UJMA

Universal Journal of Mathematics and Applications

VOLUME III
ISSUE IV

ISSN 2619-9653

http://dergipark.gov.tr/ujma

VOLUME III ISSUE IV
ISSN 2619-9653

December 2020
http://dergipark.gov.tr/ujma

UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

UJMA

Contents

Universal Journal of Mathematics and Applications, 3 (4) (2020) 133-137
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.647951

A Bilocal Problem Associated to a Fractional Differential Inclusion of Caputo-Fabrizio Type

Aurelian Cernea[1]

[1] *Faculty of Mathematics and Computer Science, University of Bucharest and Academy of Romanian Scientists, Bucharest, Romania*

Article Info

Keywords: *Differential inclusion, Fixed point, Fractional derivative, Selection.*
2010 AMS: *26A33, 34A08, 34A60.*
Received: *18 November 2019*
Accepted: *16 October 2020*
Available online: *23 December 2020*

Abstract

A fractional differential inclusion defined by Caputo-Fabrizio fractional derivative with bilocal boundary conditions is studied. A nonlinear alternative of Leray-Schauder type, Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and Covitz-Nadler set-valued contraction principle are employed in order to obtain the existence of solutions when the set-valued map that define the problem has convex or non convex values.

1. Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order [1–3]. The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [4] allows to use Cauchy conditions which have physical meanings.

Recently, a new fractional order derivative with regular kernel has been introduced by Caputo and Fabrizio [5]. The Caputo-Fabrizio operator is useful for modeling several classes of problems with the dynamics having the exponential decay law. This new definition is able to describe better heterogeneousness, systems with different scales with memory effects, the wave movement on surface of shallow water, the heat transfer model, mass-spring-damper model [6]. Another good property of this new definition is that using Laplace transform of the fractional derivative the fractional differential equation turns into a classical differential equation of integer order. Properties of this definition have been studied in [5–8]. Several recent papers are devoted to qualitative results for fractional differential equations and inclusions defined by Caputo-Fabrizio fractional derivative [9–12].

The aim of the present paper is to study the set-valued framework for problems defined by Caputo-Fabrizio operator. More exactly, we consider the following boundary value problem

$$D_{CF}^{\sigma}x(t)\in F(t,x(t)) \quad a.e.\ ([0,1]), \quad x(0)=x_0,\ x(1)=x_1, \tag{1.1}$$

where $F(.,.):[0,1]\times\mathbf{R}\to\mathscr{P}(\mathbf{R})$ is a set-valued map, $x_0,x_1\in\mathbf{R}$ and D_{CF}^{σ} denotes Caputo-Fabrizio's fractional derivative of order $\sigma\in(1,2)$.

Our goal is to present several existence results for problem (1.1). The results are essentially based on a nonlinear alternative of Leray-Schauder type, on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and on Covitz and Nadler set-valued contraction principle. Even if we apply usual methods in the theory of existence of solutions for differential inclusions (e.g., [13]) the results obtained in the present paper are new in the framework of Caputo-Fabrizio fractional differential inclusions. As far as we know, in the literature there exists one paper dealing with fractional differential inclusions defined by Caputo-Fabrizio operator, namely [9]. In [9] it is considered a Cauchy problem, instead of a boundary value problem as in our approach.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

In this section we sum up some basic facts that we are going to use later. Let (X,d) be a metric space with the corresponding norm $|.|$ and denote $I=[0,1]$. Denote by $\mathscr{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathscr{P}(X)$ the family of all nonempty subsets

***Email address and ORCID number*: acernea@fmi.unibuc.ro, 0000-0002-9174-9855 (A. Cernea)**

of X and by $\mathscr{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A,B \subset X$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, where $d^*(A,B) = \sup\{d(a,B); a \in A\}$ and $d(x,B) = \inf_{y\in B} d(x,y)$.
As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$, by $AC(I,X)$ the Banach space of all absolutely continuous functions $x(.) : I \to X$ and by $L^p(I,X)$ the Banach space of all (Bochner) p-integrable functions $x(.) : I \to X$; in particular, $L^1(I,X)$ is the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_I |x(t)|\mathrm{d}t$. A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathscr{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$.
Consider $M : X \to \mathscr{P}(X)$ a set-valued map. A point $x \in X$ is called a fixed point for $M(.)$ if $x \in M(x)$. $M(.)$ is said to be bounded on bounded sets if $M(B) := \cup_{x\in B}M(x)$ is a bounded subset of X for all bounded sets B in X. $M(.)$ is said to be compact if $M(B)$ is relatively compact for any bounded sets B in X. $M(.)$ is said to be totally compact if $\overline{M(X)}$ is a compact subset of X. $M(.)$ is said to be upper semicontinuous if for any $x_0 \in X$, $M(x_0)$ is a nonempty closed subset of X and if for each open set D of X containing $M(x_0)$ there exists an open neighborhood V_0 of x_0 such that $M(V_0) \subset D$. Let E a Banach space, $Y \subset E$ a nonempty closed subset and $M(.) : Y \to \mathscr{P}(E)$ a multifunction with nonempty closed values. $M(.)$ is said to be lower semicontinuous if for any open subset $D \subset E$, the set $\{y \in Y; M(y) \cap D \neq \emptyset\}$ is open. $M(.)$ is called completely continuous if it is upper semicontinuous and totally compact on X. It is well known that a compact set-valued map $M(.)$ with nonempty compact values is upper semicontinuous if and only if $M(.)$ has a closed graph (e.g., [14]).
The next results are key tools in the proof of our theorems. We recall, first, the following nonlinear alternative of Leray-Schauder type proved in [15] and its consequences.

Theorem 2.1. *Let D and $\overline{D}$ be the open and closed subsets in a normed linear space X such that $0 \in D$ and let $M : \overline{D} \to \mathscr{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*
i) the inclusion $x \in M(x)$ has a solution, or
ii) there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in M(x)$ for some $\lambda > 1$.

Corollary 2.2. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $M : \overline{B_r(0)} \to \mathscr{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*
i) the inclusion $x \in M(x)$ has a solution, or
ii) there exists $x \in X$ with $|x| = r$ and $\lambda x \in M(x)$ for some $\lambda > 1$.

Corollary 2.3. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $M : \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*
i) the equation $x = M(x)$ has a solution, or
ii) there exists $x \in X$ with $|x| = r$ and $x = \lambda M(x)$ for some $\lambda < 1$.

If $G(.,.) : I \times X \to \mathscr{P}(X)$ is a set-valued map with compact values we define $S_G : C(I,X) \to \mathscr{P}(L^1(I,X))$ by $S_G(x) := \{g \in L^1(I,X); \quad g(t) \in G(t,x(t)) \quad a.e.\ (I)\}$. We say that $G(.,.)$ is of *lower semicontinuous type* if $S_G(.)$ is lower semicontinuous with nonempty closed and decomposable values. The next result is proved in [16].

Theorem 2.4. *Let S be a separable metric space and $G(.) : S \to \mathscr{P}(L^1(I,X))$ be a lower semicontinuous set-valued map with closed decomposable values. Then $G(.)$ has a continuous selection (i.e., there exists a continuous mapping $g(.) : S \to L^1(I,X)$ such that $g(s) \in G(s) \quad \forall\, s \in S$).*

A set-valued map $G : I \to \mathscr{P}(X)$ with nonempty compact convex values is said to be *measurable* if for any $x \in X$ the function $t \to d(x, G(t))$ is measurable. A set-valued map $G(.,.) : I \times X \to \mathscr{P}(X)$ is said to be *Carathéodory* if $t \to G(t,x)$ is measurable for any $x \in X$ and $x \to G(t,x)$ is upper semicontinuous for almost all $t \in I$. Moreover, $G(.,.)$ is said to be L^1-*Carathéodory* if for any $r > 0$ there exists $p_r(.) \in L^1(I, \mathbf{R})$ such that $\sup\{|v|; v \in G(t,x)\} \le p_r(t)$ a.e. (I), $\forall\, x \in \overline{B_r(0)}$. The following theorem is proved in [17].

Theorem 2.5. *Let X be a Banach space, let $G(.,.) : I \times X \to \mathscr{P}(X)$ be a L^1-Carathéodory set-valued map with $S_G(x) \neq \emptyset$ for all $x(.) \in C(I,X)$ and let $\Gamma : L^1(I,X) \to C(I,X)$ be a linear continuous mapping. Then the set-valued map $\Gamma \circ S_G : C(I,X) \to \mathscr{P}(C(I,X))$ defined by*

$$(\Gamma \circ S_G)(x) = \Gamma(S_G(x))$$

has compact convex values and has a closed graph in $C(I,X) \times C(I,X)$.

Note that if $dim X < \infty$, and $G(.,.)$ is as in Theorem 2.5, then $S_G(x) \neq \emptyset$ for any $x(.) \in C(I,X)$ (e.g., [17]).
The next definitions have been introduced by Caputo and Fabrizio in [5].

Definition 2.6. *a) Caputo-Fabrizio integral of order $\alpha \in (0,1)$ of a function $f \in AC_{loc}([0,\infty), \mathbf{R})$ (which means that $f'(.)$ is integrable on $[0,T]$ for any $T > 0$) is defined by*

$$I^{\alpha}_{CF} f(t) = (1-\alpha)f(t) + \alpha \int_0^t f(s)ds.$$

b) Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ of f is defined for $t \ge 0$ by

$$D^{\alpha}_{CF} f(t) = \frac{1}{1-\alpha}\int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s)ds.$$

c) Caputo-Fabrizio fractional derivative of order $\sigma = \alpha + n$, $\alpha \in (0,1)$ $n \in \mathbf{N}$ of f is defined by

$$D^{\sigma}_{CF} f(t) = D^{\alpha}_{CF}(D^n_{CF} f(t)).$$

In particular, if $\sigma = \alpha + 1$, $\alpha \in (0,1)$ $D^{\sigma}_{CF} f(t) = \frac{1}{1-\alpha}\int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f''(s)ds.$

Definition 2.7. *A mapping $x(.) \in AC(I,\mathbf{R})$ is called a solution of problem (1.1) if there exists a function $f(.) \in L^1(I,\mathbf{R})$ such that $f(t) \in F(t,x(t))$ a.e. (I), $D^{\alpha}_{CF}x(t) = f(t)$, $t \in I$ and $x(0) = x_0$, $x(1) = x_1$.*

In order to prove our results we also need the next result proved in [11] (namely, Theorem 3.4).

Lemma 2.8. *For $\sigma = \alpha + 1$, $\alpha \in (0,1)$ and $f(.) \in L^1(I,\mathbf{R})$ the boundary value problem*

$$D^{\sigma}_{CF}x(t) = f(t), \quad x(0) = x_0,\ x(1) = x_1,$$

has a unique solution given by

$$x(t) = x_0 + (x_1 - x_0)t + (1-\alpha)(1-t)\int_0^t f(s)ds + \alpha(1-t)\int_0^t sf(s)ds - (1-\alpha)t\int_t^1 f(s)ds - \alpha t\int_t^1 (1-s)f(s)ds. \tag{2.1}$$

Remark 2.9. *If we define*

$$\mathscr{G}(t,s) = [(1-\alpha)(1-t) + \alpha(1-t)s]\chi_{[0,t]}(s) - [(1-\alpha)t + \alpha t(1-s)]\chi_{[t,1]}(s)$$

then the solution in (2.1) may be written as

$$x(t) = x_0 + (x_1 - x_0)t + \int_0^1 \mathscr{G}(t,s)f(s)ds.$$

Moreover, for any $s,t \in I$, $|\mathscr{G}(t,s)| \leq (1-\alpha) + \alpha + (1-\alpha) + \alpha = 2$.

3. The results

We present now the existence results for problem (1.1). We consider, first, the case when $F(.,.)$ is convex valued and is upper semicontinuous in the state variable.

Hypothesis 1. i) $F(.,.) : I \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ has nonempty compact convex values and is Carathéodory.
ii) There exists $\varphi(.) \in L^1(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\{|v|; \quad v \in F(t,x)\} \leq \varphi(t)\psi(|x|) \quad a.e.\ (I), \quad \forall\, x \in \mathbf{R}.$$

Theorem 3.1. *Assume that Hypothesis 1 is satisfied and there exists $r > 0$ such that*

$$r > |x_0| + |x_1 - x_0| + 2|\varphi|_1\psi(r). \tag{3.1}$$

Then problem (1.1) has at least one solution $x(.)$ such that $|x(.)|_C < r$.

Proof. Consider $X = C(I,\mathbf{R})$ and let $r > 0$ be as in (3.1). From Definition 2.7 and Remark 2.9, the existence of solutions to problem (1.1) reduces to the existence of the solutions of the integral inclusion

$$x(t) \in x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)F(s,x(s))ds, \quad t \in I. \tag{3.2}$$

Defined the set-valued map $M : \overline{B_r(0)} \to \mathscr{P}(C(I,\mathbf{R}))$ by

$$M(x) := \{v(.) \in C(I,\mathbf{R}); v(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds, f \in S_F(x)\}. \tag{3.3}$$

We show that $M(.)$ satisfies the hypotheses of Corollary 2.2. First, we show that $M(x) \subset C(I,\mathbf{R})$ is convex for any $x \in C(I,\mathbf{R})$. If $v_1, v_2 \in M(x)$ then there exist $f_1, f_2 \in S_F(x)$ such that for any $t \in I$ one has $v_i(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f_i(s)ds$, $i = 1,2$.
Let $0 \leq \alpha \leq 1$. Then for any $t \in I$ we have $(\alpha v_1 + (1-\alpha)v_2)(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)[\alpha f_1(s) + (1-\alpha)f_2(s)]ds$. The values of $F(.,.)$ are convex; thus, $S_F(x)$ is a convex set and hence, $\alpha f_1 + (1-\alpha)f_2 \in M(x)$.
We show, secondly, that $M(.)$ is bounded on bounded sets of $C(I,\mathbf{R})$. Let $B \subset C(I,\mathbf{R})$ be a bounded set. Then there exist $m > 0$ such that $|x|_C \leq m\ \forall x \in B$. If $v \in M(x)$ there exists $f \in S_F(x)$ such that $v(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds$. One has for any $t \in I$

$$|v(t)| \leq |x_0| + |x_1 - x_0| + \int_0^t |\mathscr{G}(t,s)|.|f(s)|ds \leq |x_0| + |x_1 - x_0| + \int_0^t |\mathscr{G}(t,s)|\varphi(s)\psi(|x(t)|)ds$$

and therefore, $|v|_C \leq |x_0| + |x_1 - x_0| + 2|\varphi|_1\psi(m) \quad \forall\, v \in M(x)$, i.e., $M(B)$ is bounded.
Next we prove that $M(.)$ maps bounded sets into equi-continuous sets. Let $B \subset C(I,\mathbf{R})$ be a bounded set as before and $v \in M(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds$. Then for any $t, \tau \in I$ we have

$$|v(t) - v(\tau)| \leq |x_0 + (x_1 - x_0)t - a(\tau)| + |\int_0^t \mathscr{G}(t,s)f(s)ds - \int_0^t \mathscr{G}(\tau,s)f(s)ds| + |\int_\tau^t \mathscr{G}(\tau,s)f(s)ds| \leq |x_0 + (x_1 - x_0)t - a(\tau)| + 2\int_\tau^t \varphi(s)\psi(m)ds + \int_0^t |\mathscr{G}(t,s) - \mathscr{G}(\tau,s)|\varphi(s)\psi(m)ds.$$

Thus, $|v(t) - v(\tau)| \to 0$ as $\tau \to t$. It follows that $M(B)$ is an equi-continuous set in $C(I,\mathbf{R})$. It remains to apply Arzela-Ascoli's theorem to deduce that $M(.)$ is completely continuous on $C(I,\mathbf{R})$.
At the last step of the proof we prove that $M(.)$ has a closed graph. Let $x_n \in C(I,\mathbf{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in M(x_n)\ \forall\, n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in M(x^*)$. Since $v_n \in M(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f_n(s)ds$. Define $\Gamma : L^1(I,\mathbf{R}) \to C(I,\mathbf{R})$ by $(\Gamma(f))(t) := \int_0^t \mathscr{G}(t,s)f(s)ds$. One has $\max_{t\in I}|v_n(t) - x_0 - (x_1 - x_0)t - (v^*(t) - x_0 - (x_1 - x_0)t)| = |v_n(.) - v^*(.)|_C \to 0$ as $n \to \infty$. We apply Theorem 2.5 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we obtain $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of $f^* \in S_F(x^*)$ such that $v^*(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f^*(s)ds$.
Therefore, $M(.)$ is upper semicontinuous and compact on $\overline{B_r(0)}$. We apply Corollary 2.2 to deduce that either i) the inclusion $x \in M(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $\lambda x \in M(x)$ for some $\lambda > 1$.
Assume that ii) is true. With the same arguments as in the second step of our proof we get $r = |x(.)|_C \leq |x_0| + |x_1 - x_0| + 2|\varphi|_1\psi(r)$ which contradicts (3.1). Hence, only i) is valid and theorem is proved. □

We consider, now, the case when $F(.,.)$ is not necessarily convex valued. In the first approach, $F(.,.)$ is lower semicontinuous in the state variable and, in this case, the existence result is based on the Leray-Schauder alternative for single valued maps and on Bressan-Colombo selection theorem.

Hypothesis 2. i) $F(.,.) : I \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ has compact values, $F(.,.)$ is $\mathscr{L}(I) \otimes \mathscr{B}(\mathbf{R})$ measurable and $x \to F(t,x)$ is lower semicontinuous for almost all $t \in I$.
ii) There exists $\varphi(.) \in L^1(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\{|v|; \quad v \in F(t,x)\} \le \varphi(t)\psi(|x|) \quad a.e.\ (I), \quad \forall\, x \in \mathbf{R}.$$

Theorem 3.2. *Assume that Hypothesis 2 is satisfied and there exists $r > 0$ such that condition (3.1) is satisfied. Then problem (1.1) has at least one solution on I.*

Proof. We point out, first, that if Hypothesis 2 is satisfied then $F(.,.)$ is of lower semicontinuous type (e.g., [18]). Therefore, by Theorem 2.4 applied with $S = C(I,\mathbf{R})$ and $G(.) = S_F(.)$ we find a continuous mapping $f(.) : C(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ such that $f(x) \in S_F(x)\ \forall\, x \in C(I,\mathbf{R})$. Consider problem

$$x(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(x(s))ds, \quad t \in I \tag{3.4}$$

in the space $X = C(I,\mathbf{R})$. By Definition 2.7 and Remark 2.9, if $x(.) \in C(I,\mathbf{R})$ is a solution of the problem (3.4) then $x(.)$ is a solution to problem (1.1). Let $r > 0$ that satisfies condition (3.1) and define $M : \overline{B_r(0)} \to C(I,\mathbf{R})$ by

$$(M(x))(t) := x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(x(s))ds.$$

The integral equation (3.4) is equivalent with the operator equation

$$x(t) = (M(x))(t), \quad t \in I. \tag{3.5}$$

We show, next, that $M(.)$ satisfies the hypotheses of Corollary 2.3. We note that $M(.)$ is continuous on $\overline{B_r(0)}$. By Hypotheses 2 ii) we have $|f(x(t))| \le \varphi(t)\psi(|x(t)|)$ $a.e.\ (I)$ for all $x(.) \in C(I,\mathbf{R})$. Consider $x_n, x \in \overline{B_r(0)}$ such that $x_n \to x$. Then $|f(x_n(t))| \le \varphi(t)\psi(r)$ $a.e.\ (I)$. Using Lebesgue's dominated convergence theorem and the continuity of $f(.)$ we obtain, for all $t \in I$, $\lim_{n\to\infty} \int_0^t \mathscr{G}(t,s)f(x_n(s))ds = \int_0^t \mathscr{G}(t,s)f(x(s))ds$ which provides the continuity of $M(.)$ on $\overline{B_r(0)}$.
As in the proof of Theorem 3.1, it follows that $M(.)$ is compact on $\overline{B_r(0)}$. With Corollary 2.3 we deduce that either i) the equation $x = M(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $x = \lambda M(x)$ for some $\lambda < 1$. Repeating the argument as in the proof of Theorem 3.1, if the statement ii) holds true, then we obtain a contradiction to (3.1). Thus, only the statement i) is true and problem (1.1) has a solution $x(.) \in C(I,\mathbf{R})$ with $|x(.)|_C < r$. □

The second approach concerns the situation when the set-valued map is Lipschitz in the state variable. In order to obtain an existence result for problem (1.1) by using the set-valued contraction principle we introduce the following hypothesis on F.

Hypothesis 3. i) $F : I \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ has nonempty compact values is integrably bounded and for every $x \in \mathbf{R}$, $F(.,x)$ is measurable.
ii) There exists $l \in L^1(I,\mathbf{R}_+)$ such that for almost all $t \in I$,

$$d_H(F(t,x_1),F(t,x_2)) \le l(t)|x_1 - x_2| \quad \forall\ x_1,x_2 \in \mathbf{R}.$$

iii) There exists $L \in L^1(I,\mathbf{R}_+)$ such that for almost all $t \in I$, $d(0,F(t,0)) \le L(t)$.

Theorem 3.3. *Assume that Hypothesis 3 is satisfied and $2|l|_1 < 1$. Then problem (1.1) has a solution.*

Proof. We consider problem (1.1) as a fixed point problem. More precisely, define the set-valued map $M : C(I,\mathbf{R}) \to \mathscr{P}(C(I,\mathbf{R}))$ by

$$M(x) := \{v(.) \in C(I,\mathbf{R}); v(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds, f \in S_F(x)\}.$$

The multifunction $t \to F(t,x(t))$ is measurable; thus, with the measurable selection theorem it has a measurable selection $f : I \to \mathbf{R}$. At the same time, since F is integrably bounded, $f \in L^1(I,\mathbf{R})$. Hence, $S_F(x) \neq \emptyset$. The fixed points of M are solutions of problem (1.1). We show, next, that M verifies the assumptions of Covitz-Nadler contraction principle ([19]). Since $S_F(x) \neq \emptyset$, it follows that $M(x) \neq \emptyset$ for any $x \in C(I,\mathbf{R})$.
Now, we prove that $M(x)$ is closed for any $x \in C(I,\mathbf{R})$. Let $\{x_n\}_{n\ge 0} \in M(x)$ such that $x_n \to x^*$ in $C(I,\mathbf{R})$. Then $x^* \in C(I,\mathbf{R})$ and there exists $f_n \in S_F(x_n)$ such that $x_n(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f_n(s)ds$, $t \in I$. From Hypothesis 3 and the fact that the values of F are compact, one may pass to a subsequence to obtain that f_n converges to $f \in L^1(I,\mathbf{R})$ in $L^1(I,\mathbf{R})$. In particular, $f \in S_F(x)$ and for any $t \in I$ we have $x_n(t) \to x^*(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds$, i.e., $x^* \in M(x)$ and $M(x)$ is closed.
It remains to prove that M is a contraction on $C(I,\mathbf{R})$. Let $x_1, x_2 \in C(I,\mathbf{R})$ and $v_1 \in T(x_1)$. Then, there exists $f_1 \in S_F(x_1)$ such that $v_1(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f_1(s)ds$, $t \in I$. Consider the multifunction

$$S(t) := F(t,x_2(t)) \cap \{x \in \mathbf{R};\ |f_1(t) - x| \le l(t)|x_1(t) - x_2(t)|\}, \quad t \in I.$$

Taking into account Hypothesis 3, one has

$$d_H(F(t,x_1(t)),F(t,x_2(t))) \le l(t)|x_1(t) - x_2(t)|, \quad t \in I,$$

i.e., S has nonempty closed values. On the other hand, S is measurable; thus, there exists f_2 a measurable selection of S. It follows that $f_2 \in S_F(x_2)$ and for any $t \in I$, $|f_1(t) - f_2(t)| \le l(t)|x_1(t) - x_2(t)|$. Define

$$v_2(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s) f_2(s) ds, \quad t \in I.$$

One has $|v_1(t) - v_2(t)| \le \int_0^t |\mathscr{G}(t,s)||f_1(s) - f_2(s)|ds \le 2\int_0^t l(s)|x_1(s) - x_2(s)|ds \le 2|l|_1|x_1 - x_2|_C$. Therefore, $|v_1 - v_2|_C \le 2|l|_1|x_1 - x_2|_C$. By interchanging the roles of x_1 and x_2 we deduce

$$d_H(M(x_1), M(x_2)) \le 2|l|_1|x_1 - x_2|_C.$$

Thus, M has a fixed point which is a solution to problem (1.1). □

4. Conclusions

In this paper we obtained several existence results for solutions of a bilocal problem associated to a fractional differential inclusion defined by Caputo-Fabrizio operator. In the case when the values of the set-valued map that defines the differential inclusion are convex and the set-valued map is upper semicontinuous in the state variable, the proof is based on a nonlinear alternative of Leray-Schauder type; in the situation when the values of the set-valued map are not necessarily convex and the set-valued map is lower semicontinuous in the state variable, the proof relies on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values. Also, if the multifunction has non convex values and is Lipischitz in the state variable an existence result is provided by applying Covitz and Nadler set-valued contraction principle. Such kind of results, that are new in the framework of Caputo-Fabrizio fractional differential inclusions, may be useful, afterwards, in order to obtain qualitative properties concerning the solutions of the problem considered.

References

[1] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.
[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[3] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[4] M. Caputo, *Elasticità e Dissipazione*, Zanichelli, Bologna, 1969.
[5] M. Caputo, M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progr. Fract. Differ. Appl., **1** (2015), 1-13.
[6] M.A. Refai, K. Pal, *New aspects of Caputo-Fabrizio fractional derivative*, Progr. Fract. Differ. Appl., **5** (2019), 157-166.
[7] T.M. Atanacković, S. Pilipović, D. Zorica, *Properties of the Caputo-Fabrizio fractional derivative and its distributional settings*, Frac. Calc. App. Anal., **21** (2018), 29-44.
[8] M. Caputo, M. Fabrizio, *Applications of new time and spatial fractional derivatives with exponential kernels*, Progr. Fract. Differ. Appl., **2** (2016), 1-11.
[9] D. Băleanu, S. Rezapour, Z. Saberpour, *On fractional integro-differential inclusions via the extended fractional Caputo-Fabrizio derivation*, Boundary Value Problems, **219**(79) (2019), 1-17.
[10] A. Shaikh, A. Tassaddiq, K.S. Nisar, D. Băleanu, *Analysis of differential equations involving Caputo-Fabrizio fractional operator and its applications to reaction-diffusion equations*, Adv. Difference Equations, **2019**(178) (2019), 1-14.
[11] Ş. Toprakseven, *The existence and uniqueness of initial-boundary value problems of the Caputo-Fabrizio differential equations*, Universal J. Math. Appl., **2** (2019), 100-106.
[12] S. Zhang, L. Hu, S. Sun, *The uniqueness of solution for initial value problems for fractional differential equations involving the Caputo-Fabrizio derivative*, J. Nonlinear Sci. Appl., **11** (2018), 428-436.
[13] A. Cernea, *On a Sturm-Liouville type differential inclusion of fractional order*, Fract. Differ. Calc., **7** (2017) 385-393.
[14] J.P. Aubin, H. Frankowska, *Set-valued Analysis*, Birkhauser, Basel, 1990.
[15] D. O' Regan, *Fixed point theory for closed multifunctions*, Arch. Math. (Brno), **34** (1998), 191-197.
[16] A. Bressan, G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math., **90** (1988), 69-86.
[17] A. Lasota, Z. Opial, *An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. Math., Astronom. Physiques, **13** (1965), 781-786.
[18] M. Frignon, A. Granas, *Théorèmes d'existence pour les inclusions différentielles sans convexité*, C. R. Acad. Sci. Paris, Ser. I, **310** (1990), 819-822.
[19] H. Covitz, S.B. Nadler jr., *Multivalued contraction mapping in generalized metric spaces*, Israel J. Math., **8** (1970), 5-11.

Universal Journal of Mathematics and Applications, 3 (4) (2020) 138-143
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.743949

On the $\Delta^f_{\Lambda^2}$-Statistical Convergence on Product Time Scale

Bayram Sözbir[1*], Selma Altundağ[1] and Metin Başarır[1]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, 54050, Sakarya, Turkey*
[*] *Corresponding author*

Article Info

Keywords: *Delta measure, Density, Modulus function, Product time scale, Statistical convergence, Strong Cesaro summability.*
2010 AMS: *40G15, 40A35, 46A45, 26E70, 34N05.*
Received: *28 May 2020*
Accepted: *22 October 2020*
Available online: *23 December 2020*

Abstract

In this paper, we first define a new density of a Δ-measurable subset of a product time scale Λ^2 with respect to an unbounded modulus function. Then, by using this definition, we introduce the concepts of $\Delta^f_{\Lambda^2}$-statistical convergence and $\Delta^f_{\Lambda^2}$-statistical Cauchy for a Δ-measurable real-valued function defined on product time scale Λ^2 and also obtain some results about these new concepts. Finally, we present the definition of strong $\Delta^f_{\Lambda^2}$-Cesaro summability on Λ^2 and investigate the connections between these new concepts.

1. Introduction

The idea of statistical convergence of number sequences was formally introduced by Fast [1] and also independently Steinhaus [2]. This concept is a generalization of classical convergence and has a close relation with the concept of density of the subset of natural numbers $\mathbb{N}$. The natural density of $K \subseteq \mathbb{N}$ is defined by $\delta(K) = \lim_n n^{-1} |\{k \leq n : k \in K\}|$ if the limit exists, where and throughout the paper $|K|$ denotes the cardinality of K. A sequence $x = (x_k)$ is said to be statistically convergent to L if, for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

and we denote this by $st - \lim x = L$. In later years, statistical convergence has taken a very important place in mathematical analysis and has been studied by many researchers, see [3–12]. Another notion that can be of importance is modulus function which was first given by Nakano [13]. The readers can consult the works [14–16] for more on this function. We remind here that a modulus $f : [0,\infty) \to [0,\infty)$ is a function which satisfies
i) $f(x) = 0$ if and only if $x = 0$,
ii) $f(x+y) \leq f(x) + f(y)$ for every $x \geq 0$, $y \geq 0$,
iii) f is increasing,
iv) f is continuous from right at 0.
We can easily see that a modulus function f is continuous everywhere on $[0,\infty)$ from above properties (ii) and (iv). A modulus function may be bounded or unbounded. As in example, $f(x) = \frac{x}{1+x}$ is bounded, while $f(x) = x^p$ is unbounded where $0 < p \leq 1$.
In [17], by means of an unbounded modulus function, Aizpuru et al. firstly presented a new idea of density for the subset of $\mathbb{N}$. With this way, they also defined a new convergence idea with the name f-statistical convergence and show that it is between classical convergence and statistical convergence. The readers can found further works using this concept in the references [18, 19].
A time scale is an arbitrary closed subset of the real numbers $\mathbb{R}$ and it is denoted by the symbol $\mathbb{T}$. We here suppose that it has the subspace topology which is inherited from $\mathbb{R}$ with the standart topology. The calculus of time scales was constructed by Hilger [20], and it allows to the unification of continuous and discrete cases. After that, this theory has received much attention [21–26] as it has tremendous potential for applications. Moreover, the idea of statistical convergence has been studied on time scales in [27] and [28], independently. Later, by inspiring from these works, various researchers have done many studies using the time scale on the summability theory in the literature, see [29–39]. Let's now remember some necessary concepts about the time scale calculus before proceeding further.

Email addresses and ORCID numbers: **bayramsozbir@gmail.com, 0000-0002-9475-7180 (B. Sözbir), scaylan@sakarya.edu.tr, 0000-0002-5893-9868 (S. Altundağ), basarir@sakarya.edu.tr, 0000-0002-4341-4399 (M. Başarır)**

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. Here we take $\inf\emptyset = \sup\mathbb{T}$, where $\emptyset$ is an empty set. For $a \leq b$, a closed interval in $\mathbb{T}$ is defined by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. Similarly, half-open intervals or open intervals can be defined on time scales. Let F_1 denote the family of all intervals of $\mathbb{T}$ having the form $[a,b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t < b\}$ with $a,b \in \mathbb{T}$ and $a \leq b$. Then the set function $m_1 : F_1 \to [0,\infty)$ define as $m_1([a,b)_{\mathbb{T}}) = b - a$ is a countably additive measure on F_1. The Caratheodory extension of the set function m_1 associated with family F_1 is said to be the Lebesgue $\Delta-$measure on $\mathbb{T}$ and also this is denoted by μ_Δ, see [23]. Also from the work [23] by Guseinov, one knows that if $a \in \mathbb{T}\setminus\{\max\mathbb{T}\}$, then the single point set $\{a\}$ is $\Delta-$measurable and $\mu_\Delta(\{a\}) = \sigma(a) - a$. If $a,b \in \mathbb{T}$ and $a \leq b$, then $\mu_\Delta([a,b)_{\mathbb{T}}) = b - a$ and $\mu_\Delta((a,b)_{\mathbb{T}}) = b - \sigma(a)$. If $a,b \in \mathbb{T}\setminus\{\max\mathbb{T}\}$ and $a \leq b$, then $\mu_\Delta((a,b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$ and $\mu_\Delta([a,b]_{\mathbb{T}}) = \sigma(b) - a$.
Turan and Başarır [36] gave Δ_f-convergence by combining the ideas of Seyyidoğlu and Tan [27], Turan and Duman [28], and Aizpuru et al. [17] as in the following:

Definition 1.1. *[36] Let $\mathbb{T}$ be a time scale such that $\inf\mathbb{T} = \alpha > 0$ and $\sup\mathbb{T} = \infty$ and let f be a modulus function. A $\Delta-$measurable function $g : \mathbb{T} \to \mathbb{R}$ is Δ_f- convergent to a number L on $\mathbb{T}$, if for every $\varepsilon > 0$*

$$\lim_{t\to\infty} \frac{f(\mu_\Delta(\{s \in [\alpha,t]_{\mathbb{T}} : |g(s) - L| \geqslant \varepsilon\}))}{f(\mu_\Delta([\alpha,t]_{\mathbb{T}}))} = 0,$$

which is denoted by $\Delta_f - \lim_{t\to\infty} g(t) = L$

Quite recently, Çınar et al. [32] carried statistical convergence and its related concepts which are given on 1-dimensional time scales to an arbitrary product time scales. Before remembering these definitions, let's give some necessary concepts and notations that we will use throughout this study. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be a time scale. Consider the Cartesian product

$$\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (t_1,t_2) : t_1 \in \mathbb{T}_1 \text{ and } t_2 \in \mathbb{T}_2\}.$$

Then Λ^2 is called an 2-dimensional time scale or product time scale. Here, we are interested in a product time scale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$ such that $\inf\mathbb{T}_1 = t_0$ and $\sup\mathbb{T}_1 = \infty$; $\inf\mathbb{T}_2 = r_0$ and $\sup\mathbb{T}_2 = \infty$. For convenience, we denote $A := \left\{[t_0,t]_{\mathbb{T}_1} \times [r_0,r]_{\mathbb{T}_2}\right\}$ for $(t,r) \in \Lambda^2$. Thanks to the work [25] given by Bohner and Guseinov, it is clear that $\mu_\Delta(A) = \mu_\Delta\left([t_0,t]_{\mathbb{T}_1}\right).\mu_\Delta\left([r_0,r]_{\mathbb{T}_2}\right)$.

Definition 1.2. *[32] Let $g : \Lambda^2 \to \mathbb{R}$ be a Δ-measurable function. Then g is said to be statistically convergent to L on Λ^2 , if for every $\varepsilon > 0$,*

$$\lim_{(t,r)\to\infty} \frac{\mu_\Delta(\{(s,u) \in A : |g(s,u) - L| \geq \varepsilon\})}{\mu_\Delta(A)} = 0,$$

which is denoted by $st_{\Lambda^2} - \lim_{(t,r)\to\infty} g(t,r) = L$.

Definition 1.3. *[32] Let $g : \Lambda^2 \to \mathbb{R}$ be a Δ-measurable function and $0 < p < \infty$. Then we say that g is strongly p-double Cesaro summable to L on Λ^2, if*

$$\lim_{(t,r)\to\infty} \frac{1}{\mu_\Delta(A)} \iint_A |g(s,u) - L|^p \Delta s \Delta u = 0.$$

We write $\left[w_p\right]_{\Lambda^2}$ for the set of all strongly p-double Cesaro summable functions on Λ^2.

The aim of this study is to extend the concept of f-statistical convergence and its related notions to any product time scale, in light of works Aizpuru et al. [17], Turan and Başarır [36] and Çınar et al. [32].
This paper has the following order. In Section 2, we introduce the new notions such as $\Delta^f_{\Lambda^2}$-density, $\Delta^f_{\Lambda^2}$-statistical convergence and $\Delta^f_{\Lambda^2}$-statistical Cauchy on product time scales, where f is any unbounded modulus. We also establish some results related to these new concepts. In Section 3, the definition of strong $\Delta^f_{\Lambda^2}$-Cesaro summability on any product time scale is presented, and we examine the connections between strong $\Delta^f_{\Lambda^2}$-Cesaro summability and $\Delta^f_{\Lambda^2}$-statistical convergence, Cesaro summability.

2. $\Delta^f_{\Lambda^2}$-Density, $\Delta^f_{\Lambda^2}$-Statistical Convergence and $\Delta^f_{\Lambda^2}$-Statistical Cauchy on Product Time Scale

We first define a new type of density on a product time scale Λ^2, namely $\Delta^f_{\Lambda^2}$-density, by using the idea of Aizpuru et al. [17]. Then, with the aid of this definition, the new concepts such as $\Delta^f_{\Lambda^2}$-statistical convergence and $\Delta^f_{\Lambda^2}$-statistical Cauchy on any product time scale are introduced. Throughout the paper let f be an unbounded modulus function.

Definition 2.1. *Let Ω be a Δ-measurable subset of Λ^2. Then, the $\Delta^f_{\Lambda^2}$-density of Ω on Λ^2 is defined by*

$$\delta^f_{\Lambda^2}(\Omega) = \lim_{(t,r)\to\infty} \frac{f(\mu_\Delta(\Omega(t,r)))}{f(\mu_\Delta(A))}$$

if this limit exists, where $\Omega(t,r) = \{(s,u) \in A : (s,u) \in \Omega\}$ for $(t,r) \in \Lambda^2$.

Definition 2.2. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, we say that g is $\Delta^f_{\Lambda^2}$-statistically convergent to L on Λ^2, if for every $\varepsilon>0$,*

$$\delta^f_{\Lambda^2}\left(\left\{(t,r)\in\Lambda^2:|g(t,r)-L|\geq\varepsilon\right\}\right)=0$$

holds, i.e.,

$$\lim_{(t,r)\to\infty}\frac{f(\mu_\Delta(\{(s,u)\in A:|g(s,u)-L|\geq\varepsilon\}))}{f(\mu_\Delta(A))}=0,$$

which is denoted by $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$. Also, we denote the set of all $\Delta^f_{\Lambda^2}$-statistically convergent functions on Λ^2 by $S^f_{\Lambda^2}$.

Remark 2.3. *If we choose $f(x)=x$ in Definition 2.2, then $\Delta^f_{\Lambda^2}$-statistical convergence is reduced to statistical convergence given in Definition 1.2.*

Proposition 2.4. *If $g:\Lambda^2\to\mathbb{R}$ is $\Delta^f_{\Lambda^2}$-statistically convergent function, then its limit is unique.*

Proof. The proof can be carried out by using similar techniques to Proposition 2.4 in [32]. □

Proposition 2.5. *Let $g,h:\Lambda^2\to\mathbb{R}$ be Δ-measurable functions with $st^f_{\Lambda^2}-\lim g(t,r)=L_1$ and $st^f_{\Lambda^2}-\lim h(t,r)=L_2$. Then, we have:*
i) $st^f_{\Lambda^2}-\lim(g(t,r)+h(t,r))=L_1+L_2$,
ii) $st^f_{\Lambda^2}-\lim(cg(t,r))=cL_1$ for any $c\in\mathbb{R}$.

Proof. The proof can be carried out by using similar techniques to Proposition 2.5 in [32]. □

Theorem 2.6. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. If $\lim_{(t,r)\to\infty}g(t,r)=L$, then $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$.*

Proof. Suppose that $\lim_{(t,r)\to\infty}g(t,r)=L$. Then, the set $\left\{(s,u)\in\Lambda^2:|g(s,u)-L|\geqslant\varepsilon\right\}$ is bounded, for each $\varepsilon>0$. Since

$$\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}\subset\left\{(s,u)\in\Lambda^2:|g(s,u)-L|\geqslant\varepsilon\right\}$$

and modulus function f is increasing, we get

$$\frac{f(\mu_\Delta(\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}))}{f(\mu_\Delta(A))}\leqslant\frac{f\left(\mu_\Delta\left(\left\{(s,u)\in\Lambda^2:|g(s,u)-L|\geqslant\varepsilon\right\}\right)\right)}{f(\mu_\Delta(A))}.$$

Taking limit as $(t,r)\to\infty$ in here, we obtain

$$\lim_{(t,r)\to\infty}\frac{f(\mu_\Delta(\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}))}{f(\mu_\Delta(A))}=0,$$

which means that $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$. □

Theorem 2.7. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$ implies $st_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$.*

Proof. Suppose that $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$. Then, using the limit definition and also properties of subadditivity of the modulus function f, for every $p\in\mathbb{N}$, for sufficiently large $(t,r)\in\Lambda^2$, we have

$$f(\mu_\Delta(\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}))\leqslant\frac{1}{p}f(\mu_\Delta(A))\leqslant\frac{1}{p}pf\left(\frac{\mu_\Delta(A)}{p}\right)=f\left(\frac{\mu_\Delta(A)}{p}\right).$$

Also, since f is increasing, we get

$$\frac{\mu_\Delta(\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\})}{\mu_\Delta(A)}\leqslant\frac{1}{p},$$

which means that $st_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$. □

Corollary 2.8. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, we have*

$$\lim_{(t,r)\to\infty}g(t,r)=L\Rightarrow st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L\Rightarrow st_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L.$$

Theorem 2.9. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function and $h:\mathbb{R}\to\mathbb{R}$ be a continuous function at L. If $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$, then $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}h(g(t,r))=h(L)$.*

Proof. Using techniques similar to Lemma 3.11 in [28], the proof can be carried out easily and is therefore omitted. □

Definition 2.10. *A Δ-measurable function $g:\Lambda^2\to\mathbb{R}$ is $\Delta^f_{\Lambda^2}$-statistical Cauchy on Λ^2, if for every $\varepsilon>0$, there exist some numbers $t_1>t_0$ and $r_1>r_0$ such that $\delta^f_{\Lambda^2}\left(\left\{(t,r)\in\Lambda^2:|g(t,r)-g(t_1,r_1)|\geq\varepsilon\right\}\right)=0$.*

Theorem 2.11. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, the following statements are equivalent:*
i) g is $\Delta^f_{\Lambda^2}$-statistical convergent on Λ^2,
ii) g is $\Delta^f_{\Lambda^2}$-statistical Cauchy on Λ^2.

Proof. Using techniques similar to Theorem 3 in [27], the proof can be carried out easily and is therefore omitted. □

3. Strong $\Delta^f_{\Lambda^2}$-Cesaro Summability on Product Time Scale

We begin in here by presenting the last new definition, namely, strong $\Delta^f_{\Lambda^2}$-Cesaro summability on Λ^2.

Definition 3.1. *Let f be a modulus function and $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, we say that g is strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L on Λ^2, if*

$$\lim_{(t,r)\to\infty}\frac{1}{\mu_\Delta(A)}\iint\limits_A f(|g(s,u)-L|)\Delta s\Delta u=0.$$

We also denote the set of all strongly $\Delta^f_{\Lambda^2}$-Cesaro summable functions on Λ^2 by $[w]^f_{\Lambda^2}$.

Lemma 3.2. *[15] Let f be any modulus function and let $0<\delta<1$. Then, for each $x\geqslant\delta$, we have $f(x)\leqslant 2f(1)\delta^{-1}x$.*

Lemma 3.3. *[16] Let f be any modulus function. Then $\lim\limits_{t\to\infty}\frac{f(t)}{t}$ exists.*

The next theorem gives us the connection between the concepts of strong $\Delta^f_{\Lambda^2}$-Cesaro summability and strong double Cesaro summability given in Definition 1.3.

Theorem 3.4. ***i)** For any modulus function f, we have $[w]_{\Lambda^2}\subset[w]^f_{\Lambda^2}$.*
***ii)** Let f be any modulus function. If $\lim\limits_{t\to\infty}\frac{f(t)}{t}>0$, then we have $[w]^f_{\Lambda^2}\subset[w]_{\Lambda^2}$.*

Proof. **i)** Let $g\in[w]_{\Lambda^2}$ with the limit L. Then, we have

$$\lim_{(t,r)\to\infty}\frac{1}{\mu_\Delta(A)}\iint\limits_A|g(s,u)-L|\Delta s\Delta u=0.$$

Since modulus f is continuous, for any given $\varepsilon>0$, we may choose δ with $0<\delta<1$ such that $f(t)<\varepsilon$ for every t with $0\leqslant t\leqslant\delta$. Then, by Lemma 3.2, we write

$$\begin{aligned}\frac{1}{\mu_\Delta(A)}\iint\limits_A f(|g(s,u)-L|)\Delta s\Delta u&=\frac{1}{\mu_\Delta(A)}\iint\limits_{\substack{A\\|g(s,u)-L|<\delta}} f(|g(s,u)-L|)\Delta s\Delta u+\frac{1}{\mu_\Delta(A)}\iint\limits_{\substack{A\\|g(s,u)-L|\geqslant\delta}} f(|g(s,u)-L|)\Delta s\Delta u\\&\leqslant\varepsilon+2f(1)\delta^{-1}\frac{1}{\mu_\Delta(A)}\iint\limits_A|g(s,u)-L|\Delta s\Delta u.\end{aligned}$$

Taking limit as $(t,r)\to\infty$ in here, because $\varepsilon>0$ is arbitrary, we obtain that $g\in[w]^f_{\Lambda^2}$.
ii) From the proof of Proposition 1 of [16], one has $\beta=\lim\limits_{t\to\infty}\frac{f(t)}{t}=\inf\left\{\frac{f(t)}{t}:t>0\right\}$. Then, we get $f(t)\geqslant\beta t$ for all $t\geqslant 0$. Now let $g\in[w]^f_{\Lambda^2}$ with the limit L. Since $\beta>0$, we get

$$\lim_{(t,r)\to\infty}\frac{1}{\mu_\Delta(A)}\iint\limits_A f(|g(s,u)-L|)\Delta s\Delta u\geqslant\lim_{(t,r)\to\infty}\frac{\beta}{\mu_\Delta(A)}\iint\limits_A|g(s,u)-L|\Delta s\Delta u.$$

It follows that $g\in[w]_{\Lambda^2}$ and so the proof is completed. □

Before giving the last theorem of this study, we give some lemmas that will be used in the its proof.

Lemma 3.5. *[32] Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function and let*

$$\Omega(t,r)=\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}$$

for $\varepsilon>0$. Then, we have

$$\mu_\Delta(\Omega(t,r))\leqslant\frac{1}{\varepsilon}\iint\limits_{\Omega(t,r)}|g(s,u)-L|\Delta s\Delta u\leqslant\frac{1}{\varepsilon}\iint\limits_A|g(s,u)-L|\Delta s\Delta u.$$

Lemma 3.6. *Let $t_1,t_2 \in \mathbb{T}_1$, $r_1,r_2 \in \mathbb{T}_2$ and $c,d \in \mathbb{R}$ and $D = \left\{[t_1,t_2]_{\mathbb{T}_1} \times [r_1,r_2]_{\mathbb{T}_2}\right\}$. If $\phi : D \to (c,d)$ is Δ-integrable and $F:(c,d) \to \mathbb{R}$ is convex, then*

$$F\left(\frac{\iint_D \phi(s,u)\Delta s\Delta u}{\mu_\Delta(D)}\right) \leqslant \frac{\iint_D F(\phi(s,u))\Delta s\Delta u}{\mu_\Delta(D)}.$$

Proof. It can be proved by considering a similar way in the proof of Theorem 4.1 of [22]. □

Now, we construct a connection between $\Delta^f_{\Lambda^2}$-statistical convergence and strong $\Delta^f_{\Lambda^2}$-Cesaro summability in the next theorem.

Theorem 3.7. *Let $g : \Lambda^2 \to \mathbb{R}$ be a Δ-measurable function. Then, we have*
i) Let f be a convex, modulus function such that there exists a positive constant c such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0$, $y \geq 0$, and $\lim_{t\to\infty}\frac{f(t)}{t} > 0$ and $\lim_{t\to\infty}\frac{f(1/t)}{1/t} > 0$ exist. If g is strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L, then $st^f_{\Lambda^2} - \lim_{(t,r)\to\infty} g(t,r) = L$.
ii) If $st^f_{\Lambda^2} - \lim_{(t,r)\to\infty} g(t,r) = L$ and g is a bounded function, then g is strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L, for any modulus f.

Proof. i) Let g be strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L. Using the lemmas 3.5 and 3.6, for any given $\varepsilon > 0$, we obtain that

$$\begin{aligned}
\frac{1}{\mu_\Delta(A)}\iint_A f(|g(s,u)-L|)\Delta s\Delta u &\geqslant \frac{\mu_\Delta(A)}{\mu_\Delta(A)} f\left(\frac{\iint_A f(|g(s,u)-L|)\Delta s\Delta u}{\mu_\Delta(A)}\right),\\
&\geqslant f\left(\frac{\iint\limits_{\substack{A\\ |g(s,u)-L|\geqslant\varepsilon}} f(|g(s,u)-L|)\Delta s\Delta u}{\mu_\Delta(A)}\right),\\
&\geqslant f\left(\frac{\mu_\Delta(\{(s,u)\in A : |g(s,u)-L|\geqslant\varepsilon\})}{\mu_\Delta(A)}\varepsilon\right),\\
&\geqslant cf(\mu_\Delta(\{(s,u)\in A : |g(s,u)-L|\geqslant\varepsilon\}))\,f\left(\frac{\varepsilon}{\mu_\Delta(A)}\right),\\
&= c\varepsilon\frac{f(\mu_\Delta(A))}{\mu_\Delta(A)}\,\frac{f(\mu_\Delta(\{(s,u)\in A : |g(s,u)-L|\geqslant\varepsilon\}))}{f(\mu_\Delta(A))}\,\frac{f\left(\frac{\varepsilon}{\mu_\Delta(A)}\right)}{\frac{\varepsilon}{\mu_\Delta(A)}}.
\end{aligned}$$

Also, by using $\lim_{t\to\infty}\frac{f(t)}{t} > 0$ and $\lim_{t\to\infty}\frac{f(1/t)}{1/t} > 0$, since g is strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L, we get $st^f_{\Lambda^2} - \lim_{(t,r)\to\infty} g(t,r) = L$.
ii) Let g be bounded and $st^f_{\Lambda^2} - \lim_{(t,r)\to\infty} g(t,r) = L$. Then, there exists a positive number M such that $|g(s,u)-L| \leq M$ for all $(s,u) \in \Lambda^2$. For any given $\varepsilon > 0$, we get

$$\begin{aligned}
\frac{1}{\mu_\Delta(A)}\iint_A f(|g(s,u)-L|)\Delta s\Delta u &= \frac{1}{\mu_\Delta(A)}\iint\limits_{\substack{A\\ |g(s,u)-L|\geqslant\varepsilon}} f(|g(s,u)-L|)\Delta s\Delta u + \frac{1}{\mu_\Delta(A)}\iint\limits_{\substack{A\\ |g(s,u)-L|<\varepsilon}} f(|g(s,u)-L|)\Delta s\Delta u,\\
&\leqslant \frac{\mu_\Delta(\{(s,u)\in A : |g(s,u)-L|\geqslant\varepsilon\})}{\mu_\Delta(A)} f(M) + \frac{\mu_\Delta(A)}{\mu_\Delta(A)} f(\varepsilon).
\end{aligned}$$

Hence, letting $(t,r) \to \infty$ on both sides in here and then $\varepsilon \to 0$, by means of Theorem 2.7, we get

$$\frac{1}{\mu_\Delta(A)}\iint_A f(|g(s,u)-L|)\Delta s\Delta u = 0,$$

which completes the proof. □

Remark 3.8. *If we take $f(x) = x$ in Theorem 3.7, we get Theorem 2.10 of [32] for the special case $p = 1$.*

Acknowledgements

The first author would like to thank TUBİTAK (The Scientific and Technological Research Council of Turkey) for their financial supports during his doctorate studies. The authors would also like to thank the reviewers for their valuable comments which are improved the paper.

References

[1] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.
[2] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2**(1) (1951), 73–74.
[3] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.
[4] J.A. Fridy, *On statistical convergence*, Analysis, **5** (1985), 301–313.
[5] J.S. Connor, *The statistical and strong p-Cesàro convergence of sequences*, Analysis, **8** (1988), 47–63.
[6] M. Mursaleen, O.H.H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288**(1) (2003), 223–231.
[7] F. Móricz, *Statistical limits of measurable functions*, Analysis, **24**(1) (2004), 1–18.
[8] E. Dündar, Y. Sever, *Multipliers for bounded statistical convergence of double Sequences*, Int. Math. Forum, **7**(52) (2012), 2581–2587.
[9] U. Ulusu, E. Dündar, *I-lacunary statistical convergence of sequences of sets*, Filomat, **28**(8) (2014), 1567–1574, DOI 10.2298/FIL1408567U.
[10] F. Nuray, U. Ulusu, E. Dündar, *Lacunary statistical convergence of double sequences of sets*, Soft Comput., **20** (2016), 2883–2888, DOI 10.1007/s00500-015-1691-8.
[11] S. Yegül, E. Dündar, *On statistical convergence of sequences of functions in 2-normed spaces*, J. Classical Anal., **10**(1) (2017), 49–57.
[12] S. Yegül, E. Dündar, *Statistical convergence of double sequences of functions and some properties in 2-normed spaces*, Facta Univ. Ser. Math. Inform., **33**(5) (2018), 705–719.
[13] H. Nakano, *Concave modulars*, J. Math. Soc. Japan, **5** (1953), 29–49.
[14] W.H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Can. J. Math., **25** (1973), 973–978.
[15] I.J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc., **100**(1) (1986), 161–166.
[16] I.J. Maddox, *Inclusions between FK spaces and Kuttner's theorem*, Math. Proc. Cambridge Philos. Soc., **101**(3) (1987), 523–527.
[17] A. Aizpuru, M.C. Listan-Garcia, F. Rambla-Barreno, *Density by moduli and statistical convergence*, Quaest. Math., **37**(4) (2014), 525–530.
[18] A. Aizpuru, M.C. Listan-Garcia, F. Rambla-Barreno, *Double density by moduli and statistical convergence*, Bull. Belg. Math. Soc. Simon Stevin, **19**(4) (2012), 663–673.
[19] V.K. Bhardwaj, S. Dhawan, *f-statistical convergence of order α and strong Cesàro summability of order α with respect to a modulus*, J. Ineq. Appl., **2015**(332) (2015).
[20] S. Hilger, *Analysis on measure chains-a unified approach to continuous and discrete calculus*, Results Math., **18**(1-2) (1990), 18–56.
[21] M. Bohner, A. Peterson, *Dynamic Equations On Time Scales: An Introduction With Applications*, Birkhäuser, Boston, 2001.
[22] R. Agarwal, M. Bohner, A. Peterson, *Inequalities on time scales: a survey*, Math. Inequal. Appl., **4**(4) (2001), 535–557.
[23] G. S. Guseinov, *Integration on time scales*, J. Math. Anal. Appl., **285**(1) (2003), 107–127.
[24] M. Bohner, G.S. Guseinov, *Partial differentiation on time scales*, Dynam. Syst. Appl., **13** (2004) , 351–379.
[25] M. Bohner, G. S. Guseinov, *Multiple Lebesgue integration on time scales*, Adv. Difference Equ., **2006** (2006), Article ID 26391.
[26] A. Cabada, D.R. Vivero, *Expression of the Lebesgue $\Delta-$integral on time scales as a usual Lebesgue integral: Application to the calculus of $\Delta-$antiderivatives*, Math. Comput. Model., **43**(1-2) (2006), 194–207.
[27] M.S. Seyyidoğlu, N.O. Tan, *A note on statistical convergence on time scale*, J. Inequal. Appl., **2012**(219) (2012).
[28] C. Turan, O. Duman, *Statistical convergence on time scales and its characterizations*, Springer Proc. Math. Stat., **41** (2013), 57–71.
[29] C. Turan, O. Duman, *Convergence methods on time scales*, AIP Conf. Proc., **1558** (2013), 1120–1123.
[30] C. Turan, O. Duman, *Fundamental properties of statistical convergence and lacunary statistical convergence on time scales*, Filomat, **31**(14) (2017), 4455–4467.
[31] Y. Altın, H. Koyunbakan, E. Yılmaz, *Uniform statistical convergence on time scales*, J. Appl. Math., **2014** (2014).
[32] M. Çınar, E. Yılmaz, Y. Altın, T. Gülsen, *Statistical convergence of double sequences on product time scales*, Analysis, **39**(3) (2019), 71–77.
[33] B. Sözbir, S. Altundağ, *Weighted statistical convergence on time scale*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **26** (2019), 137–143.
[34] B. Sözbir and S. Altundağ, *$\alpha\beta$-statistical convergence on time scales*, Facta Univ. Ser. Math. Inform., **35**(1) (2020), 141–150.
[35] B. Sözbir, S. Altundağ, M. Başarır, *On the (Delta,f)-lacunary statistical convergence of the functions*, Maltepe J. Math., **2**(1) (2020), 1–8.
[36] N. Turan, M. Başarır, *On the Δ_g-statistical convergence of the function defined time scale*, AIP Conf. Proc., 2183, 040017 (2019), https://doi.org/10.1063/1.5136137.
[37] N. Tok, M. Başarır, *On the λ_h^α-statistical convergence of the functions defined on the time scale*, Proc. Int. Math. Sci., **1**(1) (2019), 1–10.
[38] M. Başarır, *A note on the (θ,φ)-statistical convergence of the product time scal*e, Konuralp J. Math., **8**(1) (2020), 192–196.
[39] M. Başarır, *A note on the $(\lambda;v)_h^\alpha$-statistical convergence of the functions defined on the product of time scales*, Azerbaijan Journal of Mathematics, 2020, under communication.

Universal Journal of Mathematics and Applications, 3 (4) (2020) 144-155
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.745248

Construction of Intuitionistic Fuzzy Mappings with Applications

Soheyb Milles[1*], Ergün Nart[2], Farhan Ismail[2] and Abdelkrim Latreche[3]

[1] *Laboratory of Pure and Applied Mathematics, Department of Mathematics, University of M'sila, Algeria*
[2] *Department of Mechatronics Engineering, Faculty of Technology, Sakarya University of Applied Sciences, Sakarya, Turkey*
[3] *Department of Technology, Faculty of Technology, 20 Août 1955 University, Skikda, Algeria*
[*] *Corresponding author*

Article Info

Keywords: *Intuitionistic fuzzy set, Intuitionistic fuzzy mapping, Intuitionistic fuzzy topology*
2010 AMS: *03E72, 58C07, 54A40.*
Received: *29 May 2020*
Accepted: *29 September 2020*
Available online: *23 December 2020*

Abstract

In a recent paper, Ismail and Massa'deh have introduced the notion of L-fuzzy mapping and some basic operations were proved. In this paper, we generalize this notion to the setting of intuitionistic fuzzy sets. Moreover, we study the main properties related to intuitionistic fuzzy mapping. As applications, we provide properties of intuitionistic fuzzy continuous mappings in intuitionistic fuzzy topological spaces and investigate the relation among various kinds of intuitionistic fuzzy continuity.

1. Introduction

Mappings in crisp set theory are very well known and play a prominent role in mathematical branches such as topology and its analysis approaches. They appear to enhance the concept of functional predicate in formal logic [14] and also closely related to category theory [23]. In dynamical systems, a mapping denotes an evolution function used to create discrete dynamical systems [11].

In fuzzy setting, several authors introduce and investigate the concept of fuzzy mapping in different ways. Heilpern [13] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings. Ismail and Massa'deh [10] defined L-fuzzy mappings and studied their operations, also they developed many properties of classical mappings into L-fuzzy case. Lim et al. [19] investigated the equivalence relations and mappings for fuzzy sets and relationship among them.

In 1983, Atanassov [1] introduced the concept of intuitionistic fuzzy set which is a generalization of Zadeh's fuzzy set previously introduced in [24] by using two membership functions for the elements of the universe of discourse. After that, several intuitionistic fuzzy concepts are studied by many authors. For the concept of mapping, an extended approaches are proposed based on Atanassov's intuitionistic fuzzy sets. Kang et al. [18] introduced the concept of intuitionistic fuzzy mapping and they give the decomposition of an intuitionistic fuzzy mapping by using intuitionistic fuzzy equivalence relations. Shen et al. [22] presented the notion of intuitionistic fuzzy mapping as a generalization of fuzzy mapping, and they established the decomposition and representation theorems of intuitionistic fuzzy mappings. Very recently, Gomathi and Jayanthi [12] introduced the concept of intuitionistic fuzzy $b^\sharp$ continuous mapping in intuitionistic fuzzy topological spaces and discussed some of their properties and characterizations. For more details about intuitionistic fuzzy mappings and background, the readers are referred to [16, 20, 25] and more others.

In this paper, we continue further by generalizing the notion of fuzzy mapping introduced by Ismail and Massa'deh to the intuitionistic fuzzy setting. Hereafter, the main properties related to intuitionistic fuzzy mapping are studied. Also, we generalize the notion of fuzzy topology on fuzzy sets to the intuitionistic fuzzy case to provide properties of intuitionistic fuzzy continuous mappings. To that end, the relations among intuitionistic fuzzy continuity, precontinuity and α-continuity are investigated.

This paper is structured as follows. After recalling some basic definitions and properties in Section 2, the notion of intuitionistic fuzzy mapping by construction on a set is introduced, and some basic properties are given in Section 3. As applications, some properties of intuitionistic fuzzy continuous mappings in intuitionistic fuzzy topological space are provided and relations among some kinds of intuitionistic fuzzy continuity in Section 4 are investigated. Finally, some conclusions and future research in Section 5 are presented.

Email addresses and ORCID numbers**: soheyb.milles@univ-msila.dz, 0000-0001-8801-4607 (S. Milles), enart@sakarya.edu.tr, 0000-0001-5049-609X (E. Nart), farhanismail@subu.edu.tr, 0000-0002-0822-8961 (F. Ismail), a.latreche@univ-skikda.dz, 0000-0003-1424-7197 (A. Latreche)**

2. Preliminaries

This section contains the basic definitions and properties of intuitionistic fuzzy sets, intuitionistic fuzzy relations and some related notions that will be needed throughout this paper.

2.1. Atanassov's intuitionistic fuzzy sets

In this subsection we recall some basic concepts of intuitionistic fuzzy sets.

Let X be a universe, then a fuzzy set $A=\{\langle x,\mu_A(x)\rangle \mid x\in X\}$ defined by Zadeh [24] is characterized by a membership function $\mu_A : X\to[0,1]$, where $\mu_A(x)$ is interpreted as the degree of a membership of the element x in the fuzzy subset A for each $x\in X$.

Atanassov in [1] introduced another fuzzy object, called intuitionistic fuzzy set as a generalization of the concept of fuzzy set, shown as follows

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\},$$

which is characterized by a membership function $\mu_A : X\to[0,1]$ and a non-membership function $\nu_A : X\to[0,1]$, with the condition

$$0\le \mu_A(x)+\nu_A(x)\le 1, \tag{2.1}$$

for any $x\in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of the element x in the intuitionistic fuzzy set A for each $x\in X$.

In the fuzzy set theory, the non-membership degree of an element x of the universe is defined as $\nu_A(x)=1-\mu_A(x)$ (using the standard negation) and thus it is fixed. In intuitionistic fuzzy setting, the non-membership degree is a more-or-less independent degree: the only condition is that $\nu_A(x)\le 1-\mu_A(x)$. Certainly fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x)=1-\mu_A(x)$, but not conversely.

Throughout this paper, authors denote the set of all intuitionistic fuzzy sets in a set X as $IFS(X)$ and $X,Y,Z,\ldots$ etc., will be nonempty crisp sets.

Definition 2.1. [1] *Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\}$ and $B=\{\langle x,\mu_B(x),\nu_B(x)\rangle \mid x\in X\}$, be two IFSs on a set X. Then*

(i) $A\subseteq B$ *if* $\mu_A(x)\le\mu_B(x)$ *and* $\nu_A(x)\ge\nu_B(x)$, *for all* $x\in X$,
(ii) $A=B$ *if* $\mu_A(x)=\mu_B(x)$ *and* $\nu_A(x)=\nu_B(x)$, *for all* $x\in X$,
(iii) $A\cap B=\{\langle x,\mu_A(x)\wedge\mu_B(x),\nu_A(x)\vee\nu_B(x)\rangle \mid x\in X\}$,
(iv) $A\cup B=\{\langle x,\mu_A(x)\vee\mu_B(x),\nu_A(x)\wedge\nu_B(x)\rangle \mid x\in X\}$,
(v) $\overline{A}=\{\langle x,\nu_A(x),\mu_A(x)\rangle \mid x\in X\}$,
(vi) $[A]=\{\langle x,\mu_A(x),1-\mu_A(x)\rangle \mid x\in X\}$,
(vii) $\langle A\rangle=\{\langle x,1-\nu_A(x),\nu_A(x)\rangle \mid x\in X\}$.

For more details please refer to ([1–3,21,25]).

Definition 2.2. [3] *Let A be an intuitionistic fuzzy set on universe X. The support of A is the crisp subset of X given by*

$$Supp(A)=\{x\in X \mid \mu_A(x)>0 \text{ or } (\mu_A(x)=0 \text{ and } \nu_A(x)<1)\}.$$

In the sequel, we need the following definition of level set (which is also often called (α,β)-cut) of intuitionistic fuzzy set.

Definition 2.3. [15] *Let A be an intuitionistic fuzzy set on a nonempty set X. The (α,β)-cut of A is the crisp subset*

$$A_{(\alpha,\beta)}=\{x\in X \mid \mu_A(x)\ge\alpha \text{ and } \nu_A(x)\le\beta\},$$

where $\alpha,\beta\in[0,1]$ with $\alpha+\beta\le 1$.

2.2. Intuitionistic fuzzy relations

Burillo and Bustince [4,5] introduced the concept of intuitionistic fuzzy relation as a natural generalization of fuzzy relation.

Definition 2.4. [4,5] *An intuitionistic fuzzy binary relation (An intuitionistic fuzzy relation, for short) from a universe X to a universe Y is an intuitionistic fuzzy subset in $X\times Y$, i.e., is an expression R given by*

$$R=\{\langle (x,y),\mu_R(x,y),\nu_R(x,y)\rangle \mid (x,y)\in X\times Y\},$$

where

$$\mu_R : X\times Y\to[0,1], \text{ and } \nu_A : X\times Y\to[0,1]$$

satisfy the condition

$$0\le\mu_R(x,y)+\nu_R(x,y)\le 1, \tag{2.2}$$

for any $(x,y)\in X\times Y$. The value $\mu_R(x,y)$ is called the degree of a membership of (x,y) in R and $\nu_R(x,y)$ is called the degree of a non-membership of (x,y) in R.

Next, the following definitions is needed to recall.

Definition 2.5. *Let R and P be two intuitionistic fuzzy relations from a universe X to a universe Y.*

(i) *The transpose (inverse) R^t of R is the intuitionistic fuzzy relation from the universe Y to the universe X defined by*

$$R^t = \{\langle (x,y), \mu_{R^t}(x,y), \nu_{R^t}(x,y)\rangle \mid (x,y) \in X \times Y\},$$

where

$$\begin{cases} \mu_{R^t}(x,y) = \mu_R(y,x) \\ \text{and} \\ \nu_{R^t}(x,y) = \nu_R(y,x)\,, \end{cases}$$

for any $(x,y) \in X \times Y$.

(ii) *R is said to be contained in P or we say that P contains R, denoted by $R \subseteq P$, if for all $(x,y) \in X \times Y$ it holds that $\mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$.*

(iii) *The intersection (resp.* the union*) of two intuitionistic fuzzy relations R and P from a universe X to a universe Y is an intuitionistic fuzzy relation defined as*

$$R \cap P = \{\langle (x,y), \min(\mu_R(x,y), \mu_P(x,y)), \max(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}$$

and

$$R \cup P = \{\langle (x,y), \max(\mu_R(x,y), \mu_P(x,y)), \min(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}\,.$$

The following properties are crucial in this paper (see e.g. [4, 5, 8]).

Definition 2.6. *Let R be an intuitionistic fuzzy relation from a universe X into itself.*

(i) Reflexivity*: $\mu_R(x,x) = 1$, for any $x \in X$. In this case we note that $\nu_R(x,x) = 0$, for any $x \in X$.*

(ii) Antisymmetry*: for any $x,y \in X$, $x \neq y$ then*

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x) \end{cases},$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

(iii) Perfect antisymmetry*: for any $x,y \in X$ with $x \neq y$ and*

$$\begin{cases} \mu_R(x,y) > 0 \\ \text{or} \\ \mu_R(x,y) = 0 \ \text{ and } \nu_R(x,y) < 1\,, \end{cases}$$

then

$$\begin{cases} \mu_R(y,x) = 0 \\ \text{and} \\ \nu_R(y,x) = 1\,. \end{cases}$$

(iv) Transitivity*: $R \supseteq R \circ^{\alpha,\beta}_{\lambda,\rho} R$.*

In the above definition, the composition $R \circ^{\alpha,\beta}_{\lambda,\rho} R$ used in the transitivity means that

$$R \circ^{\alpha,\beta}_{\lambda,\rho} R = \{\langle (x,z),\ \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)]\},\ \lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\}\rangle \mid x,z \in X\}\,,$$

where α, β, λ and ρ are t-norms or t-conorms taken under the intuitionistic fuzzy condition

$$0 \leq \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)]\} + \lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\} \leq 1\,,$$

for any $x,z \in X$.

The properties of this composition and the choice of α, β, λ and ρ, for which this composition fulfills a maximal number of properties, are investigated in [4]- [8].

3. Construction of intuitionistics fuzzy mappings

In crisp set theory, mappings are defined as binary relations. In this section, the notion of intuitionistic fuzzy mapping as intuitionistic fuzzy relations by construction on a set is introduced, and some basic properties are given.

Definition 3.1. *Let A be an intuitionistic fuzzy set on X and B be an intuitionistic fuzzy set on Y, let $f : Supp\,A \to Supp\,B$ be an ordinary mapping and R be an intuitionistic fuzzy relation on $X \times Y$. Then f_R is called an intuitionistic fuzzy mapping if for all $(x,y) \in Supp\,A \times Supp\,B$ the following condition is satisfied:*

$$\mu_R(x,y) = \begin{cases} min(\mu_A(x), \mu_B(f(x)) \ , \ if\ y = f(x) \\ 0\ ,\ Otherwise, \end{cases}$$

and

$$\nu_R(x,y) = \begin{cases} max(\nu_A(x), \nu_B(f(x)) \ , \ if\ y = f(x) \\ 1\ ,\ Otherwise, \end{cases}$$

with $0 \le \mu_R(x,y) + \nu_R(x,y) \le 1$

Example 3.2. *Let $X = \{\alpha, \beta\}$, $Y = \{1,2,3\}$, $A \in IFS(X)$ and $B \in IFS(Y)$ given by :*

$$A = \{\langle\alpha, 0.5, 0.2\rangle, \langle\beta, 0.1, 0.7\rangle\} \text{ and } B = \{\langle 1,0,1\rangle, \langle 2,0.1,0.5\rangle, \langle 3,0.7,0.2\rangle\}$$

.

We will construct the intuitionistic fuzzy mapping f_R by :

(i) an ordinary mapping $f : \{\alpha, \beta\} \to \{2,3\}$ such that $f(\alpha) = 2$ and $f(\beta) = 3$,

(ii) an intuitionistic fuzzy relation R defined by :
$\mu_R(\alpha, f(\alpha)) = \mu_R(\alpha,2) = \mu_A(\alpha) \wedge \mu_B(2) = 0.1$
$\mu_R(\beta, f(\beta)) = \mu_R(\beta,3) = \mu_A(\beta) \wedge \mu_B(3) = 0.1$
$\mu_R(\alpha,1) = \mu_R(\alpha,3) = \mu_R(\beta,1) = \mu_R(\beta,2) = 0$
In similar way, it holds that
$\nu_R(\alpha, f(\alpha)) = \nu_R(\alpha,2) = \nu_A(\alpha) \vee \nu_B(2) = 0.5$
$\nu_R(\beta, f(\beta)) = \nu_R(\beta,3) = \nu_A(\beta) \vee \nu_B(3) = 0.7$
$\nu_R(\alpha,1) = \nu_R(\alpha,3) = \nu_{R_I}(\beta,1) = \nu_{R_I}(\beta,2) = 1.$
Hence, $\mu_R(x,y) = \{\langle(\alpha, f(\alpha)), 0.1, 0.5\rangle, \langle(\beta, f(\beta)), 0.1, 0.7\rangle, \langle(\alpha,1),0,1\rangle,$
$\langle(\alpha,3),0,1\rangle, \langle(\beta,1),0,1\rangle, \langle(\beta,2),0,1\rangle\}$.

Thus, f_R is an intuitionistic fuzzy mapping.

Remark 3.3. *From the above definition, we can construct the intuitionistic fuzzy mapping by this method*

(i) We determine the Supp A and Supp B.
(ii) We determine the ordinary mapping from Supp A to Supp B.
(iii) We determine the intuitionistic fuzzy relation R to get the relationship degree and non-relationship degree between each element and its image.
(iv) Finally, we conclude the construction of the intuitionistic fuzzy mapping.

Definition 3.4. *Let f_R, g_S be two intuitionistic fuzzy mappings, then f_R and g_S are equal if and only if $f = g$ and $R = S$ i.e., $(\mu_R(x, f(x)) = \mu_S(x, g(x))$ and $\nu_R(x, f(x)) = \nu_S(x, g(x)))$.*

Definition 3.5. *Let A be an intuitionistic fuzzy set on X, let $f : Supp\,A \to Supp\,A$ be an ordinary mapping such that $f(x) = x$ and R be an intuitionistic fuzzy relation on $X \times X$. Then f_R is called an intuitionistic fuzzy identity mapping if for all $x, y \in Supp\,A$ the following condition is satisfied:*

$$\mu_R(x,y) = \begin{cases} \mu_A(x)\ ,\ if\ x = y \\ 0\ ,\ Otherwise, \end{cases}$$

and

$$\nu_R(x,y) = \begin{cases} \nu_A(x)\ ,\ if\ x = y \\ 1\ ,\ Otherwise, \end{cases}$$

with $0 \le \mu_R(x,y) + \nu_R(x,y) \le 1$.

Definition 3.6. *Let A, B and C are an intuitionistic fuzzy sets on X, Y and Z respectively, let $f : Supp\,A \to Supp\,B$ and $g : Supp\,B \to Supp\,C$ are an ordinary mappings and R, S are an intuitionistic fuzzy relations on $X \times Y$ and $Y \times Z$ respectively. Then $(g \circ f)_T$ is called the composition of intuitionistic fuzzy mappings f_R and g_R such that $g \circ f : Supp\,A \to Supp\,C$ and the intuitionistic fuzzy relation T is defined by*

$$\begin{cases} \mu_T(x,z) = sup_y(min(\mu_R(x,y), \mu_S(y,z))) \\ \text{and} \\ \nu_T(x,z) = inf_y(max(\nu_R(x,y), \nu_S(y,z))), \end{cases}$$

for any $(x,z) \in Supp\,A \times Supp\,C$.

Remark 3.7. *The intuitionistic fuzzy relation T in the above definition can be written as follows:*
$\mu_T(x,z) = min\{\mu_A(x),\ \mu_B(f(x)), \mu_C(g(f(x))\}$ *and* $\nu_T(x,z) = max\{\nu_A(x),\ \nu_B(f(x)), \nu_C(g(f(x))\}$.
Indeed, for any $(x,z) \in Supp\ A \times Supp\ C$, we have

$$\begin{aligned}\mu_T(x,z) &= \mu_T(x,g(f(x)))\\ &= \mu_{S\circ R}(x,g(f(x)))\\ &= sup_y\{min\{\mu_R(x,y),\ \mu_S(y,g(f(x)))\\ &= min\{\mu_R(x,f(x))),\ \mu_S(f(x)),g(f(x)))\}\\ &= min\{\mu_A(x),\ \mu_B(f(x)), \mu_C(g(f(x))\}.\end{aligned}$$

Similarly, for any $(x,z) \in Supp\ A \times Supp\ C$, it holds that

$$\begin{aligned}\nu_T(x,z) &= \nu_T(x,g(f(x)))\\ &= \nu_{S\circ R}(x,g(f(x)))\\ &= inf_y\{max\{\nu_R(x,y),\ \nu_S(y,g(f(x)))\\ &= max\{\nu_R(x,f(x))),\ \nu_S(f(x)),g(f(x)))\}\\ &= max\{\nu_A(x),\ \nu_B(f(x)), \nu_C(g(f(x))\}.\end{aligned}$$

Example 3.8. *Let $X = \mathbb{N}$, $Y = \mathbb{R}$ and $Z = \mathbb{R}$, and let $A \in IFS(X)$, $B \in IFS(Y)$ and $C \in IFS(Z)$, defined as follows :*

$$\mu_A(n) = \frac{1}{1+n} \text{ and } \nu_A(n) = \tfrac{n}{2+2n}, \text{for any } n \in \mathbb{N}$$

.

$$\mu_B(x) = \begin{cases} 0.25\ ,\ if\ x \in [-1,1] \\ 0\ ,\ Otherwise, \end{cases} \quad \text{and} \quad \nu_B(x) = \begin{cases} 0.5\ ,\ if\ x \in [-1,1] \\ 1\ ,\ Otherwise, \end{cases}$$

$$\mu_C(x) = \frac{|\ cos(x)\ |}{3} \text{ and } \nu_C(x) = \tfrac{|sin(x)|}{3}$$

for any $x \in \mathbb{R}$.
We define an intuitionistic fuzzy mappings $f_R : A \to B$ and $g_S : B \to C$ by :

(i) an ordinary mappings $f : Supp\ A \longrightarrow Supp\ B$, defined for any $n \in Supp\ A$ by :

$$f(n) = \begin{cases} 1\ ,\ if\ n\ is\ even\ number, \\ -1\ ,if\ n\ odd\ is\ number, \end{cases}$$

and $g : Supp\ B \longrightarrow Supp\ C$ defined by $g(x) = 2x$, for any $x \in [-1,1]$.

(ii) an IF-relations R and S defined by :

$\mu_R(n,f(n)) = \wedge\{\mu_A(n), \mu_B(f(n))\} = \wedge\{\frac{1}{1+n}, 0.25\}$ *and* $\nu_R(n,f(n)) = \vee\{\nu_A(n), \nu_B(f(n))\} = \vee\{\frac{n}{2+2n}, 0.5\}$ *and* $\mu_S(x,g(x)) = \wedge\{\mu_B(x), \mu_C(g(x))\} = \begin{cases} \wedge\{0.25, \frac{|cos(2x)|}{3}\}\ ,\ x \in [-1,1], \\ 0\ ,\ otherwise, \end{cases}$

and $\nu_S(x,g(x)) = \vee\{\nu_B(x), \nu_C(g(x))\} = \begin{cases} \vee\{0.5, \frac{|sin(2x)|}{3}\}\ ,\ x \in [-1,1], \\ 1\ ,\ otherwise, \end{cases}$

Then, the composition $g_S \circ f_R = (g \circ f)_T$ is defined by :

(i) an ordinary mapping $f : Supp\ A \longrightarrow Supp\ C$, defined for any $n \in Supp\ A$ by :

$$(g\circ f)(n) = \begin{cases} 2\ ,\ if\ n is\ even\ number, \\ -2\ ,\ if\ n is\ odd\ number, \end{cases}$$

(ii) an IF-relation T defined by :

$$\begin{aligned}\mu_T(n,(g\circ f)(n)) &= \begin{cases} \wedge\{\frac{1}{1+n}, 0.25, \frac{|cos(2)|}{3}\}\ ,\ if\ n is\ even\ number \\ \wedge\{\frac{1}{1+n}, 0.25, \frac{|cos(-2)|}{3}\}\ ,\ if\ n is\ odd\ number \end{cases}\\ &= \wedge\{\frac{1}{1+n}, 0.25, \frac{|\ cos(2)\ |}{3}\}\\ &= \wedge\{\frac{1}{1+n}, 0.25\},\end{aligned}$$

$$\nu_T(n,(g\circ f)(n)) = \begin{cases} \vee\{\frac{n}{2+2n}, 0.25, \frac{|sin(2)|}{3}\}, & if\ n is\ even\ number \\ \vee\{\frac{n}{2+2n}, 0.25, \frac{|sin(-2)|}{3}\}, & if\ n is\ odd\ number \end{cases}$$
$$= \vee\{\frac{n}{2+2n}, 0.25, \frac{|\ sin(2)\ |}{3}\}$$
$$= \vee\{\frac{2}{2+2n}, 0.25\}.$$

Proposition 3.9. *The composition of intuitionistic fuzzy mappings is an associative operation.*

Proof. Let A,B,C and D are an intuitionistic fuzzy sets on X,Y,Z and T respectively, let $f_{R_1}: A\to B$, $g_{R_2}: B\to C$ and $h_{R_3}: C\to D$ are an intuitionistic fuzzy mappings. We need to show that $h_{R_3}\circ(g_{R_2}\circ f_{R_1}) = (h_{R_3}\circ g_{R_2})\circ f_{R_1}$. On the one hand, it is easy to verify that $(h\circ(g\circ f)) = ((h\circ g)\circ f)$. On the one hand,

$$\begin{aligned}
\mu_{R_3\circ(R_2\circ R_1)}(x, h\circ(g\circ f)(x)) &= min\{\mu_{R_2\circ R_1}(x, g\circ f(x)),\ \mu_{R_3}(g\circ f(x)), h\circ(g\circ f)(x)\}\} \\
&= min\{min\{\mu_{R_1}(x,f(x)),\ \mu_{R_2}(f(x),g(f(x))\},\ \mu_{R_3}(g\circ f(x), h\circ(g\circ f)(x))\} \\
&= min\{\mu_{R_1}(x,f(x)),\ \mu_{R_2}(f(x),g(f(x)),\ \mu_{R_3}(g\circ f(x), h\circ(g\circ f)(x))\} \\
&= min\{\mu_A(x),\ \mu_B(f(x)),\ \mu_C(g(f(x)),\ \mu_D((h\circ g)\circ f)(x))\} \\
&= min\{\mu_{R_1}(x,f(x)),\ \mu_{R_2}(f(x),g(f(x)),\ \mu_{R_3}(g\circ f(x),(h\circ g)\circ f)(x))\} \\
&= min\{\mu_{R_1}(x,f(x)),\ min\{\mu_{R_2}(f(x),g(f(x))),\ \mu_{R_3}(g\circ f(x),(h\circ(g\circ f))(x))\}\} \\
&= min\{\mu_{R_1}(x,f(x)),\ \mu_{R_3\circ R_2}(f(x),(g\circ f)(x))\}\} \\
&= \mu_{(R_3\circ R_2)\circ R_1}(x,((h\circ g)\circ f)(x))
\end{aligned}$$

In similar way, we prove that $\nu_{R_3\circ(R_2\circ R_1)}(x, h\circ(g\circ f)(x)) = \nu_{(R_3\circ R_2)\circ R_1}(x,(h\circ g)\circ f(x))$.

□

Remark 3.10. *The intuitionistic fuzzy identity mapping Id_R is neutral for the composition of intuitionistic fuzzy mappings.*

In the sequel, we need to introduce the notion of the direct image and the inverse image of intuitionistic fuzzy set by an intuitionistic fuzzy mapping.

Definition 3.11. *Let $f_R: A\to B$ be an intuitionistic fuzzy mapping from an intuitionistic fuzzy set A to another intuitionistic fuzzy set B and $C\subseteq A$. The direct image of C by f_R is defined by $f_R(C) = \{\langle y, \mu_{f_R(C)}(y), \nu_{f_R(C)}(y)\rangle \mid y\in Y\}$, where*

$$\mu_{f_R(C)}(y) = \begin{cases} \mu_B(y), & if\ y\in f(supp(C)) \\ 0, & Otherwise, \end{cases}$$

and

$$\nu_{f_R(C)}(y) = \begin{cases} \nu_B(y), & if\ y\in f(supp(C)) \\ 1, & Otherwise, \end{cases}$$

Similarly, if $C'\subseteq B$. The inverse image of C' by f is defined by $f_R^{-1}(C') = \{\langle x, \mu_{f_R^{-1}(C')}(x), \nu_{f_R^{-1}(C')}(x)\rangle \mid x\in X\}$, where

$$\mu_{f_R^{-1}(C')}(x) = \begin{cases} \mu_A(x), & if\ x\in f^{-1}(supp(C')) \\ 0, & Otherwise, \end{cases}$$

and

$$\nu_{f_R^{-1}(C')}(x) = \begin{cases} \nu_A(x), & if\ x\in f^{-1}(supp(C')) \\ 1, & Otherwise, \end{cases}$$

Example 3.12. *Let $X = \mathscr{P}(\mathbb{R})$, $Y = \{\alpha,\beta\}$ and $A\in IFS(X)$ defined for any $S\in\mathscr{P}(\mathbb{R})$ by :*

$$\mu_A(S) = \begin{cases} 0.55, & if\ S is\ denumerable\ set \\ 0, & Otherwise, \end{cases}$$

$$\nu_A(S) = \begin{cases} 0.3, & if\ S is\ denumerable\ set \\ 1, & Otherwise. \end{cases}$$

Also, let $B\in IFS(Y)$ given by $B = \{\langle\alpha, 0.2, 0.5\rangle, \langle\beta, 0.7, 0.3\rangle\}$.
We define the intuitionistic fuzzy mapping $f_R: A\to B$ by:

(i) an ordinary mapping $f: Supp\ A \longrightarrow Supp\ B$, defined for any $S\in Supp\ A$ by
$$f(S) = \begin{cases} \alpha, & if\ S is\ finite\ set \\ \beta, & Otherwise, \end{cases}$$
(ii) an IF-relation R defined by $\mu_R(S,f(S)) = \mu_A(S)\wedge\mu_B(f(S)) = 0.55\wedge 0.2 = 0.2$ and $\nu_R(S,f(S)) = \nu_A(S)\vee\nu_B(f(S)) = 0.3\vee 0.5 = 0.5$

Now, if we take C an IF-set on X, where $C \subseteq A$ given by:

$$\mu_C(S) = \begin{cases} 0.4\ ,\ if\ S is\ finite\ set \\ 0\ ,\ Otherwise\,, \end{cases}$$

$$\nu_C(S) = \begin{cases} 0.4\ ,\ if\ S is\ finite\ set \\ 1\ ,\ Otherwise. \end{cases}$$

Then, the direct image of C by f_R is defined by :

$$\mu_{f_R(C)}(y) = \begin{cases} \mu_B(y)\ ,\ if\ y \in f(supp(C)) \\ 0\ ,\ Otherwise\,, \end{cases} = \begin{cases} 0.2\ ,\ if\ y = \alpha \\ 0\ ,\ ,\ if\ y = \beta \end{cases}$$

and

$$\nu_{f_R(C)}(y) = \begin{cases} \mu_B(y)\ ,\ if\ y \in f(supp(C)) \\ 0\ ,\ Otherwise\,, \end{cases} = \begin{cases} 0.5\ ,\ if\ y = \alpha \\ 1\ ,\ ,\ if\ y = \beta. \end{cases}$$

Moreover, it is easy to show that $f_R(C) \subseteq B$.

Next, if we take C' an IF-set on Y, where $C' \subseteq B$ given by:

$$\mu_{C'}(y) = \begin{cases} 0.1\ ,\ if\ y = \alpha \\ 0\ ,\ y = \beta\,, \end{cases} \quad and\ \ \nu_{C'}(y) = \begin{cases} 0.6\ ,\ if\ y = \alpha \\ 1\ ,\ y = \beta. \end{cases}$$

Then, the inverse image of C' by f is defined by :

$$\mu_{f_R^{-1}(C')}(S) = \begin{cases} \mu_A(S)\ ,\ if\ S \in f^{-1}(supp(C')) \\ 0\ ,\ Otherwise\,, \end{cases} = \begin{cases} 0.55\ ,\ if\ S\ is\ finite\ set \\ 0\ ,\ Otherwise\,, \end{cases}$$

and

$$\nu_{f_R^{-1}(C')}(S) = \begin{cases} \nu_A(S)\ ,\ if\ S \in f^{-1}(supp(C')) \\ 0\ ,\ Otherwise\,, \end{cases} = \begin{cases} 0.3\ ,\ if\ S\ is\ finite\ set \\ 1\ ,\ Otherwise. \end{cases}$$

Moreover, it is easy to show that $f_R^{-1}(C') \subsetneq A$ in the case of $S = \mathbb{N}$.

Definition 3.13. *Let A be an intuitionistic fuzzy set on a set X and $\sim$ be an equivalence relation over $Supp(A)$, let B an intuitionistic fuzzy set on $\mathscr{P}(X)$ defined by :*

$$\mu_B(\theta) = \begin{cases} \mu_A(x)\ ,\ if \theta = \bar{x} \in supp(A)/\sim \\ 0\ ,\ Otherwise\,, \end{cases}$$

and

$$\nu_B(\theta) = \begin{cases} \nu_A(x)\ ,\ if \theta = \bar{x} \in supp(A)/\sim \\ 1\ ,\ Otherwise. \end{cases}$$

Then, the intuitionistic fuzzy mapping $P_R : A \longrightarrow B$ defined by :

(i) an ordinary mapping $P : Supp(A) \longrightarrow Supp(B)$ such that $P(x) = \bar{x}$ for any $x \in Supp(A)$,
(ii) an intuitionistic fuzzy relation R defined by :

$$\begin{aligned} \mu_R(x,P(x)) &= min\{\mu_A(x),\ \mu_B(P(x))\} \\ &= min\{\mu_A(x),\ \mu_B(\bar{x})\} \\ &= min\{\mu_A(x),\ \mu_A(x)\} \\ &= \mu_A(x) \end{aligned}$$

and

$$\begin{aligned} \nu_R(x,P(x)) &= max\{\nu_A(x),\ \nu_B(P(x))\} \\ &= max\{\nu_A(x),\ \nu_B(\bar{x})\} \\ &= max\{\nu_A(x),\ \nu_A(x)\} \\ &= \nu_A(x) \end{aligned}$$

is called the intuitionistic fuzzy projection mapping.

Now, we define the product of intuitionistic fuzzy sets and intuitionistic fuzzy projection mappings.

Definition 3.14. *Let A be an intuitionistic fuzzy set on X and B be an intuitionistic fuzzy set on Y. The product of A and B, denoted by $A \times B$ is an intuitionistic fuzzy set on $X \times Y$ defined by :*
$\mu_{X\times Y}(x,y) = min\{\mu_A(x), \mu_B(y)\}$ *and* $\nu_{X\times Y}(x,y) = max\{\nu_A(x), \nu_B(y)\}$.
Also, we define the first intuitionistic fuzzy projection mapping $(P_1)_R : A \times B \longrightarrow A$ by:

(i) an ordinary mapping $P_1 : Supp(A \times B) \longrightarrow Supp(A)$ such that $P_1(x,y) = x$ for any $(x,y) \in Supp(A \times B)$,

(i) an intuitionistic fuzzy relation R defined by :

$$\begin{aligned}\mu_R((x,y),P_1(x,y)) &= min\{\mu_{A\times B}(x,y),\ \mu_A(P_1(x,y))\}\\ &= min\{\mu_A(x),\ \mu_B(y),\ \mu_A(x)\}\\ &= min\{\mu_A(x),\ \mu_B(y)\}\end{aligned}$$

and

$$\begin{aligned}\nu_R((x,y),P_1(x,y)) &= max\{\nu_{A\times B}(x,y),\ \nu_A(P_1(x,y))\}\\ &= max\{\nu_A(x),\ \nu_B(y),\ \nu_A(x)\}\\ &= max\{\nu_A(x),\ \nu_B(y)\}\end{aligned}$$

The second intuitionistic fuzzy projection mapping is defined analogously.

Next, we introduce the notion of disjoint union of intuitionistic fuzzy sets and intuitionistic fuzzy inclusion mappings.

Definition 3.15. *Let A be an intuitionistic fuzzy set on X and B be an intuitionistic fuzzy set on Y. The disjoint union of A and B, denoted by $A\sqcup B$ is an intuitionistic fuzzy set on $X\times\{1\}\cup Y\times\{2\}$ defined by :*

$$\mu_{A\sqcup B}(x,k)=\begin{cases}\mu_A(x)\ ,\ if\, k=1\\ \mu_B(x)\ ,\ if\, k=2\end{cases}$$

and

$$\nu_{A\sqcup B}(x,k)=\begin{cases}\nu_A(x)\ ,\ if\, k=1\\ \nu_B(x)\ ,\ if\, k=2\end{cases}$$

Also, we define the first intuitionistic fuzzy inclusion mapping $(\varphi_1)_R : A\longrightarrow A\sqcup B$ by :

(i) an ordinary mapping φ_1, defined by :

$\varphi_1 : Supp(A)\longrightarrow Supp(A\sqcup B)$ *such that* $\varphi_1(x)=(x,1)$ *for any* $x\in Supp(A)$,

(ii) an intuitionistic fuzzy relation R defined by :

$$\begin{aligned}\mu_R(x,\varphi_1(x)) &= min\{\mu_A(x),\ \mu_{A\sqcup B}(\varphi_1(x))\}\\ &= min\{\mu_A(x),\ \mu_{A\sqcup B}(x,1)\}\\ &= min\{\mu_A(x),\ \mu_A(x)\}\\ &= \mu_A(x)\end{aligned}$$

and

$$\begin{aligned}\nu_R(x,\varphi_1(x)) &= max\{\nu_A(x),\ \nu_{A\sqcup B}(\varphi_1(x))\}\\ &= max\{\nu_A(x),\ \nu_{A\sqcup B}(x,1)\}\\ &= max\{\nu_A(x),\ \nu_A(x)\}\\ &= \nu_A(x)\end{aligned}$$

The second intuitionistic fuzzy inclusion mapping is defined analogously.

4. Applications

In this section, we establish as an application the intuitionistic fuzzy continuous mapping in intuitionistic fuzzy topological spaces.

4.1. Intuitionistic fuzzy topology

This subsection is devoted to study the structure of intuitionistic fuzzy topology as a generalization of the structure of fuzzy topology given by Kandil et al. [17].

Definition 4.1. *Let A be an intuitionistic fuzzy set on the set X and $O_A=\{U$ is an IFS on $X:\ U\subseteq A\}$. We define an intuitionistic fuzzy topology on intuitionistic fuzzy set A by the family $T\subseteq O_A$ which satisfies the following conditions :*

(i) $A,\ 0_\sim\in T$;
(ii) if $U_1,U_2\in T$, then $U_1\cap U_2\in T$;
(iii) if $U_i\in T$ for all $i\in I$, then $\cup_I U_i\in T$.

T is called an intuitionistic fuzzy topology of A and the pair (A,T) is an intuitionistic fuzzy topological space (IF-TOP, for short). Every element of T is called an intuitionistic fuzzy open set (IFOS, for short).

Example 4.2. *Let X be a nonempty set and A be an intuitionistic fuzzy set on $\mathscr{P}(X)$ given by :* $\mu_A(\theta) = \begin{cases} 1\,, & if\ \theta = \emptyset \\ 0.5\,, & 0 < |\theta| < \infty, \\ 0\,, & Otherwise\,, \end{cases}$

and $\nu_A(\theta) = \begin{cases} 0\,, & if\ \theta = \emptyset \\ 0.4\,, & 0 < |\theta| < \infty, \\ 0.2\,, & Otherwise\,, \end{cases}$

Then, the family $T = \{A, 0_\sim, U\}$ where:

$$\mu_U(\theta) = \begin{cases} 0.4\,, & |\theta| < \infty, \\ 0\,, & Otherwise\,, \end{cases} \qquad and\ \nu_U(\theta) = \begin{cases} 0.6\,, & |\theta| < \infty, \\ 0.5\,, & Otherwise\,, \end{cases}$$

is an intuitioniste fuzzy topology on A.

Inspired by the notion of interior (resp. closure) on intuitionistic fuzzy topological space on a set introduced by Atanassov [3], authors define these notions in intuitionistic fuzzy topology on an intuitionistic fuzzy set.

Definition 4.3. *Let (A,T) be an intuitionistic fuzzy topological space, for every intuitionistic fuzzy subset G of X we define the interior and closure of G by:*

$$int(G) = \{\langle x, \max_{x\in X} \mu_U(x), \min_{x\in X} \nu_U(x)\rangle \mid x \in U \subseteq G\}$$

and

$$cl(G) = \{\langle x, \min_{x\in X} \mu_K(x), \max_{x\in X} \nu_K(x)\rangle \mid x \in A \text{ and } G \subseteq K\}$$

Example 4.4. *Let $X = \{a,b,c\}$ and $A,B,C,D \in IFS(X)$ such that*
$A = \{< a, 0.5, 0.1 >, < b, 0.7, 0.2 >, < c, 0.6, 0 >\}$
$B = \{< a, 0.5, 0.2 >, < b, 0.5, 0.4 >, < c, 0.4, 0.4 >\}$
$C = \{< a, 0.4, 0.5 >, < b, 0.6, 0.3 >, < c, 0.2, 0.3 >\}$
$D = \{< a, 0.5, 0.2 >, < b, 0.6, 0.3 >, < c, 0.4, 0.3 >\}$
$E = \{< a, 0.4, 0.5 >, < b, 0.5, 0.4 >, < c, 0.2, 0.4 >\}$
Then the family $T = \{A, 0_\sim, B, C, D, E\}$ is an IFT of A.
Now, we suppose that $G \in IFS(X)$ given by $G = \{< a, 0.41, 0.49), < b, 0.61, 0.29 >, < c, 0.2, 0.2 >\}$. Then, $int(G) = C \cup E = C$ and $cl(G) = 1_\sim$.

Definition 4.5. [9] *Let (A,T) be an intuitionistic fuzzy topological space and $U \in IFS(A,T)$. Then U is called :*

1. *an intuitionistic fuzzy semiopen set (IFSOS) if $U \subseteq cl(int(U))$;*
2. *an intuitionistic fuzzy α-open set (IFαOS) if $U \subseteq int(cl(int(U)))$;*
3. *an intuitionistic fuzzy preopen set (IFPOS) if $U \subseteq int(cl(U))$;*
4. *an intuitionistic fuzzy regular open set (IFROS) if $U = int(cl(U))$.*

4.2. Intuitionistic fuzzy continuous mappings

The present section contains an interesting properties of intuitionistic fuzzy continuous mappings in intuitionistic fuzzy topological space and relations between various kinds of intuitionistic fuzzy continuous mapping. First, the notion of intuitionistic fuzzy continuous mapping is introduced.

Definition 4.6. *Let (A,T) (B,L) be two intuitionistic fuzzy topological spaces. The mapping $f_R : (A,T) \to (B,L)$ is an intuitionistic fuzzy continuous if and only if the inverse of each L-open intuitionistic fuzzy set is T-open intuitionistic fuzzy set.*

Example 4.7. *Let (A,T) and (B,T') be two intuitionistic fuzzy topologies, where*
$\mu_A(x) = 0.55$ and $\nu_A(x) = 0.4$, for any $x \in \mathbb{R}$ and

$$\mu_B(y) = \begin{cases} 0.5\,, & if\ y \geq 0 \\ 0.8\,, & Otherwise\,, \end{cases}$$

and

$$\nu_B(y) = \begin{cases} 0.2\,, & if\ y \geq 0 \\ 0.1\,, & Otherwise\,, \end{cases}$$

We suppose that $T = \{A, 0_\sim, U_1\}$, where

$$\mu_{U_1}(x) = \begin{cases} 0.55\,, & if\ x \in \mathbb{R}\setminus[-2,0] \\ 0\,, & Otherwise\,, \end{cases} \qquad and \quad \nu_{U_1}(x) = \begin{cases} 0.4\,, & if\ x \in \mathbb{R}\setminus[-2,0] \\ 1\,, & Otherwise\,, \end{cases}$$

Also, we suppose that $T' = \{B, 0_\sim, U_1'\}$, where

$$\mu_{U_1'}(y) = \begin{cases} 0.5\,, & if\ y \geq 0 \\ 0\,, & Otherwise\,, \end{cases} \qquad and \quad \nu_{U_1'}(y) = \begin{cases} 0.3\,, & if\ y \geq 0 \\ 1\,, & Otherwise. \end{cases}$$

Then, the intuitionistic fuzzy mapping $f_R : A \to B$ define by :

(i) an ordinary mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = (x+1)^2 - 1$ for any $x \in \mathbb{R}$,

(i) an intuitionistic fuzzy relation R defined by :

$$\mu_R(x,f(x)) = \begin{cases} 0.5\ , \ if\ x\in\mathbb{R}\setminus[-2,0] \\ 0.55\ ,\ Otherwise, \end{cases} \qquad and\ \nu_R(x,f(x)) = 0.4.$$

is an intuitionistic fuzzy continuous mapping.
Indeed, it is easy to show that $f_R^{-1}(B) = A$ *and* $f_R^{-1}(0_\sim) = 0_\sim$ *and we have,*

$$\begin{aligned}\mu_{f_R^{-1}(U_1')}(x) &= \begin{cases} \mu_A(x)\ ,\ if\ x\in f^{-1}(supp(U_1')) \\ 0\ ,\ Otherwise, \end{cases} \\ &= \begin{cases} \mu_A(x)\ ,\ if\ x\in\mathbb{R}\setminus[-2,0] \\ 0\ ,\ Otherwise, \end{cases} \\ &= \mu_{U_1}(x),\end{aligned}$$

and

$$\begin{aligned}\nu_{f_R^{-1}(U_1')}(x) &= \begin{cases} \nu_A(x)\ ,\ if\ x\in f^{-1}(supp(U_1')) \\ 1\ ,\ Otherwise, \end{cases} \\ &= \begin{cases} \nu_A(x)\ ,\ if\ x\in\mathbb{R}\setminus[-2,0] \\ 1\ ,\ Otherwise, \end{cases} \\ &= \begin{cases} 0.4\ ,\ if\ x\in\mathbb{R}\setminus[-2,0] \\ 1\ ,\ Otherwise, \end{cases} \\ &= \nu_{U_1}(x).\end{aligned}$$

Hence, $f_R^{-1}(U_1') = U_1 \in T$. *Thus,* f_R *is an intuitionistic fuzzy continuous mapping.*

Remark 4.8. *Let* (A,T) *be an intuitionistic fuzzy topological space. Then the intuitionistic fuzzy identity mapping* $Id_R : (A,T) \to (A,T)$ *is an intuitionistic fuzzy continuous mapping.*

Next, the relations between various kinds of intuitionistic fuzzy continuous mapping are provided. First, the definitions of precontinuous mapping, α-continuous mapping introduced by Gürçay et al. [9] need to be recalled.

Definition 4.9. [9] *Let* $f_R : (A,T) \to (B,T')$ *be an intuitionistic fuzzy mapping. Then* f_R *is called :*

1. *an intuitionistic fuzzy precontinuous mapping if* $f_R^{-1}(U')$ *is an IFPOS on A for every IFOS* U' *on B;*
2. *an intuitionistic fuzzy* α*-continuous mapping if* $f_R^{-1}(U')$ *is an IF*α*OS on A for every IFOS* U' *on B.*

The following proposition shows the relationship between intuitionistic fuzzy continuous mapping and intuitionistic fuzzy α-continuous mapping.

Proposition 4.10. *Let* $f_R : (A,T) \to (B,T')$ *be an intuitionistic fuzzy mapping. If* f_R *is an intuitionistic fuzzy continuous mapping, then* f_R *is an intuitionistic fuzzy* α*-continuous mapping.*

Proof. Let U' be an IFOS in B and we need to show that $f_R^{-1}(U')$ is an IFαOS in A. The fact that f_R is an intuitionistic fuzzy continuous mapping implies that $f_R^{-1}(U')$ is an IFOS in A. From Definition 3.11, it follows that

$$\mu_{f_R^{-1}(U')}(x) = \begin{cases} \mu_A(x)\ ,\ if\ x\in f^{-1}(supp(U')) \\ 0\ ,\ Otherwise, \end{cases}$$

and

$$\nu_{f_R^{-1}(U')}(x) = \begin{cases} \nu_A(x)\ ,\ if\ x\in f^{-1}(supp(U')) \\ 1\ ,\ Otherwise. \end{cases}$$

We conclude that, $f_R^{-1}(U')$ is an IFαOS in A. Hence, f_R is an intuitionistic fuzzy α-continuous mapping. □

Remark 4.11. *The converse of the above implication does not necessarily hold. Indeed, let us consider the intuitionistic fuzzy mapping* f_R *given in Example 4.7 and* T' *be an IF-topology given by* $T' = \{0_\sim, B, U_2'\}$*, where:*

$$\mu_{U_2'}(y) = \begin{cases} 0.3\ ,\ if\ y\geq -\frac{1}{2} \\ 0\ ,\ Otherwise, \end{cases} \qquad and\ \ \nu_{U_2'}(y) = \begin{cases} 0.4\ ,\ if\ y\geq -\frac{1}{2} \\ 1\ ,\ Otherwise. \end{cases}$$

It is easy to verify that

$$\mu_{f_R^{-1}(U_2')}(x) = \begin{cases} \mu_A(x)\ ,\ if\ x\in f^{-1}(supp(U_2')) \\ 0\ ,\ Otherwise, \end{cases} \qquad = \begin{cases} 0.55\ ,\ if\ x\in\mathbb{R}\setminus[-\frac{\sqrt{2}}{2}-1,\ \frac{\sqrt{2}}{2}-1] \\ 0\ ,\ Otherwise, \end{cases}$$

and

$$\nu_{f_R^{-1}(U_2')}(x) = \begin{cases} \nu_A(x)\ ,\ if\ x\in f^{-1}(supp(U_2')) \\ 1\ ,\ Otherwise, \end{cases} \qquad = \begin{cases} 0.4\ ,\ if\ x\in\mathbb{R}\setminus[-\frac{\sqrt{2}}{2}-1,\ \frac{\sqrt{2}}{2}-1] \\ 1\ ,\ Otherwise, \end{cases}$$

Hence, $int(f_R^{-1}(U_2')) = U_1$ *and* $cl(U_1) = 1_\sim$ *and* $int(1_\sim) = A$*. Thus,* $f_R^{-1}(U_2') \subseteq int(cl(int(f_R^{-1}(U_2'))))$*. We conclude that* $f_R^{-1}(U_2')$ *is an IF*α*S but not IFOS and* f_R *is an intuitionistic fuzzy* α*-continuous but not an intuitionistic fuzzy continuous.*

The following proposition shows the relationship between intuitionistic fuzzy α-continuous mapping and intuitionistic fuzzy pre-continuous mapping.

Proposition 4.12. *Let $f_R : (A,T) \to (B,T')$ be an intuitionistic fuzzy mapping. If f_R is an intuitionistic fuzzy α-continuous mapping, then f_R is an intuitionistic fuzzy pre-continuous mapping.*

Proof. Let U' be an IFOS in B and we need to show that $f_R^{-1}(U')$ is an IFPOS in A. The fact that f_R is an intuitionistic fuzzy α-continuous mapping implies that $f_R^{-1}(U')$ is an IFαOS in A. From Definition 3.11, it follows that

$$\mu_{f_R^{-1}(U')}(x) = \begin{cases} \mu_A(x)\,,\ if\ x \in f^{-1}(supp(U')) \\ 0\,,\ Otherwise\,, \end{cases}$$
$$\nu_{f_R^{-1}(U')}(x) = \begin{cases} \nu_A(x)\,,\ if\ x \in f^{-1}(supp(U')) \\ 1\,,\ Otherwise\,, \end{cases}$$

We conclude that, $f_R^{-1}(U')$ is an IFPOS in A. Hence, f_R is an intuitionistic fuzzy pre-continuous mapping. □

Remark 4.13. *The converse of the above implication is not necessarily holds. Indeed, let us consider the intuitionistic fuzzy mapping f_R given in Example 4.7 and T' be an IF-topology given by $T' = \{0_\sim, B, U'_3\}$, where:*

$$\mu_{U'_3}(y) = \begin{cases} 0.3\,,\ if\ y \in [-1,0] \\ 0\,,\ Otherwise\,, \end{cases} \quad and \quad \nu_{U'_3}(y) = \begin{cases} 0.4\,,\ if\ y \in [-1,0] \\ 1\,,\ Otherwise. \end{cases}$$

It is easy to verify that

$$\mu_{f_R^{-1}(U'_3)}(x) = \begin{cases} \mu_A(x)\,,\ if\ x \in f^{-1}(supp(U'_3)) \\ 0\,,\ Otherwise\,, \end{cases} = \begin{cases} 0.55\,,\ if\ x \in [-2,0] \\ 0\,,\ Otherwise\,, \end{cases}$$

and

$$\nu_{f_R^{-1}(U'_3)}(x) = \begin{cases} \nu_A(x)\,,\ if\ x \in [-2,0] \\ 1\,,\ Otherwise\,, \end{cases} = \begin{cases} 0.4\,,\ if\ x \in [-2,0] \\ 1\,,\ Otherwise. \end{cases}$$

Hence, $cl(f_R^{-1}(U'_3)) = 1_\sim$ and $int(1_\sim) = A$. Thus, $f_R^{-1}(U'_3) \subseteq int(cl(f_R^{-1}(U'_3)))$. We conclude that $f_R^{-1}(U'_3)$ is an IFPOS and f_R is an intuitionistic fuzzy pre-continuous but not an intuitionistic fuzzy α-continuous.

5. Conclusion

In this work, the notion of intuitionistic fuzzy mapping based on the intuitionistic fuzzy relation as a generalization of the notion of fuzzy mapping defined by Ismail and Massa'deh is introduced and the most interesting properties are investigated. As applications, some properties of intuitionistic fuzzy continuous mappings in intuitionistic fuzzy topological space are provided and relations among various kinds of intuitionistic fuzzy continuity are investigated.
Future work is anticipated in multiple directions. We think it makes sense to study the notion of intuitionistic fuzzy mapping for other types of topologies based on the intuitionistic fuzzy sets. Moreover, we intend to extend this work to other kinds of intuitionistic fuzzy continuous mappings.

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, VII ITKRs Scientific Session, Sofia, 1983.
[2] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), 87–96, doi:10.1016/S0165-0114(86)80034-3
[3] K. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer-Verlag, Heidelberg, New York, 1999, doi:10.1007/978-3-7908-1870-3-1
[4] P. Burillo and H. Bustince, *Intuitionistic fuzzy relations (Part I)* , Mathware and computing **2**, (1995), 5–38.
[5] P. Burillo and H. Bustince, *Intuitionistic fuzzy relations (Part II)* , Mathware and computing **2**, (1995), 117–148.
[6] H. Bustince and P. Burillo, *Antisymmetrical intuitionistic fuzzy relation. Order on the referential set induced by an bi fuzzy relation* , Fuzzy Sets and Systems, Fuzzy Sets and Systems, **2** (1995), 17–22.
[7] H. Bustince and P. Burillo, *Structures on intuitionistic fuzzy relations* , Fuzzy Sets and Systems, **3** (1996), 293–303, doi:10.1016/0165-0114(96)84610-0
[8] P. Burillo, *Construction of intuitionistic fuzzy relations with predetermined properties*, Fuzzy Sets and Systems, Fuzzy Sets and Systems **3** (2000), 379–403, doi:10.1016/S0165-0114(97)00381-3
[9] H. Gürçay, D. Çoker and A.H. Eş, *On fuzzy continuity in intuitionistic fuzzy topological spaces*, Journal of Fuzzy Mathematics, **2** (2003), 365–378.
[10] I. Farhan and O.M. Mourad, *A new structure and constructions of L-fuzzy maps*, International Journal of Computational and Applied Mathematics **1** (2013), 1–10.
[11] O. Galor, *Discrete dynamical systems*, Springer, 2007, doi:10.1007/3-540-36776-4
[12] G. Gomathi and D. Jayanthi, *Intuitionistic fuzzy $b^\sharp$ continuous mapping*, Advances in Fuzzy Mathematics **1** (2018), 39–47.
[13] S. Heilpern, *Fuzzy mappings and fixed point theorem, Journal of Mathematical Analysis and Applications*, Journal of Mathematical Analysis and Applications, **83** (1981), 566–569, doi:10.1016/0022-247X(81)90141-4
[14] G.E. Hughes and M.J. Cresswell, *A New Introduction to Modal Logic*, London: Routledge, (2012), doi:10.4324/9780203028100
[15] K. Hur, S.Y. Jang and H.M. Kang, *Intuitionistic fuzzy subgroupoids*, International Journal of Fuzzy Logic and Intelligent Systems, **1** (2003), 72–77, doi:10.5391/IJFIS.2003.3.1.072
[16] D. Jayanthi, *Intuitionistic Fuzzy Generalized Beta Continuous*, Indian Journal of Applied Research, Mappings, **4** (2014), 1–6.
[17] A. Kandil, S. Saleh and M.M. Yakout, *Fuzzy topology on fuzzy sets: regularity and separation axioms*, American Academic and Scholarly Research Journal, **2** (2012).
[18] H.W. Kang, J-G. Lee and K. Hur, *Intuitionistic fuzzy mappings and intuitionistic fuzzy equivalence relations*, Annals of Fuzzy Mathematics and Informatics, **1** (2012), 61–87.
[19] K. Lim, G.H. Choi and H. Hur, *Fuzzy mappings and fuzzy equivalence relations*, International Journal of Fuzzy Logic and Intelligent Systems, **3** (2011), 750–749, doi:10.5391/IJFIS.2011.11.3.153
[20] A. Manimaran, K. A. Prakash, P. Thangaraj, *Intuitionistic fuzzy totally continuous and totally semi-continuous mappings in intuitionistic fuzzy topological spaces*, International journal of Advanced Scientific and Technical Research, **2** (2011), 505–509.
[21] S.K. Sardar, M. Mandal and S.K. Majumder, *Intuitionistic fuzzy ideal extensions in semigroups* , J. Pure Appl. Math, **1** (2015), 59–67.

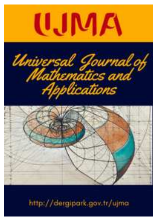
[22] Y. Shen, F. Wang and W. Chen, *A note on intuitionistic fuzzy mappings* , Iranian Journal of Fuzzy Systems, **5** (2012), 63–76.
[23] H. Simmons, *An introduction to category theory, Cambridge University Press, Cambridge* , New York Rydeheard DE, Burstall.
[24] L.A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 331–352, doi:10.1016/S0019-9958(65)90241-X
[25] L. Zedam, S. Milles and E. Rak, *The fixed point property for intuitionistic fuzzy lattices* , Fuzzy Information and Engineering, **9** (2017), 359–380, doi:10.1016/j.fiae.2017.09.007

Universal Journal of Mathematics and Applications, 3 (4) (2020) 156-159
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.730960



On Almost Generalized Weakly Symmetric α-Cosymplectic Manifolds

Mustafa Yıldırım[1*] and Selahattin Beyendi[2]

[1]*Aksaray University, Faculty of Art and Scinence, Department of Mathematics, Aksaray, Turkey*
[2]*İnönü University, Faculty of Education, 44000, Malatya, Turkey*
[*]*Corresponding author*

Article Info

Keywords: *Almost generalized weakly symmetric manifold, Almost generalized weakly Ricci-symmetric manifold, α-cosymplectic manifold.*
2010 AMS: *53C15,53C25.*
Received: *2 May 2020*
Accepted: *29 September 2020*
Available online: *23 December 2020*

Abstract

In the present paper, we study the notions of an almost generalized weakly symmetric α-cosymplectic manifolds and an almost generalized weakly Ricci-symmetrik α-cosymplectic manifolds.

1. Introduction

In 1989, L. Tamassy and T. Q. Binh intruduced the notions of weakly symmetric Riemannian manifold [10]. In the view of [5], a non flat $(2n+1)$-dimensional differantiable manifold, $n>1$, is called almost weakly pseudo symmetric manifold, if there exist A_1, B_1, C_1, D_1, (are non-zero) 1-forms on M such that

$$\begin{aligned}(\nabla_W R)(X_1,X_2,X_3,X_4) &= [A_1(W)+B_1(W)]R(X_1,X_2,X_3,X_4)+C_1(X_1)R(W,X_2,X_3,X_4)\\ &+C_1(X_2)R(X_1,W,X_3,X_4)+D_1(X_3)R(X_1,X_2,W,X_4)+D_1(X_4)R(X_1,X_2,X_3,W),\end{aligned}$$

where $R(X_1,X_2,X_3,X_4)=g(R(X_1,X_2)X_3,X_4)$, R is curvature tensor of type $(1,3)$, A_1,B_1,C_1,D_1 are non-zero 1-forms defined by $A_1(W)=g(W,\sigma_1)$, $B_1(W)=g(W,\rho_1)$, $C_1(W)=g(W,\pi_1)$, $D_1(W)=g(W,\partial_1)$ and σ_1, ρ_1, π_1, ∂_1 are vector fields metrically equivalent to the 1-forms, for all W. Also ∇ denotes Levi-Civita connection with respect to metric tensor g. A $(2n+1)$-dimensional Riemannian manifold of this kind is denoted by $(WS)_{2n+1}$-manifold.
Dubey [8] presented generalized recurrent space. In keeping with this work, we shall call a $(2n+1)$-dimensional α-cosymplectic manifold almost generalized weakly symmetric (briefly $(GWS)_{2n+1}$-manifold) if admits the equation

$$\begin{aligned}(\nabla_W R)(X_1,X_2,X_3,X_4) &= [A_1(W)+B_1(W)]R(X_1,X_2,X_3,X_4)+C_1(X_1)R(W,X_2,X_3,X_4)\\ &+C_1(X_2)R(X_1,W,X_3,X_4)+D_1(X_3)R(X_1,X_2,W,X_4)\\ &+D_1(X_4)R(X_1,X_2,X_3,W)+[A_2(W)+B_2(W)]G(X_1,X_2,X_3,X_4)\\ &+C_2(X_1)G(W,X_2,X_3,X_4)+C_2(X_2)G(X_1,W,X_3,X_4)\\ &+D_2(X_3)G(X_1,X_2,W,X_4)+D_2(X_4)G(X_1,X_2,X_3,W)\end{aligned} \tag{1.1}$$

where

$$G(X_1,X_2,X_3,X_4)=[g(X_2,X_3)g(X_1,X_4)-g(X_1,X_3)g(X_2,X_4)] \tag{1.2}$$

and A_i, B_i, C_i, D_i, $(i=1,2)$, are non-zero 1-forms defined by $A_i(W)=g(W,\sigma_i)$, $B_i(W)=g(W,\rho_i)$, $C_i(W)=g(W,\pi_i)$ and $D_i(W)=g(W,\partial_i)$. There are interesting results of such $(GWS)_{2n+1}$-manifold is that it has kind of

Email addresses and ORCID numbers: **mustafayldrm24@gmail.com, 0000-0002-7885-1492 (M. Yıldırım), selahattin.beyendi@inonu.edu.tr, 0000-0002-1037-6410 (S. Beyendi)**

i) (for $A_i = B_i = C_i = D_i = 0$), locally symmetric space in the sense of Cartan
ii) (for $A_1 \neq 0, B_i = C_i = D_i = 0$), recurrent space by Walker [13],
iii) (for $A_i \neq 0$, $B_i = C_i = D_i = 0$), generalized reccurent space by Dubey [8],
iv) (for $A_1 = B_1 = C_1 = D_1 \neq 0$ and $A_2 = B_2 = C_2 = D_2 = 0$), pseudo symmetric space by Chaki [6],
v) (for $A_1 = -B_1$, $C_1 = D_1$ and $A_2 = B_2 = C_2 = D_2 = 0$), semi-pseudo symmetric space in the sense of Tarafdar et al. [11],
vi) (for $A_1 = -B_1$, $C_1 = D_1$ and $A_2 = -B_2, C_2 = D_2 = 0$), generalized semi-pseudo symmetric space in the sense of Baishya [3],
vii) (for $A_i = B_i = C_i = D_i \neq 0$), generalized pseudo symmetric space, by Baishya [3]
viii) (for $B_1 \neq 0, A_1 = C_1 = D_1 \neq 0$ and $A_2 = B_2 = C_2 = D_2 = 0$), almost pseudo symmetric space in the sprite of Chaki et al [5],
ix) (for $B_i \neq 0, A_i = C_i = D_i \neq 0$), almost generalized pseudo symmetric space in the sense of Baishya,
x) (for $A_2 = B_2 = C_2 = D_2 = 0$), weakly symmetric space by Tamassy and Binh [10].

Recently, α-cosymplectic manifolds and almost α-cosymplectic manifolds have been studied by many different researchers ([1], [2] [4], [9]). Motivated by the above studies, we consider an almost generalized weakly symmetric α-cosymplectic manifolds and an almost generalized weakly Ricci-symmetric α-cosymplectic manifold also obtain some interesting results.

2. Preliminaries

Let M^{2n+1} be a connected almost contact metric manifold with an almost contact metric structure (φ,ξ,η,g), that is, φ is a tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\varphi\xi = 0, \quad \eta(\varphi W) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\varphi^2 W = -W + \eta(W)\xi, \quad g(W,\xi) = \eta(W), \tag{2.2}$$
$$g(\varphi W, \varphi X_1) = g(W,X_1) - \eta(W)\eta(X_1),$$

for any vector fields W and X_1 on M^{2n+1} [7].
If moreover

$$\nabla_W \xi = -\alpha\varphi^2 W, \tag{2.3}$$
$$(\nabla_W \eta)X_1 = \alpha[g(W,X_1) - \eta(W)\eta(X_1)],$$

where ∇ denotes the Riemannian connection of hold and α is a real number, then $(M^{2n+1},\varphi,\xi,\eta,g)$ is called an α-cosymplectic manifold [12]. In this case, it is well know that [9]

$$R(W,X_1)\xi = \alpha^2[\eta(W)X_1 - \eta(X_1)W], \tag{2.4}$$

$$S(W,\xi) = -2n\alpha^2\eta(W), \tag{2.5}$$

$$S(\xi,\xi) = -2n\alpha^2, \tag{2.6}$$

where S denotes the Ricci tensor. From (2.4), it easily follows that

$$R(W,\xi)X_1 = \alpha^2[g(W,X_1)\xi - \eta(X_1)W] \tag{2.7}$$
$$R(W,\xi)\xi = \alpha^2[\eta(W)\xi - W],$$

for any vector fields W, X_1, Z where R is the Riemannian curvature tensor of the manifold. An α-cosymplectic manifold is said to be an η-Einstein manifold if Ricci tensor S satisfies condition

$$S(W,X_1) = \lambda_1 g(W,X_1) + \lambda_2\eta(W)\eta(X_1),$$

where λ_1, λ_2 are certain scalars.

3. Almost generalized weakly symmetric α-cosymplectic manifold

An α-cosymplectic manifold (M^{2n+1},g) is said to be an almost generalized weakly symmetric if admits the relation (1.1), $(n \geq 1)$. Now, contracting X_1 over X_4 in both sides of (1.1), we obtain

$$\begin{aligned}(\nabla_W S)(X_2,X_3) &= [A_1(W)+B_1(W)]S(X_2,X_3) + C_1(R(W,X_2)X_3) + C_1(X_2)S(W,X_3) + D_1(X_3)S(W,X_2) + D_1(R(W,X_3)X_2) \\ &\quad + 2n[A_2(W)+B_2(W)]g(X_2,X_3) + C_2(G(W,X_2)X_3) + 2nC_2(X_2)g(W,X_3) + 2nD_2(X_3)g(X_2,W) + D_2(G(W,X_3)X_2).\end{aligned} \tag{3.1}$$

Putting $X_3 = \xi$ in (3.1) and using (1.2), (2.4), (2.5), (2.7), we have

$$\begin{aligned}(\nabla_W S)(X_2,\xi) &= (-2n\alpha^2)[A_1(W)+B_1(W)]\eta(X_2) + (-2n+1)\alpha^2 C_1(X_2)\eta(W) \\ &\quad - \alpha^2 C_1(W)\eta(X_2) + D_1(\xi)S(X_2,W) + \alpha^2 g(W,X_2)D_1(\xi) - \alpha^2 D_1(W)\eta(X_2) \\ &\quad + 2n[A_2(W)+B_2(W)]\eta(X_2) + C_2(W)\eta(X_2) - C_2(X_2)\eta(W) \\ &\quad + 2nC_2(X_2)\eta(W) + 2nD_2(\xi)g(W,X_2) + D_2(W)\eta(X_2) - D_2(\xi)g(W,X_2).\end{aligned} \tag{3.2}$$

Taking $X_3 = \xi$ in the below identity

$$(\nabla_W S)(X_2, X_3) = \nabla_W S(X_2, X_3) - S(\nabla_W X_2, X_3) - S(X_2, \nabla_W X_3)$$

and then using (2.2), (2.3), (2.5), we obtain

$$(\nabla_W S)(X_2, \xi) = 2n\alpha^2 g(X_2, W) - \alpha^2 S(X_2, W). \tag{3.3}$$

Now, using (3.3) in (3.2), we have

$$\begin{aligned} 2n\alpha^2 g(X_2, W) - \alpha^2 S(X_2, W) = & -2n\alpha^2[A_1(W) + B_1(W)]\eta(X_2) + (-2n+1)\alpha^2 C_1(X_2)\eta(W) \\ & - \alpha^2 C_1(W)\eta(X_2) + D_1(\xi)S(X_2, W) + \alpha^2 g(W, X_2)D_1(\xi) \\ & - \alpha^2 D_1(W)\eta(X_2) + 2n[A_2(W) + B_2(W)]\eta(X_2) + C_2(W)\eta(X_2) \\ & - C_2(X_2)\eta(W) + 2nC_2(X_2)\eta(W) + 2nD_2(\xi)g(W, X_2) \\ & + D_2(W)\eta(X_2) - D_2(\xi)g(W, X_2). \end{aligned} \tag{3.4}$$

Then replacing W and X_2 by ξ in (3.4) and (2.1), (2.6), we get

$$\alpha^2[A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi)] = A_2(\xi) + B_2(\xi) + C_2(\xi) + D_2(\xi). \tag{3.5}$$

In particular, if $A_2(\xi) = B_2(\xi) = C_2(\xi) = D_2(\xi) = 0$, formula (3.5) turns into

$$\alpha^2[A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi)] = 0.$$

Theorem 3.1. *In an almost generalized weakly symmetric α-cosymplectic manifold (M^{2n+1}, g), $n \geq 1$, the relation (3.5) hold good.*

Again from (3.1), putting $X_2 = \xi$, we have

$$\begin{aligned} -2n\alpha^3 g(X_3, W) - \alpha S(X_3, W) = & [A_1(W) + B_1(W)]S(\xi, X_3) + C_1(R(W, \xi)X_3) + C_1(\xi)S(W, X_3) \\ & + D_1(X_3)S(W, \xi) + D_1(R(W, X_3)\xi) + 2n[A_2(W) + B_2(W)]g(\xi, X_3) \\ & + C_2(W)g(\xi, X_3) - C_2(\xi)g(W, X_3) + 2nC_2(\xi)g(W, X_3) \\ & + 2nD_2(X_3)g(\xi, W) + D_2(W)g(\xi, X_3) - D_2(X_3)g(\xi, W). \end{aligned} \tag{3.6}$$

Using (2.4), (2.5), (2.7) in (3.6), we obtain

$$\begin{aligned} -2n\alpha^3 g(X_3, W) - \alpha S(X_3, W) = & -2n\alpha^2[A_1(W) + B_1(W)]\eta(X_3) + \alpha^2 C_1(\xi)g(W, X_3) - \alpha^2\eta(X_3)C_1(W) \\ & + C_1(\xi)S(W, X_3) - 2n\alpha^2\eta(W)D_1(X_3) + \alpha^2\eta(W)D_1(X_3) \\ & - \alpha^2\eta(X_3)D_1(W) + 2n[A_2(W) + B_2(W)]\eta(X_3) + C_2(W)\eta(X_3) \\ & + 2nD_2(X_3)\eta(W) + D_2(W)\eta(X_3) - D_2(X_3)\eta(W). \end{aligned} \tag{3.7}$$

Putting $X_3 = \xi$ in (3.7), we get

$$\begin{aligned} \alpha^2[2n(A_1(W) + B_1(W)) + C_1(W) + D_1(W)] + (2n-1)\alpha^2[C_1(\xi) + D_1(\xi)]\eta(W) = & 2n[A_2(W) + B_2(W)] + C_2(W) + D_2(W) \\ & + (2n-1)[C_2(\xi) + D_2(\xi)]\eta(W). \end{aligned} \tag{3.8}$$

Using $W = \xi$ in (3.7), we obtain

$$\begin{aligned} & 2n\alpha^2[A_1(\xi) + B_1\xi + C_1(\xi)]\eta(X_3) + \alpha^2 D_1(\xi)\eta(X_3) + (2n-1)\alpha^2 D_1(X_3) \\ & = 2n[A_2(\xi) + B_2(\xi) + C_2(\xi)]\eta(X_3) + D_2(\xi)\eta(X_3) + (2n-1)D_2(X_3). \end{aligned}$$

Replacing X_3 by W in the above equation and using (3.5), we have

$$\alpha^2 D_1(\xi)\eta(W) - \alpha^2 D_1(W) = D_2(\xi)\eta(W) - D_2(W). \tag{3.9}$$

Again, putting $W = \xi$ in (3.4), we get

$$\begin{aligned} & 2n\alpha^2[A_1(\xi) + B_1(\xi) + D_1(\xi)]\eta(X_2) + \alpha^2 C_1(\xi)\eta(X_2) + (2n-1)\alpha^2 C_1(X_2) \\ & = 2n[A_2(\xi) + B_2(\xi) + D_2(\xi)]\eta(X_2) + C_2(\xi)\eta(X_2) + (2n-1)C_2(X_2). \end{aligned} \tag{3.10}$$

Replacing X_2 by W in (3.10) and using (3.5), we obtain

$$\alpha^2 C_1(\xi)\eta(W) - \alpha^2 C_1(W) = C_2(\xi)\eta(W) - C_2(W). \tag{3.11}$$

Subtracting (3.9), (3.11) from (3.8)

$$\alpha^2[A_1(W) + B_1(W) + C_1(W) + D_1(W)] = [A_2(W) + B_2(W) + C_2(W) + D_2(W)]. \tag{3.12}$$

Next, in view of $A_2 = B_2 = C_2 = D_2 = 0$, the relation (3.12) yields

$$\alpha^2[A_1(W) + B_1(W) + C_1(W) + D_1(W)] = 0.$$

This motivates us to state the followings

Theorem 3.2. *In an almost generalized weakly symmetric α-cosymplectic manifold (M^{2n+1}, g) $(n \geq 1)$, the sum of the associated 1-forms is given by (3.12).*

Theorem 3.3. *There does not exist an α-cosymplectic manifold which is*
(i) recurrent,
(ii) generalized recurrent provided the 1-forms are collinear,
(iii) pseudo symmetric,
(iv) generalized semi-pseudo symmetric provided the 1-forms are collinear,
(v) generalized almost-pseudo symmetric provided the 1-forms are collinear.

4. Almost generalized weakly Ricci-symmetric α-cosymplectic manifold

An α-cosymplectic manifold (M^{2n+1},g) $(n\geq 1)$, is said to be almost generalized weakly Ricci-symmetric if there exist 1-forms, $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{C}_i$ and $\tilde{D}_i$ which satisfy the condition

$$\begin{aligned}(\nabla_W S)(X_2,X_3) &= [\tilde{A}_1(W)+\tilde{B}_1(W)]S(X_2,X_3)+\tilde{C}_1(X_2)S(W,X_3)+\tilde{D}_1(X_3)S(X_2,W)+[\tilde{A}_2(W)+\tilde{B}_2(W)]g(X_2,X_3)\\ &+\tilde{C}_2(X_2)g(W,X_3)+\tilde{D}_2(X_3)g(X_2,W).\end{aligned} \tag{4.1}$$

Putting $X_3=\xi$ in (4.1), and using (2.1), (2.5), we get

$$\begin{aligned}(\nabla_W S)(X_2,\xi) &= -2n\alpha^2[\tilde{A}_1(W)+\tilde{B}_1(W)]\eta(X_2)-2n\alpha^2\tilde{C}_1(X_2)\eta(W)+\tilde{D}_1(\xi)S(X_2,W)+[\tilde{A}_2(W)+\tilde{B}_2(W)]\eta(X_2)\\ &+\tilde{C}_2(X_2)\eta(W)+\tilde{D}_2(\xi)g(X_2,W).\end{aligned} \tag{4.2}$$

Using equation (3.3) in (4.2) we get,

$$\begin{aligned}-2n\alpha^3g(X_2,W)-\alpha S(X_2,W) &= -2n\alpha^2[\tilde{A}_1(W)+\tilde{B}_1(W)]\eta(X_2)-2n\alpha^2\tilde{C}_1(X_2)\eta(W)+\tilde{D}_1(\xi)S(X_2,W)+[\tilde{A}_2(W)+\tilde{B}_2(W)]\eta(X_2)\\ &+\tilde{C}_2(X_2)\eta(W)+\tilde{D}_2(\xi)g(X_2,W).\end{aligned} \tag{4.3}$$

Putting $W=X_2=\xi$ in (4.3), we have

$$2n\alpha^2[\tilde{A}_1(\xi)+\tilde{B}_1(\xi)+\tilde{C}_1(\xi)+\tilde{D}_1(\xi)]=\tilde{A}_2(\xi)+\tilde{B}_2(\xi)+\tilde{C}_2(\xi)+\tilde{D}_2(\xi). \tag{4.4}$$

Then, taking $W=\xi$ in (4.3), we obtain

$$2n\alpha^2[\tilde{A}_1(\xi)+\tilde{B}_1(\xi)+\tilde{D}_1(\xi)]\eta(X_2)+2n\alpha^2\tilde{C}_1(X_2)=[\tilde{A}_2(\xi)+\tilde{B}_2(\xi)+\tilde{D}_2(\xi)]\eta(X_2)+\tilde{C}_2(X_2). \tag{4.5}$$

Using $X_2=\xi$ in (4.3), we get

$$2n\alpha^2[\tilde{A}_1(\xi)+\tilde{B}_1(\xi)+\tilde{D}_1(\xi)]\eta(W)+2n\alpha^2\tilde{C}_1(W)=[\tilde{A}_2(\xi)+\tilde{B}_2(\xi)+\tilde{D}_2(\xi)]\eta(W)+\tilde{C}_2(W). \tag{4.6}$$

Replacing X_2 by W in (4.5) and adding with (4.6), we have

$$\begin{aligned}2n\alpha^2[\tilde{A}_1(W)+\tilde{B}_1(W)+\tilde{C}_1(W)]-[\tilde{A}_2(W)+\tilde{B}_2(W)+\tilde{C}_2(W)] &= -2n\alpha^2[\tilde{A}_1(\xi)+\tilde{B}_1(\xi)+\tilde{C}_1(\xi)+\tilde{D}_1(\xi)]\eta(W)\\ &+[\tilde{A}_2(\xi)+\tilde{B}_2(\xi)+\tilde{C}_2(\xi)+\tilde{D}_2(\xi)]\eta(W)-2n\alpha^2\tilde{D}_1(\xi)\eta(W)\\ &-\tilde{D}_2(\xi)\eta(W).\end{aligned} \tag{4.7}$$

In view of (4.4) the relation (4.7) becomes

$$2n\alpha^2[\tilde{A}_1(W)+\tilde{B}_1(W)\tilde{C}_1(W)]+2n\alpha^2\tilde{D}_1(\xi)\eta(W)=[\tilde{A}_2(W)+\tilde{B}_2(W)+\tilde{C}_2(W)]-\tilde{D}_2(\xi)\eta(W). \tag{4.8}$$

Then, taking $W=X_2=\xi$ in (4.1), we obtain

$$2n\alpha^2[\tilde{A}_1(\xi)+\tilde{B}_1(\xi)+\tilde{C}_1(\xi)]\eta(X_3)+2n\alpha^2\tilde{D}_1(X_3)=[\tilde{A}_2(\xi)+\tilde{B}_2(\xi)+\tilde{C}_2(\xi)]\eta(X_3)+\tilde{D}_2(X_3). \tag{4.9}$$

In view of (4.4), replacing X_3 by W in (4.9) and then adding the resultant with (4.8),

$$\begin{aligned}2n\alpha^2\{[\tilde{A}_1(W)+\tilde{B}_1(W)+\tilde{C}_1(W)+\tilde{D}_1(W)]+[\tilde{A}_1(\xi)+\tilde{B}_1(\xi)+\tilde{C}_1(\xi)+\tilde{D}_1(\xi)]\eta(W)\} &= [\tilde{A}_2(W)+\tilde{B}_2(W)+\tilde{C}_2(W)+\tilde{D}_2(W)]\\ &+[\tilde{A}_2(\xi)+\tilde{B}_2(\xi)+\tilde{C}_2(\xi)+\tilde{D}_2(\xi)]\eta(W).\end{aligned} \tag{4.10}$$

Next, putting (4.4) in (4.10), we get

$$2n\alpha^2[\tilde{A}_1(W)+\tilde{B}_1(W)+\tilde{C}_1(W)+\tilde{D}_1(W)]=\tilde{A}_2(W)+\tilde{B}_2(W)+\tilde{C}_2(W)+\tilde{D}_2(W). \tag{4.11}$$

Theorem 4.1. *In an almost generalized weakly Ricci-symmetric α-cosymplectic manifold (M^{2n+1},g), $n\geq 1$, the relation (4.11) hold good.*

Theorem 4.2. *There does not exist an almost generalized weakly Ricci symmetric α-cosymplectic manifold which is*

i) recurrent,
ii) generalized recurrent provided the 1-forms are collinear,
iii) pseudo symmetric,
iv) generalized semi-pseudo symmetric provided the 1-forms are collinear,
v) generalized almost-pseudo symmetric provided the 1-forms are collinear.

References

[1] N. Aktan, M. Yıldırım, C. Murathan,*Almost f-cosymplectic manifolds*, Mediterr. J. Math., **11**(2014), 775-787.
[2] G. Ayar, S.K. Chaubey, *M-Projective curvature tensor over cosymplectic manifolds*, Differ. Geom. Dyn. Syst., **21**(2019), 23-33.
[3] K.K. Baishya, P.R. Chowdhury, J. Mikes, P. Peska, *On almost generalized weakly symmetric Kenmotsu manifolds*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica, **55**(2016), 2, 5-15.
[4] S. Beyendi, G. Ayar, N. Aktan, *On a type of α-cosymplectic manifolds*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **68**(1)(2019), 852-861.
[5] M.C. Chaki, T. Kawaguchi, *On almast pseudo Ricci symmetric manifolds*, Tensor, **68**(1)(2017), 10-14.
[6] M. C. Chaki, *On pseudo Ricci symmetric manifolds*, , Bulg. J. Physics, **15**(1998), 526-531.
[7] D.E. Blair, *Contact manifolds in Riemannian geometry*, , Lecture Notes in Math. 509, (1976), Springer-Verlag, Berlin.
[8] R.S.D. Dubey, *Generalized recurrent spaces*, Indian J. Pure Appl. Math., **10**(1979), 1508-1513.
[9] H. Öztürk, C. Murathan, N. Aktan, A.T. Vanli, *Almost α-cosymplectic f-manifolds*, (2014), An. Stiint. Univ. Al. I. Cuza Iasi Inform. (N.S.) Matematica, Tomul LX, f.1.
[10] L.Tamassy, T.Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Coll. Math. Soc., J. Bolyai, **56**(1989), 663-670.
[11] M. Tarafdar, M.A.A. Jawarneh, *Semi-pseudo Ricci symmetric manifold*, J. Indian. Inst. of Science., **73**(1993), 591-596.
[12] T.W. Kim, H.K. Pak, *Canonical foliations of certain classes of almost contact metric structures*, , Acta Math, Sinica, Eng. Ser. Aug., **21**(4)(2005), 841-846.
[13] A.G. Walker, *On Ruse's space of recurrent curvature*, Proc. of London Math. Soc. **52**(1950), 36-54.

Universal Journal of Mathematics and Applications, 3 (4) (2020) 160-166
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.707355

Soft Topological Space in Virtue of Semi* Open Sets

Paulraj Gnanachandra[1*], Lellis Thivagar[2] and Muneesh Kumar Arumugam[1]

[1]*Centre for Research and Post Graduate Studies in Mathematics, Ayya Nadar Janaki Ammal College(Autonomous), Sivakasi-626 124, Tamil Nadu, India.*
[2]*School of Mathematics, Madurai Kamaraj University, Madurai-625021, Tamil Nadu, India.*
[*]*Corresponding author*

Article Info

Keywords: *Soft semi*-closure, Soft semi*-connected, Soft semi*-compact soft semi*-interior, Soft generalized closure, Soft generalized interior.*
2010 AMS: *06D72, 54A05.*
Received: *22 March 2020*
Accepted: *15 December 2020*
Available online: *23 December 2020*

Abstract

The ultimate purpose of this research article is to originate and examine some new kind of open sets in soft topological spaces such as soft semi* - open and soft semi* - closed sets using generalized closure operator with illustrating counter examples.

1. Introduction

In our day-to-day life, we look out problems with unreliabilities. To handle the lack of unreliability and to solve the problems related to uncertainity, a short time ago numberless theories have been developed like Rough Sets, Fuzzy Sets and Vague Sets. However, these methodologies have their own risks. To circumvent these difficulties, Molodtsov [5] developed Soft set theory to deal with unreliability. The development of Soft Set theory is whistle stop now-a-days. Soft set theory has a wider application and its progress is very rapid in different fields [see [19], [20] and [10]]. The approach of Soft topological spaces was codified by Shabir et al. [12]. Many researchers defined some basic notions on soft topology and studied many properties see [[4], [13], [17], [16], [22], [7], [8] and [9]]. In this milieu, we define penetration of soft semi*-open and soft semi*-closed sets in soft topological spaces and then these are used to study properties of semi* - interior, semi* - closure of soft sets in soft topological spaces. Further the behavior of these concepts under various soft functions has obtained. Also we introduce and study soft semi*-connectedness and soft semi* - compactness using soft semi* - open sets.

2. Preliminaries

We roll call the following definitions with illustrated examples for the outpouring of this article.

Let $\mathcal{U}$ indicates initial universe set and let $\mathcal{E}$ be parameters proportionate to $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ denote the power set of $\mathcal{U}$, and let $\mathcal{A} \subseteq \mathcal{E}$. A subset A of a space (X, τ) is said to be generalized closed [15] (briefly g-closed), if $cl(\mathsf{A}) \subseteq \mathcal{U}$ whenever $\mathsf{A} \subseteq \mathcal{U}$ and $\mathcal{U}$ is open. The intersection of all g-closed sets containing A is called the g - closure of A and denoted by $cl^*(\mathsf{A})$ [21]. A subset A of a space (X, τ) is said to be generalized open if its complement is generalized closed and union of all g - open sets contained in A is called the g - interior of A and is denoted by $int^*(\mathsf{A})$. A subset S of a topological space (S, τ) is said to semi*-open if $\mathsf{S} \subseteq (cl^*(int(\mathsf{S}))$ [18]. The complement of a semi*-open set is semi*-closed. It is well known that a subset S is semi*-closed if and only if $int^*(cl(\mathsf{S})) \subseteq \mathsf{S}$ [3].

Definition 2.1. *[5] A soft set $\mathscr{F}_{\mathsf{A}}$ on the universe $\mathcal{U}$ is defined by the set of ordered pairs $\mathscr{F}_{\mathsf{A}} = \{(x, f_{\mathsf{A}}(x)) | x \in \mathcal{E}, f_{\mathsf{A}}(x) \in \mathcal{P}(\mathcal{U})\}$ where $\mathcal{E}$ is a set of parameters, $\mathsf{A} \subseteq \mathcal{E}$, $\mathcal{P}(\mathcal{U})$ is the power set of $\mathcal{U}$ and $f_{\mathsf{A}} : \mathsf{A} \to \mathcal{P}(\mathcal{U})$ such that $f_{\mathsf{A}}(x) = \emptyset$ if $x \notin \mathsf{A}$. Here f_{A} is called an approximate function of the soft set $\mathscr{F}_{\mathsf{A}}$. The value of $f_{\mathsf{A}}(x)$ may be arbitrary, some of them may be empty and some may have non-empty intersection. Note that the set of all soft sets over $\mathcal{U}$ is denoted by $\mathcal{SS}(\mathcal{U})_{\mathcal{E}}$.*
For illustration, we consider an example which we present below:

***Email addresses and ORCID numbers*: pgchandra07@gmail.com, https://orcid.org/0000-0001-6089-6441, mlthivagar@yahoo.co.in, https://orcid.org/0000-0001-5997-5185, muneeshkumarar@gmail.com, https://orcid.org/0000-0001-7648-0440**

Example 2.2. *Suppose $\mathcal{U}$=set of all real numbers on the closed interval $[a,b]$.*
$\mathcal{E}$=set of parameters. Each parameter is a word or a sentence.
$\mathcal{E}$={Compact, Closed, Connected, Open}
In this case, to define a soft set means to point out closed set, connected set and so on. Let we consider below the same example in more detail. $\mathcal{U}=\{x : a \leq x \leq b\}$ and $\mathcal{E}=\{e_1,e_2,e_3,e_4\}$ where

$e_1 \to$ *'compact',*
$e_2 \to$ *'closed',*
$e_3 \to$ *'connected',*
$e_4 \to$ *'open'.*

Suppose that

$f(e_1)=\{\mathsf{A} \subseteq [a,b] :$ *Every open cover for* A *in* $[a,b]$ *has finite subcover* $\}$.
$f(e_2)=\{[\alpha,\beta] \subseteq [a,b] : \alpha,\beta \in R\}$
$f(e_3)=\{A \subseteq [a,b] :$ *Separation does not exists for A in* $[a,b]\}$
$f(e_4)=\{(\alpha,\beta) \subseteq [a,b] : \alpha,\beta \in R\}$

$\mathscr{F}_{\mathsf{A}} \to$ parametrized family of subsets of the set $\mathcal{U}$. Consider the mapping f in which $f(e_1) \to$ subsets of $\mathcal{U}$ which are compact whose functional value is the set $\{\mathsf{A} \subseteq [a,b]$: Every open cover for A in $[a,b]$ has finite subcover $\}$. Hence the soft set $\mathscr{F}_{\mathsf{A}}$ is the collection of approximations given below:

$\{(compact, \{\mathsf{A} \subseteq [a,b] :$ *Every open cover for* A *in* $[a,b]$ *has finite subcover* $\}), (Closed, \{[\alpha,\beta] \subseteq [a,b] : \alpha,\beta \in R\}), (Connected, \{\mathsf{A} \subseteq [a,b]$ *:separation does not exist for* A *in* $R\}), (Open, \{(\alpha,\beta) \subseteq [a,b] : \alpha,\beta \in R\})\} = \mathscr{F}_{\mathsf{A}}$

Definition 2.3. *[12] Let $\tilde{\tau}$ be a collection of soft sets over a universe $\mathcal{U}$ with a fixed set $\mathcal{E}$ of parameters, then $\tilde{\tau} \subseteq SS(\mathcal{U})_{\mathcal{E}}$ is called a soft topology on $\mathcal{U}$ with a fixed set $\mathcal{E}$ if*

i. *$\phi_{\mathcal{E}}$, $\mathcal{U}_{\mathcal{E}}$ belong to $\tilde{\tau}$.*
ii. *The union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.*
iii. *The intersection of any finite number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.*

The pair $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is called a soft topological space.

Definition 2.4. *[1] Let $\mathcal{U}$ be a universe and $\mathcal{E}$ a set of parameters. Then the collection $\mathcal{SS}(\mathcal{U})_{\mathcal{E}}$ of all soft sets over $\mathcal{U}$ with parameters from $\mathcal{E}$ is called a soft class.*

Definition 2.5. *[1] Let $\mathcal{S}(\mathcal{U})_{\mathcal{E}}$ and $\mathcal{S}(\mathcal{V})_{\mathcal{E}'}$ be two soft classes. Then $u : \mathcal{U} \mapsto \mathcal{V}$ and $p : \mathcal{E} \mapsto \mathcal{E}'$ be two functions. Then a function $f : \mathcal{S}(\mathcal{U})_{\mathcal{E}} \mapsto \mathcal{S}(\mathcal{V})_{\mathcal{E}'}$ and its inverse are defined as*

(i) *Let $\mathcal{L}_{\mathcal{A}}$ be a soft set in $\mathcal{SS}(\mathcal{U})_{\mathcal{E}}$ where $\mathcal{A}\tilde{\subseteq}\mathcal{E}$. The image of $\mathcal{L}_{\mathcal{A}}$ under a function f is a soft set in $\mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ such that $f(\mathcal{L}_{\mathcal{A}})(\beta) = \left(\cup_{\alpha \in p^{-1}(\beta)\cap\mathcal{A}} \mathcal{L}(\alpha)\right)$ for $\beta \in \mathcal{B} = p(\mathcal{A})\tilde{\subseteq}\mathcal{E}'$.*
(ii) *Let $\mathcal{G}$ be the soft set in $\mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ where $\mathcal{C}\tilde{\subseteq}E'$. Then the inverse image of $\mathcal{G}_{\mathcal{C}}$ under f is a soft set in $\mathcal{SS}(\mathcal{U})_{\mathcal{E}}$ such that $f^{-1}(\mathcal{G}_{\mathcal{C}})(\alpha) = u^{-1}(\mathcal{G}(p(\alpha)))$ for $\alpha \in p^{-1}(\mathcal{C})\tilde{\subseteq}\mathcal{E}$.*

3. Semi*-open and semi*-closed soft sets

In this chunk, we expound soft semi*-closure and soft semi*-interior of a soft set are defined in terms of soft semi*-closed and soft semi*-open sets.

Definition 3.1. *In a soft topological space $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ a soft set*

(i) *$\mathcal{G}_{\mathsf{C}}$ is termed as semi*-open soft set if there exists an open soft set $\mathcal{H}_{\mathsf{B}}$ such that $\mathcal{H}_{\mathsf{B}}\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}cl^*(\mathcal{H}_{\mathsf{B}})$.*
(ii) *$\mathcal{L}_{\mathsf{A}}$ is termed as semi*-closed soft set if there exists an closed soft set $\mathcal{K}_{\mathsf{D}}$ such that $int^*(\mathcal{K}_{\mathsf{D}})\tilde{\subseteq}\mathcal{L}_{\mathsf{A}}\tilde{\subseteq}\mathcal{K}_{\mathsf{D}}$.*

We denote the set of all semi - closed Soft sets (respectively, Semi* - open Soft sets) over $\mathcal{U}$ by $\mathcal{S}^*\mathcal{CSS}(\mathcal{U})_{\mathcal{E}}$ (respectively, $\mathcal{S}^*\mathcal{OSS}(\mathcal{U})_{\mathcal{E}}$)*

Theorem 3.2. *Let $\mathcal{G}_{\mathsf{C}}$ be a soft set in a soft topological space $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$. Then the subsequent are equivalent:*

1. *$\mathcal{G}_{\mathsf{C}}$ is a semi*-closed soft set.*
2. *$int^*(cl(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}$.*
3. *$cl^*(int(\mathcal{G}^c_{\mathsf{C}}))\tilde{\supseteq}\mathcal{G}^c_{\mathsf{C}}$.*
4. *$\mathcal{G}^c_{\mathsf{C}}$ is a semi*-open soft set.*

Proof. (1) ⇒(2): If $\mathcal{G}_{\mathsf{C}}$ is a semi-closed soft set, then there exists a closed soft set $\mathcal{H}_{\mathsf{B}}$ such that $int^*(\mathcal{H}_{\mathsf{B}})\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}\mathcal{H}_{\mathsf{B}}$. Also $cl(\mathcal{G}_{\mathsf{C}})$ is a smallest closed soft set that contains $\mathcal{G}_{\mathsf{C}}$. Therefore, $\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}cl(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}\mathcal{H}_{\mathsf{B}}$ which implies $int^*(cl(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}int^*(\mathcal{H}_{\mathsf{B}})\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}$.*

(2)⇒(3): Assume that $int^*(cl(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}$. Now $\mathcal{G}^c_{\mathsf{C}}\tilde{\subseteq}(int^*(cl(\mathcal{G}_{\mathsf{C}})))^c$. This implies $\mathcal{G}^c_{\mathsf{C}}\tilde{\subseteq}cl^*(cl(\mathcal{G}_{\mathsf{C}}))^c$. This implies $\mathcal{G}^c_{\mathsf{C}}\tilde{\subseteq}cl^*(int(\mathcal{G}^c_{\mathsf{C}}))$. Hence $cl^*(int(\mathcal{G}^c_{\mathsf{C}}))\tilde{\supseteq}\mathcal{G}^c_{\mathsf{C}}$.

(3)⇒(4):Take $\mathcal{H}_{\mathsf{B}} = int(\mathcal{G}^c_{\mathsf{C}})$. Then $\mathcal{H}_{\mathsf{B}}$ is an open soft set such that $int(\mathcal{G}^c_{\mathsf{C}})\tilde{\subseteq}(\mathcal{G}^c_{\mathsf{C}})\tilde{\subseteq}cl^*(int(\mathcal{G}^c_{\mathsf{C}}))$ and hence $\mathcal{H}_{\mathsf{B}}\tilde{\subseteq}\mathcal{G}^c_{\mathsf{C}}\tilde{\subseteq}cl^*(\mathcal{H}_{\mathsf{B}})$, where $\mathcal{H}_{\mathsf{B}}$ is an open soft set. Therefore $\mathcal{G}^c_{\mathsf{C}}$ is a semi-open soft set.*

(4)⇒(1): Suppose $\mathcal{G}_{\mathsf{C}}^{c}$ is a semi-open soft set. Then there exists an open soft set $\mathcal{H}_{\mathsf{B}}$ such that $\mathcal{H}_{\mathsf{B}}\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}^{c}\tilde{\subseteq}cl^*(\mathcal{H}_{\mathsf{B}})$. Hence $(cl^*(\mathcal{H}_{\mathsf{B}}))^c\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}(\mathcal{H}_{\mathsf{B}})^c$ and hence $(int^*(\mathcal{H}_{\mathsf{B}})^c)\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}(\mathcal{H}_{\mathsf{B}})^c$. As $\mathcal{H}_{\mathsf{B}}$ is an open soft set,$(\mathcal{H}_{\mathsf{B}})^c$ is a closed soft set. Therefore, there exists a closed soft set $(\mathcal{H}_{\mathsf{B}})^c$ such that $(int^*(\mathcal{H}_{\mathsf{B}})^c)\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}(\mathcal{H}_{\mathsf{B}})^c$. Hence $\mathcal{G}_{\mathsf{C}}$ is a semi*-closed soft set.*

Theorem 3.3. *In a soft topological space $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$, every open soft set in a soft topological space is a semi*-open soft set.*
Proof. Let $\mathcal{G}_{\mathsf{A}}$ be an open soft set. Since $\mathcal{G}_{\mathsf{A}}$ is an open set $int(\mathcal{G}_{\mathsf{A}})=\mathcal{G}_{\mathsf{A}}$. Now $\mathcal{G}_{\mathsf{A}}=int(\mathcal{G}_{\mathsf{A}})\tilde{\subseteq}cl^*(int(\mathcal{G}_{\mathsf{A}}))$ and hence $\mathcal{G}_{\mathsf{A}}\tilde{\subseteq}cl^*(int(\mathcal{G}_{\mathsf{A}}))$. Then $\mathcal{G}_{\mathsf{A}}$ is a semi-open soft set.*

Theorem 3.4. *Every closed soft set in a soft topological space $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is a semi*-closed soft set.*
Proof. Let $\mathcal{G}_{\mathsf{A}}$ be a closed soft set. Since $\mathcal{G}_{\mathsf{A}}$ is closed $\mathcal{G}_{\mathsf{A}}=cl(\mathcal{G}_{\mathsf{A}})$. Now $int^*(cl(\mathcal{G}_{\mathsf{A}}))=int^*(\mathcal{G}_{\mathsf{A}})\tilde{\subseteq}\mathcal{G}_{\mathsf{A}}$.Then $\mathcal{G}_{\mathsf{A}}$ is a semi-closed soft set.*

Theorem 3.5. *Every semi*-open soft set is a semi-open soft set.*
Proof. Let $\mathcal{G}_{\mathsf{A}}$ be a semi-open soft set. Then $\mathcal{G}_{\mathsf{A}}\tilde{\subseteq}cl^*(int(\mathcal{G}_{\mathsf{A}}))$. Also we see that, $cl^*(int(\mathcal{G}_{\mathsf{A}}))\tilde{\subseteq}cl(int(\mathcal{G}_{\mathsf{A}}))$. That is $\mathcal{G}_{\mathsf{A}}\tilde{\subseteq}cl(int(\mathcal{G}_{\mathsf{A}}))$. Hence $\mathcal{G}_{\mathsf{A}}$ is a soft semi-open set.*

Corollary 3.6. *Every semi*-closed soft set is a semi-closed soft set.*

Theorem 3.7. *The arbitrary union of semi*-open soft sets is a semi*-open soft set.*
Proof. Let $\{(\mathcal{G}_{\mathsf{C}_{\lambda:\in\Lambda}})\}$ be a collection of semi-open soft sets of a soft topological space. Then there exist open soft sets $(\mathcal{H}_{\mathsf{B}})_\lambda$ such that $(\mathcal{H}_{\mathsf{B}})_\lambda\tilde{\subseteq}(\mathcal{G}_{\mathsf{C}})_\lambda\tilde{\subseteq}cl^*(\mathcal{H}_{\mathsf{B}})_\lambda$ for each λ. Hence $\tilde{\cup}(\mathcal{H}_{\mathsf{B}})_\lambda\tilde{\subseteq}\tilde{\cup}(\mathcal{G}_{\mathsf{C}})_\lambda\tilde{\subseteq}\tilde{\cup}(cl^*(\mathcal{H}_{\mathsf{B}})_\lambda=cl^*(\tilde{\cup}(\mathcal{H}_{\mathsf{B}})_\lambda$. Therefore $\cup(\mathcal{G}_{\mathsf{C}})_\lambda$ is a semi*-open soft set.*

Corollary 3.8. *The arbitrary intersection of semi*-closed soft sets is a semi*-closed soft set.*

Theorem 3.9. *Let $\mathcal{G}_{\mathsf{C}}$ be a semi* - open soft set and $\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}\mathcal{K}_{\mathsf{D}}\tilde{\subseteq}cl^*(\mathcal{G}_{\mathsf{C}})$, then $\mathcal{K}_{\mathsf{D}}$ is also a semi*-open soft set.*
Proof. Let $\mathcal{G}_{\mathsf{C}}$ be a semi-open soft set. Then there exists an soft open set $\mathcal{H}_{\mathsf{B}}$ such that $\mathcal{H}_{\mathsf{B}}\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}cl^*(\mathcal{H}_{\mathsf{B}})$. By our assumption $\mathcal{H}_{\mathsf{B}}\tilde{\subseteq}\mathcal{K}_{\mathsf{D}}$ and $cl^*(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}cl^*(\mathcal{H}_{\mathsf{B}})$ which implies $\mathcal{K}_{\mathsf{D}}\tilde{\subseteq}cl^*(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}cl^*(\mathcal{H}_{\mathsf{B}})$. That is $\mathcal{H}_{\mathsf{B}}\tilde{\subseteq}\mathcal{K}_{\mathsf{D}}\tilde{\subseteq}cl^*(\mathcal{H}_{\mathsf{B}})$. Therefore $\mathcal{K}_{\mathsf{D}}$ is a semi*-open soft set.*

Theorem 3.10. *If a semi*-closed soft set $\mathcal{L}_{\mathsf{A}}$ is such that $int^*\mathcal{L}_{\mathsf{A}}\tilde{\subseteq}\mathcal{K}_{\mathsf{D}}\tilde{\subseteq}\mathcal{L}_{\mathsf{A}}$, then $\mathcal{K}_{\mathsf{D}}$ is also semi*-closed.*
Proof. Similar to the above theorem.

Definition 3.11. *Let $\mathcal{G}_{\mathsf{C}}$ be a soft set in a soft topological space.*

(i) The soft semi-closure of $\mathcal{G}_{\mathsf{C}}$ is*
$ss^*cl(\mathcal{G}_{\mathsf{C}})=\tilde{\cap}\{\mathcal{S}_{\mathsf{F}}/\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}\mathcal{S}_{\mathsf{F}}$ and $\mathcal{S}_{\mathsf{F}}\in S^*C\mathcal{S}\mathcal{S}(\mathcal{U})_{\mathcal{E}}\}$ is a soft set.

(ii) The soft semi-interior of $\mathcal{G}_{\mathsf{C}}$ is*
$ss^*int(\mathcal{G}_{\mathsf{C}})=\tilde{\cup}\{\mathcal{S}_{\mathsf{F}}/\mathcal{S}_{\mathsf{F}}\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}$ and $\mathcal{S}_{\mathsf{F}}\in S^*O\mathcal{S}\mathcal{S}(\mathcal{U})_{\mathcal{E}}\}$ is a soft set.

In short, $ss^*cl(\mathcal{G}_{\mathsf{C}})$ is the smallest semi-closed soft set containing $\mathcal{G}_{\mathsf{C}}$ and $ss^*int(\mathcal{G}_{\mathsf{C}})$ is the largest semi*-open soft set contained in $\mathcal{G}_{\mathsf{C}}$.*

Theorem 3.12. *Let $\mathcal{G}_{\mathsf{C}}$ be a soft set in a soft topological space $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$. Then the soft point $\ell_{\mathcal{F}}\in ss^*cl(\mathcal{G}_{\mathsf{C}})$ if and only if every soft semi*-open set containing $\ell_{\mathcal{F}}$ intersects $\mathcal{G}_{\mathsf{C}}$.*
Proof. We transform each implication to its contrapositve by $\ell_{\mathcal{F}}\notin ss^*cl(\mathcal{G}_{\mathsf{C}})$ if and only if there exists a soft semi-open set $\mathcal{H}_{\mathsf{B}}$ containing $\ell_{\mathcal{F}}$ that does not intersect $\mathcal{G}_{\mathsf{C}}$*
Suppose assume that $\ell_{\mathcal{F}}\tilde{\notin}ss^*cl(\mathcal{G}_{\mathsf{C}})$. Then $\ell_{\mathcal{F}}\tilde{\in}(ss^*cl(\mathcal{G}_{\mathsf{C}}))^c$. Then $(ss^*cl(\mathcal{G}_{\mathsf{C}}))^c$ is a soft semi-open set containing $\ell_{\mathcal{F}}$ that does not intersect $\mathcal{G}_{\mathsf{C}}$. Conversely if there exists a soft semi*-open set $\mathcal{H}_{\mathsf{B}}$ containing $\ell_{\mathcal{F}}$ which does not intersect $\mathcal{G}_{\mathsf{C}}$. Then $(\mathcal{H}_{\mathsf{B}})^c$ is a soft semi*-open set containing $\mathcal{G}_{\mathsf{C}}$. By the definition of soft semi*-closure, $ss^*cl(\mathcal{G}_{\mathsf{C}})$ is contained in $(\mathcal{H}_{\mathsf{B}})^c$. Hence $\ell_{\mathcal{F}}$ cannot be in $ss^*cl(\mathcal{G}_{\mathsf{C}})$.*

Theorem 3.13. *Let $\mathcal{G}_{\mathsf{C}}$ and $\mathcal{K}_{\mathsf{D}}$ be two soft sets in a soft topological space. Then*

(i) $\mathcal{G}_{\mathsf{C}}\in S^*C\mathcal{S}\mathcal{S}(\mathcal{U})_{\mathcal{E}}$ *if and only if* $\mathcal{G}_{\mathsf{C}}=ss^*cl(\mathcal{G}_{\mathsf{C}})$.

(ii) $\mathcal{G}_{\mathsf{C}}\in S^*O\mathcal{S}\mathcal{S}(\mathcal{U})_{\mathcal{E}}$ *if and only if* $\mathcal{G}_{\mathsf{C}}=ss^*int(\mathcal{G}_{\mathsf{C}})$.

(iii) $(ss^*cl(\mathcal{G}_{\mathsf{C}}))^c=ss^*int(\mathcal{G}_{\mathsf{C}}^c)$.

(iv) $(ss^*int(\mathcal{G}_{\mathsf{C}}))^c=ss^*cl(\mathcal{G}_{\mathsf{C}}^c)$.

(v) $\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}\mathcal{K}_{\mathsf{D}}$ *implies* $ss^*int(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}ss^*int(\mathcal{K}_{\mathsf{D}})$.

(vi) $\mathcal{G}_{\mathsf{C}}\tilde{\subseteq}\mathcal{K}_{\mathsf{D}}$ *implies* $ss^*cl(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}ss^*cl(\mathcal{K}_{\mathsf{D}})$.

(vii) $ss^*cl(\phi_{\mathcal{E}})=\phi_{\mathcal{E}},ss^*cl(\mathcal{U}_{\mathcal{E}})=\mathcal{U}_{\mathcal{E}}$.

(viii) $ss^*int(\phi_{\mathcal{E}})=\phi_{\mathcal{E}},ss^*int(\mathcal{U}_{\mathcal{E}})=\mathcal{U}_{\mathcal{E}}$.

(ix) $ss^*int(\mathcal{G}_{\mathsf{C}}\tilde{\cap}\mathcal{K}_{\mathsf{D}})=ss^*int(\mathcal{G}_{\mathsf{C}})\tilde{\cap}ss^*int(\mathcal{K}_{\mathsf{D}})$.

(x) $ss^*cl(\mathcal{G}_{\mathsf{C}}\tilde{\cap}\mathcal{K}_{\mathsf{D}})\tilde{\subseteq}ss^*cl(\mathcal{G}_{\mathsf{C}})\tilde{\cap}ss^*cl(\mathcal{K}_{\mathsf{D}})$.

(xi) $ss^*int(\mathcal{G}_{\mathsf{C}}\tilde{\cup}\mathcal{K}_{\mathsf{D}})\tilde{\supseteq}ss^*int(\mathcal{G}_{\mathsf{C}})\tilde{\cup}ss^*int(\mathcal{K}_{\mathsf{D}})$.

(xii) $ss^*cl(ss^*cl(\mathcal{G}_C)) = ss^*cl(\mathcal{G}_C)$.

(xiii) $ss^*int(ss^*int(\mathcal{G}_C)) = ss^*int(\mathcal{G}_C)$.

Proof.

(i) Let $\mathcal{G}_C$ be a semi*-closed soft set. Then it is a smallest semi*-closed soft set containing itself. Then by the definition of soft semi*-closure we have $\mathcal{G}_C = ss^*cl(\mathcal{G}_C)$.
Conversely let $\mathcal{G}_C = ss^*cl(\mathcal{G}_C)$. since $ss^*cl(\mathcal{G}_C)$ is the intersection of all soft semi*-closed sets and by using Corollary 3.8, $ss^*cl(\mathcal{G}_C) \in S^*CSS(\mathcal{U})_\mathcal{E}$. Hence $\mathcal{G}_C \in S^*CSS(\mathcal{U})_\mathcal{E}$.

(ii) Let $\mathcal{G}_C$ be a semi*-open soft set. Then it is a largest semi*-open soft set contained in itself. Then by the definition of soft semi*-interior $\mathcal{G}_C = ss^*int(\mathcal{G}_C)$ Conversely let $\mathcal{G}_C = ss^*int(\mathcal{G}_C)$ As $ss^*int(\mathcal{G}_C)$ is the union of all soft semi*-open sets and by using Theorem 3.7, $ss^*int(\mathcal{G}_C) \in S^*OSS(\mathcal{U})_\mathcal{E}$. This implies $\mathcal{G}_C \in S^*OSS(\mathcal{U})_\mathcal{E}$.

(iii) $ss^*int(\mathcal{G}_C) = \tilde{\bigcup}\{(\mathcal{H}_D)^c : \mathcal{H}_D$ is a semi*-closed soft set and $(\mathcal{G}_C)^c \tilde{\subseteq} \mathcal{H}_D\}$ That is $ss^*int(\mathcal{G}_C) = [\tilde{\bigcap}\{\mathcal{H}_D : \mathcal{H}_D$ is a semi*-closed soft set and $(\mathcal{G}_C)^c \tilde{\subseteq} \mathcal{H}_D\}]^c$. This implies $ss^*int(\mathcal{G}_C) = [ss^*cl(\mathcal{G}_C^c)]^c$. Hence $(ss^*int(\mathcal{G}_C))^c = ss^*cl(\mathcal{G}_C^c)$

(iv) Similar to (iii).

(v) $ss^*int(\mathcal{G}_C) \tilde{\subseteq} \mathcal{G}_C \tilde{\subseteq} \mathcal{K}_D$ implies that $ss^*int(\mathcal{G}_C) \tilde{\subseteq} \mathcal{K}_D$. As $ss^*int(\mathcal{K}_D)$ is the largest semi*-open soft set contained in $\mathcal{K}_D$, $ss^*int(\mathcal{G}_C) \tilde{\subseteq} ss^*int(\mathcal{K}_D)$.

(vi) $\mathcal{K}_D \tilde{\subseteq} ss^*cl(\mathcal{K}_D)$. This implies $\mathcal{G}_C \tilde{\subseteq} \mathcal{K}_D \tilde{\subseteq} ss^*cl(\mathcal{K}_D)$. Hence $\mathcal{G}_C \tilde{\subseteq} ss^*cl(\mathcal{K}_D)$. As $ss^*cl(\mathcal{G}_C)$ is the smallest semi*-closed soft set contains $\mathcal{G}_C$, $ss^*cl(\mathcal{G}_C) \tilde{\subseteq} ss^*cl(\mathcal{K}_D)$.

(vii) Since $\phi_\mathcal{E}$ and $\mathcal{U}_\mathcal{E}$ are semi*-closed soft set by (i), $ss^*cl(\phi_\mathcal{E}) = \phi_\mathcal{E}$ and $ss^*cl(\mathcal{U}_\mathcal{E}) = \mathcal{U}_\mathcal{E}$.

(viii) Similar to (vii).

(ix) $\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D \tilde{\subseteq} \mathcal{G}_C$ and $\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D \tilde{\subseteq} \mathcal{K}_D$.

Hence by (v)

$ss^*int(\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D) \tilde{\subseteq} ss^*int(\mathcal{G}_C)$ and $ss^*int(\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D) \tilde{\subseteq} ss^*int(\mathcal{K}_D)$. This implies

$ss^*int(\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D) \tilde{\subseteq} ss^*int(\mathcal{G}_C) \tilde{\bigcap} ss^*int(\mathcal{K}_D)$. Let $\ell_\mathscr{F} \tilde{\notin} ss^*int(\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D)$. Then $\ell_\mathscr{F} \tilde{\notin} \tilde{\bigcup}_{\lambda \in \Lambda} (\mathcal{H}_B)_\lambda$ where $(\mathcal{H}_B)_\lambda \in S^*OSS(\mathcal{U})_\mathcal{E}$ such that $(\mathcal{H}_B)_\lambda \tilde{\subseteq} \mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D$ for all $\lambda \in \Lambda$. This implies $\ell_\mathscr{F} \tilde{\notin} \tilde{\bigcup}_{\lambda \in \Lambda} (\mathcal{H}_B)_\lambda$ where $(\mathcal{H}_B)_\lambda \in S^*OSS(\mathcal{U})_\mathcal{E}$ such that $(\mathcal{H}_B)_\lambda \tilde{\subseteq} \mathcal{G}_C$ and $(\mathcal{H}_B)_\lambda \tilde{\subseteq} \mathcal{K}_D$ for all $\lambda \in \Lambda$ Hence $\ell_\mathscr{F} \tilde{\notin} \tilde{\bigcup}_{\lambda \in \Lambda} (\mathcal{H}_B)_\lambda$ where $(\mathcal{H}_B)_\lambda \in S^*OSS(\mathcal{U})_\mathcal{E}$ such that $(\mathcal{H}_B)_\lambda \tilde{\subseteq} \mathcal{G}_C$ and $\ell_\mathscr{F} \tilde{\notin} \tilde{\bigcup}_{\lambda \in \Lambda} (\mathcal{H}_B)_\lambda$ where $(\mathcal{H}_B)_\lambda \in S^*OSS(\mathcal{U})_\mathcal{E}$ such that $(\mathcal{H}_B)_\lambda \tilde{\subseteq} \mathcal{K}_D$ for all $\lambda \in \Lambda$. This implies

$\ell_\mathscr{F} \tilde{\notin} ss^*int(\mathcal{G}_C)$ and $\ell_\mathscr{F} \notin ss^*int(\mathcal{K}_D)$. Then $\ell_\mathscr{F} \tilde{\notin} ss^*int(\mathcal{G}_C) \tilde{\bigcap} ss^*int(\mathcal{K}_D)$.

Hence $ss^*int(\mathcal{G}_C) \tilde{\bigcap} ss^*int(\mathcal{K}_D) \tilde{\subseteq} ss^*int(\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D)$. Therefore $ss^*int(\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D) = ss^*int(\mathcal{G}_C) \tilde{\bigcap} ss^*int(\mathcal{K}_D)$.

(x) $\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D \tilde{\subseteq} \mathcal{G}_C$ and $\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D \tilde{\subseteq} \mathcal{K}_D$.

Hence by (vi), $ss^*cl(\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D) \tilde{\subseteq} ss^*cl(\mathcal{G}_C)$ and $ss^*cl(\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D) \tilde{\subseteq} ss^*cl(\mathcal{K}_D)$.

This implies

$ss^*cl(\mathcal{G}_C \tilde{\bigcap} \mathcal{K}_D) \tilde{\subseteq} ss^*cl(\mathcal{G}_C) \tilde{\bigcap} ss^*cl(\mathcal{K}_D)$.

(xi) $\mathcal{G}_C \tilde{\subseteq} \mathcal{G}_C \tilde{\bigcup} \mathcal{K}_D$ and $\mathcal{K}_D \tilde{\subseteq} \mathcal{G}_C \tilde{\bigcup} \mathcal{K}_D$.Then by (v),$ss^*int(\mathcal{G}_C) \tilde{\subseteq} ss^*int(\mathcal{G}_C \tilde{\bigcup} \mathcal{K}_D)$ and $ss^*int(\mathcal{K}_D) \tilde{\subseteq} ss^*int(\mathcal{G}_C \tilde{\bigcup} \mathcal{K}_D)$.

Therefore

$ss^*int(\mathcal{G}_C \tilde{\bigcup} \mathcal{K}_D) \tilde{\supseteq} ss^*int(\mathcal{G}_C) \tilde{\bigcup} ss^*int(\mathcal{K}_D)$.

(xii) Since $ss^*cl(\mathcal{G}_C) \in S^*CSS(\mathcal{U})_\mathcal{E}$, by (i) $ss^*cl(ss^*cl(\mathcal{G}_C)) = ss^*cl(\mathcal{G}_C)$.

(xiii) Since $ss^*int(\mathcal{G}_C) \in S^*OSS(\mathcal{U})_\mathcal{E}$, by (ii) $ss^*int(ss^*int(\mathcal{G}_C)) = ss^*int(\mathcal{G}_C)$.

4. Functions using soft semi*-open sets

On this spot, we elucidate generalizations of soft functions in soft topological spaces and investigate their properties.

Definition 4.1. *A soft function $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ is said to be*

(i) *soft semi*-continuous if for each soft open set $\mathcal{G}_{\mathsf{C}}$ of $\mathcal{V}_{\mathcal{E}'}$, the inverse image $f^{-1}(\mathcal{G}_{\mathsf{C}})$ is a semi*-open soft set of $\mathcal{U}_{\mathcal{E}}$.*

(ii) *soft semi*-open function if for each open soft set $\mathcal{L}_{\mathsf{A}}$ of $\mathcal{U}_{\mathcal{E}}$, the image is a semi*-open soft set of $\mathcal{V}'_{\mathcal{E}}$.*

(iii) *soft semi*-closed function if for each closed soft set $\mathcal{K}_{\mathsf{D}}$ of $\mathcal{U}_{\mathcal{E}}$, the image $f(\mathcal{K}_{\mathsf{D}})$ is a semi*-closed soft set of $\mathcal{V}'_{\mathcal{E}}$.*

(iv) *soft semi*-irresolute if for each soft open set $\mathcal{G}_{\mathsf{C}}$ of $\mathcal{V}_{\mathcal{E}'}$, the inverse image $f^{-1}(\mathcal{G}_{\mathsf{C}})$ is a semi*-open soft set of $\mathcal{U}_{\mathcal{E}}$.*

Definition 4.2. *A soft function $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ is soft semi* -continuous if for each closed soft set $K_{D'}$ of $\mathcal{V}_{\mathcal{E}'}$,the inverse image $f^{-1}(K_{D'})$ is a semi*-closed soft set of $\mathcal{U}_{\mathcal{E}}$.*

Theorem 4.3. *A soft function $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ is soft semi* -continuous if and only if $f(ss^*cl(\mathcal{L}_{\mathsf{A}}))\tilde{\subseteq}cl(f(\mathcal{L}_{\mathsf{A}}))$, for every soft set $\mathcal{L}_{\mathsf{A}}$ of $\mathcal{U}_{\mathcal{E}}$.*

Proof. *Let $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ be a soft semi* -continuous function.Now $cl(f(\mathcal{L}_{\mathsf{A}}))$ is a soft closed set of $\mathcal{V}_{\mathcal{E}'}$. By using soft semi*-continuity of f, $f^{-1}(cl(f(\mathcal{L}_{\mathsf{A}})))$ is a semi*-closed soft set of $\mathcal{U}_{\mathcal{E}}$. Also $f(\mathcal{L}_{\mathsf{A}})\tilde{\subseteq}cl(f(\mathcal{L}_{\mathsf{A}}))$. This implies $\mathcal{L}_{\mathsf{A}}\tilde{\subseteq}f^{-1}(cl(f(\mathcal{L}_{\mathsf{A}})))$. Here $f^{-1}(cl(f(\mathcal{L}_{\mathsf{A}})))$ is a semi*-closed soft set containing $\mathcal{L}_{\mathsf{A}}$. But $ss^*cl(\mathcal{L}_{\mathsf{A}})$ is a smallest semi*-closed soft set containing $\mathcal{L}_{\mathsf{A}}$. Now $\mathcal{L}_{\mathsf{A}}\tilde{\subseteq}ss^*cl(\mathcal{L}_{\mathsf{A}})\tilde{\subseteq}f^{-1}(cl(f(\mathcal{L}_{\mathsf{A}})))$.$ss^*cl(\mathcal{L}_{\mathsf{A}})\tilde{\subseteq} f^{-1}(cl(f(\mathcal{L}_{\mathsf{A}})))$ which implies $f(ss^*cl(\mathcal{L}_{\mathsf{A}})) \tilde{\subseteq}cl(f(\mathcal{L}_{\mathsf{A}}))$.*

Conversely, assume that $f(ss^*cl(\mathcal{L}_{\mathsf{A}}))\tilde{\subseteq}cl(f(\mathcal{L}_{\mathsf{A}}))$.Let $\mathcal{G}_{\mathsf{C}}$ be any soft closed set of $\mathcal{V}_{\mathcal{E}'}$. Therefore $f^{-1}(\mathcal{G}_{\mathsf{C}}) \in \mathcal{U}_{\mathcal{E}}$ which implies $f(ss^*cl(f^{-1}(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq} cl(f(f^{-1}(\mathcal{G}_{\mathsf{C}})) = cl(\mathcal{G}_{\mathsf{C}}) = \mathcal{G}_{\mathsf{C}} \Rightarrow ss^*cl(f^{-1}(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}f^{-1}(\mathcal{G}_{\mathsf{C}})$. Always $f^{-1}(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}ss^*cl(f^{-1}(\mathcal{G}_{\mathsf{C}})$ and $f^{-1}(\mathcal{G}_{\mathsf{C}}) = ss^*cl(f^{-1}(\mathcal{G}_{\mathsf{C}}))$. Therefore,$f^{-1}(\mathcal{G}_{\mathsf{C}})$ is a semi-closed soft set. By using definition 4.2, f is a semi*-continuous soft function.*

Theorem 4.4. *A soft function $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ is semi*-continuous if and only if $f^{-1}(int\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}ss^*int(f^{-1}(\mathcal{G}_{\mathsf{C}}))$ for every soft set $\mathcal{G}_{\mathsf{C}}$ of $\mathcal{V}'_{\mathcal{E}}$.*

Proof. *Suppose $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ is a soft semi*-continuous function. Now $int(\mathcal{G}_{\mathsf{C}})$ is a soft open set of $\mathcal{V}_{\mathcal{E}'}$. As f is a soft semi*-continuous function, $f^{-1}(int(\mathcal{G}_{\mathsf{C}}))$ is a soft semi*-open set of $\mathcal{U}_{\mathcal{E}}$. Also $int(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}$ implies $f^{-1}(int(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}f^{-1}(\mathcal{G}_{\mathsf{C}})$. As $ss^*int(f^{-1}(\mathcal{G}_{\mathsf{C}}))$ is a largest soft semi*-open set contained in $f^{-1}(\mathcal{G}_{\mathsf{C}})$, $f^{-1}(int(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}ss^*int(f^{-1}(\mathcal{G}_{\mathsf{C}}))$.*

Conversely assume that $f^{-1}(int(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}ss^*int(f^{-1}(\mathcal{G}_{\mathsf{C}}))$. Let $\mathcal{G}_{\mathsf{C}}$ be an soft open set of $\mathcal{V}_{\mathcal{E}'}$. Then $f^{-1}(\mathcal{G}_{\mathsf{C}}) = f^{-1}(int(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}ss^*int(f^{-1}(\mathcal{G}_{\mathsf{C}}))$. This implies $f^{-1}(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}ss^*int(f^{-1}(\mathcal{G}_{\mathsf{C}}))$. Always $ss^*int(f^{-1}(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}f^{-1}(\mathcal{G}_{\mathsf{C}})$. Hence $f^{-1}(\mathcal{G}_{\mathsf{C}}) = ss^*int(f^{-1}(\mathcal{G}_{\mathsf{C}}))$. That is $f^{-1}(\mathcal{G}_{\mathsf{C}})$ is a soft semi-open set. Hence f is a soft semi*-continuous function.*

Theorem 4.5. *A soft function $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ is soft semi*-open if and only if $f(int\mathcal{L}_{\mathsf{A}})\tilde{\subseteq}ss^*int(f(\mathcal{L}_{\mathsf{A}}))$ for every soft set $\mathcal{L}_{\mathsf{A}}$ of $\mathcal{U}_{\mathcal{E}}$.*

Proof.Suppose $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ is soft semi-open. Now $int(\mathcal{L}_{\mathsf{A}})$ is a soft open set in $\mathcal{U}_{\mathcal{E}}$ as f is soft semi*-open $f(int(\mathcal{L}_{\mathsf{A}}))$ is a soft semi*-open set. Also $int(\mathcal{L}_{\mathsf{A}})\tilde{\subseteq}\mathcal{L}_{\mathsf{A}}$. Hence $f(int(\mathcal{L}_{\mathsf{A}}))\tilde{\subseteq}f(\mathcal{L}_{\mathsf{A}})$. As $ss^*int(f(\mathcal{L}_{\mathsf{A}}))$ is the largest semi*-open soft set contained in $f(\mathcal{L}_{\mathsf{A}})$, $f(int(\mathcal{L}_{\mathsf{A}}))\tilde{\subseteq}ss^*int(f(\mathcal{L}_{\mathsf{A}}))$.*

Conversely assume that $f(int(\mathcal{L}_{\mathsf{A}}))\tilde{\subseteq}ss^*int(f(\mathcal{L}_{\mathsf{A}}))$. for every soft set $\mathcal{L}_{\mathsf{A}}$ of $\mathcal{U}_{\mathcal{E}}$. Let $\mathcal{G}_{\mathsf{C}}$ be a soft open set in $\mathcal{U}_{\mathcal{E}}$. Hence $f(\mathcal{G}_{\mathsf{C}}) = f(int(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}ss^*int(f(\mathcal{G}_{\mathsf{C}}))$. Always $ss^*int(f(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}f(\mathcal{G}_{\mathsf{C}})$. Therefore $f(\mathcal{G}_{\mathsf{C}})$ is a semi-open soft set in $\mathcal{V}_{\mathcal{E}'}$. Hence f is a semi*-open soft function.*

Theorem 4.6. *A soft function $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ is soft semi*-closed if and only if $ss^*cl(f(\mathcal{L}_{\mathsf{A}}))\tilde{\subseteq}f(cl(\mathcal{L}_{\mathsf{A}}))$ for every soft set $\mathcal{L}_{\mathsf{A}}$ of $\mathcal{U}_{\mathcal{E}}$.*

Proof. *Let $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ is a soft semi*-closed function. Since $cl(\mathcal{L}_{\mathsf{A}})$ is a soft closed set in $\mathcal{U}_{\mathcal{E}}$,$f(cl(\mathcal{L}_{\mathsf{A}}))$ is a soft semi*-closed set in $\mathcal{V}_{\mathcal{E}'}$. Also note that $\mathcal{L}_{\mathsf{A}}\tilde{\subseteq}cl(\mathcal{L}_{\mathsf{A}})$. This implies that $f(\mathcal{L}_{\mathsf{A}})\tilde{\subseteq}f(cl(\mathcal{L}_{\mathsf{A}}))$ .Since $ss^*cl(f(\mathcal{L}_{\mathsf{A}}))$ is the smallest semi*-closed soft set contains $f(\mathcal{L}_{\mathsf{A}})$, $ss^*cl(f(\mathcal{L}_{\mathsf{A}}))\tilde{\subseteq}f(cl(\mathcal{L}_{\mathsf{A}}))$. Conversely let $ss^*cl(f(\mathcal{L}_{\mathsf{A}}))\tilde{\subseteq}f(cl(\mathcal{L}_{\mathsf{A}}))$ for every soft set $\mathcal{L}_{\mathsf{A}}$ of $\mathcal{U}_{\mathcal{E}}$. Let $\mathcal{G}_{\mathsf{C}}$ be a soft closed set in $\mathcal{U}_{\mathcal{E}}$ .Then $\mathcal{G}_{\mathsf{C}} = cl(\mathcal{G}_{\mathsf{C}})$. This implies $f(\mathcal{G}_{\mathsf{C}}) = f(cl(\mathcal{G}_{\mathsf{C}}))$. Hence by our assumption $ss^*cl(f(\mathcal{G}_{\mathsf{C}}))\tilde{\subseteq}f(cl(\mathcal{G}_{\mathsf{C}})) = f(\mathcal{G}_{\mathsf{C}})$ Always $f(\mathcal{G}_{\mathsf{C}})\tilde{\subseteq}ss^*cl(f(\mathcal{G}_{\mathsf{C}}))$. Hence $f(\mathcal{G}_{\mathsf{C}}) = ss^*cl(f(\mathcal{G}_{\mathsf{C}}))$. This implies $f(\mathcal{G}_{\mathsf{C}})$ is a soft semi*-closed set. Hence f is a soft semi*-closed function.*

5. Soft semi*-compactness

In this tract, we define semi*-compactness in soft topological spaces and investigate some of its characteristics.

Definition 5.1. *A family ψ of soft sets is a cover of a soft set $\mathcal{F}_{\mathcal{A}}$ if $\mathcal{F}_{\mathcal{A}}\tilde{\subseteq}\tilde{\cup}\{(\mathcal{F}_i)_{\mathcal{A}} : (\mathcal{F}_i)_{\mathcal{A}} \in \psi, i \in I\}$. A subcover of ψ is a subfamily of ψ which is also a cover.*

Definition 5.2. *A soft topological space $(\mathcal{U}_{\mathcal{E}}, \tau)$ is said to be semi*-compact if each semi*-open soft cover of $\mathcal{U}_{\mathcal{E}}$ has a finite subcover.*

Theorem 5.3. *A soft topological space $(\mathcal{U}_{\mathcal{E}}, \tau)$ is semi*-compact if and only if each family of semi*-closed soft sets in $\mathcal{U}_{\mathcal{E}}$ with the finite intersection property has a non empty intersection.*

Proof. *Assume that $(\mathcal{U}_{\mathcal{E}}, \tilde{\tau})$ is a semi*-compact soft topological space. Let $\{(\mathcal{L}_{\mathsf{A}})_{\lambda} : \lambda \in \Lambda\}$ be a collection of semi*-closed soft sets with the finite intersection property. If possible, assume that $\tilde{\bigcap}_{\lambda\in\Lambda} (\mathcal{L}_{\mathsf{A}})_{\lambda} = \phi_{\mathcal{E}}$. This implies $\tilde{\bigcup}_{\lambda\in\Lambda} (\mathcal{L}_{\mathsf{A}})_{\lambda^c} = \mathcal{U}_{\mathcal{E}}$ So the collection $\{(\mathcal{L}_{\mathsf{A}})_{\lambda^c:\lambda\in\Lambda}\}$*

forms a soft semi-open cover of $\mathcal{U}_{\mathcal{E}}$, which is soft semi*-compact. So, there exists a finite sub collection Δ of Λ which also covers $\mathcal{U}_{\mathcal{E}}$. That is $\tilde{\bigcup}_{\lambda\in\Lambda}((\mathcal{L}_{A})_{\lambda^c} = \mathcal{U}_{\mathcal{E}}$. This implies $\tilde{\bigcup}_{\lambda\in\Lambda}((L_A))^c = \phi_{\mathcal{E}}$. This is a contradiction to the finite intersection property. Hence $\tilde{\bigcap}_{\lambda\in\Lambda}(L_A)_{\lambda} \neq \phi_{\mathcal{E}}$. Conversely, assume that each family of semi*-closed soft sets in $\mathcal{U}_{\mathcal{E}}$ with the finite intersection property has a non empty intersection. If possible let us assume $(\mathcal{U}_{\mathcal{E}},\tau)$ is not semi*-compact. Then there exists a soft semi*-open cover $\{(\mathcal{G}_{C})_{\lambda:\in\Lambda}\}$ of $\mathcal{U}_{\mathcal{E}}$ such that for every finite sub collection Δ of Λ we have $\tilde{\bigcup}_{\lambda\in\Delta}(\mathcal{G}_{C})_{\lambda} \neq \mathcal{U}_{\mathcal{E}}$. Implies $\tilde{\bigcap}_{\lambda\in\Delta}((\mathcal{G}_{C})_{\lambda^c} \neq \phi_{\mathcal{E}}$. Hence $\{((\mathcal{G}_{C})_{\lambda^c:\in\Lambda}\}$ has a finite intersection property. So, by hypothesis $\tilde{\bigcap}_{\lambda\in\Lambda}((\mathcal{G}_{C})_{\lambda^c} \neq \phi_{\mathcal{E}}$. Which implies $\tilde{\bigcup}_{\in\Lambda}(\mathcal{G}_{C})_{\lambda} \neq \mathcal{U}_{\mathcal{E}}$. This is a contradiction to our assumption. Therefore $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is a semi*-compact soft topological space.*

Theorem 5.4. *A soft topological space $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is semi*-compact if and only if for every family ψ of soft sets with finite intersection property, $\tilde{\bigcap}_{\mathcal{G}_C\in\psi} ss^*cl(\mathcal{G}_{C}) \neq \phi_{\mathcal{E}}$.*

Proof. *Let $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ be a semi*-compact soft topological space. If possible let us assume that $\tilde{\bigcap}_{\mathcal{G}_C\in\psi} ss^*cl(\mathcal{G}_{C}) = \phi_{\mathcal{E}}$ for some family ψ of soft sets with the finite intersection property. So $\tilde{\bigcup}_{\mathcal{G}_C\in\psi}(ss^*cl(\mathcal{G}_{C}))^c = \mathcal{U}_{\mathcal{E}}$. Hence $\Gamma = \{(ss^*cl(\mathcal{G}_{C}))^c : \mathcal{G}_{C}\in\psi\}$ forms an soft semi*-open cover for $\mathcal{U}_{\mathcal{E}}$. Then by semi*-compactness of $\mathcal{U}_{\mathcal{E}}$ there exists a finite subcover ω of ψ such that $\tilde{\bigcup}_{\mathcal{G}_C\in\omega}(ss^*cl(\mathcal{G}_{C}))^c = \mathcal{U}_{\mathcal{E}}$. We have $\mathcal{G}_{C}\tilde{\subseteq} ss^*cl(\mathcal{G}_{C})$. Then $\mathcal{U}_{\mathcal{E}}\tilde{\subseteq}\tilde{\bigcup}_{\mathcal{G}_C\in\omega}(\mathcal{G}_{C})^c$ and hence $\mathcal{U}_{\mathcal{E}} = \tilde{\bigcup}_{\mathcal{G}_C\in\omega}(\mathcal{G}_{C})^c$. Therefore $\tilde{\bigcap}_{\mathcal{G}_C\in\omega}\mathcal{G}_{C} = \phi_{\mathcal{E}}$. This is contradiction to the finite intersection property. Hence $\tilde{\bigcap}_{\mathcal{G}_C\in\psi} ss^*cl(\mathcal{G}_{C}) \neq \phi_{\mathcal{E}}$.*

Conversely, assume that $\tilde{\bigcap}_{\mathcal{G}_C\in\psi} ss^*cl(\mathcal{G}_{C}) \neq \phi_{\mathcal{E}}$ for every family ψ of soft sets with finite intersection property. Suppose assume that $(\mathcal{U}_{\mathcal{E}},\tau)$ is not soft semi-compact. Then there exists a family Γ of semi*-open soft sets covering $\mathcal{U}_{\mathcal{E}}$ without a finite subcover. So for every finite sub family ω of Γ we have $\tilde{\bigcup}_{\mathcal{G}_C\in\omega}\mathcal{G}_{C} \neq \mathcal{U}_{\mathcal{E}}$. This implies $\tilde{\bigcap}_{\mathcal{G}_C\in\omega}(\mathcal{G}_{C})^c \neq \phi_{\mathcal{E}}$.This implies $\{(\mathcal{G}_{C})^c : \mathcal{G}_{C}\in\Gamma\}$ is a family of soft sets with finite intersection property. Now $\tilde{\bigcup}_{\mathcal{G}_C\in\Gamma}\mathcal{G}_{C} = \mathcal{U}_{\mathcal{E}}$. This implies $\tilde{\bigcap}_{\mathcal{G}_C\in\Gamma}(\mathcal{G}_{C})^c = \phi_{\mathcal{E}}$. Since $\mathcal{G}_{C}\tilde{\subseteq} ss^*cl(\mathcal{G}_{C})$, $\tilde{\bigcap}_{\mathcal{G}_C\in\Gamma} ss^*cl(\mathcal{G}_{C})^c\tilde{\subseteq}\phi_{\mathcal{E}}$. Hence $\tilde{\bigcap}_{\mathcal{G}_C\in\Gamma} ss^*cl(\mathcal{G}_{C})^c = \phi_{\mathcal{E}}$. This is a contradiction. Therefore $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is semi*-compact soft topological space.*

Theorem 5.5. *Semi*-continuous image of a soft semi*-compact space is soft compact.*

Proof. *Let $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ be a semi*-continuous function where $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is a semi*-compact soft topological space and $(\mathcal{V}_{\mathcal{E}'},\delta)$ is another soft topological space. Let $\{(\mathcal{G}_{C})_{\lambda:\lambda\in\Lambda}\}$ be a soft open cover of $\mathcal{V}_{\mathcal{E}'}$. Since f is semi*-continuous,$\{f^{-1}(\mathcal{G}_{C})_{\lambda:\in\Lambda}\}$ forms a soft semi*-open cover for $\mathcal{U}_{\mathcal{E}}$. This implies there exists a finite subset Δ of Λ such that $\{f^{-1}(\mathcal{G}_{C})_{\lambda:\lambda\in\Delta}\}$ forms a soft semi*-open cover of $\mathcal{U}_{\mathcal{E}}$. Hence $\{(\mathcal{G}_{C})_{\lambda:\lambda\in\Delta}\}$ forms a finite soft subcover of $\mathcal{V}_{\mathcal{E}'}$.*

Theorem 5.6. *Semi*-closed subspace of a semi*-compact soft topological space is soft semi*-compact.*

Proof. *Let $\mathcal{V}_{B}$ be a semi*-closed subspace of a semi*-compact soft topological space $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ and $\{(\mathcal{G}_{C})_{\lambda:\lambda\in\Lambda}\}$ be a soft semi*-open cover for $\mathcal{V}_{B}$. As $\mathcal{V}_{B}$ is semi*-closed soft set $\mathcal{V}_{B}^c$ is a semi*-open soft set. Hence $\Gamma = \{(\mathcal{G}_{C})_{\lambda:\lambda\in\Lambda}\}\tilde{\bigcup}\mathcal{V}_{B}^c$ forms a semi*-open soft cover for $\mathcal{U}_{\mathcal{E}}$. As $\mathcal{U}_{\mathcal{E}}$ is soft semi*-compact Λ has a finite sub family Δ such that $\mathcal{U}_{\mathcal{E}} = \mathcal{V}_{B}^c\tilde{\bigcup}\{(\mathcal{G}_{C})_{\lambda:\lambda\in\Delta}\}$. Then $\mathcal{V}_{B} = \{(\mathcal{G}_{C})_{\lambda:\lambda\in\Delta}\}$.*

Theorem 5.7. *Semi*-irresolute image of a semi*-compact soft topological space is semi*-compact.*

Proof. *Let $f : \mathcal{SS}(\mathcal{U})_{\mathcal{E}} \to \mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ be a semi*-irresolute soft function where $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is a semi*-compact soft topological space and $(\mathcal{V}'_{\mathcal{E}},\delta)$ be a soft topological space. Let $\{(\mathcal{G}_{C})_{\lambda:\lambda\in\Lambda}\}$ be a soft semi*-open cover for $\mathcal{V}_{\mathcal{E}'}$. As f is a semi*-irresolute function $f^{-1}(\mathcal{G}_{C})_{\lambda}$ is a soft semi*-open set for each $\lambda\in\Lambda$. Hence $\{f^{-1}(\mathcal{G}_{C})_{\lambda:\lambda\in\Lambda}\}$ forms a semi*-open cover for $\mathcal{U}_{\mathcal{E}}$. Since $(\mathcal{U}_{\mathcal{E}},\tau)$ is a semi*-compact, there exists a finite subfamily Δ of Λ such that $\{f^{-1}(\mathcal{G}_{C})_{\lambda:\lambda\in\Delta}\}$ covers $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$. Hence $\{(\mathcal{G}_{C})_{\lambda:\lambda\in\Delta}\}$ forms a finite subcover of $f(\mathcal{U}_{\mathcal{E}})$. Hence $f(\mathcal{U}_{\mathcal{E}})$ is soft semi*-compact.*

6. Soft semi*-connectedness

Here, we come out with semi* - connectedness in soft topological spaces put into action with semi* - open soft sets and scrutinate its basic properties.

Definition 6.1. *[5] Two soft sets $\mathcal{L}_{A}$ and $\mathcal{H}_{B}$ are said to be disjoint if $\mathcal{L}_{A}(a)\tilde{\cap}\mathcal{H}_{B}(b) = \phi$ for all $a\in A, b\in B$*

Definition 6.2. *A soft semi*-separation of soft topological $(\mathcal{U}_{\mathcal{E}},\tau)$ is a pair $\mathcal{L}_{A}$, $\mathcal{H}_{B}$ of disjoint non null semi*-open sets whose union is $\mathcal{U}_{\mathcal{E}}$. If there does not exists a soft semi*-separation of $\mathcal{U}_{\mathcal{E}}$, then the soft topological space is said to be soft semi*-connected otherwise soft semi*-disconnected.*

Example 6.3. *Consider the soft topological space $(\mathcal{U}_{\mathcal{E}},\tau)$, where $U = \{h_1,h_2\}, E = \{e_1,e_2\}$, and $\tau = \{\phi_{\mathcal{E}},\mathcal{U}_{\mathcal{E}},(e_1,\{h_1\}),(e_2,\{h_1,h_2\}),$ $\{(e_1,\{h_1\}),(e_2.\{h_1,h_2\})\}$.The semi*-open soft sets are $\phi_{\mathcal{E}},\mathcal{U}_{\mathcal{E}},(e_1,\{h_1\}),(e_1,\{h_1,h_2\}),\{(e_1,\{h_1\}),(e_2,\{h_1,h_2\})\}$, $(e_2,\{h_1,h_2\}),\{(e_1,\{h_2\}),$ $(e_2,\{h_1,h_2\})\}$. Here there does not exists a Soft semi* - separation of $\mathcal{U}_{\mathcal{E}}$. Therefore, $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is Soft semi*-connected.*

Theorem 6.4. *If the soft sets $\mathcal{L}_{A}$ and $\mathcal{G}_{C}$ form a soft semi*-separation of $\mathcal{U}_{\mathcal{E}}$ and if $\mathcal{V}_{B}$ is a soft semi*-connected subspace of $\mathcal{U}_{\mathcal{E}}$ then $\mathcal{V}_{B}\tilde{\subseteq}\mathcal{L}_{A}$ or $\mathcal{V}_{B}\tilde{\subseteq}\mathcal{G}_{C}$.*

Proof. *Given $\mathcal{L}_{A}$ and $\mathcal{G}_{C}$ form a soft semi*-separation of $\mathcal{U}_{\mathcal{E}}$ Since $\mathcal{L}_{A}$ and $\mathcal{G}_{C}$ are disjoint semi*-open soft sets $\mathcal{L}_{A}\tilde{\cap}\mathcal{V}_{B}$ and $\mathcal{G}_{C}\tilde{\cap}\mathcal{V}_{B}$ are also semi*-open soft sets and their soft union gives $\mathcal{V}_{B}$.That is they would constitute a soft semi*-separation of $\mathcal{V}_{B}$. This is a contradiction. Hence one of $\mathcal{L}_{A}\tilde{\cap}\mathcal{V}_{B}$ and $\mathcal{G}_{C}\tilde{\cap}\mathcal{V}_{B}$ is empty. Therefore $\mathcal{V}_{B}$ is entirely contained in one of them.*

Theorem 6.5. *Let $\mathcal{V}_{\mathsf{B}}$ be a soft semi*-connected subspace of $\mathcal{U}_{\mathcal{E}}$ and $\mathcal{K}_{\mathsf{D}}$ be a soft set in $\mathcal{U}_{\mathcal{E}}$ such that $\mathcal{V}_{\mathsf{B}}\tilde{\subseteq}\mathcal{K}_{\mathsf{D}}\tilde{\subseteq}cl(\mathcal{V}_{\mathsf{B}})$ then $\mathcal{K}_{\mathsf{D}}$ is also soft semi*-connected.*

Proof. Let the soft set $\mathcal{K}_{\mathsf{D}}$ satisfies the hypothesis. If possible, let $\mathscr{F}_{\mathsf{A}}$ and $\mathcal{G}_{\mathsf{C}}$ form a soft semi-separation of $\mathcal{K}_{\mathsf{D}}$. Then by the theorem 5.4, $\mathcal{V}_{\mathsf{B}}\tilde{\subseteq}\mathscr{F}_{\mathsf{A}}$ or $\mathcal{V}_{\mathsf{B}}\tilde{\subseteq}\mathcal{G}_{\mathsf{C}}$. Let $\mathcal{V}_{\mathsf{B}}\tilde{\subseteq}\mathscr{F}_{\mathsf{A}}$. This implies $ss^{*}cl(\mathcal{V}_{\mathsf{B}})\tilde{\subseteq}ss^{*}cl(\mathscr{F}_{\mathsf{A}})$. Since $ss^{*}cl(\mathscr{F}_{\mathsf{A}})$ and $\mathcal{G}_{\mathsf{C}}$ are disjoint, $\mathcal{V}_{\mathsf{B}}$ cannot intersects $\mathcal{G}_{\mathsf{C}}$. This is a contradiction. Hence $\mathcal{K}_{\mathsf{D}}$ is soft semi*-connected.*

Theorem 6.6. *A soft topological space $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is soft semi*-disconnected if and only if there exists a non null proper soft subset of $\mathcal{U}_{\mathcal{E}}$ which is both soft semi*-open and soft semi*-closed.*

Let $\mathcal{U}_{\mathcal{E}}$ be soft semi-disconnected. Then there exist non null soft subsets $\mathcal{K}_{\mathsf{D}}$ and $\mathcal{H}_{\mathsf{C}}$ Such that $ss^{*}cl(\mathcal{K}_{\mathsf{D}})\tilde{\bigcap}\mathcal{H}_{\mathsf{C}}=\phi_{\mathcal{E}}$, $\mathcal{K}_{\mathsf{D}}\tilde{\bigcap}ss^{*}cl(\mathcal{H}_{\mathsf{C}})=\phi_{\mathcal{E}}$ and $\mathcal{K}_{\mathsf{D}}\tilde{\bigcup}\mathcal{H}_{\mathsf{C}}=\mathcal{U}_{\mathcal{E}}$ .Now $\mathcal{K}_{\mathsf{D}}\tilde{\subseteq}ss^{*}cl(\mathcal{K}_{\mathsf{D}})$ and $ss^{*}cl(\mathcal{K}_{\mathsf{D}})\tilde{\bigcap}\mathcal{H}_{\mathsf{C}}=\phi_{\mathcal{E}}$ .This implies $\mathcal{K}_{\mathsf{D}}\tilde{\bigcap}\mathcal{H}_{\mathsf{C}}=\phi_{\mathcal{E}}$, that is $\mathcal{H}_{\mathsf{C}}\tilde{\subseteq}(\mathcal{K}_{\mathsf{D}})^{c}$. Then $\mathcal{K}_{\mathsf{D}}\tilde{\bigcup}ss^{*}cl(\mathcal{H}_{\mathsf{C}})=\mathcal{U}_{\mathcal{E}}$ and $\mathcal{K}_{\mathsf{D}}\tilde{\bigcap}ss^{*}cl(\mathcal{H}_{\mathsf{C}})=\phi_{\mathcal{E}}$ this implies $\mathcal{K}_{\mathsf{D}}=(ss^{*}cl(\mathcal{H}_{\mathsf{C}}))^{c}$ similarly $\mathcal{H}_{\mathsf{C}}=(ss^{*}cl(\mathcal{K}_{\mathsf{D}}))^{c}$. Hence $\mathcal{K}_{\mathsf{D}}$ and $\mathcal{H}_{\mathsf{C}}$ are semi*-open soft sets being the complements of semi*-closed soft sets. Also $\mathcal{H}_{\mathsf{C}}\tilde{\subseteq}(\mathcal{K}_{\mathsf{D}})^{c}$. This implies $\mathcal{K}_{\mathsf{D}}$ and $\mathcal{H}_{\mathsf{C}}$ are also semi*-closed soft sets.*

Conversely, let $\mathcal{K}_{\mathsf{D}}$ be a non null proper soft subset of $\mathcal{U}_{\mathcal{E}}$ which is both semi-open and semi*-closed. Now let $\mathcal{H}_{\mathsf{C}}\tilde{\subseteq}(\mathcal{K}_{\mathsf{D}})^{c}$ is non null proper subset of $\mathcal{U}_{\mathcal{E}}$ which is also both semi*-open and semi*-closed. This implies $\mathcal{U}_{\mathcal{E}}$ can be expressed as the soft union of two semi*-separated soft sets $\mathcal{K}_{\mathsf{D}}$ and $\mathcal{H}_{\mathsf{C}}$. Hence $\mathcal{U}_{\mathcal{E}}$ is semi*-disconnected.*

Theorem 6.7. *Semi*-irresolute image of a soft semi*-connected soft topological space is soft semi*-connected.*

Let Let $f:\mathcal{SS}(\mathcal{U})_{\mathcal{E}}\to\mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ be a semi-irresolute soft function where $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is a semi*-connected soft topological space. Our aim is to prove is soft semi*-connected. Suppose assume that $f(\mathcal{U}_{\mathcal{E}})$ soft semi*-disconnected. Let $\mathcal{K}_{\mathsf{D}}$ and $\mathcal{H}_{\mathsf{C}}$ be non null disjoint semi*-open soft sets whose union is $f(\mathcal{U}_{\mathcal{E}})$. Since f is semi*-irresolute soft function $f^{-1}(\mathcal{K}_{\mathsf{D}})$ and $f^{-1}(\mathcal{H}_{\mathsf{C}})$ are semi*-open soft sets. Also they form a soft semi*-separation for $\mathcal{U}_{\mathcal{E}}$. This is a contradiction to the fact that $\mathcal{U}_{\mathcal{E}}$ is soft semi*-connected. Hence $f(\mathcal{U}_{\mathcal{E}})$ is soft semi*-connected.*

Theorem 6.8. *Semi*-continuous image of a soft semi*-connected soft topological space is soft connected.*

Let $f:\mathcal{SS}(\mathcal{U})_{\mathcal{E}}\to\mathcal{SS}(\mathcal{V})_{\mathcal{E}'}$ be a semi-continuous function where $(\mathcal{U}_{\mathcal{E}},\tilde{\tau})$ is a semi*-connected soft topological space and $(\mathcal{V}_{\mathcal{E}'},\delta)$ is a soft topological space. Our aim is to prove $f(\mathcal{U}_{\mathcal{E}})$ is soft connected. Suppose assume that $f(\mathcal{U}_{\mathcal{E}})$ is soft disconnected. Let $f(\mathcal{U}_{\mathcal{E}})=\mathcal{K}_{\mathsf{D}}\tilde{\bigcup}\mathcal{H}_{\mathsf{C}}$ be a soft separation that is $\mathcal{K}_{\mathsf{D}}$ and $\mathcal{H}_{\mathsf{C}}$ are disjoint soft open sets whose union is $f(\mathcal{U}_{\mathcal{E}})$. This implies $f^{-1}(\mathcal{K}_{\mathsf{D}})$ and $f^{-1}(\mathcal{H}_{\mathsf{C}})$ form a soft semi*-separation of $\mathcal{U}_{\mathcal{E}}$. This is a contradiction. Hence $f(\mathcal{U}_{\mathcal{E}})$ is soft connected.*

7. Conclusion

Topology and Soft sets are playing vital role in Pure and Applied Mathematics and gives more applications in real life using various Mathematical tools. Recently scientists have studied soft set theory, which is originated by a Mathematician Molodtsov and easily applied to the theory of uncertainties.In the present work, we have continued the study of soft sets and soft topological spaces.We investigate the behavior of Soft Semi*-open and Soft Semi*-closed sets, which is a step forward to further investigate the strong base of soft topological spaces. Further we planned to introduce and investigate soft semi*-separation Axioms using soft semi*-open and soft semi*-closed sets. We assure that the belongings in this paper will help researchers move into the new direction and promote the future work in soft topological spaces.

Acknowledgement

The authors thank **Dr. T.M. Al-Shami**, Professor, Department of Mathematics, Sana'a University, Yemen for his keen interest about this article and valuable suggestions .

References

[1] A. Kharal, B. Ahamad, *Mappings on soft classes,* New Math.Nat.Comput.**7**(3) (2011), 471-481.
[2] A. Robert, S.Pious Missier, *A new class of nearly open Sets*, Int. J. Math. Arch., **3**(7)(2012), 2575-2582.
[3] A. Robert, S.Pious Missier, *On semi*-closed sets* , Asian J. Eng. Maths., **1**(4) (2012), 173-176.
[4] Ç. Gündüz Aras, S. Bayramov, *On the Tietze extension theorem in soft topological spaces,* Proc. Inst. Math. Mech., **43** (1) (2017), 105-115.
[5] D. Molodtsov, *Soft set theory-first results,* Comp. Math. Appl., **37** (1999), 19-31.
[6] E. Peyghan, B. Samadi, A. Tayebi, *On soft connectedness,* arXiv:1202.1668V1 [math.GN], 8 Feb 2012.
[7] İ. Demir, *An approach to the concept of soft vietoris topology,* Int. J. Anal. Appl., **12** (2016), 198-206.
[8] İ. Demir, O.B. Özbakir, İ. Yıldız, *A contribution to the study of soft proximity spaces,* Filomat, **31** (2017) 2023 - 2034.
[9] İ. Demir, O. B. Özbakir, *An extension of Lowen's uniformity to the fuzzy soft sets,* Konuralp J. Math., **6** (2018) 321 - 331.
[10] I. Zorlutuna, M. Akdağ, W.K. Min, S. Atmaca, *Remarks on soft topological spaces,* Ann. Fuzzy Math. Inform., **3** (2) (2012), 171-185.
[11] K. Kannan *Soft generalized closed sets in soft topological spaces,* J. Theoret. Appl. Tech., **37** (2012), 17-21.
[12] M. Shabir, M. Naz, *On Soft topological spaces,* Comp. Math. Appl., **61**(2011), 1786-1799.
[13] M.E. El-Shafei, M. Abo-Elhamayel, T.M. Al-Shami, *Two notes on "On soft Hausdorff spaces",* Ann. of Fuzzy Math. Inform., **16** (3) (2018), 333-336.
[14] N. Levine, *Semi-open sets and Semi- continuity in topological spaces,* Amer. Math. Monthly, **70**(1) (1963), 36-41.
[15] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, **19** (1970),89–96.
[16] P.K.Maji, R.Biswas, R.Roy *Soft set theory*, Computer and Mathematics with Applications **45**(2003) 555-562.
[17] S.Hussain, B.Ahmed,*Some Properties of Soft Topological Spaces,* Comput. Math. Appl., **62** (2011), 4048-4067.
[18] S. Pious Missier, A. Robert *On semi*-open sets* , International Journal of Mathematics and soft computing, **2**(2)(2012)95-105.
[19] S.M.Khalil, M.Ulrazaq, S.Abdul-Ghani, A.F. Al-Musawi, *σ-Algebra and σ-Baire in Fuzzy Soft Setting,* Advances in Fuzzy Systems , Volume 2018, Article ID 5731682, 10 pages. **7**(3),(2011),471-481.
[20] S.M.Khalil, *Decision making using algebraic operations on soft effect matrix as new category of similarity measures and study their application in medical diagnosis problems* , Journal of Intelligent and Fuzzy Systems, **37**(2019) ,1865-1877.
[21] W. Dunham, *A New Closure Operator for Non-T1 Topologies,* Kyungpook Math. J. **22** (1982), 55–60.
[22] W.K.Min, *A Note on Soft Topological Spaces*, Comput. Math. Appl., **62**(2011),3524-3528.

Universal Journal of Mathematics and Applications, 3 (4) (2020) 167-172
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.799576

On Submanifolds of $N(k)$-Quasi Einstein Manifolds with a Type of Semi-Symmetric Metric Connection

İnan Ünal[1]

[1]*Department of Computer Engineering, Faculty of Engineering, Munzur University, Tunceli, Turkey*

Article Info

Keywords: *$N(k)-$quasi Einstein manifolds, Totally geodesic, Totally umbilical, Para-Kenmotsu*
2010 AMS: *53C15, 53C25, 53D10*
Received: *24 September 2020*
Accepted: *2 November 2020*
Available online: *23 December 2020*

Abstract

In this study, we consider the $N(k)-$quasi Einstein manifolds with respect to a type of semi-symmetric metric connection. We suppose that the generator of $N(k)-$quasi-Einstein manifolds is parallel with respect to semi-symmetric metric connection and we classify such manifolds. In addition, we consider the submanifolds of a $N(k)-$quasi Einstein manifold and we obtain some conditions on the totally geodesic and the totally umbilic submanifolds. Finally, we consider a para-Kenmotsu space form as an example of $N(k)-$quasi-Einstein manifolds.

1. Introduction

An Einstein manifold is a Riemannian manifold (M,g) satisfying Einstein fields equation. We determine such manifold by $Ric = \lambda g$, for the Ricci curvature Ric of M non-zero constant λ. In differential geometry, there are many kind of manifolds which satisfy this relation. Einstein manifolds are widely studied by researchers from mathematics and physics. A well known generalization of Einstein manifolds is the notion of quasi-Einstein manifolds defined by Chaki in [5]. Similar to Einstein manifolds , quasi-Einstein manifolds are also occur in the solutions of Einstein field equations. In this manner, quasi-Einstein manifolds have some applications in the general relativity. An example is Robertson-Walker space times [8]. A quasi-Einstein manifold is a Riemannian manifold (M,g) which has the following relation on the Ricci tensor of M;

$$Ric(\Omega_1,\Omega_2) = ag(\Omega_1,\Omega_2)+b\eta(\Omega_1)\eta(\Omega_2) \tag{1.1}$$

for some smooth functions a and b, arbitrary vector fields $\Omega_1,\Omega_2 \in \Gamma(TM)$, where η is a non-zero $1-$form on M such that $g(\Omega_1,\xi) = \eta(\Omega_1)$, $\eta(\xi) = 1$ for a vector field $\xi \in \Gamma(TM)$. We call η by associated $1-$ form and ξ by the generator of the manifold. If a $(2m+1)$-dimensional Riemannian manifold M has an almost contact metric structure (ϕ,ξ,η,g) and Ricci tensor satisfies (1.1) then M is called by an $\eta-$Einstein manifold [1]. So, an $\eta-$Einstein manifold is an example of quasi-Einstein manifolds. Also, a generalized Sasakian space form is a quasi-Einstein manifold [6].
k-nullity distribution of a quasi Einstein manifold is defined as

$$N(k): p \longrightarrow N_p(k) = \left[\Omega_3 \in \Gamma(T_pM) : Rim(\Omega_1,\Omega_2)\Omega_3 = k\left\{g(\Omega_2,\Omega_3)\Omega_1 - g(\Omega_1,\Omega_3)\Omega_2\right\}\right], \tag{1.2}$$

for any Ω_1, $\Omega_2 \in \Gamma(T_pM)$ and $k \in \mathbb{R}$, where Rim is the Riemannian curvature tensor of M. If the generator vector field ξ belongs to $k-$nullity distribution then M is called $N(k)$-quasi Einstein manifold ($NK(QE)_m$) [5]. A quasi Einstein manifold is an $NK(QE)_m$ manifold if it is conformally flat [15]. In 2004 De and Ghosh [7] prove the existence of $NK(QE)_m$ manifolds and presented some results. In 2008 Özgür [3] examined $NK(QE)_m$ manifolds under some certain curvature conditions. Yıldız et al. [4] worked on $NK(QE)_m$ manifolds with some semi-symmetry conditions and gave examples. The Riemannian geometry of $N(k)-$ quasi-Einstein manifolds have been studied by many researchers in [3, 6, 10, 12, 16].
In this work, we consider a $NK(QE)_m$ manifold admitting a type of semi-symmetric metric connection (SSMC) and we obtain some results on the submanifolds of such manifolds. Also, we present a classification of $NK(QE)_m$ manifold admitting SSMC. We proved some theorems on the totally geodesic and totally umbilical submanifolds. Finally, we consider a para-Kenmotsu space form as an example.

Email address and ORCID number: **inanunal@munzur.edu.tr, https:// orcid.org/0000-0003-1318-9685 (İ. Ünal)**

2. N(k)-quasi Einstein manifolds with a type of semi-symmetric metric connection

In the Riemannian geometry, we know that the Levi-Civita connection have no torsion and it is a metric connection. Also, there are many type of connections which has torsion and not symmetric. One of them is a semi-symmetric metric connection (SSMC) . In the [17] Yano defined a type of SSMC. Murathan and Özgür [3] studied Riemannian manifolds with this connection under some semi-symmetry conditions. The authors consider the parallel unit vector field with respect to the Levi-Civita connection. In this section, we consider a $NK(QE)_m$ manifold with the parallel vector field ξ with respect to SSMC. We present some results related to SSMC.
Let M be an $m-$dimensional $NK(QE)_m$ manifold and define a map on M by

$$\overset{\approx}{\nabla}_{\Omega_1}\Omega_2=\widetilde{\nabla}_{\Omega_1}\Omega_2+\eta(\Omega_2)\Omega_1-g(\Omega_1,\Omega_2)\xi \tag{2.1}$$

for all $\Omega_1,\Omega_2\in\Gamma(TM)$, where $\widetilde{\nabla}$ is the Levi-Civita connection (LCC) on M. The map $\overset{\approx}{\nabla}$ on M defines a semi-symmetric metric connection [17]. The Riemannian curvature of M with respect to $\overset{\approx}{\nabla}$ was obtained in [17] as;

$$\begin{aligned}\overset{\approx}{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)=\widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)-\omega(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)+\omega(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)\\-g(\Omega_2,\Omega_3)\omega(\Omega_1,\Omega_4)+g(\Omega_1,\Omega_3)\omega(\Omega_2,\Omega_4)\end{aligned} \tag{2.2}$$

for all $\Omega_1,\Omega_2,\Omega_3,\Omega_4\in\Gamma(TM)$, where ω is defined as

$$\omega(\Omega_1,\Omega_2)=(\widetilde{\nabla}_{\Omega_1}\eta)\Omega_2-\eta(\Omega_1)\eta(\Omega_2)+\frac{1}{2}g(\Omega_1,\Omega_2).$$

From (2.1) we obtain

$$\overset{\approx}{\nabla}_{\Omega_1}\xi=\widetilde{\nabla}_{\Omega_1}\xi+\Omega_1-\eta(\Omega_1)\xi.$$

Suppose that $\overset{\approx}{\nabla}_{\Omega_1}\xi=0$. Then, we recall ξ by parallel vector field with respect to SSMC. Thus, we get

$$\widetilde{\nabla}_{\Omega_1}\xi=-\Omega_1+\eta(\Omega_1)\xi. \tag{2.3}$$

On the other hand, we have

$$(\widetilde{\nabla}_{\Omega_1}\eta)\Omega_2=\widetilde{\nabla}_{\Omega_1}\eta(\Omega_2)-\eta(\widetilde{\nabla}_{\Omega_1}\Omega_2).$$

Since, $\widetilde{\nabla}$ is a metric connection i.e $(\widetilde{\nabla}_{\Omega_1}g)(\Omega_2,\Omega_3)=g(\widetilde{\nabla}_{\Omega_1}\Omega_2,\Omega_3)+g(\Omega_3,\widetilde{\nabla}_{\Omega_1}\Omega_2)$, from (2.3) we get

$$(\widetilde{\nabla}_{\Omega_1}\eta)\Omega_2=-g(\Omega_1,\Omega_2)+\eta(\Omega_1)\eta(\Omega_2).$$

Thus, we obtain $\omega(\Omega_1,\Omega_2)=-\frac{1}{2}g(\Omega_1,\Omega_2)$ and so from (2.2), we get

$$\overset{\approx}{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)=\widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)+g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4). \tag{2.4}$$

In [2] it was proved that in a $NK(QE)_m$ manifold $k=\frac{a+b}{m-1}$. Thus, from (1.2) , we obtain

$$\overset{\approx}{R}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)=(\frac{a+b}{m-1}+1)[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)] \tag{2.5}$$

Finally, we state that

Theorem 2.1. *Let M be a $NK(QE)_m$ manifold with respect to a SSMC $\overset{\approx}{\nabla}$ and ξ be a parallel vector field with respect to $\overset{\approx}{\nabla}$. We have following classifications;*

- *If $a+b=1-m$ then M is locally isometric to $m-$dimensional Euclidean space $\mathbb{E}^m$,*
- *If $a+b>1-m$ then M is locally isometric to $m-$dimensional sphere $S^m(\frac{a+b}{m-1}+1)$,*
- *If $a+b<1-m$ then M is locally isometric to $m-$dimensional hyperbolic space $H^n(\frac{a+b}{m-1}+1)$.*

Let take an orthonormal basis of M as $\{E_1,E_2,...,E_{m-1},E_m=\xi\}$. Then with taking sum over $1\leq i\leq m$ in (2.4) we obtain

$$\sum_{i=1}^{m}\overset{\approx}{Rim}(\Omega_1,E_i,E_i,\Omega_4)=\sum_{i=1}^{m}\{\widetilde{Rim}(\Omega_1,E_i,E_i,\Omega_4)+g(E_i,E_i)g(\Omega_1,\Omega_4)-g(\Omega_1,E_i)g(E_i,\Omega_4)\}$$

and so, we get

$$\overset{\approx}{Ric}(\Omega_1,\Omega_4)=\widetilde{Ric}(\Omega_1,\Omega_4)+(m-1)g(\Omega_1,\Omega_4)$$

for all $\Omega_1,\Omega_2\in\Gamma(TM)$. Then from (1.1), we obtain

$$\overset{\approx}{Ric}(\Omega_1,\Omega_4)=(a+(m-1))g(\Omega_1,\Omega_4)+bm\eta(\Omega_1)\eta(\Omega_2)$$

Finally, we conclude that;

Theorem 2.2. *Let M be an $NK(QE)_m$ manifold with respect to a LCC $\tilde{\nabla}$ and ξ be a parallel vector field with respect to SSMC $\overset{\approx}{\nabla}$. Then M is an $NK(QE)_m$ manifold with respect to $\overset{\approx}{\nabla}$.*

3. Submanifolds of N(k)-quasi Einstein manifolds with a type of semi-symmetric metric connection

Let M be an $m-$dimensional $NK(QE)_m$ manifold with respect to SSMC $\tilde{\tilde{\nabla}}$and N be an $n-$dimensional submanifold of M. Suppose that the generator vector field ξ tangent to N. Thus, we have two subbundles of TM as TN and $TN^{\perp}$ such that $TM = TN \oplus TN^{\perp}$.The subbundles TN and $TN^{\perp}$ are called tangent bundle and normal bundle of N, respectively. Let recall some classical equations from the submanifold theory. For details we refer to reader [1].
The Gauss equation is given by

$$\tilde{\nabla}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \sigma(\Omega_1,\Omega_2)$$

for all $\Omega_1,\Omega_2 \in \Gamma(TN)$, where $\sigma(\Omega_1,\Omega_2)$ denote the second fundamental form, and $\tilde{\nabla},\nabla$ are the Levi-Civita connections on M and N, respectively.
The Weingarten equation is

$$\tilde{\nabla}_{\Omega_1}W = -A_W\Omega_1 + \nabla^{\perp}_{\Omega_1}W$$

for all $\Omega_1 \in \Gamma(TN)$ and $W \in \Gamma(TN^{\perp})$, where A_W is the shape operator related to W, $\nabla^{\perp}$ is the induced normal connection on the normal bundle $TN^{\perp}$. Consider the definition of SSMC $\tilde{\tilde{\nabla}}$ and using the Gauss equation, we get

$$\tilde{\tilde{\nabla}}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1,\Omega_2)\xi + \sigma(\Omega_1,\Omega_2). \tag{3.1}$$

Suppose that ξ is parallel with respect to $\tilde{\tilde{\nabla}}$, then we obtain

$$\nabla_{\Omega_1}\xi = -\Omega_1 + \eta(\Omega_1)\xi - \sigma(\Omega_1,\xi).$$

Hence, we provide the following lemma.

Lemma 3.1. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\tilde{\nabla}$, N be a submanifold of M, and ξ be a parallel vector field with respect to SSMC $\tilde{\tilde{\nabla}}$. Then, we get*

$$\nabla_{\Omega_1}\xi = -\Omega_1 + \eta(\Omega_1)\xi\ ,\ \ \sigma(\Omega_1,\xi) = 0$$

for all $\Omega_1 \in \Gamma(TN)$, where $\xi \in \Gamma(TN)$.

Also, we know that

$$(\tilde{\nabla}_{\Omega_1}\sigma)(\Omega_2,\Omega_3) = \nabla^{\perp}_{\Omega_1}(\sigma(\Omega_1,\Omega_2)) - \sigma(\nabla_{\Omega_1}\Omega_2,\Omega_3) - \sigma(\Omega_2,\nabla_{\Omega_1}\Omega_3) \tag{3.2}$$

for all $\Omega_1,\Omega_2,\Omega_3 \in \Gamma(TN)$ [1].

Definition 3.2. *Let M be an $NK(QE)_m$ manifold and N be submanifold of M. If the covariant derivation of the second fundamental form vanishes, then N is called parallel submanifold [1].*

Theorem 3.3. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\tilde{\tilde{\nabla}}$, N be a submanifold of M and ξ be a parallel vector field with respect to SSMC $\tilde{\tilde{\nabla}}$. If N is parallel submanifold with respect to LCC $\tilde{\nabla}$ then it is not parallel submanifold with respect to SSMC $\tilde{\tilde{\nabla}}$.*

Proof. From the definition of SSMC $\tilde{\tilde{\nabla}}$, we have

$$\begin{aligned}(\tilde{\tilde{\nabla}}_{\Omega_1}\sigma)(\Omega_2,\Omega_3) = {} & \tilde{\nabla}_{\Omega_1}\sigma(\Omega_1,\Omega_2) - \sigma(\tilde{\nabla}_{\Omega_1}\Omega_2,\Omega_3) - \eta(\Omega_2)\sigma(\Omega_1,\Omega_3) - g(\Omega_1,\Omega_2)\sigma(\xi,Z)\\ & - \sigma(\Omega_2,\tilde{\nabla}_{\Omega_1}\Omega_3) - \eta(\Omega_3)\sigma(\Omega_1,\Omega_2) - g(\Omega_1,\Omega_3)\sigma(\Omega_2,\xi).\end{aligned}$$

Since ξ is parallel with respect to SSMC $\tilde{\tilde{\nabla}}$, by using Lemma 3.1 we obtain

$$(\tilde{\tilde{\nabla}}_{\Omega_1}\sigma)(\Omega_2,\Omega_3) = \nabla^{\perp}_{\Omega_1}(\sigma(\Omega_1,\Omega_2)) - \sigma(\nabla_{\Omega_1}\Omega_2,\Omega_3) - \sigma(\Omega_2,\nabla_{\Omega_1}\Omega_3) - \eta(\Omega_2)\sigma(\Omega_1,\Omega_3) - \eta(\Omega_3)\sigma(\Omega_1,\Omega_2).$$

Suppose that, N is parallel with respect to LCC $\tilde{\nabla}$. Then, from (3.2) we have

$$(\tilde{\tilde{\nabla}}_{\Omega_1}\sigma)(\Omega_2,\Omega_3) = -\eta(\Omega_2)\sigma(\Omega_1,\Omega_3) - \eta(\Omega_3)\sigma(\Omega_1,\Omega_2).$$

Thus N is not parallel with respect to SSMC $\tilde{\tilde{\nabla}}$. □

We also state following result.

Corollary 3.4. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\tilde{\tilde{\nabla}}$, N be a submanifold of M and ξ be a parallel vector field with respect to SSMC $\tilde{\tilde{\nabla}}$. If N is parallel with respect to SSMC $\tilde{\tilde{\nabla}}$ then it is not parallel with respect to LCC $\tilde{\nabla}$.*

The Codazzi equation for N is given by

$$\widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) = Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4)+g(\sigma(\Omega_1,\Omega_3),\sigma(\Omega_2,\Omega_4))-g(\sigma(\Omega_2,\Omega_3),\sigma(\Omega_1,\Omega_4)) \tag{3.3}$$

for all $\Omega_1,\Omega_2,\Omega_3,\Omega_4 \in \Gamma(TN)$, where $\widetilde{Rim}$ is the Riemannian curvature tensor of M and Rim is the Riemannian curvature tensor of N [1].
Let M be an $NK(QE)_m$ manifold with respect to SSMC $\overline{\overline{\nabla}}$, ξ be a parallel vector field with respect to SSMC $\overline{\overline{\nabla}}$ and N be a submanifold of M. From (2.4) and (3.2), we get

$$\begin{aligned}\overline{\overline{Rim}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4)+g(\sigma(\Omega_1,\Omega_3)\sigma(\Omega_2,\Omega_4))-g(\sigma(\Omega_2,\Omega_3)\sigma(\Omega_1,\Omega_4)\\&+g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4).\end{aligned}$$

Thus , by using (2.5) we obtain

$$Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) = \frac{a+b}{m-1}[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)+g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)]-g(\sigma(\Omega_1,\Omega_3),\sigma(\Omega_2,\Omega_4))+g(\sigma(\Omega_2,\Omega_3),\sigma(\Omega_1,\Omega_4)]$$

Finally, we state the following theorem.

Theorem 3.5. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\overline{\overline{\nabla}}$, N be a submanifold of M and ξ be a parallel vector field with respect to SSMC $\overline{\overline{\nabla}}$. If N is totally geodesic , then N is an $NK(QE)_m$ manifold with $k=\frac{a+b}{m-1}$.*
On the other hand if N is totally umbilical, i.e. $\sigma(\Omega_1,\Omega_2) = Hg(\Omega_1,\Omega_2)$, then we get

$$Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) = (\frac{a+b}{m-1}+g(H,H))[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)+g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)].$$

where H is the mean curvature of N. Therefore we can state following theorem.

Theorem 3.6. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\overline{\overline{\nabla}}$, N be a submanifold of M and ξ be a parallel vector field with respect to SSMC $\overline{\overline{\nabla}}$. If N is totally umbilical , then N is a generalized real space form.*

Example 3.7. *Let M be a $(2m+1)-$dimensional smooth manifold. (ϕ,ξ,η) is called an almost para-contact structure on M such that*

$$\phi^2\Omega = \Omega-\eta(\Omega)\xi,\ \ \phi(\xi)=0,\ \ \ \eta\circ\phi=0,\ \ \ \eta(\xi)=1 \tag{3.4}$$

where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a $1-$ form, and Ω is an arbitrary vector field on M [18]. M is called a para-Kenmotsu (PK) manifold if we have

$$\left(\widetilde{\nabla}_{\Omega_1}\phi\right)\Omega_2 = -g(\phi\Omega_1,\Omega_2)\xi+\eta(\Omega_2)\phi\Omega_1 \tag{3.5}$$

for all $\Omega_1,\Omega_2 \in \Gamma(TM)$ [14]. Thus on M, we have

$$\widetilde{\nabla}_{\Omega_1}\xi = -\phi^2\Omega_1 \tag{3.6}$$

for all $\Omega_1 \in \Gamma(TM)$.
Let $\overline{\overline{\nabla}}$ be a SSMC defined in (2.1) on M. Thus, we get $\overline{\overline{\nabla}}_{\Omega_1}\xi=0$, i.e ξ is parallel with respect to SSMC $\overline{\overline{\nabla}}$.
The $\phi-$sectional curvature of PK-manifold is defined as the sectional curvature of plane section spanned by Ω_1 and $\phi\Omega_1$, for unit vector field Ω_1. If M has constant $\phi-$sectional curvature c then we have

$$\begin{aligned}\widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \left(\frac{c-3}{4}\right)[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)],\\&+\left(\frac{c+1}{4}\right)[g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)-g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)+2g(\Omega_1,\phi\Omega_2)g(\phi\Omega_3,\Omega_4),\\&+\eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4)-\eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4)+g(\Omega_1,\Omega_3)\eta(\Omega_2)\eta(\Omega_4)-g(\Omega_2,\Omega_3)\eta(\Omega_1)\eta(\Omega_4)].\end{aligned} \tag{3.7}$$

A PK-manifold M with above curvature relation is called a PK-space form. For details see [13]. The Ricci curvature of a PK-space forms is given by

$$\widetilde{Ric}(\Omega_1,\Omega_2) = (\frac{(m+1)(c+1)}{4}-(m-1))g(\Omega_1,\Omega_2)-\frac{(m+1)(c+1)}{4}\eta(\Omega_1)\eta(\Omega_2). \tag{3.8}$$

This shows M is a quasi-Einstein manifold with $a=\frac{(m+1)(c+1)}{4}-(m-1)$, $b=\frac{(m+1)(c+1)}{4}$. On a PK-manifold we have

$$(\widetilde{\nabla}_{\Omega_1}\eta)\Omega_2 = g(\Omega_1,\Omega_2)-\eta(\Omega_1)\eta(\Omega_2), \tag{3.9}$$

thus we obtain

$$\omega(\Omega_1,\Omega_2) = \frac{3}{2}g(\Omega_1,\Omega_2)-2\eta(\Omega_1)\eta(\Omega_2). \tag{3.10}$$

By using (2.2), the curvature of a PK-manifold admitting SSMC given in (2.1) is

$$\begin{aligned}\overline{\overline{Rim}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)-3(g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4),\\&+\eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4)-\eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4)+\eta(\Omega_2)\eta(\Omega_4)g(\Omega_1,\Omega_3)-\eta(\Omega_1)\eta(\Omega_4)g(\Omega_2,\Omega_3)).\end{aligned}$$

Also, from (3.7), on a PK-space form we get

$$\begin{aligned}\widetilde{\widetilde{Rim}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \left(\frac{c-15}{4}\right)(g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)\\ &+\left(\frac{c-11}{4}\right)\eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4)-\eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4)+(\eta(\Omega_2)\eta(\Omega_4)g(\Omega_1,\Omega_3)\\ &+\eta(\Omega_1)\eta(\Omega_4)g(\Omega_2,\Omega_3))\\ &+\left(\frac{c+1}{4}\right)[g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)-g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)+2g(\Omega_1,\phi\Omega_2)g(\phi\Omega_3,\Omega_4)].\end{aligned} \tag{3.11}$$

A generalized para-Sasakian space form (GPSSF) is an almost para-contact metric manifold (M,ϕ,ξ,η,g) with the following curvature relation;

$$\begin{aligned}\widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= F_1[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)]\\ &+F_2(-g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)+g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)-2g(\Omega_1,\phi\Omega_2)g(\phi\Omega_3,\Omega_4))\\ &\times F_3(\eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4)-\eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4)+g(\Omega_1,\Omega_3)\eta(\Omega_2)\eta(\Omega_4)-g(\Omega_2,\Omega_3)\eta(\Omega_1)\eta(\Omega_4)).\end{aligned}$$

for all $\Omega_1,\Omega_2,\Omega_3,\Omega_4$ vector fields.

Corollary 3.8. *A PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ is a GPSSF with $F_1=\frac{c-15}{4}$, $F_2=-\frac{c-11}{4}$ and $F_3=\frac{c+1}{4}$.*

Let take an orthonormal basis of M by $E_1,E_2,...E_n,E_{m+1}=\phi E_1,...,E_{2m}=\phi E_m,\xi$. By choosing $\Omega_2=\Omega_3=E_i$ and taking sum over i such that $1\le i\le 2m$ in (3.11) then, we obtain

$$\widetilde{\widetilde{Ric}}(\Omega_1,\Omega_2)=(\frac{m(c-15)-2}{2})g(\Omega_1,\Omega_2)+\frac{c-11}{4}(1-2m)\eta(\Omega_1)\eta(\Omega_4).$$

Thus, M is a quasi-Einstein manifold. So, we state;

Corollary 3.9. *A PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ is a quasi-Einstein manifold.*

This is compatible with Theorem 2.2.
Let N be a submanifold of PK-space form M with respect to $\overline{\overline{\nabla}}$. Then, we have

$$\begin{aligned}Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \widetilde{\widetilde{Rim}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)-g(\sigma(\Omega_1,\Omega_3)\sigma(\Omega_2,\Omega_4))+g(\sigma(\Omega_2,\Omega_3)\sigma(\Omega_1,\Omega_4)\\ &-g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)+g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)\end{aligned}$$

and from (3.11) we get

$$\begin{aligned}Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \left(\frac{c-19}{4}\right)(g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)\\ &+\left(\frac{c-11}{4}\right)(\eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4)-\eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4)\\ &+\eta(\Omega_2)\eta(\Omega_4)g(\Omega_1,\Omega_3)-\eta(\Omega_1)\eta(\Omega_4)g(\Omega_2,\Omega_3))\\ &+\left(\frac{c+1}{4}\right)(g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,,\Omega_4)-g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,,\Omega_4)+2g(\Omega_1,\phi\Omega_2)g(\phi\Omega_3,\Omega_4))\\ &-g(\sigma(\Omega_1,\Omega_3)\sigma(\Omega_2,\Omega_4))+g(\sigma(\Omega_2,\Omega_3)\sigma(\Omega_1,\Omega_4)\end{aligned}$$

for all $\Omega_1,\Omega_2,\Omega_3,\Omega_4\in\Gamma(TN)$.
Suppose that ξ is normal to N and N is an anti-invariant submanifold i.e. $\phi\Omega_1\in\Gamma(TN^{\perp})$, for $\Omega_1\in\Gamma(TN)$. Then, we get

$$Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4)=(\frac{c-19}{4})(g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)+g(\sigma(\Omega_1,\Omega_3)\sigma(\Omega_2,\Omega_4))-g(\sigma(\Omega_2,\Omega_3)\sigma(\Omega_1,\Omega_4).$$

Thus, we state following results.

Corollary 3.10. *Let M be a PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ and N be an anti-invariant submanifold of M with ξ is normal to N. If N is totally geodesic, then N is $N(k)-$manifold.*

Corollary 3.11. *Let M be a PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ and N be an anti-invariant submanifold of M with ξ is normal to N. If N is totally umbilical, then N is a reel space form.*

Corollary 3.12. *Let M be a PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ and N be an anti-invariant submanifold of M with ξ is normal to N. If N is totally geodesic. Then N is an Einstein manifold.*

Let M be a PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ and N be a submanifold of M. If ξ is tangent to submanifold N, then Lemma 3.1 is verified. Also, for the same submanifold the Theorem 3.3 is verified.

References

[1] K. Yano, M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, World Scientific, **3**, 1984.
[2] C. Özgür , M. M. Tripathi, *On the concircular curvature tensor of an $N(\kappa)$-quasi Einstein manifold*, Math. Pannon., **18**(1), (2007), 95-100.
[3] C. Özgür, *$N(\kappa)$-quasi Einstein manifolds satisfying certain conditions*, Chaos Solitons Fractals, **38**(5) (2008), 1373-1377.
[4] A. Yıldız, U.C. De, A. Çetinkaya, *On some classes of $N(\kappa)$-quasi Einstein manifolds*, Proc. Natl. Acad. Sci. India A, **83**(3) (2013), 239-245.
[5] M.C. Chaki, *On quasi Einstein manifolds*, Publ. Math. Debr., **57** (2000), 297-306.
[6] S.K. Chaubey, *Existence of $N(\kappa)$-quasi Einstein manifolds*, Facta universitatis Nis. Ser. Math.Inform., **32**(3) (2017), 369-385.
[7] U.C. De, G.C.Ghosh, *On quasi Einstein manifolds*, Period. Math. Hung., **48** (2004), 223-231.
[8] U. C. De, S. Shenawy, *Generalized quasi-Einstein GRW space-times*, Int. J. Geom. Methods Mod. Phys., **16**(08) (2019), 1950124.
[9] G.C. Ghosh, U.C. De, T.Q. Binh, *Certain curvature restrictions on a quasi Einstein manifolds*, Publ. Math. Debr. **69** (2006), 209-217.
[10] A.T. Kotamkar, A. Tarini, T. Brajendra, *Certain curvature conditions catisfied by $N(\kappa)$-quasi Einstein manifolds*, Int. J. Innov. Res. Adv. Eng. G. , **1**(9) (2015), 1-9.
[11] C. Murathan, C. Özgür, *Riemannian manifolds with a semi-symmetric metric connection satisfying some semi-symmetry conditions*, Proc. Est. Acad. Sci., **57**(4) (2008), 210–216.
[12] H.G. Nagaraja, K. Venu, *On Ricci solitons in $N(\kappa)$-quasi Einstein manifolds*, NTMSCI, **5**(3) (2017), 46-52.
[13] G. Pitiş, *Geometry of Kenmotsu Manifolds*, Editura Universitatii Transilvania, 2007.
[14] B.B. Sinha, K. L. Sai Prasad, *A class of almost para contact metric manifolds*, Bull. Cal. Math. Soc., **87** (1995), 307–312.
[15] M.M. Tripathi, J. Kim, *On $N(\kappa)-$quasi Einstein manifolds*, Commun. Korean Math. Soc., **22** (2007), 411-417.
[16] A. Taleshian, A. A. Hosseinzadeh, *Investigation of some conditions on $N(\kappa)$-quasi Einstein manifolds*, Bull. Malaysian Math. Sci. Soc, **34**(3) (2011), 455-464.
[17] K. Yano, *On semi-symmetric connection*, Revue Roumaine Math. Pures Appl., **15** (1970), 1570-1586.
[18] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom., **36**(1) (2008), 37–60.