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Monte Carlo and Quasi Monte Carlo Approach to Ulam's Method for Position Dependent Random Maps

Md Shafiqul Islam [1]*

Abstract
We consider position random maps $T = \{\tau_1(x), \tau_2(x), \dots, \tau_K(x); p_1(x), p_2(x), \dots, p_K(x)\}$ on $I = [0,1]$, where $\tau_k, k = 1,2,\dots,K$ is non-singular map on $[0,1]$ into $[0,1]$ and $\{p_1(x), p_2(x), \dots, p_K(x)\}$ is a set of position dependent probabilities on $[0,1]$. We assume that the random map T posses a density function f^* of the unique absolutely continuous invariant measure (acim) μ^*. In this paper, first, we present a general numerical algorithm for the approximation of the density function f^*. Moreover, we show that Ulam's method is a special case of the general method. Finally, we describe a Monte-Carlo and a Quasi Monte Carlo implementations of Ulam's method for the approximation of f^*. The main advantage of these methods is that we do not need to find the inverse images of subsets under the transformations of the random map T.

Keywords: Dynamical systems, Invariant measure, Invariant density, Monte Carlo approach, Position dependent random maps, Quasi Monte Carlo approach, Ulam's method.
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[1] *School of Mathematical and Computational Sciences, University of Prince Edward Island, 550 University Ave, Charlottetown, PE, C1A 4P3, Canada*
***Corresponding author**: sislam@upei.ca
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1. Introduction

A position dependent random map is a special type of random dynamical system involving a set of non-singular transformations on the state space and a set of position dependent probabilities on the state space. In each iteration of the process, one map from the set of maps with one position dependent probability from the set of probabilities [1] is selected and applied. There are applications of random maps in many areas of science and engineering [2]-[3]-[4]-[5]-[6]. In [2] the author applied the theory of random dynamical systems in the study of fractals. In [3], Boyarsky and Góra applied the theory of random dynamical systems in modelling interference effects in quantum mechanics. The authors in [4] applied random maps for computing metric entropy. Random maps have application in forecasting the financial markets [5] and in economics [6].

Invariant measures describe the statistical behaviour of trajectories of position dependent random maps [1]. In particular, invariant measures of random maps which are absolutely continuous with respect to Lebesgue measure are very useful for the study of chaotic nature of random dynamical systems [7]. The Frobenius-Perron operator [1, 8] of a random map is one of the important tools for the study of invariant measures. A Fixed point f^* of the Frobenius-Perron operator of a position dependent random maps are the density function f^* of invariant measures μ^* [1, 7]. It is difficult to solve the fixed point equation or the Frobenius–Perron equation [1] for a position dependent random map because it is a complicated functional equation except for some simple cases. Therefore, finite dimensional approximation of the Frobenius-Perron operator is necessary to approximate invariant measures for position dependent random maps. In [9], Lasota and Yorke proved the existence of absolutely continuous invariant measures (acims) for one dimensional deterministic dynamical systems. In his pioneering work[10], Ulam suggested finite dimensional approximation of the Frobenious Perron operator of dynamical systems for the approximation of invariant measures. It was T-Y Li who first proved in [11] the convergence of Ulam's approximation for piecewise expanding transformations τ on $[0,1]$. In [8], Pelikan proved a Lasota–Yorke type inequality random maps with i. i. d. probabilities using bounded variation techniques. Then, he used the Lasota-Yorke type inequality for proving the existence of absolutely continuous invariant measures for i. i. d. random maps. Góra and Boyarsky [1] proved the existence of absolutely continuous invariant measures (acim) for position dependent random maps. Moreover, they proved the convergence of Ulam's method for position dependent random maps.

Ulam's method is a simple, easy to implement and very useful method for approximating invariant measures for deterministic and random maps [1]-[14]. Note that each of the map $\tau_k, k=1,2,\ldots,K$ of a position dependent random map $T=\{\tau_1,\tau_2,\cdots,\tau_K;p_1(x),p_2(x),\cdots,p_K(x)\}$ is a piecewise monotonic map on a finite partition $\mathscr{P}=\{I_1,I_2,\cdots,I_q\}$. The entries of an Ulam's matrix for a random map T are related to inverse images of the transformations $\tau_k, k=1,2,\ldots,K$. For non-linear $\tau_k, k=1,2,\ldots,K$, it is difficult to find inverse images under $\tau_k, k=1,2,\ldots,K$, and hence the computation of Ulam's matrix becomes challenging and complicated. In this paper, we describe a Monte Carlo method and a Quasi Monte Carlo approach to Ulam's method for approximating the entries of Ulam's matrix. The main advantage of Monte-Carlo method and Quasi Monte Carlo approach to Ulam's method is that we do not need to find the inverse images of subsets under the transformations of the random map T. Moreover, the evaluation of an entry of the Ulam's matrix is independent of their entries [12].

2. Invariant Measures for Position Dependent Random Maps and Ulam's Method

In this section, we review position dependent random maps, the Frobenius-Perron operator, density function of absolutely continuous invariant measures and Ulam's method. We closely follow [1, 13, 14].

2.1 Position dependent random maps and their invariant measures

Let $(I=[0,1],\mathscr{B},\lambda)$ be a measure space and $\tau_k:[0,1]\to[0,1], k=1,2,\cdots,K$, be piecewise one-to-one and differentiable, non-singular maps on a common partition $\mathscr{I}=\{I_1,I_2,\cdots,I_q\}$ of $[0,1]$. We denote $V(.)$ for the standard one dimensional variation of a function, and $BV([0,1])$ for the space of functions of bounded variation on I equipped with the norm $\|\,.\,\|_{BV}=V(.)+\|\,.\,\|_1$, where $\|\,.\,\|_1$ denotes the L^1 norm of a function. A position dependent random map T on I with position dependent probabilities is defined as

$$T=\{\tau_1,\tau_2,\cdots,\tau_K;p_1(x),p_2(x),\cdots,p_K(x)\}$$

where $\{p_1(x),p_2(x),\cdots,p_K(x)\}$ is a set of position dependent probabilities on I. For any $x\in I, T(x)=\tau_k(x)$ with probability $p_k(x)$ and, for any non-negative integer $N, T^N(x)=\tau_{k_N}\circ\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x)$ with probability $p_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))p_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\ldots p_{k_1}(x)$. It is shown in [1] that a measure μ is invariant under the

$$\mu(A)=\sum_{k=1}^{K}\int_{\tau_k^{-1}(A)}p_k(x)d\mu(x) \tag{2.1}$$

for any $A\in\mathscr{B}$.

The Frobenius–Perron operator of the position dependent random map T is given by [1]:

$$(P_T f)(x)=\sum_{k=1}^{K}\left(P_{\tau_k}(p_k f)\right)(x) \tag{2.2}$$

where P_{τ_k} in (2.2) is the Frobenius-Perron operator of τ_k [14] defined by

$$P_{\tau_k}f(x)=\sum_{z\in\{\tau_k^{-1}(x)\}}\frac{f(z)}{|\tau_k'(z)|} \tag{2.3}$$

where, for any x, the set $\{\tau_k^{-1}(x)\}$ consists of at most q points. The Frobenius-Perron operator P_T has the following properties

(i) $P_T : L^1([0,1]) \to L^1([0,1])$ is a linear operator;
(ii) P_T is non-negative, i.e., $f \in L^1([0,1])$ and $f \geq 0 => P_T f \geq 0$;
(iii) P_T is a contractive, i.e., $\| P_T f \|_1 \leq \| f \|_1$, for any $f \in L^1([0,1])$;
(iv) P_T satisfies the composition property, i.e., if T and R are two position dependent random maps on $[0,1]$, then $P_{T\circ R} = P_T \circ P_R$. In particular, for any $n \geq 1, P_T^n = P_{T^n}$;
(v) $P_T f = f$ if and only if $\mu = f \cdot \lambda$ is T-invariant.

The following Lemmas (Lemma 2.1 and Lemma 2.2) are key Lemmas for proving the existence of invariant measures for position dependent random maps. These Lemmas are proved by Bahsoun and Góra in[13].

Lemma 2.1. *[13] Consider the position dependent random maps $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1(x), p_2(x), ..., p_K(x)\}$, where $\tau_k : [0,1] \to [0,1], k = 1,2,...,K$ are piecewise one-to-one and differentiable, nonsingular maps on a common partition $\mathscr{J} = \{J_1, J_2, ..., J_q\}$ of $[0,1]$. Let $g_k(x) = \frac{p_k(x)}{|\tau_k'(x)|}, k = 1,2,...,K$. Assume that the random map T satisfies the following conditions: (i) $\sum_{k=1}^K g_k(x) < \alpha < 1, x \in [0,1]$; (ii) $g_k \in BV([0,1]), k = 1,2,...,K$. Then, for any $f \in BV([0,1])$, P_T satisfies the following Lasota-Yorke type inequality:*

$$V_{[0,1]} P_T f \leq A V_{[0,1]} f + B \| f \|_1 \tag{2.4}$$

where $A = 3\alpha + \max_{1\leq i\leq q} \sum_{k=1}^{K} V_{J_i} g_k$ *and* $B = 2\beta\alpha + \beta \max_{1\leq i\leq q} \sum_{k=1}^{K} V_{J_i} g_k$ *with* $\beta = \max_{1\leq i\leq q} \frac{1}{\lambda(J_i)}$.

Proof. See [13] □

Note that for $x \in [0,1]$ and for any $N \geq 1$ we have, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)$ with probability

$$p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \ldots p_{k_1}(x).$$

For $\omega \in \{1,2,\ldots,K\}^N$, define

$$\begin{aligned} T_\omega(x) &= T^N(x), \\ p_\omega &= p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \ldots p_{k_1}(x), \\ g_\omega &= \frac{p_\omega}{|T_\omega'(x)|}, W_N = \max_{L \in \mathscr{J}^{(N)}} \sum_{\omega \in \{1,2,\ldots,K\}^N} V_L g_\omega. \end{aligned}$$

Based on Lemma 2.1, Bahsoun and Góra [13] have proved the following Lemma for the iterates of P_T:

Lemma 2.2. *Let T be a random map satisfying conditions of Lemma 2.1 and N be a positive integer such that $A_N = 3\alpha^N + W_N < 1$. Then*

$$V_{[0,1]} P_T^N f \leq A_N V_{[0,1]} f + B_N \| f \|_1 \tag{2.5}$$

where $B_N = \beta_N \left(2\alpha^N + W_N\right), \beta_N = \max_{L \in \mathscr{J}^{(N)}} \frac{1}{\lambda(L)}$.

In the following Theorem (Theorem 2.3), Bahsoun and Góra proved the existence of invariant measures for position dependent random maps. The proof of this Theorem is based on the above Lemmas (Lemma 2.1 and Lemma 2.2) which is proved in [13].

Theorem 2.3. *[13] Consider the position dependent random map $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1(x), p_2(x), \ldots, p_K\}$. Assume that the random map T satisfies conditions of Lemma 2.1. Then, T possesses an invariant measure which is absolutely continuous with respect to Lebesgue measure. Moreover, the operator P_T is quasi-compact in $BV(I)$.*

2.2 Ulam's Method for Position Dependent Random Maps

In [1] Góra and Boyarsky described Ulam's method for position dependent random maps. Moreover, they proved the convergence of Ulam's method. For the convenience of readers, we review the Ulam's method for position dependent random maps. Let $T=\{\tau_1(x),\tau_2(x),\dots,\tau_K(x);p_1(x),p_2(x),\dots,p_K(x)\}$ be a position dependent random map and the random map T satisfies conditions of Theorem 2.3. Then, by the Theorem 2.3, the random map T has an absolutely continuous invariant measure. We also assume that the random map has a unique acim μ^* with density function f^*. In the following we describe Ulam's method for T.

Consider the partition $\mathscr{P}^{(N)}=\{J_1,J_2,\dots,J_N\}$ of $[0,1]$ into N subintervals such that $\max_{J_i\in\mathscr{P}^{(N)}}\lambda(J_i)$ goes to 0 as $N\to\infty$. For each $1\le k\le K$, construct the matrix

$$M_k^{(N)}=\left(\frac{\lambda\left(\tau_k^{-1}(J_j)\cap J_i\right)}{\lambda(J_i)}\right)_{1\le i,j\le N}.$$

Let $L^{(N)}$ be the set of functions f in $L^1([0,1],\lambda)$ such that f is constant on elements of the partition $\mathscr{P}^{(N)}$. Any $f\in L^{(n)}$ can be treated as a vector: vector $f=[f_1,f_2,\dots,f_N]$ corresponds to the function $f=\sum_{i=1}^N f_i\chi_{J_i}$. Let $Q^{(N)}$ be the isometric projection of L^1 onto $L^{(N)}$:

$$Q^{(N)}(f)=\sum_{i=1}^N\left(\frac{1}{\lambda(J_i)}\int_{J_i}fd\lambda\right)\chi_{J_i}=\left[\frac{1}{\lambda(J_1)}\int_{J_1}fd\lambda,\dots,\frac{1}{\lambda(J_N)}\int_{J_N}fd\lambda\right].$$

Let $p_k^{(N)}=Q^{(N)}p_k=\left[p_{k,1}^{(n)},p_{k,2}^{(N)},\dots,p_{k,N}^{(n)}\right]$. Let $f=[f_1,f_2,\dots,f_N]\in L^{(N)}$. Let the subscript **c** denotes the transpose of a matrix. We define the operator $P_T^{(N)}:L^{(N)}\to L^{(N)}$ by

$$P_T^{(n)}f \quad = \quad \sum_{k=1}^K\left(\mathbb{M}_k^{(n)}\right)^{\mathrm{c}}\mathrm{diag}\left(\left[p_{k,1}^{(N)}f_1,p_{k,2}^{(N)}f_2,\dots,p_{k,N}^{(N)}f_N\right]\right) \tag{2.6}$$

as a finite dimensional approximation to the operator P_T. Ulam's matrix with respect to the partition $\mathscr{P}^{(N)}$ is

$$\mathbb{M}_{\mathscr{P}^{(N)}}^{*(N)}=\sum_{k=1}^K\left(\mathbb{M}_k^{(N)}\right)^{\mathrm{c}}\mathrm{diag}\left[p_{k,1}^{(N)},p_{k,2}^{(N)},\dots,p_{k,N}^{(N)}\right]. \tag{2.7}$$

The following theorem is proved in[1] (see Theorem 3 in [1]).

Theorem 2.4. *Let α be sufficiently large where α is in Theorem 1 in [1]. Let f_N^* be is a normalized fixed point of $P_T^{(N)}$, $N=1,2,\dots$. Then the sequence $\{f_N^*\}_{N=1}^\infty$ is pre-compact in L^1. Any limit point f^* of the sequence $\{f_N^*\}_{N=1}^\infty$ is a fixed point of P_T.*

3. A General Algorithm for Finite Dimensional Approximation of the Frobenius-Perron Operator for Position Dependent Random Maps

Let $T=\{\tau_1,\tau_2,\dots,\tau_K;p_1(x),p_2(x),\dots,p_K(x)\}$ be a position dependent random map which satisfies the following assumptions:

there exists $A=3\alpha+\max_{1\le i\le q}\sum_{k=1}^K V_{J_i}g_k<1$ and $B=2\beta\alpha+\beta\ \max_{1\le i\le q}\sum_{k=1}^K V_{J_i}g_k>0$ with $\beta=\max_{1\le i\le q}\frac{1}{\lambda(I_i)}$ such that $\forall f\in BV([0,1])$.

$$V_{[0,1]}P_Tf\le AV_{[0,1]}f+B\parallel f\parallel_1. \tag{3.1}$$

We also assume that T has a unique acim μ^* with density f^*.

Note that the invariant density f^* of the unique acim μ^* is the fixed point of the Frobenius-Perron operator P_T. In the following we describe a general approximation algorithm for f^*. Our general algorithm is a generalization of the algorithm in [12] for single deterministic map to an algorithm for position dependent random maps.

For each $k=1,2,\dots,K$, let $U_{\tau_k}:L^\infty([0,1])\to L^\infty([0,1])$ be the Koopman operator of τ_k defined by

$$(U_{\tau_k}g)(x)=g(\tau_k(x)). \tag{3.2}$$

Note that each U_{τ_k} is the dual of the Frobenius-Perron operator P_{τ_k} of τ_k.

Definition 3.1. *A sequence $\{\phi_n\}_{n=1}^{\infty}$ of functions in $L^{\infty}([0,1])$ is said to be a complete sequence if for any $f \in L^1(0,1)$ with $\int_0^1 \phi_n(x) f(x) d\lambda(x) = 0,\ n = 1,2,\cdots$ implies $f = 0$.*

Proposition 3.2. *Let $T = \{\tau_1, \tau_2, \cdots, \tau_K; p_1(x), p_2(x), \ldots, p_K\}$ be a position dependent random map which has a unique acim μ^* with density f^*. Let P_T be the Frobenius-Perron operator of the random map T. Let $\{\phi_n\}_{n=1}^{\infty}$ be a complete sequence of functions. Then, f^* is a fixed point of P_T if and only if*

$$\int_I \left[\phi_n(x) - \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x))\right] f^*(x) d\lambda(x) = 0,\ n = 1,2,\cdots. \tag{3.3}$$

Proof. Suppose that f^* the unique invariant density of the random maps T. In other words,

$$(P_T f^*)(x) = f^*(x). \tag{3.4}$$

Then for $n = 1,2,\cdots,$

$$\begin{aligned}
\int_I f^*(x)\phi_n(x) d\lambda(x) &= \int_I (P_T f^*)(x)\phi_n(x) d\lambda(x) \\
&= \int_I \sum_{k=1}^{K} (P_{\tau_k}(p_k f^*))(x)\phi_n(x) d\lambda(x) \\
&= \sum_{k=1}^{K} \int_I (P_{\tau_k}(p_k f^*))(x)\phi_n(x) d\lambda(x) \\
&= \sum_{k=1}^{K} \int_I (p_k f^*)(x) U_{\tau_k}(\phi_n(x)) d\lambda(x) \\
&= \int_I f^*(x) \left[\sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x))\right] d\lambda(x).
\end{aligned}$$

Thus,

$$\int_I \left[\phi_n(x) - \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x))\right] f^*(x) d\lambda(x) = 0,\ n = 1,2,\cdots.$$

Conversely, suppose that f^* satisfies (3.3), that is,

$$\int_I \phi_n(x) f^*(x) d\lambda(x) = \int_I f^*(x) \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x)) d\lambda(x).$$

Now,

$$\begin{aligned}
\int_I f^*(x)\phi_n(x) d\lambda(x) &= \int_I f^*(x) \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x)) d\lambda(x) \\
&= \int_I f^*(x) \sum_{k=1}^{K} p_k(x) U_{\tau_k}(\phi_n(x)) d\lambda(x) \\
&= \sum_{k=1}^{K} \int_I f^*(x) p_k(x) U_{\tau_k}(\phi_n(x)) d\lambda(x) \\
&= \sum_{k=1}^{K} \int_I (P_{\tau_k}(p_k f^*))(x)\phi_n(x) d\lambda(x) \\
&= \int_I \sum_{k=1}^{K} (P_{\tau_k}(p_k f^*))(x)\phi_n(x) d\lambda(x) \\
&= \int_I (P_T f^*)(x)\phi_n(x) d\lambda(x).
\end{aligned}$$

Thus,

$$\int_I (f^*(x) - (P_T f^*)(x))\phi_n(x)d\lambda(x) = 0,\ n = 1, 2, \cdots.$$

From Definition 3.1, $f^*(x) - (P_T f^*)(x) = 0$. This proves that

$$(P_T f^*)(x) = f^*(x).$$

□

Thus, the fixed point problem (3.4) of the Frobenius-Perron operator P_T for the position dependent random map T is equivalent to homogeneous moment problem (3.3). We propose the following general algorithm for computing fixed point of P_T.

General Algorithm: Consider two complete sequences of functions ϕ_n and ψ_n. Let N be a positive integer. Construct the $N \times N$ matrix $A = (a_{ij})_{1 \le i,j \le N}$ given by

$$a_{ij} = \int_0^1 \left(\phi_i(x) - \sum_{k=1}^{K} p_k(x)\phi_i(\tau_k(x)) \right) \psi_j(x)d\lambda(x), i, j = 1, 2, \ldots, N. \tag{3.5}$$

Solve the homogeneous linear system of equation $Av = 0$ for nonzero $v = (v_1, v_2, \ldots, v_N)$ with $\| \sum_{i=1}^N v_i\psi_i \|_{L^1} = 1$. Then, $f_N = \sum_{i=1}^N v_i\psi_i$ is a normalized approximation of the fixed point f^* of P_T.

Lemma 3.3. *$Av = 0$ has a nontrivial solution v.*

Proof. For a nonzero vector $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$, the constant function $g(x) = 1$ can be written as $g(x) = 1 = \sum_{i=1}^N \eta_i\phi_i$. Moreover, for each $k = 1, 2, \ldots, K$, $U_{\tau_k}1(x) = 1(\tau_k(x)) = 1$. For each $j = 1, 2, \ldots, N$,

$$\begin{aligned}
\sum_{i=1}^{N} a_{ij}\eta_i &= \sum_{i=1}^{N} \eta_i \int_0^1 \left(\phi_i(x) - \sum_{k=1}^{K} p_k(x)\phi_i(\tau_k(x)) \right) \psi_j(x)d\lambda(x) \\
&= \int_0^1 \left(\sum_{i=1}^{N} \eta_i\phi_i(x) - \sum_{k=1}^{K} p_k(x)U_{\tau_k}(\sum_{i=1}^{N} \eta_i\phi_i(x)) \right) \psi_j(x)d\lambda(x) \\
&= \int_0^1 \left(1 - \sum_{k=1}^{K} p_k(x)U_{\tau_k}(1(x)) \right) \psi_j(x)d\lambda(x) \\
&= \int_0^1 \left(1 - \sum_{k=1}^{K} p_k(x)1 \right) \psi_j(x)d\lambda(x) \\
&= \int_0^1 (1-1)\, \psi_j(x)d\lambda(x) \\
&= 0.
\end{aligned}$$

Thus, $A^c\eta = 0$, where A^c is the transpose of A. Thus, A is singular. □

Remark 3.4. *The main purpose of the above general algorithm is to find a normalized function $f \in \mathrm{span}\,\{\psi_1, \psi_2, \ldots, \psi_\mathrm{N}\}$ such that*

$$\int_I \left[\phi_n(x) - \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x)) \right] f(x)d\lambda(x) = 0,\ n = 1, 2, \cdots.$$

Let N be a positive integer. Divide the interval $I = [0, 1]$ into N subintervals $J_i = [\frac{i-1}{N}, \frac{i}{N}], i = 1, 2, \ldots, N$. Let λ be the Lebesgue measure on I. For each $j = 1, 2, \ldots, N$, let χ_{J_i} be the characteristic function on J_i. As before, Let $L^{(N)}$ be the subspace of $L^1([0, 1])$ consisting of functions which are piecewise constant on the subinterval $J_i, i = 1, 2, \ldots, N$.
For each $i = 1, 2, \ldots, N$, let

$$\psi := \mathbf{1}_i = N\chi_{J_i},\ \ \phi_i = \chi_{J_i}.$$

Then, $\{\psi_i\}_{i=1}^N$ is a density basis of $L^{(N)}$. Thus, $f = \sum_{i=1}^N v_i\psi_i$ is a density if and only if $v \geq 0$ and $\| v \|_1 = \sum_{i=1}^N |v_i| = 1$. In the following, we show that (i,j) element of the matrix A in the above general algorithm is the (i,j) element of the Ulam's matrix described in the previous section.

$$
\begin{aligned}
a_{ij} &= \int_0^1 \left(\phi_i(x) - \sum_{k=1}^K p_k(x)\phi_i(\tau_k(x))\right)\psi_j(x)d\lambda(x) \\
&= \int_0^1 \left(\chi_{J_i}(x) - \sum_{k=1}^K p_k(x)\chi_{J_i}(\tau_k(x))\right)\mathbf{1}_j(x)d\lambda(x) \\
&= \int_0^1 \chi_{J_i}(x)\mathbf{1}_j(x)\lambda(x) - \int_0^1 \sum_{k=1}^K p_k(x)\chi_{J_i}(\tau_k(x))\mathbf{1}_j(x)d\lambda(x) \\
&= \int_0^1 \chi_{J_i}(x)\mathbf{1}_j(x)\lambda(x) - \sum_{k=1}^K \int_0^1 p_k(x)\chi_{J_i}(\tau_k(x))\mathbf{1}_j(x)d\lambda(x) \\
&= N\int_{I_i\cap I_j} d\lambda(x) - \sum_{k=1}^K \int_0^1 p_k(x)\chi_{\tau_k^{-1}(J_i)}(x)\mathbf{1}_j(x)d\lambda(x) \\
&= \delta_{ij} - \sum_{k=1}^K \frac{\lambda(J_j\cap\tau_k^{-1}(J_i))}{\lambda(I_j)}\cdot p_{k,j}^{(N)},
\end{aligned}
$$

where $p_{k,j}^{(N)}$ is the restriction of $Q^{(N)}(p_k(x))$ on I_j for the isometric projection $Q^{(N)}$ of L^1 into $L^{(N)}$ defined in the previous section. Hence $Av = 0$ if and only if $v^c = v^c\mathbb{M}_N$, where v^c is the transpose of v and

$$
\mathbb{M}_N = (m_{ij}),\;\; m_{ij} = \sum_{k=1}^K \frac{\lambda(J_j\cap\tau_k^{-1}(J_j))}{\lambda(I_i)}\cdot p_{k,i}^{(N)}. \tag{3.6}
$$

$\mathbb{M}_N$ in Equation (3.6) is exactly the Ulam's matrix $\mathbb{M}^{*(N)}_{\mathscr{P}^{(N)}}$ for position dependent random maps T described in Equation (2.7).

4. Monte Carlo and Quasi Monte Carlo approach to Ulam's Method for Position Dependent Random Maps

In this section, we present a generalization of Monte Carlo and Quasi Monte Carlo approach to Ulam's method described in [12] and [15] of single deterministic maps to Monte Carlo and Quasi Monte Carlo approach to Ulam's method for position dependent random maps.

4.1 Monte Carlo-Ulam approach to Ulam's method for position dependent random maps

Recall from Section 2.2, Ulam's matrix $\mathbb{M}^{*(N)}_{\mathscr{P}^{(N)}}$ for a position dependent random map

$$
T = \{\tau_1(x),\tau_2(x),\ldots,\tau_K(x);p_1(x),p_2(x),\ldots,p_K(x)\}
$$

with respect to the partition $\mathscr{P}^{(N)}$ is given by

$$
\mathbb{M}^{*(N)}_{\mathscr{P}^{(N)}} = \sum_{k=1}^K \left(\mathbb{M}_k^{(N)}\right)^{\mathrm{c}} \operatorname{diag}\left[p_{k,1}^{(N)},p_{k,2}^{(N)},\ldots,p_{k,N}^{(N)}\right], \tag{4.1}
$$

where for each $k = 1,2,\ldots,K$,

$$
M_k^{(N)} = \left(\frac{\lambda\left(\tau_k^{-1}(J_j)\cap J_i\right)}{\lambda(J_i)}\right)_{1\leq i,j\leq N}.
$$

Computation of $\mathbb{M}^{*(N)}_{\mathscr{P}^{(N)}}$ involves computations of K matrices $M_k^{(N)} = \left(\frac{\lambda\left(\tau_k^{-1}(J_j)\cap J_i\right)}{\lambda(J_i)}\right)_{1\leq i,j\leq N}$ where inverse images of sets (intervals) under τ_k are necessary to compute. If $\tau_k, k = 1,2,\ldots K$ has a complicated formula, then in many cases inverse images of tau_k are difficult to obtain and the computation of the Ulam's matrix becomes complicated and challenging. The Monte Carlo approach to Ulam's method simplifies the above difficulties and makes the numerical method more efficient. The Monte Carlo approach to Ulam's method allows us to approximate the entries of the matrices $M_k^{(N)}, k = 1,2,\ldots,K$. In the following we describe the Monte Carlo approach to Ulam's method:

1. Choose N (a positive integer) and and consider the partition $\{J_1, J_2, \ldots, J_N\}$ of subintervals of equal lengths, where $J_i = [x_{i-1}, x_i], h = \lambda(J_i) = \frac{1}{N}, j = 1, 2, \ldots, N$.
2. for each $k = 1, 2, K$ do
   (a) Choose M (M is a positive integer, same M for each k);
   (b) for $i = 1, 2, \ldots, N$ do
      i. Choose M points $\{z_{i,1}, z_{i,2}, \ldots, z_{i,M}\}$ randomly from the interval J_i with uniform distribution.
      ii. for $j = 1, 2, \ldots, N$ do
         A. Let q_{ij} be the number of points $\{\tau_k(z_{i,1}), \tau_k(z_{i,2}), \ldots, \tau_k(z_{i,M})\}$ in J_j
         B. Let $\frac{q_{ij}}{M}$ be an approximation of the (i, j)-th entry of the matrix $M_k^{(N)}$
   (c) Compute $\left[p_{k,1}^{(N)}, p_{k,2}^{(N)}, \ldots, p_{k,N}^{(N)}\right]$
3. Compute the Ulam's matrix $\mathbb{M}^{*(N)}_{\mathscr{P}(N)} = \sum_{k=1}^{K} \left(\mathbb{M}_k^{(N)}\right)^{\mathrm{c}} \operatorname{diag}\left[p_{k,1}^{(N)}, p_{k,2}^{(N)}, \ldots, p_{k,N}^{(N)}\right]$.
4. Compute a eigenvector v (a normalized eigenvecto) of $\mathbb{M}^{*(N)}_{\mathscr{P}(N)}$ with eigen value 1.
5. Compute $f^{(N)} = \sum_{i=1}^{N} v_i \cdot \chi_{J_i}(x)$ as an approximation of the actual density function f^* of the absolutely continuous invariant measure μ^* for the position dependent random map $T = \{\tau_1(x), \tau_2(x), \ldots, \tau_K(x); p_1(x), p_2(x), \ldots, p_K(x)\}$.

Note that the computation of i-th row of each matrix $M_k^{(N)}$ is independent of the computation of other rows. Therefore, for each $k = 1, 2, \ldots N$ one can use p processors to calculate l rows (here, $N = pl$).

4.2 Quasi Monte Carlo-Ulam Parallel Algorithm for Position Dependent Random Maps

In a Monte Carlo approach to Ulam's methods, M points in each interval $J_i, i = 1, 2, \ldots, N$ are randomly chosen with uniform distribution. In a Quasi Monte Carlo method M points $\{z_{i,1}, z_{i,2}, \ldots, z_{i,M}\}$ are chosen deterministically as follows:

$$z_{i,m} = x_{i-1} + \frac{m}{M} h, m = 1, 2, \ldots, N.$$

All other steps are similar to Monte Carlo Method in Section 4.1. This type of deterministic selections makes the numerical method more efficient as we will see the next section with examples.

5. Numerical Examples

In this section, we consider position dependent random maps T satisfying conditions of Theorem 2.3 with unique invariant density f^* and we apply Monte Carlo method and Quasi Monte Carlo approaches to Ulam's method described in the previous section. Moreover, we find the L^1 norms $\| f^* - f_N \|_1$, for some $N \geq 1$ where f_N is an approximation of f^*. Monte Carlo and Quasi Monte Carlo approach to Ulam's method can be applied to any position dependent map satisfying conditions of Theorem 2.3. However, first we consider a simple position dependent random map T, where the density f^* of the invariant measure μ^* is known. In the first example, the component maps of the position dependent random map T are piecewise linear and Markov and the probabilities are position dependent piecewise constants. The main reason for the consideration of a such a simple position dependent random map is that the actual density is known in this case and we can compare our numerically approximate densities with the actual density. In the second example, we consider a position dependent random map where the component maps are non-Markov and the actual density is not known.

Example 5.1. *Consider the position dependent random map $T = \{\tau_1(x), \tau_2(x); p_1(x), p_2(x)\}$ where $\tau_1, \tau_2 : [0,1] \to [0,1]$ are defined by*

$$\tau_1(x) = \begin{cases} 3x + \frac{1}{4}, & 0 \leq x < \frac{1}{4}, \\ 3x - \frac{3}{4}, & \frac{1}{4} \leq x < \frac{1}{2}, \\ 4x - 2, & \frac{1}{2} \leq x < \frac{3}{4}, \\ 4x - 3, & \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$\tau_2(x) = \begin{cases} 4x, & 0 \le x < \frac{1}{4}, \\ 4x-1, & \frac{1}{4} \le x < \frac{1}{2}, \\ 3x-\frac{3}{2}, & \frac{1}{2} \le x < \frac{3}{4}, \\ 3x-\frac{9}{4}, & \frac{3}{4} \le x \le 1, \end{cases}$$

and the position dependent probabilities $p_1, p_2 : [0,1] \to [0,1]$ *are defined by*

$$p_1(x) = \begin{cases} \frac{1}{4}, & 0 \le x < \frac{1}{4}, \\ \frac{1}{4}, & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{3}{4}, & \frac{1}{2} \le x < \frac{3}{4}, \\ \frac{3}{4}, & \frac{3}{4} \le x \le 1 \end{cases}$$

and

$$p_2(x) = \begin{cases} \frac{3}{4}, & 0 \le x < \frac{1}{4}, \\ \frac{3}{4}, & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{1}{4}, & \frac{1}{2} \le x < \frac{3}{4}, \\ \frac{1}{4}, & \frac{3}{4} \le x \le 1 \end{cases}$$

If $x \in [0, \frac{1}{4})$, *then* $\sum_{k=1}^{2} g_k(x) = \sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{1}{4}}{3} + \frac{\frac{3}{4}}{4} = \frac{13}{48} < 1.$
If $x \in [\frac{1}{4}, \frac{1}{2})$, *then* $\sum_{k=1}^{2} g_k(x) = \sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{1}{4}}{3} + \frac{\frac{3}{4}}{4} = \frac{13}{48} < 1.$
If $x \in [\frac{1}{2}, \frac{3}{4})$, *then* $\sum_{k=1}^{2} g_k(x) = \sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{3}{4}}{4} + \frac{\frac{1}{4}}{3} = \frac{13}{48} < 1.$
If $x \in [\frac{1}{2}, \frac{3}{4})$, *then* $\sum_{k=1}^{2} g_k(x) = \sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{3}{4}}{4} + \frac{\frac{1}{4}}{3} = \frac{13}{48} < 1.$

Moreover,

$A = 3\alpha + \max_{1\le i\le q} \sum_{k=1}^{K} V_{J_i} g_k = 3 \cdot \frac{13}{48} + 0 = \frac{39}{48} < 1$. *Here,* $B = 2\beta\alpha + \beta \ \max_{1\le i\le q} \sum_{k=1}^{K} V_{J_i} g_k > 0$ *with* $\beta = \max_{1\le i\le q} \frac{1}{\lambda(J_i)}$. *Thus, the random map* T *satisfies condition of Theorem 2.3.*

From the Lasota–Yorke result ([9]), both τ_1 *and* τ_2 *has acim. Moreover,* τ_1 *and* τ_2 *are piecewise linear, expanding and Markov. The Frobenius-Perron matrix* P_{τ_1} *of* τ_1 *is the transpose of* M_{τ_1} *where*

$$M_{\tau_1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

The matrix representation of the Frobenius–Perron operator P_{τ_2} is the transpose of M_{τ_2} where

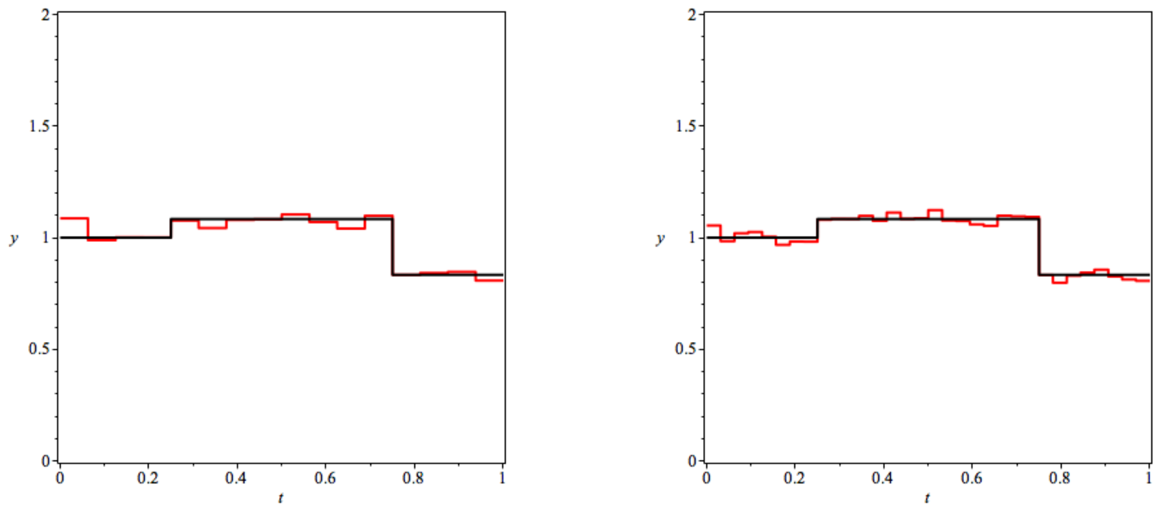
$$M_{\tau_2} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

It is easy to show that both τ_1 and τ_2 have unique acim. Thus, the random map $T = \{\tau_1(x), \tau_2(x); p_1(x), p_2(x)\}$ also has a unique acim (see Proposition 1 in [1]). The matrix representation of the Frobenius–Perron operator $P_T f = \sum_{k=1}^{2} P_{\tau_k}(p_k f)(x)$ is the transpose of the matrix M_T, where

$$M_T = \begin{bmatrix} \frac{3}{16} & \frac{13}{16} & \frac{13}{16} & \frac{13}{16} \\ \frac{13}{16} & \frac{13}{16} & \frac{13}{16} & \frac{3}{16} \\ \frac{13}{16} & \frac{13}{16} & \frac{13}{16} & \frac{3}{16} \\ \frac{13}{16} & \frac{13}{16} & \frac{13}{16} & \frac{13}{16} \end{bmatrix}.$$

The normalized density f^* of the unique acim of the random map T is the left eigenvector of the matrix M_T associated with the eigenvalue 1 (after adding the normalizing condition). In fact, $f^* = \left[1, \frac{13}{12}, \frac{13}{12}, \frac{5}{6}\right]$.

Monte Carlo approach to Ulam's method: *In Figure 5.1 (a) and 5.1 (b) we have plotted the actual density and approximate density for Monte Carlo approach to Ulam's method.*



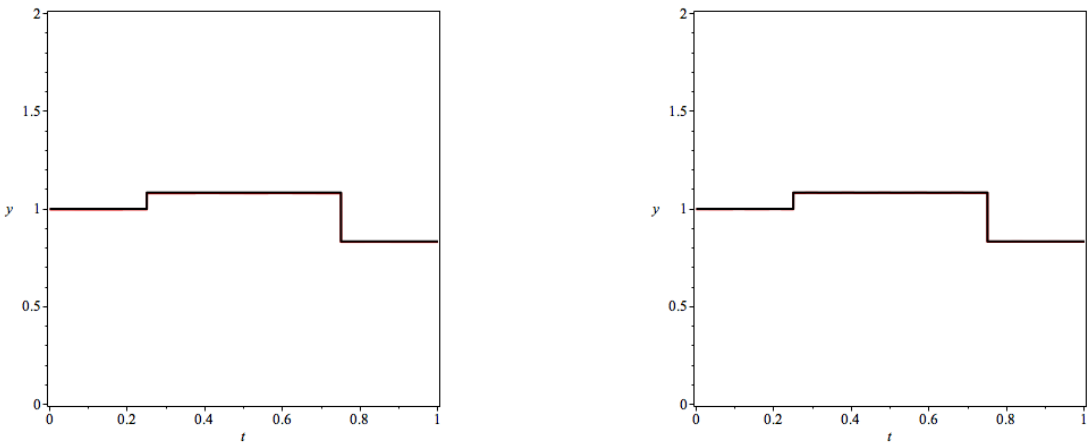
(a) **(b)**

Figure 5.1. Monte Carlo approach to Ulam's method for the random map T: Figure 5.1 (a) the graph of the approximate density function f_{16} (Monte Carlo -Ulam's method with $N = 16, K = 1000$:red curve) and the actual density function f^* (black curve); Figure 5.1 (b) the graph of the approximate density function f_{32} (Monte Carlo -Ulam's method with $N = 32, K = 1000$:red curve) and the actual density function f^* (black curve);

The L^1–norm $\| f_N - f^* \|_1$ is measured (with Maple 15) to estimate the convergence of the approximate density f_N to the actual density f^* for our Monte Carlo approach to Ulam's method.

N	$\parallel f_N - f^* \parallel_1$
16	0.01815903102
32	0.01630702912

Quasi Monte Carlo approach to Ulam's method: *In Figure 5.2 (a) and 5.2 (b) we have plotted the actual density and approximate density for Quasi Monte Carlo-Ulam's method.*



(a) **(b)**

Figure 5.2. Quasi Monte Carlo approach to Ulam's method for the random map T: Figure 2 (a)the graph of the approximate density function f_{16} (Quasi Monte Carlo -Ulam's method with $N = 16, K = 1000$:red curve) and the actual density function f^* (black curve); Figure 2 (b) Figure 1 (a)the graph of the approximate density function f_{32} (Quasi Monte Carlo approach to Ulam's method with $N = 32, K = 1000$: red curve) and the actual density function f^* (black curve);

The L^1−norm $\parallel f_N - f^* \parallel_1$ is measured (with Maple 18) to estimate the convergence of the approximate density f_N to the actual density f^* for our Quasi Monte Carlo- Ulam's method.

N	$\parallel f_N - f^* \parallel_1$
16	0.002504794762
32	0.001252884246

Example 5.2. *We consider the position dependent random map $T = \{\tau_1(x), \tau_2(x); p_1(x), p_2(x)\}$ where $\tau_1, \tau_2 : [0,1] \to [0,1]$ are defined by (see Example 5.2 of [5] for this random map)*

$$\tau_1(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2}, \\ \frac{5}{4}x + \frac{1}{10}, & \frac{1}{2} \le x \le \frac{2}{3}, \\ \frac{3}{4}x + \frac{1}{4}, & \frac{2}{3} < x \le 1, \end{cases}$$

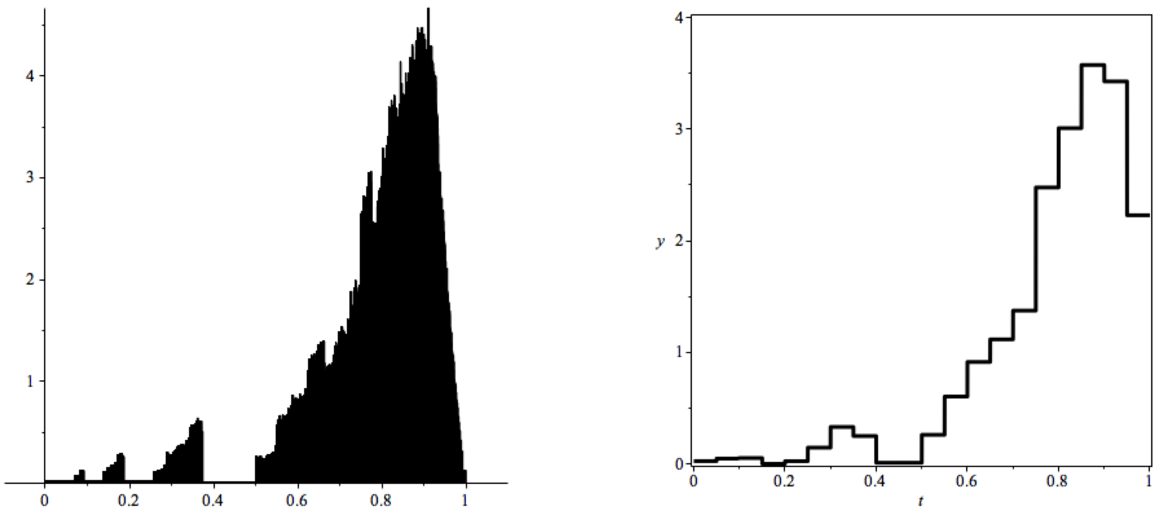
$$\tau_2(x) = \begin{cases} \frac{1}{2}x, & 0 \le x < \frac{1}{2}, \\ \frac{3}{4}x - \frac{1}{8}, & \frac{1}{2} \le x \le \frac{2}{3}, \\ \frac{3}{2}x - \frac{1}{2}, & \frac{2}{3} < x \le 1, \end{cases}$$

and the position dependent probabilities $p_1, p_2 : [0,1] \to [0,1]$ are defined by

$$p_1(x) = \begin{cases} 0.8, & 0 \le x < \frac{1}{2}, \\ 0.725, & \frac{1}{2} \le x \le \frac{2}{3}, \\ 0.4, & \frac{2}{3} < x \le 1, \end{cases}$$

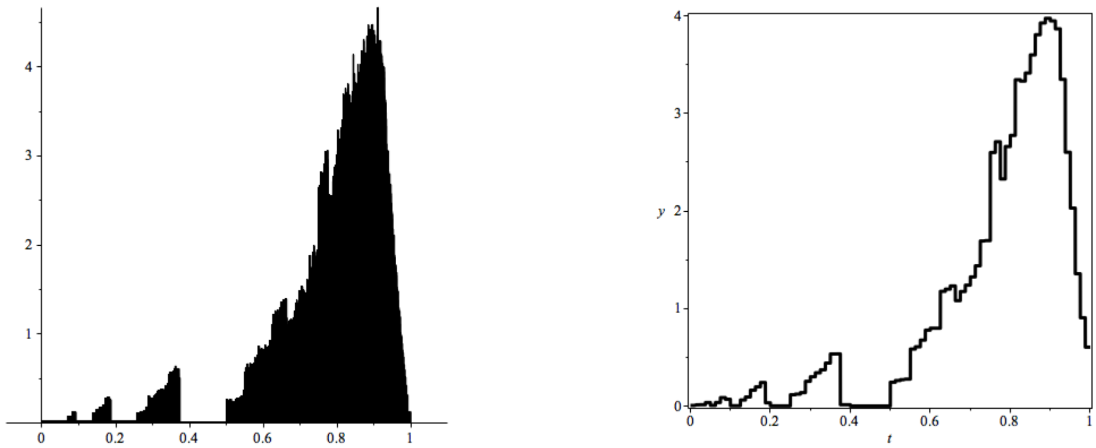
and $p_2(x) = 1 - p_1(x)$.

It can be easily shown that the random map T *satisfies condition of Theorem 2.3. Thus,* T *has an acim. Unfortunately, we do not know the actual density of the acim. Góra and Boyarsky [1] presented a Markov approximation of the random map* T *then they presented the density of the Markov random maps. Note that the density obtained from a Markov approximation of the random maps is only an approximate density. In Figure 5.3 and Figure 5.4 we have presented histogram and approximate densities via Monte Carlo approach to Ulam's and Quasi Monte Carlo approach to Ulam's method.*



(a) (b)

Figure 5.3. Histigram and Monte Carlo approach to Ulam's method: Figure 5.3 (a) the histogram of the density function of the random map T with 500,000 points on the trajectory of the random map T with 1000 subintervals for $[0,1]$.; Figure 5.3 (b) Monte Carlo approach to Ulam's method for the random map T: The graph of the approximate density function f_{20} with $K = 1000$.



(a) (b)

Figure 5.4. Histigram and Quasi Monte Carlo approach to Ulam's method: Figure 5.4 (a) the histogram of the density function of the random map T with 500,000 points on the trajectory of the random map T with 1000 subintervals for $[0,1]$.; Figure 5.4 (b) Quasi Monte Carlo approach to Ulam's method for the random map T: The graph of the approximate density function f_{80} with $K = 1000$.

6. Conclusion

In this paper, we study numerical computations of invariant measures for position dependent random maps. First, we present the Frobenius-Perron operator and the existence of invariant measures for position dependent random maps. We present the Ulam's method for the computation of invariant measures for position dependent random maps. A general algorithm for approximating fixed points of the Frobenius-Perron operator for position dependent random maps is presented. Then we present the Monte Carlo and the Quasi Monte Carlo approach to Ulam's method for the computation of invariant measures for position dependent random maps. Finally, we present two examples of position dependent random maps along with the numerical computations of invariant measures using the Monte Carlo and the Quasi Monte Carlo approach to Ulam's method. In the first example, we present L^1 norm errors between the numerical approximation of the density of the invariant measure and analytical density of invariant measures for the random map. The numerical examples show that the Monte Carlo approach and the Quasi Monte Carlo approach to Ulam's method are useful tools for the computation of invariant measures for position dependent random maps. Our numerical schemes are generalizations of numerical schemes described in [12] and [15] of single deterministic maps to numerical schemes for position dependent random maps. In future, we plan on studying the speed of convergence of the Monte Carlo approach and the Quasi Monte Carlo approach to Ulam's method.

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References

[1] P. Góra, A. Boyarsky, *Absolutely continuous invariant measures for random maps with position dependent probabilities*, Math. Anal. and Appl., **278** (2003), 225 – 242.

[2] M. Barnsley, *Fractals Everywhere*, Academic Press, London. 1998.

[3] A. Boyarsky, P. Góra, *A dynamical model for interference effects and two slit experiment of quantum physics*, Phys, Lett. A., **168** (1992), 103 – 112.

[4] W. Slomczynski, J. Kwapien, K. Zyczkowski, *Entropy computing via integration over fractal measures*, Chaos, **10** (2000), 180-188.

[5] W. Bahsoun, P. Góra, S. Mayoral, M. Morales, *Random dynamics and finance: constructing implied binomial trees from a predetermined stationary density,* Appl. Stochastic Models Bus. Ind., **23** (2007), 181– 212.

[6] K.R. Schenk-Hoppe, *Random Dynamical Systems in Economics*, working paper series, Institute of empirical research in economics, University of Zurich, ISSN 1424-0459 (2000).

[7] M.S. Islam, *Existence, approximation and properties of absolutely continuous invariant measures for random maps*, PhD thesis, Concordia University, 2004.

[8] S. Pelikan, *Invariant densities for random maps of the interval*, Proc. Amer. Math. Soc., **281** (1984), 813 – 825.

[9] A. Lasota, J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc., **186** (1973) , 481– 488.

[10] S.M. Ulam, *A collection of Mathematical problems Interscience Tracts in pure and Applied Math.*, **8**, Interscience, New York, 1960.

[11] T-Y. Li, *Finite approximation for the Frobenius-Perron operator: A solution to Ulam's conjecture*, J. Approx. Theory. **17** (1976), 177 – 186.

[12] J. Ding, Z. Wang, *Parallel Computation of Invariant Measures*,Annals of Operations Research, **103** (2001), 283 – 290.

[13] W. Bahsoun, P. Góra, *Position Dependent Random Maps in One and Higher Dimensions*, Studia Math., **166** (2005), 271 – 286.

[14] A. Boyarsky, P. Góra, *Laws of Chaos: Invariant Measures And Dynamical Systems In One Dimension*, Birkhaüser, 1997.

[15] F.Y. Hunt, *A Monte Carlo approach to the approximation of invariant measure,* Random Comput. Dynam., **2**(1) (1994), 111 – 133.

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On Generalized Fibonacci Numbers

Fidel Ochieng Oduol[1], Isaac Owino Okoth[2]*

Abstract
Fibonacci numbers and their polynomials have been generalized mainly by two ways: by maintaining the recurrence relation and varying the initial conditions, and by varying the recurrence relation and maintaining the initial conditions. In this paper, we introduce and derive various properties of r-sum Fibonacci numbers. The recurrence relation is maintained but initial conditions are varied. Among results obtained are Binet's formula, generating function, explicit sum formula, sum of first n terms, sum of first n terms with even indices, sum of first n terms with odd indices, alternating sum of n terms of $r-$sum Fibonacci sequence, Honsberger's identity, determinant identities and a generalized identity from which Cassini's identity, Catalan's identity and d'Ocagne's identity follow immediately.

Keywords: Binet's formula, Fibonacci sequence, generating function, r-sum Fibonacci sequence
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[1] Department of Pure and Applied Mathematics, Maseno University, Maseno, Kenya, ORCID: 0000-0002-1228-6339
[2] Department of Pure and Applied Mathematics, Maseno University, Maseno, Kenya, ORCID: 0000-0003-4503-4733
*Corresponding author: ookoth@maseno.ac.ke
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1. Introduction

Fibonacci sequence is the most studied sequence in the history of mathematics. In [14], the said sequence is given by A000045. The sequence is generated by a recursive formula $f_n = f_{n-1} + f_{n-2}$, for $n \geq 3$ with $f_1 = 0$ and $f_2 = 1$. The sequence has many interesting properties. For example, the ratio $\dfrac{f_{n+1}}{f_n}$ converges to the golden ratio $\dfrac{1+\sqrt{5}}{2}$ as n tends to infinity.

Various generalizations of the aforementioned sequence have been derived since it was first discovered by Fibonacci in the 13$^{\text{th}}$ century. Fibonacci sequence has been generalized mainly by two ways: by maintaining the recurrence relation and varying the initial conditions [1, 3, 4, 5, 7, 9, 10], and by varying the recurrence relation and maintaining the initial conditions [2, 4, 8, 9, 11, 13, 12, 15]. Some of the properties that have been obtained by various researchers are not limited to finding a closed form for the n^{th} term of the sequence, sum of the first n terms of the sequence, sum of the first n terms with odd (or even) indices of the sequence, explicit sum formula, Catalan's identity, Cassini's identity, d'Ocagne's identity, Honsberger's identity, determinant identities, and generating function among many others.

Let f_n be the n^{th} term of Fibonacci sequence. Binet's formula gives a closed formula for f_n as

$$f_n = \frac{1}{\alpha - \beta}\left(\alpha^{n-1} - \beta^{n-1}\right), \tag{1.1}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

Companion to Fibonacci numbers are Lucas numbers with the same recurrence relation as Fibonacci numbers except for initial conditions which are 2 and 1. Binet's formula for Lucas numbers, l_n, is given by

$$l_n = \alpha^{n-1} + \beta^{n-1}. \tag{1.2}$$

Here, l_n is the n^{th} Lucas number.

Some properties of Fibonacci sequences explored in this paper include the sum of the first n terms of Fibonacci sequence,

$$f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1, \tag{1.3}$$

and the Honsberger's identity

$$f_{n+m} = f_n f_m + f_{n+1} f_{m+1}, \tag{1.4}$$

for all $n \geq 1$ and $n > m$.

Definition 1.1. *The n^{th} term of* r-sum Fibonacci sequence*, $h_{n,r}$, is given by*

$$h_{n,r} = f_n + f_{n+1} + \cdots + f_{n+r-1}. \tag{1.5}$$

Using Definition 1.1, it follows that the first term

$$h_{1,r} = f_1 + f_2 + \cdots + f_r = f_{r+2} - 1$$

and the second term

$$h_{2,r} = f_2 + f_3 + \cdots + f_{r+1} = f_{r+3} - 1.$$

As with Fibonacci sequence, the $r-$sum Fibonacci sequence satisfies the recurrence relation

$$h_{n,r} = h_{n-1,r} + h_{n-2,r}, \tag{1.6}$$

for $n \geq 3$, with initial conditions $h_{1,r} = f_{r+2} - 1$ and $h_{2,r} = f_{r+3} - 1$.

Few entries of $h_{n,r}$ are given in Table 1 below.

Table 1. r-Sum Fibonacci numbers

r	$h_{1,r}$	$h_{2,r}$	$h_{3,r}$	$h_{4,r}$	$h_{5,r}$	$h_{6,r}$	$h_{7,r}$	$h_{8,r}$	$h_{9,r}$	$h_{10,r}$	$h_{11,r}$
1	0	1	1	2	3	5	8	13	21	34	55
2	1	2	3	5	8	13	21	34	55	89	144
3	2	4	6	10	16	26	42	68	110	178	288
4	4	7	11	18	29	47	76	123	199	322	521
5	7	12	19	31	50	81	131	212	343	555	898
6	12	20	32	52	84	136	220	356	576	932	1508
7	20	33	53	86	139	225	364	589	953	1542	2495

When $r = 1, 2$, we get Fibonacci sequence with different initial conditions. For $r \geq 3$, we get Fibonacci-like numbers. We also note that when $r = 4$, we obtain Lucas numbers.

This paper is organized as follows: Some basic properties of $h_{n,r}$ are given in Section 2. In Section 3, we obtain Binet's formula and generating function for these numbers. Further properties of these numbers are presented in Section 4. Moreover, determinant identities are presented in Section 5. We conclude the paper in Section 6.

2. Preliminary Results

We start off, with these important and interesting properties:

Lemma 2.1. *For $n \geq 1$, we have $h_{n,3} = 2h_{n,2}$.*

Proof. From (1.5), we have

$$\begin{aligned}
h_{n,3} &= f_n + f_{n+1} + f_{n+2},\\
&= f_n + f_{n+1} + f_n + f_{n+1},\\
&= 2(f_n + f_{n+1}),\\
&= 2h_{n,2}.
\end{aligned}$$

□

Proposition 2.2. *The n^{th} term of $r-$sum Fibonacci number, $h_{n,r}$, can be expressed as $h_{n,r} = f_{n+r+1} - f_{n+1}$, for all $r \geq 1$.*

Proof. From recurrence relation (1.6) and equation (1.5), we have

$$\begin{aligned}
h_{n,r} &= h_{n-1,r} + h_{n-2,r},\\
&= (f_{n-1} + f_n + \cdots + f_{n+r-2}) + (f_{n-2} + f_{n-1} + \cdots + f_{n+r-3}),\\
&= [(f_1 + f_2 + \cdots + f_{n+r-2} - (f_1 + f_2 + \cdots + f_{n-2})] + [(f_1 + f_2 + \cdots + f_{n+r-3}) - (f_1 + f_2 + \cdots + f_{n-3})].
\end{aligned}$$

By equation (1.3), we get

$$\begin{aligned}
h_{n,r} &= [(f_{n+r} - 1) - (f_n - 1)] + [(f_{n+r-1} - 1) - (f_{n-1} - 1)]\\
&= f_{n+r} - f_n + f_{n+r-1} - f_{n-1}\\
&= f_{n+r+1} - f_{n+1}.
\end{aligned}$$

□

Proposition 2.3.

$$h_{n,r} = f_{r+1}f_{n+2} + f_{n+1}\sum_{i=1}^{r-2} f_i.$$

Proof. By Proposition 2.2, we have that

$$h_{n,r} = f_{n+r+1} - f_{n+1}. \tag{2.1}$$

Now, by Honsberger's identity (1.4), we have

$$f_{n+r+1} = f_{r+1}f_{n+2} + f_r f_{n+1}.$$

Substituting this sum in (2.1), we obtain

$$\begin{aligned}
h_{n,r} &= f_{r+1}f_{n+2} + f_r f_{n+1} - f_{n+1}\\
&= f_r f_{n+2} + f_{n+1}(f_r - 1).
\end{aligned}$$

Since $\sum_{i=1}^{n} f_i = f_{n+2} - 1$, then $h_{n,r} = f_{r+1}f_{n+2} + f_{n+1}\sum_{i=1}^{r-2} f_i$. □

Theorem 2.4. *The numbers, $h_{n,r}$, can be expressed in terms of Fibonacci and Lucas numbers as:*

$$h_{n,r} = \begin{cases}
\sum_{i=1}^{m} l_{n+4i-1} & \text{if} \quad r = 4m,\\
\sum_{i=1}^{m} l_{n+4i-1} + f_{n+4m} & \text{if} \quad r = 4m+1,\\
\sum_{i=1}^{m} l_{n+4i-1} + f_{n+4m+2} & \text{if} \quad r = 4m+2,\\
\sum_{i=1}^{m} l_{n+4i-1} + 2f_{n+4m+2} & \text{if} \quad r = 4m+3.
\end{cases}$$

Proof. If $r = 4m$, then

$$\begin{aligned} h_{n,r} &= f_n + f_{n+1} + \cdots + f_{n+4m-1} \\ &= f_{n+2} + f_{n+4} + \cdots + f_{n+4m} \\ &= l_{n+3} + l_{n+7} + \cdots + l_{n+4m-1}. \end{aligned}$$

If $r = 4m+1$, then

$$\begin{aligned} h_{n,r} &= f_n + f_{n+1} + \cdots + f_{n+4m} \\ &= f_{n+2} + f_{n+4} + \cdots + f_{n+4m-2} + f_{n+4m} + f_{n+4m} \\ &= l_{n+3} + l_{n+7} + \cdots + l_{n+4m-1} + f_{n+4m}. \end{aligned}$$

If $r = 4m+2$, then

$$\begin{aligned} h_{n,r} &= f_n + f_{n+1} + f_{n+2} + \cdots + f_{n+4m+1} \\ &= f_{n+2} + f_{n+4} + \cdots + f_{n+4m+2} \\ &= l_{n+3} + l_{n+7} + \cdots + l_{n+4m-1} + f_{n+4m+2}. \end{aligned}$$

If $r = 4m+3$, then

$$\begin{aligned} h_{n,r} &= f_n + f_{n+1} + \cdots + f_{n+4m+2} \\ &= f_{n+2} + f_{n+4} + \cdots + f_{n+4m} + f_{n+4m+2} + f_{n+4m+2} \\ &= l_{n+3} + l_{n+7} + \cdots + l_{n+4m-1} + 2f_{n+4m+2}. \end{aligned}$$

□

Remark 2.5. *We note that:*

1. *For $r = 1,2$, the $r-$sum Fibonacci numbers, $h_{n,r}$, are themselves Fibonacci numbers.*
2. *We have $h_{n,3}$ as a sum of Fibonacci numbers for all $n \geq 1$.*
3. *The numbers, $h_{n,4}$, are Lucas numbers for all integers $n \geq 1$.*
4. *The numbers, $h_{n,4m}$, are sums of Lucas numbers for all integers $m \geq 1$ and $n \geq 1$.*
5. *For all $m \in \mathbb{N}$ and $n \geq 1$, we have that $h_{n,4m+1}$, $h_{n,4m+2}$, and $h_{n,4m+3}$ are sums of Fibonacci and Lucas numbers.*

Proposition 2.6. *Let $m \geq 1$. Then the n^{th} term of $4m-$sum Fibonacci sequence, $h_{n,4m}$, satisfies the equation*

$$h_{n,4m} = f_{2m+1} l_{n+2m+1}.$$

Proof. By Binet's formulas for Fibonacci numbers (1.1) and Lucas numbers (1.2) and by equation (2.1), we have

$$\begin{aligned} h_{n,4m} &= f_{n+4m+1} - f_{n+1} \\ &= \frac{1}{\alpha-\beta}(\alpha^{n+4m} - \beta^{n+4m}) - \frac{1}{\alpha-\beta}(\alpha^n - \beta^n). \end{aligned}$$

Since $\alpha\beta = -1$ then, $(\alpha\beta)^{2m} = 1$, and

$$\begin{aligned} h_{n,4m} &= \frac{1}{\alpha-\beta}(\alpha^{n+4m} - (\alpha\beta)^{2m}\alpha^n - \beta^{n+4m} + (\alpha\beta)^{2m}\beta^n) \\ &= \frac{1}{\alpha-\beta}(\alpha^{n+4m} - \beta^{2m}\alpha^{n+2m} - \beta^{n+4m} + \alpha^{2m}\beta^{n+2m}) \\ &= \frac{1}{\alpha-\beta}(\alpha^{n+2m}(\alpha^{2m} - \beta^{2m}) + \beta^{n+2m}(\alpha^{2m} - \beta^{2m})) \\ &= \frac{1}{\alpha-\beta}(\alpha^{2m} - \beta^{2m})(\alpha^{n+2m} + \beta^{n+2m}) \\ &= f_{2m+1} l_{n+2m+1}. \end{aligned}$$

□

Setting $m = 1$ in Proposition 2.6, we get:

Corollary 2.7. $h_{n,4} = l_{n+3}$, *for all* $n \geq 1$.

Proposition 2.8.

$$\sum_{k=2}^{n} h_{k,r}^2 = h_{n,r}h_{n+1,r} - h_{1,r}h_{2,r}.$$

Proof. Since $h_{n,r} = h_{n+1,r} - h_{n-1,r}$ then,

$$h_{n,r}^2 = h_{n,r}h_{n+1,r} - h_{n-1,r}h_{n,r}.$$

Now, we have

$$\begin{aligned}
h_{2,r}^2 &= h_{2,r}h_{3,r} - h_{1,r}h_{2,r} \\
h_{3,r}^2 &= h_{3,r}h_{4,r} - h_{2,r}h_{3,r} \\
h_{4,r}^2 &= h_{3,r}h_{5,r} - h_{3,r}h_{4,r} \\
&\vdots \\
h_{n-1,r}^2 &= h_{n-1,r}h_{n,r} - h_{n-2,r}h_{n-1,r} \\
h_{n,r}^2 &= h_{n,r}h_{n+1,r} - h_{n-1,r}h_{n,r}.
\end{aligned}$$

Adding up these equations, we get

$$h_{2,r}^2 + h_{3,r}^2 + h_{4,r}^2 + \cdots + h_{n-1,r}^2 + h_{n,r}^2 = h_{n,r}h_{n+1} - h_{1,r}h_{2,r}.`$$

□

Proposition 2.9. *For every positive integer* $n \geq 2$,

$$h_{n,r}^2 - h_{n-1,r}^2 = h_{n+1,r}h_{n-2,r}.$$

Proof. Since

$$h_{n-1,r}^2 = h_{n-1,r}h_{n,r} - h_{n-1,r}h_{n-2,r}$$

then,

$$\begin{aligned}
h_{n,r}^2 - h_{n-1,r}^2 &= h_{n,r}^2 - h_{n-1,r}h_{n,r} + h_{n-1,r}h_{n-2,r} \\
&= h_{n,r}(h_{n,r} - h_{n-1,r}) + h_{n-1,r}h_{n-2,r} \\
&= h_{n,r}h_{n-2,r} + h_{n-1,r}h_{n-2,r} \\
&= h_{n-2,r}(h_{n,r} + h_{n-1,r}) \\
&= h_{n+1,r}h_{n-2,r}.
\end{aligned}$$

□

3. Binet's Formula and Generating Function

We start by getting a closed formula for $h_{n,r}$.

Theorem 3.1 (Binet's Formula)**.** *The* n^{th} *term of r-sum Fibonacci sequence,* $h_{n,r}$*, is given by*

$$h_{n,r} = \frac{1}{\alpha - \beta}\left[(h_{2,r} - \beta h_{1,r})\alpha^{n-1} - (h_{2,r} - \alpha h_{1,r})\beta^{n-1}\right], \tag{3.1}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ *and* $\beta = \dfrac{1-\sqrt{5}}{2}$.

Proof. Let $n \geq 2$, then r-sum Fibonacci numbers are defined by the recurrence relation

$$h_{n,r} = h_{n-1,r} + h_{n-2,r},$$

with initial conditions $h_{1,r} = f_{r+2} - 1$ and $h_{2,r} = f_{r+3} - 1$, for all $r > 0$. The characteristic equation of the recurrence relation is $\lambda^2 - \lambda - 1 = 0$. We solve this equation to get its roots as

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

These roots are real and distinct and thus the solution of the recurrence relation is of the form

$$h_{n,r} = A\alpha^n + B\beta^n, \tag{3.2}$$

where A and B are constants.

Setting $n = 1$ and $n = 2$ in (3.2), we obtain

$$A\alpha + B\beta = h_{1,r}$$

and

$$A\alpha^2 + B\beta^2 = h_{2,r}$$

respectively. Solving these equations simultaneously, we get

$$A = \frac{h_{2,r} - \beta h_{1,r}}{\alpha(\alpha-\beta)}$$

and

$$B = \frac{\alpha h_{1,r} - h_{2,r}}{\beta(\alpha-\beta)}.$$

Thus the result follows. □

Corollary 3.2. *The n^{th} term of the $r-$sum Fibonacci sequence satisfies the equation $h_{n,r} = h_{2,r}f_n + h_{1,r}f_{n-1}$.*

Proof. From Binet's formula (3.1), we have

$$h_{n,r} = \frac{1}{\alpha-\beta}\left[h_{2,r}(\alpha^{n-1} - \beta^{n-1}) - h_{1,r}(\alpha\beta)(\alpha^{n-2} - \beta^{n-2})\right],$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Since $\alpha\beta = -1$, then

$$\begin{aligned} h_{n,r} &= \frac{1}{\alpha-\beta}\left[h_{2,r}(\alpha^{n-1} - \beta^{n-1}) + h_{1,r}(\alpha^{n-2} - \beta^{n-2})\right] \\ &= h_{2,r}f_n + h_{1,r}f_{n-1}. \end{aligned}$$

□

The following formula is rediscovered immediately upon setting $r = 1$ in (3.1).

Corollary 3.3 (Binet's formula)**.** *The n^{th} Fibonacci number, f_n, is given explicitly as*

$$f_n = \frac{1}{\alpha-\beta}\left[\alpha^{n-1} - \beta^{n-1}\right],$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$.

Corollary 3.4. *The sequence of ratio of successive r-sum Fibonacci numbers* $\dfrac{h_{n+1,r}}{h_{n,r}}$ *converges to the golden ratio, i.e.,* $\lim\limits_{n\to\infty}\dfrac{h_{n+1,r}}{h_{n,r}} = \dfrac{1+\sqrt{5}}{2}$.

Proof. From Binet's formula (3.1), we have

$$\lim_{n\to\infty}\frac{h_{n+1,r}}{h_{n,r}} = \lim_{n\to\infty}\frac{\frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\alpha^n-(h_{2,r}-\alpha h_{1,r})\beta^n\right]}{\frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\alpha^{n-1}-(h_{2,r}-\alpha h_{1,r})\beta^{n-1}\right]},$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.
Factorizing α^{n-1}, we obtain

$$\lim_{n\to\infty}\frac{h_{n+1},r}{h_{n,r}} = \lim_{n\to\infty}\frac{\frac{1}{\alpha-\beta}\alpha^{n-1}\left[(h_{2,r}-\beta h_{1,r})\alpha-(h_{2,r}-\alpha h_{1,r})\alpha^{-(n-1)}\beta^n\right]}{\frac{1}{\alpha-\beta}\alpha^{n-1}\left[(h_{2,r}-\beta h_{1,r})-(h_{2,r}-\alpha h_{1,r})\alpha^{-(n-1)}\beta^{n-1}\right]},$$

which simplifies to

$$\lim_{n\to\infty}\frac{h_{n+1,r}}{h_{n,r}} = \lim_{n\to\infty}\frac{(h_{2,r}-\beta h_{1,r})\alpha-(h_{2,r}-\alpha h_{1,r})\left(\frac{\beta}{\alpha}\right)^{n-1}\beta}{(h_{2,r}-\beta h_{1,r})-(h_{2,r}-\alpha h_{1,r})\left(\frac{\beta}{\alpha}\right)^{n-1}}.$$

Since $|\frac{\beta}{\alpha}|<1$, we have $\lim\limits_{n\to\infty}\left(\dfrac{\beta}{\alpha}\right)^{n-1}=0$ so that

$$\lim_{n\to\infty}\frac{h_{n+1,r}}{h_{n,r}} = \lim_{n\to\infty}\frac{(h_{2,r}-\beta h_{1,r})\alpha}{(h_{2,r}-\beta h_{1,r})} = \alpha = \frac{1+\sqrt{5}}{2}.$$

□

We now obtain the generating function for r-sum Fibonacci sequence.

Theorem 3.5. *Let* $H_r(t)$ *be the generating function for* $r-$*sum Fibonacci sequence, then*

$$H_r(t) = \frac{h_{1,r}t+t^2(h_{2,r}-h_{1,r})}{1-t-t^2}. \tag{3.3}$$

Proof. Let $H_r(t)=\sum\limits_{n=1}^{\infty}h_{n,r}t^n$ be the generating function for $r-$sum Fibonacci numbers, then from $h_{n,r}=h_{n-1,r}+h_{n-2,r}$, we have

$$\sum_{n\geq 3}h_{n,r}t^n = \sum_{n\geq 3}h_{n-1,r}t^n+\sum_{n\geq 3}h_{n-2,r}t^n.$$

This is the same as

$$\sum_{n\geq 1}h_{n,r}t^n-h_{2,r}t^2-h_{1,r}t = t\sum_{n\geq 2}h_{n,r}t^n+t^2\sum_{n\geq 1}h_{n,r}t^n$$

or

$$\sum_{n\geq 1}h_{n,r}t^n-h_{2,r}t^2-h_{1,r}t = t\left(\sum_{n\geq 1}h_{n,r}t^n-h_{1,r}t\right)+t^2\sum_{n\geq 1}h_{n,r}t^n.$$

Substituting $H_r(t) = \sum_{n=1}^{\infty} h_{n,r} t^n$ we get,

$$H_r(t) - h_{2,r}t^2 - h_{1,r}t = t(H_r(t) - h_{1,r}t) + t^2 H_r(t).$$

Thus,

$$H_r(t) = \frac{h_{1,r}t + t^2(h_{2,r} - h_{1,r})}{1 - t - t^2}.$$

□

4. Properties of *r*-Sum Fibonacci Numbers

In this section, we obtain further properties of $r-$sum Fibonacci numbers.

Proposition 4.1 (Sum of first n terms)**.** *The sum of first n terms of r-sum Fibonacci numbers is given by* $h_{n+2,r} - h_{2,r}$.

Proof. By Binet's formula (3.1), we have

$$\begin{aligned}
\sum_{k=1}^{n} h_{k,r} &= \frac{1}{\alpha-\beta}[(h_{2,r} - \beta h_{1,r})\alpha^0 - (h_{2,r} - \alpha h_{1,r})\beta^0 + (h_{2,r} - \beta h_{1,r})\alpha^1 - (h_{2,r} - \alpha h_{1,r})\beta^1 + \cdots + (h_{2,r} - \beta h_{1,r})\alpha^{n-1} \\
&\quad - (h_{2,r} - \alpha h_{1,r})\beta^{n-1}] \\
&= \frac{1}{\alpha-\beta}\left[(h_{2,r} - \beta h_{1,r})\left(1 + \alpha + \cdots + \alpha^{n-1}\right) - (h_{2,r} - \alpha h_{1,r})\left(1 + \beta + \cdots + \beta^{n-1}\right)\right] \\
&= \frac{1}{\alpha-\beta}\left[(h_{2,r} - \beta h_{1,r})\frac{\alpha^n - 1}{\alpha - 1} - (h_{2,r} - \alpha h_{1,r})\frac{\beta^n - 1}{\beta - 1}\right].
\end{aligned}$$

Since $\alpha - 1 = -\beta$ and $\beta - 1 = -\alpha$, we have

$$\sum_{k=1}^{n} h_{k,r} = \frac{1}{\alpha-\beta}\left[\frac{(h_{2,r} - \beta h_{1,r})(\alpha^n - 1)\alpha - (h_{2,r} - \alpha h_{1,r})(\beta^n - 1)\beta}{-\alpha\beta}\right].$$

Since $-\alpha\beta = 1$, we get

$$\begin{aligned}
\sum_{k=1}^{n} h_{k,r} &= \frac{1}{\alpha-\beta}[(h_{2,r} - \beta h_{1,r})(\alpha^{n+1} - \alpha) - (h_{2,r} - \alpha h_{1,r})(\beta^{n+1} - \beta)] \\
&= \frac{(h_{2,r} - \beta h_{1,r})\alpha^{n+1} - (h_{2,r} - \alpha h_{1,r})\beta^{n+1}}{\alpha-\beta} - \frac{(h_{2,r} - \beta h_{1,r})\alpha - (h_{2,r} - \alpha h_{1,r})\beta}{\alpha-\beta} \\
&= h_{n+2,r} - h_{2,r}.
\end{aligned}$$

□

Proposition 4.2 (Sum of first n terms with odd indices)**.** *The sum of the first n terms with odd indices of* $r-$*sum Fibonacci numbers is given by* $h_{2n,r} - h_{2,r} + h_{1,r}$.

Proof. By Binet's formula (3.1), we have

$$\begin{aligned}
\sum_{k=0}^{n-1} h_{2k+1,r} &= \frac{1}{\alpha-\beta}[(h_{2,r} - \beta h_{1,r})\alpha^0 - (h_{2,r} - \alpha h_{1,r})\beta^0 + (h_{2,r} - \beta h_{1,r})\alpha^2 - (h_{2,r} - \alpha h_{1,r})\beta^2 \\
&\quad + \cdots + (h_{2,r} - \beta h_{1,r})\alpha^{2n-2} - (h_{2,r} - \alpha h_{1,r})\beta^{2n-2}] \\
&= \frac{1}{\alpha-\beta}[(h_{2,r} - \beta h_{1,r})(1 + \alpha^2 + \cdots + \alpha^{2n-2}) - (h_{2,r} - \alpha h_{1,r})(1 + \beta^2 + \cdots + \beta^{2n-2})] \\
&= \frac{1}{\alpha-\beta}\left[\frac{(h_{2,r} - \beta h_{1,r})(\alpha^{2n} - 1)}{\alpha^2 - 1} - \frac{(h_{2,r} - \alpha h_{1,r})(\beta^{2n} - 1)}{\beta^2 - 1}\right].
\end{aligned}$$

Since $\alpha^2-1=\alpha$ and $\beta^2-1=\beta$, we have

$$\begin{aligned}\sum_{k=0}^{n-1} h_{2k+1} &= \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})(\alpha^{2n-1}-\alpha^{-1})-(h_{2,r}-\alpha h_{1,r})(\beta^{2n-1}-\beta^{-1})\right]\\ &= \frac{(h_{2,r}-\beta h_{1,r})\alpha^{2n-1}-(h_{2,r}-\alpha h_{1,r})\beta^{2n-1}}{\alpha-\beta}-\frac{(h_{2,r}-\beta h_{1,r})\alpha^{-1}-(h_{2,r}-\alpha h_{1,r})\beta^{-1}}{\alpha-\beta}\\ &= h_{2n,r}-h_{0,r}\\ &= h_{2n,r}-h_{2,r}+h_{1,r}.\end{aligned}$$

‘

□

Proposition 4.3 (Sum of first n terms with even indices)**.** *The sum of the first n terms with even indices of r−sum Fibonacci numbers is given by* $h_{2n+1,r}-h_{1,r}$.

Proof. By Binet's formula (3.1), we have

$$\begin{aligned}\sum_{k=1}^{n} h_{2k,r} &= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})\alpha^1-(h_{2,r}-\alpha h_{1,r})\beta^1+(h_{2,r}-\beta h_{1,r})\alpha^3-(h_{2,r}-\alpha h_{1,r})\beta^3+\cdots+(h_{2,r}-\beta h_{1,r})\alpha^{2n-1}\\ &\quad-(h_{2,r}-\alpha h_{1,r})\beta^{2n-1}]\\ &= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})(\alpha+\alpha^3+\cdots+\alpha^{2n-1})-(h_{2,r}-\alpha h_{1,r})(\beta+\beta^3+\cdots+\beta^{2n-1})]\\ &= \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\frac{\alpha(\alpha^{2n}-1)}{\alpha^2-1}-(h_{2,r}-\alpha h_{1,r})\frac{\beta(\beta^{2n}-1)}{\beta^2-1}\right].\end{aligned}$$

Since $\alpha^2-1=\alpha$ and $\beta^2-1=\beta$, we get

$$\begin{aligned}\sum_{k=1}^{n} h_{2k,r} &= \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})(\alpha^{2n}-1)-(h_{2,r}-\alpha h_{1,r})(\beta^{2n}-1)\right]\\ &= \frac{(h_{2,r}-\beta h_{1,r})\alpha^{2n}-(h_{2,r}-\alpha h_{1,r})\beta^{2n}}{\alpha-\beta}-\frac{(h_{2,r}-\beta h_{1,r})-(h_{2,r}-\alpha h_{1,r})}{\alpha-\beta}\\ &= h_{2n+1,r}-h_{1,r}.\end{aligned}$$

□

Proposition 4.4. *For every positive integer n,*

$$h_{1,r}+h_{4,r}+h_{7,r}+\cdots+h_{3n-2,r}=\frac{1}{2}(h_{3n,r}-h_{2,r}+h_{1,r}).$$

Proof. By Binet's formula (3.1), we get

$$\begin{aligned}\sum_{k=1}^{n} h_{3k-2,r} &= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})\alpha^0-(h_{2,r}-\alpha h_{1,r})\beta^0+(h_{2,r}-\beta h_{1,r})\alpha^3-(h_{2,r}-\alpha h_{1,r})\beta^3\\ &\quad+\cdots+(h_{2,r}-\beta h_{1,r})\alpha^{3n-3}-(h_{2,r}-\alpha h_{1,r})\beta^{3n-3}]\\ &= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})(1+\alpha^3+\cdots+\alpha^{3n-3})-(h_{2,r}-\alpha h_{1,r})(1+\beta^3+\cdots+\beta^{3n-3})]\\ &= \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\frac{\alpha^{3n}-1}{\alpha^3-1}-(h_{2,r}-\alpha h_0)\frac{\beta^{3n}-1}{\beta^3-1}\right].\end{aligned}$$

Since $\alpha^3 - 1 = 2\alpha$ and $\beta^3 - 1 = 2\beta$, the above equation simplifies to

$$\begin{aligned}
\sum_{k=1}^{n} h_{3k-2,r} &= \frac{1}{2(\alpha-\beta)}\left[(h_{2,r}-\beta h_{1,r})(\alpha^{3n-1}-\alpha^{-1})-(h_{2,r}-\alpha h_{1,r})(\beta^{3n-1}-\beta^{-1})\right] \\
&= \frac{1}{2}\left[\frac{(h_{2,r}-\beta h_{1,r})\alpha^{3n-1}-(h_{2,r}-\alpha h_{1,r})\beta^{3n-1}}{\alpha-\beta}-\frac{(h_{2,r}-\beta h_{1,r})\alpha^{-1}-(h_{2,r}-\alpha h_{1,r})\beta^{-1}}{\alpha-\beta}\right] \\
&= \frac{1}{2}(h_{3n,r}-h_{0,r}) \\
&= \frac{1}{2}(h_{3n,r}-h_{2,r}+h_{1,r}).
\end{aligned}$$

□

Proposition 4.5. *For every positive integer n,*

$$h_{2,r}+h_{5,r}+h_{8,r}+\cdots+h_{3n-1,r} = \frac{1}{2}(h_{3n+1,r}-h_{1,r}).$$

Proof. By Binet's formula (3.1), we have

$$\begin{aligned}
\sum_{k=1}^{n} h_{3k-1,r} &= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})\alpha-(h_{2,r}-\alpha h_{1,r})\beta+(h_{2,r}-\beta h_{1,r})\alpha^4-(h_{2,r}-\alpha h_{1,r})\beta^4 \\
&\quad +\cdots+(h_{2,r}-\beta h_{1,r})\alpha^{3n-2}-(h_{2,r}-\alpha h_{1,r})\beta^{3n-2}] \\
&= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})(\alpha+\alpha^4+\cdots+\alpha^{3n-2})-(h_{2,r}-\alpha h_{1,r})(\beta+\beta^4+\cdots+\beta^{3n-2})] \\
&= \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\frac{\alpha^{3n+1}-\alpha}{\alpha^3-1}-(h_{2,r}-\alpha h_{1,r})\frac{\beta^{3n+1}-\beta}{\beta^3-1}\right].
\end{aligned}$$

Since $\alpha^3 - 1 = 2\alpha$ and $\beta^3 - 1 = 2\beta$, then

$$\begin{aligned}
\sum_{k=1}^{n} h_{3k-1,r} &= \frac{1}{2(\alpha-\beta)}\left[(h_{2,r}-\beta h_{1,r})(\alpha^{3n}-1)-(h_{2,r}-\alpha h_{1,r})(\beta^{3n}-1)\right] \\
&= \frac{1}{2}\left[\frac{(h_{2,r}-\beta h_{1,r})\alpha^{3n}-(h_{2,r}-\alpha h_{1,r})\beta^{3n}}{\alpha-\beta}-\frac{(h_{2,r}-\beta h_{1,r})-(h_{2,r}-\alpha h_{1,r})}{\alpha-\beta}\right] \\
&= \frac{1}{2}(h_{3n+1,r}-h_{1,r}).
\end{aligned}$$

□

Proposition 4.6. *For every positive integer n,*

$$h_{3,r}+h_{6,r}+h_{9,r}+\cdots+h_{3n,r} = \frac{1}{2}(h_{3n+2,r}-h_{2,r}).$$

Proof. By Binet's formula (3.1), we obtain

$$\begin{aligned}
\sum_{k=1}^{n} h_{3k,r} &= \frac{1}{\alpha-\beta}[(h_{1,n}-\beta h_{1,r})\alpha^2-(h_{2,r}-\alpha h_{1,r})\beta^2+(h_{2,r}-\beta h_{1,r})\alpha^5-(h_{2,r}-\alpha h_{1,r})\beta^5+\cdots+(h_{2,r}-\beta h_{1,r})\alpha^{3n-1} \\
&\quad -(h_{2,r}-\alpha h_{1,r})\beta^{3n-1}] \\
&= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})(\alpha^2+\alpha^5+\cdots+\alpha^{3n-1})-(h_{2,r}-\alpha h_{1,r})(\beta^2+\beta^5+\cdots+\beta^{3n-1})] \\
&= \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\frac{\alpha^{3n+2}-\alpha^2}{\alpha^3-1}-(h_{2,r}-\alpha h_{1,r})\frac{\beta^{3n+2}-\beta^2}{\beta^3-1}\right]
\end{aligned}$$

Since $\alpha^3 - 1 = 2\alpha$ and $\beta^3 - 1 = 2\beta$, we get

$$\begin{aligned}\sum_{k=1}^{n} h_{3k,r} &= \frac{1}{2(\alpha-\beta)}\left[(h_{2,r}-\beta h_{1,r})(\alpha^{3n+1}-\alpha)-(h_{2,r}-\alpha h_{1,r})(\beta^{3n+1}-\beta)\right]\\ &= \frac{1}{2}\left[\frac{(h_{2,r}-\beta h_{1,r})\alpha^{3n+1}-(h_{2,r}-\alpha h_{1,r})\beta^{3n+1}}{\alpha-\beta}-\frac{(h_{2,r}-\beta h_{1,r})\alpha-(h_{2,r}-\alpha h_{1,r})\beta}{\alpha-\beta}\right]\\ &= \frac{1}{2}(h_{3n+2,r}-h_{2,r}).\end{aligned}$$

□

Proposition 4.7 (Alternating sum formula)**.** *For every positive integer n,*

$$\sum_{k=1}^{n}(-1)^{k+1}h_{k,r} = (-1)^{n-1}h_{n-1,r}+2h_{1,r}-h_{2,r}.$$

Proof. By Binet's formula (3.1), we get

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k+1}h_{k,r} &= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})-(h_{2,r}-\alpha h_{1,r})-((h_{2,r}-\beta h_{1,r})\alpha-(h_{2,r}-\alpha h_{1,r})\beta)\\ &\quad+\cdots+(-1)^{n+1}((h_{2,r}-\beta h_{1,r})\alpha^{n-1}-(h_{2,r}-\alpha h_{1,r})\beta^{n-1})]\\ &= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})(1-\alpha+\cdots+(-1)^{n+1}\alpha^{n-1})-((h_{2,r}-\alpha h_{1,r})(1-\beta+\cdots+(-1)^{n+1}\beta^{n-1})]\\ &= \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\frac{((-\alpha)^n-1)}{-\alpha-1}-(h_{2,r}-\alpha h_{1,r})\frac{((-\beta)^n-1)}{-\beta-1}\right].\end{aligned}$$

Since $-\alpha-1=-\alpha^2$ and $-\beta-1=-\beta^2$, we have

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k+1}h_{k,r} &= \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\frac{(-1)^n\alpha^n-1}{-\alpha^2}-(h_{2,r}-\alpha h_{1,r})\frac{(-1)^n\beta^n-1}{-\beta^2}\right]\\ &= (-1)^{n-1}\frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\alpha^{n-2}-(h_{2,r}-\alpha h_{1,r})\beta^{n-2}\right]\\ &\quad+\frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\alpha^{-2}-(h_{2,r}-\alpha h_{1,r})\beta^{-2}\right]\\ &= (-1)^{n-1}h_{n-1,r}+\frac{1}{\alpha-\beta}\left[\frac{(h_{2,r}-\beta h_{1,r})}{\alpha^2}-\frac{(h_{2,r}-\alpha h_{1,r})}{\beta^2}\right].\end{aligned}$$

This gives,

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k+1}h_{k,r} &= (-1)^{n-1}h_{n-1,r}+\frac{1}{(\alpha\beta)^2(\alpha-\beta)}\left[(h_{2,r}-\beta h_{1,r})\beta^2-(h_{2,r}-\alpha h_{1,r})\alpha^2\right]\\ &= (-1)^{n-1}h_{n-1,r}+\left[h_{1,r}\left(\frac{\alpha^3-\beta^3}{\alpha-\beta}\right)-h_{2,r}\left(\frac{\alpha^2-\beta^2}{\alpha-\beta}\right)\right]\\ &= (-1)^{n-1}h_{n-1,r}+\left[h_{1,r}\left(\frac{2(\alpha-\beta)}{\alpha-\beta}\right)-h_{2,r}\left(\frac{\alpha^2-\beta^2}{\alpha-\beta}\right)\right].\end{aligned}$$

Since $\alpha-\beta=\sqrt{5}$ and $\alpha^2-\beta^2=\sqrt{5}$, then

$$\sum_{k=1}^{n}(-1)^{k+1}h_{k,r} = (-1)^{n-1}h_{n-1,r}+2h_{1,r}-h_{2,r}.$$

□

Proposition 4.8. *For every positive integer n,*

$$h_{2n,r}=\sum_{k=0}^{n}\binom{n}{k}h_{k,r}.$$

Proof. By Binet's formula (3.1), we get

$$h_{2n,r} = \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})\alpha^{2n-1}-(h_{2,r}-\alpha h_{1,r})\beta^{2n-1}]$$
$$= \frac{1}{\alpha-\beta}[(h_{2,r}-\beta h_{1,r})\frac{\alpha^{2n}}{\alpha}-(h_{2,r}-\alpha h_{1,r})\frac{\beta^{2n}}{\beta}].$$

Since $\alpha^2 = 1+\alpha$ and $\beta^2 = 1+\beta$, then

$$h_{2n,r} = \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\frac{(1+\alpha)^n}{\alpha}-(h_{2,r}-\alpha h_{1,r})\frac{(1+\beta)^n}{\beta}\right].$$

Since $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, we have

$$h_{2n,r} = \frac{1}{\alpha-\beta}\left[(h_{2,r}-\beta h_{1,r})\sum_{k=0}^{n}\binom{n}{k}\alpha^{k-1}-(h_{2,r}-\alpha h_{1,r})\sum_{k=0}^{n}\binom{n}{k}\beta^{k-1}\right]$$
$$= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{(h_{2,r}-\beta h_{o,r})\alpha^{k-1}-(h_{2,r}-\alpha h_{1,r})\beta^{k-1}}{\alpha-\beta}\right]$$
$$= \sum_{k=0}^{n}\binom{n}{k}h_{k,r}.$$

□

Proposition 4.9 (Explicit sum formula)**.** *For every positive integer n,*

$$h_{n,r} = h_{1,r}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-k-1}{k}+(h_{2,r}-h_{1,r})\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-k-2}{k}, \tag{4.1}$$

where $\lfloor n\rfloor$ is the greatest integer less than or equal to n.

Proof. By generating function (3.3), we have

$$\sum_{n=1}^{\infty}h_{n,r}t^n = \frac{h_{1,r}t+t^2(h_{2,r}-h_{1,r})}{1-t-t^2}$$
$$= t\,[h_{1,r}+t(h_{2,r}-h_{1,r})](1-t-t^2)^{-1}$$
$$= t\,[h_{1,r}+t(h_{2,r}-h_{1,r})][1-(t+t^2)]^{-1}$$
$$= t\,[h_{1,r}+t(h_{2,r}-h_{1,r})]\sum_{n=1}^{\infty}t^{n-1}(1+t)^{n-1}$$
$$= [h_{1,r}+t(h_{2,r}-h_{1,r})]\sum_{n=1}^{\infty}t^n\sum_{k=0}^{n-1}\binom{n-1}{k}t^k$$
$$= [h_{1,r}+t(h_{2,r}-h_{1,r})]\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-k-1)!}t^{n+k}.$$

Replacing n by $n+k+1$, we get

$$\sum_{n=1}^{\infty}h_{n,r}t^n = [h_{1,r}+t(h_{2,r}-h_{1,r})]\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(n+k)!}{k!n!}t^{n+2k+1}.$$

Now, replacing n by $n-2k-1$, we have

$$\sum_{n=1}^{\infty}h_{n,r}t^n = [h_{1,r}+t(h_{2,r}-h_{1,r})]\sum_{n=1}^{\infty}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(n-k-1)!}{k!(n-2k-1)!}t^n$$
$$= h_{1,r}\left[\sum_{n=1}^{\infty}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(n-k-1)!}{k!(n-2k-1)!}\right]t^n+(h_{2,r}-h_{1,r})\sum_{n=1}^{\infty}\left[\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(n-k-1)!}{k!(n-2k-1)!}\right]t^{n+1}.$$

Equating the coefficients of t^n, we obtain

$$h_{n,r} = h_{1,r} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} + (h_{2,r} - h_{1,r}) \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-k-2}{k}.$$

Hence the proof follows. □

Proposition 4.10. *For every positive integer n,*

$$h_{-n,r} = (-1)^n (h_{2,r} f_{n+2} - h_{1,r} f_{n+3}).$$

Proof. By Binet's formula (3.1), we have

$$\begin{aligned} h_{-n,r} &= \frac{1}{\alpha-\beta}\left[(h_{2,r} - \beta h_{1,r})\alpha^{-n-1} - (h_{2,r} - \alpha h_{1,r})\beta^{-n-1}\right] \\ &= \frac{1}{\alpha-\beta}\left[(h_{2,r} - \beta h_{1,r})\frac{1}{\alpha^{n+1}} - (h_{2,r} - \alpha h_{1,r})\frac{1}{\beta^{n+1}}\right]. \end{aligned}$$

Since $\frac{1}{\alpha} = -\beta$ and $\frac{1}{\beta} = -\alpha$, we have

$$\begin{aligned} h_{-n,r} &= \frac{1}{\alpha-\beta}\left[(h_{2,r} - \beta h_{1,r})(-1)^{n+1}\beta^{n+1} - (h_{2,r} - \alpha h_{1,r})(-1)^{n+1}\alpha^{n+1}\right] \\ &= \frac{(-1)^{n+1}}{\alpha-\beta}\left[(h_{2,r} - \beta h_{1,r})\beta^{n+1} - (h_{2,r} - \alpha h_{1,r})\alpha^{n+1}\right] \\ &= \frac{(-1)^{n+1}}{\alpha-\beta}\left[h_{2,r}\beta^{n+1} - h_{1,r}\beta^{n+2} - h_{2,r}\alpha^{n+1} + h_{1,r}\alpha^{n+2}\right] \\ &= \frac{(-1)^{n+2}}{\alpha-\beta}\left[h_{2,r}(\alpha^{n+1} - \beta^{n+1}) - h_{1,r}(\alpha^{n+2} - \beta^{n+2})\right] \\ &= (-1)^{n+2}\left[\frac{h_{2,r}(\alpha^{n+1} - \beta^{n+1})}{\alpha-\beta} - \frac{h_{1,r}(\alpha^{n+2} - \beta^{n+2})}{\alpha-\beta}\right] \\ &= (-1)^{n+2}(h_{2,r} f_{n+2} - h_{1,r} f_{n+3}) \\ &= (-1)^{n}(h_{2,r} f_{n+2} - h_{1,r} f_{n+3}). \end{aligned}$$

□

Proposition 4.11 (Honsberger's identity)**.** *If $n > m$ then*

$$h_{n+m,r} = h_{n,r} f_m + h_{n+1,r} f_{m+1},$$

for all $m \geq 0$ and $n > 0$.

Proof. Since by Corollary 3.2, we have

$$h_{n+m,r} = h_{2,r} f_{n+m} + h_{1,r} f_{n+m-1},$$

then by Honsberger's identity of Fibonacci numbers (1.4), we get

$$\begin{aligned} h_{n+m,r} &= h_{2,r}(f_n f_m + f_{n+1} f_{m+1}) + h_{1,r}(f_{n-1} f_m + f_n f_{m+1}) \\ &= f_m(h_{2,r} f_n + h_{1,r} f_{n-1}) + f_{m+1}(h_{2,r} f_{n+1} + h_{1,r} f_n). \end{aligned}$$

Applying Corollary 3.2 again we have, $h_{n+m,r} = h_{n,r} f_m + h_{n+1,r} f_{m+1}$. □

Corollary 4.12. *We have:*

1. $h_{2n,r} = h_{n,r} f_n + h_{n+1,r} f_{n+1}$,

2. $h_{2n-1,r} = h_{n,r}f_{n-1} + h_{n+1,r}f_n$,
3. $h_{2n-2,r} = h_{n,r}f_{n-2} + h_{n+1}f_{n-1}$,
4. $h_{2n-k,r} = h_{n,r}f_{n-k} + h_{n+1,r} + f_{n-k+1}$.

Proof. The results follow from Proposition 4.11 upon setting $m = n$, $m = n-1$, $m = n-2$, and $m = n-k$ in that order. □

Proposition 4.13. *For every $n \geq 2$, we have*

$$h_{2,r}h_{3,r} + h_{3,r}h_{4,r} + \cdots + h_{2n-1,r}h_{2n,r} = h_{2n,r}^2 - h_{2,r}^2.$$

Proof. We induct on n. For base case, $n = 2$:
The left hand side gives

$$h_{2,r}h_{3,r} + h_{3,r}h_{4,r} = h_{3,r}(h_{2,r} + h_{4,r})$$

while the right hand side gives

$$h_{4,r}^2 - h_{2,r}^2 = (h_{4,r} - h_{2,r})(h_{4,r} + h_{2,r}) = h_{3,r}(h_{4,r} + h_{2,r}).$$

Since the left hand side equals to the right hand side, the base case holds.
For the induction step, we will assume the formula holds true for n and prove that it holds true for $n+1$.
Since by inductive hypothesis

$$h_{2,r}h_{3,r} + h_{3,r}h_{4,r} + \cdots + h_{2n-1,r}h_{2n,r} = h_{2n,r}^2 - h_{2,r}^2,$$

then

$$\begin{aligned}
h_{2,r}h_{3,r} + h_{3,r}h_{4,r} + \cdots + h_{2n-1,r}h_{2n,r} + h_{2n}h_{2n+1,r} + h_{2n+1,r}h_{2n+2,r} &= h_{2n,r}^2 - h_{2,r}^2 + h_{2n,r}h_{2n+1,r} + h_{2n+1,r}h_{2n+2,r} \\
&= h_{2n,r}^2 + h_{2n,r}h_{2n+1,r} - h_{2,r}^2 + h_{2n+1,r}h_{2n+2,r} \\
&= h_{2n,r}(h_{2n,r} + h_{2n+1,r}) + h_{2n+1,r}h_{2n+2,r} - h_{2,r}^2 \\
&= h_{2n,r}h_{2n+2,r} + h_{2n+1,r}h_{2n+2,r} - h_{2,r}^2 \\
&= h_{2n+2,r}(h_{2n,r} + h_{2n+1,r}) - h_{2,r}^2 \\
&= h_{2n+2,r}^2 - h_{2,r}^2.
\end{aligned}$$

By the principle of mathematical induction, the result follows. □

Lemma 4.14. *The n^{th} Fibonacci number, f_n, is given by*

$$f_n = \frac{h_{2,r}h_{n,r} - h_{1,r}h_{n+1,r}}{h_{2,r}^2 - h_{1,r}h_{3,r}}.$$

Proof. We have, by Binet's formula (3.1), that

$$\begin{aligned}
h_{2,r}h_{n,r} - h_{1,r}h_{n+1,r} &= h_{2,r}\left[\frac{(h_{2,r} - \beta h_{1,r})\alpha^{n-1}}{\alpha - \beta} + \frac{(\alpha h_{1,r} - h_{2,r})\beta^{n-1}}{\alpha - \beta}\right] - h_{1,r}\left[\frac{(h_{2,r} - \beta h_{1,r})\alpha^n}{\alpha - \beta} + \frac{(\alpha h_{1,r} - h_{2,r})\beta^n}{\alpha - \beta}\right] \\
&= h_{2,r}\left[h_{2,r}\frac{(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} + h_{1,r}\frac{(\alpha^{n-2} - \beta^{n-2})}{\alpha - \beta}\right] - h_{1,r}\left[h_{2,r}\frac{(\alpha^n - \beta^n)}{\alpha - \beta} + h_{1,r}\frac{(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}\right] \\
&= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\left[h_{2,r}^2 - h_{1,r}^2\right] + \frac{h_{1,r}h_{2,r}}{\alpha - \beta}\left[\alpha^{n-2} - \beta^{n-2} - \alpha^n + \beta^n\right] \\
&= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\left[h_{2,r}^2 - h_{1,r}^2 - h_{1,r}h_{2,r}\right].
\end{aligned}$$

Now,

$$f_n = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = \frac{h_{2,r}h_{n,r} - h_{1,r}h_{n+1,r}}{h_{2,r}^2 - h_{1,r}h_{3,r}}.$$

□

Theorem 4.15 (Generalized identity). *Let $h_{n,r}$ be the n^{th} term of $r-$sum Fibonacci sequence then*

$$h_{m,r}h_{n,r}-h_{m-k,r}h_{n+k,r}=\frac{(-1)^{m-k-1}}{h_{2,r}^2-h_{1,r}h_{3,r}}\left[(h_{2,r}h_{k+1,r}-h_{1,r}h_{k+2,r})(h_{2,r}h_{n-m+k+1,r}-h_{1,r}h_{n-m+k+2,r})\right], \tag{4.2}$$

where $n\geq m$ and $k\geq 1$.

Proof. By Binet's formula (3.1), we have

$$h_{n,r}=A\alpha^{n-1}+B\beta^{n-1}$$

where $A=\dfrac{h_{2,r}-\beta h_{1,r}}{\alpha-\beta}$, $B=\dfrac{\alpha h_{1,r}-h_{2,r}}{\alpha-\beta}$, $\alpha=\dfrac{1+\sqrt{5}}{2}$ and $\beta=\dfrac{1-\sqrt{5}}{2}$.

Now,

$$\begin{aligned}
h_{m,r}h_{n,r}-h_{m-k,r}h_{n+k,r}&=\left(A\alpha^{m-1}+B\beta^{m-1}\right)\left(A\alpha^{n-1}+B\beta^{n-1}\right)-\left(A\alpha^{m-k-1}+B\beta^{m-k-1}\right)\left(A\alpha^{n+k-1}+B\beta^{n+k-1}\right)\\
&=AB\left(\ \alpha^k-\beta^k\right)\left[\frac{\alpha^{m-1}\beta^{n-1}}{\alpha^k}-\frac{\alpha^{n-1}\beta^{m-1}}{\beta^k}\right]\\
&=AB(-1)^{-k}\left(\alpha^k-\beta^k\right)\left(\alpha^{m-1}\beta^{m-1}\right)\left(\beta^{n-m+k}-\alpha^{n-m+k}\right)\\
&=-AB(-1)^{m-k-1}\left(\alpha^k-\beta^k\right)\left(\alpha^{n-m+k}-\beta^{n-m+k}\right).
\end{aligned}$$

Since $-AB=\dfrac{h_{2,r}^2-h_{1,r}h_{3,r}}{(\alpha-\beta)^2}$, then

$$\begin{aligned}
h_{m,r}h_{n,r}-h_{m-k,r}h_{n+k,r}&=\frac{h_{2,r}^2-h_{1,r}h_{3,r}}{(\alpha-\beta)^2}(-1)^{m-k-1}\left[(\alpha^k-\beta^k)(\alpha^{n-m+k}-\beta^{n-m+k})\right]\\
&=\left(h_{2,r}^2-h_{1,r}h_{3,r}\right)(-1)^{m-k-1}\left[\frac{\alpha^k-\beta^k}{\alpha-\beta}\left(\frac{\alpha^{n-m+k}-\beta^{n-m+k}}{\alpha-\beta}\right)\right].
\end{aligned}$$

By Lemma 4.14, we have

$$f_{k+1}=\frac{\alpha^k-\beta^k}{\alpha-\beta}=\frac{h_{2,r}h_{k+1,r}-h_{1,r}h_{k+2,r}}{h_{2,r}^2-h_{1,r}h_{3,r}}$$

and

$$f_{n-m+k+1}=\frac{\alpha^{n-m+k}-\beta^{n-m+k}}{\alpha-\beta}=\frac{h_{2,r}h_{n-m+k+1,r}-h_{1,r}h_{n-m+k+2,r}}{h_{2,r}^2-h_{1,r}h_{3,r}}.$$

So

$$h_{m,r}h_{n,r}-h_{m-k,r}h_{n+k,r}=(-1)^{m-k-1}\left[\frac{\left(h_{2,r}h_{k+1,r}-h_{1,r}h_{k+2,r}\right)\left(h_{2,r}h_{n-m+k+1,r}-h_{1,r}h_{n-m+k+2}\right)}{h_{2,r}^2-h_{1,r}h_{3,r}}\right].$$

Hence the proof follows. □

Corollary 4.16 (Catalan's identity). *If we take $m=n$ in the generalized identity (4.2), we obtain*

$$h_{n,r}^2-h_{n-k,r}h_{n+k,r}=\frac{(-1)^{n-k-1}}{h_{2,r}^2-h_{1,r}h_{3,r}}\left[h_{2,r}h_{k+1,r}-h_{1,r}h_{k+2,r}\right]^2,$$

for all $n>k\geq 1$.

Corollary 4.17 (Cassini's identity). *If $m=n$ and $k=1$ in the generalized identity (4.2), then*

$$h_{n,r}^2-h_{n-1,r}h_{n+1,r}=(-1)^{n-2}\left(h_{2,r}^2-h_{1,r}h_{3,r}\right),$$

for all $n\geq 1$.

Corollary 4.18 (d'Ocagne's identity). *If $n=m$, $m=n+1$ and $k=1$ in the generalized identity (4.2), then*

$$h_{n+1,r}h_{m,r}-h_{n,r}h_{m+1,r}=(-1)^{n-1}[h_{2,r}h_{m-n+1,r}-h_{1,r}h_{m-n+2,r}],$$

where $m>n\geq 0$.

5. Determinant Identities

Determinants play a significant role in various areas in mathematics. For instance, they are quite useful in analysis and solution of systems of linear equations. T. Koshy [6] devoted two chapters of his book to the use of matrices and determinants in Fibonacci numbers. In this section, we obtain further properties of $r-$sum Fibonacci numbers involving determinants.

Proposition 5.1. *For every positive integer n,*

$$\begin{vmatrix} h_{n+1,r} & h_{n+2,r} & h_{n+3,r} \\ h_{n+4,r} & h_{n+5,r} & h_{n+6,r} \\ h_{n+7,r} & h_{n+8,r} & h_{n+9,r} \end{vmatrix} = 0.$$

Proof. Applying column reduction $C_1 \longrightarrow C_1 + C_2$ to the matrix, i.e., replace the entries of column 1 with the sum of the entries of columns 1 and 2, we get that two columns are identical and hence the determinant of the matrix is zero. □

Proposition 5.2. *For every positive integer n,*

$$\begin{vmatrix} h_{n,r}+h_{n+1,r} & h_{n+1,r}+h_{n+2,r} & h_{n+2,r}+h_{n,r} \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Proof. Applying $R_1 \longrightarrow R_1 + R_2$, we get that the determinant of the matrix is

$$\begin{vmatrix} 2h_{n+2,r} & 2h_{n+2,r} & 2h_{n+2,r} \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ 1 & 1 & 1 \end{vmatrix} = 2h_{n+2,r} \begin{vmatrix} 1 & 1 & 1 \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ 1 & 1 & 1 \end{vmatrix}.$$

Since two rows are identical, the determinant is zero. □

Proposition 5.3. *Let n be a positive integer, then*

$$\begin{vmatrix} h_{n,r} & f_n & 1 \\ h_{n+1,r} & f_{n+1} & 1 \\ h_{n+2,r} & f_{n+2} & 1 \end{vmatrix} = f_n h_{n+1,r} - f_{n+1} h_{n,r}.$$

Proof. Applying $R_1 \longrightarrow R_2 - R_1$ and $R_2 \longrightarrow R_3 - R_2$, we get that

$$\begin{vmatrix} h_{n,r} & f_n & 1 \\ h_{n+1,r} & f_{n+1} & 1 \\ h_{n+2,r} & f_{n+2} & 1 \end{vmatrix} = \begin{vmatrix} h_{n+1,r}-h_{n,r} & f_{n+1}-f_n & 0 \\ h_{n,r} & f_n & 0 \\ h_{n+2,r} & f_{n+2} & 1 \end{vmatrix}.$$

The result is thus immediate. □

Proposition 5.4. *For every positive integer n,*

$$\begin{vmatrix} h_{n,r} & l_n & 1 \\ h_{n+1,r} & l_{n+1} & 1 \\ h_{n+2,r} & l_{n+2} & 1 \end{vmatrix} = l_n h_{n+1,r} - l_{n+1} h_{n,r}.$$

Proof. The proof follows as in the proof of Proposition 5.3. □

Proposition 5.5. *For every positive integer n,*

$$\begin{vmatrix} 1+h_{n,r} & h_{n+1,r} & \cdots & h_{n+p,r} \\ h_{n,r} & 1+h_{n+1,r} & \cdots & h_{n+p,r} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n,r} & h_{n+1,r} & \cdots & 1+h_{n+p,r} \end{vmatrix} = 1 + h_{n,r} + h_{n+1,r} + \cdots + h_{n+p,r}.$$

Proof. The proof follows by induction on n and making use of column reductions. □

Proposition 5.6. *Let n be a positive integer, then*

$$\begin{vmatrix} h_{n,r} & h_{n+1,r} & h_{n+2,r} \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ h_{n+1,r} & h_{n+2,r} & h_{n,r} \end{vmatrix} = 2(h_{n,r}^3 + h_{n+1,r}^3).$$

Proof. We have

$$\begin{aligned} \begin{vmatrix} h_{n,r} & h_{n+1,r} & h_{n+2,r} \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ h_{n+1,r} & h_{n+2,r} & h_{n,r} \end{vmatrix} &= h_{n,r}(h_{n,r}^2 - h_{n+1,r}h_{n+2,r}) + h_{n+1,r}(h_{n+1,r}^2 - h_{n,r}h_{n+2,r}) + h_{n+2,r}(h_{n+2,r}^2 - h_{n,r}h_{n+1,r}) \\ &= h_{n,r}^3 + h_{n+1,r}^3 + h_{n+2,r}^3 - 3h_{n,r}h_{n+1,r}h_{n+2,r}. \end{aligned}$$

Substituting $h_{n+2,r} = h_{n,r} + h_{n+1,r}$ and expanding, we obtain the desired result. □

6. Conclusion

In this paper, we have derived Binet's formula (3.1) and generating function (3.3) for the $r-$sum Fibonacci sequence. Further, we have obtained explicit sum formula, sum of first n terms, sum of first n terms with even indices, sum of first n terms with odd indices, alternating sum of n terms of $r-$sum Fibonacci sequence, Honsberger's identity, determinant identities and generalized identity (4.2) from which Cassini's identity, Catalan's identity and d'Ocagne's identity are simple cases.

References

[1] S. Falcon, A. Plaza, *On the Fibonacci K-numbers,* Chaos Solution Fractals, **32**(5) (2007), 1615–1624.

[2] Y.K Gupta, M. Singh, O. Sikhwal, *Generalized Fibonacci-Like sequence associated with Fibonacci and Lucas sequences*, Turkish J. Anal. Number Theory, **2**(6) (2014), 233–238.

[3] A.F. Horadam, *A generalized Fibonacci sequence,* Amer. Math. Monthly, **68**(1961), 455–459.

[4] A.F. Horadam, *Basic properties of a certain generalized sequence of numbers,* Fib. Quart, **3**(3) (1965),161–176.

[5] D. Kalma, R. Mena, *The Fibonacci Numbers-Exposed,* Math. Mag., **2** (2002).

[6] T. Koshy, *Fibonacci and Lucas Numbers with Applications,* Wisley-Interscience Publications, New York, 2011.

[7] Y.K. Panwar, M. Singh, *Certain properties of generalized Fibonacci sequence,* Turkish J. Anal. Number Theory, **2**(1) (2014), 6–8.

[8] G.P.S Rathore, O. Sikhwal, R. Choudhary, *Generalized Fibonacci-like sequence and some identities,* SCIREA J. Math., **1**(1)(2016), 107–118.

[9] O. Sikhwal, Y. Vyas, *Generalized Fibonacci-type sequence and its Properties,* Int. J. Sci. Res., **5**(12) (2016), 2043–2047.

[10] B. Singh, S. Bhatnagar, *Fibonacci-like sequence and its properties,* Int. J. Contemp. Math. Sci., **5**(18) (2010), 859–868.

[11] B. Singh, S. Bhatnagar, O. Sikhwal, *Fibonacci-like sequence,* Int. J. Adv. Math. Sci., **1**(3)(2013), 145–151.

[12] M. Singh, Y. Gupta, O. Sikhwal, *Identities of generalized Fibonacci-like sequence,* Turkish J. Anal. Number Theory, **2**(5) (2014), 170–175.

[13] B. Singh, O. Sikhwal, Y. K Gupta, *Generalized Fibonacci-Lucas sequence,* Turkish J. Anal. Number Theory, **2**(6)(2014), 193–197.

[14] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences (OEIS),* Available at `http://oeis.org`.

[15] A. Wani, G. P. S. Rathore, K. Sisodiya, *On the properties of Fibonacci-Like sequence,* Int. J. Math. Trends Tech., **29**(2) (2016), 80–86.

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The Existence of Positive Solutions of Singular Initial-Value Problem for Second Order Differential Equations-2

Afgan Aslanov [1]*

Abstract
We consider the singular initial value problem for the second order differential equation. We interested in the existence of positive solutions and proved an easily applicable theorem on the existence of positive solutions to initial-value problems for second-order nonlinear singular differential equations. The previously established result on the existence of positive solution by Agarwal and O'Regan is not easy for the applications due to very complex definition of a new function and their properties in the statement of their theorem. We used the Lebesgue dominated convergence theorem and the Schauder–Tychonoff theorem in the proof of the main result. The main result can be easily applied for the singular and regular type of problems.

Keywords: Second order equations, Existence, Lane-Emden equation, Emden-Fowler equation, Fixed points
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[1] *Computer Engineering Department, Istanbul Esenyurt University, Istanbul, Turkey*
***Corresponding author**: afganaslanov@yahoo.com
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1. Introduction

In the mathematical literature there are many existence results on singular differential equations [3–8]. We are mainly motivated by [1] and the references therein.

We consider the problem

$$\begin{aligned} (py')' + pqg(y) &= 0, t \in [0,T], \\ y(0) &= a > 0, \\ \lim_{t\to 0+} p(t)y'(t) &= 0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} (py')' + pqg(y) &= 0, t \in [0,T], \\ y(0) &= a > 0, \\ y'(0) &= 0, \end{aligned} \tag{1.2}$$

where $0 < T < \infty$, $p \geq 0$, $q \geq 0$ and $g : [0,\infty) \to [0,\infty)$.

Agarwal and O'Regan [1] proved the next existence theorem for the positive solution of the problem (1.1) and (1.2).

Theorem 1.1. *[1]Suppose the following conditions are satisfied*

$$p \in C[0,T) \cap C^1(0,T) \ \text{with } p > 0 \text{ on } (0,T), \tag{1.3}$$

$$q \in L_p^1[0,t^*] \ \text{for any } t^* \in (0,T) \text{ with } q > 0 \text{ on } (0,T). \tag{1.4}$$

Here $L_r^1[0,a]$ *is the space of functions* $u(t)$ *with* $\int_0^a |u(t)|\, r(t)dt < \infty$,

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)dxds < \infty \ \text{for any } t^* \in (0,T), \tag{1.5}$$

and

$$g : [0,\infty) \to [0,\infty) \text{ is continuous, nondecreasing on } [0,\infty) \text{ and } g(u) > 0 \text{ for } u > 0.$$

Let

$$H(z) = \int_z^a \frac{dx}{g(x)} \ \text{for } 0 < z \le a$$

and

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)\tau(x)dxds < a \ \text{for any } t^* \in (0,T). \tag{1.6}$$

Here

$$\tau(x) = g\left(H^{-1}\left(\int_0^x \frac{1}{p(w)} \int_0^w p(z)q(z)dzdw\right)\right).$$

Then equation (1) has a solution $y \in C[0,T)$ *with* $py' \in C[0,T)$, $(py')' \in L_{pq}^1(0,T)$ *and* $0 < y(t) \le a$ *for* $t \in [0,T)$. *In addition, if either*

$$p(0) \neq 0$$

or

$$p(0) = 0 \text{ and } \lim_{t \to 0+} \frac{p(t)q(t)}{p'(t)} = 0$$

holds, then y *is a solution of* (1.2).

The condition (1.6) in connection with the definition of the function $\tau(x)$, makes this theorem difficult for application. In [2], we proved the theorem:

Theorem 1.2. *Suppose* (1.3)-(1.5) *hold. In addition, we assume*

$$\begin{aligned} \int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)g(a-\varphi(x))dxds &\le a - \varphi(x), \\ \int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)g(\varphi(x))dxds &\ge \varphi(x) \end{aligned}$$

for some $\varphi(x) \in C[0,T]$, *with* $0 \le \varphi(x) \le a$. *Then* (1.1) *has a solution* $y \in C[0,T]$ *with* $py' \in C[0,T]$, $(py')' \in L_{pq}^1(0,T)$ *and* $0 < y(t) \le a$ *for* $t \in [0,T]$.

The main purpose of this paper is to establish more easily applicable theorem on the existence of positive solution of the problem (1.1).

2. Main Results

The following theorem is the main result of this article.

Theorem 2.1. *Suppose the following conditions are satisfied*

$$\begin{array}{rcl} p & \in & C[0,T)\cap C^1(0,T) \text{ with } p>0 \text{ on } (0,T], \\ q & \geq & 0, \end{array}$$

$$\int_0^{t^*}\frac{1}{p(s)}\int_0^s p(x)q(x)dxds<\infty \text{ for any } t^*\in(0,T],$$

$$g:[0,\infty)\to[0,\infty) \text{ is continuous, nondecreasing on } [0,\infty),$$

and assume

$$\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(a-\varphi(t))dxds\leq\varphi(t),$$

for some $\varphi(t)\in C[0,T]$, *with* $0\leq\varphi(t)\leq a,\varphi(0)=a$. *Then equation (1) has a solution* $y\in C[0,T]$ *with* $py'\in C[0,T]$, $(py')'\in L^1_{pq}(0,T)$ *and* $0<y(t)\leq a$ *for* $t\in[0,T]$. *In addition, if either*

$$p(0)\neq 0$$

or

$$p(0)=0 \text{ and } \lim_{t\to 0+}\frac{p(t)q(t)}{p'(t)}=0$$

holds, then y is a solution of (1.2).

Proof. We construct a sequence of functions such that the subsequences of odd-numbered terms and even-numbered terms are convergent. By using the limits of these sequences we construct a new set and the operator in this set. Then we use Schauder-Tychonoff theorem to show that this operator has a fixed point.

Consider the sequence $\{y_n(t)\}, n=0,1,2,...$ with $y_0(t)\equiv\varphi(t)$,

$$y_n(t)=\varphi(t)-\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_{n-1}(x))dxds,\ n=1,2,...,\ t\leq T.$$

We have

$$\begin{array}{rcl} y_0(t) & \equiv & a-\varphi(t), \\ y_1(t) & = & a-\displaystyle\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_0(x))dxds\geq y_0(x), \end{array}$$

and in like manner

$$\begin{array}{rcl} y_2(t) & = & a-\displaystyle\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_1(x))dxds \\ & \leq & a-\displaystyle\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_0(x))dxds \\ & = & y_1(t), \\ y_3(t) & = & a-\displaystyle\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_2(x))dxds\geq y_2(t), \\ y_4(t) & = & a-\displaystyle\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_3(x))dxds\leq y_3(t),\ldots \\ y_{2n-1}(t) & \geq & y_{2n-2}(t), \\ y_{2n}(t) & \leq & y_{2n-1}(t),\ldots \end{array}$$

The sequences $\{y_{2n}(t)\}$ and $\{y_{2n+1}(t)\}$ are equicontinuous. Indeed, we have

$$|y_n(t)-y_n(r)| = \int_r^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_{n-1}(x))dxds \le M \int_r^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds, \tag{2.1}$$

where

$$M = \max\{g(u) : 0 \le u \le a\}$$

and the right hand side of (2.1) can be taken $< \varepsilon$ for $|t-r| < \delta$, regardless of the choice of t and r: the function $\int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds$ is (uniformly) continuous on $[0,T]$. It follows from Ascoli Arzela Theorem that the sequence $\{y_{2n}(t)\}$ has the (uniformly) convergent subsequence, $y_{2n_k}(t) \to u(t)$. The Lebesgue dominated theorem guarantees that

$$\begin{aligned}
y_{2n_k+1}(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_{2n_k}(x))dxds \to v(t), \\
v(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(u(x))dxds, \\
\text{and } u(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(v(x))dxds.
\end{aligned}$$

If $u(t) = v(t)$; we have that the function $u(t)$ is the solution of the problem (1.1). Indeed, it follows from

$$u(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(u(x))dxds$$

that

$$\begin{aligned}
u'(t) &= -\frac{1}{p(t)} \int_0^t p(x)q(x)g(u(x))dx, \\
pu' &= -\int_0^t p(x)q(x)g(u(x))dx, \\
\left(pu'\right)' &= -pqg(u).
\end{aligned}$$

So, we suppose $u(t) \ne v(t)$. We have $u(0) = v(0) = a$ and if for example, $u(t) > v(t)$ on the interval $(0,b)$, then we obtain

$$u(b) - v(b) = \int_0^b \frac{1}{p(s)} \int_0^s p(x)q(x)\left[g(u(x)) - g(v(x)\right] dxds > 0$$

and therefore $u(t) > v(t)$ on the whole interval $(0,T]$. The same holds for all points of intersections $t_0 : u(t_0) = v(t_0)$. That is if $u(t_0) = v(t_0)$, then for any $\varepsilon > 0$ there are infinitely many points $t_n \in [t_0, t_0+\varepsilon)$ such that $u(t_n) = v(t_n)$. Therefore, $u(t) > v(t)$ (or ¡) on $(t_0,T]$. Without loss of generality, let us suppose $u(t) > v(t)$ on $(0,T]$ and consider the operator $N : C[0,T] \to C[0,T]$ defined by

$$Ny(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y(x))dxds.$$

Next, let

$$K = \{y \in C[0,T] : v(t) \le y(t) \le u(t) \text{ for } t \in [0,T]\}.$$

The set K is closed, convex and bounded subset of $C[0,T]$ and clearly $N : K \to K$. Let us show that $N : K \to K$ is continuous and compact operator. Continuity follows from Lebesgue dominated convergence theorem: if $y_n(t) \to y(t)$, then $Ny_n(t) \to Ny(t)$. To show that N is completely continuous let $y(t) \in K$, then

$$|Ny(t) - Ny(r)| \le M \left| \int_r^t \frac{1}{p(x)} \int_0^x p(z)q(z)dzds \right| \text{ for } t,r \in [0,T],$$

that is N completely continuous on $[0,T]$. It follows from Schauder-Tychonoff theorem that N has a fixed point $w \in K$, i.e. w is a solution of (1.1). It follows from

$$w'(t) = -\frac{1}{p(t)} \int_0^t p(x)q(x)g(w(x))dx, \tag{2.2}$$

that if $p(0) \neq 0$; then $w'(0) = 0$. Now if $p(0) = 0$ but $\lim_{t\to 0+} \frac{p(t)q(t)}{p'(t)} = 0$ using L'Hôpital's rule we obtain from (8)

$$\begin{aligned} w'(0+) &= -\lim_{t\to 0+} \frac{\int_0^t p(x)q(x)g(w(x))dx}{p(t)} \\ &= -\lim_{t\to 0+} \frac{p(t)q(t)g(w(t))}{p'(t)} = 0, \end{aligned}$$

that is w is a solution of (1.2). The proof is complete. □

3. Conclusion

The theorem in [1] seems difficult for applications. The condition (1.6) very restrictive and decreases the sphere of applicability of the Theorem 1.1. We do not require the existence of the inverse H^{-1}. We proved an easily applicable theorem on the existence of positive solutions to initial-value problems for second-order nonlinear singular differential equations. The main result can be easily applied for the singular and regular type of problems.

Conflict of interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] R. P. Agarwal, D. O'Regan, *Second order initial value problems of Lane-Emden type*, Appl. Math. Lett., **20** (2007), 1198-1205.

[2] A. Aslanov, *A new theorem on the existence of positive solutions of singular initial-value problem for second order differential equations*, Commun. Adv. Math. Sci., **2**(1) (2019), 22-26.

[3] A. Constantin, *On the existence of positive solutions of second-order differential equations*, Ann. Mat. Pura Appl., **184**(2005), 131-138.

[4] T. Ertem, A. Zafer, *Asymptotic integration of second-order nonlinear differential equations via principal and nonprincipal solutions*, Appl. Math. Comput., **219** (2013), 5876–5886.

[5] T. Ertem, A. Zafer, *Existence of solutions for a class of nonlinear boundary value problems on half-line*, Bound. Value Probl., **2012**(2012), Article Number 43.

[6] T. Ertem, A. Zafer, *Monotone positive solutions for a class of second-order nonlinear differential equations*, . J. Comput. Appl. Math., **259**(2014), 672-681.

[7] Z. Yin, *Monotone positive solutions of second-order nonlinear differential equations*, Nonlinear Anal., **54** (2003), 391-403.

[8] Z. Zhao, *Positive Solutions of Nonlinear Second Order Ordinary Differential Equations*, Proc. Amer. Math. Soc., **121** (1994), 465-469.

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A Study on Some Multi-Valued Interpolative Contractions

Seher Sultan Yeşilkaya[1]*, Cafer Aydın[2] and Yaşar Aslan[3]

Abstract
In the present study, we introduce a new approach to interpolative mappings in fixed point theory by combining the ideas of Nadler [1], Karapınar et. al.[2, 3], Jleli and Samet [4]. We introduce some fixed point theorems for interpolative single and multi-valued Kannan type and Reich Rus Ćirić type θ-contractive mappings on complete metric spaces and prove some fixed point results for these mappings. These results extend the main results of many comparable results from the current literature. Also, we give an example to show that our main theorems are applicable.

Keywords: Fixed point, Metric spaces, Multi-valued contractive mapping
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[1] *Institute of Science and Technology, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, 46040, Turkey ORCID: 0000-0002-1748-2398*
[2] *Department of Mathematics, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, 46040, Turkey ORCID: 0000-0002-3707-5837*
[3] *Department of Mathematics, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, 46040, Turkey ORCID: 0000-0002-8827-2128*
***Corresponding author**: sultanseher20@gmail.com
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1. Introduction

Banach [5] introduced a famous fundamental fixed point theorem, also known as the Banach contraction principle. There are various extensions and generalizations of the Banach contraction principle in the literature see for example Kannan's [6], Reich [7] and see also Ćirić's [8].

In 1968, Kannan [6] proved a new fixed point theorem and considered the following contractive type:

$$d(F\eta, F\omega) \leq \lambda[d(\eta, F\eta) + d(\omega, F\omega)] \tag{1.1}$$

where $\lambda \in [0, \frac{1}{2})$. In [2], the notion of an interpolation Kannan type contractive was introduced. On the other hand, Reich, Rus and Ćirić [7, 9, 10, 11, 12, 13, 14] combined and improved both Banach and Kannan fixed point theorems. Recently, Karapınar et. al., [3] proved interpolative Reich Rus Ćirić type contractive mappings on partial metric spaces.

In 1969, using Pompeiu-Hausdorff metric, Nadler [1] introduced the notion of multi-valued contraction mapping and proved a multi-valued version of the well known Banach contraction principle. Denote by $P(X)$ the family of all nonempty subsets of X, $C(X)$ the family of all nonempty, closed subsets of X, $CB(X)$ the family of all nonempty, closed and bounded subsets of X and $K(X)$ the family of all nonempty compact subsets of X. It is clear that, $K(X) \subseteq CB(X) \subseteq C(X) \subseteq P(X)$. It is well known that, $H : CB(X) \times CB(X) \to \mathbb{R}$ is defined by, for every $F, G \in CB(X)$,

$$H(F,G) = \max\left\{\sup_{f\in F} d(f,G), \sup_{g\in G} d(g,F)\right\}$$

is a metric on $CB(X)$, which is called the Pompeiu–Hausdorff metric induced by d, where $D(f,G) = \inf\{d(f,g) : g \in G\}$ and $D(F,G) = \sup\{D(f,G) : f \in F\}$. Additionally, we will use the following lemma:

Lemma 1.1. *Let (X,d) metric spaces and F compact subsets of X. Afterwards, for $x \in X$, there exist $f \in F$, such that*

$$d(x,f) = D(x,F).$$

Lemma 1.2. *[1] Let F and G be nonempty closed and bounded subsets of a metric space. Therefore, for any $f \in F$,*

$$D(f,G) \le H(F,G).$$

Lately, Jleli and Samet [4] introduced a new type of contractions called θ-contraction. They introduced the family of all functions, $\theta : (0,\infty) \to (1,\infty)$ supplying the following particulars by Θ:

(Θ_1) θ is nondecreasing;

(Θ_2) For each sequence $\{s_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(s_n) = 1$ if and only if $\lim_{n\to\infty} s_n = 0^+$;

(Θ_3) There exist $m \in (0,1)$ and $z \in (0,\infty]$ such that $\lim_{s\to 0^+} \frac{\theta(s)-1}{s^m} = z$.

In section 1, some basic definitions and theorem in the literature that will be used in the paper are given. In section 2, by using the approach of Nadler [1], Jleli and Samet [4] and Karapınar et. al.[2, 3], we introduce the notion of extended interpolative single and multi-valued Kannan type and Reich Rus Ćirić type θ-contractive mappings.

2. Main Results

Firstly, let us start with the definition of interpolative Kannan type θ-contractive mapping.

Definition 2.1. *Let (X,d) be a complete metric space and $\theta \in \Theta$. A mapping $F : X \to X$ is said to be an interpolative Kannan type θ-contractive mapping if $\theta \in \Theta$ and there exist $\lambda \in [0,1)$, $\alpha \in (0,1)$ such that*

$$\theta(d(F\eta, F\omega)) \le [\theta(d(\eta, F\eta))]^{\lambda\alpha} [\theta(d(\omega, F\omega))]^{\lambda(1-\alpha)} \tag{2.1}$$

for all $\eta, \omega \in X$.

Theorem 2.2. *Let (X,d) be a complete metric space and $F : X \to X$ be an interpolative Kannan type θ-contractive, then F has a fixed point in X.*

Proof. Starting from $\eta_0 \in X$, consider $\{\eta_n\}$ given as $\eta_n = F\eta_{n-1}$ for all positive integer n. If there is n_0 so that $\eta_{n0} = \eta_{n0+1}$ then η_{n0} is a fixed point of F. Assume that $\eta_n \ne \eta_{n+1}$ for all $n \ge 0$. Taking $\eta = \eta_{n-1}$ and $\omega = \eta_n$ in (2.1), one writes

$$\theta(d(\eta_n, \eta_{n+1})) \le [\theta(d(\eta_{n-1}, \eta_n))]^{\lambda\alpha} [\theta(d(\eta_n, \eta_{n+1}))]^{\lambda(1-\alpha)}. \tag{2.2}$$

If

$$d(\eta_{n-1}, \eta_n) < d(\eta_n, \eta_{n+1}),$$

then, from (2.2) we obtain

$$\theta(d(\eta_n, \eta_{n+1})) \le [\theta(d(\eta_n, \eta_{n+1}))]^{\lambda(1-\alpha+\alpha)} = [\theta(d(\eta_n, \eta_{n+1}))]^{\lambda}$$

which is a contradiction. Thus, for all $n \in \mathbb{N}$

$$\theta(d(\eta_n, \eta_{n+1})) \le [\theta(d(\eta_{n-1}, \eta_n))]^{\lambda}. \tag{2.3}$$

Using (2.3) we have

$$\theta(d(\eta_n, \eta_{n+1})) \le [\theta(d(\eta_{n-1}, \eta_n))]^{\lambda} \le [\theta(d(\eta_{n-2}, \eta_{n-1}))]^{\lambda^2} \le \cdots \le [\theta(d(\eta_0, \eta_1))]^{\lambda^n}. \tag{2.4}$$

Letting $n \to \infty$ in (2.4) we obtain

$$\lim_{n\to\infty} \theta(d(\eta_n, \eta_{n+1})) = 1, \tag{2.5}$$

From (Θ_2) we get

$$\lim_{n\to\infty} d(\eta_n,\eta_{n+1}) = 0^+,$$

and from (Θ_3), there exist $a \in (0,1)$ and $b \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(\eta_n,\eta_{n+1}))-1}{(d(\eta_n,\eta_{n+1}))^a} = b. \tag{2.6}$$

Suppose that $b<\infty$. In this case, let $S=\frac{b}{2}>0$. Using the definition of the limit, there exist $n_0\in\mathbb{N}$ such that

$$\left|\frac{\theta(d(\eta_n,\eta_{n+1}))-1}{(d(\eta_n,\eta_{n+1}))^a} - b\right| \le S, \text{ for all } n\ge n_0.$$

This implies that

$$\frac{\theta(d(\eta_n,\eta_{n+1}))-1}{(d(\eta_n,\eta_{n+1}))^a} \ge b-S=S, \text{ for all } n\ge n_0.$$

Then

$$n(d(\eta_n,\eta_{n+1}))^a \le Rn[\theta(d(\eta_n,\eta_{n+1}))-1],$$

for all $n\ge n_0$ where $R=\frac{1}{S}$. Now suppose that $b=\infty$ and $S>0$ be an arbitrary positive number. Using the definition the limit, there exist $n_0\in\mathbb{N}$ such that

$$\frac{\theta(d(\eta_n,\eta_{n+1}))-1}{(d(\eta_n,\eta_{n+1}))^a} \ge S,$$

for all $n\ge n_0$. This implies that

$$n(d(\eta_n,\eta_{n+1}))^a \le Rn[\theta(d(\eta_n,\eta_{n+1}))-1],$$

for all $n\ge n_0$, where $R=\frac{1}{S}$.

Therefore, in all cases, there exist $R>0$ and $n_0\in\mathbb{N}$ such that

$$n(d(\eta_n,\eta_{n+1}))^a \le Rn[\theta(d(\eta_n,\eta_{n+1}))-1],$$

for all $n\ge n_0$. Using (2.4), we can write

$$n(d(\eta_n,\eta_{n+1}))^a \le Rn([\theta(d(\eta_0,\eta_1))]^{\lambda^n}-1), \tag{2.7}$$

for all $n\ge n_0$. Letting $n\to\infty$ in (2.7) we get

$$\lim_{n\to\infty} n(d(\eta_n,\eta_{n+1}))^a = 0.$$

Hence, there exist $n_1\in\mathbb{N}$ such that

$$d(\eta_n,\eta_{n+1}) \le \frac{1}{n^{\frac{1}{a}}}, \text{ for all } n\ge n_1. \tag{2.8}$$

For what follows, we shall prove that $\{\eta_n\}$ is a Cauchy sequence by employing standard tools, For any $n,m\in\mathbb{N}$ with $m>n\ge n_0$ we obtain

$$\begin{aligned} d(\eta_n,\eta_m) \le& d(\eta_n,\eta_{n+1})+d(\eta_{n+1},\eta_{n+2})+\cdots+d(\eta_{m-1},\eta_m)\\ \le& \sum_{i=n}^{m-1}\frac{1}{i^{\frac{1}{a}}}. \end{aligned}$$

Since the last term of the above inequality tends to zero as $n,m \to \infty$, we have $d(\eta_n,\eta_m) \to 0$. As (X,d) is a complete metric spaces, the sequence $\{\eta_n\}$ converges to some point $u \in X$, that is,

$$\lim_{n\to\infty} \eta_n = u. \tag{2.9}$$

As a next step make evident that the limit η of the iterative sequence $\{\eta_n\}$ is a fixed point of the given mapping F. Suppose that $\eta \neq F\eta$, then $d(\eta,F\eta) > 0$. By letting $\eta = \eta_n$ and $\omega = \eta$ in (2.1), we obtain that

$$d(\eta_{n+1},F\eta) = d(F\eta_n,F\eta) \leq [\theta(d(\eta_n,F\eta_n))]^{\lambda\alpha}[\theta(d(\eta,F\eta))]^{\lambda(1-\alpha)}.$$

Letting $n \to \infty$ in the above inequality, we obtain, $\eta = F\eta$. Thus the proof is completed.

□

Remark 2.3. *Taking $\theta(t) = e^t$ in inequality (2.1), then it turns to Kannan contraction mapping with $\lambda\alpha \in [0,\frac{1}{2})$ and $\lambda(1-\alpha) \in [0,\frac{1}{2})$.*

Definition 2.4. *Let (X,d) be a complete metric space and $\theta \in \Theta$. A mapping $F : X \to K(X)$ is said to be an interpolative multi-valued Kannan type θ-contractive mapping if $\theta \in \Theta$ and there exist $\lambda \in [0,1)$, $\alpha \in (0,1)$ such that*

$$\theta(H(F\eta,F\omega)) \leq [\theta(D(\eta,F\eta))]^{\lambda\alpha}[\theta(D(\omega,F\omega))]^{\lambda(1-\alpha)} \tag{2.10}$$

for all $\eta,\omega \in X$.

Theorem 2.5. *Let (X,d) be a complete metric space and $F : X \to K(X)$ be an interpolative multi-valued Kannan type θ-contractive, then F has a fixed point in X.*

Proof. Let η_0 be an arbitrary point of X and choose a $\eta_1 \in X$ such that $\eta_1 \in F\eta_0$. Suppose that $\eta_1 \in F\eta_1$, that is, η_1 is a fixed point of F. Then, let $\eta_1 \notin F\eta_1$. Since $F\eta_1$ is closed, we have $D(\eta_1,F\eta_1) > 0$ for all $\eta \in X$. On the other hand, from

$$0 < D(\eta_1,F\eta_1) \leq H(F\eta_0,F\eta_1),$$

so, from (2.10), and considering (Θ_1),

$$\theta(D(\eta_1,F\eta_1)) \leq \theta(H(F\eta_0,F\eta_1)) \leq [\theta(D(\eta_0,F\eta_0))]^{\lambda\alpha}[\theta(D(\eta_1,F\eta_1))]^{\lambda(1-\alpha)}. \tag{2.11}$$

Since $F\eta_1$ is compact, there exist $\eta_2 \in F\eta_1$ such that $d(\eta_0,\eta_1) = D(\eta_0,F\eta_0)$ and $d(\eta_1,\eta_2) = D(\eta_1,F\eta_1)$. From (2.11), we get

$$\theta(d(\eta_1,\eta_2)) \leq \theta(H(F\eta_0,F\eta_1)) \leq [\theta(d(\eta_0,\eta_1))]^{\lambda\alpha}[\theta(d(\eta_1,\eta_2))]^{\lambda(1-\alpha)}. \tag{2.12}$$

Therefore, continuing recursively, we get $\eta_n \in X$ such that $\eta_n \in F\eta_{n-1}$, $\eta_{n+1} \in F\eta_n$ and

$$\theta(d(\eta_n,\eta_{n+1})) \leq [\theta(d(\eta_{n-1},\eta_n))]^{\lambda\alpha}[\theta(d(\eta_n,\eta_{n+1}))]^{\lambda(1-\alpha)}. \tag{2.13}$$

If

$$d(\eta_{n-1},\eta_n) < d(\eta_n,\eta_{n+1}),$$

then, from (2.13) we obtain

$$\theta(d(\eta_n,\eta_{n+1})) \leq [\theta(d(\eta_n,\eta_{n+1}))]^{\lambda(1-\alpha+\alpha)} = [\theta(d(\eta_n,\eta_{n+1}))]^{\lambda}$$

which is a contradiction. Thus, for all $n \in \mathbb{N}$

$$\theta(d(\eta_n,\eta_{n+1})) \leq [\theta(d(\eta_{n-1},\eta_n))]^{\lambda}. \tag{2.14}$$

Denote

$$\mu_n = d(\eta_n,\eta_{n+1}),$$

for all $n \in \mathbb{N}$. Then, $\mu_n > 0$ and using (2.14) we have

$$\theta(\mu_n) \leq [\theta(\mu_{n-1})]^{\lambda} \leq [\theta(\mu_{n-2})]^{\lambda^2} \leq \cdots \leq [\theta(\mu_0)]^{\lambda^n}. \tag{2.15}$$

Letting $n \to \infty$ in (2.15) we obtain

$$\lim_{n\to\infty} \theta(\mu_n) = 1, \tag{2.16}$$

From (Θ_2) we get

$$\lim_{n\to\infty} \mu_n = 0^+,$$

and so from (Θ_3), there exist $a \in (0,1)$ and $b \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\mu_n)-1}{(\mu_n)^a} = b. \tag{2.17}$$

Assume that $b < \infty$. In this case, let $S = \frac{b}{2} > 0$. From the definition of the limit, there exist $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(\mu_n)-1}{(\mu_n)^a} - b \right| \leq S, \text{ for all } n \geq n_0.$$

This implies that

$$\frac{\theta(\mu_n)-1}{(\mu_n)^a} \geq b - S = S, \text{ for all } n \geq n_0.$$

Thus

$$n(\mu_n)^a \leq Rn[\theta(\mu_n)-1],$$

for all $n \geq n_0$ where $R = \frac{1}{S}$. Now assume that $b = \infty$ and $S > 0$ be an arbitrary positive number. From the definition the limit, there exist $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(\mu_n)-1}{(\mu_n)^a} \geq S,$$

for all $n \geq n_0$. This implies that

$$n(\mu_n)^a \leq Rn[\theta(\mu_n)-1],$$

for all $n \geq n_0$, where $R = \frac{1}{S}$.

Therefore, in all cases, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$n(\mu_n)^a \leq Rn[\theta(\mu_n)-1],$$

for all $n \geq n_0$. Using (2.15), we obtain

$$n(\mu_n)^a \leq Rn([\theta(\mu_0)]^{\lambda^n} - 1), \tag{2.18}$$

for all $n \geq n_0$. Letting $n \to \infty$ in (2.18) we get

$$\lim_{n\to\infty} n(\mu_n)^a = 0.$$

Therefore, there exist $n_1 \in \mathbb{N}$ such that

$$\mu_n \leq \frac{1}{n^{\frac{1}{a}}}, \text{ for all } n \geq n_1. \tag{2.19}$$

For what follows, we shall prove that $\{\eta_n\}$ is a Cauchy sequence by employing standard tools, For any $n,m \in \mathbb{N}$ with $m > n \geq n_0$ we obtain

$$\begin{aligned} d(\eta_n, \eta_m) \leq & d(\eta_n, \eta_{n+1}) + d(\eta_{n+1}, \eta_{n+2}) + \cdots + d(\eta_{m-1}, \eta_m) \\ = & \mu_n + \mu_{n+1} + \cdots + \mu_{m-1} \\ \leq & \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{a}}}. \end{aligned}$$

Since the last term of the above inequality tends to zero as $n,m \to \infty$, we have $d(\eta_n, \eta_m) \to 0$. As (X,d) is a complete metric spaces, the sequence $\{\eta_n\}$ converges to some point $u \in X$, that is,

$$\lim_{n\to\infty} \eta_n = u. \tag{2.20}$$

Case 1: There is a subsequence $\{\eta_{n_r}\}$ such that $F\eta_{n_r} = Fu$ for all $r \in \mathbb{N}$. In this case,

$$D(u,Fu) = \lim_{n\to\infty} D(\eta_{n_{r+1}}, Fu) \leq \lim_{n\to\infty} H(F\eta_{n_r}, Fu) = 0.$$

Case 2: There is a natural number N such that $F\eta_n \neq Fu$ for all $n \geq N$. In this cases applying (2.10) for $u = \eta_n$ and $\omega = u$ we have

$$\theta(D(\eta_{n+1}, Fu)) \leq \theta(H(F\eta_n, Fu)) \leq [\theta(D(\eta_n, F\eta_n))]^{\lambda\alpha} [\theta(D(u,Fu))]^{\lambda(1-\alpha)}. \tag{2.21}$$

Then assume that

$$D(\eta_n, F\eta_n) < D(u,Fu),$$

letting $n \to \infty$ in (2.21) we obtain,

$$\theta(D(u,Fu)) \leq [\theta(D(u,Fu))]^{\lambda}$$

which is a contradiction. Then we obtain

$$D(u,Fu) \leq D(\eta_n, F\eta_n),$$

so, we get

$$\theta(D(u,Fu)) \leq [\theta(D(\eta_n, F\eta_n))]^{\lambda}. \tag{2.22}$$

Since $F\eta_n$ is compact, there exist $\eta_{n+1} \in F\eta_n$ such that $d(\eta_n, \eta_{n+1}) = D(\eta_n, F\eta_n)$. Since (2.22), we get

$$\theta(D(u,Fu)) \leq [\theta(D(\eta_n, \eta_{n+1}))]^{\lambda} \tag{2.23}$$

letting $n \to \infty$ in (2.23) we obtain, $u \in Fu$. Thus the proof is completed.

□

Hançer et al. [15], showed that we can take "$CB(X)$" instead of "$K(X)$", by adding the condition (θ_4) on $\theta : (0,\infty) \to (1,\infty)$, as follows:

(θ_4) $\theta(\inf M) = \inf \theta(M)$ for all $M \subset (0,\infty)$ with $\inf M > 0$.

Take in the consideration if θ is right continuous and satisfies (θ_1), in that case (θ_4) founds. Let Ξ be the family of all functions θ satisfying $(\theta_1) - (\theta_4)$.

Corollary 2.6. *Let (X,d) be a complete metric space and $F : X \to CB(X)$ be a mapping. Given that there are $\theta \in \Xi$, $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that*

$$\theta(H(F\eta, F\omega)) \leq [\theta(D(\eta, F\eta))]^{\lambda\alpha} [\theta(D(\omega, F\omega))]^{\lambda(1-\alpha)} \tag{2.24}$$

for all $\eta, \omega \in X$ then F has a fixed point in X.

Proof. Let η_0 be an arbitrary point of X and choose a $\eta_1 \in X$ such that $\eta_1 \in F\eta_0$. Assume that $\eta_1 \in F\eta_1$, that is, η_1 is a fixed point of F. Then, let $\eta_1 \notin F\eta_1$. As $F\eta_1$ is closed, we obtain $D(\eta_1, F\eta_1) > 0$ for all $\eta \in X$. So, from (2.24), and considering (Θ_1),

$$\theta(D(\eta_1,F\eta_1)) \le \theta(H(F\eta_0,F\eta_1)) \le [\theta(D(\eta_0,F\eta_0))]^{\lambda\alpha}[\theta(D(\eta_1,F\eta_1))]^{\lambda(1-\alpha)}. \tag{2.25}$$

Considering condition (θ_4), we obtain $\theta(D(\eta_1,F\eta_1)) = \inf_{z\in F\eta_1}\theta(d(\eta_1,z))$. Then we have

$$\begin{aligned}\inf_{z\in F\eta_1}\theta(d(\eta_1,z)) &\le [\theta(D(\eta_0,F\eta_0))]^{\lambda\alpha}[\theta(D(\eta_1,F\eta_1))]^{\lambda(1-\alpha)}\\ &< [\theta(D(\eta_0,F\eta_0))]^{\lambda_1\alpha}[\theta(D(\eta_1,F\eta_1))]^{\lambda_1(1-\alpha)}.\end{aligned} \tag{2.26}$$

where $\lambda_1 \in (\lambda, 1)$. Then, from (2.26), there exist $\eta_1 \in F\eta_0$ and $\eta_2 \in F\eta_1$ such that

$$\theta(d(\eta_1,\eta_2)) \le [\theta(d(\eta_0,\eta_1))]^{\lambda_1\alpha}[\theta(d(\eta_1,\eta_2))]^{\lambda_1(1-\alpha)}. \tag{2.27}$$

The rest of the proof can be completed as in the proof of Theorem 2.5. □

Definition 2.7. *Let (X,d) be a complete metric space and $\theta \in \Theta$. A mapping $F : X \to X$ is said to be an interpolative Reich Rus Ćirić type θ-contractive mapping if $\theta \in \Theta$ and there exist $\lambda \in [0,1)$, $\beta, \alpha \in (0,1)$ with $\beta + \alpha < 1$ such that*

$$\theta(d(F\eta,F\omega)) \le [\theta(d(\eta,\omega))]^{\lambda\beta}[\theta(d(\eta,F\eta))]^{\lambda\alpha}[\theta(d(\omega,F\omega))]^{\lambda(1-\beta-\alpha)} \tag{2.28}$$

for all $\eta, \omega \in X$.

Theorem 2.8. *Let (X,d) be a complete metric space and $F : X \to X$ be an interpolative Reich Rus Ćirić type θ-contractive, then F has a fixed point in X.*

Proof. Starting from $\eta_0 \in X$, consider $\{\eta_n\}$ given as $\eta_n = F\eta_{n-1}$ for all positive integer n. If there is n_0 so that $\eta_{n0} = \eta_{n0+1}$ then η_{n0} is a fixed point of F. Assume that $\eta_n \neq \eta_{n+1}$ for all $n \ge 0$. Taking $\eta = \eta_{n-1}$ and $\omega = \eta_n$ in (2.28), one writes

$$\theta(d(\eta_n,\eta_{n+1})) \le [\theta(d(\eta_{n-1},\eta_n))]^{\lambda\beta}[\theta(d(\eta_{n-1},\eta_n))]^{\lambda\alpha}[\theta(d(\eta_n,\eta_{n+1}))]^{\lambda(1-\beta-\alpha)}. \tag{2.29}$$

If

$$d(\eta_{n-1},\eta_n) < d(\eta_n,\eta_{n+1}),$$

then, from (2.29) we obtain

$$\theta(d(\eta_n,\eta_{n+1})) \le [\theta(d(\eta_n,\eta_{n+1}))]^{\lambda(\beta+\alpha+1-\alpha-\beta)} = [\theta(d(\eta_n,\eta_{n+1}))]^{\lambda}$$

which is a contradiction. Thus, for all $n \in \mathbb{N}$

$$\theta(d(\eta_n,\eta_{n+1})) \le [\theta(d(\eta_{n-1},\eta_n))]^{\lambda}. \tag{2.30}$$

From (2.30) we have

$$\theta(d(\eta_n,\eta_{n+1})) \le [\theta(d(\eta_{n-1},\eta_n))]^{\lambda} \le [\theta(d(\eta_{n-2},\eta_{n-1}))]^{\lambda^2} \le \cdots \le [\theta(d(\eta_0,\eta_1))]^{\lambda^n}. \tag{2.31}$$

Then, it can be seen that the $\{\eta_n\}$ is a Cauchy with similar operations in Theorem 2.2. As (X,d) is a complete metric spaces, the sequence $\{\eta_n\}$ converges to some point $u \in X$, that is,

$$\lim_{n\to\infty} d(\eta_n,\eta_{n+1}) = u. \tag{2.32}$$

As a next step make evident that the limit η of the iterative sequence $\{\eta_n\}$ is a fixed point of the given mapping F. Suppose that $\eta \neq F\eta$, then $d(\eta,F\eta) > 0$. By letting $\eta = \eta_n$ and $\omega = \eta$ in (2.28), we obtain that

$$d(\eta_{n+1},F\eta) = d(F\eta_n,F\eta) \le [\theta(d(\eta_n,\eta))]^{\lambda\beta}[\theta(d(\eta_n,F\eta_n))]^{\lambda\alpha}[\theta(d(\eta,F\eta))]^{\lambda(1-\alpha-\beta)}.$$

Letting $n \to \infty$ in the above inequality, we obtain, $\eta = F\eta$. Thus the proof is completed.

□

Remark 2.9. *Taking $\theta(t) = e^t$ in inequality (2.28), then it turns to Reich-Rus-Ćirić contraction mapping with $a,b,c \in (0,1)$ such that $\lambda\beta = a$, $\lambda\alpha = b$ and $c = 1-\beta-\alpha$, $0 \leq a+b+c < 1$.*

Definition 2.10. *Let (X,d) be a complete metric space and $\theta \in \Theta$. A mapping $F : X \to K(X)$ is said to be an interpolative multi-valued Reich Rus Ćirić type θ-contractive mapping if $\theta \in \Theta$ and there exist $\lambda \in [0,1)$, $\beta,\alpha \in (0,1)$ with $\beta+\alpha < 1$ such that*

$$\theta(H(F\eta,F\omega)) \leq [\theta(d(\eta,\omega))]^{\lambda\beta}[\theta(D(\eta,F\eta))]^{\lambda\alpha}[\theta(D(\omega,F\omega))]^{\lambda(1-\beta-\alpha)} \tag{2.33}$$

for all $\eta,\omega \in X$.

Theorem 2.11. *Let (X,d) be a complete metric space and $F : X \to K(X)$ be an interpolative multi-valued Reich Rus Ćirić type θ-contractive, then F has a fixed point in X.*

Proof. Let $\eta_0 \in X$. Since $F\eta$ is nonempty for all $\eta_0 \in X$, we can chose a $\eta_1 \in X$. Assume that $\eta_1 \in F\eta_1$, that is, η_1 is a fixed point of F. Now, let $\eta_1 \notin F\eta_1$. As $F\eta_1$ is closed, we obtain $D(\eta_1,F\eta_1) > 0$ for all $\eta \in X$. Moreover, as

$$0 < d(\eta_1,F\eta_1) \leq H(F\eta_0,F\eta_1),$$

from (2.33) and considering (Θ_1), we can write that

$$\begin{aligned}\theta(d(\eta_1,F\eta_1)) \leq& \theta(H(F\eta_0,F\eta_1))\\ \leq& [\theta(d(\eta_0,\eta_1))]^{\lambda\beta}[\theta(d(\eta_0,F\eta_0))]^{\lambda\alpha}[\theta(d(\eta_1,F\eta_1))]^{\lambda(1-\beta-\alpha)}.\end{aligned} \tag{2.34}$$

As $F\eta_1$ is compact, there exist $\eta_2 \in F\eta_1$ such that $d(\eta_0,\eta_1) = d(\eta_0,F\eta_0)$ and $d(\eta_1,\eta_2) = d(\eta_1,F\eta_1)$. From (2.34), we obtain

$$\begin{aligned}\theta(d(\eta_1,\eta_2)) \leq& \theta(H(F\eta_0,F\eta_1))\\ \leq& [\theta(d(\eta_0,\eta_1))]^{\lambda\beta}[\theta(d(\eta_0,\eta_1))]^{\lambda\alpha}[\theta(d(\eta_1,\eta_2))]^{\lambda(1-\beta-\alpha)}.\end{aligned} \tag{2.35}$$

Therefore, continue recursively, we get $\eta_n \in X$ such that $\eta_n \in F\eta_{n-1}$, $\eta_{n+1} \in F\eta_n$, and

$$\theta(d(\eta_n,\eta_{n+1})) \leq [\theta(d(\eta_{n-1},\eta_n))]^{\lambda\beta}[\theta(d(\eta_{n-1},\eta_n))]^{\lambda\alpha}[\theta(d(\eta_n,\eta_{n+1}))]^{\lambda(1-\beta-\alpha)}. \tag{2.36}$$

Suppose that

$$d(\eta_{n-1},\eta_n) < d(\eta_n,\eta_{n+1}),$$

then from (2.36) we obtain

$$\theta(d(\eta_n,\eta_{n+1})) \leq [\theta(d(\eta_n,\eta_{n+1}))]^{\lambda(\beta+\alpha+1-\alpha-\alpha)} = [\theta(d(\eta_n,\eta_{n+1}))]^{\lambda}$$

which is a contradiction. Therefore for all $n \in \mathbb{N}$

$$\theta(d(\eta_n,\eta_{n+1})) \leq [\theta(d(\eta_{n-1},\eta_n))]^{\lambda}. \tag{2.37}$$

Let

$$\mu_n = d(\eta_n,\eta_{n+1}),$$

for all $n \in \mathbb{N}$. Thus, $\mu_n > 0$ and handling (2.37) we get

$$\theta(\mu_n) \leq [\theta(\mu_{n-1})]^{\lambda} \leq [\theta(\mu_{n-2})]^{\lambda^2} \leq \cdots \leq [\theta(\mu_0)]^{\lambda^n}. \tag{2.38}$$

Then, it can be seen that the $\{\eta_n\}$ is a Cauchy with similar operations in Theorem 2.5.

Since (X,d) is a complete metric spaces, the sequence $\{\eta_n\}$ converges to some point $u \in X$, that is,

$$\lim_{n\to\infty} \eta_n = u. \tag{2.39}$$

Case 1: There is a subsequence $\{\eta_{n_r}\}$ such that $F\eta_{n_r} = Fu$ for all $r \in \mathbb{N}$. Therefore,

$$D(u,Fu) = \lim_{n\to\infty} D(\eta_{n_{r+1}},Fu) \leq \lim_{n\to\infty} H(F\eta_{n_r},Fu) = 0.$$

Case 2: There is a natural number N such that $F\eta_n \neq Fu$ for all $n \geq N$. In this cases applying (2.33) for $u = \eta_n$ and $\omega = u$ we have

$$\begin{aligned}\theta(D(\eta_{n+1}, Fu)) \leq &\theta(H(F\eta_n, Fu))\\ \leq &[\theta(D(\eta_n, u))]^{\lambda\beta}[\theta(D(\eta_n, F\eta_n))]^{\lambda\alpha}[\theta(D(u, Fu))]^{\lambda(1-\beta-\alpha)}.\end{aligned} \tag{2.40}$$

Hence, suppose that

$$D(\eta_n, F\eta_n) < D(u, Fu),$$

letting $n \to \infty$ in (2.40) we get,

$$\theta(D(u, Fu)) \leq [\theta(D(u, Fu))]^{\lambda(1-\beta)}$$

which is a contradiction. So we obtain

$$D(u, Fu) \leq D(\eta_n, F\eta_n),$$

then, we get

$$\theta(D(u, Fu)) \leq [\theta(D(\eta_n, F\eta_n))]^{\lambda(1-\beta)}. \tag{2.41}$$

As $F\eta_n$ is compact, there exist $\eta_{n+1} \in F\eta_n$ such that $d(\eta_n, \eta_{n+1}) = D(\eta_n, F\eta_n)$. From (2.41), we obtain

$$\theta(D(u, Fu)) \leq [\theta(D(\eta_n, \eta_{n+1}))]^{\lambda(1-\beta)} \tag{2.42}$$

letting $n \to \infty$ in (2.42) we get, $u \in Fu$. Therefore the proof is completed.

□

Example 2.12. *Let $X = [0, \infty)$ and define $d(\eta, \omega) = |\eta - \omega|$, for all $\eta, \omega \in X$. (X, d) is a complete metric space. Also defined $F : X \to K(X)$ a mapping, where*

$$F\eta = \begin{cases} \{0\}, & \text{if } \eta \in [0,1) \\ \{\frac{\eta}{6}\}, & \text{if } \eta \in [1, \infty). \end{cases}$$

Let $\lambda = \frac{1}{\sqrt{2}}$, $\beta = \frac{1}{2}$, $\alpha = \frac{1}{3}$ and $\theta(m) = e^m$ pertain to Θ. Without loss of generality, we may assume that $\eta \geq \omega$. Thus, through a series of standard calculations, we can proved that

$$\theta(H(F\eta, F\omega)) \leq [\theta(D(\eta, \omega))]^{\lambda\beta}[\theta(D(\eta, F\eta))]^{\lambda\alpha}[\theta(D(\omega, F\omega))]^{\lambda(1-\beta-\alpha)}$$

for all $\eta, \omega \in X$. So, this is satisfying the condition of Theorem 2.11. F has fixed points. Since similar process are performed, the condition of Theorem 2.5 is satisfied.

Corollary 2.13. *Let (X, d) be a complete metric space and $F : X \to CB(X)$ be a mapping. Suppose that there are $\theta \in \Xi$, $\lambda \in [0, 1)$ and $\beta, \alpha \in (0, 1)$ with $\beta + \alpha < 1$ such that*

$$\theta(H(F\eta, F\omega)) \leq [\theta(d(\eta, \omega))]^{\lambda\beta}[\theta(d(\eta, F\eta))]^{\lambda\alpha}[\theta(d(\omega, F\omega))]^{\lambda(1-\beta-\alpha)} \tag{2.43}$$

for all $\eta, \omega \in X$ then F has a fixed point in X.

Proof. Let η_0 be an arbitrary point of X and choose a $\eta_1 \in X$ such that $\eta_1 \in F\eta_0$. Assume that $\eta_1 \in F\eta_1$, that is, η_1 is a fixed point of F. Therefore, let $\eta_1 \notin F\eta_1$. Since $F\eta_1$ is closed, we obtain $D(\eta_1, F\eta_1) > 0$ for all $\eta \in X$. Hence, from (2.43), and considering (Θ_1), we can write

$$\begin{aligned}\theta(D(\eta_1, F\eta_1)) \leq &\theta(H(F\eta_0, F\eta_1))\\ \leq &[\theta(d(\eta_0, \eta_1))]^{\lambda\beta}[\theta(d(\eta_0, F\eta_0))]^{\lambda\alpha}[\theta(d(\eta_1, F\eta_1))]^{\lambda(1-\beta-\alpha)}.\end{aligned} \tag{2.44}$$

Considering condition (θ_4), we get $\theta(D(\eta_1, F\eta_1)) = \inf_{z \in F\eta_1} \theta(d(\eta_1, z))$. Thus, we have

$$\begin{aligned}\inf_{z \in F\eta_1} \theta(d(\eta_1, z)) \leq &[\theta(d(\eta_0, \eta_1))]^{\lambda\beta}[\theta(d(\eta_0, F\eta_0))]^{\lambda\alpha}[\theta(d(\eta_1, F\eta_1))]^{\lambda(1-\beta-\alpha)}\\ < &[\theta(d(\eta_0, \eta_1))]^{\lambda_1\beta}[\theta(d(\eta_0, F\eta_0))]^{\lambda_1\alpha}[\theta(d(\eta_1, F\eta_1))]^{\lambda_1(1-\beta-\alpha)}\end{aligned} \tag{2.45}$$

where $\lambda_1 \in (\lambda, 1)$. Then, from (2.45), there exist $\eta_1 \in F\eta_0$ and $\eta_2 \in F\eta_1$ such that

$$\theta(d(\eta_1, \eta_2)) \leq [\theta(d(\eta_0, \eta_1))]^{\lambda_1\beta}[\theta(d(\eta_0, \eta_1))]^{\lambda_1\alpha}[\theta(d(\eta_1, \eta_2))]^{\lambda_1(1-\beta-\alpha)}. \tag{2.46}$$

The rest of the proof can be completed as in the proof of Theorem 2.11. □

3. Conclusion

We aimed to present new some results to the fixed point theory by combining the ideas of Nadler, Karapınar et. al., Jleli and Samet. We introduce the concept of interpolative single and multi-valued Kannan type and Reich Rus Ćirić type θ-contractive mappings metric spaces and prove some fixed point results for such mappings.

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References

[1] S.B. Nadler, *Multivalued contraction mappings*, Pacific Journal of Mathematics, **30** (1969), 475-488.

[2] E. Karapınar, *Revisiting the Kannan type contractions via interpolation*, Advances in the Theory of Nonlinear Analysis and its Applications, **2** (2018), 85-87.

[3] E. Karapınar, R.P. Agarwal, H. Aydi, *Interpolative Reich-Rus-Ćirić Type Contractions on Partial Metric Spaces*, Mathematics , **6**(11) (2018), 256.

[4] M. Jleli, B. Samet, *A new generalization of the Banach contraction principle*, Journal of Inequalities and Applications, **38** (2014), 1-8.

[5] S. Banach, *Sur les operations dans les ensembles abstracits et leur application aux equations integrales*, Fund. Math., **3** (1922), 133-181.

[6] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc., **60** (1968), 71-76.

[7] S. Reich, *Fixed point of contractive functions*, Boll. Unione Mat. Ital., **5** (1972), 26-42.

[8] L. B. Ćirić, *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Beograd)(NS), **12** (1971), 19-26.

[9] S. Reich, *Some remarks concerning contraction mappings*, Canadian Mathematical Bulletin, **14** (1971), 121–124.

[10] L.B, Ćirić, *On contraction type mappings*, Math. Balk., **1** (1971), 52-57.

[11] L.B, Ćirić, *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Belgr.), **12** (1971), 19-26.

[12] S. Reich, *Kannan's fixed point theorem*, Boll. Unione Mat. Ital., **4** (1971), 1–11.

[13] I.A. Rus, *Principles and applications of the fixed point theory*, Editura Dacia, Clui-Napoca, Romania, (1979).

[14] I.A. Rus, *Generalized contractions and applications*; Cluj University Press: Clui-Napoca, Romania, (2001).

[15] H. A. Hançer, G. Mınak, I. Altun, *On a broad category of multivalued weakly Picard operators*, Fixed point theory, **18** (2017) 229-236.

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Some Characterizations of h-Regular Γ-Hemiring in terms of Cubic h-Ideals

Debabrata Mandal[1]*

Abstract
The aim of this paper is to study h-hemiregular and h-intra-hemiregular Γ-hemiring using the combined concept of cubic set and h-ideals.We have defined two types of compositions of cubic sets and used these to obtain some characterizations of h-hemiregular and h-intra-hemiregular Γ-hemiring.

Keywords: Γ-hemiring, Cubic (h-, h-bi-, h-quasi-) ideal, h-intra-hemiregular, h-hemiregular.
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[1] *Department of Mathematics, Raja Peary Mohan College, Uttarpara, Hooghly,India -712258. ORCID: 0000-0001-8605-7865*
***Corresponding author**: dmandaljumath@gmail.com
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1. Introduction

Semirings [3], a common generalization of rings and distributive lattices, arise naturally in graph theory, automata theory, mathematical modelling, functional analysis etc.. We know that ideals of ring are a very important tool to describe the structure theory and it is useful in various purposes. If we try to obtain similar results in case of semiring, we find that there are some limitations. To make the connection between the ring ideals and semiring ideals Henriksen [4] defined a special kind of ideals, called k-ideal. Iizuka [5] extended the results to a similar kind of ideals called h-ideals. As a continuation of this, La Torre [10] studied h-ideals and k-ideals in hemirings and tried to investigate the gap between the ring ideals and semiring ideals.
In 1965, Zadeh [16] proposed the theory of fuzzy sets. After that we have seen that it is a very useful mathematical tool for describing the vague or complex or illdefined systems. Rosenfeld [15] used the concept to study of fuzzy algebraic structure. Since then many researchers have developed these ideas. Jun et al [6] applied the concept to initiate the study of fuzzy h-ideals in hemiring. Sardar et al [13, 14], Ma et al [11] extended some of these results in more general setting of hemiring i.e. Γ-hemiring. Jun et al [7, 8] initiated the study of cubic subgroups and cubic sets. Khan et al [9] applied this in case of cubic h-ideals of hemirings. Chinnadurai [1, 2] used this notion to study cubic bi-ideals and cubic lateral ideals in near-ring and ternary near-ring respectively.
As a continuation of this, the main aim of this paper is to study h-hemiregularity and h-intra-hemiregularity criterion of Γ-hemiring using cubic h-ideals, cubic h-bi-ideals and cubic h-quasi-ideals.

2. Primary Ideas

We know that a hemiring is a nonempty set S on which operations addition and multiplication have been defined such that $(S,+)$ is a commutative monoid with identity 0, $(S,\cdot)$ is a semigroup and multiplication distributes over addition from either side. In addition to that, $0\cdot s = 0 = s\cdot 0$ for all $s \in S$. As an extension of this Γ-hemiring can be defined as follows:

For two additive commutative semigroups with zero, S and Γ, there exists a mapping $S\times\Gamma\times S \to S$ ($(a,\alpha,b) \mapsto a\alpha b$) which satisfy the following conditions:

i) $(a+b)\alpha c = a\alpha c + b\alpha c$,

ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,

iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,

iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$,

v) $0_S \alpha a = 0_S = a\alpha 0_S$,

vi) $a0_\Gamma b = 0_S = b0_\Gamma a$

for all $a,b,c \in S$ and for all $\alpha, \beta \in \Gamma$.

We now summon up the definitions of several types of ideals.
A subset I of a Γ-hemiring S is called a left(resp. right) ideal of S if I is closed under addition and $S\Gamma I \subseteq I$ (resp.$I\Gamma S \subseteq I$).
A subset Q of a Γ-hemiring S is called a quasi-ideal of S if Q is closed under addition and $S\Gamma Q \cap Q\Gamma S \subseteq Q$.
A subset B of a Γ-hemiring S is called a bi-ideal if B is closed under addition and $B\Gamma S\Gamma B \subseteq B$.
A left ideal H of S is called a left h-ideal if $x,z \in S$, $a,b \in H$ and $x+a+z = b+z$ implies $x \in H$. A right h-ideal is defined analoguesly.

We now remind the definition of cubic set and characteristic cubic set. For a non-empty set X, a cubic set C in X is a structure $C =< \widetilde{\mu}, f >$ where $\widetilde{\mu} = [\mu^-, \mu^+]$ is an interval valued fuzzy set and f is a fuzzy set in X. For any non-empty set G of a set X, the characteristic cubic set of G is defined to be the structure $\chi_G(x) =< x, \widetilde{\zeta}_{\chi_G}(x), \eta_{\chi_G}(x) : x \in X >$ where

$$\widetilde{\zeta}_{\chi_G}(x) = \begin{cases} [1,1] \approx \widetilde{1} \text{ if } x \in G \\ [0,0] \approx \widetilde{0} \text{ otherwise.} \end{cases}$$

and

$$\eta_{\chi_G}(x) = \begin{cases} 0 \;\text{ if } x \in G \\ 1 \;\text{ otherwise.} \end{cases}$$

3. Cubic h-Ideals

In this section, we recall some definitions and results from [12] which will be used to develop the main portion of the paper.

Definition 3.1. *Let $< \widetilde{\mu}, f >$ be a non empty cubic subset of a Γ-hemiring S. Then $< \widetilde{\mu}, f >$ is called a cubic left ideal [respectively, cubic right ideal] of S if*

(i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x+y) \le \max\{f(x), f(y)\}$ and

(ii) $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(y)$, $f(x\gamma y) \le f(y)$ [respectively, $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(x), f(x\gamma y) \le f(x)$].

for all $x,y \in S$, $\gamma \in \Gamma$.
Note: For cubic left or right ideal $< \widetilde{\mu}, f >$ of a Γ-hemiring S, $\widetilde{\mu}(0) \supseteq \widetilde{\mu}(x)$ and $f(0) \le f(x)$ for all $x \in S$.

Definition 3.2. *A cubic left ideal $< \widetilde{\mu}, f >$ of a Γ-hemiring S is called a cubic left h-ideal if for all $a,b,x,z \in S$, $x+a+z = b+z$ then $\widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$ and $f(x) \le \max\{f(a), f(b)\}$.*

Definition 3.3. *Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\theta}, g >$ be two cubic sets of a $\Gamma-$hemiring S. Define intersection of A and B by*

$$A \cap B =< \widetilde{\mu}, f > \cap < \widetilde{\theta}, g > =< \widetilde{\mu} \cap \widetilde{\theta}, f \cup g > .$$

Definition 3.4. *Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\theta}, g >$ be two cubic sets of a $\Gamma-$hemiring S. Define composition of A and B by*

$$A\Gamma_{ch}B =< \widetilde{\mu}, f > \Gamma_{ch} < \widetilde{\theta}, g > =< \widetilde{\mu}\Gamma_{ch}\widetilde{\theta}, f\Gamma_{ch}g >$$

where

$$\begin{aligned} \widetilde{\mu}\Gamma_{ch}\widetilde{\theta}(x) &= \bigcup_{x+a_1\gamma b_1+z=a_2\delta b_2+z}[\cap\{\widetilde{\mu}(a_1), \widetilde{\mu}(a_2), \widetilde{\theta}(b_1), \widetilde{\theta}(b_2)\}] \\ &= \widetilde{0}, \text{if } x \text{ cannot be expressed as } x+a_1\gamma b_1+z = a_2\delta b_2+z. \end{aligned}$$

and

$$f\Gamma_{ch}g(x) = \inf_{x+a_1\gamma b_1+z=a_2\delta b_2+z}\{\max\{f(a_1),f(a_2),g(b_1),g(b_2)\}\}$$
$$= 1, \text{if } x \text{ cannot be expressed as above}$$

for $x,z,a_1,a_2,b_1,b_2 \in S$ *and* $\gamma,\delta \in \Gamma$.

Definition 3.5. *Let* $A =< \widetilde{\mu}, f >$ *and* $B =< \widetilde{\theta}, g >$ *be two cubic sets of a* $\Gamma-$*hemiring S. Define generalized composition of A and B by*

$$Ao_{ch}B =< \widetilde{\mu}, f > o_{ch} < \widetilde{\theta}, g >=< \widetilde{\mu}o_{ch}\widetilde{\theta}, fo_{ch}g >$$

where

$$\widetilde{\mu}o_{ch}\widetilde{\theta}(x) = \cup_{x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}[\bigcap_i\{\cap\{\widetilde{\mu}(a_i),\widetilde{\mu}(c_i),\widetilde{\theta}(b_i), \ \widetilde{\theta}(d_i)\}\}]$$
$$= \widetilde{0}, \text{if } x \text{ cannot be expressed as above}$$

and

$$fo_{ch}g(x) = \inf_{x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}[\max_i\{\max\{f(a_i),f(c_i),g(b_i),g(d_i)\}\}]$$
$$= 1, \text{if } x \text{ cannot be expressed as above}$$

where $x,z,a_i,b_i,c_i,d_i \in S$ *and* $\gamma_i,\delta_i \in \Gamma$, *for* $i \in \{1,...,n\}$.

Lemma 3.6. *Let* $A =< \widetilde{\mu}_1, f >$, $B =< \widetilde{\mu}_2, g >$ *be two cubic h-ideal of a* $\Gamma-$*hemiring S. Then* $A\Gamma_{ch}B \subseteq Ao_{ch}B \subseteq A\cap B \subseteq A($ *and* $B)$.

Definition 3.7. *A cubic subset* $< \widetilde{\mu}, f >$ *of a* Γ*-hemiring S is called cubic h-bi-ideal if for all* $x,y,z,a,b \in S$ *and* $\alpha,\beta \in \Gamma$ *we have*

i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}$, $f(x+y) \leq \max\{f(x),f(y)\}$

ii) $\widetilde{\mu}(x\alpha y) \supseteq \cap\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}$, $f(x\alpha y) \leq \max\{f(x),f(y)\}$

iii) $\widetilde{\mu}(x\alpha y\beta z) \supseteq \cap\{\widetilde{\mu}(x),\widetilde{\mu}(z)\}$, $f(x\alpha y\beta z) \leq \max\{f(x),f(z)\}$

iv) If $x+a+z = b+z$ *then* $\widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a),\widetilde{\mu}(b)\}$, $f(x) \leq \max\{f(a),f(b)\}$

Definition 3.8. *A cubic subset* $< \widetilde{\mu}, f >$ *of a* Γ*-hemiring S is called cubic h-quasi-ideal if for all* $x,y,z,a,b \in S$ *we have*

i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}$, $f(x+y) \leq \max\{f(x),f(y)\}$

ii) $(\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})\cap(\widetilde{\zeta}_{\chi_S}o_{ch}\widetilde{\mu}) \subseteq \widetilde{\mu}$, $(fo_{ch}\eta_{\chi_S})\cup(\eta_{\chi_S}o_{ch}f) \supseteq f$,

iii) If $x+a+z = b+z$ *then* $\widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a),\widetilde{\mu}(b)\}$, $f(x) \leq \max\{f(a),f(b)\}$

Lemma 3.9. *Any cubic h-quasi-ideal of S is a cubic h-bi-ideal of S.*

4. Cubic *h*-Hemiregularity and Cubic *h*-Intra-Hemiregularity

In this section, we study the concept of h-hemiregularity and h-intra-hemiregularity in Γ-hemiring by using cubic h-ideal, cubic h-bi-ideal and cubic h-quasi-ideal.

Definition 4.1. *[11] A* Γ*-hemiring S is said to be h-hemiregular if for each* $x \in S$, *there exist* $a,b \in S$ *and* $\alpha,\beta,\gamma,\delta \in \Gamma$ *such that* $x+x\alpha a\beta x+z = x\gamma b\delta x+z$.

We now try to find some characterizations of h-hemiregular Γ-hemiring in terms of cubic h-ideals.

Theorem 4.2. *Let S be an h-hemiregular Γ-hemiring. Then for any cubic right h-ideal $A =< \widetilde{\mu}, f >$ and any cubic left h-ideal $B =< \widetilde{\nu}, g >$ of S, we have $A\Gamma_{ch}B = A \cap B$.*

Proof. Let S be an h-hemiregular Γ-hemiring. By Lemma 3.6, we have $A\Gamma_{ch}B \subseteq A \cap B$.
Since S is h-hemiregular, for any $a \in S$, there exist $z, x_1, x_2 \in S$ and $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \Gamma$ such that $a + a\alpha_1 x_1 \beta_1 a + z = a\alpha_2 x_2 \beta_2 a + z$.
Now for any $e, b, c, d \in S$ and $\gamma, \delta \in \Gamma$, the general expression of a as $a + e\gamma b + z = c\delta d + z$ implies that

$$\begin{aligned}
(\widetilde{\mu}\Gamma_{ch}\widetilde{\nu})(a) &= \bigcup_{a+e\gamma b+z=c\delta d+z}\{\cap\{\widetilde{\mu}(e),\ \widetilde{\mu}(c), \widetilde{\nu}(b), \widetilde{\nu}(d)\}\} \\
&\supseteq \bigcap_{a+a\alpha_1x_1\beta_1a+z=a\alpha_2x_2\beta_2a+z}\{\widetilde{\mu}(a\alpha_1x_1), \widetilde{\mu}(a\alpha_2x_2), \widetilde{\nu}(a)\} \\
&\supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(a), \widetilde{\nu}(a)\} \\
&= \cap\{\widetilde{\mu}(a), \widetilde{\nu}(a)\} = (\widetilde{\mu} \cap \widetilde{\nu})(a).
\end{aligned}$$

$$\begin{aligned}
(f\Gamma_{ch}g)(a) &= \inf_{a+e\gamma b+z=c\delta d+z}\{\max\{f(e), f(c), g(b), g(d)\}\} \\
&\leq \max_{a+a\alpha_1x_1\beta_1a+z=a\alpha_2x_2\beta_2a+z}\{f(a\alpha_1x_1), f(a\alpha_2x_2), g(a)\} \\
&\leq \max\{f(a), f(a), g(a)\} \\
&= \max\{f(a), g(a)\} = (f \cup g)(a).
\end{aligned}$$

Therefore $(A \cap B) \subseteq (A\Gamma_{ch}B)$.
Hence $A\Gamma_{ch}B = A \cap B$. □

Corollary 4.3. *If S be a h-hemiregular Γ-hemiring, then for any cubic right h-ideal $A =< \widetilde{\mu}, f >$ and any cubic left h-ideal $B =< \widetilde{\nu}, g >$ of S we have $Ao_{ch}B = A \cap B$.*

Theorem 4.4. *Let S be a h-hemiregular Γ-hemiring. Then*
(i) $A \subseteq Ao_{ch}\chi_S o_{ch}A$ for every cubic h-bi-ideal $A =< \widetilde{\mu}, f >$ of S.
(ii) $A \subseteq Ao_{ch}\chi_S o_{ch}A$ for every cubic h-quasi-ideal $A =< \widetilde{\mu}, f >$ of S.

Proof. Suppose that $A =< \widetilde{\mu}, f >$ be any cubic h-bi-ideal of S and x be any element of S. Since S is h-hemiregular there exist $a, b, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$.

Now for any general expression of x as $x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z$, where $a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, we have

$$\begin{aligned}
&(\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S}o_{ch}\widetilde{\mu})(x) \\
&= \bigcup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}(\cap\{(\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})(a_i), (\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})(c_i), \widetilde{\mu}(b_i), \widetilde{\mu}(d_i)\}) \\
&\supseteq \bigcap_{x+x\alpha a\beta x+z=x\gamma b\delta x+z}\{(\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})(x\alpha a), (\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})(x\gamma b), \widetilde{\mu}(x)\} \\
&= \cap\{\bigcup_{x\alpha a+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}(\cap\{(\widetilde{\mu}(a_i), \widetilde{\mu}(c_i))\}), \bigcup_{x\gamma b+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}(\cap\{(\widetilde{\mu}(a_i), \widetilde{\mu}(c_i))\}), \\
&\widetilde{\mu}(x)\} \\
&\supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(x), \widetilde{\mu}(x)\}(\text{since } x\alpha a + x\alpha a\beta x\alpha a + z\alpha a = x\gamma b\delta x\alpha a + z\alpha a \text{ and } x\gamma b + x\alpha a\beta x\gamma b + z\gamma b = x\gamma b\delta x\gamma b + z\gamma b). \\
&= \widetilde{\mu}(x).
\end{aligned}$$

$$\begin{aligned}
&(fo_{ch}\eta_{\chi_S}o_{ch}f)(x) \\
&= \inf_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}(\max\{(fo_{ch}\eta_{\chi_S})(a_i), (fo_{ch}\eta_{\chi_S})(c_i), f(b_i), f(d_i)\})
\end{aligned}$$

$\leq \max_{x+x\alpha a\beta x+z=x\gamma b\delta x+z}\{(fo_{ch}\eta_{\chi_S})(x\alpha a),(fo_{ch}\eta_{\chi_S})(x\gamma b),f(x)\}$
$= \max\{\inf_{x\alpha a+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}(\max\{(f(a_i),f(c_i))\}), \inf_{x\gamma b+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}(\max\{(f(a_i),f(c_i))\}), f(x)\}$
$\leq \max\{f(x),f(x),f(x)\}$(since $x\alpha a+x\alpha a\beta x\alpha a+z\alpha a=x\gamma b\delta x\alpha a+z\alpha a$ and $x\gamma b+x\alpha a\beta x\gamma b+z\gamma b=x\gamma b\delta x\gamma b+z\gamma b$).
$=f(x)$.
This implies that $A\subseteq Ao_{ch}\chi_S o_{ch}A$.
(i)⇒(ii) By Lemma 3.9 "Any cubic h-quasi-ideal of S is a cubic h-bi-ideal of S". Thus, if the proof of (ii) is made, it is straightforward to see that (i) is true by Lemma 3.9. □

Theorem 4.5. *Let S be a h-hemiregular Γ-hemiring. Then*
(i) $A\cap B\subseteq Ao_{ch}Bo_{ch}A$ for every cubic h-bi-ideal $A=<\widetilde{\mu},f>$ and every cubic h-ideal $B=<\widetilde{\nu},g>$ of S.
(ii) $A\cap B\subseteq Ao_{ch}Bo_{ch}A$ for every cubic h-quasi-ideal $A=<\widetilde{\mu},f>$ and every cubic h-ideal $B=<\widetilde{\nu},g>$ of S.

Proof. Suppose S is a h-hemiregular Γ-hemiring and $A=<\widetilde{\mu},f>$, $B=<\widetilde{\nu},g>$ be any cubic h-bi-ideal and cubic$h$-ideal of S, respectively and x be any element of S. Since S is h-hemiregular, there exist $a,b,z\in S$ and $\alpha,\beta,\gamma,\delta\in\Gamma$ such that $x+x\alpha a\beta x+z=x\gamma b\delta x+z$.

Now for any general expression of x as $x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$, where $a_i,b_i,c_i,d_i\in S$ and $\gamma_i,\delta_i\in\Gamma$, we have

$(\widetilde{\mu}o_{ch}\widetilde{\nu}o_{ch}\widetilde{\mu})(x)$
$=\cup_{x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}(\cap\{(\widetilde{\mu}o_{ch}\widetilde{\nu})(a_i),(\widetilde{\mu}o_{ch}\widetilde{\nu})(c_i),\widetilde{\mu}(b_i),\widetilde{\mu}(d_i)\})$
$\supseteq\cap_{x+x\alpha a\beta x+z=x\gamma b\delta x+z}\{(\widetilde{\mu}o_{ch}\widetilde{\nu})(x\alpha a),(\widetilde{\mu}o_{ch}\widetilde{\nu})(x\gamma b),\widetilde{\mu}(x)\}$
$=\cap\{\cup_{x\alpha a+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}(\cap\{(\widetilde{\mu}(a_i),\widetilde{\mu}(c_i),\widetilde{\nu}(b_i),\widetilde{\nu}(d_i))\}),\cup_{x\gamma b+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}(\cap\{(\widetilde{\mu}(a_i),\widetilde{\mu}(c_i),\widetilde{\nu}(b_i),\widetilde{\nu}(d_i))\}),$
$\widetilde{\mu}(x)\}$
$\supseteq\cap\{\cap\{\widetilde{\mu}(x),\widetilde{\nu}(a\beta x\alpha a),\widetilde{\nu}(b\delta x\alpha a)\},\cap\{\widetilde{\mu}(x),\widetilde{\nu}(a\beta x\gamma b),\widetilde{\nu}(b\delta x\gamma b)\},\widetilde{\mu}(x)\}$
(since $x\alpha a+x\alpha a\beta x\alpha a+z\alpha a=x\gamma b\delta x\alpha a+z\alpha a$ and $x\gamma b+x\alpha a\beta x\gamma b+z\gamma b=x\gamma b\delta x\gamma b+z\gamma b$)
$\supseteq\cap\{\widetilde{\mu}(x),\widetilde{\nu}(x)\}=(\widetilde{\mu}\cap\widetilde{\nu})(x)$.

$(fo_{ch}go_{ch}f)(x)$
$=\inf_{x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}(\max\{(fo_{ch}g)(a_i),(fo_{ch}g)(c_i),f(b_i),g(d_i)\})$
$\leq\max_{x+x\alpha a\beta x+z=x\gamma b\delta x+z}\{(fo_{ch}g)(x\alpha a),(fo_{ch}g)(x\gamma b),f(x)\}$
$=\max\{\inf_{x\alpha a+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}(\max\{(f(a_i),f(c_i),g(b_i),g(d_i))\}),\inf_{x\gamma b+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}(\max\{(f(a_i),f(c_i),g(b_i),g(d_i))\}),$
$f(x)\}$
$\leq\max\{\max\{f(x),g(a\beta x\alpha a),g(b\delta x\alpha a)\},\cap\{f(x),g(a\beta x\gamma b),g(b\delta x\gamma b)\},f(x)\}$
(since $x\alpha a+x\alpha a\beta x\alpha a+z\alpha a=x\gamma b\delta x\alpha a+z\alpha a$ and $x\gamma b+x\alpha a\beta x\gamma b+z\gamma b=x\gamma b\delta x\gamma b+z\gamma b$)
$\supseteq\max\{f(x),g(x)\}=(f\cup g)(x)$.
(i)⇒(ii) This is straightforward using the Lemma 3.9. □

Definition 4.6. *[13]A Γ-hemiring S is said to be h-intra-hemiregular if for each $x\in S$, there exist $z,a_i,a_i^{'},b_i,b_i^{'}\in S$, and $\alpha_i,\beta_i,\gamma_i,\delta_i,\eta\in\Gamma$, $i\in\mathbf{N}$, such that $x+\sum_{i=1}^{n}a_i\alpha_i x\eta x\beta_i a_i^{'}+z=\sum_{i=1}^{n}b_i\gamma_i x\eta x\delta b_i^{'}+z$.*

We now try to find a characterization of h-intrahemiregular Γ-hemiring in terms of cubic h-ideals.

Theorem 4.7. *Let S be a h-intra-regular Γ-hemiring. Then $A \cap B \subseteq Ao_{ch}B$ for every cubic left h-ideal $A =< \widetilde{\mu}, f >$ and every cubic right h-ideal $A =< \widetilde{\nu}, g >$ of S.*

Proof. Suppose S is h-intra-hemiregular. Let $A =< \widetilde{\mu}, f >$ and $A =< \widetilde{\nu}, g >$ be any cubic left h-ideal and cubic right h-ideal of S respectively. Now let $x \in S$. Then by hypothesis there exist $z, a_i, a_i^{'}, b_i, b_i^{'} \in$S, and $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, i$\in$ **N**, the set of natural numbers, such that $x + \sum_{i=1}^{n} a_i\alpha_i x\eta x\beta_i a_i^{'} + z = \sum_{i=1}^{n} b_i\gamma_i x\eta x\delta b_i^{'} + z$. Now for any general expression of x as $x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z$, where $a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, we have

$$
\begin{aligned}
(\widetilde{\mu} o_{ch} \widetilde{\nu})(x) &= \bigcup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\bigcap_i\{\cap\{\widetilde{\mu}(a_i), \widetilde{\mu}(c_i), \widetilde{\nu}(b_i), \ \widetilde{\nu}(d_i)\}\}] \\
&\supseteq \bigcap_{i}[\cap\{\widetilde{\mu}(a_i\alpha_i x), \widetilde{\mu}(b_i\gamma_i x), \widetilde{\nu}(x\beta_i a_i^{'}), \widetilde{\nu}(x\delta_i b_i^{'})\}]_{x+\sum_{i=1}^{n} a_i\alpha_i x\eta x\beta_i a_i^{'} + z = \sum_{i=1}^{n} b_i\gamma_i x\eta x\delta b_i^{'} + z} \\
&\supseteq \cap\{\widetilde{\mu}(x), \widetilde{\nu}(x)\} = (\widetilde{\mu} \cap \widetilde{\nu})(x).
\end{aligned}
$$

$$
\begin{aligned}
(f o_{ch} g)(x) &= \inf_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\max_i\{\max\{f(a_i), f(c_i), g(b_i), g(d_i)\}\}] \\
&\leq \max_i[\max\{f(a_i\alpha_i x), f(b_i\gamma_i x), g(x\beta_i a_i^{'}), g(x\delta_i b_i^{'})\}]_{x+\sum_{i=1}^{n} a_i\alpha_i x\eta x\beta_i a_i^{'} + z = \sum_{i=1}^{n} b_i\gamma_i x\eta x\delta b_i^{'} + z} \\
&\leq \max\{f(x), g(x)\} = (f \cup g)(x).
\end{aligned}
$$

Hence the proof is completed. ☐

We now combine the concepts of h-hemiregularity and h-intra-hemiregularity of a Γ-hemiring and obtain a characterization.

Theorem 4.8. *Let S be both h-hemiregular and h-intra-hemiregular Γ-hemiring. Then*
(i) $A = Ao_{ch}A$ for every cubic h-bi-ideal $A =< \widetilde{\mu}, f >$ of S.
(ii) $A = Ao_{ch}A$ for every cubic h-quasi-ideal $A =< \widetilde{\mu}, f >$ of S.

Proof. Suppose S be both h-hemiregular and h-intra-hemiregular Γ-hemiring. Let $x \in S$ and $A =< \widetilde{\mu}, f >$ be any cubic h-bi-ideal of S. Since S is both h-hemiregular and h-intra-hemiregular there exist $z, a_i, b_i, c_i, d_i \in S$ and $\alpha_i, \beta_i, \alpha_i^{'}, \beta_i^{'}, \gamma_i, \delta_i, \gamma_i^{'}, \delta_i^{'}, \eta \in \Gamma$, $i \in$ **N** such that $x + \sum_{i=1}^{n} x\alpha_i a_i\alpha_i^{'} x\eta x\beta_i^{'} b_i\beta_i x + z = \sum_{i=1}^{n} x\gamma_i c_i\gamma_i^{'} x\eta x\delta_i^{'} d_i\delta_i x + z$.

Now for any general expression of x as $x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z$, where $a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, we have

$$
\begin{aligned}
&(\widetilde{\mu} o_{ch} \widetilde{\mu})(x) \\
&= \bigcup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\bigcap_i\{\cap\{\widetilde{\mu}(a_i), \widetilde{\mu}(c_i), \widetilde{\mu}(b_i), \ \widetilde{\mu}(d_i)\}\}] \\
&\supseteq \bigcap_i[\cap\{\widetilde{\mu}(x\alpha_i a_i\alpha_i^{'} x), \widetilde{\mu}(x\beta_i^{'} b_i\beta_i x), \widetilde{\mu}(x\gamma_i c_i\gamma_i^{'} x), \widetilde{\mu}(x\delta_i^{'} d_i\delta_i x)\}]_{x+\sum_{i=1}^{n} x\alpha_i a_i\alpha_i^{'} x\eta x\beta_i^{'} b_i\beta_i x + z = \sum_{i=1}^{n} x\gamma_i c_i\gamma_i^{'} x\eta x\delta_i^{'} d_i\delta_i x + z} \\
&\supseteq \widetilde{\mu}(x).
\end{aligned}
$$

$$
\begin{aligned}
&(fo_{ch}f)(x) \\
&= \inf[\max_i\{\max\{f(a_i), f(c_i), f(b_i), f(d_i)\}\}] \\
&\qquad x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z \\
&\leq \max_i[\max\{f(x\alpha_i a_i\alpha_i^{'}x), f(x\beta_i^{'}b_i\beta_i x), f(x\gamma_i c_i\gamma_i^{'}x), f(x\delta_i^{'}d_i\delta_i x)\}] \\
&\qquad x+\sum_{i=1}^{n} x\alpha_i a_i\alpha_i^{'}x\eta x\beta_i^{'}b_i\beta_i x + z = \sum_{i=1}^{n} x\gamma_i c_i\gamma_i^{'}x\eta x\delta_i^{'}d_i\delta_i x + z \\
&\leq f(x).
\end{aligned}
$$

Now $Ao_{ch}A \subseteq Ao_{ch}\chi_S \subseteq A$. Hence $Ao_{ch}A = A$ for every cubic h-bi-ideal $A$of S.
(i)⇒(ii) This is straightforward using the Lemma 3.9. □

5. Conclusion

In this paper, I have studied some properties h-hemiregular and h-intra-hemiregular Γ-hemiring using the concept of cubic h-ideal, cubic h-bi-ideal and cubic h-quasi-ideal. At the end section, I also acquire some characterizations of h-hemiregular and h-intra-hemiregular Γ-hemiring. Interested reader may find some other feature of these types of Γ-hemiring and extend the obtained result using the concept of neutrosophic set and neutrosophic ideal.

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References

[1] V. Chinnadurai, K. Bharathivelan, *Cubic bi-ideals in near-rings*, International Journal of computer and Mathematical Sciences, **5**(2)(2016), 44 - 52.

[2] V. Chinnadurai, K. Bharathivelan, *Cubic Lateral Ideals in Ternary Near - Rings*, International Advanced Research Jou rnal in Science, Engineering and Technology, **3**(11)(2016), 209 - 215 .

[3] J.S.Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, (1999).

[4] M. Henriksen, *Ideals in semirings with commutative addition*, Amer. Math. Soc. Notices, **6**(1958), 321.

[5] K. Iizuka, *On the Jacobson radical of semiring*, Tohoku Math.J., **11**(2) (1959), 409-421

[6] Y.B. Jun, M.A.Öztürk, S.Z.Song, *On Fuzzy h-ideals in hemiring*, Inform. Sci., **162**(2004), 211-226.

[7] Y.B. Jun, S.T. Jung and M.S. Kim, *Cubic subgroups*, An. Fuzzy Math. Inform., **2**(2011), 9 - 15.

[8] Y.B. Jun, C.S. Kim and K.O. Yang, *Cubic sets*, An. Fuzzy Math. Inform., **4**(2012), 83 - 98.

[9] A. Khan, Y.B. Jun, S.I.A. Shah, M. Ali, *Characterizations of hemirings in terms of cubic h-ideals*, Soft Comput., DOI 10.1007/s00500-014-1396-4.

[10] D.R.La Torre, *On h-ideals and k-ideals in hemirings*, Publ. Math. Debrecen, **12**(1965), 219-226.

[11] X. Ma, J. Zahn, *Fuzzy h-ideals in h-hemiregular and h-semisimple Γ-hemirings*, Neural Comput and Applic, **19**(2010), 477-485 .

[12] D. Mandal, O*n Cubic h-ideals of* Γ*-hemiring*, Bull. Int. Math. Virtual Inst., **10**(3) (2020), 567-579.

[13] S.K. Sardar, *D. Mandal, On fuzzy h-ideals in h-regular* Γ*-hemiring and h-duo* Γ*-hemiring*, Gen. Math. Notes,**2**(1),(2011), 64-85.

[14] S.K. Sardar, D.Mandal, *On fuzzy h-ideal in* Γ*-hemiring*, Int. J. Pure. Appl. Math, **56**(3) (2009), 439-450.

[15] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., **35**(1971), 512-517.

[16] L.A. Zadeh, *Fuzzy Sets*, Inf. Control, **8**(1965), 338-353.

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Commutable Matrix-Valued Functions and Operator-Valued Functions

Abdelaziz Maouche[1*]

Abstract
A simple expression is established for an analytic commutable matrix-valued function. Then a characterization of two by two functional commutative matrices is proven. Finally, a family of analytic normal compact operators on a Hilbert space, which commute with their derivatives, is shown to be functionally commutative.

Keywords: Analytic matrix-valued function, Commutable matrices, Eigenvalue, Holomorphic operator-valued function, Resolvent, Riesz projection, Spectrum
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[1] *Department of Mathematics, Faculty of Science, Sultan Qaboos University, Oman*
[*]**Corresponding author**: maouche@squ.edu.om
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1. Introduction

The theory of analytic perturbation is historically the first subject in perturbation theory. It is mainly concerned with the behaviour of isolated eigenvalues and eigenvectors (or eigenprojections) of an operator depending on a parameter holomorphically. Once the notion of holomorphic dependence of an (in general unbounded) operator on a parameter is introduced, it is rather straightforward to show that the results obtained in the finite-dimensional case can be extended, at least for isolated eigenvalues, without essential modification. This is exactly what we aim to achieve in our present work. Indeed, in the first part of this paper we study matrix-valued analytic functions which commute with their derivatives. We recall the main theorems obtained by many authors such as Schwerdtfeger [15], Dieudonné [4], Goff [7] and Evard [6]. First we comment some of these results by adding some remarks and examples to see if there is a possibility to extend some of them or not. We recall that Theorem 1 in [4] is an extension of the main theorem of [15] from analytic matrix functions to continuously differentiable functions. Then we state J. Claude Evard's theorem on continuously differentiable and diagonalizable matrices with a constant number of eigenvalues [6]. Our first result, namely Theorem 3.14, extends this theorem to the case of an analytic family of matrices which are not supposed to be diagonalizable. Then in after a preparatory lemma we give a very simple expression of a functionally commutable analytic matrix-valued function on a real interval and in Theorem 3.17, we study functional commutativity in the case of two by two matrices.

The second part of the paper deals with the extension of some of the above results from the case of matrix-valued analytic functions to the infinite-dimensional situation of analytic operator-valued functions on a Hilbert space. We first recall our main result in the case of analytic compact and self-adjoint oprator functions on a Hilbert space ([12], Theorem 2). Then we show that this result can be extended to the more general situation of analytic normal compact operator-valued functions on a Hilbert space. Indeed, the proof of Theorem 2 in[12] is based on the Spectral Theorem for Compact self-adjoint operators, but this spectral representation is still valid for normal compact operators on a Hilbert space as established recently in Theorem 3.3 of [11]. We end this paper by two open questions worth studying for their importance in applications to problems on differential equations and other topics as can be seen from the historical overview given in [6] about problems related to functional commutativity in general.

2. Preliminaries

Our interest in Section 3 focuses on analytic matrix-valued functions, these are functions that can be represented by Taylor series at some point of a real interval I. The main point is whether these matrices are diagonalizable or not, that is everything boils down to the nature of the eigenvalues of these matrices, in other words, to their spectrum. We start with Example 1, which exhibits some features of general analytic matrices through a simple two by two matrix. A neat idea of what causes difficulties in the problem of functional commutativity is clearly highlighted by this example. To overcome such situations involving singularities, we present a theorem from [14] dealing with diagonalizability of analytic hermitian matrix functions on a real interval (Theorem 4). Then we move to the core of the present paper, that is functional commutativity of analytic function matrices commuting with their derivatives. The first theorem in this direction was due to Shwertdfeger [15]. It shows that every analytic diagonalizable matrix function is functionally commutative. This is followed by a more general result, which was generalized by Evard [6], who showed that any continuously differentiable family of diagonalizable matrices with a constant number of eigenvalues is functionally commutative. At this point, one can ask if this theorem can be extended, in other words can we drop the condition on the constancy number of eigenvalues. The answer is affirmative, of course not on all the interval I, but on the interval minus an exceptional set. Here we make use of a powerful theorem on the scarcity of elements with finite spectrum in Banach algebras ([1], Theorem 3.4.25). This new result is presented in Theorem 3.13. Finally, we prove that matrices which are functionally commutative can be expressed in a very simple form, namely as linear combination of constant matrices with coefficients given by analytic scalar functions as done in Lemma 1. We close the section by a nice characterization of functionally commutative two by two matrices. This is presented in Theorem 3.17. Then we turn our attention to the case of compact operators on a separable Hilbert space. First, we recall that analytic hermitian function matrices which commute with their derivative on some real interval I, i.e, $[A(t),A'(t)] = 0$ for all $t \in I$, where we use the bracket notation $[A(t),A'(t)] = 0$ instead of $A(t)A'(t) = A(t)A(t)$, were studied in [7]. As a main result, it was shown that these matrices are functionally commutative on I, i.e., $[A(s),A(t)] = 0$ for all $s,t \in I$ ([7], Theorem 3.6). Subsequently in [6], the study of the nonlinear differential equation $[A(t), \frac{dA(t))}{dt}] = 0, t \in \Lambda$, where Λ is an open interval in $\mathbb{R}$ and A is a differentiable map from Λ into the $\mathbb{C}$-Banach space M_n of all $n \times n$ matrices $(\alpha_{i,j})$, with $\alpha_{i,j} \in \mathbb{C}$ for $i, j \in \{1,\cdots,n\}$, led the author to consider the more general problem where Λ is an open connected subset of a Banach space on $\mathbb{R}$ or $\mathbb{C}$. Thus obtaining Theorem 4.3 in [6], which generalizes the main theorem of [7]. Moreover, [6] contains both a comprehensive historical summary on the motivations behind the problem on matrix functions commuting with their derivatives and further paths of investigations such as the one of interest to us. Indeed this is illustrated by our main result, Theorem 2 in [12], which extends the finite dimensional result of Goff [7] to the infinite-dimensional situation of compact self-adjoint operators on a Hilbert space. In section 4, we study analytic families of normal compact operators, on a complex Hilbert space, which commute with their derivative on some real interval I. Our main result establishes that these operators must be functionally commutative on I, that is, $[A(s),A(t)] = 0$ for all $s,t \in I$, thus extending the main result of [12].

To make the paper as self-contained as possible, we include the proofs of some well known theorems. We denote by $\mathscr{B}(\mathscr{H})$ the Banach algebra of bounded operators on the Hilbert space $\mathscr{H}$.

2.1 Adjoint of bounded operators in Hilbert spaces

Proposition 2.1. *Let $T \in \mathscr{B}(\mathscr{H})$ be a bounded operator. For all $x \in \mathscr{H}$, there exists a unique $T^*x \in \mathscr{H}$ such that*

$$\forall y \in \mathscr{H},\ < Ty, x >=< y, T^*x > \text{ and } < x, Ty >=< T^*x, y > .$$

The application $T^* : \mathscr{H} \longrightarrow \mathscr{H}$ is a bounded operator called the adjoint of T.

An important definition can be given now.

Definition 2.2. *Let $T \in \mathscr{B}(\mathscr{H})$, then:*

- *T is* self-adjoint *if $T^* = T$;*
- *T is* normal *if $T^*T = TT^*$, i.e. it commutes with its adjoint;*
- *T is* unitary *if $TT^* = T^*T = I_{\mathscr{H}}$, where $I_{\mathscr{H}}$ is the identity operator on $\mathscr{H}$, i.e. T is invertible and its inverse $T^{-1} = T^*$.*

Note that any self-adjoint operator is normal, any unitary operator is normal.

2.2 The resolvent and spectrum of a bounded operator

Let E be a complex Banach space and let $T \in \mathscr{B}(E)$. The *spectrum* of T can be seen as the generalization in infinite dimension of the notion of eigenvalues in finite dimension. Let us start by the complementary set.

Definition 2.3. *Let $T \in \mathscr{B}(E)$. The* resolvent set *of T is the set*

$$\rho(T) := \{\lambda \in \mathbb{C} : \lambda - T \textit{ is invertible}\}.$$

The *resolvent* is the map

$$\rho(T) \longrightarrow \mathscr{B}(E) : \lambda \longmapsto R_\lambda(T) = (\lambda - T)^{-1}.$$

By $\lambda - T$ we mean $\lambda I_E - T$, where I_E is the identity operator of E. Note that, as a consequence of the closed Graph theorem, if T is bounded and invertible, then its inverse is automatically bounded.

Definition 2.4. *Let $T \in \mathscr{B}(E)$. The spectrum of T is*

$$\mathrm{Sp(T)} := \mathbb{C} \backslash \rho(\mathrm{T}) = \{\lambda \in \mathbb{C} : \lambda - \mathrm{T} \textit{ is invertible}\}.$$

Definition 2.5. *An eigenvalue of T is a number $\lambda \in \mathbb{C}$ such that* $\ker(T - \lambda) \neq \{0\}$. *The set formed by the eigenvalues, denoted by* $\mathrm{Sp_p(T)}$, *is called the point spectrum.*

We have $\mathrm{Sp_p(T)} \subset \mathrm{Sp(T)}$.

Proposition 2.6. *In finite dimension, the spectrum and the point spectrum coincide.*

Proof. In finite dimension, the operator $T - z$ is injective if and only if it is surjective, whereas the continuity is always guaranteed. □

Theorem 2.7. *Let $T \in \mathscr{B}(E)$ where E is a Banach space. Then $\rho(T)$ is an open subset of $\mathbb{C}$.*

2.3 Digression: The notion of analyticity

Let Ω be a non-empty subset in $\mathbb{C}$. We say that $f : \Omega \longrightarrow E$ is *holomorphic* when, for all $z \in \Omega$ the limit

$$\lim_{w \longrightarrow z} \frac{f(w) - f(z)}{w - z}$$

exists in the norm of E. It is denoted $f'(z)$.

Proposition 2.8. *Let $f : \Omega \longrightarrow E$. Then f is holomorphic if and only if it is weakly holomorphic, i.e. $\ell \circ f$ is holomorphic on Ω for all $\ell \in E'$ (i.e. the dual space of E).*

Proof. See Chapter 1 in [8]. □

Corollary 2.9. *Let $f : \mathbb{C} \longrightarrow E$ be holomorphic. If f is bounded, then it is constant.*

Proof. Assume that we can find $z_0 \in \mathbb{C}$ and $z_1 \in \mathbb{C}$ such that $f(z_0) \neq f(z_1)$. Then by the Hahn-Banach theorem, there exists some $\ell \in E'$ such that $\ell \circ f(z_0) \neq \ell \circ f(z_1)$. By the classical Liouville's theorem, it must be constant. This is a contradiction. □

The mathematical theory of Banach space valued analytic functions parallels the classical theory of analytic functions as is well presented in Chapter I of [8]. For example, if γ is a closed path in a simply connected domain Ω, then

$$\oint_\gamma f(z)\, dz = 0. \tag{2.1}$$

(The integral defined in the usual way by the norm convergent Riemann sums.) To prove (2.1), note that for $\ell \in E'$,

$$\ell(\oint_\gamma f(z)\, dz) = \oint_\gamma \ell(f(z)), dz = 0.$$

Since E' separates points in E, (2.1) holds.

Starting with (2.1) one obtains in the usual way the Cauchy integral formula

$$\frac{1}{2\pi i}\oint_{|w-z|<r}\frac{f(w)}{w-z}\,dw = f(z). \tag{2.2}$$

Starting with the Cauchy integral formula one proves that for $w \in \Omega$,

$$f(z) = \Sigma_{n=0}^{\infty} a_n (z-w)^n \tag{2.3}$$

where $a_n \in E$. The power series converges and the representation holds in the largest open disk centered at w and contained in Ω. An excellent reference on this topic is [8].

As an example, the resolvent map is analytic.

Theorem 2.10. *Let $T \in \mathscr{B}(E)$ where E is a Banach space. Then the map $\lambda \mapsto R_\lambda(T)$ is a holomorphic function from $\rho(T)$ into $\mathscr{B}(E)$.*

Another useful application of holomorphy is in obtaining the so called Riesz projections.

Proposition 2.11. *Let us consider a bounded operator $T \in \mathscr{B}(E)$ where E is a Banach space and $\lambda \in \mathbb{C}$ as an isolated element of* $\mathrm{Sp(T)}$. *Let $\Gamma_\lambda \subset \rho(T)$ be a contour that enlaces only λ as element of the spectrum of T. Define*

$$P_\lambda := \frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-T)^{-1}\,dz.$$

The bounded operator $P_\lambda : E \longrightarrow \mathrm{Dom(T)} \subset \mathrm{F}$ commutes with T and does not depend on the choice of Γ_λ. The operator P_λ is a projection and

$$P_\lambda - Id = \frac{1}{2\pi i}\oint_{\Gamma_\lambda}(\zeta-\lambda)^{-1}(T-\lambda)(\zeta-T)^{-1}\,d\zeta.$$

It is appropriate to recall at this point that most of the results on analytic matrix functions that we use in the next section can be obtained by using finite-dimensional operators on a Hilbert space. We have already seen above that the spectrum of an operator in finite dimension coincides with the point spectrum. Next, we look at the notion of finite-algebraic multiplicity. For the proofs of the next results and more details see [9].

Proposition 2.12. *Assume that the Hilbert space $\mathscr{H}$ is of finite dimension. Fix T in $\mathscr{B}(\mathscr{H})$ and let $\lambda \in$* $\mathrm{Sp(T)}$. *Then, λ is an eigenvalue. If Γ_λ is a contour enlacing only λ, then P_λ is the projection on the algebraic eigenspace associated with λ.*

Proof. It is well known that $\mathscr{H}$ can be written as a sum of the eigenspaces $\mathscr{H}_j$ associated with the distinct eigenvalues of T. The eigenspaces $\mathscr{H}_j$ are stable under T. We can assume that $\mathscr{H}_1$ is associated with λ. There exists a basis of $\mathscr{H}$ such that the matrix of T is block diagonal $(T_1, \cdots, T_k)$ where the T_j is the (upper triangular) matrix of $T_{\mathscr{H}_j}$. In this adapted basis, the matrix of P_λ is block diagonal $(P_{\lambda,1}, \cdots, P_{\lambda,k})$. By holomorphy, we have $P_{\lambda,j} = 0$ when $j \neq 1$. To simplify, assume that $\dim \mathscr{H}_1 = 2$ (the other cases being similar) so that

$$T_1 := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad P_{\lambda,1} := \frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-T_1)^{-1}\,dz.$$

where Γ_λ is (for example) the circle of center λ and radius 1. Let $n \in \mathbb{N}$. Recall that

$$\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)^{-n}\,dz = \frac{1}{2\pi i}\oint_{\theta=0}^{2\pi} e^{i(1-n)\theta}d\theta = \begin{cases} 1 & \text{if} \quad n=1 \\ 0 & \text{if} \quad n \neq 1. \end{cases}$$

It follows that

$$P_{\lambda,1} := \frac{1}{2i\pi}\oint_{\Gamma_\lambda}\begin{pmatrix} (z-\lambda)^{-1} & -(z-\lambda)^{-2} \\ 0 & (z-\lambda)^{-1} \end{pmatrix} dz = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Id}_{\mathscr{H}_1}.$$

The application P_λ is indeed the projection on $\mathscr{H}_1$.

□

Corollary 2.13. *If $\lambda \in$* $\mathrm{Sp(T)}$ *is isolated with a finite algebraic multiplicity, then it is necessarily an eigenvalue.*

3. Analytic Matrix Functions

3.1 n-Dimensional perturbation theory

Let $X(t)$ be a complex valued $n\times n$ matrix function defined on a real interval I.

The following results are taken from [2,13,14] where proofs and more results on this topic can be found.

Definition 3.1. *When we say that $X(t)$ is analytic in a neighborhood of $t=t_0$, we mean that each element of $X(t)$ is representable as a Taylor series centered at t_0 which converges in some neighborhood of t_0.*

Example 3.2. *Consider the matrix valued analytic matrix*

$$X(\beta)=\begin{bmatrix}1 & \beta\\ \beta & -1\end{bmatrix}.$$

Its eigenvalues are

$$\lambda_{\pm}(\beta)=\pm\sqrt{\beta^2+1}.$$

We notice the following features:

1. *$X(\beta)$ is an entire function but the eigenvalues are not entire (they have singularities as functions of β); the singularities are not on the real axis where $X(\beta)$ is self-adjoint but occur at non real β, namely at $\beta=\pm i$.*
2. *at the singular values of β, that is, at $\beta=\pm i$ there are fewer distinct eigenvalues, namely one, than at other points where there are two.*
3. *at the singular values of β the matrix $X(\beta)$ is not diagonalizable.*

If $X(\beta)$ is a matrix-valued analytic function in a connected region R of the complex plane, then the eigenvalues of $X(\beta)$ are solutions of the equation

$$\det(X(\beta)-\lambda I)=(-1)^n(\lambda^n+a_1(\beta)\lambda^{n-1}+\cdots+a_n(\beta)=0. \tag{3.1}$$

Theorem 3.3. *Let*

$$F(\beta,\lambda)=\lambda^n+a_1(\beta)\lambda^{n-1}+\cdots+a_n(\beta)$$

be a polynomial of degree n in λ whose leading coefficient is one and whose other coefficients are all analytic functions of β.

1. *Suppose that $\lambda=\lambda_0$ is a* simple root *of $F(\beta_0,\lambda)$. Then for β near β_0, there is exactly one root $\lambda(\beta)$ of $F(\beta,\lambda)$ near λ_0, and $\lambda(\beta)$ is analytic in β near $\beta=\beta_0$.*
2. *Suppose that $\lambda=\lambda_0$ is a root of multiplicity m of $F(\beta_0,\lambda)$. Then for β near β_0, there are exactly m roots (counting multiplicities) of $F(\beta,\lambda)$ near β_0, and these roots are branches of one or more multivalued analytic functions with at most algebraic points at $\beta=\beta_0$. Explicitly, there exist $p_1,\cdots,p_k$ with $\sum_{i=1}^{k}p_i=m$ and multivalued analytic functions $\lambda_1,\cdots,\lambda_k$ (not necessarily distinct) with convergent Puiseux series*

$$\lambda_i(\beta)=\lambda_0+\sum_{j=1}^{\infty}\alpha_j(\beta-\beta_0)^{\frac{j}{p}}$$

so that the m roots near λ_0 are given by the p_1 values of λ_1, the p_2 values of $\lambda_2,\cdots$, etc.

Corollary 3.4. *Let $X(\beta)$ be a matrix-valued analytic function near β_0.*

1. *If λ_0 is a* simple root *of $X(\beta_0)$, then for β near β_0, $X(\beta)$ has exactly one root $\lambda_0(\beta)$ near λ_0; ($\lambda_0(\beta)$ is a simple and analytic eigenvalue if β is near $\beta=\beta_0$.*
2. *If λ_0 is an eigenvalue of $T(\beta_0)$ of algebraic multiplicity m, then for β near β_0, $X(\beta)$ has exactly m eigenvalues (counting multiplicity) near λ_0. These eigenvalues are all the branches of one or more multivalued functions analytic near β_0 with at most algebraic singularities at β_0.*

3.2 Analytic perturbation of self-adjoint and hermitian matrices

If X and Y are self-adjoint, the perturbed eigenvalues of $X+\beta Y$ are analytic at $\beta=0$ even if X has degenerate eigenvalues. That the branch points allowed by the last theorem do not occur in this case is a theorem of F. Rellich in [14]. The example at the beginning of this section shows that the branch points can occur for non real β even in the self-adjont case, $X(\beta)^*=X(\bar{\beta})$.

Theorem 3.5 ([14])**.** *Suppose that $X(\beta)$ is a matrix-valued analytic function in a region R containing a section of the real axis, and that $X(\beta)$ is* self-adjoint *for β real. Let λ_0 be an eigenvalue of $X(\beta_0)$ of multiplicity m. If β_0 is real, there are distinct functions $\lambda_1(\beta),\cdots,\lambda_p(\beta)$, single-valued and analytic in a neighborhood of β_0, which are all the eigenvalues.*

For analytic Hermitian matrices the next explicit formulation of Theorem 4 holds.

Theorem 3.6 ([14])**.** *Let $X(\lambda)$ be an $n\times n$ Hermitian matrix function which is analytic on a real interval (a,b) in which* $\det X(\lambda)\neq 0$. *There exists an analytic unitary matrix function $U(\lambda)$, $\lambda\in(a,b)$ and analytic real-valued functions $\mu_i(\lambda)$, $\lambda\in(a,b)$, such that*

$$X(\lambda)=U(\lambda)\begin{bmatrix}\mu_1(\lambda) & 0 & \cdots & \cdots & 0\\ 0 & \mu_2(\lambda) & \cdots & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \ddots & 0\\ 0 & 0 & \cdots & 0 & \mu_n(\lambda)\end{bmatrix}U(\lambda)^{-1}$$

Remark 3.7. *The interesting aspect of this theorem is its validity even at points where the multiplicity of the eigenvalues changes. When $X(\lambda)$ is not hermitian its eigenvalues usually have branch points. Moreover, [14] contains a counterexample which shows that the theorem cannot be extended to infinitely differentiable, but not holomorphic functions. In that counterexample, $X(\lambda)$ and the eigenvalues are still everywhere differentiable, but all eigenvectors are discontinuous at the point $\lambda=0$. It will be seen later that the sharpest results, such as in [7] for hermitian matrices, are obtained when we deal with conditions close to the ones satisfied in F. Rellich [14]. Otherwise, Example 1 and the previous discussion show some of the difficulties we encounter when we consider general analytic matrix-valued functions (and similarly for operator-valued functions), which force us to impose extra conditions, like the constancy on the number of eigenvalues or constancy on the number of Jordan blocks and other similar conditions (see [6] for examples and a more complete list of references on matrices commuting with their derivatives).*

3.3 Analytic matrix functions commuting with their derivatives

We start with a result on analytic matrices which commute with their derivatives.

Theorem 3.8 ([15])**.** *Let $X(t)$ be an $n\times n$ analytic matrix function defined on a real interval I. Suppose that for all $t\in I$, the eigenvalues of $X(t)$ are distinct, and $[X(t),X'(t)]=0$ on I. Then $[X(s),X(t)]=0$ for all $s\neq t$ in I.*

Proof. If for $t\in I$, eigenvalues of $X(t)$ are distinct, then there exists a constant invertible matrix U such that

$$X(t)=UD(t)U^{-1},$$

where $D(t)$ is diagonal. Thus

$$[X(s),X(t)]=0$$

for $s\neq t$ in I. □

Remark 3.9. *The paper [15] contains examples showing that the conclusion fails to hold if the eigenvalues of $X(t)$ are not all distinct. Jean Dieudonné proved in [4] that Theorem 3.8 holds if $X(t)$ is only continuously differentiable.*

Next we recall an important theorem contained in [6].

Theorem 3.10 ([6])**.** *If X is an analytic matrix function, defined on an open interval Λ of $\mathbb{R}$, taking its values in $\mathscr{M}_n$ and commuting with its derivative on Λ, and if $t_0\in\Lambda$ and $\mu_1,\cdots,\mu_p$, are the distinct eigenvalues of $X(t_0)$ of respective algebraic multiplicities $m_1,\cdots,m_p$, then on a neighborhood $\Lambda_{t_0}\cap\Lambda$ of t_0, A has the form*

$$X(t)=U\,\mathrm{diag}(\boldsymbol{m}_1(t),\cdots,\boldsymbol{m}_p(t))\,U^{-1}$$

where $t\in\Lambda_{t_0}$, U is an invertible matrix, and for each $k\in\{1,2,\cdots,p\}$, $\boldsymbol{m}_K$ is an analytic matrix function from Λ_{t_0} into $\mathscr{M}_{m_k}$ commuting with its derivative on Λ_{t_0}, such that $\mathrm{Sp}(\boldsymbol{m}_k(t_0))=\{\mu_k\}$.

3.3.1 Diagonalizable matrix functions

As in [6], we say that a matrix X from a set Λ into $\mathscr{M}_n$ is pointwise diagonalizable on Λ if, for every $t \in \Lambda$, there exists an invertible matrix $U(t)$ such that

$$U(t)^{-1}X(t)U(t)$$

is diagonal, whereas we say that A is globally diagonalizable on Λ if there exists an invertible matrix U such that, for all $t \in \Lambda$, the matrix

$$U^{-1}X(t)U$$

is diagonal.

A family $(X(t))_{t\in\Lambda}$ of matrices of $\mathscr{M}_n$ is functionally commutative if

$$[X(t),X(s)] = 0$$

for all $s,t \in \Lambda$.

Theorem 3.11 ([7])**.** *If $(X(s))$ is an* analytic *family of* hermitian *matrices such that $[X(s),X'(s)] = 0$ for all s in an interval I of $\mathbb{R}$, then the family $(X(s))$ is* commutative*, i.e. $[X(s),X(t)] = 0$ for all $s,t \in I$.*

Proof. It is well known that $X(t) = \sum_{i=1}^{r} \lambda_i(t)G_i$, where the G_i are projection matrices such that $G_i^2 = G_i$ and $G_iG_j = G_jG_i = 0$ if $i \neq j$. Now $[X(t),X'(t)] = 0$ yields that G_i is constant for each i. Hence,

$$X(s)X(t) = \sum_{i=1}^{r} \lambda_i(s)\lambda_i(t)G_i = X(t)X(s).$$

□

Remark 3.12. *The condition of analyticity cannot be relaxed in Theorem 8. Indeed, let X and Y two $n\times n$ constant non-commutative hermitian matrices and define $A(t) = X\exp(-t^2)$ if $t < 0, A(0) = 0, A(t) = Y\exp(-t^2)$ if $t > 0$. Then $A(t)$ is hermitian, of class $\mathscr{C}^\infty$, and not analytic, and commutes with its derivative on $\mathbb{R}$. However if $s < 0$ and $t > 0$, then $A(s)A(t) \neq A(t)A(s)$.*

The next theorem is a generalization of Theorem 3.8 to continuously differentiable matrices commuting with their derivatives.

Theorem 3.13 ([6])**.** *If $(X(s))$ is a continuously* differentiable *family of* diagonalizable *matrices with a constant number of eigenvalues, such that*

$$[X(s),X'(s)] = 0$$

for all s in an interval I of $\mathbb{R}$, then the family $(X(s))$ is functionally commutative, i.e.

$$[X(s),X(t)] = 0$$

for all $s,t \in I$.

Question 1. Is it possible to extend the previous theorem to the case of an analytic family $(X(s))$ of matrices (without the condition of diagonalizability)?

If we replace ‘differentiable’ by ‘analytic’ and remove ‘constant number of eigenvalues’ in Theorem 3.13, we obtain the following result.

Theorem 3.14. *If $X(s)$ is an analytic family of diagonalizable matrices, such that $[X(s),X'(s)] = 0$ for all s in a real interval I, then $X(s)$ is functionally commutative on $I\backslash S$ where S is an exceptional set containing a finite number of points, i.e. $[X(s),X(t)] = 0$ for all $s,t \in I\backslash S$.*

Proof. Follows from Theorem 3.4.25 in [1]. □

3.4 Characterization of functional commutativity

Definition 3.15. *An analytic matrix function X defined on a contour Γ is said to be functionnally commutative if $[X(s),X(t)]=0$ whenever $s,t\in\Gamma$.*

It is natural to try to express matrix-valued functions which commute with their derivative in their simplest form possible. The following lemma and theorem are adapted from [3], where similar results were proven for measurable functions.

Lemma 3.16. *The matrix function G is functionally commutative if and only if*

$$G(t)=\sum_{j=1}^{m}\phi_j(t)G_j$$

where $m\le n^2$ and G_j are pairwise commuting constant matrices and ϕ_j are analytic scalar functions.

Proof. ($\Leftarrow$) The sufficiency of the conditions of the lemma is evident.

($\Rightarrow$) Let $\mathscr{L}$ be the linear hull of the set of all matrices of the form $G(t), t\in\Gamma$ in the space of all constant matrices of nth order and choose in it a basis $G_1,\dots,G_m$. Since the space of $n\times n$ matrices has dimension n^2, then $m\le n^2$. Furthermore $G(t_1)G(t_2)=G(t_2)G(t_1)$ implies that any two matrices from $\mathscr{L}$ commute. Consequently, all matrices $G_1,\dots,G_m$ commute pairwise. Finally, as $G(t)\in\mathscr{L}$ for all $t\in\Gamma$ and $\{G_j\}_{j=1}^m$ is a basis in $\mathscr{L}$, then there exists a unique representation $G(t)$ in the form of a linear combination of the G_j's. Supposing $\phi_j(t)$ to be equal to the jth coefficient of this linear combination, we have $G(t)=\sum\phi_j(t)G_j$. □

Next, we seek some relations between the entries of matrix-valued functions which are functionally commutative. To do so, let us answer the simple question: what is functional commutativity of two by two matrices?

Theorem 3.17. *The matrix function $G=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}$ is functionally commutative if and only if $g_{11}-g_{22}$, g_{12} and g_{21} are scalar multiples of one and the same function ϕ, i.e. $g_{12}(t)=\alpha\phi(t)$, $g_{21}(t)=\beta\phi(t)$, $g_{11}(t)-g_{22}(t)=\gamma\phi(t)$ where α, β, γ are numbers.*

Proof. ($\Leftarrow$) If the conditions

$$g_{12}(t)=\alpha\phi(t),\, g_{21}(t)=\beta\phi(t),\, g_{11}(t)-g_{22}(t)=\gamma\phi(t)$$

are fulfilled, then

$$G(t)=\begin{pmatrix} g_{22}(t)+\gamma\phi(t) & \alpha\phi(t\\ \beta\phi(t & g_{22}(t)\end{pmatrix}.$$

A direct verification shows that, for any $s,t\in\Gamma$, we have

$$G(s)G(t)=G(t)G(s).$$

($\Rightarrow$) let the matrix function G be functionally commutative. At first, suppose that $g_{11}(t)=g_{22}(t)$. If in addition, $g_{12}=g_{21}=0$, then the conditions

$$g_{12}(t)=\alpha\phi(t),\, g_{21}(t)=\beta\phi(t),\, g_{11}(t)-g_{22}(t)=\gamma\phi(t)$$

are satisfied for $\alpha=0$, $\beta=0$, $\gamma=0$.

In the opposite case, a point $t_0\in\Gamma$ can be found such that at least one of the functions g_{12} or g_{21} is different from zero. Equating the diagonal elements of the matrices $G(t)G(t_0)$ and $G(t_0)G(t)$, we get $g_{12}(t)g_{21}(t_0)=g_{12}(t_0)g_{21}(t)$. If $g_{21}(t_0)\neq 0$, we may take $\phi=g_{21}$, $\beta=1$, $\gamma=\frac{g_{12}(t_0)}{g_{21}(t_0)}$, and if $g_{12}(t_0)\neq 0$, then we can choose $\phi=g_{12}$, $\alpha=1$, $\gamma=0$, $\beta=\frac{g_{21}(t_0)}{g_{12}(t_0)}$. Now, let the functions g_{11} and g_{22} be different. Then there exists a point $t_0\in\Gamma$ such that $g_{11}(t_0)\neq g_{22}(t_0)$. Equating the elements outside the diagonals of the matrices $G(t)G(t_0)$ and $G(t_0)G(t)$, we obtain

$$(g_{11}(t)-g_{22}(t))(g_{12}(t_0))=(g_{11}(t_0)-g_{22}(t_0))(g_{12}(t)),$$

and

$$(g_{11}(t)-g_{22}(t))(g_{21}(t_0))=(g_{11}(t_0)-g_{22}(t_0))(g_{21}(t)),$$

so that we may set

$$\phi=g_{11}-g_{22},\,\gamma=1,\,\alpha=\frac{g_{12}(t_0)}{(g_{11}(t_0)-g_{22}(t_0))},\,\beta=\frac{g_{21}(t_0)}{(g_{11}(t_0)-g_{22}(t_0))},$$

and the theorem is proved.

□

4. Extension to Infinite-Dimensional Case

4.1 Spectral theorem for normal compact operators

We recall that $\mathscr{B}(\mathscr{H})$ denotes the Banach algebra of all bounded operators on the complex Hilbert space $\mathscr{H}$ and by definition the *spectrum* of T, denoted by $\mathrm{Sp}(T)$, is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible in $\mathscr{B}(\mathscr{H})$. We recall the important theorem on the spectrum of a compact operator on a Hilbert space.

Theorem 4.1. *Let $T \subset \mathscr{B}(\mathscr{H})$ be compact. Then* $\mathrm{Sp(T)}$ *is a non-empty compact subset of $\mathbb{C}$ with no accumulation point other than zero. Each non-zero $\lambda \in \mathrm{Sp}(T)$ is an isolated eigenvalue of T with finite algebraic multiplicity.*

Theorem 4.2 (Spectral theorem for compact operators)**.** *If an operator $T \in \mathscr{B}[\mathscr{H}]$ on a non-zero Hilbert space $\mathscr{H}$ is compact and normal, then there exists a unique countable resolution of the identity $\{E_k\}$ on $\mathscr{H}$ and a bounded set of scalars λ_k for which $T = \sum_{k=1}^{\infty} \lambda_k E_k$, where $\{\lambda_k\} = \mathrm{Sp}_p(T)$ is the (non-empty) set of all (distinct) eigenvalues of T and each E_k is the orthogonal projection onto the eigenspace $\mathscr{N}(\lambda_k I - T)$ (i.e., $\mathscr{R}(E_k) = \mathscr{N}(\lambda_k I - T)$). If the above countable weighted sum of projections is infinite, then it converges in the (uniform) topology of $\mathscr{B}(\mathscr{H})$.*

Proof. See [11], Theorem 3.3 on page 58. □

Remark 4.3. *It is worth mentioning here that according to Proposition 4.K in [11], the projections E_k coincide with the Riesz projections associated to spectral values λ_k as usually obtained by the Holomorphic Functional Calculus.*

Definition 4.4. *An operator-valued function $T : \Omega \mapsto \mathscr{B}(\mathscr{H})$, defined on an open subset Ω of the complex plane $\mathbb{C}$, is said to be analytic at $z_0 \in \Omega$ if there are operators $T_n \in \mathscr{B}(\mathscr{H})$ and a positive number δ such that*

$$T(z) = \sum_{n=0}^{\infty} (z - z_0)^n T_n,$$

where the power series on the right-hand side converges with respect to the operator norm on $\mathscr{B}(\mathscr{H})$ in a disc $|z - z_0| < \delta$ for some $\delta > 0$. We say that T is analytic or holomorphic in Ω if it is analytic at every point in Ω.

4.2 Riesz projections depending on a parameter

Let $A(t)$ denote an analytic family of normal compact operators on a complex Hilbert space and defined on a real interval $I = (a,b)$ with $a < 0 < b$. Let λ be an eigenvalue of $A(0)$. More generally $A(t)$ can be an analytic family of bounded normal operators on $\mathscr{H}$ with the property that λ is an isolated point of the spectrum $\mathrm{Sp}A(0)$, and such that the $\lambda-$eigenspace of $A(0)$ is finite-dimensional. Let D be a closed disk centered at λ such that $\mathrm{Sp}A(0) \cap D = \{\lambda\}$. It follows that, for t sufficiently small, $\mathrm{Sp}A(t) \cap \gamma = \emptyset$ where $\gamma = \partial D$ is the boundary of D. For such t, we have the *orthogonal* Riesz projections

$$P_\lambda(t) = \frac{1}{2\pi i} \int_\gamma (\xi I - A(t))^{-1} d\xi$$

with range $\mathscr{H}(t)$, depending analytically on t, such that $P(0)$ is the orthogonal projection of $\mathscr{H}$ onto the $\lambda-$eigenspace of $A(0)$. Our main result will be based essentially on the possibility of decomposing an operator like $A(t)$ into a sum

$$\sum_{k=1}^{\infty} \mu_k P_k(t),$$

where the $P_k(t)$ are mutually orthogonal analytic projections, such that

$$[A(t), P_k(t)] = 0.$$

It is a spectral decomposition with the added condition of analyticity. For more details on the spectrum, riesz idempotents/projections, spectral decomposition of operators and related questions to spectral theory see [1], [10],[11], [13].

4.3 Analytic family of compact operators on a Hilbert space

Our goal is to extend Theorem 3.11 from matrices to compact operators on a Hilbert space. We proved in [12] the following theorem for self-adjoint compact operators.

Theorem 4.5 ([12])**.** *If $(A(s))$ is an analytic family of compact self-adjoint operators on a Hilbert space, such that $[A(s),A'(s)] = 0$ for all s in an interval I of $\mathbb{R}$, then $(A(s))$ is functionally commutative, i.e. $[A(s),A(t)] = 0$ for all $s,t \in I$.*

Thanks to Theorem 3.3 of [11] we can extend our previous theorem to compact normal operators on a Hilbert space.

Theorem 4.6. *If $(A(s))$ is an analytic family of normal compact operators on a Hilbert space, such that $[A(s),A'(s)] = 0$ for all s in an interval I of $\mathbb{R}$, then $(A(s))$ is functionally commutative, i.e. $[A(s),A(t)] = 0$ for all $s,t \in I$.*

4.4 Analytic normal compact operators on a Hilbert space commuting with their derivatives

The following results may be proved in much the same way as their equivalent ones in [12].

Lemma 4.7. *Let $A(t)$ be an analytic family of normal compact operators on a Hilbert space $\mathscr{H}$ which commute with its derivative. Then the projections associated to the eigenvalues of $A(t)$ commute with their derivative.*

Proof. Similar to the proof of Lemma 2 in [12] if we replace self-adjoint by normal. □

Lemma 4.8. *If a family of projections $P(t)$ commutes with its derivative on an interval $I \subset \mathbb{R}$, then $P(t)$ is constant.*

Proof. Similar to the proof of Lemma 3 in [12] if we replace self-adjoint by normal. □

Theorem 4.9. *Let $A(t)$ be an analytic family of normal compact operators on a Hilbert space H. Suppose that $A(t)$ commutes with its derivative for all $t \in I \subset \mathbb{R}$. Then $A(t)$ is functionally commutative, i.e. $[A(s),A(t)] = 0$ for all $s,t \in I$.*

Proof. Similar to the proof of Theorem 2 in [12] if we replace self-adjoint by normal. □

It remains to solve the following two more general extensions by dropping either compactness of the operators or normality.

Problem 4.10. *If $(A(s))$ is an analytic family of compact operators, without quasi-nilpotent component (this is the case for self-adjoint operators on a Hilbert space), on a Banach space, such that $[A(s),A'(s)] = 0$ for all s in an interval I of $\mathbb{R}$, then $(A(s))$ is functionally commutative, i.e. $[A(s),A(t)] = 0$ for all $s,t \in I$.*

Problem 4.11. *If $(A(s))$ is an analytic family of self-adjoint operators on a Hilbert space, such that $[A(s),A'(s)] = 0$ for all s in an interval I of $\mathbb{R}$, then the family $(A(s))$ is functionally commutative, i.e. $[A(s),A(t)] = 0$ for all $s,t \in I$.*

Final comments. In analytic perturbation theory, we are concerned with the analytic dependence of various quantities on the parameter x, assuming that the given family $T(x)$ of operators is analytic. Among the quantities that have been considered so far, there are the resolvent $R_\lambda(T) = (\lambda - T)^{-1}$, the isolated eigenvalues λ_n and the associated spectral eigenprojections or Riesz idempotents. The general form of the spectral theorem ([11], Theorem 3.15) for self-adjoint operators furnishes other functions to be considered. One of them is the *spectral family* $E(\lambda,x)$ for $T(x)$, defined for real x, where $T(x)$ is a self-adjoint family (see [10] for more details). This combined with Theorem 3.15 in [11], might perhaps be the path to explore towards a possible solution of the second problem.

References

[1] B. Aupetit, *A Primer On Spectral Theory*, Universitext, Springer-Verlag, 1991.

[2] H. Baumgartel, *Analytic Perturbation Theory for Matrices and Operators, Operator Theory: Advances and Applications*, **15**, Birkhäuser, 1985.

[3] K. Clancy, I. Gohberg, *Factorization of matrix functions and singular integral operators*, Oper. Theory Adv. Appl., **3**, Birkhäuser Verlag (Basel) 1981.

[4] J. Dieudonné, *Sur un théorème de Schwertfeger*, Ann. Polon. Math. **24**(1974), 87 - 88.

[5] J.C. Evard, *Conditions for a vector subspace $E(t)$ and for a projector $P(t)$ not to depend on t.* , Lin. Alg. Appl. **91**(1987), 121-131.

[6] J. C. Evard, *On matrix functions which commute with their derivative*, *Lin. Alg. Appl.* **68**(1985), 145 - 178.

[7] S. Goff, *Hermitian function matrices which commute with their derivative*, *Lin. Alg. Appl.* **36**(1981), 33 - 40.

[8] I. Gohberg, J. Leiter, *Holomorphic Operator Functions of one Variable and Applications*, Oper. Theory Adv. Appl., **192**, 2009.

[9] I. Gohberg, S. Goldberg, M.A. Kaashoek, *Classes of linear operators,* **1**, Birkhauser 1990.

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1980.

[11] C.S. Kubrusly, *Spectral Theory of Bounded Linear Operators*, Birkhauser 2020.

[12] A. Maouche, *Functional commutativity of analytic families of self adjoint compact operators on a Hilbert space*, *Commun. Adv. Math. Sci.,* **3**(1) (2020), 9 - 12.

[13] M. Reed, B. Simon, *Modern Methods of Mathematical Physics*, Academic Press, 1975.

[14] F. Rellich, *Perturbation Theory of Eigenvalue Problems*, Institute of Mathematical Sciences, New York, 1950.

[15] H. Schwertfeger, *Sur les matrices permutables avec leur derivée*, *Rend. Sem. Mat. Univ. Politec. Torino*. **11**(1952), 329 - 333.