. Let $f_x = v_{x+1} - v_x$ for $x \in \{1, 2, ..., t-1\}$ with $x \neq m$. As similar as the proof of Theorem 2.1 we conclude $f_x \leq 5$ for each $x \in \{1, 2, ..., t-1\}$ with $x \neq m$. This yields

$$\sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \leq 5(t-2).$$

Since $v_1 = 3$ and $v_{m+1} = n+2$ in all cases of n, it follows $\sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x = v_m + v_t - (n+5)$.

If $n \equiv 0 \pmod 5$, then $v_t = 2n-3$ and $v_m = n-1$. Thus,

$$2n-9 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \leq 5(t-2),$$

and hence $|T| = t \geq \lceil \frac{2n+1}{5} \rceil$. This implies that $\gamma_t^d(D_2(P_n)) \geq \lceil \frac{2n+2}{5} \rceil$.

If $n \equiv 1, 3 \pmod 5$, then $v_t = 2n-1$ and $v_m = n$. Thus,

$$2n-6 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \leq 5(t-2),$$

and hence $|T| = t \geq \lceil \frac{2n+4}{5} \rceil$. This implies that $\gamma_t^d(D_2(P_n)) \geq \lceil \frac{2n+2}{5} \rceil + 1$ for $n \equiv 1 \pmod 5$ and $\gamma_t^d(D_2(P_n)) \geq \lceil \frac{2n+2}{5} \rceil$ for $n \equiv 3 \pmod 5$.

If $n \equiv i$ (mod 5) for $i \in \{2,4\}$, then $v_t = 2n-i+2$ and $v_m = n-1$. Thus,

$$2n-i-4 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \leq 5(t-2),$$

and hence $|T| = t \geq \lceil \frac{2n-i+6}{5} \rceil$. This implies that $\gamma_t^d(D_2(P_n)) \geq \lceil \frac{2n+2}{5} \rceil$.

The proof is completed by combining the lower and upper bounds for $\gamma_t^d(D_2(P_n))$. □

Theorem 2.3. *Let $K_{1,s}, W_{1,n}, K_{r,s}$ denote a star, a wheel and a complete bipartite graph, respectively, and if $G \cong H$, where $H \in \{K_{1,s}, W_n, K_{r,s}\}$, then $\gamma_t^d(D_2(G)) = 2$.*

Proof. Since $diam(D_2(G)) = 2$ for $G \cong H$, where $H \in \{K_{1,s}, W_n, K_{r,s}\}$, the result follows from Observation 1.2. □

Theorem 2.4. *For $n \geq 6$,*

$$\gamma_t^d(D_{sd}(P_n,\{2\})) = \begin{cases} \lceil \frac{3n}{8} \rceil + 1, & \text{if } n \equiv 5 \text{ (mod 8)} \\ \lceil \frac{3n}{8} \rceil, & \text{otherwise.} \end{cases}$$

Proof. We first establish the upper bound for $\gamma_t^d(D_{sd}(P_n,\{2\}))$. Let

$$S = \left\{\{8i+3, 8i+5\} \mid 0 \leq i \leq \left\lceil \frac{n-4}{8} \right\rceil - 1\right\} \cup \left\{n+8i+6 \mid 0 \leq i \leq \left\lceil \frac{n-5}{8} \right\rceil - 1\right\}.$$

If $n \equiv 0,6,7$ (mod 8), then let $S' = S$; if $n \equiv 1,5$ (mod 8), then let $S' = S \cup \{2n-1\}$; if $n \equiv 2$ (mod 8), then let $S' = S \cup \{2n-2\}$; if $n \equiv 3$ (mod 8), then let $S' = S \cup \{n-2, 2n-1\}$ and if $n \equiv 4$ (mod 8), then let $S' = S \cup \{n-3, 2n-2\}$. The set S' is a DTD-set of $D_{sd}(P_n,\{2\})$ in all cases. Thus, if $n \equiv 5$ (mod 8), then $\gamma_t^d(D_{sd}(P_n,\{2\})) \leq |S'| = \lceil \frac{3n}{8} \rceil + 1$ and for other cases $\gamma_t^d(D_{sd}(P_n,\{2\})) \leq |S'| = \lceil \frac{3n}{8} \rceil$.

Let T be a γ_t^d-set of $D_{sd}(P_n,\{2\})$ to prove the lower bound. Assume that $T = \{v_1, v_2, ..., v_i, ..., v_j, ..., v_t\}$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \leq v_i \leq n$ for $i \in \{1,2,...,m\}$ and $n+1 \leq v_j \leq 2n$ for $j \in \{m+1, m+2, ..., t\}$. Let $f_x = v_{x+2} - v_x$ for $x \in \{1,2,...,m-2\}$ and $f_y = v_{y+1} - v_y$ for $y \in \{m+1, m+2, ..., t-1\}$. We must prove $f_x \leq 8$ for $x \in \{1,2,...,m-2\}$ and $f_y \leq 8$ for $y \in \{m+1, m+2, ..., t-1\}$. Suppose that at least one inequality is not true. Without loss of generality, let $f_y > 8$ for at least one y. We claim that $f_{m+1} = 9$ for $y = m+1$. In accordance with this claim, one of the set can be constructed is

$$\{n+6\} \cup \left\{\{8i+3, 8i+5\} \mid 0 \leq i \leq \left\lceil \frac{n-4}{8} \right\rceil - 1\right\} \cup \left\{n+8i+7 \mid 0 \leq i \leq \left\lceil \frac{n}{8} \right\rceil - 2\right\}.$$

However, all vertices of this set are not DT-dominated. Therefore, $f_x \leq 8$ for each $x \in \{1,2,...,m-2\}$ and $f_y \leq 8$ for each $y \in \{m+1, m+2, ..., t-1\}$. This yields $\sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3)$.

Since $v_1 = 3$, $v_2 = 5$ and $v_{m+1} = n+6$ in all cases of n, it follows $\sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y = v_{m-1} + v_m + v_t - (n+14)$.

If $n \equiv i$ (mod 8) for $i \in \{1,2\}$, then $v_{m-1} = n-i-5$, $v_m = n-i-3$ and $v_t = 2n-i$. Thus,

$$3n-3i-22 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3)$$

and hence $|T| = t \geq \lceil \frac{3n-3i+2}{8} \rceil$. This implies $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$.

If $n \equiv i$ (mod 8) for $i \in \{3,4\}$, then $v_{m-1} = n-i-3$, $v_m = n-i+1$ and $v_t = 2n-i+2$. Thus,

$$3n-3i-14 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-3i+10}{8} \rceil$. This implies $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$.

If $n \equiv i \pmod 8$ for $i \in \{5,6\}$, then $v_{m-1} = n-i+3$, $v_m = n-i+5$ and $v_t = 2n+i-6$. Thus,

$$3n-i-12 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-i+12}{8} \rceil$. This implies that $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil + 1$ for $n \equiv 5 \pmod 8$ and $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$ for $n \equiv 6 \pmod 8$.

Let $n \equiv i \pmod 8$ for $i \in \{0,7\}$. We take $i = 8$ for $n \equiv 0 \pmod 8$. Then $v_{m-1} = n-i+3$, $v_m = n-i+5$ and $v_t = 2n-i+6$. Thus,

$$3n-3i = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-3i+24}{8} \rceil$. This implies $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$.

Consequently, the proof follows from the lower and upper bounds. □

Theorem 2.5. *For $n \geq 3$,*

$$\gamma_t^d(D_{sd}(C_n,\{2\})) = \begin{cases} \lceil \frac{3n}{8} \rceil + 1, & \text{if } n \equiv 3,4,5 \pmod 8 \\ \lceil \frac{3n}{8} \rceil, & \text{otherwise.} \end{cases}$$

Proof. For the upper bound on $\gamma_t^d(D_{sd}(C_n,\{2\}))$, let

$$S = \left\{ \{8i+1, 8i+3\} \mid 0 \leq i \leq \left\lceil \frac{n-2}{8} \right\rceil - 1 \right\} \cup \left\{ n+8i+4 \mid 0 \leq i \leq \left\lceil \frac{n-3}{8} \right\rceil - 1 \right\}.$$

If $n \equiv 1 \pmod 8$, then let $S' = S \cup \{n\}$; if $n \equiv 2 \pmod 8$, then let $S' = S \cup \{n-1\}$; if $n \equiv 3 \pmod 8$, then let $S' = S \cup \{2n\}$ and otherwise let $S' = S$. The set S' is a DTD-set of $D_{sd}(C_n,\{2\})$ in all cases. Thus, if $n \equiv 3,4,5 \pmod 8$, then $\gamma_t^d(D_{sd}(C_n,\{2\})) \leq |S'| = \lceil \frac{3n}{8} \rceil + 1$ and for other cases $\gamma_t^d(D_{sd}(C_n,\{2\})) \leq |S'| = \lceil \frac{3n}{8} \rceil$.

Now, we need to prove the lower bound to complete the proof. Let T be a γ_t^d-set of $D_{sd}(C_n,\{2\})$. Assume that $T = \{v_1, v_2, ..., v_i, ..., v_j, ..., v_t\}$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \leq v_i \leq n$ for $i \in \{1,2,...,m\}$ and $n+1 \leq v_j \leq 2n$ for $j \in \{m+1, m+2, ..., t\}$. As similar as the proof of Theorem 2.4, we define functions $f_x = v_{x+2} - v_x$ for $x \in \{1,2,...,m-2\}$ and $f_y = v_{y+1} - v_y$ for $y \in \{m+1, m+2, ..., t-1\}$. It is easily seen that $f_x \leq 8$ for each $x \in \{1,2,...,m-2\}$ and $f_y \leq 8$ for each $y \in \{m+1, m+2, ..., t-1\}$ as in the proof of Theorem 2.4. This means that $\sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3)$.

Since $v_1 = 1$, $v_2 = 3$ and $v_{m+1} = n+4$ in all cases of n, it follows $\sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y = v_{m-1} + v_m + v_t - (n+8)$.

If $n \equiv i \pmod 8$ for $i \in \{1,2\}$, then $v_{m-1} = n-i-5$, $v_m = n-i+1$ and $v_t = 2n-i-4$. Thus,

$$3n-3i-16 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-3i+8}{8} \rceil$. This means that $\gamma_t^d(D_{sd}(C_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$.

If $n \equiv 3 \pmod 8$, then $v_{m-1} = n-2$, $v_m = n$ and $v_t = 2n$. Thus,

$$3n-10 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n}{8} \rceil + 1$.

Let $n \equiv i \pmod 8$ for $i \in \{0,4,5,6,7\}$. We take $i = 8$ for $n \equiv 0 \pmod 8$. Then $v_{m-1} = n-i+1$, $v_m = n-i+3$ and $v_t = 2n-i+4$. Thus,

$$3n-3i = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-3i+24}{8} \rceil$. This implies that $\gamma_t^d(D_{sd}(C_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil + 1$ for $n \equiv 4,5 \pmod 8$ and $\gamma_t^d(D_{sd}(C_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$ for otherwise. □

Theorem 2.6. *For $r \geq 1$ and $s \geq 2$, $\gamma_t^d(D_{sd}(K_{r,s},\{2\})) = 3$.*

Proof. Let $V(D_{sd}(K_{r,s},\{2\})) = V(K_{r,s}) \cup V(K'_{r,s})$ be vertex set of $D_{sd}(K_{r,s},\{2\})$, in which $V(K_{r,s}) = \{u_1,u_2,...,u_r,v_1,v_2,...,v_s\}$ and $V(K'_{r,s}) = \{u'_1,u'_2,...,u'_r,v'_1,v'_2,...,v'_s\}$. We first establish the upper bound for $\gamma_t^d(D_{sd}(K_{r,s},\{2\}))$. If $S = \{u_1,u_2,v_1\}$, then the set S is a DTD-set of $D_{sd}(K_{r,s},\{2\})$. Thus, $\gamma_t^d(D_{sd}(K_{r,s},\{2\})) \leq 3$.

For the lower bound, let T be a $\gamma_t^d(D_{sd}(K_{r,s},\{2\}))$-set. Suppose that $|T| = 2$, this means that the vertices of T are adjacent. Then we have the following cases.

Case 1. Let $T = \{u_i,v_j\}$ for any $i \in \{1,2,...,r\}$ and $j \in \{1,2,...,s\}$ (The case $T = \{u'_i,v'_j\}$ for any $i \in \{1,2,...,r\}$ and $j \in \{1,2,...,s\}$ is similar). All vertices except u'_i and u'_j are totally dominated by the vertices of T. However, since $d(u'_i,v_j) = 2$ and $d(u'_i,u_i) = 3$, the vertex u'_i is not DT-dominated by the vertices of T.

Case 2. Let $T = \{u_i,u'_j\}$ for any $i,j \in \{1,2,...,r\}$ and $i \neq j$. (The case $T = \{v_i,v'_j\}$ for any $i,j \in \{1,2,...,s\}$ and $i \neq j$ is similar.) Since $d(u_j,u_i) = 2$ and $d(u_j,u'_j) = 3$, the vertex u_j is not DT-dominated by the vertices of T.

Therefore, $\gamma_t^d(D_{sd}(K_{r,s},\{2\})) = |T| \geq 3$, and this concludes the proof. □

Theorem 2.7. *For $n \geq 3$, $\gamma_t^d(D_{sd}(W_{1,n},\{2\})) = 3$.*

Proof. Let $V(D_{sd}(W_{1,n},\{2\})) = V(W_{1,n}) \cup V(W'_{1,n})$ be vertex set of $D_{sd}(W_{1,n},\{2\})$ in which $V(W_{1,n}) = \{c,u_1,u_2,...,u_n\}$ and $V(W'_{1,n}) = \{c',u'_1,u'_2,...,u'_n\}$, where c is the center vertex of $W_{1,n}$. We first establish the upper bound for $\gamma_t^d(D_{sd}(W_{1,n},\{2\}))$. If $S = \{c,u_1,u'_2\}$, then the set S is a DTD-set of $D_{sd}(W_{1,n},\{2\})$. Thus, $\gamma_t^d(D_{sd}(W_{1,n},\{2\})) \leq 3$.

Now, we need to prove the lower bound. Let T be a $\gamma_t^d(D_{sd}(W_{1,n},\{2\}))$-set. Suppose that $|T| = 2$. We have following cases.

Case 1. Let $T = \{c,u_i\}$ for $i \geq 1$. (The case $T = \{c'_1,u'_i\}$ for $i \geq 1$ is similar.) Since $d(u'_i,u_i) = 3$ and $d(c'_1,c_1) = 3$, then vertices c' and u'_i are not DT-dominated.

Case 2. Let $T = \{u_i,u_{i+1}\}$ for $i \in \{1,2,...,n-1\}$. (The case $T = \{u'_i,u'_{i+1}\}$ for $i \in \{1,2,...,n-1\}$ is similar.) Since $d(u'_i,u_i) = 3$ and $d(u'_{i+1},u_{i+1}) = 3$, vertices u'_i and u'_{i+1} are not DT-dominated.

Case 3. Let $T = \{u_i,u'_j\}$ for $j \notin \{i,(i-1)(\text{mod } n),(i+1)(\text{mod } n)\}$. Note that we take $j = n$ when $j = 0$. In this case, since $d(u'_i,u_i) = 3$ and $d(u_j,u'_j) = 3$, vertices u'_i and u_j are not DT-dominated.

In all cases, the assumption is false and $\gamma_t^d(D_{sd}(W_{1,n},\{2\})) = |T| \geq 3$, which completes the proof. □

Theorem 2.8. *For $n \geq 14$,*

$$\gamma_t^d(D_{sd}(P_n,\{3\})) = \begin{cases} \frac{n}{3}+2, & \text{if } n \equiv 0 \ (\text{mod } 6) \\ \lceil \frac{n}{3} \rceil + 1, & \text{otherwise.} \end{cases}$$

Proof. For the upper bound for $\gamma_t^d(D_{sd}(P_n,\{3\}))$, let $D = \{3\} \cup \{\{6i+5, n+6i+5\} \mid 0 \leq i \leq \lceil \frac{n-4}{6} \rceil - 2\}$. If $n \equiv 0$ (mod 6), then let $S = D \cup \{n-1, 2n-3, 2n-1\}$; if $n \equiv 0,2$ (mod 6), then let $S = D \cup \{n-3, 2n-3, 2n-1\}$; if $n \equiv 1$ (mod 6), then let $S = D \cup \{n-2, 2n, 2n-2\}$; if $n \equiv 3$ (mod 6), then let $S = D \cup \{n-4, 2n-4, 2n-2\}$ and if $n \equiv 4$ (mod 6), then let $S = D \cup \{n-5, n-2, 2n-5, 2n-2\}$ and if $n \equiv 5$ (mod 6), then let $S = D \cup \{n-3, 2n-3\}$. Then the set S is a DTD-set of $D_{sd}(P_n,\{3\})$ in all cases of n. Thus, if $n \equiv 0$ (mod 6), then $\gamma_t^d(D_{sd}(P_n,\{3\})) = |S| \leq \frac{n}{3}+2$ and otherwise $\gamma_t^d(D_{sd}(P_n,\{3\})) = |S| \leq \lceil \frac{n}{3} \rceil + 1$.

Now, we prove the lower bound for $\gamma_t^d(D_{sd}(P_n,\{3\}))$. Let $T = \{v_1,v_2,...,v_i,...,v_j,...,v_t\}$ be a γ_t^d-set of $D_{sd}(P_n,\{3\})$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \leq v_i \leq n$ for $i \in \{1,2,...,m\}$ and $n+1 \leq v_j \leq 2n$ for $j \in \{m+1,m+2,...,t\}$. Assume that $f_1 = v_2 - v_1$ and $f_y = v_{y+1} - v_y$ for $y \in \{2,...,t-1\}$ with $y \neq m$. We will show that $f_1 \leq 2$ and $f_y \leq 6$ for each y. Suppose first that $f_1 \geq 3$. In order to DT-dominate v_1, the condition $f_y \leq 6$ must be hold for at least one y. Thus, the set

$$T' = \{2,5,n+4\} \cup \{\{6i+8, n+6i+8\} \mid 0 \leq i \leq \lceil \frac{n-7}{6} \rceil - 2\}$$

is constructed. However, this contradicts with our upper bound. For example, if $n \equiv 1$ (mod 6), then $T = T' \cup \{n-5, n-2, 2n-5, 2n-2\}$ and $|T| = \frac{n+7}{3}$, a contradiction.

Suppose now that $f_1 \leq 2$ and $f_y \geq 7$ for at least one y. Then the set

$$T' = \{3,5,n+5,n+7,12,n+12\} \cup \{\{6i+15,n+6i+15\} \mid 0 \leq i \leq \lceil \frac{n-14}{6} \rceil - 1\}$$

is constructed. However, this contradicts with our upper bound. For example, if $n \equiv 1 \pmod 6$, then $T = T' \cup \{2n-2\}$ and $|T| = \frac{n+8}{3}$, a contradiction.

Therefore, $f_1 \leq 2$ and $f_y \leq 6$ for each $y \in \{2,...,t-1\}$. This yields $f_1 + \sum_{y=2}^{t-1} f_y \leq 2+6(t-3)$. Since $v_2 = 5$ and $v_{m+1} = n+5$, it follows $2 + \sum_{y=2}^{t-1} f_y = 2 + v_m - v_1 + v_t - v_{m+1} = v_m + v_t - (n+8)$.

If $n \equiv 0 \pmod 6$, then $v_m = n-1$ and $v_t = 2n-1$. This yields

$$2n-10 = 2 + \sum_{y=2}^{t-1} f_y \leq 2+6(t-3),$$

and hence $|T| = t \geq \lceil \frac{2n+6}{6} \rceil$.

If $n \equiv i \pmod 6$ for $i \in \{1,2,3\}$, then $v_m = n-i-1$ and $v_t = 2n-i+1$. This yields

$$2n-2i-8 = 2 + \sum_{y=2}^{t-1} f_y \leq 2+6(t-3),$$

and hence $|T| = t \geq \lceil \frac{2n-2i+8}{6} \rceil$.

If $n \equiv 4 \pmod 6$, then $v_m = n-2$ and $v_t = 2n-2$. This yields

$$12(\lceil \frac{n-4}{6} \rceil - 1) + 8 = 2 + \sum_{y=2}^{t-1} f_y \leq 8+6(t-5),$$

and hence $|T| = t \geq \frac{n+5}{3}$.

If $n \equiv 5 \pmod 6$, then $v_m = n-3$ and $v_t = 2n-3$. This yields

$$2n-14 = 2 + \sum_{y=2}^{t-1} f_y \leq 2+6(t-3),$$

and hence $|T| = t \geq \lceil \frac{2n+4}{6} \rceil$.

Consequently, if $n \equiv 0 \pmod 6$, then $\gamma_t^d(D_{sd}(P_n,\{3\})) \geq \frac{n}{3}+2$ and otherwise $\gamma_t^d(D_{sd}(P_n,\{3\})) \geq \lceil \frac{n}{3} \rceil + 1$. This completes the proof. □

Since $D_{sd}(P_n,\{3\}) \cong C_8$ for $n=4$, by Theorem 1.1 we have $\gamma_t^d(D_{sd}(P_4,\{3\})) = 4$. Therefore, we give the result of $\gamma_t^d(D_{sd}(P_n,\{3\}))$ for $5 \leq n \leq 13$ in Table 1.

n	5	6	7	8	9	10	11	12	13
$\gamma_t^d(D_{sd}(P_n,\{3\}))$	4	4	4	4	4	4	4	5	5

Table 1: The values of $\gamma_t^d(D_{sd}(P_n,\{3\}))$ for $5 \leq n \leq 13$

Theorem 2.9. *For $n \geq 15$,*

$$\gamma_t^d(D_{sd}(C_n,\{3\})) = \begin{cases} \frac{n+5}{3}, & \text{if } n \equiv 1 \ (mod\ 6) \\ \lceil \frac{n+2}{3} \rceil, & \text{otherwise.} \end{cases}$$

Proof. For the upper bound for $\gamma_t^d(D_{sd}(C_n,\{3\}))$, let $D = \left\{\{6i+3,n+6i+3\} \mid 0 \leq i \leq \lceil \frac{n-2}{6} \rceil - 2\right\}$. If $n \equiv 0 \pmod 6$, then let $S = D \cup \{n-3,2n-3,2n-1\}$; if $n \equiv 1 \pmod 6$, then let $S = D \cup \{n-4,2n-4,n,2n\}$; if $n \equiv 2 \pmod 6$, then let $S = D \cup \{n-5,2n-5,n-1,2n-1\}$; if $n \equiv 3 \pmod 6$, then let $S = D \cup \{n,2n\}$; if $n \equiv 4 \pmod 6$, then let $S = D \cup \{n-1,2n-1\}$ and if $n \equiv 5 \pmod 6$, then let $S = D \cup \{n,n-3,2n-6\}$. The set S is a DTD-set of $D_{sd}(C_n,\{3\})$ in all cases of n. Thus, if $n \equiv 1 \pmod 6$, then $\gamma_t^d(D_{sd}(C_n,\{3\})) = |S| \leq \frac{n+5}{3}$ and otherwise $\gamma_t^d(D_{sd}(C_n,\{3\})) = |S| \leq \lceil \frac{n+2}{3} \rceil$.

We need to prove the opposite inequality to complete the proof. Let $T = \{v_1, v_2, ..., v_i, ..., v_j, ..., v_t\}$ be a γ_t^d-set of $D_{sd}(C_n, \{3\})$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \le v_i \le n$ for $i \in \{1, 2, ..., m\}$ and $n+1 \le v_j \le 2n$ for $j \in \{m+1, m+2, ..., t\}$. Assume that $f_x = v_{x+1} - v_x$ for $x \in \{1, 2, ..., t-1\}$ with $x \ne m$. As similar as the proof of Theorem 2.1 we can show that $f_x \le 6$ for each $x \in \{1, 2, ..., t-1\}$ with $x \ne m$. This yields

$$\sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 6(t-2).$$

Since $v_1 = 3$ and $v_{m+1} = n+3$ in all cases of n, we have $\sum\limits_{x=1}^{m-1} f_x + \sum\limits_{x=m+1}^{t-1} f_x = v_m - v_1 + v_t - v_{m+1} = v_m + v_t - (n+6)$.

If $n \equiv 0 \pmod 6$, then $v_m = n-3$ and $v_t = 2n-1$. This yields

$$2n - 10 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 6(t-2),$$

and hence $|T| = t \ge \lceil \frac{2n+2}{6} \rceil$.

If $n \equiv i \pmod 6$ for $i \in \{1, 3\}$, then $v_m = n$ and $v_t = 2n$. This yields

$$2n - 6 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 6(t-2),$$

and hence $|T| = t \ge \lceil \frac{2n+6}{6} \rceil$.

If $n \equiv i \pmod 6$ for $i \in \{2, 4\}$, then $v_m = n-1$ and $v_t = 2n-1$. This yields

$$2n - 8 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 6(t-2),$$

and hence $|T| = t \ge \lceil \frac{2n+4}{6} \rceil$.

If $n \equiv 5 \pmod 6$, then $v_m = n$ and $v_t = 2n-6$. This yields

$$12\left(\left\lceil \frac{n-2}{6} \right\rceil - 2\right) + 10 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 10 + 6(t-5),$$

and hence $|T| = t \ge \frac{n+4}{3}$.

As a consequence, if $n \equiv 1 \pmod 6$, then $\gamma_t^d(D_{sd}(C_n, \{3\})) \ge \frac{n+5}{3}$ and otherwise $\gamma_t^d(D_{sd}(C_n, \{3\})) \ge \lceil \frac{n+2}{3} \rceil$, and this completes the proof. □

For $n = 4$, since diameter of $D_{sd}(C_n, \{3\})$ is two, it is clear that $\gamma_t^d(D_{sd}(C_n, \{3\})) = 2$. For $5 \le n \le 14$, the result is given in Table 2.

n	5	6	7	8	9	10	11	12	13	14
$\gamma_t^d(D_{sd}(C_n, \{3\}))$	3	3	4	4	4	4	4	5	5	5

Table 2: The values of $\gamma_t^d(D_{sd}(C_n, \{3\}))$ for $5 \le n \le 14$

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