JOURNAL OF MATHEMATICAL SCIENCES AND MODELLING

ISSN: 2636-8692

VOLUME III
ISSUE III

VOLUME III ISSUE III
ISSN 2636-8692

December 2020
http://dergipark.gov.tr/jmsm

JOURNAL OF MATHEMATICAL SCIENCES AND MODELLING

Editors

Editorial Board of Journal of Mathematical Sciences and Modelling

Contents

Journal of Mathematical Sciences and Modelling, 3 (3) (2020) 102-119
Research Article

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm
ISSN 2636-8692
DOI: http://dx.doi.org/10.33187/jmsm.748724

Dynamics and Bifurcation of a Second Order Quadratic Rational Difference Equation

Shahd Herzallah[1] and Mohammad Saleh[1*]

[1]*Department of Mathematics, Birzeit University, P.O.Box 14, West Bank, Palestine*
[*]*Corresponding author*

Article Info

Keywords: *Fixed point, Neimark-Sacker bifurcation, Stability.*
2010 AMS: *3A50, 39A23, 39A28, 39A30.*
Received: *6 June 2020*
Accepted: *18 December 2020*
Available online: *29 December 2020*

Abstract

In this paper, we study the dynamics and bifurcation of

$$x_{n+1}=\frac{\alpha+\beta x^2_{n-1}}{A+Bx_n+Cx^2_{n-1}},\ n=0,\ 1,\ 2,\ \ldots$$

with positive parameters α, β, A, B, C, and non-negative initial conditions.
Among others, we investigate local stability, invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points.

1. Introduction

In this paper, we consider the second order, quadratic rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x^2_{n-1}}{A+Bx_n+Cx^2_{n-1}},\ n=0,\ 1,\ 2,\ \ldots$$

with positive parameters α, β, A, B, C, and non-negative initial conditions.
We focus on local stability, invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points.
Global asymptotic stability and Neimark-Sacker bifurcation of the difference equation

$$x_{n+1}=\frac{F}{bx_nx_{n-1}+Cx_{n-1}{}^2+f},\ n=0,\ 1,\ 2,$$

have been investigated by M. R. S. Kulenović et al. [1], with non-negative parameters and non-negative initial conditions such that the denominator is always positive.
Y. Kostrov and Z. Kudlak in [2] studied the boundedness character, local and global stability of solutions of the following second-order rational difference equation with quadratic denominator,

$$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{B+Dx_nx_{n-1}+x_{n-1}},\ n=0,\ 1,\ 2,$$

where the coefficients are positive numbers, and the initial conditions are non-negative numbers such that the denominator is nonzero.
S. Moranjkić, and Z. Nurkanović [3] investigated local and global dynamics of difference equation

$$x_{n+1}=\frac{Bx_nx_{n-1}+Cx_{n-1}{}^2+F}{bx_nx_{n-1}+cx_{n-1}{}^2+f},\ n=0,\ 1,\ 2,$$

with positive parameters and nonnegative initial conditions. Other higher ordered rational difference equations have been recently studied in [4], [5],[6],[7],[8],[9],[10],[11].

Email addresses and ORCID numbers**: shoshoherzallah@gmail.com, https://orcid.org/0000-0001-9771-1703 (S. Herzallah), msaleh@birzeit.edu, https://orcid.org/0000-0002-4254-2540 (M. Saleh)**

2. Preliminaries

Before studying the behavior of solutions of this rational difference equation, we will review some definitions and basic results that will be used throughout this paper.
Consider the second order difference equation,

$$x(n+1) = f(x(n), x(n-1)),\ n = 0,\ 1,\ 2,\ ... \tag{2.1}$$

where $f : I \times I \to I$ is a continuously differentiable function, and I is an interval of real numbers. Then for every set of initial conditions $x_{-1},\ x_0 \in I$ the difference equation (2.1) has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Definition 2.1. *[12] A point $\bar{x} \in I$ is an equilibrium point of equation (2.1) if $f(\bar{x}, \bar{x}) = \bar{x}$.*

Definition 2.2. *[12] Consider the difference equation (2.1). Then the linearized equation associated with this difference equation is*

$$y_{n+1} = ay_n + by_{n-1},\ n = 0,\ 1,\ 2,\ ...$$

where $a = \frac{\partial f}{\partial u}(\bar{x}, \bar{x})$, and $b = \frac{\partial f}{\partial v}(\bar{x}, \bar{x})$.
And the characteristic equation of (2.1) is

$$\lambda^2 - a\lambda - b = 0 \tag{2.2}$$

Theorem 2.3. *[13] (Linearized Stability)*
Consider the characteristic equation (2.2).

1. *If both characteristic roots of (2.2) lie inside the unit disk in the complex plane, then the equilibrium $\bar{x}$ of (2.1) is locally asymptotically stable.*
2. *If at least one characteristic root of (2.2) is outside the unit disk in the complex plane, the equilibrium point $\bar{x}$ is unstable.*
3. *If one characteristic root of (2.2) is on the unit disk and the other characteristic root is either inside or on the unit disk, then the equilibrium point $\bar{x}$ may be stable, unstable, or asymptotically stable.*
4. *A necessary and sufficient condition for both roots of (2.2) to lie inside the unit disk in the complex plane, is*

$$|a| < 1 - b < 2.$$

Let $A = Jf(\bar{x})$ be the Jacobian matrix of f at $\bar{x}$, where

$$Jf(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} |_{\bar{x}}$$

An important way to determine the stability of fixed points is given in the following result.

Theorem 2.4. *[14] Consider the map $f : H \subset \mathbb{R}^2 \to \mathbb{R}^2$, and let $A = Jf(\bar{x})$, with spectral norm $\rho(A)$. Then $\rho(A) < 1$, if and only if*

$$|tr(A)| - 1 < det(A) < 1$$

where $tr(A)$ is the trace of A, and $det(A)$ is the determinant of A.

The following theorem will be used to investigate global stability of fixed points.

Theorem 2.5. *[12] Let $[a,b]$ be an interval of real numbers and assume that $f : [a,b] \times [a,b] \to [a,b]$ is a continuous function satisfying the following properties:*

1. *$f(x,y)$ is non-increasing in $x \in [a,b]$ for each $y \in [a,b]$, $f(x,y)$ is non-decreasing in $y \in [a,b]$ for each $x \in [a,b]$.*
2. *The difference equation (2.1) has no solutions of prime period two in $[a,b]$.*
   Then (2.1) has a unique equilibrium $\bar{x} \in [a,b]$ and every solution of (2.1) converges to $\bar{x}$.

There are several types of bifurcation, the saddle-node bifurcation, period-doubling bifurcation, Neimark-Sacker bifurcation. For more information on types of bifurcation, the readers can refer to [15].

3. Dynamics of $x_{n+1} = \frac{\alpha + \beta x^2_{n-1}}{A + Bx_n + Cx^2_{n-1}}$

In this section, we consider the second order quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x^2_{n-1}}{A + Bx_n + Cx^2_{n-1}},\ n = 0,\ 1,\ 2,\ ... \tag{3.1}$$

with positive parameters $\alpha,\ \beta,\ A,\ B,\ C$, and non-negative initial conditions.

3.1. Change of variables

The change of variables

$$x_n = \frac{A}{B} y_n.$$

reduces equation (3.1) to the difference equation

$$y_{n+1} = \frac{p + q{y^2}_{n-1}}{1 + y_n + r{y^2}_{n-1}}, \; n = 0,\ 1,\ 2,\ \ldots$$

Where $p = \alpha \frac{B}{A^2}$, $q = \frac{\beta}{B}$, and $r = \frac{CA}{B^2}$.

3.2. Equilibrium points

To find the equilibrium point of

$$y_{n+1} = \frac{p + q{y^2}_{n-1}}{1 + y_n + r{y^2}_{n-1}}, \; n = 0,\ 1,\ 2,\ \ldots \tag{3.2}$$

with positive parameters $p,\ q,\ r$, and non-negative initial conditions. We solve the following equation

$$\bar{y} = \frac{p + q\bar{y}^2}{1 + \bar{y} + r\bar{y}^2.}$$

Hence,

$$r\bar{y}^3 + (1-q)\bar{y}^2 + \bar{y} - p = 0. \tag{3.3}$$

can be considered as two curves with behavior

$$r\bar{y}^2 + (1-q)\bar{y} = \frac{p}{\bar{y}} - 1.$$

Equation (3.2) has a unique positive equilibrium point $\bar{y}$, which can be obtained as an intersection point of these two curves. From Figure 3.1 and Figure 3.2 we obtain the required conclusion.

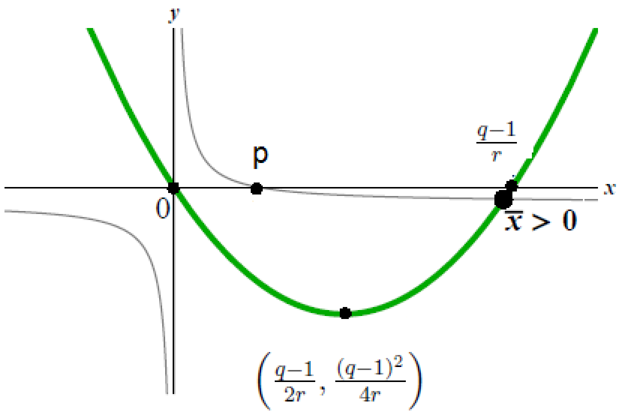

Figure 3.1: The equilibrium of (4.2.1), $q > 1$.

And then we choose the positive root to be $\bar{y}$.

3.3. Linearized equation

To find the linearized equation of (3.2) about the equilibrium point $\bar{y}$, let

$$f(x,y) = \frac{p + qy^2}{1 + x + ry^2}$$

We have

$$\frac{\partial f}{\partial x}(\bar{y}, \bar{y}) = \frac{-\bar{y}}{1 + \bar{y} + r\bar{y}^2}.$$

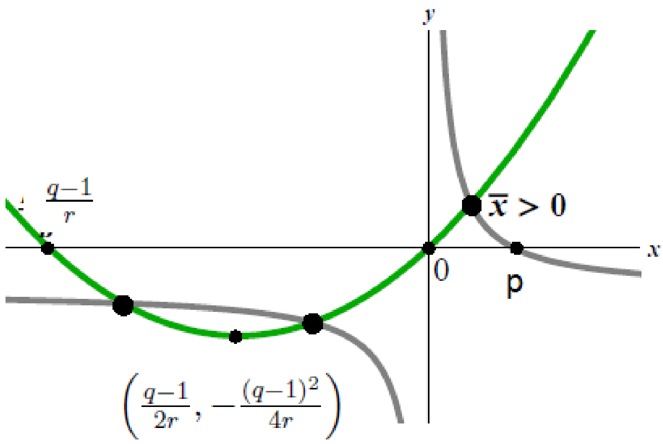

Figure 3.2: The equilibrium of (4.2.1), $0 < q < 1$.

And

$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}.$$

The linearized equation is

$$y_{n+1} = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2}y_n + \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}y_{n-1}.$$

And the characteristic equation is

$$\lambda^2 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}\lambda - \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 0.$$

3.4. Local stability

To check when the unique positive equilibrium point $\bar{y}$ of equation (3.2) is locally asymptotically stable, let

$$a = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2},\ b = \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}$$

Using Theorem 2.3 (4), a sufficient condition for asymptotic stability of $\bar{y}$ is $|a| < 1-b < 2$, which is equivalent to

$$-b < 1, \tag{3.4}$$
$$\textit{and } |a| < 1-b. \tag{3.5}$$

(3.4) holds when

$$q > \frac{-1+r\bar{y}^2-\bar{y}}{2\bar{y}}.$$

And (3.5) is equivalent to

$$a > -1+b, \tag{3.6}$$
$$\textit{and } a < 1-b. \tag{3.7}$$

(3.6) holds when

$$q < \frac{1+3r\bar{y}^2}{2\bar{y}}.$$

And (3.7) holds when

$$q < \frac{1+2\bar{y}+3r\bar{y}^2}{2\bar{y}}.$$

Hence a sufficient conditions for asymptotic stability of $\bar{y}$ is

$$q > \frac{-1+r\bar{y}^2-\bar{y}}{2\bar{y}}.$$

$$q < \frac{1+3r\bar{y}^2}{2\bar{y}}. \tag{3.8}$$

$$q < \frac{1+2\bar{y}+3r\bar{y}^2}{2\bar{y}}. \tag{3.9}$$

Note that if (3.8) holds, then (3.9) holds, thus $\frac{-1+r\bar{y}^2-\bar{y}}{2\bar{y}} < q < \frac{1+3r\bar{y}^2}{2\bar{y}}$ is a sufficient condition for asymptotic stability of $\bar{y}$.

3.5. Invariant intervals

Consider the difference equation (3.2), and $\{y_n\}_{n=-1}^{\infty}$ as a solution. Then $[0, \frac{q}{r}]$ when $pr \leq q$ is an invariant interval.

Proof. Assume that $pr \leq q$, and $y_{N-1},\ y_N \in [0, \frac{q}{r}]$ for some integer N.

$$\begin{aligned} y_{N+1} &= \frac{p+qy^2{}_{N-1}}{1+y_N+ry^2{}_{N-1}} \\ &= \frac{q(\frac{p}{q}+y^2{}_{N-1})}{r(\frac{1}{r}+\frac{1}{r}y_N+y^2{}_{N-1})} \\ &\leq \frac{q(\frac{1}{r}+y^2{}_{N-1})}{r(\frac{1}{r}+y^2{}_{N-1})} \\ &= \frac{q}{r} \end{aligned}$$

And working inductively we complete the proof. □

3.6. Boundedness

We will show that every solution of the difference equation (3.2) is bounded. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of (3.2). then we have for $n = 0,\ 1,\ 2,\ ...$

$$\begin{aligned} 0 < y_{n+1} &= \frac{p+qy^2{}_{n-1}}{1+y_n+ry^2{}_{n-1}} \\ &= \frac{p}{1+y_n+ry^2{}_{n-1}} + \frac{qy^2{}_{n-1}}{1+y_n+ry^2{}_{n-1}} \\ &\leq \frac{p}{1} + \frac{qy^2{}_{n-1}}{ry^2{}_{n-1}} \\ &= p + \frac{q}{r}. \end{aligned}$$

Hence, the solution is bounded, since it is bounded from below and from above.

3.7. Period two cycles

In general, we say that the solution $\{y_n\}_{n=-1}^{\infty}$ has a prime period two if the solution eventually takes the form:

$$...,\ \phi,\ \psi,\ \phi,\ \psi,\ ...$$

where ϕ *and* ψ are positive, and $\phi \neq \psi$.

Theorem 3.1. *Assume that Equation (3.2) has a two periodic cycle $\{\phi,\ \psi\}$, where ϕ and ψ are positive, and $\phi \neq \psi$. Then q must satisfy the following condition:*

$$q > \frac{1+r(\phi^2+\psi^2)}{\phi+\psi}.$$

Proof. Assume $\{\phi,\ \psi\}$ is a prime period two solution of Equation (3.2), then $\phi,\ \psi$ satisfy:

$$\phi = \frac{p+q\phi^2}{1+\psi+r\phi^2} \tag{3.10}$$

and,

$$\psi = \frac{p+q\psi^2}{1+\phi+r\psi^2}. \tag{3.11}$$

From Equation (3.10), we have

$$\phi+\phi\psi+r\phi^3 = p+q\phi^2, \tag{3.12}$$

and from Equation (3.11), we have

$$\psi+\psi\phi+r\psi^3 = p+q\psi^2. \tag{3.13}$$

Subtracting Equation (3.13) from (3.12), we get:

$$(\phi-\psi)+r(\phi^3-\psi^3) = q(\phi^2-\psi^2).$$

Since, $\phi \neq \psi$, the last equation can be divided by $(\phi - \psi)$, and we get

$$1 + r(\phi^2 + \phi\psi + \psi^2) = q(\phi + \psi). \tag{3.14}$$

So,

$$\phi\psi = \frac{-1 - r(\phi^2 + \psi^2) + q(\phi + \psi)}{r}.$$

But, $\psi\phi \geq 0$, so

$$-1 - r(\phi^2 + \psi^2) + q(\phi + \psi) \geq 0.$$

Hence,

$$q > \frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi}$$

which complete the proof. Note that from (3.14), we get:

$$\phi + \psi = \frac{r(\phi^2 + \phi\psi + \psi^2) + 1}{q}$$

which is always positive. □

3.8. Global stability

Now, we will investigate a result about the global stability of the positive equilibrium point of (3.2) $\bar{y}$.

Theorem 3.2. *Assume $pr \leq q \leq \frac{\sqrt{r}}{\sqrt{2}}$. Then the positive equilibrium point $\bar{y}$ on the interval $S = [0, \frac{q}{r}]$ is globally asymptotically stable.*

Proof: This proof can be easily done depending on Theorem 2.5. Assume $pr \leq q$, and consider the function

$$f(x, y) = \frac{p + qy^2}{1 + x + ry^2}.$$

Note that S is an invariant interval and all non-negative solutions of Equation (3.2) lie in this interval. And $f(x, y)$ on S is non-increasing function in x, and non-decreasing in y.
Now, we need to show that the difference equation (3.2) has no solution of prime period two in S.
For seek of contradiction, assume that the difference equation (3.2) has a solution of prime period two $\{\phi, \psi\} \in S$. Then q must satisfy

$$q > \frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi},$$

but since $\{\phi, \psi\} \in S$

$$\frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi} \geq \frac{1 + 0}{\frac{q}{r} + \frac{q}{r}},$$

hence

$$q > \frac{r}{2q},$$

so,

$$q^2 > \frac{r}{2},$$

which is a contradiction, since $q \leq \frac{\sqrt{r}}{\sqrt{2}}$.
So, Equation (3.2) has no solution of prime period two in S. Then both conditions of Theorem 2.5 hold, so (3.2) has a unique positive equilibrium point $\bar{y} \in S$, and it is globally asymptotically stable. □

4. Bifurcation of $y_{n+1} = \frac{p+qy^2{}_{n-1}}{1+y_n+ry^2{}_{n-1}}$

In this section we study types of bifurcation that occur at $q = q^*$ as q is the bifurcation parameter.
In order to convert Equation (3.2) to a second dimensional system with three parameters p, q, and r, let

$$z_n = y_{n-1},$$

and

$$v_n = y_n.$$

We get the following system

$$z_{n+1} = v_n$$

$$v_{n+1} = \frac{p+qz^2{}_n}{1+v_n+rz^2{}_n},\ n = 0,\ 1,\ 2,\ ...$$

This system has the unique fixed point $(\bar{z},\bar{v})^T = (\bar{y},\bar{y})^T$. Convert this system into a second dimensional map

$$F\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} f_1(z,v) \\ f_2(z,v) \end{pmatrix} = \begin{pmatrix} v \\ \frac{p+qz^2}{1+v+rz^2} \end{pmatrix}. \tag{4.1}$$

So, the Jacobian matrix of $F(z,v)$ at $(\bar{y},\bar{y})$ is

$$JF(z,v)|_{(\bar{y},\bar{y})} = \begin{pmatrix} 0 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}$$

So,

$$\det(JF(\bar{y},\bar{y})) = -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2},$$

and,

$$tr(JF(\bar{y},\bar{y})) = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2}.$$

Theorem 4.1. *The fixed point $(\bar{y},\bar{y})$ of the system (4.1) undergoes a saddle-node bifurcation, when $q = \frac{3r\bar{y}^2+2\bar{y}+1}{2\bar{y}}$.*

Proof: Saddle-node bifurcation happens when,

$$-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} - 1$$

thus,

$$q = \frac{3r\bar{y}^2+2\bar{y}+1}{2\bar{y}}.$$

So, saddle-node bifurcation happens if $q = \frac{3r\bar{y}^2+2\bar{y}+1}{2\bar{y}}$. □

Theorem 4.2. *The fixed point $(\bar{y},\bar{y})$ of the system (4.1) undergoes a period-doubling bifurcation, when $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$.*

Proof: Period-doubling bifurcation happens when,

$$\det(J) = -tr(J) - 1.$$

So, the fixed point $(\bar{y},\bar{y})$ of the system (4.1) undergoes a period-doubling bifurcation if

$$-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} - 1$$

thus,

$$q = \frac{3r\bar{y}^2+1}{2\bar{y}}.$$

So, period-doubling bifurcation happens if $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$. □

Theorem 4.3. *The fixed point $(\bar{y},\bar{y})$ of the system (4.1) undergoes Neimark-Sacker bifurcation when, $q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}$, if $r > \frac{1+\bar{y}}{\bar{y}^2}$.*

Proof: Assume $r > \frac{1+\bar{y}}{\bar{y}^2}$. Neimark-Sacker bifurcation which happens when,

$$\det(J) = 1$$

and,

$$-2 < tr(J) < 2.$$

So, the system (4.1) undergoes Neimark-Sacker bifurcation when,

$$\det(JF(\bar{y},\bar{y})) = 1 \tag{4.2}$$

and,

$$-2 < tr(JF(\bar{y},\bar{y})) < 2.$$

The last inequality always holds, since it is equivalent to

$$-2 < \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} < 2,$$

which can be splitted into two inequalities, namely

$$-2 < \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2},$$

and,

$$\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} < 2,$$

The first inequality

$$-2-2\bar{y}-2r\bar{y}^2 < -\bar{y},$$

implies

$$-2-\bar{y}-2r\bar{y}^2 < 0,$$

which always holds. And $\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} < 2$ implies

$$2+3\bar{y}+2r\bar{y}^2 > 0.$$

which also always holds.
Now, Equation (4.2) holds if

$$-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 1$$

so,

$$-2\bar{y}(q-r\bar{y}) = 1+\bar{y}+r\bar{y}^2$$

thus,

$$q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}.$$

Which is positive since $r > \frac{1+\bar{y}}{\bar{y}^2}$. So the system (4.1) undergoes Neimark-sacker bifurcation at $(\bar{y},\bar{y})$ when $q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}$. □

4.1. Direction of the period-doubling (flip) bifurcation

In this subsection, we will find the direction of Flip bifurcation of system (4.1) at $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$.
We need at first to shift the fixed point $(\bar{y},\bar{y})$ to the origin. Let

$$w_n = z_n - \bar{y},\ u_n = v_n - \bar{y}.$$

System (4.1) will be

$$w_{n+1} = u_n$$

$$u_{n+1} = \frac{p+q(w_n+\bar{y})^2}{1+(u_n+\bar{y})+r(w_n+\bar{y})^2},\ n = 0,\ 1,\ 2,\ \dots$$

Or,

$$Y_{n+1} = AY_n + G(Y_n),$$

where,

$$A=\begin{pmatrix}0 & 1\\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2}\end{pmatrix},\ Y_n=\begin{pmatrix}w_n\\ u_n\end{pmatrix},$$

and,

$$G(Y)=\frac{1}{2}B(Y,Y)+\frac{1}{6}C(Y,Y,Y)+O(\|Y\|^4)$$

$$B(Y,Y)=\begin{pmatrix}B_1(Y,Y)\\ B_2(Y,Y)\end{pmatrix}\ and,\ C(Y,Y,Y)=\begin{pmatrix}C_1(Y,Y,Y)\\ C_2(Y,Y,Y)\end{pmatrix}$$

where,

$$B_i(x,y)=\sum_{k,j=1}^{n}\frac{\partial^2 Y_i(\eta)}{\partial\eta_k\partial\eta_j}|_{\eta=0}(x_k y_j)$$

and,

$$C_i(x,y,z)=\sum_{l,k,j=1}^{n}\frac{\partial^3 Y_i(\eta)}{\partial\eta_l\partial\eta_k\partial\eta_j}|_{\eta=0}(x_l y_k z_j).$$

So, $B_1(\psi,\phi)=0$ and $C_1(\psi,\phi,\xi)=0$,

$$B_2(\psi,\phi)=\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_1)-\frac{2\bar{y}(2r\bar{y}+q)}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_2+\psi_2\phi_1)+\frac{2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_2\phi_2),$$

and

$$C_2(\psi,\phi,\xi)=\frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_1)+\frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_2+\psi_1\phi_2\xi_1+\psi_2\phi_1\xi_1)+\frac{4\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_1+\psi_2\phi_1\xi_2+\psi_1\phi_2\xi_2)+\frac{-6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_2).$$

Now, we find the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda=-1$ at the bifurcation point $q=\frac{3r\bar{y}^2+1}{2\bar{y}}$. Let $\hat{q}$ and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda=-1$ respectively. So, we have

$$A\hat{q}=-\hat{q},\ \text{and}\ A^Tp^*=-p^*.$$

Or,

$$(A+I)\hat{q}=0 \tag{4.3}$$
$$(A^T+I)p^*=0. \tag{4.4}$$

From Equation (4.3), we get $\hat{q}\sim\begin{pmatrix}1\\-1\end{pmatrix}$.

And from equation (4.4), we get $p^*\sim\begin{pmatrix}\frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}\\ 1\end{pmatrix}$.

Now, we normalize p^* and $\hat{q}$,

$$\langle p^*,\hat{q}\rangle=\frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}-1.$$

Take $\hat{p}=\eta\begin{pmatrix}\frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}\\ 1\end{pmatrix}$, $\eta=-\frac{1+\bar{y}+r\bar{y}^2}{1+(2q+1)\bar{y}-r\bar{y}^2}$.

The critical eigenspace T^c corresponding to $\lambda=-1$ is a one-dimensional and spanned by an eigenvector $\hat{q}$. Let T^{su} denote a one-dimensional linear eigenspace of A corresponding to all eigenvalues other than λ. Note that the matrix $(A-\lambda I_n)$ has common invariant spaces with the matrix A, so we conclude that $y\in T^{su}$ if and only if $\langle\hat{p},y\rangle=0$.

So, to find $c(0)$ which is given by the following invariant formula:

$$c(0)=\frac{1}{6}\langle\hat{p},C(\hat{q},\hat{q},\hat{q})\rangle-\frac{1}{2}\langle\hat{p},B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle.$$

We evaluate

$$B(\hat{q},\hat{q}) = \begin{pmatrix} 0 \\ \frac{2\bar{y}(3q+1+4r\bar{y})+2q-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix}.$$

$$C(\hat{q},\hat{q},\hat{q}) = \begin{pmatrix} 0 \\ \frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3} - 3\frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{12\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3} \end{pmatrix}.$$

$$\langle \hat{p}, C(\hat{q},\hat{q},\hat{q})\rangle = -\left(\frac{1+\bar{y}+r\bar{y}^2}{1+(2q+1)\bar{y}-r\bar{y}^2}\right)\left[\frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3} - 3\frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{12\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3}\right].$$

$$(A-I)^{-1} = \begin{pmatrix} -1 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & -1+\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}^{-1} = \frac{1+\bar{y}+r\bar{y}^2}{2\bar{y}+r\bar{y}^2}\begin{pmatrix} -1+\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} & -1 \\ -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & -1 \end{pmatrix}.$$

$$(A-I)^{-1}B(\hat{q},\hat{q}) = \frac{1+\bar{y}+r\bar{y}^2}{2\bar{y}+r\bar{y}^2}\begin{pmatrix} \frac{-2\bar{y}(3q+1+4r\bar{y})-2q+2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2} \\ \frac{-2\bar{y}(3q+1+4r\bar{y})-2q+2r(p+2\bar{y}(2q\bar{y}+2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix}.$$

$$B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q})) = \frac{1+\bar{y}+r\bar{y}^2}{2\bar{y}+r\bar{y}^2}\begin{pmatrix} 0 \\ m \end{pmatrix},$$

where

$$m = \left(\frac{-2\bar{y}(3q+1+4r\bar{y})-2q+2r(p+2\bar{y}(2q\bar{y}+2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2}\right)\times \left(\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}\right).$$

$$\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle = \left(\left[\frac{2\bar{y}(3q+1+4r\bar{y})+2q-2r(p+2\bar{y}(2q\bar{y}+2r\bar{y}^2))}{(2\bar{y}+r\bar{y}^2)(1+(2q+1)\bar{y}-r\bar{y}^2)}\right]\left[\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}\right]\right).$$

If $c(0)>0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q=\frac{3r\bar{y}^2+1}{2\bar{y}}$.

4.2. Direction and stability of Neimark-Sacker bifurcation

We will first show when the Neimark-Sacker bifurcation conditions are satisfied.

Theorem 4.4. *If $q=q^*=\frac{r\bar{y}^2-1-\bar{y}}{2\bar{y}}$, and $r>\frac{1+\bar{y}}{\bar{y}^2}$, then the characteristic equation of (3.2) has two complex conjugate roots that lie on the unit circle. Moreover, the Neimark-Sacker bifurcation conditions are satisfied.*

Proof: At the beginning, we will show that the characteristic equation of (3.2)

$$\lambda^2 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}\lambda - \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 0. \tag{4.5}$$

has two complex roots. The roots of (4.5) are

$$\lambda_{1,2} = \frac{-\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \pm \sqrt{\Delta}}{2},$$

where,

$$\Delta = \frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} + 4\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}.$$

Substituting $q=q^*$, we get

$$\Delta = \frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} + 4\frac{-1-\bar{y}-r\bar{y}^2}{1+\bar{y}+r\bar{y}^2}.$$

So,

$$\Delta = \frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} - 4.$$

Thus, (4.5) has two complex roots if $\Delta < 0$, which is equivalent to

$$\frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} - 4 < 0,$$

which implies

$$\bar{y}^2 < 4(1+\bar{y}+r\bar{y}^2)^2,$$

so,

$$4(1+2\bar{y}+\bar{y}^2+2(1+\bar{y})r\bar{y}^2+r^2\bar{y}^4) - \bar{y}^2 > 0,$$

thus, $\Delta(q^*) < 0$ if

$$4+8\bar{y}+3\bar{y}^2+8(1+\bar{y})r\bar{y}^2+4r^2\bar{y}^4 > 0,$$

which always holds.
Next, we show that (4.5) has two conjugate complex roots on the unit circle when $q = q^*$.
Since, $\lambda_{1,2}$ are the roots of (4.5), we have

$$\lambda_1\lambda_2 = -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}.$$

Substituting $q = q^*$ we get

$$\lambda_1\lambda_2 = 1.$$

But, $\lambda_1\lambda_2 = |\lambda_{1,2}|^2 = 1$. Thus, the two complex roots are on the unit circle.
Assume the roots of (4.5) at $q = q^*$ are $e^{\pm i\theta}$. So, we have

$$e^{i\theta} + e^{-i\theta} = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2},$$

but, $e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$. Thus,

$$\cos(\theta) = -\frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$

Note that $\frac{-1}{2} < \cos(\theta) < 0$. So, there exists $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that

$$\theta_0 = \cos^{-1}\left(-\frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}\right).$$

And, $e^{ik\theta_0} \neq 1$ for $k = 1,\ 2,\ 3,\ 4$.
Next, we will show that $\frac{d|\lambda|^2}{dq}|_{q=q^*} \neq 0$.

$$|\lambda|^2 = -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2},$$

differentiate with respect to q, we get

$$\frac{d|\lambda|^2}{dq} = -\frac{(1+\bar{y}+r\bar{y}^2)(2\bar{y}(1-r\frac{d\bar{y}}{dq})+(q-r\bar{y})2\frac{d\bar{y}}{dq}) - (2\bar{y}(q-r\bar{y}))(\frac{d\bar{y}}{dq}+2r\bar{y}\frac{d\bar{y}}{dq})}{(1+\bar{y}+r\bar{y}^2)^2}.$$

To find $\frac{d\bar{y}}{dq}$, we differentiate equation (3.3) with respect to q

$$\frac{d}{dq}(r\bar{y}^3+(1-q)\bar{y}^2+\bar{y}-p) = 0,$$

so,

$$3r\bar{y}^2\frac{d\bar{y}}{dq} + (1-q)2\bar{y}\frac{d\bar{y}}{dq} + \bar{y}^2(-1) + \frac{d\bar{y}}{dq} = 0,$$

thus,

$$\frac{d\bar{y}}{dq} = \frac{\bar{y}^2}{3r\bar{y}^2+(1-q)2\bar{y}+1}.$$

Substituting $q = q^*$, we get

$$\frac{d\bar{y}}{dq}|_{q=q^*} = \frac{\bar{y}^2}{2r\bar{y}^2+3\bar{y}+2}.$$

So,

$$\frac{d|\lambda|^2}{dq}|_{q=q^*} = -\frac{(3r\bar{y}^3+6\bar{y}^2+3\bar{y})(r\bar{y}^2+\bar{y}+1)}{(2r\bar{y}^2+3\bar{y}+2)(1+\bar{y}+r\bar{y}^2)^2} < 0,$$

So the Neimark-Sacker bifurcation conditions are satisfied.

□

As in the previous subsection, we shift the fixed point $(\bar{y},\bar{y})$ to the origin. We get

$$Y_{n+1} = AY_n + G(Y_n), \tag{4.6}$$

where,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix},\ Y_n = \begin{pmatrix} w_n \\ u_n \end{pmatrix},$$

and,

$$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(\|Y\|^4)$$

$$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ B_2(Y,Y) \end{pmatrix}\ and,\ C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ C_2(Y,Y,Y) \end{pmatrix}$$

where,

$$B_i(x,y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial\eta_k\partial\eta_j}|_{\eta=0}(x_k y_j)$$

and,

$$C_i(x,y,z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial\eta_l\partial\eta_k\partial\eta_j}|_{\eta=0}(x_l y_k z_j).$$

So $B_1(\psi,\phi) = 0$ and $C_1(\psi,\phi,\xi) = 0$,

$$B_2(\psi,\phi) = \frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_1) - \frac{2\bar{y}(2r\bar{y}+q)}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_2+\psi_2\phi_1) + \frac{2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_2\phi_2),$$

and,

$$C_2(\psi,\phi,\xi) = \frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_1) + \frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_2+\psi_1\phi_2\xi_1+\psi_2\phi_1\xi_1) + \frac{4\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_1+\psi_2\phi_1\xi_2+\psi_1\phi_2\xi_2) + \frac{-6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_2).$$

Now, we find the eigenvectors of A and A^T corresponding to the eigenvalue $e^{\pm i\theta_0}$ at the bifurcation point $q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}$. Let $\hat{q}$ and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $e^{\pm i\theta_0}$ respectively. So, we have

$$A\hat{q} = e^{i\theta_0}\hat{q},\ \text{ and }\ A^T p^* = e^{-i\theta_0}p^*.$$

Or,

$$(A - e^{i\theta_0}I)\hat{q} = 0 \tag{4.7}$$

$$(A^T - e^{-i\theta_0}I)p^* = 0. \tag{4.8}$$

From Equation (4.7), we get $\hat{q} \sim \begin{pmatrix} 1 \\ e^{i\theta_0} \end{pmatrix}$.

And Equation (4.8) gives $p^* \sim \begin{pmatrix} 1+e^{i\theta_0}\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \\ e^{i\theta_0} \end{pmatrix}$.

Now, we normalize p^* and $\hat{q}$, take $\hat{p}=\eta\begin{pmatrix}1+e^{i\theta_0}\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}\\ e^{i\theta_0}\end{pmatrix}$, $\eta=\frac{1}{2+\frac{e^{-i\theta_0}\bar{y}}{1+\bar{y}+r\bar{y}^2}}$.

So, to determine the direction of the Neimark-sacker bifurcation, we compute $a(0)$ as given in ([15]).
where,

$$g_{20}=\frac{e^{i\theta_0}(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0})}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$
$$g_{11}=\frac{e^{i\theta_0}[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))+2\bar{y}-4\bar{y}(2r\bar{y}+q)\cos(\theta_0)]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$
$$g_{02}=\frac{e^{i\theta_0}[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{-i\theta_0}+2\bar{y}e^{-2i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

And,

$$g_{21}=\langle\hat{p},C(\hat{q},\hat{q},\bar{\hat{q}})\rangle+2\langle\hat{p},B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle+$$
$$\langle\hat{p},B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle+\frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{-i\theta_0}}\langle\hat{p},B(\hat{q},\hat{q})\rangle\langle\hat{p},B(\hat{q},\bar{\hat{q}})\rangle$$
$$-\frac{2}{1-e^{-i\theta_0}}|\langle\hat{p},B(\hat{q},\bar{\hat{q}})\rangle|^2-\frac{e^{i\theta_0}}{e^{3i\theta_0}-1}|\langle\hat{p},B(\bar{\hat{q}},\bar{\hat{q}})\rangle|^2.$$

where,

$$\langle\hat{p},C(\hat{q},\hat{q},\bar{\hat{q}})\rangle=\frac{e^{i\theta_0}[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2}+$$
$$\frac{e^{i\theta_0}[(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos(\theta_0)+e^{i\theta_0})+4\bar{y}(q-3r\bar{y})(2+e^{2i\theta_0})-6\bar{y}e^{i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2}.$$

And,

$$\langle\hat{p},B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle=\frac{e^{i\theta_0}M(1+\bar{y}+r\bar{y}^2)}{2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y}}.$$

$$s=\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))+2\bar{y}-4\bar{y}(2r\bar{y}+q)\cos(\theta_0)}{(1+2\bar{y}-2q\bar{y}+3r\bar{y}^2)(1+\bar{y}+r\bar{y}^2)}.$$

$$M=s\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}(2r\bar{y}+q)(1+e^{i\theta_0})+2\bar{y}e^{i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2}.$$

Finally,

$$\langle\hat{p},B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle=\frac{e^{i\theta_0}L[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}+\frac{e^{i\theta_0}L[-2\bar{y}(2r\bar{y}+q)(e^{2i\theta_0}+e^{-i\theta_0})+2\bar{y}e^{2i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

where,

$$L=\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0}}{(e^{4i\theta_0}(1+\bar{y}+r\bar{y}^2)+e^{2i\theta_0}\bar{y}-2\bar{y}(q-r\bar{y}))(1+\bar{y}+r\bar{y}^2)}.$$

So,

$$a(0)=\frac{1}{2}Re\left(e^{-i\theta_0}\langle\hat{p},C(\hat{q},\hat{q},\bar{\hat{q}})\rangle\right)+Re\left(e^{-i\theta_0}\langle\hat{p},B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle\right)$$
$$+\frac{1}{2}Re\left(e^{-i\theta_0}\langle\hat{p},B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle\right).$$

Let

$$B_1=Re\left(e^{-i\theta_0}\langle\hat{p},C(\hat{q},\hat{q},\bar{\hat{q}})\rangle\right),\ B_2=Re\left(e^{-i\theta_0}\langle\hat{p},B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle\right),$$

and

$$B_3=Re\left(e^{-i\theta_0}\langle\hat{p},B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle\right).$$

To find B_1,

$$B_1 = Re\left(\frac{[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2}+\frac{e^{i\theta_0}[(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos(\theta_0)+e^{i\theta_0})+4\bar{y}(q-3r\bar{y})(2+e^{2i\theta_0})-6\bar{y}e^{i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2}\right).$$

Multiplying and dividing by the conjugate of the complex part of the denominator, the denominator becomes,

$$A_1 = (4(1+\bar{y}+r\bar{y}^2)^2+4(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(\theta_0)+\bar{y}^2)(1+\bar{y}+r\bar{y}^2)^2.$$

Multiplying the numerator by the conjugate of the complex part of the denominator and taking the real part of the numerator, we get,

$$\begin{aligned}A_2 = {} & 2(1+\bar{y}+r\bar{y}^2)[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]+\\ & \bar{y}[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]\cos(\theta_0)\\ & +4(1+\bar{y}+r\bar{y}^2)(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos(\theta_0))+\\ & \bar{y}(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos^2(\theta_0))+\\ & \bar{y}(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)\cos(2\theta_0)+\\ & 16(1+\bar{y}+r\bar{y}^2)\bar{y}(q-3r\bar{y})+8(1+\bar{y}+r\bar{y}^2)\bar{y}(q-3r\bar{y})(\cos(2\theta_0))+\\ & 8\bar{y}^2(q-3r\bar{y})(\cos(\theta_0))+4\bar{y}^2(q-3r\bar{y})(\cos(3\theta_0))-12\bar{y}(1+\bar{y}+r\bar{y}^2)\cos(\theta_0)\\ & -6\bar{y}^2\cos(2\theta_0).\end{aligned}$$

So, we have $B_1 = \frac{A_2}{A_1}$.
To find B_2:

$$B_2 = Re\left(s\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}(2r\bar{y}+q)(1+e^{i\theta_0})}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}+\frac{2\bar{y}e^{i\theta_0}}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}\right).$$

Multiplying and dividing by the conjugate of the complex part of the denominator, the denominator becomes,

$$A_3 = (4(1+\bar{y}+r\bar{y}^2)^2+4(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(\theta_0)+\bar{y}^2)(1+\bar{y}+r\bar{y}^2).$$

Multiplying the numerator by the conjugate of the complex part of the denominator and taking the real part of the numerator, we get,

$$\begin{aligned}A_4 = {} & s[2(1+\bar{y}+r\bar{y}^2)(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)))\\ & +\bar{y}(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)))\cos(\theta_0)-4(1+\bar{y}+r\bar{y}^2)\bar{y}(2r\bar{y}+q)\\ & -4(1+\bar{y}+r\bar{y}^2)\bar{y}(2r\bar{y}+q)\cos(\theta_0)-2\bar{y}^2(2r\bar{y}+q)\cos(\theta_0)\\ & -2\bar{y}^2(2r\bar{y}+q)\cos(2\theta_0)+4\bar{y}(1+\bar{y}+r\bar{y}^2)\cos(\theta_0)+2\bar{y}^2\cos(2\theta_0)].\end{aligned}$$

We have $B_2 = \frac{A_4}{A_3}$.
To find B_3

$$B_3 = Re\left(\frac{[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}+\frac{[-2\bar{y}(2r\bar{y}+q)(e^{2i\theta_0}+e^{-i\theta_0})+2\bar{y}e^{2i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}\times\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0}}{(e^{4i\theta_0}(1+\bar{y}+r\bar{y}^2)+e^{2i\theta_0}\bar{y}-2\bar{y}(q-r\bar{y}))(1+\bar{y}+r\bar{y}^2)}\right).$$

Multiplying and dividing by the conjugate of the complex part of the denominator, the denominator becomes,

$$\begin{aligned}A_5 = {} & (4(1+\bar{y}+r\bar{y}^2)^2+4(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(\theta_0)+\bar{y}^2)(1+\bar{y}+r\bar{y}^2)^2[(1+\bar{y}+r\bar{y}^2)^2+\bar{y}^2\\ & +4\bar{y}^2(q-r\bar{y})^2+2(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(2\theta_0)\\ & -4\bar{y}(q-r\bar{y})(1+\bar{y}+r\bar{y}^2)\cos(4\theta_0)-4\bar{y}(q-r\bar{y})\cos(2\theta_0)].\end{aligned}$$

Multiplying the numerator by the conjugate of the complex part of the denominator and taking the real part of the numerator, we get,

$$\begin{aligned}A_6 = {} & (a_8b_6+a_9b_1)\cos(5\theta_0)+((a_1+a_6+a_{12})b_1+a_9b_4+a_8b_3+a_5b_6)\cos(4\theta_0)+\\ & ((a_1+a_6+a_{12})b_4+(a_2+a_7+a_{10})b_1+(a_3+a_4+a_{11})b_6+a_5b_3+a_8b_5+a_9b_2)\cos(3\theta_0)+\\ & ((a_1+a_6+a_{12})b_2+(a_2+a_7+a_{10})b_4+(a_2+a_7+a_{10})b_6+(a_3+a_4+a_{11})b_1\\ & +(a_3+a_4+a_{11})b_3+a_8b_2+a_9b_5)\cos(2\theta_0)+((a_1+a_6+a_{12})b_5\\ & +(a_1+a_6+a_{12})b_6+(a_2+a_7+a_{10})b_2+(a_2+a_7+a_{10})b_3+(a_3+a_4+a_{11})b_4\\ & +(a_3+a_4+a_{11})b_5+a_5b_1+a_5b_2+a_5b_5+a_8b_4+a_9b_3)\cos(\theta_0)\\ & +(a_1+a_6+a_{12})b_3+(a_2+a_7+a_{10})b_5+(a_3+a_4+a_{11})b_2+a_5b_4+a_8b_1+a_9b_6,\end{aligned}$$

where,

$$a_1 = [2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]^2.$$
$$a_2 = -4\bar{y}(2r\bar{y}+q)[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))].$$

$$a_3 = 2[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]\bar{y}.$$
$$a_4 = -2\bar{y}(2r\bar{y}+q)[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))].$$

$$a_5 = a_6 = 8\bar{y}^2(2r\bar{y}+q)^2, \;\; a_7 = a_8 = -4\bar{y}^2(2r\bar{y}+q).$$

$$a_9 = -2[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]\bar{y}(2r\bar{y}+q).$$
$$a_{10} = 2\bar{y}[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))].$$

$$a_{11} = -8\bar{y}^2(2r\bar{y}+q), \;\; a_{12} = 4\bar{y}^2.$$

$$b_1 = 2(1+\bar{y}+r\bar{y}^2)^2, \;\; b_2 = 2(1+\bar{y}+r\bar{y}^2)\bar{y}.$$

$$b_3 = -4(1+\bar{y}+r\bar{y}^2)\bar{y}(q-r\bar{y}), \;\; b_4 = (1+\bar{y}+r\bar{y}^2)\bar{y}.$$

$$b_5 = \bar{y}^2, \;\; b_6 = -2\bar{y}^2(q-r\bar{y}).$$

And,

$$\cos(\theta_0) = -\frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$
$$\cos(2\theta_0) = 2\cos^2(\theta_0) - 1 = \frac{\bar{y}^2}{2(1+\bar{y}+r\bar{y}^2)^2} - 1.$$

$$\cos(3\theta_0) = 4\cos^3(\theta_0) - 3\cos(\theta_0) = -\frac{\bar{y}^3}{2(1+\bar{y}+r\bar{y}^2)^3} + \frac{3\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$
$$\cos(4\theta_0) = 2\cos^2(2\theta_0) - 1 = 2\left(\frac{\bar{y}^2}{2(1+\bar{y}+r\bar{y}^2)^2} - 1\right)^2 - 1.$$
$$\cos(5\theta_0) = 2\cos(2\theta_0)\cos(3\theta_0) - \cos(\theta_0)$$
$$= 2\left(\frac{\bar{y}^2}{2(1+\bar{y}+r\bar{y}^2)^2} - 1\right)\left(-\frac{\bar{y}^3}{2(1+\bar{y}+r\bar{y}^2)^3} + \frac{3\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}\right) + \frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$

So, $B_3 = \frac{A_6}{A_5}$. And

$$a(0) = \frac{1}{2}B_1 + B_2 + \frac{1}{2}B_3.$$

Theorem 4.5. *If $a(0) < 0$ (respectively, > 0), then Neimark-Saker bifurcation of system (4.6) at $q = q^*$ is supercritical (respectively, subcritical) and there exists a unique invariant closed curve bifurcates from the positive fixed point $\bar{y}$ which is asymptotically stable (respectively, unstable).*

5. Numerical discussions

In this section some numerical examples which support our results are given.

Example 5.1. *Consider the difference equation (3.2). Fix p, r, and consider q as bifurcation parameter. Take $p = 0.5$, $r = 1.8$, and $0 < q \leq 10$. Equation (3.2) becomes*

$$y_{n+1} = \frac{0.5 + q{y^2}_{n-1}}{1 + y_n + 1.8{y^2}_{n-1}}, \; n = 0, \; 1, \; 2, \; ... \tag{5.1}$$

Which is equivalent to

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{0.5+qy_1(n)^2}{1+y_2(n)+1.8y_1(n)^2} \end{pmatrix}.$$

The positive equilibrium point $\bar{y}$ of (5.1) satisfies

$$1.8\bar{y}^3 + (1-q)\bar{y}^2 + \bar{y} - 0.5 = 0. \tag{5.2}$$

Theorem 4.2 shows that the fixed point undergoes a period-doubling bifurcation at $q^* = \frac{3\times 1.8\bar{y}^2+1}{2\bar{y}}$. *So Equation (5.2) at* q^* *becomes*

$$-0.9\bar{y}^3+\bar{y}^2+0.5\bar{y}-0.5=0.$$

Which has two positive roots, so we have two values of q^*.
Thus the first value of q^* *gives the following fixed point of (5.1)*

$$\bar{y}=0.6495.$$

Substituting the value of $\bar{y}$ *in* q^* *we get*

$$q^*=2.5235.$$

Now to determine the direction of period-doubling bifurcation we find $c(0)$.

$$c(0)=0.9539>0$$

So this shows that a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q^*=2.5235$.
The second value of q^* *gives the following fixed point of (5.1)*

$$\bar{y}=1.1840.$$

Substituting the value of $\bar{y}$ *in* q^* *we get*

$$q^*=3.6192.$$

Now to determine the direction of period-doubling bifurcation we find $c(0)$.

$$c(0)=-0.4132$$

So this shows that no stable period-two cycle bifurcates from the fixed point at the bifurcation point $q^*=3.6192$.
Figure 5.1 shows the stable period-two cycle.

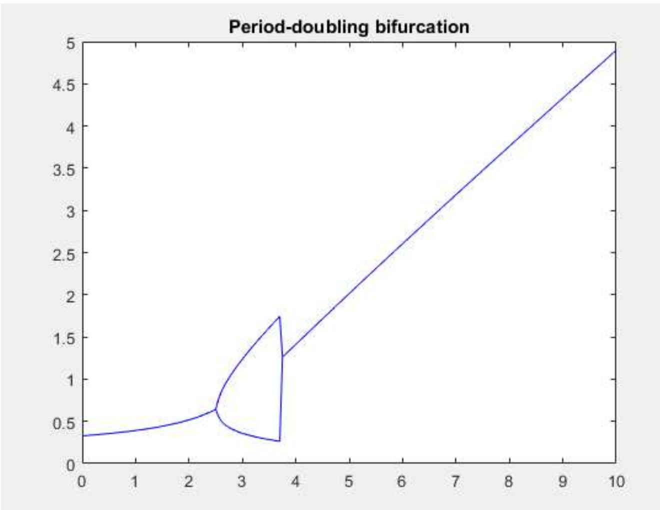

Figure 5.1: Period-doubling bifurcation of $y_{n+1}=\frac{0.5+qy^2_{n-1}}{1+y_n+1.8y^2_{n-1}}$.

Example 5.2. *Consider the difference equation (3.2). Fix* p, r, *and consider* q *as bifurcation parameter. Take* $p=2$, $r=9$, *and* $0<q\leq 10$. *Equation (3.2) becomes*

$$y_{n+1}=\frac{2+qy^2_{n-1}}{1+y_n+9y^2_{n-1}},\ n=0,\ 1,\ 2,\ \ldots \tag{5.3}$$

Which is equivalent to

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1)\end{pmatrix}=\begin{pmatrix} y_2(n) \\ \frac{2+qy_1(n)^2}{1+y_2(n)+9y_1(n)^2}\end{pmatrix}.$$

The positive equilibrium point $\bar{y}$ *of (5.3) satisfies*

$$9\bar{y}^3+(1-q)\bar{y}^2+\bar{y}-2=0. \tag{5.4}$$

Theorem5.3 shows that the fixed point undergoes a Neimark-Sacker bifurcation at $q^*=\frac{9\bar{y}^2-\bar{y}-1}{2\bar{y}}$. *So Equation (5.4) at* q^* *becomes*

$$4.5\bar{y}^3+1.5\bar{y}^2+1.5\bar{y}-2=0.$$

Which has one positive roots.

Thus the value of q^* gives the following fixed point of (5.3)

$$\bar{y} = 0.5462.$$

Substituting the value of $\bar{y}$ in q^* we get

$$q^* = 1.0424.$$

Now to determine the direction of period-doubling bifurcation we find $a(0)$.

$$a(0) = 11.7658 > 0$$

So this shows that the Neimark- Sacker bifurcation at $q^* = 1.0424$ is subcritical.

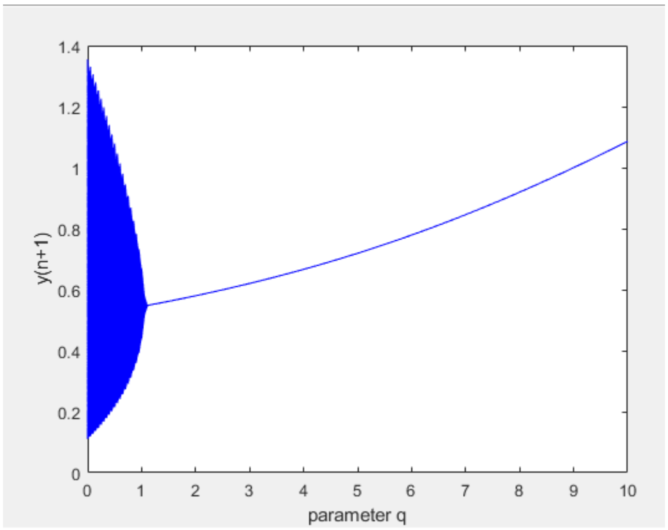

Figure 5.2: Neimark-Sacker bifurcation of $y_{n+1} = \frac{2+qy^2_{n-1}}{1+y_n+9y^2_{n-1}}$.

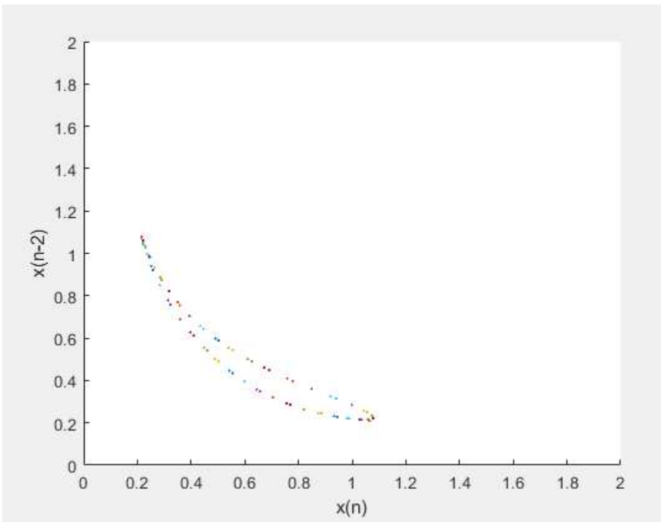

Figure 5.3: Phase portraits of the map $y_{n+1} = \frac{2+qy^2_{n-1}}{1+y_n+9y^2_{n-1}}$ at $q = 0.5$.

Figure 5.2 shows that the positive fixed point $\bar{y}$ is asymptotically stable for $q > q^*$ and change its stability at Neimark-Sacker bifurcation value q^*. Figure 5.3 shows phase portraits associated with Figure 5.2 at $q = 0.5$.

References

[1] M. Kulenovic, et al., *Naimark-Sacker bifurcation of second order rational difference equation with quadratic terms*, J. Nonlinear Sci. Appl., **10**(7) (2017), 3477-3489.
[2] Y. Kostrov, Z. Kudlak, *On a second-order rational difference equation with a quadratic term*, Int. J. Difference Equ., (2016), 179-202.
[3] S. Moranjkić, Z. Nurkanović, *Local and global dynamics of certain second-order rational difference equations containing quadratic terms*, Adv. Dyn. Syst. Appl., (2017), 123-157.
[4] M. Abu Alhalawa M, M.Saleh, *Dynamics of higher order rational difference equation* $x_{n+1} = \frac{\alpha x_n + beta x_n}{A+Bx_n+Cx_{n-k}}$, Int. J. Nonlinear Anal. Appl. **8**(2) (2017), 363-379.

[5] A. Jafar, M. Saleh, *Dynamics of nonlinear difference equation* $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}$, J. Appl. Math. Comput., **57** (2018), 493-522.

[6] M. Saleh, N. Alkoumi, A. Farhat, *On the dynamics of a rational difference equation* $x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}$, Chaos Soliton, **96** (2017), 76-84.

[7] M. Saleh, A. Farhat, *Global asymptotic stability of the higher order equation* $x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$, J. Appl. Math. Comput., **55** (2017), 135-148.

[8] M. Saleh, A. Asad, *Dynamics of kth order rational difference equation*, J. Appl. Nonlinear Dynam., (2021), 125-149, DOI 10.5890/JAND.2021.03.008.

[9] M. Saleh, S.Hirzallah, *Dynamics and bifurcation of a second order rational difference equation with quadratic terms*, J. Appl. Nonlinear Dynam., (in press).

[10] C. Wang, X. Fang, R. Li, *On the solution for a system of two rational difference equations*, J. Comput. Anal. Appl., **20**(1) (2016), 175-186.

[11] C. Wang, X. Fang, R. Li, *On the dynamics of a certain four-order fractional difference equations*, J. Comput. Anal. Appl., **22**(5) (2017), 968-976.

[12] M. Kulenovic, G. Ladas, *Dynamics of Second Order Rational Difference Equations With Open Problems and Conjectures,* Chapman. Hall/CRC, Boca Raton, 2002.

[13] S. Elaydi, *An Introduction to Difference Equations,* 3rd edition. Springer, 2000.

[14] S. Elaydi, *Discrete Chaos With Applications In Science And Engineering*, 2nd edition. Chapman Hall/CRC.

[15] Y. Kuznetsov, *Elements of Applied Bifurcation Theory*, 2nd edition, Springer-Verlag, 1998.

Journal of Mathematical Sciences and Modelling, 3 (3) (2020) 120-129
Research Article

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm
ISSN 2636-8692
DOI: http://dx.doi.org/10.33187/jmsm.776898

Boolean Hypercubes, Mersenne Numbers, and the Collatz Conjecture

Ramon Carbó-Dorca

Institut de Química Computacional i Catàlisi / Universitat de Girona / Girona 17003 (Catalonia) Spain

Article Info

Keywords: *Boolean hypercubes, Collatz algorithm, Collatz conjecture, Collatz operator, Mersenne numbers, Mersenne twins, Recursive construction of natural numbers, $(3x+1)$ conjecture.*

2010 AMS: *03E10, 11Z05.*

Received: *4 August 2020*

Accepted: *18 December 2020*

Available online: *29 December 2020*

Abstract

This study is based on the trivial transcription of the vertices of a Boolean N-Dimensional Hypercube $\mathbf{H}_N$ into a subset $\mathbb{S}_N$ of the decimal natural numbers $\mathbb{N}$. Such straightforward mathematical manipulation permits to achieve a recursive construction of the whole set $\mathbb{N}$. In this proposed scheme, the Mersenne numbers act as upper bounds of the iterative building of $\mathbb{S}_N$. The paper begins with a general description of the Collatz or $(3x+1)$ algorithm presented in the $\mathbb{S}_N \subset \mathbb{N}$ iterative environment. Application of a defined *ad hoc* Collatz operator to the Boolean Hypercube recursive partition of $\mathbb{N}$, permits to find some hints of the behavior of natural numbers under the $(3x+1)$ algorithm, and finally to provide a scheme of the Collatz conjecture partial resolution by induction.

1. Introduction

The Collatz or $(3x+1)$ conjecture, (see the work of Lagarias [1] for a comprehensive review and reference [2] for a decade-old discussion of this problem by several authors), seems that it still has to be proven true since its description around the first third of the last century. More information on the Collatz conjecture can be easily obtained from reference [3]. Recent advances towards the problem solution can be found thanks to the insight of Tao in reference [4]. More information on the so-called Collatz or $(3x+1)$conjecture can be found in references contained within the previous publications and in the references [5, 6].

Some previous work on the application of N-Dimensional Boolean Hypercubes, $\mathbf{H}_N$, has been carried on by the present author. For instance, to describe some inner structure of vector spaces [7], and discuss several mathematical, physical, biological, and chemical problems, which can be found in references [8]-[22]. The study of computational multinomial combinatorics for colorings of Hypercubes for all irreducible representations with applications to non-rigid molecules has been recently published by Balasubramanian [23, 24].

In the present paper, Boolean Hypercubes will be chosen as the cornerstone tool to study the Collatz conjecture, because apparently, no discussion of this old problem includes the use of the *iterative construction* of natural numbers in terms of the sequence of decimal numbers, associated to the 2^N binary vertices of an N-Dimensional Boolean Hypercube $\mathbf{H}_N$.

It is intriguing that, at least as far as the author knows, no such simple computational-geometric structure has been considered to approach the Collatz conjecture.

The scheme which this paper will follow consists of the following parts: 1) Firstly, we shall provide the definitions of an appropriate Collatz algorithm and an *ad hoc* operator able to be employed in the development of a proof of the $(3x+1)$ problem. 2) Next, it will be discussed the recursive construction of Boolean Hypercubes and their translation into an iterative buildup of natural numbers. 3) Then, a section will be devoted to a computational study of the Mersenne numbers from the Collatz algorithm perspective. 4) After this, some extensions of Mersenne numbers will be studied, and the characteristic behavior of them in front of the $(3x+1)$ problem described. 5) Finally, will be studied the Collatz conjecture through the perspective of an inductive procedure.

2. The collatz algorithm

It is straightforward to describe the problem associated to the Collatz conjecture. Provided the set of natural numbers:$\mathbb{N}$, then one can expose any natural number to the simple algorithm:

***Email address and ORCID number*: ramoncarbodorca@gmail.com, http://orcid.org/0000-0002-9219-0686 (R. Carbó-Dorca)**

Algorithm 1: Collatz or$(3x+1)$procedure

$\forall m \in \mathbb{N}$;
$I = 0$;
$Define\ C_I[m]$;
while $m > 1$;
  $I = I+1$; $c \leftarrow m/2$;
  if $2 * c \neq m :\ m \leftarrow 3m+1$; else : $m \leftarrow c$;

When using the Collatz algorithm above, the variable I indicates the number of times (here they will be called Collatz *iterations*) that the algorithm has been applied until termination is reached, resulting in reaching the natural unit, bearing a Collatz iteration number $I = end$. When such termination occurs, a so-called Collatz *orbit* has been completed.
The name orbit possibly comes from the generalization of the Collatz conjecture with the chance that the transformation $(px+1)$, being p a prime number, is used instead of the usual $(3x+1)$ transform. While in this $(3x+1)$study no never-ending divergence nor loop has been found, a simple test with $p = 5 \wedge x = 5$ yields a closed loop.
Because the sequence of natural numbers, provided until a $(3x+1)$ Collatz iteration completion is reached, has not a closed structure, perhaps better than naming such a sequence as an orbit, one might call it a *complete path*. Also, by a *partial path,* one can understand any non-terminal sequence of natural numbers originated by the Collatz **Algorithm 1**. One can simply refer to a *path* if the context makes it well-defined, not needing more information.
For example, the complete path, made of seven iterations, generated by Collatz **Algorithm 1** applied to the natural number 3 becomes:

$$\begin{array}{ll} Path: & \{3 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1\} \\ Iteration: & [0 \cdots 1 \cdots 2 \cdots 3 \cdots 4 \cdots 5 \cdots 6 \cdots 7] \end{array}, \tag{2.1}$$

which can be also imagined as a graph, whose vertices are constituted by the Collatz sequence of natural numbers generated from the application of **Algorithm 1**.

2.1. Collatz conjecture

The final stage or the path completion associated with the essence of the Collatz conjecture, affecting the whole set of natural numbers $\mathbb{N}$, might be expressed as:

$$\forall m \in \mathbb{N} : C_{end}[m] = 1. \tag{2.2}$$

Also, any natural number n fulfilling:

$$n \in \mathbb{N} \wedge C_{end}[n] = 1$$

will be called Collatz *compliant*.

3. Development of the collatz operator

Collatz **Algorithm 1** as defined above might be supposedly constructed by a non-linear operator, which conducts a Collatz path flux from a starting natural number down to the natural unit. Thus, to discuss and study the behavior of the Collatz compliance of natural numbers it is interesting to design the structure of a Collatz operator, as follows.

3.1. Definitions

One can formally use the Collatz operator as a symbolic structure, allowing to represent the path evolution under **Algorithm 1** applied to any natural number. It can be obtained via an indeterminate number of **Algorithm 1** iterations I. For instance, a partial path of the Collatz operator could be written like:

$$\forall m \in \mathbb{N} : C_I[m] = m_I.$$

Moreover, using the equation (2.2), then any complete path generated by the Collatz operator can be easily described.
To grasp the meaning of the previous notation which will be used from now on, one might suppose that the basic natural numbers $\{0,1\}$ are also Collatz compliant, or:

$$C_{end}[0] = 1 \wedge C_{end}[1] = 1.$$

3.2. Extension to subsets of natural numbers

If a natural number subset $\mathbb{S}$ of cardinality: $Card[\mathbb{S}] = N$ is defined as: $\mathbb{S} \subset \mathbb{N}$, then one can define the partial path application of the Collatz algorithm over such a natural number subset, using as much as I iterations, noting:

$$C_I[\mathbb{S}] = \mathbb{S}_I$$

and meaning that:

$$\forall s \in \mathbb{S} : C_I[s] = s_I \in \mathbb{S}_I.$$

The application of the Collatz algorithm over a natural number subset as above defined, until the outcome of a Collatz complete path is reached, could be symbolically expressed with:

$$C_{\langle end|}[\mathbb{S}] = \langle 1_N|,$$

where the row vector on the right is defined as the unity vector of dimension N, that is:

$$\langle 1_N| = (1,1,1,...,1).$$

Therefore, the Collatz conjecture involving the whole natural set can be easily expressed using alternatively the equation:

$$C_{\langle end|}[\mathbb{N}] = \langle 1_\infty|. \tag{3.1}$$

Note that in this case, the iteration subindex indicator: $\langle end|$ must be considered as a vector of dimension N, bearing everyone of ending number of iterations of each element of the set S, upon application of Collatz algorithm. In the case of equation (3.1), the $\langle end|$ vector has to be supposed of ∞ dimension.

3.3. Application to natural vector spaces

This scheme, as above described, suggests that if a natural vector space of dimension N: $\mathbb{V}_N(\mathbb{N})$ is defined (see for more details references [8,9]) then one can also write an alternative to the above-defined symbolic application of the Collatz algorithm, like:

$$\forall \langle s| \in \mathbb{V}_N(\mathbb{N}) : C_{\langle end|}[\langle s|] = \langle 1_N|;$$

additionally, considering the natural number set as an infinite-dimensional row perfect vector: $\langle \mathbb{N}| \in \mathbb{V}_\infty(\mathbb{N})$, then the Collatz conjecture could be also rewritten as:

$$C_{\langle end|}[\langle \mathbb{N}|] = \langle 1_\infty|.$$

3.4. A trivial property

Here it will be considered an important but trivial property of the symbolic application of the Collatz operator. Suppose now that the Collatz operator is applied to a natural number, in such a way that after some number of iterations I the result of the generated partial path appears to be:

$$\exists n \in \mathbb{N} : C_I[n] = m_I \in \mathbb{N}.$$

If the resulting natural number m_I is *a priori* known to be Collatz compliant, then necessarily the initial number n is Collatz compliant. Because one can write:

$$C_{end}[m_I] = 1 \Rightarrow C_{end}[n] = 1.$$

Such an obvious consequence of the definition of the Collatz operator will be of further use to discuss a possible Collatz conjecture incomplete demonstration track.

3.5. Powers of 2 as collatz compliant natural numbers

When observing **Algorithm 1**, it is immediate to consider the powers of 2 as a trivial Collatz compliant set of natural numbers, and thus one can write:

$$\forall N \in \mathbb{N} : C_N\left[2^N\right] = 1. \tag{3.2}$$

Hence, because of the trivial property discussed in the previous paragraph 3.4, whenever in a Collatz path development, attached to a specific natural number, there appears a power of 2 in the path sequence, then such an initial natural number is Collatz compliant:

$$\exists P, m, N \in \mathbb{N} : C_P[m] = 2^N \Rightarrow C_{end}[m] = 1.$$

The path of number 3, as shown in the equation (2.1) is an example of this situation, as at the third Collatz iteration, $2^4 = 16$ is reached.

Powers of 2 transcribed into binary vectors have a leading role in the definition of the vertices of Boolean Hypercubes. Indeed, they correspond to a binary *canonical* basis set, see for instance references [10-22], where every vector has one bit 1 in some position of the Boolean vertex vector and the rest of elements are made by 0 bits.

3.6. Forbidden partial paths

It appears that some vertices of Collatz partial paths cannot be present if Collatz conjecture in equations (2.2) or (3.1) is accepted as true. For instance, suppose that:

$$\exists m, n, I, J \in \mathbb{N} : C_I[m] = n \wedge C_{I+J}[m] = n \Rightarrow \tilde{\exists} C_{end}[m]$$

Then, if this is the case, **Algorithm 1** will enter an unending loop, and Collatz conjecture will be false.

To avoid this kind of situation, if the Collatz conjecture has to be true, the natural number sequence of a complete path generated by **Algorithm 1** must be made in any case by completely *different* natural elements.

4. N-dimensional boolean hypercubes and natural number sequences

The concatenation of Boolean Hypercubes (see for more information references [9]-[22]) constitute the fundament where a possible demonstration of Collatz conjecture will be built. To briefly introduce the useful structure of Boolean Hypercubes, one can start choosing a set of natural numbers of cardinality 2^N: $\mathbb{S}_N$. This set might be constructed as the ordered sequence of elements of the natural number subset, which can be associated with the decimal transcription of the set of 2^N vertices of an N-Dimensional Boolean Hypercube: $\mathbf{H}_N$.
Simply, the set $\mathbb{S}_N$ corresponds to the natural number subset:

$$\mathbb{S}_N = \left\{ m = 0, 2^N - 1 : \Delta = 1 \right\} = \left\{ 0, 1, ..., 2^N - 1 \right\},$$

where the last number in the set $\mathbb{S}_N$ corresponds to the so-called *Mersenne number*: $\mu_2(N) = 2^N - 1$.
At the same time, and generally speaking, any decimal Mersenne number is transcribed into a Boolean vector, a bit string, constituting a characteristic vertex of an N-Dimensional Boolean Hypercube $\mathbf{H}_N$, the *binary unity vector* of the adequate dimension: $\langle 1_N | = (1, 1, ..., 1)$ made with all its elements as the 1 bit. It can be called the Mersenne vertex of the Boolean Hypercube and might be considered as the farthest vertex from the Boolean zero vertex: $\langle 0_N | = (0, 0, ..., 0)$.
The most interesting property of this kind of partition of the natural numbers is the trait by which the natural number sequences $\mathbb{S}_N$ resemble a fractal structure. Indeed, the decimal translation of the 2^{N+1} vertices of an (N+1)-Dimensional Boolean Hypercube $\mathbf{H}_{N+1}$ can be calculated iteratively, once the sequence $\mathbb{S}_N$, attached in turn to a previous N-Dimensional Boolean Hypercube $\mathbf{H}_N$ is known. This extension can be obtained just by adding to each member of $\mathbb{S}_N$ the N-th power of two: 2^N.
That is, the above-mentioned recursion involving a Boolean Hypercube structure can be easily written with the following Algorithm:
Algorithm 2: Knowing the Initial Natural Number Subset $\mathbb{S}_N$ Iterate Towards the Natural Number Sequence $\mathbb{S}_{N+1}$

$$\begin{aligned} \mathbb{S}_{N+1} &= \mathbb{S}_N \cup \left[2^N \oplus \mathbb{S}_N \right] \equiv \mathbb{S}_N \cup \mathbb{A}_{N+1} \\ &\equiv \left\{ 0, 1, ..., 2^N - 1 \right\} \cup \left\{ 2^N, 2^N + 1, ..., 2^{N+1} - 1 \right\} \\ &= \left\{ 0, 1, ..., 2^N - 1, 2^N, ..., 2^{N+1} - 1 \right\} \end{aligned} \tag{4.1}$$

In **Algorithm 2** the summation symbol $\oplus$ means that the power 2^N has to be summed to every element of the set $\mathbb{S}_N$. **Algorithm 2** above is also revealing a convenient *iterative construction* of natural numbers. When stepping one dimension up from a known N-Dimensional Boolean Hypercube $\mathbf{H}_N$, obtaining $\mathbf{H}_{N+1}$, the new decimal translation of his bit string vertices can be collected in the set: $\mathbb{S}_{N+1}$. Then the set $\mathbb{S}_{N+1}$ corresponds to the associated natural numbers, which are the decimal transcription of the Boolean vertices of the new $(N+1)$-Dimensional Boolean Hypercube $\mathbf{H}_{N+1}$ in the upgraded sequence.
The natural set $\mathbb{S}_{N+1}$ also can be expressed as the union of two parts. The first one is coincident with the old $\mathbb{S}_N$ sequence, associated with the entire set of vertices of the N-Dimensional Boolean Hypercube $\mathbf{H}_N$. The second part is the set, which could be called: $\mathbb{A}_{N+1}$, say, just corresponding to the old set $\mathbb{S}_N$, that to each of its elements has been added the value of 2^N.
The set $\mathbb{A}_{N+1}$ characterizes the decimal transcription of half the set of vertices of the $(N+1)$-Dimensional Boolean Hypercube $\mathbf{H}_{N+1}$. It corresponds to an entirely new set, which is not included in the decimal transcription $\mathbb{S}_N$ of the Boolean vertices within the initial Boolean Hypercube $\mathbf{H}_N$. From the Boolean point of view one can also write the concatenation:

$$\langle e_{N+1} | = \{1\} \cup \langle 0_N | = (1, 0, 0, ..., 0)$$

as a new element of the Boolean canonical basis set, then:

$$\mathbf{H}_{N+1} = \mathbf{H}_N \cup \{ \langle e_{N+1} | \oplus \mathbf{H}_N \}. \tag{4.2}$$

In the equation (4.2) above, the symbol $\oplus$ means that the Boolean vector $\langle e_{N+1} |$ is summed to every vertex of the Boolean Hypercube $\mathbf{H}_N$. Of course, the same rule applies when going downwards on the Boolean Hypercube dimension, wherein this case at the end, the sequence: $\mathbb{A}_{N+1} = \left[2^N \oplus \mathbb{S}_N \right]$ shall be overridden.

5. Mersenne numbers and their extensions

5.1. Mersenne numbers and their twins

As one can easily see, the Mersenne numbers become important when, as discussed in the previous section 4, this kind of partition and iterative construction of natural numbers via Boolean Hypercubes is considered. Mersenne numbers mark the end of the recursive sequences, associated with the decimal transcription of the vertices of the Boolean Hypercubes.
Also, Mersenne numbers might appear when other kinds of natural numbers are studied. For instance, the so-called *perfect* numbers, which can be defined as:

$$p_2(N) = 2^N \cdot \left(2^{N+1} - 1 \right) = 2^N \cdot \mu_2(N+1). \tag{5.1}$$

Perfect numbers, from the Collatz conjecture and Algorithm 1 viewpoint, iterate in the same manner as the associated Mersenne number. That is, one can write:

$$C_{end_p}[p_2(N)] \equiv C_{end_\mu}[\mu_2(N+1)],$$

because the multiplicative term 2^N in the equation (5.1), being even, transforms into the unit after N Collatz iterations, as previously discussed in the equation (3.2). Thus, this is the same as to consider that:

$$end_p = end_\mu + N.$$

At the same time, one can observe the trivial extension of the Mersenne numbers (here will be named *twin* Mersenne numbers), corresponding to the second term (the first one is 2^N, which is Collatz compliant) in the sequence $\mathbb{A}_N = \left[2^N \oplus \mathbb{S}_N\right]$ defined in the equation (4.1):

$$\nu_2(N) = 2^N + 1. \tag{5.2}$$

Thus, these new numbers: the Mersenne twins (5.2), are defined in this way. Then, it can be easily deduced that:

$$\mu_2(N)\,\nu_2(N) = 2^{2N} - 1 = \mu_2(2N).$$

5.2. Mersenne twins and base 3 Mersenne-like numbers

It is easy to prove the Collatz path connection between Mersenne numbers and similar powers of 3. For instance, using the symbol $C_I[\mu_2(N)]$ to indicate the application of the Collatz algorithm at the I-th iteration. That is, a partial path made of *I-th* iterations, then one can write:

$$\begin{array}{l} C_1[\mu_2(N)] = 3\cdot 2^N - 3 + 1 = 3\cdot 2^N - 2 \to \\ C_2[\mu_2(N)] = 3\cdot 2^{N-1} - 1 \to \\ C_3[\mu_2(N)] = 3^2\cdot 2^{N-1} - 3 + 1 = 3^2\cdot 2^{N-1} - 2 \to \\ C_4[\mu_2(N)] = 3^2\cdot 2^{N-2} - 1 \to \ldots \\ C_{2N}[\mu_2(N)] = 3^N - 1 = \mu_3(N) \end{array} \tag{5.3}$$

Meaning that, after a partial path containing a finite number of *2N* iterations, *any* Mersenne number submitted to the Collatz **Algorithm 1** converges to an equivalent number:

$$\mu_3(N) = 3^N - 1,$$

providing a Mersenne-like number associated with a base 3 instead of 2. Note that $\mu_3(N)$are even natural numbers.
This result implies that the number of iterations needed to complete the respective paths can be related by:

$$end_2 = end_3 + 2N.$$

Unfortunately, it cannot be shown the same property with the numbers defined in the equation (5.2), that is, one has in general:

$$C_K[\nu_2(N)] \neq 3^N + 1 = \nu_3(N).$$

However, one can study the Mersenne twins with even and odd powers of the base 2, in such a way that:

1. $N = 2n$.

Then one can write the Mersenne twin as:

$$\nu_2(N) = \nu_2(2n) = 4^n + 1 = \nu_4(n)$$

and the Collatz algorithm transforms the base 4 number into:

$$C[\nu_2(2n)] = C[\nu_4(n)] = \nu_3(n).$$

A particular case of Mersenne twins of this kind are the so-called Fermat numbers, which can be written as:

$$\nu_2\left(2^N\right) = 2^{2^N} + 1$$

but one can easily write:

$$\nu_2\left(2^N\right) = \nu_4\left(2^{N-1}\right) \to C\left[\nu_4\left(2^{N-1}\right)\right] = \nu_3\left(2^{N-2}\right) = 3^{2^{N-2}} + 1$$

1. $N = 2n+1$.

Then one can also write:

$$\nu_2(N) = \nu_2(2n+1) = 2\cdot 4^n + 1 = \eta_4(n)$$

thus, it is easy to see that in this case the Collatz algorithm transforms the number $\eta_4(n)$ in a similar manner as in case a):

$$C[\eta_4(n)] = 2\cdot 3^n + 1 = \eta_3(n).$$

Some numerical calculations have been performed on several $\nu_2(N)$ and $\nu_3(N)$numbers. It has been found that the number of iterations to complete the Collatz paths might vary with varying the power N:

$$C_{end_{\nu_2}}[\nu_2(N)] \wedge C_{end_{\nu_3}}[\nu_3(N)] \to end_{\nu_3} > end_{\nu_2},$$

that is, the number of iterations to arrive at the completion of Collatz **Algorithm 1** appears reversed to the case of Mersenne numbers $\mu_2(N)$ and $\mu_3(N)$.

5.3. Computational tests over Mersenne primes

Numerical tests over Mersenne numbers with powers yielding prime numbers, which can be obtained from the list posted in reference [25], prove that several prime Mersenne numbers containing an exceptionally large number of digits are Collatz compliant. The same occurs with their twins.
Of course, the number of Collatz iterations is increasing with the digits of the Mersenne prime and his twin, subject to the Collatz Algorithm 1, as Table 1 indicates.
The interest of the associated Mersenne numbers of base 3, $\mu_3(N)$, which is apparent from the equation (5.3), has raised the curiosity to test them numerically from Collatz **Algorithm 1** point of view. As a result, all have been found Collatz compliant, but differing from the Mersenne numbers $\mu_2(N)$, yielding a lesser number of iterations towards Collatz completion. The difference between the complete path iterations happened to be systematically equal to $2N$, as a consequence of the demonstrated property shown in the equation (5.3).
Table 1 has not been augmented with exponents larger than the displayed in the table, because while the information of first elements was obtained in a reasonable time, the last one has taken several days to complete the Collatz path, using a desktop computer with an i7 processor.
Every element of the tested sample of prime Mersenne numbers, as shown in Table 1, corresponds to a Collatz **Algorithm 1** starting on a natural number containing an exceptionally large set of decimal figures.

N	#Iterations $\mu_2(N)$	#Iterations $\nu_2(N)$
44497	598067	317327
86243	1158876	618898
216091	2906179	1562363
756839	10197081	5486483
859433	11568589	6209600
2976221	40055567	21532695
3021377	40663017	21846558
6972593	93778449	50464352

Table 1: Values of the Power N of assorted prime Mersenne numbers, $\mu_2(N)$, see reference [25], and their twin numbers, $\nu_2(N)$, and the necessary Collatz iterations to obtain complete paths

Collatz **Algorithm 1** has been also tested for the same assorted exponents employed in the Mersenne sequence, but on the numbers $\nu_2(N)$, and after this also on $\nu_3(N)$ numbers, which were also computed with the same powers as those of Table 1, to study if the $2N$ rule of the equation (5.3) was also fulfilled in these cases. Results of the termination iterations of $\nu_2(N)$ within the Collatz **Algorithm 1** can be also seen in Table 1.

5.4. Equal iteration numbers on complete path of collatz*Algorithm 1*

Due to the earlier mentioned Collatz connection between Mersenne numbers and the powers of base 3, also, as already commented, the twin of $\mu_3(N)$, $\nu_3(N)$ has been tested for the same exponents as those shown in Table 1. The result was intriguing because, without exception, it has been found the *same* number of iterations on both twins, obtained with the exponents of Table 1, to complete the Collatz **Algorithm 1**.
That is, it was found:

$$C_{end_\mu}[\mu_3(N)] = C_{end_\nu}[\nu_3(N)] \rightarrow end_\mu = end_\nu. \tag{5.4}$$

Such a result was unexpected; thus, an extra search was performed to shed as much light as possible into such a peculiarity, its possible extension, and the possibility that this was a general feature of these numbers.
At the light of the obtained results, one could test the conjecture that the twin pairs $\{\mu_3(N), \nu_3(N)\}$ possessed the property described in the equation (5.4) for any value of the power N. Alternatively, one could consider that the results were obtained just by chance and that the tested twins have shown this property because only the limited number of powers of Table 1 were used.
Now one must stress that even if the iteration termination numbers become equal in these twins of base 3, this does not preclude that the iterative sequences of Collatz paths become the same. In fact, for both twins, Collatz **Algorithm 1** already starts differently, because: $\mu_3(N)$ and $\nu_3(N)$ are both different even natural numbers. It is interesting to obtain the result of the product of both twins for any power:

$$\mu_3(N)\nu_3(N) = \left(3^N - 1\right)\left(3^N + 1\right) = 3^{2N} - 1 = \mu_3(2N) \tag{5.5}$$

which yields the same result that the Mersenne twins of base 2.

5.5. Numerical tests to search for equal iteration numbers in complete paths of base 3 Mersenne twins

To test the possibility of the presence of the property of equal terminating iterations in a large range of powers, a numerical prospection has been made for the twin pairs of base 3, $\{\mu_3(N), \nu_3(N)\}$ using powers lying in the interval $\{2, ..., 8192 = 2^{13}\}$.
Results revealed that there exists a **large** percentage of base 3 twin numbers bearing equal Collatz iteration termination numbers. Base 3 twins possess in the tested interval above an abundant number of twins having the same terminal iteration numbers. The results yielded around **93%** of the total tested twins within the above interval.
Unfortunately, this property **is not a general one**, as **7%** of base 3 twins **do not** possess the property of possessing complete paths with equal iteration numbers. Even bearing this curious property, the Collatz complete paths appear essentially different in both twins but arrive at the terminal completion unit at the **same** iteration time. Then, as the percent of equal iterations to reach complete paths in the studied power range is so high, no wonder that the small sample of base 3 twins of Table 1 present all of them the same complete path iteration numbers.

Contrarily to the base 3 case, the true Mersenne twin numbers: $\{\mu_2(N), \nu_2(N)\}$ both in Table 1 and also studied within the power interval: $\{2-8192\}$ do not present at all coincident Collatz **Algorithm 1** complete path termination iteration numbers, although all of them are Collatz compliant. This was also a remarkable result in full contrast with the already discussed base 3 twins.

5.6. Mersenne-like twin pairs involving prime number base

One must note here that while Mersenne twins are odd, base 3 twins are even, and perhaps the equal iteration numbers shown in base 3 twins is due to this kind of odd-even startup of the Collatz **Algorithm 1**.
The property of the equation (5.5) can be extended to any base number, as one can straightforwardly write:

$$\mu_B(N)\,\nu_B(N) = \left(B^N - 1\right)\left(B^N + 1\right) = B^{2N} - 1 = \mu_B(2N) \tag{5.6}$$

A property that might be related to the behavior of these Mersenne prime twins concerning **Algorithm 1**.
Tests performed with several **even** number bases show the same behavior as Mersenne twins: **none or a scarce** number of equal termination iterations of Collatz paths.
However, tests on **prime or odd** number bases provided a similar behavior as the base 3 results. Many powers show equal Collatz complete paths termination iterations. When the number of powers in a tested sample increases, this produces an increase of twins' percent possessing equal termination iterations.
Table 2 shows the results of complete path Collatz iterations coincidences in several prime bases, calculated in a sample of powers contained within the interval: $\{2-8192\}$.

B	3	5	7	11	13	17	19	23	29	31	37	41	43	47
%	93	95	91	95	96	92	96	93	96	92	96	96	96	93

Table 2: Percentages of coincident Collatz iterations to reach the complete path using the fourteen first prime base twins $\{\mu_B(N), \nu_B(N)\}$ computed with the powers in the interval$\{2-8192\}$.

From Table 2 one can conclude that the percentages fluctuate with the prime base number, but these fluctuations keep the amount of equal terminating iterations for twins on several bases greater than 90%, with an average of 94% of tested cases. This means that there is a high chance to find out odd base twins with equal terminating iterations within a large set of powers.
This might explain the results found when calculating Table 1. Several previous tests have shown a clear trend of obtaining larger percentages of equal Collatz complete path iterations as the set of tested power range increases.
This can be illustrated by the base 7 behavior, which provides, in the power range $\{2-1024\}$, a 77% of equal termination iterations, but in the power range $\{2-2048\}$, the obtained results deliver 83%, while in the power range $\{2049-4096\}$ this percentage rises to 92%. The value for this base 7 presented in Table 2, corresponds to a raising percentage, weighted by the increasing trend associated with larger exponents.

5.7. Numerical proof of Collatz conjecture

One must also mention that the results of Table 2 can also be considered as a large sample test (around 160,000 elements) of Collatz **Algorithm 1** convergence, obtained within a large interval of natural numbers: $\left\{\left(3^2 \pm 1\right) \rightarrow \left(47^{8192} \pm 1\right)\right\}$.
In all the studied cases included in this paper, no Collatz **Algorithm 1** cyclic loops or divergences associated with any tested natural number appeared. Statistically speaking, the obtained numerical results look like one can safely accept Collatz conjecture as true, at least from a computational point of view.

6. On trying to prove Collatz conjecture

Besides the already shown computational results, one might attempt to prove the Collatz conjecture. This could be tested by using the recursive structure of the natural number set. That is, constructing Boolean Hypercube using concatenation and afterward transforming the Boolean Hypercube vertices into natural numbers, as earlier discussed in section 4 and as shown in **Algorithm 2**.

6.1. Some preliminary considerations

Under the extensive computational tests to show the Collatz compliance of the natural numbers, as it has been already commented, no natural number checked in the described numerical tests have shown cyclic or divergent Collatz **Algorithm 1** behavior. In the sampling of the commented properties in section *5.7* above **no exception** to overall convergence has been found.
Nevertheless, these previous computational-experimental findings, as many others described in reference [26], and still ongoing tests, see reference [27] for example, by themselves *do not prove* the Collatz conjecture. However, they can be surely taken as tentative evidence of the general Collatz compliance of the natural number set as a whole.
Another comment one can put forward is the obtained logical fact showing that, considering some natural numbers made as powers of some natural number base B: B^N, the number of Collatz **Algorithm 1** iterations increases as the base number B grows keeping the same exponent, and also the number of Collatz **Algorithm 1** iterations increases when keeping the base number constant and increasing the exponent.
Therefore, it seems a bit useless to try a systematic search of Collatz compliance using as much as a possibly large set of natural numbers. Like the computational endeavor promoted in reference [27], which constitutes a web page dedicated to the $(3x+1)$ problem, studied via a large computing effort. As it seems that, according to the recent study of Tao [4], it is sufficient to consider some numeric sequences, which can be associated with the Collatz compliance problem.
This is the case of Mersenne numbers $\mu_2(N)$, or the $\nu_2(N)$, $\mu_3(N)$ and $\nu_3(N)$ numbers when studied with large power sequences.

Also, in the twin sequences $\{\mu_B(N), \nu_B(N)\}$ with the first ten prime natural numbers chosen as a base B, the iteration numbers to reach the complete path of the Collatz **Algorithm 1** could be huge, as huge are the natural powers tested, which can be of the order of 10^{8000}or greater. In these cases of extremely large natural numbers submitted to Collatz **Algorithm 1** testing, no cyclic nor convergence problems have been encountered, except the challenging sizable amount of computing time needed to finish the Collatz complete paths using a standard desktop CPU computer.

6.2. Are Mersenne numbers collatz compliant?

It has been demonstrated, see the equation (5.3), that Collatz **Algorithm 1** applied to Mersenne numbers, becomes equivalent to the transformation providing a partial path via $2N$iterations:$\mu_2(N) \rightarrow \mu_3(N)$, into Mersenne-like numbers with base 3.
Thus, the Collatz compliance problem of Mersenne numbers is equivalent to the compliance of the Mersenne-like numbers of base 3. The numbers $\mu_3(N)$ as already commented are even though. Therefore, if it can be proven that even natural numbers are Collatz compliant, then this will also show that odd true Mersenne numbers $\mu_2(N)$ are Collatz compliant.

6.3. Collatz compliance

When trying to understand the general case of the Collatz **Algorithm 1** application to any natural number, the iterative construction shown in the equation (4.1) can be recalled as an adequate step to develop a test of possible inductive proof of the Collatz conjecture.
Hence, if one has found that at some Boolean Hypercube dimension step N, all the elements of the transcribed natural number sequence $\mathbb{S}_N$ have been found Collatz compliant; that is, supposing that:

$$\forall m \in \mathbb{S}_N \rightarrow C_{end}[m] = 1$$

or using a more compact notation as previously noted:

$$C_{\langle end|}[\mathbb{S}_N] = \langle 1_{2^N}|.$$

Assuming moving one step further, with the natural transcription of the vertices of a Boolean Hypercube of dimension $N+1$, then one can study from the Collatz **Algorithm 1** the behavior of the elements of the additional set, which truly conform the transcribed augmented natural number set $\mathbb{S}_{N+1}$as described in the equation (4.1).
Proving that the elements of the set: $\mathbb{A}_{N+1} = \left[2^N \oplus \mathbb{S}_N\right]$ are Collatz compliant, then the Collatz compliance of any natural number set, constructed in the way proposed in this work, leads to consider Collatz conjecture to be inductively proved.

6.4. Collatz compliance of natural even numbers

Moreover, the set $\mathbb{A}_{N+1}$ can be divided into the two disjoint sets of even: $\mathbb{E}_{N+1}$ and odd $\mathbb{O}_{N+1}$ elements, that is:

$$\mathbb{A}_{N+1} = \mathbb{E}_{N+1} \bigcup \mathbb{O}_{N+1} \Leftarrow \mathbb{E}_{N+1} \bigcap \mathbb{O}_{N+1} = \emptyset. \tag{6.1}$$

Now, one can easily prove that all the even elements of the set $\mathbb{E}_{N+1}$, after the first Collatz iteration, they transform into elements of the set $\mathbb{S}_N$, which have been supposedly considered as Collatz compliant.
Indeed, one can write a trivial sequence:

$$\begin{aligned}
&\forall m_{N+1} \in \mathbb{E}_{N+1} \wedge \forall k \in \mathbb{S}_{N-1} : m_{N+1} = 2^N + 2k \rightarrow \\
&C_1[m_{N+1}] = 2^{N-1} + k = m_N \in \mathbb{S}_N \rightarrow \\
&C_{end_N}[m_N] = 1 \rightarrow C_{end_{N+1}}[m_{N+1}] = 1
\end{aligned} \tag{6.2}$$

The sequence in the equation (6.2) demonstrates that all the even numbers $\mathbb{E}_{N+1}$ contained in the set $\mathbb{A}_{N+1}$ are Collatz compliant. This is so since, after iteration 1, the resultant number in the partial path belongs to the previous step set of natural numbers, and it could be considered by hypothesis Collatz compliant. Then, using the property of section 3.4, the even numbers contained in the set $\mathbb{E}_{N+1}$ are Collatz compliant. Thus, the induction proof apparently might permit to deduce that the whole set of natural even numbers are Collatz compliant.
However, the induction proof appears *incomplete*. Because it doesn't use the whole natural set $\mathbb{A}_{N+1}$ contained in the equation (6.1), but just the even elements $\mathbb{E}_{N+1}$, and lacks to include the odd number set $\mathbb{O}_{N+1}$, which is necessary to deduce in the next induction step that even numbers are Collatz compliant.
Thus, it is **not** possible within the present analysis by induction to deduce that all even numbers are Collatz compliant.
One can use instead, this potential property as a new conjecture.
Conjecture 1: Partial Collatz compliance
"All even natural numbers are Collatz compliant".
As one can write the natural number set as the union of even and odd numbers:

$$\mathbb{N} = \mathbb{E} \cup \mathbb{O} \wedge \mathbb{E} \cap \mathbb{O} = \emptyset,$$

the induction procedure result, performed over the even numbers whole set could be now **conjectured** to behave as:

$$\forall e \in \mathbb{E} \rightarrow C_{end}[e] = 1,$$

6.5. Collatz compliance of Mersenne numbers

The **Conjecture 1** about partial Collatz compliance of the even natural numbers is sufficient to deduce that Mersenne numbers are Collatz compliant, even if they are odd natural numbers. As it has been commented in section 5.2.
Recalling equation (5.3), one can easily write:

$$C_{2N}\left[\mu_2(N)\right] = \mu_3(N) \in \mathbb{E}_{N+1} \rightarrow C_{end}\left[\mu_3(N)\right] = 1 \rightarrow C_{end}\left[\mu_2(N)\right] = 1,$$

showing that *any* Mersenne number is Collatz compliant.
One can also consider the results shown in Table 2, as an indirect computational proof of the Collatz compliance of even numbers, besides of the curious property of twin Mersenne numbers, having a large probability to converge in the same number of iterations but via different paths. One must keep in mind, though, that computational behavior, even if consistent over a large set of even Mersenne twins, constructed with base prime numbers, cannot be a complete proof of **Conjecture 1**.
This is like this because it cannot be discarded that perhaps does exist a unique natural even number, which is not Collatz compliant and has not yet computationally found. Although, besides the huge computational effort of reference [27], there are also contemporary attempts to set up a proof of the Collatz conjecture by computational means [28]. Perhaps they are a pair of useless efforts.

6.6. Collatz compliance of odd natural numbers

Thus, there is only left to try if it is possible to make use of **Conjecture 1** that if true, one can demonstrate that the elements of the odd subset $\mathbb{O}_{N+1}$ are also Collatz compliant. That is, only the natural numbers in the sequence:

$$\forall m_{N+1} \in \mathbb{O}_{N+1} \wedge \forall k \in \mathbb{S}_{N-1} : m_{N+1}(k) = 2^N + (2k+1) \tag{6.3}$$

need to be considered now, once the even natural numbers $\mathbb{E}$ are supposedly Collatz compliant.
Then, the *first* iteration of Collatz **Algorithm 1** applied to the numbers belonging to the odd set $\mathbb{O}_{N+1}$ of $\mathbb{A}_{N+1}$, generally represented by the equation (6.3), will yield:

$$\begin{aligned} C_1\left[m_{N+1}(k)\right] &= 3m_{N+1}(k) + 1 \\ &= 3 \cdot 2^N + 3 \cdot 2k + 4 = 2 \cdot \left(3 \cdot 2^{N-1} + 3 \cdot k + 2\right) \in \mathbb{E} \end{aligned} \tag{6.4}$$

which yields an even number. Then one can also write:

$$C_2\left[m_{N+1}(k)\right] = 3 \cdot 2^{N-1} + 3 \cdot k + 2 = \left(2^N + 2^{N-1} + 2k + 2\right) + k$$

therefore, one can see that further application of Collatz **Algorithm 1** will depend on the odd or even nature of the number $k \in \mathbb{S}_{N-1}$.
The equation (6.4) can be seen as demonstrating that odd numbers in $\mathbb{O}_{N+1}$ are Collatz compliant if and only if all the *whole set of even* natural numbers can be compliant. **Conjecture 1** can again be invoked to write, as it is the case to be true:

$$\forall o \in \mathbb{O} \rightarrow C_{end}[o] = 1,$$

which allows to finally write the Collatz conjecture:

$$\forall n \in \mathbb{N} \rightarrow C_{end}[n] = 1.$$

7. Conclusion

A large set of numerical experiments reported here have yielded no natural number presenting compliance problems when tested with Collatz **Algorithm 1**. The set of examples tested has been mainly focused on Mersenne numbers, Mersenne-like, and related number sets, bearing diverse base and powers.
With odd base numbers and for a large range of powers Mersenne-like twins appear which follow, in a percentage larger than 90%, different paths to completion, but present the same number of Collatz iterations to achieve the respective complete paths.
The number of numerical experiments has been sufficiently large and varied allowing to say that Collatz conjecture holds from the statistical point of view. However, this is an inconclusive proof of the conjecture.
After using the recursive building of the whole natural number set, provided by the Boolean Hypercube concatenation and the following Boolean vertex transcription into decimal numbers, it has been tested if it was possible to construct a general inductive proof of the Collatz compliance of the whole natural number set.
Results indicate that one can reach an incomplete verification of the Collatz conjecture by an inductive proof, based on a recursive construction of natural numbers. But even if this attempt was inconclusive, it leads to a new partial **Conjecture 1** involving just the even natural numbers. The attempted proof even incomplete sheds light on the fact that, if the whole set of even natural numbers can be proved Collatz compliant, then the whole natural set is compliant.

Acknowledgement

The author wants to express his gratitude to Professor Carlos C. Perelman for a patient discussion on the former manuscript version contents. Also, Professor Jacek Karwowski's deep insight and fruitful discussion about the present manuscript has enlightened the initial contents and polished the nuances on the proof of Collatz conjecture.

Compliance with ethical standards

Conflict of interest: The author states that there is no conflict of interest related to this work.

References

[1] J. C. Lagarias, *The 3x+1 problem and its generalizations*, Am. Math. Monthly, **92** (1985), 3-23.
[2] J. C. Lagarias (Editor), *The Ultimate Challenge: The 3x+1 Problem*, American Mathematical Society, 2010.
[3] *Collatz conjecture*, available at https://en.wikipedia.org/wiki/
[4] T. Tao, *Almost all orbits of the collatz map attain almost bounded values*, (2019), arXiv:1909.03562v2 [math.PR].
[5] J. C. Lagarias, *The 3x+1 problem: An Annotated Bibliography, I* (1963-1999), (2009), arXiv:math/0309224v12 [math.NT].
[6] J. C. Lagarias, *The 3x+1 problem: An Annotated Bibliography, II* (2000-2009), (2009), arXiv:math/0608208v5 [math.NT].
[7] R. Carbó-Dorca, *Boolean hypercubes and the structure of vector spaces*, J. Math. Sci. Modelling, **1** (2018), 1-14.
[8] R. Carbó-Dorca, *Natural vector spaces, (Inward power and Minkowski norm of a natural vector, natural Boolean hypercubes) and Fermat's last theorem*, J. Math. Chem., **55** (2017), 914-940.
[9] R. Carbó-Dorca, C. Muñoz-Caro, A. Niño, S. Reyes, *Refinement of a generalized Fermat's last theorem conjecture in natural vector spaces*, J. Math. Chem., **55** (2017), 1869-1877.
[10] R. Carbó-Dorca, *Cantor-like infinity sequences and Gödel-like incompleteness revealed by means of Mersenne infinite dimensional Boolean hypercube concatenation*, J. Math. Chem., **58** (2020), 1-5.
[11] R. Carbó-Dorca, *Fuzzy sets and Boolean tagged sets*, J. Math. Chem., **22** (1997), 143-147.
[12] R. Carbó, B. Calabuig, *Molecular similarity and quantum chemistry*, M. A. Johnson, G. M. Maggiora (editors) Chapter 6 in Concepts and Applications of Molecular Similarity, John Wiley & Sons Inc., New York, 1990, pp. 147-171.
[13] R. Carbó, B. Calabuig, *Molecular Quantum Similarity Measures and N-Dimensional Representation of Quantum Objects II. Practical Applications (3F-Propanol conformer taxonomy among other examples)*, Intl. J. Quant. Chem., **42** (1992), 1695-1709.
[14] R. Carbó-Dorca, *About Erdös discrepancy conjecture*, J. Math. Chem., **54** (2016), 657-660.
[15] R. Carbó-Dorca, *N-dimensional Boolean hypercubes and the Goldbach conjecture*, J. Math. Chem., **54** (2016), 1213-1220.
[16] R. Carbó-Dorca, *A study on Goldbach conjecture*, J. Math. Chem., **54** (2016), 1798-1809.
[17] R. Carbó-Dorca, *Boolean hypercubes as time representation holders*, J. Math. Chem., **56** (2018), 1349-1352.
[18] R. Carbó-Dorca, *DNA, unnatural base pairs and hypercubes*, J. Math. Chem., **56** (2018), 1353-1356.
[19] R. Carbó-Dorca, *Transformation of Boolean hypercube vertices into unit interval elements: QSPR workout consequences*, J. Math. Chem., **57** (2019), 694-696.
[20] R. Carbó-Dorca, *Role of the structure of Boolean hypercubes when used as vectors in natural (Boolean) vector semispaces*, J. Math. Chem., **57** (2019), 697-700.
[21] R. Carbó-Dorca, T. Chakraborty, *Hypercubes defined on n-ary sets, the Erdös-Faber-Lovász conjecture on graph coloring, and the polypeptides and RNA description spaces*, J. Math. Chem., **57** (2019), 2182-2194.
[22] J. Chang, R. Carbó-Dorca, *Fuzzy hypercubes and their time-like evolution*, J. Math. Chem., **58** (2020), 1337–1344.
[23] K. Balasubramanian, *Combinatorial multinomial generators for colorings of 4D-hypercubes and their applications*, J. Math. Chem., **56** (2018), 2707-2723.
[24] K. Balasubramanian, *Computational multinomial combinatorics for colorings of 5D-hypercubes for all irreducible representations and applications*, J. Math. Chem., **57** (2018), 655-689.
[25] https://www.mersenne.org/primes/
[26] A.V. Kontorovich, J. C. Lagarias, *Stochastic models for the 3x+1 and 5x+1 problems*, (2009), arXiv:0910.1944v1 [math.NT].
[27] http://www.ericr.nl/wondrous/
[28] W. Ren *A new approach on proving collatz conjecture*, Hindawi J. Math., (2019), Article ID 6129836, 1-12.

Journal of Mathematical Sciences and Modelling, 3 (3) (2020) 130-134
Research Article

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm
ISSN 2636-8692
DOI: http://dx.doi.org/10.33187/jmsm.808041

Early Childhood Children in COVID-19 Quarantine Days and Multiple Correspondence Analysis

Nihat Topaç[1], Musa Bardak[2], Seda Bağdatlı Kalkan[3] and Murat Kirişci[4*]

[1]*Department of Early Childhood Education, Istanbul University-Cerrahpaşa Istanbul, Turkey*
[2]*Department of Early Childhood Education, Istanbul Sabahattin Zaim University, Istanbul, Turkey*
[3]*Department of Statistics, Istanbul Commerce University Istanbul, Turkey*
[4]*Department of Biostatistics and Medical Informatics, Istanbul University-Cerrahpaşa Istanbul, Turkey*
[*]*Corresponding author*

Article Info

Keywords: *Cognitive development, COVID-19, Early childhood, Multiple correspondence analysis.*
2010 AMS: *62H17, 62P15.*
Received: *9 October 2020*
Accepted: *18 December 2020*
Available online: *29 December 2020*

Abstract

Multiple correspondence analysis is an extension of correspondence analysis that consent one to examine the stencil of intercourses of several categorical dependent variables. The aim of this study is to analyze the cognitions, feelings, and thoughts of early childhood children who stayed at home during the quarantine process due to coronavirus with multiple correspondence analysis. The theory and commentary of multiple correspondence analysis in the case of two and more than two variables are provided through an example. The result from multiple correspondence analysis is a graphical monitor of the rows and columns of a contingency table that is conceived to permission visualization of the prominent correlations among the variable responses in a low-dimensional space. Such a presentment discloses a more global picture of the correlations among row-column pairs.

1. Introduction

From time to time, the world is faced with natural or human-induced disasters. Various political, economic, health and social consequences of these disasters affect people over different periods of time. Sawada and Takasaki [1] state that the most important damage caused by disasters is the loss of life as well as the serious blows to the economies of the states. These disasters affect humans, then animals and plants, relatively the most from living groups. However, people and countries can be affected by disasters not only economically but also in different ways. *OECD* [2] has collected the effects of natural disasters under four main headings: individual impact, physical impact, sociological impact, public health, and mental health effects.

Children are one of the disadvantaged groups regarding public health and mental health. The effects of natural disasters can be seen more prominently, especially in children in early childhood, where development is faster than in other periods. In this case, mental health disorders and developmental retardation can be seen in children.

Especially children may face troubling situations due to the prolongation of epidemic diseases, therefore quarantine and these measures limit people. Ericson [3] mentioned the importance of the environment in his developmental theory and stated that the lack of opportunities to prepare children for developmental tasks may cause dysfunctional behaviors. In some cases, people, especially children, are subject to some restrictions because they are in quarantine, even if they are not physically harmed. While the child will be able to play in different environments, due to the quarantine, he will only have limited experience within the scope of his home. However, play is extremely important in the development of the child. While making his moment enjoyable, it also plays a role in affecting his whole life. In this context, Topac, Bardak, and Ünal [4] stated that play improves the quality of a child's life, affects his career and private life to a great extent, and children whose right to play is denied will become unhappy individuals in the coming years because of they cannot live their childhood sufficiently.

***Email addresses and ORCID numbers*: ntopac@istanbul.edu.tr, https://orcid.org/0000-0001-9364-4072 (N. Topaç), musa.bardak@izu.edu.tr, https://orcid.org/0000-0001-5585-8002 (M. Bardak),sbagdatli@ticaret.edu.tr, https://orcid.org/0000-0003-3002-2983 (S. B. Kalkan), mkirisci@hotmail.com, https://orcid.org/0000-0003-4938-5207 (M. Kirişci)**

Open perceptions and readiness of children about new information in the learning process affect their learning speed. The stimuli that are exposed during the pandemic process direct the perception of children to this issue. In this case, it can be said that children will pay more attention to the pandemic, quarantine, and related stimuli. With these effects, children may show introversion, communication, and speech disorders, anxious, obsessive, or compulsive behaviors.

Topac and Bardak [5] stated that school is an important factor in gaining the habit of cleaning in early childhood. In this context, children who stay away from school may not acquire the ideal cleaning habit due to disruption or dropping out. In addition, exaggeration behaviors related to cleaning can be seen through stimuli in the environment. Different tools have been developed in order to protect the mental health of various layers of people during times of disaster. One of them, Public Child and Family Disaster Communication, has been defined as a public health tool that can be used to cope with the post-disaster situation / encourage resilience and improve incompatible child responses. It has also been stated that schools are an important (promising) system for child and family disaster communication [6]. In this context, Dadds, Holland, Laurens, Mullins, Barrett, and Spence [7] found that a school-based intervention shows a permanent and effective result in reducing children's anxiety in a study they conducted to prevent anxiety in children.

The purpose of this study is to reveal the cognitions, feelings, and thoughts of early childhood children during the Covid-19 quarantine process with the method of multiple correspondence analysis.

2. Method

2.1. Multiple correspondence analysis

Multiple correspondence analysis (MCA) is an extension of correspondence analysis (CA) which allows one to analyze the pattern of relationships of several categorical dependent variables. As such, it can also be seen as a generalization of principal component analysis when the variables to be analyzed are categorical instead of quantitative.

MCA is obtained by using a standard correspondence analysis on an indicator matrix (i.e., a matrix whose entries are 0 or 1). The percentages of explained variance need to be corrected, and the correspondence analysis interpretation of interpoint distances needs to be adapted.

MCA is used to analyze a set of observations described by a set of nominal variables. Each nominal variable comprises several levels, and each of these levels is coded as a binary variable.

CA is a categorical data analysis method that takes place within the optimal scaling techniques. Simple CA is used in the analysis of two-way contingency tables. MCA or homogeneity analysis (Homogeneity Analysis by Alternating Least Squares-HOMALS) is used in the analysis of multi-directional contingency tables.MCA is similar to the Principal Component Analysis (PCA) in terms of the application purpose of the analysis method. However, all variables are categorical in MCA. The purpose of this analysis method is to reveal which categories of categorical variables are compatible with each other. Thus, relations that are difficult to interpret with the analysis of chi-square and contingency tables can be easily interpreted through graphs, and relations between categories can be revealed [8], [9], [10]. MCA also shows similarities with multidimensional scaling. However, since it shows the relationship between categories in the same space, it differs from multidimensional scaling [11]. In MCA, the difference between variables is defined by a loss function. The loss function is minimized using the alternating least squares method, and object scores and quantifications that achieve maximum homogeneity between variables are achieved. Another important point in the MCA is the explained variance value. The variance explained as in the PCA is expressed by eigenvalues. Eigenvalues express the value of inertia in MCA. The most important point in MCA is the interpretation of the graph obtained. In the graphic, the distance of the category point to the origin shows the importance of the category. If the direction of one point is opposite to the direction of other points, there is a negative correlation between them. If the direction is the same, the correlation is positive. In addition, if the angle between the lines showing the distance of the two category points to the origin is small, that is, if the points are close to each other, the correlation between them is high, if the angle is large, the correlation is low.

In the literature, there are studies in many different areas related to MCA. In the work of Parchomenko [12], the method of MCA was used to assess 63 Circular Economy metrics and 24 features relevant to Circular Economy, such as recycling efficiency, longevity, and stock availability.

2.2. Model

In this study, the survey model was used. The survey was prepared in the Likert type and the opinions of the participants were determined. In this study, a Likert questionnaire suitable for development levels was prepared.

The linguistic terms and their numeric labels are: Questions asked to children: Yes (1), maybe/some (2), no (3). The survey included the following questions:

- Do you know Corona-virus?
- Are you afraid of Corona-virus?
- Does Corona-virus harm people?
- Does Corona-virus harm animals?
- Can Corona-virus be prevented?
- Do you think it's nice not to go to school?
- Are you upset that you can't go to school?
- Is the obligation to stay home boring?

- Can we be protected from Corona-virus by staying at home?
- Do you think you can go to school from now on?

In this study, from Turkey, 201 children ages 5-6 units were the participants.

3. Results

In the study, the data of the children were divided into two groups and multiple compliance analysis was performed. Chi-square analysis was applied to determine the variable groups and the variables that were related to each other were determined. In the first analysis, do you know the Coronavirus? (Q1) Are you afraid of coronavirus? (Q2) Does Coronavirus harm people? (Q3) Does Coronavirus harm animals? (Q4) Can coronavirus be prevented? (Q5).

When Figure 3.1 is analyzed, those who answer yes for all questions form a group.So

- 1. knowing coronavirus,
- 2. afraid,
- 3. who thinks they can be prevented
- 4. that it harms people and animals

thinking children constitute a group.

Another group is

- 1. Are you afraid of corona virus,
- 2. Does coronavirus harm animals?
- 3. Corona virus can be prevented?

he/she evaluates his/her questions as maybe/some answer.

When an assessment is made in terms of those who answered no,

- 1. coronavirus does not harm animals
- 2. cororavirus cannot be prevented

the children who are saying are located close to each other.

In the second analysis, do you think it's good not to go to school? (Q6) Are you upset that you can't go to school? (Q7) Is the obligation to stay at home boring? (Q8), can we get rid of the coronavirus by staying at home? (Q9), do you think you can go to school from now on? (Q10) variables were used and Figure 3.2 was created as a result of MCA.

When Figure 3.2 is examined, it is clearly seen that the children who gave maybe/some answers for all questions are together. It is observed that children who do not feel sorry for not going to school and who do not find it boring to stay at home are also a group. In addition, those who say yes to the question of not going to school and children who say no to the question of whether you can go to school from now on are a group. Another group shows that there are children who say no to the question of not going to school and say yes to all other questions.

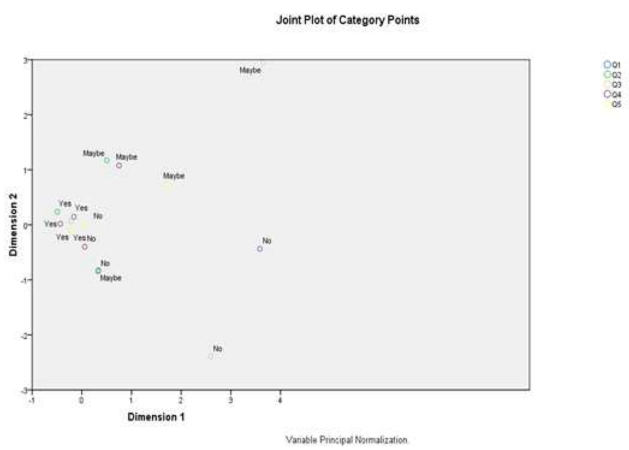

Figure 3.1: Joint Plot for first analysis

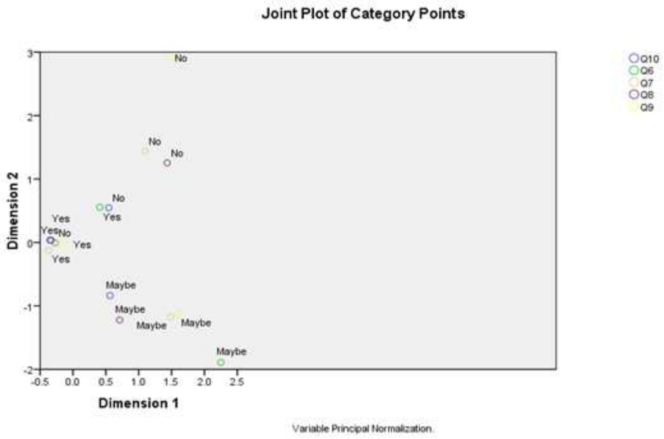

Figure 3.2: Joint Plot for second analysis

4. Discussion

Awareness and cognition about disaster and emergency planning should be developed in all parts and levels of the health system, regardless of physical or mental health. As can be understood from here, both adults and children can exhibit wrong behaviors by having wrong cognitions during epidemic times. Even in quarantine, people can develop false cognitions and act accordingly.

Every new experience means new knowledge and is a situation that needs to be learned. In particular, children should learn from natural disasters such as epidemics with correct experiences. The information must be coded correctly and transformed into behavior. For this, it is important for the administrators to inform the public with correct information and thinking about their psychology. It is important to take precautions and inform the public in a timely manner so that the public does not panic. It should not be forgotten that people can cause the spread of the disease with panic and wrong information, and they may behave in ways that can harm themselves and the society they live in. In this context, it can be said that children's cognition and behavior may also be affected by the negative aspects of the process.

In this study, we examined how the quarantine affects the cognitive and behavioral states of early childhood children with the multiple correspondence analysis methods within the scope of the measures taken to prevent the spread of the epidemic caused by the coronavirus. Questions directed to children were analyzed in two groups. As a result of the analysis, subgroups different from the answers given by the participants emerged in the group with the first five questions. Different groups of children were formed in the part where the other five questions were found. It can be said that this diversity coincides with the characteristics of the early childhood period where individual differences are high. However, the stimuli interacted with are also an important factor. The density of clusters formed as a result of the analysis can be explained by the level of interaction of the children in the relevant group with these factors.

Children in the first group, which consists of the answers given to the first five questions, are children who have knowledge about the epidemic, fear the virus, think that the disease can be prevented, and think that this virus harms people and animals. In other words, the children in this group answered yes to all questions. Children of this age can develop their cognition in line with the information they receive from the stimuli around them. However, these cognitions may not turn into healthy feelings and behaviors. In the second group, there are those who give maybe/a little answers to the questions of whether the are you afraid of coronavirus, does the corona virus harm animals and can the coronavirus be prevented. As a third group, children who say that coronavirus does not harm animals and coronavirus cannot be prevented are seen. Finally, those who know a little about coronavirus, and those who are not afraid of the virus are also a group. These situations can also be associated with the information that children acquire from different stimuli in the environment. In this context, Pearson and Degotardi [13] emphasized that the interaction of a child with his environment at an early age is extremely important in terms of observing the environment, establishing a cause-effect relationship, and reaching the competence to continue a discussion about a problem. In this way, the child can be in both an influencing and affected state while performing these skills. However, according to the ecological theory put forward by Bronfenbrenner [14], the family and home environment are the first circles around the child. Therefore, it can be said that the answers given by the children within the scope of this study are not independent of parental influence.

The first group consisting of the answers given to the second five questions is the children who give maybe/a little answer to all questions. In another group, there are children who are not upset about not being able to go to school and who do not find the obligation to stay home boring. It can be concluded that children in this group do not like school. In another group, there are those who say yes to the question of whether it is good to not go to school and children who say no to the question of whether you can go to school from now on. Again, it can be said that these children responded in this direction because they did not want to go to school and hoped not to. Finally, the children who said no to the question of whether it is good to not go to school and said yes to all other questions formed a group. It can be interpreted that these children love school and want to go to school. In addition, it can be said that the attitudes, behaviors, and discourses of the parents may have guided the cognition, feelings, and thoughts of the children in this period. Learning as Bandura [15] stated in his Social Learning Theory is the product of the dynamic relationship of the triangle of environment, behavior, and personal factors. Children's responses can be explained by the learning experiences that are the product of this relationship.

As seen in the findings, most of the participating children stated that they were afraid of coronavirus. Some of them think that the coronavirus cannot be prevented. There can be many factors that affect this condition. One of these factors, as mentioned earlier, is the likelihood that one or more family members may present a negative profile. According to the National Academies of Sciences, Engineering and Medicine [16], having a parent with untreated mental illness or addiction has genetic risks, epigenetic changes, negative behavioral modeling, negative social learning and It can lead to a difficult negative childhood experience involving many mechanisms that can also lead to relational skills. Therefore, the underlying cause of some of these cognitions of children can be fed from different sources, but the effect of the current period should also be taken into account as a trigger.

References

[1] Y. Sawada, Y. Takasaki, *Natural Disaster, Poverty, andDevelopmen: An Introduction*, World Development, **94**, (2017), 2–15.
[2] OECD, Guidelines on DisasterMitigation. (1994) http://www.oecd.org/dac/environment-development/1887740.pdf (Access 04.04.2020)
[3] E. H. Erikson, *Identity: Youth and crisis*, New York: Norton, 1968.
[4] N. Topaç, M. Bardak, D. D. Ünal *The Right to Play and Reasons for Not Accessing the Child's Right to Play*. Proceedings of 5th International Social Humanities and Educational Sciences Congress (ISBN: 978-605-7594-40-2), December, 19-20, 2019, 675–680 (Turkish).
[5] N. Topaç, M. Bardak, *The Place of Early Childhood Education Institutions in Child Getting the Habit of Cleaning*. Proceedings of 5th International Social Humanities and Educational Sciences Congress (ISBN: 978-605-7594-40-2). December, 19-20, 2019, 681-686 (Turkish).
[6] J.B. Houston, J. First, M.L. Spialek, M.E. Sorenson, M. Koch, *Public disaster communication and child and family disaster mental health: A review of theoretical frameworks and empirical evidence*. Current Psychiatry Reports, **18**(54) (2016), 1–9.
[7] M.R. Dadds, D.E. Holland, K.R. Laurens, M. Mullins, P.M. Barrett, S.H. Spence, *Early intervention and prevention of anxiety disorders in children: Results at 2-year follow-up*. Journal of Consulting and Clinical Psychology, **67** (1) (1999), 145–150.
[8] N. Sourial, C. Wolfson, B. Zhu, J. Quai, J. Fletcher, S. Karunananthan, K. Banden-Roche, F. Beland, H. Bergman, *Correspondence analysis is a useful tool to uncover the relationships among categorical variables*. Journal of Clinical Epidemiology, **63**(6) (2010), 638–646. https://doi.org/10.1016/j.jclinepi.2009.08.008
[9] V. Audigier, F. Husson, J. Josse, *MIMCA: multiple imputation for categorical variables with multiple correspondence analysis*. Statistics and Computing, **27**, (2017), 501–518.
[10] G. Di Franco, *Multiple correspondence analysis: one only or several techniques?*. Quality & Quantity, **50**, (2016), 1299–1315
[11] D. Hoffman, J. De Leeuw, *Interpreting multiple correspondence analysis as a multidimensional scaling method*. Marketing Letters, **3**(3) (1992), 259–272.
[12] A. Parchomenko, D. Nelen, J. Gillabel, H. Rechberger, *Measuring the circular economy — A Multiple Correspondence Analysis of 63 metrics*. J. Clean. Prod. **210** (2019), 200–216.
[13] E. Pearson, S. Degotardi, *Education for sustainable development in early childhood education: A global solution to local concerns?*. International Journal of Early Childhood, **41**(2) (2009), 97–111.
[14] U. Bronfenbrenner, *Ecological systems theory*. Annals of Child Development, **6**, (1989), 187–249.
[15] A. Bandura, *Social Learning Theory*, Englewood Cliffs, NJ: Prentice Hall, 1977.
[16] National Academies of Sciences, Engineering and Medicine. Vibrant and Healthy Kids: Aligning Science, Practice, and Policy to Advance Health Equity, Washington DC, The National Academies Press, 2019.

Journal of Mathematical Sciences and Modelling, 3 (3) (2020) 135-138
Research Article

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm
ISSN 2636-8692
DOI: http://dx.doi.org/10.33187/jmsm.774123

Fitting an Epidemiological Model to Transmission Dynamics of COVID-19

Endalew Getnet Tsega

Department of Mathematics, College of Science, Bahir Dar University Bahir Dar, Ethiopia

Article Info

Keywords: *Covid-19, Epidemic, Model parameters, Reported data, SIRD model, Worldwide.*
2010 AMS: *97N40, 93A30, 92D30.*
Received: *26 July 2020*
Accepted: *24 December 2020*
Available online: *29 December 2020*

Abstract

A rapid increase in daily new cases was reported in the world from February 19 to April 3, 2020. In this study, a susceptible-infected-recovered-dead (SIRD) was developed to analyse the dynamics of the global spread of COVID-19 during the above-mentioned period of time. The values of the model parameters fitted the reported data were estimated by minimizing the sum of squared errors using the Levenberg-Marquardt optimization algorithm. A time-dependent infection rate was considered. The set of differential equations in the model was solved using the fourth order Runge-Kutta method. It was observed that a time-dependent parameter gives a better fit to a dynamic data. Based on the fitted model, the average value of basic reproduction number (*R0*) for COVID-19 trasmission was estimated to be 2.8 which shows that the spread of COVID-19 disease in the world was growing exponentially. This may indicate that the control measures implemented worldwide could not decrease the COVID-19 transmission.

1. Introduction

First cases of COVID-19 were reported at the end of 2019 in Wuhan city of China [1]. It has been rapidly spreading in the world and affecting the lives of millions of people, the global economy and educational systems. Governments have been taking drastic measures to stop the transmission of the disease. The control measures include tracing close contacts, promoting social distancing, imposing lockdown, quarantining the infected cases and self-protection using protective equipment. Unfortunately, these measures have not been able to stop the spreading of COVID-19, which caused several outbreaks around the globe [2, 3, 4]. Many studies have been done to understand the dynamics of the COVID-19 disease in different regions or countries to create prevention awareness among people and support healthcare authorities in taking appropriate control measures [3, 4, 5, 6]. Compartmental models are commonly used to describe the spread of infectious disease including COVID-19 [3, 7, 8]. In these models, the population under study is subdivided into a number of compartments based on infection status. The flows from one compartment to another are described by ordinary differential equations (or difference equations) [9, 10, 11]. The parameters in the differential equations are determined by fitting the model to available data using optimization methods [12, 13].

The novel coronavirus (COVID-19) daily new cases was increasing dramatically in the world from February 19 to April 3, 2020. The objective of this study was to develop a compartmental (SIRD) model describing this phenomenon mathematically. An exponential function was intended to characterize the number of infected cases [14]. The data source to the COVID-19 cases for this study was *Worldometer* [15].

2. Mathematical model

Figure 2.1 shows the COVID-19 daily new cases from February 1 to May 10, 2020. The increment was very high from February 19 to April 3, 2020. The cases were 516 on February 19 and 96352 on April 3. The increment was relatively continuous during this time. In this study, a deterministic SIRD model [4, 9] was used to describe the dynamics of COVID-19 spread.
The model divides the world population into four separate compartments: Susceptible (S), Infected (I), Recovered (R) and Dead (D). The underlying assumptions (constant population size and well-mixed population) of SIR model of Kermack and MacKendrick [16] were

Email address and ORCID number**: endalebdumath2016@gmail.com, https://orcid.org/0000-0002-7526-3003 (E. G. Tsega)**

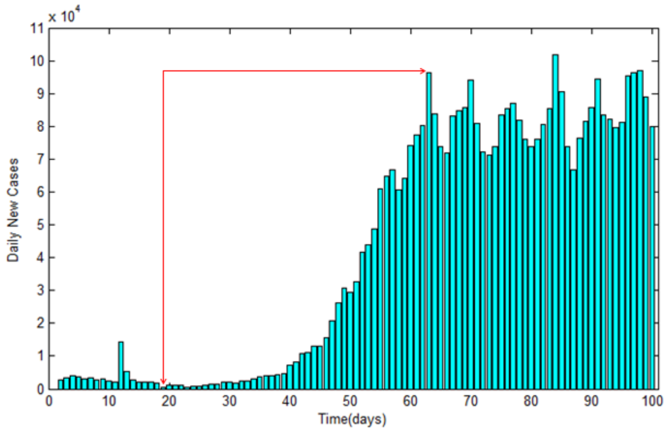

Figure 2.1: COVID-19 daily new cases in the world from February 1 to May 10, 2020.

considered. The control measures being taken were not considered in the modeling. The system of differential equations in SIRD model is given by

$$\frac{dS}{dt} = -\frac{\alpha}{N}SI \tag{2.1}$$

$$\frac{dI}{dt} = \frac{\alpha}{N}SI - \beta I - \gamma I \tag{2.2}$$

$$\frac{dR}{dt} = \beta I \tag{2.3}$$

$$\frac{dD}{dt} = \gamma I \tag{2.4}$$

where the parameters α, β and γ represent the infection rate, recovery rate and death rate respectively. The total confirmed cases T is calculated as $T = I + R + D$. In this model, the parameter α was consider to be time-dependent [3, 4] and expressed by an exponential function as

$$\alpha(t) = A(1 - e^{-kt^n}) \tag{2.5}$$

where A, k and n are parameters to be determined. The basic reproduction number, R_0(the number of persons that an infected number will infect) is estimated from the SIRD model as [17]:

$$R_0 = \frac{\alpha}{\beta + \gamma} \tag{2.6}$$

In epidemiology, the basic reproduction number is considered to be very important to describe the spreading nature of a disease [12].

3. Estimation of model parameters

The model parameters were estimated by minimizing the sum of squared errors [18, 19]

$$SSE(A,k,n,\beta,\gamma) = \|Y - \hat{Y}\| \tag{3.1}$$

where

$$Y = \begin{bmatrix} S(t_1) & I(t_1) & R(t_1) & D(t_1) \\ S(t_2) & I(t_2) & R(t_2) & D(t_2) \\ \vdots & \vdots & \vdots & \vdots \\ S(t_n) & I(t_n) & R(t_n) & D(t_n) \end{bmatrix}$$

is the matrix of the COVID-19 data for compartments,

$$\hat{Y} = \begin{bmatrix} S(\hat{t_1}) & I(\hat{t_1}) & R(\hat{t_1}) & D(\hat{t_1}) \\ S(\hat{t_2}) & I(\hat{t_2}) & R(\hat{t_2}) & D(\hat{t_2}) \\ \vdots & \vdots & \vdots & \vdots \\ S(\hat{t_n}) & I(\hat{t_n}) & hatR(t_n) & D(\hat{t_n}) \end{bmatrix}$$

is the matrix of the corresponding estimates predicted by the model and $\|\|$ is the Euclidean norm. The MATLAB function *ode45* was used to solve the model differential equations in Eq.(2.1)-Eq.(2.4). The MATLAB function *lsqnonlin* with Levenberg-Marquardt algorithm was used for minimization of SSE. Based on *Worldometer*, the world population was taken to be $N = 778359130$. The initial estimates of the parameters A, β and γ were calculated by replacing the derivatives in Eq.(2.1), Eq.(2.3) and Eq.(2.4) by forward difference approximation and employing least squares parameter estimation using respective data and time step size of $\Delta t = 1 day$ [9]. A MATLAB code was developed to estimate the values of all parameters with the given range and to display the results of this study.

4. Results and discussion

Figure 4.1 displays the SIRD model results of COVID-19 in the world from February 19 to April 3, 2020. It compares the actual data with the infected, recovered, dead and total cases predicted by the model as [4, 19]. From the best fit optimization, the values of the model parameters α, β and γ were estimated to be 0.09572, 0.02793 and 0.006268, respectively. The values of the parameters A, k and n used in the exponential model of the infection rate were also obtained as 0.1414, 0.00008087 and 3.4446, respectively. Using Eq.(2.6) and the model parameters α, β and γ, the average value of basic reproduction number ($R0$) was estimated to be 2.8. This indicates that COVID-19 was spreading exponentially in the world population [14]. The model results showed that all the infected, recovered and dead cases were increasing during the period of time. As we can see in Figure 4.1, COVID-19 daily new cases for upcoming 37 days (from April 4 to May 10, 2020), has different distribution from the previous days. Thus, the model cannot be used to forecast the distribution for the upcoming days. The predicted graphs of the infected cases and total confirmed cases are almost similar. We can observe that the resulting SIRD model prediction agreed well with the actual data for all four cases. The relative error of the model fit [18, 20] to the data for total confirmed cases is shown in Figure 4.2. The time-dependent infection rate gave a better fit to the reported COVID-19 data.

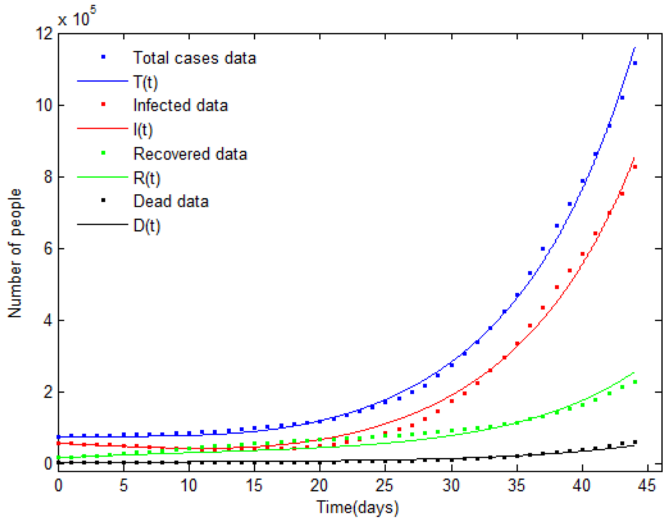

Figure 4.1: The SIRD model fitted to COVID-19 reported data from February 19 to April 3, 2020.

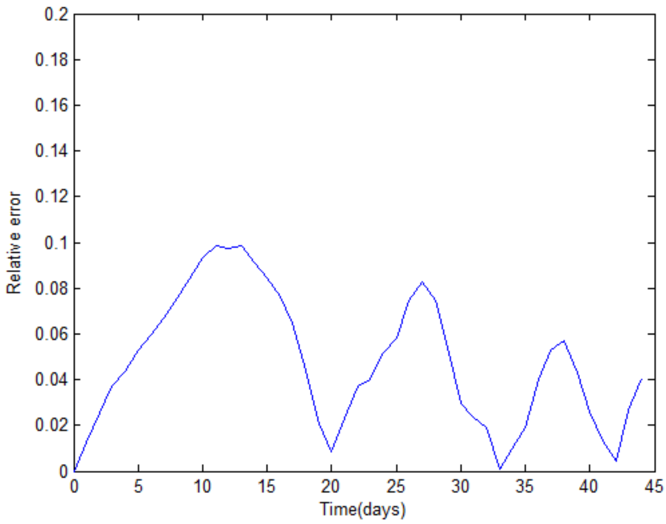

Figure 4.2: Plot of the relative error between the reported data and results of SIRD model for total confirmed cases.

5. Conclusion

In this study, SIRD epidemiological model was fitted to worldwide COVID-19 reported data from February 19 to April 3, 2020. A time-dependent infection rate was considered. Model parameters were estimated using nonlinear least squares fit by minimizing the sum of squared errors. The model results ensured that COVID-19 was spreading exponentially in the world population during this time. The number of infected cases and total confirmed cases showed similar distributions. This study may indicate that the transmission of COVID-19 was not slow down by the control measures implemented globally. The application of mathematics in describing real phenomena may be appreciated.

Conflict of interest statement

The author declares that there is no conflict of interest regarding the publication of this article.

References

[1] World Health Organization,*Coronavirus disease (COVID-19) outbreak*, 2020, available at https://www.who.int/health-topics/Coronavirus.
[2] A. J. Kucharski, T. W. Russell, C. Diamond, Y. Liu, J. Edmunds, S. Funk, R. M. Eggo, F. Sun, M. Jit, J. D. Munday, N. Davies, *Early dynamics of transmission and control of COVID-19: a Mathematical modeling study*, The Iancet infectious diseases, 2020.
[3] A. L. Kapetanovic, D. Poljack, *Modeling the epidemic outbreak and dynamics of COVID-19 in Croatia*, (2020), pp.arXiv-2005.
[4] D. Caccavo,*Chinese and Italian COVID-19 outbreaks can be correctly described by a modified SIRD model*, available at https://doi.org/10.1101/2020.03.19.20039388.
[5] F. Ndairou, I. Area, J. J. Nieto, D.F. Torres,*Mathematical modeling of COVID-19 transmission dynamics with a case study of Wuhan*, Chaos, Solitons Fractals, (2020), p.109846.
[6] R. Sameni, *Mathematical modeling of epidemic diseases; a case study of the COVID-19 coronavirus*, (2020), arXiv:2003.11371.
[7] J. C. Blackwood, L. M. Childs, *An introduction compartmental modeling for the budding infectious disease modeller*, Letters in Biomathematics, **5**(1)(2018), 195-221.
[8] E. L. Piccolomiini, F. Zama, *Monitoring Italian COVID-19 spread by an adaptive SEIRD model*, medRxiv, 2020.
[9] C. Anastassopoulou, L. Russo, A. Tsakris, C. Siettos, *Data-based analysis, modelling and forecasting of the COVID-19 outbreak*, PloS one, **15**(3) (2020), p.e0230405.
[10] P. Khrapov, A. Loginova, *Mathematical modeling of the dynamics of coronavirus COVID-19 epidemic development in China*, Int. J. Open Inf. Technol., **8**(4)(2020), 13-16.
[11] F. Brauer, C. Castillo-Chavez, Z. Feng, *Mathematical Model in, Epidemiology*, Springer, New York, 2019.
[12] M. Y. Li, *An Introduction to Mathematical Modeling Infectious Diseases*, (Vol.2) Cham: Springer, 2018.
[13] M. A. A. Al-qaness, A. A. Ewees, H. Fan, M. E. Abd Elaziz , *Optimization methods for Forecasting confirmed cases of COVID-19 in China*, J. Clinical Medicine, **9**(3) (2020), pp.674.
[14] J. Fernandez-Villaverde, C. I. Jones, *Estimating and Simulating a SIRD Model of COVID-19 for Many Countries, States and Cities(No.w27128)*, National Bureau of Economic Research, 2020.
[15] https://www.worldometers.info/coronavirus/.
[16] W. O. Kermack, A. G. MacKendrick, *A contribution to the mathematical theory of epidemics*, In: Proceedings of the Royal Society of London, vol. A, p. 700-721, 1927.
[17] L. Ferrari, G. Gerardi, G. Manzi, A. Micheletti, F. Nicolussi, S. Salini, *Modeling provisional COVID-19 epidemic data in Italy using an adjusted time-dependent SIRD model*, (2020), arXiv :2005.12170.
[18] E. L. Piccolomiini, F. Zama, *Monitoring Italian COVID-19 spread by an adaptive SEIRD model*, medRxiv, 2020.
[19] A. Ianni, Rossi, N., *Describing the COVID-19 Outbreak Fitting Modified SIR models to Data*, medRxiv, 2020.
[20] G. Chowell, *Fitting dynamic models to epidemic outbreaks with quantified uncertainty: A primer for parameter uncertainty, identifiability, and forecasts. Infectious Disease Modelling*, 2(2017), 379-398.

Journal of Mathematical Sciences and Modelling, 3 (3) (2020) 139-143
Research Article

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm
ISSN 2636-8692
DOI: http://dx.doi.org/10.33187/jmsm.810655

On Bicomplex Jacobsthal-Lucas Numbers

Serpil Halici[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Pamukkale University, Denizli, Turkey*

Article Info

Keywords: *Bicomplex numbers, Jacobsthal sequence, Recurrences.*
2010 AMS: *11B37, 11B39, 11R52.*
Received: *14 October 2020*
Accepted: *28 December 2020*
Available online: *29 December 2020*

Abstract

In this study we introduced a sequence of bicomplex numbers whose coefficients are chosen from the sequence of Jacobsthal-Lucas numbers. We also present some identities about the known some fundamental identities such as the Cassini's, Catalan's and Vajda's identities.

1. Introduction and preliminaries

Quaternionic numbers, defined by Hamilton in 1843, led to the existence of an algebraic structure with all the properties of real and complex numbers, except the property of change of multiplication. There are also many studies on quaternions' coefficients(see, [1],[2],[3]). The recognition and identification of bicomplex numbers was made by James Cockle[4]. Cockle defined a bicomplex number as $b = z_1 + \mathbf{j}z_2$ using the new unit **j**, which Hamilton described, inspired by the definition of quaternions. In 1892, Segre gave different interpretations of this algebra by studying again on bicomplex numbers algebra[5]. In [6], Price, while dealing with the field properties of bicomplex numbers, some authors [7], [8], [9],[10] made some important studies on bicomplex holomorphic functions. In the last years, some studies have been done on bicomplex numbers. In particular, one can refer to the related sources for some studies made using different representations of these numbers [11], [9]. Furthermore, there are some studies on bicomplex numbers and their algebraic, geometric, topological and dynamic properties. For some of these studies, the reader can look at [12], [13], [4],[14], [15]. Recently, the studies conducted by selecting the coefficients of bicomplex numbers from different sequences have attracted attention. Because it is easier to work with these new sequences created using the properties of the selected a sequence, it is also easier to find application areas in other fields. One of these studies, the coefficients by selecting from the Fibonacci sequence and using idempotent notation, by made Halici[11]. Jacobsthal and Jacobsthal-Lucas sequences have a rich history, especially in view of its relationship to the Fibonacci numbers these are studied by some authors. Cerin, in [13], examined the products of Jacobsthal numbers and gave sum of their squares. A. F. Horadam studied the Jacobsthal representation numbers and Fibonacci quaternions [16],[17]. Szynal-Liana and Włoch worked on Jacobsthal quaternions[18]. Diana, in [19], considered Fibonacci octonions and generalized Fibonacci-Lucas octonions. Jacobsthal and Jacobsthal-Lucas sequences, respectively, are defined by the aid of the following recursive relations[20]: For $n \geq 2$

$$J_n = J_{n-1} + 2J_{n-2}; \quad J_0 = 0, \quad J_1 = 1$$

and

$$j_n = j_{n-1} + 2j_{n-2}; \quad j_0 = 2, \quad j_1 = 1.$$

Moreover, these sequences can be also given by the following formulas: For $n \geq 0$

$$J_n = \frac{1}{3}\{2^n - (-1)^n\}, \quad j_n = 2^n + (-1)^n.$$

We listed some important identities used some of them:

$$\sum_{i=1}^{\infty} j_i x^{i-1} = (1+4x)(1-x-2x^2)^{-1}.$$

Email address and ORCID number: **shalici@pau.edu.tr, https://orcid.org/0000-0002-8071-0437 (S. Halıcı)**

$$j_{n+1}+j_n=3.2^n,\ n\geq 0.$$

$$j_{n+1}-j_n=2^n-2(-1)^n,\ n\geq 0.$$

$$j_{n+r}+j_{n-r}=2^{n-r}(2^{2r}+1)+2(-1)^{n-r},\ n\geq r.$$

$$j_{n+r}-j_{n-r}=2^{n-r}(2^{2r}-1),\ n\geq r.$$

$$j_n=J_{n+1}+2J_{n-1}.$$

$$j_nJ_n=J_{2n},\ n\geq 0.$$

$$J_mj_n+J_nj_m=2J_{m+n};\quad J_mj_n-J_nj_m=(-1)^n2^{n+1}J_{m-n}.$$

The set of bicomplex numbers is as follows:

$$BC=\left\{z_1+z_2\mathbf{j}|z_1,\ z_2\in C,\ \mathbf{j}^2=-1\right\}.$$

Here, $\mathbf{i},\mathbf{j}$ are different and commuting imaginary units, $\mathbf{ij}=\mathbf{ji},\mathbf{ii}=\mathbf{jj}=-1$. Hyperbolic unit $\mathbf{k}$ arises from the multiplication of the two imaginary units $\mathbf{i}$ and $\mathbf{j}$; $\mathbf{ij}=\mathbf{k}$. In the set BC, the addition operation is component-wise and the multiplication operation is done taking into account the multiplication of the base elements.

Now, let's define a new bicomplex sequence whose coefficients are selected from Jacobsthal Lucas sequence. If we denote nth bicomplex Jacobsthal number by jBQ_n, then we can write it as follows: For $n\geq 0$

$$jBQ_n=j_n+\mathbf{i}j_{n+1}+\mathbf{j}j_{n+2}+\mathbf{ij}j_{n+3},$$

where j_n is nth Jacobsthal-Lucas number. Let us write the set of such numbers as follows:

$$BC_j=\left\{j_{C_n}+j_{C_{n+2}}\mathbf{j}|j_{C_n}=j_n+\mathbf{i}j_{n+1},j_{C_{n+2}}=j_{n+2}+\mathbf{i}j_{n+3},\mathbf{j}^2=-1\right\}.$$

Then one can write

$$BC_j=\{jBQ_0,jBQ_1,jBQ_2,\ldots,jBQ_n,\ldots\}.$$

Notice that the nth element of this sequence satisfy the following recurrence relation

$$jBQ_n=jBQ_{n-1}+2jBQ_{n-2}.$$

For any two elements jBQ_n,jBQ_m the algebraic operations are as follows:

$$jBQ_n+jBQ_m=(j_n+j_m)+(j_{n+1}+j_{m+1})\mathbf{i}+(j_{n+2}+j_{m+2})\mathbf{j}+(j_{n+3}+j_{m+3})\mathbf{ij},$$

$$jBQ_n-jBQ_m=(j_n-j_m)+(j_{n+1}-j_{m+1})\mathbf{i}+(j_{n+2}-j_{m+2})\mathbf{j}+(j_{n+3}-j_{m+3})\mathbf{ij},$$

$$jBQ_njBQ_m=sc(jBQ_njBQ_m)+vec(jBQ_njBQ_m),$$

where the scalar and vectorial parts are follows.

$$sc(jBQ_njBQ_m)=(j_nj_m-j_{n+1}j_{m+1}-j_{n+2}j_{m+2}+j_{n+3}j_{m+3})$$

$$vec(jBQ_njBQ_m)=(j_nj_{m+1}+j_{n+1}j_m-j_{n+2}j_{m+3}-j_{n+3}j_{m+2})\mathbf{i}$$

$$+(j_nj_{m+2}-j_{n+1}j_{m+3}+j_{n+2}j_m-j_{n+3}j_{m+1})\mathbf{j}$$

$$+(j_nj_{m+3}+j_{n+1}j_{m+2}+j_{n+2}j_{m+1}+j_(n+3)j_m)\mathbf{ij}$$

respectively.

2. Main identities

In this section we provide some fundamental identities such as the Cassini's, Catalan's, Vajda's identities. The elements of the sequence $\{jBQ_n\}_{n\geq 0}$ can be seen as the coefficients of the power series of the corresponding generating function.

Now, let us give the generating function for the bicomplex Jacobsthal-Lucas numbers.

Theorem 2.1. *The generating function for the numbers jBQ_n is*

$$G(t)=\frac{jBQ_0(1-t)+jBQ_1t}{1-t-2t^2}.$$

Proof. Suppose $G(t)$ is a generating function for jBQ_n

$$G(t)=\sum_{i=0}^{\infty} jBQ_i t^i.$$

Multiply this function by t and t^2, respectively, to take advantage of the recursive relation. Thus, we write

$$tG(t)=jBQ_0 t+jBQ_1 t^2+\ldots+jBQ_n t^{n+1}+\ldots.$$

$$t^2G(t)=jBQ_0 t^2+jBQ_1 t^3+\ldots+jBQ_n t^{n+2}+\ldots.$$

Using the characteristic equation and making needed arrangements, we get

$$G(t)(1-t-2t^2)=jBQ_0+t(jBQ_1-jBQ_0),$$

$$G(t)=\sum_{i=0}^{\infty} jBQ_i t^i$$

which is desired result. □

Theorem 2.2. *For integers n,r such that $n\geq r$, we have*

$$i)\ jBQ_n+jBQ_{n+1}=3\underline{\alpha}\alpha^n.$$

$$ii)\ jBQ_{n+r}-jBQ_{n-r}=\underline{\alpha}(2^{n+r}-2^{n-r}).$$

Proof. Using the Binet formula the accuracy of the desired equations can be easily seen. □

Theorem 2.3. *For the bicomplex Jacobsathal-Lucas sequence jBQ_n, the following equality is then provided:*

$$\sum_{s=1}^{n} jBQ_s=\frac{1}{2}(jBQ_{n+2}-jBQ_2).$$

Proof. In order to prove the claim we will use the following equalities:

$$\{j_n\}_{n\geq 0}=\{2,1,5,7,17,31,65,127,257,511,\ldots\}$$

and

$$\sum_{i=1}^{n} j_i=\frac{j_{n+2}-5}{2}.$$

So, we have

$$\sum_{s=1}^{n} jBQ_s=\sum_{s=1}^{n} j_s+\mathbf{i}\sum_{s=1}^{n} j_{s+1}+\mathbf{j}\sum_{s=1}^{n} j_{s+2}+\mathbf{ij}\sum_{s=1}^{n} j_{s+3}.$$

This implies that

$$\sum_{s=1}^{n} jBQ_s=\frac{1}{2}\left\{(j_{n+2}+\mathbf{i}j_{n+3}+\mathbf{j}j_{n+4}+\mathbf{ij}j_{n+5})-(5+7\mathbf{i}+17\mathbf{j}+31\mathbf{ij})\right\}.$$

This proves our result. That is,

$$\sum_{s=1}^{n} jBQ_s=\frac{1}{2}(jBQ_{n+2}-jBQ_2).$$

The following formula gives any element of the sequence BC_j and this formula is known as the Binet formula. □

Theorem 2.4. *(Binet's Formula) For every positive integer n, the following equality holds.*

$$jBQ_n=\underline{\alpha}\alpha^n+\underline{\beta}\beta^n$$

where

$$\underline{\alpha}=(1+2\boldsymbol{i}+4\boldsymbol{j}+8\boldsymbol{ij})\ \text{and}\ \underline{\beta}=(1-\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{ij}).$$

Proof. The general term of the sequence $\{jBQ_n\}_{n\geq 0}$ is $jBQ_n=A\alpha^n+B\beta^n$. Using the roots α,β of the characteristic equation associated with the recurrence $jBQ_n=jBQ_{n-1}+2jBQ_{n-2}$ we get the values A and B as follows:

$$A=\frac{jBQ_1-jBQ_0\beta}{3},\quad B=\frac{-jBQ_1+jBQ_0\alpha}{3}.$$

Writing these values in the equation $jBQ_n=A\alpha^n+B\beta^n$, we have

$$jBQ_n=\frac{jBQ_1-jBQ_0\beta}{3}\alpha^n+\frac{-jBQ_1+jBQ_0\alpha}{3}\beta^n,$$

$$jBQ_n=(1+2\mathbf{i}+4\mathbf{j}+8\mathbf{ij})\alpha^n+(1-\mathbf{i}+\mathbf{j}-\mathbf{ij})\beta^n$$

which is desired. □

Theorem 2.5. *For the integers n,m such that $n \geq m$, Catalan's identity is follows:*

$$jBQ_{n+m}jBQ_{n-m} - jBQ_n^2 = \underline{\alpha}\underline{\beta}(\alpha\beta)^{n-m}\left\{(\alpha\beta)^{2m} - 2(\alpha\beta)^m + 1\right\}.$$

Proof. From the Binet formula, we write

$$jBQ_{n+m}jBQ_{n-m} - jBQ_n^2 = (\underline{\alpha}\alpha^{n+m} + \underline{\beta}\beta^{n+m})(\underline{\alpha}\alpha^{n-m} + \underline{\beta}\beta^{n-m}) - (\underline{\alpha}\alpha^n + \underline{\beta}\beta^n)^2.$$

If we use the fact $\underline{\alpha}\underline{\beta} = \underline{\beta}\underline{\alpha}$, then we get

$$jBQ_{n+m}jBQ_{n-m} - jBQ_n^2 = \underline{\alpha}\underline{\beta}(\alpha^{n+m}\beta^{n-m} + \alpha^{n-m}\beta^{n+m} - 2\alpha^n\beta^n).$$

$$jBQ_{n+m}jBQ_{n-m} - jBQ_n^2 = \underline{\alpha}\underline{\beta}(\alpha^n\beta^n)\left\{\frac{\alpha^m}{\beta^m - 1} + \frac{\beta^m}{\alpha^m - 1}\right\}.$$

Hence, after the some calculations we obtain the following equality which is desired result:

$$jBQ_{n+m}jBQ_{n-m} - jBQ_n^2 = \underline{\alpha}\underline{\beta}(\alpha\beta)^{n-m}\left\{(\alpha\beta)^{2m} - 2(\alpha\beta)^m + 1\right\}.$$

□

Theorem 2.6. *For $n \geq 1$, Cassini's identity is follows:*

$$jBQ_{n-1}jBQ_{n+1} - jBQ_n^2 = 9\underline{\alpha}\underline{\beta}(\alpha\beta)^{n-1}.$$

Proof. One can see that $\underline{\alpha}\underline{\beta} = \underline{\beta}\underline{\alpha}$. Also, by considering the multiplication rules of the base elements we write

$$jBQ_{n-1}jBQ_{n+1} - (jBQ_n)^2 = \underline{\alpha}\underline{\beta}(\alpha^{n-1}\beta^{n+1} + \alpha^{n+1}\beta^{n-1} - 2\alpha^n\beta^n).$$

By making the necessary adjustments and calculations, we obtain

$$jBQ_{n-1}jBQ_{n+1} - jBQ_n^2 = 9\underline{\alpha}\underline{\beta}(\alpha\beta)^{n-1}.$$

Thus, we completed the proof. □

Notice that Theorem 2.6 is an immediate consequence of Theorem 2.5. In this case, $m = 1$, Catalan's formula induces the Cassini's formula:

$$jBQ_{n-1}jBQ_{n+1} - jBQ_n^2 = 9\underline{\alpha}\underline{\beta}(\alpha\beta)^{n-1}.$$

Theorem 2.7. *(d'Ocagne Identity). For the elements jBQ_n, we have*

$$jBQ_m jBQ_{n+1} - jBQ_n jBQ_{m+1} = 3\underline{\alpha}\underline{\beta}(\alpha^n\beta^m - \alpha^m\beta^n).$$

Proof. From the Binet's formula, the equality

$$jBQ_m jBQ_{n+1} - jBQ_n jBQ_{m+1}$$

is equal to this:

$$\underline{\alpha}\underline{\beta}\left\{\alpha^m\beta^n(\beta - \alpha) + \alpha^n\beta^m(\alpha - \beta)\right\}.$$

Hence, we get

$$jBQ_m jBQ_{n+1} - jBQ_n jBQ_{m+1} = 3\underline{\alpha}\underline{\beta}(\alpha^n\beta^m - \alpha^m\beta^n).$$

□

3. Conclusion

In this study, we introduced the bicomplex number sequence whose coefficients are chosen from the Jacobsthal-Lucas sequence. We gave algebraic properties of the elements of this sequence and obtained their Binet formula. We also obtained the well known some generalized identities related with these numbers such as Cassini and Catalan identities. It should be note that due their applications in different areas, the study of these numbers can give other properties and applications.

References

[1] S. Halici, *On fibonacci quaternions*. Adv. Appl. Clifford Algebr., **22**(2) (2012), 321-327.
[2] S. Halici, *On Complex Fibonacci Quaternions*. Adv. Appl. Clifford Algebr., **23**(1) (2013), 105-112.
[3] M. Akyigit, H. H. Kösal, M. Tosun, *Fibonacci generalized quaternions*, Adv. Appl. Clifford Algebr., **24**(3) (2014), 631-641.
[4] J. Cockle, *LII. On systems of algebra involving more than one imaginary; and on equations of the fifth degree*. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, **35**(238) (1849), 434-437.
[5] C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici*, Math. Ann., **40** (1892), 413-467.
[6] G. B. Price, *An Introduction to Multicomplex Spaces and Functions, Monographs and Textbooks in Pure and Applied Mathematics*, 140, Marcel Dekker, Inc., New York, 1991.
[7] G. Dragoni, *Sulle funzioni olomorfe di una variable bicomplessa*, Reale Acad.d'Italia Mem. Class Sci. Fic. Mat.Nat., **5** (1934),597-665.
[8] M. Futagawa, *On the theory of functions of quaternary variable-I*, Tohoku Math. J., **29** (1928), 175-222.
[9] A. A. Pogorui, R. M. Rodriguez-Dagnino, *On the set of zeros of bicomplex polynomials*, Complex Var. Elliptic Equ., **51** 7 (2006), 725-730.
[10] J. Ryan, *Complexified Clifford analysis*, Complex Var. Elliptic Equ., **1** (1982), 119-149.
[11] S. Halici, *On Bicomplex Fibonacci Numbers and Their Generalization*, Models and Theories in Soc. Syst., Springer Nature, Studies in Systems, Decision and Control Series, 2019.
[12] M.E. Luna-Elizarraras, E. M. Shapiro, D. C. Struppa, A. Vajiac *Bicomplex numbers and their elementary functions*, Cubo (Temuco), 14(2) (2012), 61-80.
[13] Z. Cerin, *Sums of squares and products of Jacobsthal numbers*, J. Integer Seq., **10**(07.2) (2007), 5.
[14] D. Rochon, M. Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers*, An. Univ. Oradea Fasc. Mat., **11** (2004),71-110.
[15] S. Halici, Ş. Curuk, *On Some Matrix Representations of Bicomplex Numbers*, Konuralp Journal of Mathematics, **7**(2), (2019), 449-455.
[16] A. F. Horadam, *Jacobsthal representation numbers*, Significance, **2** (1996), 2-8.
[17] A. F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, Amer. Math. Monthly, **70**(3) (1963), 289-291.
[18] A. Szynal-Liana, I. Włoch, *A note on jacobsthal quaternions*, Adv. Appl. Clifford Algebr., **26**(1) (2016), 441-447.
[19] D. Savin, *Some properties of Fibonacci numbers, Fibonacci octonions, and generalized Fibonacci-Lucas octonions*, Adv. Difference Equ., **2015**(1) (2015), 298.
[20] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Vol 1, John Wiley and Sons, 2001.