CONFERENCE PROCEEDINGS OF SCIENCE AND TECHNOLOGY

HTTP://DERGIPARK.GOV.TR/CPOST

VOLUME III
ISSUE II
IECMSA 2020

Conference Proceedings of Science and Technology

ISSN 2651-544X

Conference Proceeding of 9th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2020) Skopje - North Macedonia

CONFERENCE PROCEEDINGS OF SCIENCE AND TECHNOLOGY

ISSN: 2651-544X

Preface

It is my great pleasure and honour to welcome you at the 9th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2020) which has been organized in cooperation with Sakarya University and International Balkan University.

Unfortunately, in 2020 humanity has faced an unusual, dangerous challenge connected with the new COVID-19 and one impact of this virus has placed constraints on the ability of researchers to join a face-to-face meeting. As the health and safety of everyone is our priority, IECMSA will proceed with our annual gathering this year through a virtual conference, instead of an in-person event. The decision to hold IECMSA-2020 as a virtual conference on the original dates has appeared preferable to a postponed meeting face to face in Skopje, especially during this uncertain time. Thus, the virtual conference format will allow us to present our studies whilst still providing many of the benefits of a face to face meeting. Besides, virtual presentations will be more widely available, yielding a greater exposure to our studies.

Established since 2012, the series of IECMSA features the latest developments in the field of mathematics and applications. The previous conferences were held as follows: IECMSA-2012, Prishtine, Kosovo, IECMSA-2013, Sarajevo, Bosnia and Herzegovina, IECMSA-2014, Vienna, Austria, IECMSA-2015, Athens, Greece, IECMSA-2016, Belgrade, Serbia, IECMSA-2017, Budapest, Hungary, IECMSA-2018, Kyiv, Ukraine, and IECMSA-2019, Baku, Azerbaijan. These conferences gathered a large number of international world-renowned participants.

Now in IECMSA-2020, the scientific committee members and the external reviewers invested significant time in analyzing and assessing multiple papers, consequently, they hold and maintain a high ii standard of quality for this conference. The scientific committee accepted 116 virtual presentations. Despite the effects of coronavirus, 136 participants are attending the conference from 23 different countries. The scientific program of the conference features keynote talks, followed by contributed presentations in two parallel sessions.

The conference program represents the efforts of many people. I would like to express my gratitude to all members of the scientific committee, external reviewers, sponsors and, honorary committee for their continued support to the IECMSA. I also thank the invited speakers for presenting their talks on current researches. Also, the success of IECMSA depends on the effort and talent of researchers in mathematics and its applications that have written and submitted papers on a variety of topics. So, I would like to sincerely thank all participants of IECMSA-2020 for contributing to this great meeting in many different ways. I believe and hope that each of you will get the maximum benefit from the conference.

Wish you all health and safety during this difficult time
Prof. Dr. Murat TOSUN
Chairman
On behalf of the Organizing Committee

Editor in Chief

Murat Tosun
Department of Mathematics, Faculty of Science and Arts, Sakarya University, Sakarya-TÜRKİYE
tosun@sakarya.edu.tr

Managing Editors

Emrah Evren Kara
Department of Mathematics,
Faculty of Science and Arts, Düzce University,
Düzce-TÜRKİYE
eevrenkara@duzce.edu.tr

Mahmut Akyiğit
Department of Mathematics,
Faculty of Science and Arts, Sakarya University,
Sakarya-TÜRKİYE
makyigit@sakarya.edu.tr

Merve İlkhan Kara
Department of Mathematics,
Faculty of Science and Arts, Düzce University,
Düzce-TÜRKİYE
merveilkhan@duzce.edu.tr

Fuat Usta
Department of Mathematics,
Faculty of Science and Arts, Düzce University,
Düzce-TÜRKİYE
fuatusta@duzce.edu.tr

Editorial Board of Conference Proceedings of Science and Technology

Contents

Conference Proceeding Science and Technology, 3(2), 2020, 215–221

Conference Proceeding of 9th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2020).

On Integral Transforms of Some Special Functions

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Serpil HALICI[1,*] *Şule ÇÜRÜK*[2]

[1] *Pamukkale Uni., Depart. of Math. Denizli,Turkey, ORCID: 0000-0002-8071-0437*
[2] *Pamukkale Uni., Depart. of Math. Denizli,Turkey, ORCID: 0000-0002-4514-6156*
* *Corresponding Author E-mail: shalici@pau.edu.tr, scuruk10@posta.pau.edu.tr*

Abstract: In this study, known integral transforms such as Fourier and Hartley are studied and these integral transforms are studied in detail for bicomplex numbers. In addition, the properties of the bicomplex Hartley transform have been investigated. Also, the relation between Hartley and Fourier transform for bicomplex numbers is given.

Keywords: Bicomplex functions, Fourier transformations, Integral transformations.

1 Introduction

The concept of bicomplex number was first defined by Corrada Segre $(1863 - 1924)$ in 1892 . This number system has been considered as a generalization of complex numbers and is defined as $\mathbb{BC} = \{z_1 + z_2 j | z_1,\ z_2 \in \mathbb{C},\ j^2 = -1\}$ where j plays a role as an imaginary unit [10]. Any bicomplex number Z can be written as a linear combination of bases e_1 and e_2: $Z = (z_1 - z_2 i)e_1 + (z_1 + z_2 i)e_2$, where the relationships between bases e_1 and e_2 are $e_1 + e_2 = 1, e_1 e_2 = e_2 e_1 = 0$. The idempotent coefficients are $z_1 - iz_2$ and $z_1 + iz_2$.
The aim of this study is to first recall the Fourier and Hartley transformations and then examine these transformations in a set of bicomplex numbers. As well known that the Fourier transform plays an important role in solving differential equations and integral equations. Especially in mathematical statistics, statistical mechanics problems, problems related to free vibration, diffusion, in geophysical engineering; measuring the resistance and strength of fault lines, the two-dimensional wave equation or Cauchy problems, solution of unknown $f(x)$ functions in integral equations, Fourier transforms are used effectively.
The Fourier transform of a function $f(x)$ with $k \in \mathbb{R}$ is represented by the symbol $\mathfrak{F}\{f(x)\}$, and it is defined as

$$\mathfrak{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx. \tag{1}$$

This integral is called complex Fourier transform or exponential Fourier transform. The only condition for obtaining the Fourier transform of a function $f(x)$ is its absolute integration.
The inverse Fourier transform is represented by the symbol $\mathfrak{F}^{-1}\{F(k)\} = f(x)$ and is given as

$$\mathfrak{F}^{-1}\{F(k)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k) dk \tag{2}$$

where, $\mathfrak{F}^{-1}$ is called the inverse Fourier transform operator. Let a and b be any real constants. If the Fourier transforms of functions $f(x)$ and $g(x)$ are denoted by $\mathfrak{F}\{f(x)\} = F(k)$ and $\mathfrak{F}\{g(x)\} = G(k)$, respectively, the properties of the Fourier transform associated with these functions can be given in the following table.

$\mathfrak{F}\{af(x) + bg(x)\} = a\mathfrak{F}\{f(x)\} + b\mathfrak{F}\{g(x)\} = aF(k) + bG(k)$	Linearity property
$\mathfrak{F}\{e^{iax} f(x)\} = F(k - a)$	Translation Property
$\mathfrak{F}\{f(x - a)\} = e^{-ika} F(k)$	Shifting Property
$\mathfrak{F}\{f(ax)\} = \frac{1}{\lvert a\rvert} F(\frac{k}{a})$	Scaling property
$\mathfrak{F}\{x^n f(x)\} = (i)^n \frac{d^n}{dx^n} \{F(k)\}$	The property of multiplication by x^n
$If\ it's\ f^n(x) \to 0\ for\ \lvert x\rvert \to \infty,\ then\ \mathfrak{F}\{f^n(x)\} = (ik)^n F(k)$	Fourier Transform of Derivative

Table 1 Properties of the Fourier transform

Example. If $f(x) = e^{-a|x|}$ with $a \in \mathbb{R}$ and $w = u + iv \in \mathbb{C}$, then we find the Fourier transform of the function $f(x)$.

$$\hat{f}(w) = \int_{-\infty}^{\infty} e^{iwx} f(x)dx = \int_{-\infty}^{\infty} e^{iwx} e^{-a|x|}dx$$

$$\hat{f}(w) = \int_{-\infty}^{0} e^{iwx} e^{ax}dx + \int_{0}^{\infty} e^{iwx} e^{-ax}dx$$

$$\hat{f}(w) = \int_{-\infty}^{0} e^{(iw+a)x}dx + \int_{0}^{\infty} e^{(iw-a)x}dx.$$

If we write $u + iv$ instead of w, we get the following equality.

$$\hat{f}(w) = \int_{-\infty}^{0} e^{(iu-v+a)x}dx + \int_{0}^{\infty} e^{(iu-v-a)x}dx.$$

When the necessary operations are done, we get the following equality:

$$\hat{f}(w) = \frac{2v}{a^2 + u^2 - v^2 + 2iuv}. \tag{3}$$

Now, let us mention from bicomplex Fourier transform. In this section, our aim is to extend the Fourier transform $\mathfrak{F} : \mathbb{D} \subset \mathbb{R} \to \mathbb{BC}$ in bicomlex variables from its complex version and examine its fundamental properties.

2 Bicomplex Fourier Transform

There are some studies on bicomplex Fourier transform in the literature. One of them belongs to Banerjee. In this section, we will first discuss and remind you of these transformations. Let $f(t)$ be a real valued continuous function for the values t, $-\infty < t < \infty$. Accordingly,

$$|f(t)| \leq c_1 e^{-\alpha t}, \ t \geq 0, \ \alpha > 0 \tag{4}$$

$$|f(t)| \leq c_2 e^{\beta t}, \ t \leq 0, \ \beta > 0. \tag{5}$$

The above equations show that f is absolutely integrable in the whole real plane [1].
The complex Fourier transform of $f(t)$ satisfying the condition $|\hat{f}_1(w_1)| < \infty$ where w_1 is a complex frequency, is defined as

$$\hat{f}_1(w_1) = \mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{iw_1 t} f(t)dt, \quad w_1 = x + iy.$$

Then,

$$|\hat{f}_1(w_1)| = |\int_{-\infty}^{\infty} e^{iw_1 t} f(t)dt| \leq \int_{-\infty}^{\infty} |e^{-yt} f(t)|dt = \int_{-\infty}^{0} e^{-yt}|f(t)|dt + \int_{0}^{\infty} e^{-yt}|f(t)|dt.$$

Thus, we write

$$|\hat{f}_1(w_1)| \leq c_2 \int_{-\infty}^{0} e^{(\beta-y)t}dt + c_1 \int_{0}^{\infty} e^{-(\alpha+y)t}dt = \frac{c_2}{\beta - y} + \frac{c_1}{\alpha + y}.$$

Note that in order to be $|\hat{f}_1(w_1)| < \infty$ it must be $\breve{}\alpha < y < \beta$.
That is, $|\hat{f}_1(w_1)|$ is holomorphic in the Ω_1 region below:

$$\Omega_1 = \{w_1 \in \mathbb{C} : -\infty < Re(w_1) < \infty, \ -\alpha < Im(w_1) < \beta\}. \tag{6}$$

Similarly, the complex Fourier transform of $f(t)$ associated with the other complex frequency w_2 is as follows:

$$\hat{f}_2(w_2) = \mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{iw_2 t} f(t)dt, \quad w_2 \in \mathbb{C}.$$

$\hat{f}_2(w_2)$ is holomorphic in the Ω_2 region below:

$$\Omega_2 = \{w_2 \in \mathbb{C} : -\infty < Re(w_2) < \infty, \ -\alpha < Im(w_2) < \beta\}. \tag{7}$$

The complex functions $\hat{f}_1(w_1)$ and $\hat{f}_2(w_2)$ can be written as follows with the help of idempotent bases e_1 and e_2:

$$\hat{f}_1(w_1)e_1 + \hat{f}_2(w_2)e_2 = \int_{-\infty}^{\infty} e^{iw_1 t} f(t)dte_1 + \int_{-\infty}^{\infty} e^{iw_2 t} f(t)dte_2$$

$$\hat{f}_1(w_1)e_1 + \hat{f}_2(w_2)e_2 = \int_{-\infty}^{\infty} e^{i(w_1 e_1 + w_2 e_2)t} f(t)dt = \int_{-\infty}^{\infty} e^{iwt} f(t)dt = \hat{f}(w).$$

Since $\hat{f}_1(w_1)$ and $\hat{f}_2(w_2)$ are complex holomorphic functions in Ω_1 and Ω_2, respectively, then the bicomplex function $\hat{f}(w)$ will be holomorphic in the region Ω:

$$\Omega = \{w \in \mathbb{BC} : w = w_1 e_1 + w_2 e_2, \ w_1 \in \Omega_1 \ and \ w_2 \in \Omega_2\}. \tag{8}$$

The complex-valued holomorphic functions $\hat{f}_1(w_1)$ and $\hat{f}_2(w_2)$ are absolutely convergent in the regions Ω_1 and Ω_2, respectively. Then the region of absolute convergence of $\hat{f}(w)$ is region Ω.

Let $w_1 = x_1 + ix_2,\ \ w_2 = y_1 + iy_2,\ \ x_1, x_2, y_1, y_2 \in \mathbb{R}$ be with $w_1, w_2 \in \mathbb{C}$. For $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$

$$-\infty < x_1, y_1 < \infty,\ \ -\alpha < x_2 < \beta,\ \ -\alpha < y_2 < \beta.$$

Using the last inequalities and the idempotent elements e_1, e_2, w transforms to $4-$ component form as follows:

$$w = \frac{x_1 + y_1}{2} + i\frac{x_2 + y_2}{2} + j\frac{y_2 - x_2}{2} + ij\frac{x_1 - y_1}{2} = a_0 + ia_1 + ja_2 + ija_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For the elements x_2, y_2 we have three possible states with respect to these. That is

1) If $x_2 = y_2$, then

$$-\alpha < a_1 < \beta,\ \ a_2 = 0.$$

2) If $x_2 > y_2$, then

$$-\alpha - a_2 < a_1 < \beta + a_2,\ \ -\frac{\alpha + \beta}{2} < a_2 < 0.$$

3) If $x_2 < y_2$, then

$$-\alpha + a_2 < a_1 < \beta - a_2,\ \ 0 < a_2 < \frac{\alpha + \beta}{2}.$$

In the above all three cases we have $-\infty < a_0, a_3 < \infty$. Thus, considering these inequalities we get the following results:

$$-\infty < a_0, a_3 < \infty$$

$$-\alpha + |a_2| < a_1 < \beta - |a_2| \quad 0 \leq |a_2| < \frac{\alpha + \beta}{2}.$$

And so, the convergence region Ω of $\hat{f}(w)$ is as follows:

$$\Omega = \{w \in \mathbb{BC} : -\infty < a_0, a_3 < \infty,\ \alpha + |a_2| < a_1 < \beta - |a_2|,\ \ 0 \leq |a_2| < \frac{\alpha + \beta}{2}\}. \tag{9}$$

Now, let us investigate the existence of the bicomplex Fourier transform $\hat{f}(w)$:

If $w = a_0 + ia_1 + ja_2 + ija_3 \in \Omega$, then we have

$$-\infty < a_0, a_3 < \infty,\ \alpha + |a_2| < a_1 < \beta - |a_2| \quad 0 \leq |a_2| < \frac{\alpha + \beta}{2}.$$

If we can write the number w using idempotent bases, that is

$$w = \{(a_0 + a_3) + i(a_1 - a_2)\}e_1 + \{(a_0 - a_3) + i(a_1 + a_2)\}e_2 = w_1e_1 + w_2e_2,$$

then we get the following inequalities:

1) If $a_2 = 0, -\alpha < a_1 < \beta$, then

$$-\alpha < a_1 - a_2 < \beta,\ \ -\alpha < a_1 + a_2 < \beta.$$

2) If $a_2 < 0$, using by the first side of inequality $-\alpha - a_2 < a_1 < \beta + a_2$, we get

$$-\alpha < a_1 + a_2,\ \ a_1 - a_2 < \beta.$$

Since $a_2 < 0$, if we combine these two results, we get

$$-\alpha < a_1 + a_2 < a_1 - a_2 < \beta.$$

From here, this state can be interpreted as

$$-\alpha < a_1 + a_2 < a_1 - a_2,\ \ a_1 + a_2 < a_1 - a_2 < \beta.$$

3) If $a_2 > 0$, from the first side of inequality we get

$$-\alpha + a_2 < a_1 < \beta - a_2,\ \ -\alpha < a_1 - a_2.$$

And $a_1 + a_2 < \beta$ is obtained from the end side. Since $a_2 > 0$, if we combine the two result, the we have

$$-\alpha < a_1 - a_2 < a_1 + a_2 < \beta.$$

So, we can write as

$$-\alpha < a_1 - a_2 < a_1 + a_2,\ \ a_1 - a_2 < a_1 + a_2 < \beta.$$

Let $f(t)$ be a real valued continuous function in the interval $(-\infty, \infty)$ that satisfies (4) and (5). Then the Fourier transform of $f(t)$ is as follows:

$$\hat{f}(w) = \mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{iwt} f(t)dt, \quad w \in \mathbb{BC}. \tag{10}$$

We know that the Fourier transform $\hat{f}(w)$ exists and holomorphic for all $w \in \Omega$, where Ω is the convergence region of $\hat{f}(w)$ [1].
That is if the function $f(t)$ satisfies (4) and (5) is a continuous and real-valued function for $-\infty < t < \infty$, then the Fourier transform $\hat{f}(w)$

exist in the region (9). Let's give this theorem.

Theorem 1. *[1] Let the Fourier transforms of $f(t)$ and $g(t)$ functions be $\hat{f}(w)$ and $\hat{g}(w)$, respectively. If $\hat{f}(w) = \hat{g}(w)$, then*

$$f(t) = g(t). \tag{11}$$

Now, let's give some of basic Properties of the Fourier transform.

1. **Linearity Property**. Let the Fourier transforms of $f(t)$ and $g(t)$ be $\hat{f}(w)$ and $\hat{g}(w)$, respectively. Then linearity property can be given as follows:

$$\mathfrak{F}\{af(t) + bg(t)\} = a\hat{f}(w) + b\hat{g}(w)$$

where a and b are constants in the region Ω [1].

2. **Shifting Property**. Let $\hat{f}(w)$ be the Fourier transform of the continuous function $f(t)$. Then, $\mathfrak{F}\{f(t-a)\} = e^{iwa}\hat{f}(w)$ with $t - a \in \Omega$ [1].

3. **Scaling Property**. Let $\hat{f}(w)$ be the Fourier transform of the continuous function $f(t)$. Then, with $a \neq 0$,

$$\mathfrak{F}\{f(at)\} = \frac{1}{|a|}\hat{f}(\frac{w}{a}).$$

Before giving the convolution theorem for the Fourier transform, let us give the convolution property.
4. **Convolution Property.** If $f(t)$ and $g(t)$ functions are piecewise continuous in the interval $[0, \infty)$, then convolution of f and g is as follows and denoted by $f * g$:

$$f(t) * g(t) = \int_0^t f(x)g(t-x)dx.$$

Theorem 2. *(Convolution Theorem) [1] If the Fourier transforms of $f(t)$ and $g(t)$ functions be $\hat{f}(w)$ and $\hat{g}(w)$ respectively, then*

$$\mathfrak{F}f(t) * g(t) = \mathfrak{F}\{\int_{-\infty}^{\infty} f(u)g(t-a)du\} = \hat{f}(w)\hat{g}(w).$$

Theorem 3. *[1] If $f(t)$ and $t^r f(t)$ functions are integrable in the interval $-\infty < t < \infty$ with $r = 1, 2, \ldots, n$, then*

$$\mathfrak{F}\{t^n f(t)\} = i^n \frac{d^n}{dw^n}\hat{f}(w). \tag{12}$$

Where, $\hat{f}(w)$ is the Fourier transform of the function $f(t)$.

Proof: For $n = 1$, let's examine the right side of the equation.

$$\frac{d}{dw}\hat{f}(w) = \frac{\partial}{w_1}\hat{f}_1(w_1)e_1 + \frac{\partial}{w_2}\hat{f}_2(w_2)e_2$$

$$= \frac{\partial}{w_1}\int_{-\infty}^{\infty} e^{iw_1 t} f(t)dte_1 + \frac{\partial}{w_2}\int_{-\infty}^{\infty} e^{iw_2 t} f(t)dte_2$$

When the necessary operations are done, we obtain the following equality:

$$\frac{d}{dw}\hat{f}(w) = i\mathfrak{F}tf(t).$$

Thus, $\mathfrak{F}\{tf(t)\} = -i\frac{d}{dw}\hat{f}(w)$. Similarly, let's see for $n = 2$.

$$\frac{d^2}{dw^2}\hat{f}(w) = \frac{d}{dw}[\frac{d}{dw}\hat{f}(w)] = i\frac{d}{dw}[\int_{-\infty}^{\infty} e^{iwt} f(t)dt]$$

$$= i\frac{\partial}{w_1}\int_{-\infty}^{\infty} e^{iw_1 t} tf(t)dte_1 + i\frac{\partial}{w_2}\int_{-\infty}^{\infty} e^{iw_2 t} tf(t)dte_2$$

When the necessary operations are done, we obtain the following equality:

$$\frac{d^2}{dw^2}\hat{f}(w) = -\mathfrak{F}t^2 f(t).$$

Thus, $\mathfrak{F}\{t^2 f(t)\} = (-i)^2\frac{d^2}{dw^2}$. If this is continued, the following result will be obtained:

$$\mathfrak{F}\{t^n f(t)\} = i^n \frac{d^n}{dw^n}\hat{f}(w).$$

$\square$

Theorem 4. *[1] If $f(t)$ and $f^{(r)}(t)$, $r = 1, 2, ..., n$, are piecewise smooth and tend to 0 as $|t| \to \infty$ and f with its derivatives of order up to n are integrable in $-\infty < t < \infty$, then*

$$\mathfrak{F}\{f^n(t)\} = (-iw)^n \hat{f}(w), \tag{13}$$

where $\hat{f}(w)$ is the Fourier transform of the function $f(t)$ and $f^r(t) = \frac{d^r}{dt^r} f(t)$.

Example. If $f(t) = \begin{cases} e^{-2t}, & t > 0 \\ 0, & x \leq 0 \end{cases}$, where $\alpha = 2$, β any positive number,

then we get $\hat{f}(w) = \frac{1}{2-iw}$. Actually,

$$\hat{f}(w) = \int_0^\infty e^{-2t} e^{iwt} dt = \int_0^\infty e^{-t(2-iw)} dt = [\frac{e^{-t(2-iw)}}{2-iw}]_0^\infty = \frac{1}{2-iw}.$$

Its the convergent region is

$$\Omega = \{w = a_0 + ia_1 + ja_2 + ija_3 \in \mathbb{BC} : 0 \leq |a_2| < \frac{2+\beta}{2},\ a_1 > -2\}$$

where $\beta \in \mathbb{Z}^+$.

Example. If $f(x) = e^{-a|x|}$ with $a \in \mathbb{R}$ and $w = w_1 + jw_2 \in \mathbb{BC}$, then we find the Fourier transform of the function $f(x)$. Where w_1 is $F_n + iF_{n+1}$, w_2 is $F_{n+2} + iF_{n+3}$ and F_n is nth Fibonacci number.
The Fourier transform of $\hat{f}(w)$ could be written as follows with the help of idempotent bases:

$$\hat{f}(w) = \hat{f}_1(w_1)e_1 + \hat{f}_2(w_2)e_2.$$

Let's first find the $\hat{f}_1(w_1)$ Fourier transform.

$$\hat{f}_1(w_1) = \int_{-\infty}^{\infty} e^{iw_1x} e^{-a|x|} dx$$

$$= \int_{-\infty}^{0} e^{iw_1x} e^{ax} dx + \int_0^\infty e^{iw_1x} e^{-ax} dx$$

$$= \int_{-\infty}^{0} e^{(iw_1+a)x} dx + \int_0^\infty e^{(iw_1-a)x} dx.$$

If we write $F_n + iF_{n+1}$ instead of w_1, we get the following equality:

$$\hat{f}_1(w_1) = \int_{-\infty}^{0} e^{(iF_n - F_{n+1} + a)x} dx + \int_0^\infty e^{(iF_n - F_{n+1} - a)x} dx.$$

When the necessary operations are done, we get the following equality:

$$\hat{f}_1(w_1) = \frac{2a}{a^2 + F_n^2 - F_{n+1}^2 + 2iF_nF_{n+1}}.$$

Now, let's find the $\hat{f}_2(w_2)$ Fourier transform.

$$\hat{f}_2(w_2) = \int_{-\infty}^{\infty} e^{iw_2x} e^{-a|x|} dx$$

$$= \int_{-\infty}^{0} e^{iw_2x} e^{ax} dx + \int_0^\infty e^{iw_2x} e^{-ax} dx$$

$$= \int_{-\infty}^{0} e^{(iw_2+a)x} dx + \int_0^\infty e^{(iw_2-a)x} dx.$$

If we write $F_{n+2} + iF_{n+3}$ instead of w_2, we get the following equality:

$$\hat{f}_2(w_2) = \int_{-\infty}^{0} e^{(iF_{n+2} - F_{n+3} + a)x} dx + \int_0^\infty e^{(iF_{n+2} - F_{n+3} - a)x} dx.$$

When the necessary operations are done, we get the following equality:

$$\hat{f}_2(w_2) = \frac{2a}{a^2 + F_{n+2}^2 - F_{n+3}^2 + 2iF_{n+2}F_{n+3}}.$$

Thus,

$$\hat{f}(w) = 2a(\frac{1}{a^2 + F_n^2 - F_{n+1}^2 + 2iF_nF_{n+1}} e_1 + \frac{1}{a^2 + F_{n+2}^2 - F_{n+3}^2 + 2iF_{n+2}F_{n+3}} e_2).$$

3 Bicomplex Hartley Transform

Let us briefly mention about the Hartley transformation. The Hartley transform is an integral transformation that maps to a real valued frequency function through the kernel $casvx = cosvx + sinvx$. This new version of the Fourier transform was discovered by Ralph Vinton Lyon Hartley in 1942. Sinusoids are waves that acting between certain frequencies and certain amplitudes that repeat. The Fourier transform involves the complex sum of real and imaginary numbers and sinusoidal functions. The Hartley transform includes the real sum of real numbers and sinusoidal functions. Both transformations were effective in solving fluctuating events.Hartley transform is a transform obtained by taking the function cas instead of the exponential kernel in the Fourier transform. The Hartley transformation is as

$$H(f) = \int_{-\infty}^{\infty} caswtV(t)dt = \int_{-\infty}^{\infty} cas2\pi ftV(t)dt. \tag{14}$$

Where t is time, $V(t)$ is a absolute integrable function in the $(-\infty, \infty)$ depends on t, f is frequency, w is angular frequency and $cast = cost + sint$. This integral transform is called the Hartley transform of the $V(t)$ function. Additionally, $w = 2\pi f$. $V(t)$ function is found as

$$V(t) = \int_{-\infty}^{\infty} caswtH(f)dt = \int_{-\infty}^{\infty} cas2\pi ftH(f)dt. \tag{15}$$

Now, let's mention about the bicomplex Hartley transformation. So, let us first give the complex Hartley transform.
Let's define the complex Hartley transform $\mathbb{H} : \mathbb{D} \subset \mathbb{R} \to \mathbb{C}$. We can define the complex Hartley transformation of $V(t)$ satisfying the condition $|H_1(w_1)| < \infty$, where w_1 is a complex frequency as

$$H_1(w_1) = \int_{-\infty}^{\infty} cas2\pi w_1 tV(t)dt.$$

The transform $H_1(w_1)$ is holomorphic in the region

$$\Omega_1 = w_1 \in \mathbb{C} : -\infty < Re(w_1) < \infty.$$

Similarly, the complex Hartley transform of $V(t)$ associated with the other complex frequency w_2 is as follows:

$$H_2(w_2) = \int_{-\infty}^{\infty} cas2\pi w_2 tV(t)dt.$$

The transform $H_2(w_2)$ is holomorphic in the region

$$\Omega_2 = w_2 \in \mathbb{C} : -\infty < Re(w_2) < \infty.$$

The complex functions $H_1(w_1)$ and $H_2(w_2)$ are written with the help of idempotent bases e_1 and e_2 as follows:

$$H_1(w_1)e_1 + H_2(w_2)e_2 = \int_{-\infty}^{\infty} cas2\pi w_1 tV(t)dte_1 + \int_{-\infty}^{\infty} cas2\pi w_2 tV(t)dte_2$$

$$= \int_{-\infty}^{\infty} (cos2\pi w_1 t + sin2\pi w_1 t)V(t)e_1 + \int_{-\infty}^{\infty} (cos2\pi w_2 t + sin2\pi w_2 t)V(t)e_2$$

$$= \int_{-\infty}^{\infty} cas2\pi wtV(t)dt = H(w).$$

Since $H_1(w_1)$ and $H_2(w_2)$ are complex holomorphic functions in the regions Ω_1 and Ω_2, respectively, the $H(w)$ bicomplex function is holomorphic in the region

$$\Omega = w \in \mathbb{BC} : w = w_1e_1 + w_2e_2, w_1 \in \Omega_1 \ and \ w_2 \in \Omega_2. \tag{16}$$

The complex valued $H_1(w_1)$ and $H_2(w_2)$ holomorphic functions are convergent in the regions Ω_1 and Ω_2, respectively. Then, the convergence region of $H(w)$ is region Ω.
Let $w_1 = x_1 + ix_2,\ w_2 = y_1 + iy_2,\ \ x_1, x_2, y_1, y_2 \in \mathbb{R}$ be with $w_1, w_2 \in \mathbb{C}$.

Where, $-\infty < x_1, y_1 < \infty$ for $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$. Using the $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ equations, w is transformed into a 4-component form.

$$w = \frac{x_1 + y_1}{2} + i\frac{x_2 + y_2}{2} + j\frac{y_2 - x_2}{2} + ij\frac{x_1 - y_1}{2} = a_0 + ia_1 + ja_2 + ija_3.$$

Where, $a_0, a_1, a_2, a_3 \in \mathbb{R}$. There are three possible cases:

1. If $x_1 = y_1$, then

$$a_3 = 0.$$

2. If $x_1 > y_1$, then

$$\frac{x_1 - y_1}{2} > 0.$$

That is, $a_3 > 0$.
3. If $x_1 < y_1$,then

$$\frac{x_1 - y_1}{2} < 0.$$

That is, $a_3 < 0$.

Thus, from these three cases we get the following inequalities:

$$-\infty < a_0 < \infty, \ -\infty < a_3 < \infty. \tag{17}$$

So, the region of convergence of $H(w)$ is as follows:

$$\Omega = \{w \in \mathbb{BC} : -\infty < a_0 < \infty, -\infty < a_3 < \infty\}. \tag{18}$$

Let $V(t)$ be a real valued continuous function in the $(-\infty, \infty)$. The Hartley transformation of $V(t)$ is as follows:

$$H(w) = \int_{-\infty}^{\infty} cas 2\pi wt V(t) dt, \quad w \in \mathbb{BC}. \tag{19}$$

The Hartley transform $H(w)$ exists and holomorphic for all $w \in \Omega$, where Ω is the convergence region of $H(w)$. We give the following theorems without proof.

Theorem 5. *If the function $V(t)$ is a continuous and real-valued function for $-\infty < t < \infty$, Then the Hartley transform $H(w)$ exists in the region (18).*

Theorem 6. *Let Hartley transformations of $V_1(t)$ and $V_2(t)$ functions be $H_1(w)$ and $H_2(w)$, respectively. If $H_1(w) = H_2(w)$, then $V_1(t) = V_2(t)$.*

4 Conclusion

In future studies, different properties of these special integral transforms, which we have defined and examined here, can be examined.

5 References

1 A. Banerjee, S. K. Datta, M. A. Hoque, *Fourier transform for functions of bicomplex variables*, arXiv preprint arXiv: 1404.4236, (2014).
2 M. Futagawa, *On the theory of functions of a quaternary variable*, Tohoku Mathematical Journal, First Series, **29** (1928), 175-222.
3 M. Futagawa, *On the theory of functions of a quaternary variable (Part II)*, Tohoku Mathematical Journal, First Series, **35**, (1932), 69-120.
4 K. Koklu, *İntegral Dönüşüm ve Uygulamaları*, Papatya Yayıncılık, İstanbul, 2018.
5 M. E. Luna-Elizarraras, M. Shapiro, D. C. Struppa, A. Vajiac, *Bicomplex numbers and their elementary functions.* Cubo (Temuco), **14**(2) (2012), 61-80.
6 G. B. Price, *An introduction to multicomplex spaces and functions.*, M. Dekker, New York, 1991.
7 J. D. Riley, *Contributions to the theory of functions of a bicomplex variable.* Tohoku Mathematical Journal, Second Series, **5**(2) (1953), 132-165.
8 D. Rochon, M. Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers*, Anal. Univ. Oradea, fasc. math, **11**(71), (2004), 110.
9 S. Rönn, *Bicomplex algebra and function theory*, (2001). arXiv preprint math/0101200.
10 C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti peralgebrici.* Mathematische Annalen, **40**(3), (1892), 413-467.
11 P. Usta Puhl, *Hartley dönüşümleri ve uygulamaları*, MsC Thesis, Yildiz Technical Uni, 2016.

Conference Proceeding Science and Technology, 3(2), 2020, 222–226

Conference Proceeding of 9th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2020).

Global Existence and General Decay of Solutions for Quasilinear System with Degenerate Damping Terms

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Erhan Pişkin[1] *Fatma Ekinci*[2,*]

[1] *Dicle University, Department of Mathematics, Diyarbakır, Turkey, ORCID: 0000-0001-6587-4479*
[2] *Dicle University, Department of Mathematics, Diyarbakır, Turkey, ORCID: 0000-0002-9409-3054*
* *Corresponding Author E-mail: episkin@dicle.edu.tr, ekincifatma2017@gmail.com*

Abstract: In this work, we investigate a quasilinear system of two viscoelastic equations with degenarete damping, dispersion and source terms under Dirichlet boundary conditions. Under suitable conditions on the relaxation function h_i $(i = 1, 2)$ and initial data, we establish global existence and general decay results. This work generalizes and improves earlier results in the literature.

Keywords: General decay, Viscoelastic equations, Degenerate damping, Quasilinear equations.

1 Introduction

In this work, we considere the following quasilinear system of two viscoelastic equations with degenerate damping, dispersion and source terms:

$$\begin{cases} |u_t|^{\eta} u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + \left(|u|^k + |v|^l\right) |u_t|^{j-1} u_t = f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ |v_t|^{\eta} v_{tt} - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + \left(|v|^{\theta} + |u|^{\varrho}\right) |v_t|^{s-1} v_t = f_2(u,v), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with a sufficiently smooth boundary in R^n $(n \geq 1)$, $j, s \geq 1$, $\eta > 0$, $k, l, \theta, \varrho \geq 0$; $h_i(.) : R^+ \to R^+$ $(i = 1, 2)$ are positive relaxation functions which will be specified later. $\left(|(.)|^a + |(.)|^b\right) |(.)_t|^{\tau-1} (.)_t$ and $-\Delta (.)_{tt}$ are the degenerate damping term and the dispersion term, respectively.

By taking

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\kappa+1)}(u+v) + b|u|^{\kappa} u |v|^{\kappa+2}, \\ f_2(u,v) &= a|u+v|^{2(\kappa+1)}(u+v) + b|v|^{\kappa} v |u|^{\kappa+2}, \end{aligned}$$

in which $a > 0$, $b > 0$, and

$$1 < \kappa < +\infty \quad \text{if } n = 1, 2 \text{ and } 1 < \kappa \leq \frac{3-n}{n-2} \text{ if } n \geq 3. \tag{2}$$

It is easy to show that

$$uf_1(u,v) + vf_2(u,v) = 2(\kappa+2) F(u,v),\ \forall (u,v) \in R^2, \tag{3}$$

where

$$F(u,v) = \frac{1}{2(\kappa+2)} \left[a|u+v|^{2(\kappa+2)} + 2b|uv|^{\kappa+2} \right]. \tag{4}$$

To motivate our problem (1) can trace back to the initial boundary value problem for the single viscoelastic equation of the form

$$|u_t|^{\eta} u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds - \Delta u_{tt} + |u_t|^{j-2} u_t = |u|^{p-2} u \tag{5}$$

which was studied by Wu [1]. The author established a general uniform decay result under some appropriate assumptions on the relaxation function h and the initial data. Then in [2], the author investigated same problem and obtained general decay result for $j = 2$.

For a coupled system, He [3] looked into the following problem

$$\begin{cases} |u_t|^{\eta} u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + |u_t|^{j-2} u_t = f_1(u,v), \\ |v_t|^{\eta} v_{tt} - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + |v_t|^{s-2} v_t = f_2(u,v), \end{cases} \tag{6}$$

where $\eta > 0$, $j, s \geq 2$. The author studied general decay results and a blow-up result. Then, in [4], the author investigated same problem without damping term and established a general decay result of solutions.

The rest paper is arranged as follows: In Section 2, as preliminaries, we give necessary assumptions and lemmas that will be used later. In section 3, we prove the global existence of solution. In last section, we studied the general decay of solutions.

2 Preliminaries

In this section, we will present some assumptions, notations, and lemmas that will be used later for our main results. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$ norm $\|.\|_p = \|.\|_{L^p(\Omega)}$.

To state and prove our result, we need some assumptions:

$(\mathbf{A1})$ Regarding $h_i : [0,\infty) \to (0,\infty)$, $(i = 1,2)$ are C^1 functions and satisfy

$$h_i(\alpha) > 0, \;\; h_i'(\alpha) \leq 0, \;\;\; 1 - \int_0^\infty h_i(\alpha)d\alpha = l_i > 0, \; \alpha \geq 0$$

and non-increasing differentiable positive C^1 functions ς_1 and ς_2 such that

$$h_i'(t) \leq -\varsigma_i(t)h_i^{\rho_i}(t), \; t \geq 0, \; 1 \leq \rho_i < \frac{3}{2} \text{ for } i = 1,2.$$

$(\mathbf{A2})$ For the nonlinearity, we assume that

$$\begin{cases} 1 \leq j, s \;\text{ if } n = 1,2, \\ 1 \leq j, s \leq \frac{n+2}{n-2} \;\text{ if } n \geq 3. \end{cases}$$

$(\mathbf{A3})$ Assume that η satisfies

$$\begin{cases} 0 < \eta \;\text{ if } n = 1,2, \\ 0 < \eta \leq \frac{2}{n-2} \text{ if } n \geq 3. \end{cases}$$

In addition, we present some notations:

$$(h_i^s \diamond \nabla w)(t) = \int_0^t h_i^s(t-s) \|\nabla w(t) - \nabla w(s)\|^2 ds,$$

$$l = \min\{l_1, l_2\}.$$

Remark 1. *(A1) is need to guarantee the hyperbolicity of the system* (1)*. Conditions $\rho_i < \frac{3}{2}$, $(i = 1,2)$ are imposed so that $\int_0^\infty h_i(s)\,ds < \infty$, $(i = 1,2)$.*

Lemma 2. *(Sobolev-Poincare inequality) [7]. Let q be a number with $2 \leq q < \infty$ $(n = 1,2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$, then there is a constant $C_* = C_*(\Omega, q)$ such that*

$$\|u\|_q \leq C_* \|\nabla u\| \textit{ for } u \in H_0^1(\Omega).$$

Lemma 3. *[8] Suppose that* (4) *holds. Then there exist $\rho > 0$ such that for the solution (u,v)*

$$\|u+v\|_{2(\kappa+2)}^{2(\kappa+2)} + 2\|uv\|_{\kappa+2}^{\kappa+2} \leq \rho(l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2)^{\kappa+2}. \tag{7}$$

Now, we state the local existence theorem that can be established by combining arguments of [1]-[6].

Theorem 4. *Assume that (A1), (A2), (A3) and (2) hold. Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ are given. Then, for some $T_m > 0$, problem* (1) *has a weak solution in the following class:*

$$u, v \in C\left([0,T_m); H_0^1(\Omega)\right),$$

$$u_t, v_t \in C\left([0,T_m); L^2(\Omega)\right).$$

We define the energy function as follows

$$
\begin{aligned}
E(t) &= \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}\right)+\frac{1}{2}\left[(h_1\diamond\nabla u)(t)+(h_2\diamond\nabla v)(t)+\|\nabla u_t\|^2+\|\nabla v_t\|^2\right] \\
&+\frac{1}{2}\left[(1-\int_0^t h_1(s)ds)\,\|\nabla u(t)\|^2+(1-\int_0^t h_2(s)ds)\,\|\nabla v(t)\|^2\right]-\int_\Omega F(u,v)\,dx. \qquad (8)
\end{aligned}
$$

Also, we define

$$
\begin{aligned}
I(t) &= \|\nabla u_t\|^2+\|\nabla v_t\|^2+(1-\int_0^t h_1(s)ds)\,\|\nabla u(t)\|^2+(1-\int_0^t h_2(s)ds)\,\|\nabla v(t)\|^2 \\
&+(h_1\diamond\nabla u)(t)+(h_2\diamond\nabla v)(t)-2(\kappa+2)\int_\Omega F(u,v)\,dx \qquad (9)
\end{aligned}
$$

and

$$
\begin{aligned}
J(t) &= \frac{1}{2}\left[(1-\int_0^t h_1(s)ds)\,\|\nabla u(t)\|^2+(1-\int_0^t h_2(s)ds)\,\|\nabla v(t)\|^2\right] \\
&+\frac{1}{2}\left[(h_1\diamond\nabla u)(t)+(h_2\diamond\nabla v)(t)+\frac{1}{2}(\|\nabla u_t\|^2+\|\nabla v_t\|^2)\right] \\
&-\int_\Omega F(u,v)\,dx. \qquad (10)
\end{aligned}
$$

By computation, we get

$$
\begin{aligned}
\frac{d}{dt}E(t) &\leq \frac{1}{2}\left[(h_1'\diamond\nabla u)(t)+(h_2'\diamond\nabla v)(t)\right] \\
&-\frac{1}{2}\left(h_1(t)\,\|\nabla u\|^2+h_2(t)\,\|\nabla v\|^2\right) \\
&-\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}\,dx-\int_\Omega\left(|v|^\theta+|u|^\varrho\right)|v_t|^{s+1}\,dx \\
&\leq 0. \qquad (11)
\end{aligned}
$$

3 Global Existence

This section is devoted to prove the global existence of solution (1).

Lemma 5. *[5]. Let $(u_0,v_0)\in H_0^1(\Omega)$, $(u_1,v_1)\in L^2(\Omega)$. Suppose that $(A1)-(A3)$ hold. If*

$$
I(0)>0 \text{ and } \beta=\rho\left(\frac{2(\kappa+2)}{\kappa+1}E(0)\right)^{\kappa+1}<1, \qquad (12)
$$

then

$$
I(t)>0,\ \forall t>0.
$$

Theorem 6. *Suppose that the conditions of Lemma 5 hold, then the solution (1) is bounded and global in time.*

Proof: It suffices to show that

$$
\|(u,v)\|_H := \|\nabla u(t)\|^2+\|\nabla v(t)\|^2+\|\nabla u_t\|^2+\|\nabla v_t\|^2
$$

is bounded independently of t. For this pupose, we apply (8), (10) and (11) to get

$$
\begin{aligned}
E(0) &\geq E(t)=J(t)+\frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}\right) \\
&\geq \frac{\kappa+1}{2(\kappa+2)}\left(l_1\,\|\nabla u(t)\|^2+l_2\,\|\nabla v(t)\|^2+\|\nabla u_t\|^2+\|\nabla v_t\|^2\right. \\
&\left.+(h_1\diamond\nabla u)(t)+(h_2\diamond\nabla v)(t)\right) \\
&+\frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}\right). \qquad (13)
\end{aligned}
$$

Thus,

$$
\|(u,v)\|_H \leq CE(0),
$$

where positive constant C, which depends only on κ, l_1, l_2. □

4 General Decay of Solutions

This section is devoted to prove the decay of solution (1). Set

$$\Gamma(t) := ME(t) + \varepsilon\Phi(t) + \digamma(t), \tag{14}$$

where M and ε are some positive constants to be specified later and

$$\begin{aligned} \Phi(t) \quad = \quad & \delta_1(t)\left[\frac{1}{\eta+1}\int_\Omega |u_t|^\eta u_t u dx + \int_\Omega \nabla u_t \nabla u dx\right] \\ & + \delta_2(t)\left[\frac{1}{\eta+1}\int_\Omega |v_t|^\eta v_t v dx + \int_\Omega \nabla v_t \nabla v dx\right], \end{aligned} \tag{15}$$

$$\begin{aligned} \digamma(t) \quad = \quad & \delta_1(t)\left[\int_\Omega \left(\Delta u_t - \frac{|u_t|^\eta u_t}{\eta+1}\right)\int_0^t h_1(t-s)(u(t)-u(s))dsdx\right] \\ & + \delta_2(t)\left[\int_\Omega \left(\Delta v_t - \frac{|v_t|^\eta v_t}{\eta+1}\right)\int_0^t h_2(t-s)(v(t)-v(s))dsdx\right]. \end{aligned} \tag{16}$$

Lemma 7. *For ε small enough while M large enough, the relation*

$$\alpha_1\Gamma(t) \le E(t) \le \alpha_2\Gamma(t), \quad \forall t \ge 0. \tag{17}$$

holds for two positive constants α_1 and α_2.

Proof: As references [9]-[5], it is easy to see that $\Gamma(t)$ and $E(t)$ are equivalent in the sense that α_1 and α_2 are positive constants, depending on ε and M. □

Lemma 8. *[1] Assume that (12) holds. Let (u, v) be the solution of problem (1). Then, for $\sigma \ge 0$, we get*

$$\begin{cases} \int_\Omega \left(\int_0^t h_1(t-s)(u(t)-u(s))ds\right)^{\sigma+2} dx \le (1-l_1)^{\sigma+1} c_*^{\sigma+2}\left(\frac{2(\kappa+2)E(0)}{l_1(\kappa+1)}\right)^{\frac{\sigma}{2}}(h_1 \diamond \nabla u)(t) \\ \int_\Omega \left(\int_0^t h_2(t-s)(v(t)-v(s))ds\right)^{\sigma+2} dx \le (1-l_2)^{\sigma+1} c_*^{\sigma+2}\left(\frac{2(\kappa+2)E(0)}{l_2(\kappa+1)}\right)^{\frac{\sigma}{2}}(h_2 \diamond \nabla v)(t) \end{cases}. \tag{18}$$

Lemma 9. *Let $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$ be given and satisfying (12). Assume that $(A1)-(A3)$ hold. Then, for any t_0, the functional $\Gamma(t)$ verifies, along solution of (1),*

$$\Gamma'(t) \le -\xi_1 E(t) + \xi_2\left[(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t)\right] \tag{19}$$

for some $\xi_i > 0,\ (i = 1, 2)$.

Proof: As references [9]-[5]-[1], it is easy to obtain desired result. We omit it. □

Now, we are ready to state our stability result.

Theorem 10. *Assume that (4), $(A1)-(A3)$ hold and that $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and satisfy $E(0) < E_1$ and*

$$\left(l_1\|\nabla u_0\|^2 + l_2\|\nabla v_0\|^2\right)^{\frac{1}{2}} < \alpha_*. \tag{20}$$

Then for each , there exist two positive constants K and k such that the energy of (1) satisfies

$$E(t) \le Ke^{-k\int_{t_0}^t \delta(s)ds}, \quad t \ge t_0 \tag{21}$$

where $\delta(t) := \min\{\delta_1(t), \delta_2(t)\}$.

Proof: Multiplying (19) by $\delta(t)$, we have

$$\delta(t)\Gamma'(t) \leq -\xi_1\delta(t)E(t) + \xi_2\delta(t)\left[(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t)\right].$$

Since $(A2)$ and $\delta(t) := \min\{\delta_1(t), \delta_2(t)\}$ and using the fact that $-\left[(h_1' \diamond \nabla u)(t) + (h_2' \diamond \nabla v)(t)\right] \leq -2E'(t)$ by (11), we get

$$\begin{aligned} \delta(t)\Gamma'(t) &\leq -\xi_1\delta(t)E(t) - \xi_2\delta(t)\left[(h_1' \diamond \nabla u)(t) + (h_2' \diamond \nabla v)(t)\right] \\ &\leq -\xi_1\delta(t)E(t) - 2\xi_2 E'(t),\ \forall t \geq t_0. \end{aligned} \tag{22}$$

That is

$$G'(t) \leq -c_*\delta(t)E(t) \leq -k\delta(t)G(t),\ \forall t \geq t_0, \tag{23}$$

where $G(t) = \delta(t)\Gamma(t) + CE(t)$ is equivalent to $E(t)$ due to (17) and k is a positive constant. A simple integration of (23) leads to

$$G(t) \leq G(t_0)e^{-k\int_{t_0}^{t}\delta(s)ds},\ \forall t \geq t_0 \tag{24}$$

This completes the proof. □

5 Conclusion

As far as we know, there is not any global existence and general decay results in the literature known for quasilinear viscoelastic equations with degenerate damping terms. Our work extends the works for some quasilinear viscoelastic equations treated in the literature to the quasilinear viscoelastic equation with degenerate damping terms.

6 References

1 ST. Wu, *General decay of solutions for a viscoelastic equation with nonlinear damping and source terms*, Acta Math Sci., (318), (2011), 1436-1448.
2 ST. Wu, *General decay of energy for a viscoelastic equation with damping and source terms*, Taiwan J Math., **16** (1), (2012), 113-128.
3 L. He, *On decay and blow-up of solutions for a system of equations*, Appl Anal., (2019), 1-30. Doi: 10.1080/00036811.2019.1689562
4 L. He, *On decay of solutions for a system of coupled viscoelastic equations*, Acta Appl Math. (167), (2020), 171-198.
5 E. Pişkin, F. Ekinci, *General decay and blow-up of solutions for coupled viscoelastic equation of Kirchhoff type with degenerate damping terms*, Math Meth Appl Sci., **42** (16), (2019), 5468-5488.
6 ST. Wu, *General decay of solutions for a nonlinear system of viscoelastic wave equations with degenerate damping and source terms*, J Math Anal Appl., (406), (2013), 34-48.
7 Adams RA, Fournier JJF. *Sobolev Spaces*. Academic Press, New York, 2003.
8 B. Said-Houari, SA. Messaoudi, A. Guesmia, *General decay of solutions of a nonlinear system of viscoelastic wave equations*, NoDEA Nonlinear Differential Equations Appl., (2011), (18), 659-684.
9 W. Liu, *General decay and blow-up of solution for a quasilinear viscoelastic problem with nonlinear source*, Nonlinear Anal. (73), (2010), 1890-1904.

Conference Proceeding Science and Technology, 3(2), 2020, 227–235

Conference Proceeding of 9th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2020).

Generalized Spherical Fuzzy Einstein Aggregation Operators: Application to Multi-Criteria Group Decision-Making Problems

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Elif Güner[1,*] *Halis Aygün*[2]

[1] *Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli, Turkey, ORCID: 0000-0002-6969-400X*
* *Corresponding Author E-mail: elif.guner@kocaeli.edu.tr*
[2] *Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli, Turkey, ORCID: 0000-0003-3263-3884*

Abstract: The aim of this paper is to present the extension of a concept related to aggregation operators from spherical fuzzy sets to generalized spherical fuzzy sets. We first introduce Einstein sum, product and scalar multiplication for generalized spherical fuzzy sets based on Einstein triangular norm and triangular conorm. Then we give the generalized spherical fuzzy Einstein weighted averaging and generalized spherical fuzzy Einstein weighted geometric operators, namely generalized spherical fuzzy Einstein aggregation operators, constructed on these operations. After investigating some fundamental properties of these operators, we develop a model for generalized spherical fuzzy Einstein aggregation operators to solve the multiple attribute group decision-making problems. Finally, we give a numerical example to demonstrate that the developed method is suitable and effective for the decision process.

Keywords: Generalized spherical fuzzy number, Einstein aggregation operators, Multi-criteria group decision making.

1 Introduction

In 1965, Zadeh [32] introduced the theory of fuzzy set (FS) in which is discussed the degree of membership (positive-membership) of an element to a set. FS theory has a wide range of applications in numerous fields such as artificial intelligence, engineering, economics, computer science and etc. [7]-[32]-[33]-[34]. Also, the study of multi-criteria decision-making was started in fuzzy environment by Bellman and Zadeh [7] in 1970. While the studies and developments in the field of FS theory were continued, Atanassov [6] observed that there are some deficiencies in this theory and defined the concept of intuitionistic fuzzy set (IFS) as a generalization of FS. Since each element is expressed by a positive-membership degree and a negative-membership degree in the IFS theory, this theory is a more powerful tool to deal with vagueness than the FS theory. Then basing on the well-known weighted averaging (WA) operator [16] and the ordered weighted averaging (OWA) operator [30], Xu [29] investigated some aggregation operators in the intuitionistic fuzzy environment and studied their applications to the multi-criteria decision-making process. Another idea which is an extension of the FS theory is the soft set (SS) theory introduced by Molodtsov [22] to deal with the vagueness and uncertainties of many problems that arise in engineering, social science, medical science, economics and etc. A first practical application of soft sets to the decision-making problems was given by Maji et al. [20]-[21]. They have also initiated the concept of a fuzzy soft set (FSS) which is a combination of FS and SS and obtained some of its properties. In literature, there are many significant applications of the SS and FSS theories. (see [5]-[9]-[24]-[25]).

After, Yager [31] extended the IFS theory to Pythagorean fuzzy set (PyFS) theory by considering the condition that the sum of the squares of its positive-membership and negative-membership degrees is less than or equal to 1. IFS theory and PyFS theory have been successfully used in many distinct areas, but it is not enough to use these theories when we face human opinions involving more answers of types such as yes, no, abstain, refusal. Voting in a democratic election is a good example of such a case since the voters may be divided into four groups of those who: vote for, vote against, abstain, refusal of the voting. So, Cuong [8] proposed the notion of picture fuzzy set (PFS) which is an extension of IFS. PFS gives the positive-membership degree, the neutral-membership degree and the negative-membership degree of an element to a set. The PFS theory resolved the voting problem successfully and was applied to decision-making problems by many authors in different ways [27]-[28]-[3]-[4]-[10]. But there are some situations that PFS theory may not be handled in some uncertain and unstable data. For example, if a person said their opinions about the situation in terms of yes is 0.7, abstained is 0.3, and no is 0.5, then we obtain that $0.7 + 0.5 + 0.3 \nleq 1$. So PFS theory is not able to handle under such types of cases. To use in these types of situations, Mahmood et al. [19] initiated the concept of spherical fuzzy set (SFS) and T-spherical fuzzy set (T-SFS) as an extension of FS, IFS and PFS. While in SFS theory the sum of the square of the three membership degrees is less than or equal to 1, in T-SFS theory the nth power of the three membership degrees is less than or equal to 1. Also, Mahmood et al. [19] defined some fundamental operations of SFSs and T-SFSs along with spherical fuzzy relations and presented medical diagnostics and decision-making problems in the SFSs and T-SFSs environments as practical applications. Then, Ashraf and Abdullah [1] extended different strict Archimedean triangular norms and triangular conorms to aggregate the spherical fuzzy information and also defined some spherical aggregation operators and applied these operators to multi-criteria group decisionâĂŘmaking problems. Different types of aggregation operators for SFSs can be found in [2]-[19]-[23]. Later, Jin et al. [17] proposed a new method to solve the spherical fuzzy multi-criteria group decision-making problems by investigating logarithmic operations of spherical fuzzy sets. Also, GÃijndogdu and Kahraman [12]-[13]-[14] extended the TOPSIS, VIKOR and WASPAS methods to the spherical fuzzy environment. Recently, Haque et al. [15] presented the notion of generalized spherical fuzzy set (GSFS) as an expansion of the SFS in which the sum of the square of the three membership degrees is less than or equal to 3. They established a new exponential operational law for GSFS and investigated its various algebraic properties. They also developed a multi-criteria group decision-making method in the generalized spherical fuzzy environment by using the established exponential operational law. Peng et al. [26] introduced the Pythagorean fuzzy soft set (PyFSS) along with various binary operations and also proposed an algorithm for decision making. Then, Cuong[8] proposed the notion of picture fuzzy soft set (PFSS) as a combination of PFS and SS and also discussed various properties and operations in the theory of PFSS. Guleria and Bajaj [11] extend the concept of PFSS by proposing the T-spherical fuzzy soft set (T-SFSS) along with various aggregation operators and applications.

The main purpose of this paper is to establish the generalized spherical fuzzy Einstein aggregation operators and develop a model for generalized spherical fuzzy Einstein aggregation operators to solve the multiple attribute group decision-making problems. This paper is contained in the following sections: In Section 2, we recollect some basic notions and relevant concepts that are used in the main section. In Section 3, we introduce Einstein sum, product and scalar multiplication for GSFSs based on Einstein triangular norm and triangular conorm. Then, we give the generalized spherical fuzzy Einstein weighted averaging (GSEWA) and generalized spherical fuzzy Einstein weighted geometric (GSEWG) operators, namely generalized spherical fuzzy Einstein aggregation operators, constructed on the Einstein sum, product and scalar multiplication for GSFSs. Also, we investigate some fundamental properties of these operators. In section 4, we develop a model to solve the multiple attribute group decision-making problems in generalized spherical fuzzy environment. Then, we give a medical treatment selection problem as an example which demonstrates that the developed method is effective and suitable for the decision-making process. Finally, we give a brief summary in Section 5.

2 Preliminaries

In this section, we recall some fundamental definitions which will be used in the main sections. Throughout this paper U will denote the set of the universe.

Definition 1. *[6]-[31] Let $\mu : U \to [0,1]$ and $\nu : U \to [0,1]$ be two mappings. A set $I = \{< x, \mu(x), \nu(x) > | x \in U\}$ is called*
(i) Intuitionistic fuzzy set (IFS) if the condition $0 \leq \mu(x) + \nu(x) \leq 1$ hold for all $x \in U$.
(ii) Pythagorean fuzzy set (PyFS) if the condition $0 \leq \mu^2(x) + \nu^2(x) \leq 1$ hold for all $x \in U$.
The values $\mu(x), \nu(x) \in [0,1]$ denote the degree of positive-membership and negative-membership of x to I, respectively.
The pair $I =< \mu, \nu >$ where $\mu, \nu \in [0,1]$ and $\mu + \nu \leq 1$ (or $\mu^2 + \nu^2 \leq 1$), is called a intuitionistic fuzzy number (IFN) (or Pythagorean fuzzy number (PyFN)).

Remark 1. *[31] The set of intuitionistic fuzzy numbers is the subset of the set of Pythagorean fuzzy numbers.*

Definition 2. *[1]-[8]-[15] Let $\mu : U \to [0,1]$, $\iota : U \to [0,1]$ and $\nu : U \to [0,1]$ be three mappings. A set $G = \{< x, \mu(x), \iota(x), \nu(x) > | x \in U\}$ is called*
(i) Picture fuzzy set (PFS) if the condition $0 \leq \mu(x) + \iota(x) + \nu(x) \leq 1$ hold for all $x \in U$.
(ii) Spherical fuzzy set (SFS) if the condition $0 \leq \mu^2(x) + \iota^2(x) + \nu^2(x) \leq 1$ hold for all $x \in U$.
(ii) Generalized spherical fuzzy set (GSFS) if the condition $0 \leq \mu^2(x) + \iota^2(x) + \nu^2(x) \leq 3$ hold for all $x \in U$.
The values $\mu(x), \iota(x), \nu(x) \in [0,1]$ denote the degree of positive-membership, neutral-memberhip and negative-membership of x to G, respectively.
The triplet $G =< \mu, \iota, \nu >$ where $\mu, \iota, \nu \in [0,1]$ and $\mu^2 + \iota^2 + \nu^2 \leq 3$ (or $\mu + \iota + \nu \leq 1$ and $\mu^2 + \iota^2 + \nu^2 \leq 1$, resp.), is called a generalized spherical fuzzy number (GSFN) (or Picture fuzzy number (PFN) and Spherical fuzzy number (SFN), resp.).

Remark 2. *[15] (1) The set of spherical fuzzy numbers is the subset of the set of generalized spherical fuzzy numbers and the set of picture fuzzy numbers is the subset of the set of spherical fuzzy numbers.*
(2) In PFN, since the sum of the three membership functions (positive, neutral and negative) is less than or equal to 1, the sum is taken as linearly and it represents a plane in space. But in the case of SFN and GSFN, it is considered the non-linear form of membership functions which represents a sphere in space.

Definition 3. *[15] Let $G =< \mu, \iota, \nu >, G_1 =< \mu_1, \iota_1, \nu_1 >$, $G_2 =< \mu_2, \iota_2, \nu_2 >$ be three GSFNs and $a \geq 0$. Then the operations between generalized spherical fuzzy numbers are defined as follows:*
(i) $G^c =< \nu, \iota, \mu >$,
(ii) $G_1 \leq G_2$ iff $\mu_1 \leq \mu_2, \iota_1 \geq \iota_2$ and $\nu_1 \geq \nu_2$,
(iii) $G_1 = G_2$ iff $G_1 \leq G_2$ and $G_2 \leq G_1$,
(iv) $G_1 + G_2 =< \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \iota_1\iota_2, \nu_1\nu_2 >$,
(v) $aG =< \sqrt{1 - (1 - \mu^2)^a}, \iota^a, \nu^a >$,
(vi) $G^a =< \mu^a, \iota^a, \sqrt{1 - (1 - \nu^2)^a} >$.

Lemma 1. *[15] Let $G_1 =< \mu_1, \iota_1, \nu_1 >$, $G_2 =< \mu_2, \iota_2, \nu_2 >$ be two GSFNs and $a, a_1, a_2 \geq 0$. Then the following properties hold:*
(i) $G_1 + G_2 = G_2 + G_1$,
(ii) $a(G_1 + G_2) = aG_1 + aG_2$,
(iii) $(a_1 + a_2)G_1 = a_1G_1 + a_2G_2$,
(vi) $(G_1^{a_1})^{a_2} = G_1^{a_1a_2}$.

Definition 4. *[15] Let $\mathscr{G}$ be the collection of all GSFNs and $G \in \mathscr{G}$ where $G =< \mu, \iota, \nu >$.*
(i) A score function $SF : \mathscr{G} \to [-1,1]$ is defined as $SF(G) = \frac{3\mu^2 - 2\iota^2 - \nu^2}{3}$.
(ii) An accuracy function $AF : \mathscr{G} \to [0,1]$ is defined as $AF(G) = \frac{1+3\mu^2-\nu^2}{4}$.

Definition 5. *[15] Let $G_1 =< \mu_1, \iota_1, \nu_1 >$ and $G_2 =< \mu_2, \iota_2, \nu_2 >$ be two GSFNs. Then the ranking method (comparison technique) as follows:*
(i) If $SF(G_1) < SF(G_2)$, then $G_1 < G_2$,
(ii) If $SF(G_1) > SF(G_2)$, then $G_1 > G_2$,
(iii) $SF(G_1) = SF(G_2)$, then
(a) $AF(G_1) < AF(G_2)$, then $G_1 < G_2$,
(b) $AF(G_1) > AF(G_2)$, then $G_1 > G_2$,
(c) $AF(G_1) = AF(G_2)$, then $G_1 = G_2$.

3 Generalized Spherical Fuzzy Einstein Aggregation Operators

In this section, we introduce the Einstein sum, product and scalar multiplication for generalized spherical fuzzy sets based on Einstein triangular norm and triangular conorm. Then we define the generalized spherical fuzzy Einstein weighted averaging and generalized spherical fuzzy Einstein weighted geometric operators based on these operations. Also, we investigate some fundamental properties of these operators.

Definition 6. *Let $G =< \mu, \iota, \nu >, G_1 =< \mu_1, \iota_1, \nu_1 >$ and $G_2 =< \mu_2, \iota_2, \nu_2 >$ be three GSFNs and $a \geq 0$. Then the Einstein operations are defined over the GSFNs as follow:*

(i) $G_1 \oplus_E G_2 = \left\langle \sqrt{\frac{\mu_1^2+\mu_2^2}{1+\mu_1^2.\mu_2^2}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2.\nu_2^2}{1+(1-\nu_1^2)(1-\nu_2^2)}} \right\rangle,$

(ii) $G_1 \odot_E G_2 = \left\langle \sqrt{\frac{\mu_1^2.\mu_2^2}{1+(1-\mu_1^2)(1-\mu_2^2)}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2+\nu_2^2}{1+\nu_1^2.\nu_2^2}} \right\rangle,$

(iii) $a \cdot_E G = \left\langle \sqrt{\frac{(1+\mu^2)^a-(1-\mu^2)^a}{(1+\mu^2)^a+(1-\mu^2)^a}}, \sqrt{\frac{2\iota^{2a}}{(2-\iota^2)^a+\iota^{2a}}}, \sqrt{\frac{2\nu^{2a}}{(2-\nu^2)^a+\nu^{2a}}} \right\rangle,$

(iv) $G^{\wedge_E a} = \left\langle \sqrt{\frac{2\mu^{2a}}{(2-\mu^2)^a+\mu^{2a}}}, \sqrt{\frac{2\iota^{2a}}{(2-\iota^2)^a+\iota^{2a}}}, \sqrt{\frac{(1+\nu^2)^a-(1-\nu^2)^a}{(1+\nu^2)^a+(1-\nu^2)^a}} \right\rangle.$

Lemma 2. *Let $G_1 =< \mu_1, \eta_1, \nu_1 >$, $G_2 =< \mu_2, \eta_2, \nu_2 >$ be two GSFNs and $a, a_1, a_2 \geq 0$. Then the following properties hold:*
(i) $G_1 \oplus_E G_2 = G_2 \oplus_E G_1$,
(ii) $a \cdot_E (G_1 \oplus_E G_2) = a \cdot_E G_1 \oplus_E a \cdot_E G_2$,
(iii) $(a_1 + a_2) \cdot_E G_1 = a_1 \cdot_E G_1 \oplus_E a_2 \cdot_E G_2$,
(iv) $G_1 \odot_E G_2 = G_2 \odot_E G_1$,
(v) $(G_1 \odot_E G_2)^{\wedge_E a} = G_1^{\wedge_E a} \odot_E G_2^{\wedge_E a}$,
(vi) $G^{\wedge_E a_1} \odot_E G^{\wedge_E a_2} = G^{\wedge_E a_1 + a_2}$,
(vii) $(G_1^{\wedge_E a_1})^{\wedge_E a_2} = G_1^{\wedge_E a_1 a_2}$.

Proof: Proofs of (i) and (iv) are trivial.

(ii) We can write the equation $G_1 \oplus_E G_2 = \left\langle \sqrt{\frac{\mu_1^2+\mu_2^2}{1+\mu_1^2.\mu_2^2}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2.\nu_2^2}{1+(1-\nu_1^2)(1-\nu_2^2)}} \right\rangle$ in the following way:

$$G_1 \oplus_E G_2 = \left\langle \sqrt{\frac{(1+\mu_1^2)(1+\mu_2^2)-(1-\mu_1^2)(1-\mu_2^2)}{(1+\mu_1^2)(1+\mu_2^2)+(1-\mu_1^2)(1-\mu_2^2)}}, \sqrt{\frac{2.\iota_1^2.\iota_2^2}{(2-\iota_1^2)(2-\iota_2^2)+\iota_1^2.\iota_2^2}}, \sqrt{\frac{2.\nu_1^2.\nu_2^2}{(2-\nu_1^2)(2-\nu_2^2)+\nu_1^2.\nu_2^2}} \right\rangle.$$

If we take $s = (1 + \mu_1^2)(1 + \mu_2^2)$, $t = (1 - \mu_1^2)(1 - \mu_2^2)$, $u = \iota_1^2.\iota_2^2$, $x = (2 - \iota_1^2)(2 - \iota_2^2)$, $y = \nu_1^2.\nu_2^2$ and $z = (2 - \nu_1^2)(2 - \nu_2^2)$, by the Einstein operational law (ii), we have that

$$a \cdot_E (G_1 \oplus_E G_2) = a \cdot_E \left\langle \sqrt{\frac{s-t}{s+t}}, \sqrt{\frac{2u}{x+u}}, \sqrt{\frac{2y}{z+y}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1+\frac{s-t}{s+t})^a - (1-\frac{s-t}{s+t})^a}{(1+\frac{s-t}{s+t})^a + (1-\frac{s-t}{s+t})^a}}, \sqrt{\frac{2(\frac{2u}{x+u})^a}{(2-\frac{2u}{x+u})^a + (\frac{2u}{x+u})^a}}, \sqrt{\frac{2(\frac{2y}{z+y})^a}{(2-\frac{2y}{z+y})^a + (\frac{2y}{z+y})^a}} \right\rangle$$

$$= \left\langle \sqrt{\frac{s^a - t^a}{s^a + t^a}}, \sqrt{\frac{2u^a}{x^a + u^a}}, \sqrt{\frac{2y^a}{z^a + y^a}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1+\mu_1^2)^a(1+\mu_2^2)^a - (1-\mu_1^2)^a(1-\mu_2^2)^a}{(1+\mu_1^2)^a(1+\mu_2^2)^a + (1-\mu_1^2)^a(1-\mu_2^2)^a}}, \sqrt{\frac{2\iota_1^{2a}.\iota_2^{2a}}{(2-\iota_1^2)^a(2-\iota_2^2)^a + \iota_1^{2a}.\iota_2^{2a}}}, \sqrt{\frac{2\nu_1^{2a}.\nu_2^{2a}}{(2-\nu_1^2)^a(2-\nu_2^2)^a + \nu_1^{2a}\nu_2^{2a}}} \right\rangle.$$

Also, if we take $s_1 = (1 + \mu_1^2)^a$, $t_1 = (1 - \mu_1^2)^a$, $u_1 = \iota_1^{2a}$, $x_1 = (2 - \iota_1^2)^a$, $y_1 = \nu_1^{2a}$, $z_1 = (2 - \nu_1^2)^a$, $s_2 = (1 + \mu_2^2)^a$, $t_2 = (1 - \mu_2^2)^a$, $u_2 = \iota_2^{2a}$, $x_2 = (2 - \iota_2^2)^a$, $y_2 = \nu_2^{2a}$ and $z_2 = (2 - \nu_2^2)^a$, then we have that

$a \cdot_E G_1 = \left\langle \sqrt{\frac{s_1-t_1}{s_1+t_1}}, \sqrt{\frac{2u_1}{x_1+u_1}}, \sqrt{\frac{2y_1}{z_1+u_1}} \right\rangle$ and $a \cdot_E G_2 = \left\langle \sqrt{\frac{s_2-t_2}{s_2+t_2}}, \sqrt{\frac{2u_2}{x_2+u_2}}, \sqrt{\frac{2y_2}{z_2+u_2}} \right\rangle.$

By the Einstein operational law (i), we get that

$$a\cdot_E G_1 \oplus_E a\cdot_E G_2 = \left\langle \sqrt{\frac{(\frac{s_1-t_1}{s_1+t_1})+(\frac{s_2-t_2}{s_2+t_2})}{1+(\frac{s_1-t_1}{s_1+t_1})(\frac{s_2-t_2}{s_2+t_2})}}, \sqrt{\frac{(\frac{2u_1}{x_1+u_1})(\frac{2u_2}{x_2+u_2})}{1+(1-\frac{2u_1}{x_1+u_1})(1-\frac{2u_2}{x_2+u_2})}}, \sqrt{\frac{(\frac{2y_1}{z_1+y_1})(\frac{2y_2}{z_2+y_2})}{1+(1-\frac{2y_1}{z_1+y_1})(1-\frac{2y_2}{z_2+y_2})}} \right\rangle$$

$$= \left\langle \sqrt{\frac{s_1s_2-t_1t_2}{s_1s_2+t_1t_2}}, \sqrt{\frac{2u_1u_2}{x_1x_2+u_1u_2}}, \sqrt{\frac{2y_1y_2}{z_1z_2+y_1y_2}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1+\mu_1^2)^a(1+\mu_2^2)^a-(1-\mu_1^2)^a(1-\mu_2^2)^a}{(1+\mu_1^2)^a(1+\mu_2^2)^a+(1-\mu_1^2)^a(1-\mu_2^2)^a}}, \sqrt{\frac{2\iota_1^{2a}\iota_2^{2a}}{(2-\iota_1^2)^a(2-\iota_2^2)^a+\iota_1^{2a}\iota_2^{2a}}}, \sqrt{\frac{2\nu_1^{2a}\nu_2^{2a}}{(2-\nu_1^2)^a(2-\nu_2^2)^a+\nu_1^{2a}\nu_2^{2a}}} \right\rangle$$

Hence, we satisfy that $a\cdot_E(G_1 \oplus_E G_2) = a\cdot_E G_1 \oplus_E a\cdot_E G_2$.

The proofs of (iii), (v), (vi) and (vii) can be completed similar to the proof of (ii). □

Definition 7. *Let $\mathscr{G}$ be a collection of all GSFNs and $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$ where $G_i =< \mu_i, \iota_i, \nu_i >$ for all $i = 1, 2, ..., n$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ be the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$. A mapping $GSEWA_\alpha : \mathscr{G}^n \to \mathscr{G}$ is said to be a generalized spherical fuzzy Einstein weighted averaging (GSEWA) operator and is defined by*

$$GSEWA_\alpha(G_1, G_2, ..., G_n) = \alpha_1\cdot_E G_1 \oplus_E \alpha_2\cdot_E G_2 \oplus_E ...\alpha_n\cdot_E G_n = \oplus_{i=1}^n \alpha_i\cdot_E G_i \tag{1}$$

Theorem 1. *Let $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$. Then the aggregated value $GSEWA_\alpha(G_1, G_2, ..., G_n)$ is also a GSFN and is calculated by*

$$\begin{aligned} GSEWA_\alpha(G_1, G_2, ..., G_n) &= \left\langle \sqrt{\frac{\prod_{i=1}^n(1+\mu_i^2)^{\alpha_i}-\prod_{i=1}^n(1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^n(1+\mu_i^2)^{\alpha_i}+\prod_{i=1}^n(1-\mu_i^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota_i^{2\alpha_i}}{\prod_{i=1}^n(2-\iota_i^2)^{\alpha_i}+\prod_{i=1}^n \iota_i^{2\alpha_i}}}\right. \\ &\left., \sqrt{\frac{2\prod_{i=1}^n \nu_i^{2\alpha_i}}{\prod_{i=1}^n(2-\nu_i^2)^{\alpha_i}+\prod_{i=1}^n \nu_i^{2\alpha_i}}} \right\rangle \end{aligned} \tag{2}$$

Proof The above result given in equation (2) can be proved by using induction method on n as follows:
Step I: For $n = 2$, we have

$$GSEWA_\alpha(G_1, G_2) = \alpha_1\cdot_E G_1 \oplus_E \alpha_2\cdot_E G_2.$$

Since $\alpha_1\cdot_E G_1$ and $\alpha_2\cdot_E G_2$ are GSFNs, then $\alpha_1\cdot_E G_1 \oplus_E \alpha_2\cdot_E G_2$ is also a GSFN. Then, we obtain

$$GSEWA_\alpha(G_1, G_2) = \alpha_1._E G_1 \oplus_E \alpha_2\cdot_E G_2$$

$$= \left\langle \sqrt{\frac{(1+\mu_1^2)^{\alpha_1}-(1-\mu_1^2)^{\alpha_1}}{(1+\mu_1^2)^{\alpha_1}+(1-\mu_1^2)^{\alpha_1}}}, \sqrt{\frac{2\iota_1^{2\alpha_1}}{(2-\iota_1^2)^{\alpha_1}+\iota_1^{2\alpha_1}}}, \sqrt{\frac{2\nu_1^{2\alpha_1}}{(2-\nu_1^2)^{\alpha_1}+\nu_1^{2\alpha_1}}} \right\rangle$$

$$\oplus_E \left\langle \sqrt{\frac{(1+\mu_2^2)^{\alpha_2}-(1-\mu_2^2)^{\alpha_2}}{(1+\mu_2^2)^{\alpha_2}+(1-\mu_2^2)^{\alpha_2}}}, \sqrt{\frac{2\iota_2^{2\alpha_2}}{(2-\iota_2^2)^{\alpha_2}+\iota_2^{2\alpha_2}}}, \sqrt{\frac{2\nu_2^{2\alpha_2}}{(2-\nu_2^2)^{\alpha_2}+\nu_2^{2\alpha_2}}} \right\rangle$$

$$= \left\langle \sqrt{\frac{\frac{(1+\mu_1^2)^{\alpha_1}-(1-\mu_1^2)^{\alpha_1}}{(1+\mu_1^2)^{\alpha_1}+(1-\mu_1^2)^{\alpha_1}}+\frac{(1+\mu_2^2)^{\alpha_2}-(1-\mu_2^2)^{\alpha_2}}{(1+\mu_2^2)^{\alpha_2}+(1-\mu_2^2)^{\alpha_2}}}{1+\left[\frac{(1+\mu_1^2)^{\alpha_1}-(1-\mu_1^2)^{\alpha_1}}{(1+\mu_1^2)^{\alpha_1}+(1-\mu_1^2)^{\alpha_1}}\right]\left[\frac{(1+\mu_2^2)^{\alpha_2}-(1-\mu_2^2)^{\alpha_2}}{(1+\mu_2^2)^{\alpha_2}+(1-\mu_2^2)^{\alpha_2}}\right]}}, \sqrt{\frac{\frac{2\iota_1^{2\alpha_1}}{(2-\iota_1^2)^{\alpha_1}+\iota_1^{2\alpha_1}}\cdot\frac{2\iota_2^{2\alpha_2}}{(2-\iota_2^2)^{\alpha_2}+\iota_2^{2\alpha_2}}}{1+[1-\frac{2\iota_1^{2\alpha_1}}{(2-\iota_1^2)^{\alpha_1}+\iota_1^{2\alpha_1}}][1-\frac{2\iota_2^{2\alpha_2}}{(2-\iota_2^2)^{\alpha_2}+\iota_2^{2\alpha_2}}]}}\right.$$

$$\left., \sqrt{\frac{\frac{2\nu_1^{2\alpha_1}}{(2-\nu_1^2)^{\alpha_1}+\nu_1^{2\alpha_1}}\cdot\frac{2\nu_2^{2\alpha_2}}{(2-\nu_2^2)^{\alpha_2}+\nu_2^{2\alpha_2}}}{1+[1-\frac{2\nu_1^{2\alpha_1}}{(2-\nu_1^2)^{\alpha_1}+\nu_1^{2\alpha_1}}][1-\frac{2\nu_2^{2\alpha_2}}{(2-\nu_2^2)^{\alpha_2}+\nu_2^{2\alpha_2}}]}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1+\mu_1^2)^{\alpha_1}.(1+\mu_2^2)^{\alpha_2}-(1-\mu_1^2)^{\alpha_1}.(1-\mu_2^2)^{\alpha_2}}{(1+\mu_1^2)^{\alpha_1}.(1+\mu_2^2)^{\alpha_2}+(1-\mu_1^2)^{\alpha_1}.(1-\mu_2^2)^{\alpha_2}}}, \sqrt{\frac{2\iota_1^{2\alpha_1}.\iota_2^{2\alpha_2}}{(2-\iota_1^2)^{\alpha_1}.(2-\iota_2^2)^{\alpha_2}+\iota_1^{2\alpha_1}.\iota_2^{2\alpha_2}}}\right.$$

$$\left., \sqrt{\frac{2\nu_1^{2\alpha_1}.\nu_2^{2\alpha_2}}{(2-\nu_1^2)^{\alpha_1}.(2-\nu_2^2)^{\alpha_2}+\nu_1^{2\alpha_1}.\nu_2^{2\alpha_2}}} \right\rangle$$

$$= \left\langle \sqrt{\frac{\prod_{i=1}^2(1+\mu_i^2)^{\alpha_i}-\prod_{i=1}^2(1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^2(1+\mu_i^2)^{\alpha_i}+\prod_{i=1}^2(1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^2 \iota_i^{2\alpha_i}}{\prod_{i=1}^2(2-\iota_i^2)^{\alpha_i}+\prod_{i=1}^2 \iota_i^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^2 \nu_i^{2\alpha_i}}{\prod_{i=1}^2(2-\nu_i^2)^{\alpha_i}+\prod_{i=1}^2 \nu_i^{2\alpha_i}}} \right\rangle$$

Hence, the equation (2) hold for $n = 2$.

Step II: Now, we suppose that the equation (2) hold for $n = k$, that is

$$\begin{aligned} & GSEWA_\alpha(G_1, G_2, ..., G_k) = \alpha_1 \cdot_E G_1 \oplus_E \alpha_2 \cdot_E G_2 \oplus_E ... \oplus_E \alpha_k \cdot_E G_k \\ = & \left\langle \sqrt{\frac{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^k (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^k (1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^k \iota_i^{2\alpha_i}}{\prod_{i=1}^k (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^k \iota_i^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^k \nu_i^{2\alpha_i}}{\prod_{i=1}^k (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^k \nu_i^{2\alpha_i}}} \right\rangle \end{aligned}$$

Similarly, we show that the equation (2) hold for $n = k+1$. Then, we have

$$\begin{aligned} & GSEWA_\alpha(G_1, G_2, ..., G_k, G_{k+1}) = GSEWA_\alpha(G_1, G_2, ..., G_k) \oplus_E \alpha_{k+1} G_{k+1} \\ = & \left\langle \sqrt{\frac{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^k (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^k (1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^k \iota_i^{2\alpha_i}}{\prod_{i=1}^k (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^k \iota_i^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^k \nu_i^{2\alpha_i}}{\prod_{i=1}^k (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^k \nu_i^{2\alpha_i}}} \right\rangle \\ \oplus_E & \left\langle \sqrt{\frac{(1+\mu_{k+1}^2)^{\alpha_{k+1}} - (1-\mu_{k+1}^2)^{\alpha_{k+1}}}{(1+\mu_{k+1}^2)^{\alpha_{k+1}} + (1-\mu_{k+1}^2)^{\alpha_{k+1}}}}, \sqrt{\frac{2\iota_{k+1}^{2\alpha_{k+1}}}{(2-\iota_{k+1}^2)^{\alpha_{k+1}} + \iota_{k+1}^{2\alpha_{k+1}}}}, \sqrt{\frac{2\nu_{k+1}^{2\alpha_1}}{(2-\nu_{k+1}^2)^{\alpha_{k+1}} + \nu_{k+1}^{2\alpha_{k+1}}}} \right\rangle \\ = & \left\langle \sqrt{\frac{\frac{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^k (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^k (1-\mu^2)^{\alpha_i}} + \frac{(1+\mu_{k+1}^2)^{\alpha_{k+1}} - (1-\mu_{k+1}^2)^{\alpha_{k+1}}}{(1+\mu_{k+1}^2)^{\alpha_{k+1}} + (1-\mu_{k+1}^2)^{\alpha_{k+1}}}}{1 + \left(\frac{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^k (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^k (1-\mu^2)^{\alpha_i}}\right)\left(\frac{(1+\mu_{k+1}^2)^{\alpha_{k+1}} - (1-\mu_{k+1}^2)^{\alpha_{k+1}}}{(1+\mu_{k+1}^2)^{\alpha_{k+1}} + (1-\mu_{k+1}^2)^{\alpha_{k+1}}}\right)}} \right. \\ & , \sqrt{\frac{\left(\frac{2\prod_{i=1}^k \iota_i^{2\alpha_i}}{\prod_{i=1}^k (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^k \iota_i^{2\alpha_i}}\right)\left(\frac{2\iota_{k+1}^{2\alpha_{k+1}}}{(2-\iota_{k+1}^2)^{\alpha_{k+1}} + \iota_{k+1}^{2\alpha_{k+1}}}\right)}{1 + \left(1 - \frac{2\prod_{i=1}^k \iota_i^{2\alpha_i}}{\prod_{i=1}^k (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^k \iota_i^{2\alpha_i}}\right)\left(1 - \frac{2\iota_{k+1}^{2\alpha_{k+1}}}{(2-\iota_{k+1}^2)^{\alpha_{k+1}} + \iota_{k+1}^{2\alpha_{k+1}}}\right)}} \\ & \left. , \sqrt{\frac{\left(\frac{2\prod_{i=1}^k \nu_i^{2\alpha_i}}{\prod_{i=1}^k (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^k \nu_i^{2\alpha_i}}\right)\left(\frac{2\nu_{k+1}^{2\alpha_{k+1}}}{(2-\nu_{k+1}^2)^{\alpha_{k+1}} + \nu_{k+1}^{2\alpha_{k+1}}}\right)}{1 + \left(1 - \frac{2\prod_{i=1}^k \nu_i^{2\alpha_i}}{\prod_{i=1}^k (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^k \nu_i^{2\alpha_i}}\right)\left(1 - \frac{2\nu_{k+1}^{2\alpha_{k+1}}}{(2-\nu_{k+1}^2)^{\alpha_{k+1}} + \nu_{k+1}^{2\alpha_{k+1}}}\right)}} \right\rangle \end{aligned} \tag{3}$$

Thus, the equation (2) hold for $n = k+1$. Hence, by induction method the equation (2) hold for all $n \in \mathbb{N}$.

Lemma 3. *(Idempotency of $GSEWA_\alpha$ operator) If $G_i = G$ for all $i = 1, 2, ..., n$ where $G =< \mu, \iota, \nu >$, $G_i =< \mu_i, \iota_i, \nu_i >$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ is the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$, then $GSEWA_\alpha = G$.*

Proof Let $G_i = G$ for all $i = 1, 2, ..., n$ where $G =< \mu, \iota, \nu >$ and $G_i =< \mu_i, \iota_i, \nu_i >$. Suppose that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ is the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$. SĂřnce $G_i = G$ for all $i = 1, 2, ..., n$, we have that $\mu_i = \mu$, $\iota_i = \iota$ and $\nu_i = \nu$ for all $i = 1, 2, ..., n$. Then

$$\begin{aligned} & GSEWA_\alpha(G_1, G_2, ..., G_n) = \left\langle \sqrt{\frac{\prod_{i=1}^n (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^n (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^n (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^n (1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota_i^{2\alpha_i}}{\prod_{i=1}^n (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^n \iota_i^{2\alpha_i}}} \right. \\ & \left. , \sqrt{\frac{2\prod_{i=1}^n \nu_i^{2\alpha_i}}{\prod_{i=1}^n (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^n \nu_i^{2\alpha_i}}} \right\rangle \\ = & \left\langle \sqrt{\frac{\prod_{i=1}^n (1+\mu^2)^{\alpha_i} - \prod_{i=1}^n (1-\mu^2)^{\alpha_i}}{\prod_{i=1}^n (1+\mu^2)^{\alpha_i} + \prod_{i=1}^n (1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota^{2\alpha_i}}{\prod_{i=1}^n (2-\iota^2)^{\alpha_i} + \prod_{i=1}^n \iota^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \nu^{2\alpha_i}}{\prod_{i=1}^n (2-\nu^2)^{\alpha_i} + \prod_{i=1}^n \nu^{2\alpha_i}}} \right\rangle \\ = & \left\langle \sqrt{\frac{(1+\mu^2)^{\sum_{i=1}^n \alpha_i} - (1-\mu^2)^{\sum_{i=1}^n \alpha_i}}{(1+\mu^2)^{\sum_{i=1}^n \alpha_i} + (1-\mu^2)^{\sum_{i=1}^n \alpha_i}}}, \sqrt{\frac{2\iota^{\sum_{i=1}^n 2\alpha_i}}{(2-\iota^2)^{\sum_{i=1}^n \alpha_i} + \iota^{\sum_{i=1}^n 2\alpha_i}}}, \sqrt{\frac{2\nu^{\sum_{i=1}^n 2\alpha_i}}{(2-\nu^2)^{\sum_{i=1}^n \alpha_i} + \nu^{\sum_{i=1}^n 2\alpha_i}}} \right\rangle \\ = & < \mu, \iota, \nu >= G \end{aligned}$$

Lemma 4. *(Boundedness of $GSEWA_\alpha$ operator) Let $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$ and $i \in \{1, 2, ..., n\}$. Then,*

$$\min_i G_i \leq GSEWA_\alpha(G_1, G_2, ..., G_n) \leq \max_i G_i$$

where $\min_i G_i =< \min_i \mu_i, \max_i \iota_i, \max_i \nu_i >$ *and* $\max_i G_i =< \max_i \mu_i, \min_i \iota_i, \min_i \nu_i >$.

Proof The proof is easily obtained from Definition 6 and equation (2).

Lemma 5. *(Monotonicity of $GSEWA_\alpha$ operator) Let $(G_1, G_2, ..., G_n), (G'_1, G'_2, ..., G'_n) \in \mathscr{G}^n$. If $G_i \le G'_i$ for all $i = 1, 2, .., n$, then*

$$GSEWA_\alpha(G_1, G_2, ..., G_n) \le GSEWA_\alpha(G'_1, G'_2, ..., G'_n).$$

Proof The proof is easily obtained from Definition 6 and equation (2).

Definition 8. *Let $\mathscr{G}$ be a collection of all GSFNs and $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$ where $G_i =< \mu_i, \iota_i, \nu_i >$ for all $i = 1, 2, ..., n$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ be the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \ge 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$. A mapping $GSEWG_\alpha : \mathscr{G}^n \to \mathscr{G}$ is said to be a generalized spherical fuzzy Einstein weighted geometric (GSEWG) operator and is defined by*

$$GSEWG_\alpha(G_1, G_2, ..., G_n) = G_1^{\wedge_E \alpha_1} \odot_E G_2^{\wedge_E \alpha_2} \odot_E ... \odot_E G_n^{\wedge_E \alpha_n} = \odot_{i=1}^n G_i^{\wedge_E \alpha_i} \tag{4}$$

Theorem 2. *Let $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$. Then the aggregated value $GSEWG_\alpha(G_1, G_2, ..., G_n)$ is also a GSFN and is calculated by*

$$\begin{aligned} GSEWG_\alpha(G_1, G_2, ..., G_n) &= \Bigg\langle \sqrt{\frac{2\prod_{i=1}^n \mu_i^{2\alpha_i}}{\prod_{i=1}^n (2-\mu_i^2)^{\alpha_i} + \prod_{i=1}^n \mu_i^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota_i^{2\alpha_i}}{\prod_{i=1}^n (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^n \iota_i^{2\alpha_i}}} \\ &\quad , \sqrt{\frac{\prod_{i=1}^n (1+\nu_i^2)^{\alpha_i} - \prod_{i=1}^n (1-\nu_i^2)^{\alpha_i}}{\prod_{i=1}^n (1+\nu_i^2)^{\alpha_i} + \prod_{i=1}^n (1-\nu^2)^{\alpha_i}}} \Bigg\rangle \end{aligned} \tag{5}$$

Proof The proof is obtained similar to the proof of Theorem 1.

Lemma 6. *(Idempotency of $GSEWG_\alpha$ operator) If $G_i = G$ for all $i = 1, 2, ..., n$ where $G =< \mu, \iota, \nu >$, $G_i =< \mu_i, \iota_i, \nu_i >$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ is the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \ge 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$, then $GSEWG_\alpha = G$.*

Proof The proof is easily obtained from equation (5).

Lemma 7. *(Boundedness of $GSEWG_\alpha$ operator) Let $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$ and $i \in \{1, 2, ..., n\}$. Then,*

$$\min_i G_i \le GSEWG_\alpha(G_1, G_2, ..., G_n) \le \max_i G_i$$

where $\min_i G_i =< \min_i \mu_i, \max_i \iota_i, \max_i \nu_i >$ *and* $\max_i G_i =< \max_i \mu_i, \min_i \iota_i, \min_i \nu_i >$.

Proof The proof is easily obtained from Definition 6 and equation (5).

Lemma 8. *(Monotonicity of $GSEWG_\alpha$ operator) Let $(G_1, G_2, ..., G_n), (G'_1, G'_2, ..., G'_n) \in \mathscr{G}^n$. If $G_i \le G'_i$ for all $i = 1, 2, .., n$, then*

$$GSEWG_\alpha(G_1, G_2, ..., G_n) \le GSEWG_\alpha(G'_1, G'_2, ..., G'_n).$$

Proof The proof is easily obtained from Definition 6 and equation (5).

4 An Application of Generalized Spherical Fuzzy Einstein Aggregation Operators to Multi-Criteria Group Decision-Making Problems

In this section, we develop a method for multi-criteria group decision-making problems under the generalized spherical fuzzy environment using the defined GSEWA and GSEWG operators and then we give a numerical example to explain this method.

4.1 Methodology

Let $A = \{A_1, A_2, ..., A_m\}$ be the set of m different options and $E = \{E_1, E_2, ..., E_n\}$ be the set of n different attributes. Assume that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is the weight vector of the attribute E_i $(i = 1, 2, ..., n)$ where $\alpha_i \ge 0$ for all $i = 1, 2, ..., n$ and $\sum_{i=1}^n \alpha_i = 1$. Also suppose that $D = \{D_1, D_2, ..., D_k\}$ is the set of n distinct decision-makers with the options whose weight vector is expressed as $\delta = (\delta_1, \delta_2, ..., \delta_k)$ where $\delta_i \ge 0$ for all $i = 1, 2, ..., k$ and $\sum_{i=1}^k \delta_i = 1$. This vector ($\delta$) has been handled according to the age, experience, education, thinking ability, and knowledge power of the decision-maker. Actually, as a first step decision matrices associated with options to attribute values are built on considering the preference of the decision-makers. But here, we consider the entity of the decision matrices as GSFNs and are given by $B_{ij}^r =< \mu_{ij}^r, \iota_{ij}^r, \nu_{ij}^r >$ $(i = 1, 2, ..., m), (j = 1, 2, ..., n), (r = 1, 2, ..., k)$ and the associated decision matrix is given as follows:

$$\mathbf{D^r} = \begin{pmatrix} B_{11}^r & B_{12}^r & \dots & B_{1n}^r \\ B_{21}^r & B_{22}^r & \dots & B_{2n}^r \\ \vdots & \vdots & \dots & \vdots \\ B_{m1}^r & B_{m2}^r & \dots & B_{mn}^r \end{pmatrix}.$$

Now, we develop the multi-criteria group decision-making procesure under the generalized spherical fuzzy environment as the following steps:

Step I: Use the either GSEWA or GSEWG operator on each decision matrix D^r to get the following matrix:

$$\mathbf{F^r_{m\times 1}} = \begin{pmatrix} C^r_{11} \\ C^r_{21} \\ \vdots \\ C^r_{m1} \end{pmatrix}$$

where $C^r_{i1} = GSEWA_\alpha(B^r_{i1}, B^r_{i2}, ..., B^r_{in})$ (or $C^r_{i1} = GSEWG_\alpha(B^r_{i1}, B^r_{i2}, ..., B^r_{in})$) for $i = 1, 2, ..., m$ and $r = 1, 2, ..., k$.

Step II: Apply the decision-maker's weight vector (δ) under the scalar multiplication, addition and power of GSFNs to evolute the final matrix $D = \sum_{i=1}^{k} \delta_i F^i_{m\times 1}$ when GSEWA operator is used and $D = \sum_{i=1}^{k} (F^i_{m\times 1})^{\delta_i}$ where $(F^i_{m\times 1})^{\delta_i} = \begin{pmatrix} (C^i_{11})^{\delta_i} \\ (C^i_{21})^{\delta_i} \\ \vdots \\ (C^i_{m1})^{\delta_i} \end{pmatrix}$ when GSEWG operator is used. Denote this matrix as follows:

$$\mathbf{D} = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \\ \vdots \\ \tilde{A}_m \end{pmatrix}.$$

Step III: Calculate the score values $SF(\tilde{A}_i)$ $(i = 1, 2, ..., m)$ of the cumulative overall preference value. If two score values $SF(\tilde{A}_i)$ and $SF(\tilde{A}_j)$ are same for any $i, j = 1, 2, ..., m$, then it is found the accuracy values $AF(\tilde{A}_i)$ $(i = 1, 2, ..., m)$.

Step IV: Rank the options A_i $(i = 1, 2, ..., m)$ and choose the best option which has the maximum score value.

4.2 A numerical Example

The following medical treatment choice problem is given to demonstrate the suitability, validity and efficiency of the observed multi-criteria group decision making method and is handled from [18].

There is a 50-year-old man patient who was diagnosed with the acute inflammatory demyelinating polyneuropathy disease by his specialist doctor. Acute inflammatory demyelinating polyneuropathy disease is an autoimmune process that is characterized by progressive areflexic weakness and mild sensory changes. The first symptoms of this disease usually include varying degrees of weakness or tingling sensations in the legs. In many instances, the weakness and abnormal sensations ascend and spread to the arms and upper body. And also it can cause life-threatening complications by affecting the peripheral nervous system. Most patients can rescue from this disease with convenient treatment within a few months to a year. But minor by-effects may continue such as areflexia. The doctor chosen four treatment options, including steroid therapy (A_1), plasmapheresis (A_2), intravenous immunoglobulin (A_3) and immunosuppressive medicines (A_4), based on his current physical conditions and medical history.

To satisfy the patient and his family's understanding of the advantages and disadvantages of each treatment choice, the doctor described the treatment options using three criteria that including the side effects (E1), the probability of a Cure (E2), and cost (E3). A prioritization relationship among the criteria E_i $(i = 1, 2, 3)$ which satisfies $E_2 > E_1 > E_3$ was determined according to the patientâĂŹs preferences and his current financial situation. So, assume that $\alpha = (0.3, 0.45, 0.25)$ is the weight vector of the attribute $\{E_1, E_2, E_3\}$. In order to choose the optimum treatment, the patient (D1), the doctor (D2) and the patientâĂŹs family (D3), with a prioritization relationship among the decision-makers D_i $(i = 1, 2, 3)$ satisfying $D_2 > D_1 > D_3$, evaluated the four treatment options based on these criteria considering the generalized spherical fuzzy Einstein aggregation operators. Take the decision-makers weight vector as $\delta = (0.35, 0.45, 0.2)$. The decision matrices are shown as follows:

$$\mathbf{D^1} = \begin{pmatrix} <0.6, 0, 8, 0, 2> & <0.4, 0.3, 0.7> & <0.2, 0.7, 0.4> \\ <0.55, 0.2, 0.8> & <0.8, 0.75, 0.65> & <0.9, 0.8, 0.2> \\ <0.7, 0.4, 0.4> & <0.55, 0.2, 0.45> & <0.5, 0.7, 0.8> \\ <0.35, 0.6, 0.5> & <0.7, 0.8, 0.55> & <0.8, 0.6, 0.5> \end{pmatrix}$$

$$\mathbf{D^2} = \begin{pmatrix} <0.85, 0.7, 0.8> & <0.4, 0.75, 0.8> & <0.6, 0.8, 0.5> \\ <0.3, 0.4, 0.4> & <0.8, 0.2, 0.45> & <0.5, 0.6, 0.8> \\ <0.9, 0, 8, 0, 2> & <0.4, 0.8, 0.7> & <0.8, 0.7, 0.4> \\ <0.75, 0.3, 0.5> & <0.8, 0.5, 0.45> & <0.5, 0.6, 0.8> \end{pmatrix}$$

$$\mathbf{D^3} = \begin{pmatrix} <0.75, 0.4, 0.5> & <0.8, 0.8, 0.45> & <0.8, 0.6, 0.8> \\ <0.9, 0, 6, 0, 4> & <0.4, 0.6, 0.9> & <0.2, 0.7, 0.4> \\ <0.55, 0.5, 0.8> & <0.8, 0.75, 0.85> & <0.6, 0.8, 0.2> \\ <0.75, 0.4, 0.8> & <0.4, 0.8, 0.45> & <0.8, 0.6, 0.6> \end{pmatrix}$$

Step I: Using the GSEWA operator on each decision matrix D^r with the weight vector $\alpha = (0.3, 0.45, 0.25)$, we get the following matrices:

$$\mathbf{F^1_{4\times 1}} = \begin{pmatrix} <0.4396, 0.5141, 0.4299> \\ <0.7841, 0.5376, 0.5345> \\ <0.5914, 0.3442, 0.5093> \\ <0.6606, 0.6874, 0.5221> \end{pmatrix}$$

$$\mathbf{F^2_{4\times 1}} = \begin{pmatrix} <0.6436, 0.7473, 0.7196> \\ <0.6380, 0.3287, 0.5093> \\ <0.7330, 0.7748, 0.4299> \\ <0.7309, 0.4525, 0.5431> \end{pmatrix}$$

$$\mathbf{F^3_{4\times 1}} = \begin{pmatrix} <0.7861, 0.6162, 0.5431> \\ <0.6305, 0.6243, 0.6> \\ <0.6962, 0.6819, 0.6151> \\ <0.6515, 0.6162, 0.5826> \end{pmatrix}$$

Step II: Applying the decision-maker's weight vector $\delta = (0.35, 0.45, 0.2)$, we get the following matrix:

$$\mathbf{D} = \begin{pmatrix} <0.6312, 0.6308, 0.5680> \\ <0.6989, 0.4439, 0.5352> \\ <0.6837, 0.5685, 0.49> \\ <0.6932, 0.5572, 0.5432> \end{pmatrix}$$

Step III: Now, we calculate the score values $SF(\tilde{A_i})$ $(i = 1, 2, 3, 4)$. Here, we have that $SF(\tilde{A_1}) = 0.03$, $SF(\tilde{A_2}) = 0.26$, $SF(\tilde{A_3}) = 0.1719$ and $SF(\tilde{A_4}) = 0.1752$.

Step IV: The ranking order of score values is that $SF(\tilde{A_2}) > SF(\tilde{A_4}) > SF(\tilde{A_3}) > SF(\tilde{A_1})$. Hence, according to the Definition 5, the ranking order of the options is that $A_2 > A_4 > A_3 > A_1$. Therefore, the best option is A_2.

4.3 Sensitivity analysis of the numerical example

The aim of sensitivity analysis is to observe the weights of the decision-makers keeping the rest of the other terms are fixed in the problem. So, the sensitivity analysis is given to understand how decision-makers' weight affects the final matrix and its ranking. The sensitivity analysis result for the problem given in the above section is shown respect to the GSEWA and GSEWG operators in Table 1 and Table 2, respectively.

Weights of the decision-makers	Final decision matrix	Ranking order
$<0.35, 0.45, 0.2>$	$\begin{pmatrix} <0.6312, 0.6308, 0.5680> \\ <0.6989, 0.4439, 0.5352> \\ <0.6837, 0.5685, 0.49> \\ <0.6932, 0.5572, 0.5432> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$<0.35, 0.4, 0.25>$	$\begin{pmatrix} <0.6412, 0.62471, 0.5600> \\ <0.6987, 0.4584, 0.5396> \\ <0.6816, 0.5649, 0.4989> \\ <0.6892, 0.5658, 0.5451> \end{pmatrix}$	$A_2 > A_3 > A_4 > A_1$
$<0.35, 0.36, 0.29>$	$\begin{pmatrix} <0.6490, 0.6199, 0.5538> \\ <0.6984, 0.4703, 0.5432> \\ <0.6799, 0.5620, 0.5061> \\ <0.6860, 0.5729, 0.5466> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$<0.3, 0.5, 0.2>$	$\begin{pmatrix} <0.6388, 0.6427, 0.5828> \\ <0.6909, 0.4331, 0.5339> \\ <0.6902, 0.5920, 0.4859> \\ <0.6967, 0.5456, 0.5443> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$<0.3, 0.55, 0.15>$	$\begin{pmatrix} <0.6287, 0.6489, 0.5910> \\ <0.6912, 0.4194, 0.5296> \\ <0.6922, 0.5958, 0.4773> \\ <0.7006, 0.5372, 0.5424> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$

Table 1 Sensitivity analysis under GSEWA operator

Weights of the decision-makers	Final decision matrix	Ranking order
$< 0.35, 0.45, 0.2 >$	$\begin{pmatrix} < 0.9910, 0.6308, 0.05263 > \\ < 0.9893, 0.4439, 0.06192 > \\ < 0.9920, 0.5685, 0.05885 > \\ < 0.9922, 0.5572, 0.03185 > \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$< 0.35, 0.4, 0.25 >$	$\begin{pmatrix} < 0.9898, 0.6247, 0.0557 > \\ < 0.9882, 0.4584, 0.0650 > \\ < 0.9910, 0.5649, 0.0584 > \\ < 0.9914, 0.5658, 0.0373 > \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$< 0.35, 0.36, 0.29 >$	$\begin{pmatrix} < 0.9893, 0.6199, 0.0571 > \\ < 0.9878, 0.4703, 0.0661 > \\ < 0.9906, 0.5620, 0.0565 > \\ < 0.9911, 0.5729, 0.0409 > \end{pmatrix}$	$A_2 > A_3 > A_4 > A_1$
$< 0.3, 0.5, 0.2 >$	$\begin{pmatrix} < 0.9913, 0.6427, 0.0511 > \\ < 0.9899, 0.4331, 0.0605 > \\ < 0.9923, 0.5920, 0.0648 > \\ < 0.9925, 0.5456, 0.0334 > \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$< 0.3, 0.55, 0.15 >$	$\begin{pmatrix} < 0.9929, 0.6489, 0.0462 > \\ < 0.9916, 0.4194, 0.0552 > \\ < 0.9937, 0.5958, 0.0618 > \\ < 0.9938, 0.5373, 0.0264 > \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$

Table 2 Sensitivity analysis under GSEWG operator

5 Conclusion and Future Work

In this paper, we give the generalized spherical fuzzy Einstein weighted averaging and generalized spherical fuzzy Einstein weighted geometric operators constructed on Einstein sum, product and scalar multiplication for generalized spherical fuzzy sets which are based on Einstein triangular norm and triangular conorm. We also investigate some fundamental properties of these operators and develop a model for generalized spherical fuzzy Einstein aggregation operators to solve the multi-criteria group decision-making problems. Further, we give a numerical example related to the medical treatment choosing to demonstrate that the developed method is suitable and effective for the decision process. For future work, we propose to develop the methods by considering different types of operators to solve the multi-criteria group decision-making problems under the generalized spherical fuzzy environment and also we aim to compare all obtained operators in terms of their results.

6 References

1 S. Ashraf, S. Abdullah, *Spherical aggregation operators and their application in multiattribute group decision-making*, International Journal of Intelligent Systems, **34**(3) (2019), 493-523.

2 S. Ashraf, S. Abdullah, T. Mahmood, *Spherical fuzzy Dombi aggregation operators and their application in group decision making problems*, Journal of Ambient Intelligence and Humanized Computing, **11**, (2020), 2731-2749.

3 S. Ashraf, T. Mahmood, S. Abdullah, Q. Khan, *Different approaches to multi-criteria group decision making problems for picture fuzzy environment*, Bull Braz Math Soc., **50** (2), (2018), 373-397.

4 S. Ashraf, T. Mahmood, S. Abdullah, Q, Khan, *Picture fuzzy linguistic sets and their applications for multi-attribute group*, Nucleus, **55** (2), (2018), 66-73.

5 A. Aygünoğlu, H. Aygün, *Some notes on soft topological spaces*, Neural computing and Applications, **21** (1), (2012), 113-119.

6 K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**,(1), (1986), 87-96.

7 R. Bellman, L. A. Zadeh, *Decision-making in a fuzzy environment*, Manage Sci. **17**, (1970), 141-154.

8 B. Cuong, *Picture fuzzy sets-first results*, Seminar on neuroâĂŞfuzzy systems with applications Institute of Mathematics, Hanoi, (2013).

9 V. Çetkin, A. Aygünoğlu, H. Aygün, *A new approach in handling soft decision making problems*, J. Nonlinear Sci. Appl **9** (2016), 231-239.

10 H. Garg, *Some picture fuzzy aggregation operators and their applications to multicriteria decision-making*, Arabian Journal for Science and Engineering, **42** (12), (2017), 5275-5290.

11 A. Guleria, R. K. Bajaj, *T-spherical Fuzzy Soft Sets and its Aggregation Operators with Application in Decision Making*, Scientia Iranica , (2019).

12 F. Kutlu GÃijndoÄ§du, C. Kahraman, *A novel VIKOR method using spherical fuzzy sets and its application to warehouse site selection*, Journal of Intelligent and Fuzzy Systems, **37**(1), (2019), 1197-1211.

13 F. Kutlu Gundogdu, C. Kahraman, *Extension of WASPAS with spherical fuzzy sets*, Informatica, **30** (2, (2019), 269-292.

14 F. Kutlu Gündoğdu, C. Kahraman, *Spherical fuzzy sets and spherical fuzzy TOPSIS method*, Journal of Intelligent and Fuzzy Systems, **36** (1), (2019), 337-352.

15 T. S. Haque, A. Chakraborty, S. P. Mondal, S. Alam, *Approach to solve multi-criteria group decision-making problems by exponential operational law in generalized spherical fuzzy environment*, CAAI Transactions on Intelligence Technology, **5** (2), (2020), 106-114.

16 J. C. Harsanyi, *Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility*, J. Political Econ. (63), (1955), 309âĂŞ321.

17 Y. Jin, S. Ashraf, S. Abdullah, *Spherical fuzzy logarithmic aggregation operators based on entropy and their application in decision support systems*, Entropy, **21**(7), (2019), 628.

18 Y. X. Ma, J. Q. Wang, J. Wang, X. H. Wu, *An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options*, Neural Computing and Applications, **28** (9), (2017), 2745-2765.

19 T. Mahmood, U. Kifayat, Q. Khan, N. Jan, *An approach toward decision making and medical diagnosis problems using the concept of spherical fuzzy sets*, Neural Computing and Applications, **31**, (2018), 7041-7053.

20 P. K. Maji, A. R. Roy, *An application of soft set in decision making problem*, Comput. Math. Appl., **44**, (2002), 1077-1083.

21 P. K. Maji, R. Biswas, A. R. Roy, *Soft set theory*, Comput. Math. Appl., **45** (2003), 555-562.

22 D. Molodtsov, *Soft set theory-first results*, Computers Mathematics with Appl., **37**, (1999), 19-31.

23 M. Munir, H. Kalsoom, K. Ullah, T. Mahmood, Y. M. Chu, *T-spherical fuzzy Einstein hybrid aggregation operators and their applications in multi-attribute decision making problems*, Symmetry, **12** (3), (2020), 365-389.

24 B. Pazar Varol, H. Aygün, *Fuzzy soft topology*, Hacettepe Journal of Mathematics and Statistics, **41** (3), (2012), 407-419.

25 B. Pazar Varol, A. Sostak, H. Aygün, *A new approach to soft topology*, Hacettepe Journal of Mathematics and Statistics, **41** (5), (2012), 731-741.

26 X. Peng, Y. Yang, J. Song, Y. Jiang, *Pythagorean Fuzzy Soft Set and Its Application*, Computer Engineering, **41**, (2015), 224âĂŞ229.

27 G. Wei, *Picture fuzzy aggregation operators and their application to multiple attribute decision making*, Journal of Intelligent and Fuzzy Systems, **33**(2), (2017), 713-724.

28 G. Wei, *Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making*, Fundamenta Informaticae, **157** (3), (2018), 271-320.

29 Z. Xu, *Intuitionistic fuzzy aggregation operators*, IEEE Trans. Fuzzy Syst. **15**, (2007), 1179-1187.

30 R. R. Yager, *On ordered weighted averaging aggregation operators in multi-criteria decision making*, IEEE Trans. Syst., Man, Cybern., **18**, (1988), 183âĂŞ190.

31 R. R. Yager, *Pythagorean fuzzy subsets*, In Proceedings of Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, (2013), 57-61.

32 L. A. Zadeh, *Fuzzy sets*, Information and Control 8, (1965), 338-353.

33 L. A. Zadeh, *Similarity relations and fuzzy orderings*, Inf. Sci., **3** (1971), 177-206.

34 L. A. Zadeh, *A fuzzy-alogorithmic approach to the definition of complex or imprecise concepts*, Int. J. Man. Mach. Stud. **8**, (1976), 249-291.

Conference Proceeding Science and Technology, 3(2), 2020, 236–241

Conference Proceeding of 9th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2020).

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Risk Assessment of Cognitive Development of Early Childhood Children in Quarantine Days: A New AHP Approach

Murat Kirişci[1,*] *Nihat Topaç*[2] *Musa Bardak*[3]

[1] *Department of Biostatistics and Medical Informatics, Istanbul University-Cerrahpaşa, Istanbul, Turkey, ORCID: 0000-0003-4938-5207*
** Corresponding Author E-mail: mkirisci@hotmail.com*
[2] *Istanbul University-Cerrahpasa, HAYEF, Department of Early Childhood Education, Istanbul, Turkey, ORCID: 0000-0001-9364-4072*
[3] *Istanbul Sabahattin Zaim University, Department of Early Childhood Education, Istanbul, Turkey, ORCID: 0000-0001-5585-8002*

Abstract: The world is faced with disasters caused by natural or human effects from time to time. The various political, economic, health, and social consequences of these disasters affect people for different periods of time. In natural disasters and especially in epidemic diseases, some measures are taken to protect people from the negative effects of the situation. One of the measures that can be taken is quarantine. The target audience of this study is children aged 5-6 in the early childhood. Children of this age group are in the process of gaining skills in expressing their feelings during this period. In addition, the emotional responses of these children can be noticed by a careful observer or even an expert. The purpose of this study is to evaluate the risks of the effects of quarantine status related to COVID-19 pandemic on cognition and behavior of children staying at home. A new AHP technique was used to assess the risks of the quarantine process in early childhood children.

Keywords: COVID-19, early childhood, risk assessment, Pythagorean fuzzy set, analytic hierarchy process, cognitive development, behavioural development.

1 Introduction

Risk is always related to what can happen in the future. In today's world, risk can be accurately defined, analyzed and managed in a rational way. People knows that the probability of harmful events is a natural part of life. Such events can be caused by natural forces, such as flooding, lightning, natural disasters, or earthquake. Some harmful events can be foreseen and readily addressed, while other come unexpectedly because they appear foreseeable or have only a very remote likelihood of occurrence. Risk analysis is used to identify the causes of harmful events, to determine the possible consequences of harmful events, to identify and prioritize barriers, and to form a basis for deciding whether or not the risk related to a system is tolerable.

The aim of this study is to investigate the effects of quarantine status due to the COVID-19 pandemic on the cognitions and behaviours of children who stay at home. In this study, we first identified what possible cognitive an behavioral hazardous of COVID-19 quarantine could be in early childhood children. Secondly, we determined the answers to the probability of cognitive and behavioral changes in children and the possible consequences of their harms with qualitative statements. Finally, we assigned the cognitive and behavioral harmful states of children of the COVID-19 quarantine.

2 Preliminaries

Throughout the paper, the initial universe, parameters sets will denote $\mathcal{U}$, $\mathcal{P}$, respectively.

The FS has emerged as a generalization of the classical set concept. If we choose a non-empty set $\mathcal{U}$, then a function $m_{\mathcal{A}} : \mathcal{U} \to [0,1]$ is called *FS* on $\mathcal{U}$ and represented by

$$\mathcal{A} = \{(x_i, m_{\mathcal{A}}(x_i)) : m_{\mathcal{A}}(x_i) \in [0,1]; \forall x_i \in \mathcal{U}\}.$$

FS $\mathcal{A}$ on $\mathcal{U}$ can be expressed by set of ordered pair as follows:

$$\mathcal{A} = \{(x, m_{\mathcal{A}}(x)) : x \in \mathcal{U}\}.$$

The set

$$\mathcal{B} = \{(x, m_{\mathcal{B}}(x), n_{\mathcal{B}}(x)) : x \in \mathcal{U}\}$$

is called an *intuitionistic fuzzy set*(IFS) $\mathcal{B}$ on $\mathcal{U}$, where, $m_{\mathcal{B}} : \mathcal{U} \to [0, 1]$ and $n_{\mathcal{B}} : \mathcal{U} \to [0, 1]$ such that $0 \leq m_{\mathcal{B}}(x) + n_{\mathcal{B}}(x) \leq 1$ for any $x \in \mathcal{U}$ [1].

The degree of indeterminacy $p_{\mathcal{B}} = 1 - m_{\mathcal{B}}(x) - n_{\mathcal{B}}(x)$.

An *Pythagorean fuzzy set*(PFS) $\mathcal{C}$ in $\mathcal{U}$ is given by

$$\mathcal{C} = \{(x, m_{\mathcal{C}}(x), n_{\mathcal{C}}(x)) : x \in \mathcal{U}\},$$

where $m_{\mathcal{C}} : \mathcal{U} \to [0, 1]$ denotes the degree of membership and $n_{\mathcal{C}} : \mathcal{U} \to [0, 1]$ denotes the degree of non-membership of the element $x \in \mathcal{U}$ to the set $\mathcal{C}$, respectively, with the condition that $0 \leq [m_{\mathcal{C}}(x)]^2 + [n_{\mathcal{C}}(x)]^2 \leq 1$ [7–9].

The degree of indeterminacy $p_{\mathcal{C}} = \sqrt{1 - [m_{\mathcal{C}}(x)]^2 - [n_{\mathcal{C}}(x)]^2}$.

An interval valued PFS defined by Peng and Yang [5] as follows:

$$D = \{(x, m_{\mathcal{D}}(x), n_{\mathcal{D}}(x)) : x \in \mathcal{U}\}$$

where $m_{\mathcal{D}}(x) = \left[m^I_{\mathcal{D}}(x), m^J_{\mathcal{D}}(x)\right] \subset [0, 1]$ and $n_{\mathcal{D}}(x) = \left[n^I_{\mathcal{D}}(x), n^J_{\mathcal{D}}(x)\right] \subset [0, 1]$.

Cosine similarity is an angle-based measure of similarity between two vectors of n dimensions using the cosine of the angle between them. It measures the similarity between two vectors based only on the direction [2]. The cosine value of the angle between the two vectors expresses the similarity between the two vectors. Cosine similarity measures the angle between two vectors and calculates by dividing the inner product of those vectors by multiplication of their length [4]. In cosine similarity, the similarity between two users is between 0 and 1. The cosine similarity between two vectors d_i and d_j is formulated as follows:

$$sim_{cos}(d_i, d_j) = \frac{\overrightarrow{d_i}.\overrightarrow{d_j}}{\|\overrightarrow{d_i}\|.\|\overrightarrow{d_j}\|}. \tag{1}$$

3 Method

Weighted scales for PFAHP method are given in Table 1 [3, 6], where Linguistic terms Certainly Low Importance, Very Low Importance, Low Importance, Below Average Importance, Average Importance, Above Average Importance, High Importance, Very High Importance, Certainly High Importance, Exactly Equal are shown as α, β, γ, δ, ε, η, θ, λ, μ, φ, respectively.

The steps of Pythagorean fuzzy AHP are presented as follows:

Step 1. Construct the pairwise comparison matrix $E = (e_{ik})_{m\times m}$ as given in according to experts' evaluations based on Table 1.
Step 2. Obtain the differences matrix $F = (f_{ik})_{m\times m}$ with respect to lower and upper values of the membership and non-membership functions using Equations 2 and 3.
Step 3. Obtain the interval multiplicative matrix $G = (g_{ik})_{m\times m}$ by using Equations 4 and 5.
Step 4. Compute the determinacy value $H = (h_{ik})_{m\times m}$ of the e_{ik} by employing Equation 6.
Step 5. Multiply the determinacy values with $G = (g_{ik})_{m\times m}$ matrix to obtain the weights matrix before normalization, $T = (t_{ik})_{m\times m}$, by using Equation 7.
Step 6. Calculate the normalized priority weights ω_i by adopting Equation 8.

The new PFAHP method [6] is as follows:

Step 1. Identify work activities to determine the potential hazards and the interval valuation scale.
Step 2. Classify those hazards to form a hierarchy by consulting the partners and transform the problem into a hierarchy of goals and criteria.
Step 3. Generate binary comparison matrices for the criteria using range-valued sets based Table 1.
Step 4. Calculate the normalized criteria weights using the recommended interval valuation scale.
Step 4.1. The values in each column of the matrix are collected.
Step 4.2. After selecting the highest values for each parameter, each parameter is divided by the highest value selected.
Step 4.3. Calculate the average of each sequence to calculate the priority vectors.
Step 4.4. The steps above are repeated for each criterion and weight vectors are obtained for all. These procedures were repeated to obtain the priority weights of these criteria.
Step 5. Apply the cosine similarity measure between each alternative pair using the obtained priority weights.
Step 6. Using the linear regression function, the corresponding AHP score is assigned.
Step 7. Alternative weights were obtained based on the classic AHP steps.
Step 8. To be able to use the L matrix method, the probability and severity criteria of alternatives are rated according to alternative weights.
Step 9. Apply the L matrix method using the obtained grades.

Table 1 Weighted scales for the PFAHP

Linguistic terms	PFN equivalents IVPF numbers			
	m_I	m_J	n_I	n_J
α	0	0	0.9	1
β	0.1	0.2	0.8	0.9
γ	0.2	0.35	0.65	0.8
δ	0.35	0.45	0.55	0.65
ε	0.45	0.55	0.45	0.55
η	0.55	0.65	0.35	0.45
θ	0.65	0.8	0.2	0.35
λ	0.8	0.9	0.1	0.2
μ	0.9	1	0	0
φ	0.195	0.195	0.195	0.195

$$f_{ikI} = m_{ikI}^2 - n_{ikI}^2 \tag{2}$$

$$f_{ikU} = m_{ikJ}^2 - n_{ikJ}^2 \tag{3}$$

$$g_{ikI} = \sqrt{1000^{f_{ikI}}} \tag{4}$$

$$g_{ikJ} = \sqrt{1000^{f_{ikJ}}} \tag{5}$$

$$h_{ik} = 1 - \left(m_{ikJ}^2 - m_{ikI}^2\right) - \left(n_{ikJ}^2 - n_{ikI}^2\right) \tag{6}$$

$$t_{ik} = \left\{\frac{g_{ikI} + g_{ikJ}}{2}\right\} h_{ik} \tag{7}$$

$$\omega_i = \frac{\sum_{k=1}^{m} t_{ik}}{\sum_{i=1}^{m}\sum_{k=1}^{m} t_{ik}} \tag{8}$$

4 COVID-19 Quarantine Application

In order to determine the criteria to be measured, the cognitive and behavioral status of children should be taken into account when performing risk analysis in accordance with their attitudes in quarantine practice. For the weighting procedure, an aggregate of expert opinions consisting of evaluations of Early Childhood experts will be taken. After this stage, the sub-criteria and their weights will be used as inputs for both AHP methods to prioritize the goals and take the final decision. The experts in this study are people working on Early Childhood. Experts compare the criteria determined according to the cognitive and behavioral attitudes of these age children and express their evaluations.

The linguistic terms and their numeric labels are: For Questions to be asked to the child: Yes (1), maybe/some (2), no (3). For Questions to be asked to parents: too much (1), much (2), some (3), too little (4), none (5). The survey was prepared to be answered on the internet. Survey questions were asked to children aged 5-6 and their families. The survey included the following questions:

Questions to be asked to the child:

E1 Do you know Corona-virus?
E2 Does Corona-virus harm people?
E3 Does Corona-virus harm animals?
E4 Can Corona-virus be prevented?
E5 Are you afraid of Corona-virus?
E6 Do you think it's nice not to go to school?
E7 Are you upset that you can't go to school?
E8 Is the obligation to stay home boring?
E9 Can we be protected from Corona-virus by staying at home?
E10 Do you think you can go to school from now on?

Questions to be asked to parents:

P1 Does your child pay more attention to cleaning after Corona-virus?
P2 Has your child's sleep pattern been impaired after Corona-virus?
P3 Have there been changes in your child's nutritional habits after Corona-virus?
P4 Does your child behave anxiously after Corona-virus?
P5 Is your child afraid when a conversation about Corona-virus has passed?
P6 Does your child ask about Corona-virus?
P7 Did your child develop undesirable behaviour after Corona-virus?
P8 Is your child happy because she/he can't go to school?
P9 Has the time your child spent on the Internet after Corona-virus increased?
P10 Has the time your child spent in front of the TV increased after Corona-virus?

Table 2 Classifications of hazards about children's cognition

Current status information(CSI)	Children's COVID-19 knowledge	E1
	The idea of COVID-19 harming people	E2
	The idea of COVID-19 harming animals	E3
	Knowledge of to prevent COVID-19	E4
Affecting children's emotions(ACE)	Children's fear of COVID-19	E5
	Nice not to go to school	E6
	It's sad to not go to school	E7
	The boringness of staying in the compulsory home	E8
Affecting children's thoughts(ACT)	Being protected from COVID-19 by staying at home	E9
	To think that schools can be reopened	E10

Table 3 Classifications of hazards about children's behaviour

Change of basic habits of children(CBHC) in the quarantine period	Change in cleaning habits after COVID-19	P1
	Disruption in sleep pattern after COVID-19	P2
	Change in nutritional habits after COVID-19	P3
Change in behavioural after COVID-19(CB)	Anxiety increase after COVID-19	P4
	The emergence of fear when COVID-19 is spoken	P5
	Asking questions about COVID-19	P6
	Development of undesirable behavior after COVID-19	P7
	The idea that it is good not to go to school	P8
Change in behavior related to Information Technologies(CBIT)	increase in time spent on the internet	P9
	Increase in time spent in front of TV	P10

The cognitive and behavioral distributions of questions are as follows:

For children's cognition;

C1 Do children know about the current situation? (4 questions)
C2 Does the current situation affect children's emotions? (4 questions)
C3 Does the current situation affect children's thoughts? (2 questions)

For children's behavioral;

B1 Has Corona-virus changed the basic habits of children? (3 questions)
B2 Did behavior change occur in children after quarantine? (5 questions)
B3 Did children's behavior regarding information technologies increase after quarantine? (2 questions)

In this study, from Turkey, 201 children ages 5-6 units and 201 parents were the participants. Opinions of each child and each parent about the questions asked were got. The effect of quarantine on their own cognition in line with the answers given by the children and the effect of the behaviour of their children in line with the observations of the parents have been revealed.

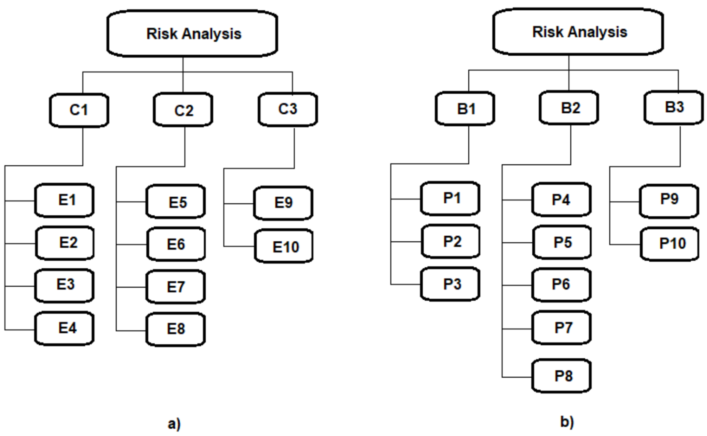

Fig. 1: Risk analysis a) for children's cognition, b) for children's behavioural

Risk factors were identified as a result of interviews and evaluations with Early Childhood experts. Basic problem and sub-problems related to this problem were created and data were obtained. The evaluations of early childhood experts were obtained for the weights with the acquired data. The risk analysis structure of children's and parents' evaluations is given in Figure 1. Cognitive and behavioral risks that can be classified

Table 4 Linguistic evaluations for CSI

	E1	E2	E3	E4
E1	φ	μ	θ	λ
E2	λ	φ	η	μ
E3	θ	δ	φ	γ
E4	μ	λ	ε	φ

Table 5 Linguistic evaluations for CB

	P4	P5	P6	P7	P8
P4	φ	ε	μ	μ	θ
P5	μ	φ	λ	μ	ε
P6	η	θ	φ	ε	γ
P7	μ	λ	θ	φ	ε
P8	β	γ	α	γ	φ

Table 6 Comparison matrix for probability

	E1	E2	E3	E4	sum
E1	0.195	0.8	0.5	0.2	1.695
	0.195	0.9	0.35	0.6	2.045
	0.195	0.65	0.8	0.55	2.195
E2	0.8	0.195	0.0	0.9	1.895
	0.9	0.195	0.45	1	2.545
	0.8	0.195	0.5	0.65	2.145
E3	0.8	0.45	0.195	0.8	2.245
	0.8	0.65	0.195	0.35	1.995
	0.65	0.5	0.195	0.6	1.945
E4	1	0.2	0.0	0.195	1.395
	0.9	0.4	0.55	0.195	2.045
	0.0	0.65	0.2	0.195	

Table 7 Normalized comparison matrix for probability

	E1	E2	E3	E4
E1	0.095	0.4	0.24	0.1
	0.095	0.44	0.17	0.3
	0.095	0.4	0.4	0.27
E2	0.31	0.08	0.0	0.35
	0.35	0.08	0.18	0.4
	0.31	0.08	0.2	0.255
E3	0.36	0.2	0.09	0.36
	0.36	0.29	0.09	0.16
	0.29	0.22	0.09	0.27
E4	0.49	0.1	0.0	0.1
	0.44	0.2	0.27	0.1
	0.0	0.32	0.1	0.1

Table 8 Weight for probability

	Weights		
E1	0.165	0.178	0.08
E2	0.107	0.136	0.093
E3	0.09	0.014	0.11
E4	0.055	0.08	0.142

in children are classified in Table 2 and Table 3. Linguistic evaluations for CSI and for CB are shown in Table 4 and Table 5, respectively.

For the weighting procedure, the sum of the assessments of the three experts was taken. After this step, in order to determine the priorities of the aims and make the final decision, the sub-problems and the their weights as PFAHP inputs are studied. Experts are early childhood employees and can compare specified problems, report results and indicate their evaluations.

Comparison matrices for probability owing to PFAHP are given in Table 6. The results in Table 6 indicate the experts evaluations about the criteria. Normalized comparison matrices for probability owing to PFAHP are given in Table 7.

Weights for probability are given in Table 8. It demonstrates the weight value for the criteria for CSI.

The same processes are carried out for severity. In addition, these procedures will be done for each classification of hazards. Table 9 will be obtained as a result of these processes.

Table 9 Weight for probability

Weights	Probablity	Severity	probability(P)	severity(S)	P × S
E1	0.01	0.021	1	2	2
E2	0.022	0.024	2	2	4
E3	0.013	0.029	1	3	3
E4	0.03	0.033	3	3	9
E5	0.038	0.042	4	4	16
E6	0.033	0.039	3	4	12
E7	0.021	0.018	2	2	4
E8	0.032	0.041	3	4	12
E9	0.028	0.03	3	3	9
E10	0.033	0.027	3	3	9
P1	0.024	0.018	2	2	4
P2	0.034	0.039	3	4	12
P3	0.027	0.042	3	4	12
P4	0.044	0.041	4	4	16
P5	0.039	0.042	4	4	16
P6	0.033	0.029	3	3	9
P7	0.032	0.04	3	4	12
P8	0.016	0.023	2	2	4
P9	0.034	0.043	3	4	12
P10	0.031	0.04	3	4	12

The results of the proposed approach owing to PFAHP are given in Table 9. According to these results, the criterion (E5) for cognition development and the criterions (P4) and (P5) for behaviour development have been found to be most critical factors. Further, the criterion (E1) for cognition development and the criterion (P8) for behaviour development have been found to the least critical factors.

5 References

1 K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst., **20**, (1986), 87–96.
2 Bolturk E, and Kahraman C. *A novel interval-valued neutrosophic AHP with cosine similarity measure*. Soft Comput **22**, (2018), 4941–4958.
3 Esra Ilbahar, Ali Karaşan, Selcuk Cebi, Cengiz Kahraman, *A novel approach to risk assessment for occupational health and safety using Pythagorean fuzzy AHP & fuzzy inference system*, Safety Science 103 (2018) 124–136. https://doi.org/10.1016/j.ssci.2017.10.025
4 Madylova A, and Oguducu SG. A taxonomy based semantic similarity of documents using the cosine measure. 24th International Symposium on Computer and Information Sciences 129:34, 2009.
5 Peng X, Yang Y. *Fundamental proporties of interval valued pythagorean fuzzy aggregation operators*, Int J Intell Syst 2016;31(5):444–87.
6 Serap Tepe, İhsan Kaya (2020) *A fuzzy-based risk assessment model for evaluations of hazards with a real-case study*, Human and Ecological Risk Assessment: An International Journal, 26:2, 512-537, DOI: 10.1080/10807039.2018.1521262
7 R. R. Yager, *Pythagorean fuzzy subsets*, In: Proc Joint IFSA World Congress and NAFIPS Annual M eeting, Edmonton, Canada; (2013), 57–61.
8 R. R. Yager, *Pythagorean membership grades in multicriteria decision making* IEEE Trans Fuzzy Syst., **22**, (2014), 958–965.
9 R. R. Yager, A. M. Abbasov, *Pythagorean membership grades, complex numbers, and decision makin*, Int J Intell Syst. **28**, (2013), 436–2.

Conference Proceeding Science and Technology, 3(2), 2020, 242–246

Conference Proceeding of 9th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2020).

On a Different Method For Determining the Primary Numbers

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Serpil Halici[1,] Hamit Cacur[2] Nazli Koca[3]*

[1] *Pamukkale Universitesi, Department of Mathematics, Denizli, Turkey, ORCID: 0000-0002-8071-0437*
[2] *Pamukkale Universitesi, Department of Mathematics, Denizli, Turkey,*
[3] *Pamukkale Universitesi, Department of Mathematics, Denizli, Turkey, ORCID: 0000-0002-4663-6253*
* *Corresponding Author E-mail: shalici@pau.edu.tr, nkoca17@posta.pau.edu.tr*

Abstract: In this study, firstly, ordinal numbers of the odd integers were defined to reach prime numbers, and a new prime number sieve was obtained with the help of these numbers. A general formula was given to examine this sieve. Then, by representing the ordinal numbers with the help of matrices, some properties of these numbers were also examined.

Keywords: Prime Numbers, Prime Number Sieves, Applications of Sieves Methods.

1 Introduction and Preliminaries

While identifying prime numbers is simple, finding a new and large prime number is difficult. To date, the distribution of prime numbers, the next major prime number, prime number calculation algorithms, etc. Mathematicians have developed many theories on topics. The study of the most common and known prime numbers is the study of Euclid in about 300 BC and this also constitutes the basic principles of arithmetic [1]. The Eratosthenes sieve, handled by the Greek mathematicians, is not sufficient in calculating large numbers in terms of cost and time . Then, in the $17th$ century, Fermat and Euler worked on the properties of prime numbers and to make them more understandable. Today, efforts are being made to find the largest prime number within the scope of the Great Internet Mersenne Prime Search (GIMPS) project [2]. As is known, Euclid proved that the prime numbers are infinite. However, no exact formula has yet been found to find these prime numbers. Therefore, studies are still ongoing on prime numbers. Since it is not easy to find prime numbers or to distinguish them from composite numbers, prime numbers are grouped according to some properties [1].
To date, a formula found and proven to generate prime numbers is not yet available in the literature [3]. Through trial-and-error, the search for prime numbers using mathematical calculations and algorithms continues. In this study, we gave a screening algorithm different from previous studies. We supported this method first with the help of tables.
Since we will work on set of the odd integers in our study, and every element of this set is likely to be prime, we named this set of numbers as a possible set of prime numbers and denoted this set with the letter M_s. To further narrow the single set of integers than the probability of being prime, we examined the finale digits of potentially prime numbers. It should be noted that the numbers in the last digit should end with one of the numbers 1, 3, 7 and 9, but the numbers that end in each of the last digits with one of these numbers are not prime. In this method, unlike other prime number tests, we have defined and used the numbers of possible prime numbers. Let's give some definitions that we use and which are not available in the literature.

Definition 1. *The numbers obtained by subtracting 1 from M_s numbers and dividing by 2 are called the number of rows of possible prime numbers and are indicated M_{ss}. For example,*
$M_{ss}(3) = 1, M_{ss}(5) = 2, M_{ss}(7) = 3, \cdots$

It is not possible to find prime numbers directly in M_s array. Therefore, it will be found that the non-prime numbers are found and the remaining numbers are prime numbers. These operations will be done using the row numbers of possible numbers. For $b \in Z^{odd}$ and $s, a \in Z$ the number s shows the number of rows of the number b. Let's define the following numbers with the help of this number b:

$$a = (s, s+1, s+2, s+3, \cdots) = (s+n); \;\; n \in N.$$

Using these numbers, let's call the numbers described below as the number of rows of non-prime numbers.

$$M'_{ss} = s\,(1, 1, 1, \ldots) + b\,(s, s+1, s+2, \cdots)\,.$$

$Number\ of\ rows\ with\ multiples\ of\ 3$										
M'_{ss}	4	7	10	13	16	19	22	25	28	31
s	1	1	1	1	1	1	1	1	1	1
a	1	2	3	4	5	6	7	8	9	10
$Number\ of\ rows\ with\ multiples\ of\ 5$										
M'_{ss}	12	17	22	27	32	37	42	47	52	57
s	2	2	2	2	2	2	2	2	2	2
a	2	3	4	5	6	7	8	9	10	11
$Number\ of\ rows\ with\ multiples\ of\ 7$										
M'_{ss}	24	31	38	45	52	59	66	73	80	87
s	3	3	3	3	3	3	3	3	3	3
a	3	4	5	6	7	8	9	10	11	12
$Number\ of\ rows\ with\ multiples\ of\ 9$										
M'_{ss}	40	49	58	67	76	85	94	103	112	121
s	4	4	4	4	4	4	4	4	4	4
a	4	5	6	7	8	9	10	11	12	13
$Number\ of\ rows\ with\ multiples\ of\ 11$										
M'_{ss}	60	71	82	93	104	115	126	137	148	159
s	5	5	5	5	5	5	5	5	5	5
a	5	6	7	8	9	10	11	12	13	14
$Number\ of\ rows\ with\ multiples\ of\ 13$										
M'_{ss}	84	97	110	123	136	149	162	175	188	201
s	6	6	6	6	6	6	6	6	6	6
a	6	7	8	9	10	11	12	13	14	15
$Number\ of\ rows\ with\ multiples\ of\ 15$										
M'_{ss}	112	127	142	157	172	187	202	217	232	247
s	7	7	7	7	7	7	7	7	7	7
a	7	8	9	10	11	12	13	14	15	16
$Number\ of\ rows\ with\ multiples\ of\ 17$										
M'_{ss}	144	161	178	195	212	229	246	263	280	297
s	8	8	8	8	8	8	8	8	8	8
a	8	9	10	11	12	13	14	15	16	17

For example,
for $s \in Z^+$, $b = 3, 5, 7, 9$ a sequence of numbers as follows is obtained using the formula above, respectively.

$$(1, 1, 1, \cdots) + 3(1, 2, 3, \cdots) = (4, 7, 10, \cdots),$$

$$(2, 2, 2, \cdots) + 5(2, 3, 4, \cdots) = (12, 17, 22, \cdots),$$

$$(3, 3, 3, \cdots) + 7(3, 4, 5, \cdots) = (24, 31, 38, \cdots),$$

$$(4, 4, 4, \cdots) + 9(4, 5, 6, \cdots) = (40, 49, 58, \cdots).$$

The first sequence $(4, 7, 10, \cdots)$ here is the sequence numbers of the odd multiples of the number 3, respectively $(12, 17, 22, \cdots)$, respectively. The sequence numbers of the odd multiples floors of the number 5, $(24, 31, 38, \cdots)$, respectively, gives the number of rows of odd multiples of 7, respectively. Tables for M'_{ss} numbers can also be edited:

According to the table above, the following theorem can be given.

Theorem 1. *The following formula is true for the ordinal numbers of non-prime numbers.*

$$a \times b + s = M'_{ss}$$

By looking at the M'_{ss} numbers in the table above, we can give information about whether a given number is a prime number: If the given number is included in this table, it is not a prime number. For example, let's examine whether the number 299 is prime or not with the above table. First, let's write the number of rows of 299 using.

Definition 2. M'_{ss}=149. *Since the number 149 is included in above table, this number cannot be a prime number. Non-prime numbers can also be obtained on the matrix by using M'_{ss} numbers, that is, the ordinal numbers of non-prime numbers. If the sequence numbers, M'_{ss} of non-prime numbers for the integer $s \leq 8$ are placed on the matrix, then the following equation can be written using the above theorem.*

$$M_{ss}^{'}=\begin{bmatrix}1&0&0&0&0&0&0&0\\2&2&0&0&0&0&0&0\\3&3&3&0&0&0&0&0\\4&4&4&4&0&0&0&0\\5&5&5&5&5&0&0&0\\6&6&6&6&6&6&0&0\\7&7&7&7&7&7&7&0\\8&8&8&8&8&8&8&8\end{bmatrix}\times\begin{bmatrix}3&0&0&0&0&0&0&0\\0&5&0&0&0&0&0&0\\0&0&7&0&0&0&0&0\\0&0&0&9&0&0&0&0\\0&0&0&0&11&0&0&0\\0&0&0&0&0&13&0&0\\0&0&0&0&0&0&15&0\\0&0&0&0&0&0&0&17\end{bmatrix}+\begin{bmatrix}1&0&0&0&0&0&0&0\\1&2&0&0&0&0&0&0\\1&2&3&0&0&0&0&0\\1&2&3&4&0&0&0&0\\1&2&3&4&5&0&0&0\\1&2&3&4&5&6&0&0\\1&2&3&4&5&6&7&0\\1&2&3&4&5&6&7&8\end{bmatrix}$$

$$\begin{bmatrix}4&0&0&0&0&0&0&0\\7&12&0&0&0&0&0&0\\10&17&24&0&0&0&0&0\\13&22&31&40&0&0&0&0\\16&27&38&49&60&0&0&0\\19&32&45&58&71&84&0&0\\22&37&52&67&82&97&112&0\\25&42&59&76&93&110&127&144\end{bmatrix}$$

The diagonal matrix b in this last formula consists of odd integers. Matrix a is obtained from the row numbers of the numbers in matrix b. This $M_{ss}^{'}$ matrix gives us the ordinal numbers of non-prime numbers. Also, the formed $M_{ss}^{'}$ matrix is a sub-triangular matrix, and the numbers on the prime diagonal of the $M_{ss}^{'}$ matrix give the sequence numbers of the exact squares of the odd non-prime numbers. For example, the main diagonal of the matrix $M_{ss}^{'}$; 12, 24, 40, 60, 84, 112, 144 non-prime rank numbers correspond to 9, 25, 49, 81, 121, 169, 225, 289 respectively. Here, the $M_{ss}^{'}$ matrix can be further generalized if desired. With the help of a matrix with an infinite number of rows and columns, formulas of the number of rows of non-prime numbers will also be given in the following theorem.

Theorem 2. *$k, n \in Z^{+}$, k is the number of lines, we have*

$$M_{ss}^{'row}=2n(n+1)+(k-1)(2n+1),\ \ k\geq 1.$$

We will call this formula the matrix row formula for non-prime numbers. Let's examine this formula on an example below. Line 1 formula is as follows, so for $k=1$ we have $M_{ss}^{'row}=2n^2+2n$. For $k=2$,

$$M_{ss}^{'row}=2n^2+2n+1+1.(2n+1)=2n^2+4n+1.$$

Continuing in this way, the desired equations for lines $k=3,4,5,\ldots$ can be written similarly. These operations can also be written in matrix form as follows:

$$\begin{array}{r c}
 & \begin{array}{cccc}\text{n=1} & \text{n=2} & \text{n=3} & \cdots\end{array}\\
\begin{array}{r}2n^2+2n\\2n^2+4n+1\\2n^2+6n+2\\\vdots\end{array} & \begin{bmatrix}4&12&24&\cdots\\7&17&31&\ldots\\10&22&38&\ldots\\\vdots&\vdots&\vdots&\ddots\end{bmatrix}
\end{array}$$

The formula given here is the generalized form of the formula $a\times b+s$=$M_{ss}^{'}$. The first column in this matrix returns the number of rows of odd multiples of the number 3. The second column gives the number of rows of odd multiples of number 5. The third column gives the number of rows of odd multiples of number 7, and so on. Now, let's give the column formula below for the expanded $M_{ss}^{'}$ matrix. The sequence numbers of non-prime numbers from the matrix row formula can be written in the matrix form, or the same operations can be arranged as a column formula:

Theorem 3. *For $a, b \in Z^{+}$, $b\geq 1$, we have*

$$M_{ss}^{'column}=(3a+1)+(b-1)(2a+1).$$

Where b is the number of columns.

Let's examine this formula for some values: According to the formula, when the 1st column formula, $b=1$, the following equation is obtained.

$$M_{ss}^{'column}=(3a+1).$$

For $b=2$, we get

$$M_{ss}^{'column}=(3a+1)+1(2a+1)=(5a+2).$$

Continuing with the same idea, similar equations can be written for b=3 ,4, 5 $\ldots$ lines. These equations can be written again in matrix form as follows:

$$\begin{array}{c} \\ 3a+1 \\ 5a+2 \\ 7a+3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} a=1 & a=2 & a=3 & \cdots \end{array} \\ \begin{bmatrix} 4 & 7 & 10 & \cdots \\ 7 & 12 & 17 & \dots \\ 10 & 17 & 24 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Note that this matrix has a diagonal and symmetrical matrix feature. Lower and upper triangular matrices can also be created with the help of symmetrical matrix.

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 7 & 12 & 0 & 0 \\ 10 & 17 & 24 & 0 \\ 13 & 22 & 31 & 40 \end{bmatrix} or \begin{bmatrix} 4 & 7 & 10 & 13 \\ 0 & 12 & 17 & 22 \\ 0 & 0 & 24 & 31 \\ 0 & 0 & 0 & 40 \end{bmatrix}$$

The formula given in the theorem can be considered as the formula that gives the number of rows in the column. Therefore, we will call this formula the matrix column formula for non-prime numbers. Similarly, we will create a matrix of row numbers of prime numbers, and we will show the matrix of row numbers of these prime numbers as M_{ss}. First, some prime numbers are given below.

$$\{3, 5, 7, 11, 13, 17, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, \ldots\}.$$

The line numbers of these prime numbers are as follows.

$$M_{ss} = \{1, 2, 3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 21, 23, 26, 28, 30, 33, \ldots\}.$$

Here, the elements of the following x and y matrices are integers, and the columns of the z matrix are the following formula, consisting of the ordinal numbers of the prime numbers:

$$M_{ss} = xy + z$$

$$M_{ss}^{'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 & 8 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 10 & 24 \\ 1 & 2 & 3 & 5 & 6 \\ 2 & 6 & 6 & 10 & 12 \\ 1 & 4 & 6 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

$$M_{ss}^{'} = \begin{bmatrix} 1 & 2 & 3 & 5 & 6 \\ 2 & 6 & 6 & 15 & 30 \\ 3 & 8 & 9 & 20 & 36 \\ 5 & 14 & 15 & 30 & 48 \\ 6 & 18 & 21 & 35 & 54 \end{bmatrix}$$

$M_{ss}^{'}$ and M_{ss} matrices are applications on the matrices, except for the table representation of non-prime numbers and the sequence numbers of prime numbers, respectively.

Definition 3. *The formula giving the $M_{ss}^{'}$ numbers on the prime diagonal can be given as follows. K_{ss}=M_{ss} $(1 + M_s)$. Here K_{ss} is the sequence number of the square of non-prime numbers. For example,*

$$M_s = 7,\ M_{ss} = 3,\ K_{ss} = 3(1 + 7) = 24,\ M_{ss}^{'}(24) = 49 = 7^2$$

$$M_s = 11,\ M_{ss} = 5,\ K_{ss} = 5(1 + 11) = 60,\ M_{ss}^{'}(60) = 121 = 11^2$$

Definition 4. *Let $M_s = x$. Then, we write*

$$M_{ss} = \frac{(x-1)}{2}\ and\ K_{ss} = (\frac{x-1}{2})(1+x) = \frac{(x^2-1)}{2}.$$

Here, the following definition can be given if the formula obtained to find non-prime numbers in the row and column in the graph of the formulas K_{ss} and $M_{ss}^{('column)}$ is developed.

Theorem 4. *For $b \in Z^{odd}$ and $b > 2$, we have*

$$\frac{(b^2-1)}{2},\ \frac{(b^2+2b-1)}{2},\ \frac{(b^2+4b-1)}{2},\ \frac{(b^2+6b-1)}{2},\ \ldots \tag{1}$$

For $b = 3$, the ordinal numbers of the multiples of 3 are obtained. So, $4, 7, 10, 13, 16, 19, 22, 25, \ldots$ Accordingly, since the number of rows is 13, the number itself is 39, this number is not prime. For $b = 5$, that is, the number of rows of multiples of 5.

$$12, 17, 22, 27, 32, 37, 42, \ldots$$

The same operations are repeated for other odd numbers.

b	$\frac{b^2-1}{2}$	$\frac{b^2+2b-1}{2}$	$\frac{b^2+4b-1}{2}$	$\frac{b^2+6b-1}{2}$	$\frac{b^2+8b-1}{2}$	$\frac{b^2+10b-1}{2}$
2	1,5	3,5	5,5	7,5	9,5	11,5
3	4	7	10	13	16	19
4	7,5	11,5	15,5	19,5	23,5	27,5
5	12	17	22	27	32	37
6	17,5	23,5	29,5	35,5	41,5	47,5
7	24	31	38	45	52	59
8	31,5	39,5	47,5	55,5	63,5	71,5
9	40	49	58	67	76	85
10	49,5	59,5	69,5	79,5	89,5	99,5
11	60	71	82	93	104	115
12	71,5	83,5	95,5	107,5	119,5	131,5
13	84	97	110	123	136	149

Observation results

1) The number of rows of no prime numbers is found in the table above.
2) When an even number is given instead of b, it is seen that even numbers correspond to the number of rows.
3) When odd number values are given instead of b, it is seen that the result is odd non-prime numbers.
4) When odd numbers are given instead of b, single multiples of the given number are obtained, respectively.

2 References

1 J.J O'Connor, J.J., Robertson, E.F., *Prime Numbers*, (2019), http://www-history.mcs.st-andrews.ac.uk/HistTopics/Primenumbers.html.
2 https://www.mersenne.org/. Access date: 13.07.2020.
3 C. Özgü, *A Study on Prime Number Patterns and Goldbach Conjecture*, Master Thesis, Ege University Institute of Science, International Computer Program, (2002).
4 T. Yerlikaya, T., Kara, O., *Prime Number Testing Algorithms Used in Cryptology*, Trakya Uni.J.Eng.Sci., **18**(1), (2017).

Conference Proceeding Science and Technology, 3(2), 2020, 247–251

Conference Proceeding of 9th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2020).

On Excellent Safe Primary Numbers and Encryption

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Nazlı Koca[1,*] *Serpil Halıcı*[2]

[1] *Pamukkale Uni., Depart. of Math. Denizli,Turkey, ORCID: 0000-0002-4663-6253*
[2] *Pamukkale Uni., Depart. of Math. Denizli,Turkey, ORCID: 0000-0002-8071-0437*
* *Corresponding Author E-mail: nkoca17@posta.pau.edu.tr*

Abstract: In this study, we firstly included the definition of perfectly safe prime numbers that we created. We then examined the RSA and Rabin cryptosystem, which are the techniques of implementing prime numbers in encrypting any message. Finally, we used these perfectly safe prime numbers, which were first defined, in RSA and Rabin's encryption methodsssss.

Keywords: Prime Numbers, Distribution of Primes, Applications of Prime Numbers.

1 Introduction and Preliminaries

Prime numbers are also used to send encrypted messages in the field Cryptography in the past and today. In cryptography, each user has two different keys, one open and one secret for encryption and decryption. The public key is public and anyone can see it. The secret key is kept confidential and should not be known to anyone other than its owner. Encryption is done with a public key, while the decryption is done with a private key. These encryption keys are used in binary and the other key decodes the information that one encrypts. The most known and used encryption method today is the RSA encryption method, the public key encryption method. The name of this system carries the surnames of scientists named Ron Rivest, Adi Shamir and Leonard Adlemen in 1978, namely the method was named after factoring the integers [1]. Another public key encryption method is Rabin's encryption technique. The Rabin's encryption method is a crypto system found in 1979 by Michael Rabin. This cryptosystem is based on the asymmetric encryption technique. Rabin's cryptosystem, which is a different type of RSA method, takes advantage of the difficulty of factoring the compound numbers as in RSA [2].In this study, we will start with the definition of perfectly safe prime numbers that we will use in encryption with prime numbers. Then we will use the perfect numbers that we have created in RSA and Rabin encryptions.

Definition 1. *When applying the formula* $\frac{(k-1)}{2}$ *for* $k > 0$ *and* $k \in \mathbb{Z}^+$ *the lower row number of* k *is obtained.*

For example, the lower row number of 17 is 8.

Definition 2. *When applying the formula* $\frac{(k+1)}{2}$ *for* $k > 0$ *and* $k \in \mathbb{Z}^+$ *the upper row number of* k *is obtained.*

For example, the upper row number of 21 is 11.

Definition 3. *Let* p *be the prime number. So that if* p*'s lower row number and upper row number is prime number,* p *number is called perfect safe prime number.*

For example, $p = 5$ prime number is six row number 2 and top row number is 7. So, $p = 5$ is the perfect safe prime number, since the lower row number and the upper row number are prime. Some perfectly safe prime numbers are: $5, 11, 23, 83, 179, 359, 719, \cdots$.
In the table 1.2 below, consecutive odd numbers are given in the white region. The upper row numbers are written on the odd numbers in this white section. To the next higher row, the upper row number of the number written below will be written. Also, the lower row number of these numbers is written in the lower part of the numbers in the white part. If the result is an even number when writing down the lower row number, it is not continued. Because we will do our operations on odd numbers. Numbers indicated in red are prime numbers. The prime number in the middle of 3 consecutive prime numbers constitutes perfectly safe prime numbers. We highlighted the perfectly safe prime number in green.

255	383	511	639	767	895	1023	1151	1279	1407	1535	1663	1791
127	191	255	319	383	447	511	575	639	703	767	831	895
63	95	127	159	191	223	255	287	319	351	383	415	447
31	47	63	79	95	111	127	143	159	175	191	207	223
15	23	31	39	47	55	63	71	79	87	95	103	111
7	11	15	19	23	27	31	35	39	43	47	51	55
3	5	7	9	11	13	15	17	19	21	23	25	27
1	2	3	4	5	6	7	8	9	10	11	12	13
		1		2		3		4		5		6
						1				2		

Table 1.1 Perfect safe prime numbers

1919	2047	2175	2303	2431	2559	2687	2815	2943	3071	3199	3327	3479
959	1023	1087	1151	1215	1279	1343	1407	1471	1535	1599	1663	1739
479	511	543	575	607	639	671	703	735	767	799	831	869
239	255	271	287	303	319	335	351	367	383	399	415	431
119	127	135	143	151	159	167	175	183	191	199	207	215
59	63	67	71	75	79	83	87	91	95	99	103	107
29	31	33	35	37	39	41	43	45	47	49	51	53
14	15	16	17	18	19	20	21	22	23	24	25	26
	7		8		9		10		11		12	
	3				4				5			
	1								2			

Table 1.2 Perfect safe prime numbers

Let's give the above table with the following theorem.

Theorem 1. *Perfect numbers are increasing rapidly as prime numbers progress irregularly.*

From here we can say that it is not easy to reach Perfect prime numbers.
In the following sections, we will use this feature of perfect numbers to use to encrypt messages. Modular mathematics is used in the RSA algorithm. Below we n ,coutline the RSA algorithm.

2 Encryption with Primary Numbers

Public Key Generation Algorithm. Each person A creates their public key as follows:
Selects two prime numbers, p and q, which are different, random and have the same number of digits. Calculates $n = pq$ and $\varphi = (p-1) \times (q-1)$.
Selects a random c number such that $1 < c < \varphi$ and $(c, \varphi(n)) = 1$. Using the Euclidean algorithm, it calculates the number d satisfying the condition.
$1 < d < \varphi(n)$ and $cd \equiv 1(mod\varphi(n))$
Person A has a public key (n, c) and his secret key becomes d.
So, encryption can generally be given as follows: The pair $(n, c)$the public key, is known to the person sending the information.
The explicit text is converted to an integer such that $M \in [0, n-1]$.
The encrypted text is calculated as $C \equiv M^c\,(modn)$.The person sending the information sends the C encrypted text to the person receiving the information [3].

$\boxed{\text{PLAIN TEXT} \Rightarrow \text{ENCRYPTION} \Rightarrow \text{ENCRYPED TEXT} \Rightarrow \text{DECODING} \Rightarrow \text{PLAIN TEXT}}$

Algorithm 2.1 Encryption and decryption algorithm

The flow chart for the RSA algorithm is given in the figure below.

SENDER
$\Downarrow$
OPEN TEXT (M)
$\Downarrow$
NUMBER BLOCKS $(M_1, M_2, ...)$
$\Downarrow$
ENCRYPED TEXT $C_1 \equiv M_1^e modn \quad C_2 \equiv M_2^e modn$
$\Downarrow$
DECODED TEXT $M_1 \equiv C_1^e$ mod n $\quad M_2 \equiv C_2^e$ mod n
$\Downarrow$
RECEIVER
$\Downarrow$
RANDOM NUMBER (e)
HIDDEN KEY$\hookleftarrow \quad \hookrightarrow$ OPEN KEY

Algorithm 2.2 Flow Chart of RSA Algorithm

Encryption Algorithm. If person B wants to send an m message to A, person B does the following to encrypt the message m:

- First, he learns the public key (n, e) of person A.
- Writes the message m in the range $[0, n-1]$.
- Then it calculates

$C \equiv M^c\ (modn)$.

- Sends the formed C password to person A.

To find clear text from encrypted C text, person A does the following:

- By using the secret key d and executing $m \equiv cd(modn)$, m reaches the open text [4].

For example, choose prime numbers $p = 11$, $q = 17$.

Then, $n = 11 \times 17 = 187$ and $\varphi(n) = (11-1)(17-1) = 160$ For $1 < c < \varphi(n)$, choose $c = 3$. And let $d = 107$ satisfy $dc \equiv 1(mod\varphi(n))$. According to this, $dc \equiv 321 \equiv 1(mod\varphi(n))$.

Definition 4. *Let p be the prime number. So that if p is the number of the previous row and the number of next row is the prime number, p number is called the perfect safe prime number.*

For example, p is the perfectly safe prime number since $p = 5$ is the prime number, and the next 7 is the prime number. Some perfectly safe prime numbers are 5, 11, 23, 83, 179, 359, 719,...

3 RSA Method and Perfect Prime Numbers

Now, in this section, an application of the RSA security algorithm for perfectly safe prime numbers will be given. Let p and q be two consecutive perfectly safe prime numbers: $p = 11, q = 23$ then we write $n = (2q+1)(2q+1) = 23.47 = 1081$ $\varphi(n)$=$\frac{(p-1)}{2} \times \frac{(q-1)}{2}$= 5×11=55. Let's choose a random number e. And $e = 19$ with $1 < e < n$ and $(e, \varphi(n)) = 1$. $d = 29$ can be taken as a number that satisfies $de \equiv 1(mod\varphi(n))$. Since, $de \equiv 551 \equiv 1(mod\varphi(n))$ the public key becomes $(n, e) = (1081, 19)$. Thus, the closed key is $(n, d) = (1081, 29)$.
As a example let us choose $p = 5, q = 11$ consecutive perfectly safe prime numbers. $n = (2p+1) \times (2q+1) = 253$ $\varphi(n) = \frac{(p-1)}{2}.\frac{(q-1)}{2} = 10$.
Let's choose a random number e. Let this number be prime between $1 < e < n$ and $(e, \varphi(n)) = 1$. If we take $e = 3$, then we have $de \equiv 1(mod\varphi(n)), d = 7$ and $de = 21 \equiv 1(mod\varphi(n))$. The public key is
$(n, e) = (253, 3)$ and the closed key is $(n, e) = (253, 7)$.
Now, let the text you want to send be "SIFRE". If the letters are numbered respectively,$S = 1$, $I = 2$, $F = 3$, $R = 4$, $E = 5$. The person to whom the message will be sent and anyone who reads the message is given the numbers n calculated above and randomly selected e. Calculation is made according to
$e = 3$ and $(mod10)$ for $SIFRE = 12345$.

$$
\begin{gathered}
1^3 \equiv 1\ (mod10) \\
2^3 \equiv 8\ (mod10) \\
3^3 \equiv 7\ (mod10) \\
4^3 \equiv 4\ (mod10) \\
5^3 \equiv 5\ (mod10).
\end{gathered}
$$

So the password to be sent will be 18745. Deciphering is completed by converting this number into "SIFRE" text using our codes. For example, letters are matched to numbers between integer values in $[0, \varphi(n-1)]$.

A	I	K	M	N	O	R	S	U	Y
0	1	2	3	4	5	6	7	8	9

Table 3.1 Encryption table of some letters.

To be able to send a message using the letters we have numbered above, let's encrypt our message first. We find the values required for encryption as $\varphi(n) = 10$ and $e = 3$.

$Message$	M	I	R	A	Y	K	O	N	U	S	U	Y	O	R
$Number\,Value$	3	1	6	0	9	2	5	4	8	7	8	9	5	6
$Power\ of\ e$	3^3	1^3	6^3	0^3	9^3	2^3	5^3	4^3	8^3	7^3	8^3	9^3	5^3	6^3
$Value$	27	1	216	0	729	8	125	64	512	343	512	729	125	216
$Remaining$	7	1	6	0	9	8	5	4	2	3	2	9	5	6
$Password\ message$	S	I	R	A	Y	U	O	N	K	M	K	Y	O	R

Table 3.2 message encryption table.

The person who receives our encrypted message will do the following to turn this message into understandable text.

$Password\ message$	S	I	R	A	Y	U	O	N	K	M	K	Y	O	R
$Power\ of\ d$	7^7	1^7	6^7	0^7	9^7	8^7	5^7	4^7	2^7	3^7	2^7	9^7	5^7	6^7
$Value$	27	1	216	0	729	8	125	64	512	343	512	729	125	216
$Remaining$	3	1	6	0	9	2	5	4	8	7	8	9	5	6
$Message$	M	I	R	A	Y	K	O	N	U	S	U	Y	O	R

Table 3.3 encryption message decoding.

It should be noted that for open the encrypted message we need to know the secret key $d = 7$.

In the section below, we have examined a different encryption method using the special prime numbers we have defined.

Rabin's Encryption Algorithm. This method is a cryptosystem method given by Michael Rabin in 1979. This cryptosystem is an asymmetric encryption technique. The Rabin's cryptosystem is a version of RSA. But there are some differences between them. Some of these are given below.

It is a great advantage over RSA that it is not easy for any text in the Rabin's cryptosystem to be able to obtain the encrypted text by someone who can factor only the n public key. The decoding of the Rabin cryptosystem proved to be equivalent to the factor of dividing integers. Therefore, the Rabin method is safer than the RSA method. When $e = 3$ is taken, there is a modular multiplication and a modular squared operation in the RSA method. However, since there is only one modular squaring process in the Rabin method, the processes are faster than the RSA method [2]. The disadvantage of the Rabin's cryptosystem is difficult to find out which of the four different keys is correct. This is a disadvantage compared to the RSA method, although it is attempted to select the correct output with the addition method [3].

The key generation in the Rabin cryptosystem is done as follows:

Approximately the same size, large and random p, q prime numbers are selected. $n = p \times q$ is calculated. Therefore, the number n is the public key and the (p, q) ordered pair is the secret key. When encrypting in the Rabin's cryptosystem method, someone who wants to encrypt the m message is created with the number m not less than n. The sender's public key is reached. And the equivalence of $c \equiv m^2 (modn)$ is calculated. The sender sends the encrypted c message to the person he wants.

Password decoding with the Rabin's cryptosystem method is done as follows:

The person receiving the encrypted message calculates the m value using his private key.

Accordingly, there are two solutions of m_p and m_q . Using the extended Euclidean algorithm, $y_p \times p + y_q \times q = 1$ is calculated. Then the roots of m_1, m_2, m_3, m_4 are calculated as follows.

$m_1 = (y_p \times p \times m_q + m_q \times q \times m_p)(modn)$, $m_2 = n - m_1$, $m_3 = (y_p.p.m_q - m_q.q.m_p)(modn)$, $m_4 = n - m_3$

Here, one of the roots m_1, m_2, m_3, m_4 is the key for us to open the sent text. For example, for person A key generation; $p = 277$and $q = 331$ by choosing the prime numbers, he finds $n = 91687$. The public key of person A is $n = 91687$ and the secret key is $p = 277$, $q = 331$. For the encryption process, the message of person A to be encrypted is $m = 40569$. Then, $c \equiv 40569^2 (mod 91687) = 62111$ is obtained. This found c message is sent to person B.

Decryption. Person B calculates c encrypted text with $n = 91687$ public key, four square roots of c according to $(modn)$ The following four roots are available from different roots:

$m_1 = 69654$, $m_2 = 22033$, $m_3 = 40569$, $m_4 = 51118$.

Here m_3 opens the message c, which is the encrypted text, and reaches the original, real message. Let's examine the Rabin's method using the perfect prime numbers in the example below:

Our message to be encrypted: get MRY.

M	1
I	0
R	5
A	7
y	8

Table 3.4 Coding table of some letters.

We matched the letters with the numbers we chose arbitrarily. Now let's choose perfectly safe primes $p = 11$ and $q = 23$. From here, $n = p.q = 253$. Our public key is $n = 253$, and our private key is $(p, q) = (11, 23)$.
Let's choose the number m as $m = 158$ so that $m < n$. So, $c \equiv 158^2(mod253) = 170$.
Our secret message we send is MAI. The recipient of our secret message with MAI, using his public key, $n = 253$ $m_{11} \equiv 170^{\frac{(11+1)}{4}}(mod11) = 4$ and $m_{23} \equiv 170^{\frac{(23+1)}{4}}(mod23) = 3$ finds their numbers. Using the extended Euclidean algorithm, $y_p \times p + y_q \times q = 1$. There are unknown things from equality:
$y_p = -2$, $y_q = 1$. The roots are follows:
$m_1 = (-2 \times 11 \times 3 + 1 \times 23 \times 4)(mod253) = 26$,
$m_2 = 253 - 26 = 227$,
$m_3 = (-2 \times 11 \times 3 - 1 \times 23 \times 4)(mod253) = 95$,
$m_4 = 253 - 95 = 158$.
The root that opens the real message here is the number m_4. So, our encrypted MAI secret message is calculated by the public key, $n = 253$, the actual message $m_4 = 158$ found; MRY opens.

4 Conclusions

As a result, we have created perfectly safe prime numbers by taking advantage of the irregular progression of the prime numbers. Thus, we used the perfectly secure prime numbers we obtained in RSA and RABIN cryptosystems and thus made encryption more secure.

5 References

1 Akbar, A. A., *Asal Sayıların Sifreleme Teorisindeki Uygulamalari*, Yüksek Lisans Tezi, Atatürk Uni. Fen Bil. Enst., Erzurum, (2015).
2 http://bilgisayarkavramlari.sadievrenseker.com, 2009, **6**(4), /rabin-sifreleme. 1.06.2020.
3 Beskirli, A., Ozdemir, D., Beskirli, M., *Sifreleme Yontemleri ve RSA Algoritması Uzerine Bir Inceleme*, Avrupa Bilim ve Teknoloji Derg., Ozel sayi, 2019, 284-291.
4 Zuckerman, H.S., Niven, I. And Monygomery H.L., *An Introduction To The Theory Of Numbers*, New York: John Wiley Sons, 1991, 25-26.